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Θ-Completeness for Classifying Stacks

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Abstract

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We provide a criterion for the classifying stack BG to be Θ-complete over an algebraically closed field of characteristic 0. It is known that the classifying stack BG is Θ-complete when G is a unipotent algebraic group or reductive group (hence the product of a unipotent group and a reductive group). We prove that this is in fact a necessary condition. That is the Θ-completeness of the classifying stack BG implies that G must be the trivial product of a unipotent group and a reductive group.

DEDICATION

to my family

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Abstract

We provide a criterion for the classifying stack $\mathbf{B}\mathsf{G}$ to be Θ-complete over an algebraically closed field of characteristic 0. It is known that the classifying stack $\mathbf{B}\mathsf{G}$ is Θ-complete when G is a unipotent algebraic group or reductive group (hence the product of a unipotent group and a reductive group). We prove that this is in fact a necessary condition. That is the Θ-completeness of the classifying stack $\mathbf{B}\mathsf{G}$ implies that G must be the trivial product of a unipotent group and a reductive group.

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1 Introduction

Moduli problems are a fundamental concept in algebraic geometry, which is a branch of mathematics that studies geometric objects defined by polynomial equations. Moduli problems are concerned with the classification and parameterization of geometric objects.

Given a moduli problem described by an algebraic stack $\mathcal{X}$, it is known that if automorphism groups are trivial, then $\mathcal{X}$ is representable by an algebraic space (or a scheme.) Hence, $\mathcal{X}$ admits a *fine moduli space.* The assumption that the automorphism groups are trivial is important because without it $\mathcal{X}$ will never be representable by an algebraic space. However, it is shown in [KM97] that an algebraic stack $\mathcal{X}$ with finite inertia and in particular finite automorphism groups admits a *coarse moduli space.*

In [Alp13], the notion of *good moduli spaces* is introduced. By definition, a quasi-compact morphism $\phi : \mathcal{X} \to X$ from an algebraic stack to an algebraic space is a *good moduli space* if the push-forward functor on quasi-coherent sheaves is exact and the induced morphism on sheaves $\mathcal{O}_X \to \phi_* \mathcal{O}_{\mathcal{X}}$ is an isomorphism.

In [AHLH22], the notion of S-complete and Θ-complete morphisms of algebraic stacks is introduced as a part of a general criterion for the existence of *good moduli space.* A special case of [AHLH22, Theorem A] is that for an algebraic group G over a field k of characteristic 0, the classifying stack $\mathbf{B}G$ admits a separated good moduli space if and only if $\mathbf{B}G$ is Θ-complete and S-complete. With our result (Theorem A) and [Alp22, Proposition 6.7.48], we can conclude that the classifying stack $\mathbf{B}G$ admits a good moduli space if and only if G is a reductive group.

It is also known that for an algebraic group G over a field, the classifying stack $\mathbf{B}G$ is S-complete if and only if G is reductive (see [Alp22, Proposition 6.7.48].) There is an application of this fact in the proof of [ABHLX20, Theorem 1.3], which states that if (X, D) is a K-polystable log Fano pair, then $\operatorname{Aut}(X, D)$ is reductive.

Our main result gives a necessary and sufficient condition under which the classifying stack $\mathbf{B}G$ is Θ-complete.

Theorem A (Theorem 7.1)**.** *Let G be a connected affine algebraic group over an algebraically closed field k of characteristic 0. Then the classifying stack $\mathbf{B}G$ is Θ-complete if and only if $G \cong U \times R$ for some unipotent group U and reductive group R.*

We sketch the main ideas in the proof of Theorem A. By Mostow's Theorem, any connected algebraic group G is isomorphic to a semi-direct product of a unipotent group U and a reductive group R defined by an action morphism $\sigma : R \times U \to U$. This implies that Theorem A is equivalent to the statement that for any semi-direct product $G = U \rtimes_\sigma R$ of a unipotent group U and a reductive group R, the classifying stack $\mathbf{B}G$ is Θ-complete if and only if the product is trivial, i.e., σ is the trivial action.

If the product is trivial, then it suffices to show that $\mathbf{B}U$ and $\mathbf{B}R$ are Θ-complete. As the product of Θ-complete stacks is Θ-complete and $\mathbf{B}U \times \mathbf{B}R \cong \mathbf{B}G$, Θ-completeness of $\mathbf{B}U$ and $\mathbf{B}R$ implies that of $\mathbf{B}G$. Proposition 6.14 (resp. Proposition 6.11) shows that if an algebraic group H is unipotent (resp. reductive), then the classifying stack $\mathbf{B}H$ is Θ-complete.

The proof of the other direction consists of two parts. We start by assuming that the reductive group R is $\mathbb{G}_m$. If the product is not trivial, we show that there exists a one-parameter subgroup $\lambda : \mathbb{G}_m \to \mathbb{G}_m$ such that $G_\lambda^+ \subseteq G$ and the quotient G/G_λ^+ is not projective. By Proposition 6.13, $\mathbf{B}G$ is not Θ-complete.

To prove the case when R is any reductive group, we realize that there exists a one-parameter subgroup $\lambda : \mathbb{G}_m \to R$ such that the restricted action $\tau : \mathbb{G}_m \times U \xrightarrow{\lambda \times \mathrm{id}_U} R \times U \xrightarrow{\sigma} U$ is not trivial. Such λ exists because the R-action on U is not trivial. With $G' := U \rtimes_\tau \mathbb{G}_m$, the classifying stack $\mathbf{B}G'$ is not Θ-reductive. In fact, the induced morphism $\mathbf{B}G' \to \mathbf{B}G$ of classifying stacks is affine. Since the pullback of a Θ-complete stack via an affine morphism is a Θ-complete stack (Proposition 6.6), the classifying stack $\mathbf{B}G$ is not Θ-complete.

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2 Algebraic Groups

The field k is always assumed to be algebraically closed and of characteristic 0.

2.1 Group Schemes and Algebraic Groups

Definition 2.1. A *group scheme* (G, μ) *over a scheme* S is a morphism $\pi : G \to S$ of schemes and a multiplication morphism $\mu : G \times_S G \to G$ of S-schemes such that there exist an identity morphism $e : S \to G$ and an inverse morphism $\iota : G \to G$ of S-schemes such that the following diagrams commute:

$$\begin{tikzcd} G \times_S G \times_S G \arrow[r, "\mathrm{id}_G \times \mu"] \arrow[d, "\mu \times \mathrm{id}_G"'] & G \times_S G \arrow[d, "\mu"] \\ G \times_S G \arrow[r, "\mu"] & G \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[r, "(e \circ \pi, \mathrm{id}_G)"] \arrow[d, "(\mathrm{id}_G, e \circ \pi)"'] \arrow[dr, "\mathrm{id}_G"] & G \times_S G \arrow[d, "\mu"] \\ G \times_S G \arrow[r, "\mu"] & G \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[r, "(\mathrm{id}_G, \iota)"] \arrow[d, "(\iota, \mathrm{id}_G)"'] \arrow[dr, "e \circ \pi"] & G \times_S G \arrow[d, "\mu"] \\ G \times_S G \arrow[r, "\mu"] & G. \end{tikzcd}$$

Associativity Law of identity Law of inverse

For any $g_1, g_2 \in G$, we often write $g_1 \cdot g_2$ (or even $g_1 g_2$) to denote $\mu(g_1, g_2)$ and g_1^{-1} to denote $\iota(g_1)$.
For group schemes (H, μ_H) and (G, μ_G), a *morphism of group schemes* is a morphism $\phi : H \to G$ of S-schemes such that the following diagram commutes:

$$\begin{tikzcd} H \times_S H \arrow[r, "\mu_H"] \arrow[d, "\phi \times \phi"'] & H \arrow[d, "\phi"] \\ G \times_S G \arrow[r, "\mu_G"] & G. \end{tikzcd}$$

Remark 2.2. Let G be a scheme over a scheme S. There exists a morphism $\mu : G \times_S G \to G$ of S-schemes such that (G, μ) is a group scheme over S if and only if the functor $h_G : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Sets}$ factors through the forgetful functor $\mathrm{Grps} \hookrightarrow \mathrm{Sets}$. To see this, suppose $\mu : G \times_S G \to G$ is a morphism of S-schemes such that (G, μ) is a group scheme over S. Let $e : S \to G$ and $\iota : G \to G$ be an identity morphism and an inverse morphism. These morphisms induce morphisms $h_e : h_S \to h_G$, $h_\iota : h_G \to h_G$, and $h_\mu : h_G \times h_G \to h_G$ of functors. (Note that $h_G \times h_G$ is canonically isomorphic to $h_{G \times_S G}$.) The associativity, μ induces a morphism of functors $h_{G \times_S G} \to h_G$. As $h_G \times h_G$ is canonically isomorphic to $h_{G \times_S G}$, we have a morphism $h_\mu : h_G \times h_G \to h_G$ of functors. The commutativity of the three diagrams in Definition 2.1 implies that $h_G : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Sets}$ is a group-valued functor, i.e., it factors through the forgetful functor $\mathrm{Grps} \hookrightarrow \mathrm{Sets}$.

On the other hand, suppose $h_G : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Sets}$ factors through the forgetful functor $\mathrm{Grps} \hookrightarrow \mathrm{Sets}$. This implies that there is a multiplication morphism $F : h_G \times h_G \to h_G$ of functors. Consider morphisms $\bar{e} : h_S \to h_G$ and $\bar{\iota} : h_G \to h_G$ such that $\bar{e}_X(*)$ is the identity of $h_G(X)$ and $\bar{\iota}_X(g) = g^{-1}$ for any

S-scheme X and $g \in h_G(X)$. (Here $*$ denotes the only element in $h_S(X) = \{*\}$.) Hence, the following diagrams of functors commute:

$$\begin{tikzcd}
h_G \times h_G \times h_G \arrow[r, "\mathrm{id}\times F"] \arrow[d, "F\times \mathrm{id}"'] & h_G \times h_G \arrow[d, "F"] \\
h_G \times h_G \arrow[r, "F"] & h_G
\end{tikzcd}
\qquad
\begin{tikzcd}
h_G \arrow[r, "(\bar{e}\circ h_\pi, \mathrm{id})"] \arrow[d, "(\mathrm{id}, \bar{e}\circ h_\pi)"'] \arrow[dr, "\mathrm{id}"] & h_G \times h_G \arrow[d, "F"] \\
h_G \times h_G \arrow[r, "F"] & h_G
\end{tikzcd}
\qquad
\begin{tikzcd}
h_G \arrow[r, "(\mathrm{id}, \bar{\iota})"] \arrow[d, "(\bar{\iota}, \mathrm{id})"'] \arrow[dr, "\bar{e}\circ h_\pi"] & h_G \times h_G \arrow[d, "F"] \\
h_G \times h_G \arrow[r, "F"] & h_G
\end{tikzcd}$$

Associativity Law of identity Law of inverse

As $h_G \times h_G$ is canonically isomorphic to $h_{G\times_S G}$, we have a morphism $\bar{F} : h_{G\times_S G} \to h_G$ of functors (the composition of the canonical isomorphism and F). By Yoneda Lemma, the functors $\bar{F}, \bar{e}$, and $\bar{\iota}$ correspond to morphisms $\mu : G \times_S G \to G$, $e : S \to G$, and $\iota : G \to G$ of S-schemes. In particular, the commutativity of the three diagrams above implies that (G, μ) is a group scheme over S with e and ι as an identity morphism and an inverse morphism respectively.

Remark 2.3. To define a group scheme structure on G, it is equivalent to define a functor $F : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Grps}$ such that the following diagram of categories commutes:

$$\begin{tikzcd}
(\mathrm{Sch}/S)^{\mathrm{op}} \arrow[r, "F"] \arrow[dr, "h_G"'] & \mathrm{Grps} \arrow[d, hook] \\
& \mathrm{Sets}
\end{tikzcd}$$

(See Remark 2.2)

Remark 2.4. Let (G, μ) be a group scheme over S. An identity morphism $e : S \to G$ and an inverse morphism $\iota : G \to G$ are unique. The morphisms e and ι are in one-to-one correspondence with $\bar{e} : h_S \to h_G$ and $\bar{\iota} : h_G \to h_G$ defining the identity of an abstract group and the inverse of an element in an abstract group (Remark 2.2). As the identity and the inverse of an abstract group are unique, so are e and ι.

Definition 2.5. An *algebraic group over a field* k is an affine group scheme (G, μ) of finite type over k. A morphism $\phi : (H, \mu_H) \to (G, \mu_G)$ of k-schemes is a *morphism of algebraic groups* if it is a morphism of group schemes. An algebraic group is connected if it is connected as a scheme.

Remark 2.6. Let $(\pi : G \to \mathrm{Spec}\,(k), \mu)$ be an algebraic group over a field k. Let $A = \Gamma(G, \mathcal{O}_G)$. The multiplication morphism μ, the identity morphism e and the inverse morphism ι induce the co-multiplication morphism $\mu^* : A \to A \otimes_R A$, the co-identity morphism $e^* : A \to R$ and the co-inverse morphism $\iota^* : A \to A$ of k-algebras. In particular, the commutative diagrams defining the group structure of (G, μ) translate to the following commutative diagrams of k-algebras:

$$\begin{tikzcd}
A \otimes_k A \otimes_k A & A \otimes_k A \arrow[l, "\mathrm{id}_A \otimes \mu^*"'] \\
A \otimes_k A \arrow[u, "\mu^* \otimes \mathrm{id}_A"] & A \arrow[l, "\mu^*"] \arrow[u, "\mu^*"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A & A \otimes_k A \arrow[l, "(\pi^*\circ e^*)\cdot(\mathrm{id}_A)"'] \\
A \otimes_k A \arrow[u, "(\mathrm{id}_A)\cdot(\pi^*\circ e^*)"] & A \arrow[l, "\mu^*"] \arrow[u, "\mu^*"'] \arrow[ul, "\mathrm{id}_A"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A & A \otimes_k A \arrow[l, "(\mathrm{id}_A)\cdot(\iota^*)"'] \\
A \otimes_k A \arrow[u, "(\iota^*)\cdot(\mathrm{id}_A)"] & A \arrow[l, "\mu^*"] \arrow[u, "\mu^*"'] \arrow[ul, "\pi^*\circ e^*"']
\end{tikzcd}$$

In fact, any morphisms μ^*, e^*, and ι^* of k-algebras making the above three diagrams commute induce the group structure on $\pi : G \to \mathrm{Spec}\,(k)$.

Definition 2.7. A *Hopf algebra* over a field k is a k-algebra A together with a morphism $\mu^* : A \to A \otimes_k A$ such that the following diagrams of k-algebras commute:

$$\begin{tikzcd}
A \otimes_k A \otimes_k A & & A \otimes_k A \arrow[ll, "\mathrm{id}_A \otimes \mu^*"'] \\
A \otimes_k A \arrow[u, "\mu^* \otimes \mathrm{id}_A"] & & A \arrow[ll, "\mu^*"] \arrow[u, "\mu^*"']
\end{tikzcd}$$

Co-associativity

$$\begin{tikzcd}
A & & A \otimes_k A \arrow[ll, "(\pi^* \circ e^*) \cdot (\mathrm{id}_A)"'] \\
A \otimes_k A \arrow[u, "(\mathrm{id}_A) \cdot (\pi^* \circ e^*)"] & & A \arrow[ll, "\mu^*"] \arrow[u, "\mu^*"'] \arrow[llu, "\mathrm{id}_A"']
\end{tikzcd}$$

Law of co-identity

$$\begin{tikzcd}
A & & A \otimes_k A \arrow[ll, "(\mathrm{id}_A) \cdot (\iota^*)"'] \\
A \otimes_k A \arrow[u, "(\iota^*) \cdot (\mathrm{id}_A)"] & & A. \arrow[ll, "\mu^*"] \arrow[u, "\mu^*"'] \arrow[llu, "\pi^* \circ e^*"']
\end{tikzcd}$$

Law of co-inverse

For Hopf algebras (A, μ_A^*) and (B, μ_B^*) over k, a *morphism of Hopf algebras* is a morphism $\phi : A \to B$ of k-algebras such that the following diagram of k-algebras commutes:

$$\begin{tikzcd}
A \arrow[r, "\phi"] \arrow[d, "\mu_A^*"] & B \arrow[d, "\mu_B^*"] \\
A \otimes A \arrow[r, "\phi \otimes \phi"'] & B \otimes B.
\end{tikzcd}$$

Example 2.8. Let k be a field.

- The *multiplicative group over* k is $\mathbb{G}_m = \mathrm{Spec}\,(k[t]_t)$ with the multiplication morphism $\mu : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ sending $(a, b) \mapsto ab$. The co-multiplication morphism $\mu^* : k[t]_t \to k[t]_t \otimes k[t']_{t'}$ sends $t \mapsto t \otimes t'$. $\mathbb{G}_m$ can also be thought of functorially. For any k-scheme X, $h_{\mathbb{G}_m}(X) = \Gamma(X, \mathcal{O}_X)^\times$ and the multiplication morphism $\Gamma(X, \mathcal{O}_X)^\times \times \Gamma(X, \mathcal{O}_X)^\times \to \Gamma(X, \mathcal{O}_X)^\times$ sends $(r, s) \mapsto rs$.

- The *additive group over* k is $\mathbb{G}_a = \mathrm{Spec}\,(k[t])$ with the multiplication morphism $\mu : \mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a$ sending $(a, b) \mapsto a+b$. The co-multiplication morphism is $\mu^* : k[t] \to k[t] \otimes k[t']$ sends $t \mapsto t \otimes 1 + 1 \otimes 1'$. $\mathbb{G}_a$ can also be thought of functorially. For any k-scheme X, $h_{\mathbb{G}_a}(X) = \Gamma(X, \mathcal{O}_X)$ and the multiplication morphism $\Gamma(X, \mathcal{O}_X) \times \Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathcal{O}_X)$ sends $(r, s) \mapsto r + s$.

- The *general linear group* is $\mathrm{GL}_n = \mathrm{Spec}\,\big(k[x_{ij}]_{\det(x_{ij})}\big)$ with the multiplication morphism $\mu : \mathrm{GL}_n \times \mathrm{GL}_n \to \mathrm{GL}_n$ sending $(A, B) \mapsto AB$. The co-multiplication morphism $\mu^* : k[x_{ij}]_{\det(x_{ij})} \to k[x_{ij}]_{\det(x_{ij})} \otimes k[x'_{ij}]_{\det(x'_{ij})}$ sends $x_{ij} \mapsto \sum_k x_{ik} \otimes x'_{kj}$. GL_n can also be thought of functorially. For any k-scheme X, $\mathrm{GL}_n(X) = \Gamma(X, \mathcal{O}_X)[x_{ij}]_{\det(x_{ij})}$ and the multiplication morphism $\Gamma(X, \mathcal{O}_X)[x_{ij}]_{\det(x_{ij})} \times \Gamma(X, \mathcal{O}_X)[x_{ij}]_{\det(x_{ij})} \to \Gamma(X, \mathcal{O}_X)[x_{ij}]_{\det(x_{ij})}$ sends $(R, S) \mapsto RS$.

Remark 2.9. In Definition 2.5 of algebraic groups, they are assumed to be affine. Hence, an algebraic group is separated as a scheme.

Proposition 2.10. *Every algebraic group is smooth.*

Proof. See [Mil17, Theorem 3.23]. Note that $\mathrm{char}(k) = 0$. $\square$

Remark 2.11. If $\mathrm{char}(k) > 0$, then an algebraic group is not necessarily smooth, e.g., $G = \mathrm{Spec}\,(k[x]/(x^p))$ for $\mathrm{char}(k) = p$.

2.2 Group Actions and Representations

Definition 2.12. Let $(\pi : G \to \mathrm{Spec}\,(k)\,, \mu)$ be an algebraic group over a field k and $e : \mathrm{Spec}\,(k) \to G$ be the identity morphism. An *action* of G on a k-scheme $p : X \to \mathrm{Spec}\,(k)$ is a morphism $\sigma : G \times_k X \to X$ of k-schemes such that the following diagrams commute:

$$\begin{tikzcd}
G \times_k G \times_k X \arrow[r, "\mathrm{id}_G \times \sigma"] \arrow[d, "\mu \times \mathrm{id}_G"'] & G \times_k X \arrow[d, "\sigma"] \\
G \times_k X \arrow[r, "\sigma"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "(e \circ p, \mathrm{id}_X)"] \arrow[dr, "\mathrm{id}_X"'] & G \times_k X \arrow[d, "\sigma"] \\
& X
\end{tikzcd}$$

Compatibility

Law of identity

For any $g \in G$, $x \in X$, we often write $g \cdot x$ (or even gx) to denote $\sigma(g, x)$.
The morphism σ is often called *the action morphism.* The action is trivial if σ is simply the projection onto X.
Suppose G acts on k-schems X and Y via the action morphisms σ_X and σ_Y respectively. A morphism $f : X \to Y$ of k-schemes is G-*equivalent* if the following diagram commutes:

$$\begin{tikzcd}
G \times_k X \arrow[r, "\sigma_X"] \arrow[d, "\mathrm{id}_G \times f"'] & X \arrow[d, "f"] \\
G \times_k Y \arrow[r, "\sigma_Y"] & Y
\end{tikzcd}$$

In addition, if G acts trivially on Y, then f is G-*invariant.*

Example 2.13. Let (G, μ) be an algebraic group over a field k.

- G acts on itself by multiplication, i.e., μ is the action morphism.
- G acts on itself by conjugation. Precisely, $\sigma : G \times G \to G$ sends $(g, x) \mapsto gxg^{-1}$.

Remark 2.14. Let $\sigma : G \times X \to X$ be a group action on algebraic group G on a scheme X. A homomorphism $\lambda : H \to G$ of algebraic groups induces a group action of H on X. Explicitly, the action morphism $H \times X \to X$ is

$$H \times X \xrightarrow{\lambda \times \mathrm{id}_X} G \times X \xrightarrow{\sigma} X.$$

Example 2.15. Let $(H, \mu_H) \subseteq (G, \mu_G)$ be a subgroup of an algebraic group G. From Example 2.13 and Remark 2.14,there are group actions of H on G.

(a) For any $h \in H$ and $g \in G$, $h \cdot g = hg$. The action morphism of this action is
$$H \times G \xrightarrow{i \times \mathrm{id}_G} G \times G \xrightarrow{\mu_G} G,$$
where $i : H \hookrightarrow G$ is an inclusion and $\mu_G : G \times G \to G$ is the multiplication morphism.

(b) For any $h \in H$ and $g \in G$, $h \cdot g = hgh^{-1}$.

