Explicit solutions to linear, second-order, initial and boundary value problems with variable coefficients

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Abstract

Explicit solution to linear, second-order, initial and
boundary value problems with variable coefficients

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I derive explicit solution representations for linear, second-order Initial-Boundary Value Problems (IBVPs) with coefficients that are spatially varying, with linear, constant-coefficient, two-point boundary conditions. I accomplish this by considering the variable-coefficient problem as the limit of a constant-coefficient interface problem, previously solved using the Unified Transform Method of Fokas. Our method produces an explicit representation of the solution, allowing us to determine properties of the solution directly. I prove that these representations are solutions to *fully* and *partially dissipative* problems under general conditions. As explicit examples, I demonstrate the solution procedure for different IBVPs of variations of the heat equation, and the linearized complex Ginzburg-Landau (CGL) equation (with periodic boundary conditions). The solution can be used to find the eigenvalues of second-order linear operators (including non-self-adjoint ones) as roots of a transcendental function, and their eigenfunctions may be written explicitly in terms of the eigenvalues.

Contents

1. Introduction

1.1 Introduction

The Unified Transform Method (UTM), or Method of Fokas, was first developed as a generalization of the Inverse Scattering Transform (IST) method to solve Initial-Boundary Value Problems (IBVPs) for integrable nonlinear equations. It was later realized that it was particularly convenient and straightforward for linear, constant-coefficient IBVPs. The UTM leads to many new insights on PDEs and IBVPs, see for instance [1, 7, 11, 12, 13, 14, 15, 30], and references therein. Especially relevant for this work, the method has been used to explicitly solve interface problems with piecewise-constant coefficients, see [5, 6, 18, 24, 25, 26, 27]. The purpose of this thesis is to generalize the UTM to solve variable-coefficient IBVPs.

The classical approach of separation of variables is useful if the associated ODE is a second-order, self-adjoint problem on a finite domain, for which there is regular Sturm-Liouville theory [3], but does not generalize well to problems that are not self adjoint, of higher order, or posed on an unbounded domain. In [30], Fokas and Treharne use a Lax Pair approach to analyze variable-coefficient IBVPs. Their approach reduces the problem from solving a *Partial* Differential Equation to solving an *Ordinary* Differential Equation (ODE) by writing the solution of the PDE as an integral over the solutions to a non-autonomous ODE.

In this approach to variable-coefficient IBVPs, the domain is divided into N parts and the equation is approximated by a constant-coefficient equation on each part. The resulting interface problem is solved using the UTM as shown in [5, 6, 18, 24, 25, 26, 27]. Using Cramer's rule, the solution in each part is found as a ratio of determinants. Through the nontrivial steps of obtaining an explicit[1] expression for the determinants and taking the limit as N goes to infinity, a complicated but explicit solution expression is obtained. The limit is taken non-rigorously and the results are justified independently by proving that they are solutions to the given IBVP.

As in previous applications of the UTM (*e.g.,* [1, 14] for constant-coefficient problems, [6, 28] for interface problems), one of the benefits of this approach is characterizing which boundary conditions give rise to a well-posed IBVP. In particular, for the finite-interval problem, this work is consistent with Locker's work on Birkhoff regularity, *e.g.,* [17]. Since the UTM is generalizable to large classes of varying boundary conditions, IBVPs of higher order, including non-self-adjoint problems, this work is expected to generalize in these same directions as well.

These formulae may seem complicated; however, they are similar to the solutions found in [22], which have been used to prove a variety of properties of solutions to ODEs and eigenvalue problems. Indeed, the notation used here is inspired by this book. While the solutions here are similar, the methods are entirely different. The reader may also find these expressions reminiscent of path integrals [29], although those are usually used to propagate in time, unlike the spatial "discretization" approach used here.

1.2 Assumptions and Definitions

Throughout this thesis, the following linear, second-order evolution equation with spatially variable coefficients is considered:

$$q_t = \alpha(x)\left(\beta(x)q_x\right)_x + \gamma(x)q + f(x,t), \qquad x \in \mathcal{D} \subseteq \mathbb{R}, \quad t > 0, \tag{1.2.1a}$$

$$q(x,0) = q_0(x), \qquad x \in \mathcal{D}, \tag{1.2.1b}$$

on different domains $\mathcal{D}$ with (possibly) some functions $f_0(t)$, $f_1(t)$ prescribed at the boundary of $\mathcal{D}$.

For any of the IBVPs corresponding to (1.2.1), the coefficient of the largest derivative of (1.2.1a), $\alpha(x)\beta(x)$, is required to have a non-negative real part, *i.e.,* $\text{Re}(\alpha(x)\beta(x)) \geq 0$ or $|\arg(\alpha(x)\beta(x))| \leq \pi/2$. Problems with $\sup_{x\in\mathcal{D}} |\arg(x)\beta(x)| < \pi/2$ are referred to as *fully dissipative*, problems with $|\arg(x)\beta(x)| = \pi/2$ as *fully dispersive*, and problems with $\sup_{x\in\mathcal{D}} |\arg(x)\beta(x)| = \pi/2$ but $\inf_{x\in\mathcal{D}} |\arg(x)\beta(x)| < \pi/2$ as the *mixed dissipative–dispersive*

[1]By explicit, we mean a solution written down entirely in terms of summations and integrations.

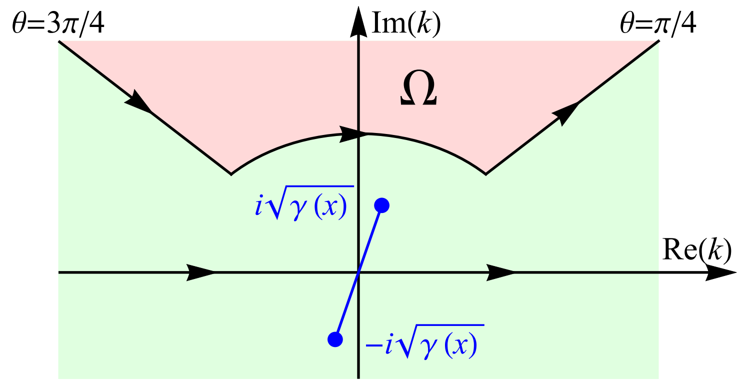


Figure 1.1: The region Ω with the branch cuts for $\mathfrak{g}(k,x)$.

case or as *partially dissipative/dispersive.* This thesis covers the *fully dissipative case* for the three domains, the whole line, half line, and finite interval, and the *mixed dissipative-dispersive case* on the finite interval.

In all cases, the solution is written as

$$q(x,t) = \frac{1}{2\pi}\int_{\partial\Omega} \frac{\Phi(k,x,t)}{\Delta(k)} e^{-k^2 t}\, dk, \tag{1.2.2}$$

where the functions $\Phi(k,x,t)$ and $\Delta(k)$ depend on $\mathcal{D}$ and the initial and boundary conditions provided. The region $\Omega = \{k \in \mathbb{C} : |k| > r \text{ and } \pi/4 < \arg(k) < 3\pi/4\}$, for some $r > \sqrt{M_\gamma}$, where $M_\gamma = \|\gamma\|_\infty$, as shown in Figure 1.1. In this section, some notation is established and the assumptions on the functions in (1.2.1) and on the boundary functions $f_m(t)$ $(m = 0,1)$ that we use throughout the thesis is introduced.

The argument of complex variables is defined as $\arg(\,\cdot\,) \in [-\pi/2, 3\pi/2)$. The domain of each problem is denoted as $\mathcal{D}$, so that $\mathcal{D} = \mathbb{R}$, $\mathcal{D} = (x_l, \infty)$, and $\mathcal{D} = (x_l, x_r)$ for the whole-line, half-line, and the finite-interval problems, respectively. The domain $\mathcal{D}$ is given by the open set, and the closure by $\overline{\mathcal{D}}$. The L^1–norm over the domain $\mathcal{D} \subseteq \mathbb{R}$ is denoted by $\|\cdot\|_{\mathcal{D}}$. When used on a function of multiple variables, a supremum norm on the other variables is implicitly assumed, *e.g.,* for a function $f(k,x)$ for $k \in \Omega \subseteq \mathbb{C}$ and $x \in \mathcal{D}$,

$$\|f\|_{\mathcal{D}} = \sup_{k\in\Omega} \int_{\mathcal{D}} |f(k,x)|\, dx. \tag{1.2.3}$$

In this way, the norms always represent fixed numbers, never functions. The notation $\mathrm{AC}(\,\cdot\,)$ represents the space of locally absolutely continuous functions on the closure of the domain, and $\mathrm{AC}^1(\,\cdot\,)$ represents the space of functions that have locally absolutely continuous derivatives on the closure of the domain. Throughout the thesis, the 'big-oh' notation $O(\,\cdot\,)$ and the 'little-oh' notation $o(\,\cdot\,)$ is used, as described in [20].

Consider (1.2.1) on the finite interval $x_l < x < x_r$. Two linear, constant-coefficient boundary conditions are specified:

$$f_0(t) = a_{11}q(x_l,t) + a_{12}q_x(x_l,t) + b_{11}q(x_r,t) + b_{12}q_x(x_r,t), \qquad t > 0, \tag{1.2.4a}$$
$$f_1(t) = a_{21}q(x_l,t) + a_{22}q_x(x_l,t) + b_{21}q(x_r,t) + b_{22}q_x(x_r,t), \qquad t > 0. \tag{1.2.4b}$$

The concatenated matrix of coefficients is denoted by

$$(a:b) = \begin{pmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \end{pmatrix}, \tag{1.2.5}$$

and the determinant of the 2×2 minor with columns at i and j [23] is denoted by $(a:b)_{i,j} = \det((a:b)_{\{1,2\},\{i,j\}})$. The rank of $(a:b)$ is required to be 2 and one of the following *Boundary Cases* is required.

Definition 1. *For the finite-interval problem, for $x \in \mathcal{D} = (x_l, x_r)$, define the functions*

$$\mu(x) = \frac{1}{\sqrt{\alpha(x)\beta(x)}} \qquad \textit{and} \qquad \mathfrak{u}(x) = \frac{1}{\mu(x)}\left(\frac{\beta'(x)}{\beta(x)} - \frac{\alpha'(x)}{\alpha(x)}\right), \tag{1.2.6}$$

and the constants $\mathfrak{u}_\pm = \mathfrak{u}(x_r) \pm \mathfrak{u}(x_l)$, *and*

$$m_{\mathfrak{c}_0} = \frac{(a:b)_{1,4}}{\mu(x_l)} - \frac{(a:b)_{2,3}}{\mu(x_r)}, \qquad m_{\mathfrak{c}_1} = \frac{(a:b)_{1,4}}{\mu(x_l)} + \frac{(a:b)_{2,3}}{\mu(x_r)}, \qquad \text{and} \qquad m_{\mathfrak{s}} = \frac{(a:b)_{1,3}}{\mu(x_l)\mu(x_r)}. \tag{1.2.7}$$

Define the following Boundary Cases:

1. $(a:b)_{2,4} \neq 0$,
2. $(a:b)_{2,4} = 0$ *and* $m_{\mathfrak{c}_0} \neq 0$,
3. $(a:b)_{2,4} = 0$, $m_{\mathfrak{c}_0} = 0$, $m_{\mathfrak{c}_1} = 0$, *and* $(a:b)_{1,3} \neq 0$,
4. $(a:b)_{2,4} = 0$, $m_{\mathfrak{c}_0} = 0$, $m_{\mathfrak{c}_1} \neq 0$, *and* $m_{\mathfrak{c}_1}\mathfrak{u}_+ - 8m_{\mathfrak{s}} \neq 0$.

Different assumptions are required for different classes of problem. As noted above, the PDE (1.2.1a) is separated into *fully dissipative*, *fully dispersive*, and *mixed dissipative–dispersive* problems, based on $\arg(\alpha(x)\beta(x))$, and the finite-interval IBVP is separated into the four Boundary Cases 1–4. Boundary Case 4 requires additional assumptions. The IBVPs are also separated into two classes *regular* and *irregular* problems. *Regular* problems consist of the whole-line problems, the half-line problems, and the finite-interval problems under Boundary Cases 1, 2, and the subcase of Boundary Case 3, if $(a:b)_{1,2} = 0 = (a:b)_{3,4}$. *Irregular* problems consist of the finite-interval Boundary Case 4 and the alternate subcase of Boundary Case 3, if $(a:b)_{1,2} \neq 0$ or $(a:b)_{3,4} \neq 0$. The assumptions required for each type of problem are given below.

Assumption 2. *The following are always assumed about the coefficient functions* α, β, γ*:*

1. $\sup_{x\in\mathcal{D}} |\arg(\alpha(x)\beta(x)| \leq \pi/2$,
2. $\alpha, \beta \in \mathrm{AC}(\mathcal{D})$,
3. $m_{\alpha\beta} = \inf_{x\in D} |\alpha(x)\beta(x)| > 0$,
4. $\alpha\beta$, $\gamma \in L^\infty(\mathcal{D})$, *and we define* $M_{\alpha\beta} = \|\alpha\beta\|_\infty$ *and* $M_\gamma = \|\gamma\|_\infty$,
5. $(\beta'/\beta - \alpha'/\alpha)$, $\gamma' \in L^1(\mathcal{D})$.

For mixed dissipative–dispersive *problems or for Boundary Case 4, the following is also required:*

6. $\beta'/\beta - \alpha'/\alpha \in \mathrm{AC}(\mathcal{D})$.

For Boundary Case 4 of a mixed dissipative–dispersive *problem, the following is required:*

7. γ', $\beta'/\beta + \alpha'/\alpha$, $(\beta'/\beta - \alpha'/\alpha)' \in \mathrm{AC}(\mathcal{D})$.

Assumption 3. *The following are assumed about the inhomogeneous, initial, and boundary functions* f, q_0, f_m*:*

1. *For the inhomogeneous function* $f(x,t)$, $f(x,\cdot) \in \mathrm{AC}((0,T))$ *for each* $x \in \overline{\mathcal{D}}$, *and*
$$\|f\|_{\mathcal{D}} = \sup_{t\in[0,T]} \int_{\mathcal{D}} |f(x,t)|\,dx < \infty \qquad \text{and} \qquad \|f_t\|_{\mathcal{D}} = \sup_{t\in[0,T]} \int_{\mathcal{D}} |f_t(x,t)|\,dx < \infty.$$
2. *For the initial condition* $q_0(x)$, $q_0 \in L^1(\mathcal{D})$.
3. *For the boundary functions* $f_m(t)$, $m = 0,1$, $f_m \in \mathrm{AC}((0,T))$ *and* $f_m' \in L^\infty((0,T))$.

For partially dissipative *problems or for Boundary Case 4, the following is required:*

4. *For the boundary functions* $f_m(t)$, $m = 0,1$, $f_m \in \mathrm{AC}^1((0,T))$ *and* $f_m'' \in L^\infty((0,T))$.

For irregular, partially dissipative *problems, the following is also required:*

5. *For the inhomogeneous function* $f(x,t)$, *we assume* $f(x,\cdot) \in \mathrm{AC}^1((0,T))$ *for each* $x \in \overline{\mathcal{D}}$, *and*
$$\|f_t\|_{\mathcal{D}} = \sup_{t\in[0,T]} \int_{\mathcal{D}} |f_t(x,t)|\,dx < \infty \qquad \text{and} \qquad \|f_{tt}\|_{\mathcal{D}} = \sup_{t\in[0,T]} \int_{\mathcal{D}} |f_{tt}(x,t)|\,dx < \infty.$$

Assumption 4. *For* irregular, partially dissipative *problems on the finite-interval, the boundary data $f_m(t)$, $m = 0, 1$, and the initial condition $q_0(x)$ are both required to satisfy*

$$(a:b)_{1,2}q_0(x_l) - (a:b)_{2,3}q_0(x_r) = a_{22}f_0(0) - a_{12}f_1(0), \tag{1.2.8a}$$
$$(a:b)_{1,4}q_0(x_l) + (a:b)_{3,4}q_0(x_r) = b_{22}f_0(0) - b_{12}f_1(0), \tag{1.2.8b}$$

which we call the compatibility conditions.

Remark 5.

- *Assumption 2.1 is a common necessary requirement [14]. In Chapter 4, the* fully dissipative *problems are considered and in Chapter 5, the* mixed dissipative–dispersive *problems on the finite interval are considered.*
- *Assumption 2.2 may seem odd considering that the derivation is through an interface problem, see Chapter 3. However, in that section, the mean value theorem is used as the limit of the number of interfaces N is taken to infinity, and thus the continuity of our functions is assumed. This section can be amended to include piecewise continuous functions, but makes the solution formulas even more complicated. For simplicity, the restriction to continuous functions is made. Alternatively, distribution theory could be employed to extend the current results to discontinuous functions.*
- *Assumptions 2.3 and 2.4 are physically natural conditions to impose. Assumption 2.5 ensures that the solutions are well defined. It may be possible to extend this to other L^p spaces or other more general spaces with some more work.*
- *Assumption 2.7 (with Assumptions 2.2 and 2.6) implies that $\alpha', \beta' \in \mathrm{AC}(\mathcal{D})$.*
- *The usual definition of* compatibility conditions *is that the boundary data $f_m(t)$, $m = 0, 1$, and the initial condition $q_0(x)$ both satisfy the boundary conditions (1.2.4) at the origin $(x, t) = (0, 0)$,* i.e.,

  $$f_0(0) = a_{11}q_0(x_l) + a_{12}q_0'(x_l) + b_{11}q_0(x_r) + b_{12}q_0'(x_r), \tag{1.2.9a}$$
  $$f_1(0) = a_{21}q_0(x_l) + a_{22}q_0'(x_l) + b_{21}q_0(x_r) + b_{22}q_0'(x_r). \tag{1.2.9b}$$

  For irregular problems, *if q_0 is differentiable and these are satisfied, then (1.2.8) are also satisfied. Alternatively if q_0 is not differentiable, we could impose*

  $$f_0(0) = a_{11}q_0(x_l) + b_{11}q_0(x_r), \tag{1.2.10a}$$
  $$f_1(0) = a_{21}q_0(x_l) + b_{21}q_0(x_r), \tag{1.2.10b}$$

  which also satisfy (1.2.8).

Remark 6. *Non-zero terms are denoted by underline in this remark. Further, row reduction and the fact that the order of equations (1.2.4) is irrelevant is used. For Boundary Case:*

1. *If $(a:b)_{2,4} \neq 0$, the most general form of the matrix $(a:b)$ in (1.2.5) is*

   $$(a:b) = \begin{pmatrix} a_{11} & \underline{a_{12}} & b_{11} & 0 \\ a_{21} & 0 & b_{21} & \underline{b_{22}} \end{pmatrix}.$$

   This case includes the classical Neumann and Robin boundary conditions at both boundaries. We refer to these as Robin-type boundary conditions. *In the case of constant coefficients, this is Birkhoff regular [17].*

2. *If $(a:b)_{2,4} = 0$ and $m_{\mathfrak{c}_0} \neq 0$, the most general form of the matrix $(a:b)$ in (1.2.5) is*

   $$(a:b) = \begin{pmatrix} a_{11} & \underline{a_{12}} & 0 & 0 \\ a_{21} & 0 & \underline{b_{21}} & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & b_{11} & \underline{b_{12}} \\ \underline{a_{21}} & 0 & b_{21} & 0 \end{pmatrix}, \quad \begin{pmatrix} \underline{a_{11}} & 0 & 0 & 0 \\ 0 & a_{22} & b_{21} & \underline{b_{22}} \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & \underline{b_{11}} & 0 \\ a_{21} & \underline{a_{22}} & 0 & b_{22} \end{pmatrix};$$

   or

   $$(a:b) = \begin{pmatrix} \underline{a_{11}} & 0 & \underline{b_{11}} & 0 \\ 0 & \underline{a_{22}} & b_{21} & \underline{b_{22}} \end{pmatrix}, \qquad \text{where} \qquad \frac{\underline{a_{11}}\,\underline{b_{22}}}{\mu(x_l)} + \frac{\underline{a_{22}}\,\underline{b_{11}}}{\mu(x_r)} \neq 0.$$

   This case includes a Robin boundary condition on the left (or right) and a Dirichlet boundary condition on the right (or left). It also includes the classical periodic 'boundary conditions'. These are referred to as mixed-type or periodic-type boundary conditions. *In the case of constant coefficients, these are Birkhoff regular [17].*

3. *If $(a:b)_{2,4}=0$, $m_{\mathfrak{c}_0}=0$, $m_{\mathfrak{c}_1}=0$, and $(a:b)_{1,3}\neq 0$, the most general form of the matrix $(a:b)$ in (1.2.5) is*
$$(a:b)=\begin{pmatrix}\underline{a_{11}} & 0 & 0 & b_{12}\\ 0 & 0 & \underline{b_{21}} & 0\end{pmatrix}\qquad or \qquad (a:b)=\begin{pmatrix}\underline{a_{11}} & 0 & 0 & 0\\ 0 & a_{22} & \underline{b_{21}} & 0\end{pmatrix}.$$
This case includes the case of the classical Dirichlet boundary conditions on both ends. These are referred to as Dirichlet-type boundary conditions. *In the case of constant coefficients, this is Birkhoff regular for the case of Dirichlet boundary conditions* (i.e., *if $a_{22}=0=b_{12}$ or, equivalently, if $(a:b)_{1,2}=0=(a:b)_{3,4}$) and is Birkhoff irregular if $a_{22}\neq 0$ or $b_{12}\neq 0$ (or, equivalently, if $(a:b)_{1,2}\neq 0$ or $(a:b)_{3,4}\neq 0$) [17].*

4. *If $(a:b)_{2,4}=0$, $m_{\mathfrak{c}_0}=0$, $m_{\mathfrak{c}_1}\neq 0$, and $m_{\mathfrak{c}_1}\mathfrak{u}_+-8m_{\mathfrak{s}}\neq 0$, the most general form of the matrix $(a:b)$ is*
$$(a:b)=\begin{pmatrix}\underline{a_{11}} & 0 & \underline{b_{11}} & 0\\ a_{21} & \underline{a_{22}} & 0 & \underline{b_{22}}\end{pmatrix},\qquad where\qquad \frac{\underline{a_{11}}\,\underline{b_{22}}}{\mu(x_l)}+\frac{\underline{a_{22}}\,\underline{b_{11}}}{\mu(x_r)}=0\qquad and\qquad a_{21}\neq-\frac{1}{4}\mu(x_l)\underline{a_{22}}\mathfrak{u}_+.$$
This case does not include any classical boundary conditions. Instead, it is an interface problem on a circle. In the case of constant coefficients, this is Birkhoff irregular [17].

We now introduce some common notations and definitions that the solution formulas given in Chapter 2 require.

Definition 7. *For $x\in\overline{\mathcal{D}}$, since $\alpha(x)$ and $\beta(x)$ are continuous by Assumption 2.2, the arguments of $\alpha(x)$ and $\beta(x)$ are defined to be $\theta_\alpha(x)$ and $\theta_\beta(x)$, chosen to be continuous*[2]*, so that*
$$\alpha(x)=|\alpha(x)|e^{i\theta_\alpha(x)}\qquad and \qquad \beta(x)=|\beta(x)|e^{i\theta_\beta(x)}. \tag{1.2.11}$$
Using this, the branch cuts for $\mu(x)$ (1.2.6), for $x\in\overline{\mathcal{D}}$, are defined as in
$$\mu(x)=\frac{1}{\sqrt{|\alpha(x)\beta(x)|}}e^{-\frac{i}{2}(\theta_\alpha(x)+\theta_\beta(x))}, \tag{1.2.12}$$
and, for $x\in\overline{\mathcal{D}}$ and $k\in\mathbb{C}$, define
$$\mathfrak{g}(k,x)=\sqrt{1+\frac{\gamma(x)}{k^2}}=\sqrt{\left|1+\frac{\gamma(x)}{k^2}\right|}e^{\frac{i}{2}\arg(1+\gamma(x)/k^2)}. \tag{1.2.13}$$
Note that the branch cut of $\mathfrak{g}(k,x)$ is shown in Figure 1.1. Also define $\mathfrak{n}(k,x)=\mu(x)\mathfrak{g}(k,x)$, $(\beta\mu)(x)=\beta(x)\mu(x)$, and $(\beta\mathfrak{n})(k,x)=\beta(x)\mathfrak{n}(k,x)$,
$$\sqrt{(\beta\mu)(x)}=\sqrt[4]{\left|\frac{\beta(x)}{\alpha(x)}\right|}e^{\frac{i}{4}(\theta_\beta(x)-\theta_\alpha(x))},\qquad \sqrt{\mathfrak{g}(k,x)}=\sqrt[4]{\left|1+\frac{\gamma(x)}{k^2}\right|}e^{\frac{i}{4}\arg(1+\gamma(x)/k^2)}, \tag{1.2.14}$$
and $\sqrt{(\beta\mathfrak{n})(k,x)}=\sqrt{(\beta\mu)(x)}\sqrt{\mathfrak{g}(k,x)}$. For any function $g(s)$ or $g(x,s)$ for $s\in[0,T]$ (and $x\in\overline{\mathcal{D}}$), define the linear transform
$$\mathcal{G}[g](k^2,t)=\int_0^t g(s)e^{k^2s}\,ds\qquad or\qquad \mathcal{G}[g](k^2,x,t)=\int_0^t g(x,s)e^{k^2s}\,ds. \tag{1.2.15}$$
Denote
$$q_\alpha(x)=\frac{q_0(x)}{\alpha(x)},\qquad f_\alpha(x,t)=\frac{f(x,t)}{\alpha(x)},\qquad \tilde{f}_\alpha(k^2,x,t)=\mathcal{G}[f_\alpha](k^2,x,t), \tag{1.2.16}$$
and $\psi_\alpha(k^2,x,t)=q_\alpha(x)+\tilde{f}_\alpha(k^2,x,t)$. Finally, define
$$\mathcal{D}_n^{(a,b)}=\{\mathbf{y}_{n+2}\in(a,b)^{n+2}:a=y_0<y_1<\cdots<y_n<y_{n+1}=b\}, \tag{1.2.17}$$
and the functions $\mathcal{E}_0^{(a,b)}(k)=1$, $\tilde{\mathcal{E}}_0^{(a,b)}(k)=1$, and for $n\geq 1$,
$$\mathcal{E}_n^{(a,b)}(k)=\frac{1}{2^n}\int_{\mathcal{D}_n^{(a,b)}}\left(\prod_{p=1}^n\frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)}\right)\exp\left(ik\sum_{p=0}^n(1-(-1)^{n-p})\int_{y_p}^{y_{p+1}}\mathfrak{n}(k,\xi)\,d\xi\right)d\mathbf{y}_n, \tag{1.2.18a}$$
$$\tilde{\mathcal{E}}_n^{(a,b)}(k)=\frac{1}{2^n}\int_{\mathcal{D}_n^{(a,b)}}\left(\prod_{p=1}^n\frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)}\right)\exp\left(ik\sum_{p=0}^n(1-(-1)^{p})\int_{y_p}^{y_{p+1}}\mathfrak{n}(k,\xi)\,d\xi\right)d\mathbf{y}_n, \tag{1.2.18b}$$

[2]Note that not necessarily $\theta_\alpha(x)=\arg(\alpha(x))$, given how the range of $\arg(\cdot)$ is defined above, because of the continuity requirement. For instance, if $\alpha(x)=\exp(ix)$ (and say $\beta(x)=\exp(-ix)$), $\theta_\alpha(x)=x\neq\arg(\alpha(x))$ for $x\notin[-\pi/2,3\pi/2)$.

where $d\mathbf{y}_n = dy_1 \cdots dy_n$ *and the prime denotes the derivative with respect to the second variable. Similarly, define* $\mathcal{C}_0^{(a,b)}(k) = 1$, $\mathcal{S}_0^{(a,b)}(k) = 0$, *and for* $n \geq 1$,

$$\mathcal{C}_n^{(a,b)}(k) = \frac{1}{2^n} \int_{\mathcal{D}_n^{(a,b)}} \left(\prod_{p=1}^{n} \frac{(\beta\mathfrak{n})'(k, y_p)}{(\beta\mathfrak{n})(k, y_p)} \right) \cos \left(k \sum_{p=0}^{n} (-1)^p \int_{y_p}^{y_{p+1}} \mathfrak{n}(k, \xi)\, d\xi \right) d\mathbf{y}_n, \tag{1.2.19a}$$

$$\mathcal{S}_n^{(a,b)}(k) = \frac{1}{2^n} \int_{\mathcal{D}_n^{(a,b)}} \left(\prod_{p=1}^{n} \frac{(\beta\mathfrak{n})'(k, y_p)}{(\beta\mathfrak{n})(k, y_p)} \right) \sin \left(k \sum_{p=0}^{n} (-1)^p \int_{y_p}^{y_{p+1}} \mathfrak{n}(k, \xi)\, d\xi \right) d\mathbf{y}_n. \tag{1.2.19b}$$

2. Solution statements

2.1 The whole-line problem

Consider (1.2.1) for $x \in \mathcal{D} = \mathbb{R}$ and with decay at infinity,

$$q_t = \alpha(x)\left(\beta(x)q_x\right)_x + \gamma(x)q + f(x,t), \qquad x \in \mathbb{R}, \quad t > 0, \tag{2.1.1a}$$

$$q(x,0) = q_0(x), \qquad x \in \mathbb{R}, \tag{2.1.1b}$$

$$\lim_{|x|\to\infty} q(x,t) = 0, \qquad t > 0. \tag{2.1.1c}$$

Theorem 8. *Under Assumptions 2 and 3, the IVP (2.1.1) has the solution*

$$q(x,t) = \frac{1}{2\pi}\int_{\partial\Omega} \frac{\Phi(k,x,t)}{\Delta(k)} e^{-k^2 t}\, dk, \tag{2.1.2}$$

where Ω is shown in Figure 1.1. Here

$$\Phi(k,x,t) = \int_{-\infty}^{\infty} \frac{\Psi(k,x,y)\psi_\alpha(k^2,y,t)}{\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,y)}}\, dy \qquad \text{and} \qquad \Delta(k) = \sum_{n=0}^{\infty} \mathcal{E}_{2n}^{(-\infty,\infty)}(k), \tag{2.1.3}$$

with, for $y < x$,

$$\Psi(k,x,y) = \exp\left(ik\int_y^x \mathfrak{n}(k,\xi)\, d\xi\right) \sum_{n=0}^{\infty}\sum_{\ell=0}^{n} (-1)^\ell \tilde{\mathcal{E}}_{n-\ell}^{(-\infty,y)}(k)\mathcal{E}_\ell^{(x,\infty)}(k), \tag{2.1.4}$$

and for $y > x$, $\Psi(k,x,y) = \Psi(k,y,x)$. The functions $\psi_\alpha(k^2,y,t)$, $\mathfrak{n}(k,x)$, $(\beta\mathfrak{n})(k,x)$, $\mathcal{E}_n^{(a,b)}(k)$, and $\tilde{\mathcal{E}}_n^{(a,b)}(k)$ are defined in Definition 7. The function $\mathcal{E}_n^{(a,b)}(k)$ is defined for $b = \infty$ and if n is even, for $a = -\infty$. The function $\tilde{\mathcal{E}}_n^{(a,b)}(k)$ is defined for $a = -\infty$ and if n is even, for $b = \infty$.

Proof. The formal derivation is given in Chapter 3, and its validity is proven in Chapters 4 and 5. □

2.1.1 Example: The partially lumped heat equation

Consider the heat equation with partial lumping analysis [21], describing the temperature $T(x,t)$ in a body with minimal temperature variation in the y and z directions with ambient temperature T_∞, heat transfer coefficient h_0,

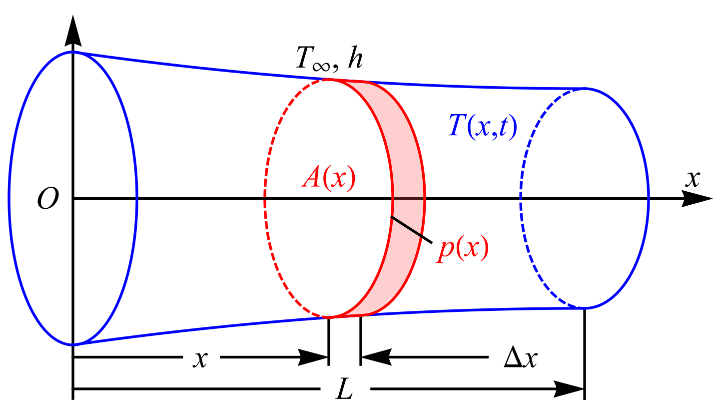


Figure 2.1: Terminology for the derivation of the partially lumped heat equation [21].

thermal conductivity k_0, cross-sectional area $A(x)$, and perimeter $p(x)$, see Figure 2.1. Assume the length L is much greater than the width in the y and z directions. Ignoring temperature deviations in the y and z-directions, this IBVP takes the form

$$\theta_t = \frac{1}{A(x)}\big(A(x)\theta_x\big)_x - C(x)\theta, \qquad x \in \mathbb{R}, \quad t > 0, \tag{2.1.5a}$$

$$\theta(x,0) = \theta_0(x), \qquad x \in \mathbb{R}, \tag{2.1.5b}$$

$$\lim_{|x|\to\infty} \theta(x,t) = 0, \qquad t > 0. \tag{2.1.5c}$$

Here $\theta(x,t) = T(x,t) - T_\infty$ represents the difference of the temperature in the body $T(x,t)$ and the ambient temperature T_∞, the function $C(x) = h_0 p(x)/(k_0 A(x)) > 0$, and the thermal diffusivity is set to 1. Comparing this to (2.1.1), $\alpha(x) = 1/A(x)$, $\beta(x) = A(x)$, $\gamma(x) = -C(x)$, and $f(x,t) \equiv 0$. Requiring the absolute continuity of $A(x) > 0$, the boundedness of $C(x)$, and the absolute integrability of $A'(x)/A(x)$ and $C'(x)$ guarantees Assumption 2 is satisfied. Then the solution is given as

$$\theta(x,t) = \frac{1}{2\pi}\int_{\partial\Omega} \frac{\Phi(k,x,t)}{\Delta(k)} e^{-k^2 t}\, dk, \tag{2.1.6}$$

where Ω is shown in Figure 1.1,

$$\mathfrak{n}(k,x) = \sqrt{1 - \frac{C(x)}{k^2}}, \tag{2.1.7}$$

and $\psi_\alpha(k^2,x,t) = A(x)\theta_0(x)$. The functions $\Phi(k,x,t)$ and $\Delta(k)$ are given in (2.1.3).

2.1.2 A note about the integrability conditions.

A variable coefficient PDE in the form

$$q_t = a(x)q_{xx} + b(x)q_x + c(x)q, \tag{2.1.8}$$

can always be written in the form of (2.1.1a) as

$$q_t = a(x)\exp\left(-\int_{x_0}^x \frac{b(y)}{a(y)}\,dy\right)\left[\exp\left(\int_{x_0}^x \frac{b(y)}{a(y)}\,dy\right) q_x\right]_x + c(x)q, \tag{2.1.9}$$

which gives

$$\alpha(x) = a(x)\exp\left(-\int_{x_0}^x \frac{b(y)}{a(y)}\,dy\right), \qquad \beta(x) = \exp\left(\int_{x_0}^x \frac{b(y)}{a(y)}\,dy\right), \qquad \text{and} \qquad \gamma(x) = c(x). \tag{2.1.10}$$

From this,

$$\frac{(\beta\mathfrak{n})'(k,x)}{(\beta\mathfrak{n})(k,x)} = \frac{1}{2}\left(\frac{\beta'(x)}{\beta(x)} - \frac{\alpha'(x)}{\alpha(x)} + \frac{\gamma'(x)}{k^2+\gamma(x)}\right) = \frac{1}{2}\left(\frac{2b(x)}{a(x)} - \frac{a'(x)}{a(x)} + \frac{c'(x)}{k^2+c(x)}\right), \tag{2.1.11}$$

which is not integrable (over an infinite or semi-infinite domain) if a, b, c are constants with $ab \neq 0$. This presents a problem for the solution (2.1.2). However, making the change of variables,

$$q(x,t) = \exp\left(-\int_{x_0}^x \frac{b(y)}{2a(y)}\,dy\right) u(x,t), \tag{2.1.12}$$

the PDE becomes

$$u_t = a(x)u_{xx} + \left(\frac{a'(x)b(x) - a(x)b'(x)}{2a(x)} - \frac{b(x)^2}{4a(x)} + c(x)\right)u, \tag{2.1.13}$$

for which,

$$\alpha(x) = a(x), \qquad \beta(x) = 1, \qquad \gamma(x) = \frac{a'(x)b(x) - a(x)b'(x)}{2a(x)} - \frac{b(x)^2}{4a(x)} + c(x), \tag{2.1.14}$$

and

$$\frac{(\beta\mathfrak{n})'(k,x)}{(\beta\mathfrak{n})(k,x)} = \frac{1}{2}\left(-\frac{a'(x)}{a(x)} + \frac{\gamma'(x)}{k^2+\gamma(x)}\right). \tag{2.1.15}$$

In the case of constant coefficients, the integrability condition, Assumption 2.5, is satisfied and the solution (2.1.2) is well defined.

2.1.2.1 Example: The constant-coefficient, advected heat equation

Consider the constant-coefficient IBVP

$$q_t = q_{xx} + cq_x, \qquad x \in \mathbb{R}, \quad t > 0, \tag{2.1.16a}$$

$$q(x,0) = q_0(x), \qquad x \in \mathbb{R}, \tag{2.1.16b}$$

$$\lim_{|x|\to\infty} q(x,t) = 0, \qquad t > 0. \tag{2.1.16c}$$

This problem is well posed for $c \in \mathbb{R}$ [14]. The PDE (2.1.16a) can be written in the form (2.1.1a) as

$$q_t = e^{-cx}\left(e^{cx} q_x\right)_x, \tag{2.1.17}$$

with $\alpha(x) = e^{-cx}$, $\beta(x) = e^{cx}$, and $\gamma(x) = 0$. Since $\beta'/\beta - \alpha'/\alpha = 2c$ is not absolutely integrable over the real line, and Assumption 2.5 is not satisfied. With the change of variables $q(x,t) = e^{-cx/2}u(x,t)$, the IBVP (2.1.16) becomes

$$u_t = u_{xx} - \frac{c^2}{4}u, \qquad x \in \mathbb{R}, \quad t > 0, \tag{2.1.18a}$$

$$u(x,0) = e^{cx/2} q_0(x), \qquad x \in \mathbb{R}, \tag{2.1.18b}$$

$$\lim_{|x|\to\infty} u(x,t) = 0, \qquad t > 0. \tag{2.1.18c}$$

Now $\alpha(x) = 1$, $\beta(x) = 1$, and $\gamma(x) = -c^2/4$, so that $\beta'/\beta - \alpha'/\alpha = 0$ and $\gamma' = 0$, and Assumption 2 is satisfied. This example shows that, although all evolution equations can be written in the form (2.1.1a), a transformation may be needed before the integrability conditions are met and the solution expression (2.1.2) applies.

2.2 The half-line problem

Consider (1.2.1) on the half line $x \in \mathcal{D} = (x_l, \infty)$ with a Robin boundary condition at $x = x_l$ and decay at infinity,

$$q_t = \alpha(x)\left(\beta(x) q_x\right)_x + \gamma(x) q + f(x,t), \qquad x > x_l, \quad t > 0, \tag{2.2.1a}$$

$$q(x,0) = q_0(x), \qquad x > x_l, \tag{2.2.1b}$$

$$f_0(t) = a_0 q(x_l,t) + a_1 q_x(x_l,t), \qquad t > 0, \tag{2.2.1c}$$

$$\lim_{x\to\infty} q(x,t) = 0, \qquad t > 0, \tag{2.2.1d}$$

with $(a_0, a_1) \neq (0,0)$.

