Brownian Motion, Quasiconformal Mappings and the Beltrami Equation

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Abstract

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Consider a Jordan domain Ω in the plane with 3 distinct points marked on its boundary. These 3 points split $\partial\Omega$ into 3 arcs. For each $z \in \Omega$, we can assign it the harmonic coordinates by taking the harmonic measures from z to each of these 3 arcs on the boundary. We showed that these harmonic coordinates uniquely characterize the interior points of Ω. In particular, we can define these harmonic coordinates for points in the unit disk $\mathbb{D}$ with 3 distinct points marked on its boundary. We further showed that we can define a map ϕ from Ω onto $\mathbb{D}$ by sending each point in Ω to the point in $\mathbb{D}$ with the same harmonic coordinates, and this map is one-to-one and analytic. We also gave an elementary proof of the conformal invariance of Brownian motion, and since harmonic measure can be defined in terms of Brownian motion, this gives us a new perspective on the Riemann mapping theorem.

Furthermore, this strategy could potentially be generalized to prove the measurable Riemann mapping theorem. In this case, the map ϕ from Ω to $\mathbb{D}$ is assumed to be a quasiconformal map satisfying the Beltrami equation $\phi_{\overline{z}} = \mu(z)\phi_z$ almost everywhere for certain Beltrami coefficient $\mu : \Omega \to \mathbb{C}$ measurable with $||\mu||_\infty \leq k < 1$. However, a Brownian motion is in general not preserved under quasiconformal maps. In order to define the correct coordinates for this strategy to work, it is necessary to consider the harmonic measure of stochastic processes which get mapped to a Brownian motion under quasiconformal maps. In this direction, we proved that we can explicitly construct the stochastic processes (up

to a time-change) whose images under certain quasiconformal maps are Brownian motions. Moreover, this construction requires only information about the Beltrami coefficient.

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Contents

1 Introduction

For a domain $\Omega \subset \mathbb{C}$, a map $\phi : \Omega \to \mathbb{C}$ is called analytic if, for all points in Ω, it agrees with a power series in a neighborhood around that point. Due to the nature of the power series, analytic maps are infinitely differentiable [17, Thm II 4.3]. If furthermore the map is one-to-one, then it defines a smooth homeomorphism from Ω onto its image. Moreover, it preserves angles (and their orientations). So one-to-one analytic maps are also called conformal maps in complex analysis. In fact, angle-preserving (including the orientation of the angles) property characterizes a conformal map under the assumption that this map is C^1 and one-to-one [17, Thm X.2.5].

Conformal maps are particularly interesting because on the one hand they are rigid in the sense of supporting various of conformal invariances. To list some examples we care about, they preserve Brownian motions (up to a time-change), harmonicity and quasiconformality. We will explain each of these later in this paper. On the other hand, they are versatile in the plane by virtue of the well-known Riemann mapping theorem. The Riemann mapping theorem assures that given any simply connected open proper subset of the plane, we can always find a conformal map mapping it onto the unit disk.

Theorem 1.1 (Riemann mapping theorem)**.** *Suppose $\Omega \subset \mathbb{C}$ is simply connected and $\Omega \neq \mathbb{C}$, then there exists a one-to-one analytic map ϕ from Ω onto the unit disk $\mathbb{D} = \{z : |z| < 1\}$.*

A direct generalization of conformal maps are the quasiconformal maps.

Definition 1.2. For a region $\Omega \subset \mathbb{C}$ and $K \geq 1$, a homeomorphism $\phi : \Omega \to \Omega' \subset \mathbb{C}$ is called *K-quasiconformal* if it is orientation-preserving, absolutely continuous on almost every vertical or horizontal line segments in Ω (i.e. $\phi \in ACL(\Omega)$) with partial derivatives in L^2_{loc}, and they satisfy

$$\max_{\alpha} |\partial_\alpha \phi(z)| \leq K \min_{\alpha} |\partial_\alpha \phi(z)| \tag{1.1}$$

for almost every $z \in \Omega$, where

$$\partial_\alpha \phi(z) := \cos(\alpha)\phi_x(z) + \sin(\alpha)\phi_y(z), \quad \alpha \in [0, 2\pi)$$

is the directional derivative.

We say a map $\phi : \Omega \to \Omega'$ is *quasiconformal* if it is K-quasiconformal for some $K \geq 1$.

Intuitively, assuming for now that ϕ is differentiable, this is saying that the ratio between the maximum stretch and the minimum stretch of a quasiconformal map at any point is uniformly bounded. If we specifically look at an infinitesimal circle inside Ω, its image under ϕ will be an infinitesimal ellipse. The orientation and size of this ellipse can vary depending on the location of the infinitesimal circle and the actual derivative of ϕ, but the ratio between its major and minor semi-axis is uniformly bounded above by K. Taking more and more such infinitesimal circles, we will see a clearer and clearer picture of the behavior of ϕ. A slightly less intuitive but more frequently considered perspective is to take the preimage of infinitesimal circles. Since the inverse of a K-quasiconformal map is again K-quasiconformal (see Theorem 2.4), the preimage of infinitesimal circles are infinitesimal ellipses. Given a K-quasiconformal map ϕ on Ω, this defines a field of infinitesimal ellipses whose image under ϕ is a field of infinitesimal circles. We call this ellipse field the *ellipse field corresponding to* ϕ.

If one adopts the complex analysis notation $\phi_z = \frac{1}{2}(\phi_x - i\phi_y)$ and $\phi_{\overline{z}} = \frac{1}{2}(\phi_x + i\phi_y)$, and we realize that

$$\max_\alpha |\partial_\alpha \phi(z)| = |\phi_z(z)| + |\phi_{\overline{z}}(z)|$$
$$\min_\alpha |\partial_\alpha \phi(z)| = |\phi_z(z)| - |\phi_{\overline{z}}(z)|,$$

then (1.1) becomes

$$|\phi_z(z)| + |\phi_{\overline{z}}(z)| \leq K(|\phi_z(z)| - |\phi_{\overline{z}}(z)|).$$

This is equivalent to

$$|\phi_{\overline{z}}(z)| \leq \frac{K-1}{K+1}|\phi_z(z)|. \tag{1.2}$$

Note that since ϕ is orientation preserving, its Jacobian determinant $J = |\phi_z|^2 - |\phi_{\overline{z}}|^2 > 0$ almost everywhere. So ϕ_z is non-zero almost everywhere. Thus we can write

$$\mu(z) = \frac{\phi_{\overline{z}}(z)}{\phi_z(z)},$$

and we get that a homeomorphism $\phi : \Omega \to \Omega'$ that is in ACL is K-quasiconformal if and only if

$$\phi_{\overline{z}}(z) = \mu(z)\phi_z(z) \tag{1.3}$$

for almost every $z \in \Omega$, where μ, called the *Beltrami coefficient*, is a complex-valued measurable function satisfying

$$||\mu||_\infty \leq \frac{K-1}{K+1} < 1.$$

Equation (1.3) is called the *Beltrami equation*. See [2, Thm 2.5.4] for more details.

The Beltrami coefficient μ encodes certain information about ϕ but not all. It specifies the ratio $\frac{\max_\alpha |\partial_\alpha \phi|}{\min_\alpha |\partial_\alpha \phi|}$ and the directions in which the maximum and the minimum are attained. In terms of the ellipse field we mentioned above, it specifies the eccentricity and the orientation of the ellipses. More precisely, for the infinitesimal ellipse at $z \in \Omega$, the ratio of the lengths of its major and minor axes is given by $\frac{1+|\mu(z)|}{1-|\mu(z)|}$, which is also equal to $\frac{\max_\alpha |\partial_\alpha \phi(z)|}{\min_\alpha |\partial_\alpha \phi(z)|}$. Its minor semi-axis makes an angle of $\frac{\arg \mu(z)}{2}$ with the positive real axis, which is also the direction in which ϕ attains its maximum stretch. The solution ϕ to equation 1.3 is a homeomorphism that infinitesimally maps this field of ellipses to a field of round circles almost everywhere.

However, the Beltrami coefficient does not encode the relative size of these ellipses and their image circles or the directions in which the major and minor semi-axes get mapped to, as multiplying ϕ by a non-zero complex constant does not change its Beltrami coefficient at all. When ϕ is conformal, $\phi_{\overline{z}}(z) = 0$ and $\phi_z(z) \neq 0$ for all $z \in \Omega$. Thus the Beltrami coefficient corresponding to a conformal map is constantly 0.

We have seen how to compute the Beltrami coefficient μ of a quasiconformal map ϕ, and it is natural to ask whether it is possible to find a quasiconformal map prescribing certain Beltrami coefficient. This is answered by the measurable Riemann mapping theorem.

Theorem 1.3 (measurable Riemann mapping theorem)**.** *Let $\mu : \mathbb{C} \to \mathbb{C}$ be a measurable function with $|\mu(z)| \leq k < 1$ for almost every $z \in \mathbb{C}$. Then there exists a solution $\phi : \mathbb{C} \to \mathbb{C}$ to the Beltrami equation satisfying $\phi_{\overline{z}}(z) = \mu(z)\phi_z(z)$ for almost every $z \in \mathbb{C}$, and ϕ is a K-quasiconformal map for all $K \geq \frac{1+k}{1-k}$ on $\mathbb{C}$.*

Remark 1.4. As we pointed out above, multiplying a non-zero constant factor does not affect the Beltrami coefficient of a quasiconformal map, and thus the solution to the Beltrami equation is not unique. But if we only consider solutions with the normalization $\phi(0) = 0$ and $\phi(1) = 1$, then the solution is unique. See [2, Thm 5.3.4] for details. In this paper we will always take this normalization when talking about the solutions to the Beltrami equation unless otherwise specified.

This theorem generalizes the Riemann mapping theorem in the following sense. The Riemann mapping theorem can be phrased as saying for any simply connected proper subset Ω of the plane, we can find a quasiconformal map on Ω with Beltrami coefficient μ equal to 0 which maps Ω to the unit disk. Now the measurable Riemann mapping theorem assures the existence of a quasiconformal map ϕ with any complex-valued measurable function μ with $||\mu||_\infty < 1$ as its Beltrami coefficient (up to a set of measure 0). If we post-compose ϕ with a well-chosen conformal map, then the composed map maps Ω to the unit disk and has the same μ as its Beltrami coefficient.

The measurable Riemann mapping theorem was first proven by Lars Alfors and Lipman Bers in 1960 [1]. In this project we are going to look at the measurable Riemann mapping theorem from a different perspective. By investigating the relationship between quasiconformal maps and Brownian motion, we aim at providing a probabilistic proof of the measurable Riemann mapping theorem by explicitly constructing the "Riemann" map ϕ which solves the Beltrami equation 1.3 using Brownian motions. With this as the our initial motivation, we have made several discoveries along the way trying to tackle it down. In this paper we would like to share our strategies to approach this problem, several results, many of which are interesting on its own, as well as future directions.

Before we start, we would like to point out that getting an explicit solution to the Beltrami equation is in general delicate. A few examples where we can explicitly write down the solutions are the following. Let $c \in \mathbb{C}$ with $|c| < 1$. For a constant $\mu(z) = c$ for all $z \in \mathbb{C}$, we have the map

$$\phi(z) = \frac{1}{1+c}(z + c\bar{z})$$

to be its normalized solution. For constant Beltrami coefficient supported on the unit disk, i.e. $\mu(z) = c \cdot 1_{\mathbb{D}}(z)$, the map

$$\phi(z) = \begin{cases} \frac{1}{1+c}(z + c\bar{z}), \text{ if } z \in \mathbb{D} \\ \frac{1}{1+c}(z + c/z), \text{ if } z \in \mathbb{D}^c \end{cases}$$

solves the corresponding Beltrami equation. But if instead we take a constant Beltrami coefficient supported on the square $S = (-1, 1) \times (-1, 1)$, then the solution becomes complicated - the image of S under its solution becomes a topological quadrilateral with all 4 corners twisted into spirals. More details and some nice pictures can be found in [4]. The numerical computation of quasiconformal maps is of great interest in computational conformal geometry, see for instance [5] and the references therein.

To illustrate the main idea of our strategy to prove the measurable Riemann mapping theorem, we will start with carrying out a proof in the special case when $\mu = 0$. This turns out to be a constructive proof of the Riemann mapping theorem in which we will construct the Riemann map explicitly. Currently we do not intend to extract the weakest condition for this proof to work, so for the convenience of illustrating the main idea, we assume Ω to be relatively nice by assuming it is a bounded simply connected domain and $\partial\Omega$ is a closed analytic curve (See 2.12 for the definition).

The construction of the Riemann map is as follows. Let a, b, c be three distinct points on the $\partial\Omega$. They split the $\partial\Omega$ into 3 arcs, namely α, β and γ, where arc α is the arc between b and c and etc. Then for each point $z \in \Omega$, we can associate a coordinate

$$H(z) = H_{\Omega,a,b,c}(z) = (\omega(z, \alpha, \Omega), \omega(z, \beta, \Omega), \omega(z, \gamma, \Omega)),$$

where ω denotes the harmonic measure. We call $H(z)$ the harmonic coordinates of z in Ω with boundary points a, b and c, or just the harmonic coordinates when the domain is clear from the context. In particular, we can define the harmonic coordinates for points in the unit disk $\mathbb{D}$ with distinct boundary point a', b' and c'.

Now for each $z \in \Omega$, we pick $w \in \mathbb{D}$ such that $H_{\mathbb{D},a',b',c'}(w) = H_{\Omega,a,b,c}(z)$ and set $\phi(z) = w$. We claim this ϕ is the conformal map from Ω to $\mathbb{D}$ which maps the boundary points a, b and c to a', b' and c' respectively.

Theorem 1.5. *The map ϕ on Ω is well-defined. It is one-to-one and analytic onto $\mathbb{D}$.*

To be clear, this theorem is easy if we assume the Riemann mapping theorem and argue that ϕ coincides with the Riemann map ψ. The point here is that we can construct the Riemann map ψ this way and obtain the Riemann mapping theorem. We will present a proof of these in section 4 below.

So far we are still in the case where μ is constantly 0 and so ϕ is conformal. But we would also like to prove the measurable Riemann mapping theorem in the general case where ϕ is in general a quasiconformal map. Recall that we constructed the map ϕ by matching a triple of 3 Brownian hitting probabilities in the domain Ω and those inside the unit disk. The reason we use the Brownian motions in both domains is that Brownian motions are preserved (up to a time-change) under conformal maps and thus their harmonic measures are also preserved. However this is no longer true under general quasiconformal maps. For example, consider the linear map

$$\phi(z) = 2 \operatorname{Re} z + i \operatorname{Im} z.$$

Under this map, a Brownian motion will be stretched by a factor of 2 in the real direction while kept unchanged in the imaginary direction. The resulting process is not a Brownian motion with or without time-changes. Thus in order to still carry out the construction of ϕ using the harmonic measures in the general case, it is necessary to understand the processes which get mapped to a Brownian motion under quasiconformal maps. And for our purpose it is crucial to be able to understand their harmonic measures using only the information provided in the Beltrami coefficient. For this reason, we will define the notion of quasi-Brownian motions in section 5 which describes these processes. And then we will prove that we can construct such processes (up to a time-change) using only the information encoded in the Beltrami coefficient by providing two different constructions from very different perspectives.

In addition, one of these constructions provides us a way to simulate these quasi-Brownian motions, which can be used to numerically compute the Riemann map in the measurable Riemann mapping theorem.

The rest of this paper will be structured in the following way. We will go through some useful preliminaries in Section 2. In Section 3 we will present an elementary proof to the conformal invariance of Brownian motion. After that we will give the full proof of Theorem 1.5 in section 4. In Section 5, we will define the notion of quasi-Brownian motion and introduce its locality property, which generalizes the conformal invariance of Brownian motion. Section 6 will be dedicated to the constructions of quasi-Brownian motions based on the Beltrami coefficient. In this section we will see two successful constructions from different perspectives: one by Øksendal using Dirichlet forms and the other one our square-wise construction, as well as several attempts that failed. In Section 7 we will discuss future directions, together with the comparison between Øksendal's construction and the square-wise construction, and numerical simulations.

2 Preliminaries

In this section we recall some useful definitions and relevant results.

Quasiconformal mapping

In the preceding section we defined the notion of quasiconformal map. Here we will list some of its properties. First we quote a differentiability result on open mappings. Recall that a map is open if it maps open sets to open sets.

Theorem 2.1. *[2, Thm 3.3.2] Let $\Omega \subset \mathbb{C}$ be a domain and $\phi : \Omega \to \mathbb{C}$ be a continuous open map.*

Then f is differentiable almost everywhere in Ω if and only if ϕ has finite first partial derivatives almost everywhere.

We assume a quasiconformal map to have partial derivatives in L^2_{loc}. In particular, this requires its partial derivatives to be finite almost everywhere. Thus this theorem implies the following.

Corollary 2.2 (almost everywhere differentiablity)**.** *If ϕ is quasiconformal in Ω, then ϕ is differentiable almost everywhere in Ω.*

Therefore, our discussion right after Definition 1.2 actually makes sense almost everywhere for any quasiconformal map.

The next result concerns the regularity of quasiconformal maps.

Theorem 2.3. *Let $\Omega \subset \mathbb{C}$ be a domain and $\phi : \Omega \to \Omega' \subset \mathbb{C}$ be a quasiconformal map. Then*

$$\phi \in W^{1,2}_{loc}(\Omega).$$

Proof. This follows directly from our definition and Lemma A.5.2 in [2]. □

Quasiconformality can be equivalently defined in multiple different ways. In this paper we do not intend to go into the equivalence of those definitions, but for the purpose of introducing properties of quasiconformal maps, it is helpful to be aware of some of them. For example, the preceding theorem in particular implies our definition of quasiconformality is equivalent to that Definition 3.1.1 in [2]. Thus the following theorem from [2] holds under our definition of quasiconformal map as well.

Theorem 2.4. *[2, Thm 3.1.2] Let $\Omega \subset \mathbb{C}$ be a domains and $\phi : \Omega \to \Omega' \subset \mathbb{C}$ be a K-quasiconformal map. Then*

- *$\phi^{-1} : \Omega' \to \Omega$ is K-quasiconformal.*
- *For all measurable set $E \subset \Omega$, $\text{Area}(E) = 0$ if and only if $\text{Area}\,\phi(E) = 0$.*
- *The Jacobian determinant $J_\phi > 0$ almost everywhere in Ω.*

In the previous section we mentioned that a conformal map is a quasiconformal map with Beltrami coefficient 0 almost everywhere. The next theorem states that the converse also holds true. Note that (1.2) in particular implies a homeomorphism ϕ is 1-quasiconformal if and only if its Beltrami coefficient $||\mu||_\infty = 0$.

Theorem 2.5 (1-quasiconformal is conformal)**.** *[16, Thm I.5.1] Let* $\Omega \subset \mathbb{C}$ *be a domain. If* $\phi : \Omega \to \Omega' \subset \mathbb{C}$ *is* 1*-quasiconformal, then* ϕ *is conformal.*

As a remark, this theorem is not trivial as it might seem. One way to prove it is to first realize that 1-quasiconformal maps preserve the modulus of topological quadrilaterals, which is defined as the modulus of the family of curves inside this topological quadrilateral joining a particular pair of its opposite sides. Then we use the fact that if two topological quadrilaterals are aligned together to form a rectangle, the sum of the modulus of these topological quadrilaterals is equal to the modulus of the rectangle only if both topological quadrilaterals are actually rectangles. Interested readers may see [16] for the details.

Lastly we quote a theorem relating the locally uniform convergence of quasiconformal maps to the almost everywhere convergence of their Beltrami coefficients.

Theorem 2.6 (Good approximation lemma)**.** *[2, Lem 5.3.5] Given a sequence of Beltrami coefficients* $\{\mu_n\}_{n\in\mathbb{N}}$ *such that*

$$||\mu_n||_\infty \leq k < 1, \forall n \in \mathbb{N},$$

and such that the point-wise limit

$$\mu(z) = \lim_{n\to\infty} \mu_n(z)$$

exists almost everywhere. If $\phi_n : \mathbb{C} \to \mathbb{C}$ *are homeomorphisms in* $W^{1,2}_{loc}(\mathbb{C})$ *satisfying*

$$(\phi_n)_{\overline{z}}(z) = \mu_n(z)(\phi_n)_z(z)$$

and normalized so that $\phi_n(0) = 0$ *and* $\phi_n(1) = 1$ *for all* $n \in \mathbb{N}$*. Then the limit* $\phi(z) = \lim_{n\to\infty} \phi_n(z)$ *exists. The convergence is uniform on compact subsets of* $\mathbb{C}$ *and* ϕ *satisfies the Beltrami equation*

$$\phi_{\overline{z}}(z) = \mu(z)\phi_z(z) \text{ almost everywhere }.$$

Harmonic function

Definition 2.7. A continuous function u defined in a domain $\Omega \subset \mathbb{C}$, with values in $[-\infty, \infty)$, is called *subharmonic* in Ω if for each $z \in \Omega$ there is an $r_z > 0$ such that

$$u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{it})dt$$

for all $r < r_z$.

We say u is *harmonic* in Ω if both u and $-u$ are subharmonic in Ω.

Harmonic functions enjoy the following nice properties.

Theorem 2.8 (maximum principle). *[17, Thm X 1.3] If u is subharmonic in a region $\Omega \subset \mathbb{C}$, and if u attains its maximum in Ω, then u is constant.*

Theorem 2.9 (smoothness). *[7, Thm 2.2.3.6] If u is harmonic in a domain Ω, then*

$$u \in C^{\infty}(\Omega).$$

Lemma 2.10 (Hopf lemma). *[15, Lem 5.3.1] Let Ω be a bounded domain in $\mathbb{C}$ with smooth boundary. Let f be a real-valued function that is harmonic in Ω and continuous in $\overline{\Omega}$. If x is a boundary point such that $f(x) > f(y)$ for all y in Ω sufficiently close to x, then the (one-sided) directional derivative of f in the direction of the outward pointing normal to the boundary at x is strictly positive.*

Harmonic functions in a simply connected domain in the plane are closely related to analytic functions by virtue of the following theorem.

Theorem 2.11. *[17, Thm X 2.2] If u is harmonic in a simply connected domain Ω, then there exists a harmonic function v on Ω such that*

$$f := u + iv$$

is analytic on Ω.

Such function v is called a *harmonic conjugate* of u.

Definition 2.12. A simple closed curve γ is called *analytic* if there exists a one-to-one analytic map f defined in a neighborhood of $\partial\mathbb{D}$ such that $f(\partial\mathbb{D}) = \gamma$.

Theorem 2.13 (Schwarz reflection principle). *[17, Thm XI 2.2] Let $\Omega \subset \mathbb{C}$ be a simply connected domain and suppose $\partial\Omega$ is an analytic closed curve. If u is harmonic in Ω, continuous in $\overline{\Omega}$ and $u = 0$ on $\partial\Omega$, then u extends to be harmonic in a neighborhood of $\overline{\Omega}$.*

Planar Brownian motion

Definition 2.14. A real-valued stochastic process $B = \{B_t : t \geq 0\}$ is called a *linear Brownian motion* starting at $x_0 \in \mathbb{R}$ if the following holds:

1. $B_0 = x_0$,

2. For times $0 \leq t_1 \leq t_2 \leq ... \leq t_n$, the increments $B_{t_2} - B_{t_1}, B_{t_3} - B_{t_2}, ..., B_{t_n} - B_{t_{n-1}}$ are independent normal random variables with mean 0 and variance $t_2 - t_1, t_3 - t_2, ..., t_n - t_{n-1}$ respectively.

3. Almost surely the function $t \to B_t$ is continuous.

The existence of such process can be established in multiple ways. For a constructive proof, see [18, Thm 1.3]. In this project we will focus on 2-dimensional Brownian motion taking values in the complex plane $\mathbb{C}$:

Definition 2.15. If B^1, B^2 are two independent linear Brownian motions started at x_1, x_2, then the stochastic process $\{B_t : t \geq 0\}$ given by

$$B_t = B_t^1 + iB_t^2$$

is called a *planar Brownian motion*, or just a *Brownian motion* in this paper, starting at $z_0 = x_1 + ix_2$. Here i is the imaginary unit.

Given a Brownian motion $\{B_t : t \geq 0\}$ starting at z_0 and $s > 0$, one can check that the process $\{B_{t+s} - B_s : t \geq 0\}$ is also a Brownian motion starting at 0. This is called the *Markov property* of Brownian motion. Furthermore, one can generalize this to a stopping time τ.