Remark 2.16. Let G be an algebraic group over a field k and X be a k-scheme. A similar statement to Remark 2.2 can also be made about group actions. To define an action of G on X, it is equivalent to define an action of the functor h_G on the functor h_X. By Yoneda lemma, there is a one-to-one correspondence between morphisms $\sigma : G \times_k X \to X$ of k-schemes and morphisms $h_{G \times_k X} \to h_X$ of functors. By universal property of the fiber product, the functor $h_{G \times_k X}$ is canonically isomorphic to the functor $h_G \times h_X$. Lastly, the commutativity of the two diagrams in Definition 2.12 defining the action of G on X make the morphism $h_G \times h_X \to h_X$ of functors an action morphism associated to an action of h_G on h_X. Hence, there is a natural bijection from the set of actions of G on X to the set of actions of the functor h_G on the functor h_X:

$$\{\text{actions of } G \text{ on } X\} \xrightarrow{\sim} \{\text{actions of the functor } h_G \text{ on the functor } h_X\}$$

Remark 2.17. Let k be a field. Suppose an algebraic group $(G = \mathrm{Spec}\,(A), \mu)$ acts on an affine scheme $X = \mathrm{Spec}\,(R)$. The action morphism σ induces the *co-action* morphism $\rho : R \to R \otimes A$ of k-algebras. In particular, the commutative diagrams defining the action of G on X translate to the following commutative diagrams of k-algebras:

$$\begin{tikzcd}
R \otimes_k A \otimes_k A & R \otimes_k A \arrow[l, "\rho \otimes \mathrm{id}_A"'] \\
R \otimes_k A \arrow[u, "\mathrm{id}_R \otimes \mu^*"] & R \arrow[l, "\rho"] \arrow[u, "\rho"']
\end{tikzcd}
\qquad
\begin{tikzcd}
R & R \otimes_k A \arrow[l, "\mathrm{id}_R \otimes (p^* \circ e^*)"'] \\
 & R \arrow[u, "\rho"'] \arrow[ul, "\mathrm{id}_R"]
\end{tikzcd},$$

Compatibility Law of identity

where $e^* : A \to k$ is the co-identity morphism and $p^* : k \to R$ is the structure morphism. In fact, any morphisms $\rho : R \to R \otimes A$ of k-algebras making the above diagrams commute induces a group action of G on X.

Example 2.18 (The standard action of $\mathbb{G}_m$ on $\mathbb{A}^1$)**.** Consider a morphism $\rho : k[x] \to k[x] \otimes k[t]_t$ sending $x \mapsto x \otimes t$. This is a co-action map of the action of $\mathbb{G}_m = \mathrm{Spec}\,(k[t]_t)$ on $\mathbb{A}^1 = \mathrm{Spec}\,(k[x])$. We say that $\mathbb{G}_m$ acts on $\mathbb{A}^1$ by multiplication.

Remark 2.19. A $\mathbb{G}_m$-action on an affine k-scheme X gives a $\mathbb{Z}$-grading on $\mathcal{O}(X)$. Let $\mathbb{G}_m$ act on $X = \mathrm{Spec}\,(A)$ with the coaction morphism $\rho : A \to A \otimes_R \mathcal{O}(\mathbb{G}_m) = A \otimes k[t]_t$. Consider $A_i := \{a \in A : \rho(a) = a \otimes t^i\}$ for $i \in \mathbb{Z}$. This gives a morphism $A_i \hookrightarrow A$ of vector spaces over k. Hence, there is an induced morphism $\phi : \bigoplus_{i\in\mathbb{Z}} A_i \to A$ of vector spaces over k. As $\cap_{i\in\mathbb{Z}} A_i = 0$, ϕ is injective. The surjectivity of ϕ can be shown in two steps. Firstly, for $a \in A$, we can write $\rho(a) = \sum_{i\in\mathbb{Z}} a_i \otimes t^i$ (finite sum). By Law of identity of the $\mathbb{G}_m$-action, we can show that $a = \sum_{i\in\mathbb{Z}} a_i$. Secondly, we show that $a_i \in A_i$. This follows immediately from the compatibility of the $\mathbb{G}_m$-action. Therefore, $A \cong \bigoplus_{\mathbb{Z}} A_i$ as vector spaces over k, where $A_i := \left\{a \in A : \rho(a) = a \otimes t^i\right\}$.

Definition 2.20. Let G and H be algebraic groups over a field k such that G acts on H by the action morphism $\sigma : G \times H \to H$. The *semi-direct product* of G and H defined by σ is the functor $G \rtimes_\sigma H : (\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{Grps}$ sending $\mathrm{Spec}\,(R)$ to the semi-direct product $G(R) \rtimes_{\sigma_R} H(R)$ as abstract groups.

Remark 2.21. The semi-direct product of abstract groups has a natural group structure. Thus, the functor $G \rtimes_\sigma H$ defined in 2.20 is a group functor. In particular, as G and H are affine schemes, the underlying scheme $G \rtimes_\sigma H$ is also affine. Hence, $G \rtimes_\sigma H$ is an algebraic group.

Remark 2.22. Suppose that an algebraic group R acts on an algebraic group U and $\sigma : R \times U \to U$ is the action morphism. Let $G := U \rtimes_\sigma R$ be the semi-direct product. Consider the action of R on G by conjugation. Namely, for $s \in R$ and $(u, r) \in G$,

$$s \cdot (u, r) = (1, s)(u, r)(1, s)^{-1} = (\sigma(s, u), srs^{-1}).$$

If $r = 1$, then

$$s \cdot (u, 1) = (\sigma(s, u), 1).$$

This means that the data of the group action $\sigma : R \times U \to U$ is the same as that of the conjugation action of R on G restricted to U.

Definition 2.23. Let k be a field and V be a vector space over k. The functor $GL_V : (\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{Grps}$ is defined as

$$X \mapsto \mathrm{Aut}_{\Gamma(X,\mathcal{O}_X)-\mathrm{linear}}\left(V \otimes_R \Gamma(X, \mathcal{O}_X)\right).$$

Remark 2.24. Let k be a field and V be a finite-dimensional vector space over k. The functor GL_V is isomorphic to GL_n, where $n = \dim V$. (Note that GL_n is thought as the functor represented by $\mathrm{Spec}\left(k[x_{ij}]_{\det(x_{ij})}\right)$.) To see this, let (e_i) be a basis for V. We consider a morphism $\Phi : GL_n \to GL_V$ as follows: for any k-scheme X,

$$\Phi_X : GL_n(X) \to GL_V(X) \quad (f \mapsto (e_i \otimes 1 \mapsto \sum_j e_j \otimes f^*(x_{ji}))).$$

We will see that this morphism is an isomorphism. First, we note that

$$GL_V(X) := \mathrm{Aut}\left(V \otimes \Gamma(X, \mathcal{O}_X)\right) \subseteq \mathrm{End}\left(V \otimes \Gamma(X, \mathcal{O}_X)\right)$$

which is the set of $\Gamma(X, \mathcal{O}_X)$-linear endomorphisms of $V \otimes \Gamma(X, \mathcal{O}_X)$. Such endomorphism is determined by the image of $e_i \otimes 1$ for $i = 1, \ldots, n$. Also note that a morphism $f \in GL_n(X) := \mathrm{Hom}_k\left(X, \mathrm{Spec}\left(k[x_{ij}]_{\det(x_{ij})}\right)\right)$ is in one-to-one correspondence with a morphism $f^* : k[x_{ij}]_{\det(x_{ij})} \to \Gamma(X, \mathcal{O}_X)$ of k-algebras. Thus, $\det(f^*(x_{ij})) = f^*(\det(x_{ij})) \in \Gamma(X, \mathcal{O}_X)^{\times}$. Hence, the image of f via Φ_X is an automorphism of $V \otimes \Gamma(X, \mathcal{O}_X)$. In fact, Φ_X is injective because $\Phi_X(f_1) = \Phi_X(f_2)$ if and only if $f_1^*(x_{ij}) = f_2^*(x_{ij})$ for all $i, j = 1, \ldots, n$, which is true if and only if $f_1 = f_2$. Lastly, Φ_X is surjective. Indeed, any automorphism of $V \otimes \Gamma(X, \mathcal{O}_X)$ sending $e_i \mapsto \sum_j e_j \otimes a_{ji}$ is the image under Φ_X of the morphism $f^* : k[x_{ij}]_{\det(x_{ij})} \to \Gamma(X, \mathcal{O}_X)$ sending $x_{ij} \mapsto a_{ij}$. Thus, Φ is an isomorphism of functors.

It is also worth to note that if V is finite-dimensional, the isomorphism $\Phi : GL_n \to GL_V$ of functors also preserves the algebraic group structure. In particular, GL_V is an algebraic group.

Definition 2.25. Let (G, μ) be an algebraic group over a field k. A *representation* (V, r) *of* G is a vector space V over k together with a morphism $r : h_G \to GL_V$ of group functors.

We say that a subspace $W \subseteq V$ is *stable under* G if $r_X(g)(W \otimes \Gamma(X, \mathcal{O}_X)) \subseteq W \otimes \Gamma(X, \mathcal{O}_X)$ for every k-scheme X and $g \in h_G(X)$. Thus, r induces a group homomorphism $r|_W : h_G \to GL_W$ of functors. In particular, $(W, r|_W)$ is a representation of G. We then say that W is a *subrepresentation* of (V, r).

Example 2.26. Let k be a field and V be a vector space over k. Write $\mathbb{G}_m = \operatorname{Spec}(k[t]_t)$. Consider a $k[t]_t$-linear automorphism of $V \otimes k[t]_t$ sending $v \otimes 1 \mapsto v \otimes t$. By Yoneda Lemma, this gives a morphism $r : \mathbb{G}_m \to GL_V$ of group functors. Hence, (V, r) is a representation of G.

Example 2.27. Let $(G = \operatorname{Spec}(A), \mu)$ be an algebraic group over a field k. Considering A as a k-vector space, a morphism $\phi : A \otimes_k A \to A \otimes_k A$ sending $a \otimes b \mapsto b\mu^*(a)$ is an A-linear automorphism of $A \otimes A$. In particular, $\phi \in GL_A(G)$. By Yoneda lemma, let $r : h_G \to GL_A$ be a morphism of functors corresponding to ϕ. As μ^* is the co-multiplication morphism, r is a group homomorphism. In fact, (A, r) is a representation of G. This representation (A, r) is called the *regular representation.*

Definition 2.28. Let k be a field and V be a vector space over k. We define the functor $V_a : (\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{Sets}$ as

$$X \mapsto V \otimes_k \Gamma(X, \mathcal{O}_X)$$

Remark 2.29. Let (G, μ) be an algebraic group over a field k. For a vector space over k, there is a bijection between representations (V, r) of G and actions of the functor h_G on the functor V_a. Suppose (V, r) is a representation of G. For a k-scheme X, we have a morphism $h_G(X) \times V_a(X) \to V_a(X)$ sending $(g, v) \mapsto r_X(g)(v)$ for $g \in h_G(X)$ and $v \in V_a(X)$. This gives an action of h_G on V_a.

On the other hand, let $\sigma : h_G \times V_a \to V_a$ be an action morphism of functors. We define a morphism $r : h_G \to GL_V$ of functors as follows: for any k-scheme X, the morphism $r_X : h_G(X) \to GL_V(X)$ sends

$$g \mapsto (v \mapsto \sigma(g, v))$$

As σ is an action morphism, one can check that r is a group homomorphism. Thus, there is a bijection between representations (V, r) of G and actions of the functor h_G on the functor V_a:

$$\{\text{representations } (V, r) \text{ of } G\} \xrightarrow{\sim} \{\text{actions of the functor } h_G \text{ on the functor } V_a\}$$

Definition 2.30. Let (V, r) be a representation of an algebraic group G. For an algebraic subgroup H of G, we define the *subspace fixed by* H, denoted V^H, to be $\{v \in V : h \cdot v = v \text{ for } h \in H\}$.

Remark 2.31. Let G be an algebraic group over k and N be its normal subgroup, i.e., $h_N(X)$ is a normal subgroup of $h_G(X)$ for every k-scheme X. For any representation (V, r) of G, the subspace V^N is stable under G. To see this, consider a k-scheme X, a vector $v \in V^N \otimes_k \Gamma(X, \mathcal{O}_X)$, and $g \in h_G(X)$. For any $h \in h_N(X)$, we know that

$$g^{-1}hg \in h_N(X).$$

Hence,

$$(g^{-1}hg) \cdot v = v \quad \Rightarrow \quad h \cdot (g \cdot v) = (g \cdot v).$$

Therefore, $g \cdot v \in V^N \otimes_k \Gamma(X, \mathcal{O}_X)$.

Remark 2.32. Let k be a field and V be a finite-dimensional vector space over k. The functor V_a is representable by the k-scheme $\operatorname{Spec}\left(\operatorname{Sym}_k^*\left(V^\vee\right)\right)$. From Remark 2.16 and 2.29, we can conclude that there is a bijection between representations (V, r) of G and actions of G on $\operatorname{Spec}\left(\operatorname{Sym}_k^*\left(V^\vee\right)\right)$:

$$\{\text{representations } (V, r) \text{ of } G\} \xrightarrow{\sim} \left\{\text{actions of } G \text{ on } \operatorname{Spec}\left(\operatorname{Sym}_k^*\left(V^\vee\right)\right)\right\}$$

Definition 2.33. Let $(G = \operatorname{Spec}(A), \mu)$ be an algebraic group over a field k. An A-*comodule* (V, ρ) is a k-vector space V together with a morphism $\rho : V \to V \otimes A$ of k-vector spaces such that the following diagrams of k-vector spaces commute:

$$\begin{tikzcd}
V \otimes_k A \otimes_k A & V \otimes_k A \arrow[l, "\rho \otimes \mathrm{id}_A"'] & & V & V \otimes_k A \arrow[l, "\mathrm{id}_V \otimes e^*"'] \\
V \otimes_k A \arrow[u, "\mathrm{id}_R \otimes \mu^*"] & V \arrow[l, "\rho"] \arrow[u, "\rho"'] & & & V \arrow[ul, "\mathrm{id}_V"] \arrow[u, "\rho"'] ,
\end{tikzcd}$$

where $e^* : A \to k$ is the co-identity morphism.
The morphism ρ is often called the *co-action* morphism.

A subspace $W \subseteq V$ is an A-*subcomodule of* V if $\rho(W) \subseteq W \otimes_k A$.

Remark 2.34. Let k be a field and V be a vector space over k. For an algebraic group $(G = \operatorname{Spec}(A), \mu)$, there is a ono-to-one correspondence between representations (V, r) and A-comodules (V, ρ). Suppose $r : G \to GL_V$ is a morphism of functors. By Yoneda lemma, let $\phi \in GL_V(G)$ be an element corresponding to r. Since $GL_V(G) = \operatorname{Aut}_{A-\text{linear}}(V \otimes A)$, the morphism ϕ is an A-linear automorphism of $V \otimes A$. In particular, ϕ is uniquely determined by its restriction to a k-linear morphism $\rho : V \to V \otimes A$ which sends $v \mapsto \phi(v \otimes 1)$. Reality check shows that r is a group homomorphism if and only if ρ makes the diagrams in Definition 2.33 commute (See [Mil17, Chapter 4a].) Therefore, there is a bijection between representations (V, r) of G and A-comdules (V, ρ):

$$\{\text{representations } (V, r) \text{ of } G\} \xrightarrow{\sim} \{A\text{-comodules}(V, \rho)\}$$

Example 2.35. Let $(G = \operatorname{Spec}(A), \mu)$ be an algebraic group over a field k. The regular representation (Example 2.27) corresponds in the sense of Remark 2.34 to the A-comodule (A, μ^*).

Proposition 2.36. *Let* (V, r) *be a representation of an algebraic group* $(G = \operatorname{Spec}(A), \mu)$ *over a field* k*. Suppose* $\rho : V \to V \otimes A$ *be the corresponding co-action morphism. A subspace* $W \subseteq V$ *is an* A*-subcomodule if and only if* W *is stable under* G*.*

Proof. Let W be a subspace of V.
($\Rightarrow$) Assume that W is an A-subcomodule of V. Hence, we have the commutative diagram of k-vector spaces:

$$\begin{tikzcd}
V \arrow[r, "\rho"] & V \otimes_k A \\
W \arrow[u, hook] \arrow[r, "\rho_{|W}"'] & W \otimes_k A \arrow[u, hook]
\end{tikzcd}$$

This implies that the following diagram of A-modules commute:

$$\begin{tikzcd}
V \otimes_k A \arrow[r, "r_G(\mathrm{id}_G)"] & V \otimes_k A \\
W \otimes_k A \arrow[u, hook] \arrow[r, "r_G(\mathrm{id}_G)_{|(W \otimes_k A)}"'] & W \otimes_k A \arrow[u, hook]
\end{tikzcd}$$

In particular, for any k-scheme X, $g \in h_G(X)$, we have the following commutative diagram:

$$\begin{tikzcd}
 & V \otimes_k A \arrow[rr, "r_G(\mathrm{id}_G)"] \arrow[dl, "\mathrm{id}_V \otimes g^*"'] & & V \otimes_k A \arrow[dl, "\mathrm{id}_V \otimes g^*"] \\
V \otimes_k \Gamma(X, \mathcal{O}_X) \arrow[rr, "r_X(g)"] & & V \otimes_k \Gamma(X, \mathcal{O}_X) & \\
 & W \otimes_k A \arrow[uu, hook] \arrow[rr, "r_G(\mathrm{id}_G)|_{(W \otimes_k A)}"'] \arrow[dl, "\mathrm{id}_W \otimes g^*"'] & & W \otimes_k A \arrow[uu, hook] \arrow[dl, "\mathrm{id}_W \otimes g^*"] \\
W \otimes_k \Gamma(X, \mathcal{O}_X) \arrow[uu, hook] \arrow[rr, "r_X(g)|_{(W \otimes_k \Gamma(X, \mathcal{O}_X))}"'] & & W \otimes_k \Gamma(X, \mathcal{O}_X) \arrow[uu, hook] &
\end{tikzcd}$$

Hence, W is stable under G.
($\Leftarrow$) Assume that W is stable under G. This implies that for any k-scheme X and $g \in h_G(X)$, the following diagram of $\Gamma(X, \mathcal{O}_X)$-modules commutes:

$$\begin{tikzcd}
V \otimes_k \Gamma(X, \mathcal{O}_X) \arrow[r, "r_X(g)"] & V \otimes_k \Gamma(X, \mathcal{O}_X) \\
W \otimes_k \Gamma(X, \mathcal{O}_X) \arrow[u, hook] \arrow[r, "r_X(g)_{|(W \otimes_k \Gamma(X, \mathcal{O}_X))}"'] & W \otimes_k \Gamma(X, \mathcal{O}_X) \arrow[u, hook]
\end{tikzcd}$$

By letting $X = G$ and $g = \mathrm{id}_G$, we obtain the commutative diagram of A-modules

$$\begin{tikzcd}
V \otimes_k A \arrow[r, "r_G(\mathrm{id}_G)"] & V \otimes_k A \\
W \otimes_k A \arrow[u, hook] \arrow[r, "r_G(\mathrm{id}_G)_{|(W \otimes_k A)}"'] & W \otimes_k A \arrow[u, hook]
\end{tikzcd} .$$

This means that $\rho(W) \subseteq W \otimes_k A$. Hence, W is an A-subcomodule. ☐

Proposition 2.37. *Let* $(G = \mathrm{Spec}\,(A)\,, \mu)$ *be an algebraic group over a field* k. *Every* A*-comodule* (V, ρ) *is a union of its finite-dimensional* A*-subcomodules.*

Proof. It suffices to show that for $v \in V$, v is in a finite-dimensional A-subcomodule. Let $\{e_i\}_{i \in I}$ be a basis for A as a k-vector space. Write

$$\rho(v) = \sum_i v_i \otimes e_i$$

for some $v_i \in V$. Note that the sum is finite. Write $\mu^*(e_i) = \sum_{j,k} a_{ijk}(e_j \otimes e_k)$ for some $a_{ijk} \in k$. Since (V, ρ) is an A-comodule, there is a commutative diagram

$$\begin{tikzcd}
V \otimes_k A \otimes_k A & V \otimes_k A \arrow[l, "\rho \otimes \mathrm{id}_A"'] \\
V \otimes_k A \arrow[u, "\mathrm{id}_R \otimes \mu^*"] & V \arrow[l, "\rho"] \arrow[u, "\rho"']
\end{tikzcd}$$

Thus,

$$\begin{aligned}\sum_k \rho(v_k)\otimes e_k &= (\rho\otimes \mathrm{id}_A)(\sum_k v_k\otimes e_k)\\ &= (\rho\otimes \mathrm{id}_A)(\rho(v))\\ &= (\mathrm{id}_V\otimes \mu^*)(\rho(v)) \qquad \text{(commutativity of the diagram)}\\ &= (\mathrm{id}_V\otimes \mu^*)\left(\sum_i v_i\otimes e_i\right)\\ &= \sum_i v_i\otimes\left(\sum_{j,k} a_{ijk}(e_j\otimes e_k)\right)\\ &= \sum_{i,j,k} a_{i,j,k}(v_i\otimes e_j\otimes e_k)\\ &= \sum_k\left(\sum_{i,j} a_{i,j,k}v_i\otimes e_j\right)\otimes e_k.\end{aligned}$$

Comparing coefficients of $1\otimes 1\otimes e_k$, we have

$$\rho(v_k)=\sum_{i,j} a_{i,j,k}(v_i\otimes e_j)$$

Let W be a subspace spanned by v and v_k. Hence, $\rho(W)\subseteq W\otimes A$. In particular, W is an A-subcomodule containing v. ☐

Corollary 2.38. *Let* $(G=\mathrm{Spec}\,(A),\mu)$ *be an algebraic group over a field* k. *Every representation of* G *is a union of its finite-dimensional subrepresentations.*

Proof. Let (V,r) be a representation of G and $\rho: V\to V\otimes A$ be the corresponding co-action morphism. By Proposition 2.37, V is a union of finite-dimensional A-subcomodules. By Proposition 2.36, V is a union of finite-dimensional subrepresentations. ☐

Proposition 2.39. *Every algebraic group is isomorphic to a closed algebraic subgroup of* GL_n *for some* n.

Proof. Let $(G=\mathrm{Spec}\,(A),\mu)$ be an algebraic group over a field k. Consider the regular representation (Example 2.27). Since A is finitely generated as k-algebras, there is a finite set of generators for A. By Proposition 2.37, there is a finite-dimensional A-subcomodule V containing the generators of A. Let (e_i) be a basis of V as a vector space over k. By Remark 2.24, there is a group homomorphism $\mathrm{GL}_V\cong\mathrm{GL}_n$. Thus, there exists a group homomorphism $r: h_G\to\mathrm{GL}_V\cong\mathrm{GL}_n$ of functors, where $h_G\to\mathrm{GL}_V$ corresponds to the comodule (V,μ^*). Write $\mu^*(e_i)=\sum_j e_j\otimes a_{ji}$ for $a_{ji}\in A$. By Yoneda Lemma, $r: h_G\to\mathrm{GL}_n$ corresponds to a morphism $f: G\to\mathrm{Spec}\left(k[x_{ij}]_{\det(x_{ij})}\right)$ of algebraic groups such that $f^*(x_{ij})=a_{ij}$. To show that G is isomorphic to a closed algebraic subgroup of GL_n, it suffices to show that f^* is surjective. Indeed, let $e^* : A \to k$ be the co-identiy morphism of G. As μ^* is a co-multiplication morphism, we have

$$e_i = ((e^* \otimes \mathrm{id}_A) \circ \mu^*)(e_i) = (e^* \otimes \mathrm{id}_A)\left(\sum_j e_j \otimes a_{ji}\right) = \sum_j e^*(e_j) a_{ji}.$$

This means that V is contained in the image of f^* and so is A. In particular, f^* is a surjective morphism of k-algebras and G is isomorphic to a closed algebraic subgroup of GL_n. ☐

Corollary 2.40. *Let G be an algebraic group. There exists a faithful finite-dimensional representation (V, r) of G.*

Proof. By Proposition 2.39, G is isomorphic to a closed algebraic subgroup of GL_n. Let $r : G \to \mathsf{GL}_n$ be the isomorphism and V be the vector space generated by vectors $v_i \in k^n$, where $v_i = [0 \cdots 1 \cdots 0]^T$ (1 in the i^{th}-entry.) Hence, (V, r) is a faithful finite-dimensional representation. ☐

Definition 2.41. A representation (V, r) of an algebraic group G is called *simple* if only subrepersentations of V are 0 and itself.