Theorem 9. *Under Assumptions 2 and 3, the IBVP (2.2.1) has the solution*

$$q(x,t) = \frac{1}{2\pi}\int_{\partial\Omega} \frac{\Phi(k,x,t)}{\Delta(k)} e^{-k^2 t}\, dk, \tag{2.2.2}$$

where Ω is shown in Figure 1.1. Here

$$\Delta(k) = 2\sum_{n=0}^{\infty}\left(\frac{(-1)^n i a_0}{k\mathfrak{n}(k,x_l)} - a_1\right)\mathcal{E}_n^{(x_l,\infty)}(k), \tag{2.2.3}$$

and

$$\Phi(k,x,t) = \mathcal{B}_0(k,x) F_0(k^2,t) + \Phi_\psi(k,x,t). \tag{2.2.4}$$

The boundary term $\mathcal{B}_0(k,x)$ is defined by

$$\mathcal{B}_0(k,x) = \frac{4\beta(x_l)\exp\left(ik\int_{x_l}^{x}\mathfrak{n}(k,\xi)\,d\xi\right)}{\sqrt{(\beta\mathfrak{n})(k,x_l)}\sqrt{(\beta\mathfrak{n})(k,x)}}\sum_{n=0}^{\infty}(-1)^n\mathcal{E}_n^{(x,\infty)}(k), \tag{2.2.5}$$

and

$$\Phi_\psi(k,x,t) = \int_{x_l}^{\infty}\frac{\Psi(k,x,y)\psi_\alpha(k^2,y,t)}{\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,y)}}\, dy. \tag{2.2.6}$$

For fully dissipative *problems,*

$$F_m(k^2,t) = \mathcal{G}[f_m](k^2,t), \qquad m = 0, 1, \tag{2.2.7a}$$

and, for partially dissipative *problems,*

$$F_m(k^2,t) = -\frac{1}{k^2}\big(f_m(0) + \mathcal{G}[f_m'](k^2,t)\big), \qquad m = 0, 1. \tag{2.2.7b}$$

Note that we only need $F_0(k^2,t)$ for the half-line problem, but we will use $F_1(k^2,t)$ in the finite-interval problem in Section 2.3. For $x_l < y < x$,

$$\Psi(k,x,y) = 4\exp\left(ik\int_{x_l}^{x}\mathfrak{n}(k,\xi)\,d\xi\right)\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}(-1)^{\ell}\left(\frac{a_0}{k\mathfrak{n}(k,x_l)}\mathcal{S}_{n-\ell}^{(x_l,y)}(k) - a_1\mathcal{C}_{n-\ell}^{(x_l,y)}(k)\right)\mathcal{E}_{\ell}^{(x,\infty)}(k), \tag{2.2.8}$$

and $\Psi(k,x,y) = \Psi(k,y,x)$ for $x_l < x < y$. The functions $\mathfrak{n}(k,x)$, $(\beta\mathfrak{n})(k,x)$, $\psi_\alpha(k^2,x,t)$, $\mathcal{E}_n^{(a,b)}(k)$, $\mathcal{C}_n^{(a,b)}(k)$, and $\mathcal{S}_n^{(a,b)}(k)$ are defined in Definition 7.

2.2.1 Example: The advected heat equation

Consider the advected heat equation on the half line with spatially variable thermal conductivity $\sigma^2(x) > 0$ and velocity $c(x)$, without forcing and with homogeneous Dirichlet boundary conditions, i.e.,

$$q_t = \big(\sigma^2(x)q_x\big)_x - c(x)q_x, \qquad x > 0, \quad t > 0, \tag{2.2.9a}$$

$$q(x,0) = q_0(x), \qquad x > 0, \tag{2.2.9b}$$

$$q(0,t) = 0, \qquad t > 0, \tag{2.2.9c}$$

$$\lim_{x\to\infty} q(x,t) = 0, \qquad t > 0. \tag{2.2.9d}$$

Here $x_l = 0$, $a_0 = 1$ and $a_1 = 0$. Further,

$$\alpha(x) = \exp\left(\int_0^x \frac{c(\xi)}{\sigma^2(\xi)}\,d\xi\right), \qquad \beta(x) = \sigma^2(x)\exp\left(-\int_0^x \frac{c(\xi)}{\sigma^2(\xi)}\,d\xi\right), \qquad \gamma(x) = 0, \tag{2.2.10}$$

$f(x,t) = 0$, and $f_0(t) = 0$. The absolute continuity of $\sigma(x)$ and boundedness of $c(x)$ is required, and since

$$\frac{\beta'(x)}{\beta'(x)} - \frac{\alpha'(x)}{\alpha(x)} = \frac{2\sigma'(x)}{\sigma(x)} - \frac{2c(x)}{\sigma^2(x)}, \tag{2.2.11}$$

the absolute integrability of $\sigma'(x)/\sigma(x)$ and $c(x)$ are also required, so that Assumption 2 is satisfied. Note that if $\sigma(x)$ is absolutely continuous and $\sigma'(x)/\sigma(x)$ is absolutely integrable, then $\sigma(x)$ is bounded above and below. This problem has the solution (2.2.2), where Ω is shown in Figure 1.1, $\mathfrak{n}(k,x) = 1/\sigma(x)$,

$$k\mathfrak{n}(k,0)\Delta(k) = 2i\sum_{n=0}^{\infty}(-1)^n\mathcal{E}_n^{(0,\infty)}(k). \tag{2.2.12}$$

Since $\mathcal{B}_0(k,x,t) = 0$ and $\psi(k^2,y,t) = q_0(y)$,

$$\Phi(k,x,t) = \int_0^{\infty}\exp\left(\frac{1}{2}\int_y^x \frac{c(\xi)}{\sigma^2(\xi)}\,d\xi\right)\frac{\Psi(k,x,y)q_0(y)}{\sqrt{\sigma(x)\sigma(y)}}\,dy, \tag{2.2.13}$$

and for $0 < y < x$,

$$k\mathfrak{n}(k,0)\Psi(k,x,y) = 4\exp\left(ik\int_0^x \frac{d\xi}{\sigma(\xi)}\right)\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}(-1)^{\ell}\mathcal{S}_{n-\ell}^{(0,y)}(k)\mathcal{E}_{\ell}^{(x,\infty)}(k), \tag{2.2.14}$$

and $\Psi(k,x,y) = \Psi(k,y,x)$ for $0 < x < y$.

2.3 The finite-interval problem

Theorem 10. *Consider* (1.2.1) *for* $x \in \mathcal{D} = (x_l, x_r)$ *and with the general two-point boundary condition* (1.2.4),

$$\begin{aligned} q_t &= \alpha(x)\left(\beta(x)q_x\right)_x + \gamma(x)q + f(x,t), && x \in (x_l,x_r), \quad t>0, && \text{(2.3.1a)}\\ q(x,0) &= q_0(x), && x\in(x_l,x_r), && \text{(2.3.1b)}\\ f_0(t) &= a_{11}q(x_l,t) + a_{12}q_x(x_l,t) + b_{11}q(x_r,t) + b_{12}q_x(x_r,t), && t>0, && \text{(2.3.1c)}\\ f_1(t) &= a_{21}q(x_l,t) + a_{22}q_x(x_l,t) + b_{21}q(x_r,t) + b_{22}q_x(x_r,t), && t>0. && \text{(2.3.1d)} \end{aligned}$$

Under Assumptions 2 and 3, the IBVP (2.3.1) has the solution

$$q(x,t) = \frac{1}{2\pi}\int_{\partial\Omega} \frac{\Phi(k,x,t)}{\Delta(k)} e^{-k^2 t}\, dk, \tag{2.3.2}$$

where Ω *is shown in Figure 1.1. Define*

$$\Xi(k) = \exp\left(ik \int_{x_l}^{x_r} \mathfrak{n}(k,\xi)\, d\xi \right), \tag{2.3.3}$$

where $\mathfrak{n}(k,x)$ *is defined in Definition 7. Then*

$$\Delta(k) = 2i\,\Xi(k)\left(\mathfrak{a}(k) + \sum_{n=0}^{\infty} \mathfrak{c}_n(k)\mathcal{C}_n^{(x_l,x_r)}(k) + \sum_{n=0}^{\infty}\mathfrak{s}_n(k)\mathcal{S}_n^{(x_l,x_r)}(k) \right), \tag{2.3.4}$$

with

$$\begin{aligned} \mathfrak{a}(k) &= \frac{\beta(x_r)(a:b)_{1,2} + \beta(x_l)(a:b)_{3,4}}{k\sqrt{(\beta\mathfrak{n})(k,x_l)}\sqrt{(\beta\mathfrak{n})(k,x_r)}}, && \text{(2.3.5a)}\\ \mathfrak{c}_n(k) &= (-1)^n \frac{(a:b)_{1,4}}{k\mathfrak{n}(k,x_l)} - \frac{(a:b)_{2,3}}{k\mathfrak{n}(k,x_r)}, && \text{(2.3.5b)}\\ \mathfrak{s}_n(k) &= (-1)^n (a:b)_{2,4} + \frac{(a:b)_{1,3}}{k^2\mathfrak{n}(k,x_l)\mathfrak{n}(k,x_r)}. && \text{(2.3.5c)} \end{aligned}$$

The numerator of (2.3.2) is

$$\Phi(k,x,t) = \mathcal{B}_0(k,x)F_0(k^2,t) + \mathcal{B}_1(k,x)F_1(k^2,t) + \Phi_\psi(k,x,t), \tag{2.3.6a}$$

where

$$\Phi_\psi(k,x,t) = \int_{x_l}^{x_r} \frac{\Psi(k,x,y)\psi_\alpha(k^2,y,t)}{\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,y)}}\, dy. \tag{2.3.6b}$$

The functions $\psi_\alpha(k^2,x,t)$ *and* $(\beta\mathfrak{n})(k,x)$ *are defined in Definition 7,* $F_m(k^2,t)$ *is defined in* (2.2.7)*, and the boundary terms* $\mathcal{B}_0(k,x)$ *and* $\mathcal{B}_1(k,x)$ *are given by*

$$\begin{aligned} \mathcal{B}_{2-j}(k,x) = (-1)^j \frac{4\Xi(k)}{\sqrt{(\beta\mathfrak{n})(k,x)}} \Bigg\{ & \frac{\beta(x_r)}{\sqrt{(\beta\mathfrak{n})(k,x_r)}} \left[-\frac{a_{j1}}{k\mathfrak{n}(k,x_l)} \sum_{n=0}^{\infty} \mathcal{S}_n^{(x_l,x)}(k) + a_{j2}\sum_{n=0}^{\infty}\mathcal{C}_n^{(x_l,x)}(k) \right] \\ & + \frac{\beta(x_l)}{\sqrt{(\beta\mathfrak{n})(k,x_l)}} \left[\frac{b_{j1}}{k\mathfrak{n}(k,x_r)} \sum_{n=0}^{\infty}\mathcal{S}_n^{(x,x_r)}(k) + b_{j2}\sum_{n=0}^{\infty}(-1)^n \mathcal{C}_n^{(x,x_r)}(k) \right] \Bigg\}, \end{aligned} \tag{2.3.6c}$$

with j=1,2. Further, for $x_l < y < x < x_r$,

$$\begin{aligned} \Psi(k,x,y) = 4\Xi(k) \Bigg\{ & -(a:b)_{2,4} \sum_{n=0}^{\infty}\sum_{\ell=0}^{n} (-1)^\ell \mathcal{C}_{n-\ell}^{(x_l,y)}(k)\mathcal{C}_\ell^{(x,x_r)}(k) + \frac{(a:b)_{1,3}}{k^2\mathfrak{n}(k,x_l)\mathfrak{n}(k,x_r)} \sum_{n=0}^{\infty}\sum_{\ell=0}^{n} \mathcal{S}_{n-\ell}^{(x_l,y)}(k)\mathcal{S}_\ell^{(x,x_r)}(k) \\ & + \frac{(a:b)_{1,4}}{k\mathfrak{n}(k,x_l)} \sum_{n=0}^{\infty}\sum_{\ell=0}^{n} (-1)^\ell \mathcal{S}_{n-\ell}^{(x_l,y)}(k)\mathcal{C}_\ell^{(x,x_r)}(k) - \frac{(a:b)_{2,3}}{k\mathfrak{n}(k,x_r)} \sum_{n=0}^{\infty}\sum_{\ell=0}^{n} \mathcal{C}_{n-\ell}^{(x_l,y)}(k)\mathcal{S}_\ell^{(x,x_r)}(k) \\ & - \frac{\beta(x_r)(a:b)_{1,2}}{k\sqrt{(\beta\mathfrak{n})(k,x_l)}\sqrt{(\beta\mathfrak{n})(k,x_r)}} \sum_{n=0}^{\infty} \mathcal{S}_n^{(y,x)}(k) \Bigg\}, \end{aligned} \tag{2.3.7a}$$

and, for $x_l < x < y < x_r$,

$$\begin{aligned}\Psi(k,x,y) = 4\Xi(k)\Bigg\{&-(a:b)_{2,4}\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}(-1)^{\ell}\mathcal{C}_{n-\ell}^{(x_l,x)}(k)\mathcal{C}_{\ell}^{(y,x_r)}(k) + \frac{(a:b)_{1,3}}{k^2\mathfrak{n}(k,x_l)\mathfrak{n}(k,x_r)}\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}\mathcal{S}_{n-\ell}^{(x_l,x)}(k)\mathcal{S}_{\ell}^{(y,x_r)}(k)\\ &+\frac{(a:b)_{1,4}}{k\mathfrak{n}(k,x_l)}\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}(-1)^{\ell}\mathcal{S}_{n-\ell}^{(x_l,x)}(k)\mathcal{C}_{\ell}^{(y,x_r)}(k) - \frac{(a:b)_{2,3}}{k\mathfrak{n}(k,x_r)}\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}\mathcal{C}_{n-\ell}^{(x_l,x)}(k)\mathcal{S}_{\ell}^{(y,x_r)}(k)\\ &-\frac{\beta(x_l)(a:b)_{3,4}}{k\sqrt{(\beta\mathfrak{n})(k,x_l)}\sqrt{(\beta\mathfrak{n})(k,x_r)}}\sum_{n=0}^{\infty}\mathcal{S}_{n}^{(x,y)}(k)\Bigg\}.\end{aligned} \tag{2.3.7b}$$

Note that $\Psi(k,x,y) \neq \Psi(k,y,x)$ *unless* $\beta(x_r)(a:b)_{1,2} = \beta(x_l)(a:b)_{3,4}$. *The functions* $\mathcal{C}_n^{(a,b)}(k)$ *and* $\mathcal{S}_n^{(a,b)}(k)$ *are defined in* (1.2.19).

2.3.1 Example: The heat equation with homogeneous, Dirichlet boundary conditions

Consider the heat equation on the finite interval with spatially varying thermal conductivity $\sigma^2(x)$ without forcing and with homogeneous Dirichlet boundary conditions, *i.e.*,

$$q_t = \left(\sigma^2(x)q_x\right)_x, \qquad x\in(0,1), \quad t>0, \tag{2.3.8a}$$

$$q(x,0) = q_0(x), \qquad x\in(0,1), \tag{2.3.8b}$$

$$q(0,t) = 0, \qquad t>0, \tag{2.3.8c}$$

$$q(1,t) = 0, \qquad t>0. \tag{2.3.8d}$$

We let $x_l = 0$, $x_r = 1$, $\alpha(x) = 1$, $\beta(x) = \sigma^2(x)$, $\gamma(x) = 0$, $f(x,t) = 0$, $f_m(t) = 0$ $(m = 0,1)$, and

$$(a:b) = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}. \tag{2.3.9}$$

Since $(a:b)_{2,4} = 0$, $m_{\mathfrak{c}_0} = 0$, $m_{\mathfrak{c}_1} = 0$, and $(a:b)_{1,3} = 1 \neq 0$, this is an example of the *regular* version of Boundary Case 3. Absolute continuity of $\sigma(x)$, integrability of $q_0(x)$, and absolutely integrability of $\sigma'(x)/\sigma(x)$ are required. This has the solution

$$q(x,t) = \frac{1}{2\pi}\int_{\partial\Omega}\frac{\Phi(k,x)}{\Delta(k)}e^{-k^2t}\,dk, \tag{2.3.10}$$

where Ω is shown in Figure 1.1. Since $\mathfrak{n}(k,x) = 1/\sigma(x)$, $\mathfrak{a}(k) = 0$, $\mathfrak{c}_n(k) = 0$, and $\mathfrak{s}_n(k) = \sigma(0)\sigma(1)/k^2$, then

$$\frac{k^2\Delta(k)}{2i\sigma(0)\sigma(1)} = \exp\left(ik\int_0^1\frac{d\xi}{\sigma(\xi)}\right)\sum_{n=0}^{\infty}\mathcal{S}_n^{(0,1)}(k), \tag{2.3.11}$$

and since $\mathcal{B}_0(k,x) = \mathcal{B}_1(k,x) = 0$ and $\psi_\alpha(k^2,y,t) = q_0(y)$, we have

$$\Phi(k,x) = \int_0^1\frac{\Psi(k,x,y)q_0(y)}{\sqrt{\sigma(x)\sigma(y)}}\,dy, \tag{2.3.12}$$

where, for $0<y<x<1$,

$$\frac{k^2\Psi(k,x,y)}{4\sigma(0)\sigma(1)} = \sum_{n=0}^{\infty}\sum_{\ell=0}^{n}\mathcal{S}_{n-\ell}^{(0,y)}(k)\mathcal{S}_{\ell}^{(x,1)}(k), \tag{2.3.13}$$

and $\Psi(k,x,y) = \Psi(k,y,x)$ for $0<x<y<1$. This is the same solution given in [10]. It reduces to the solution given in [14] for constant $\sigma(x)$.

2.3.1.1 Numerics: The heat equation with homogeneous, Dirichlet boundary conditions

Setting $q_0(x) = x(1-x)$ and $\sigma^2(x) = (3-(2x-1)^2)/24$, we have the exact solution $q(x,t) = x(1-x)e^{-t}$. We construct an approximation to the solution (2.3.10) such that $q_N(x,t)\to q(x,t)$ as $N\to\infty$ (the index N does not denote differentiation):

$$q_N(x,t) = \frac{1}{i\pi}\int_\gamma\frac{\exp\left(ik\int_0^1\frac{d\xi}{\sigma(\xi)}\right)\Phi_N(k,x)}{\exp\left(ik\int_0^1\frac{d\xi}{\sigma(\xi)}\right)\Delta_N(k)}e^{-k^2t}\,dk, \tag{2.3.14}$$

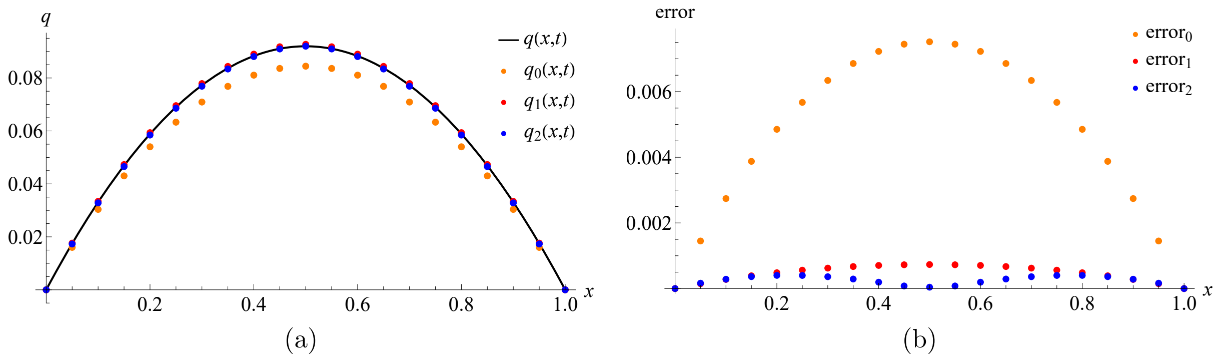


Figure 2.2: (a) The exact solution $q(x,t) = x(1-x)e^{-t}$ and its successive approximations (2.3.14) for $N = 0, 1, 2$. (b) The error of the approximations.

where we multiply denominator and numerator by the exponential so that both are decaying in the upper-half complex k plane, and where we truncate each series up to $n = N$. The contour γ is used instead of $\partial\Omega$ to aid convergence as the factor $\exp(-k^2 t)$ decays along it. The results are shown in Figure 2.2.

2.3.2 Example: The CGL equation with periodic boundary conditions

The complex Ginzburg-Landau (CGL) equation is the nonlinear PDE

$$A_t = (1 + ia(x))A_{xx} + A - (1 + ib(x))|A|^2 A, \tag{2.3.15}$$

where a, b are real functions of x. In the special case $a(x) = 0 = b(x)$, (2.3.15) is the real Ginzburg-Landau equation. If $a(x), b(x) \to \infty$, (2.3.15) becomes the Nonlinear Schrödinger (NLS) equation [2]. Consider the linearized (about $A = 0$), CGL equation with periodic boundary conditions:

$$A_t = (1 + ia(x))A_{xx} + A, \qquad x \in (0,1), \quad t > 0, \tag{2.3.16a}$$
$$A(x,0) = A_0(x), \qquad x \in (0,1), \tag{2.3.16b}$$
$$A(0,t) = A(1,t), \qquad t > 0, \tag{2.3.16c}$$
$$A_x(0,t) = A_x(1,t), \qquad t > 0. \tag{2.3.16d}$$

Here $x_l = 0$, $x_r = 1$, $\alpha(x) = 1 + ia(x)$, $\beta(x) = 1$, $\gamma(x) = 1$, $f(x,t) = 0$, $f_0(t) = 0 = f_1(t)$, and

$$(a : b) = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}. \tag{2.3.17}$$

Assuming $a(x) \in \mathbb{R}$ and $a \in \mathrm{AC}(\mathcal{D})$ satisfies Assumption 2. Here,

$$\mu(x) = \frac{e^{-\frac{i}{2}\arctan(a(x))}}{\sqrt[4]{1 + a(x)^2}} \qquad \text{and} \qquad \mathfrak{g}(k) = \sqrt{1 + \frac{1}{k^2}}, \tag{2.3.18}$$

and $\mathfrak{n}(k,x) = \mu(x)\mathfrak{g}(k)$, where the square root in $\mathfrak{g}(k)$ is defined in (1.2.13). Since $(a : b)_{2,4} = 0$ and $m_{\mathfrak{c}_0} \neq 0$, this is a Boundary Case 2 example, which is *regular*. For simplicity, assume that $a(x)$ is periodic, *i.e.,* $a(0) = a(1)$. This problem has the solution

$$q(x,t) = -\frac{1}{2\pi i}\int_{\partial\Omega} \frac{\tilde{\Phi}(k,x)}{\tilde{\Delta}(k)} e^{-k^2 t}\, dk, \tag{2.3.19}$$

where we define $\tilde{\Delta}(k) = k\mathfrak{n}(k,0)\Xi(-k)\Delta(k)/(4i)$ and $\tilde{\Phi}(k,x) = -k\mathfrak{n}(k,0)\Xi(-k)\Phi(k,x)/4$, and where Ω is shown in Figure 1.1. Here, $\mathfrak{a}(k) = 2/(k\mathfrak{n}(k,0))$, $\mathfrak{c}_n(k) = -(1 + (-1)^n)/(k\mathfrak{n}(k,0))$, $\mathfrak{s}_n(k) = 0$, and since $\mathcal{B}_0(k,x,t) = 0$, $\mathcal{B}_1(k,x,t) = 0$ and $\psi_\alpha(k^2,x,t) = A_0(x)/(1 + ia(x))$,

$$\tilde{\Delta}(k) = 1 - \sum_{n=0}^{\infty} \mathcal{C}_{2n}^{(0,1)}(k) \qquad \text{and} \qquad \tilde{\Phi}(k,x) = \int_0^1 \frac{\tilde{\Psi}(k,x,y)A_0(y)}{(1 + ia(y))\sqrt{\mathfrak{n}(k,x)}\sqrt{\mathfrak{n}(k,y)}}\, dy. \tag{2.3.20}$$

Define $\tilde{\Psi}(k,x,y) = -k\mathfrak{n}(k,0)\Xi(-k)\Psi(k,x)/4$,

$$\tilde{\Psi}(k,x,y) = \sum_{n=0}^{\infty}\sum_{\ell=0}^{n}(-1)^{\ell}\mathcal{S}_{n-\ell}^{(0,y)}(k)\mathcal{C}_{\ell}^{(x,1)}(k) + \sum_{n=0}^{\infty}\sum_{\ell=0}^{n}\mathcal{C}_{n-\ell}^{(0,y)}(k)\mathcal{S}_{\ell}^{(x,1)}(k) + \sum_{n=0}^{\infty}\mathcal{S}_{n}^{(y,x)}(k), \tag{2.3.21}$$

for $0<y<x<1$, and $\tilde{\Psi}(k,x,y) = \tilde{\Psi}(k,y,x)$ for $0<x<y<1$.

2.3.3 Sturm-Liouville Problems: Eigenvalues and Eigenfunctions

Theorem 11. *The Sturm–Liouville problem*

$$\alpha(x)\left(\beta(x)y'\right)' + \gamma(x)y = \lambda y, \tag{2.3.22}$$

with boundary conditions

$$a_{11}y(x_l) + a_{12}y'(x_l) + b_{11}y(x_r) + b_{12}y'(x_r) = 0, \tag{2.3.23a}$$
$$a_{21}y(x_l) + a_{22}y'(x_l) + b_{21}y(x_r) + b_{22}y'(x_r) = 0, \tag{2.3.23b}$$

has the eigenfunctions

$$X_m(x) = \frac{C_m}{\sqrt{(\beta\mathfrak{n})(\kappa_m,x)}}\sum_{n=0}^{\infty}\mathcal{C}_n^{(x_l,x)}(\kappa_m) + \frac{S_m}{\sqrt{(\beta\mathfrak{n})(\kappa_m,x)}}\sum_{n=0}^{\infty}\mathcal{S}_n^{(x_l,x)}(\kappa_m), \tag{2.3.24}$$

corresponding to the eigenvalues $\lambda_m = -\kappa_m^2$, where $\{\kappa_m\}_{m=1}^{\infty}$ are the zeros of $\Delta(k)$ (2.3.4). Here,

$$C_m = -\frac{a_{12}\kappa_m\mathfrak{n}(\kappa_m,x_l)}{\sqrt{(\beta\mathfrak{n})(\kappa_m,x_l)}} - \frac{b_{11}}{\sqrt{(\beta\mathfrak{n})(\kappa_m,x_r)}}\sum_{n=0}^{\infty}\mathcal{S}_n^{(x_l,x_r)}(\kappa_m) - \frac{b_{12}\kappa_m\mathfrak{n}(\kappa_m,x_r)}{\sqrt{(\beta\mathfrak{n})(\kappa_m,x_r)}}\sum_{n=0}^{\infty}(-1)^n\mathcal{C}_n^{(x_l,x_r)}(\kappa_m), \tag{2.3.25a}$$

$$S_m = \frac{a_{11}}{\sqrt{(\beta\mathfrak{n})(\kappa_m,x_l)}} + \frac{b_{11}}{\sqrt{(\beta\mathfrak{n})(\kappa_m,x_r)}}\sum_{n=0}^{\infty}\mathcal{C}_n^{(x_l,x_r)}(\kappa_m) - \frac{b_{12}\kappa_m\mathfrak{n}(\kappa_m,x_r)}{\sqrt{(\beta\mathfrak{n})(\kappa_m,x_r)}}\sum_{n=0}^{\infty}(-1)^n\mathcal{S}_n^{(x_l,x_r)}(\kappa_m). \tag{2.3.25b}$$

Proof. Using (4.2.1) in (2.3.24) gives that the eigenfunctions solve the eigenvalue equation (2.3.22). Inserting (2.3.24) into the boundary conditions (2.3.23a) gives

$$a_{11}X_m(x_l) + a_{12}X_m'(x_l) + b_{11}X_m(x_r) + b_{12}X_m'(x_r) = C_mS_m - S_mC_m = 0. \tag{2.3.26}$$

For (2.3.23b),

$$\begin{aligned}
&a_{21}X_m(x_l) + a_{22}X_m'(x_l) + b_{21}X_m(x_r) + b_{22}X_m'(x_r)\\
&= C_m\left[\frac{a_{21}}{\sqrt{(\beta\mathfrak{n})(\kappa_m,x_l)}} + \frac{b_{21}}{\sqrt{(\beta\mathfrak{n})(\kappa_m,x_r)}}\sum_{n=0}^{\infty}\mathcal{C}_n^{(x_l,x_r)}(\kappa_m) - \frac{b_{22}\kappa_m\mathfrak{n}(\kappa_m,x_r)}{\sqrt{(\beta\mathfrak{n})(\kappa_m,x_r)}}\sum_{n=0}^{\infty}(-1)^n\mathcal{S}_n^{(x_l,x_r)}(\kappa_m)\right]\\
&\quad + S_m\left[\frac{a_{22}\kappa_m\mathfrak{n}(\kappa_m,x_l)}{\sqrt{(\beta\mathfrak{n})(\kappa_m,x_l)}} + \frac{b_{21}}{\sqrt{\beta\mathfrak{n})(\kappa_m,x_r)}}\sum_{n=0}^{\infty}\mathcal{S}_n^{(x_l,x_r)}(\kappa_m) + \frac{b_{22}\kappa_m\mathfrak{n}(\kappa_m,x_r)}{\sqrt{(\beta\mathfrak{n})(\kappa_m,x_r)}}\sum_{n=0}^{\infty}(-1)^n\mathcal{C}_n^{(x_l,x_r)}(\kappa_m)\right].
\end{aligned} \tag{2.3.27}$$

Expanding this,

$$\begin{aligned}
&a_{21}X_m(x_l) + a_{22}X_m'(x_l) + b_{21}X_m(x_r) + b_{22}X_m'(x_r)\\
&= \frac{(a:b)_{1,2}\kappa_m}{\beta(x_l)} + \frac{(a:b)_{3,4}\kappa_m}{\beta(x_r)}\left[\sum_{n=0}^{\infty}(-1)^n\mathcal{C}_n^{(x_l,x_r)}(\kappa_m)\sum_{n=0}^{\infty}\mathcal{C}_n^{(x_l,x_r)}(\kappa_m) + \sum_{n=0}^{\infty}(-1)^n\mathcal{S}_n^{(x_l,x_r)}(\kappa_m)\sum_{n=0}^{\infty}\mathcal{S}_n^{(x_l,x_r)}(\kappa_m)\right]\\
&\quad + \frac{\kappa_m^2\mathfrak{n}(\kappa_m,x_l)\mathfrak{n}(\kappa_m,x_r)}{\sqrt{(\beta\mathfrak{n})(\kappa_m,x_l)}\sqrt{(\beta\mathfrak{n})(\kappa_m,x_r)}}\left[\sum_{n=0}^{\infty}\mathfrak{c}_n(k)\mathcal{C}_n^{(x_l,x_r)}(k) + \sum_{n=0}^{\infty}\mathfrak{s}_n(k)\mathcal{S}_n^{(x_l,x_r)}(k)\right].
\end{aligned} \tag{2.3.28}$$

Using the identity

$$1 = \sum_{n=0}^{\infty}(-1)^n\mathcal{C}_n^{(x_l,x_r)}(k)\sum_{n=0}^{\infty}\mathcal{C}_n^{(x_l,x_r)}(k) + \sum_{n=0}^{\infty}(-1)^n\mathcal{S}_n^{(x_l,x_r)}(k)\sum_{n=0}^{\infty}\mathcal{S}_n^{(x_l,x_r)}(k), \tag{2.3.29}$$

in (2.3.28), this becomes

$$a_{21}X_m(x_l)+a_{22}X_m'(x_l)+b_{21}X_m(x_r)+b_{22}X_m'(x_r)=\frac{\kappa_m^2\mathfrak{n}(\kappa_m,x_l)\mathfrak{n}(\kappa_m,x_r)}{\sqrt{(\beta\mathfrak{n})(\kappa_m,x_l)}\sqrt{(\beta\mathfrak{n})(\kappa_m,x_r)}}\Delta(\kappa_m)=0, \tag{2.3.30}$$

and the second boundary condition (2.3.23b) is satisfied.

To prove (2.3.29), define the right-hand side as $\mathfrak{e}_1$ and rewrite it as a Cauchy product, obtaining

$$\mathfrak{e}_1=\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}(-1)^\ell\left[\mathcal{C}_\ell^{(x_l,x_r)}(k)\mathcal{C}_{n-\ell}^{(x_l,x_r)}(k)+\mathcal{S}_\ell^{(x_l,x_r)}(k)\mathcal{S}_{n-\ell}^{(x_l,x_r)}(k)\right]. \tag{2.3.31}$$

Letting $\ell\to n-\ell$ in the inner sum, it is concluded that

$$\sum_{\ell=0}^{n}(-1)^\ell\left[\mathcal{C}_\ell^{(x_l,x_r)}(k)\mathcal{C}_{n-\ell}^{(x_l,x_r)}(k)+\mathcal{S}_\ell^{(x_l,x_r)}(k)\mathcal{S}_{n-\ell}^{(x_l,x_r)}(k)\right]=0, \tag{2.3.32}$$

for odd n. The $n=0$ term is 1. The $n\geq 2$ even terms are 0 and thus gives $\mathfrak{e}_1=1$. For $n=2$, we show

$$\begin{aligned}0=&\int_{x_l}^{x_r}dz_1\int_{z_1}^{x_r}dz_2\,\cos\left(\int_{x_l}^{x_r}-\int_{x_l}^{z_1}+\int_{z_1}^{z_2}-\int_{z_2}^{x_r}\nu(k,\xi)\,d\xi\right)\\&-\int_{x_l}^{x_r}dy_1\int_{x_l}^{x_r}dz_1\,\cos\left(\int_{x_l}^{y_1}-\int_{y_1}^{x_r}-\int_{x_l}^{z_1}+\int_{z_1}^{x_r}\nu(k,\xi)\,d\xi\right)\\&+\int_{x_l}^{x_r}dy_1\int_{y_1}^{x_r}dy_2\,\cos\left(\int_{x_l}^{y_1}-\int_{y_1}^{y_2}+\int_{y_2}^{x_r}-\int_{x_l}^{x_r}\nu(k,\xi)\,d\xi\right).\end{aligned} \tag{2.3.33}$$

Let I_j denote the three integrals above, in order. Since the first and the last term are equal and equal to

$$I_1=I_3=\int_{x_l}^{x_r}dy_1\int_{y_1}^{x_r}dy_2\,\cos\left(2\int_{y_1}^{y_2}\nu(k,\xi)\,d\xi\right), \tag{2.3.34}$$

and since the second term is

$$\begin{aligned}I_2&=-\int_{x_l}^{x_r}dy_1\int_{x_l}^{x_r}dz_1\,\cos\left(2\int_{z_1}^{y_1}\nu(k,\xi)\,d\xi\right)\\&=-\int_{x_l}^{x_r}dy_1\int_{x_l}^{y_1}dz_1\,\cos\left(2\int_{z_1}^{y_1}\nu(k,\xi)\,d\xi\right)-\int_{x_l}^{x_r}dy_1\int_{y_1}^{x_r}dz_1\,\cos\left(2\int_{z_1}^{y_1}\nu(k,\xi)\,d\xi\right)\\&=-2\int_{x_l}^{x_r}dy_1\int_{y_1}^{x_r}dz_1\,\cos\left(2\int_{z_1}^{y_1}\nu(k,\xi)\,d\xi\right),\end{aligned} \tag{2.3.35}$$

and so the $n=2$ term is 0. The other n terms are similar. □

Comparing to Pöschel and Trubowitz [22] ($\alpha(x)=\beta(x)=1$, $\gamma(x)=-q(x)$, $\lambda=k^2$), it must be, by uniqueness, that

$$y_1(x,k^2,q)=\sqrt{\frac{\mathfrak{n}(k,0)}{\mathfrak{n}(k,x)}}\sum_{n=0}^{\infty}\mathcal{C}_n^{(0,x)}(k)\qquad\text{and}\qquad y_2(x,k^2,q)=\frac{1}{k\sqrt{\mathfrak{n}(k,0)}\sqrt{\mathfrak{n}(k,x)}}\sum_{n=0}^{\infty}\mathcal{S}_n^{(0,x)}(k), \tag{2.3.36}$$

where $\mathfrak{n}(k,x)=\sqrt{1-q(x)/k^2}$.

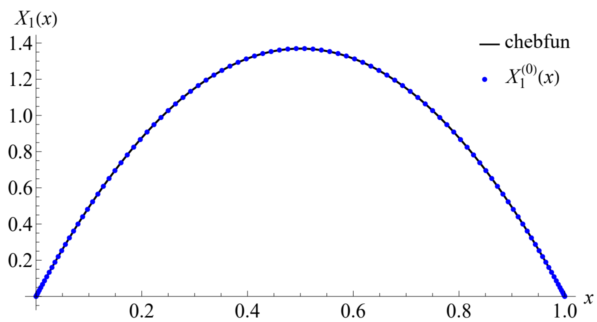
2.3.3.1 Example: Eigenvalues and eigenfunctions for the heat equation with homogeneous, Dirichlet boundary conditions

We revisit the heat equation with homogeneous, Dirichlet boundary conditions in Section 2.3.1. The associated eigenvalue problem is of the form

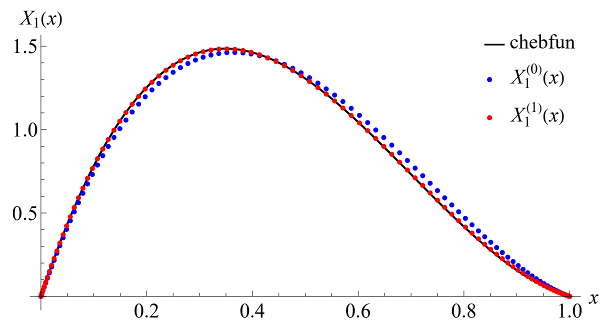
$$\left(\sigma^2(x)y'\right)'=\lambda y,\qquad y(0)=0=y(1). \tag{2.3.37}$$

Method:	λ_1	λ_2	λ_3	λ_4
chebfun	-1.0000	-4.2540	-9.6812	-17.2800
NDEigenvalues	-1.0000	-4.2540	-9.6818	-17.2834
FindRoot: $\Delta_0(k)$	-1.0856	-4.3423	-9.7702	-17.3692
FindRoot: $\Delta_1(k)$	-0.9917	-4.2474	-9.6749	-17.2737
FindRoot: $\Delta_2(k)$	-1.0006	-4.2542	-9.6814	-17.2801

Table 2.1: Eigenvalues of the system (2.3.37) calculated using the chebfun package [9] in MATLAB, compared to using Mathematica's NDEigenvalues and Mathematica's FindRoot on $\Delta_N(k)$ for $N = 0, 1, 2$.



(a)



(b)

Figure 2.3: (a) The successive approximations to the eigenfunctions (2.3.39) shown with the numerically computed eigenfunctions using MATLAB's chebfun package. (b) The same, using $\sigma^2(x) = \frac{6337-252\sqrt{111}-4500x^2(11+x(-14+5x))}{9000(11+6x(-7+5x))}$.

The eigenvalues $\lambda_m = -\kappa_m^2$ are related to the nonzero roots κ_m $(m = 1, 2, \ldots)$ of

$$\tilde{\Delta}(k) = \sum_{n=0}^{\infty} \mathcal{S}_n^{(0,1)}(k). \tag{2.3.38}$$

Since $\tilde{\Delta}(k)$ is odd in k, if κ_m is a root, so is $-\kappa_m$, and each gives rise to the same eigenvalue. The eigenfunctions are given by

$$X_m(x) = \frac{1}{\sqrt{\sigma(x)}} \sum_{n=0}^{\infty} \mathcal{S}_n^{(0,x)}(\kappa_m), \qquad m = 1, 2, \ldots \tag{2.3.39}$$

To demonstrate the computation of the eigenvalues and eigenfunctions, we consider (2.3.37) with the same $\sigma(x)$ as in Section 2.3.1.1. To find the eigenvalues, we use Mathematica's FindRoot command on $\Delta_N(k)$. The results are shown in Table 2.1. We see that our method converges to the eigenvalues and outperforms Mathematica's built-in NDEigenvalues command for $n = 4$. We denote the order-N truncated eigenfunctions (2.3.39) as $X_m^{(N)}(x)$, similar to (2.3.14). These are shown in Figure 2.3. For the simple $\sigma(x)$, given above, the order-0 truncation is quite accurate. For a more complicated $\sigma(x)$, the order-1 truncation gives an accurate representation.