Theorem 2.16. *[18, Thm 2.16] Given a Brownian motion $\{B_t : t \geq 0\}$ and a stopping time (See [18, Sec 2.2] for the definition) τ such that $0 \leq \tau < \infty$ almost surely. Then the process*

$$\{B_{t+\tau} - B_\tau : t \geq 0\}$$

is a Brownian motion starting at 0.

We call this the *strong Markov property* of Brownian motion.

Theorem 2.17. *[18, Thm 3.8] Suppose $\Omega \subset \mathbb{C}$ is a bounded domain. Let $\{B_t : t \geq 0\}$ be a planar Brownian motion and $\tau = \inf\{t \geq 0 : B_t \in \partial\Omega\}$ the first exit time of Ω. Let $f : \partial\Omega \to \mathbb{R}$ be measurable, and such that the function $u : \Omega \to \mathbb{R}$ with*

$$u(z) = E_z[f(B_\tau)1_{[\tau<\infty]}],$$

where E_z denotes the expectation associated to the law of a Brownian motion starting at z, is locally bounded. Then u is a harmonic function.

Lastly, we would like to highlight the remarkable result that Brownian motions are preserved under conformal maps, up to a change in its time-parametrization.

Theorem 2.18 (Conformal invariance of Brownaian motion)**.** *[18, Thm 7.20] Let* $\Omega \in \mathbb{C}$ *be a domain,* $z_0 \in \Omega$*, and let* $\phi : \Omega \to \Omega'$ *be conformal. Let* $\{B_t : t \geq 0\}$ *be a Brownian motion starting at* z_0 *and* $\tau = \inf\{t \geq 0, B_t \in \partial\Omega\}$*. Then the process* $\{\phi(B_t) : 0 \leq t \leq \tau\}$ *is a time-changed Brownian motion, i.e. there exists a Brownian motion* $\{\tilde{B}_t : t \geq 0\}$ *such that for any* $t \in [0, \tau)$*,*

$$\phi(B_t) = \tilde{B}_{\zeta(t)},$$

where

$$\zeta(t) = \int_0^t |\phi'(B_s)|^2 ds.$$

Moreover, $\zeta(\tau)$ *is the first exit time from* Ω' *by* $\{\tilde{B}(t) : t \geq 0\}$*.*

In Section 5, this theorem will be the real hero behind Theorem 5.3, which plays a key role in our construction of the quasi-Brownian motion. This theorem is well-understood, but in most of the textbooks it is proven based on Itô calculus. In our research, we found that this theorem can be proven based on the fact that locally a conformal map is well-approximated by a complex linear map, and Brownian motions are preserved under these linear maps. It is often found in mathematics that using advanced theory allows one to write down elegant proofs, but at the same time it tends to hide the basic ideas further away from the readers. So we decide to still present this proof, for it depending only on elementary complex analysis and probability theory. Section 3 will be dedicated to this proof.

Green's function

Definition 2.19. Let $\Omega \subset \mathbb{C}$ be a domain and $z_0 \in \Omega$ be a point. We say Ω admits a Green's function with pole at z_0 if we can find a real valued function $G(\cdot, z_0)$ on Ω with the following properties.

1. $G(\cdot, z_0)$ is harmonic in $\Omega \setminus \{z_0\}$.

2. $G(z, z_0) > 0$ for all $z \in \Omega$.

3. If $a \subset \partial\Omega$, then $\lim_{z \to a} G(z, z_0) = 0$.

4. $G(\cdot, z_0) + \log|\cdot - z_0|$ extends to be harmonic in Ω.

The next theorem states that a bounded simply connected domain with analytic boundary in the plane always admits a Green's function. This theorem can be established in multiple different ways. For a analytic proof, see Chapter XIII in [17]. For a probabilistic approach, see [20] Sections 3.1, 3.3.

Theorem 2.20. *If a domain $\Omega \subset \mathbb{C}$ is bounded and simply connected, and $\partial\Omega$ is an analytic curve. Let $z_0 \in \Omega$, then Ω admits a Green's function with pole at z_0.*

Harmonic measure

Definition 2.21. Let $\Omega \subset \mathbb{C}$ be a bounded domain, and $\{B_t : t \geq 0\}$ be a Brownian motion starting at $z_0 \in \Omega$. For a set $\alpha \subset \partial\Omega$ Borel, define

$$\omega(z_0, \alpha, \Omega) = \mathbb{P}[B_\tau \in \alpha],$$

where $\tau = \inf\{t \geq 0, B_t \in \Omega^c\}$. We call $\omega(z_0, \cdot, \Omega)$ the *harmonic measure of Brownian motion* in Ω at z_0.

Note that a linear Brownian motion is almost surely unbounded, so for a bounded domain Ω, $\tau = \inf\{t \geq 0, B_t \in \Omega^c\}$ is almost surely finite [18]. Thus the expression $\mathbb{P}[B_\tau \in \alpha]$ makes sense. One can check that $\omega(z_0, \cdot, \Omega)$ is a probability measure supported on $\partial\Omega$. We call this measure the *harmonic measure* in Ω based at z_0. If we fix a Borel set $\alpha \subset \partial\Omega$ and consider $\omega(\cdot, \alpha, \Omega)$ as a function on Ω, then it is harmonic by Theorem 2.17. Moreover, since a Brownian motion is conformally invariant up to a time-change, and the time-change does not affect where each sample path exits the domain, the harmonic measure is also invariant under conformal maps.

Next we will quote a useful theorem due to Beurling, which provides us an estimation of the harmonic measure in a disk.

Theorem 2.22 (Beurling projection theorem)**.** *[10, Thm 9.2] If $E \subset \mathbb{D}\backslash\{0\}$ and $E^* = \{|z| : z \in E\}$ is the circular projection of E, then*

$$\omega(z, E, \mathbb{D} \setminus E) \geq \omega(-|z|, E^*, \mathbb{D} \setminus E^*).$$

Regarding stochastic processes other than the Brownian motion, for example the Quasi-Brownian motion we will define in Section 5, by their harmonic measure we will mean their exit distribution, provided their exiting distribution is well-defined. In this case, the harmonic measure is not necessarily harmonic as a function of the starting point any more.

Weak convergence of probability measures on the Wiener space

A major component of this project is to construct the stochastic processes which we will call quasi-Brownian motions in Section 5. The construction involves generating a sequence of stochastic processes which "approximate" the desired process. To make this precise, we recall the following notion of convergence.

Definition 2.23 (weak convergence of probability measures)**.** Let (E, d) be a metric space and $\mathcal{B}$ be the Borel σ-algebra on it. For probability measures P_n, $n \in \mathbb{N}$, and P on $(E, \mathcal{B})$, we say P_n *converges weakly* to P if for every bounded continuous function $f : E \to \mathbb{R}$ we have

$$\lim_{n\to\infty} \int f dP_n = \int f dP.$$

To show a sequence of probability measures converges weakly, a frequently used strategy is to first show the sequence is weakly relatively compact and then show the subsequential limits are identical. By "weakly relatively compact" we mean the following.

Definition 2.24 (weakly relative compactness)**.** If π is a family of probability measures on $(E, \mathcal{B})$, we say it is *weakly relative compact* if every sequence in π contains a weakly convergent subsequence.

In this project we are mostly interested in the 2-dimensional Wiener space, $W^2 := C(\mathbb{R}^+, \mathbb{C})$, equipped with the uniform convergence on compact subsets topology. This space is complete and metrizable with the metric

$$d(w, w') = \sum_{n=1}^{\infty} 2^{-n} \frac{\sup_{t \le n} |w(t) - w'(t)|}{1 + \sup_{t \le n} |w(t) - w'(t)|} \quad \text{for } w, w' \in W^2.$$

See [14, Thm A2.3] for details.

Now for $w \in W^2$, we define

$$V^N(w, \delta) := \sup\{|w(t) - w(t')| : |t - t'| < \delta, t, t' < N\},$$

then we have the following criteria for weakly relatively compactness on probability measures on W^2.

Proposition 2.25. *[20, Prop XIII.1.5] A sequence $\{P_n\}$ of probability measures on W^2 is weakly relatively compact if and only if the following two conditions hold:*

1. *for every $\epsilon > 0$, there exist a number A and an integer n_0 such that*

$$P_n[|w(0)| > A] < \epsilon \quad \text{for every } n > n_0;$$

2. *for every $\eta, \epsilon > 0$ and $N \in \mathbb{N}$, there exists a number δ and integer n_0 such that*

$$P_n[V^N(\cdot, \delta) > \eta] < \epsilon \quad \text{for every } n > n_0.$$

Here we phrased the Wiener space in the plane but all of these hold in $\mathbb{R}^d$ for $d \in \mathbb{N}$ as well. See [20, Chapter XIII] for details.

Topology

Lastly, the following topological characterization of a global homeomorphism turns out to be useful.

Theorem 2.26. *[12, Thm 2] Let Ω be a path connected topological space and Ω' be a simply connected Hausdorff topological space. A local homeomorphism $\phi : E \to \Omega'$ is a global homeomorphism if and only if the map ϕ is proper.*

3 An elementary proof of the conformal invariance of Brownain motion

In this section we will give an elementary proof of the conformal invariance of Brownian motion, which does not involve stochastic calculus or any advanced characterizations of a Brownian motion. First we state the version of the law of large numbers we will use.

Theorem 3.1 (LLN for triangular array [1]). *Let $(X_{n,i})_{n\in\mathbb{N}, i=1,2,\ldots,n}$ be a triangular array of scalar random variables such that for each n, the row $X_{n,1}, X_{n,2}, \ldots, X_{n,n}$ is a collection of independent random variables. For each n, we form the partial sum $S_n = \sum_{i=1}^n X_{n,i}$.*

1. *If all $X_{n,i}$'s have mean μ and $\sup_{n,i} E|X_{n,i}|^2 < \infty$, then $\frac{S_n}{n} \to \mu$ in probability.*

[1] This is stated as an exercise in Terence Tao's blog [22, Exer 12].

2. If all $X_{n,i}$'s have mean μ and $\sup_{n,i} E|X_{n,i}|^4 < \infty$, then $\frac{S_n}{n} \to \mu$ almost surely.

Proof. By considering $Y_{n,i} = X_{n,i} - \mu$, we may without loss of generality assume $\mu = 0$.

For 1: Let $M_2 = \sup_{n,i} E\left[|X_{n,i}|^2\right] < \infty$. Then

$$E\left[\left|\frac{S_n}{n}\right|^2\right] = \frac{\sum_{i=1}^n E\left[|X_{n,i}|^2\right]}{n^2} \leq \frac{M_2}{n}.$$

By the Markov inequality, we have

$$\mathbb{P}\left[\left|\frac{S_n}{n}\right| \geq \epsilon\right] \leq \frac{E\left[\left|\frac{S_n}{n}\right|^2\right]}{\epsilon^2} \leq \frac{M_2}{n\epsilon^2} \to 0 \text{ as } n \to \infty.$$

For 2: Let $M_2 = \sup_{n,i} E|X_{n,i}|^2$ and $M_4 = \sup_{n,i} E|X_{n,i}|^4$. Since $M_4 < \infty$, we have $M_2 < \infty$ as well by the Jensen's inequality. Now we compute

$$E\left[\left|\frac{S_n}{n}\right|^4\right] = \frac{\sum_{i=1}^n E[X_{n,i}^4] + 3\sum_{i \neq j} E[X_{n,i}^2]E[X_{n,j}^2]}{n^4} \leq \frac{nM_4 + 3n(n-1)M_2^2}{n^4}.$$

By the Markov inequality, we have for all $\epsilon > 0$,

$$\mathbb{P}\left[\left|\frac{S_n}{n}\right| > \epsilon\right] \leq \frac{E\left[\left|\frac{S_n}{n}\right|^4\right]}{\epsilon^4} \leq \frac{nM_4 + 3n(n-1)M_2^2}{\epsilon^4 n^4},$$

which has finite sum over n. By the Borel-Cantelli theorem,

$$\mathbb{P}\left[\left|\frac{S_n}{n}\right| > \epsilon \text{ for infinitely many } n\right] = 0.$$

Thus we conclude $\lim_{n\to\infty} \frac{S_n}{n} = 0$ almost surely. □

The main idea of our proof is the following. A conformal map is necessarily complex differentiable with non-vanishing first derivative everywhere. Thus locally it can be well approximated by a linear function of the form $\phi(z) = a_1(z - z_0) + a_0$, where a_0 and a_1 are its corresponding Taylor coefficients. In terms of linear transformations, this linear map consists of a rotation, a dilation and a translation. It is well understood that a Brownian motion is preserved (up to a time-change) under all of these transformations. Ideally, if we can "assemble" the image of a Brownian motion under a conformal map with the images of this Brownian motion under its linear approximations, then we get (with a lot of details to be filled in) its image under this conformal map is again a Brownian motion. The issue here is that a linear approximation comes with some error. This error is only guaranteed to be small in a sufficiently small neighborhood around the point we take this linear approximation. But to cover the whole path of the Brownian motion, a large number of such neighborhoods are needed, especially when they are small. Thus we need to make sure the small errors in each neighborhood do not accumulate into something nonn-negligible due to the increasing number of neighborhoods we need to take.

To make this precise, let $\Omega \subset \mathbb{C}$ be a bounded domain and $z_0 \in \Omega$ is a point. For $z \in \mathbb{C}$ and $r > 0$, we use the notation $D(z, r)$ to denote a open disk centered at z with radius r. Now consider a planar Brownian motion B starting at $z_0 \in \Omega$, and let

$$\tau = \inf\{t > 0 : B_t \notin \Omega\}$$

be its first exiting time from Ω. From now on and throughout this section, fix a sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$ which decreases to 0. For each n define the sequence of stopping times $\{\tau_{n,i}\}_{i\in\mathbb{N}}$ by

$$\begin{aligned}
\tau_{n,1} &= \inf\{t > 0 : B_t \notin D(z_0, \epsilon_n)\} \\
\tau_{n,2} &= \inf\{t > \tau_{n,1} : B_t \notin D(B_{\tau_{n,1}}, \epsilon_n)\} \\
\tau_{n,3} &= \inf\{t > \tau_{n,2} : B_t \notin D(B_{\tau_{n,2}}, \epsilon_n)\} \\
&\vdots
\end{aligned}$$

Now consider $N_\tau = N_\tau(n, \omega) = \inf\{k : \tau_{n,k} > \tau\}$, which denotes the number of times B escapes a disk of radius ϵ_n before leaving Ω. First we prove that N_τ grows like $\frac{1}{\epsilon_n^2}$ as $n \to \infty$.

Lemma 3.2. *Almost surely,*

$$0 < \lim_{n\to\infty} \epsilon_n^2 N_\tau(\epsilon_n, \omega) < \infty.$$

Proof. Without loss of generality assume $z_0 = 0$. Recall that $\tau_{1,1}$ and $\tau_{2,1}$ are the first time a Brownian motion escapes a disk of radius ϵ_1 and ϵ_2 respectively. Since $\epsilon_1 > \epsilon_2$, $\tau_{1,1}$ and $\tau_{2,1}$ do not have the same mean. This prevents us from applying the law of large numbers directly. To deal with this problem, for each ϵ_n, we scale everything by the map $f_n(z) = \frac{1}{\epsilon_n} z$ to normalize $\tau_{n,i}$ in n. To be precise, for each n, we know that $f_n(B_t)$ is a Brownian motion with some time-change. Let

$\tilde{B}$ denote this Brownian motion (without applying the time-change) and define

$$\begin{aligned}
\tilde{\tau}_{n,0} &= 0 \\
\tilde{\tau}_{n,1} &= \inf\{t > 0 : \tilde{B}_t \notin D(0,1)\} \\
\tilde{\tau}_{n,2} &= \inf\{t > \tilde{\tau}_{n,1} : \tilde{B}_t \notin D(\tilde{B}_{\tilde{\tau}_{n,1}}, 1)\} \\
\tilde{\tau}_{n,3} &= \inf\{t > \tilde{\tau}_{n,2} : \tilde{B}_t \notin D(\tilde{B}_{\tilde{\tau}_{n,2}}, 1)\} \\
&\vdots
\end{aligned}$$

Then, using these stopping times of $\tilde{B}$, we can express $N_\tau = \inf\{k : \tilde{\tau}_{n,k} > \frac{\tau}{\epsilon_n^2}\}$.

For each $i, n \in \mathbb{N}$, let $X_{n,i} = \tilde{\tau}_{n,i} - \tilde{\tau}_{n,i-1}$. Then $\{X_{n,i}\}_{n\in\mathbb{N}, i=1,2,\ldots N_\tau}$ forms a subsequence of a triangular array of scalar random variables. Moreover, each $\tilde{\tau}_{n,i}$ are identically distributed with their density function decaying exponentially. Thus $E|\tilde{\tau}_{n,i}|^4 < \infty$.

Since Ω is bounded, $\tau < \infty$ almost surely. Thus that for each n, we have almost surely

$$\tilde{\tau}_{n,N_\tau-1} \leq \frac{\tau}{\epsilon_n^2} \leq \tilde{\tau}_{n,N_\tau}.$$

That is

$$\sum_{i=1}^{N_\tau-1} X_{n,i} \leq \frac{\tau}{\epsilon_n^2} \leq \sum_{i=1}^{n} X_{N_\tau}.$$

Dividing by N_τ, we get

$$\frac{\sum_{i=1}^{N_\tau-1} X_{n,i}}{N_\tau} \leq \frac{\frac{\tau}{\epsilon_n^2}}{N_\tau} \leq \frac{\sum_{i=1}^{N_\tau} X_{n,i}}{N_\tau}.$$

Now if we take the limit as $n \to \infty$, clearly we have $N_\tau \to \infty$. Then by the law of large numbers (Theorem 3.1), we get

$$\lim_{n\to\infty} \frac{\frac{\tau}{\epsilon_n^2}}{N_\tau} = E[\tilde{\tau}_{1,1}] \text{ , almost surely.}$$

Thus,

$$\lim_{n\to\infty} \frac{N_\tau}{\frac{1}{\epsilon_n^2}} = \frac{\tau}{E[\tilde{\tau}_{1,1}]} \text{ , almost surely.}$$

□

Remark. The same proof works for a deterministic time t, instead of the exit time τ, as well. To be precise, let $t > 0$ and let $N_t = N_t(\epsilon_n, \omega) = \inf\{k : \tau_k > t\}$ be the number of times B escapes from a disk of radius ϵ before time t. Using exactly the same proof, we get that, almost surely,

$$0 < \lim_{n\to\infty} \epsilon_n^2 N_t(\epsilon_n, \omega) < \infty.$$

Now we are ready to present our proof of the conformal invariance of Brownian motion.

Proof of Theorem 2.18. By taking a compact exhaustion $\{K_m\}$ of Ω and proving this theorem on $\Omega_m = \text{int}(K_m)$ for each m, we may without loss of generality assume that Ω is bounded, and ϕ extends one-to-one and analytically to a neighborhood of $\overline{\Omega}$. In particular this implies ϕ' is uniformly continuous, bounded and bounded away from 0, and ϕ'' is bounded in Ω.

Fix $n \in \mathbb{N}$ with ϵ_n small enough such that ϕ can be extended analytically to $\Omega^{2\epsilon} = \{z \in \mathbb{C} : \text{dist}(z, \Omega) < 2\epsilon\}$. First we define $\tau_1 = \inf\{t > 0 : B_t \notin D(z_0, \epsilon_n)\}$ and let $z_1 = B_{\tau_1}$. Consider B on the interval $[0, \tau_1]$. If we define the linear approximation $L_0(z) = \phi(z_0) + \phi'(z_0)(z - z_0)$ of ϕ at z_0. Then we have

$$|\phi(z) - L_0(z)| < C\epsilon_n^2, \text{ for all } z \in \overline{D(z_0, \epsilon_n)}$$

for some constant $C = C(\Omega, \phi)$. In particular, we have for all $t \in [0, \tau_1]$,

$$|\phi(B_t) - L_0(B_t)| < C\epsilon_n^2.$$

To be more careful at the end point $z_1 = B_{\tau_1}$, we have

$$\phi(z_1) - L_0(z_1) = \phi''(z_0) \cdot (z_1 - z_0)^2 + O(\epsilon_n^3).$$

Now we define $\tau_2 = \inf\{t > \tau_1 : B_t \notin D(B_{\tau_1}, \epsilon_n)\}$ and let $z_2 = B_{\tau_2}$. This time consider B on the interval $[\tau_1, \tau_2]$. If we take the linear approximation $L_1(z) = \phi(z_1) + \phi'(z_1)(z - z_1)$ of ϕ at z_1. Then we also have

$$|\phi(z) - L_1(z)| < C\epsilon_n^2, \text{ for all } z \in \overline{D(z_1, \epsilon_n)}$$

for the same constant C. In particular, we have for all $t \in [\tau_1, \tau_2]$,

$$|\phi(B_t) - L_1(B_t)| < C\epsilon_n^2.$$

At the end point $z_2 = B_{\tau_2}$, we have

$$\phi(z_2) - L_1(z_2) = \phi''(z_1) \cdot (z_2 - z_1)^2 + O(\epsilon_n^3).$$

Recall that we would like to assemble $\phi(B)$ using the linear approximations $L_k(B)$. However here we encounter the problem $L_0(z_1)$ does not equal to $L_1(z_1)$ in general, and as a result these two pieces of linear image of B do not line up continuously. To deal with this problem, we apply a translation by $L_0(z_1) - L_1(z_1)$ to L_1 to concatenate them. To do this we define the process $\{(X_1)_t : t \in [0, \tau_2)\}$ by

$$(X_1)_t = \begin{cases} L_0(B_t) & \text{for } t \in [0, \tau_1) \\ L_1(B_t) + L_0(z_1) - L_1(z_1) & \text{for } t \in [\tau_1, \tau_2) \end{cases}$$

Then X_1 is continuous and it satisfies that

$$(X_1)_t - \phi(B_t) = \begin{cases} O(\epsilon_n^2) & \text{for } 0 \le t < \tau_1 \\ O(\epsilon_n^2) + \phi''(z_1) \cdot (z_2 - z_1)^2 + O(\epsilon_n^3) & \text{for } \tau_1 \le t < \tau_2 \end{cases}$$

Moreover, due to the strong Markov property and translation/rotation/scaling invariance of Brownian motion, X_1 is a concatenation of two independent time-changed Brownian motions, thus it is itself a time-changed Brownian motion. We will be precise about the time-parametrization later.

Following this pattern, for each $k = 2, 3, ..., N_\tau - 2$, we define $\tau_{k+1} = \inf\{t > \tau_k : B_t \notin D(B_{\tau_k}, \epsilon_n)\}$, $z_{k+1} = B_{\tau_{k+1}}$, and $L_k = \phi(z_k) + \phi'(z_k)(z - z_k)$. Using this, we extend our definition of X_{k-1} by

$$(X_k)_t = \begin{cases} (X_{k-1})_t & \text{for } t \in [0, \tau_k) \\ L_k(B_t) + \sum_{j=1}^{k} (L_{j-1}(z_k) - L_j(z_k)) & \text{for } t \in [\tau_k, \tau_{k+1}) \end{cases}$$

In the last disk in which B exits the domain Ω, or equivalently, when $k = N_\tau - 1$, we extend the definition of X_k in the same way but only up to the exit time τ. That is, for $k = N_{\tau-1}$, let

$$(X_k)_t = \begin{cases} (X_{k-1})_t & \text{for } t \in [0, \tau_k) \\ L_k(B_t) + \sum_{j=1}^{k} (L_{j-1}(z_k) - L_j(z_k)) & \text{for } t \in [\tau_k, \tau] \end{cases}$$

Finally, take a independent planar Brownian motion B' and concatenate it (the same one) to both $\{X_k : t \in [0, \tau)\}$ and $\{\phi(B)_t : t \in [0, \tau]\}$ and call the resulting processes X and Y respectively. This way we have

$$X_t = \begin{cases} (X_{k_{N_\tau}-1})_t & \text{for } t < \tau \\ B'_{t-\tau} - B'_0 + (X_{k_{N_\tau - 1}})_\tau & \text{for } t \ge \tau \end{cases}$$

and

$$Y_t = \begin{cases} \phi(B_t) & \text{for } t < \tau \\ B'_{t-\tau} - B'_0 + \phi(B_\tau) & \text{for } t \ge \tau \end{cases}$$

Here ϕ is to be understood as its continuous extension to $\overline{\Omega}$. This completes our construction.