Proposition 2.42. *Let (V, r) be a representation of an algebraic group G. If $V = \sum_i V_i$, where each V_i is a simple subrepresentation, then the sum is direct, i.e,*

$$V = \bigoplus_i V_i.$$

Proof. Assume that $V = \sum_i V_i$, where each V_i is a simple subrepresentation. Let V_i and V_j be any simple subrepresentations. If there exists nonzero $v \in V_i \cap V_j$, then Gv is a nonzero subrepresentation of both V_i and V_j. Since V_i and V_j are simple, $V_i = V_j$. Thus, $V_i \cap V_j = 0$ or $V_i = V_j$. In particular,

$$V = \bigoplus_i V_i.$$

☐

Remark 2.43. Every one-dimensional representation of an algebraic group is simple.

3 $\mathbb{G}_m$-actions and One-parameter Subgroups

3.1 The Multiplicative Group $\mathbb{G}_m$

Let k be an algebraically closed field of characteristic 0. Recall that the multiplicative group, $(\mathbb{G}_m, \mu)$, is Spec $(k[t]_t)$ with the multiplication morphism $\mu : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ sending $(a, b) \mapsto ab$ for $a, b \in \mathbb{G}_m(k)$. If $(k[t]_t, m)$ is the corresponding Hopf algebra, then $m : k[t]_t \to k[t]_t \otimes k[s]_s$ sends $t \mapsto t \otimes s$.

For an integer $n \in \mathbb{Z}$, there is a group homomorphism $\phi : \mathbb{G}_m \to \mathbb{G}_m$ sending $a \mapsto a^n$ for $a \in \mathbb{G}_m(k)$. In fact, every group endomorphism of $\mathbb{G}_m$ arises from some integer.

Proposition 3.1. *If* $\phi : \mathbb{G}_m \to \mathbb{G}_m$ *is a group homomorphism, then there exists an integer* $n \in \mathbb{Z}$ *such that* $\phi(a) = a^n$ *for every* $a \in \mathbb{G}_m(k)$.

Proof. Let $(k[t]_t, m)$ be the corresponding Hopf algebra of $\mathbb{G}_m$. Hence, there is a commutative diagram of k-algebras

$$\begin{tikzcd}
k[t]_t \arrow[r, "\phi^*"] \arrow[d, "m"] & k[t]_t \arrow[d, "m"] \\
k[t]_t \otimes k[s]_s \arrow[r, "\phi^* \otimes \phi^*"'] & k[t]_t \otimes k[s]_s
\end{tikzcd}$$

If $\phi^*(t) = \sum_{i \in \mathbb{Z}} a_i t^i$ (finite sum) for $a_i \in k$, then the above diagram implies that

$$\sum_{i,j \in \mathbb{Z}} a_i a_j (t^i \otimes s^j) = \sum_{i,j \in \mathbb{Z}} a_i^2 (t^i \otimes s^i).$$

Hence,

$$a_i^2 = a_i \quad \text{and} \quad a_i a_j = 0 \quad \text{for } i \neq j.$$

As $\phi^*(t)$ must be a unit, there exists an integer n such that $a_n \neq 0$. In particular, $a_n = 1$ and $a_j = 0$ for $j \neq n$. This shows that $\phi^*(t) = t^n$. Hence, $\phi(a) = a^n$ for every $a \in \mathbb{G}_m(k)$. ☐

For any integer $n \in \mathbb{Z}$, we let $\phi_n : \mathbb{G}_m \to \mathbb{G}_m$ be a group homomorphism such that $\phi(a) = a^n$ for $a \in \mathbb{G}_m(k)$. If $n_1, n_2 \in \mathbb{Z}$, then we see that the group homomorphism $\phi_{n_1+n_2} : \mathbb{G}_m \to \mathbb{G}_m$ is the same as the composition $\mu \circ (\phi_{n_1}, \phi_{n_2})$, i.e., there is a commutative diagram of algebraic groups

$$\begin{tikzcd}
\mathbb{G}_m \arrow[r, "(\phi_{n_1}, \phi_{n_2})"] \arrow[dr, "\phi_{n_1+n_2}"'] & \mathbb{G}_m \times \mathbb{G}_m \arrow[d, "\mu"] \\
 & \mathbb{G}_m
\end{tikzcd}$$

Thus, there is a natural group structure on $\mathrm{Hom}_k(\mathbb{G}_m, \mathbb{G}_m)$. Explicitly, for $\phi, \psi \in \mathrm{Hom}_k(\mathbb{G}_m, \mathbb{G}_m)$, the group operation of $\mathrm{Hom}_k(\mathbb{G}_m, \mathbb{G}_m)$ is defined as $\phi \cdot \psi := \mu \circ (\phi, \psi)$. This suggests that $\mathrm{Hom}_k(\mathbb{G}_m, \mathbb{G}_m)$ is isomorphic to $(\mathbb{Z}, +)$ as groups.

Proposition 3.2. $\mathrm{Hom}_k(\mathbb{G}_m, \mathbb{G}_m)$ *is isomorphic to* $(\mathbb{Z}, +)$ *as groups.*

Proof. For any integer $n \in \mathbb{Z}$, let $\phi_n : \mathbb{G}_m \to \mathbb{G}_m$ be a group homomorphism sending $a \mapsto a^n$ for $a \in \mathbb{G}_m(k)$. Consider a morphism $\Phi : \mathbb{Z} \to \mathrm{Hom}_k(\mathbb{G}_m, \mathbb{G}_m)$ sending $n \mapsto \phi_n$. Reality check shows that Φ is a group homomorphism. The surjectivity of Φ follows from Proposition 3.1. To see the injectivity of Φ, suppose $\phi_{n_1} = \phi_{n_2}$ for some integers n_1, n_2. This means $a^{n_1 - n_2} = 1$ for all $a \in \mathbb{G}_m(k)$. Thus, $n_1 = n_2$. In particular, Φ is a group isomorphism. $\square$

The identity element of $\mathrm{Hom}_k(\mathbb{G}_m, \mathbb{G}_m)$ is the trivial morphism, ϕ_0. The inverse element of ϕ_n is ϕ_{-n}.

Definition 3.3. Let (V, r) be a representation of $\mathbb{G}_m$ and $\rho : V \to V \otimes k[t]_t$ be the corresponding coaction morphism. For $i \in \mathbb{Z}$,

$$V_i := \{v \in V : \rho(v) = v \otimes t^i\}.$$

It is easy to see that V_i is a subspace of V. Indeed, for $v, w \in V_i$ and $a \in k$, we have

$$\rho(v + w) = \rho(v) + \rho(w) = v \otimes t^i + w \otimes t^i = (v + w) \otimes t^i,$$

and

$$\rho(av) = a\rho(v) = a(v \otimes t^i) = av \otimes t^i.$$

Lastly, $0 \in V_i$. Thus, V_i is a subspace of V.

Proposition 3.4. *Let* (V, r) *be a representation of* $\mathbb{G}_m$ *and* $\rho : V \to V \otimes k[t]_t$ *be the corresponding coaction morphism. Then,*

$$V = \bigoplus_{i \in \mathbb{Z}} V_i.$$

Proof. Let $v \in V$. Write $\rho(v) = \sum_i v_i \otimes t^i$. As (V, ρ) is a $k[t]_t$-module, we have

$$(\mathrm{id}_V \otimes \Delta) \circ \rho = (\rho \otimes \mathrm{id}_{\mathcal{O}(\mathbb{G}_m)}) \circ \rho \quad \text{and} \quad \mathrm{id}_V = (\mathrm{id}_V \otimes \epsilon) \circ \rho,$$

where δ is the co-multiplication morphism and ϵ is the co-identity morphism. The first equation implies that

$$\sum v_i \otimes t^i \otimes t^i = \sum \rho(v_i) \otimes t^i.$$

Hence, $\rho(v_i) = v_i \otimes t^i$ and so $v_i \in V_i$. The other equation implies that

$$v = (\mathrm{id}_V \otimes \epsilon) \left(\sum_i v_i \otimes t^i \right) = \sum_i v_i.$$

Thus, $V = \sum_i V_i$. To see that the sum is direct, let $w \in V_i \cap V_j$, where i and j are different integers. Hence,

$$w \otimes t^i = \rho(w) = w \otimes t^j.$$

This is not true unless $w = 0$. Thus,

$$V = \bigoplus_i V_i.$$

□

3.2 $\mathbb{G}_m$-actions

Definition 3.5. Let X be a scheme of finite type over k equipped with an action of $\mathbb{G}_m$ defined by an action morphism $\sigma : \mathbb{G}_m \times X \to X$. Define functors $X^0_\sigma, X^+_\sigma : (\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{Sets}$:

$$\begin{aligned} X^0_\sigma &:= \underline{\mathrm{Hom}}^{\mathbb{G}_m}_k(\mathrm{Spec}\,(k), X) \\ X^+_\sigma &:= \underline{\mathrm{Hom}}^{\mathbb{G}_m}_k(\mathbb{A}^1, X) \end{aligned}$$

Note that $\mathbb{G}_m$ acts on $\mathbb{A}^1$ by multiplication (weight 1.) If there is no confusion on the $\mathbb{G}_m$-action of X, we write X^0 and X^+ instead.

Remark 3.6. Let X be a separated k-scheme of finite type equipped with a $\mathbb{G}_m$-action $\sigma : \mathbb{G}_m \times X \to X$. Consider the $\mathbb{G}_m$-action on $\mathbb{A}^1 \setminus 0$ by left multiplication, i.e., $t \cdot a = ta$, where $t \in \mathbb{G}_m(k)$ and $a \in \mathbb{A}^1 \setminus 0$.

For any point $x \in X(k)$, there is a unique $\mathbb{G}_m$-equivalent morphism $f_x : \mathbb{A}^1 \setminus 0 \to X$, where $1 \mapsto x$. We say that $\lim_{t\to 0} t \cdot x$ *(or* $\lim_{t\to 0} \sigma(t, x)$*) exists* if the morphism f_x can be extended to a $\mathbb{G}_m$-equivalent morphism $f : \mathbb{A}^1 \to X$, i.e, the following diagram of k-schemes commutes:

$$\begin{tikzcd} \mathbb{A}^1 \arrow[r, dashed, "\exists f"] & X \\ \mathbb{A}^1 \setminus 0 \arrow[u, hook] \arrow[ur, "f_x"'] & \end{tikzcd}$$

Since X is separated, the extended morphism is unique if exists.

Consider another $\mathbb{G}_m$-action on X where $t \cdot x := \sigma(t^{-1}, x)$ for $t \in \mathbb{G}_m$ and $x \in X$. Let $\tau : \mathbb{G}_m \times X \to X$ be the corresponding action morphism. We say that $\lim_{t\to\infty} t \cdot x$ *(or* $\lim_{t\to\infty} \sigma(t, x)$*) exists* if $\lim_{t\to 0} \tau(t, x)$ exists. We will also define the functor $X^-_\sigma := X^+_\tau$.

Definition 3.7. An action of a group scheme G on a scheme X over a field k is **locally affine** if X admits a covering by G-invariant open affine subschemes.

Remark 3.8. Not every group action is locally affine. Consider $\mathbb{P}^1$ with a non-trivial action of $\mathbb{G}_m$. Let Y be the nodal cubic, i.e., the scheme obtained by gluing the origin and the point at infinity of $\mathbb{P}^1$ together. Any open neighborhood U of the glued point contains a nonzero point. Hence, for U to be $\mathbb{G}_m$-invariant, it must contain the whole scheme Y. However, Y is not affine. Thus, this $\mathbb{G}_m$-action on Y is not locally affine. By Sumihiro's theorem [Sum74], every normal scheme of finite type equipped with an action of $\mathbb{G}_m$ admits a covering by $\mathbb{G}_m$-invariant open affine subschemes.

Remark 3.9. Recall that $X \cong \underline{\mathrm{Hom}}_k(\mathrm{Spec}(k), X)$ as functors. Hence, there is a morphism of functors $X^0 \to X$, where the action of $\mathbb{G}_m$ is forgotten.

Remark 3.10. Consider the morphism of schemes $\mathrm{Spec}(k) \to \mathrm{Spec}(k[x]) = \mathbb{A}^1$, where $x \mapsto 1$ on the algebras. Precomposing any $\mathbb{G}_m$-morphism from $\mathbb{A}^1$ to X with this morphism gives a morphism of functors $i_1 : X^+ \to X$.

Remark 3.11. Consider that the morphism of schemes $\mathrm{Spec}(k) \to \mathrm{Spec}(k[x]) = \mathbb{A}^1$, where $x \mapsto 0$ on the algebras. This morphism is $\mathbb{G}_m$-equivalent. In particular, (pre)composing any $\mathbb{G}_m$-morphism from $\mathbb{A}^1$ to X with this morphism gives a morphism of functors $i_0 : X^+ \to X^0$.

Example 3.12. Consider $X = \mathbb{A}^1$ equipped with the standard action of $\mathbb{G}_m$. Then, $X^0 = \mathrm{Spec}(k)$ and $X^+ = \mathbb{A}^1$. The morphism $X^0 \to X$ is the inclusion of the origin. The morphism $i_1 : X^+ \to X$ is the identity and $i_0 : X^+ \to X^0$ is trivial.

Example 3.13. Consider $X = \mathbb{A}^2$ equipped with the action of $\mathbb{G}_m$ defined by $t \cdot (x, y) := (tx, t^{-1}y)$. Then, $X^0 = \mathrm{Spec}(k)$ and $X^+ = \mathbb{A}^1$ (the x-axis). The morphism $X^0 \to X$ is the inclusion of the origin. The morphism $i_1 : X^+ \to X$ is the inclusion of the x-axis and $i_0 : X^+ \to X^0$ is trivial.

Example 3.14. Consider $X = \mathbb{P}^1$ equipped with the action of $\mathbb{G}_m$ defined by $t \cdot [x : y] := [tx : y]$. Then, $X^0 = \mathrm{Spec}(k) \sqcup \mathrm{Spec}(k)$ and $X^+ = \mathbb{A}^1 \sqcup \mathrm{Spec}(k)$. The morphism $X^0 \to X$ is the inclusion of the origin and the point at infinity. The morphism $i_1 : X^+ \to X$ is the inclusion of $\mathbb{A}^1$ into $\mathbb{P}^1$ and $\mathrm{Spec}(k)$ to the point at infinity. The morphism $i_0 : X^+ \to X^0$ is trivial on each component.

We now show that the functors X^0 and X^+ are represented by schemes.

Proposition 3.15. *Let X be an affine scheme of finite type over k equipped with an action of $\mathbb{G}_m$. Then, the functor X^0 is represented by an affine scheme. In particular, if $X = \mathrm{Spec}(A)$ and $\rho : A \to A \otimes \mathcal{O}(\mathbb{G}_m) = A \otimes k[t]_t$ is the coaction morphism, then X^0 is represented by $\mathrm{Spec}\left(A/(\sum_{i\neq 0} A_i)\right)$, where $A_i = \left\{a \in A : \rho(a) = a \otimes t^i\right\}$.*

Proof. Let $X = \mathrm{Spec}(A)$ be an affine scheme of finite type over k equipped with an action of $\mathbb{G}_m$ and $\rho : A \to A \otimes \mathcal{O}(\mathbb{G}_m) = A \otimes k[t]_t$ be the coaction morphism. For a k-algebra R, let $\mathrm{id}_R \otimes 1 : R \to R \otimes k[t]_t$ be the coaction morphism of $\mathbb{G}_m$ corresponding to the trivial action on $\mathrm{Spec}(R)$. We have

$$\begin{aligned} X^0(\mathrm{Spec}(R)) &= \mathrm{Hom}_k^{\mathbb{G}_m}(\mathrm{Spec}(R), \mathrm{Spec}(A)) \\ &= \left\{ \phi : A \to R \ k\text{-algebra map} \;\middle|\; \begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\rho"] & R \arrow[d, "\mathrm{id}_R \otimes 1"] \\ A \otimes k[t]_t \arrow[r, "\phi \otimes \mathrm{id}"] & R \otimes k[t]_t \end{tikzcd} \right\}. \end{aligned}$$

Next, we show that

$$\left\{ \phi : A \to R \text{ k-algebra map} \;\middle|\; \begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\rho"] & R \arrow[d, "\mathrm{id}_R \otimes 1"] \\ A \otimes k[t]_t \arrow[r, "\phi \otimes \mathrm{id}"] & R \otimes k[t]_t \end{tikzcd} \right\} = \mathrm{Hom}_k \left(A/(\sum_{i \neq 0} A_i), R \right).$$

Indeed, given $\phi : A \to R$ such that the diagram commutes and $a \in A_i$, we have

$$\phi(a) \otimes t^i = \phi(a \otimes t^i) = \phi(\rho(a)) = (\mathrm{id}_R \otimes 1)(\phi(a)) = \phi(a) \otimes 1.$$

Thus, either $i = 0$ or $\phi(a) = 0$. In particular, ϕ factors as $A \to A/(\sum_{i \neq 0} A_i) \to R$. For the same reason, any morphism $A/(\sum_{i \neq 0}) \to R$ obtained by precomposing with the quotient $A \to A/(\sum_{i \neq 0} A_i)$ makes the diagram commute. Then,

$$X^0(\mathrm{Spec}\,(R)) = \mathrm{Hom}_k \left(A/(\sum_{i \neq 0} A_i), R \right) = \mathrm{Hom}_k \left(\mathrm{Spec}\,(R), \mathrm{Spec} \left(A/(\sum_{i \neq 0} A_i) \right) \right).$$

For any k-scheme T of finite type, we can write $T = \mathrm{colim}_j \mathrm{Spec}\,(R_j)$. In particular,

$$\begin{aligned} X^0(T) &= \mathrm{Hom}_k^{\mathbb{G}_m} \left(\mathrm{colim}_j \mathrm{Spec}\,(R_j), \mathrm{Spec}\,(A) \right) \\ &= \lim_j \mathrm{Hom}_k^{\mathbb{G}_m} \left(\mathrm{Spec}\,(R_j), \mathrm{Spec}\,(A) \right) \\ &= \lim_j \mathrm{Hom}_k \left(A/(\sum_{i \neq 0} A_i), R_j \right) \\ &= \mathrm{Hom}_k \left(A/(\sum_{i \neq 0} A_i), \lim_j R_j \right) \\ &= \mathrm{Hom}_k \left(A/(\sum_{i \neq 0} A_i), \Gamma(T, \mathcal{O}_T) \right) \\ &= \mathrm{Hom}_k \left(T, \mathrm{Spec} \left(A/(\sum_{i \neq 0} A_i) \right) \right). \end{aligned}$$

Thus, X^0 is represented by an affine scheme Spec $\left(A/(\sum_{i \neq 0} A_i) \right)$. □

We extend this result to a more general case, i.e. when X is any finite type scheme over k. To do so, we need the following lemma.

Lemma 3.16. *Let X be a scheme of finite type over k equipped with an action of $\mathbb{G}_m$. If U is a $\mathbb{G}_m$-invariant open subscheme of X, then $U^0 = X^0 \times_X U$ as the fiber product of functors. In particular, the projection $U^0 \to X^0$ realizes U^0 as an open subfunctor of X^0.*

Proof. Let R be a k-algebra. We have

$$\begin{aligned} X^0(R) \times_{X(R)} U(R) &= \left\{ (f,g) \;\middle|\; \begin{tikzcd} \mathrm{Spec}\,(R) \arrow[rr, "f\ (\mathbb{G}_m\text{-morphism})"] \arrow[dr, "g"] & & X \\ & U \arrow[ur, hook] & \end{tikzcd} \right\} \\ &= \mathrm{Hom}_k^{\mathbb{G}_m}(\mathrm{Spec}\,(R), U) \\ &= U^0(R). \end{aligned}$$

Next, we show that U^0 is an open subfunctor of X^0. Let $\mathrm{Spec}\,(T)$ be an affine scheme and $\mathrm{Spec}\,(T) \to X^0$ be a morphism of functors. Hence, we have the following diagram of functors.

$$\begin{tikzcd} \mathrm{Spec}\,(T) \times_{X^0} U^0 \arrow[r] \arrow[d] & U^0 \arrow[r] \arrow[d] & U \arrow[d, hook] \\ \mathrm{Spec}\,(T) \arrow[r] & X^0 \arrow[r] & X \end{tikzcd}$$

Since the second square is a cartesian diagram of functors, $\mathrm{Spec}\,(T) \times_{X^0} U^0 \cong \mathrm{Spec}\,(T) \times_X U$. Moreover, $\mathrm{Spec}\,(T), X$ and U are representable by schemes and so is $\mathrm{Spec}\,(T) \times_X U$. Lastly, $\mathrm{Spec}\,(T) \times_X U \to \mathrm{Spec}\,(T)$ is an open embedding because $U \hookrightarrow X$ is. ☐

We now prove that the functor X^0 is represented by a scheme.

Proposition 3.17. *Let* X *be a scheme of finite type over* k *equipped with a locally affine action of* $\mathbb{G}_m$. *Then, the functor* X^0 *is represented by a closed subscheme of* X.

Proof. Let $\{U_i\}$ be a $\mathbb{G}_m$-invariant open affine covering of X. By Lemma 3.16, we have the following cartesian diagram of functors

$$\begin{tikzcd} U_i^0 \arrow[r] \arrow[d, hook] & U_i \arrow[d, hook] \\ X^0 \arrow[r] & X \end{tikzcd}.$$

By Proposition 3.15, U_i^0 is represented by a closed subscheme of U_i. Thus, X^0 admits a representable open covering of functors. Hence, X^0 is representable by a scheme. In particular, each U_i^0 is a closed subscheme of U and so X^0 is a closed subscheme of X. ☐

A similar conclusion is true for the functor X^+. We begin with the affine case.

Proposition 3.18. *Let* X *be an affine scheme of finite type over* k *equipped with an action of* $\mathbb{G}_m$. *Then, the functor* X^+ *is represented by an affine scheme. In particular, if* $X = \mathrm{Spec}\,(A)$ *and* $\rho : A \to A \otimes \mathcal{O}(\mathbb{G}_m) = A \otimes k[t]_t$ *is the coaction morphism, then* X^+ *is represented by* $\mathrm{Spec}\,\left(A/(\sum_{i<0} A_i)\right)$, *where* $A_i = \left\{a \in A : \rho(a) = a \otimes t^i\right\}$.