2.3.3.2 Example: Eigenvalues for the CGL equation with periodic boundary conditions

We revisit the complex Ginzburg-Landau equation described in Section 2.3.2, setting $a(x) = x\sin(2\pi x)$. The associated eigenvalue problem is of the form

$$(1 + ia(x))y'' + y = \lambda y, \qquad y(0) = y(1), \qquad y'(0) = y'(1). \tag{2.3.40}$$

The eigenvalues $\lambda_m = -\kappa_m^2$ are related to the zeroes κ_m $(m = 0, 1, 2, \ldots)$ of $\tilde{\Delta}(k)$ (2.3.20). Since $\tilde{\Delta}(k)$ is even in k, if κ_m is a root, so is $-\kappa_m$, and each gives rise to the same eigenvalue. Since $\mathfrak{g}(\pm i) = 0$, and

$$\mathcal{C}_n^{(0,1)}(0) = \frac{1}{2^n} \int_{\mathcal{D}_n^{(0,1)}} \left(\prod_{p=1}^{n} \frac{\mu'(y_p)}{\mu(y_p)} \right) d\mathbf{y}_n = \frac{1}{2^n n!} \left(\int_0^1 \frac{\mu'(y)}{\mu(y)}\, dy \right)^n = \frac{1}{2^n n!} \left(\log\left(\frac{\mu(1)}{\mu(0)} \right) \right)^n = 0, \tag{2.3.41}$$

Method:	λ_1	λ_2	λ_3	λ_4
chebfun	$-41.585+3.3357i$	$-41.689+7.7171i$	$-170.71+19.919i$	$-170.62+23.463i$
NDEigenvalues	$-41.585+3.3364i$	$-41.689+7.7167i$	$-170.73+19.929i$	$-170.65+23.464i$
Hill's Method	$-41.585+3.3358i$	$-41.689+7.7171i$	$-170.71+19.919i$	$-170.62+23.463i$
FindRoot: $\Delta_0(k)$	$-42.012+5.3928i$	$-42.012+5.3928i$	$-171.05+21.571i$	$-171.05+21.571i$
FindRoot: $\Delta_1(k)$	$-41.595+3.3501i$	$-41.671+7.7097i$	$-170.73+19.949i$	$-170.60+23.434i$
FindRoot: $\Delta_2(k)$	$-41.585+3.3356i$	$-41.689+7.7172i$	$-170.70+19.916i$	$-170.63+23.466i$

Table 2.2: Eigenvalues of the system (2.3.22) calculated using the chebfun package [9] in MATLAB, Mathematica's NDEigenvalues, Hill's method [4], and a root finding algorithm on $\tilde{\Delta}_N(k)$ for $N = 0, 1, 2$.

then $\tilde{\Delta}(\pm i) = 0$, and $\kappa_0 = i$ (and $-i$) is an exact double root of $\tilde{\Delta}(k)$, and $\lambda_0 = -\kappa_0^2 = 1$ is an exact eigenvalue of the problem, which can be confirmed directly (with the constant eigenfunction). Define

$$\mathfrak{m}(x) = \int_0^x \mu(\xi)\, d\xi \qquad \text{and} \qquad \eta(y) = \frac{(\beta\mathfrak{n})'(k,x)}{(\beta\mathfrak{n})(k,x)} = \frac{\mu'(x)}{\mu(x)} = \frac{2\pi x\cos(2\pi x) + \sin(2\pi x)}{2i - 2x\sin(2\pi x)}. \tag{2.3.42}$$

Truncate (2.3.20) at order $n = N$ and denote as $\tilde{\Delta}_N(k)$. Denoting $\kappa = k\mathfrak{g}(k)$, the zeroth-order approximations of the roots of $\tilde{\Delta}(k)$ are

$$\tilde{\Delta}_0(k) = 1 - \cos\left(\mathfrak{m}(1)\kappa\right) = 0 \qquad \Rightarrow \qquad \kappa_m^{(0)} = \pm\frac{\sqrt{4m^2\pi^2 - \mathfrak{m}(1)^2}}{\mathfrak{m}(1)}, \qquad m = 1, 2, 3, \ldots. \tag{2.3.43}$$

As in the case $\kappa_0 = \pm i$, these approximations are double roots. However, the actual eigenvalues are simple roots that are near these points. The next-order approximations $\kappa_m^{(1)}$ are the roots of

$$0 = \tilde{\Delta}_1(k) = 1 - \cos\left(\mathfrak{m}(1)\kappa\right) - \mathcal{C}_2^{(0,1)}(k). \tag{2.3.44}$$

In order to compute $\mathcal{C}_2^{(0,1)}(k)$, we use an interpolation function for $\mathfrak{m}(x)$, and rewrite

$$k\sum_{p=0}^{n}(-1)^p \int_{y_p}^{y_{p+1}} \mathfrak{n}(k,\xi)\, d\xi = k\mathfrak{g}(k)\sum_{p=0}^{n}(-1)^p(\mathfrak{m}(y_{p+1}) - \mathfrak{m}(y_p)) = \kappa\left(\mathfrak{m}(1) - 2\sum_{p=0}^{n}(-1)^p\mathfrak{m}(y_p)\right). \tag{2.3.45}$$

Then we use (1.2.19a) to compute the $\tilde{\Delta}_1(k)$. We use a root finding algorithm to find the roots, using that

$$\partial_\kappa \mathcal{C}_n^{(0,1)}(k) = \frac{1}{2^n}\int_{0<\cdots<1}\left(\prod_{p=1}^{n}\eta(y_p)\right)\cos\left(\kappa\left(\mathfrak{m}(1) - 2\sum_{p=0}^{n}(-1)^p\mathfrak{m}(y_p)\right)\right)\left(\mathfrak{m}(1) - 2\sum_{p=0}^{n}(-1)^p\mathfrak{m}(y_p)\right)d\mathbf{y}_n. \tag{2.3.46}$$

The results are shown in Table 2.2.

3. Derivations

In this chapter, the solution expressions for the finite-interval, half-line, and whole-line IBVPs, in that order, are derived. The solution for the finite-interval problem for $x \in (x_l, x_r)$ is first derived through an interface problem. The solution to the half-line problem (for $x \in (x_l, \infty)$) is obtained from the solution to the finite-interval problem by taking the limit as $x_r \to \infty$. Similarly, the solution to the whole-line problem is obtained from the solution of the half-line problem by taking the limit as $x_l \to -\infty$.

It is possible to derive the solutions for the whole-line and half-line problems in the same way as for the finite-interval, *i.e.,* through an interface problem. The key difference is a non-uniform partition is required. The solutions (2.1.2) and (2.2.2) are the same.

3.1 The finite-interval problem

To consider the finite-interval IBVP with variable coefficients,

$$q_t = \alpha(x)\left(\beta(x)q_x\right)_x + \gamma(x)q + f(x,t), \qquad x \in (x_l, x_r), \quad t > 0, \tag{3.1.1a}$$
$$q(x,0) = q_0(x), \qquad x \in (x_l, x_r), \tag{3.1.1b}$$
$$f_0(t) = a_{11}q(x_l,t) + a_{12}q_x(x_l,t) + b_{11}q(x_r,t) + b_{12}q_x(x_r,t), \qquad t > 0, \tag{3.1.1c}$$
$$f_1(t) = a_{21}q(x_l,t) + a_{22}q_x(x_l,t) + b_{21}q(x_r,t) + b_{22}q_x(x_r,t), \qquad t > 0. \tag{3.1.1d}$$

form a partition $\{x_j,\ j = 0, \dots, N\}$ of the interval $[x_l, x_r]$, see Figure 3.1. For simplicity, assume that the partition is evenly spaced, *i.e.,* $\Delta x_j = \Delta x = (x_r - x_l)/N$ for $j = 1, \dots, N$, although this assumption may be relaxed easily. On each subinterval, approximate the evolution equation (3.1.1a) with constant-coefficients α_j, β_j, γ_j, $j = 1, \dots, N$ for $\alpha(x)$, $\beta(x)$, and $\gamma(x)$ (such that $\alpha_j \to \alpha(x_j)$, *etc.*, in the limit as $N \to \infty$), with the initial condition restricted to the subinterval. At each interface x_j, $j = 1, \dots, N-1$, continuity of the solution and a jump discontinuity on the derivative, corresponding to the evolution equation, are imposed. This yields the following interface problem:

$$q_t^{(j)} = \alpha_j\beta_j q_{xx}^{(j)} + \gamma_j q^{(j)} + f(x,t), \qquad x \in (x_{j-1}, x_j), \quad t > 0, \quad j = 1, \dots, N, \tag{3.1.2a}$$
$$q^{(j)}(x,0) = q_0(x), \qquad x \in (x_{j-1}, x_j), \quad t > 0, \quad j = 1, \dots, N, \tag{3.1.2b}$$
$$q^{(j)}(x_j,t) = q^{(j+1)}(x_j,t), \qquad t > 0, \quad j = 1, \dots, N-1, \tag{3.1.2c}$$
$$\beta_j q_x^{(j)}(x_j,t) = \beta_{j+1} q_x^{(j+1)}(x_j,t), \qquad t > 0, \quad j = 1, \dots, N-1, \tag{3.1.2d}$$

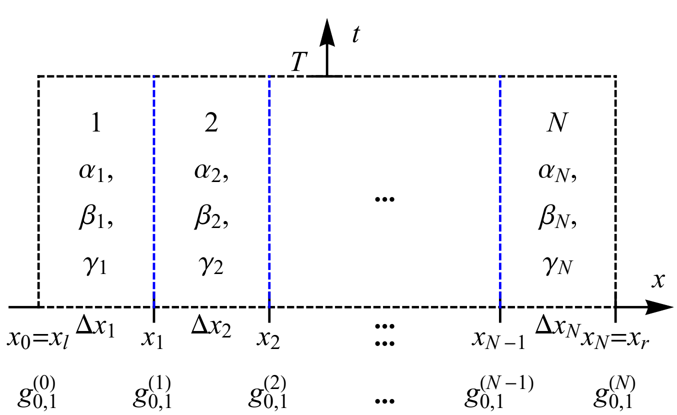


Figure 3.1: A partition of the finite interval $[x_l, x_r]$.

with the boundary conditions

$$a_{11}q^{(1)}(x_l,t)+a_{12}q_x^{(1)}(x_l,t)+b_{11}q^{(N)}(x_r,t)+b_{12}q_x^{(N)}(x_r,t)=f_0(t), \qquad t>0, \tag{3.1.3a}$$
$$a_{21}q^{(1)}(x_l,t)+a_{22}q_x^{(1)}(x_l,t)+b_{21}q^{(N)}(x_r,t)+b_{22}q_x^{(N)}(x_r,t)=f_1(t), \qquad t>0. \tag{3.1.3b}$$

The jump discontinuity in the derivative (3.1.2d) can be derived by dividing the PDE (3.1.1a) by $\alpha(x)$ and integrating over a small interval containing x_j. Following [5, 6, 24, 25, 26, 27], the *local relations* are

$$\left(e^{-i\kappa x+w_jt}q^{(j)}(x,t)\right)_t=\alpha_j\beta_j\left(e^{-i\kappa x+w_jt}\left(q_x^{(j)}(x,t)+i\kappa q^{(j)}(x,t)\right)\right)_x+e^{-i\kappa x+w_jt}f(x,t), \tag{3.1.4}$$

for $x\in(x_{j-1},x_j)$, $1\le j\le N$, and $w_j(\kappa)=\alpha_j\beta_j\kappa^2-\gamma_j$. This is equivalent to (3.1.2a), seen by differentiating (3.1.4). Define the "transforms"

$$\hat{q}_0^{(j)}(k)=\frac{1}{\alpha_j}\int_{x_{j-1}}^{x_j}e^{-iky}q_0(y)\,dy, \qquad j=1,\dots,N, \tag{3.1.5a}$$
$$\hat{q}^{(j)}(k,t)=\frac{1}{\alpha_j}\int_{x_{j-1}}^{x_j}e^{-iky}q^{(j)}(y,t)\,dy, \qquad j=1,\dots,N, \tag{3.1.5b}$$
$$\tilde{f}_j(k,t)=\frac{1}{\alpha_j}\int_0^t ds\int_{x_{j-1}}^{x_j}e^{-iky+Ws}f(y,s)\,dy, \qquad j=1,\dots,N, \tag{3.1.5c}$$
$$F_m(W,t)=\int_0^t e^{Ws}f_m(s)\,ds, \qquad m=0,1, \tag{3.1.5d}$$
$$g_m^{(j)}(W,t)=\int_0^t e^{Ws}q_{mx}^{(j)}(x_j,s)\,ds, \qquad j=0,\dots,N, \quad m=0,1, \tag{3.1.5e}$$

with $q_{mx}^{(0)}(x_l,t)=q_{mx}^{(1)}(x_l,t)$, for consistency at $j=0$. Using the interface conditions (3.1.2c) and (3.1.2d),

$$g_0^{(j)}(W,t)=\int_0^t e^{Ws}q^{(j+1)}(x_j,s)\,ds, \qquad g_1^{(j)}(W,t)=\frac{\beta_{j+1}}{\beta_j}\int_0^t e^{Ws}q_x^{(j+1)}(x_j,s)\,ds, \qquad j=0,\dots,N-1, \tag{3.1.6}$$

where $\beta_0=\beta_1$ is again defined for consistency. From the boundary conditions (3.1.3),

$$a_{11}g_0^{(0)}(k^2,t)+a_{12}g_1^{(0)}(k^2,t)+b_{11}g_0^{(N)}(k^2,t)+b_{12}g_1^{(N)}(k^2,t)=F_0(k^2,t), \tag{3.1.7a}$$
$$a_{21}g_0^{(0)}(k^2,t)+a_{22}g_1^{(0)}(k^2,t)+b_{21}g_0^{(N)}(k^2,t)+b_{22}g_1^{(N)}(k^2,t)=F_1(k^2,t). \tag{3.1.7b}$$

Integrating the local relations (3.1.4) over $D_j=(x_{j-1},x_j)\times(0,T)$ gives

$$\alpha_j\tilde{f}_j(\kappa,T)=\int_0^T dt\int_{x_{j-1}}^{x_j}dx\,\left[\left(e^{-i\kappa x+w_jt}q^{(j)}\right)_t-\alpha_j\beta_j\left(e^{-i\kappa x+w_jt}\left(q_x^{(j)}+i\kappa q^{(j)}\right)\right)_x\right], \qquad j=1,\dots,N. \tag{3.1.8}$$

Using Green's theorem,

$$\begin{aligned}\alpha_j\tilde{f}_j(\kappa,T)=&-e^{w_jT}\int_{x_{j-1}}^{x_j}e^{-i\kappa x}q^{(j)}(x,T)\,dx+\alpha_je^{-i\kappa x_j}\int_0^T e^{w_jt}\left(\beta_jq_x^{(j)}(x_j,t)+i\beta_j\kappa q^{(j)}(x_j,t)\right)dt\\&+\int_{x_{j-1}}^{x_j}e^{-i\kappa x}q_0(x)\,dx-\alpha_je^{-i\kappa x_{j-1}}\int_0^T e^{w_jt}\left(\beta_jq_x^{(j)}(x_{j-1},t)+i\beta_j\kappa q^{(j)}(x_{j-1},t)\right)dt,\end{aligned} \tag{3.1.9}$$

for $j=1,\dots,N$. These are rewritten as *global relations* using (3.1.5),

$$\begin{aligned}e^{w_jt}\hat{q}^{(j)}(\kappa,t)=&\,\hat{q}_0^{(j)}(\kappa)-\tilde{f}_j(\kappa,t)+e^{-i\kappa x_j}\left(\beta_jg_1^{(j)}(w_j,t)+i\beta_j\kappa g_0^{(j)}(w_j,t)\right)\\&-e^{-i\kappa x_{j-1}}\left(\beta_{j-1}g_1^{(j-1)}(w_j,t)+i\beta_j\kappa g_0^{(j-1)}(w_j,t)\right), \qquad j=1,\dots,N.\end{aligned} \tag{3.1.10}$$

As in [18, 25, 27], it is convenient for the first arguments of $g_m^{(j)}(w_j,t)$ to be identical. To this end, transform the independent variable κ in the jth equation as

$$\kappa=\nu_j(k)=\frac{k}{\sqrt{\alpha_j\beta_j}}\sqrt{1+\frac{\gamma_j}{k^2}}, \qquad j=1,\dots,N. \tag{3.1.11}$$

The resulting branch cuts in the solution are defined in Section 1.2 and proven to be correct in Chapters 4 and 5. Since $\gamma(x)$ is assumed to be bounded, see Assumption 2.4, there are no branch cuts for $|k| > \sqrt{M_\gamma}$ for $k \in \Omega$ where $M_\gamma = \|\gamma\|_\infty < \infty$. Until the limit is taken, the k dependence of $\nu_j(k)$ is suppressed. The global relations (3.1.10) become

$$\begin{aligned} e^{k^2 t}\hat{q}^{(j)}(\nu_j,t) &= \hat{q}_0^{(j)}(\nu_j) - \tilde{f}_j(\nu_j,t) + e^{-i\nu_j x_j}\left(\beta_j g_1^{(j)}(k^2,t) + i\beta_j\nu_j g_0^{(j)}(k^2,t)\right) \\ &\quad - e^{-i\nu_j x_{j-1}}\left(\beta_{j-1} g_1^{(j-1)}(k^2,t) + i\beta_j \nu_j g_0^{(j-1)}(k^2,t)\right), \qquad j = 1,\ldots,N. \end{aligned} \tag{3.1.12}$$

These relations are valid for $k \in \mathbb{C}$, since the domains are bounded. Letting $k \mapsto -k$, (and $\nu_j \mapsto -\nu_j$), gives $2N$ equations, along with (3.1.7) for $2N+2$ unknowns. This linear system of equations may be written in matrix form as

$$\mathcal{A}_N(k)X_N(k^2,t) = Y_N(k,t) - e^{k^2 t}\mathcal{Y}_N(k,t), \tag{3.1.13}$$

where

$$X_N(k^2,t) = \left(g_0^{(0)}(k^2,t),\ldots,g_0^{(N)}(k^2,t),\beta_0 g_1^{(0)}(k^2,t),\ldots,\beta_N g_1^{(N)}(k^2,t)\right)^\top, \tag{3.1.14a}$$

$$\begin{aligned} Y_N(k,t) &= \left(0,\hat{q}_0^{(1)}(\nu_1),\ldots,\hat{q}_0^{(N)}(\nu_N),\hat{q}_0^{(1)}(-\nu_1),\ldots,\hat{q}_0^{(N)}(-\nu_N),0\right)^\top \\ &\quad - \left(-F_0(k^2,t),\ \tilde{f}_1(\nu_1,t),\ldots,\tilde{f}_N(\nu_N,t),\tilde{f}_1(-\nu_1,t),\ldots,\tilde{f}_N(-\nu_N,t),\ -F_1(k^2,t)\right)^\top, \end{aligned} \tag{3.1.14b}$$

$$\mathcal{Y}_N(k,t) = \left(0,\hat{q}^{(1)}(\nu_1,t),\ldots,\hat{q}^{(N)}(\nu_N,t),\hat{q}^{(1)}(-\nu_1,t),\ldots,\hat{q}^{(N)}(-\nu_N,t),0\right)^\top, \tag{3.1.14c}$$

and $\mathcal{A}_N(k)$ is the $(2N+2)\times(2N+2)$ matrix

$$\mathcal{A}_N(k) = \left(\begin{array}{ccccc:ccccc} a_{11} & 0 & \cdots & 0 & b_{11} & a_{12}/\beta_0 & 0 & \cdots & 0 & b_{12}/\beta_N \\ \hdashline & & A_{N,N+1}(k) & & & & & B_{N,N+1}(k) & & \\ \hdashline & & A_{N,N+1}(-k) & & & & & B_{N,N+1}(-k) & & \\ \hdashline a_{21} & 0 & \cdots & 0 & b_{21} & a_{22}/\beta_0 & 0 & \cdots & 0 & b_{22}/\beta_N \end{array}\right). \tag{3.1.15}$$

Here, $A_{N,N+1}(k)$ is the $N\times(N+1)$ matrix

$$A_{N,N+1}(k) = \begin{pmatrix} i\beta_1\nu_1 e^{-i\nu_1 x_0} & -i\beta_1\nu_1 e^{-i\nu_1 x_1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & i\beta_N\nu_N e^{-i\nu_N x_{N-1}} & -i\beta_N\nu_N e^{-i\nu_N x_N} \end{pmatrix}, \tag{3.1.16}$$

and $B_{N,N+1}(k)$ is the $N\times(N+1)$ matrix

$$B_{N,N+1}(k) = \begin{pmatrix} e^{-i\nu_1 x_0} & -e^{-i\nu_1 x_1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & e^{-i\nu_N x_{N-1}} & -e^{-i\nu_N x_N} \end{pmatrix}. \tag{3.1.17}$$

Since the contribution involving $\mathcal{Y}_N(k,t)$ along the contour $\partial\Omega$, see Figure 1.1, is zero [27], it suffices to solve $\mathcal{A}_N(k)X_N(k^2,t) = Y_N(k,t)$ for the unknown functions $g_m^{(j)}(k^2,t)$. This is further justified in Chapters 4 and 5. Using Cramer's rule,

$$X_N^{(j)}(k^2,t) = g_0^{(j-1)}(k^2,t) = \frac{\det\left(\mathcal{A}_N^{(j)}(k)\right)}{\det\left(\mathcal{A}_N(k)\right)}, \qquad j = 1,\ldots,N+1, \tag{3.1.18}$$

where the matrix $\mathcal{A}_N^{(j)}(k)$ is $\mathcal{A}_N(k)$ with the jth column replaced by $Y_N(k,t)$. If we multiply this equation by $ke^{-k^2 t}$ and integrate over $\partial\Omega$, where $\Omega = \{k \in \mathbb{C} : |k| > r \text{ and } \pi/4 < \arg(k) < 3\pi/4\}$ for some $r > \sqrt{M_\gamma}$, see Figure 1.1, the time "transform" $g_0^{(j-1)}(k^2,t)$ (3.1.5e) can be inverted [14] to find

$$q^{(j)}(x_{j-1},t) = \frac{1}{i\pi}\int_{\partial\Omega} \frac{\det(\mathcal{A}_N^{(j)}(k))}{\det(\mathcal{A}_N(k))} k e^{-k^2 t}\, dk, \qquad j = 1,\ldots,N+1. \tag{3.1.19}$$

This inversion can be done formally by inserting (3.1.5e) into (3.1.18), using the continuity condition (3.1.2c), multiplying by ke^{-k^2t}, integrating over $\partial\Omega$, switching the order of integration, using the transformation $k^2 = iz$, and using the integral representation

$$\delta(x-y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-y)} f(z)\,dz. \tag{3.1.20}$$

Equation (3.1.19) gives the solution at the interface boundary points, which is all that is needed to consider the limit $q(x,t) = \lim_{N\to\infty} q^{(j)}(x_{j-1},t)$, where the N dependence of $q^{(j)}(x_{j-1},t)$ is implicit. Alternatively, the full solution of the interface problem can be computed as in [5, 25, 26, 27] and the limit of the full solution can be taken. This gives the same result.

Define

$$D_N(k) = i^{N+1}\frac{\det(\mathcal{A}_N(k))}{\nu_1\nu_N}\left(\prod_{p=1}^{N-1}\frac{1}{\Lambda_p^+}\right), \tag{3.1.21}$$

with

$$\Lambda_p^{\boldsymbol{\ell}} = (\beta\nu)_{p+1} + (-1)^{\ell_p+\ell_{p+1}}(\beta\nu)_p \qquad \text{and} \qquad \Lambda_p^{\pm} = (\beta\nu)_{p+1} \pm (\beta\nu)_p, \tag{3.1.22}$$

where $\boldsymbol{\ell}\in\{0,1\}^N$, so that ℓ_p, $\ell_{p+1}\in\{0,1\}$ and $(\beta\nu)_j = \beta_j\nu_j$. Note that $\Lambda_p^{\boldsymbol{\ell}} = \Lambda_p^+$ when $\ell_p = \ell_{p+1}$ and $\Lambda_p^{\boldsymbol{\ell}} = \Lambda_p^-$ when $\ell_p \neq \ell_{p+1}$. For $N\le 8$, it can be explicitly verified using Mathematica that

$$\begin{aligned} D_N(k) = 2i\Bigg\{&\frac{\beta_N(a:b)_{1,2}+\beta_1(a:b)_{3,4}}{\sqrt{(\beta\nu)_1}\sqrt{(\beta\nu)_N}}\frac{\sqrt{(\beta\nu)_N}}{\sqrt{(\beta\nu)_1}}\left(\prod_{p=1}^{N-1}\frac{2(\beta\nu)_p}{\Lambda_p^+}\right) + (a:b)_{2,4}\mathfrak{S}_{N,1}^{(1,N)}(k) + \frac{(a:b)_{1,3}}{\nu_1\nu_N}\mathfrak{S}_{N,0}^{(1,N)}(k) \\ &+\frac{(a:b)_{1,4}}{\nu_1}\mathfrak{C}_{N,1}^{(1,N)}(k) - \frac{(a:b)_{2,3}}{\nu_N}\mathfrak{C}_{N,0}^{(1,N)}(k)\Bigg\}, \end{aligned} \tag{3.1.23}$$

where $(a:b)_{i,j} = \det((a:b)_{\{1,2\},\{i,j\}})$ is the determinant of the minor of maximal size with columns at i and j [23] of the concatenated matrix $(a:b)$ (1.2.5). Define

$$\mathfrak{C}_{N,\lambda}^{(q,s)}(k) = \sum_{\substack{\boldsymbol{\ell}\in\{0,1\}^{s-q+1}\\ \ell_q=0}} (-1)^{\lambda\ell_s}\left(\prod_{p=q}^{s-1}\frac{\Lambda_p^{\boldsymbol{\ell}}}{\Lambda_p^+}\right)\cos\left(\sum_{p=q}^{s}(-1)^{\ell_p}\nu_p\Delta x\right), \tag{3.1.24a}$$

$$\mathfrak{S}_{N,\lambda}^{(q,s)}(k) = \sum_{\substack{\boldsymbol{\ell}\in\{0,1\}^{s-q+1}\\ \ell_q=0}} (-1)^{\lambda\ell_s}\left(\prod_{p=q}^{s-1}\frac{\Lambda_p^{\boldsymbol{\ell}}}{\Lambda_p^+}\right)\sin\left(\sum_{p=q}^{s}(-1)^{\ell_p}\nu_p\Delta x\right), \tag{3.1.24b}$$

where $\lambda = 0,1$. This result is not proven here for general N. Its justification follows indirectly from the proofs in Chapters 4 and 5. Using Taylor series, it can be shown that

$$\prod_{p=1}^{N-1}\frac{2(\beta\nu)_p}{\Lambda_p^+} = \exp\left(\sum_{p=1}^{N-1}\big(\ln(2(\beta\nu)_p) - \ln(\Lambda_p^+)\big)\right) = \frac{\sqrt{(\beta\nu)_1}}{\sqrt{(\beta\nu)_N}} + O(\Delta x), \tag{3.1.25}$$

as $N\to\infty$ and $\Delta x\to 0^+$. Similarly,

$$\frac{\Lambda_p^-}{\Lambda_p^+} = \frac{1}{2}\frac{(\beta\mathfrak{n})'(k,x_p)}{(\beta\mathfrak{n})(k,x_p)}\Delta x + O\big((\Delta x)^2\big) \qquad \text{and} \qquad \prod_{p=\ell}^{m-1}\frac{2(\beta\nu)_p}{\Lambda_p^+} = \frac{\sqrt{(\beta\mathfrak{n})(k,x_\ell)}}{\sqrt{(\beta\mathfrak{n})(k,x_m)}} + O(\Delta x), \tag{3.1.26}$$

as $\ell, m, N\to\infty$, $\Delta x\to 0^+$, and where the prime denotes the derivative with respect to the second variable, and $\mathfrak{n}(k,x)$, $(\beta\mathfrak{n})(k,x)$ are defined in Definition 7. Note that to use (3.1.26), it is assumed that $(\beta\mathfrak{n})(k,x)$ is a smooth function of x. If this function has a countable number of discontinuities, it is possible to proceed, but the jumps would need to be accounted for.

Consider (3.1.23) as $N\to\infty$ (*i.e.*, $\Delta x\to 0^+$). To this end, break up the sum in (3.1.24a) by the number of times n the entries of the vector $\boldsymbol{\ell} = (\ell_q,\ldots,\ell_s)$ switch from 0 to 1 or from 1 to 0, *e.g.*, $(0,\ldots,0,1,\ldots,1)$ switches once, so $n=1$. Sum over where the possible switches $y_1,\ldots,y_n$, of each order n can occur *i.e.*, $q-1 < y_1 < y_2 < \cdots < y_n < s$.

At the location of each switch, $\Lambda_p^{\boldsymbol{\ell}}/\Lambda_p^+ = \Lambda_p^-/\Lambda_p^+$, whereas $\Lambda_p^{\boldsymbol{\ell}}/\Lambda_p^+ = 1$ otherwise. Defining $y_0 = q-1$ and $y_{n+1} = s$, this gives

$$\mathfrak{C}_{N,\lambda}^{(q,s)}(k) = \sum_{n=0}^{s-q} \sum_{y_0<y_1<\cdots<y_n<y_{n+1}} (-1)^{\lambda n} \left(\prod_{p=1}^{n} \frac{\Lambda_{y_p}^-}{\Lambda_{y_p}^+} \right) \cos\left(\sum_{p=0}^{n} (-1)^p \sum_{r=y_p+1}^{y_{p+1}} \nu_p \Delta x \right). \tag{3.1.27}$$

Using (3.1.26), this yields a sum of n-dimensional Riemann sums which limit to n-dimensional integrals, giving

$$\mathfrak{C}_{N,\lambda}^{(q,s)}(k) = \sum_{n=0}^{\infty} (-1)^{\lambda n} \mathcal{C}_n^{(x_q,x_s)}(k) + O(\Delta x), \tag{3.1.28a}$$

where $\mathcal{C}_n^{(a,b)}(k)$ is defined in (1.2.19a). Similarly, the limit of (3.1.24b) is

$$\mathfrak{S}_{N,\lambda}^{(q,s)}(k) = \sum_{n=0}^{\infty} (-1)^{\lambda n} \mathcal{S}_n^{(x_q,x_s)}(k) + O(\Delta x), \tag{3.1.28b}$$

obtained the same way, with $\mathcal{S}_n^{(a,b)}(k)$ defined in (1.2.19b). No more rigor is required at this point, as the results are proven to be solutions in Chapters 4 and 5 under less restrictive assumptions needed to justify these steps.

Using (3.1.25) and (3.1.28) in (3.1.23), we have that

$$\Delta(k) = \lim_{N\to\infty} \Xi(k) D_N(k), \tag{3.1.29}$$

gives (2.3.4). For the numerator of (3.1.19), similar to $D_N(k)$ in (3.1.21), define

$$E_N(k,j,t) = i^N \frac{2\det(\mathcal{A}_N^{(j)}(k))}{\nu_1 \nu_N} \left(\prod_{p=1}^{N-1} \frac{1}{\Lambda_p^+} \right), \tag{3.1.30}$$

and use a cofactor expansion along the jth column of $\mathcal{A}_N^{(j)}(k)$, so that

$$E_N(k,j,t) = \sum_{m=1}^{N+1} Y_N^{(m)} M_N^{(m,j)}(k) + \sum_{m=1}^{N+1} Y_N^{(m+N+1)} M_N^{(m+N+1,j)}(k), \tag{3.1.31}$$

where $M_N^{(m,j)}(k)$ are cofactors of the matrix $\mathcal{A}_N^{(j)}(k)$, scaled by the same factor as in (3.1.30). For a fixed x with $x = x_j = j\Delta x = j/N$, let

$$\mathcal{B}_0(k,x) = k\Xi(k) \lim_{N\to\infty} M_N^{(1,j)}(k) \qquad \text{and} \qquad \mathcal{B}_1(k,x) = k\Xi(k) \lim_{N\to\infty} M_N^{(2N+2,j)}(k). \tag{3.1.32}$$

Since, for $m = 1, \ldots, 2N$,

$$Y_N^{(m+1)} = \hat{q}_0^{(m)}(\nu_m) - \tilde{f}_m(\nu_m,t) = \frac{e^{-i\nu_m x_m}}{\alpha_m} \left(q_0(x_m) - \int_0^t f(x_m,s) e^{k^2 s}\, ds \right) \Delta x + O\big((\Delta x)^2\big), \tag{3.1.33}$$

so that, for $m = 1, \ldots, N$,

$$Y_N^{(m+1)} = \frac{e^{-i\nu_m x_m} \psi_N^{(m)}(k^2,t)}{\alpha_m} \Delta x + O\big((\Delta x)^2\big) \quad \text{and} \quad Y_N^{(m+N+1)} = \frac{e^{i\nu_m x_m} \psi_N^{(m)}(k^2,t)}{\alpha_m} \Delta x + O\big((\Delta x)^2\big), \tag{3.1.34}$$

which define $\psi_N^{(m)}(k^2,t)$. Then

$$\Phi(k,x,t) = \lim_{N\to\infty} k\Xi(k) E_N(k,j,t), \tag{3.1.35}$$

which gives (2.3.6a). Here

$$\Phi_\psi(k,x,t) = \lim_{N\to\infty} k\Xi(k) \sum_{m=1}^{N} \frac{\psi_N^{(m)}(k^2,t)}{\alpha_m} \left(e^{-i\nu_m x_m} M_N^{(m+1,j)}(k) + e^{i\nu_m x_m} M_N^{(m+N+1,j)}(k) \right) \Delta x, \tag{3.1.36}$$

where $y = x_m = m\Delta x = m/N$ is kept fixed. This gives (2.3.6b), after defining

$$\Psi(k,x,y) = \lim_{N\to\infty} \Psi_N^{(j,m)}(k) = \lim_{N\to\infty} \Xi(k)\sqrt{(\beta\nu)_m}\sqrt{(\beta\nu)_j}\left(e^{-i\nu_m x_m} M_N^{(m+1,j)}(k) + e^{i\nu_m x_m} M_N^{(m+N+1,j)}(k)\right), \tag{3.1.37a}$$

which defines $\Psi_N^{(j,m)}(k)$, and defining

$$\psi_\alpha(k^2,y,t) = \lim_{N\to\infty} \frac{\psi_N^{(m)}(k^2,t)}{\alpha_m} = \frac{q_0(y)}{\alpha(y)} - \int_0^t \frac{f(y,s)}{\alpha(y)} e^{k^2 s}\, ds. \tag{3.1.37b}$$

For the boundary term $\mathcal{B}_0(k,x)$, similar to (3.1.23), it can be explicitly verified using Mathematica for $N \le 8$ and for $1 = m < j \le N$ that

$$M_N^{(1,j)}(k) = \frac{4}{\sqrt{(\beta\nu)_{j-1}}} \left\{ \frac{\beta_N}{\sqrt{(\beta\nu)_{j-1}}} \left(\prod_{p=j-1}^{N-1} \frac{2(\beta\nu)_p}{\Lambda_p^+} \right) \left[-\frac{a_{21}}{\nu_1} \mathfrak{S}_{N,0}^{(1,j-1)}(k) + a_{22}\mathfrak{C}_{N,0}^{(1,j-1)}(k) \right] \right.$$
$$\left. + \frac{\sqrt{(\beta\nu)_{j-1}}}{\nu_1} \left(\prod_{p=1}^{j-1} \frac{2(\beta\nu)_p}{\Lambda_p^+} \right) \left[\frac{b_{21}}{\nu_N} \mathfrak{S}_{N,0}^{(j,N)}(k) + b_{22}\mathfrak{C}_{N,1}^{(j,N)}(k) \right] \right\}, \tag{3.1.38}$$

and using (3.1.26), (3.1.28a), (3.1.28b), and (3.1.32), this yields (2.3.6c) for $j=2$. Similarly, for the other boundary term $\mathcal{B}_1(k,x)$, taking the limit of $M_N^{(2N+2,j)}(k)$ gives (2.3.6c) for $j=1$.

For the remaining terms, for $1 \le m < j \le N$, it can be verified that

$$\Psi_N^{(j,m)}(k) = 4\Xi(k)\frac{\sqrt{(\beta\nu)_j}}{\sqrt{(\beta\nu)_m}} \left(\prod_{p=m}^{j-1} \frac{2(\beta\nu)_p}{\Lambda_p^+} \right) \left[-(a:b)_{2,4}\mathfrak{C}_{N,0}^{(1,m)}(k)\mathfrak{C}_{N,1}^{(j,N)}(k) + \frac{(a:b)_{1,3}}{\nu_1\nu_N}\mathfrak{S}_{N,0}^{(1,m)}(k)\mathfrak{S}_{N,0}^{(j,N)}(k) \right.$$
$$\left. + \frac{(a:b)_{1,4}}{\nu_1}\mathfrak{S}_{N,0}^{(1,m)}(k)\mathfrak{C}_{N,1}^{(j,N)}(k) - \frac{(a:b)_{2,3}}{\nu_N}\mathfrak{C}_{N,0}^{(1,m)}(k)\mathfrak{S}_{N,0}^{(j,N)}(k) \right]$$
$$- \frac{4(a:b)_{1,2}\beta_N}{\sqrt{(\beta\nu)_1}\sqrt{(\beta\nu)_N}} \frac{\sqrt{(\beta\nu)_j}}{\sqrt{(\beta\nu)_{j-1}}} \frac{\sqrt{(\beta\nu)_m}}{\sqrt{(\beta\nu)_1}} \left(\prod_{p=1}^{m} \frac{(\beta\nu)_p}{\Lambda_p^+} \right) \frac{\sqrt{(\beta\nu)_N}}{\sqrt{(\beta\nu)_{j-1}}} \left(\prod_{p=j-1}^{N-1} \frac{(\beta\nu)_p}{\Lambda_p^+} \right) \mathfrak{S}_{N,0}^{(m+1,j-1)}(k). \tag{3.1.39}$$

Taking the limit using (3.1.26), (3.1.28a), and (3.1.28b), as before letting $x_j \to x$ and $x_m \to y$, gives that for $x_r \le y < x \le x_\ell$,

$$\Psi(k,x,y) = 4\Xi(k) \left\{ -(a:b)_{2,4} \left(\sum_{n=0}^{\infty} \mathcal{C}_n^{(x_l,y)} \right) \left(\sum_{n=0}^{\infty} (-1)^n \mathcal{C}_n^{(x,x_r)} \right) + \frac{(a:b)_{1,3}}{k^2\mathfrak{n}(k,x_l)\mathfrak{n}(k,x_r)} \left(\sum_{n=0}^{\infty} \mathcal{S}_n^{(x_l,y)} \right) \left(\sum_{n=0}^{\infty} \mathcal{S}_n^{(x,x_r)} \right) \right.$$
$$+ \frac{(a:b)_{1,4}}{k\mathfrak{n}(k,x_l)} \left(\sum_{n=0}^{\infty} \mathcal{S}_n^{(x_l,y)} \right) \left(\sum_{n=0}^{\infty} (-1)^n \mathcal{C}_n^{(x,x_r)} \right) - \frac{(a:b)_{2,3}}{k\mathfrak{n}(k,x_r)} \left(\sum_{n=0}^{\infty} \mathcal{C}_n^{(x_l,y)} \right) \left(\sum_{n=0}^{\infty} \mathcal{S}_n^{(x,x_r)} \right)$$
$$\left. - \frac{(a:b)_{1,2}\beta(x_r)}{k\sqrt{(\beta\mathfrak{n})(k,x_l)}\sqrt{(\beta\mathfrak{n})(k,x_r)}} \sum_{n=0}^{\infty} \mathcal{S}_n^{(y,x)} \right\}, \tag{3.1.40}$$

which may be rewritten as (2.3.7a). Similar computations for $x_l \le x < y \le x_r$ yields (2.3.7b). Finally,

$$q(x,t) = \lim_{N\to\infty} \frac{1}{i\pi} \int_{\partial\Omega} \frac{\det(\mathcal{A}_N^{(j)}(k))}{\det(\mathcal{A}_N(k))} k e^{-k^2 t}\, dk = \lim_{N\to\infty} \frac{1}{2\pi} \int_{\partial\Omega} \frac{k\Xi(k)E_N(k,j,t)}{\Xi(k)D_N(k)} e^{-k^2 t}\, dk, \tag{3.1.41}$$

which gives (2.3.2).

3.2 The half-line problem

The solution of the half-line problem is obtained by taking the limit as $x_r \to \infty$ of the solution of the finite-interval problem (2.3.1) with $f_1(t) = 0$ and

$$(a:b) = \begin{pmatrix} a_0 & a_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3.2.1}$$

In this limit, (2.3.2) becomes (2.2.2) with the same Ω, shown in Figure 1.1. This process is detailed below.