Now we compare the processes X and Y. Before applying any time-changes, we have

$$X_t - Y_t = \begin{cases} O(\epsilon_n^2) & \text{for } 0 \le t < \tau_1 \\ O(\epsilon_n^2) + \phi''(z_1) \cdot (z_2 - z_1)^2 + O(\epsilon_n^3) & \text{for } \tau_1 \le t < \tau_2 \\ \vdots & \quad \vdots \\ O(\epsilon_n^2) + \sum_{j=1}^{N_\tau - 1}[\phi''(z_j) \cdot (z_{j+1} - z_j)^2 + O(\epsilon_n^3)] & \text{for } \tau_{N_\tau - 1} \le t < \infty. \end{cases}$$

Here the last expression is valid for all t up to infinity because we are attaching the same Brownian motion B' to the end of both X and Y starting from time τ. Thus starting from time τ, the difference between X and Y stays constant.

Now fix a $t > 0$. On the event $\{t \ge \tau\}$ and for each $n \in \mathbb{N}$,

$$|X_t - Y_t| \le |O(\epsilon_n^2)| + |\sum_{j=1}^{N_\tau}[\phi''(z_j) \cdot (z_{j+1} - z_j)^2]| + |N_\tau \cdot O(\epsilon_n^3)|.$$

Here, $|O(\epsilon_n^2)| \to 0$ as n goes to ∞; By Lemma 3.2, N_τ growths as $\frac{1}{\epsilon_n^2}$, therefore $|N_\tau \cdot O(\epsilon_n^3)| \to 0$ as $n \to \infty$ as well.

Consider $|\sum_{j=1}^{N_\tau}[\phi''(z_j) \cdot (z_{j+1} - z_j)^2]|$. Note that for each n, $\left\{\frac{(z_{j+1} - z_j)}{\epsilon_n}\right\}_j$ is a sequence of independent uniform random variable on the unit circle, thus so is $\left\{\frac{(z_{j+1} - z_j)^2}{\epsilon_n^2}\right\}_j$. Since ϕ'' is bounded on Ω,

$$E\left[\left|\phi''(z_j) \cdot \frac{(z_{j+1} - z_j)^2}{\epsilon_n^2}\right|^4\right] < M < \infty,$$

for some $M > 0$ independent of n and j. By Lemma 3.2, and Theorem 3.1 applied to both the real part and the imaginary part, we have almost surely,

$$\lim_{n \to \infty} \sum_{j=1}^{N_\tau}[\phi''(z_j) \cdot (z_{j+1} - z_j)^2] = \lim_{n \to \infty} \frac{\sum_{j=1}^{N_\tau}[\frac{\phi''(z_j) \cdot (z_{j+1} - z_j)^2}{\epsilon_n^2}]}{N_\tau} \cdot (N_\tau \cdot \epsilon_n^2) = 0.$$

Thus $|X_t - Y_t| \to 0$ almost surely.

In case of the event $\{t < \tau\}$, then for each $n \in \mathbb{N}$ large enough, we have

$$|X_t - Y_t| \le |O(\epsilon_n^2)| + |\sum_{j=1}^{N_t}[\phi''(z_j) \cdot (z_{j+1} - z_j)^2]| + |N_\tau \cdot O(\epsilon_n^3)|.$$

The same argument above, with N_τ replaced by N_t, together with the remark after Lemma 3.2 yields, almost surely,

$$\lim_{n\to\infty}\sum_{j=1}^{N_t}[\phi''(z_j)\cdot(z_{j+1}-z_j)^2] = \lim_{n\to\infty}\frac{\sum_{j=1}^{N_t}[\frac{\phi''(z_j)\cdot(z_{j+1}-z_j)^2}{\epsilon_n^2}]}{N_t}\cdot(N_t\cdot\epsilon_n^2) = 0.$$

Combine these two events, we have

$$|X_t - Y_t| \to 0 \text{ as } n\to\infty, \text{ almost surely.} \tag{3.1}$$

Moreover, if we take a finite collection of times $0 \le t_1 \le t_2 \le ... \le t_m < \infty$, we have

$$\sup_k |X_{t_k} - Y_{t_k}| \to 0 \text{ as } n\to\infty, \text{ almost surely.} \tag{3.2}$$

Now we consider the time parametrization. Fix $n \in \mathbb{N}$ large enough. Since X is a concatenation of independent time-changed Brownian motions, it is a time-changed Brownian motion. By the Brownian scaling, the correct time-change is given by the following. Let

$$f(t) = \begin{cases} \sum_{j=0}^{N_t-2}[|\phi'(z_j)|^2(\tau_{j+1}-\tau_j)] + |\phi'(z_{N_t-1})|^2(t-\tau_{N_t-1}) & \text{if } t<\tau \\ \sum_{j=0}^{N_\tau-2}[|\phi'(z_j)|^2(\tau_{j+1}-\tau_j)] + |\phi'(z_{N_\tau-1})|^2(\tau-\tau_{N_\tau-1}) + (t-\tau) & \text{if } t\ge\tau \end{cases}$$

and define

$$s(t) = \inf\{u \ge 0 : f(u) \ge t\}.$$

Then the process $\{X_{s(t)}\}_{t\ge 0}$ is a Brownian motion.

One the other hand, let

$$g(t) = \begin{cases} \int_0^t |\phi'(B_t)|^2 dt & \text{if } t<\tau \\ \int_0^\tau |\phi'(B_t)|^2 dt + (t-\tau) & \text{if } t\ge\tau \end{cases}$$

and define

$$p(t) = \inf\{u \ge 0 : g(u) \ge t\}.$$

Our goal is to show the process $\{Y_{p(t)}\}_{t\ge 0}$ is a Brownian motion.

Note that ϕ' is uniformly continuous in Ω, that is, for any $\delta > 0$, there exists ϵ' such that $|\phi'(z) - \phi'(w)| < \delta$ wherever $|z-w| < \epsilon$. Take n large enough, such that $\epsilon_n < \epsilon'$. Then we have

$$
\begin{aligned}
\sup_{t<\tau}|f-g| &\le \sum_{j=0}^{N_\tau-2}\left[\int_{\tau_j}^{\tau_{j+1}}|\phi'(Y_t)-\phi'(Y_{\tau_j})|dt\right]+\int_{\tau_{N_\tau-1}}^{\tau}|\phi'(Y_t)-\phi'(Y_{\tau_{N_\tau-1}})|dt\\
&\le \sum_{j=0}^{N_\tau-2}[\delta(\tau_{j+1}-\tau_j)]+\delta(\tau-\tau_{N_\tau-1})\\
&\le \delta\tau.
\end{aligned}
$$

Moreover, since their difference $f-g$ stays constant for $t>\tau$, we actually have $||f-g||_{\sup}\le\tau\delta$. As $\tau<\infty$ almost surely and δ is arbitrary, this means almost surely $||f-g||_{\sup}\to 0$ as $n\to\infty$.

Note that since ϕ' is bounded and bounded away from 0, all of the maps f, g, s, and p are homeomorphisms from $[0,\infty)$ to $[0,\infty)$. By Theorem 1 in [3], $s\to p$ locally uniformly as well.

Now fix $t\ge 0$, then $|Y_{p(t)}-X_{s(t)}|\le|Y_{p(t)}-Y_{s(t)}|+|Y_{s(t)}-X_{s(t)}|$. This almost surely goes to 0 as $n\to\infty$ due to the fact that Y is almost surely continuous and (3.1). Using (3.2), the same also holds true simultaneously on a finite collection of times $0\le t_1\le t_2\le ...\le t_m<\infty$. This implies the finite dimensional distributions of $\{Y_{p(t)}\}_{t\ge0}$ are arbitrarily close to that of a Brownian motion, namely $\{X_{s(t)}\}_{t\ge0}$, for some n large enough. Thus $\{Y_{p(t)}\}_{t\ge0}$ itself is a Brownian motion.

Lastly, since ϕ is conformal, it is proper. So $\phi(B_t)$ tends to the boundary of $\phi(\Omega)$ as $t\to\tau$. Thus $f(\tau)$ is the exit time of $\phi(\Omega)$ by $\{Y_{p(t)}\}_{t\ge0}$.

□

4 Another proof of the Riemann mapping theorem

One of the most important theorems in complex analysis is the Riemann mapping theorem. Although it is usually proven using the complex analysis techniques, it is not necessary for one to know complex analysis to understand the statement if we state it in the following way.

Theorem 4.1 (Riemann mapping theorem). *Given a simple connected domain Ω in the plane that is not the entire plane, we can find a continuously differentiable homeomorphism from Ω to $\mathbb{D}$ that preserves angles.*

In this section, we will present a proof of Theorem 1.5, which leads to a new proof of the Riemann mapping theorem. As we mentioned earlier, Theorem 1.5 is easy to prove if we use the Riemann mapping theorem and notice that the Riemann map preserves the harmonic coordinates.

The goal of this section is to provide a proof of Theorem 1.5 that does *not* make use of the Riemann map, aiming at a probabilistic proof of the Riemann mapping theorem. Moreover, we hope this proof does not rely heavily on the complex analysis machinery so that it is accessible to people with limited complex analysis background. For example, actually we will show the map ϕ is "angle preserving" directly, and "analyticity" is just a fact follows. But in this sense we have been only partially successful since complex analysis still plays a few important roles, for example, in proving Lemma 4.2, and our proof uses several elements from the classical proof of the Riemann mapping theorem.

Now let us start with presenting several useful lemmas.

Lemma 4.2. *Let $\Omega \subset \mathbb{C}$ be a simply connected domain and $z_0 \in \Omega$ be a point. Suppose that Ω admits a Green's function with pole at z_0, which we denote $G(\cdot, z_0)$, then $|\nabla_1 G(z, z_0)| > 0$ for all $z \in \Omega \setminus \{z_0\}$.*

Proof. By our assumption $G(z, z_0) + \log|z - z_0|$ extends to be harmonic on Ω, which is a simply connected domain, so we can find an analytic function f defined on Ω such that

$$\operatorname{Re} f(z) = G(z, z_0) + \log|z - z_0|.$$

Hence the function

$$\phi(z) = (z - z_0)e^{-f(z)}$$

is analytic, with $|\phi(z)| = e^{-G(z,z_0)} < 1$ for all $z \in \Omega$ and $\phi(z_0) = 0$.

We claim ϕ is one-to-one. Clearly, if $\phi(z) = 0$ then $z = z_0$. Now let $z_1 \in \Omega \setminus \{z_0\}$, then $|\phi(z_1)| < 1$. Define

$$\phi_1(z) = \frac{\phi(z) - \phi(z_1)}{1 - \overline{\phi(z_1)}\phi(z)}.$$

Clearly it is analytic on Ω and $|\phi_1| < 1$. Now consider the family $\mathcal{F}_{z_1}$ of all subharmonic functions on $\Omega \setminus \{z_1\}$ satisfying

1. $v = 0$ on $\Omega \setminus K$ for some compact $K \subset \Omega$ with $K \neq \Omega$, and

2. $\limsup_{z \to z_1}(v(z) + \log|z|) < \infty$

If $v \in \mathcal{F}_{z_1}$, then by the maximum principle, for all $\epsilon > 0$, we have

$$v + (1 + \epsilon)\log|\phi_1| \leq 0,$$

thus we get

$$\sup_{v \in \mathcal{F}_{z_1}} v < \infty.$$

Therefore the Green's function $G(z, z_1)$ exists. Moreover, taking the limit as $\epsilon \to 0$ we see that for all $z \in \Omega \setminus \{z_1\}$,

$$G(z, z_1) + \log|\phi_1(z)| \leq 0.$$

Choose $z = z_0$, we get

$$G(z_0, z_1) \leq -\log|\phi_1(z_0)| = -\log|\phi(z_1)| = G(z_1, z_0).$$

Switching the role of z_0 and z_1 we get $G(z_0, z_1) = G(z_1, z_0)$. Now if $\phi(z_2) = \phi(z_1)$, then $\phi_1(z_2) = 0$. It follows that $G(z_2, z_1) = \infty$ and thus $z_2 = z_1$. This proves our claim and therefore ϕ' does not vanish and neither does $\nabla_z G(z, z_0)$. □

Note. *This proof is modified from the proof for Thm XV 2.4 in [17].*

The same proof also proves the next lemma.

Lemma 4.3. *Let $\Omega \subset \mathbb{C}$ be a simply connected domain and $z_0 \in \Omega$ be a point. Suppose that Ω admits a Green's function with pole at z_0, then Ω also admits a Green's function with pole at any other interior points.*

From now on we assume our domain Ω to be a bounded simply connected domain whose boundary is an analytic curve.

Lemma 4.4. *Let z_0 be a point in Ω which admits a Green's function. Then $\forall z \in \overline{\Omega} \setminus \{z_0\}$, there exists a unique gradient (ascending) flow curve from z, denoted γ_z, driven by the vector field $\nabla_1 G(z, z_0)$. Moreover, they end at z_0.*

Proof. First assume $z \in \Omega \setminus \{z_0\}$, then γ_z is the solution to the initial value problem

$$\begin{cases} \dot{\gamma}_z(t) = \nabla_1 G(\gamma_z(t), z_0) \\ \gamma_z(0) = z. \end{cases}$$

Here since $G(\cdot, z_0)$ is harmonic, it is smooth in $\Omega \setminus \{z_0\}$. The classical existence and uniqueness theorem i.e. [23, Thm 2.2] and extension theorem for solutions to ODE [23, Thm 2.14] apply, which guarantee the existence and uniqueness of γ_z.

Now if $z \in \partial\Omega$, since $\partial\Omega$ is an analytic curve and $G(\cdot, z_0) = 0$ on $\partial\Omega$, $G(\cdot, z_0)$ extends to be harmonic across $\partial\Omega$ by the Schwarz reflection theorem. Thus the existence and uniqueness of γ_z follows from the same argument as above.

To prove it ends at z_0, recall that by lemma 4.2, $\nabla_1 G(\gamma_z(t), z_0)$ does not vanish inside $\Omega \setminus \{z_0\}$. Since $G(\cdot, z_0)$ extends to be harmonic across $\partial\Omega$, it is continuously differentiable on $\overline{\Omega} \setminus \{z_0\}$. As $G(\cdot, z_0) = 0$ on $\partial\Omega$ and

$$G(\cdot, z_0) > 0$$

on $\Omega \setminus \{z_0\}$, $\nabla_1 G(z, z_0)$ does not vanish on $\partial\Omega$ by lemma 2.10. Also note that as $z \to z_0$, $|\nabla_1 G(z, z_0)| \to \infty$. If follows that $\exists \delta > 0$ such that $|\nabla_1 G(z, z_0)| > \delta$ for all $z \in \overline{\Omega} \setminus \{z_0\}$. Now let $[0, T_z)$ be the maximal interval which γ_z can be extended to, where T_z could potentially be $+\infty$. Since for each $t \in [0, T_z)$,

$$\begin{aligned} G(\gamma(t)) &= \int_0^t \nabla_1 G(\gamma_z(s), z_0) \cdot \gamma'(s) ds \\ &= \int_0^t \nabla_1 G(\gamma_z(s), z_0) \cdot \nabla_1 G(\gamma_z(s), z_0) ds \\ &\geq t\delta^2, \end{aligned}$$

if $G(\gamma_z(T_z)) < \infty$, then $T_z < \infty$, but then it follows from [23, Thm 2.14] that we can further extend γ_z to $[0, T_z + \epsilon)$for some $\epsilon > 0$. This contradicts the fact that $[0, T_z)$ is the maximal interval we can extend γ_z. Thus

$$\lim_{t \to T_z} \gamma_z(t) = z_0.$$

□

As a corollary to the proof, we get the following conclusion about the flow lines.

Corollary 4.5. *In the same setting as above, if z_1, z_2 are two points in $\overline{\Omega} \setminus \{z_0\}$, without loss of generality $G(z_1, z_0) \leq G(z_2, z_0)$, then either γ_{z_1} goes through z_2 or γ_{z_1} and γ_{z_2} are disjoint.*

Proof. Suppose $\gamma_{z_1} \cap \gamma_{z_2} \neq \emptyset$. At every point of intersection z, since the initial value problem

$$\begin{cases} \dot{\gamma}_z(t) = \nabla_1 G(\gamma_z(t), z_0) \\ \gamma_z(0) = z \end{cases}$$

has a unique solution, they will flow along identical path after that point. From this intersection point z we can also drive the descending flow according to the initial value problem

$$\begin{cases} \dot{\gamma}_z(t) = -\nabla_1 G(\gamma_z(t), z_0) \\ \gamma_z(0) = z \end{cases} .$$

As this initial problem also has a unique solution, and the solution is just a time reversal of the ascending flow from both z_1 and z_2, both z_1 and z_2 will be on this curve. Thus γ_{z_1} goes through z_2. □

To elaborate a bit more on lemma 4.4. This lemma tells us that from each point in the closure of Ω, except for the preset point z_0, we can define a gradient flow curve driven by the vector field $\nabla_1 G(z, z_0)$. Since $\nabla_1 G(z, z_0)$ does not vanish as proven in lemma 4.2, the value of the Green's function strictly increases along each gradient flow curve. With this idea, we can define a map f from $\Omega \setminus \{z_0\}$ into $\Omega \setminus \{z_0\}$ if we choose a "flowing distance" for each point. Here our choice is to flow each point so that the Green's function increases by a fixed number. To be precise, fix $k > 0$, for each point $z \in \Omega \setminus \{z_0\}$, define the function

$$f_k(z) := \gamma_z(t(z, k)),$$

where $t(z, k)$ is chosen so that

$$G(f_k(z), z_0) = G(z, z_0) + k.$$

The existence of such $t(z, k)$ is guaranteed by the intermediate value theorem.

Lemma 4.6. *Let z_0 be a point in Ω and suppose Ω admits a Green's function $G(z, z_0)$ at z_0. Then for each $k \geq 0$, the map f_k defined above extends to be a conformal map from Ω onto $\Omega_k := \{z \in \Omega : G(z, z_0) \in (k, +\infty]\}$, which is continuous up to the boundary.*

Proof. Take a point $a \in \partial\Omega$. By lemma 4.4 we can define a gradient flow curve γ_a going from a to z_0.

First we show that $\Omega \setminus \overline{\gamma_a}$ is connected. For any two points z_1, z_2 in $\overline{\Omega} \setminus \{z_0\}$, define $z_1 \sim z_2$ if $z_1 \in \gamma_{z_2}$ or $z_2 \in \gamma_{z_1}$. By the virtue of lemma 4.5, $\sim$ is a equivalence relation. Denote the corresponding quotient map by q. Moreover, every point in $\Omega \setminus \overline{\gamma_a}$ is equivalent to a boundary point of Ω as the gradient descending flow will exit Ω on its boundary. Thus we have

$$(\overline{\Omega} \setminus \{z_0\})/\sim \quad \cong \partial\Omega \cong \mathbb{S}^1$$

As removing any single point from $\mathbb{S}^1$ does not destroy its connectivity, it follows that $q(\overline{\Omega} \setminus \{z_0\}) \setminus q(a)$ is connected. It is also clear that each fiber in this quotient, which are the flowing curves, is connected and the claim follows.

We also claim that $\Omega \setminus \overline{\gamma_a}$ is simply connected. To see this, take a closed curve inside $\Omega \setminus \overline{\gamma_a}$. First we shrink each point on this curve radially along its flowing curve to a level set of $G(z, z_0)$. As γ_a can only go across this level set at one point, the remains of this level set is still path-connected, thus we can further shrink the closed curve along this level set continuously to a point.

Now we have that the set $\Omega \setminus \overline{\gamma_a}$ is simply connected. As $G(\cdot, z_0)$ is harmonic on it, we can find its harmonic conjugate $\tilde{G}(\cdot, z_0)$. Then the map

$$g(z) = G(z, z_0) + i\tilde{G}(z, z_0)$$

is analytic on $\Omega \setminus \overline{\gamma_a}$.

Now we check the boundary values of g. Clearly we have

$$\lim_{z \to z_0} g(z) = \infty$$

For $p \in \partial\Omega$, by assumption

$$\lim_{z \to p} G(z, z_0) = 0.$$

As G can be extended harmonically across $\partial\Omega$, $\tilde{G}$ can be defined continuously on $\partial\Omega \setminus \{a\}$. Thus

$$\lim_{z \to p} g(z) \in \{z : \operatorname{Re} z = 0\}.$$

For $p \in \gamma_a$ and $p \neq a$ or z_0, since G can is harmonic across γ_a, $\tilde{G}$ can be defined across γ_a from each side individually. Moreover, γ_a is the gradient flow curve by $\nabla_1 G$, thus is also the level curve of $\tilde{G}$. Thus, from each side of γ_a,

$$\lim_{z \to p \text{ from that side}} \tilde{G}(z, z_0)$$

exists and is constant. Write

$$A = \lim_{\substack{z \to p \\ \text{counterclockwise}}} \tilde{G}(z, z_0),$$

$$B = \lim_{\substack{z \to p \\ \text{clockwise}}} \tilde{G}(z, z_0),$$

and let $\{l(t), 0 < t < 1\}$ be a smooth curve in Ω going from p to p winding around z_0 counterclockwise. Then

$$
\begin{aligned}
A - B &= \lim_{\epsilon \to 0} \int_{\epsilon}^{1-\epsilon} \nabla_1 \tilde{G}(l(t), z_0) \cdot T_l(t) |l'(t)| dt && (4.1)\\
&= \lim_{\epsilon \to 0} \int_{\epsilon}^{1-\epsilon} \nabla_1 G(l(t), z_0) \cdot \eta_l(t) |l'(t)| dt && (4.2)\\
&= \lim_{\epsilon \to 0} \int_{\epsilon}^{1-\epsilon} \nabla_1 (\log |z - z_0| + u(z)) \cdot \eta_l(t) |l'(t)| dt && (4.3)\\
&= -2\pi && (4.4)
\end{aligned}
$$

where T_l denotes the unit tangent vector, η_l denotes the outer unit normal vector, $arc(\epsilon)$ denotes the portion of the $\partial\mathbb{B}(z_0, r)$ that lies between $\gamma_{l(1-\epsilon)}$ and $\gamma_{l(\epsilon)}$. Here the second equality is obtained by rotating both vectors in the dot product by $-\pi/2$. The last equality holds since $u(z)$ is harmonic in Ω and thus the integral

$$\oint_l \nabla u(z) \cdot \eta(t) ds = 0$$

by the divergence theorem.

Thus we have that $g(z)$ tends to the boundary of the half strip $S := \{z : \operatorname{Re} z > 0, B < \operatorname{Im} z < A\}$. Since $\nabla_1 G$ does not vanish, f' does not vanish either. So g is locally homeomorphic. We just showed that $g : \Omega \setminus \overline{\gamma_a} \to S$ is proper. By lemma 2.26, g is globally homeomorphic and thus g is a conformal.

Now, for $k > 0$, consider the map $f_k = g^{-1} \circ T_k \circ g$, where $T_k(z) = z + k$ is a translation map. This map agrees with the map we described above on $\Omega \setminus \overline{\gamma_a}$. Moreover f_k is a conformal map from $\Omega \setminus \overline{\gamma_a}$ onto $\Omega_k \setminus \overline{\gamma_a}$, where $\Omega_k := \{z \in \Omega : G(z, z_0) > k\}$. It is easy to see from its construction that f_k can be extended continuously across γ_a, thus it extends analytically also. Lastly around z_0, f_k is bounded, thus f extends to be analytic on Ω by Riemann's theorem on removable singularity. Also it is clear that $f(z_0) = z_0$, $f(\gamma_a) \subset \gamma_a$ and f restricted to γ_a is one-to-one, so f is a conformal map from Ω to Ω_k.

Lastly, to show f extends continuously to the boundary, note that at each point in $\partial\Omega \setminus \{a\}$, $G(z, z_0)$ can be extended harmonically across $\partial\Omega$ by the Schwarz reflection theorem. Thus g can be extended continuous to $\partial\Omega \setminus \{a\}$, and so f can be extended to $\partial\Omega \setminus \{a\}$ also. Similarly by taking another cut, we see that f can be continuously extended to $\partial\Omega$. □

Now that for each $k > 0$ we have a conformal map from Ω to Ω_k, where Ω_k is the "sup-level set" of the Green's function $G(\cdot.z_0)$, i.e. $\Omega_k = \{z \in \Omega : G(z, z_0) > k\}$. The next lemma we will present tells us that Ω_k is essentially a disk when k is large.

Lemma 4.7. *As $k \to \infty$, $\{\Omega_k\}$ converges to a round disk centered at z_0 in the following sense. For any $\epsilon > 0$, we can find $k > 0$ and a homeomorphism $h_k : \partial(ce^k(\Omega_k - z_0)) \to \partial\mathbb{D}$ with*

$$\sup_{z \in \partial\Omega_k} |h_k(z) - z| < \epsilon,$$

for some constant $c > 0$ independent of k. Moreover, let $b \in \partial\Omega$ be a point. So $f_k(b) \in \Omega_k$, where f_k is the map mentioned in lemma 4.6. Then $h_k(ce^k(f_k(b) - z_0))$ is constant for all $k > 0$.