Proof. Let $X = \mathrm{Spec}\,(A)$ be an affine scheme of finite type over k equipped with an action of $\mathbb{G}_m$ and $\rho : A \to A \otimes \mathcal{O}(\mathbb{G}_m) = A \otimes k[t]_t$ be the coaction morphism. For a k-algebra R, let $\mathrm{id}_R \otimes t : R[x] \to R[x] \otimes k[t]_t$ ($x \mapsto x \otimes t$) be the coaction morphism of $\mathbb{G}_m$ corresponding to the standard action on $\mathrm{Spec}\,(R[x])$. We have

$$\begin{aligned} X^+(R) &= \mathrm{Hom}_k^{\mathbb{G}_m}\left(\mathrm{Spec}\,(R[x])\,, \mathrm{Spec}\,(A)\right) \\ &= \left\{ \phi : A \to R[x] \text{ k-algebra map} \;\middle|\; \begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\rho"] & R[x] \arrow[d, "\mathrm{id}_R \otimes t"] \\ A \otimes k[t]_t \arrow[r, "\phi \otimes \mathrm{id}"] & R[x] \otimes k[t]_t \end{tikzcd} \right\}. \end{aligned}$$

Next, we show that

$$\left\{ \phi : A \to R[x] \text{ k-algebra map} \;\middle|\; \begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\rho"] & R[x] \arrow[d, "\mathrm{id}_R \otimes t"] \\ A \otimes k[t]_t \arrow[r, "\phi \otimes \mathrm{id}"] & R[x] \otimes k[t]_t \end{tikzcd} \right\} \cong \mathrm{Hom}_k\left(A/(\sum_{i<0} A_i), R\right)$$

as sets. Given $\phi : A \to R[x]$ such that the diagram commutes, consider $a \in A_i$ and write $\phi(a) = \sum_{j \geq 0} r_j x^j$. The commutativity of the diagram implies that

$$\sum_{j \geq 0} r_j x^j \otimes t^i = \sum_{j \geq 0} r_j x^j \otimes t^j.$$

If $i < 0$, then $\phi(a) = 0$. Otherwise, $\phi(a) = r x^i$ for some $r \in R$. Hence, ϕ factors as $A \to A/(\sum_{i<0} A_i) \to R[x]$. By composing $A/(\sum_{i \leq 0} A_i) \to R[x]$ with $R[x] \to R$ sending $x \to 1$, we have a morphism of k-algebras $\bar{\phi} : A/(\sum_{i<0} A_i) \to R$.

On the other hand, given a morphism of k-algebras $\psi : A/(\sum_{i<0} A_i) \to R$, we define $\hat{\psi} : A \to R[x]$ by sending $a \in A_i$ to $\psi(a + \sum_{i<0} A_i) x^i$. This morphism is a well-defined morphism of k-algebras. In particular, $\hat{\psi}$ is compatible with the coaction of $\mathbb{G}_m$.

Reality check shows that the operations $\hat{\cdot}$ and $\bar{\cdot}$ are the inverses of each other. Thus,

$$X^+(R) = \mathrm{Hom}_k\left(A/(\sum_{i<0} A_i), R\right) = \mathrm{Hom}_k\left(\mathrm{Spec}\,(R)\,, \mathrm{Spec}\left(A/(\sum_{i<0} A_i)\right)\right).$$

For any k-scheme T of finite type, we can write $T = \mathrm{colim}_j \mathrm{Spec}\,(R_j)$. In particular,

$$\begin{aligned}
X^+(T) &= \mathrm{Hom}_k^{\mathbb{G}_m}\left(\mathbb{A}^1 \times \mathrm{colim}_j \mathrm{Spec}\,(R_j)\,, \mathrm{Spec}\,(A)\right)\\
&= \mathrm{Hom}_k^{\mathbb{G}_m}\left(\mathrm{colim}_j\left(\mathbb{A}^1 \times \mathrm{Spec}\,(R_j)\right), \mathrm{Spec}\,(A)\right)\\
&= \lim_j \mathrm{Hom}_k^{\mathbb{G}_m}\left(\mathrm{Spec}\,(R_j[x])\,, \mathrm{Spec}\,(A)\right)\\
&= \lim_j \mathrm{Hom}_k\left(A/(\sum_{i<0} A_i), R_j\right)\\
&= \mathrm{Hom}_k\left(A/(\sum_{i<o} A_i), \lim_j R_j\right)\\
&= \mathrm{Hom}_k\left(A/(\sum_{i<0} A_i), \Gamma(T, \mathcal{O}_T)\right)\\
&= \mathrm{Hom}_k\left(T, \mathrm{Spec}\left(A/(\sum_{i<0} A_i)\right)\right).
\end{aligned}$$

Thus, X^+ is represented by an affine scheme $\mathrm{Spec}\left(A/(\sum_{i<0} A_i)\right)$. □

Remark 3.19. Under the same assumption as in Proposition 3.18, X^- is represented by $\mathrm{Spec}\left(A/(\sum_{i>0} A_i)\right)$. (The proof is essentially the same as that in Proposition 3.18.)

Example 3.20. Consider $X = \mathrm{Spec}\,(A)$, where $A = k[x_1, \dots, x_n]$. Let $\mathbb{G}_m = \mathrm{Spec}\,(k[t]_t)$ act on X by a weight $w = (w_1, \dots, w_n) \in \mathbb{Z}^n$ that is $\rho(x_i) = t^{w_i} \otimes x_i$, where $\rho : A \to A \otimes k[t]_t$ is the coaction morphism. Hence, $X^+ = \mathrm{Spec}\left(A/(\sum_{i<0} A_i)\right)$, where $A_i = \left\{a \in A : \rho(a) = t^i \otimes a\right\}$. We then see that $X^+ = X$ if and only if $\sum_{i<0} A_i = 0$. Note that $\sum_{i<0} A_i = 0$ if and only if $w_i \geq 0$ for every i. Indeed, if $w_i \geq 0$ for every i, then for any monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$,

$$\rho(x_1^{\alpha_1} \cdot x_n^{\alpha_n}) = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \otimes t^{\sum_{i=1}^n \alpha_i w_i},$$

Then, $x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in A_s$ for $s = \sum_{i=1}^n \alpha_i w_i \geq 0$. In particular, $A_i = 0$ for all $i < 0$. On the other hand, if $\sum_{i<0} A_i = 0$, then $A_i = 0$ for every $i < 0$. Since $\rho(x_i) = t^{w_i} \otimes x_i \in A_{w_i}$, $w_i \geq 0$ for every i.

Similar to X^0, we extend this result to a more general case by using the following lemma.

Lemma 3.21. *Let X be a scheme of finite type over k equipped with an action of $\mathbb{G}_m$. If U is a $\mathbb{G}_m$-invariant open subscheme of X, then $U^+ = X^+ \times_{X^0} U^0$ as the cartesian product of functors;*

$$\begin{tikzcd}
U^+ \arrow[r] \arrow[d] & U^0 \arrow[d, hook] \\
X^+ \arrow[r, "i_0"] & X^0
\end{tikzcd}$$

In particular, the projection $U^+ \to X^+$ *realizes* U^+ *as an open subfunctor of* X^+*.*

Proof. Consider the following diagram of functors:

$$\begin{tikzcd}
X^+ \times_{X^0} U^0 \arrow[r] \arrow[d] & U^0 \arrow[r, hook] \arrow[d, hook] & U \arrow[d, hook] \\
X^+ \arrow[r, "i_0"] & X^0 \arrow[r, hook] & X
\end{tikzcd}$$

By Lemma 3.16, the second square is a cartesian diagram. Hence, $X^+ \times_{X^0} U^0 \cong X^+ \times_X U$ and so $X^+ \times_X U$ will be computed instead. Let R be a k-algebra. We have

$$X^+(R) \times_{X(R)} U(R) = \left\{ (f, g) \;\middle|\; \begin{tikzcd}
\operatorname{Spec}(R) \arrow[r, "g"] \arrow[d, "x=0"] & U \arrow[d, hook] \\
\operatorname{Spec}(R[x]) \arrow[r, "f\ (\mathbb{G}_m\text{-morphism})"] & X
\end{tikzcd} \right\}$$

For any $(f, g) \in X^+(R) \times_{X(R)} U(R)$, since U is $\mathbb{G}_m$-invariant and f is a $\mathbb{G}_m$-morphism, $f^{-1}(U)$ is a $\mathbb{G}_m$-invariant open subscheme of $\mathbb{A}^1_R$. The commutativity of the diagram implies that $f^{-1}(U)$ must contain the origin of $\mathbb{A}^1_R$. Hence, $f^{-1}(U) = \mathbb{A}^1_R$ and so f factors through U. Thus, $f \in U^+(R)$ and $g = f|_{x=0}$. Hence, the map $\phi : X^+(R) \times_{X(R)} U(R) \to U^+(R)$ defined by $(f, g) \mapsto f$ is well-defined. In particular, if $(f', g'), (f, g) \in X^+(R) \times_{X(R)} U(R)$ are such that $\phi(f, g) = \phi(f', g')$, then $f = f'$ and $g = f|_{x=0} = f'|_{x=0} = g'$. Thus, ϕ is injective.

On the other hand, if $f \in U^+(R)$, then $(\operatorname{Spec}(R[x]) \xrightarrow{f} U \hookrightarrow X, f|_{x=0}) \in X^+(R) \times_{X(R)} U(R)$ is mapped to f as shown in the following diagram

$$\begin{tikzcd}
\operatorname{Spec}(R) \arrow[rr, "f|_{x=0}"] \arrow[d, "x=0"] & & U \arrow[d, hook] \\
\operatorname{Spec}(R[x]) \arrow[urr, "f"] & & X
\end{tikzcd} .$$

Thus, ϕ is also surjective and hence bijective.
Next, we show that U^+ is an open subfunctor of X^+. Let $\operatorname{Spec}(T)$ be an affine scheme and $\operatorname{Spec}(T) \to X^+$ be a morphism of functors. Hence, we have the following diagram of functors.

$$\begin{tikzcd}
\operatorname{Spec}(T) \times_{X^+} U^+ \arrow[r] \arrow[d] & U^+ \arrow[r] \arrow[d] & U^0 \arrow[d, hook] \\
\operatorname{Spec}(T) \arrow[r] & X^+ \arrow[r, "i_0"] & X^0
\end{tikzcd}$$

Since the second square is a cartesian diagram of functors, $\operatorname{Spec}(T) \times_{X^+} U^+ \cong \operatorname{Spec}(T) \times_{X^0} U^0$. Moreover, $\operatorname{Spec}(T), X^0$ and U^0 are representable by schemes and so is Spec $(\mathsf{T}) \times_{\mathsf{X}^0} \mathsf{U}^0$. Lastly, Spec $(\mathsf{T}) \times_{\mathsf{X}^0} \mathsf{U}^0 \to$ Spec (T) is an open embedding because $\mathsf{U} \hookrightarrow \mathsf{X}$ is. Thus, U^+ is an open subfunctor of X^+. ☐

We now prove that the functor X^+ is representable by a scheme.

Proposition 3.22. *Let* X *be a scheme of finite type over* k *equipped with a locally affine action of* $\mathbb{G}_m$*. Then, the functor* X^+ *is represented by a scheme. In particular,* $i_0 : \mathsf{X}^+ \to \mathsf{X}^0$ *is an affine morphism of schemes.*

Proof. Let $\{\mathsf{U}_i\}$ be a $\mathbb{G}_m$-invariant open affine covering of X. By Lemma 3.21, we have the following cartesian diagram of functors

$$\begin{tikzcd}
\mathsf{U}_i^+ \arrow[r] \arrow[d, hook] & \mathsf{U}_i^0 \arrow[d, hook] \\
\mathsf{X}^+ \arrow[r, "i_0"] & \mathsf{X}^0
\end{tikzcd}.$$

By Proposition 3.18, U_i^+ is represented by an affine scheme. Thus, X^+ admits a representable open covering of functors. Hence, X^+ is representable by a scheme. In particular, $i_0 : \mathsf{X}^+ \to \mathsf{X}$ is an affine morphism. ☐

3.3 Białlynicki-Birula Decomposition

The goal of this section is to give a functorial proof to the Białlynicki-Birula decomposition.

Proposition 3.23 (Białynicki-Birula)**.** *Let* X *be a scheme of finite type over* k *equipped with a locally affine action of* $\mathbb{G}_m$*. Then, the functors* X^0 *and* X^+ *are represented by schemes. Moreover,*

(1) $\mathsf{X}^0 \to \mathsf{X}$ *is a closed embedding of schemes.*

(2) If X *is separated, then* $i_1 : \mathsf{X}^+ \to \mathsf{X}$ *is a monomorphism of schemes.*

(3) If X *is proper, then* $i_1 : \mathsf{X}^+ \to \mathsf{X}$ *is a surjective morphism of schemes.*

Write $\mathsf{X}^0 = \bigsqcup \mathsf{F}_i$*, where each* $\mathsf{F}_i \subset \mathsf{X}^0$ *is a connected component and* $\mathsf{X}_i := i_1^{-1}(\mathsf{F}_i)$*. Then,*

(4) If X *is smooth, then* X^0 *is smooth and* $\mathsf{X}^+ \to \mathsf{X}$ *is an affine fibration.*

(5) Assume further with one of the followings:

  (a) X *is affine,*
  (b) X *is separated and smooth,*
  (c) There exists a $\mathbb{G}_m$*-morphism locally closed embedding* $\mathsf{X} \hookrightarrow \mathbb{P}(\mathsf{V})$*.*

Then, $\mathsf{X}_i \hookrightarrow \mathsf{X}$ *is a locally closed embedding.*

Remark 3.24. The classical proof of Theorem 3.23 is given in [BB73].

Proof of (1). This is done in Proposition 3.17. □

Proof of (2). Assume that X is separated. To show that $i_1 : X^+ \to X$ is a monomorphism of schemes, it is equivalent to showing that the corresponding morphism of functors $X^+ \to X$ is injective. Let R be a k-algebra and $f, g \in X^+(R)$ be such that $i_1(f) = i_1(g)$. Then, $f(1) = g(1)$. Let x be the coordinate of $\mathbb{A}^1_R$. Since f and g are $\mathbb{G}_m$-morphisms, $f|_{\mathrm{Spec}(R[x]_x)} = g|_{\mathrm{Spec}(R[x]_x)}$. Next, we consider the following cartesian diagram of schemes

$$\begin{tikzcd}
V \arrow[r] \arrow[d] & X \arrow[d, "\delta"] \\
\mathbb{A}^1_R \arrow[r, "{(f,g)}"] & X \times_k X
\end{tikzcd}$$

Since X is separated, the diagonal morphism $\delta : X \to X \times_k X$ is a closed embedding. Thus, V is a closed subscheme of $\mathbb{A}^1_R$, where f and g agree. In particular, Spec $(R[x]_x)$ is an open subscheme of V. Hence, $IR[x]_x = 0$, where $I \subseteq R[x]$ is the ideal corresponding to V. Since x is not a zero-divisor, $I = 0$. Thus, $V = \mathbb{A}^1_R$, and $f = g$. In particular, $X^+(R) \to X(R)$ is injective. □

To see that separation hypothesis is necessary, consider the affine line with double origin. The scheme X^+ will be the disjoint union of two copies of $\mathbb{A}^1$ and the morphism $i_1 : X^+ \to X$ is collapsing on the nonzero points.

Proof of (3). Assume that X is proper. Denote the action map of $\mathbb{G}_m$ on $\mathbb{A}^1$ (resp. X) by $\sigma_{\mathbb{A}^1}$ (resp. σ_X). To show that $i_1 : X^+ \to X$ is a surjective morphism of schemes, it is equivalent to showing that the corresponding map of sets $X^+(K) \to X(K)$ is surjective for any field extension K over k. Let K be a field extension of k and $p \in X(K)$. The goal is to show that there exists $f \in X^+(K)$ such that $i_1(f) = p$. Note that Spec $(K[x]_x)$ is a $\mathbb{G}_m$-invariant subscheme of $\mathbb{A}^1_K$. Consider the morphism $g : \mathrm{Spec}\,(K[x]_x) \cong \mathbb{G}_m \times \mathrm{Spec}\,(K) \xrightarrow{(\mathrm{id},p)} \mathbb{G}_m \times X \xrightarrow{\sigma_X} X$. The morphism g is a $\mathbb{G}_m$-morphism. Indeed, consider the following diagram:

$$\begin{tikzcd}
\mathbb{G}_m \times \mathrm{Spec}\,(K[x]_x) \arrow[r, "{(\mathrm{id}_{\mathbb{G}_m},\, \cong)}"] \arrow[d, "\sigma_{\mathbb{A}^1}\big|_{x \neq 0}"'] & \mathbb{G}_m \times \mathbb{G}_m \times \mathrm{Spec}\,(K) \arrow[r, "{(\mathrm{id}_{\mathbb{G}_m},\, \mathrm{id}_{\mathbb{G}_m},\, p)}"] \arrow[d, "{(m,\, \mathrm{id}_{\mathrm{Spec}(K)})}"'] & \mathbb{G}_m \times \mathbb{G}_m \times X \arrow[r, "{(\mathrm{id}_{\mathbb{G}_m},\, \sigma_X)}"] \arrow[d, "{(m,\, \mathrm{id}_X)}"] & \mathbb{G}_m \times X \arrow[d, "\sigma_X"] \\
\mathrm{Spec}\,(K[x]_x) \arrow[r, "\sim"] & \mathbb{G}_m \times \mathrm{Spec}\,(K) \arrow[r, "{(\mathrm{id}_{\mathbb{G}_m},\, p)}"] & \mathbb{G}_m \times X \arrow[r, "\sigma_X"] & X
\end{tikzcd}$$

Reality check on the algebras shows that the first square commutes. The second square clearly commutes. The commutativity of the last square follows directly from the axiom of group action on X. Hence, $g : \mathrm{Spec}\,(K[x]_x) \to X$ is a $\mathbb{G}_m$-morphism. Next, we are going to use the properness of X via the valuative criterion of properness. Consider the following commutative diagram

$$\begin{tikzcd}
\operatorname{Spec}(K(x)) \arrow[r] \arrow[d] & \operatorname{Spec}(K[x]_x) \arrow[r, "g"] & X \arrow[d] \\
\operatorname{Spec}\left(K[x]_{(x)}\right) \arrow[rru, dashed, "\exists\ h"] \arrow[rr] & & \operatorname{Spec}(k)
\end{tikzcd}$$

By valuative criterion of properness, there exists $h : \operatorname{Spec}(K[x]_x) \to X$. Let U be a $\mathbb{G}_m$-invariant affine open subscheme of X containing the image of the closed point via h. Hence, $g^{-1}(U)$ is a non-empty $\mathbb{G}_m$-invariant open subscheme of $\operatorname{Spec}(K[x]_x)$. Thus, $g^{-1}(U) = \operatorname{Spec}(K[x]_x)$. In particular, we have the following commutative diagram

$$\begin{tikzcd}
 & & U \\
\operatorname{Spec}(K[x]_x) \arrow[r] \arrow[rru, bend left, "g"] & \mathbb{A}^1_K \arrow[ru, dashed, "\exists\ f"] & \\
\operatorname{Spec}(K(x)) \arrow[r] \arrow[u] & \operatorname{Spec}\left(K[x]_{(x)}\right) \arrow[u] \arrow[ruu, bend right, "h"'] &
\end{tikzcd}$$

Since $\mathbb{A}^1_K$ is the pushout of this diagram in the category of affine schemes, there exists $f : \mathbb{A}^1_K \to U$. By reality check on the algebras, f is a $\mathbb{G}_m$-morphism. By composing with the inclusion of U into X. We have a $\mathbb{G}_m$-morphism from $\mathbb{A}^1_K$ to X as desired. □

Recall Example 3.14 of $X = \mathbb{P}^1$ with the action $t \cdot [x : y] = [tx : y]$ of $\mathbb{G}_m$. $X^+ = \mathbb{A}^1 \sqcup \operatorname{Spec}(k)$.

To prove (4) of the proposition, we need the following lemmas

Lemma 3.25. *Let* $\mathbb{G}_m$ *act on* $\mathbb{A}^n$ *linearly. Then,* $(\mathbb{A}^n)^0 \hookrightarrow \mathbb{A}^n$ *is smooth and* $(\mathbb{A}^n)^+ \to (\mathbb{A}^n)^0$ *is an affine fibration.*

Proof of Lemma 3.25. The result follows directly from Proposition 3.15 and Proposition 3.18. □

Lemma 3.26. *Let* X *and* Y *be schemes of finite type over* k *equipped with an action of* $\mathbb{G}_m$ *and* $f : X \to Y$ *be an étale* $\mathbb{G}_m$*-morphism of schemes. Then, each* *square of the diagram*

$$\begin{tikzcd}
X^+ \arrow[r] \arrow[d] & Y^+ \arrow[d] \\
X^0 \arrow[r] \arrow[d, hook] & Y^0 \arrow[d, hook] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

is a fibered square.

Proof of Lemma 3.26. We begin with the bottom square. Let R be a k-algebra. Thinking of them as functors, we define $X^0(R) \to X(R) \times_{Y(R)} Y^0(R)$ by $g \mapsto (g, f \circ g)$. This map is well-defined and injective. Thus, surjectivity is left to show. Let $(g, f \circ g) \in X(R) \times_{Y(R)} Y^0(R)$. We want to show that g is a $\mathbb{G}_m$-morphism, i.e., the diagram

$$\begin{tikzcd}
\mathbb{G}_m \times \operatorname{Spec}(R) \arrow[r, "\mathrm{id} \times g"] \arrow[d, "\mathrm{pr}_2"] & \mathbb{G}_m \times X \arrow[d, "\sigma_x"] \\
\operatorname{Spec}(R) \arrow[r, "g"] & X
\end{tikzcd}$$

commutes. Consider the cartesian diagram

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] & \mathbb{G}_m \times \operatorname{Spec}(R) \arrow[d, "(g \circ \mathrm{pr}_2, \sigma_X \circ (\mathrm{id} \times g))"] \\
X \arrow[r, "\delta"] & X \times_Y X
\end{tikzcd}$$

Since $f : X \to Y$ is étale, $\delta : X \to X \times_Y X$ is an open embedding [Sta20, Tag 02G3] and so is $Z \to \mathbb{G}_m \times \operatorname{Spec}(R)$. Thus, we may assume that R is a field. In fact, it is equivalent to show that for any $x \in X$, if $f(x) \in Y^0$, then $x \in X^0$. Étaleness of f implies that the closed embedding $\mathbb{G}_{m,x} \hookrightarrow \mathbb{G}_{m,f(x)}$ is such that the index $\mathbb{G}_{m,f(x)}/\mathbb{G}_{m,x}$ is finite. By assumption, $\mathbb{G}_{m,f(x)} = \mathbb{G}_m$ and so $\mathbb{G}_{m,x} = \mathbb{G}_m$. Thus, $x \in X^0$. This shows surjectivity. Thus, $X^0 = X \times_Y Y^0$.