Using (3.2.1), the coefficients become $\mathfrak{a}(k)=0$, $\mathfrak{c}_n(k)=-a_1/(k\mathfrak{n}(k,x_r))$, and $\mathfrak{s}_n(k)=a_0/(k^2\mathfrak{n}(k,x_l)\mathfrak{n}(k,x_r))$. Defining $\tilde{\Delta}(k)=k\mathfrak{n}(k,x_r)\Delta(k)$, (2.3.4) becomes

$$\tilde{\Delta}(k)=2i\Xi(k)\left\{\frac{a_0}{k\mathfrak{n}(k,x_l)}\sum_{n=0}^{\infty}\mathcal{S}_n^{(x_l,x_r)}(k)-a_1\sum_{n=0}^{\infty}\mathcal{C}_n^{(x_l,x_r)}(k)\right\}. \tag{3.2.2}$$

It follows that $\mathcal{B}_1(k,x)=0$, and for $j=2$, (2.3.6c) becomes

$$\tilde{\mathcal{B}}_0(k,x)=k\mathfrak{n}(k,x_r)\mathcal{B}_0(k,x)=\frac{4\beta(x_l)\Xi(k)}{\sqrt{(\beta\mathfrak{n})(k,x_l)}\sqrt{(\beta\mathfrak{n})(k,x)}}\sum_{n=0}^{\infty}\mathcal{S}_n^{(x,x_r)}(k). \tag{3.2.3}$$

For $x_l\le y<x\le x_r$, (2.3.7a) gives

$$\tilde{\Psi}(k,x,y)=k\mathfrak{n}(k,x_r)\Psi(k,x,y)=4\Xi(k)\left\{\frac{a_0}{k\mathfrak{n}(k,x_l)}\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}\mathcal{S}_\ell^{(x,x_r)}(k)\mathcal{S}_{n-\ell}^{(x_l,y)}(k)-a_1\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}\mathcal{S}_\ell^{(x,x_r)}(k)\mathcal{C}_{n-\ell}^{(x_l,y)}(k)\right\}, \tag{3.2.4}$$

and, for $x_l<x<y<x_r$, $\tilde{\Psi}(k,x,y)=\tilde{\Psi}(k,y,x)$. From (2.3.6a),

$$\tilde{\Phi}(k,x,t)=k\mathfrak{n}(k,x_r)\Phi(k,x,t)=\tilde{\mathcal{B}}_0(k,x)+\tilde{\Phi}_\psi(k,x,t), \tag{3.2.5}$$

where

$$\tilde{\Phi}_\psi(k,x,t)=k\mathfrak{n}(k,x_r)\Phi_\psi(k,x,t)=\int_{x_l}^{x_r}\frac{\tilde{\Psi}(k,x,y)\psi_\alpha(k^2,y,t)}{\sqrt{(\beta\mathfrak{n})(k,x)(\beta\mathfrak{n})(k,y)}}\,dy, \tag{3.2.6}$$

and $\psi_\alpha(k^2,y,t)$ is defined in Definition 7.

To take the limit as $x_r\to\infty$, using (1.2.19b), write

$$\exp\left(\int_a^{x_r}ik\mathfrak{n}(k,\xi)\,d\xi\right)\mathcal{S}_n^{(a,x_r)}(k)=\frac{1}{2i\cdot 2^n}\int_{\mathcal{D}_n^{(a,x_r)}}\left(\prod_{p=1}^{n}\frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)}\right)\left[\exp\left(\sum_{p=0}^{n}(1+(-1)^p)\int_{y_p}^{y_{p+1}}ik\mathfrak{n}(k,\xi)\,d\xi\right)\right.$$
$$\left.-\exp\left(\sum_{p=0}^{n}(1-(-1)^p)\int_{y_p}^{y_{p+1}}ik\mathfrak{n}(k,\xi)\,d\xi\right)\right]d\mathbf{y}_n. \tag{3.2.7}$$

For the *fully dissipative* problems, $\operatorname{Re}(ik\mathfrak{n}(k,x))<0$ for all $k\in\Omega$ and all $x>x_l$, see Lemma 15 in Section 4.1, it follows that

$$\exp\left(\int_{y_n}^{x_r}ik\mathfrak{n}(k,\xi)\,d\xi\right)\to 0, \tag{3.2.8}$$

as $x_r\to\infty$. Thus the term in (3.2.7) which does not contain the $p=n$ term survives. Considering even and odd n separately yields

$$\exp\left(\int_a^{x_r}ik\mathfrak{n}(k,\xi)\,d\xi\right)\mathcal{S}_n^{(a,x_r)}(k)\to-\frac{(-1)^n}{2i}\mathcal{E}_n^{(a,\infty)}(k),\quad\text{as}\quad x_r\to\infty, \tag{3.2.9}$$

where $\mathcal{E}_n^{(a,b)}(k)$ is defined in (1.2.18b). Similarly,

$$\exp\left(\int_a^{x_r}ik\mathfrak{n}(k,\xi)\,d\xi\right)\mathcal{C}_n^{(a,x_r)}(k)\to\frac{1}{2}\mathcal{E}_n^{(a,\infty)}(k),\quad\text{as}\quad x_r\to\infty. \tag{3.2.10}$$

Therefore, as $x_r\to\infty$,

$$-2i\tilde{\Delta}(k)\to 2\sum_{n=0}^{\infty}\left(\frac{(-1)^n ia_0}{k\mathfrak{n}(k,x_l)}-a_1\right)\mathcal{E}_n^{(x_l,\infty)}(k), \tag{3.2.11}$$

and

$$-2i\tilde{\mathcal{B}}_0(k,x)\to\frac{4\beta(x_l)\exp\left(\int_{x_l}^{x}ik\mathfrak{n}(k,\xi)\,d\xi\right)}{\sqrt{(\beta\mathfrak{n})(k,x_l)}\sqrt{(\beta\mathfrak{n})(k,x)}}\sum_{n=0}^{\infty}(-1)^n\mathcal{E}_n^{(x,\infty)}(k), \tag{3.2.12}$$

and, for $x_l \leq y < x$,

$$-2i\tilde{\Psi}(k,x,y) \to 4\exp\left(\int_{x_l}^{x} i\nu(k,\xi)\,d\xi\right)\sum_{n=0}^{\infty}\sum_{n=0}^{n}(-1)^{\ell}\left(\frac{a_0}{k\mathfrak{n}(k,x_l)}\mathcal{S}_{n-\ell}^{(x_l,y)}(k) - a_1\mathcal{C}_{n-\ell}^{(x_l,y)}(k)\right)\mathcal{E}_{\ell}^{(x,\infty)}(k), \tag{3.2.13}$$

and similarly for $x_l \leq x < y$. These final results combine to give (2.2.2).

Note that these solutions were derived for *fully dissipative problems*, but Chapter 5 proves that they are also valid for *partially dispersive problems* on the finite-interval.

3.3 The whole-line problem

Repeating the process from the previous section, now letting $x_l \to -\infty$, gives the whole-line solution (2.1.2). Starting from the half-line solution (2.2.2) with $f_0(t) = 0$, $a_0 = 1$, and $a_1 = 0$, the denominator in (2.2.2) is determined by

$$k\mathfrak{n}(k,x_l)\Delta(k) = 2i\sum_{n=0}^{\infty}(-1)^n\mathcal{E}_n^{(x_l,\infty)}(k). \tag{3.3.1}$$

Since $\mathcal{B}_0(k,x,t) = 0$, from (2.2.4),

$$\Phi(k,x,t) = \int_{-\infty}^{\infty}\frac{\Psi(k,x,y)\psi_\alpha(k^2,y,t)}{\sqrt{(\beta\mathfrak{n})(k,x)(\beta\mathfrak{n})(k,y)}}\,dy, \tag{3.3.2}$$

where $\psi_\alpha(k^2,y,t)$ is defined in Definition 7. For $x_l < y < x$, (2.2.8) becomes

$$k\mathfrak{n}(k,x_l)\Psi(k,x,y) = 4\exp\left(\int_{x_l}^{x} ik\mathfrak{n}(k,\xi)\,d\xi\right)\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}(-1)^{\ell}\mathcal{S}_{n-\ell}^{(x_l,y)}(k)\mathcal{E}_{\ell}^{(x,\infty)}(k), \tag{3.3.3}$$

and $\Psi(k,x,y) = \Psi(k,y,x)$ for $x_l < x < y$. Since $\mathcal{E}_n^{(x_l,\infty)}(k) \to 0$ if n is odd, and from (3.2.7),

$$\exp\left(\int_{x_l}^{b} ik\mathfrak{n}(k,\xi)\,d\xi\right)\mathcal{S}_n^{(x_l,b)}(k) \to -\frac{1}{2i}\tilde{\mathcal{E}}_n^{(-\infty,b)}(k), \tag{3.3.4}$$

as $x_l \to -\infty$. Therefore,

$$k\mathfrak{n}(k,x_l)\Delta(k) \to 2i\sum_{\substack{n=0\\ n\text{ even}}}^{\infty}\mathcal{E}_n^{(-\infty,\infty)}(k). \tag{3.3.5}$$

For $x_l < y < x$,

$$k\mathfrak{n}(k,x_l)\Psi(k,x,y) = 2i\exp\left(\int_{y}^{x} ik\mathfrak{n}(k,\xi)\,d\xi\right)\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}(-1)^{\ell}\tilde{\mathcal{E}}_{n-\ell}^{(-\infty,y)}(k)\mathcal{E}_{\ell}^{(x,\infty)}(k), \tag{3.3.6}$$

and $\Psi(k,x,y) = \Psi(k,y,x)$ for $x_l < x < y$. Combining these results gives (2.1.2).

4. Proofs: fully dissipative problems

4.1 The solution expressions are well defined

Prior to proving that the solution expression (1.2.2) solves the evolution equation (1.2.1a) and satisfies the initial and boundary conditions for the problem considered, it is shown in this appendix that this expression is well defined for all problems considered. We refer to the whole-line, half-line, and *regular* finite-interval problems as *regular problems*, and the *irregular* finite-interval problems as *irregular problems*. Throughout, we need Assumptions 2 and 3 from Section 1.2.

In this section, the r dependence of Ω is denoted explicitly as $\Omega(r) = \{k \in \mathbb{C} : |k| > r \text{ and } \pi/4 < \arg(k) < 3\pi/4\}$ and the definition $\mathbb{C}^+(r) = \{k \in \mathbb{C} : |k| > r \text{ and } 0 < \arg(k) < \pi\}$ is also used. Define $\theta = \arg(k)$, and for *regular* problems, $b = 0$ and for *irregular* problems, $b = 1$.

The following lemma characterizes some properties of the coefficient functions α, β that follow from the assumptions.

Lemma 12. *If $\alpha(x)\beta(x)$ is not identically zero, the following are equivalent:*

a. $\alpha\beta \in L^\infty(\mathcal{D})$,

b. $\alpha \in L^\infty(\mathcal{D})$,

c. $\beta \in L^\infty(\mathcal{D})$,

as are the following:

i. $m_{\alpha\beta} = \inf_{x\in\mathcal{D}} |\alpha(x)\beta(x)| > 0$,

ii. $m_\alpha = \inf_{x\in\mathcal{D}} |\alpha(x)| > 0$,

iii. $m_\beta = \inf_{x\in\mathcal{D}} |\beta(x)| > 0$.

Proof. Under Assumption 2.3, there exists an $x_0 \in \mathcal{D}$ such that $0 < |\alpha(x_0)\beta(x_0)| < \infty$. From Assumptions 2.2 and 2.5,

$$\left|\frac{\beta(x)}{\beta(x_0)}\right| = \left|\frac{\alpha(x)}{\alpha(x_0)} \exp\left(\int_{x_0}^{x} \frac{\beta'(y)}{\beta(y)} - \frac{\alpha'(y)}{\alpha(y)}\, dy\right)\right| \leq \left|\frac{\alpha(x)}{\alpha(x_0)}\right| \exp\left(\left\|\frac{\beta'}{\beta} - \frac{\alpha'}{\alpha}\right\|_{\mathcal{D}}\right) = E\left|\frac{\alpha(x)}{\alpha(x_0)}\right|, \tag{4.1.1a}$$

with $E = \exp(\|\beta'/\beta - \alpha'/\alpha\|_{\mathcal{D}})$. It follows that $b \Rightarrow c$, $b \Rightarrow a$, $a \Rightarrow c$, $iii \Rightarrow ii$, $iii \Rightarrow i$, and $i \Rightarrow ii$. Similarly,

$$\left|\frac{\alpha(x)}{\alpha(x_0)}\right| \leq E\left|\frac{\beta(x)}{\beta(x_0)}\right|, \tag{4.1.1b}$$

so that $c \Rightarrow b$, $c \Rightarrow a$, $a \Rightarrow b$, $ii \Rightarrow iii$, $ii \Rightarrow i$, and $i \Rightarrow iii$. □

Next, Lemmas 13–15 present some properties of the functions $\mathfrak{n}(k, x)$ and $(\beta\mathfrak{n})(k, x)$.

Lemma 13. *For $|k| \geq r > \sqrt{M_\gamma}$, where M_γ is defined in Assumption 2.4,*

$$m_{\mathfrak{n}} = \frac{1}{\sqrt{M_{\alpha\beta}}}\sqrt{1 - \frac{M_\gamma}{r^2}} \leq |\mathfrak{n}(k, x)| \leq \frac{1}{\sqrt{m_{\alpha\beta}}}\sqrt{1 + \frac{M_\gamma}{r^2}} = M_{\mathfrak{n}}, \tag{4.1.2}$$

which defines $m_{\mathfrak{n}}, M_{\mathfrak{n}} > 0$. From this, there is also $m_{\mathfrak{n}} \leq |\mu(x)| \leq M_{\mathfrak{n}}$.

Proof. The proof is trivial from the definition of $\mathfrak{n}(k, x)$ in Definition 7 using Assumptions 2.3 and 2.4. □

Lemma 14. *For $|k| \geq r > \sqrt{M_\gamma}$, $(\beta\mathfrak{n})'/(\beta\mathfrak{n}) \in L^1(\mathcal{D})$, and under Assumption 2.6, $\mathfrak{u} \in \mathrm{AC}(\mathcal{D})$.*

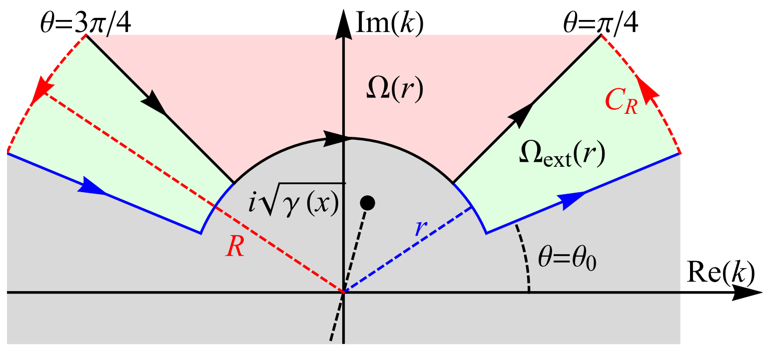


Figure 4.1: The region $\Omega_{\rm ext}(r) = \{k \in \mathbb{C} : |k| > r \text{ and } \theta_0 < \arg(k) < \pi - \theta_0\}$ described in Lemma 15 ($\Omega(r) \cup$ the green regions) and the contour $C_R = \{k \in \mathbb{C} : |k| = R \text{ and } \theta_0 < \arg(k) < \pi/4 \text{ or } 3\pi/4 < \theta < \pi - \theta_0\}$.

Proof. The function

$$\frac{(\beta\mathfrak{n})'(k,x)}{(\beta\mathfrak{n})(k,x)} = \frac{1}{2}\left(\frac{\beta'(x)}{\beta(x)} - \frac{\alpha'(x)}{\alpha(x)} + \frac{\gamma'(x)}{k^2+\gamma(x)}\right) \in L^1(\mathcal{D}), \tag{4.1.3}$$

for $|k| \geq r > \sqrt{M_\gamma}$, by Assumption 2.5. By Assumptions 2.2 and 2.3, $\mu \in \mathrm{AC}(\mathcal{D})$ and, from Assumption 2.6, $\mathfrak{u} \in \mathrm{AC}(\mathcal{D})$. □

Lemma 15. *There exists an $r > \sqrt{M_\gamma}$, $m_{i\mathfrak{n}} > 0$, and $0 < \theta_0 < \pi/4$ such that*

$$\mathrm{Re}(ik\mathfrak{n}(k,x)) \leq -m_{i\mathfrak{n}}|k|, \tag{4.1.4}$$

for $k \in \Omega_{\rm ext}(r)$, where $\Omega_{\rm ext}(r) = \{k \in \mathbb{C} : |k| > r \text{ and } \theta_0 < \arg(k) < \pi - \theta_0\}$, see Figure 4.1.

Proof. With $\phi = \arg(k\mathfrak{n}(k,x))$, $\Theta = \sup_{x\in\mathcal{D}} |\arg(\alpha(x)\beta(x))|$, $\psi = \arg(1+\gamma(x)/k^2)$, (and $\theta = \arg(k)$), from the definition of $\mathfrak{n}(k,x)$ in Definition 7,

$$\theta - \frac{1}{2}(\Theta - \psi) \leq \phi \leq \theta + \frac{1}{2}(\Theta + \psi), \tag{4.1.5}$$

see Figure 4.2a. Using Assumption 2.4 and Figure 4.2b,

$$|\psi| \leq \arcsin\left(\left|\frac{\gamma(x)}{k^2}\right|\right) \leq \arcsin\left(\frac{M_\gamma}{r^2}\right) \leq \frac{2M_\gamma}{r^2}. \tag{4.1.6}$$

Since $0 \leq \Theta < \pi/2$, $r > \sqrt{M_\gamma}$ can be chosen large enough so that

$$|\psi| \leq \frac{1}{2}\left(\frac{\pi}{2} - \Theta\right), \tag{4.1.7}$$

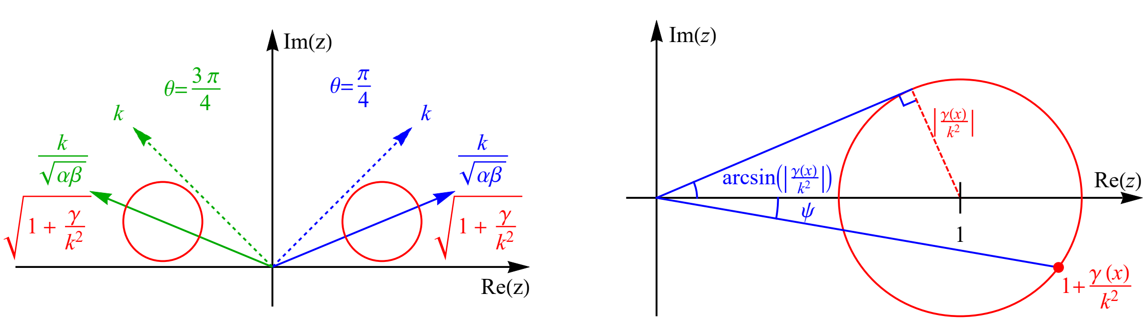


Figure 4.2: (a) The arguments of $k\mathfrak{n}(k,x)$ and its components, (b) $\psi = \arg\left(1+\gamma(x)/k^2\right)$.

which gives, from (4.1.5),

$$\theta-\theta_1=\theta-\frac{1}{4}\left(\Theta+\frac{\pi}{2}\right)\le\theta-\frac{1}{2}\left(\Theta+|\psi|\right)\le\phi\le\theta+\frac{1}{2}\left(\Theta+|\psi|\right)\le\theta+\frac{1}{4}\left(\Theta+\frac{\pi}{2}\right)=\theta+\theta_1, \tag{4.1.8}$$

which defines $0<\theta_1<\pi/4$. For $|k|>r$ and $\theta_1\le\arg(k)\le\pi-\theta_1$, it follows that $0\le\phi\le\pi$, so that $\mathrm{Re}(ik\mathfrak{n}(k,x))\le 0$. In particular, $\mathrm{Re}(ik\mathfrak{n}(k,x))<0$ for $k\in\Omega_{\mathrm{ext}}(r)$. More specifically, using that $\sin(\phi)\ge\phi(\pi-\phi)/\pi$ for $0\le\phi\le\pi$, then for $\theta_1\le\theta\le\pi-\theta_1$,

$$\mathrm{Re}(ik\mathfrak{n}(k,x))=-|k\mathfrak{n}(k,x)|\sin(\phi)\le-\frac{1}{\pi}m_{\mathfrak{n}}|k|\phi(\pi-\phi)\le-\frac{1}{\pi}m_{\mathfrak{n}}|k|(\theta-\theta_1)(\pi-\theta_1-\theta). \tag{4.1.9}$$

Finally, (4.1.4) follows from choosing θ_0 such that $0\le\theta_1<\theta_0<\pi/4$ and letting $m_{i\mathfrak{n}}=m_{\mathfrak{n}}(\theta_0-\theta_1)/4$. □

Having established some properties of the coefficient functions α,β and the *dispersion functions* $\mathfrak{n}$, $(\beta\mathfrak{n})$, we define a generalization $\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)$ of the functions $\mathcal{E}_n^{(a,b)}(k)$, $\tilde{\mathcal{E}}_n^{(a,b)}(k)$, $\mathcal{C}_n^{(a,b)}(k)$, and $\mathcal{S}_n^{(a,b)}(k)$, and we show some relations between these functions. Further, we show these functions are bounded and well defined, and we find their large-k asymptotics.

Definition 16. *For some $r>\sqrt{M_\gamma}$, for $(a,b)\subseteq\mathcal{D}$, $k\in\Omega_{\mathrm{ext}}(r)$, and integer $n>0$, define the function*

$$\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)=\frac{1}{2^n}\int_{\mathcal{D}_n^{(a,b)}}\left(\prod_{p=1}^{n}\frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)}\right)\exp\left(\sum_{p=0}^{n}\sigma_{p,n}\int_{y_p}^{y_{p+1}}ik\mathfrak{n}(k,\xi)\,d\xi\right)d\mathbf{y}_n, \tag{4.1.10a}$$

where $\sigma_{p,n}$ is a non-negative integer-valued function of n and $p=0,1,\dots,n$. Here we require for any p that $\sigma_{p,n}\neq\sigma_{p+1,n}$, and $\sigma_{p,n}\le M_\sigma$ for all p and n. For $n<0$, we define $\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)=0$, and for $n=0$, we define

$$\mathcal{J}_0^{(a,b)}[\sigma_{0,0}](k)=\exp\left(\sigma_{0,0}\int_a^b ik\mathfrak{n}(k,\xi)\,d\xi\right). \tag{4.1.10b}$$

The function $\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)$ is defined as $a\to-\infty$ if $\sigma_{0,n}=0$, and as $b\to\infty$ if $\sigma_{n,n}=0$ (if $\mathcal{D}$ is unbounded).

Finally, we define

$$\mathfrak{C}_n^{(a,b)}(k)=\exp\left(\int_a^b ik\mathfrak{n}(k,\xi)\,d\xi\right)\mathcal{C}_n^{(a,b)}(k)\qquad\text{and}\qquad\mathfrak{S}_n^{(a,b)}(k)=\exp\left(\int_a^b ik\mathfrak{n}(k,\xi)\,d\xi\right)\mathcal{S}_n^{(a,b)}(k), \tag{4.1.11}$$

where $\mathcal{C}_n^{(a,b)}(k)$ and $\mathcal{S}_n^{(a,b)}(k)$ are defined in (1.2.19).

Lemma 17. *With $\mathcal{E}_n^{(a,b)}(k)$ and $\tilde{\mathcal{E}}_n^{(a,b)}(k)$ defined in (1.2.18), and $\mathfrak{C}_n^{(a,b)}(k)$ and $\mathfrak{S}_n^{(a,b)}(k)$ defined in (4.1.11), we have the following relations:*

$$\mathcal{E}_n^{(a,b)}(k)=\mathcal{J}_n^{(a,b)}[1-(-1)^{n-p}](k), \tag{4.1.12a}$$

$$\tilde{\mathcal{E}}_n^{(a,b)}(k)=\mathcal{J}_n^{(a,b)}[1-(-1)^{p}](k), \tag{4.1.12b}$$

$$\mathfrak{C}_n^{(a,b)}(k)=\frac{1}{2}\left[\mathcal{J}_n^{(a,b)}[1+(-1)^{p}](k)+\mathcal{J}_n^{(a,b)}[1-(-1)^{p}](k)\right], \tag{4.1.12c}$$

$$\mathfrak{S}_n^{(a,b)}(k)=\frac{1}{2i}\left[\mathcal{J}_n^{(a,b)}[1+(-1)^{p}](k)-\mathcal{J}_n^{(a,b)}[1-(-1)^{p}](k)\right]. \tag{4.1.12d}$$

Proof. The proofs follow immediately from the definitions in (1.2.18) and (1.2.19). □

The next two lemmas give bounds and asymptotics for the function $\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)$.

Lemma 18. *For $(x,y)\subseteq(a,b)\subseteq\mathcal{D}$, $k\in\Omega_{\mathrm{ext}}(r)$, and r from Lemma 15,*

$$\left|\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)\right|\le\frac{1}{2^n n!}\left\|\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right\|_{(a,b)}^n\quad\text{and}\quad\left|\sum_{\ell=0}^{n}(-1)^{\lambda\ell}\mathcal{J}_{n-\ell}^{(a,x)}[\sigma_{p,n-\ell}](k)\mathcal{J}_\ell^{(y,b)}[\tilde{\sigma}_{p,\ell}](k)\right|\le\frac{1}{2^n n!}\left\|\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right\|_{(a,b)}^n, \tag{4.1.13}$$

where $\lambda=0,1$. These inequalities hold as $a\to-\infty$ and $b\to\infty$, provided the functions are defined. Thus, $\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)$ is well defined. The same bounds hold for $\mathcal{E}_n^{(a,b)}(k)$, $\tilde{\mathcal{E}}_n^{(a,b)}(k)$, $\mathfrak{C}_n^{(a,b)}(k)$, and $\mathfrak{S}_n^{(a,b)}(k)$.

Proof. By Lemma 15, the exponentials in (4.1.10) are bounded by 1 for $k \in \Omega_{\text{ext}}(r)$. Using Lemma 14,

$$\left|\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)\right| \le \frac{1}{2^n} \int_{a<y_1<\cdots<y_n<b} \left| \prod_{p=1}^{n} \frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)} \right| dy_1 \cdots dy_n = \frac{1}{2^n n!} \left\| \frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})} \right\|_{(a,b)}^n, \tag{4.1.14a}$$

so that

$$\begin{aligned}\left| \sum_{\ell=0}^{n} (-1)^{\lambda\ell} \mathcal{J}_{n-\ell}^{(a,x)}[\sigma_{p,n-\ell}](k) \mathcal{J}_{\ell}^{(y,b)}[\sigma_{p,\ell}](k) \right| &\le \sum_{\ell=0}^{n} \frac{1}{2^n (n-\ell)!\ell!} \left\| \frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})} \right\|_{(a,x)}^{n-\ell} \left\| \frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})} \right\|_{(y,b)}^{\ell} \\ &= \frac{1}{2^n n!} \left(\left\| \frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})} \right\|_{(a,x)} + \left\| \frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})} \right\|_{(y,b)} \right)^n \le \frac{1}{2^n n!} \left\| \frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})} \right\|_{(a,b)}^n. \end{aligned} \tag{4.1.14b}$$

For $\mathcal{E}_n^{(a,b)}(k)$, $\tilde{\mathcal{E}}_n^{(a,b)}(k)$, $\mathcal{C}_n^{(a,b)}(k)$, and $\mathcal{S}_n^{(a,b)}(k)$, the result follows from (4.1.12). These bounds hold as $a \to -\infty$ or $b \to \infty$, provided the functions are defined. ☐

Lemma 19. *On the finite-interval, we have*

$$\exp\left(ik \int_a^b \mathfrak{n}(k,\xi)\, d\xi \right) = \exp\left(ik \int_a^b \mu(\xi)\, d\xi \right) \left(1 + O(k^{-1})\right). \tag{4.1.15}$$

Proof. Using the definition of $\mathfrak{n}(k,x)$ in Definition 7, we have

$$\exp\left(ik \int_a^b \mathfrak{n}(k,\xi)\, d\xi \right) = \exp\left(ik \int_a^b \mu(\xi)\, d\xi \right) \exp\left(ik \int_a^b \mu(\xi) \left(\mathfrak{g}(k,\xi) - 1\right)\, d\xi \right), \tag{4.1.16}$$

which gives (4.1.15). ☐

Lemma 20. *There exists $r > \sqrt{M_\gamma}$, such that for any $(a,b) \subseteq \mathcal{D}$ and $n \ge 1$,*

$$\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k) \to 0, \qquad \text{as } |k| \to \infty, \qquad k \in \Omega_{\text{ext}}(r). \tag{4.1.17a}$$

This result holds as $a \to -\infty$ and $b \to \infty$, provided the functions are defined. The result extends to $\mathcal{E}_n^{(a,b)}(k)$, $\tilde{\mathcal{E}}_n^{(a,b)}(k)$, $\mathcal{C}_n^{(a,b)}(k)$, and $\mathcal{S}_n^{(a,b)}(k)$.

Next, we define $\lambda_{p,n} = \sigma_{p-1,n} - \sigma_{p,n}$. Using Assumption 2.6 and since $\lambda_{p,n} \neq 0$ (see Definition 16), we have

$$\mathcal{J}_1^{(a,b)}[\sigma_{p,n}](k) = \frac{1}{4\lambda_{1,1} ik} \left[\mathfrak{u}(b) \exp\left(\sigma_{0,1} \int_a^b ik\mu(\xi)\, d\xi \right) - \mathfrak{u}(a) \exp\left(\sigma_{1,1} \int_a^b ik\mu(\xi)\, d\xi \right) \right] + o(k^{-1}). \tag{4.1.17b}$$

There exists $r > \sqrt{M_\gamma}$ and $C > 1$ such that, for any $(a,b) \subseteq \mathcal{D}$ and $k \in \Omega_{\text{ext}}(r)$,

$$\left|\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)\right| \le \frac{C^n}{|k|^{\lfloor \frac{n+1}{2} \rfloor}}, \tag{4.1.17c}$$

where $\lfloor \cdot \rfloor$ is the floor function.

Proof. From Lemma 15, for any $r > \sqrt{M_\gamma}$ and for all $k \in \Omega_{\text{ext}}(r)$, we have

$$\begin{aligned}\left|\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)\right| &= \frac{1}{2^n} \left| \int_{\mathcal{D}_n^{(a,b)}} \left(\prod_{p=1}^{n} \frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)} \right) \exp\left(\sum_{p=0}^{n} \sigma_{p,n} \int_{y_p}^{y_{p+1}} ik\mathfrak{n}(k,\xi)\, d\xi \right) dy_1 \cdots dy_n \right| \\ &\le \frac{1}{2^n} \int_{\mathcal{D}_n^{(a,b)}} \left(\prod_{p=1}^{n} \left| \frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)} \right| \right) \exp\left(-m_{\mathfrak{in}} |k| \sum_{p=0}^{n} \sigma_{p,n} (y_{p+1} - y_p) \right) dy_1 \cdots dy_n, \end{aligned} \tag{4.1.18}$$

and since $\sigma_{p,n} \ge 0$ and $\sigma_{p,n} \neq \sigma_{p+1,n}$ for any p, the argument of the exponential is strictly negative. Thus, by Lemma 18 and the Dominated Convergence Theorem (DCT), we have (4.1.17a). For $\mathcal{E}_n^{(a,b)}(k)$, $\tilde{\mathcal{E}}_n^{(a,b)}(k)$, $\mathcal{C}_n^{(a,b)}(k)$, and $\mathcal{S}_n^{(a,b)}(k)$, the result follows from (4.1.12).

Using Assumption 2.5, (1.2.6), (4.1.3), and (4.1.15) in (4.1.10) for $n = 1$, we have

$$\mathcal{J}_1^{(a,b)}[\sigma_{p,n}](k) = \frac{1 + O(k^{-1})}{4} \int_a^b \mathfrak{u}(y)\mu(y) \exp\left(\sigma_{0,1} \int_a^y + \sigma_{1,1} \int_y^b ik\mu(\xi)\, d\xi\right) dy + O(k^{-2}). \tag{4.1.19}$$

By Lemma 14, $\mathfrak{u} \in \mathrm{AC}(\mathcal{D})$ and integration by parts gives

$$\begin{aligned}\mathcal{J}_1^{(a,b)}[\sigma_{p,n}](k) = \frac{1}{4\lambda_{1,1} ik} &\left(\mathfrak{u}(b) \exp\left(\sigma_{0,1} \int_a^b ik\mu(\xi)\, d\xi\right) - \mathfrak{u}(a) \exp\left(\sigma_{1,1} \int_a^b ik\mu(\xi)\, d\xi\right)\right) \\ &- \frac{1}{4\lambda_{1,1} ik} \int_a^b \mathfrak{u}'(y) \exp\left(\sigma_{0,1} \int_a^y + \sigma_{1,1} \int_y^b ik\mu(\xi)\, d\xi\right) dy + O(k^{-2}).\end{aligned} \tag{4.1.20}$$

By Lemma 14 and the DCT, we obtain (4.1.17b).

Inequality (4.1.17c) for $n = 0$ and $n = 1$ follows from (4.1.10b) and (4.1.17b), respectively. Using (1.2.6), (4.1.3), and (4.1.15) in (4.1.10) for $n \geq 2$, we have

$$\begin{aligned}\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k) = \frac{1 + O(k^{-1})}{2^{n+1}} &\int_{\mathcal{D}_n^{(a,b)}} \left(\prod_{p=1}^{n-1} \frac{(\beta\mathfrak{n})'(k, y_p)}{(\beta\mathfrak{n})(k, y_p)}\right) \mathfrak{u}(y_n)\mu(y_n) \exp\left(\sum_{p=0}^{n} \sigma_{p,n} \int_{y_p}^{y_{p+1}} ik\mu(\xi)\, d\xi\right) d\mathbf{y}_n \\ + \frac{1 + O(k^{-1})}{2^{n+1}} &\int_{\mathcal{D}_n^{(a,b)}} \left(\prod_{p=1}^{n-1} \frac{(\beta\mathfrak{n})'(k, y_p)}{(\beta\mathfrak{n})(k, y_p)}\right) \left(\frac{\gamma'(y_n)}{k^2 + \gamma(y_n)}\right) \exp\left(\sum_{p=0}^{n} \sigma_{p,n} \int_{y_p}^{y_{p+1}} ik\mu(\xi)\, d\xi\right) d\mathbf{y}_n.\end{aligned} \tag{4.1.21}$$

Let $\mathcal{I}_n^{(a,b)}(k)$ denote the integral in the first line of (4.1.21):

$$\mathcal{I}_n^{(a,b)}(k) = \int_{\mathcal{D}_n^{(a,b)}} \left(\prod_{p=1}^{n-1} \frac{(\beta\mathfrak{n})'(k, y_p)}{(\beta\mathfrak{n})(k, y_p)}\right) \mathfrak{u}(y_n)\mu(y_n) \exp\left(\sum_{p=0}^{n} \sigma_{p,n} \int_{y_p}^{y_{p+1}} ik\mu(\xi)\, d\xi\right) d\mathbf{y}_n, \tag{4.1.22}$$

with

$$\mathcal{I}_0^{(a,b)}(k) = \exp\left(ik \int_a^b \mu(\xi)\, d\xi\right). \tag{4.1.23}$$

Integration by parts with respect to $y_n \in (y_{n-1}, b)$ gives

$$\begin{aligned}\mathcal{I}_n^{(a,b)}(k) = \frac{\mathfrak{u}(b)}{ik\lambda_{n,n}} &\int_{\mathcal{D}_{n-1}^{(a,b)}} \left(\prod_{p=1}^{n-1} \frac{(\beta\mathfrak{n})'(k, y_p)}{(\beta\mathfrak{n})(k, y_p)}\right) \exp\left(\sum_{p=0}^{n-1} \sigma_{p,n} \int_{y_p}^{y_{p+1}} ik\mu(\xi)\, d\xi\right) d\mathbf{y}_{n-1} \\ &- \int_{\mathcal{D}_{n-1}^{(a,b)}} \left(\prod_{p=1}^{n-1} \frac{(\beta\mathfrak{n})'(k, y_p)}{(\beta\mathfrak{n})(k, y_p)}\right) \frac{\mathfrak{u}(y_{n-1})}{ik\lambda_{n,n}} \exp\left(\sigma_{n,n} \int_{y_{n-1}}^{b} ik\mu(\xi)\, d\xi + \sum_{p=0}^{n-2} \sigma_{p,n} \int_{y_p}^{y_{p+1}} ik\mu(\xi)\, d\xi\right) d\mathbf{y}_{n-1} \\ &- \int_{\mathcal{D}_n^{(a,b)}} \left(\prod_{p=1}^{n-1} \frac{(\beta\mathfrak{n})'(k, y_p)}{(\beta\mathfrak{n})(k, y_p)}\right) \frac{\mathfrak{u}'(y_n)}{ik\lambda_{n,n}} \exp\left(\sum_{p=0}^{n-1} \sigma_{p,n} \int_{y_p}^{y_{p+1}} ik\mu(\xi)\, d\xi\right) d\mathbf{y}_n.\end{aligned} \tag{4.1.24}$$

In the second line of (4.1.24), we integrate over $y_{n-1} \in (a, b)$ last, and performing the remaining integral over $a = y_0 < y_1 < \cdots < y_{n-2} < y_{n-1}$ first. Similarly, in the third line of (4.1.24), we integrate over $y_n \in (a, b)$ last and leave the remaining integral over $a = y_0 < y_1 < \cdots < y_{n-1} < y_n$ to be done first. Returning to (4.1.21) yields

$$\begin{aligned}\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k) &= \frac{1+O(k^{-1})}{2^{n+1}}\frac{\mathfrak{u}(b)}{\lambda_{n,n}ik}\int_{\mathcal{D}_{n-1}^{(a,b)}}\left(\prod_{p=1}^{n-1}\frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)}\right)\exp\left(\sum_{p=0}^{n-1}\sigma_{p,n}\int_{y_p}^{y_{p+1}}ik\mu(\xi)\,d\xi\right)d\mathbf{y}_{n-1}\\ &\quad-\frac{1+O(k^{-1})}{2^{n+1}}\int_a^b dy_{n-1}\,\frac{\mathfrak{u}(y_{n-1})}{\lambda_{n,n}ik}\frac{(\beta\mathfrak{n})'(k,y_{n-1})}{(\beta\mathfrak{n})(k,y_{n-1})}\exp\left(\sigma_{n,n}\int_{y_{n-1}}^{b}ik\mu(\xi)\,d\xi\right)\times\\ &\qquad\times\int_{\mathcal{D}_{n-2}^{(a,y_{n-1})}}\left(\prod_{p=1}^{n-2}\frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)}\right)\exp\left(\sum_{p=0}^{n-2}\sigma_{p,n}\int_{y_p}^{y_{p+1}}ik\mu(\xi)\,d\xi\right)d\mathbf{y}_{n-2}\\ &\quad-\frac{1+O(k^{-1})}{2^{n+1}}\int_a^b dy_n\left(\frac{\mathfrak{u}'(y_n)}{\lambda_{n,n}ik}+\frac{\gamma'(y_n)}{k^2+\gamma(y_n)}\right)\exp\left(\sigma_{n,n}\int_{y_n}^{b}ik\mu(\xi)\,d\xi\right)\times\\ &\qquad\times\int_{\mathcal{D}_{n-1}^{(a,y_n)}}\left(\prod_{p=1}^{n-1}\frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)}\right)\exp\left(\sum_{p=0}^{n-1}\sigma_{p,n}\int_{y_p}^{y_{p+1}}ik\mu(\xi)\,d\xi\right)d\mathbf{y}_{n-1},\end{aligned}\tag{4.1.25}$$

which gives the asymptotic recurrence relation

$$\begin{aligned}\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k) &= \frac{1+O(k^{-1})}{4}\frac{\mathfrak{u}(b)}{\lambda_{n,n}ik}\mathcal{J}_{n-1}^{(a,b)}[\sigma_{p,n}](k)\\ &\quad-\frac{1+O(k^{-1})}{8}\int_a^b\frac{\mathfrak{u}(y_{n-1})}{\lambda_{n,n}ik}\frac{(\beta\mathfrak{n})'(k,y_{n-1})}{(\beta\mathfrak{n})(k,y_{n-1})}\exp\left(\sigma_{n,n}\int_{y_{n-1}}^{b}ik\mu(\xi)\,d\xi\right)\mathcal{J}_{n-2}^{(a,y_{n-1})}[\sigma_{p,n}](k)\,dy_{n-1}\\ &\quad-\frac{1+O(k^{-1})}{4}\int_a^b\left(\frac{\mathfrak{u}'(y_n)}{\lambda_{n,n}ik}+\frac{\gamma'(y_n)}{k^2+\gamma(y_n)}\right)\exp\left(\sigma_{n,n}\int_{y_n}^{b}ik\mu(\xi)\,d\xi\right)\mathcal{J}_{n-1}^{(a,y_n)}[\sigma_{p,n}](k)\,dy_n.\end{aligned}\tag{4.1.26}$$

Assuming (4.1.17c) holds for $n = 0, 1, \ldots, m-1$, and using that $|\lambda_{p,n}| \geq 1$, we find

$$\left|\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)\right| \leq \frac{1+O(k^{-1})}{4}\left[\frac{\|\mathfrak{u}\|_\infty}{|k|}\frac{C^{n-1}}{|k|^{\lfloor\frac{n}{2}\rfloor}}+\frac{\|\mathfrak{u}\|_\infty}{2|k|}\left\|\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right\|_{\mathcal{D}}\frac{C^{n-2}}{|k|^{\lfloor\frac{n-1}{2}\rfloor}}+\left(\frac{\|\mathfrak{u}'\|_{\mathcal{D}}}{|k|}+\frac{\|\gamma'\|_{\mathcal{D}}}{|k|^2-M_\gamma}\right)\frac{C^{n-1}}{|k|^{\lfloor\frac{n}{2}\rfloor}}\right],\tag{4.1.27}$$

which, using Lemma 14, gives (4.1.17c) for $n \geq 0$ by induction. □

Having defined the function $\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)$ and established some of its properties, we prove that the function $\Delta(k)$ is bounded and well defined, and that the "transforms" $\Phi_0(k,x)$, $\Phi_f(k,x,t)$ and $\mathcal{B}_m(k,x)$, of the initial condition $q_0(x)$, the inhomogeneous function $f(x,t)$, and the boundary functions $f_m(t)$, respectively, are bounded and well defined.