Proof. Without loss of generality we assume $z_0 = 0$. Since $\Omega \setminus \overline{\gamma_a}$ is simply connected, we can define $\log(z)$ analytically on it. Then

$$\log(z) = \log|z| + i \operatorname{Im} \log(z).$$

Recall that the Green's function takes the form

$$G(z, 0) = -\log|z| + u(z),$$

where u is a harmonic function on Ω. So the function

$$\tilde{G}(z, 0) := -\operatorname{Im} \log(z) + \tilde{u}(z)$$

is one of its harmonic conjugates. Here we normalize so that $\tilde{u}(0) = 0$. Then the function

$$g(z) = -\log(z) + u(z) + i\tilde{u}(z)$$

is analytic on $\Omega \setminus \overline{\gamma_a}$. As we argued in the proof of lemma 4.6, g maps $\Omega \setminus \overline{\gamma_a}$ conformally onto a half-stripe. Moreover, g maps the level set $L_k \setminus \gamma_a$ onto a vertical segment of length 2π. Thus g^{-1} defines a parametrization for $L_k \setminus \gamma_a$. As L_k is a closed curve, this parametrization can be extended periodically to be on $\mathbb{R}$.

Similarly, consider the map

$$g^*(z) = -\log(z) + u(0).$$

It maps each circular loop $\partial\mathbb{B}(0, r)$ to a vertical segment, and its inverse

$$(g^*)^{-1}(z) = e^{u(0)-z}$$

can be extended to define a parametrization of $\partial\mathbb{B}(0,r)$ by $\mathbb{R}$.

Combining these two observations, we get that the map $(g^*)^{-1}\circ g$ maps L_k to a circle. Furthermore, since L_k shrinks in size as $k\to\infty$, we compensate this effect by multiplying the proper normalization factor e^k. Then we get that for all z such that $e^{-k}z\in L_k$,

$$\begin{aligned} h_k(z) :&= e^k(g^*)^{-1}(g(e^{-k}z)) \\ &= e^k(e^{-k}z)e^{u(0)-u(e^{-k}z)-i\tilde{u}(e^{-k}z)} \\ &\to z \end{aligned}$$

as $k\to\infty$. Since $\partial\Omega_k$ is compact, $h_k(z)\to z$ uniformly. At this stage, h_k maps $e^k\Omega_k$ to a disk of radius $e^{u(0)}$. We can renormalize h according this scaling factor so that the image has radius 1. For the last claim, since $\tilde{G}$ is constant on γ_b, $\operatorname{Arg}((g^*)^{-1})\circ g$ is constant on it as well. Thus all points on γ_b are mapped to the same point on ∂D. □

Lemma 4.8. *Let Ω_k be a sequence of bounded domains converging to $\mathbb{D}$ in the sense that, for any $\epsilon>0$, there exists $k>0$ and a homeomorphism $h_k:\partial\Omega_k\to\partial\mathbb{D}$ such that*

$$\sup|h_k(z)-z|<\epsilon.$$

If α^* is a connected arc on $\partial\mathbb{D}$ with end points b^* and c^*, then we have

$$\nabla\omega(0,h_k^{-1}(\alpha^*),\Omega_k)\to\nabla\omega(0,\alpha^*,\mathbb{D}),$$

as $k\to\infty$.

Note. *The Ω_k's in this lemma should be understood to be the Ω_k's in the previous lemmas with proper translation and scaling.*

Proof. For the convenience of notation, denote $\alpha_k=h_k^{-1}(\alpha^*)$, $b_k=h_k^{-1}(b^*)$ and $c_k=h_k^{-1}(c^*)$. Take $z\in\partial\mathbb{B}(0,1/2)$ and consider a Brownian motion $\{B_t\}$ starting from z. Here we assume k is large enough such that $\overline{\mathbb{B}(z_0,1/2)}\subset\Omega_k$.

Let

$$\tau:=\inf\{t>0,B_t\in\partial\Omega_k\},$$

$$\tau^*:=\inf\{t>0,B_t\in\partial\mathbb{D}\}.$$

Also consider events

$$E := \{B_{\tau_k} \in \alpha_k\},$$

$$F := \{B_{\tau^*} \in \alpha^*\},$$

and

$$N := \{\text{dist}(B_{\tau_k}, \{b_k, c_k\}) > \sqrt{\epsilon} + \epsilon \text{ if } \tau_k \leq \tau^* \text{ OR } \text{dist}(B_{\tau^*}, \{b^*, c^*\}) > \sqrt{\epsilon} + \epsilon \text{ if } \tau_k > \tau^*\}.$$

Then we have

$$\begin{aligned}
&|\omega(z, \alpha_k, \Omega_k) - \omega(z, \alpha^*, \mathbb{D})| \\
=&|\mathbb{P}^z(E) - \mathbb{P}^z(F)| \\
\leq&\mathbb{P}^z(E \setminus F) + \mathbb{P}^z(F \setminus E) \\
\leq&\mathbb{P}^z[(E \setminus F) \cap \{\tau_k \leq \tau^*\} \cap N] + \mathbb{P}^z[(E \setminus F) \cap \{\tau_k > \tau^*\} \cap N] \\
&+ \mathbb{P}^z[(F \setminus E) \cap \{\tau_k \leq \tau^*\} \cap N] + \mathbb{P}^z[(E \setminus F) \cap \{\tau_k > \tau^*\} \cap N] + \mathbb{P}^z(N^c)
\end{aligned}$$

Now we claim all the terms in the last line goes to 0 as $\epsilon \to 0$.

First consider the event $(E \setminus F) \cap \{\tau_k \leq \tau^*\} \cap N$. In this event, we have $\tau_k \leq \tau^*$ and $B_{\tau_k} \in \alpha_k$. Moreover, we have $\text{dist}(B_{\tau_k}, \{b_k, c_k\}) > \sqrt{\epsilon} + \epsilon$, which means that α^* is a curve starting and ending at points at least $\sqrt{\epsilon}$ away from B_{τ_k} but $\text{dist}(B_{\tau_k}, \alpha^*) < \epsilon$. If $B_{\tau_k} \in \alpha^*$ then clearly $B_{\tau_k} \in \partial\mathbb{D} \setminus \alpha^*$. Otherwise, by the Beurling projection theorem, we have

$$\omega(B_{\tau_k}, \alpha^*, \mathbb{B}(B_{\tau_k}, \sqrt{\epsilon})) \geq \omega(0, [\epsilon, \sqrt{\epsilon}), \mathbb{B}(0, \sqrt{\epsilon})) \to 1$$

as $\epsilon \to 0$. Thus we have

$$\mathbb{P}^z[F | E \cap \{\tau_k \leq \tau^*\} \cap N] \to 1,$$

and so

$$\mathbb{P}^z[F^c | E \cap \{\tau_k \leq \tau^*\} \cap N] \to 0.$$

Therefore,

$$\mathbb{P}^z[(E \setminus F) \cap \{\tau_k \leq \tau^*\} \cap N] \to 0,$$

as $\epsilon \to 0$. Repeating this argument for the remaining terms, we get the same result for $\mathbb{P}^z[(E \setminus F) \cap \{\tau_k > \tau^*\} \cap N]$, $\mathbb{P}^z[(F \setminus E) \cap \{\tau_k \leq \tau^*\} \cap N]$ and $\mathbb{P}^z[(E \setminus F) \cap \{\tau_k > \tau^*\} \cap N]$. Lastly, we have

$$\mathbb{P}^z(N^c) < \omega(0, \partial[\mathbb{B}(b^*, \sqrt{\epsilon} + 2\epsilon) \cup \mathbb{B}(c^*, \sqrt{\epsilon} + 2\epsilon)], \mathbb{D}) \to 0,$$

as $\epsilon \to 0$. Therefore,

$$|\omega(z, \alpha_k, \Omega_k) - \omega(z, \alpha^*, \mathbb{D})| \to 0$$

as $k \to \infty$. Taking the maximum we have

$$\max_{|z|=1/2} |\omega(z, \alpha_k, \Omega_k) - \omega(z, \alpha^*, \mathbb{D})| \to 0$$

as $k \to \infty$. Thus by the maximum principle,

$$\omega(z, \alpha_k, \Omega_k) \to \omega(z, \alpha^*, \mathbb{D})$$

uniformly on all compact subsets of $\mathbb{B}(0, 1/2)$. We also have that $\nabla_z \omega(z, \alpha_k, \Omega_k) \to \nabla_z \omega(z, \alpha^*, \mathbb{D})$ uniformly on all compact subsets. In particular,

$$\nabla_z \omega(0, \alpha_k, \Omega_k) \to \nabla_z \omega(0, \alpha^*, \mathbb{D})$$

. □

Now we collect all these results to give a proof of Theorem 1.5.

Proof. Recall that a, b, c are three distinct points on $\partial\Omega$ and a', b' and c' are three distinct points on $\partial\mathbb{D}$. We claim the map $\phi : \Omega \to \mathbb{D}$ defined by $\phi(z) = w$ whenever $H_{\Omega,a,b,c}(z) = H_{\mathbb{D},a',b',c'}(w)$ is well-defined and it is an angle-preserving homeomorphism.

First we show the map ϕ is well-defined. That is, for any triple of real number $0 < A, B, C < 1$ with $A + B + C = 1$, we can find exactly one point $w \in \mathbb{D}$ such that $H_{\mathbb{D},a',b',c'}(w) = (A, B, C)$.

To show existence, let $M(z) \in \mathrm{Aut}(\mathbb{D})$ be a Möbius map which maps the boundary points $1, e^{2\pi Ci}, e^{2\pi(A+C)i}$ to a', b' and c'. Choose $w = M(0)$, then by the conformal invariance of harmonic measure,

$$H_{\mathbb{D},a',b',c'}(w) = H_{\mathbb{D},1,e^{2\pi Ci},e^{2\pi(A+C)i}}(0) = (A, B, C).$$

To show uniqueness, let $w' \in \mathbb{D}$ be another point with $H_{\mathbb{D},a',b',c'}(w') = (A, B, C)$. Take two Möbius maps $M_1, M_2 \in \mathrm{Aut}(\mathbb{D})$ mapping w and w' to 0 respectively, and fixing a' on the boundary. Then both M_1 and M_2 map a', b', c' to 3 points on the circle which split the circle into arcs of lengths $2\pi A, 2\pi B, 2\pi C$ in the same order. That is M_1 and M_2 maps a', b', c' to the same places. Thus $M_1 = M_2$ and so $w = w'$.

The argument above also shows that the map

$$H_{\mathbb{D},a',b',c'} : \mathbb{D} \to \{(A, B, C) \in \mathbb{R}^3 : 0 < A, B, C < 1, A + B + C = 1\}$$

is bijective. Moreover, the projection of H onto each coordinate is harmonic, thus smooth. So H is a smooth map. Since the harmonic measures have explicit formula on $\mathbb{D}$, one can verify by direct computation that $H'_{\mathbb{D},a',b',c'}$ is invertible, so $H^{-1}_{\mathbb{D},a',b',c'}$ is also smooth by the inverse function theorem. Thus

$$\phi = H^{-1}_{\mathbb{D},a',b',c'} \circ H_{\Omega,a,b,c}$$

is smooth.

To prove ϕ is analytic, it suffices to show it preserves angles. We now proceed to prove ϕ preserves the angle between the gradients of the harmonic measures $\omega(z_0, \alpha, \Omega)$, $\omega(z_0, \beta, \Omega)$ and $\omega(z_0, \gamma, \Omega)$.

Under our assumption about Ω, it admits a Green's function. Take f_k to be the conformal map in lemma 4.6 mapping Ω onto Ω_k. Since the conformal maps f_k preserve angles, it suffices to show the angles between $\nabla\omega(z_0, \alpha_k, \Omega_k)$, $\nabla\omega(z_0, \beta_k, \Omega_k)$ and $\nabla\omega(z_0, \gamma_k, \Omega_k)$ are correct, where α_k, β_k and γ_k denotes $f_k(\alpha)$, $f_k(\beta)$ and $f_k(\gamma)$. By lemma 4.7, for any $\epsilon > 0$. we can find a scaling factor c_k so that $c_k\Omega_k$ is close to $\mathbb{D}$ for k large, in the sense that we can find a homeomorphism $h_k : \partial c_k\Omega_k \to \partial\mathbb{D}$ and

$$\sup_{z\in\partial\Omega_k} |h_k(z) - z| < \epsilon.$$

Moreover, h_k maps $c_k\alpha_k$, $c_k\beta_k$ and $c_k\gamma_k$ to fixed arcs, denoted α^*, β^* and γ^* respectively, on $\partial\mathbb{D}$.

By lemma 4.8, we have as $k \to \infty$,

$$c_k\nabla\omega(z_0, \alpha_k, \Omega_k) \to \nabla\omega(0, \alpha^*, \mathbb{D})$$
$$c_k\nabla\omega(z_0, \beta_k, \Omega_k) \to \nabla\omega(0, \beta^*, \mathbb{D})$$
$$c_k\nabla\omega(z_0, \gamma_k, \Omega_k) \to \nabla\omega(0, \gamma^*, \mathbb{D})$$

Since on the right hand side $\nabla\omega(0, \alpha^*, \mathbb{D})$, $\nabla\omega(z_0, \beta_k, \Omega_k)$, and $\nabla\omega(0, \gamma^*, \mathbb{D})$ are all non-zero, in particular this implies $\nabla\omega(z_0, \alpha_k, \Omega_k)$, $\nabla\omega(z_0, \beta_k, \Omega)$ and $\nabla\omega(z_0, \gamma, \Omega_k)$ are non-zero for k large. As $f_k : \Omega \to \Omega_k$ is conformal, $\nabla\omega(z_0, \alpha, \Omega)$, $\nabla\omega(z_0, \beta, \Omega)$ and $\nabla\omega(z_0, \gamma, \Omega)$ are all non-zero as well. So the level sets of $\omega(z_0, \alpha, \Omega)$, $\omega(z_0, \beta, \Omega)$ and $\omega(z_0, \gamma, \Omega)$ are locally smooth curves near z_0. Taking the angle between them we have that

$$\angle(\nabla\omega(z_0, \alpha, \Omega), \nabla\omega(z_0, \beta, \Omega)) = \angle(\nabla\omega(z_0, \alpha_k, \Omega_k), \nabla\omega(z_0, \beta_k, \Omega_k)) \to \angle(\nabla\omega(0, \alpha^*, \mathbb{D}), \nabla\omega(0, \beta^*, \mathbb{D}))$$
$$\angle(\nabla\omega(z_0, \beta, \Omega), \nabla\omega(z_0, \gamma, \Omega)) = \angle(\nabla\omega(z_0, \beta_k, \Omega_k), \nabla\omega(z_0, \gamma_k, \Omega_k)) \to \angle(\nabla\omega(0, \beta^*, \mathbb{D}), \nabla\omega(0, \gamma^*, \mathbb{D}))$$
$$\angle(\nabla\omega(z_0, \gamma, \Omega), \nabla\omega(z_0, \alpha, \Omega)) = \angle(\nabla\omega(z_0, \gamma_k, \Omega_k), \nabla\omega(z_0, \alpha_k, \Omega_k)) \to \angle(\nabla\omega(0, \gamma^*, \mathbb{D}), \nabla\omega(0, \alpha^*, \mathbb{D}))$$

where $\angle(v_1, v_2)$ stands for the angle between vectors v_1 and v_2. Thus

$$\angle(\nabla\omega(z_0,\alpha,\Omega),\nabla\omega(z_0,\beta,\Omega)) = \angle(\nabla\omega(0,\alpha^*,\mathbb{D}),\nabla\omega(0,\beta^*,\mathbb{D}))$$
$$\angle(\nabla\omega(z_0,\beta,\Omega),\nabla\omega(z_0,\gamma,\Omega)) = \angle(\nabla\omega(0,\beta^*,\mathbb{D}),\nabla\omega(0,\gamma^*,\mathbb{D}))$$
$$\angle(\nabla\omega(z_0,\gamma,\Omega),\nabla\omega(z_0,\alpha,\Omega)) = \angle(\nabla\omega(0,\gamma^*,\mathbb{D}),\nabla\omega(0,\alpha^*,\mathbb{D}))$$

Let us return to the map ϕ defined in Theorem 1.5. Clearly ϕ maps the level curves of $\omega(\cdot,\alpha,\Omega)$, $\omega(\cdot,\beta,\Omega)$ and $\omega(\cdot,\gamma,\Omega)$ to the level curves of $\omega(\cdot, arc(b'c'),\mathbb{D})$, $\omega(\cdot, arc(c'a'),\mathbb{D})$ and $\omega(\cdot, arc(a'b'),\mathbb{D})$. For $z_0 \in \Omega$, by taking an Möbius map of the disk, we see that the angle between the level curves of $\omega(\cdot, arc(b'c'),\mathbb{D})$, $\omega(\cdot, arc(c'a'),\mathbb{D})$ and $\omega(\cdot, arc(a'b'),\mathbb{D})$ going through $\phi(z_0)$ are exactly

$$\angle(\nabla\omega(0,\alpha^*,\mathbb{D}),\nabla\omega(0,\beta^*,\mathbb{D})),$$
$$\angle(\nabla\omega(0,\beta^*,\mathbb{D}),\nabla\omega(0,\gamma^*,\mathbb{D})),$$
$$\angle(\nabla\omega(0,\gamma^*,\mathbb{D}),\nabla\omega(0,\alpha^*,\mathbb{D})).$$

Thus we showed that ϕ is angle preserving, which completes the proof.

□

5 Quasi-Brownian motion

The proof we presented in the preceding section is encouraging, as it serves as an evidence that our general strategy is effective at least in a special case of the measurable Riemann mapping theorem. Now we would like to shift our attention to the general case. Recall that the map ϕ in Theorem 1.5 is constructed by matching a triple of 3 Brownian hitting probabilities in the domain Ω and that inside the disk. The reason we use Brownian motions for both domains is that a Brownian motion is preserved (up to a time-change) under a conformal map. However this is no longer true under general quasiconformal maps. For example, consider the linear map

$$\phi(z) = 2\,\mathrm{Re}\, z + i\,\mathrm{Im}\, z.$$

Under this map, a Brownian motion will be stretched by a factor of 2 in the real direction while kept unchanged in the imaginary direction. The resulting process is not a Brownian motion with or without time-changes. Thus in order to still carry out the construction of ϕ in a similar manner in the general case, it is necessary to understand the image process of a Brownian motion under quasiconformal maps. For this reason, we define the notion of quasi-Brownian motions as follows.

Definition 5.1. Let $\Omega \subset \mathbb{C}$ be a region and let $f : \Omega \to \Omega' \subset \mathbb{C}$ be a quasiconformal map and $z_0 \in \Omega$ be a point. We say a stochastic process $\{X_t, t \geq 0\}$ is a quasi-Brownian motion, or in short a QBM, starting at z_0 with respect to f if the process $\{f(X_t), t \geq 0\}$ is a Brownian motion in Ω' starting at $f(z_0)$.

The idea to study quasiconformal maps using stochastic processes and particularly quasiconformal images of Brownian motion is due to Bernt Øksendal, see [19].

An observation about QBM that is crucial for our program and that does not appear in Øksendal's work is the fact that the "shape" (versus the "time" in which they are parametrized) of their paths depends only on the Beltrami coefficient of the quasiconformal maps. To see this, first we quote a useful lemma about the Beltrami coefficient of composition of quasiconformal maps.

Lemma 5.2 (Beltrami chain rule)**.** *[2, Thm 5.6.6.] Suppose $f : \Omega \to \Omega''$ and $g : \Omega \to \Omega'$ are quasiconformal, with Beltrami coefficients μ_f and μ_g respectively. Then the composition $f \circ g^{-1}$ is quasiconformal in Ω', with Beltrami coefficient*

$$\mu_{f\circ g^{-1}} \circ g(z) = \frac{\mu_f(z) - \mu_g(z)}{1 - \mu_f(z)\overline{\mu_g(z)}} \cdot \frac{g_z(z)}{\overline{g_z(z)}}.$$

Proof. Let $h = f \circ g^{-1}$, then $f = h \circ g$. Using the chain rule we have

$$f_z = h_z \circ g \cdot g_z + h_{\overline{z}} \circ g \cdot \overline{g_{\overline{z}}}$$

$$f_{\overline{z}} = h_z \circ g \cdot g_{\overline{z}} + h_{\overline{z}} \circ g \cdot \overline{g_z}.$$

Solving for $h_z \circ g$ and $h_{\overline{z}} \circ g$ we get

$$h_z \circ g = \frac{f_z\overline{g_z} - f_{\overline{z}}\overline{g_{\overline{z}}}}{g_z\overline{g_z} - g_{\overline{z}}\overline{g_{\overline{z}}}}$$

$$h_{\overline{z}} \circ g = -\frac{f_z g_{\overline{z}} - f_{\overline{z}} g_z}{g_z\overline{g_z} - g_{\overline{z}}\overline{g_{\overline{z}}}}$$

Thus we have

$$\begin{aligned}\mu_h \circ g &= \frac{h_{\overline{z}} \circ g}{h_z \circ g} \\ &= -\frac{f_z g_{\overline{z}} - f_{\overline{z}} g_z}{f_z\overline{g_z} - f_{\overline{z}}\overline{g_{\overline{z}}}} \\ &= \frac{\mu_f - \mu_g}{1 - \mu_f\overline{\mu_g}} \cdot \frac{g_z}{\overline{g_z}}\end{aligned}$$

□

In particular, if $\mu_f = \mu_g$, lemma 5.2 asserts that $\mu_{f\circ g^{-1}} = 0$ and thus $f \circ g^{-1}$ is conformal. Using the conformal invariance of Brownian motion, we get the following theorem, which turns out to be crucial to our attempt to build the QBM using only the Beltrami coefficient of quasiconformal maps.

Theorem 5.3 (Locality)**.** *Let $f_1,\ f_2 : \mathbb{C} \to \mathbb{C}$ be two quasiconformal maps with Beltrami coefficient μ_1 and μ_2 respectively. Let $U \subset \mathbb{C}$ be a domain and $z_0 \in U$. Let $X_1,\ X_2$ be quasi-Brownian motions starting at z_0 with respect to $f_1,\ f_2$ respectively. If $\mu_1 = \mu_2$ almost everywhere in U, then X_1 is a time change of X_2 up to the hitting time τ_U.*

Proof. It suffices to prove $f_2(X_1)$ is a time-changed Brownian motion up to τ_U. Note that $f_2(X_1) = f_2 \circ f_1^{-1}(B_t)$, where B is a planar Brownian motion. Since $\mu_1 = \mu_2$ almost everywhere in U, by the chain rule (Theorem 5.2), $f_2 \circ f_1^{-1}$ is quasiconformal with dilatation 0 almost everywhere in U. Thus, $f_2 \circ f_1^{-1} : f_1(U) \to f_2(U)$ is conformal. By Theorem 2.18, $f_2(X_1)$ is a time-changed Brownian motion, with the time-change given by $\zeta(t) = \int_0^t |(f_2 \circ f_1^{-1})'(B_s)|^2 ds$. □

6 Constructions of QBM: successes and failures

In this section we will focus on the construction of quasi-Brownian motions. More specifically, recall that the only information provided in a Beltrami equation is the Beltrami coefficient μ. For the purpose of solving the Beltrami equation, we would like to construct quasi-Brownian motions using only the Beltrami coefficient, without knowing the quasiconformal map itself.

One caveat here is that the Beltrami coefficient does not contain all the information about a quasiconformal map. If we post-compose a quasiconformal map with any conformal map, its Beltrami coefficient stays unchanged. However, post-composing a conformal map will potentially force us to apply a time-change to the image process in order for it to still be a Brownian motion. For this reason, we only expect to be able to construct the quasi-Brownian motion up to a time-change and this is already good enough for our purpose to define the exiting probabilities.

This section is structured as follows. First we will introduce an approach to this problem by Bernt Øksendal based on Oshima, Takeda and Fukushima's work in [9] using Dirichlet forms. Then we will introduce our constructive approach, starting from the easy case with a square and then move to the general case. Lastly we will present several constructions we attempted that ended up in failures. Although each of these attempts failed the way they did, I think the intuition they deliver helped a lot in leading us to the correct approach and we think it is worthwhile to also include them in this paper.