We show the top square is cartesian. Let R be a k-algebra. Thinking of them as functors, we define $X^+(R) \to X^0(R) \times_{Y^0(R)} Y^+(R)$ by $g \mapsto (g|_{x=0}, f \circ g)$, where x is the coordinate of $\operatorname{Spec}(R[x])$. This map is well-defined and injective. Again, surjectivity is left to show. It is then equivalent to show the following: for any $\mathbb{G}_m$-morphism $p : \operatorname{Spec}(R) \to X$ and $h : \operatorname{Spec}(R) \times \mathbb{A}^1_k \to Y$ such that $\operatorname{Spec}(R) \xrightarrow{p} X \xrightarrow{f} Y = \operatorname{Spec}(R) \xhookrightarrow{x=0} \operatorname{Spec}(R) \times \mathbb{A}^1_k \xrightarrow{h} Y$, there exists a $\mathbb{G}_m$-morphism $\tilde{h} : \operatorname{Spec}(R) \times \mathbb{A}^1_k \to X$ such that the diagram

$$\begin{tikzcd}
\operatorname{Spec}(R) \arrow[r, "p"] \arrow[d, hook, "x=0"'] & X \arrow[d, "f"] \\
\operatorname{Spec}(R) \times \mathbb{A}^1_k \arrow[r, "h"] \arrow[ur, dashed, "\exists \tilde{h}"] & Y
\end{tikzcd}$$

commutes. We will use Tannaka Duality [HR19] which states that for any k-algebra R and a scheme X equipped with an action of $\mathbb{G}_m$, the canonical morphism

$$\Phi : \mathrm{Hom}_k^{\mathbb{G}_m}\left(\mathrm{Spec}\,(R) \times \mathbb{A}_k^1, X\right) \to \lim_n \mathrm{Hom}_k^{\mathbb{G}_m}\left(\mathrm{Spec}\,(R) \times \mathrm{Spec}\,(k[x]/(x^n)), X\right)$$

is an isomorphism. First, we denote h_n for $\mathrm{Spec}\,(R[x]/(x^n)) \hookrightarrow \mathrm{Spec}\,(R) \times \mathbb{A}_k^1 \xrightarrow{h} Y$. Consider the diagram

$$\begin{tikzcd}
\mathrm{Spec}\,(R) \arrow[r, "p"] \arrow[d, hook] & X \arrow[d, "f"] \\
\mathrm{Spec}\,\left(R[x]/(x^2)\right) \arrow[ru, dashed, "\exists p_2"] \arrow[r, "h_2"] & Y
\end{tikzcd}$$

By étaleness of f, there exists $p_2 : \mathrm{Spec}\,\left(R[x]/(x^2)\right) \to Y$. With this fashion, we construct morphisms $p_n : \mathrm{Spec}\,(R[x]/(x^n)) \to X$.

$$\begin{tikzcd}
\mathrm{Spec}\,(R) \arrow[rr, "p"] \arrow[d, hook] & & X \arrow[d, "f"] \\
\mathrm{Spec}\,\left(R[x]/(x^2)\right) \arrow[rru, "p_2"] \arrow[rr, "h_2"] \arrow[d, hook] & & Y \\
\mathrm{Spec}\,\left(R[x]/(x^3)\right) \arrow[rruu, "p_3"] \arrow[rru, "h_3"] \arrow[d, hook] & & \\
\mathrm{Spec}\,\left(R[x]/(x^4)\right) \arrow[rruuu, "p_4"] \arrow[rruu, "h_4"] & & \\
 & \vdots &
\end{tikzcd}$$

Hence, $(p_i) \in \lim_n \mathrm{Hom}_k^{\mathbb{G}_m}\left(\mathrm{Spec}\,(R) \times \mathrm{Spec}\,(k[x]/(x^n)), X\right)$ and $\tilde{h} := \Phi^{-1}((p_i))$ is our desired $\mathbb{G}_m$-morphism. This shows surjectivity. Thus, $X^+ = X^0 \times_{Y^0} Y^+$. ☐

Proof of (4). Assume that X is smooth. Smoothness is local and so we may assume that X is affine. Suppose $X = \mathrm{Spec}\,(A)$ and let $x \in X^0 \subseteq X$. We will show that X^0 is smooth at x. Let $\mathfrak{m} \subseteq A$ be the maximal ideal of x. Thus, $\mathfrak{m} \twoheadrightarrow \mathfrak{m}/\mathfrak{m}^2$ is a surjection of $\mathbb{G}_m$-representations. Since $\mathbb{G}_m$ is linearly reductive, there exists a section $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}$. In particular, we have the morphism $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m} \to A$ of k-vector spaces. This induces $\mathrm{Sym}^*_k\left(\mathfrak{m}/\mathfrak{m}^2\right) \to A$ and so gives the morphism $\phi : \mathrm{Spec}\,(A) \to \mathrm{Tgt}_{X,x}$ of k-schemes. By construction, ϕ is a $\mathbb{G}_m$-morphism and is étale at x. Let U be an open neighborhood of x such that $\phi|_U$ is étale. By Lemma 3.26, we have the cartesian diagram

$$\begin{tikzcd}
U^0 \arrow[r] \arrow[d, hook] & \mathrm{Tgt}^0_{X,x} \arrow[d, hook] \\
U \arrow[r, "\phi|_U"] & \mathrm{Tgt}_{X,x}
\end{tikzcd}$$

By Lemma 3.25, $\mathrm{Tgt}^0_{X,x} \hookrightarrow \mathrm{Tgt}_{X,x}$ is smooth and so is $U^0 \hookrightarrow U$.

Next, we show that $i_0 : U^+ \to U^0$ is an affine fibration. By lemma 3.25, $i_0 : \mathrm{Tgt}^+_{X,x} \to \mathrm{Tgt}^0_{X,x}$ is an affine fibration, i.e., there exists an open subscheme $x \in W \subseteq \mathrm{Tgt}^0_{X,x}$ such that

$$\begin{tikzcd}
\mathbb{A}^n \times W \arrow[r, "\cong"] \arrow[dr, "\mathrm{pr}_2"'] & i_0^{-1}(W) \arrow[r, hook] \arrow[d] & \mathrm{Tgt}^+_{X,x} \arrow[d, "i_0"] \\
 & W \arrow[r, hook] & \mathrm{Tgt}^0_{X,x}
\end{tikzcd}.$$

By Lemma 3.26, we have the cartesian diagram

$$\begin{tikzcd}
U^+ \arrow[r] \arrow[d] & \mathrm{Tgt}^+_{X,x} \arrow[d] \\
U^0 \arrow[r, "\phi|_{U^0}"] & \mathrm{Tgt}^0_{X,x}
\end{tikzcd}$$

Hence, we have

$$\begin{tikzcd}
 & \mathbb{A}^n \times W \arrow[rr, "\cong"] \arrow[ddrr, bend right, "\mathrm{pr}_2"] & & i_0^{-1}(W) \arrow[rr, hook] \arrow[dd] & & \mathrm{Tgt}^+_{X,x} \arrow[dd, "i_0"] \\
\mathbb{A}^n \times \phi^{-1}(W) \arrow[ur] \arrow[rr, "\cong"] \arrow[ddrr, bend right] & & i_0^{-1}(\phi^{-1}(W)) \arrow[ur] \arrow[rr, hook] \arrow[dd] & & U^+ \arrow[ur] \arrow[dd, "i_0"] & \\
 & & & W \arrow[rr, hook] & & \mathrm{Tgt}^0_{X,x} \\
 & & \phi^{-1}(W) \arrow[ur] \arrow[rr, hook] & & U^0 \arrow[ur, "\phi|_{U^0}"'] &
\end{tikzcd}$$

Therefore, $i_0 : U^+ \to U^0$ is an affine fibration. □

Proof of (5)(a). Suppose $X = \mathrm{Spec}\,(A)$ is affine. By Proposition 3.18, X^+ is a closed subscheme of X. If F_i is a connected component of X^0, then $X_i := i_1^{-1}(F_i)$ is a closed subscheme of X^+. In particular, X_i is a (locally) closed subscheme of X. ☐

To show (5)(b), we need the following lemma.

Lemma 3.27. *Let X be a separated scheme of finite type over k equipped with a locally affine action of $\mathbb{G}_m$. Then, for any point $x \in X^+$, there exists a $\mathbb{G}_m$-invariant open subscheme U of X such that $x \in U^+$ and the induced morphism $U^+ \to i_1^{-1}(U)$ is an open and closed embedding.*

$$\begin{tikzcd}
U^+ \arrow[r] \arrow[dr, hook] & i_1^{-1}(U) \arrow[r] \arrow[d, hook] & U \arrow[d, hook] \\
 & X^+ \arrow[r, "i_1"] & X
\end{tikzcd}$$

Moreover, if $Z \subseteq X^+$ is an irreducible component, then $Z \hookrightarrow X^+ \xrightarrow{i_1} X$ is a locally closed embedding.

Proof of Lemma 3.27. Assume that X is separated. Let U be a $\mathbb{G}_m$-invariant affine open subscheme U of X containing $i_0(x)$. By Proposition 3.18, $U^+ \to U$ is a closed embedding. By (2), $i_1 : X^+ \to X$ is a monomorphism and so is $i_1^{-1}(U) \to U$. Thus, $U^+ \to i_1^{-1}(U)$ is a closed embedding. By Lemma 3.21, $U^+ = U^0 \times_{X^0} X^+$ and so $x \in U^+$. Thus, $U^+ \to i_1^{-1}(U)$ is also an open embedding. If $x \in Z \subseteq X^+$ is an irreducible component. Then, $Z \cap U^+$ is a nonempty open and closed subscheme of the irreducible scheme $Z \cap i_1^{-1}(U)$. Thus, $Z \cap U^+ = Z \cap i_1^{-1}(U)$ and $Z \cap i_1^{-1}(U) \hookrightarrow U$ is a closed embedding. This is true for any point $x \in Z$. Therefore, $Z \hookrightarrow X^+ \xrightarrow{i_1} X$ is a locally closed embedding. ☐

Remark 3.28. A proof of Lemma 3.27 is also in [AHR20].

Proof of (5)(b). Suppose X is separated and smooth. By (4), X_i is smooth and connected and so irreducible. By Lemma 3.27, $X_i \hookrightarrow X$ is a locally closed embedding. ☐

Proof of (5)(c). First, we reduce to the case $X = \mathbb{P}(V)$. Consider the following diagram

$$\begin{tikzcd}
 & X^+ \arrow[r, dashed] \arrow[rr, bend left] \arrow[dl, "i_0"'] \arrow[dr] & i_1^{-1}(X) \arrow[r] \arrow[d] & X \arrow[d, hook] \\
X^0 \arrow[dr] & & \mathbb{P}(V)^+ \arrow[r, "i_1"] \arrow[dl, "i_0"] & \mathbb{P}(V) \\
 & \mathbb{P}(V)^0 & &
\end{tikzcd}$$

By (2), $i_1 : \mathbb{P}(V)^+ \to \mathbb{P}(V)$ is a monomorphism. Note also that X is separated. Therefore, $i_1; X^+ \to X$ is a monomorphism. Then, $X^+ \to i_1^{-1}(X)$ is a closed embedding. Hence, if $\mathbb{P}(V)^+ \to \mathbb{P}(V)$ is a locally closed embedding on connected components, then $X_i \to X$ is a locally closed embedding. For $X = \mathbb{P}(V)$, a direct computation shows that each X_i is of the form $\mathbb{P}(W) \setminus \mathbb{P}(W')$ for some linear subspaces $W' \subseteq W \subseteq V$ [AHR20]. ☐

3.4 One-parameter subgroups

Definition 3.29. A *one-parameter subgroup* of an algebraic group G is an algebraic group homomorphism $\lambda : \mathbb{G}_m \to G$.

Definition 3.30. Let $\lambda : \mathbb{G}_m \to G$ be a one-parameter subgroup of G. For any $g \in G$, there is a morphism $\lambda^g : \mathbb{A}^1 \setminus 0 \to G$ of schemes sending $t \mapsto \lambda(t) g \lambda(t)^{-1}$. We say that $\lim_{t\to 0} \lambda(t) g \lambda(t)^{-1}$ *exists* if λ^g can be uniquely extended to a morphism $\mathbb{A}^1 \to G$, i.e., there exists a unique morphism $\bar{\lambda}^g : \mathbb{A}^1 \to G$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{A}^1 \setminus 0 \arrow[r, dashed, "\bar{\lambda}^g"] \arrow[d, hook] & G \\
\mathbb{A}^1 . \arrow[ur, "\lambda^g"'] &
\end{tikzcd}$$

Definition 3.31. For any one-parameter subgroup $\lambda : \mathbb{G}_m \to G$, define the subgroup

$$G_\lambda^+ := \{g \in G : \lim_{t\to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists } \}$$

Proposition 3.32. *Let U be an algebraic group equipped with a $\mathbb{G}_m$-action $\sigma : \mathbb{G}_m \times U \to U$. Suppose $G := U \rtimes_\sigma \mathbb{G}_m$ is the semi-direct product of U and $\mathbb{G}_m$ corresponding to σ. If $\lambda_1 : \mathbb{G}_m \to G$ be a one-parameter subgroup such that $\lambda_1(t) = (1, t)$, then*

$$G_{\lambda_1}^+ = U^+ \rtimes_\sigma \mathbb{G}_m.$$

Similarly, if $\lambda_{-1} : \mathbb{G}_m \to G$ be a one-parameter subgroup such that $\lambda_1(t) = (1, t^{-1})$, then

$$G_{\lambda_{-1}}^+ = U^- \rtimes_\sigma \mathbb{G}_m.$$

Proof. Suppose $(u, r) \in G_{\lambda_1}^+$. For $t \in \mathbb{G}_m$, we have

$$\lambda_1(t)(u, r)\lambda_1(t)^{-1} = (1, t)(u, r)(1, t)^{-1} = (\sigma(t, u), trt^{-1}) = (\sigma(t, u), r).$$

Hence, $\lim_{t\to 0} \lambda_1(t)(u, r)\lambda_1(t)^{-1}$ exists if and only if $\lim_{t\to 0} \sigma(t, u)$ exists. In particular,

$$G_{\lambda_1}^+ = U^+ \rtimes_\sigma \mathbb{G}_m.$$

Similarly, suppose $(u, r) \in G_{\lambda_{-1}}^+$. For $t \in \mathbb{G}_m$, we have

$$\lambda_{-1}(t)(u, r)\lambda_{-1}(t)^{-1} = (1, t^{-1})(u, r)(1, t) = (\sigma(t^{-1}, u), t^{-1}rt) = (\sigma(t^{-1}, u), r).$$

Hence, $\lim_{t\to 0}\lambda_{-1}(t)(u,r)\lambda_{-1}(t)^{-1}$ exists if and only if $\lim_{t\to\infty}\sigma(t,u)$ exists. In particular,

$$G^{+}_{\lambda_{-1}} = U^{-} \rtimes_{\sigma} \mathbb{G}_m.$$

□

4 Unipotent and Reductive Groups

4.1 Unipotent Representations

Definition 4.1. An algebraic group U is *unipotent* if every nonzero representation of U has a nonzero fixed vector.

Remark 4.2. If every nonzero finite-dimensional representation (V, r) of an algebraic group G has a nonzero fixed vector, then G is unipotent. This is true because every representation is a union of its finite-dimensional representations.

Remark 4.3. A quotient of a unipotent group is unipotent. Indeed, let Q be a quotient of a unipotent algebraic group G. If (V, r) is a nonzero representation of Q, then $(V, r \circ \pi)$ is a nonzero representation of G, where $\pi : G \to Q$ is the quotient morphism. Thus, V has a nonzero fixed vector and Q is unipotent.

Definition 4.4. We define *the upper triangular matrix group*, denoted $\mathbb{U}_n$, to be Spec $(k[x_{ij}])/I$, where $I = (x_{ii} - 1, x_{ij}; i > j)$, with the multiplication morphism $\mu : \mathbb{U}_n \times \mathbb{U}_n \to \mathbb{U}_n$ sending $(A, B) \mapsto AB$ for $A, B \in \mathbb{U}_n(k)$.

Remark 4.5. The upper triangular matrix group $\mathbb{U}_n$ is an algebraic subgroup of GL_n. The comultiplication morphism $\Delta : k[x_{ij}]/I \to k[xij]/I \otimes k[y_{ij}]/J$ sends $x_{ij} \mapsto \sum_k x_{ik} \otimes y_{kj}$, where the ideal $J = (y_{ii} - 1, y_{ij}; i > j)$.

Definition 4.6. A finite-dimensional representation (V, r) of an algebraic group G is *unipotent* if there is a basis of V such that $r(G) \subseteq \mathbb{U}_n$ for some n.

Proposition 4.7. *An algebraic group G is unipotent if and only if every finite-dimensional representation (V, r) of G is unipotent.*

Proof. Let (V, r) be a finite-dimensional representation of an algebraic group G. ($\Rightarrow$) Suppose that G is unipotent. We first show that there exists an ascending sequence

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V$$

of vector spaces such that V_{i+1}/V_i is one-dimensional and G acts trivially on V_{i+1}/V_i for $i = 0, \ldots, n-1$. As G is unipotent, there exists a nonzero vector $v_1 \in V$ such that $r(g) \cdot v_1 = v_1$ for all $g \in G$. If $V_1 :=< v_1 >$ is the subspace spanned by v_1, the vector space V/V_1 is stable under the G-action. In particular, V/V_1 is a representation of G whose dimension is $\dim V - 1$. If $V/V_1 = 0$, we have the desire sequence

$$0 = V_0 \subset V_1 = V$$

If $V/V_1 \neq 0$, then there exist a nonzero vector $w_2 \in V/V_1$ such that $r(g) \cdot w_2 = w_2$ in V/V_1. Let $v_2 \in V$ be a vector whose image in V/V_1 is w_2. Thus, $v_2 \notin V_1$ and V_2/V_1 is a one-dimensional representation of G. In particular, the vector space V/V_2 is stable under the G-action. This implies that V/V_2 is once again a representation of G whose dimension is $\dim V - 2$. If we continue this process, we will have an ascending sequence

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V$$

of subspaces of V such that V_{i+1}/V_i is one-dimensional and G acts trivially on V_{i+1}/V_i for $i = 0, \dots, n-1$. (The process will terminate as $\dim V < \infty$.) Most importantly, the set $\{v_1, \dots, v_n\}$ forms a basis of V such that $r(G) \subseteq \mathbb{U}_n$.
($\Leftarrow$) Suppose (V, r) is unipotent. Hence, there exists a basis $\{v_1, \dots, v_n\}$ of V such that $r(G) \subseteq \mathbb{U}_n$. This implies that v_1 is a nonzero vector fixed by G. Hence, G is unipotent. □

Remark 4.8. An algebraic group G is unipotent if and only if for every finite-dimensional representation $r : G \to \mathrm{GL}_V$ of G, there exists an ascending sequence

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V$$

of subspaces of V such that V_{i+1}/V_i is one-dimensional and G acts trivially on V_{i+1}/V_i for $i = 0, \dots, n-1$. The forward direction of this statement is shown in Proposition 4.7. To see the other direction, suppose there exists such ascending sequence. Hence, G acts on V_1 trivially. Thus, there exists a nonzero vector $v \in V_1$ fixed by G. In particular, G is unipotent.

Proposition 4.9. *A unipotent algebraic group G is isomorphic to an algebraic subgroup of $\mathbb{U}_n$ for some n.*

Proof. Let G be a unipotent algebraic group. By Corollary 2.40, there exists a faithful finite-dimensional representation (V, r) of G. By Proposition 4.7, (V, r) is unipotent. Hence, there exists a basis of V such that $r(G) \subseteq \mathbb{U}_n$ for some n. In particular, G is isomorphic to an algebraic subgroup of $\mathbb{U}_n$. □

Remark 4.10. The converse of Proposition 4.9 is also true, i.e., an algebraic subgroup of $\mathbb{U}_n$ is unipotent. See [Mil17, Theorem 14.5].

4.2 One-parameter Subgroups of Unipotent Groups

Recall that a one-parameter subgroup of an algebraic group G is a group homomorphism $\lambda : \mathbb{G}_m \to G$. The goal of this section is to show that any one-parameter subgroup of a unipotent group is trivial. We begin with a lemma.

Proposition 4.11. *Any group homomorphism $\lambda : \mathbb{G}_m \to \mathbb{U}_n$ is trivial.*

Proof. Let $\lambda : \mathbb{G}_m \to \mathbb{U}_n$ be a group homomorphism. Write $\mathbb{G}_m = \operatorname{Spec}(k[t]_t)$ and $\mathbb{U}_n = \operatorname{Spec}(k[x_{ij}]/I)$, where $I = (x_{ii} - 1, x_{ij}; i > j)$. The data of λ is equivalent to a morphism $\phi : k[x_{ij}]/I \to k[t]_t$ of k-algebras such that the following diagram of k-algebras commute:

$$\begin{tikzcd}
k[x_{ij}]/I \arrow[r, "\phi"] \arrow[d, "\Delta_{\mathbb{U}_n}"'] & k[t]_t \arrow[d, "\Delta_{\mathbb{G}_m}"] \\
k[x_{ij}]/I \otimes k[y_{ij}]/J \arrow[r, "\phi\otimes\phi"'] & k[t]_t \otimes k[s]_s,
\end{tikzcd}$$

where $J = (y_{ii}, y_{ij} - 1; i > j)$. (Explicitly, the morphism $\Delta_{\mathbb{U}_n}$ denotes the comultiplication morphism of $\mathbb{U}_n$ sending $x_{ij} \mapsto \sum_k x_{ik} \otimes y_{kj}$. The morphism

$\Delta_{\mathbb{G}_m}$ denotes the comultiplication morphism of $\mathbb{G}_m$ sending $t \mapsto t \otimes s$.) For $i, j \in \{1, 2, \ldots, n\}$, write

$$\phi(x_{ij}) = f_{ij}(t).$$

Hence, $f_{ii} = 1$ and $f_{ij} = 0$ for $i > j$. More importantly, by the commutativity of the diagram above, we have the equation

$$f_{ij}(t \otimes s) = \sum_k f_{ik}(t) \otimes f_{kj}(s) = \sum_{k=i}^{n} f_{ik}(t) \otimes f_{kj}(s) = \sum_{k=0}^{j-i} f_{i,i+k}(t) \otimes f_{i+k,j}(s),$$

for any $j > i$. In particular,

$$f_{i,i+l}(t \otimes s) = \sum_{k=0}^{l} f_{i,i+k}(t) \otimes f_{i+k,i+l}(s)$$

We show by induction that for a fixed i, $f_{i,i+l} = 0$ for any $l > 0$. For $l = 1$, we see that

$$f_{i,i+1}(t \otimes s) = 1 \otimes f_{i,i+1}(s) + f_{i,i+1}(t) \otimes 1.$$

Comparing coefficients, we see that $f_{i,i+1} = 0$. Next, suppose that $f_{i,i+1} = \cdots = f_{i,i+l-1} = 0$. Hence,

$$f_{i,i+l}(t \otimes s) = \sum_{k=0}^{l} f_{i,i+k}(t) \otimes f_{i+k,i+l}(s) = 1 \otimes f_{i,i+l}(s) + f_{i,i+l}(t) \otimes 1$$

Comparing coefficients, we see that $f_{i,i+l} = 0$. Thus, we conclude that $\phi(x_{ij}) = 1$ if $i = j$ and 0 otherwise. In particular, $\lambda : \mathbb{G}_m \to \mathbb{U}_n$ is the trivial group homomorphism. ☐

Corollary 4.12. *If* G *is a unipotent group, then any one-parameter subgroup* $\lambda : \mathbb{G}_m \to U$ *is trivial.*

Proof. Let U be a unipotent group. By Proposition 4.9, U is isomorphic to a subgroup of $\mathbb{U}_n$ for some n, i.e., there is an injective group homomorphism $i : U \hookrightarrow \mathbb{U}_n$. If $\lambda : \mathbb{G}_m \to G$ is a one-parameter subgroup of U, then $i \circ \lambda : \mathbb{G}_m \to \mathbb{U}_n$ is a one-parameter subgroup of $\mathbb{U}_n$. By Proposition 4.11, $i \circ \lambda$ is trivial. Hence, λ is trivial. ☐

4.3 Unipotent Radical

Proposition 4.13. *Let* N *be a normal subgroup of an algebraic group* G*. If the subgroup* N *and the quotient* G/N *are unipotent then* G *is unipotent*

Proof. Let G be an algebraic group and N be its normal subgroup such that both N and G/N are unipotent. Let (V, r) be a nonzero finite-dimensional representation of G. By Remark 2.31, the subspace V^N is stable under G. Hence, V^N can be regarded as a representation of G. Let $s : G \to GL_{V^N}$ be the group homomorphism associated to the representation V^N of G. As N acts on V^N

trivially, the homomorphism s factors through G/N making V^N a representation of G/N also. Since V is nonzero and N is unipotent, V^N is nonzero. Since G/N is unipotent, $(V^N)^{G/N}$ is nonzero. Therefore, $V^G = (V^N)^{G/N}$ is nonzero and G is unipotent. □

Corollary 4.14. *If* H *and* N *are unipotent algebraic subgroups of an algebraic group* G *with* N *normal, then* HN *is also unipotent.*

Proof. Let H and N be unipotent algebraic subgroups of an algebraic group G with N normal. Hence, the quotient $H/H \cap N$ is unipotent (Remark 4.3.) By [Mil17, Theorem 5.52], $H/H \cap N \cong HN/N$ and so HN/N is unipotent. By Proposition 4.13, HN is unipotent. □

Proposition 4.15. *Every algebraic group* G *contains a unique largest connected normal unipotent subgroup* H*. In particular, any connected normal unipotent subgroup of the quotient* G/H *is trivial.*

Proof. Let G be an algebraic group. Since the trivial group is connected normal and unipotent, G contains a connected normal unipotent subgroup. Hence, there is a maximal connected normal unipotent subgroup of G. Suppose H and N are maximal connected normal unipotent subgroups of G. By Corollary 4.14, HN is also maximal connected normal unipotent. Thus, $H = HN = N$. Let H be the largest connected normal unipotent subgroup of G. If the quotient G/H had a nontrivial connected normal unipotent subgroup, then its inverse image in G would properly contain H. That is a contradiction. Thus, any connected normal unipotent subgroup of the quotient G/H is trivial. □

Definition 4.16. Let G be a connected algebraic group. We define the *unipotent radical*, denoted $R_u(G)$, to be the largest connected normal unipotent subgroup of G.