Definition 21. *We define*

$$\Phi_0(k,x) = \int_{\mathcal{D}}\frac{\Psi(k,x,y)q_\alpha(y)}{\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,y)}}\,dy,\tag{4.1.28a}$$

$$\Phi_f(k,x,t) = \int_{\mathcal{D}}\frac{\Psi(k,x,y)\tilde{f}_\alpha(k^2,y,t)}{\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,y)}}\,dy,\tag{4.1.28b}$$

so that $\Phi_\psi(k,x,t) = \Phi_0(k,x) + \Phi_f(k,x,t)$. We define the corresponding parts of the solution as

$$q_0(x,t) = \frac{1}{2\pi}\int_{\partial\Omega(r)}\frac{\Phi_0(k,x)}{\Delta(k)}e^{-k^2t}\,dk,\tag{4.1.29a}$$

$$q_f(x,t) = \frac{1}{2\pi}\int_{\partial\Omega(r)}\frac{\Phi_f(k,x,t)}{\Delta(k)}e^{-k^2t}\,dk,\tag{4.1.29b}$$

$$q_{\mathcal{B}_m}(x,t) = \frac{1}{2\pi}\int_{\partial\Omega(r)}\frac{\mathcal{B}_m(k,x)}{\Delta(k)}F_m(k^2,t)e^{-k^2t}\,dk,\qquad m=0,1,\tag{4.1.29c}$$

where we define $\mathcal{B}_m(k,x) = 0$ $(m = 0,1)$ for the whole-line problem and $\mathcal{B}_1(k,x) = 0$ for the half-line problem. Thus $q(x,t) = q_0(x,t) + q_f(x,t) + q_{\mathcal{B}_0}(x,t) + q_{\mathcal{B}_1}(x,t)$ for the finite-interval, half-line and whole-line problems.

Lemma 22. *For all three problems, there exists $r > \sqrt{M_\gamma}$ and $M_\Delta > 0$, so that for all $k \in \Omega_{\text{ext}}(r)$,*

$$\Delta(k) = \mathfrak{b}_0(k)(1+\varepsilon(k)) \qquad \text{and} \qquad \frac{1}{2}|\mathfrak{b}_0(k)| \le |\Delta(k)| \le M_\Delta, \tag{4.1.30}$$

where $|\varepsilon(k)| < 1/2$. For the whole-line problem,

$$\mathfrak{b}_0(k) = 1; \tag{4.1.31}$$

for the half-line problem, if $a_1 \neq 0$,

$$\mathfrak{b}_0(k) = -2a_1, \tag{4.1.32a}$$

if $a_1 = 0$,

$$\mathfrak{b}_0(k) = -\frac{2a_0}{ik\mu(x_l)}; \tag{4.1.32b}$$

and, for the finite-interval problem,

1. *if $(a:b)_{2,4} \neq 0$, then*

$$\mathfrak{b}_0(k) = -(a:b)_{2,4}; \tag{4.1.33a}$$

2. *if $(a:b)_{2,4} = 0$ and $m_{\mathfrak{c}_0} \neq 0$, then*

$$\mathfrak{b}_0(k) = \frac{im_{\mathfrak{c}_0}}{k}; \tag{4.1.33b}$$

3. *if $(a:b)_{2,4} = 0$, $m_{\mathfrak{c}_0} = 0$, $m_{\mathfrak{c}_1} = 0$, and $(a:b)_{1,3} \neq 0$, then*

$$\mathfrak{b}_0(k) = -\frac{m_{\mathfrak{s}}}{k^2}; \tag{4.1.33c}$$

4. *with Assumption 2.6, if $(a:b)_{2,4} = 0$, $m_{\mathfrak{c}_0} = 0$, $m_{\mathfrak{c}_1} \neq 0$, and $m_{\mathfrak{c}_1}\mathfrak{u}_+ - 8m_{\mathfrak{s}} \neq 0$, then*

$$\mathfrak{b}_0(k) = \frac{1}{8k^2}\left(m_{\mathfrak{c}_1}\mathfrak{u}_+ - 8m_{\mathfrak{s}}\right). \tag{4.1.33d}$$

Proof. For the whole-line problem,

$$\Delta(k) = 1 + \sum_{n=1}^{\infty} \mathcal{E}_{2n}^{(-\infty,\infty)}(k) = 1 + \varepsilon(k). \tag{4.1.34}$$

By Lemmas 18 and 20 and the DCT,

$$\varepsilon(k) = \sum_{n=1}^{\infty} \mathcal{E}_{2n}^{(-\infty,\infty)}(k) \to 0, \tag{4.1.35}$$

as $|k| \to \infty$. Thus, we can choose r sufficiently large so that for $k \in \Omega_{\text{ext}}(r)$, $|\varepsilon(k)| < 1/2$, and

$$\frac{1}{2} \le 1 - |\varepsilon(k)| \le |\Delta(k)| \le 1 + |\varepsilon(k)| < \frac{3}{2}. \tag{4.1.36}$$

For the half-line problem, if $a_1 \neq 0$, we write

$$\Delta(k) = 2\sum_{n=0}^{\infty}\left(\frac{(-1)^n ia_0}{k\mathfrak{n}(k,x_l)} - a_1\right)\mathcal{E}_n^{(x_l,\infty)}(k) = -2a_1\left[1 - \frac{ia_0}{a_1 k\mathfrak{n}(k,x_l)} + \sum_{n=1}^{\infty}\left(-\frac{(-1)^n ia_0}{a_1 k\mathfrak{n}(k,x_l)} + 1\right)\mathcal{E}_n^{(x_l,\infty)}(k)\right]. \tag{4.1.37}$$

Then

$$|\varepsilon(k)| = \left|-\frac{ia_0}{a_1 k\mathfrak{n}(k,x_l)} + \sum_{n=1}^{\infty}\left(-\frac{(-1)^n a_0}{a_1 k\mathfrak{n}(k,x_l)} - 1\right)\mathcal{E}_n^{(x_l,\infty)}(k)\right| \le \frac{|a_0|}{m_{\mathfrak{n}}|a_1 k|} + \left(\frac{|a_0|}{m_{\mathfrak{n}}|a_1 k|} + 1\right)\sum_{n=1}^{\infty}\left|\mathcal{E}_n^{(x_l,\infty)}(k)\right| \to 0, \tag{4.1.38}$$

by the DCT. We choose $r > \sqrt{M_\gamma}$ large enough such that $|\varepsilon(k)| < 1/2$ for $k \in \Omega_{\text{ext}}(r)$. On the other hand, if $a_1 = 0$, then $a_0 \neq 0$, and

$$\Delta(k) = 2\sum_{n=0}^{\infty}\frac{(-1)^n ia_0}{k\mathfrak{n}(k,x_l)}\mathcal{E}_n^{(x_l,\infty)}(k) = \frac{2ia_0}{k\mu(x_l)}\left[1 + \frac{\mu(x_l)}{\mathfrak{n}(k,x_l)} - 1 + \sum_{n=1}^{\infty}\frac{(-1)^n\mu(x_l)}{\mathfrak{n}(k,x_l)}\mathcal{E}_n^{(x_l,\infty)}(k)\right]. \tag{4.1.39}$$

Then since $\mu(x_l)/\mathfrak{n}(k,x_l) = 1 + O(k^{-2})$,

$$|\varepsilon(k)| = \left| \frac{\mu(x_l)}{\mathfrak{n}(k,x_l)} - 1 + \sum_{n=1}^{\infty} \frac{(-1)^n \mu(x_l)}{\mathfrak{n}(k,x_l)} \mathcal{E}_n^{(x_l,\infty)}(k) \right| \to 0, \tag{4.1.40}$$

by the DCT. We choose $r > \sqrt{M_\gamma}$ large enough such that $|\varepsilon(k)| < 1/2$ for $k \in \Omega_{\text{ext}}(r)$.

For the finite-interval problem, since

$$\mathcal{C}_0^{(x_l,x_r)}(k) = \Xi(k)\mathcal{C}_0^{(x_l,x_r)}(k) = \frac{1}{2}\left(\Xi(k)^2 + 1\right) \qquad \text{and} \qquad \mathcal{S}_0^{(x_l,x_r)}(k) = \Xi(k)\mathcal{S}_0^{(x_l,x_r)}(k) = \frac{1}{2i}\left(\Xi(k)^2 - 1\right), \tag{4.1.41}$$

where $\Xi(k)$ is defined in (2.3.3) and $\mathcal{C}_n^{(a,b)}(k)$ and $\mathcal{S}_n^{(a,b)}(k)$ are defined in (4.1.11), we factor out the $n = 0$ term in (2.3.4) and write

$$\Delta(k) = 2i\mathfrak{a}(k)\Xi(k) + i\mathfrak{c}_0(k)\left(\Xi(k)^2 + 1\right) + \mathfrak{s}_0(k)\left(\Xi(k)^2 - 1\right) + 2i\sum_{n=1}^{\infty}\left(\mathfrak{c}_n(k)\mathcal{C}_n^{(x_l,x_r)}(k) + \mathfrak{s}_n(k)\mathcal{S}_n^{(x_l,x_r)}(k)\right). \tag{4.1.42}$$

Since $\Xi(k) \to 0$ exponentially fast, we have

$$\Delta(k) = i\mathfrak{c}_0(k) - \mathfrak{s}_0(k) + 2i\sum_{n=1}^{\infty}\left(\mathfrak{c}_n(k)\mathcal{C}_n^{(x_l,x_r)}(k) + \mathfrak{s}_n(k)\mathcal{S}_n^{(x_l,x_r)}(k)\right) + o(k^{-2}). \tag{4.1.43}$$

1. If $(a:b)_{2,4} \neq 0$, then we can write (4.1.30) with $\mathfrak{b}_0(k)$ defined in (4.1.33a), where

$$\varepsilon(k) = \frac{-1}{(a:b)_{2,4}}\left[i\mathfrak{c}_0(k) - \mathfrak{s}_0(k) + (a:b)_{2,4} + 2i\sum_{n=1}^{\infty}\left(\mathfrak{c}_n(k)\mathcal{C}_n^{(x_l,x_r)}(k) + \mathfrak{s}_n(k)\mathcal{S}_n^{(x_l,x_r)}(k)\right)\right] + o(k^{-2}). \tag{4.1.44}$$

   Since $\mathfrak{c}_n(k) = O(k^{-1})$, $\mathfrak{s}_0(k) = (a:b)_{2,4} + O(k^{-2})$, $\mathfrak{s}_n(k) = O(k^0)$, and because both $\mathcal{C}_n^{(x_l,x_r)}(k) \to 0$ and $\mathcal{S}_n^{(x_l,x_r)}(k) \to 0$ by Lemma 20 and both are bounded (see Lemma 18), we can choose $r > \sqrt{M_\gamma}$ sufficiently large so that $|\varepsilon(k)| < 1/2$ for $k \in \Omega_{\text{ext}}(r)$, by the DCT. We have

$$\frac{1}{2}|(a:b)_{2,4}| \leq |\Delta(k)| \leq \frac{3}{2}|(a:b)_{2,4}|. \tag{4.1.45}$$

2. If $(a:b)_{2,4} = 0$ and $m_{\mathfrak{c}_0} \neq 0$, then we can write (4.1.30) with $\mathfrak{b}_0(k)$ defined in (4.1.33b), where

$$\varepsilon(k) = \frac{k}{im_{\mathfrak{c}_0}}\left[i\mathfrak{c}_0(k) - \frac{im_{\mathfrak{c}_0}}{k} - \mathfrak{s}_0(k) + 2i\sum_{n=1}^{\infty}\left(\mathfrak{c}_n(k)\mathcal{C}_n^{(x_l,x_r)}(k) + \mathfrak{s}_n(k)\mathcal{S}_n^{(x_l,x_r)}(k)\right)\right] + o(k^{-1}). \tag{4.1.46}$$

   Since $\mathfrak{c}_0(k) = m_{\mathfrak{c}_0}/k + O(k^{-3})$, $\mathfrak{s}_n(k) = O(k^{-2})$, $\mathfrak{c}_n(k) = O(k^{-1})$, and since $\mathcal{C}_n^{(x_l,x_r)}(k) \to 0$ and $\mathcal{S}_n^{(x_l,x_r)}(k) \to 0$ and both are bounded (see Lemma 18), we can choose $r > \sqrt{M_\gamma}$ large enough such that $|\varepsilon(k)| < 1/2$ for $k \in \Omega_{\text{ext}}(r)$, by the DCT. We have

$$\frac{|m_{\mathfrak{c}_0}|}{2|k|} \leq |\Delta(k)| \leq \frac{3|m_{\mathfrak{c}_0}|}{2r}. \tag{4.1.47}$$

3. If $(a:b)_{2,4} = 0$, $m_{\mathfrak{c}_0} = 0$, $m_{\mathfrak{c}_1} = 0$, and $(a:b)_{1,3} \neq 0$, then $\mathfrak{c}_0(k) = \mathfrak{c}_1(k) = 0$ and we can write (4.1.30) with $\mathfrak{b}_0(k)$ defined in (4.1.33c), where

$$\varepsilon(k) = -\frac{k^2}{m_{\mathfrak{s}}}\left[-\mathfrak{s}_0(k) + \frac{m_{\mathfrak{s}}}{k^2} + 2i\sum_{n=1}^{\infty}\mathfrak{s}_n(k)\mathcal{S}_n^{(x_l,x_r)}(k)\right] + o(k^0). \tag{4.1.48}$$

   Since $\mathfrak{s}_n(k) = m_{\mathfrak{s}}/k^2 + O(k^{-4})$, and since $\mathcal{S}_n^{(x_l,x_r)}(k) \to 0$ and is bounded (see Lemma 18), we can choose $r > \sqrt{M_\gamma}$ sufficiently large so that $|\varepsilon(k)| < 1/2$ for $k \in \Omega_{\text{ext}}(r)$, by the DCT. We have

$$\frac{|m_{\mathfrak{s}}|}{2|k|^2} \leq |\Delta(k)| \leq \frac{3|m_{\mathfrak{s}}|}{2r^2}. \tag{4.1.49}$$

4. If $(a:b)_{2,4}=0$, $m_{\mathfrak{c}_0}=0$, $m_{\mathfrak{c}_1}\neq 0$, and $m_{\mathfrak{c}_1}\mathfrak{u}_+-8m_{\mathfrak{s}}\neq 0$, we can write (4.1.30) with $\mathfrak{b}_0(k)$ defined in (4.1.33d), where

$$\varepsilon(k)=\frac{8k^2}{m_{\mathfrak{c}_1}\mathfrak{u}_+-8m_{\mathfrak{s}}}\left[i\mathfrak{c}_0(k)+2i\mathfrak{c}_0(k)\mathcal{C}_2^{(x_l,x_r)}(k)-\mathfrak{s}_0(k)+2i\mathfrak{c}_1(k)\mathcal{C}_1^{(x_l,x_r)}(k)-\frac{m_{\mathfrak{c}_1}\mathfrak{u}_+-8m_{\mathfrak{s}}}{8k^2}\right.$$
$$\left.+2i\sum_{n=3}^{\infty}\mathfrak{c}_n(k)\mathcal{C}_n^{(x_l,x_r)}(k)+2i\Xi(k)\sum_{n=1}^{\infty}\mathfrak{s}_n(k)\mathcal{S}_n^{(x_l,x_r)}(k)\right]+o(k^0). \tag{4.1.50}$$

By Lemmas 17 and 20, we have

$$\begin{aligned}\mathcal{C}_1^{(x_l,x_r)}(k)&=\frac{1}{2}\left[\mathcal{J}_1^{(x_l,x_r)}[1+(-1)^p](k)+\mathcal{J}_1^{(x_l,x_r)}[1-(-1)^p](k)\right]\\&=-\frac{1}{16ik}\left(\mathfrak{u}(x_r)+\mathfrak{u}(x_l)\right)\left(1-\exp\left(\int_{x_l}^{x_r}2ik\mu(\xi)\,d\xi\right)\right)+o(k^{-1})=-\frac{1}{16ik}\mathfrak{u}_++o(k^{-1}).\end{aligned} \tag{4.1.51}$$

For $n>2$, from Lemma 20, we have

$$\sum_{n=3}^{\infty}\left|k\mathcal{C}_n^{(x_l,x_r)}(k)\right|\leq\sum_{n=3}^{\infty}\frac{|k|C^n}{|k|^{\lfloor\frac{n+1}{2}\rfloor}}=\frac{C^4+C^3}{k-C^2}=O(k^{-1}). \tag{4.1.52}$$

Since $\mathfrak{c}_0(k)=O(k^{-3})$, $\mathfrak{c}_1(k)=-m_{\mathfrak{c}_1}/k+O(k^{-3})$, $\mathfrak{s}_n(k)=m_{\mathfrak{s}}/k^2+O(k^{-4})$, and since $\mathcal{S}_n^{(x_l,x_r)}(k)\to 0$ and $\mathcal{S}_n^{(x_l,x_r)}(k)$ is bounded (see Lemma 18), we can choose $r>\sqrt{M_\gamma}$ sufficiently large so that $|\varepsilon(k)|<1/2$ for $k\in\Omega_{\text{ext}}(r)$, by the DCT. We have

$$\frac{1}{16|k|^2}\left|m_{\mathfrak{c}_1}\mathfrak{u}_+-8m_{\mathfrak{s}}\right|\leq|\Delta(k)|\leq\frac{3}{16r^2}\left|m_{\mathfrak{c}_1}\mathfrak{u}_+-8m_{\mathfrak{s}}\right|. \tag{4.1.53}$$

□

Remark 23. *Note that for constant-coefficient IBVPs (α, β, γ constant), the denominator $\Delta(k)$ reduces to*

$$\Delta(k)=2i\mathfrak{a}(k)\Xi(k)+i\mathfrak{c}_0(k)\left(\Xi(k)^2+1\right)+\mathfrak{s}_0(k)\left(\Xi(k)^2-1\right). \tag{4.1.54}$$

If $(a:b)_{2,4}=0$ and $m_{\mathfrak{c}_0}=0$ (i.e., $\mathfrak{c}_0(k)=0$ and $\mathfrak{s}_0(k)=(a:b)_{1,3}$), then we require $(a:b)_{1,3}\neq 0$, so that $\Delta(k)\not\to 0$ exponentially fast (or is not identically zero). Thus, Boundary Cases 1–4 are the only allowable cases giving rise to a well-defined solution for constant-coefficient problems. If the coefficients are not constant, it may be possible to go out to higher order in the asymptotics of Lemma 22, e.g., *$(a:b)_{2,4}=0$, $m_{\mathfrak{c}_0}=0$, $m_{\mathfrak{c}_1}\neq 0$, and $m_{\mathfrak{c}_1}\mathfrak{u}_+-8m_{\mathfrak{s}}=0$, and additional allowable boundary conditions may be identified. This requires further investigation.*

Lemma 24. *Consider the finite-interval, half-line, and whole-line problems. For all three, there exists an $r>\sqrt{M_\gamma}$ and $M_\Psi>0$ such that, for $k\in\Omega_{\text{ext}}(r)$, $x\in\overline{\mathcal{D}}$, and $y\in\overline{\mathcal{D}}$,*

$$|\Psi(k,x,y)|\leq M_\Psi. \tag{4.1.55a}$$

For the regular *problems,*

$$\left|\frac{\Psi(k,x,y)}{\Delta(k)}\right|\leq M_\Psi, \tag{4.1.55b}$$

and for the irregular *problems,*

$$\left|\frac{\Psi(k,x,y)}{\Delta(k)}\right|\leq M_\Psi\left(1+|k|\left(e^{-m_{\text{in}}|k|(x-x_l)}+e^{-m_{\text{in}}|k|(x_r-x)}\right)\right)\leq 3M_\Psi|k|. \tag{4.1.55c}$$

Thus $\Psi(k,x,y)$ and $\Psi(k,x,y)/\Delta(k)$ are well-defined functions.

Proof. For the whole-line problem, from (2.1.4) and Lemma 15,

$$|\Psi(k,x,y)|\leq e^{-m_{\text{in}}|k||x-y|}\sum_{n=0}^{\infty}\frac{1}{2^n n!}\left\|\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right\|_{\mathbb{R}}^n\leq\exp\left(\frac{1}{2}\left\|\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right\|_{\mathcal{D}}\right), \tag{4.1.56}$$

and (4.1.55a) follows. From Lemma 22, (4.1.55b) follows. For the half-line problem, with $x_l < y < x$, from (2.2.8)

$$\begin{aligned}|\Psi(k,x,y)| &\le 4\left|\exp\left(\int_y^x ik\mathfrak{n}(k,\xi)\,d\xi\right)\right|\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}\left|\left(\frac{a_0}{k\mathfrak{n}(k,x_l)}\mathcal{S}_{n-\ell}^{(x_l,y)}(k)-a_1\mathcal{C}_{n-\ell}^{(x_l,y)}(k)\right)\mathcal{E}_\ell^{(x,\infty)}(k)\right|\\ &\le 4e^{-m_{i\mathfrak{n}}|k||x-y|}\left(\frac{|a_0|}{m_{\mathfrak{n}}r}+|a_1|\right)\exp\left(\frac{1}{2}\left\|\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right\|_{\mathcal{D}}\right),\end{aligned} \tag{4.1.57}$$

and similarly for $x_l < x < y$. Therefore (4.1.55a) follows. From Lemma 22, we have

$$\left|\frac{\Psi(k,x,y)}{\Delta(k)}\right| \le \frac{4\left(|a_1|+\frac{|a_0|}{|k\mathfrak{n}(k,x_l)|}\right)}{\left||a_1|-\left|\frac{a_0}{k\mathfrak{n}(k,x_l)}\right|\right|}\exp\left(\frac{1}{2}\left\|\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right\|_{\mathcal{D}}\right) \le 4A\exp\left(\frac{1}{2}\left\|\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right\|_{\mathcal{D}}\right), \tag{4.1.58}$$

where

$$\left|\frac{\frac{(-1)^n ia_0}{k\mathfrak{n}(k,x_l)}-a_1}{\frac{ia_0}{k\mathfrak{n}(k,x_l)}-a_1}\right| \le \frac{|a_1|+\frac{|a_0|}{m_{\mathfrak{n}}r}}{\left||a_1|-\frac{|a_0|}{m_{\mathfrak{n}}r}\right|} = A < \infty, \tag{4.1.59}$$

This gives (4.1.55b).

For the finite-interval problem:

1. if $(a:b)_{2,4} \neq 0$, from (2.3.7a), we find for $x_l < y < x < r_r$,

$$\begin{aligned}|\Psi(k,x,y)| \le{}& 4\left(|(a:b)_{2,4}|+\frac{|(a:b)_{1,3}|}{m_{\mathfrak{n}}^2 r^2}+\frac{|(a:b)_{1,4}|+|(a:b)_{2,3}|}{m_{\mathfrak{n}}r}\right)\exp\left(\frac{1}{2}\left\|\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right\|_{\mathcal{D}}\right)\\ &+\frac{4M_\beta|(a:b)_{1,2}|}{m_\beta m_{\mathfrak{n}}|k|}\exp\left(\frac{1}{2}\left\|\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right\|_{\mathcal{D}}\right)e^{-m_{i\mathfrak{n}}|k|(x_r-x_l-|x-y|)},\end{aligned} \tag{4.1.60}$$

   and similarly for $x_l < x < y < x_r$. Thus (4.1.55a) follows. From Lemma 22, (4.1.55b) follows.

2. If $(a:b)_{2,4} = 0$ and $m_{\mathfrak{c}_0} \neq 0$, from (2.3.7a), we find for $x_l < y < x < r_r$,

$$\begin{aligned}|\Psi(k,x,y)| \le{}& \frac{4}{|k|}\left(\frac{|(a:b)_{1,4}|+|(a:b)_{2,3}|}{m_{\mathfrak{n}}}+\frac{|(a:b)_{1,3}|}{m_{\mathfrak{n}}^2 r}\right)\exp\left(\frac{1}{2}\left\|\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right\|_{\mathcal{D}}\right)\\ &+\frac{4M_\beta|(a:b)_{1,2}|}{m_\beta m_{\mathfrak{n}}|k|}\exp\left(\frac{1}{2}\left\|\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right\|_{\mathcal{D}}\right)e^{-m_{i\mathfrak{n}}|k|(x_r-x_l-|x-y|)},\end{aligned} \tag{4.1.61}$$

   and similarly for $x_l < x < y < x_r$. This gives (4.1.55a). From Lemma 22, (4.1.55b) follows.

3. If $(a:b)_{2,4} = 0$, $m_{\mathfrak{c}_0} = 0$, $m_{\mathfrak{c}_1} = 0$, and $(a:b)_{1,3} \neq 0$, then for $x_l < y < x < x_r$,

$$|\Psi(k,x,y)| \le \frac{4}{|k|^2}\left(\frac{|(a:b)_{1,3}|}{m_{\mathfrak{n}}^2}+\frac{4M_\beta|(a:b)_{1,2}||k|}{m_\beta m_{\mathfrak{n}}}e^{-m_{i\mathfrak{n}}|k|(x_r-x_l-|x-y|)}\right)\exp\left(\frac{1}{2}\left\|\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right\|_{\mathcal{D}}\right), \tag{4.1.62}$$

   and similarly for $x_l < x < y < x_r$. This gives (4.1.55a). This Boundary Case is regular if both $(a:b)_{1,2} = 0$ and $(a:b)_{3,4} = 0$ and irregular if either $(a:b)_{1,2} \neq 0$ or $(a:b)_{3,4} \neq 0$, see Remark 6. Lemma 22 gives (4.1.55b) or (4.1.55c).

4. If $(a:b)_{2,4} = 0$, $m_{\mathfrak{c}_0} = 0$, $m_{\mathfrak{c}_1} \neq 0$, and $m_{\mathfrak{c}_1}\mathfrak{u}_+ - 8m_{\mathfrak{s}} \neq 0$, then,

$$\frac{(a:b)_{1,4}}{\mu(x_l)} = \frac{(a:b)_{2,3}}{\mu(x_r)} = \frac{m_{\mathfrak{c}_1}}{2}. \tag{4.1.63}$$

   From this $(a:b)_{1,4}/\mathfrak{n}(k,x_l) = m_{\mathfrak{c}_1}/2 + O(k^{-2})$ and $(a:b)_{2,3}/\mathfrak{n}(k,x_r) = m_{\mathfrak{c}_1}/2 + O(k^{-2})$. Using Lemma 20, there exists an $r > C^2$ such that

$$\left|\sum_{n=1}^{\infty}\sum_{\ell=0}^{n}(-1)^{\lambda\ell}\mathcal{J}_{n-\ell}^{(x_l,y)}[\sigma_{p,n-\ell}](k)\mathcal{J}_\ell^{(x,x_r)}[\overline{\sigma}_{p,\ell}](k)\right| \le \sum_{n=1}^{\infty}\sum_{\ell=0}^{n}\frac{C^{n-\ell}}{|k|^{\lfloor\frac{n-\ell+1}{2}\rfloor}}\frac{C^\ell}{|k|^{\lfloor\frac{\ell+1}{2}\rfloor}} = \frac{(k+C)^2}{(k-C^2)^2}-1 = O(k^{-1}), \tag{4.1.64}$$

for $k \in \Omega_{\text{ext}}(r)$. For $x_l < y < x < x_r$, the $n = 0$ terms involving $(a : b)_{1,4}$ and $(a : b)_{2,3}$ combine to give

$$\begin{aligned}&\frac{(a:b)_{1,4}}{k\mathfrak{n}(k,x_l)}\mathcal{S}_0^{(x_l,y)}(k)\mathcal{C}_0^{(x,x_r)}(k) - \frac{(a:b)_{2,3}}{k\mathfrak{n}(k,x_r)}\mathcal{C}_0^{(x_l,y)}(k)\mathcal{S}_0^{(x,x_r)}(k)\\ &\quad= \frac{m_{\mathfrak{c}_1}}{2k}\left(\mathcal{S}_0^{(x_l,y)}(k)\mathcal{C}_0^{(x,x_r)}(k) - \mathcal{C}_0^{(x_l,y)}(k)\mathcal{S}_0^{(x,x_r)}(k)\right) + O(k^{-3})\\ &\quad= \frac{m_{\mathfrak{c}_1}}{2k}\sin\left(\int_{x_l}^{y} - \int_{x}^{x_r} k\mathfrak{n}(k,\xi)\,d\xi\right) + O(k^{-3}),\end{aligned} \tag{4.1.65}$$

so that, for $x_l < y < x < x_r$,

$$|\Psi(k,x,y)| \le 4\left\{\frac{|m_{\mathfrak{c}_1}|}{4|k|}\left(e^{-m_{in}|k|(x-x_l)} + e^{-m_{in}|k|(x_r-x)}\right) + O(k^{-2})\right\} + \frac{4M_\beta|(a:b)_{1,2}|}{m_\beta m_{\mathfrak{n}}|k|}e^{-m_{in}|k|(x_r-x)}. \tag{4.1.66}$$

This gives (4.1.55a). Using Lemma 22, we arrive at (4.1.55c). The same can be shown for $x_l < x < y < x_r$.

□

Lemma 25. *Consider the finite-interval and half-line problems. There exists an $r > \sqrt{M_\gamma}$ and $M_{\mathcal{B}} > 0$ such that for $k \in \Omega_{\text{ext}}(r)$ and $x \in \overline{\mathcal{D}}$, for both the half-line ($m = 0$) and the finite-interval problem ($m = 0, 1$),*

$$|\mathcal{B}_m(k,x)| \le M_{\mathcal{B}}. \tag{4.1.67a}$$

Further, for the half-line problem ($m = 0$),

$$\left|\frac{\mathcal{B}_0(k,x)}{\Delta(k)}\right| \le M_{\mathcal{B}}|k|e^{-m_{in}|k|(x-x_l)}, \tag{4.1.67b}$$

and for the finite-interval problem ($m = 0, 1$),

$$\left|\frac{\mathcal{B}_m(k,x)}{\Delta(k)}\right| \le M_{\mathcal{B}}|k|^{b+1}\left(e^{-m_{in}|k|(x_r-x)} + e^{-m_{in}|k|(x-x_l)}\right). \tag{4.1.67c}$$

Here, $b = 0$ for regular *boundary conditions, and $b = 1$ for* irregular *boundary conditions. It follows that the functions $\mathcal{B}_m(k,x)$ and $\mathcal{B}_m(k,x)/\Delta(k)$ are well defined for the half-line and finite-interval problems.*

Proof. For the half-line problem, using Lemmas 15 and 18 in (2.2.5), we have

$$|\mathcal{B}_0(k,x)| \le \frac{4M_\beta}{m_\beta m_{\mathfrak{n}}}\exp\left(\frac{1}{2}\left\|\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right\|_{\mathcal{D}}\right)e^{-m_{in}|k|(x-x_l)}, \tag{4.1.68}$$

which gives (4.1.67a). Lemma 22 gives (4.1.67b). Similarly, for the finite-interval problem, using Lemmas 15 and 18 in (2.3.6c), we have

$$|\mathcal{B}_{2-j}(k,x)| \le \frac{4M_\beta}{m_\beta m_{\mathfrak{n}}}\exp\left(\frac{1}{2}\left\|\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right\|_{\mathcal{D}}\right)\left\{\left(\frac{|a_{j1}|}{m_{\mathfrak{n}}|k|} + |a_{j2}|\right)e^{-m_{in}|k|(x_r-x)} + \left(\frac{|b_{j1}|}{m_{\mathfrak{n}}|k|} + |b_{j2}|\right)e^{-m_{in}|k|(x-x_l)}\right\}, \tag{4.1.69}$$

for $j = 1, 2$, which gives (4.1.67a). For the finite-interval problem with Boundary Case 1 or 2 and for the irregular boundary conditions, (4.1.67c) follows from the above and Lemma 22. For the regular version of Boundary Case 3, we have $a_{ij} = 0$ for all $i, j = 1, 2$, except for a_{11} and b_{21}, see Remark 6. Thus,

$$|\mathcal{B}_{2-j}(k,x)| \le \frac{4M_\beta}{m_\beta m_{\mathfrak{n}}^2|k|}\exp\left(\frac{1}{2}\left\|\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right\|_{\mathcal{D}}\right)\left(|a_{j1}|e^{-m_{in}|k|(x_r-x)} + |b_{j1}|e^{-m_{in}|k|(x-x_l)}\right), \tag{4.1.70}$$

from which (4.1.67c) follows, using Lemma 22. □

Lemma 26. *Consider the finite-interval, half-line, and whole-line problems. For all three, there exists an $r > \sqrt{M_\gamma}$ and $M_\Phi > 0$ such that for $k \in \Omega_{\text{ext}}(r)$ and $x \in \overline{\mathcal{D}}$,*

$$|\Phi_0(k,x)| \le M_\Phi\|q_0\|_{\mathcal{D}}. \tag{4.1.71a}$$

For the regular problems*,*

$$\left|\frac{\Phi_0(k,x)}{\Delta(k)}\right| \leq M_\Phi \|q_0\|_{\mathcal{D}}, \tag{4.1.71b}$$

and for the irregular problems*,*

$$\left|\frac{\Phi_0(k,x)}{\Delta(k)}\right| = M_\Phi \|q_0\|_{\mathcal{D}} \left(1+|k|\left(e^{-m_{in}|k|(x-x_l)}+e^{-m_{in}|k|(x_r-x)}\right)\right) \leq 3M_\Phi|k|\|q_0\|_{\mathcal{D}}. \tag{4.1.71c}$$

It follows that $\Phi_0(k,x)$ *and* $\Phi_0(k,x)/\Delta(k)$ *are well-defined functions.*

Proof. The inequalities (4.1.71) follow directly from Lemma 24. □

Lemma 27. *Consider the finite-interval, half-line, and whole-line problems. For all three, there exists an* $r > \sqrt{M_\gamma}$ *and* $M_f > 0$ *such that for* $k \in \Omega_{\text{ext}}(r)\backslash\Omega(r)$ *(the green region of Figure 4.1), for* $x \in \overline{\mathcal{D}}$*, and for* $t \in [0,T]$*,*

$$\left|\Phi_f(k,x,t)e^{-k^2t}\right| \leq M_f\|f\|_{\mathcal{D}}. \tag{4.1.72a}$$

Further, for the regular problems*,*

$$\left|\frac{\Phi_f(k,x,t)e^{-k^2t}}{\Delta(k)}\right| = M_f\|f\|_{\mathcal{D}}, \tag{4.1.72b}$$

and for the irregular problems*,*

$$\left|\frac{\Phi_f(k,x,t)e^{-k^2t}}{\Delta(k)}\right| = M_f\|f\|_{\mathcal{D}} \left(1+|k|\left(e^{-m_{in}|k|(x-x_l)}+e^{-m_{in}|k|(x_r-x)}\right)\right) \leq 3M_f|k|\|f\|_{\mathcal{D}}. \tag{4.1.72c}$$

Thus, $\Phi_f(k,x,t)$ *and* $\Phi_f(k,x,t)/\Delta(k)$ *are well-defined functions.*

Proof. For $k \in \Omega_{\text{ext}}(r)\backslash\Omega(r)$, $|e^{-k^2(t-s)}| < 1$. It follows from (1.2.16) and Assumption 3.1 that

$$\int_{\mathcal{D}} \left|\tilde{f}_\alpha(k^2,x,t)e^{-k^2t}\right| dx \leq \int_{\mathcal{D}}\int_0^t |f_\alpha(x,s)|\, ds dy \leq \frac{T\|f\|_{\mathcal{D}}}{m_\alpha}. \tag{4.1.73}$$

Using this and (4.1.55) in (4.1.28b), we obtain (4.1.72) for any $x \in \mathcal{D}$ and for $t \in [0,T]$. □

Lemma 28. *There exists an* $r > \sqrt{M_\gamma}$ *so that for* $x \in \mathcal{D}$ *and* $t \in [0,T]$*,* $\mathcal{J}_n^{(a,b)}(k)$*,* $\Delta(k)$*,* $\Psi(k,x,y)$*, and* $\Phi_0(k,x)$ *are analytic in* k*, for* $k \in \Omega_{\text{ext}}(r)$*. The functions* $\Phi_f(k,x,t)e^{-k^2t}$ *and* $\mathcal{B}_m(k,x)e^{-k^2t}$ *are analytic in* k *for* $k \in \Omega_{\text{ext}}(r)\backslash\Omega(r)$*.*

Proof. Consider a closed contour $\Gamma \in \Omega_{\text{ext}}(r)$. Then

$$\oint_\Gamma \mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)\, dk = \frac{1}{2^n}\int_{\mathcal{D}_n^{(a,b)}} d\mathbf{y}_n \oint_\Gamma dk \left(\prod_{p=1}^n \frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)}\right)\exp\left(\sum_{p=0}^n \sigma_{p,n}\int_{y_p}^{y_{p+1}} ik\mathfrak{n}(k,\xi)\, d\xi\right) = 0, \tag{4.1.74}$$

by Cauchy's theorem. We can switch the order of integration by Fubini's theorem and Lemma 18. Therefore, by Morera's theorem, $\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)$ is analytic for $k \in \Omega_{\text{ext}}(r)$. For all three types of IBVPs considered, the same argument applies for the $\Delta(k)$, $\Psi(k,x,y)$, and the $\Phi_0(k,x)$ functions by Lemmas 22, 24, and 26, and for the $\mathcal{B}_m(k,x)$ and $\Phi_f(k,x,t)$ functions by Lemma 25 and 27. □

The following lemmas prove that the different parts of the solution are well defined.

Lemma 29. *For the half-line problem* ($m=0$) *and the finite-interval problem* ($m=0,1$)*, there exists an* $r > \sqrt{M_\gamma}$ *such that, for any* $x \in \overline{\mathcal{D}}$ *and* $t \in (0,T)$*, the function* $q_{\mathcal{B}_m}(x,t)$ (4.1.29c) *can be written as*

$$q_{\mathcal{B}_m}(x,t) = \frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}(r)} \frac{\mathcal{B}_m(k,x)}{\Delta(k)}\mathfrak{F}_m(k^2,t)e^{-k^2t}\, dk, \tag{4.1.75a}$$

where

$$\mathfrak{F}_m(k^2,t) = -\frac{f_m(0)}{k^2} - \frac{\mathcal{G}[f_m'](k^2,t)}{k^2}, \tag{4.1.75b}$$

with the bound

$$\left|\mathfrak{F}_m(k^2,t)e^{-k^2t}\right| \le \frac{\|f_m\|_\infty e^{-|k|^2\cos(2\theta)t}}{|k|^2} + \frac{\|f_m'\|_\infty\left(1-e^{-|k|^2\cos(2\theta)t}\right)}{|k|^4\cos(2\theta)}. \tag{4.1.76}$$

The function $q_{\mathcal{B}_m}(x,t)$ is well defined.

Proof. From (2.2.7) and Assumption 3.3, for $k \in \Omega_{\text{ext}}(r)\backslash\Omega(r)$,

$$\left|F_m(k^2,t)e^{-k^2t}\right| \le \left|\int_0^t e^{-k^2(t-s)} f_m(s)\,ds\right| \le T\|f_m\|_\infty. \tag{4.1.77}$$

Therefore, for $x \in \mathcal{D}$, we have exponential decay of the integrand of $q_{\mathcal{B}_m}(x,t)$ from Lemma 25. Using Lemma 28, we can deform the contour of (4.1.29c) from $\Omega(r)$ to $\Omega_{\text{ext}}(r)$. Assumption 3.3 allows us to integrate (2.2.7) by parts so that

$$F_m(k^2,t)e^{-k^2t} = \frac{f_m(t)}{k^2} - \frac{f_m(0)e^{-k^2t}}{k^2} - \frac{\mathcal{G}[f_m'](k^2,t)e^{-k^2t}}{k^2}, \tag{4.1.78}$$

which gives (4.1.75), after using Cauchy's theorem on the $f_m(t)$ term. Equation (4.1.76) follows from (4.1.75b) and Assumption 3.3. From Lemma 25, for the half-line problem,

$$|q_{\mathcal{B}_m}(x,t)| \le \frac{M_{\mathcal{B}}}{2\pi}\int_{\partial\Omega_{\text{ext}}(r)} |k|e^{-m_{in}|k|(x-x_l)}\left|\mathfrak{F}_m(k^2,t)e^{-k^2t}\right|dk, \tag{4.1.79a}$$

and for the finite-interval problem,

$$|q_{\mathcal{B}_m}(x,t)| \le \frac{M_{\mathcal{B}}}{2\pi}\int_{\partial\Omega_{\text{ext}}(r)} |k|^{b+1}\left(e^{-m_{in}|k|(x_r-x)} + e^{-m_{in}|k|(x-x_l)}\right)\left|\mathfrak{F}_m(k^2,t)e^{-k^2t}\right|dk. \tag{4.1.79b}$$

From (4.1.79), we see that $q_{\mathcal{B}_m}(x,t)$ is well defined for $x \in \overline{\mathcal{D}}$ and for $t \in (0,T)$. □

Lemma 30. *Consider the finite-interval, half-line, and whole-line problems. There exists an $r > \sqrt{M_\gamma}$ so that for $x \in \overline{\mathcal{D}}$ and $t \in (0,T)$, $q_0(x,t)$ (4.1.29a) can be written as*

$$q_0(x,t) = \frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}(r)} \frac{\Phi_0(k,x)}{\Delta(k)}e^{-k^2t}\,dk, \tag{4.1.80}$$

which is well defined.