6.1 Øksendal's construction

In 1988, based on Fukushima, Oshima, and Takeda's work in [9], Øksendal gave a construction of the quasi-Brownian motions using Dirichlet forms. The details are in [19]. In his construction he formed the Dirichlet form using the first order derivatives of the quasiconformal map, which requires more information than what is provided in the Beltrami equation. We realized that one particular version of his construction depends only on the information encoded in the Beltrami coefficient. With this observation, this construction fulfills our requirement that only information encoded in the Beltrami coefficient should be used.

To understand his construction, we first present a key result relating symmetric forms and Hunt processes together with some relevant definitions. Since the full theory is lengthy, we only attempt to provide a very brief summary of the result. Readers who are interested should refer to the original textbook [9] for more details.

Definition 6.1. [9, Section A.2] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A strong Markov process

$$M = (\Omega, \mathcal{F}, \{X_t\}_{t\in[0,\infty]}, \{\mathbb{P}^x\}_{x\in S})$$

with state space $(S_\Delta, \mathcal{B}_\Delta(S))$ and cemetery state $\Delta \in S_\Delta$ is called a *Hunt process* if the following conditions are satisfied:

1. $X_\infty(\omega) = \Delta$ for all $\omega \in \Omega$,

2. $X_t(\omega) = \Delta$ for all $t \geq \inf\{s \geq 0 : X_s(\omega) = \Delta\}$,

3. for each $t \in [0, \infty]$, there is a map $\theta_t : \Omega \to \Omega$ for which

   $$X_s \circ \theta_t = X_{t+s}$$

   for all $s \geq 0$,

4. $X_t(\omega)$ is right continuous on $[0, \infty)$ and has left limit inside $(0, \infty)$ on S_Δ for each $w \in \Omega$.

5. M is quasi-left continuous, which means that for any stopping time τ, and any sequence, $\{\tau_n\}$, of stopping times increasing to τ,

$$\mathbb{P}^x[\lim_{n\to\infty} X_{\tau_n} = X_\tau, \tau < \infty] = \mathbb{P}^x[\tau < \infty]$$

Definition 6.2. Let H be a Hilbert space with inner product $(\cdot,\cdot)$. We call $\mathcal{E}$ a *symmetric form* on H if the following conditions are satisfied:

1. $\mathcal{E}$ is defined on $\mathcal{D}[\mathcal{E}] \times \mathcal{D}[\mathcal{E}]$ with values in $\mathbb{R}$, where $\mathcal{D}[\mathcal{E}]$ is a dense linear subspace of H,
2. for all $u, v, w \in \mathcal{D}[\mathcal{E}]$ and $a \in \mathbb{R}$ we have $\mathcal{E}(u,v) = \mathcal{E}(v,u)$, $\mathcal{E}(u+v,w) = \mathcal{E}(u,w) + \mathcal{E}(v,w)$, $a\mathcal{E}(u,v) = \mathcal{E}(au,v)$, and $\mathcal{E}(u,u) \geq 0$.

Note that the inner product $(\cdot,\cdot)$ itself is a symmetric form defined on H. Moreover, given a symmetric form $\mathcal{E}$ on H, for each $\alpha > 0$,

$$\begin{cases} \mathcal{E}_\alpha(u,v) = \mathcal{E}(u,v) + \alpha(u,v), \quad u,v \in \mathcal{D}[\mathcal{E}] \\ \mathcal{D}[\mathcal{E}_\alpha] = \mathcal{D}[\mathcal{E}] \end{cases}$$

defines a new symmetric form on H. Note that the space $\mathcal{D}[\mathcal{E}]$ is a pre-Hilbert space with inner product $\mathcal{E}_\alpha$. Here we say $\mathcal{E}$ is *closed* if $\mathcal{D}[\mathcal{E}]$ is complete with respect to the inner product $\mathcal{E}_\alpha$ for some (and thus all) $\alpha > 0$.

Let $(X, \mathcal{B}, m)$ be a σ-finite measure space. A symmetric form $\mathcal{E}$ defined on $H = L^2(X;m)$ is called *Markovian* if the following holds:

For each $\epsilon > 0$, there exists a real valued function $f_\epsilon(t)$ defined on $\mathbb{R}$ such that

1. $f_\epsilon(t) = t, \quad t \in [0,1]$
2. $-\epsilon \leq f_\epsilon(t) \leq 1 + \epsilon, \quad \forall t \in \mathbb{R}$
3. $0 \leq f_\epsilon(t') - f_\epsilon(t) \leq t' - t$ whenever $t < t'$

and we have

$$u \in \mathcal{D}[\mathcal{E}] \implies f_\epsilon(u) \in \mathcal{D}[\mathcal{E}], \text{ with } \mathcal{E}(\phi_\epsilon(u), \phi_\epsilon(v)) \leq \mathcal{E}(u,v)$$

A symmetric form on $L^2(X;m)$ that is both closed and Markovian is called a *Dirichlet form.*

A symmetric form $\mathcal{E}$ is called *regular* if we can find a subset $\mathcal{C}$ of $\mathcal{D}[\mathcal{E}] \cap C_0(X)$ such that $\mathcal{C}$ is dense in $\mathcal{D}$ with respect to $\mathcal{E}_\alpha$ norm and dense in $C_0(X)$ with respect to the uniform norm.

With these definitions in mind, the key result states the following.

Theorem 6.3. *[9, Thm 7.21] Let X be a locally compact metric space and m a positive Radon measure on X supported on all of X. Given a regular Dirichlet form $\mathcal{E}$ on $L^2(X, m)$, there exists an m-symmetric Hunt process M on $(X, \mathcal{B}(X))$ whose Dirichlet form is $\mathcal{E}$.*

Now we are ready to introduce Øksendal's construction. let ϕ be a quasiconformal map on Ω. Then since ϕ is orientation-preserving, we have that its Jacobian determinant $J_\phi = |\phi_z|^2 - |\phi_{\overline{z}}|^2 > 0$ almost everywhere. Thus we can define the 2×2 matrix a by

$$a = a_\phi = J_\phi \cdot (\phi')^{-1} \cdot ((\phi')^{-1})^T$$

almost everywhere in Ω, and the following symmetric bilinear form on the Hilbert space $H = L^2(\Omega, dA)$:

$$\tilde{\mathcal{E}}(u, v) = \tilde{\mathcal{E}}_\phi(u, v) = \frac{1}{2} \int_\Omega (\nabla u)^T \cdot a \cdot (\nabla v) dA,$$

for $u, v \in C_c^\infty(\Omega)$, regarding ∇u as a 2×1 matrix.

Theorem 6.4. *[19, Lem 2.1] $\tilde{\mathcal{E}}_\phi$ is a regular Dirichlet form on $L^2(\Omega, dA)$.*

Thus we can associate a Hunt process $M = (\Omega, \mathcal{F}, \{X_t\}_{t \in [0,\infty]}, \{\mathbb{P}^z\}_{z \in \Omega})$ to $\tilde{\mathcal{E}}_\phi$, and the generator A of M is related to $\tilde{\mathcal{E}}_\phi$ by

$$\tilde{\mathcal{E}}_\phi(u, v) = -(Au, v)_H, \tag{6.1}$$

for $u \in \mathcal{D}(A) \subset \mathcal{D}(\tilde{\mathcal{E}}_\phi), v \in \mathcal{D}(\tilde{\mathcal{E}}_\phi) \subset L^2(\Omega, dA)$. See [9] for details.

Regarding the Hunt process M, Øksendal proved that it is the pre-image of a time-changed Brownian motion under the quasiconformal map ϕ:

Theorem 6.5. *[19, Thm 2.3, 5.3] Let $\phi : \Omega \subset \mathbb{C} \to \mathbb{C}$ be a quasiconformal map. Let $M = (\Omega, \mathcal{F}, \{X_t\}_{t \in [0,\infty]}, \{\mathbb{P}^z\}_{z \in \Omega})$ be the Hunt process associated to the Dirichlet form $\tilde{\mathcal{E}}$, and $\{B_t : t \geq 0\}$ a Brownian motion starting at 0. Then for any compact subset $K \subset \Omega$, the process*

$$Z_t = \begin{cases} \phi(X_t), & t < \tau_K \\ B_{t - \tau_K} + \phi(X_{\tau_K}), & t \geq \tau_K \end{cases}$$

where X_t has law $\mathbb{P}^z$ and τ_k is the first exit time of K by X, is a Brownian motion starting at $\phi(z)$ with a time-change, for all $z \in \Omega$.

Remark. In [19] this time-change is explicitly provided. It is given by $a_t = \inf\{s : b_s > t\}$, where $b_s = \int_0^s J_\phi(X_r)dr$. But this time-change requires information about J_ϕ, which is beyond what the Beltrami coefficient encodes. So we only get the QBM up to some time-change when assuming only the Beltrami coefficient.

This theorem states that the process that comes out of this construction is indeed what we called the Quasi-Brownian motion with respect to ϕ. Moreover, it turns out that this construction depends only on the Beltrami coefficient of ϕ, although the derivative $(\phi')^{-1}$ is used to define the matrix a. To see this, we first elaborate more on what information the Beltrami coefficient encodes in terms of linear transformations. Recall that geometrically the Beltrami coefficient tells us the eccentricity and the orientation of ellipses which get mapped to circles, but neither the relative size between the ellipses and circles nor the direction of the image of the major axis of the ellipses is encoded. Consider a linear map, represented by the 2×2 real-valued matrix M, mapping a ellipse to a circle. Taking a singular value decomposition, we have

$$M = U\Sigma V,$$

where U, V are rotation matrices and Σ is a diagonal matrix with real coefficients. This means that M is equivalent to a rotation(V) followed by a coordinate scaling(Σ) and then another rotation(U). In order to map a ellipse to a circle with these three steps, we need to first rotate this ellipse so that its major and minor axes are parallel to the basis vectors. This determines V. Then we need to scale the major and minor axes according to their ratio. This nails down the ratio of the two nonzero entries in Σ. As the Beltrami coefficient does not tell us the relative size between this ellipse and this circle, we only know the ratio but not their values. Lastly, since we don't know the direction of the image of the major axis of the ellipse, U remains completely unknown to us.

Fortunately, the matrix a depends only on the information known to us. For $z \in \mathbb{C}$ where ϕ^{-1} is differentiable, let $M = \phi'(\phi^{-1}(z))$, we have

$$\begin{aligned}
a(z) &= J_\phi(z) \cdot (\phi')^{-1}(z) \cdot ((\phi')^{-1})^T(z) \\
&= |\Sigma| \cdot (U\Sigma V)^{-1} \cdot ((U\Sigma V)^{-1})^T \\
&= |\Sigma| V^{-1}\Sigma^{-1}U^{-1}(U^{-1})^T(\Sigma^{-1})^T(V^{-1})^T \\
&= V^{-1} \cdot |\Sigma|(\Sigma^{-1})^2 \cdot (V^{-1})^T.
\end{aligned}$$

Note that $|\Sigma|(\Sigma^{-1})^2$ is a diagonal matrix with determinant 1, it is completely determined by the ratio of its diagonal entries. Therefore a is determined almost everywhere by the Beltrami coefficients.

6.2 Square-wise construction: constant μ supported on a square

Øksendal's construction, together with our observation that it depends only on the information encoded in the Beltrami coefficient, is successful in the sense that it indeed defined the Quasi-BM. However it utilizes the relatively advanced theory on Dirichlet forms, and the relationship between the Beltrami coefficient and the resulting Quasi-BM is somewhat abstract. Here we would like to propose a more constructive approach to this problem.

To start with the fundamental case in our construction, we first assume the Beltrami coefficient μ takes the form

$$\mu(z) = \begin{cases} c & \text{if } z \in (-1,1)\times(-1,1) \\ 0 & \text{otherwise} \end{cases}$$

where $c \in \mathbb{C}$ with $|c| < 1$.

Let ϕ be a quasiconformal map with dilatation μ, and $z_0 \in \mathbb{C}$. Our goal is to find a process Y such that $\phi(Y)$ is a time-changed Brownian motion starting at $\phi(z_0)$.

Let S denote the square $(-1,1)\times(-1,1)$, $G = \partial S$ denote its boundary, $V = \{1+i, -1+i, -1-i, 1-i\}$ denote its vertices, and $D = \{z \in \mathbb{C} : \operatorname{Re} z = \pm \operatorname{Im} z\}$ denote the set formed by its diagonal lines, extended to infinity.

Now we define some processes which will serve as the building blocks.

Define the process $X_c = \phi_c^{-1}(B)$ where B is a Brownian motion starting at 0 and ϕ_c is the mapping defined by $\phi_c(z) = \frac{1}{c+1}(z + c\bar{z})$.

Define the process $X_{0|c} = \phi_{0|c}^{-1}(B)$ where $\phi_{0|c}$ is the map

$$\phi_{0|c}(z) = \begin{cases} z & \text{if } \operatorname{Re} z \le 0 \\ \frac{1}{1-c}(z + c\bar{z}) & \text{if } \operatorname{Re} z > 0 \end{cases},$$

the process $X_{c|0} = \phi_{c|0}^{-1}(B)$ where $\phi_{c|0}$ is the map

$$\phi_{c|0}(z) = \begin{cases} \frac{1}{1-c}(z + c\bar{z}) & \text{if } \operatorname{Re} z \le 0 \\ z & \text{if } \operatorname{Re} z > 0 \end{cases},$$

the process $X_{\frac{0}{c}} = \phi_{\frac{0}{c}}^{-1}(B)$ where $\phi_{\frac{0}{c}}$ is the map

$$\phi_{\frac{0}{c}}(z) = \begin{cases} z & \text{if } \operatorname{Im} z \geq 0 \\ \frac{1}{1+c}(z + c\bar{z}) & \text{if } \operatorname{Im} z < 0 \end{cases},$$

and the process $X_{\frac{c}{0}} = \phi_{\frac{c}{0}}^{-1}(B)$ where $\phi_{\frac{c}{0}}$ is the map

$$\phi_{\frac{c}{0}}(z) = \begin{cases} \frac{1}{1+c}(z + c\bar{z}) & \text{if } \operatorname{Im} z \geq 0 \\ z & \text{if } \operatorname{Im} z < 0 \end{cases}.$$

Here the subscript "$0|c$" stands for "0 on the left half-plane and c on the right half-plane" and the subscript "$\frac{c}{0}$" stands for "c on the upper half-plane and 0 on the lower half-plane". The maps $\phi_{\_}$'s are chosen so that they are automorphisms of $\mathbb{C}$ and piece-wise linear with the corresponding dilatation within the corresponding half-planes. As a result, their distortions are easy to estimate. Here we care more about the lower bound, so for completeness we phrase it as a lemma explicitly.

Lemma 6.6. *Let c_1, $c_2 \in \mathbb{C}$ with $\max\{|c_1|, |c_2|\} \leq k < 1$, and (using the idea of our notation above)*

$$\phi_{\frac{c_1}{c_2}}(z) = \begin{cases} \frac{1}{1+c_1}(z + c_1\bar{z}) & \textit{if } \operatorname{Im} z \geq 0 \\ \frac{1}{1+c_2}(z + c_2\bar{z}) & \textit{if } \operatorname{Im} z < 0 \end{cases}.$$

Then we have

$$|\phi_{\frac{c_1}{c_2}}(z) - \phi_{\frac{c_1}{c_2}}(w)| \geq C|z - w|, \quad \forall z, w \in \mathbb{C},$$

where $C = \frac{1-k}{1+k}$.

Proof. First let $z' = \phi_{\frac{c_1}{c_2}}(z)$ and $w' = \phi_{\frac{c_1}{c_2}}(w)$. For $z, w \in \overline{\mathbb{H}}$, we have

$$\begin{aligned} |z' - w'| &= |\frac{1}{1+c_1}((z - w) + c_1(\overline{z - w}))| \\ &\geq \frac{1 - |c_1|}{|1 + c_1|}|z - w| \\ &\geq \frac{1 - |c_1|}{1 + |c_1|}|z - w| \\ &\geq \frac{1 - k}{1 + k}|z - w| \end{aligned}$$

Similarly, this holds when $z, w \in \overline{\mathbb{H}^-}$. In case $z \in \mathbb{H}$ and $w \in \mathbb{H}^-$, we have

$$|z' - w'| = |z' - r'| + |r' - w'|,$$

where $r' \in \mathbb{R}$ is the point on the segment joining z' and w', and its preimage is denoted by r. Then we have

$$\begin{aligned} |z' - w'| &= |z' - r'| + |r' - w'| \\ &\geq C|z - r| + C|r - w| \\ &\geq C|z - w|. \end{aligned}$$

Switching the role of z and w yields the same result, which concludes the proof. Note due to our normalization that 1 is mapped to 1, we actually have $r = r'$, but this normalization is not necessary for this lemma to hold.

$\square$

By looking at these definitions, it is tempting to infer symmetries among these processes. Here we would like to point out that $X_{0|c}$ and $X_{c|0}$ (and also $X_{\frac{0}{c}}$ and $X_{\frac{c}{c}}$) are indeed rotation by π of each other, but $X_{\frac{c}{0}}$ is in general not $X_{0|c}$ rotated by $\frac{\pi}{2}$. When rotating the process $X_{0|c}$, not only the region on which it has a certain dilatation is rotated, but also the dilatation itself. Rotating the process $X_{0|c}$ by $\frac{\pi}{2}$, we actually get $X_{\frac{-c}{0}}$ instead. Using this idea, we get that Lemma 6.6 holds for $\phi_{c_1|c_2}$ as well by rotating $\phi_{\frac{-c_1}{-c_2}}$ by $\frac{\pi}{2}$.

Now we start to construct the quasi-Brownian motion Y. See Figure 6.2. we start with taking a universal probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where we can define countably many independent Brownian motions. For simplicity we will define all mentioned process on this probability space. Without loss of generality, let $z_0 = 0$. First we launch a process X_c at 0, and stop it once it hits the boundary set G. Let $\tau_1 = \inf\{t > 0 : (X_c)_t \in G\}$ denote this hitting time. For $t \in [0, \tau_1]$, we set $Y_t = (X_c)_t$. This defines Y up to time τ_1, and we have $Y_{\tau_1} \in G$.

Since almost surely this hitting point will be off the corners of S, we only need to consider the cases where this hitting point is on one of the 4 sides of S. Depending on which side this hitting point is on, we continue this construction in the following ways:

If $Y_{\tau_1} \in \{1\} \times (-1, 1)$, we launch the process $X_{c|0}$ translated to start at Y_{τ_1}, time translated so that its clock starts at $t = \tau_1$. Similarly, if $Y_{\tau_1} \in \{-1\} \times (-1, 1)$, $(-1, 1) \times \{-1\}$ or $(-1, 1) \times \{1\}$, we launch the process $X_{0|c}$, $X_{\frac{c}{0}}$ or $X_{\frac{0}{c}}$ respectively with the same manner of translations. No matter what case we are in, we stop this process when it hits the diagonal set D and let $\sigma_1 = \inf\{t > \tau_1 :$ (the choice of $X)_t \in D\}$ denote this hitting time.

Now we concatenate Y with the process (the choice of X) mentioned above. This defines the process Y up to time σ_1 and we have $Y_{\sigma_1} \in D$.

Note again that Y_{σ_1} is almost surely not in V. So either Y_{σ_1} is inside of S, or it is outside of S (not on the boundary). Now depending on where Y_{σ_1} is, we proceed as follows. If $Y_{\sigma_1} \in S$, we launch an independent copy of process X_c, translated to start at Y_{σ_1} and time translated to start at time σ_1; If $Y_{\sigma_1} \in S^c$, we launch a Brownian motion translated and time translated in the same manner. No matter which case it is in, we stop it once it hits G again. We call this hitting time τ_2. We concatenate this process to Y and this defines the process Y further up to time τ_2 and we have $Y_{\tau_2} \in G$ again.

Repeating this process, we almost surely get a process Y defined on the interval $[0, \sup_k \sigma_k)$ and a sequence of stopping times $0 < \tau_1 < \sigma_1 < \tau_2 < \sigma_2 < ...$ where $Y_{\tau_k} \in G$ and $Y_{\sigma_k} \in D$ for all $k \in \mathbb{N}$.

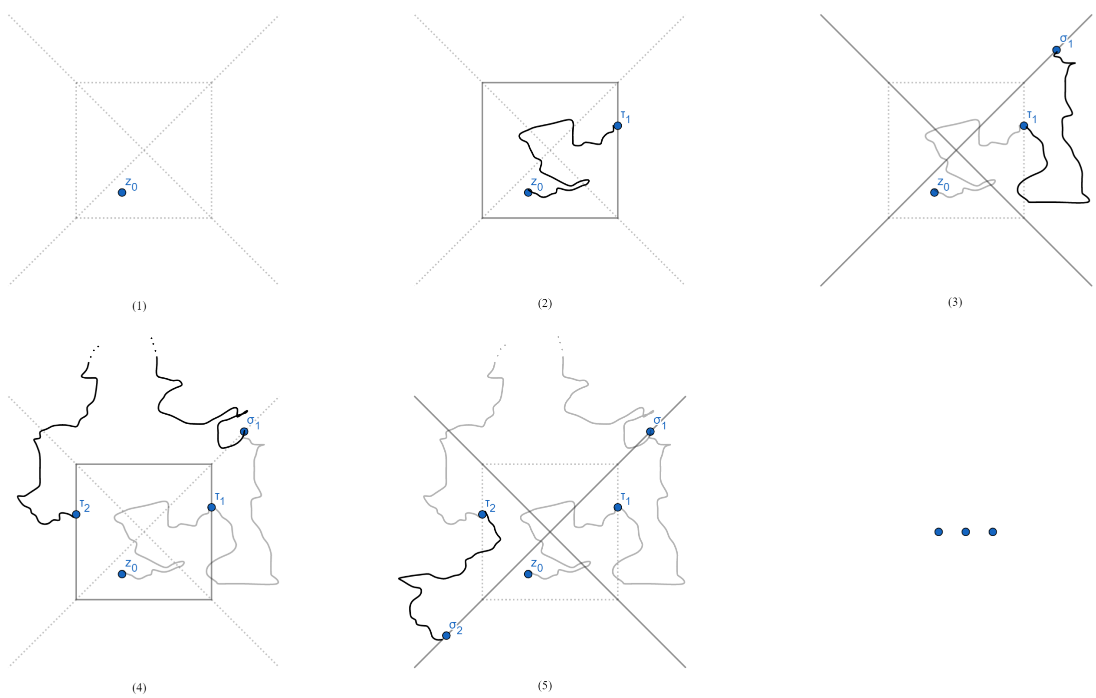

Figure 1: step-by-step construction of Y

To see why Y is the desired process, first consider Y restricted to the interval $[0, \tau_1)$. It coincides with the process X_c and by definition X_c is a QBM with respect to the map ϕ_c. Note that the dilatation of ϕ and ϕ_c agrees on S, thus by our locality theorem (Theorem 5.3), $\phi(Y)$ is a time-changed BM up to time τ_1. That is, there exists Brownian motion B' such that $B'_{s(t)} = \phi(Y_t)$ for $0 \le t < \tau_1$, where s is the time-change given in Theorem 5.3. Since $s(\tau_1)$ is the first time B' exits $\phi(S)$, it is a stopping time of B'. Now consider the process Y restricted to the interval $[\tau_1, \sigma_1)$. Without loss of generality, we assume $Y_{\tau_1} \in \{1\} \times (-1, 1)$. Then this portion of Y coincides with the process $X_{c|0}$ with a translation. By definition, $X_{c|0}$ is a QBM with respect to the map $\phi_{c|0}$. It is translated so that its dilatation agrees with that of ϕ within the sector $\{z : \arg(z) \in (-\pi/4, \pi/4)\}$. So by the locality theorem, $\phi(Y)$ restricted to the interval $[\tau_1, \sigma_1)$ is again a time-changed Brownian motion. Thus $\phi(Y)$ up to time σ_1 is a concatenation of two independent Brownian motions at a stopping time, so it is a time-changed Brownian motion (up to some finite time). Inductively, we get that the process $\{Y_t : 0 \le t < \lim_{k\to\infty} \sigma_k\}$ is a concatenation of independent time-changed Brownian motions, so itself is a time-changed Brownian motion (potentially only up to some finite time).

Now we show that $\lim_{k\to\infty} \sigma_k = \lim_{k\to\infty} \tau_k = \infty$ almost surely, so that our construction yields a process defined on $[0, \infty)$. We start with proving a 2-d analog of the reflection principle.