Remark 4.17. Let G be a connected algebraic group. By Proposition 4.15, the unipotent radical $R_u(G)$ exists. In fact, a connected algebraic group G is unipotent if and only if $G = R_u(G)$.

4.4 Reductive Groups

Definition 4.18. An algebraic group G is a *torus* if it is isomorphic to $\mathbb{G}_m^n$ for some n. A *subtorus* of an algebraic group H is an algebraic subgroup of H that is a torus. A *maximal torus* of an algebraic group H is a subtorus of H that is not properly contained in another subtorus.

Remark 4.19. This definition of tori is not standard. However, it agrees with the standard definition over an algebraically closed field.

Definition 4.20. A connected algebraic group G is *reductive* if its unipotent $R_u(G)$ is trivial.

Example 4.21. Let G be a connected algebraic group. The quotient $G/R_u(G)$ is reductive.

Proposition 4.22. *Any two maximal tori of an algebraic group* G *are conjugate.*

Proof. See [Bor91, Theorem 6.4.1] $\square$

Proposition 4.23 (Matsushima's Theorem)**.** *Let* G *be a reductive algebraic group and* H *be its subgroup. The quotient* G/H *is affine if and only if* H *is reductive.*

Proof. See [Arz08, Theorem 2.1] or [BB63, Theorem 1]. $\square$

Proposition 4.24. *A connected algebraic group* G *over* k *is reductive if and only if every finite-dimensional representation of* G *is semisimple.*

Proof. See [Mil17, Theorem 22.42]. Recall that char $\mathsf{k} = 0$ as the forward direction of this proposition is not true over a field of a positive characteristic. $\square$

Remark 4.25. Let G be a reductive group. Regarding G as a subgroup of GL_n for some n, we can consider elements of G as matrices. Proposition 4.24 implies that any element $g \in \mathsf{G}(\mathsf{k})$, when considered as a matrix, is diagonalizable. In particular, any element $g \in \mathsf{G}(\mathsf{k})$ is contained in a subtorus of G that is isomorphic to $\mathbb{G}_m$.

Proposition 4.26. *Let* G *be a connected algebraic group over* k*. There exists a unipotent algebraic group* U *equipped with a group action* $\sigma : \mathsf{R} \times \mathsf{U} \to \mathsf{U}$ *by a reductive algebraic group* R *such that*

$$\mathsf{G} \cong \mathsf{U} \rtimes_\sigma \mathsf{R}.$$

Proof. See [Con14, Proposition 5.4.1] or [McN10, Section 3]. (Recall that char $\mathsf{k} = 0$.) $\square$

5 Classifying Stacks

5.1 Algebraic Spaces

In this section, we give a review on algebraic spaces and algebraic stacks. We mainly follow [Ols16] and [Alp22].

Definition 5.1. A morphism $f : X \to Y$ of k-schemes is *étale* if for every commutative diagram of k-schemes

$$\begin{tikzcd}
\mathrm{Spec}\,(R/I) \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\
\mathrm{Spec}\,(R) \arrow[r] & Y,
\end{tikzcd}$$

where $I^2 = 0$, there exists a unique morphism $\phi : \mathrm{Spec}\,(R) \to X$ such the following diagram of k-scheme commutes:

$$\begin{tikzcd}
\mathrm{Spec}\,(R/I) \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\
\mathrm{Spec}\,(R) \arrow[r] \arrow[ur, dashed, "\exists!\phi"'] & Y.
\end{tikzcd}$$

Proposition 5.2. *Let* $f : X \to Y$ *be an étale morphism of* k*-schemes.*

(a) If $g : Y \to Z$ *is an étale morphism of* k*-schemes, then* $g \circ f : X \to Z$ *is étale.*

(b) If $g' : Y' \to Y$ *is a morphism of* k*-schemes, then* $f' : X \times_Y Y' \to Y'$ *is étale.*

Proof. Let $f : X \to Y$ be an étale morphism of k-schemes.

(a) Let $g : Y \to Z$ be an étale morphism of k-schemes. Consider a commutative diagram of k-schemes

$$\begin{tikzcd}
\mathrm{Spec}\,(R/I) \arrow[r] \arrow[dd, hook] & X \arrow[d, "f"] \\
 & Y \arrow[d, "g"] \\
\mathrm{Spec}\,(R) \arrow[r] & Z,
\end{tikzcd}$$

where $I^2 = 0$. Since g is étale, there exists a unique morphism $h_1 : \mathrm{Spec}\,(R) \to Y$ of k-schemes such that the following diagram of k-schemes commutes:

$$\begin{tikzcd}
\operatorname{Spec}(R/I) \arrow[r] \arrow[dd, hook] & X \arrow[d, "f"] \\
 & Y \arrow[d, "g"] \\
\operatorname{Spec}(R) \arrow[r] \arrow[ur, dashed, "\exists! h_1"'] & Z.
\end{tikzcd}$$

Since f is étale, there exists a unique morphism $h_2 : \operatorname{Spec}(R) \to X$ of k-schemes such that the following diagram of k-schemes commutes:

$$\begin{tikzcd}
\operatorname{Spec}(R/I) \arrow[r] \arrow[dd, hook] & X \arrow[d, "f"] \\
 & Y \arrow[d, "g"] \\
\operatorname{Spec}(R) \arrow[r] \arrow[ur, dashed, "\exists! h_1"'] \arrow[uur, dashed, "\exists! h_2"] & Z.
\end{tikzcd}$$

Thus, $g \circ f$ is étale.

(b) Let $g' : Y' \to Y$ be a morphism of k-schemes. Consider the following diagram of k-schemes

$$\begin{tikzcd}
\operatorname{Spec}(R/I) \arrow[r] \arrow[d, hook] & X \times_Y Y' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\
\operatorname{Spec}(R) \arrow[r] & Y' \arrow[r, "g'"'] & Y
\end{tikzcd}$$

Since f is étale, there exists a unique morphism $h_1 : \operatorname{Spec}(R) \to X$ of k-schemes such that the following diagram of k-schemes commutes:

$$\begin{tikzcd}
\operatorname{Spec}(R/I) \arrow[r] \arrow[d, hook] & X \times_Y Y' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\
\operatorname{Spec}(R) \arrow[r] \arrow[urr, dashed, "\exists! h_1"'] & Y' \arrow[r, "g'"'] & Y
\end{tikzcd}$$

By the universal property of the fiber product $X \times_Y Y'$, there exists a unique morphism $h_2 : \operatorname{Spec}(R) \to X$ of k-schemes such that the following diagram of k-schemes commutes:

$$\begin{tikzcd}
\operatorname{Spec}(R/I) \arrow[r] \arrow[d, hook] & X \times_Y Y' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\
\operatorname{Spec}(R) \arrow[r] \arrow[urr, dashed, "\exists! h_1"'] \arrow[ur, dashed, "\exists! h_2"] & Y' \arrow[r, "g'"'] & Y
\end{tikzcd}$$

Thus, $f' : X \times_Y Y' \to Y'$ is étale.

□

Definition 5.3. Let k be a field and $f : F \to G$ be a morphism of sheaves on Sch/k with the étale topology.

(i) f is *representable by schemes* if for every k-scheme T and a morphism $T \to G$ the fiber product $F \times_G T$ is a scheme.

(ii) Suppose P is a property of morphisms of schemes stable under base change and f is representable by schemes. We say that f *has property* P if for every k-scheme T the induced morphism of schemes $F \times_G T \to T$ has property P.

Definition 5.4. An *algebraic space* over a field k is a functor $X : (\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{Sets}$ such that the following hold:

(i) X is a sheaf with respect to the big étale topology,

(ii) the diagonal morphism $\Delta : X \to X \times X$ is representable by schemes, and

(iii) there exists a k-scheme U and a surjective étale morphism $U \to X$.

Morphism of algebraic spaces are morphisms of functors.

Remark 5.5. Given the condition (ii) in Definition 5.4, the condition (iii) makes sense. To see this, it suffices to show that assuming (ii), a morphism $f : U \to X$ from a k-scheme U to a sheaf X is representable by schemes. Let $g : T \to X$ be a morphism from a k-scheme T. Hence, we have the cartesian diagram

$$\begin{tikzcd}
U \times_X T \arrow[r] \arrow[d] & U \times_k T \arrow[d, "f \times g"] \\
X \arrow[r, "\Delta"] & X \times X
\end{tikzcd}$$

Since $\Delta : X \to X \times X$ is representable (by (ii)), the product $U \times_X T$ is a scheme. In particular, $f : U \to X$ is representable by schemes.

5.2 Groupoids

Definition 5.6. A *groupoid* is a category with every morphism is an isomorphism. For groupoids $\mathcal{C}$, $\mathcal{D}$, *a morphism of groupoids* $f : \mathcal{C} \to \mathcal{D}$ is a functor $f : \mathcal{C} \to \mathcal{D}$.

Example 5.7. The trivial category $*$.

Example 5.8. Let G be an abstract group. Consider a category $\mathbf{B}G$, where there is only one object $*$ and $\mathrm{Mor}(*, *) = G$. Hence, $\mathbf{B}G$ is a groupoid.

Example 5.9. Let X be a set equipped with a group action by an abstract group G. Consider a category $[X/G]$, where the objects are $x \in X$ and for any $x, x' \in X$, $\mathrm{Mor}(x, x') = \{g \in G : x' = g \cdot x\}$. Hence, $[X/G]$ is a groupoid. In particular, if X is trivial, then $[\{*\}/G] \cong \mathbf{B}G$ as groupoids.

Remark 5.10. Let H and G be abstract groups acting on sets Y and X respectively. Suppose $\phi : \mathsf{H} \to \mathsf{G}$ and $f : \mathsf{Y} \to \mathsf{X}$ are respectively a group homomorphism and a map of sets such that the following diagram commutes:

$$\begin{tikzcd}
\mathsf{H} \times \mathsf{Y} \arrow[r, "\sigma_Y"] \arrow[d, "\phi \times f"'] & \mathsf{Y} \arrow[d, "f"] \\
\mathsf{G} \times \mathsf{X} \arrow[r, "\sigma_X"] & \mathsf{X}
\end{tikzcd}$$

where σ_X and σ_Y are the corresponding action morphisms. Then, there is a natural morphism $F : [\mathsf{Y}/\mathsf{H}] \to [\mathsf{X}/\mathsf{G}]$ sending $y \mapsto f(y)$. In particular, if X and Y are trivial, then ϕ induces a morphism of groupoids $\Phi : \mathbf{B}\mathsf{H} \to \mathbf{B}\mathsf{G}$. If there is no confusion, we will also write $\phi : \mathbf{B}\mathsf{H} \to \mathbf{B}\mathsf{G}$.

Proposition 5.11. *Let $\phi : \mathsf{H} \to \mathsf{G}$ be a homomorphism of abstract groups. Then the following 2-diagram of groupoids is cartesian:*

$$\begin{tikzcd}
{[\mathsf{G}/\mathsf{H}]} \arrow[r, "p_1"] \arrow[d, "p_2"] \arrow[dr, phantom, "\Swarrow \alpha"] & * \arrow[d, "i"] \\
\mathbf{B}\mathsf{H} \arrow[r, "\phi"'] & \mathbf{B}\mathsf{G},
\end{tikzcd}$$

where $\alpha_g := g$ for any $g \in \mathsf{G}/\mathsf{H}$.

Proof. Suppose $p : \mathcal{T} \to *$ and $q : \mathcal{T} \to \mathbf{B}\mathsf{H}$ are morphisms of groupoids and $\kappa : f \circ p \to g \circ q$ is a natural transformation such that the following 2-diagram commutes:

$$\begin{tikzcd}
\mathcal{T} \arrow[drr, bend left, "p"] \arrow[ddr, bend right, "q"'] & & \\
& \arrow[phantom, "\overset{\kappa}{\Swarrow}"] & * \arrow[d, "i"] \\
& \mathbf{B}\mathsf{H} \arrow[r, "\phi"'] & \mathbf{B}\mathsf{G}.
\end{tikzcd}$$

For any morphism $f : t \to t'$ in $\mathcal{T}$, we have the following commutative diagram in $\mathbf{B}\mathsf{G}$:

$$\begin{tikzcd}
* \arrow[r, "\kappa_t"] \arrow[d, equal] & * \arrow[d, "\phi(q(f))"] \\
* \arrow[r, "\kappa_{t'}"'] & *.
\end{tikzcd}$$

Define a natural transformation $r : \mathcal{T} \to [\mathsf{G}/\mathsf{H}]$ as follows:

(i) for each object $t \in \mathcal{T}$, define $r(t) := \kappa_t$, and

(ii) for each morphism $f : t \to t' \in \mathcal{T}$, define $r(f) := q(f)$.

Hence, the following 2-diagram commutes:

$$\begin{tikzcd}
\mathcal{T} \arrow[rrd, bend left, "p"] \arrow[rd, "r"] \arrow[rdd, bend right, "q"'] & & \\
& {[G/H]} \arrow[r, "p_1"] \arrow[d, "p_2"] \arrow[rd, phantom, "\swarrow \alpha"] & * \arrow[d, "i"] \\
& \mathbf{B}H \arrow[r, "\phi"'] & \mathbf{B}G.
\end{tikzcd}$$

It is also straightforward to check that r is essentially unique (see Remark 5.12). $\square$

Remark 5.12. This remark contains the meaning of the universal property of fiber products of groupoids in the general settings. However, when interpreted in the settings in Proposition 5.11, most of it becomes straightforward.
Let $f : \mathcal{C} \to \mathcal{D}$ and $g : \mathcal{D}' \to \mathcal{D}$ be morphisms of groupoids. Suppose $p : \mathcal{T} \to \mathcal{C}$ and $q : \mathcal{T} \to \mathcal{D}'$ are morphisms of groupoids and $\kappa : f \circ p \to g \circ q$ is a natural transformation such that the following 2-diagram commutes:

$$\begin{tikzcd}
\mathcal{T} \arrow[rrd, bend left, "p"] \arrow[rdd, bend right, "q"'] \arrow[rd, phantom, "\kappa \swarrow"] & & \\
& & \mathcal{C} \arrow[d, "f"] \\
& \mathcal{D}' \arrow[r, "g"'] & \mathcal{D}
\end{tikzcd}$$

Then, there exist a morphism of groupoids $r : \mathcal{T} \to \mathcal{C} \times_{\mathcal{D}} \mathcal{D}'$ and natural transformations $\alpha : p \to p_1 \circ r$, $\beta : q \to p_2 \circ r$ such that the following 2-diagram commutes:

$$\begin{tikzcd}
\mathcal{T} \arrow[rrd, bend left, "p"] \arrow[rd, "\exists r"] \arrow[rdd, bend right, "q"'] & \swarrow \alpha & \\
\nearrow \beta & \mathcal{C} \times_{\mathcal{D}} \mathcal{D}' \arrow[r, "p_1"] \arrow[d, "p_2"] \arrow[rd, phantom, "\swarrow \delta"] & \mathcal{C} \arrow[d, "f"] \\
& \mathcal{D}' \arrow[r, "g"'] & \mathcal{D}
\end{tikzcd}$$

Moreover, the morphism of groupoids $r : \mathcal{T} \to \mathcal{C} \times_{\mathcal{D}} \mathcal{D}'$ is essentially unique, i.e., if there are a morphism of groupoids $r' : \mathcal{T} \to \mathcal{C} \times_{\mathcal{D}} \mathcal{D}'$, and natural transformations $\mu : p_1 r \to p_1 r'$, $\nu : p_2 r \to p_2 r'$ such that $(\delta r')(f\mu) = (g\nu)(\delta r)$, there exists a unique natural transformation $\gamma : r \to r'$ such that $p_1\gamma = \mu$ and $p_2\gamma = \nu$.

5.3 Prestacks

Definition 5.13. Let k be a field. A functor $p : \mathcal{X} \to \mathrm{Sch}/k$ is a *prestack over* k such that the followings hold.

(1) For every diagram

$$\begin{tikzcd}
\mathsf{a} \arrow[r, dashed, "\exists \phi"] \arrow[d, mapsto, dashed] & \mathsf{b} \arrow[d, mapsto] \\
\mathsf{S} \arrow[r, "\mathsf{f}"] & \mathsf{T}
\end{tikzcd}$$

of solid arrows, there exists a morphism $\phi : \mathsf{a} \to \mathsf{b}$ such that $\mathsf{p}(\phi) = \mathsf{f}$.

(2) For every diagram

$$\begin{tikzcd}
\mathsf{a} \arrow[r, dashed, "\exists ! \lambda"] \arrow[rr, bend left, "\psi"] \arrow[d, mapsto] & \mathsf{b} \arrow[r, "\phi"] \arrow[d, mapsto] & \mathsf{c} \arrow[d, mapsto] \\
\mathsf{p}(\mathsf{a}) \arrow[r, "\mathsf{h}"] & \mathsf{p}(\mathsf{b}) \arrow[r, "\mathsf{p}(\phi)"] & \mathsf{p}(\mathsf{c})
\end{tikzcd}$$

of solid arrows, there exists a unique morphism $\lambda : \mathsf{a} \to \mathsf{b}$ such that $\mathsf{p}(\lambda) = \mathsf{h}$ and $\psi = \phi \circ \lambda$.

Definition 5.14. Let k be a field and $\mathsf{p} : \mathcal{X} \to \mathrm{Sch}/\mathsf{k}$ be a prestack over k. For any k-scheme T, the *fiber category* $\mathcal{X}(\mathsf{T})$ is the category whose objects are $\mathsf{a} \in \mathcal{X}$ such that $\mathsf{p}(\mathsf{a}) = \mathsf{T}$ and whose morphisms are $\mathsf{a} \to \mathsf{b} \in \mathcal{X}$ such that $\mathsf{p}(\mathsf{a} \to \mathsf{b}) = \mathrm{id}_{\mathsf{T}}$.

Lemma 5.15. *Let* $\mathsf{p} : \mathcal{X} \to \mathrm{Sch}/\mathsf{k}$ *be a prestack. For every* k*-scheme* T*, the fiber category* $\mathcal{X}(\mathsf{T})$ *is a groupoid.*

Proof. Let $\mathsf{p} : \mathcal{X} \to \mathrm{Sch}/\mathsf{k}$ be a prestack and T be a k-scheme. For a morphism $\phi : \mathsf{a} \to \mathsf{b}$ in $\mathcal{X}(\mathsf{T})$, consider the following diagram

$$\begin{tikzcd}
\mathsf{b} \arrow[r, dashed, "\exists ! \lambda"'] \arrow[rr, bend left, equal] \arrow[d, mapsto] & \mathsf{a} \arrow[r, "\phi"'] \arrow[d, mapsto] & \mathsf{b} \arrow[d, mapsto] \\
\mathsf{T} \arrow[r, equal] & \mathsf{T} \arrow[r, equal] & \mathsf{T}
\end{tikzcd}$$

of solid arrows. There exists a unique morphism $\lambda : \mathsf{b} \to \mathsf{a}$ such that $\mathrm{id}_{\mathsf{b}} = \phi \circ \lambda$. Similarly, consider the following diagram

$$\begin{tikzcd}
\mathsf{a} \arrow[r, dashed, "\exists ! \phi'"'] \arrow[rr, bend left, equal] \arrow[d, mapsto] & \mathsf{b} \arrow[r, "\lambda"'] \arrow[d, mapsto] & \mathsf{a} \arrow[d, mapsto] \\
\mathsf{T} \arrow[r, equal] & \mathsf{T} \arrow[r, equal] & \mathsf{T}
\end{tikzcd}$$

of solid arrows. There exists a unique morphism $\phi' : \mathsf{a} \to \mathsf{b}$ such that $\mathrm{id}_{\mathsf{a}} = \lambda \circ \phi'$. Note that

$$\phi' = \mathrm{id}_{\mathsf{b}} \circ \phi' = \phi \circ \lambda \circ \phi' = \phi \circ \mathrm{id}_{\mathsf{a}} = \phi.$$

In particular, ϕ is an isomorphism and so $\mathcal{X}(\mathsf{T})$ is a groupoid. $\square$

Definition 5.16. Let $p_{\mathcal{X}} : \mathcal{X} \to \mathrm{Sch}/k$ and $p_{\mathcal{Y}} : \mathcal{Y} \to \mathrm{Sch}/k$ be prestacks.

(i) A *morphism of prestacks* $f : \mathcal{X} \to \mathcal{Y}$ is a functor $f : \mathcal{X} \to \mathcal{Y}$ such that $p_{\mathcal{X}}(a) = p_{\mathcal{Y}}(f(a))$ for every object $a \in \mathcal{X}$.

(ii) If $f, g : \mathcal{X} \to \mathcal{Y}$ are morphisms of prestacks, a *2-morphism* $\alpha : f \to g$ is a natural transformation $\alpha : f \to g$ such that for every object $a \in \mathcal{X}$, the morphism $\alpha_a : f(a) \to g(a)$ is in $\mathcal{X}(p_{\mathcal{X}}(a))$. (This makes sense as α_a is in fact an isomorphism.)

(iii) We define $\mathrm{MOR}\,(\mathcal{X}, \mathcal{Y})$ to be the category whose objects are morphisms of prestacks from $\mathcal{X}$ to $\mathcal{Y}$ and arrows are 2-morphisms.

Example 5.17. Schemes are prestacks. Let X be a scheme over a field k. Consider a functor $p : \mathcal{X}_X \to \mathrm{Sch}/k$ such that for each k-scheme T, the objects over T are elements of $X(T)$. A morphism from $x' \in X(T')$ to $x \in X(T)$ is a morphism $f : T' \to T$ such that $x' = x \circ f$.

Definition 5.18. Let G be an algebraic group acting on a k-scheme X. We define the *quotient prestack*, denoted $[X/G]^{pre}$, as $p : [X/G]^{pre} \to \mathrm{Sch}/k$ such that for each k-scheme T, the objects over T are those in the groupoid $X(T)$. A morphism from $x' \in X(T')$ to $x \in X(T)$ is the data of a morphism $f : T' \to T$ and an element $g \in G(T')$ such that $x' = g \cdot (x \circ f)$.