Proof. By Lemmas 26 and 28, $\Phi_0(k,x)/\Delta(k)$ is bounded, well defined, and analytic for $k \in \Omega_{\text{ext}}(r)$. Let $C_R = \{k \in \mathbb{C} : |k| = R \text{ and } \theta_0 < \theta < \pi/4 \text{ or } 3\pi/4 < \theta < \pi - \theta_0\}$, see Figure 4.1. For the *regular problems*, using symmetry,

$$\left|\int_{C_R} \frac{\Phi_0(k,x)}{\Delta(k)}e^{-k^2t}\,dk\right| \le 2M_\Phi\|q_0\|_{\mathcal{D}}\int_{\theta_0}^{\frac{\pi}{4}} e^{-R^2\cos(2\theta)t}R\,d\theta \le \frac{\pi M_\Phi\|q_0\|_{\mathcal{D}}\left(1-e^{-R^2t}\right)}{2Rt} \to 0, \tag{4.1.81}$$

as $R \to \infty$. Thus we can deform the contour by Cauchy's theorem to conclude (4.1.80). For the *irregular problems*, the above holds for the integral over the first term of (4.1.71c) and for $x \in \mathcal{D}$, the second term is exponentially decaying, and we again conclude (4.1.80). It follows that for all three problems

$$|q_0(x,t)| = \frac{M_\Phi\|q_0\|_{\mathcal{D}}}{2\pi}\int_{\partial\Omega_{\text{ext}}} \left|ke^{-k^2t}\right|\,|dk| < \infty. \tag{4.1.82}$$

□

Lemma 31. *Consider the finite-interval, half-line, and whole-line problems. There exists an* $r > \sqrt{M_\gamma}$ *so that for* $x \in \overline{\mathcal{D}}$ *and* $t \in (0,T)$*,* $q_f(x,t)$ (4.1.29b) *can be written as*

$$q_f(x,t) = \frac{1}{2\pi} \int_{\partial\Omega_{\text{ext}}(r)} \frac{\Phi_{\mathfrak{f}}(k,x,t)e^{-k^2 t}}{\Delta(k)} dk, \tag{4.1.83a}$$

where

$$\Phi_{\mathfrak{f}}(k,x,t) = \int_{\mathcal{D}} \frac{\Psi(k,x,y)\mathfrak{f}_\alpha(k^2,y,t)}{\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,y)}} dy, \tag{4.1.83b}$$

and

$$\mathfrak{f}_\alpha(k^2,y,t) = -\frac{f_\alpha(y,0)}{k^2} - \frac{\mathcal{G}[f_{\alpha,t}](k^2,y,t)}{k^2}. \tag{4.1.83c}$$

Further, we have the bound

$$\int_{\mathcal{D}} \left|\mathfrak{f}_\alpha(k^2,y,t)e^{-k^2 t}\right| dy \le \frac{\|f\|_{\mathcal{D}} e^{-|k|^2\cos(2\theta)t}}{m_\alpha |k|^2} + \frac{\|f_t\|_{\mathcal{D}}\left(1 - e^{-|k|^2\cos(2\theta)t}\right)}{m_\alpha |k|^4 \cos(2\theta)}. \tag{4.1.84}$$

For all three problems, there exists an $M_f > 0$ *such that*

$$\left|\Phi_{\mathfrak{f}}(k,x,t)e^{-k^2 t}\right| \le M_f \int_{\mathcal{D}} \left|\mathfrak{f}_\alpha(k^2,y,t)e^{-k^2 t}\right| dy. \tag{4.1.85a}$$

For the regular problems

$$\left|\frac{\Phi_{\mathfrak{f}}(k,x,t)e^{-k^2 t}}{\Delta(k)}\right| \le M_f \int_{\mathcal{D}} \left|\mathfrak{f}_\alpha(k^2,y,t)e^{-k^2 t}\right| dy, \tag{4.1.85b}$$

and for the irregular problems*,*

$$\left|\frac{\Phi_{\mathfrak{f}}(k,x,t)e^{-k^2 t}}{\Delta(k)}\right| \le M_f \left(1 + |k|\left(e^{-m_{\text{in}}|k|(x-x_l)} + e^{-m_{\text{in}}|k|(x_r-x)}\right)\right) \int_{\mathcal{D}} \left|\mathfrak{f}_\alpha(k^2,y,t)e^{-k^2 t}\right| dy. \tag{4.1.85c}$$

It follows that $q_f(x,t)$ *is well defined for all three problems.*

Proof. By Lemmas 27 and 28, $\Phi_f(k,x,t)e^{-k^2 t}/\Delta(k)$ is bounded, well defined, and analytic for $k \in \Omega_{\text{ext}}(r)/\Omega(r)$. Let C_R be defined as in the proof of Lemma 30, see Figure 4.1. Then, for the *regular problems*, using symmetry,

$$\left|\int_{C_R} \frac{\Phi_f(k,x,t)e^{-k^2 t}}{\Delta(k)} dk\right| \le 2M_f \|f\|_{\mathcal{D}} \int_0^{\frac{\pi}{4}} Re^{-R^2\cos(2\theta)t} d\theta \to 0, \tag{4.1.86}$$

as $R \to \infty$. Thus, we can deform the integral in (4.1.29b) from $\Omega(r)$ to $\Omega_{\text{ext}}(r)$. For the *irregular problems*, the above holds for the integral over the first term of (4.1.72c) and for $x \in \mathcal{D}$ the second term is exponentially decaying. Thus, we can still deform from $\Omega(r)$ to $\Omega_{\text{ext}}(r)$. Using Assumption 3.1, we can integrate (1.2.16) by parts, to obtain

$$\tilde{f}_\alpha(k^2,x,t) = \frac{f_\alpha(x,t)e^{k^2 t}}{k^2} - \frac{f_\alpha(x,0)}{k^2} - \frac{\mathcal{G}[f_{\alpha,t}(k^2,x,t)]}{k^2}, \tag{4.1.87}$$

which gives (4.1.83), after using Cauchy's theorem on the $f_\alpha(x,t)$ term. Equation (4.1.84) follows directly from (4.1.83c) and Assumption 3.1, and equation (4.1.85) follows from Lemma 24. From (4.1.85), we see that the integrand in $q_f(x,t)$ is absolutely integrable and is therefore well defined for all $x \in \overline{\mathcal{D}}$ and any $t \in (0,T)$ (or for any $x \in \mathcal{D}$ and for all $t \in [0,T]$). ☐

Finally, we combine all the results obtained.

Theorem 32. *There exists an* $r > \sqrt{M_\gamma}$ *such that the functions* (2.1.2)*,* (2.2.2)*, and* (2.3.2) *are well defined for all* $x \in \overline{\mathcal{D}}$ *and for any* $t \in (0,T)$*.*

Proof. Combining Lemmas 29, 30, and 31, we obtain our result. ☐

4.2 The solution expressions solve the evolution equation

In this appendix, we prove that the solution expressions (2.1.2), (2.2.2), and (2.3.2) for the whole-line, half-line, and finite-interval problems, respectively, solve the evolution equation (1.2.1) in their respective domains. Naturally, we are in need of lemmas on the derivatives of various quantities defining the solution expressions. The following lemma deals with derivatives with respect to the spatial variable.

Lemma 33. *For $n \geq 0$, the derivatives of $\mathcal{E}_n^{(x,\infty)}(k)$ and $\tilde{\mathcal{E}}_n^{(-\infty,x)}(k)$ are given by*

$$\partial_x \mathcal{E}_n^{(x,\infty)}(k) = -\frac{1}{2}\frac{(\beta\mathfrak{n})'(k,x)}{(\beta\mathfrak{n})(k,x)}\mathcal{E}_{n-1}^{(x,\infty)}(k) - (1-(-1)^n)ik\mathfrak{n}(k,x)\mathcal{E}_n^{(x,\infty)}(k), \tag{4.2.1a}$$

$$\partial_x \tilde{\mathcal{E}}_n^{(-\infty,x)}(k) = \frac{1}{2}\frac{(\beta\mathfrak{n})'(k,x)}{(\beta\mathfrak{n})(k,x)}\tilde{\mathcal{E}}_{n-1}^{(-\infty,x)}(k) + (1-(-1)^n)ik\mathfrak{n}(k,x)\tilde{\mathcal{E}}_n^{(-\infty,x)}(k), \tag{4.2.1b}$$

and those of $\mathcal{C}_n^{(a,b)}(k)$ and $\mathcal{S}_n^{(a,b)}(k)$ are

$$\partial_x \mathcal{C}_n^{(x_l,x)}(k) = \frac{1}{2}\frac{(\beta\mathfrak{n})'(k,x)}{(\beta\mathfrak{n})(k,x)}\mathcal{C}_{n-1}^{(x_l,x)}(k) - (-1)^n k\mathfrak{n}(k,x)\mathcal{S}_n^{(x_l,x)}(k), \tag{4.2.1c}$$

$$\partial_x \mathcal{C}_n^{(x,x_r)}(k) = -\frac{1}{2}\frac{(\beta\mathfrak{n})'(k,x)}{(\beta\mathfrak{n})(k,x)}\mathcal{C}_{n-1}^{(x,x_r)}(k) + k\mathfrak{n}(k,x)\mathcal{S}_n^{(x,x_r)}(k), \tag{4.2.1d}$$

$$\partial_x \mathcal{S}_n^{(x_l,x)}(k) = \frac{1}{2}\frac{(\beta\mathfrak{n})'(k,x)}{(\beta\mathfrak{n})(k,x)}\mathcal{S}_{n-1}^{(x_l,x)}(k) + (-1)^n k\mathfrak{n}(k,x)\mathcal{C}_n^{(x_l,x)}(k), \tag{4.2.1e}$$

$$\partial_x \mathcal{S}_n^{(x,x_r)}(k) = \frac{1}{2}\frac{(\beta\mathfrak{n})'(k,x)}{(\beta\mathfrak{n})(k,x)}\mathcal{S}_{n-1}^{(x,x_r)}(k) - k\mathfrak{n}(k,x)\mathcal{C}_n^{(x,x_r)}(k). \tag{4.2.1f}$$

Proof. Since $(\beta\mathfrak{n})'/(\beta\mathfrak{n}) \in L^1(\mathcal{D})$ by Lemma 14, the proof is by direct calculation of the derivatives of (1.2.18) and (1.2.19) [19]. We show one such calculation. From (1.2.18b),

$$\partial_x \tilde{\mathcal{E}}_n^{(a,x)}(k) = \frac{\partial_x}{2^n}\int_a^x dy_1 \int_{y_1}^x dy_2 \cdots \int_{y_{n-2}}^x dy_{n-1}\int_{y_{n-1}}^x dy_n \left(\prod_{p=1}^n \frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right)\exp\left(\sum_{p=0}^n (1-(-1)^p)\int_{y_p}^{y_{p+1}} ik\mathfrak{n}(k,\xi)\,d\xi\right), \tag{4.2.2}$$

so that

$$\begin{aligned}\partial_x \tilde{\mathcal{E}}_n^{(a,x)}(k) = {}& \frac{(\beta\mathfrak{n})'(k,x)}{(\beta\mathfrak{n})(k,x)}\frac{1}{2^n}\int_{\mathcal{D}_{n-1}^{(a,x)}}\left(\prod_{p=1}^{n-1}\frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)}\right)\exp\left(\sum_{p=0}^{n-1}(1-(-1)^p)\int_{y_p}^{y_{p+1}} ik\mathfrak{n}(k,\xi)\,d\xi\right)d\mathbf{y}_{n-1}\\ &+ (1-(-1)^n)\frac{ik\mathfrak{n}(k,x)}{2^n}\int_{\mathcal{D}_n^{(a,x)}}\left(\prod_{p=1}^{n}\frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)}\right)\exp\left(\sum_{p=0}^{n}(1-(-1)^p)\int_{y_p}^{y_{p+1}} ik\mathfrak{n}(k,\xi)\,d\xi\right)d\mathbf{y}_{n},\end{aligned} \tag{4.2.3}$$

which gives (4.2.1b). □

In Lemma 34, we prove a general summation identity for the generalized accumulation functions $\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)$. This identity is used to prove the problem-specific identities in Lemma 35. In turn, these are used to prove the relation between $\chi(k,x)$ (4.2.18b) and $\Delta(k)$ in Lemma 39.

Lemma 34. *Let $\overline{\sigma}_{p,n-\ell}$ and $\tilde{\sigma}_{p,\ell}$ be two non-negative integer-valued functions as described in Definition 16. Denote*

$$\sigma_{p,n} = \begin{cases}\overline{\sigma}_{p,n-\ell}, & \text{if } 0 \leq p \leq n-\ell,\\ \tilde{\sigma}_{p-(n-\ell),\ell}, & \text{if } n-\ell < p \leq n.\end{cases} \tag{4.2.4a}$$

If $\overline{\sigma}_{n-\ell,n-\ell} = \tilde{\sigma}_{0,\ell}$ and $\sigma_{p,n}$ is independent of ℓ, then

$$\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k) = \sum_{\ell=0}^n \mathcal{J}_{n-\ell}^{(a,x)}[\overline{\sigma}_{p,n-\ell}](k)\mathcal{J}_\ell^{(x,b)}[\tilde{\sigma}_{p,\ell}](k). \tag{4.2.4b}$$

Proof. Define

$$\mathfrak{j}_n^{(a,b)}(k) = \sum_{\ell=0}^{n} \mathcal{J}_{n-\ell}^{(a,x)}[\overline{\sigma}_{p,n-\ell}](k)\mathcal{J}_\ell^{(x,b)}[\tilde{\sigma}_{p,\ell}](k). \tag{4.2.5}$$

By the definition of $\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)$ (4.1.10),

$$\begin{aligned}\mathfrak{j}_n^{(a,b)}(k) = \frac{1}{2^n}\sum_{\ell=0}^{n}\int_{\mathcal{D}_{n-\ell}^{(a,x)}} \left(\prod_{p=1}^{n-\ell}\frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)}\right)\exp\left(\sum_{p=0}^{n-\ell}\overline{\sigma}_{p,n-\ell}\int_{y_p}^{y_{p+1}} ik\mathfrak{n}(k,\xi)\,d\xi\right)dy_1\cdots dy_{n-\ell}\times \\ \times\int_{\mathcal{D}_\ell^{(x,b)}}\left(\prod_{p=n-\ell+1}^{n}\frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)}\right)\exp\left(\sum_{p=n-\ell}^{n}\tilde{\sigma}_{p-(n-\ell),\ell}\int_{y_p}^{y_{p+1}} ik\mathfrak{n}(k,\xi)\,d\xi\right)dy_{n-\ell+1}\cdots dy_n.\end{aligned} \tag{4.2.6}$$

In the exponential of the first integral, for the $p = n-\ell$ term, $y_{n-\ell+1}$ is defined as x. In the exponential in the second integral, for the $p = n-\ell$ term, $y_{n-\ell} = x$. Since $\overline{\sigma}_{n-\ell,n-\ell} = \tilde{\sigma}_{0,\ell} = \sigma_{n-\ell,n}$, multiplying the exponentials and adding these terms together, we have

$$\overline{\sigma}_{n-\ell,n-\ell}\int_{y_{n-\ell}}^{x} ik\mathfrak{n}(k,\xi)\,d\xi + \tilde{\sigma}_{0,\ell}\int_{x}^{y_{n-\ell+1}} ik\mathfrak{n}(k,\xi)\,d\xi = \sigma_{n-\ell,n}\int_{y_{n-\ell}}^{y_{n-\ell+1}} ik\mathfrak{n}(k,\xi)\,d\xi, \tag{4.2.7}$$

and the two integrals are combined as

$$\mathfrak{j}_n^{(a,b)}[\sigma_{p,n}](k) = \frac{1}{2^n}\sum_{\ell=0}^{n}\int_{a<\cdots<y_{n-\ell}<x<y_{n-\ell+1}<\cdots<b}\left(\prod_{p=1}^{n}\frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)}\right)\exp\left(\sum_{p=0}^{n}\sigma_{p,n}\int_{y_p}^{y_{p+1}} ik\mathfrak{n}(k,\xi)\,d\xi\right)dy_1\cdots dy_n. \tag{4.2.8}$$

Summing over ℓ is equivalent to adding up all possibilities of x lying between one of the $y_1,\ldots,y_n$. Since $\sigma_{p,n}$ is independent of ℓ by assumption, the integrand is independent of ℓ, and

$$\mathfrak{j}_n^{(a,b)}(k) = \frac{1}{2^n}\int_{\mathcal{D}_n^{(a,b)}}\left(\prod_{p=1}^{n}\frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)}\right)\exp\left(\sum_{p=0}^{n}\sigma_{p,n}\int_{y_p}^{y_{p+1}} ik\mathfrak{n}(k,\xi)\,d\xi\right)dy_1\cdots dy_n, \tag{4.2.9}$$

which is (4.2.4). □

From the identity in Lemma 34, we can prove the following more specific forms of (4.2.4b).

Lemma 35. *For the whole-line problem, if n is even,*

$$\mathcal{E}_n^{(-\infty,\infty)}(k) = \sum_{\ell=0}^{n}\tilde{\mathcal{E}}_{n-\ell}^{(-\infty,x)}(k)\mathcal{E}_\ell^{(x,\infty)}(k). \tag{4.2.10a}$$

For the half-line problem, for any n,

$$\mathcal{E}_n^{(x_l,\infty)}(k) = \sum_{\ell=0}^{n}\left(\mathcal{C}_{n-\ell}^{(x_l,x)}(k) - (-1)^n i\mathcal{S}_{n-\ell}^{(x_l,x)}(k)\right)\mathcal{E}_\ell^{(x,\infty)}(k). \tag{4.2.10b}$$

Finally, for the finite-interval problem, for any n,

$$\mathcal{C}_n^{(x_l,x_r)}(k) = \sum_{\ell=0}^{n}\left(\mathcal{C}_{n-\ell}^{(x_l,x)}(k)\mathcal{C}_\ell^{(x,x_r)}(k) - (-1)^{n-\ell}\mathcal{S}_{n-\ell}^{(x_l,x)}(k)\mathcal{S}_\ell^{(x,x_r)}(k)\right), \tag{4.2.10c}$$

$$\mathcal{S}_n^{(x_l,x_r)}(k) = \sum_{\ell=0}^{n}\left(\mathcal{S}_{n-\ell}^{(x_l,x)}(k)\mathcal{C}_\ell^{(x,x_r)}(k) + (-1)^{n-\ell}\mathcal{C}_{n-\ell}^{(x_l,x)}(k)\mathcal{S}_\ell^{(x,x_r)}(k)\right). \tag{4.2.10d}$$

Proof. For the whole-line problem, we define $\mathfrak{e}_{\mathrm{wl}}$ as the right-hand side of (4.2.10a). Using (4.1.12), we write $\mathfrak{e}_{\mathrm{wl}}$ as

$$\mathfrak{e}_{\mathrm{wl}} = \sum_{\ell=0}^{n}\mathcal{J}_{n-\ell}^{(-\infty,x)}[1-(-1)^p](k)\mathcal{J}_\ell^{(x,\infty)}[1-(-1)^{\ell-p}](k). \tag{4.2.11}$$

From Lemma 34, if n is even, $\overline{\sigma}_{p,n-\ell} = 1-(-1)^p$ and $\tilde{\sigma}_{p,\ell} = 1-(-1)^{\ell-p}$ so that $\overline{\sigma}_{n-\ell,n-\ell} = \tilde{\sigma}_{0,\ell}$ and

$$\sigma_{p,n} = \begin{cases} \overline{\sigma}_{p,n-\ell}, & \text{if } 0 \le p \le n-\ell, \\ \tilde{\sigma}_{p-(n-\ell),\ell}, & \text{if } n-\ell < p \le n, \end{cases} = \begin{cases} 1-(-1)^p, & \text{if } 0 \le p \le n-\ell, \\ 1-(-1)^{\ell-(p-(n-\ell))}, & \text{if } n-\ell < p \le n, \end{cases} = 1-(-1)^{n-p}, \tag{4.2.12}$$

is independent of ℓ so that (4.2.10a) follows.

For the half-line problem, we define $\mathfrak{e}_{\mathrm{hl}}$ as the right-hand side of (4.2.10b). Using (4.1.12),

$$\mathfrak{e}_{\mathrm{hl}} = \frac{1}{2}\sum_{\ell=0}^{n} \Big((1-(-1)^n)\mathcal{J}_{n-\ell}^{(x_l,x)}[1+(-1)^p](k) + (1+(-1)^n)\mathcal{J}_{n-\ell}^{(x_l,x)}[1-(-1)^p](k) \Big) \mathcal{J}_{\ell}^{(x,\infty)}[1-(-1)^{\ell-p}](k), \tag{4.2.13}$$

which is simplified to

$$\mathfrak{e}_{\mathrm{hl}} = \sum_{\ell=0}^{n} \mathcal{J}_{n-\ell}^{(x_l,x)}[1-(-1)^{n-p}](k)\mathcal{J}_{\ell}^{(x,\infty)}[1-(-1)^{\ell-p}](k). \tag{4.2.14}$$

From Lemma 34, $\overline{\sigma}_{p,n-\ell} = 1-(-1)^{n-p}$ and $\tilde{\sigma}_{p,\ell} = 1-(-1)^{\ell-p}$ so that $\overline{\sigma}_{n-\ell,n-\ell} = \tilde{\sigma}_{0,\ell}$ and $\sigma_{p,n} = 1-(-1)^{n-p}$. Equation (4.2.10b) follows.

For the finite-interval problem, we define $\mathfrak{e}_c$ and $\mathfrak{e}_s$ as the right-hand side of (4.2.10c) and (4.2.10d), respectively. Using (4.1.12), we write these in terms of $\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)$, obtaining

$$\begin{aligned} \mathfrak{e}_c = {} & \frac{1}{4}\sum_{\ell=0}^{n}(1+(-1)^{n-\ell}) \Big(\mathcal{J}_{n-\ell}^{(x_l,x)}[1+(-1)^p](k)\mathcal{J}_{\ell}^{(x,x_r)}[1+(-1)^p](k) \\ & \qquad + \mathcal{J}_{n-\ell}^{(x_l,x)}[1-(-1)^p](k)\mathcal{J}_{\ell}^{(x,x_r)}[1-(-1)^p](k) \Big) \\ & + \frac{1}{4}\sum_{\ell=0}^{n}(1-(-1)^{n-\ell}) \Big(\mathcal{J}_{n-\ell}^{(x_l,x)}[1+(-1)^p](k)\mathcal{J}_{\ell}^{(x,x_r)}[1-(-1)^p](k) \\ & \qquad + \mathcal{J}_{n-\ell}^{(x_l,x)}[1-(-1)^p](k)\mathcal{J}_{\ell}^{(x,x_r)}[1+(-1)^p](k) \Big), \end{aligned} \tag{4.2.15a}$$

$$\begin{aligned} \mathfrak{e}_s = {} & \frac{1}{4i}\sum_{\ell=0}^{n}(1+(-1)^{n-\ell}) \Big(\mathcal{J}_{n-\ell}^{(x_l,x)}[1+(-1)^p](k)\mathcal{J}_{\ell}^{(x,x_r)}[1+(-1)^p](k) \\ & \qquad - \mathcal{J}_{n-\ell}^{(x_l,x)}[1-(-1)^p](k)\mathcal{J}_{\ell}^{(x,x_r)}[1-(-1)^p](k) \Big) \\ & + \frac{1}{4i}\sum_{\ell=0}^{n}(1-(-1)^{n-\ell}) \Big(\mathcal{J}_{n-\ell}^{(x_l,x)}[1+(-1)^p](k)\mathcal{J}_{\ell}^{(x,x_r)}[1-(-1)^p](k) \\ & \qquad - \mathcal{J}_{n-\ell}^{(x_l,x)}[1-(-1)^p](k)\mathcal{J}_{\ell}^{(x,x_r)}[1+(-1)^p](k) \Big), \end{aligned} \tag{4.2.15b}$$

which simplify to

$$\mathfrak{e}_c = \frac{1}{2}\sum_{\ell=0}^{n} \Big(\mathcal{J}_{n-\ell}^{(x_l,x)}[1+(-1)^p](k)\mathcal{J}_{\ell}^{(x,x_r)}[1+(-1)^{n-\ell+p}](k) + \mathcal{J}_{n-\ell}^{(x_l,x)}[1-(-1)^p](k)\mathcal{J}_{\ell}^{(x,x_r)}[1-(-1)^{n-\ell+p}](k) \Big), \tag{4.2.16a}$$

$$\mathfrak{e}_s = \frac{1}{2i}\sum_{\ell=0}^{n} \Big(\mathcal{J}_{n-\ell}^{(x_l,x)}[1+(-1)^p](k)\mathcal{J}_{\ell}^{(x,x_r)}[1+(-1)^{n-\ell+p}](k) - \mathcal{J}_{n-\ell}^{(x_l,x)}[1-(-1)^p](k)\mathcal{J}_{\ell}^{(x,x_r)}[1-(-1)^{n-\ell+p}](k) \Big). \tag{4.2.16b}$$

For the first terms of $\mathfrak{e}_c$ and $\mathfrak{e}_s$, $\overline{\sigma}_{p,n-\ell} = 1+(-1)^p$, and $\tilde{\sigma}_{p,\ell} = 1+(-1)^{n-\ell+p}$ so that $\overline{\sigma}_{n-\ell,n-\ell} = \tilde{\sigma}_{0,\ell}$ and $\sigma_{p,n} = 1+(-1)^p$. For the second terms of $\mathfrak{e}_c$ and $\mathfrak{e}_s$, $\overline{\sigma}_{p,n-\ell} = 1-(-1)^p$, and $\tilde{\sigma}_{p,\ell} = 1-(-1)^{n-\ell+p}$ so that $\overline{\sigma}_{n-\ell,n-\ell} = \tilde{\sigma}_{0,\ell}$ and $\sigma_{p,n} = 1-(-1)^p$. Equations (4.2.10c) and (4.2.10d) follow. □

Now, we begin taking derivatives of the solution expressions. In Definition 36, we introduce some functions that appear in the derivatives of the solution expressions. In Lemmas 37–39, we prove some properties of these functions.

Definition 36. *We define*

$$\overline{\Psi}(k,x,y) = \sqrt{(\beta\mathfrak{n})(k,x)}\frac{\partial}{\partial x}\left(\frac{\Psi(k,x,y)}{\sqrt{(\beta\mathfrak{n})(k,x)}}\right) = \Psi_x(k,x,y) - \frac{1}{2}\frac{(\beta\mathfrak{n})'(k,x)}{(\beta\mathfrak{n})(k,x)}\Psi(k,x,y), \quad \textit{and} \tag{4.2.17a}$$

$$\tilde{\Psi}(k,x,y) = \sqrt{(\beta\mathfrak{n})(k,x)}\frac{\partial}{\partial x}\left(\frac{(\beta\overline{\Psi})(k,x,y)}{\sqrt{(\beta\mathfrak{n})(k,x)}}\right) = (\beta\overline{\Psi})_x(k,x,y) - \frac{1}{2}\frac{(\beta\mathfrak{n})'(k,x)}{(\beta\mathfrak{n})(k,x)}\left(\beta\overline{\Psi}\right)(k,x,y), \tag{4.2.17b}$$

where we use the notation $(\beta\overline{\Psi})(k,x,y) = \beta(x)\overline{\Psi}(k,x,y)$. *We also define*

$$\Psi(k,x,x^{\pm}) = \lim_{y\to x^{\pm}}\Psi(k,x,y), \qquad \overline{\Psi}(k,x,x^{\pm}) = \lim_{y\to x^{\pm}}\overline{\Psi}(k,x,y), \tag{4.2.18a}$$

and

$$\chi(k,x) = (\beta\overline{\Psi})(k,x,x^-) - (\beta\overline{\Psi})(k,x,x^+). \tag{4.2.18b}$$

Lemma 37. *For the whole-line problem, for* $y < x$,

$$\overline{\Psi}(k,x,y) = ik\mathfrak{n}(k,x)\exp\left(\int_y^x ik\mathfrak{n}(k,\xi)\,d\xi\right)\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}\tilde{\mathcal{E}}_{n-\ell}^{(-\infty,y)}(k)\mathcal{E}_{\ell}^{(x,\infty)}(k), \tag{4.2.19a}$$

and for $x < y$,

$$\overline{\Psi}(k,x,y) = -ik\mathfrak{n}(k,x)\exp\left(\int_x^y ik\mathfrak{n}(k,\xi)\,d\xi\right)\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}(-1)^n\tilde{\mathcal{E}}_{n-\ell}^{(-\infty,x)}(k)\mathcal{E}_{\ell}^{(y,\infty)}(k). \tag{4.2.19b}$$

For the half-line problem, for $x_l < y < x$,

$$\overline{\Psi}(k,x,y) = 4ik\mathfrak{n}(k,x)\exp\left(\int_{x_l}^x ik\mathfrak{n}(k,\xi)\,d\xi\right)\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}\left(\frac{a_0}{k\mathfrak{n}(k,x_l)}\mathcal{S}_{n-\ell}^{(x_l,y)}(k) - a_1\mathcal{C}_{n-\ell}^{(x_l,y)}(k)\right)\mathcal{E}_{\ell}^{(x,\infty)}(k), \tag{4.2.20a}$$

and for $x_l < x < y$,

$$\overline{\Psi}(k,x,y) = 4k\mathfrak{n}(k,x)\exp\left(\int_{x_l}^y ik\mathfrak{n}(k,\xi)\,d\xi\right)\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}(-1)^n\left(\frac{a_0}{k\mathfrak{n}(k,x_l)}\mathcal{C}_{n-\ell}^{(x_l,x)}(k) + a_1\mathcal{S}_{n-\ell}^{(x_l,x)}(k)\right)\mathcal{E}_{\ell}^{(y,\infty)}(k). \tag{4.2.20b}$$

For the finite-interval problem, for $x_l < y < x < x_r$,

$$\begin{aligned}\overline{\Psi}(k,x,y) = 4k\mathfrak{n}(k,x)\Xi(k)\Bigg\{&-\frac{\beta(x_r)(a:b)_{1,2}}{k\sqrt{(\beta\mathfrak{n})(k,x_l)}\sqrt{(\beta\mathfrak{n})(k,x_r)}}\sum_{n=0}^{\infty}(-1)^n\mathcal{C}_n^{(y,x)}(k)\\ &-(a:b)_{2,4}\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}(-1)^{\ell}\mathcal{C}_{n-\ell}^{(x_l,y)}(k)\mathcal{S}_{\ell}^{(x,x_r)}(k) - \frac{(a:b)_{1,3}}{k^2\mathfrak{n}(k,x_l)\mathfrak{n}(k,x_r)}\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}\mathcal{S}_{n-\ell}^{(x_l,y)}(k)\mathcal{C}_{\ell}^{(x,x_r)}(k)\\ &+\frac{(a:b)_{1,4}}{k\mathfrak{n}(k,x_l)}\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}(-1)^{\ell}\mathcal{S}_{n-\ell}^{(x_l,y)}(k)\mathcal{S}_{\ell}^{(x,x_r)}(k) + \frac{(a:b)_{2,3}}{k\mathfrak{n}(k,x_r)}\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}\mathcal{C}_{n-\ell}^{(x_l,y)}(k)\mathcal{C}_{\ell}^{(x,x_r)}(k)\Bigg\},\end{aligned} \tag{4.2.21a}$$

and for $x_l < x < y < x_r$,

$$\begin{aligned}\overline{\Psi}(k,x,y) = 4k\mathfrak{n}(k,x)\Xi(k)\Bigg\{&\frac{\beta(x_l)(a:b)_{3,4}}{k\sqrt{(\beta\mathfrak{n})(k,x_l)}\sqrt{(\beta\mathfrak{n})(k,x_r)}}\sum_{n=0}^{\infty}\mathcal{C}_n^{(x,y)}(k)\\ &+(a:b)_{2,4}\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}(-1)^n\mathcal{S}_{n-\ell}^{(x_l,x)}(k)\mathcal{C}_{\ell}^{(y,x_r)}(k) + \frac{(a:b)_{1,3}}{k^2\mathfrak{n}(k,x_l)\mathfrak{n}(k,x_r)}\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}(-1)^{n-\ell}\mathcal{C}_{n-\ell}^{(x_l,x)}(k)\mathcal{S}_{\ell}^{(y,x_r)}(k)\\ &+\frac{(a:b)_{1,4}}{k\mathfrak{n}(k,x_l)}\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}(-1)^n\mathcal{C}_{n-\ell}^{(x_l,x)}(k)\mathcal{C}_{\ell}^{(y,x_r)}(k) + \frac{(a:b)_{2,3}}{k\mathfrak{n}(k,x_r)}\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}(-1)^{n-\ell}\mathcal{S}_{n-\ell}^{(x_l,x)}(k)\mathcal{S}_{\ell}^{(y,x_r)}(k)\Bigg\}.\end{aligned} \tag{4.2.21b}$$

Proof. Using (4.2.1) in (2.1.4), (2.2.8), and (2.3.7), we find (4.2.19), (4.2.20), and (4.2.21) for the whole-line, half-line, and finite-interval problems, respectively. ☐

Lemma 38. *Consider the finite-interval, half-line, and whole-line problems. There exists an* $r > \sqrt{M_\gamma}$ *and* $M_\Psi > 0$ *so that for* $k \in \Omega_{\text{ext}}(r)$*, for* $x \in \overline{\mathcal{D}}$*, and for* $y \in \overline{\mathcal{D}}$

$$|\overline{\Psi}(k, x, y)| \leq M_\Psi |k|. \tag{4.2.22a}$$

For the regular problems

$$\left|\frac{\overline{\Psi}(k, x, y)}{\Delta(k)}\right| \leq M_\Psi |k|, \tag{4.2.22b}$$

and for the irregular problems

$$\left|\frac{\overline{\Psi}(k, x, y)}{\Delta(k)}\right| \leq M_\Psi |k| \left(1 + |k| \left(e^{-m_{in}|k|(x - x_l)} + e^{-m_{in}|k|(x_r - x)}\right)\right). \tag{4.2.22c}$$

Therefore, $\overline{\Psi}(k, x, y)$ *and* $\overline{\Psi}(k, x, y)/\Delta(k)$ *are well-defined functions.*

Proof. The proof is identical to that of Lemma 24 in Appendix 4.1. Note that M_Ψ here and from Lemma 13 are identical up to a factor of $M_{\mathfrak{n}}$. Without loss of generality, we take them to be the same. ☐

Lemma 39. *For the finite-interval, half-line, and whole-line problems,*

$$\chi(k, x) = 2ik(\beta\mathfrak{n})(k, x)\Delta(k), \tag{4.2.23}$$

where $\chi(k, x)$ *is defined in* (4.2.18b).