Lemma 6.7 (2-d reflection principle)**.** *Let $\{B_t, t \ge 0\}$ be a planar Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$, and $\{\mathcal{F}_t\}$ be its natural right continuous filtration. If τ is a a.s. finite stopping time w.r.t $\{\mathcal{F}_t\}$ and $\theta = \theta(\omega)$ is $\mathcal{F}_\tau$ measurable, then the reflected Brownian motion, B^*, defined by:*

$$B^*_t = \begin{cases} B_t, & t \le \tau \\ R_\theta(B_t) + c, & t > \tau \end{cases},$$

where

$$R_\theta = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$$

is a reflection transformation and $c = c(B_\tau)$ is chosen so that B^* is continuous, is again a planar Brownian motion.

Proof. If τ is finite, then by the strong Markov property, $\{B_{t+\tau} - B_\tau, t \ge 0\}$ is a planar Brownian motion independent of $\{B_t, 0 \le t \le \tau\}$. First consider a fixed $\theta = \theta_0 \in [0, \pi)$. We can write

$$R_\theta = U \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} U^T,$$

for some orthonormal matrix $U = U_\theta$. Since a Brownian motion is rotationally invariant, and a 1-d Brownian motion satisfies the reflection principle, we have that $\{\mathbb{R}_\theta(B_{t+\tau} - B_\tau), t \ge 0\}$ is a planar Brownian motion starting at 0 independent of θ as it is only an independent rotation factor. Now for a random θ that is $\mathcal{F}_\tau$ measurable, we define a discrete approximation θ_n of θ by $\theta_n = \pi m 2^{-n}$ if $\pi m 2^{-n} \leq \theta < \pi(m+1)2^{-n}$. Let $B_n = \{R_{\theta_n}(B_{t+\tau} - B_\tau), t \geq 0\}$, and $B_{n,m} = \{R_{\pi m 2^{-n}}(B_{t+\tau} - B_\tau), t \geq 0\}$. Then for the event $\{B_n \in E\}$, and a measurable set $F \subset [0, \pi)$,

$$
\begin{aligned}
&\mathbb{P}[B_n \in E \cap \theta_n \in F] \\
&= \sum_{m=0}^{2^n-1} \mathbb{P}[B_{n,m} \in E \cap F \cap \{\theta_n = \pi m 2^{-n}\}] \\
&= \mathbb{P}[B \in E] \sum_{m=0}^{2^n-1} \mathbb{P}[F \cap \{\theta_n = \pi m 2^{-n}\}] \\
&= \mathbb{P}[B \in E]\mathbb{P}[F]
\end{aligned}
$$

Thus B_n and θ_n are independent. Taking $n \to \infty$, we get $R_\theta(B_{t+\tau} - B_\tau), t \geq 0\}$ is independent of θ, and so independent of $\mathcal{F}_\tau$. As a result, B^* is a continuous concatenation of two independent Brownian motions, and thus a Brownian motion. □

Corollary 6.8. *Let X be a planar Brownian motion reflected when it first exits the unit disk, about the line tangent to the unit disk at the exit point. Then X is a planar Brownian motion.*

Proof. Take $\tau = \inf\{t > 0, |B_t| = 1\}$ and $\theta = \arg(B_\tau) + \pi/2$ and apply lemma 6.7. □

Corollary 6.9. *Let B be a planar Brownian motion starting at 0, then $\forall r, T > 0$ we have*

$$
\begin{aligned}
\mathbb{P}[\sup_{0 \leq t \leq T} |B_t| \geq r] &\leq \mathbb{P}[|B_T| \geq r] + \mathbb{P}[r < |B_T| < 3r] \\
&< 2\mathbb{P}[|B_T| \geq r]
\end{aligned}
$$

Proof. Let B^* be the process obtained by reflecting B when it first exits the disk $\{|z| < r\}$, about the line tangent to the disk at the tangent point, then we have

$$
\begin{aligned}
&\mathbb{P}[\sup_{0 \leq t \leq T} |B_t| \geq r] \\
&= \mathbb{P}[|B_T| \geq r] + \mathbb{P}[\sup_{0 \leq t \leq T} |B_t| \geq r \text{ and } |B_T| < r] \\
&\leq \mathbb{P}[|B_T| \geq r] + \mathbb{P}[r < |B_T^*| < 3r]
\end{aligned}
$$

Since B^* is also a Brownian motion, the claim follows. □

Now we prove the following as promised.

Proposition 6.10. *Almost surely,*

$$\lim_{k\to\infty} \sigma_k = \lim_{k\to\infty} \tau_k = \infty.$$

Proof. Since τ_k and σ_k alternate, we only need to show $\lim_{k\to\infty} \sigma_k = \infty$. First we chop up the process Y as the following. For each $k \in \mathbb{N}^+$, define the processes $\{Y_t^{(k)}, 0 \le t < \sigma_k - \sigma_{k-1}\}$ by $Y_t^{(k)} = Y_{t+\sigma_{k-1}}$. Note that each $Y^{(k)}$ has to start from a point in D, then pass through G and then come back to a point on D. We will distinguish two cases, depending on whether $Y^{(k)}$ has traveled a certain distance away from its starting point or not. In the latter case, $Y^{(k)}$ necessarily has to start and end near one vertex of S.

To make this precise, for each k, we say $Y^{(k)}$ is of case 1 if

$$\sup_{0\le s<\sigma_k-\sigma_{k-1}} |Y_s^{(k)} - Y_0^{(k)}| \ge (\sigma_k - \sigma_{k-1})^{\frac{1}{4}};$$

and of case 2 if it is not of case 1. Note that for any point z on D, the distance between z and G is at least $\frac{1}{\sqrt{2}}$ the distance from z to the closest corner point. Thus in case 2, it must satisfy

$$\inf_{v\in V} |Y_0^{(k)} - v| < \sqrt{2}(\sigma_k - \sigma_{k-1})^{\frac{1}{4}}.$$

Also note that

$$\begin{aligned}
\mathbb{P}[Y^{(k)} \text{ is of case 1}] &= \mathbb{P}[\sup_{0\le s<\sigma_k-\sigma_{k-1}} |Y_s^{(k)} - Y_0^{(k)}| \ge (\sigma_k - \sigma_{k-1})^{\frac{1}{4}}] \\
&\le \mathbb{P}[\sup_{0\le s<\sigma_k-\sigma_{k-1}} |B_s - B_0| \ge C(\sigma_k - \sigma_{k-1})^{\frac{1}{4}}] \\
&\le 2\mathbb{P}[|B_{\sigma_k-\sigma_{k-1}} - B_0| \ge C(\sigma_k - \sigma_{k-1})^{\frac{1}{4}}] \\
&= 2\mathbb{P}[|B_1 - B_0| \ge C(\sigma_k - \sigma_{k-1})^{-\frac{1}{4}}] \\
&= 2\exp(-\frac{1}{2}C(\sigma_k - \sigma_{k-1})^{-\frac{1}{4}})
\end{aligned}$$

where B is a Brownian motion and $C = \frac{1-k}{1+k}$ with $k = ||\mu||_\infty$ is the minimum possible contraction factor the maps ϕ's could impose. Here the first inequality follows from Lemma 6.6 and the second inequality is due to the 2-d reflection principle(Lemma 6.7). Then the rest follows from Brownian scaling and straightforward computation.

Now assume $\lim_{k\to\infty}\sigma_k < \infty$. Then we have

$$\sum_k (\sigma_k - \sigma_{k-1}) < \infty$$

and thus

$$\lim_{k\to 0}(\sigma_k - \sigma_{k-1}) = 0.$$

Notice that $2\exp(-\frac{1}{2}x^{-\frac{1}{4}}) < x$ for all positive x sufficiently close to 0, so the above implies

$$\sum_k \mathbb{P}[Y^{(k)} \text{ is of case 1}] \leq \sum_k 2\exp\left(-\frac{1}{2}C(\sigma_k - \sigma_{k-1})^{-\frac{1}{4}}\right) < \infty.$$

By the Borel-Cantelli theorem, almost surely, case 1 happens only finitely many times and so only case 2 will happen after certain time. In other words, there exists $n \in \mathbb{N}$ such that for all $k > n$, we have

$$\sup_{0\leq s<\sigma_k-\sigma_{k-1}} |Y_s^{(k)} - Y_0^{(k)}| < (\sigma_k - \sigma_{k-1})^{\frac{1}{4}}$$

and

$$\inf_{v\in V} |Y_0^{(k)} - v| < \sqrt{2}(\sigma_k - \sigma_{k-1})^{\frac{1}{4}}.$$

Recall that under our assumption, $\sigma_k - \sigma_{k-1} \to 0$ as $k \to 0$. As points in V are discrete, for k large enough, these 2 inequalities imply that Y_s converges to one single point in V as $s \to (\lim_{k\to\infty}\sigma_k)$, and so $\phi(Y_s)$ converges to the image of that point. We argued before that $\phi(Y_s)$ is a time-changed Brownian motion, so this can only happen with probability 0. Thus this implies our assumption $\lim_{k\to\infty}\sigma_k < \infty$ happens with probability 0, which concludes the proof.

□

Now we proved that the process $\phi(Y)$, defined on $[0,\infty)$, is a time-changed BM. That is, there exists an increasing sequence of stopping times $\{s(t) : t \in [0,T)\}$, with $s(0) = 0$ and $s(t) \to \infty$ as $t \to T$ almost surely, such that $\{\phi_n[(Y_n)_{s(t)}], 0 \leq t < T(\omega)\}$ is a Brownian motion up to time T, for some $T = T(\omega)$ taking values in $(0,+\infty]$. However, at this point, we still don't know whether T is a finite or $+\infty$. This determines whether $\phi(Y)$ with certain time-change is a whole Brownian motion on $[0,+\infty)$ or just a finite portion of it. We will carefully investigate this issue in the next section when dealing with the general case. See Proposition 6.12.

6.3 Square-wise construction: general measurable μ

Now we carry out the construction based on exactly the same idea in the general case. Given $\mu : \mathbb{C} \to \mathbb{C}$ measurable with $||\mu(z)||_\infty < 1$. Let ϕ be the quasiconformal map with dilatation μ, normalized to fix 0, 1 and ∞.

Define μ_n to be the approximation of μ by averaging on squares in the $2^{-n}\mathbb{Z} \times 2^{-n}\mathbb{Z}$ grid. To be precise, for each $n \in \mathbb{N}$, define μ_n by

$$\mu_n(z) = 2^{2n} \int_{S_n(z)} \mu dA,$$

where $S_n(z)$ denotes the square in this lattice containing z as its interior point if such square exists. If such square does not exist, z has to be on the grid lines. In this case the value of $\mu_n(z)$ can be chosen arbitrarily as its value on zero measure sets will not matter. This way we get a sequence μ_n that are piece-wise constant on each square. Let ϕ_n be the quasiconformal maps with dilatation μ_n with the same normalization as ϕ.

In this more general situation, we need to extend our definitions of D and G. We denote G_n the grid set, i.e. the union of all vertical and horizontal lines forming the grid, and D_n the union of the diagonals lines of all of the squares in our grid.

As before we define several processes as building blocks. For general c, $d \in \mathbb{C}$, with $|c| < 1$, $|d| < 1$, define the process $X_c = \phi_c^{-1}(B)$ where B is a Brownian motion starting at 0 and ϕ_c is the mapping defined by $\phi_c(z) = \frac{1}{c+1}(z + c\bar{z})$.

Define the process $X_{c|d} = \phi_{c|d}^{-1}(B)$ where $\phi_{c|d}$ is the map

$$\phi_{c|d}(z) = \begin{cases} \frac{1}{1-c}(z + c\bar{z}) & \text{if } \operatorname{Re} z \leq 0 \\ \frac{1}{1-d}(z + d\bar{z}) & \text{if } \operatorname{Re} z > 0 \end{cases},$$

and the process $X_{\frac{c}{d}} = \phi_{\frac{c}{d}}^{-1}(B)$ where $\phi_{\frac{c}{d}}$ is the map

$$\phi_{\frac{c}{d}}(z) = \begin{cases} \frac{1}{1+c}(z + c\bar{z}) & \text{if } \operatorname{Im} z \geq 0 \\ \frac{1}{1+d}(z + d\bar{z}) & \text{if } \operatorname{Im} z < 0 \end{cases}.$$

As before we take a universal probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where we can define countably many independent Brownian motions and we will define all processes mentioned on this common probability space.

Let's for now fix a level n. To initiate the construction of the QBM, Y_n, corresponding to μ_n, choose a starting point $z_0 \in \mathbb{C}$ that is not a corner point of any square. If z_0 is the interior point of any square S on the grid, we launch a process X_{μ_S}, where μ_S is the value of μ_k in the square S, and stop it once it hits the grid set G_n. As before, let τ_1 denote this hitting time. For $t \in [0, \tau_1]$, we set $(Y_n)_t = (X_{\mu_S})_t$. This defines Y_n up to time τ_1, and we have $(Y_n)_{\tau_1} \in G_n$.

Depending on whether this hitting point is on a vertical line or a horizontal line of G_n, we continue this construction in the following ways:

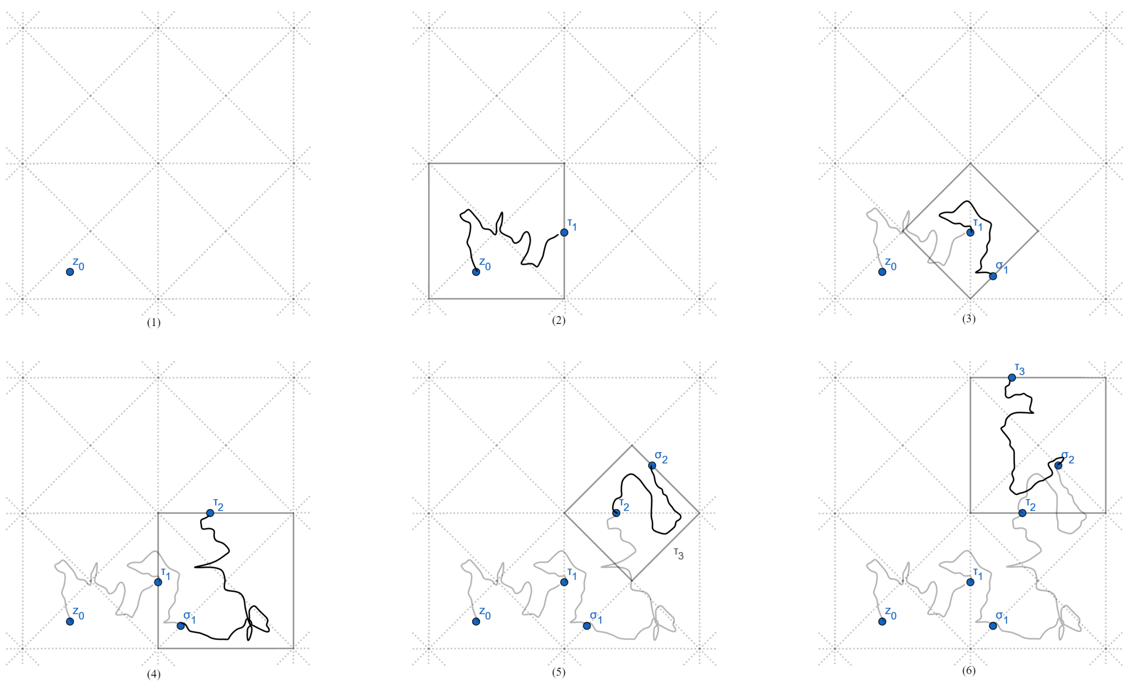

Figure 2: step-by-step construction of Y_n

If it leaves S_n on a vertical side, let μ_L be the value of μ_n in the square right to the left of the hitting point and μ_R be the value of μ_n in the square right to the right of the hitting point. We launch an independent process of $X_{\mu_L|\mu_R}$ translated so that it starts at this hitting point and time translated so that it starts at time τ_1, and stop it when it hits the diagonal set D_n; In case it leaves S on a horizontal side, let μ_T be the value of μ_n in the square right above the hitting point and μ_B be the value of μ_k in the square right below the hitting point. We launch an independent process of $X_{\frac{\mu_T}{\mu_B}}$ translated so that it starts at this hitting point and time translated so that it starts at time τ_1, and stop it when it hits the diagonal set D_n. In either case, we let $\sigma_1 = \inf\{t > \tau_1 : (\text{the choice of } X)_t \in D_n\}$ denote this hitting time on D_n. We concatenate this process of choice to what we have, then we get process Y_n extended to the interval $[0, \tau_1)$ with $(Y_n)_{\sigma_1} \in D_n$.

Note that almost surely $(Y_n)_{\sigma_1}$ is not a corner point, so that it will be an interior point of some square S. This brings us back to the situation where we started. Repeating these two steps above, we almost surely obtain a process Y_n defined on the interval $[0, \sup_k \sigma_k)$ and a sequence of stopping times $0 < \tau_1 < \sigma_1 < \tau_2 < \sigma_2 < ...$, where $(Y_n)_{\tau_k} \in G_n$ and $(Y_n)_{\sigma_k} \in D_n$ for all $k \in \mathbb{N}$.

As before, Y_n, clearly continuous a.s., restricted to each of the sub-intervals $[\sigma_k, \tau_k)$ or $[\tau_k - 1, \sigma_k)$ coincides with an independent sequence of process $X_{c|d}$ and $X_{\frac{c}{d}}$ carefully chosen so that its corresponding dilatation agrees with μ_n in the square it lives in, thus by our locality theorem (Thm 5.3), $\phi_n(Y_n)$ is a time-changed BM up to time $\sup_k \sigma_k$. Also the following holds true by adapting the proof of proposition 6.10 from four corner points to countable many.

Proposition 6.11. *For each $n \in \mathbb{N}$, almost surely,*

$$\lim_{k\to\infty} \sigma_k = \lim_{k\to\infty} \tau_k = \infty.$$

Proof. The same proof as the one for Proposition 6.10 works here, with our new definition of the diagonal set D_n and the grid set G_n instead of the old ones, and the finite vertices set V replaced by $2^{-n}\mathbb{Z}^2$. □

Thus we conclude that the process $\phi_n(Y_n)$ defined on $[0, \infty)$ is a time-changed Brownian motion. Now we prove it is the whole Brownian on $\mathbb{R}^+$ with this time change, instead of any finite portion of it.

Proposition 6.12. *There exists an increasing sequence of stopping times $\{s(t) : t \in [0, \infty)\}$, with $s(0) = 0$ and $s(t) \to \infty$ as $t \to \infty$ almost surely, such that $\{\phi_n[(Y_n)_{s(t)}], t \geq 0\}$ is a Brownian motion.*

Remark. To clarify, we already have $\phi_n(Y_n)$ defined on $[0, \infty)$ is a time-changed Brownian motion. That is, there exists an increasing sequence of stopping times $\{s(t) : t \in [0, T)\}$ as $t \to \infty$, with $s(0) = 0$ and $s(t) \to \infty$ as $t \to T$ almost surely, such that $\{\phi_n[(Y_n)_{s(t)}], 0 \leq t < T(\omega)\}$ is a Brownian motion up to time T, for some $T = T(\omega)$ taking values in $(0, +\infty]$. This proposition claims T is almost surely $+\infty$.

Proof. Since this proposition is about a fixed level n, we omit the subscript n throughout the proof for simplicity. Assume for contradiction that $f(Y)$ is a finite Brownian motion. Let $\mathcal{S}$ denote the collection of all open squares S in our grid, and define

$$\mathcal{S}' = \{S \in \mathcal{S} : Y_{\sigma_k} \in S \text{ for infinitely many } k \in \mathbb{N}\},$$

where as before the σ_k's are stopping times on which Y hits the diagonal set D. Clearly, $\mathcal{S}' \subset \mathcal{S}$.

Claim 1: Almost surely $\mathcal{S}' \neq \emptyset$:

Consider $\omega \in \Omega$ in the event $\{\mathcal{S}' = \emptyset\}$. For a compact set $K \subset \mathbb{C}$, we can find a finite collection $\{S_1, S_2, ..., S_m\} \subset \mathcal{S}$ such that $K \subset \bigcup_{j=1}^{m} \overline{S_m}$. Since $\mathcal{S}' = \emptyset$, there exist $s > 0$ such that $Y_t(\omega)$ never visits any squares in $\{S_1, S_2, ..., S_m\}$ for $t > s$. Thus $Y_t(\omega)$ never revisits K for $t > s$. Since K is arbitrary, this means $Y_t(\omega) \to \infty$ as $t \to \infty$. Thus $\phi(Y_t) \to \infty$ and we conclude that the event $\{\mathcal{S}' = \emptyset\}$ has probability 0.

Claim 2: If $S \in \mathcal{S}'$, then almost surely all four neighbors of S are in $\mathcal{S}'$:

Let $S \in \mathcal{S}'$, then by definition there exists $\sigma_{k_1}, \sigma_{k_2}, ...$ such that $Y_{\sigma_{k_j}} \in D \cap S$ for all $j \in \mathbb{N}$. If

$$\inf_j \operatorname{dist}(Y_{\sigma_{k_j}}, \partial S) = 0,$$

then Y gets arbitrarily close to one of the corner points of S as $Y_{\sigma_{k_j}}$ have to follow the diagonal. So $\phi(Y)$ gets arbitrarily close to one of the image of these corners under ϕ, but this has probability 0 for a finite Brownian motion. If

$$\inf_j \operatorname{dist}(Y_{\sigma_{k_j}}, \partial S) > 0,$$

then some compact set $K \subset S$ contains $Y_{\sigma_{k_j}}$ for all $j \in \mathbb{N}$. Starting inside K, the probability our building block process X_{μ_S} exits S from each of its four sides is bounded below. Thus, almost surely each side of S contains infinitely many $\{Y_{\tau_{k_j+1}}\}_{j \in \mathbb{N}}$. Repeat the same argument in each squares formed by the diagonal set, with these four sides as one of its diagonals, we get almost surely each square next to S contains infinitely many $\{Y_{\sigma_{k_j+1}}\}_{j \in \mathbb{N}}$. Thus claim 2 holds.

Combining claim 1 and 2, under our assumption $\phi(Y)$ is a finite Brownian motion, we have $\mathcal{S}' = \mathcal{S}$ almost surely. But almost surely a finite Brownian motion will not visit all squares in the grid. So our assumption is false and the proposition follows. □

It remains to show the processes Y_n converges to the QBM corresponding to ϕ as $n \to \infty$. However, the issue is that the time parametrizations in each Y_n is not ideal. They are not yet normalized to any sort of "quadratic variation" like the usual Brownian motion case. We will tackle down this issue in the next section. For now, in order to carry out this proof, we need to borrow the time parametrizations from the fact that $\phi_n(Y_n)$ are time-changed Brownian motions.

For each $n \in \mathbb{N}$, let $s_n : [0, \infty) \to [0, \infty)$ be the time reparametrization claimed in proposition 6.12, and define the processes $\{(Z_n)_t : t \geq 0\}$ by $(Z_n)_t = (Y_n)_{s_n(t)}$, so that $\phi_n(Z_n)$ are Brownian motions without time-changes.

Theorem 6.13. *The processes Z_n converges weakly on the Wiener space as $n \to \infty$, and the limiting process is a QBM starting at z_0 with respect to ϕ.*

Proof. Without loss of generality, by applying a translation we may assume $z_0 = \phi(z_0) = 0$. Since $||\mu||_\infty < 1$, $\mu \in L^1_{\text{loc}}$ and thus $\mu_n \to \mu$ almost everywhere by the Lebesgue differentiation Theorem [8, Thm 3.21]. By Theorem 2.6, we have $f_n \to f$ as $n \to \infty$ uniformly on compact subsets.

Now let P_n be the laws of Z_n and we first prove the family $\{P_n\}$ is weakly relatively compact. Let B be another Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$ that is independent of $\{Z_n\}$.

Fix η, $\epsilon > 0$, and $N \in \mathbb{N}$. First let $R' > 0$ be large enough such that

$$\mathbb{P}[\sup_{0 \leq t \leq N} |B_t| > R'] < \frac{\epsilon}{2}.$$

As $\phi : \mathbb{C} \to \mathbb{C}$ is a homeomorphism, we can find $R > 0$ such that

$$\phi(\mathbb{B}_R) \supset \mathbb{B}_{R'},$$

where $\mathbb{B}_r = \{z : |z| < r\}$ denotes a open ball centered at 0 with radius r. Since $\phi_n \to \phi$ uniformly on compact subsets, we can find $n_1 \in \mathbb{N}$ such that

$$\phi_n(\mathbb{B}_{2R}) \supset \mathbb{B}_{R'}$$

for all $n > n_1$.