If $X = \mathrm{Spec}\,(k)$, then we call $[\mathrm{Spec}\,(k)\,/G]^{pre}$ *the classifying prestack* $\mathbf{B}G^{pre}$.

Remark 5.19. Let $\phi : H \to G$ be a morphism of algebraic groups. For a k-scheme T, ϕ induces a morphism of groupoids $\Phi_T : [(\mathrm{Spec}\,(k))(T)/H(T)] \to [(\mathrm{Spec}\,(k))(T)/G(T)]$. Hence, there is an induced morphism of prestacks $\Phi : \mathbf{B}H^{pre} \to \mathbf{B}G^{pre}$.

Proposition 5.20. *Let* $\phi : H \to G$ *be a homomorphism of algebraic groups. Then the following diagram of prestacks is cartesian:*

$$\begin{tikzcd}
{[G/H]^{pre}} \arrow[r, "p_1"] \arrow[d, "p_2"] & \mathrm{Spec}\,(k) \arrow[d, "i"] \\
\mathbf{B}H^{pre} \arrow[r, "\phi"'] & \mathbf{B}G^{pre}.
\end{tikzcd}$$

Proof. It suffices to show that for each k-scheme T the following 2-diagram of groupoids commutes:

$$\begin{tikzcd}
{[G(T)/H(T)]} \arrow[r, "p_1"] \arrow[d, "p_2"] \arrow[dr, phantom, "\Swarrow \alpha"] & * \arrow[d, "i"] \\
\mathbf{B}H(T) \arrow[r, "\Phi_T"'] & \mathbf{B}G(T),
\end{tikzcd}$$

where $\alpha_g := g$ for any $g \in G(T)/H(T)$. By Proposition 5.11, this is true. □

5.4 Stacks

Definition 5.21. A prestack $p : \mathcal{X} \to \mathrm{Sch}/k$ over a site $\mathcal{C}$ is a *stack* if for all coverings $\{S_i \to S\}$ of a k-scheme S the followings hold.

(i) For objects $a, b \in \mathcal{X}(S)$ and morphisms $\phi_i : a|_{S_i} \to b$ such that $\phi_i|_{S_{ij}} = \phi_j|_{S_{ij}}$ as displayed in the diagram

$$\begin{tikzcd}
 & a|_{S_i} \arrow[rr] \arrow[rrr, bend left, "\phi_i"] \arrow[dd, mapsto] & & a \arrow[r, dashed] \arrow[dd, mapsto] & b \arrow[dd, mapsto] \\
a|_{S_{ij}} \arrow[ur] \arrow[rr] \arrow[dd, mapsto] & & a|_{S_j} \arrow[ur] \arrow[urr, bend right, "\phi_j"'] \arrow[dd, mapsto] & & \\
 & S_i \arrow[rr] & & S \arrow[r, equal] & S, \\
S_{ij} \arrow[ur] \arrow[rr] & & S_j \arrow[ur] & &
\end{tikzcd}$$

there exists a unique morphism $\phi : a \to b$ in $\mathcal{X}(S)$ such that $\phi|_{S_i} = \phi_i$.

(ii) For objects $a_i \in \mathcal{X}(S_i)$ and isomorphisms $\alpha_{ij} : a_i|_{S_{ij}} \to a_j|_{S_{ij}}$ in $\mathcal{X}(S_{ij})$ as displayed in the diagram

$$\begin{tikzcd}
 & a_i \arrow[rrr, dashed] \arrow[dd, mapsto] & & & a \arrow[dd, mapsto] \\
a_i|_{S_{ij}} \arrow[ur] \arrow[rr, "\alpha_{ij}"] \arrow[dd, mapsto] & & a_i|_{S_{ij}} \arrow[r] \arrow[dd, mapsto] & a_j \arrow[ur, dashed] \arrow[dd, mapsto] & \\
 & S_i \arrow[rrr] & & & S \\
S_{ij} \arrow[ur] \arrow[rr, equal] & & S_{ij} \arrow[r] & S_j \arrow[ur] &
\end{tikzcd}$$

satisfying the cocycle condition $\alpha_{ik}|_{S_{ijk}} \circ \alpha_{ij}|_{S_{ijk}} = \alpha_{ik}|_{S_{ijk}}$ in $\mathcal{X}(S_{ijk})$, then there exists an object $a \in \mathcal{X}(S)$ and isomorphisms $\phi_i : a|_{S_i} \to a_i$ in $\mathcal{X}(S_i)$ such that $\alpha_{ij} \circ \phi_i|_{S_{ij}} = \phi_j|_{S_{ij}}$ in $\mathcal{X}(S_{ij})$.

Morphisms of stacks are morphism of prestacks.

Proposition 5.22 (Stackification)**.** *Let $\mathcal{X}$ be a prestack over a site $\mathcal{C}$. There exists a stack $\mathcal{X}^{st}$ and a morphism $\mathcal{X} \to \mathcal{X}^{st}$ of prestacks such that for every stack $\mathcal{Y}$ over $\mathcal{C}$, the induced functor*

$$\mathrm{MOR}\left(\mathcal{X}^{st}, \mathcal{Y}\right) \to \mathrm{MOR}\left(\mathcal{X}, \mathcal{Y}\right)$$

is an equivalence of categories.

Proof. See [Alp22, Theorem 2.4.13] or [Ols16, Theorem 4.6.5] □

Remark 5.23. The stack $\mathcal{X}^{\mathsf{st}}$ in Proposition 5.22 is called the *stackification* of $\mathcal{X}$. In fact, the usual universal property argument gives us that the stackification commutes with taking fiber product, i.e., $(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})^{\mathsf{st}} \cong \mathcal{X}^{\mathsf{st}} \times_{\mathcal{Z}^{\mathsf{st}}} \mathcal{Y}^{\mathsf{st}}$.

Definition 5.24. Let G be an algebraic group acting on a k-scheme X. We define the *quotient stack*, denoted $[\mathsf{X}/\mathsf{G}]$, as the stackification of the quotient prestack $[\mathsf{X}/\mathsf{G}]^{\mathsf{pre}}$.

If $\mathsf{X} = \mathrm{Spec}\,(\mathsf{k})$, then we call $[\mathrm{Spec}\,(\mathsf{k})\,/\mathsf{G}]$ *the classifying stack* $\mathbf{B}\mathsf{G}$.

Proposition 5.25. *Let* $\phi : \mathsf{H} \to \mathsf{G}$ *be a homomorphism of algebraic groups. Then the following diagram of stacks is cartesian:*

$$\begin{tikzcd}
{[\mathsf{G}/\mathsf{H}]} \arrow[r] \arrow[d] & \mathrm{Spec}\,(\mathsf{k}) \arrow[d, "i"] \\
\mathbf{B}\mathsf{H} \arrow[r, "\phi"'] & \mathbf{B}\mathsf{G}.
\end{tikzcd}$$

Proof. This follows from Proposition 5.20 and Remark 5.23. □

Definition 5.26. A morphism of stacks $\mathsf{f} : \mathcal{X} \to \mathcal{Y}$ is *representable* if for every scheme U and morphism $\mathsf{U} \to \mathcal{Y}$, the fiber product

$$\mathcal{X} \times_{\mathcal{Y}} \mathsf{U}$$

is an algebraic space.

Definition 5.27. Let k be a field. A stack $\mathcal{X}/\mathsf{k}$ is an *algebraic stack* if the following hold:

(i) the diagonal morphism $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable, and

(ii) there exists a k-scheme X and a surjective smooth morphism $\mathsf{X} \to \mathcal{X}$.

Morphism of algebraic stacks are morphisms of stacks.

Remark 5.28. Given the condition (i) in Definition 5.27, the condition (ii) makes sense. To see this, it suffices to show that assuming (i), a morphism $\mathsf{f} : \mathsf{X} \to \mathcal{X}$ from a k-scheme T to a stack $\mathcal{X}$ is representable. Let $\mathsf{g} : \mathsf{T} \to \mathcal{X}$ be a morphism from a k-scheme T. Hence, we have the cartesian diagram

$$\begin{tikzcd}
\mathsf{X} \times_{\mathcal{X}} \mathsf{T} \arrow[r] \arrow[d] & \mathsf{X} \times_{\mathsf{k}} \mathsf{T} \arrow[d, "\mathsf{f} \times \mathsf{g}"] \\
\mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times \mathcal{X}
\end{tikzcd}$$

Since $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable (by (i)), the product $\mathsf{X} \times_{\mathcal{X}} \mathsf{T}$ is an algebraic space. In particular, $\mathsf{f} : \mathsf{X} \to \mathcal{X}$ is representable.

Proposition 5.29. *For an algebraic group* G*, the classifying stack* $\mathbf{B}\mathsf{G}$ *is an algebraic stack.*

Proof. See [Alp22, Theorem 3.1.11]. □

6 Θ-completeness

6.1 Θ-complete morphisms

Definition 6.1. Consider the standard $\mathbb{G}_m$-action on $\mathbb{A}^1$. We define the *quotient stack theta*, denoted Θ, to be the quotient stack $[\mathbb{A}^1/\mathbb{G}_m]$.

For any discrete valuation k-algebra R, we define

$$\Theta_R := \Theta \times \operatorname{Spec}(R) .$$

Remark 6.2. There are two orbits in the standard $\mathbb{G}_m$-action on $\mathbb{A}^1$. Hence, the quotient stack Θ has two points. In fact, there is a morphism of stacks $\mathbf{B}\mathbb{G}_m \overset{0}{\hookrightarrow} \Theta$, where the image of $\mathbf{B}\mathbb{G}_m$ is the zero orbit. There is also an open morphism of stacks $\operatorname{Spec}(k) \xrightarrow{1} \Theta$, where the image of $\operatorname{Spec}(k)$ is the non-zero orbit. To see that the morphism $\operatorname{Spec}(k) \xrightarrow{1} \Theta$ is open, consider the following cartesian diagram of algebraic stacks

$$\begin{tikzcd}
\mathbb{A}^1 \setminus 0 \arrow[r, hook] \arrow[d] & \mathbb{A}^1 \arrow[d] \\
\operatorname{Spec}(k) \arrow[r, "1"] & \Theta.
\end{tikzcd}$$

By descent, the morphism $\operatorname{Spec}(k) \xrightarrow{1} \Theta$ is open.

Remark 6.3. Let R be a discrete valuation k-algebra and K be its fraction field. If $0 \in \Theta_R$ is the unique closed point, then

$$\Theta_R \setminus 0 = \operatorname{Spec}(R) \bigcup_{\operatorname{Spec}(K)} \Theta_K .$$

Definition 6.4. Let Θ denote the stack $[\mathbb{A}^1/\mathbb{G}_m]$ over $\operatorname{Spec}(k)$ and $\Theta_R := \Theta \times \operatorname{Spec}(R)$ for every k-algebra R. A morphism $f : \mathcal{X} \to \mathcal{Y}$ of noetherian algebraic stacks is Θ-*complete* if for every discrete valuative k-algebra R and every commutative diagram of algebraic stacks:

$$\begin{tikzcd}
\Theta_R \setminus 0 \arrow[r] \arrow[d, hook] & \mathcal{X} \arrow[d, "f"] \\
\Theta_R \arrow[r] & \mathcal{Y},
\end{tikzcd}$$

there exists a unique morphism of algebraic stacks $h : \Theta_R \to \mathcal{X}$ such that the following diagram of algebraic stacks commutes:

$$\begin{tikzcd}
\Theta_R \setminus 0 \arrow[r] \arrow[d, hook] & \mathcal{X} \arrow[d, "f"] \\
\Theta_R \arrow[r] \arrow[ur, dashed, "\exists h"] & \mathcal{Y}.
\end{tikzcd}$$

An algebraic stack $\mathcal{X}$ is Θ-*complete* if the morphism $\mathcal{X} \to \operatorname{Spec}(k)$ is Θ-complete.

Proposition 6.5. *Let* $f : \mathcal{X} \to \mathcal{Y}$ *be a* Θ*-complete morphism of noetherian algebraic stacks.*

(a) *If* $g : \mathcal{Y} \to \mathcal{Z}$ *is a* Θ*-complete morphism, then* $g \circ f : \mathcal{X} \to \mathcal{Z}$ *is* Θ*-complete.*

(b) *If* $g' : \mathcal{Y}' \to \mathcal{Y}$ *is a morphism of noetherian algebraic stacks over* $\mathrm{Spec}\,(\Bbbk)$*, then* $f' : \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \to \mathcal{Y}'$ *is* Θ*-complete.*

Proof.

(a) Let R be a discrete valuative k-algebra. Consider a commutative diagram

$$\begin{tikzcd}
\Theta_R \setminus 0 \arrow[r] \arrow[dd, hook] & \mathcal{X} \arrow[d, "f"] \\
 & \mathcal{Y} \arrow[d, "g"] \\
\Theta_R \arrow[r] & \mathcal{Z}
\end{tikzcd}$$

Since $g : \mathcal{Y} \to \mathcal{Z}$ is Θ-complete, there exists a unique morphism $h_1 : \Theta_R \to \mathcal{Y}$ that makes the diagram commute, i.e., the following diagram commutes.

$$\begin{tikzcd}
\Theta_R \setminus 0 \arrow[r] \arrow[dd, hook] & \mathcal{X} \arrow[d, "f"] & & \Theta_R \setminus 0 \arrow[r] \arrow[dd, hook] & \mathcal{X} \arrow[d, "f"] \\
 & \mathcal{Y} \arrow[d, "g"] & \Rightarrow & & \mathcal{Y} \arrow[d, "g"] \\
\Theta_R \arrow[r] \arrow[ur, dashed, "\exists! h_1"'] & \mathcal{Z} & & \Theta_R \arrow[r] \arrow[ur, dashed, "\exists! h_1"'] \arrow[uur, dashed, "\exists! h_2"] & \mathcal{Z}
\end{tikzcd}$$

Similarly, since $f : \mathcal{X} \to \mathcal{Y}$ is Θ-complete, there exists a unique morphism $h_2 : \Theta_R \to \mathcal{X}$ that makes the diagram commute. Thus, $g \circ f : \mathcal{X} \to \mathcal{Z}$ is Θ-complete.

(b) Let R be a discrete valuative k-algebra. Consider a commutative diagram

$$\begin{tikzcd}
\Theta_R \setminus 0 \arrow[r] \arrow[d, hook] & \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\
\Theta_R \arrow[r] & \mathcal{Y}' \arrow[r, "g'"] & \mathcal{Y}
\end{tikzcd}$$

Since $f : \mathcal{X} \to \mathcal{Y}$ is Θ-complete, there exists $h_1 : \Theta_R \to \mathcal{X}$ that makes the diagram commute, i.e., the following diagram commutes.

$$\begin{tikzcd}
\Theta_R \setminus 0 \arrow[r] \arrow[d, hook] & \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\
\Theta_R \arrow[r] \arrow[urr, dashed, "\exists! h_1"'] & \mathcal{Y}' \arrow[r, "g'"] & \mathcal{Y}
\end{tikzcd}$$

Because of the universal property of the fiber product $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$, there exists a unique morphism $h_2 : \theta_R \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ that makes the diagram commute, i.e., the following diagram commutes.

$$\begin{tikzcd}
\Theta_R \setminus 0 \arrow[r] \arrow[d, hook] & \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\
\Theta_R \arrow[r] \arrow[ur, dashed, "\exists ! h_2"] \arrow[urr, dashed, "\exists ! h_1"] & \mathcal{Y}' \arrow[r, "g'"] & \mathcal{Y}
\end{tikzcd}$$

□

Now, we give a nice lemma regarding the pullback of Θ-completeness via an affine morphism.

Lemma 6.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ is an affine morphism of noetherian algebraic stacks. If $\mathcal{Y}$ is Θ-complete, then $\mathcal{X}$ is Θ-complete.*

Proof. Let R be a discrete valuative k-algebra. Consider the following commutative diagram of algebraic stacks:

$$\begin{tikzcd}
\Theta_R \setminus 0 \arrow[r] \arrow[dd, hook] & \mathcal{X} \arrow[d, "f"] \\
 & \mathcal{Y} \arrow[d] \\
\Theta_R \arrow[r] & \operatorname{Spec}(k)
\end{tikzcd}$$

Since $\mathcal{Y}$ is Θ-complete, there exists a morphism $h : \Theta_R \to \mathcal{Y}$ of stacks such that the following diagram of algebraic stacks commutes:

$$\begin{tikzcd}
\Theta_R \setminus 0 \arrow[r] \arrow[dd, hook] & \mathcal{X} \arrow[d, "f"] \\
 & \mathcal{Y} \arrow[d] \\
\Theta_R \arrow[r] \arrow[ur, dashed, "\exists h_1"] & \operatorname{Spec}(k) .
\end{tikzcd}$$

Note that Θ_R is regular and $0 \in \Theta_R$ is codim 2 so pushforward of structure sheaf along $\Theta_R \setminus 0 \to \Theta_R$ is the structure sheaf. Hence,

$$\begin{aligned}
\mathrm{MOR}_{\mathcal{Y}}(\Theta_R \setminus 0, \mathcal{X}) &\cong \mathrm{MOR}_{\mathcal{O}_{\mathcal{Y}}-\mathrm{alg}}\left(f_* \mathcal{O}_{\mathcal{X}}, (\mathcal{O}_R \setminus 0 \to \mathcal{Y})_* \mathcal{O}_{\Theta_R \setminus 0}\right) \\
&\cong \mathrm{MOR}_{\mathcal{O}_{\mathcal{Y}}-\mathrm{alg}}\left(f_* \mathcal{O}_{\mathcal{X}}, (\mathcal{O}_R \to \mathcal{Y})_* \mathcal{O}_{\Theta_R}\right) \\
&\cong \mathrm{MOR}_{\mathcal{Y}}(\Theta_R, \mathcal{X})
\end{aligned}$$

In particular, there exists a morphism $h_2 : \Theta_R \to \mathcal{X}$ of algebraic stacks such that the following diagram of algebraic stacks commutes:

$$\begin{tikzcd}
\Theta_R \setminus 0 \arrow[r] \arrow[dd, hook] & \mathcal{X} \arrow[d, "f"] \\
& \mathcal{Y} \arrow[d] \\
\Theta_R \arrow[r] \arrow[ur, "h_1"'] \arrow[uur, dashed, "\exists h_2"] & \mathrm{Spec}\,(k)\,.
\end{tikzcd}$$

Therefore, $\mathcal{X}$ is Θ-complete. $\square$

There is another way to think about Θ-complete stacks. Let $\mathcal{X}$ be an algebraic stack. Consider the stack $\underline{\mathrm{Mor}}\,(\Theta, \mathcal{X})$ and the morphism

$$ev_1 : \underline{\mathrm{Mor}}\,(\Theta, \mathcal{X}) \to \mathcal{X} \quad \textit{sending} \quad f \mapsto f(1)$$

of stacks. The stack $\mathcal{X}$ is Θ-complete if and only if ev_1 satisfies the valuative criterion for properness.

Proposition 6.7. *A noetherian algebraic stack $\mathcal{X}$ is Θ-complete if and only if the natural morphism $ev_1 : \underline{\mathrm{Mor}}_k\,(\Theta, \mathcal{X}) \to \mathcal{X}$ satisfies the valuative criterion of properness.*

Proof. Let $\mathcal{X}$ be a noetherian stack over k.
($\Rightarrow$) Assume that $\mathcal{X}$ is Θ-complete. Let R be a discrete valuation k-algebra and K be its fraction field. Suppose there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}\,(K) \arrow[r] \arrow[d, hook] & \underline{\mathrm{Mor}}_k\,(\Theta, \mathcal{X}) \arrow[d, "ev_1"] \\
\mathrm{Spec}\,(R) \arrow[r, "g"'] & \mathcal{X}.
\end{tikzcd}$$

Let $f : \Theta_K \to \mathcal{X}$ be a morphism of stacks corresponding to the morphism $\mathrm{Spec}\,(K) \to \underline{\mathrm{Mor}}_k\,(\Theta, \mathcal{X})$. In fact, we have the morphism of stacks

$$\Theta_R \setminus 0 = \mathrm{Spec}\,(R) \bigcup_{\mathrm{Spec}(K)} \Theta_K \to \mathcal{X}.$$

As $\mathcal{X}$ is Θ-complete, this morphism can be extended uniquely to $h : \Theta_R \to \mathcal{X}$. This morphism h is in one-to-one correspondence to a morphism $\tilde{h} : \mathrm{Spec}\,(R) \to \underline{\mathrm{Mor}}_k\,(\Theta, \mathcal{X})$. Hence, there exists a unique morphism $\tilde{h} : \mathrm{Spec}\,(R) \to \underline{\mathrm{Mor}}_k\,(\Theta, \mathcal{X})$ making the following diagram commute:

$$\begin{tikzcd}
\mathrm{Spec}\,(K) \arrow[r] \arrow[d, hook] & \underline{\mathrm{Mor}}_k\,(\Theta, \mathcal{X}) \arrow[d, "ev_1"] \\
\mathrm{Spec}\,(R) \arrow[r, "g"'] \arrow[ur, dashed, "\exists! \tilde{h}"] & \mathcal{X}.
\end{tikzcd}$$

Thus, the morphism $ev_1 : \underline{\mathrm{Mor}}_k(\Theta, \mathcal{X}) \to \mathcal{X}$ satisfies the valuative criterion for properness.

($\Leftarrow$) Assume that the natural morphism $ev_1 : \underline{\mathrm{Mor}}_k(\Theta, \mathcal{X}) \to \mathcal{X}$ satisfies the valuative criterion for properness. Let $s : \Theta_R \setminus 0 \to \mathcal{X}$ be a morphism of stacks. As $\Theta_R \setminus 0 = \mathrm{Spec}(R) \bigcup_{\mathrm{Spec}(K)} \Theta_K$, this gives us two morphisms

$$g : \mathrm{Spec}(R) \hookrightarrow \Theta_R \setminus 0 \xrightarrow{s} \mathcal{X} \quad \text{and} \quad f : \Theta_K \hookrightarrow \Theta_R \setminus 0 \xrightarrow{s} \mathcal{X}.$$

Let $\tilde{f} : \mathrm{Spec}(K) \to \underline{\mathrm{Mor}}_k(\Theta, \mathcal{X})$ be a morphism of stacks corresponding to f. The following diagram of stacks commutes:

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r, "\tilde{f}"] \arrow[d, hook] & \underline{\mathrm{Mor}}_k(\Theta, \mathcal{X}) \arrow[d, "ev_1"] \\
\mathrm{Spec}(R) \arrow[r, "g"'] & \mathcal{X}.
\end{tikzcd}$$

By the assumption, there exists a unique morphism $h : \mathrm{Spec}(R) \to \underline{\mathrm{Mor}}_k(\Theta, \mathcal{X})$ making the diagram commute:

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r, "\tilde{f}"] \arrow[d, hook] & \underline{\mathrm{Mor}}_k(\Theta, \mathcal{X}) \arrow[d, "ev_1"] \\
\mathrm{Spec}(R) \arrow[r, "g"'] \arrow[ur, dashed, "\exists ! h"] & \mathcal{X}.
\end{tikzcd}$$

Let $\hat{h} : \Theta_R \to \mathcal{X}$ be a morphism of stacks corresponding to $h : \mathrm{Spec}(R) \to \underline{\mathrm{Mor}}_k(\Theta, \mathcal{X})$. Thus, this morphism serves as the unique extension of $s : \Theta_R \setminus 0 \to \mathcal{X}$, i.e., the following diagram of stacks commutes:

$$\begin{tikzcd}
\Theta_R \setminus 0 \arrow[r, "s"] \arrow[d] & \mathcal{X} \\
\Theta_R. \arrow[ur, "\hat{h}"'] &
\end{tikzcd}$$

Hence, $\mathcal{X}$ is Θ-complete. □

6.2 Examples of Θ-complete stacks

Recall that a noetherian stack $\mathcal{X}$ over k is Θ-complete if the morphism $\mathcal{X} \to \mathrm{Spec}(k)$ is Θ-complete. We now begin with small examples.