Proof. For the whole-line problem, using Lemma 37 in (4.2.18b),

$$\chi(k, x) = ik(\beta\mathfrak{n})(k, x) \sum_{n=0}^{\infty} (1 + (-1)^n) \sum_{\ell=0}^{n} \tilde{\mathcal{E}}_{n-\ell}^{(-\infty,x)}(k)\mathcal{E}_\ell^{(x,\infty)}(k), \tag{4.2.24}$$

which gives (4.2.23), using Lemma 35. Similarly, for the half-line problem,

$$\chi(k, x) = -4k(\beta\mathfrak{n})(k, x) \exp\left(\int_{x_l}^{x} ik\mathfrak{n}(k, \xi)\, d\xi\right) \sum_{n=0}^{\infty} \left(\frac{(-1)^n a_0}{k\mathfrak{n}(k, x_l)} + ia_1\right) \sum_{\ell=0}^{n} \left(\mathcal{C}_{n-\ell}^{(x_l,x)}(k) - (-1)^n i\mathcal{S}_{n-\ell}^{(x_l,x)}(k)\right) \mathcal{E}_\ell^{(x,\infty)}(k). \tag{4.2.25}$$

Using Lemma 35,

$$\chi(k, x) = -4k(\beta\mathfrak{n})(k, x) \sum_{n=0}^{\infty} \left(\frac{(-1)^n a_0}{k\mathfrak{n}(k, x_l)} + ia_1\right) \mathcal{E}_n^{(x_l,\infty)}(k) = 2ik(\beta\mathfrak{n})(k, x)\Delta(k). \tag{4.2.26}$$

Finally, for the finite-interval problem, since $\mathcal{C}_n^{(x,x)}(k) = \delta_{0n}$ and $\mathcal{S}_n^{(x,x)}(k) = 0$,

$$\begin{aligned}\chi(k, x) = -4k(\beta\mathfrak{n})(k, x)\Xi(k) \Bigg\{\mathfrak{a}(k) + \sum_{n=0}^{\infty} \mathfrak{c}_n(k) \sum_{\ell=0}^{n} \left(\mathcal{C}_{n-\ell}^{(x_l,x)}(k)\mathcal{C}_\ell^{(x,x_r)}(k) - (-1)^{n-\ell}\mathcal{S}_{n-\ell}^{(x_l,x)}(k)\mathcal{S}_\ell^{(x,x_r)}(k)\right) \\ + \sum_{n=0}^{\infty} \mathfrak{s}_n(k) \sum_{\ell=0}^{n} \left(\mathcal{S}_{n-\ell}^{(x_l,x)}(k)\mathcal{C}_\ell^{(x,x_r)}(k) + (-1)^{n-\ell}\mathcal{C}_{n-\ell}^{(x_l,x)}(k)\mathcal{S}_\ell^{(x,x_r)}(k)\right)\Bigg\},\end{aligned} \tag{4.2.27}$$

which gives (4.2.23), using Lemma 35. ☐

Lemma 40. *For the half-line problem,*

$$\mathcal{B}_{0,x}(k, x) = \frac{4\beta(x_l)ik\mathfrak{n}(k, x) \exp\left(\int_{x_l}^{x} ik\mathfrak{n}(k, \xi)\, d\xi\right)}{\sqrt{(\beta\mathfrak{n})(k, x_l)}\sqrt{(\beta\mathfrak{n})(k, x)}} \sum_{n=0}^{\infty} \mathcal{E}_n^{(x,\infty)}(k), \tag{4.2.28a}$$

and there exists an $r > \sqrt{M_\gamma}$ *and* $M_\mathcal{B} > 0$ *so that for* $k \in \Omega_{\text{ext}}(r)$ *and* $x \in \overline{\mathcal{D}}$,

$$|\mathcal{B}_{0,x}(k,x)| \le M_\mathcal{B}|k| \qquad \text{and} \qquad \left|\frac{\mathcal{B}_{0,x}(k,x)}{\Delta(k)}\right| \le M_\mathcal{B}|k|^2 e^{-m_{\mathfrak{n}}|k|(x-x_l)}. \tag{4.2.28b}$$

For the finite-interval problem, we have for $j = 1, 2$,

$$\begin{aligned}\mathcal{B}_{2-j,x}(k,x) = -(-1)^j \frac{4k\mathfrak{n}(k,x)\Xi(k)}{\sqrt{(\beta\mathfrak{n})(k,x)}} \Bigg\{ \frac{\beta(x_r)}{\sqrt{(\beta\mathfrak{n})(k,x_r)}} \left[\frac{a_{j1}}{k\mathfrak{n}(k,x_l)} \sum_{n=0}^{\infty} (-1)^n \mathcal{C}_n^{(x_l,x)}(k) + a_{j2} \sum_{n=0}^{\infty} (-1)^n \mathcal{S}_n^{(x_l,x)}(k) \right] \\ + \frac{\beta(x_l)}{\sqrt{(\beta\mathfrak{n})(k,x_l)}} \left[\frac{b_{j1}}{k\mathfrak{n}(k,x_r)} \sum_{n=0}^{\infty} \mathcal{C}_n^{(x,x_r)}(k) - b_{j2} \sum_{n=0}^{\infty} (-1)^n \mathcal{S}_n^{(x,x_r)}(k) \right] \Bigg\},\end{aligned} \tag{4.2.29a}$$

and there exists an $r > \sqrt{M_\gamma}$ *and* $M_\mathcal{B} > 0$ *so that for* $k \in \Omega_{\text{ext}}(r)$ *and* $x \in \overline{\mathcal{D}}$,

$$|\mathcal{B}_{m,x}(k,x)| \le M_\mathcal{B}|k| \qquad \text{and} \qquad \left|\frac{\mathcal{B}_{m,x}(k,x)}{\Delta(k)}\right| \le M_\mathcal{B}|k|^{b+2}\left(e^{-m_{\mathfrak{n}}|k|(x_r-x)} + e^{-m_{\mathfrak{n}}|k|(x-x_l)}\right). \tag{4.2.29b}$$

For regular *boundary conditions* $b = 0$, *and for* irregular *boundary conditions* $b = 1$. *Therefore, the functions* $\mathcal{B}_{m,x}(x,t)$ *and* $\mathcal{B}_{m,x}(x,t)/\Delta(k)$ *are well defined for the half-line and finite-interval problems.*

Proof. Lemma 33 and a direct calculation gives (4.2.28a) and (4.2.29a). The proofs for (4.2.28b) and (4.2.29b) are identical to the proof of Lemma 25. Note that, as in Lemma 38, the $M_\mathcal{B}$'s differ only by a factor of $M_\mathfrak{n}$ (see Lemma 13). Without loss of generality, we may take them to be identical. □

Lemma 41. *Consider the finite-interval, half-line, and whole-line problems. We have*

$$\Phi_{0,x}(k,x) = \int_\mathcal{D} \frac{\overline{\Psi}(k,x,y) q_\alpha(y)}{\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,y)}}\, dy, \tag{4.2.30}$$

where $\Phi_0(k,x)$ *is defined in* (4.1.28a). *There exists an* $M_\Phi > 0$ *so that*

$$|\Phi_{0,x}(k,x)| \le M_\Phi |k| \|q_0\|_\mathcal{D} \qquad \text{and} \qquad \left|\frac{\Phi_{0,x}(k,x)}{\Delta(k)}\right| \le M_\Phi |k|^2 \|q_0\|_\mathcal{D}. \tag{4.2.31}$$

Thus $\Phi_{0,x}(k,x)$ *and* $\Phi_{0,x}(k,x)/\Delta(k)$ *are well defined for all three problems.*

Proof. Breaking up the integral over $\mathcal{D}$ in (4.1.28a) into two integrals over the regions $y < x$ and $y > x$ and using the Leibniz integral rule, we obtain

$$\Phi_{0,x}(k,x) = \frac{(\Psi(k,x,x^-) - \Psi(k,x,x^+))q_\alpha(x)}{(\beta\mathfrak{n})(k,x)} + \int_\mathcal{D} \frac{\overline{\Psi}(k,x,y) q_\alpha(y)}{\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,y)}}\, dy. \tag{4.2.32}$$

Since $\Psi(k,x,x^-) = \Psi(k,x,x^+)$, we find (4.2.30). We obtain (4.2.31) from Lemma 38. Since the integrand in (4.2.30) is absolutely integrable, differentiation under the integral is allowed. □

Lemma 42. *Consider the finite-interval, half-line, and whole-line problems. For* $k \in \Omega_{\text{ext}}$, $x \in \overline{\mathcal{D}}$, *and* $t \in (0,T)$,

$$\Phi_{\mathfrak{f},x}(k,x,t) = \int_\mathcal{D} \frac{\overline{\Psi}(k,x,y)\mathfrak{f}_\alpha(k^2,y,t)}{\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,y)}}\, dy. \tag{4.2.33}$$

Further, there exists an $M_f > 0$ *so that*

$$\left|\Phi_{\mathfrak{f},x}(k,x,t)e^{-k^2 t}\right| \le M_f |k| \int_\mathcal{D} \left|\mathfrak{f}_\alpha(k^2,y,t)e^{-k^2 t}\right| dy. \tag{4.2.34a}$$

For the regular *problems,*

$$\left|\frac{\Phi_{\mathfrak{f},x}(k,x,t)e^{-k^2 t}}{\Delta(k)}\right| \le M_f |k| \int_\mathcal{D} \left|\mathfrak{f}_\alpha(k^2,y,t)e^{-k^2 t}\right| dy, \tag{4.2.34b}$$

and for the irregular problems,

$$\left|\frac{\Phi_{\mathfrak{f}}(k,x,t)e^{-k^2t}}{\Delta(k)}\right| \le M_f|k|\Big(1+|k|\big(e^{-m_{\text{in}}|k|(x-x_l)}+e^{-m_{\text{in}}|k|(x_r-x)}\big)\Big)\int_{\mathcal{D}}\left|\mathfrak{f}_\alpha(k^2,y,t)e^{-k^2t}\right|dy. \tag{4.2.34c}$$

where $\Phi_{\mathfrak{f}}(k,x,t)$ *and* $\mathfrak{f}_\alpha(k^2,y,t)$ *are defined in* (4.1.83b) *and* (4.1.83c)*, respectively.*

Proof. Breaking up the integral over $\mathcal{D}$ in (4.1.28a) into two integrals over the regions $y<x$ and $y>x$ and using the Leibniz integral rule, we obtain

$$\Phi_{\mathfrak{f},x}(k,x,t)=\frac{(\Psi(k,x,x^-)-\Psi(k,x,x^+))\,\mathfrak{f}_\alpha(k^2,x,t)}{\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,x)}}+\int_{\mathcal{D}}\frac{\overline{\Psi}(k,x,y)\mathfrak{f}_\alpha(k^2,y,t)}{\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,y)}}\,dy. \tag{4.2.35}$$

Since $\Psi(k,x,x^-)=\Psi(k,x,x^+)$ for all three problems, we obtain (4.2.33). Equation (4.2.34) follows from (4.2.22a). Since the integrand (4.2.33) is absolutely integrable, differentiation under the integral is allowed. ☐

Lemma 43. *Consider the finite-interval, half-line, and whole-line problems. For* $x\in\mathcal{D}$ *and* $t\in(0,T)$*,*

$$q_{0,x}(x,t)=\frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}}\frac{\Phi_{0,x}(k,x)}{\Delta(k)}e^{-k^2t}\,dk, \tag{4.2.36a}$$

$$q_{f,x}(x,t)=\frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}}\frac{\Phi_{\mathfrak{f},x}(k,x,t)e^{-k^2t}}{\Delta(k)}dy, \tag{4.2.36b}$$

$$q_{\mathcal{B}_m,x}(x,t)=\frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}}\frac{\mathcal{B}_{m,x}(k,x)}{\Delta(k)}\mathfrak{F}_m(k^2,t)e^{-k^2t}\,dk, \tag{4.2.36c}$$

are well defined, i.e., *we can differentiate under the integral sign. Furthermore,* $q_{0,x}(x,t)$ *and* $q_{f,x}(x,t)$ *are well defined for* $x\in\overline{\mathcal{D}}$*. For the* regular problems, $q_{\mathcal{B}_m,x}(x,t)$ *is well defined for* $x\in\overline{\mathcal{D}}$*.*

Proof. The integrand in $q_{0,x}(x,t)$ is exponentially decaying for $t\in(0,T)$, and therefore is well defined for $x\in\overline{\mathcal{D}}$. From (4.2.34) and (4.1.84), we see that, for any $t\in(0,T)$, $q_{f,x}(x,t)$ is also well defined for $x\in\overline{\mathcal{D}}$. For $t\in(0,T)$, from (4.2.28b), (4.2.29b), and (4.1.76), we see that for $x\in\mathcal{D}$, $q_{\mathcal{B}_m,x}(x,t)$ has exponential decay and is well defined. For the *regular problems*, $q_{\mathcal{B}_m,x}(x,t)$ is absolutely integrable for $x\in\overline{\mathcal{D}}$ and is well defined. ☐

Remark. For the *irregular problems*, $q_{\mathcal{B}_m,x}(x,t)$ may be ill defined at the boundaries, but the boundary conditions (2.3.1c) and (2.3.1d) are well defined and satisfied, see Section 4.3.

Lemma 44. *Consider the finite-interval, half-line, and whole-line problems. For* $x,y\in\mathcal{D}$ *and* $k\in\Omega_{\text{ext}}$*,*

$$\tilde{\Psi}(k,x,y)=-\frac{k^2+\gamma(x)}{\alpha(x)}\Psi(k,x,y). \tag{4.2.37a}$$

For the half-line $(m=0)$ *and the finite-interval problems* $(m=0,1)$*,*

$$(\beta\mathcal{B}_{m,x})_x(k,x)=-\frac{k^2+\gamma(x)}{\alpha(x)}\mathcal{B}_m(k,x), \tag{4.2.37b}$$

for $x\in\mathcal{D}$*,* $t\in(0,T)$*, and* $k\in\Omega$*.*

Proof. For the whole-line problem, a direct calculation using Lemma 33 gives (4.2.37a) from (4.2.19a) for $y<x$ and from (4.2.19b) for $y>x$. Similarly, for the half-line problem, we obtain (4.2.37a) from (4.2.20a) for $x_l<y<x$ and from (4.2.20b) for $x_l<x<y$. Equation (4.2.37b) follows from (4.2.28a). For the finite-interval problem, we obtain (4.2.37a) from (4.2.21a) for $x_l<y<x<x_r$ and from (4.2.21b) for $x_l<x<y<x_r$. Finally, (4.2.37b) follows from (4.2.29a). ☐

Lemma 45. *Consider the finite-interval, half-line, and whole-line problems. With* $\mathfrak{f}(k^2,x,t)=\alpha(x)\mathfrak{f}_\alpha(k^2,x,t)$*,*

$$\alpha(x)(\beta\Phi_{0,x})_x(k,x)=2ik\Delta(k)q_0(x)-(k^2+\gamma(x))\Phi_0(k,x), \tag{4.2.38a}$$

$$\alpha(x)(\beta\Phi_{\mathfrak{f},x})_x(k,x,t)=2ik\Delta(k)\mathfrak{f}(k^2,x,t)-(k^2+\gamma(x))\Phi_{\mathfrak{f}}(k,x,t), \tag{4.2.38b}$$

Proof. Using Lemmas 41 and 42, we split $\mathcal{D}$ into the two parts $y < x$ and $y > x$, and the Leibniz integral rule gives

$$(\beta\Phi_{0,x})_x(k,x) = \frac{\chi(k,x)q_0(x)}{\alpha(x)(\beta\mathfrak{n})(k,x)} + \int_{\mathcal{D}} \frac{\tilde{\Psi}(k,x,y)q_0(y)}{\alpha(y)\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,y)}}\,dy, \tag{4.2.39a}$$

$$(\beta\Phi_{\mathfrak{f},x})_x(k,x,t) = \frac{\chi(k,x)\mathfrak{f}(k^2,x,t)}{\alpha(x)(\beta\mathfrak{n})(k,x)} + \int_{\mathcal{D}} \frac{\tilde{\Psi}(k,x,y)\mathfrak{f}(k^2,y,t)}{\alpha(y)\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,y)}}\,dy, \tag{4.2.39b}$$

where $\chi(k,x)$ and $\tilde{\Psi}(k,x,y)$ are defined in Definition 36. Using Lemmas 39 and 44 gives (4.2.38). □

Lemma 46. *Consider the finite-interval, half-line, and whole-line problem. For $x \in \mathcal{D}$ and $t \in (0,T)$, the t-derivatives of $q_0(x,t)$, $q_f(x,t)$ and $q_{\mathcal{B}_m}(x,t)$ are*

$$q_{0,t}(x,t) = -\frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}} \frac{k^2\Phi_0(k,x)}{\Delta(k)} e^{-k^2t}\,dk, \tag{4.2.40a}$$

$$q_{f,t}(x,t) = -\frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}(r)}} \frac{k^2\Phi_{\mathfrak{f}}(k,x,t)e^{-k^2t}}{\Delta(k)}\,dk, \tag{4.2.40b}$$

$$q_{\mathcal{B}_m,t}(x,t) = -\frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}(r)}} \frac{k^2\mathcal{B}_m(k,x)}{\Delta(k)}\mathfrak{F}_m(k^2,t)e^{-k^2t}\,dk, \qquad m = 0,1. \tag{4.2.40c}$$

These functions are well defined.

Proof. Differentiating (4.1.80) with respect to t gives (4.2.40a), since the integrand is absolutely integrable. From (4.1.83c), $\mathfrak{f}_{\alpha,t}(k^2,x,t)e^{-k^2t} = -f_{\alpha,t}(x,t)/k^2$, and differentiating (4.1.83b) with respect to t yields

$$\Phi_{\mathfrak{f},t}(k,x,t)e^{-k^2t} = -\int_{\mathcal{D}} \frac{\Psi(k,x,y)f_{\alpha,t}(y,t)}{k^2\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,y)}}\,dy, \tag{4.2.41}$$

so that, using Lemma 24,

$$\left|\frac{\Phi_{\mathfrak{f},t}(k,x,t)e^{-k^2t}}{\Delta(k)}\right| \le \frac{M_f}{|k|^2}\Big(1 + |k|\big(e^{-m_{in}|k|(x-x_l)} + e^{-m_{in}|k|(x_r-x)}\big)\Big)\|f_{\alpha,t}\|_{\mathcal{D}}. \tag{4.2.42}$$

Differentiating (4.1.83a) with respect to t, we obtain

$$q_{f,t}(x,t) = \frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}(r)}} \frac{\Phi_{\mathfrak{f},t}(k,x,t)e^{-k^2t}}{\Delta(k)}\,dk - \frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}(r)}} \frac{k^2\Phi_{\mathfrak{f}}(k,x,t)e^{-k^2t}}{\Delta(k)}\,dk. \tag{4.2.43}$$

From (4.2.42), it follows that the first contour integral can be closed in the upper half plane, implying it is zero by Cauchy's theorem, resulting in (4.2.40b). From (4.1.85), $q_{f,t}(x,t)$ is well defined, for $x \in \mathcal{D}$. Since $\mathfrak{F}_{m,t}(k^2,t)e^{-k^2t} = -f_m'(t)/k^2$, differentiating (4.1.75a) with respect to t,

$$q_{\mathcal{B}_m,t}(x,t) = -\frac{f_m'(t)}{2\pi}\int_{\partial\Omega_{\text{ext}(r)}} \frac{\mathcal{B}_m(k,x)}{k^2\Delta(k)}\,dk - \frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}(r)}} \frac{k^2\mathcal{B}_m(k,x)}{\Delta(k)}\mathfrak{F}_m(k^2,t)e^{-k^2t}\,dk. \tag{4.2.44}$$

As above, (4.1.67) allows us to close the contour of the first integral in the upper half plane, showing the first term is zero by Cauchy's theorem, obtaining (4.2.40c). From (4.1.67) and (4.1.76), $q_{\mathcal{B}_m,t}(x,t)$ is well defined for $x \in \mathcal{D}$. □

Lemma 47. *For $x \in \mathcal{D}$ and $t \in (0,T)$, the derivatives*

$$\alpha(x)(\beta q_{0,x})_x(x,t) + \gamma(x)q_0(x,t) = q_{0,t}(x,t), \tag{4.2.45a}$$

$$\alpha(x)(\beta q_{f,x})_x(x,t) + \gamma(x)q_f(x,t) + f(x,t) = q_{f,t}(x,t), \tag{4.2.45b}$$

$$\alpha(x)(\beta q_{\mathcal{B}_m,x})_x(x,t) + \gamma(x)q_{\mathcal{B}_m}(x,t) = q_{\mathcal{B}_m,t}(x,t), \qquad m = 0,1. \tag{4.2.45c}$$

are well defined, i.e., *differentiation under the integral sign is allowed.*

Proof. Direct differentiation of the results in Lemma 43 yields

$$(\beta q_{0,x})_x(x,t) = \frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}} \frac{(\beta\Phi_{0,x})_x(k,x)}{\Delta(k)} e^{-k^2 t}\,dk, \tag{4.2.46a}$$

$$(\beta q_{f,x})_x(x,t) = \frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}} \frac{(\beta\Phi_{\mathfrak{f},x})_x(k,x,t)e^{-k^2 t}}{\Delta(k)} dk, \tag{4.2.46b}$$

$$(\beta q_{\mathcal{B}_m,x})_x(x,t) = \frac{1}{2\pi}\int_{\partial\Omega} \frac{(\beta\mathcal{B}_{m,x})_x(k,x)}{\Delta(k)} \mathfrak{F}_m(k^2,t)e^{-k^2 t}\,dk, \qquad m=0,1. \tag{4.2.46c}$$

Using Lemmas 44, 45, and 46,

$$\alpha(x)(\beta q_{0,x})_x(x,t) + \gamma(x)q_0(x,t) = q_{0,t}(x,t) - \frac{q_0(x)}{i\pi}\int_{\partial\Omega_{\text{ext}}} k e^{-k^2 t}\,dk, \tag{4.2.47a}$$

$$\alpha(x)(\beta q_{f,x})_x(x,t) + \gamma(x)q_f(x,t) = q_{f,t}(x,t) - \frac{1}{i\pi}\int_{\partial\Omega_{\text{ext}}} k\mathfrak{f}(k^2,x,t)e^{-k^2 t}dk, \tag{4.2.47b}$$

$$\alpha(x)(\beta q_{\mathcal{B}_m,x})_x(x,t) + \gamma(x)q_{\mathcal{B}_m}(x,t) = q_{\mathcal{B}_m,t}(x,t), \qquad m=0,1. \tag{4.2.47c}$$

Since the integrands in (4.2.47) are absolutely integrable, the differentiation inside the integral is justified. The path for the remaining integral in (4.2.47a) can be deformed down to the real line showing it is zero. Using (4.1.83c), the remaining integral in (4.2.47b) is evaluated as

$$-\int_{\partial\Omega_{\text{ext}}} k\mathfrak{f}(k^2,x,t)e^{-k^2 t}dk = \int_{\partial\Omega_{\text{ext}}} \left(\frac{f(x,0)}{k} + \frac{\mathcal{G}[f_t](k^2,x,t)}{k}\right) e^{-k^2 t}dk, \tag{4.2.48}$$

which may also be deformed to an indented contour on the real line. The principal-value part integral is zero, while the indentation integral evaluates to

$$\frac{1}{i\pi}\int_{\partial\Omega_{\text{ext}}} k\mathfrak{f}(k^2,x,t)e^{-k^2 t}dk = \operatorname{Res}\left(\left(\frac{f(x,0)}{k} + \frac{\mathcal{G}[f_t](k^2,x,t)}{k}\right) e^{-k^2 t};\, k=0\right) = f(x,t). \tag{4.2.49}$$

Equation (4.2.47) yields (4.2.45). □

Theorem 48. *The solution expressions* (2.1.2), (2.2.2), *and* (2.3.2) *each solve the evolution equation* (1.2.1a).

Proof. Since $q(x,t) = q_0(x,t) + q_f(x,t) + q_{\mathcal{B}_0}(x,t) + q_{\mathcal{B}_1}(x,t)$, (4.2.45) gives the result. □

4.3 The solution expressions satisfy the boundary values

Definition 49. *In this appendix, $\ell = 0$ corresponds to the half-line problem, while $\ell = 1,2$ correspond to the finite-interval problem. We define, for $k \in \Omega_{\text{ext}}$ and $y \in \mathcal{D}$,*

$$\mathfrak{P}^{(0)}(k,y) = \frac{a_0\Psi(k,x_l,y) + a_1\overline{\Psi}(k,x_l,y)}{\sqrt{(\beta\mathfrak{n})(k,x_l)}}, \tag{4.3.1a}$$

$$\mathfrak{P}^{(\ell)}(k,y) = \frac{a_{\ell 1}\Psi(k,x_l,y) + a_{\ell 2}\overline{\Psi}(k,x_l,y)}{\sqrt{(\beta\mathfrak{n})(k,x_l)}} + \frac{b_{\ell 1}\Psi(k,x_r,y) + b_{\ell 2}\overline{\Psi}(k,x_r,y)}{\sqrt{(\beta\mathfrak{n})(k,x_r)}}, \qquad \ell = 1,2. \tag{4.3.1b}$$

For $k \in \Omega_{\text{ext}}$,

$$\mathfrak{B}_0^{(0)}(k) = a_0\mathcal{B}_0(k,x_l) + a_1\mathcal{B}_{0,x}(k,x_l), \tag{4.3.2a}$$

$$\mathfrak{B}_m^{(\ell)}(k) = a_{\ell 1}\mathcal{B}_m(k,x_l) + a_{\ell 2}\mathcal{B}_{m,x}(k,x_l) + b_{\ell 1}\mathcal{B}_m(k,x_r) + b_{\ell 2}\mathcal{B}_{m,x}(k,x_r), \qquad \ell=1,2, \quad m=0,1, \tag{4.3.2b}$$

and

$$\mathcal{P}_0^{(0)}(k) = a_0\Phi_0(k,x_l) + a_1\Phi_{0,x}(k,x_l), \tag{4.3.3a}$$

$$\mathcal{P}_0^{(\ell)}(k) = a_{\ell 1}\Phi_0(k,x_l) + a_{\ell 2}\Phi_{0,x}(k,x_l) + b_{\ell 1}\Phi_0(k,x_r) + b_{\ell 2}\Phi_{0,x}(k,x_r), \qquad \ell=1,2. \tag{4.3.3b}$$

For $k \in \Omega_{\text{ext}}$ and $t \in (0,T)$,

$$\mathcal{P}_f^{(0)}(k,t) = a_0 \Phi_{\mathfrak{f}}(k,x_l,t) + a_1 \Phi_{\mathfrak{f},x}(k,x_l,t), \tag{4.3.4a}$$

$$\mathcal{P}_f^{(\ell)}(k,t) = a_{\ell 1} \Phi_{\mathfrak{f}}(k,x_l,t) + a_{\ell 2} \Phi_{\mathfrak{f},x}(k,x_l,t) + b_{\ell 1} \Phi_{\mathfrak{f}}(k,x_r,t) + b_{\ell 2} \Phi_{\mathfrak{f},x}(k,x_r,t), \qquad \ell = 1,2. \tag{4.3.4b}$$

Finally, for $t \in (0,T)$,

$$\mathcal{Q}_{\mathcal{B}_m}^{(0)}(t) = a_0 q_{\mathcal{B}_m}(x_l,t) + a_1 q_{\mathcal{B}_m,x}(x_l,t), \tag{4.3.5a}$$

$$\mathcal{Q}_{\mathcal{B}_m}^{(\ell)}(t) = a_{\ell 1} q_{\mathcal{B}_m}(x_l,t) + a_{\ell 2} q_{\mathcal{B}_m,x}(x_l,t) + b_{\ell 1} q_{\mathcal{B}_m}(x_r,t) + b_{\ell 2} q_{\mathcal{B}_m,x}(x_r,t), \qquad \ell = 1,2, \tag{4.3.5b}$$

and

$$\mathcal{Q}_0^{(0)}(t) = a_0 q_0(x_l,t) + a_1 q_{0,x}(x_l,t), \tag{4.3.6a}$$

$$\mathcal{Q}_0^{(\ell)}(t) = a_{\ell 1} q_0(x_l,t) + a_{\ell 2} q_{0,x}(x_l,t) + b_{\ell 1} q_0(x_r,t) + b_{\ell 2} q_{0,x}(x_r,t), \qquad \ell = 1,2, \tag{4.3.6b}$$

and

$$\mathcal{Q}_f^{(0)}(t) = a_0 q_f(x_l,t) + a_1 q_{f,x}(x_l,t), \tag{4.3.7a}$$

$$\mathcal{Q}_f^{(\ell)}(t) = a_{\ell 1} q_f(x_l,t) + a_{\ell 2} q_{f,x}(x_l,t) + b_{\ell 1} q_f(x_r,t) + b_{\ell 2} q_{f,x}(x_r,t), \qquad \ell = 1,2. \tag{4.3.7b}$$

Lemma 50. *For both the half-line problem and the finite-interval problem, for $k \in \Omega_{\text{ext}}$ and $y \in \overline{\mathcal{D}}$,*

$$\mathfrak{P}^{(\ell)}(k,y) = 0, \quad \ell = 0,1,2. \tag{4.3.8}$$

Proof. For the half-line, using (2.2.8) (with $x_l = x < y < x_r$) and (4.2.20b) in (4.3.1a), gives (4.3.8).

$$\Psi(k,x_l,y) = -4 \exp\left(\int_{x_l}^{y} ik\mathfrak{n}(k,\xi)\, d\xi \right) \sum_{n=0}^{\infty} (-1)^n a_1 \mathcal{E}_n^{(y,\infty)}(k),$$

$$\overline{\Psi}(k,x_l,y) = 4 \exp\left(\int_{x_l}^{y} ik\mathfrak{n}(k,\xi)\, d\xi \right) \sum_{n=0}^{\infty} (-1)^n a_0 \mathcal{E}_n^{(y,\infty)}(k),$$

$$\Rightarrow \qquad \mathfrak{P}^{(0)}(k,y) = \frac{a_0 \Psi(k,x_l,y) + a_1 \overline{\Psi}(k,x_l,y)}{\sqrt{(\beta\mathfrak{n})(k,x_l)}} = 0. \tag{4.3.9}$$

Using (2.3.7) and (4.2.21) in (4.3.1b), the calculations for the finite-interval case are equally straightforward albeit more tedious. □

Lemma 51. *For the half-line problem ($m = 0$), and for the finite-interval problem ($m = 0,1$), for $k \in \Omega_{\text{ext}}$,*

$$\mathfrak{B}_m^{(\ell)}(k) = -2ik\Delta(k)\tilde{\delta}_{\ell-1,m}, \quad \ell = 0,1,2. \tag{4.3.10a}$$

Here

$$\tilde{\delta}_{\ell-1,m} = \begin{cases} 1, & \ell = 0,\ m = 0, \\ 1, & \ell \neq 0,\ m = \ell - 1, \\ 0, & \ell \neq 0,\ m \neq \ell - 1. \end{cases} \tag{4.3.10b}$$

Proof. For the half-line problem, using (2.2.5) and (4.2.28a) in (4.3.2a), we find (4.3.10):

$$\mathcal{B}_0(k,x_l) = \frac{4}{\mathfrak{n}(k,x_l)} \sum_{n=0}^{\infty} (-1)^n \mathcal{E}_n^{(x_l,\infty)}(k),$$

$$\mathcal{B}_{0,x}(k,x_l) = \frac{4ik\mathfrak{n}(k,x_l)}{\mathfrak{n}(k,x_l)} \sum_{n=0}^{\infty} \mathcal{E}_n^{(x_l,\infty)}(k),$$

$$\Rightarrow \qquad \mathfrak{B}_0^{(0)}(k) = a_0 \mathcal{B}_0(k,x_l) + a_1 \mathcal{B}_{0,x}(k,x_l) = 4 \sum_{n=0}^{\infty} \left(\frac{(-1)^n a_0}{\mathfrak{n}(k,x_l)} + a_1 ik \right) \mathcal{E}_n^{(x_l,\infty)}(k) = -2ik\Delta(k). \tag{4.3.11}$$

The finite-interval case (using (2.3.6c) and (4.2.29a) in (4.3.2b)) is similar but more tedious. Its details are omitted. □

Lemma 52. *For both the half-line problem and the finite-interval problem, for $k \in \Omega_{\text{ext}}$ and $t \in [0, T]$,*

$$\mathcal{P}_0^{(\ell)}(k) = 0, \tag{4.3.12a}$$

$$\mathcal{P}_f^{(\ell)}(k, t) = 0. \tag{4.3.12b}$$

Proof. Using (4.1.28a) and (4.2.30) in (4.3.3a) and (4.3.3b), we find

$$\mathcal{P}_0^{(\ell)}(k) = \int_{\mathcal{D}} \frac{\mathfrak{P}^{(\ell)}(k, y) q_\alpha(y)}{\sqrt{(\beta\mathfrak{n})(k, y)}}\, dy, \tag{4.3.13a}$$

which gives (4.3.12a), using Lemma 50. Similarly, using (4.1.83b) and (4.2.33) in (4.3.4a) and (4.3.4b), we find

$$\mathcal{P}_f^{(\ell)}(k, t) = \int_{\mathcal{D}} \frac{\mathfrak{P}^{(\ell)}(k, y) \mathfrak{f}_\alpha(k^2, y, t)}{\sqrt{(\beta\mathfrak{n})(k, y)}}\, dy, \tag{4.3.13b}$$

which gives (4.3.12b), using Lemma 50. □

Lemma 53. *For the half-line ($m = 0$) and the finite-interval problem ($m = 0, 1$), for $k \in \Omega_{\text{ext}}$ and $t \in [0, T]$,*

$$\mathcal{Q}_0^{(\ell)}(t) = 0, \tag{4.3.14a}$$

$$\mathcal{Q}_f^{(\ell)}(t) = 0, \tag{4.3.14b}$$

$$\mathcal{Q}_{\mathcal{B}_m}^{(\ell)}(t) = f_m(t)\tilde{\delta}_{\ell-1,m}, \tag{4.3.14c}$$

where $\tilde{\delta}_{\ell-1,m}$ is defined in (4.3.10b).

Proof. From Lemmas 30, 31, and 43, $\mathcal{Q}_0^{(\ell)}(t)$ (4.3.6) and $\mathcal{Q}_f^{(\ell)}(t)$ (4.3.7) are well-defined functions. Similarly, for the *regular problems*, $Q_{\mathcal{B}_m}^{(\ell)}(t)$ (4.3.5) is a well-defined function from Lemmas 29 and 43. For the *irregular problems*, for Boundary Case 3, $q_{\mathcal{B}_m,x}(x, t)$ may be undefined at the boundary, but the linear combination of boundary terms $\mathcal{Q}_{\mathcal{B}_m}^{(\ell)}(t)$ (4.3.5b) is well defined. For Boundary Case 4, using Assumption 2.6, $q_{\mathcal{B}_m,x}(x, t)$ is well defined at the boundary and therefore $Q_{\mathcal{B}_m}^{(\ell)}(t)$ is well defined.

For the irregular Boundary Case 3, see Remark 6.3, from (4.2.29a) for $x \approx x_l$,

$$\mathcal{B}_{2-j,x}(k, x) = (-1)^j \frac{4k\mathfrak{n}(k, x)\Xi(k)}{\sqrt{(\beta\mathfrak{n})(k, x)}} \left\{ \frac{\beta(x_l) b_{j2}}{\sqrt{(\beta\mathfrak{n})(k, x_l)}} \sum_{n=0}^{\infty} (-1)^n \mathcal{S}_n^{(x, x_r)}(k) \right\} + O(k^0). \tag{4.3.15}$$

We can prove that either (i) $b_{12} = 0 = b_{22}$, in which case $\mathcal{B}_{m,x}(k, x_l) = O(k^0)$, $\mathcal{B}_{m,x}(k, x_l)/\Delta(k) = O(k^{-2})$, and $q_{\mathcal{B}_m,x}(x_l, t)$ is well defined, see Lemma 43; or (ii) if $(b_{12}, b_{22}) \neq (0, 0)$, then $a_{12} = 0 = a_{22}$, in which case $q_{\mathcal{B}_m,x}(x, t)$ does not appear in $\mathcal{Q}_{\mathcal{B}_m}^{(\ell)}(t)$. The same holds for $x \approx x_r$. It follows that $\mathcal{Q}_{\mathcal{B}_m}^{(\ell)}(t)$ is well defined.

For Boundary Case 4, with Assumption 2.6, we integrate $\mathfrak{F}_m(k^2, t)$ (4.1.75b) by parts to obtain

$$\mathfrak{F}_m(k^2, t) = -\frac{f_m(0)}{k^2} - \frac{e^{k^2 t} f_m'(t)}{k^4} + \frac{f_m'(0)}{k^4} + \frac{\mathcal{G}[f_m''](k^2, t)}{k^4}, \tag{4.3.16}$$

so that we may write $q_{\mathcal{B}_m,x}(x, t)$ (4.2.36c) as

$$q_{\mathcal{B}_m,x}(x, t) = \frac{1}{2\pi} \int_{\partial\Omega_{\text{ext}}} \frac{\mathcal{B}_{m,x}(k, x)}{\Delta(k)} \tilde{\mathfrak{F}}_m(k^2, t) e^{-k^2 t}\, dk, \tag{4.3.17}$$

where

$$\tilde{\mathfrak{F}}_m(k^2, t) = -\frac{f_m(0)}{k^2} + \frac{f_m'(0)}{k^4} + \frac{\mathcal{G}[f_m''](k^2, t)}{k^4}, \tag{4.3.18}$$

and where the integral of the $f_m'(t)$ term is zero by Cauchy's theorem (before the x-differentiation). The first two terms of $\tilde{\mathfrak{F}}_m(k^2, t)e^{-k^2 t}$ are exponentially decaying for $t \in (0, T)$ and the last term is $O(k^{-4})$, by Assumption 3.4. Therefore $q_{\mathcal{B}_m,x}(x, t)$ is well defined for $x \in \overline{\mathcal{D}}$ and $t \in (0, T)$. Consequentially, $\mathcal{Q}_{\mathcal{B}_m}^{(\ell)}(t)$ is well defined.

Using (4.1.80) and (4.2.36a) in (4.3.6), we find

$$\mathcal{Q}_0^{(\ell)}(t) = \frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}} \frac{\mathcal{P}_0^{(\ell)}(k)}{\Delta(k)} e^{-k^2 t}\,dk, \tag{4.3.19a}$$

which gives (4.3.14a), using Lemma 52. Similarly, using (4.1.83) and (4.2.36b) in (4.3.7), we find

$$\mathcal{Q}_f^{(\ell)}(t) = \frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}} \frac{\mathcal{P}_f^{(\ell)}(k,t)e^{-k^2 t}}{\Delta(k)} dk. \tag{4.3.19b}$$

Using Lemma 52, this gives (4.3.14b). Using (4.1.75) and (4.2.36c) in (4.3.5),

$$\mathcal{Q}_{\mathcal{B}_m}^{(\ell)}(t) = \frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}} \frac{\mathfrak{B}_m^{(\ell)}(k)}{\Delta(k)} \mathfrak{F}_m(k^2,t) e^{-k^2 t}\,dk. \tag{4.3.19c}$$

Finally, using Lemma 51 and (4.1.75b), we obtain

$$\mathcal{Q}_{\mathcal{B}_m}^{(\ell)}(t) = -\frac{\tilde{\delta}_{\ell-1,m}}{i\pi}\int_{\partial\Omega_{\text{ext}}} \left(\frac{f_m(0)}{k} + \frac{\mathcal{G}[f_m'](k^2,t)}{k}\right) e^{-k^2 t}\,dk. \tag{4.3.19d}$$

Since the integrand is $O(k^{-3})$, we can deform the path of integration to the real axis. Using the oddness of the integrand, the principal value integral vanishes and only the residue contribution at the origin needs to be calculated:

$$\mathcal{Q}_{\mathcal{B}_m}^{(\ell)}(t) = \tilde{\delta}_{\ell-1,m}\operatorname{Res}\left(\left(\frac{f_m(0)}{k} + \frac{\mathcal{G}[f_m'](k^2,t)}{k}\right) e^{-k^2 t};\, k=0\right) = f_m(t)\tilde{\delta}_{\ell-1,m}. \tag{4.3.19e}$$

□

Lemma 54. *Consider any $t \in (0,T)$, fixed. Then*

$$\lim_{|x|\to\infty} q(x,t) = 0 \qquad \textit{and} \qquad \lim_{x\to\infty} q(x,t) = 0, \tag{4.3.20}$$

for the whole-line and half-line problems, respectively.

Proof. For any fixed $t \in (0,T)$, we have absolute integrability in (4.1.75), (4.1.80), and (4.1.83a). Therefore, we may switch the limit and integrals. Since, from (4.1.56) and (4.1.57),

$$\lim_{|x|\to\infty}\left|\frac{\Psi(k,x,y)}{\Delta(k)}\right| \le M_\Psi e^{-m_{i\nu}|k||x-y|} = 0, \qquad \lim_{x\to\infty}\left|\frac{\Psi(k,x,y)}{\Delta(k)}\right| \le M_\Psi e^{-m_{i\nu}|k||x-y|} = 0, \tag{4.3.21}$$

for the whole-line problem and the half-line problem, respectively, (4.3.20) follows. □

Remark 55. *Since we have absolute integrability in (4.2.36a), (4.2.36b), and in (4.2.36c), we conclude that also*

$$\lim_{|x|\to\infty} q_x(x,t) = 0 \qquad \textit{and} \qquad \lim_{x\to\infty} q_x(x,t) = 0, \tag{4.3.22}$$

for the whole-line and half-line problems, respectively.

Theorem 56. *Consider the finite-interval, the half-line, and the whole-line problems. For all three problems, the solution expression (1.2.2) satisfies the appropriate boundary conditions.*

Proof. Lemma 54 shows the boundary conditions for the whole-line problem and the right boundary condition for the half-line problem are satisfied. From Lemma 53,

$$a_0 q(x_l,t) + a_1 q(x_l,t) = \mathcal{Q}_0^{(0)}(t) + \mathcal{Q}_f^{(0)}(t) + \mathcal{Q}_{\mathcal{B}_m}^{(0)}(t) = f_0(t). \tag{4.3.23a}$$

Similarly, for the finite-interval problem,

$$\begin{aligned} a_{\ell 1} q(x_l,t) + a_{\ell 2} q_x(x_l,t) + b_{\ell 1} q(x_r,t) + b_{\ell 2} q_x(x_r,t) &= \mathcal{Q}_0^{(\ell)}(t) + \mathcal{Q}_f^{(\ell)}(t) + \mathcal{Q}_{\mathcal{B}_0}^{(\ell)}(t) + \mathcal{Q}_{\mathcal{B}_1}^{(\ell)}(t) \\ &= f_0(t)\tilde{\delta}_{\ell-1,0} + f_1(t)\tilde{\delta}_{\ell-1,1}. \end{aligned} \tag{4.3.23b}$$

□

4.4 The solution expressions satisfy the initial condition

Theorem 57. *Consider the finite-interval, half-line, and whole-line problms. For $x \in \mathcal{D}$, fixed,*

$$\lim_{t\to 0^+} q_f(x,t) = 0, \tag{4.4.1a}$$

$$\lim_{t\to 0^+} q_{\mathcal{B}_m}(x,t) = 0. \tag{4.4.1b}$$

Proof. Since the integral in (4.1.83) is absolutely convergent, we can pass the limit $t \to 0^+$ inside the integral to obtain (4.4.1a) by using Cauchy's theorem. Similarly, we move the limit in the integral in (4.1.75) to obtain (4.4.1b). □

Lemma 58. *For fixed $x \in \mathcal{D}$, for $y \in \overline{\mathcal{D}}$, for the finite-interval, half-line, and whole-line problems,*

$$\frac{\Psi(k,x,y)}{\Delta(k)} = \exp\left(\operatorname{sgn}(x-y) ik \int_y^x \mathfrak{n}(k,\xi)\, d\xi\right)\left(1 + o(k^0)\right) + o(k^{-1}), \tag{4.4.2}$$

as $|k| \to \infty$ for $k \in \Omega_{\text{ext}}$.

Proof. For the whole-line problem, from (2.1.4), for $y < x$,

$$\Psi(k,x,y) = \exp\left(\operatorname{sgn}(x-y) ik \int_y^x \mathfrak{n}(k,\xi)\, d\xi\right)\left(1 + \sum_{n=1}^{\infty}\sum_{\ell=0}^{n} (-1)^\ell \tilde{\mathcal{E}}_{n-\ell}^{(-\infty,y)}(k)\mathcal{E}_\ell^{(x,\infty)}(k)\right). \tag{4.4.3}$$

By Lemma 20 and the DCT,

$$\sum_{n=1}^{\infty}\sum_{\ell=0}^{n} (-1)^\ell \tilde{\mathcal{E}}_{n-\ell}^{(-\infty,y)}(k)\mathcal{E}_\ell^{(x,\infty)}(k) = o(k^0). \tag{4.4.4}$$

Dividing (4.4.3) by $\Delta(k)$ and using Lemma 22, we obtain (4.4.2). The proof for $x < y$ is identical.

For the half-line problem, for $x_l < y < x$, we write (2.2.8) as

$$\Psi(k,x,y) = 4\exp\left(ik\int_y^x \mathfrak{n}(k,\xi)\, d\xi\right)\left[\left(\frac{a_0}{k\mathfrak{n}(k,x_l)}\mathcal{S}_0^{(x_l,y)}(k) - a_1\mathcal{C}_0^{(x_l,y)}(k)\right) + \left(\frac{|a_0|}{m_{\mathfrak{n}}|k|} + |a_1|\right) o(k^0)\right]. \tag{4.4.5}$$

Using

$$\mathcal{C}_0^{(a,b)}(k) = \frac{1}{2}\left(\exp\left(2ik\int_a^b \mathfrak{n}(k,\xi)\, d\xi\right) + 1\right), \tag{4.4.6a}$$

$$\mathcal{S}_0^{(a,b)}(k) = \frac{1}{2i}\left(\exp\left(2ik\int_a^b \mathfrak{n}(k,\xi)\, d\xi\right) - 1\right), \tag{4.4.6b}$$

in (4.4.5), we find

$$\Psi(k,x,y) = 2\exp\left(ik\int_y^x \mathfrak{n}(k,\xi)\, d\xi\right)\left[\frac{ia_0}{k\mathfrak{n}(k,x_l)} - a_1 + \left(\frac{|a_0|}{m_{\mathfrak{n}}|k|} + |a_1|\right) o(k^0)\right] + O\left(e^{-m_{in}|k|(x-x_l)}\right), \tag{4.4.7}$$

which, from (4.1.30) with (4.1.32), gives (4.4.2). The proof is identical for $x_l < x < y$.