Now for η, $r > 0$ and a map $\psi : \mathbb{C} \to \mathbb{C}$, define

$$D_{\eta,r}(\psi) = \min_{x \in \overline{\mathbb{B}_r}} \min_y \{|\psi(y) - \psi(x)| : |y - x| = \eta\}.$$

Again since ϕ is a homeomorphism, we have $D_{\eta,2R}(f) > 0$. Since $\phi_n \to \phi$ uniformly on $\overline{\mathbb{B}_{2R+\eta}}$, we can find $n_2 > n_1$ such that

$$D_{\eta,2R}(\phi_n) > \frac{D_{\eta,2R}(\phi)}{2}$$

for all $n > n_2$.

Now we choose $\delta > 0$ such that

$$\mathbb{P}[V^N(B(\cdot), \delta) > \frac{D_{\eta,2R}(\phi)}{2}] < \frac{\epsilon}{2},$$

where $V^N(w, \delta)$ is defined as in Proposition 2.25. Then we have

$$\begin{aligned} & P_n[V^N(\cdot, \delta) > \eta] \\ &= \mathbb{P}[V^N(Z_n(\cdot), \delta) > \eta] \\ &< \mathbb{P}[\sup_{0\le t\le N} |(Z_n)_t| > 2R] + \mathbb{P}[\sup_{0\le t\le N} |(Z_n)_t| \le 2R \text{ and } V^N(Z_n(\cdot), \delta) > \eta] \\ &< \mathbb{P}[\sup_{0\le t\le N} |B_t| > R'] + \mathbb{P}[V^N(B(\cdot), \delta) > \frac{D_{\eta,2R}(\phi)}{2}] \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \end{aligned}$$

for all $n > n_2$.

Since $Z_n(0) = 0$ for all n, by Proposition 2.25 the family $\{P_n\}$ is weakly relatively compact. Now take any convergent subsequence $\{P_{n_k}\}$ and consider their finite dimensional distribution. Take $0 \le t_1 \le t_2 \le ... \le t_m$ and open rectangles $R_1, R_2, .., R_m \subset \mathbb{C}$, then

$$\begin{aligned} & \mathbb{P}[(Z_{n_k})_{t_1} \in R_1, (Z_{n_k})_{t_2} \in R_2, ..., (Z_{n_k})_{t_m} \in R_m] \\ &= \mathbb{P}[B_{t_1} \in \phi_{n_k}(R_1), B_{t_2} \in \phi_{n_k}(R_2), ..., B_{t_m} \in \phi_{n_k}(R_m)] \\ &\to \mathbb{P}[B_{t_1} \in \phi(R_1), B_{t_2} \in \phi(R_2), ..., B_{t_m} \in \phi(R_m)] \text{ as } n \to \infty \\ &= \mathbb{P}[\phi^{-1}(B_{t_1}) \in R_1, \phi^{-1}(B_{t_2}) \in R_2, ..., \phi^{-1}(B_{t_m}) \in R_m]. \end{aligned}$$

Here the first equality is because $\phi_n(Z_n)$ are Brownian motions. The limit holds because ϕ_n converges uniformly on the boundary of the squares and the finite dimensional distributions of a Brownian motion are absolutely continuous with respect to the Lebesgue measure. This shows that all subsequential limits of $\{P_n\}$ have the same finite dimensional distribution, thus the subsequential limits are identical. Moreover this limit is the law of the preimage of a Brownian motion under ϕ. □

6.4 A nice modification to the square-wise construction

Now we proved the processes Y_n, after assigning a proper time parametrization so that their image under ϕ_n are Brownian motions without time-change, converge weakly on the Wiener space to the

QBM as desired. But we are not totally satisfied with this result. It is slightly disappointing that we did not explicitly write down the time parametrization, although for this one we can make the excuse that for our purpose of defining a map using the harmonic measures of this process, its time parametrization does not matter. However, there is a bigger problem. Due to our carelessness in the time parametrization, the processes Y_n's do not enjoy the Markov property. To see this, imagine at some point of time $t_0 > 0$ on which $(Y_n)_{t_0} = z \in S_n$ for a level n dyadic square S_n, and $\mu_n(z) = c$. In order to determine how Y_n evolves at least up to the time it leaves S_n, we need to know whether (Y_n) entered this square from one of its vertical sides of horizontal sides and whether the "stopping time"(the σ's and τ's we defined in out construction) right prior to t_0 is a τ or σ. These determine whether Y_n is going to evolve as $\phi_1^{-1}(B)$ where $\phi_1(z) = \frac{1}{1+c}(z + c\bar{z})$ or as $\phi_2^{-1}(B)$ where $\phi_2(z) = \frac{1}{1-c}(z + c\bar{z})$. This violates the assumption for a Markov process that the future of the process depends only on its current state but not its history. With this issue it is more difficult for us to analyze this process because the rich theory regarding Markov processes won't apply.

Notice that ϕ_1 and ϕ_2 differ only by a constant factor. This means in these two cases Y_n are almost identical except that they evolve in different scales, or equivalently in different speeds due to the scaling invariance of a Brownian motion. In order to get a Markov process, we need to compensate for this difference. We can achieve this by applying a time-change to the Brownian motions used in building the processes X_c, $X_{c|d}$ and $X_{\frac{c}{d}}$ according to the Brownian scaling. To be precise, for c, $d \in \mathbb{C}$ with $|c| < 1$, $|d| < 1$, and B a standard planar Brownian motion, we define the modified auxiliary processes

$$X'_c = \phi_c^{-1}(B'_c)$$

where B'_c is a time-changed Brownian motion satisfying

$$(B'_c)_{s(t)} = B_t, \text{ with } s(t) = \int_0^t J_{\phi_c^{-1}}(B_r)dr$$

starting at 0 and ϕ_c is the same map defined by $\phi_c(z) = \frac{1}{c+1}(z + c\bar{z})$ as before; and the process

$$X'_{c|d} = \phi_{c|d}^{-1}(B'_{c|d})$$

where $B'_{c|d}$ is a time-changed Brownian motion satisfying

$$(B'_{c|d})_{s(t)} = B_t, \text{ with } s(t) = \int_0^t J_{\phi_{c|d}^{-1}}(B_r)dr$$

starting at 0 and $\phi_{c|d}$ is the same map as before (Almost surely B spends 0 amount of time on the imaginary axis, so the value of $J_{\phi_{c|d}^{-1}}$ there does not matter); and the process

$$X'_{\frac{c}{d}} = \phi_{\frac{c}{d}}^{-1}(B'_{\frac{c}{d}})$$

where $B'_{\frac{c}{d}}$ is a time-changed Brownian motion satisfying

$$(B'_{\frac{c}{d}})_{s(t)} = B_t, \text{ with } s(t) = \int_0^t J_{\phi_{\frac{c}{d}}^{-1}}(B_r)dr$$

starting at 0 and $\phi_{\frac{c}{d}}$ is the same map as before.

Now we carry the setup we had in the previous section over here, and implement exactly the same construction of the process Y using X''s instead of X's, and call the resulting process Y'_n. Since the only change we have made is the time-parametrization of the Brownian motion used in the construction, we get that Y'_n is just Y_n with a time-change. So we have $\phi_n(Y'_n)$ is a concatenation of time-changed Brownian motions just like Y_n.

Moreover, under our assumption that $||\mu||_\infty = k < 1$, it follows that $||\mu_n \cdot 1_S||_\infty \leq k$ in each square S also. With explicit computation we obtain that the Jacobian determinants $J_{\phi_c^{-1}}$, $J_{\phi_{c|d}^{-1}}$ and $J_{\phi_{\frac{c}{d}}^{-1}}$ are all bounded below by $\frac{1-k}{1+k}$ and above by $\frac{1+k}{1-k}$ almost everywhere. As these Jacobian determinants are factors in the time-change going between Y and Y', we have Proposition 6.11 holds true for Y'_n as well.

The way our modification makes a difference is the following. Notice that the time-changes we used to define $B'_{\frac{c}{d}}$ are carefully chosen so that their quadratic variation satisfies

$$[B'_{\frac{c}{d}}]_t = 2\int_0^t J_{\phi_{\frac{c}{d}}}(B_s)ds.$$

The factor of 2 is not surprising because we are working in 2-d where the standard planar Brownian motion has quadratic variation $[B]_t = 2t$. Heuristically, let's for a moment embrace the wrong assumption that the quadratic variation of a process scales according to the Jacobian under quasiconformal maps, then the quadratic variation of $X'_{\frac{c}{d}}$ satisfies

$$[X'_{\frac{c}{d}}]_t = 2t.$$

Similarly we get the same for X'_c and $X'_{c|d}$. This implies that Y'_n is constructed as a concatenation of processes parametrized by their quadratic variation. Therefore Y'_n itself is parametrized by its quadratic variation and so its scale, or speed depending on your perspective, no longer depends on the maps ϕ_c, $\phi_{c|d}$ or $\phi_{\frac{c}{d}}$. Using this assumption again, we have $\phi_n(X'_{\frac{c}{d}})$ has quadratic variation satisfying

$$[\phi_n(X'_{\frac{c}{d}})]_t = 2\int_0^t J_{\phi_n}((X'_{\frac{c}{d}})_r)dr.$$

As Y'_n is a concatenation of such processes, its quadratic variation also satisfies

$$[Y'_n]_t = 2\int_0^t J_{\phi_n}((Y'_n)_r)dr.$$

We have argued that Y'_n is a time-changed Brownian motion. Now with its quadratic variation explicit, the time-change also becomes explicit. Rigorously speaking, our the assumption is false in general. But thanks to Theorem 2.18, it holds when we are mapping a Brownian motion with a conformal map. Notice that because the map $\phi_n \circ \phi_\_^{-1}$ is indeed conformal in each of the squares for the right choice of $\phi_\_ = \phi_c$, $\phi_{c|d}$ or $\phi_{\frac{c}{d}}$, we can make this idea into a proof.

Theorem 6.14. *For each $n \in \mathbb{N}$, the process $\{\phi_n((Y'_n)_t) : t \geq 0\}$ is a time-changed Brownian motion with the time-change given by $s(t) = \int_0^t \phi_n((Y'_n)_r)dr$. That is, there exists a Brownian motion B such that*

$$\phi_n((Y'_n)_t) = B_{s(t)}.$$

Proof. The proof is identical for each $n \in \mathbb{N}$, so we omit all the subscripts n for simplicity. Without loss of generality, assume we start at z_0 and $z_0 \in S$ for some square S. Then

$$\phi(Y') = \phi(X'_c) + z_0 = \phi \circ \phi_c^{-1}((B'_c)_t) + z_0$$

up to time τ_1, where c is the average value of μ in S. Since $\phi \circ \phi_c^{-1}$ is conformal in S and B'_c is a time-changed BM with quadratic variation

$$[B'_c]_t = 2\int_0^t J_{\phi_c}(B_s)ds,$$

$\phi(Y')$ is a time-changed Brownian motion up to time τ_1 with quadratic variation given by

$$[\phi(Y')]_t = 2\int_0^t |(\phi \circ \phi_c^{-1})'(B_s)|^2 \cdot J_{\phi_c}(B_s)ds, \text{ for } t \in [0, \tau_1).$$

Let D denote the set of $z \in \mathbb{C}$ such that both ϕ_c^{-1} is differentiable at z and ϕ is differentiable at $\phi_c^{-1}(z)$. Then for $z \in D$,

$$|(\phi \circ \phi_c^{-1})'(z)|^2 = J_\phi(\phi_c^{-1}(z)) \cdot J_{\phi_c^{-1}}(z).$$

Moreover, since ϕ is quasiconformal, it is differentiable almost everywhere. Thus D^c has measure 0. Almost surely a Brownian motion spends 0 amount of time in D^c, thus we have

$$[\phi(Y')]_t = 2\int_0^t J_\phi(\phi_c^{-1}(B_s)) \cdot J_{\phi_c^{-1}}(B_s) \cdot J_{\phi_c}(B_s)ds = 2\int_0^t J_\phi(Y_s')ds, \text{ for } t \in [0, \tau_1).$$

Following the iteration steps in our construction of Y' and applying the same argument as above, we get

$$[\phi(Y')]_t = 2\int_0^t J_\phi(Y_s')ds$$

also for $t \in [0, \sigma_1)$, $[0, \tau_2)$, $[0, \sigma_2)$, etc. By Proposition 6.11, we have this is true for all $t \in [0, \infty)$, and the conclusion follows. □

Lastly, as Y_n' and Y_n differ only by a time-change, Theorem 6.13 applies to Y_n' as well, and the time-changes we should apply are now explicit.

6.5 Failed attempts

The square-wise construction was not our first attempt to construct the Quasi-Brownian motions. In the early stage of this project, we had several ideas which potentially led to constructions of the Quasi-Brownian motions and they seemed simpler and more natural. However, it turned out that these ideas were too naive to actually give the QBM. In this section we would like to elaborate on these ideas.

Biased random walk on lattices

As the notion of quasi-Brownian motion generalizes the usual Brownian motion, we expect some constructions or approximations of the usual Brownian motion generalizes to that of a quasi-Brownian motion with minor modifications.

One famous approximation of Brownian motion is the simple random walk. See [18, Thm 5.22] for the details about the 1-dimensional case, and this idea generalizes well to the plane where the simple random walk takes place on a $\mathbb{Z}^2$ lattice. In such a simple random walk, in each step we have a consistent and equal probability of $\frac{1}{4}$ to move to any one of the 4 neighboring points. At this point, it is natural to ask if we can adjust the 4 numbers p_U, p_D, p_L and p_R corresponding to the probabilities we move up, down, left and right based on the "value" of the Beltrami coefficient μ at the starting point of each step to yield a approximation to the quasi-Brownian motion corresponding to μ. Here the "value" should be understood as either a point-wise evaluation when μ is assumed to be nice so that point-wise evaluation makes sense, or its average value in a small neighborhood.

It turns out that this only works in a very limited number of cases. Consider the cases where μ is constant throughout the plane. In this case, as the value of the Beltrami coefficient which drives the transition probabilities is constant, we expect we can choose the probabilities p_U, p_D, p_L and p_R to be constant throughout the $\mathbb{Z}^2$ lattice. Note that a constant μ corresponds to a linear map, and the Itô formula asserts that the image of a Brownian motion under a linear map is a drift-free diffusion process. Thus we have to have

$$p_U = p_D,$$

$$p_L = p_R$$

for our random walk to also be drift-free. Together with the restriction

$$p_U + p_D + p_L + p_R = 1,$$

we have only one degree of freedom remaining, and it can be nailed down by choosing a value in $[0, +\infty]$ for the ratio

$$\frac{p_U + p_D}{p_L + p_R}.$$

It will determine the likelihood we make a vertical move versus a horizontal move. But no matter how this value is chosen, everything so far is symmetric horizontally and vertically and thus the only permissible μ's are the ones corresponding to a constant ellipse field with identical horizontal or vertical ellipses.

The problem with the $\mathbb{Z}^2$ grid is the lack of degree of freedom, so we shifted our attention to the triangular lattice. On a triangular lattice, we move to one of the six neighboring points with probability p_0, p_π, $p_{\frac{\pi}{3}}$, $p_{-\frac{\pi}{3}}$, $p_{\frac{2\pi}{3}}$, $p_{-\frac{2\pi}{3}}$. As before we have the restriction

$$p_0 + p_\pi + p_{\frac{\pi}{3}} + p_{-\frac{\pi}{3}} + p_{\frac{2\pi}{3}} + p_{-\frac{2\pi}{3}} = 1.$$

And in this case, it takes 3 degrees of freedom to guarantee we get a drift-free random walk as we have 3 directions to balance out. Note that here we do not necessarily have $p_0 = p_\pi$ and etc as before. For example, choosing

$$p_0 = p_{\frac{2\pi}{3}} = p_{-\frac{2\pi}{3}} = \frac{1}{3}$$

is allowed. Now that we have two degrees of freedom remaining, it seems to be sufficient to encode the direction and the eccentricity of the ellipse field.

Nevertheless, this approach also comes with a flaw. Imagine the case where we need to prescribe a constant ellipse field consists of very long and thin horizontal ellipses. To approximate such quasi-Brownian motion, we need each step to have high variance in the horizontal direction but low variance in the vertical direction. This can be achieved by setting

$$p_0 = p_\pi >> p_{\frac{\pi}{3}} = p_{-\frac{\pi}{3}} = p_{\frac{2\pi}{3}} = p_{-\frac{2\pi}{3}}.$$

But in case these thin ellipses are vertical we are in trouble. The best we can do to maximize the variance of a step in the vertical direction (while keeping the symmetry in the horizontal direction) is to choose

$$p_0 = p_\pi = 0$$

and

$$p_{\frac{\pi}{3}} = p_{-\frac{\pi}{3}} = p_{\frac{2\pi}{3}} = p_{-\frac{2\pi}{3}} = \frac{1}{4}.$$

The problem is this incurs a lower bound on the variance in the horizontal direction. More specifically, with a regular triangular lattice, the variance of a step in the vertical direction is at most 3 times as large as the variance in the horizontal direction. This means with a triangular lattice and restricting to constant Beltrami coefficients, we will be able to prescribe many Beltrami coefficients by manipulating these 6 transition probabilities. But still, we won't be able to prescribe every $\mu \in \mathbb{D}$ as desired. Moreover, as one can imagine, no matter how complicated a lattice we use, one can always arrange a thin ellipse oriented towards a "gap" of this lattice and a random walk will fail to prescribe such ellipse at that point. So we do not expect a different choice of the underling lattice to solve this problem.

Recall that our idea is to choose the transition probabilities based on the value of the Beltrami coefficient at the starting point. This in particular implies that we are associating a transition probability to each value a Beltrami coefficient can possibly take, i.e. each $z \in \mathbb{C}$ with $|z| < 1$. As this association cares only about the value of μ locally around each point, it does not recognize whether μ is constant or not. So the failure of this approach in the special case when μ is constant implies it will certainly not work for general μ. For this reason, we abandoned our attempts to build quasi-Brownian motions using biased random walk on discrete lattices.

However, if we give up our restriction of using only nearest-neighbor random walks with transition probabilities depending only on the value of μ at the starting point in each step, then the situation changes. Indeed, in [21] a random walk approximation of finite range on the scaled lattice $\frac{1}{n}\mathbb{Z}^d$ is constructed for symmetric diffusions with uniformly elliptic divergence form generators. This applies in particular to our setting of the Quasi-Brownian motion, and leads to a different approximation of QBM. We would like to thank Zhen-Qing Chen for providing this reference and for his explanations.

Elliptic random walk

Our attempt with discrete lattices drove us to think about a continuous state space rather than a discrete one for each step of our random walk. A natural candidate would be the ellipse itself. Recall the following random walk approximates the usual Brownian motion. In each step we take a uniform random point on a circle with fixed radius. Donsker's invariance principle implies that if we sum up these random variables and then take a proper interpolation and scaling limit, we get in law a Brownian motion out of this. Now we have a ellipse field associated to the Beltrami coefficient, it is natural to ask if we can launch a random walk with each step taken on the ellipses corresponding to the underlying ellipse field to approximate the quasi-Brownian motion.

To make this precise, let $\mu : \mathbb{C} \to \mathbb{D}$ with $||\mu||_\infty < 1$ be the measurable Beltrami coefficient we want to prescribe and $z_0 \in \mathbb{C}$ be the starting point. For simplicity, let us further assume μ is smooth. For each $c \in \mathbb{D}$, define the linear map

$$\phi_c(z) = \frac{1}{\sqrt{1-|c|^2}}(z + c\overline{z}), \quad z \in \mathbb{C}$$

so that $\frac{\overline{\partial}\phi_c}{\partial\phi_c} \equiv c$ and $|J_{\phi_c}| \equiv 1$, and let $E_c = \phi_c^{-1}(\mathbb{S}^1)$. Now we define the discrete time Markov chain $X = \{X_n, n \in \mathbb{N}\}$ by requiring

1. $X_0 = z_0$.
2. $(X_{n+1} - X_n | X_n = z) \sim \epsilon Y_z$ where Y_z is a r.v. on $E_{\mu(z)}$ such that $\phi_{\mu(z)}(Y_z)$ is uniform on $\mathbb{S}^1$ and ϵ is a fixed number called the step size.

We call this process X the *elliptic random walk* with respect to μ starting at z_0.

For a given μ, it is very tempting to believe such elliptic random walk approximates the corresponding quasi-Brownian motion if we take the limit as $\epsilon \to 0$ "in the correct way". Indeed, several experts in quasi-conformal mappings had this intuition. It is so for several reasons. First of all, this idea is true when μ is constant. Note that constant μ corresponds to linear maps. Under the corresponding linear map, the image of our elliptic random walk becomes the usual circular random walk, which approximates the Brownian motion. For non-constant μ, things become more subtle but still plausible. A popular example of a quasiconformal map is the map $\phi(z) = z|z|^{k-1}$ for $k > 0$. Viewing in the polar coordinates, this map preserves the arguments of points while applies the function $r \mapsto r^k$ to their radii. We can compute its dilatation

$$\mu_\phi(z) = k\frac{z}{\bar{z}}.$$

Its associated ellipse field consists of ellipses with one axis pointing towards the origin with length $\sqrt{k}$ and the other axis of length $1/\sqrt{k}$ when normalized to have their product equals 1. If we take a elliptic random walk on this ellipse field, we do not expect it to "directly" approximate the quasi-Brownian motion with respect to ϕ, because this elliptic random walk turned out to be transient for $0 < k < 1$ [11, Cor 3.2] and it is impossible to be the homeomorphic pre-image of a (neighborhood) recurrent process. But this is due to the lack of variation in the radial direction, so it is still plausible this elliptic random walk approximates the quasi-Brownian motion when we take the limit "in the right way", for example, in the compact subsets of the plane.

But this intuition turns out to be false. The key to see this is to realize that the elliptic random walk itself is always a local martingale, as the symmetry of a ellipse (together with the law) implies

$$E[X_{n+1} - X_n] = 0$$

for all n and any choice of μ. On the other hand, if we take a quasiconformal map ϕ that is C^2 whose coordinate maps have nontrivial Laplacian, the Itô formula asserts the corresponding quasi-Brownian motion has non-trivial drift term. Thus in such cases, elliptic random walks will not converge to the quasi-Brownian motion in any usual sense. Also, this example has given us a very clear signal. To approximate the quasi-Brownian motion, we need to consider not only the value of μ at each point, but also the variation of μ at each point relative to its values nearby, because the second order behavior of ϕ affects the drift of its corresponding quasi-Brownian motion.

6.6 Improving the elliptic random walk

Now we have seen that our square-wise construction successfully gives us an approximation of the desired QBM, while the elliptic random walk ends up in failure. At this point it would be very interesting to see what exactly is the factor contributing to this difference between the square-wise construction and the elliptic random walk.

First we would like to point out that the square-wise construction and the elliptic random walk are closely related. To see this, let us take

$$\mu(z) = \begin{cases} 0, & \text{if } z \in \mathbb{H} \\ \frac{1}{3}, & \text{if } z \in \overline{\mathbb{H}^-} \end{cases}$$

as an example, where $\mathbb{H}$ and $\mathbb{H}^-$ are the upper and lower half plane. Using our notations introduced in Section 6.2, the solution to the Beltrami equation with this μ is given by $\phi_{\frac{0}{1/3}}$ and its corresponding QBM is the process $X_{\frac{0}{1/3}}$ (here we do not restrict $X_{\frac{0}{1/3}}$ to start at 0).

Now consider a elliptic random walk E which takes steps on the ellipse field corresponding to μ. Since $\mu = 0$ in $\mathbb{H}$, E restricted to $\mathbb{H}$ is actually a circular random walk, and $X_{\frac{0}{1/3}}$ restricted to $\mathbb{H}$ is indeed a Brownian motion. Thus, restricting to $\mathbb{H}$, the elliptic random walk does approximate the QBM. In $\mathbb{H}^-$, the ellipse field corresponding to this μ consists of vertical ellipses whose vertical axis is twice as long as its horizontal axis. The map $\phi_{\frac{0}{1/3}}$ is a linear map mapping these ellipses to circles exactly (which is more than just approximately or infinitesimally). Thus restricting to $\mathbb{H}^-$ the image of E under $\phi_{\frac{0}{1/3}}$ is exactly a circular random walk. Pulling it back and we see that the corresponding elliptic random walk indeed approximates the QBM restricting to $\mathbb{H}^-$ as well. With this been said, it is a bit surprising that the elliptic random walk fails to approximate the general QBM, providing that it "almost everywhere" approximates the process $X_{\frac{0}{1/3}}$ and processes like $X_{\frac{0}{1/3}}$ are the building blocks of our approximation of the general QBM as shown earlier in Section 6.3.