Example 6.8. The stack Θ is Θ-complete.

Example 6.9. Schemes are Θ-complete. Let X be a scheme over k. Consider a discrete valuation k-algebra and its field of fractions K. Suppose there is a morphism of stacks $\phi : \Theta_R \setminus 0 \to X$. Since $\Theta_R \setminus 0 = \mathrm{Spec}(R) \bigcup_{\mathrm{Spec}(K)} \Theta_K.$, the data of ϕ is equivalent to the data of morphisms $f : \mathrm{Spec}(R) \to X$ and

$g : \Theta_K \to X$ such that they agree on Spec (K). Since there is a good moduli space Spec $(K) \to$ Spec (K), there exists a unique morphism $h :$ Spec $(K) \to X$ such that the following diagram of stacks commutes:

$$\begin{tikzcd}
\Theta_K \arrow[r, "g"] \arrow[d, "\text{g.m.s}"'] & X \\
\text{Spec}\,(K)\,. \arrow[ur, dashed, "\exists ! h"'] &
\end{tikzcd}$$

In particular, the morphism $\phi :$ Spec $(R) \bigcup_{\text{Spec}(K)} \Theta_K \to X$ factors through $f :$ Spec $(R) \to X$. Lastly, precomposing f with the projection $\Theta_R \to$ Spec (R), we have the following commutative diagram of stacks:

$$\begin{tikzcd}
\Theta_R \setminus 0 \arrow[r, equal] \arrow[d, hook] & \text{Spec}\,(R) \bigcup_{\text{Spec}(K)} \Theta_K \arrow[r, "\phi"] \arrow[d] & X \\
\Theta_R \arrow[r] & \text{Spec}\,(R)\,. \arrow[ur, "f"'] &
\end{tikzcd}$$

In particular, X is Θ-complete.

Example 6.10. The classifying stack $\mathbf{B}\mathsf{GL}_n$ is Θ-complete. Let R be a discrete valuation k-algebra. A morphism $\Theta_R \setminus 0 \to \mathbf{B}\mathsf{GL}_n$ corresponds to a vector bundle E on $\Theta_R \setminus 0$. Since Θ_R is regular and $0 \in \Theta_R$ is codim 2, the pushforward of a vector bundle E along $\Theta_R \setminus 0 \to \Theta_R$ is a vector bundle. This gives the desired extension $\Theta_R \to \mathbf{B}\mathsf{GL}_n$.

Example 6.10 above gives the question: is $\mathbf{B}R$ Θ-complete if R is a reductive group. The answer to this question is positive.

Proposition 6.11. *If R is a reductive algebraic group over a field k, then the classifying stack $\mathbf{B}R$ is Θ-complete.*

Proof. Let R be a reductive group over a field k. By Proposition 2.39, there exists n and a homomorphism $R \hookrightarrow \mathsf{GL}_n$ realizing R as a closed algebraic subgroup of GL_n. By Matsushima's Theorem (Proposition 4.23), GL_n/R is affine. Consider the cartesian diagram of algebraic stacks:

$$\begin{tikzcd}
\mathsf{GL}_n/R \arrow[r] \arrow[d] & \text{Spec}\,(k) \arrow[d] \\
\mathbf{B}R \arrow[r] & \mathbf{B}\mathsf{GL}_n
\end{tikzcd}$$

By descent, $\mathbf{B}R \to \mathbf{B}\mathsf{GL}_n$ is an affine morphism of algebraic stacks. Recall that $\mathbf{B}\mathsf{GL}_n$ is Θ-complete (see Example 6.10.) By Proposition 6.6, $\mathbf{B}R$ is Θ-complete. □

We now can ask the same question for unipotent groups. It turns out that this question has the positive answer, i.e., if U is a unipotent algebraic group, then the classifying stack $\mathbf{B}U$ Θ-complete. However, we need to state the following two propositions first.

Proposition 6.12. *Let* G *be an algebraic group over a field* k*. Then*

$$\underline{\mathrm{Mor}}\,(\Theta, \mathbf{B}\mathsf{G}) \cong \bigsqcup_{\lambda} \mathbf{B}\mathsf{G}_{\lambda}^{+},$$

where λ *is a representative of each conjugacy class of one-parameter subgroups of* G*.*

Proof. This is a special case of [HL22, Theorem 1.4.7]. ☐

Proposition 6.13. *Let* G *be an algebraic group over a field* k*. Then the classifying stack* $\mathbf{B}\mathsf{G}$ *is* Θ*-complete if and only if for every one-parameter subgroup* $\lambda : \mathbb{G}_m \to \mathsf{G}$*, the quotient* $\mathsf{G}/\mathsf{G}_{\lambda}^{+}$ *is projective.*

Proof. Let G be an algebraic group over a field k. By Remark 6.7, $\mathbf{B}\mathsf{G}$ is Θ-complete if and only if the morphism $\underline{\mathrm{Mor}}\,(\Theta, \mathbf{B}\mathsf{G}) \to \mathbf{B}\mathsf{G}$ satisfies the valuative criterion for properness. By Proposition 6.12, the morphism $\underline{\mathrm{Mor}}\,(\Theta, \mathbf{B}\mathsf{G}) \to \mathbf{B}\mathsf{G}$ satisfies the valuative criterion for properness if and only if for every one-parameter subgroup $\lambda : \mathbb{G}_m \to \mathsf{G}$, the morphism $\mathbf{B}\mathsf{G}_{\lambda}^{+} \to \mathbf{B}\mathsf{G}$ satisfies the valuative criterion for properness. Since for every one-parameter subgroup $\lambda : \mathbb{G}_m \to \mathsf{G}$, the morphism $\mathbf{B}\mathsf{G}_{\lambda}^{+} \to \mathbf{B}\mathsf{G}$ is quasi-compact, the morphism $\mathbf{B}\mathsf{G}_{\lambda}^{+} \to \mathbf{B}\mathsf{G}$ satisfies the valuative criterion for properness if and only if the morphism $\mathbf{B}\mathsf{G}_{\lambda}^{+} \to \mathbf{B}\mathsf{G}$ is proper. Consider the following cartesian diagram

$$\begin{tikzcd}
\mathsf{G}/\mathsf{G}_{\lambda}^{+} \arrow[r] \arrow[d] & \mathrm{Spec}\,(\mathsf{k}) \arrow[d] \\
\mathbf{B}\mathsf{G}_{\lambda}^{+} \arrow[r] & \mathbf{B}\mathsf{G}.
\end{tikzcd}$$

We see that $\mathbf{B}\mathsf{G}_{\lambda}^{+} \to \mathbf{B}\mathsf{G}$ is proper if and only if $\mathsf{G}/\mathsf{G}_{\lambda}^{+}$ is proper. By [Mil17, Theorem 8.43], the quotient $\mathsf{G}/\mathsf{G}_{\lambda}^{+}$ is always quasi-projective. Hence, $\mathbf{B}\mathsf{G}_{\lambda}^{+} \to \mathbf{B}\mathsf{G}$ is proper if and only if $\mathsf{G}/\mathsf{G}_{\lambda}^{+}$ is projective. ☐

With Proposition 6.13, we can now prove the following proposition regarding Θ-completeness of unipotent groups.

Proposition 6.14. *If* U *is a unipotent algebraic group over a field* k*, then the classifying stack* $\mathbf{B}\mathsf{U}$ *is* Θ*-complete.*

Proof. Let U be a unipotent group over a field k. By Proposition 6.13, $\mathbf{B}\mathsf{U}$ is Θ-complete if and only if for every one parameter group $\lambda : \mathbb{G}_m \to \mathsf{U}$, the quotient $\mathsf{U}/\mathsf{U}_{\lambda}^{+}$ is projective. Hence, it suffices to show that $\mathsf{U}/\mathsf{U}_{\lambda}^{+}$ is projective for any one-parameter subgroup $\lambda : \mathbb{G}_m \to \mathsf{U}$. By Proposition 4.11, every one parameter subgroup of a unipotent group is trivial. If $\lambda : \mathbb{G}_m \to \mathsf{U}$ is a one parameter subgroup, then $\mathsf{U}_{\lambda}^{+} = \mathsf{U}$. In particular, $\mathsf{U}/\mathsf{U}_{\lambda}^{+}$ is trivial and projective. Thus, $\mathbf{B}\mathsf{U}$ is Θ-complete. ☐

We end this section with the following proposition combining Θ-completeness of unipotent algebraic groups and linearly reductive algebraic groups.

Proposition 6.15. *If* $\mathsf{G} \cong \mathsf{U} \times \mathsf{R}$ *is the direct product of a unipotent group* U *and a reductive group* R*, then the classifying stack* $\mathbf{B}\mathsf{G}$ *is* Θ*-complete.*

Proof. Suppose $\mathsf{G} \cong \mathsf{U} \times \mathsf{R}$ is the direct product of a unipotent group U and a reductive group R. By Proposition 6.14 and 6.11, $\mathbf{B}\mathsf{U}$ and $\mathbf{B}\mathsf{R}$ are Θ-complete. Hence, consider the cartesian diagram of algebraic stacks:

$$\begin{tikzcd}
\mathbf{B}\mathsf{G} \cong \mathbf{B}\mathsf{U} \times \mathbf{B}\mathsf{R} \arrow[r] \arrow[d] & \mathbf{B}\mathsf{R} \arrow[d] \\
\mathbf{B}\mathsf{U} \arrow[r] & \mathrm{Spec}\,(\mathsf{k})
\end{tikzcd}$$

By Proposition 6.5, Θ-reducitve morphisms of algebraic stacks are preserved under base change and composition. In particular, the morphism $\mathbf{B}\mathsf{G} \to \mathbf{B}\mathsf{R}$ is Θ-complete and so is the composition $\mathbf{B}\mathsf{G} \to \mathbf{B}\mathsf{R} \to \mathrm{Spec}\,(\mathsf{k})$. Thus, $\mathbf{B}\mathsf{G}$ is Θ-complete. ☐

7 Main Result

The goal of this section is to prove our main theorem (Theorem 7.1.) Recall that k denotes an algebraically closed field of characteristic 0.

Theorem 7.1. *Let* G *be a connected algebraic group over* k*. The classifying stack* $\mathbf{B}G$ *is* Θ*-complete if and only if* $G \cong U \times R$ *for a unipotent algebraic group* U *and a reductive algebraic group* R*.*

Remark 7.1. Recall that char $k = 0$. By Proposition 4.26, any connected algebraic group G is the semi-direct product of a unipotent group U and a reductive group R corresponding to a group action $\sigma : R \times U \to U$, i.e, $G \cong U \rtimes_\sigma R$. To prove Theorem 7.1, it is equivalent to show that the classifying stack $\mathbf{B}G$ is Θ-complete if and only if σ is trivial.

For now, we assume that the reductive component of G is $\mathbb{G}_m$. In fact, the following examples provide insights on how to prove this case of the reductive component being $\mathbb{G}_m$.

Example 7.2. Consider $G = \mathbb{G}_a \rtimes \mathbb{G}_m$, where $\sigma : \mathbb{G}_m \times \mathbb{G}_a \to \mathbb{G}_a$ sends $(a, b) \mapsto ab$ (weight 1.) We can view G as a closed subgroup of GL_2 as

$$G \cong \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} | a \neq 0, b \in k \right\}.$$

Consider $\lambda_{-1} : \mathbb{G}_m \to G$ sending $t \mapsto (1, t^{-1})$. Hence,

$$G^+_{\lambda_{-1}} \cong \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \middle| \lim_{t \to 0} \begin{bmatrix} a & t^{-1}b \\ 0 & 1 \end{bmatrix} \right\} \cong \left\{ \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} | a \neq 0, \right\} \cong \mathbb{G}_m.$$

In particular, $G/G^+_{\lambda_{-1}} \cong \mathbb{A}^1$ is not projective. By Proposition 6.13, the classifying stack $\mathbf{B}G$ is not Θ-complete.

Example 7.3. Consider $G = \mathbb{G}_a \rtimes \mathbb{G}_m$, where $\sigma : \mathbb{G}_m \times \mathbb{G}_a \to \mathbb{G}_a$ sends $(a, b) \mapsto a^{-1}b$ (weight -1.) We can view G as a closed subgroup of GL_2 as

$$G \cong \left\{ \begin{bmatrix} a^{-1} & b \\ 0 & 1 \end{bmatrix} | a \neq 0, b \in k \right\}.$$

Consider $\lambda_1 : \mathbb{G}_m \to G$ sending $t \mapsto (1, t)$. Hence,

$$G^+_{\lambda_1} \cong \left\{ \begin{bmatrix} a^{-1} & b \\ 0 & 1 \end{bmatrix} \middle| \lim_{t \to 0} \begin{bmatrix} a^{-1} & t^{-1}b \\ 0 & 1 \end{bmatrix} \right\} \cong \left\{ \begin{bmatrix} a^{-1} & 0 \\ 0 & 1 \end{bmatrix} | a \neq 0, \right\} \cong \mathbb{G}_m.$$

In particular, $G/G^+_{\lambda_1} \cong \mathbb{A}^1$ is not projective. By Proposition 6.13, the classifying stack $\mathbf{B}G$ is not Θ-complete.

Remark 7.4. In Example 7.3, if we were to compute $G_{\lambda_{-1}}$, where $\lambda_{-1} : \mathbb{G}_m \to G$ sending $t \mapsto (1, t^{-1})$, we would obtain the trivial quotient, i.e., $G = G_{\lambda_{-1}}$.

The following lemma is needed to prove the general case (Proposition 7.6.)

Lemma 7.5. *Let* U *be a unipotent algebraic group equipped with an* R*-action* $\sigma : R \times U \to U$ *and* $G := U \rtimes_\sigma R$ *denotes the semi-direct product. Suppose* $i : U' \hookrightarrow U$ *is an algebraic subgroup stable under the* R*-action, i.e., there is a commutative diagram*

$$\begin{tikzcd}
R \times U' \arrow[r, "\mathrm{id}_R \times i"] \arrow[d, dashed, "\exists ! \tau"'] & R \times U \arrow[d, "\sigma"] \\
U' \arrow[r, hook, "i"'] & U.
\end{tikzcd}$$

If $G' := U' \rtimes_\tau R$*, then the quotients* G/G' *and* U/U' *are isomorphic as* k*-schemes.*

Proof. It suffices to show that G/G' and R/R' are isomorphic as functors. For each k-scheme X, consider a morphism $U(X) \to G(X)$ sending $u \mapsto (u, 1)$. The image of $U'(X)$ via this map is contained in $G'(X)$. Hence, there is an induced morphism $\phi : U/U' \to G/G'$ of functors such that $\phi_X : U(X)/U'(X) \to G(X)/G'(X)$ sending $uU'(X) \mapsto (u, 1)G'(X)$. The morphism ϕ_X is surjective. Indeed, for any $(u, r) \in G(X)$,

$$(u, r)G'(X) = (u, 1)(1, r)G'(X) = (u, 1)G'(X) = \phi(uU'(X)).$$

ϕ_X is also injective. Consider $u_1U'(X), u_2U'(X) \in U(X)/U'(X)$. We have

$$\begin{aligned}
\phi(u_1U'(X)) = \phi(u_2U'(X)) &\Leftrightarrow (u_1, 1)G'(X) = (u_2, 1)G'(X) \\
&\Leftrightarrow (u_1, 1)^{-1}(u_2, 1) \in G'(X) \\
&\Leftrightarrow (u_1^{-1}u_2, 1) \in G'(X) \\
&\Leftrightarrow u_1^{-1}u_2 \in U'(X) \\
&\Leftrightarrow u_1U'(X) = u_2U'(X)
\end{aligned}$$

Thus, ϕ_X is a bijection. In particular, the quotient G/G' and U/U' are isomorphic as k-schemes. ☐

Proposition 7.6. *Let* U *be a unipotent algebraic group equipped with a* $\mathbb{G}_m$*-action* $\sigma : \mathbb{G}_m \times U \to U$ *and* $G := U \rtimes_\sigma \mathbb{G}_m$ *denote the semi-direct product. If the classifying stack* $\mathbf{B}G$ *is* Θ*-complete, then* σ *is trivial.*

Proof. For $i = \pm 1$, consider one-parameter subgroups $\lambda_i : \mathbb{G}_m \to G$ sending $t \mapsto t^i$ for $t \in \mathbb{G}_m(k)$. By Proposition 3.32, we have

$$G^+_{\lambda_1} = U^+ \rtimes_\sigma \mathbb{G}_m \text{ and } G^+_{\lambda_{-1}} = U^- \rtimes_\sigma \mathbb{G}_m.$$

By Lemma 7.5, we have

$$G/G^+_{\lambda_1} \cong U/U^+ \text{ and } G/G^+_{\lambda_{-1}} \cong U/U^-.$$

Suppose that the classifying stacks **BG** is Θ-complete. By Proposition 6.13, the connected affine schemes U/U^+ and U/U^- are projective. Hence, $U = U^+ = U^-$. In particular,

$$U = U^+ \cap U^- = U^0.$$

Therefore, σ is trivial. $\square$

The stronger version of Proposition 7.6 is also true. We begin with an example where the reductive component of G is not $\mathbb{G}_m$.

Example 7.7. Consider the GL_n-action on $\mathbb{G}_a$ defined by

$$A \cdot b = \det(A) \cdot b$$

for any $A \in GL_n$ and $b \in \mathbb{G}_a$.
Let $G := \mathbb{G}_a \rtimes GL_n$ be the semi-direct product of $\mathbb{G}_a$ and GL_n defined by this action. We can view G as a closed algebraic subgroup of GL_{n+2}, i.e.,

$$G = \left\{ \begin{bmatrix} \det A & b & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & A & \\ 0 & 0 & & & \end{bmatrix} | b \neq 0, A \in GL_n \right\}.$$

Consider a one-parameter subgroup $\lambda : \mathbb{G}_m \to G$ defined by

$$\lambda(t) = \begin{bmatrix} t^{-n} & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & t^{-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & t^{-1} \end{bmatrix}$$

for any $t \in \mathbb{G}_m$. Hence, we have the induced GL_n-action

$$t \cdot \begin{bmatrix} \det A & b & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & A & \\ 0 & 0 & & & \end{bmatrix} = \begin{bmatrix} \det A & t^{-n}b & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & A & \\ 0 & 0 & & & \end{bmatrix},$$

for any $t \in \mathbb{G}_m$ and $(b, A) \in G$. In particular,

$$G_\lambda^+ = \left\{ \begin{bmatrix} \det A & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & A & \\ 0 & 0 & & & \end{bmatrix} | A \in GL_n \right\} \cong GL_n$$

Thus, $G/G_\lambda^+ \cong \mathbb{A}^1$ is not projective. By Proposition 6.13, **BG** is not Θ-complete.

The following lemma is needed to prove the general case (Proposition 7.9.)

Lemma 7.8. *Let* U *be a unipotent algebraic group equipped with an* R*-action* $\sigma : R \times U \to U$ *and* $G := U \rtimes_\sigma R$ *denotes the semi-direct product. Suppose* $i : R' \hookrightarrow R$ *is an algebraic subgroup and* $\tau : R' \times U \to U$ *denotes the group action*

$$R' \times U \xrightarrow{i \times \mathrm{id}_U} R \times U \xrightarrow{\sigma} U.$$

If $G' := U \rtimes_\tau R'$*, then the quotients* G/G' *and* R/R' *are isomorphic as* k*-schemes.*

Proof. It suffices to show that G/G' and R/R' are isomorphic as functors. For each k-scheme X, consider a morphism $R(X) \to G(X)$ mapping $r \mapsto (1, r)$. The image of $R'(X)$ via this map is contained in $G'(X)$. Hence, there is an induced map $\phi : R/R' \to G/G'$ of functors such that $\phi_X : R(X)/R'(X) \to G(X)/G'(X)$ mapping $rR'(X) \mapsto (1, r)G'(X)$ for each k-scheme X. ϕ_X is surjective. Indeed, for any $(u, r) \in G(X)$,

$$(u, r)G'(X) = (1, r)(\sigma(r^{-1}, u), 1)G'(X) = (1, r)G'(X) = \phi(rR'(X)).$$

ϕ_X is also injective. Consider $r_1R'(X), r_2R'(X) \in R(X)/R'(X)$. We have

$$\begin{aligned}
\phi(r_1R'(X)) = \phi(r_2R'(X)) &\Leftrightarrow (1, r_1)G'(X) = (1, r_2)G'(X) \\
&\Leftrightarrow (1, r_1)^{-1}(1, r_2)G'(X) \\
&\Leftrightarrow (1, r_1^{-1}r_2) \in G'(X) \\
&\Leftrightarrow r_1^{-1}r_2 \in R'(X) \\
&\Leftrightarrow r_1R'(X) = r_2R'(X)
\end{aligned}$$

Thus, ϕ_X is a bijection. In particular, the quotients G/G' and R/R' are isomorphic as k-schemes. □

Proposition 7.9. *Let* U *be a unipotent algebraic group equipped with an* R*-action* $\sigma : R \times U \to U$ *and* $G := U \rtimes_\sigma R$ *denote the semi-direct product. If the classifying stack* $\mathbf{B}G$ *is* Θ*-complete, then* σ *is trivial.*

Proof. Assume that σ is not trivial. There exists an injective group homomorphism $i : \mathbb{G}_m \to R$ such that the restricted action $\tau : \mathbb{G}_m \to U \xrightarrow{i \times \mathrm{id}_G} R \times U \to U$ is also non-trivial (Remark 4.25.) Let G' denote the subgroup $U \rtimes_\tau \mathbb{G}_m$. By Proposition 7.6, the classifying stack $\mathbf{B}G'$ is not Θ-complete. By Lemma 7.8,

$$G/G' \cong R/\mathbb{G}_m.$$

As R and $\mathbb{G}_m$ are reductive, by Matsushima's Theorem (Proposition 4.23), the quotient $R/\mathbb{G}_m$ is affine. Hence, the quotient G/G' is affine. Consider the following cartesian diagram of stacks

$$\begin{tikzcd}
G/G' \arrow[r] \arrow[d] & \mathrm{Spec}\,(k) \arrow[d] \\
\mathbf{B}G' \arrow[r] & \mathbf{B}G.
\end{tikzcd}$$

By descent, the morphism $\mathbf{B}\mathsf{G}' \to \mathbf{B}\mathsf{G}$ is affine. By Proposition 6.6, the classifying stack $\mathbf{B}\mathsf{G}$ is not Θ-complete. $\square$

We end this section with the proof of Theorem 7.1.

Proof of Theorem 7.1. Let G be a connected algebraic group. We can write G as the semi-direct product $\mathsf{U} \rtimes_{\sigma} \mathsf{R}$, where U is unipotent, R is reductive and $\sigma : \mathsf{R} \times \mathsf{U} \to \mathsf{U}$ is a group action. To prove the theorem, it is equivalent to show that the classifying stack $\mathbf{B}\mathsf{G}$ is Θ-complete if and only if σ is trivial. (Remark 7.1).
($\Rightarrow$) Proposition 7.9
($\Leftarrow$) Proposition 6.15 $\square$

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