The only place where this elliptic random walk E fails to approximate $X_{\frac{0}{1/3}}$ is on real line. To see this, consider the preimage of a circular random walk under the map $\phi_{\frac{0}{1/3}}$, and denote it C. Since a circular random walk approximates the Brownian motion, the preimage of that approximates the process $X_{\frac{0}{1/3}}$. When we are away from the real line, the steps of E and C have the same distribution, but at places near the real line, the steps of C and and those of E have a subtle but important difference. See Figure 3.

The steps of E are always taken either on a circle with respect to the uniform measure or a ellipse with respect to the pullback of the uniform measure on a circle by the map $\phi_{1/3}$, depending on whether the starting point lies in $\mathbb{H}$ or $\mathbb{H}^-$. As a result, E is always a local martingale as we

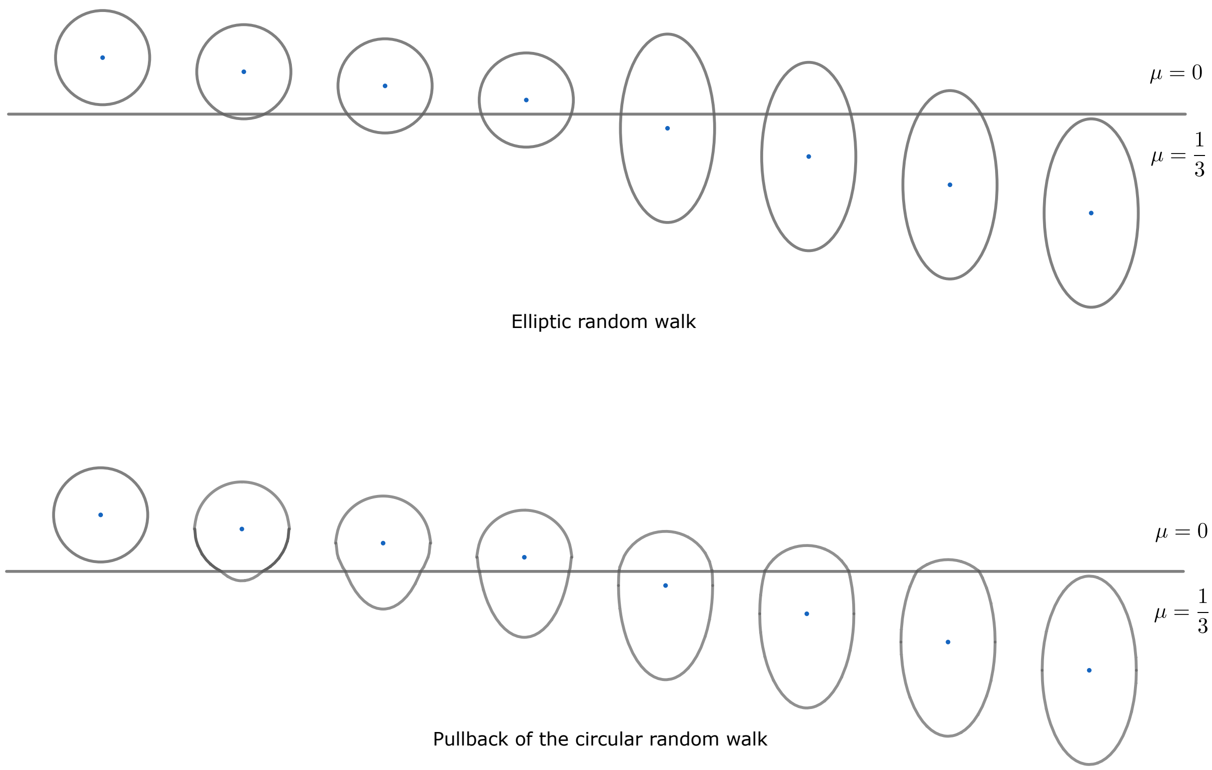

Figure 3: a step of the elliptic random walk versus that of the pullback of the circular random walk by the map $\phi_{\frac{0}{1/3}}$ with different starting points near the real line.

have shown earlier. The steps of C near the real line is more complicated. When the starting point is in $\mathbb{H}$, it takes its next step on a circle centered at the starting point with the portion of this circle invading into $\mathbb{H}^-$ stretched into a portion of an ellipse. When the starting point is in $\mathbb{H}^-$, its next step is taken on an ellipse out of the ellipse field corresponding to μ, with the portion of this ellipse inside $\mathbb{H}$ compressed back into a portion of a circle. The exact distribution of this step can be obtained by pulling back the uniform measure on a circle by $\phi_{\frac{0}{1/3}}$. As a result, the expectation of all of these steps near the real line has a negative y coordinate, which gives C a drift downward near the real line.

With this in mind, actually we can modify the steps of the elliptic random walks accordingly to turn it into a good approximation to QBM's. Given a Beltrami coefficient $\mu : \mathbb{C} \to \mathbb{C}$ measurable with $||\mu||_\infty < 1$, we define μ_n as in Section 6.3. Given a point $z_0 \in \mathbb{C}$ that is away from any vertices in the $2^{-n}\mathbb{Z} \times 2^{-n}\mathbb{Z}$ grid as the starting point, if z_0 is away from the edges so that it will certainly not travel across any edges in the next step, then we take our next step as a elliptic random walk. If the next step have a chance to go across one of the edges, then we take our next step as that of C - take the preimage of uniform random variable of a circle under the corresponding choice of $\phi_{c|d}$ or $\phi_{\frac{c}{d}}$. This way we obtain a modified elliptic random walk and it approximates the processes $X_{c|d}$ and $X_{\frac{c}{d}}$, which in turn approximates the QBM as we desired. A caveat remaining is when we are close to one of the vertices of the square lattice. In this case the next step might take place in a square that is only adjacent to the current one at a corner. In this case we do not have a good way to take the next step, but fortunately this happens with small probability when the size of the steps is far smaller than the size of the grid.

7 Future work

7.1 Øksendal's construction v.s. square-wise construction

In the previous section we have presented two different constructions of the QBM (up to a time-change) based on a Beltrami coefficient, namely the Øksendal's construction and our square-wise construction. Each of them has their own advantages and disadvantages.

For the Øksendal's construction, while being abstract in the relationship between the Beltrami coefficient and the resulting process, it has the crucial advantage that it automatically comes with its associated Dirichlet form. This provides us a good way to analyze this process by itself, without considering the image process under certain quasiconformal maps. Meanwhile, the square-wise construction is more constructive. It takes advantage of the fact that the solutions to Beltrami equations with constant Beltrami coefficient can be written down explicitly and therefore locally the relationship between the resulting QBM and the Brownian motion is explicit. Thanks to this, we can carry properties of the Brownian motion within each square back to the resulting QBM. This to some extent allows us to analyze the resulting process. More interestingly, it also allows us to numerically simulate a QBM using those welldeveloped techniques to simulate the standard Brownian motion. Moreover, we have come up with a method to efficiently simulate the exiting distribution of this QBM using the harmonic measure of the standard planar Brownian motions and conformal maps between polygonal domains. We will talk more about numerical simulations in section 7.3. On the other hand, we bear the drawback that the resulting process is more difficult to analyze.

At this point it is natural to ask whether these two constructions are equivalent, in the sense that the laws of the resulting process out of these two constructions are the same (potential up to a time-change) when using the same parameters (Ω, μ, z_0). In some sense the answer is positive. Given a bounded simply connected domain $\Omega \subset \mathbb{C}$, and $\mu : \Omega \to \mathbb{C}$ with $||\mu||_\infty < 1$ (extended by $\mu(z) = 0$ for $z \in \Omega^c$ when necessary) as the Beltrami coefficient to prescribe. Let ψ be a quasiconformal map with μ as its Beltrami coefficient, whose existence is guaranteed by the measurable Riemann mapping theorem. Then Theorem 6.5 and Theorem 6.13 showed that both constructions yield a process whose image under ψ is a time-changed Brownian motion. Note that the solution to the Beltrami equation is unique up to post-composing a conformal map, and Brownian motions are preserved under conformal maps. So in this sense, these two constructions are equivalent. However, this is based on the existence of such ψ, which relies on the measurable Riemann mapping theorem. It is still interesting to see if the equivalence between the resulting processes out of these two constructions can be established on their own, without having to resort to their images under ψ.

7.2 The measurable Riemann mapping theorem

As our initial motivation, to give a probabilistic proof of the measurable Riemann mapping theorem has been something we keep reflecting on throughout this project. In Chapter 6 we presented two different ways to construct the QBM (up to a time-change) based on a given Beltrami coefficient. With this we have made a considerable step towards it. Given a bounded simply connected domain $\Omega \subset \mathbb{C}$ and $\mu : \Omega \to \mathbb{C}$ with $||\mu||_\infty < 1$ as the Beltrami coefficient to prescribe, we can now define the QBM based on μ (extended by $\mu(z) = 0$ for $z \in \Omega^c$ when necessary) and it is clear that we should take the harmonic measure, if such harmonic measure is well-defined, of this process to define the map ϕ as in Theorem 1.5.

It remains for us to construct the map ϕ and to prove it solves the Beltrami equation with μ as its Beltrami coefficient. To be precise, for this purpose we need to prove this statement without assuming the existence of a quasiconformal map prescribing certain Beltrami coefficient, or otherwise we will be giving a circular argument by proving the measurable Riemann mapping theorem using the measurable Riemann mapping theorem. We have obtained that given a quasiconformal map ψ and its Beltrami coefficient μ, the process we constructed, using either Øksendal's construction or our square-based construction, is a QBM with respect to ψ - its image under certain a quasiconformal map ψ, with μ as its Beltrami coefficient, is a Brownian motion. But now we are only provided with a Beltrami coefficient μ from the Beltrami equation. We should not assume the existence of such ψ here because proving its existence is part of this mission. For this reason, in order to prove ϕ is the map we want, we need to extract more information purely from these processes themselves, instead of their images under ψ. For this reason, here it is easier to work with Øksendal's construction because we automatically obtain the Dirichlet form associated to the resulting process. Thus we will for now proceed with the resulting process out of Øksendal's construction.

Let X be the process we obtain by carrying out Øksendal's construction with Ω and μ as above, and a point z_0 in Ω. In other words, X is the stochastic process mentioned in Theorem 6.5. If we further assume $\text{Area}(\partial\Omega) = 0$, then the harmonic measure of the X is well-defined. This fact is essentially already proven by Øksendal in [19]. For details, see [19, Lem 3.3] and use the fact that this resulting process is neighborhood recurrent [13, Thm A] and thus the event $[\tau_\Omega < \infty]$ happens almost surely. It is important to note that, neither of these results depends on the measurable Riemann mapping theorem, thus we can build the map ϕ using the harmonic measures of the X the same way as in Theorem 1.5. This brings us one step further towards this goal.

Nevertheless, we have not been able to find a proof Theorem 1.5 in this case. To do this we need to better understand the Dirichlet form associated to X, and use it to understand the its corresponding harmonic measure (which is not actually harmonic) and Green's function. We are currently still working in this direction and we look forward to sharing any discoveries we will get.

7.3 Numerical simulation

Simulations of the QBM is an interesting topic in its own. Note that if we are given an explicit quasiconformal map ϕ, then the QBM with respect to such ϕ is easy to obtain by definition - one just need to take the preimage of a planar Brownian motion under this ϕ. This problem is more interesting when we are only provided with the Beltrami coefficient of ϕ. In this case, we do not have enough information to obtain the correct time parametrization, so we make the reasonable assumption that we do not care about getting the correct time parametrization for our simulations as well.

One of the advantages of the square-wise construction is that it is easy to simulate. Recall that the process Y_n defined in section 6.3 is a concatenation of processes X_c, $X_{c|d}$, and $X_{\frac{c}{d}}$, which individually are the images of Brownian motions under piece-wise linear maps. Simulations of the linear Brownian motions are well studied. For example, one can add up normal random variables and linearly interpolate. This works particularly well when one only cares about its finite dimensional distributions. Alternatively one could apply Lévy's construction to get uniform convergence of the approximations almost surely. Or one can take a simple random walk which works well in the large scale. Based on these, obtain a simulation of the planar Brownian motion is easy. There are also methods to directly simulate the planar Brownian motions. For example, one can take a simple random walk on a square lattice, or the circular random walk we mentioned in Section 6.5. With simulations of the Brownian motion, we can simulate the QBM using the idea we presented in this paper.

One way to go is to follow exactly the same steps in the construction of Y_n in Section 6.3 to get a simulation of it. But this procedure can be further optimized. In the construction of Y_n we also considered the diagonal set D_n and the stopping times σ's on which our process hits D_n. This setup gave us some extra structures to prove Theorem 6.12 but they are not necessary for the purpose of simulation. Also as the modification we made in Section 6.4 affects only the time parametrization, it is in general not necessary for simulations either. Thus one could follow the following optimized steps to get a simulation of the QBM, and Figure 4 gives us an illustration of these steps.

1. Determine the square lattice by first determining its size 2^{-n}, and then translate it if necessary to make sure z_0 is not on a vertex of the lattice.

2. If z_0 is in the interior of a square S: first compute the average value c of μ in S. Apply the map ϕ_c to the square S, we get a parallelogram $\phi_c(S)$. Simulate a Brownian motion in this parallelogram and let w_1 be its exiting point. Pull the Brownian motion back to S and let $z_1 = \phi_c^{-1}(w_1)$. If z_0 is on the lattice: let $z_1 = z_0$ and move to the next step.

3. With high probability z_1 is on an edge of the lattice but away from any vertices. (For a Brownian motion this should happen almost surely, but some simulation methods, for example taking simple random walks on a finer square lattice, might give us approximations of the Brownian motion that actually pass through some of the vertices. These methods are not recommended here.) If z_1 is on a horizontal edge: compute c, the average value of μ in the square right above z_1, and d, that of the square right below it. Let R be the rectangle formed by these 2 squares (and the side between them). Translate R so that z_1 is at the origin and apply the map $\phi_{\frac{c}{d}}$ to R. In the resulting hexagon $\phi_{\frac{c}{d}}(R)$ we simulate a Brownian motion starting at 0 and let w_1 be its exiting point. Pull the Brownian motion back to R and

update z_1 so that it denotes its exiting point from R; If z_1 is on a vertical edge: compute c and d the same way but with the square to the left of z_1 and to the right of z_1. Translate R so that z_1 is at the origin and apply the map $\phi_{c|d}$ to R. In the resulting hexagon $\phi_{c|d}(R)$ we simulate a Brownian motion starting at 0 and let w_1 be its exiting point. Pull the Brownian motion back to R and update z_1 so that it denotes its exiting point from R;

4. Iterate the previous step.

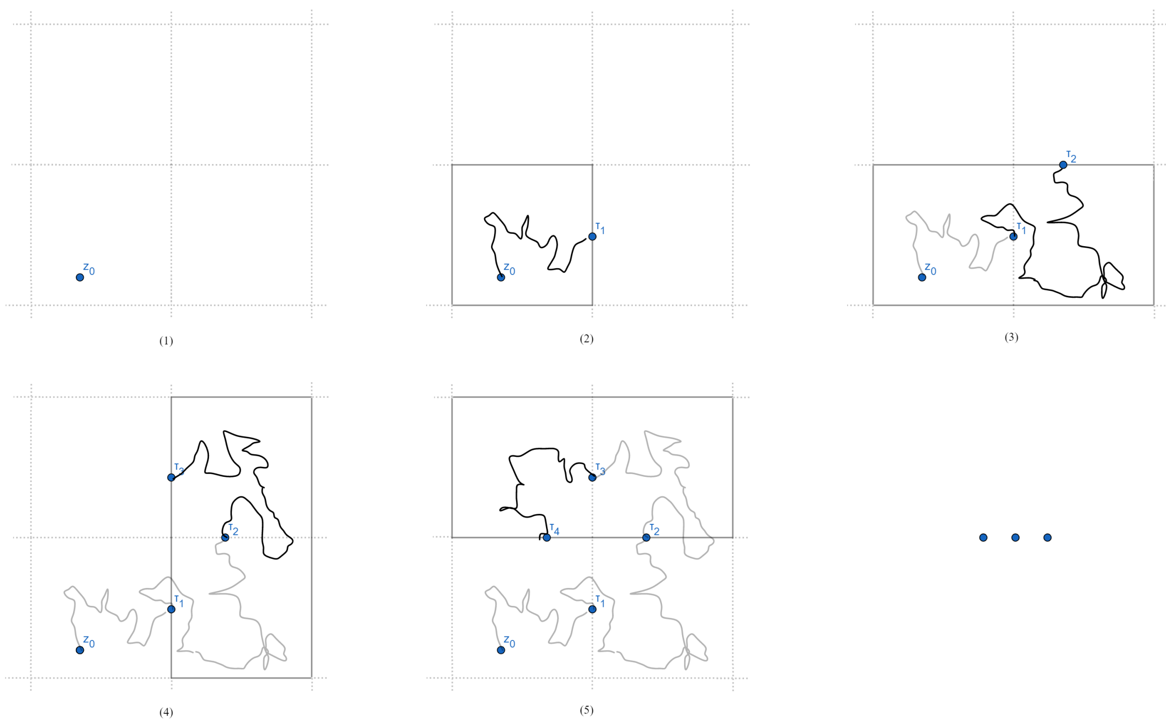

Figure 4: optimized simulation steps

Now that we know a way to simulate quasi-Brownian motions based on a given Beltrami coefficient, if we apply our idea to prove the measurable Riemann mapping theorem, we also obtained a way to numerically compute the "Riemann" map.

Let Ω be a bounded Jordan domain. Recall that the "Riemann" map in Theorem 1.5 is built using the harmonic measures of 3 arcs on the boundary. To numerically compute these harmonic measures at $z_0 \in \Omega$, we can simulate a sufficiently large number of QBM's and look at the proportion of them exiting through each of the 3 arcs. This gives us an estimate of the exit probabilities of the

3 arcs, which gives us the harmonic coordinates of z_0 in Ω. Now take the unit disk with 3 distinct points marked on its boundary, we can find a unique point in it with such harmonic coordinates. We will take this point to be the image of z_0.

We have implemented this idea to visualize the quasiconformal maps from $\mathbb{B}_2$ to $\mathbb{B}_1$ with real constant Beltrami coefficients supported on the square $S = [-1,1] \times [-1,1]$. See Figure 5. In order to have horizontal and vertical symmetries, the marked boundary points in $\mathbb{B}_2$ are chosen to be 2, $2i$ and -2, and the marked boundary points in $\mathbb{B}_1$ chosen to be 1, i and -1 respectively. Only the points in the first quadrant (including its boundary) are computed; the others are by reflection. From each point, $100,000$ QBM's are launched to estimate its harmonic coordinates.

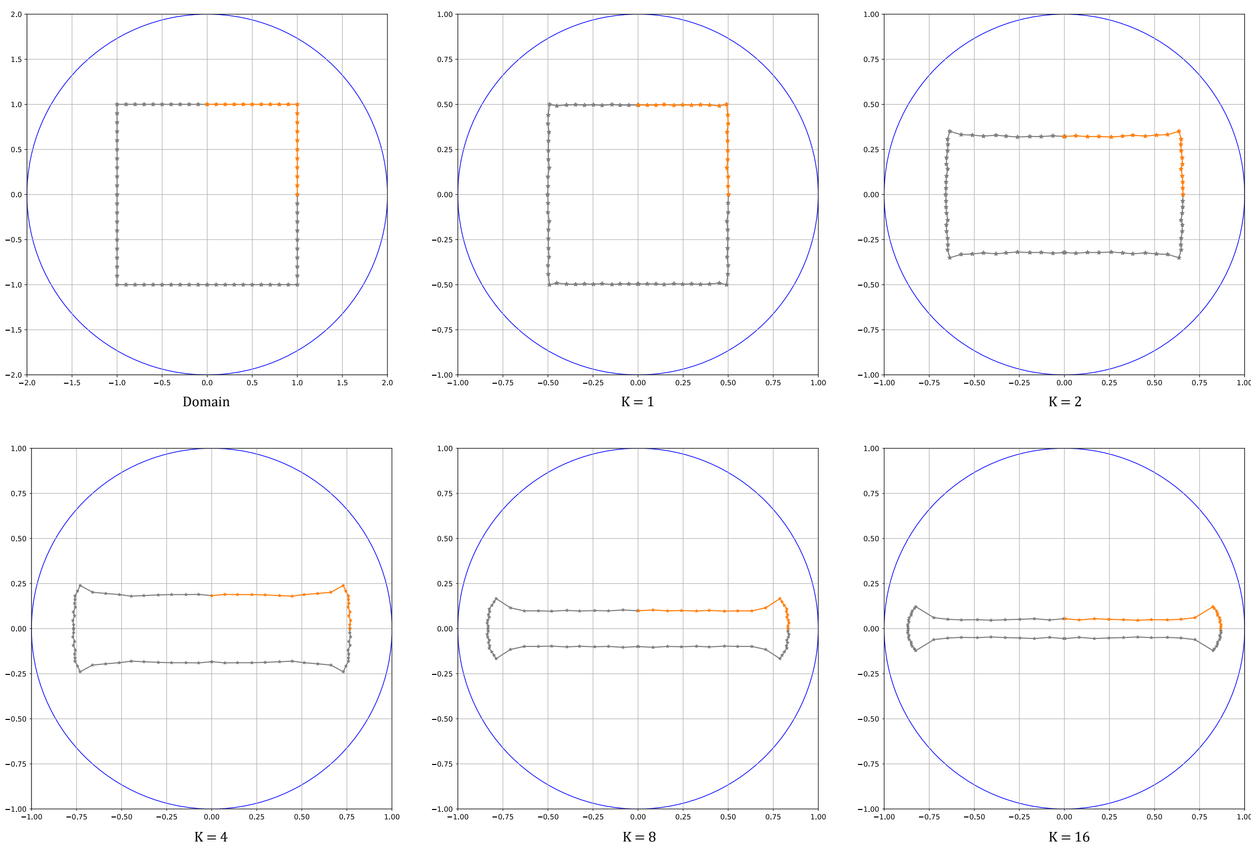

Figure 5: image of $S = [-1,1] \times [-1,1]$ under quasiconformal maps from $\mathbb{B}_2$ to $\mathbb{B}_1$ with Beltrami coefficient of the form $\mu(z) = \frac{K-1}{K+1} \cdot 1_S(z)$ for different K values.

In order to accurately compute the harmonic coordinates, it is necessary to simulate a large number of QBM's. To be precise, when we estimate the harmonic coordinates using 10 QBM's, we might for example get 2 of them exit through arc 1, 3 of them exit through arc 2 and the remaining 5 of them to exit through arc 3. In this case our estimates of the harmonic coordinates are $(0.2, 0.3, 0.5)$ and we only expect these to be accurate up to at most 1 decimal place. If we launch 100 QBM's, we expect our estimates of the harmonic coordinates to be accurate up to at most 2 decimal places, and this is even before considering the error due to randomness. Thus to get an accurate simulation of the "Riemann" map, a large amount of simulations are inevitable. This is why we plot only a few points and take advantage of the symmetries to reduce the computation workload.

Meanwhile, several optimizations can be implemented to reduce the run-time. First of all, this algorithm is optimal for parallel computing. Notice that all of the QBM's used to compute the harmonic coordinates are independent of each other, thus they can be simulated simultaneously when the hardware permits, for example, when we are using a multi-core processor. We have implemented this multicore optimization already and it reduced our run-time significantly in generating Figure 5. Moreover, as we only need to know how the QBM's exit Ω instead of their whole path, we can further optimize the procedures above. In Step 3 and 4, instead of simulating a Brownian motion in the parallelogram $\phi_c(S)$ or the hexagon $\phi_{\frac{c}{d}}(R)$, $\phi_{c|d}(R)$, we can directly compute the Riemann map from the unit disk to these domains and push forward a uniform random point on the unit circle to simulate this exiting point. Since these are all simply connected polygonal domains, the conformal map from each of them to the unit disk is given by the Schwarz-Christoffel map, which is computable numerically. For example, see [6] for an implementation written in Matlab. This way we can potentially simulate how the QBM moves between the squares totally contained within Ω more efficiently. For squares which intersect the boundary of Ω, this optimization could potentially fail. Here we could either simplify the domain Ω by taking a compact set formed by the union of all closed squares contained inside Ω, or alternatively just simulate the whole path of the QBM in these squares for the actual exiting point. When the squares are small enough, we should still obtain a good approximation. We have not yet implemented this optimization as it requires a lot of work in engineering, but we would like to test how it increases the performance in the future.

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