For the finite-interval problem we consider the 4 different cases.

1. If $(a:b)_{2,4} \neq 0$, then for $x_l < y < x < x_r$, using (4.1.30) with (4.1.33a), we write (2.3.7a) as

$$\begin{aligned}\frac{\Psi(k,x,y)}{\mathfrak{b}_0(k)} = {}& \frac{-4}{(a:b)_{2,4}}\exp\left(ik\int_y^x \mathfrak{n}(k,\xi)\, d\xi\right)\left\{-(a:b)_{2,4}\mathcal{C}_0^{(x_l,y)}(k)\mathcal{C}_0^{(x,x_r)}(k) + o(k^0)\right\} \\ &+ \frac{4\beta(x_r)(a:b)_{1,2}}{(a:b)_{2,4}k\sqrt{(\beta\mathfrak{n})(k,x_l)}\sqrt{(\beta\mathfrak{n})(k,x_r)}}\sum_{n=0}^{\infty}\Xi(k)\mathcal{S}_n^{(y,x)}(k).\end{aligned} \tag{4.4.8}$$

Using (4.4.6a) and dividing by $1+\varepsilon(k)$, we arrive at (4.4.2). The proof for $x_l < x < y < x_r$ is identical.

2. If $(a:b)_{2,4} = 0$ and $m_{\mathfrak{c}_0} \neq 0$, then for $x_l < y < x < x_r$, using (4.4.6) and (4.1.30) with (4.1.33b), we write (2.3.7a) as

$$\frac{\Psi(k,x,y)}{\mathfrak{b}_0(k)} = \frac{4k}{im_{\mathfrak{c}_0}} \exp\left(ik\int_y^x \mathfrak{n}(k,\xi)\,d\xi\right)\left\{-\frac{1}{4i}\left(\frac{(a:b)_{1,4}}{k\mathfrak{n}(k,x_l)} + \frac{(a:b)_{2,3}}{k\mathfrak{n}(k,x_r)}\right) + o(k^{-1})\right\} + o(k^{-1}). \tag{4.4.9}$$

Using $1/\mathfrak{n}(k,x) = 1/\mu(x) + O(k^{-2})$ and dividing by $1+\varepsilon(k)$, we obtain (4.4.2). The proof for $x_l < x < y < x_r$ is identical.

3. If $(a:b)_{2,4} = 0$, $m_{\mathfrak{c}_0} = 0$, $m_{\mathfrak{c}_1} = 0$, and $(a:b)_{1,3} \neq 0$, then for $x_l < y < x < x_r$, using (4.1.30) with (4.1.33c), we write (2.3.7a) as

$$\frac{\Psi(k,x,y)}{\mathfrak{b}_0(k)} = -\frac{4k^2}{m_{\mathfrak{s}}} \exp\left(ik\int_y^x \mathfrak{n}(k,\xi)\,d\xi\right)\left\{-\frac{1}{4}\frac{(a:b)_{1,3}}{k^2\mathfrak{n}(k,x_l)\mathfrak{n}(k,x_r)} + o(k^{-2})\right\} + o(k^{-1}). \tag{4.4.10}$$

Using the asymptotics for $1/\mathfrak{n}(k,x)$ and dividing by $1+\varepsilon(k)$ gives (4.4.2). The proof is identical for $x_l < x < y < x_r$.

4. If $(a:b)_{2,4} = 0$, $m_{\mathfrak{c}_0} = 0$, $m_{\mathfrak{c}_1} \neq 0$, and $m_{\mathfrak{c}_1}\mathfrak{u}_+ - 8m_{\mathfrak{s}} \neq 0$, then for $x_l < y < x < x_r$, using that

$$\sum_{n=3}^{\infty}\left|k\mathcal{C}_n^{(a,b)}(k)\right| = O(k^{-1}), \tag{4.4.11}$$

using the asymptotics of $1/\mathfrak{n}(k,x)$, the fact that $(a:b)_{1,4}/\mu(x_r) = (a:b)_{2,3}/\mu(x_l) = m_{\mathfrak{c}_1}/2$, and (4.1.30) with (4.1.33d), we write (2.3.7a) as

$$\begin{aligned}\frac{\Psi(k,x,y)}{\mathfrak{b}_0(k)} = \frac{32k^2}{m_{\mathfrak{c}_1}\mathfrak{u}_+ - 8m_{\mathfrak{s}}} \exp\left(ik\int_y^x \mathfrak{n}(k,\xi)\,d\xi\right)\Bigg\{&\frac{m_{\mathfrak{c}_1}}{2k}\sum_{n=1}^{2}\sum_{\ell=0}^{n}(-1)^\ell \mathcal{S}_{n-\ell}^{(x_l,y)}(k)\mathcal{C}_\ell^{(x,x_r)}(k)\\ &-\frac{m_{\mathfrak{c}_1}}{2k}\sum_{n=1}^{2}\sum_{\ell=0}^{n}\mathcal{C}_{n-\ell}^{(x_l,y)}(k)\mathcal{S}_\ell^{(x,x_r)}(k) + o(k^{-2}) - \frac{m_{\mathfrak{s}}}{4k^2}\Bigg\} + o(k^{-1}).\end{aligned} \tag{4.4.12}$$

Using integration by parts as in Lemma 20, we derive

$$\mathcal{C}_1^{(a,b)}(k) = \frac{1}{16ik}\mathfrak{u}_+(a,b)\left(\exp\left(2ik\int_a^b \mathfrak{n}(k,\xi)\,d\xi\right) - 1\right) + o(k^{-1}), \tag{4.4.13a}$$

$$\mathcal{S}_1^{(a,b)}(k) = -\frac{1}{16k}\mathfrak{u}_-(a,b)\left(\exp\left(2ik\int_a^b \mathfrak{n}(k,\xi)\,d\xi\right) + 1\right) + o(k^{-1}), \tag{4.4.13b}$$

$$\mathcal{C}_2^{(a,b)}(k) = \frac{1}{64ik}m_{\text{int}}(a,b)\left(\exp\left(2ik\int_a^b \mathfrak{n}(k,\xi)\,d\xi\right) - 1\right) + o(k^{-1}), \tag{4.4.13c}$$

$$\mathcal{S}_2^{(a,b)}(k) = -\frac{1}{64k}m_{\text{int}}(a,b)\left(\exp\left(2ik\int_a^b \mathfrak{n}(k,\xi)\,d\xi\right) + 1\right) + o(k^{-1}), \tag{4.4.13d}$$

where $\mathfrak{u}_\pm(a,b) = \mathfrak{u}(b) \pm \mathfrak{u}(a)$, and

$$m_{\text{int}}(a,b) = \int_a^b \frac{1}{\mu(y)}\left(\frac{(\beta\mu)'(y)}{(\beta\mu)(y)}\right)^2 dy, \tag{4.4.14}$$

with $\mathfrak{u}(x)$ defined in (1.2.6). We find

$$\begin{aligned}\frac{\Psi(k,x,y)}{\mathfrak{b}_0(k)} = \frac{32k^2}{m_{\mathfrak{c}_1}\mathfrak{u}_+ - 8m_{\mathfrak{s}}} \exp\left(ik\int_y^x \mathfrak{n}(k,\xi)\,d\xi\right)\Bigg\{&-\frac{m_{\mathfrak{s}}}{4k^2} + \frac{m_{\mathfrak{c}_1}}{64k^2}\left(\mathfrak{u}_+(x_l,y) - \mathfrak{u}_-(x_l,y)\right)\\ &+\frac{m_{\mathfrak{c}_1}}{64k^2}\left(\mathfrak{u}_+(x,x_r) + \mathfrak{u}_-(x,x_r)\right) + o(k^{-2})\Bigg\} + o(k^{-1}).\end{aligned} \tag{4.4.15}$$

Combining terms,

$$\frac{\Psi(k,x,y)}{\mathfrak{b}_0(k)} = \frac{32k^2}{m_{\mathfrak{c}_1}\mathfrak{u}_+ - 8m_{\mathfrak{s}}} \exp\left(ik\int_y^x \mathfrak{n}(k,\xi)\,d\xi\right)\left\{-\frac{m_{\mathfrak{s}}}{4k^2} + \frac{m_{\mathfrak{c}_1}}{32k^2}\mathfrak{u}_+ + o(k^{-2})\right\} + o(k^{-1}), \tag{4.4.16}$$

which, after dividing by $1+\varepsilon(k)$, gives (4.4.2). The proof is identical for $x_l < x < y < x_r$.

□

Theorem 59. *Consider the finite-interval, half-line, and whole-line problems. If $q_0 \in L^1(\mathcal{D})$, then for almost every $x \in \mathcal{D}$,*

$$\lim_{t\to 0^+} q_0(x,t) = q_0(x). \tag{4.4.17}$$

Proof. Using the change of variables $k = \lambda z$ with $\lambda = 1/\sqrt{t}$ in (4.1.80),

$$q_0(x,t) = \frac{\lambda}{2\pi}\int_{\partial\Omega_{\text{ext}}} \frac{\Phi_0(\lambda z,x)}{\Delta(\lambda z)} e^{-z^2}\,dz = \frac{\lambda}{2\pi}\int_{\partial\Omega_{\text{ext}}} \frac{e^{-z^2}}{\Delta(\lambda z)}\int_{\mathcal{D}} \frac{\Psi(\lambda z,x,y)q_\alpha(y)}{\sqrt{(\beta\mathfrak{n})(\lambda z,x)}\sqrt{(\beta\mathfrak{n})(\lambda z,y)}}\,dy\,dz. \tag{4.4.18}$$

By Lemma 24, we can use the Fubini-Tonelli theorem to write this as

$$q_0(x,t) = \frac{\lambda}{2\pi}\int_{\mathcal{D}} q_\alpha(y)\int_{\partial\Omega_{\text{ext}}} \frac{\Psi(\lambda z,x,y)}{\Delta(\lambda z)}\frac{e^{-z^2}}{\sqrt{(\beta\mathfrak{n})(\lambda z,x)}\sqrt{(\beta\mathfrak{n})(\lambda z,y)}}\,dz\,dy. \tag{4.4.19}$$

Using (4.4.2),

$$q_0(x,t) = \frac{\lambda\big(1+o(\lambda^0)\big)}{2\pi}\int_{\mathcal{D}} \frac{q_\alpha(y)}{\sqrt{(\beta\mu)(x)}\sqrt{(\beta\mu)(y)}}\int_{\partial\Omega_{\text{ext}}} \exp\left(\operatorname{sgn}(x-y)i\lambda z\int_y^x \mathfrak{n}(\lambda z,\xi)\,d\xi\right)e^{-z^2}\,dz\,dy + o(\lambda^0). \tag{4.4.20}$$

Since

$$\left|\lambda\exp\left(\operatorname{sgn}(x-y)i\lambda z\int_y^x \mu(\xi)\,d\xi\right)O(|x-y|\lambda^{-1})e^{-z^2}\right| \le O(|x-y|\lambda^0)\big|e^{-z^2}\big|, \tag{4.4.21}$$

is absolutely integrable, we may use the DCT on the remainder term from (4.1.15). Substituting this result in (4.4.20), we obtain

$$q_0(x,t) = \frac{\lambda}{2\pi}\int_{\mathcal{D}} \frac{q_\alpha(y)}{\sqrt{(\beta\mu)(x)}\sqrt{(\beta\mu)(y)}}\int_{\partial\Omega_{\text{ext}}} \exp\left(\operatorname{sgn}(x-y)i\lambda z\int_y^x \mu(\xi)\,d\xi\right)e^{-z^2}\,dz\,dy + o(\lambda^0), \tag{4.4.22}$$

as $\lambda\to\infty$. Define $M_x(y) = \int_x^y \mu(\xi)\,d\xi$. Deforming $\partial\Omega_{\text{ext}}$ down to the real axis and integrating the z-integral gives

$$q_0(x,t) = \frac{\lambda}{2\sqrt{\pi}}\int_{\mathcal{D}} \frac{q_\alpha(y)e^{-\frac14\lambda^2M_x^2(y)}}{\sqrt{(\beta\mu)(x)}\sqrt{(\beta\mu)(y)}}\,dy + o(\lambda^0). \tag{4.4.23}$$

For a fixed $x\in\mathcal{D}$, if $q_0(x)$ is finite, using that $q_\alpha(y)/(\mu(y)\sqrt{(\beta\mu)(y)}) \in L^1(\mathcal{D})$, it follows that for any $\epsilon>0$ and for each λ, there exists $\varphi \in \mathrm{AC}(\mathcal{D})\cap C_0(\mathcal{D})$ [16], so that

$$\int_{\mathcal{D}}\left|\frac{q_\alpha(y)}{\mu(y)\sqrt{(\beta\mu)(y)}} - \varphi(y)\right|dy \le \lambda^{-2}, \qquad \text{and} \qquad \left|\varphi(x) - \frac{q_\alpha(x)}{\mu(x)\sqrt{(\beta\mu)(x)}}\right| \le \frac{\epsilon}{2}. \tag{4.4.24}$$

Using this,

$$q_0(x,t) = \frac{\lambda}{2\sqrt{\pi}\sqrt{(\beta\mu)(x)}}\left[\int_{\mathcal{D}}\left(\frac{q_\alpha(y)}{\mu(y)\sqrt{(\beta\mu)(y)}} - \varphi(y)\right)\mu(y)e^{-\frac14\lambda^2M_x^2(y)}\,dy + \int_{\mathcal{D}}\varphi(y)\mu(y)e^{-\frac14\lambda^2M_x^2(y)}\,dy\right] + o(\lambda^0). \tag{4.4.25}$$

For the first integral and any $\epsilon>0$, we can find λ sufficiently large, so that

$$\left|\frac{\lambda}{2\sqrt{\pi}\sqrt{(\beta\mu)(x)}}\int_{\mathcal{D}}\left(\frac{q_\alpha(y)}{\mu(y)\sqrt{(\beta\mu)(y)}} - \varphi(y)\right)\mu(y)e^{-\frac14\lambda^2M_x^2(y)}\,dy\right| \le \frac{M_{\mathfrak{n}}}{2\sqrt{\pi m_\beta m_{\mathfrak{n}}}\lambda} \le \frac{\epsilon}{2}. \tag{4.4.26}$$

Since $\varphi \in \mathrm{AC}(\mathcal{D})$, we may integrate the second integral of (4.4.25) by parts to obtain

$$\frac{\lambda}{2\sqrt{\pi}} \int_{\mathcal{D}} \varphi(y)\mu(y)e^{-\frac{1}{4}\lambda^2 M_x^2(y)}\, dy = -\frac{1}{2} \int_{\mathcal{D}} \varphi'(y)\mathrm{erf}\left(\frac{\lambda M_x(y)}{2}\right) dy. \tag{4.4.27}$$

At this point, we may take the limit as $\lambda \to \infty$ using the DCT. Since $\arg(\mu) \in (-\pi/4, \pi/4)$, if $y > x$, then $\arg(M_x(y)) \in (-\pi/4, \pi/4)$ and the error function limits to 1 as $\lambda \to \infty$ [8]. If $y < x$, then $\arg(M_x(y)) \in (3\pi/4, 5\pi/4)$ and the error function limits to -1. It follows that

$$\lim_{\lambda\to\infty} \frac{\lambda}{2\sqrt{\pi}} \int_{\mathcal{D}} \varphi(y)\mu(y)e^{-\frac{1}{4}\lambda^2 M_x^2(y)}\, dy = \varphi(x), \tag{4.4.28}$$

and we have

$$q_0(x,t) \to \frac{\varphi(x)}{\sqrt{\beta\mu(x)}} \to q_0(x), \tag{4.4.29}$$

as $t \to 0^+$ and $\epsilon \to 0^+$. Since $q_0 \in L^1(\mathcal{D})$, $q_0(x)$ is finite for almost every $x \in \mathcal{D}$, concluding the proof. $\square$

5. Proofs: partially dissipative problems on the finite interval

5.1 The solution expressions are well defined

We now repeat the proofs for *partially dissipative* problems. Many of the results of Chapter 4 remain the same or are similar with some slightly different bounds. In this section, we again denote the r dependence of Ω explicitly as $\Omega(r) = \{k \in \mathbb{C} : |k| > r \text{ and } \pi/4 < \arg(k) < 3\pi/4\}$ and $\mathbb{C}^+(r) = \{k \in \mathbb{C} : |k| > r \text{ and } 0 < \arg(k) < \pi\}$. We define $\arg(\,\cdot\,) \in [-\pi/2, 3\pi/2)$ with $\theta = \arg(k)$. For *regular* problems, we define $b = 0$ and for *irregular* problems, we define $b = 1$.

Remark 60. *For* partially dissipative *problems, we always assume Assumption 2.6, so Lemmas 12–14 from Chapter 4 still hold.*

Lemma 61. *For $|k| \geq r > \sqrt{M_\gamma}$, under Assumption 2.7, $\mathfrak{u}'$, $\mathfrak{v} \in \mathrm{AC}(\mathcal{D})$, where we define*

$$\mathfrak{v}(k,x) = \frac{1}{\mathfrak{n}(k,x)} \frac{(\beta\mathfrak{n})'(k,x)}{(\beta\mathfrak{n})(k,x)} = \frac{1}{2\mathfrak{n}(k,x)} \left(\frac{\beta'(x)}{\beta(x)} - \frac{\alpha'(x)}{\alpha(x)} + \frac{\gamma'(x)}{k^2 + \gamma(x)} \right) = \frac{1}{2}\mathfrak{u}(x) + O(k^{-2}). \tag{5.1.1}$$

Proof. From (5.1.1) and Assumption 2.7, we have $\mathfrak{v} \in \mathrm{AC}(\mathcal{D})$. Since

$$\mathfrak{u}'(x) = -\frac{\mu'(x)}{\mu(x)^2} \left(\frac{\beta'(x)}{\beta(x)} - \frac{\alpha'(x)}{\alpha(x)} \right) + \frac{1}{\mu(x)} \left(\frac{\beta'(x)}{\beta(x)} - \frac{\alpha'(x)}{\alpha(x)} \right)' \qquad \text{and} \qquad \frac{\mu'(x)}{\mu(x)} = -\frac{1}{2} \left(\frac{\alpha'(x)}{\alpha(x)} + \frac{\beta'(x)}{\beta(x)} \right), \tag{5.1.2}$$

by Assumption 2.7 and Lemma 13, $\mu'/\mu \in \mathrm{AC}(\mathcal{D})$ and therefore $\mathfrak{u}' \in \mathrm{AC}(\mathcal{D})$. ☐

Remark 62. *We still have Definition 16 and the relations in Lemma 17.*

Lemma 63. *Similar to Lemma 18, there exists an $r > \sqrt{M_\gamma}$ such that, for $(x,y) \subseteq (a,b) \subseteq \mathcal{D}$, $k \in \mathbb{C}^+(r)$,*

$$\left| \mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k) \right| \leq \frac{1}{2^n n!} \left\| \frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})} \right\|_{(a,b)}^n \mathfrak{E}^{(a,b)}(k), \tag{5.1.3a}$$

and

$$\left| \sum_{\ell=0}^{n} (-1)^\ell \mathcal{J}_{n-\ell}^{(a,y)}[\sigma_{p,n-\ell}](k) \mathcal{J}_\ell^{(x,b)}[\tilde{\sigma}_{p,\ell}](k) \right| \leq \frac{1}{2^n n!} \left\| \frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})} \right\|_{(a,b)}^n \mathfrak{E}^{(a,b)}(k), \tag{5.1.3b}$$

where we define

$$\mathfrak{E}^{(a,b)}(k) = 2 \begin{cases} e^{M_{\mathcal{J}}(b-a)|k|(\theta_0 - \theta)}, & \text{if } 0 \leq \theta < \theta_0, \\ 1, & \text{if } \theta_0 \leq \theta \leq \pi - \theta_0, \\ e^{M_{\mathcal{J}}(b-a)|k|(\theta - \pi + \theta_0)}, & \text{if } \pi - \theta_0 < \theta \leq \pi. \end{cases} \tag{5.1.3c}$$

Thus, the $\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)$ function is well defined. We have the same bounds for $\mathcal{C}_n^{(a,b)}(k)$, and $\mathcal{S}_n^{(a,b)}(k)$. Note that we will use $\mathfrak{E}^{(a,b)}(k) \leq \mathfrak{E}^{(x_l,x_r)}(k)$.

Proof. Using (4.1.15) from Lemma 19, we have

$$\left| \exp\left(\sum_{p=0}^{n} \sigma_{p,n} \int_{y_p}^{y_{p+1}} ik\mathfrak{n}(k,\xi)\, d\xi \right) \right| = \left| \exp\left(\sum_{p=0}^{n} \sigma_{p,n} \int_{y_p}^{y_{p+1}} ik\mu(\xi)\, d\xi \right) \left(1 + O(k^{-1})\right) \right|. \tag{5.1.4}$$

Since $\arg(\mu) \in [-\pi/4, \pi/4]$, $\mathrm{Re}(ik\mu) = -|k\mu|\sin(\theta + \arg(\mu))$. If $\theta \in [0, \pi/4]$, $\sin(\theta + \arg(\mu)) \geq \theta - \pi/4$ and therefore $\mathrm{Re}(ik\mu) \leq M_{\mathfrak{n}}|k|(\pi/4 - \theta)$. Similarly, if $\theta \in [3\pi/4, \pi]$, $\mathrm{Re}(ik\mu) \leq M_{\mathfrak{n}}|k|(\theta - 3\pi/4)$. There then exists an $r > \sqrt{M_\gamma}$ such that for $k > r$

$$\left|\exp\left(\sum_{p=0}^{n} \sigma_{p,n} \int_{y_p}^{y_{p+1}} ik\mathfrak{n}(k,\xi)\, d\xi\right)\right| \leq 2 \begin{cases} \exp\left(M_{\mathfrak{n}}|k|\left(\frac{\pi}{4} - \theta\right)\sum_{p=0}^{n} \sigma_{p,n}(y_{p+1} - y_p)\right), & 0 \leq \theta \leq \frac{\pi}{4}, \\ 1, & \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}, \leq \mathfrak{E}^{(a,b)}(k), \\ \exp\left(M_{\mathfrak{n}}|k|\left(\theta - \frac{3\pi}{4}\right)\sum_{p=0}^{n} \sigma_{p,n}(y_{p+1} - y_p)\right), & \frac{3\pi}{4} \leq \theta \leq \pi, \end{cases} \tag{5.1.5}$$

where $M_{\mathcal{J}} = M_{\mathfrak{n}} M_\sigma$ (and M_σ is from Definition 16), and so

$$\left|\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)\right| \leq \frac{1}{2^n}\mathfrak{E}^{(a,b)}(k) \int_{a<y_1<\cdots<y_n<b} \left|\prod_{p=1}^{n} \frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)}\right| dy_1 \cdots dy_n. \tag{5.1.6}$$

Since the integrand is symmetric with respect to permutations of $y_1, \ldots, y_n$, we have (5.1.3a), which is finite by Lemma 14. Using this bound and the binomial theorem, we also get (5.1.3b). For $\mathcal{C}_n^{(a,b)}(k)$ and $\mathcal{S}_n^{(a,b)}(k)$, the result follows from (4.1.11). ☐

Lemma 64. *There exists an $r > \sqrt{M_\gamma}$, $m_\Xi > 0$, and $\theta_0 \in (0, \pi/4)$ such that $\Xi(k) \leq e^{-m_\Xi|k|}$ for $k \in \Omega_{\mathrm{ext}}(r)$, where $\Omega_{\mathrm{ext}}(r) = \{k \in \mathbb{C} : |k| > r \text{ and } \theta_0 < \arg(k) < \pi - \theta_0\}$, see Figure 4.1, and where $\Xi(k)$ is defined in (2.3.3).*

There exists an $r > \sqrt{M_\gamma}$ and a $C > 1$, such that, for all $n \geq 0$, for $(a,b) \subseteq \mathcal{D}$, and for $k \in \Omega_{\mathrm{ext}}(r)$,

$$\left|\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)\right| \leq \frac{C^n}{|k|^{\lfloor\frac{n+1}{2}\rfloor}}\mathfrak{E}^{(a,b)}(k) \qquad \text{and} \qquad \sum_{n=j}^{\infty}\left|\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)\right| = O\left(k^{-\lfloor\frac{j+1}{2}\rfloor}\right)\mathfrak{E}^{(a,b)}(k), \tag{5.1.7}$$

where $\lfloor\cdot\rfloor$ represents the floor function.

Defining $\lambda_{p,n} = \sigma_{p-1,n} - \sigma_{p,n}$ (where $\lambda_{p,n} \neq 0$ by Definition 16), under Assumption 2.7, we have

$$\mathcal{J}_1^{(a,b)}[\sigma_{p,n}](k) = \frac{1}{4\lambda_{1,1}ik}\left(\mathfrak{u}(b)\exp\left(\sigma_{0,1}\int_a^b ik\mu(\xi)\, d\xi\right) - \mathfrak{u}(a)\exp\left(\sigma_{1,1}\int_a^b ik\mu(\xi)\, d\xi\right)\right) + O(k^{-2})\mathfrak{E}^{(x_l,x_r)}(k). \tag{5.1.8}$$

Proof. From (2.3.3) and Lemma 19, we have

$$|\Xi(k)| = \left|\exp\left(\int_{x_l}^{x_r} ik\mu(\xi)\, d\xi\right)(1 + O(k^{-1}))\right| \leq 2\exp\left(-\int_{x_l}^{x_r} |k\mu(\xi)|\sin(\theta + \arg(\mu(\xi)))\, d\xi\right). \tag{5.1.9}$$

Let $\theta = \pi/4 + \Delta\theta$ where $-\mathfrak{d} \leq \Delta\theta \leq \mathfrak{d}$ for some $\mathfrak{d} \in (0, \pi/4)$ and $\arg(\mu(\xi)) = -\pi/4 + \Delta\mu(\xi)$ where $0 \leq \Delta\mu(\xi) \leq \pi/2$. Then

$$|\Xi(k)| \leq 2\exp\left(-\int_{x_l}^{x_r} |k\mu(\xi)|\sin(\Delta\theta + \Delta\mu(\xi))\, d\xi\right), \tag{5.1.10}$$

and expanding the sine,

$$|\Xi(k)| \leq 2\exp\left(\int_{x_l}^{x_r} |k\mu(\xi)|\left|\sin(\Delta\theta)\cos(\Delta\mu(\xi))\right| d\xi - \int_{x_l}^{x_r} |k\mu(\xi)|\cos(\Delta\theta)\sin(\Delta\mu(\xi))\, d\xi\right). \tag{5.1.11}$$

Using Lemma 13 and that $\sin(\Delta\mu) \geq 0$ and $\cos(\Delta\theta) \geq 0$,

$$|\Xi(k)| \leq 2\exp\left(M_{\mathfrak{n}}|k|\sin(\mathfrak{d})(x_r - x_l) - \frac{2}{\pi}m_{\mathfrak{n}}|k|\cos(\mathfrak{d})\int_{x_l}^{x_r} \Delta\mu(\xi)\, d\xi\right). \tag{5.1.12}$$

Denote

$$\aleph = \frac{1}{x_r - x_l}\int_{x_l}^{x_r} \Delta\mu(\xi)\, d\xi. \tag{5.1.13}$$

Since the equation is *partially dissipative* $\aleph > 0$, and we can choose $\mathfrak{d}$ small enough so that

$$\tan(\mathfrak{d}) < \frac{m_{\mathfrak{n}}\aleph}{\pi M_{\mathfrak{n}}}. \tag{5.1.14}$$

Equation (5.1.12) then yields

$$|\Xi(k)| < \exp\left(-\frac{m_{\mathfrak{n}}\aleph}{\pi}\cos(\mathfrak{d})(x_r - x_l)|k| + \ln(2)\right). \tag{5.1.15}$$

Choosing $m_\Xi < m_{\mathfrak{n}}\aleph\cos(\mathfrak{d})(x_r - x_l)/\pi$, then for r large enough, $|\Xi(k)| \le e^{-m_\Xi|k|}$ for $|k| > r$. The same logic applies to $\theta = 3\pi/4 + \Delta\theta$ and for $\pi/4 + \mathfrak{d}/2 < \theta < 3\pi/4 - \mathfrak{d}/2$. Let $\theta_0 = \pi/4 - \mathfrak{d}$ and we have the result for $\Xi(k)$.

Using (4.1.15) in (4.1.10a) for $n = 1$, we have

$$\mathcal{J}_1^{(a,b)}(k) = \frac{1 + O(k^{-2})}{4}\int_a^b \left(\mathfrak{u}(y)\mu(y) + \frac{\gamma'(y)}{k^2 + \gamma(y)}\right)\exp\left(\sigma_{0,1}\int_a^y + \sigma_{1,1}\int_y^b ik\mu(\xi)\,d\xi\right)dy. \tag{5.1.16}$$

By Lemma 14, $\mathfrak{u} \in \mathrm{AC}(\mathcal{D})$ and integration by parts gives

$$\begin{aligned}\mathcal{J}_1^{(a,b)}[\sigma_{p,n}](k) = {}& \frac{1 + O(k^{-1})}{4\lambda_{1,1}ik}\left(\mathfrak{u}(b)\exp\left(\sigma_{0,1}\int_a^b ik\mu(\xi)\,d\xi\right) - \mathfrak{u}(a)\exp\left(\sigma_{1,1}\int_a^b ik\mu(\xi)\,d\xi\right)\right)\\ &- \frac{1 + O(k^{-1})}{4}\int_a^b \left(\frac{\mathfrak{u}'(y)}{\lambda_{1,1}ik} - \frac{\gamma'(y)}{k^2 + \gamma(y)}\right)\exp\left(\sigma_{0,1}\int_a^y + \sigma_{1,1}\int_y^b ik\mu(\xi)\,d\xi\right)dy.\end{aligned} \tag{5.1.17}$$

Under Assumption 2.7, $\mathfrak{u}' \in \mathrm{AC}(\mathcal{D})$ and another integration by parts gives (5.1.8).

The first inequality in (5.1.7) for $n = 0$ and $n = 1$ follows from (4.1.15) and (5.1.8), respectively. Integration by parts of (4.1.10a), as in Lemma 20, yields the asymptotic recurrence relation

$$\begin{aligned}\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k) = {}& \frac{1 + O(k^{-1})}{4}\frac{\mathfrak{u}(b)}{\lambda_{n,n}ik}\mathcal{J}_{n-1}^{(a,b)}[\sigma_{p,n}](k)\\ &- \frac{1 + O(k^{-1})}{8}\int_a^b dy_{n-1}\,\frac{\mathfrak{u}(y_{n-1})}{\lambda_{n,n}ik}\frac{(\beta\mathfrak{n})'(k, y_{n-1})}{(\beta\mathfrak{n})(k, y_{n-1})}\exp\left(\sigma_{n,n}\int_{y_{n-1}}^b ik\mu(\xi)\,d\xi\right)\mathcal{J}_{n-2}^{(a,y_{n-1})}[\sigma_{p,n}](k)\\ &- \frac{1 + O(k^{-1})}{4}\int_a^b dy_n\left(\frac{\mathfrak{u}'(y_n)}{\lambda_{n,n}ik} + \frac{\gamma'(y_n)}{k^2 + \gamma(y_n)}\right)\exp\left(\sigma_{n,n}\int_{y_n}^b ik\mu(\xi)\,d\xi\right)\mathcal{J}_{n-1}^{(a,y_n)}[\sigma_{p,n}](k).\end{aligned} \tag{5.1.18}$$

Assuming the first inequality in (5.1.7) holds for $n = 0, 1, \ldots, m-1$, and using that $|\lambda_{p,n}| \ge 1$, we use (5.1.18) to prove that it holds for all $n \ge 0$ by induction. Using the first inequality in (5.1.7), we find

$$\sum_{n=j}^{\infty}\left|\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)\right| \le \mathfrak{E}^{(a,b)}(k)\left(\frac{C^j}{|k|^{\lfloor\frac{j+1}{2}\rfloor}}\frac{k}{k - C^2} + \frac{C^j}{|k|^{\lfloor\frac{j}{2}\rfloor}}\frac{C}{k - C^2}\right) = O\left(k^{-\lfloor\frac{j+1}{2}\rfloor}\right)\mathfrak{E}^{(a,b)}(k). \tag{5.1.19}$$

□

Corollary 65. *Under Assumption 2.7,*

$$\mathcal{C}_1^{(a,b)}(k) = \frac{\mathfrak{u}_+(a,b)}{16ik}\left(\exp\left(2ik\int_a^b \mu(\xi)\,d\xi\right) - 1\right) + O(k^{-2})\mathfrak{E}^{(x_l,x_r)}(k), \tag{5.1.20a}$$

$$\mathcal{S}_1^{(a,b)}(k) = -\frac{\mathfrak{u}_-(a,b)}{16k}\left(\exp\left(2ik\int_a^b \mu(\xi)\,d\xi\right) + 1\right) + O(k^{-2})\mathfrak{E}^{(x_l,x_r)}(k). \tag{5.1.20b}$$

where $\mathfrak{u}_\pm(a,b) = \mathfrak{u}(b) \pm \mathfrak{u}(a)$ *(so that* $\mathfrak{u}_\pm = \mathfrak{u}_\pm(x_l, x_r)$*).*

Proof. Using (5.1.8) in (4.1.11), we find (5.1.20). □

Having established some stronger bounds and asymptotics of the function $\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)$ under the stronger Assumption 2.7, we keep Definition 21 and prove stronger bounds on $\Delta(k)$, $\Psi(k,x,y)$, *etc.*

Lemma 66. *We have the same asymptotics for* $\Delta(k)$ *as in Lemma 22 with* $\varepsilon(k) = O(k^{-1})\mathfrak{E}^{(x_l,x_r)}(k)$*.*

Proof. Starting with (4.1.42) from Lemma 22, since $\Xi(k) \to 0$ exponentially fast, we find

$$\Delta(k) = i\mathfrak{c}_0(k) - \mathfrak{s}_0(k) + 2i \sum_{n=1}^{\infty} \left(\mathfrak{c}_n(k) \mathcal{C}_n^{(x_l,x_r)}(k) + \mathfrak{s}_n(k) \mathcal{S}_n^{(x_l,x_r)}(k) \right) + O(k^{-5}) \mathfrak{E}^{(x_l,x_r)}(k). \tag{5.1.21}$$

1. If $(a:b)_{2,4} \neq 0$, then, using that $\mathfrak{c}_n(k) = (-1)^n m_{\mathfrak{c}_n}/k + O(k^{-3})$ and $\mathfrak{s}_n(k) = (-1)^n (a:b)_{2,4} + m_{\mathfrak{s}}/k^2 + O(k^{-4})$, we can write (4.1.30) with $\mathfrak{b}_0(k)$ defined in (4.1.33a), where

$$\varepsilon(k) = -\frac{1}{(a:b)_{2,4}} \left[\frac{i m_{\mathfrak{c}_0}}{k} - \frac{m_{\mathfrak{s}}}{k^2} + 2i \sum_{n=1}^{2} \mathfrak{c}_n(k) \mathcal{C}_n^{(x_l,x_r)}(k) + 2i \sum_{n=1}^{4} \mathfrak{s}_n(k) \mathcal{S}_n^{(x_l,x_r)}(k) \right] + O(k^{-3}) \mathfrak{E}^{(x_l,x_r)}(k), \tag{5.1.22}$$

which, by Lemma 64, gives $\varepsilon(k) = O(k^{-1}) \mathfrak{E}^{(x_l,x_r)}(k)$.

2. If $(a:b)_{2,4} = 0$ and $m_{\mathfrak{c}_0} \neq 0$, then we can write (4.1.30) with $\mathfrak{b}_0(k)$ defined in (4.1.33b), where

$$\varepsilon(k) = \frac{k}{i m_{\mathfrak{c}_0}} \left[i\mathfrak{c}_0(k) - \frac{i m_{\mathfrak{c}_0}}{k} - \frac{m_{\mathfrak{s}}}{k^2} + 2i \sum_{n=1}^{4} \mathfrak{c}_n(k) \mathcal{C}_n^{(x_l,x_r)}(k) + 2i \sum_{n=1}^{2} \mathfrak{s}_n(k) \mathcal{S}_n^{(x_l,x_r)}(k) \right) + O(k^{-3}) \mathfrak{E}^{(x_l,x_r)}(k). \tag{5.1.23}$$

Since $\mathfrak{c}_0(k) = m_{\mathfrak{c}_0}/k + O(k^{-3})$, $\mathfrak{s}_n(k) = O(k^{-2})$, $\mathfrak{c}_n(k) = O(k^{-1})$, and since the sums over $\mathcal{C}_n^{(x_l,x_r)}(k)$ and $\mathcal{S}_n^{(x_l,x_r)}(k)$ are both $O(k^{-1}) \mathfrak{E}^{(x_l,x_r)}(k)$, we have $\varepsilon(k) = O(k^{-1}) \mathfrak{E}^{(x_l,x_r)}(k) \mathfrak{E}^{(x_l,x_r)}(k)$.

3. If $(a:b)_{2,4} = 0$, $m_{\mathfrak{c}_0} = 0$, $m_{\mathfrak{c}_1} = 0$, and $(a:b)_{1,3} \neq 0$, then $\mathfrak{c}_0(k) = \mathfrak{c}_1(k) = 0$ and we can write (4.1.30) with $\mathfrak{b}_0(k)$ defined in (4.1.33c), where

$$\varepsilon(k) = -\frac{k^2}{m_{\mathfrak{s}}} \left[-\mathfrak{s}_0(k) + \frac{m_{\mathfrak{s}}}{k^2} + 2i \frac{m_{\mathfrak{s}}}{k^2} \sum_{n=1}^{4} \mathcal{S}_n^{(x_l,x_r)}(k) \right] + O(k^{-3}) \mathfrak{E}^{(x_l,x_r)}(k). \tag{5.1.24}$$

Since $\mathfrak{s}_n(k) = m_{\mathfrak{s}}/k^2 + O(k^{-4})$, and since $\mathcal{S}_n^{(x_l,x_r)}(k) = O(k^{-1}) \mathfrak{E}^{(x_l,x_r)}(k)$, we have $\varepsilon(k) = O(k^{-1}) \mathfrak{E}^{(x_l,x_r)}(k)$.

4. If $(a:b)_{2,4} = 0$, $m_{\mathfrak{c}_0} = 0$, $m_{\mathfrak{c}_1} \neq 0$, and $m_{\mathfrak{c}_1} \mathfrak{u}_+ - 8 m_{\mathfrak{s}} \neq 0$, we can write (4.1.30) with $\mathfrak{b}_0(k)$ defined in (4.1.33d), where

$$\begin{aligned} \varepsilon(k) = \frac{8k^2}{m_{\mathfrak{c}_1} \mathfrak{u}_+ - 8 m_{\mathfrak{s}}} \Bigg[&- \frac{m_{\mathfrak{c}_1} \mathfrak{u}_+ - 8 m_{\mathfrak{s}}}{8k^2} + i\mathfrak{c}_0(k) - \mathfrak{s}_0(k) + 2i\mathfrak{c}_0(k) \mathcal{C}_2^{(x_l,x_r)}(k) + 2i\mathfrak{c}_1(k) \sum_{\substack{n=1 \\ n \text{ odd}}}^{5} \mathcal{C}_n^{(x_l,x_r)}(k) \\ &+ 2i \frac{m_{\mathfrak{s}}}{k^2} \sum_{n=1}^{4} \mathcal{S}_n^{(x_l,x_r)}(k) \Bigg] + O(k^{-3}) \mathfrak{E}^{(x_l,x_r)}(k). \end{aligned} \tag{5.1.25}$$

Since $\mathfrak{c}_0(k) = O(k^{-3})$, $\mathfrak{c}_1(k) = -m_{\mathfrak{c}_1}/k + O(k^{-3})$, $\mathfrak{s}_n(k) = m_{\mathfrak{s}}/k^2 + O(k^{-4})$, $\mathcal{C}_1^{(x_l,x_r)}(k) = -\mathfrak{u}_+/(16k) + O(k^{-2}) \mathfrak{E}^{(x_l,x_r)}(k)$, and $\mathcal{S}_n^{(x_l,x_r)}(k) = O(k^{-1}) \mathfrak{E}^{(x_l,x_r)}(k)$, we have $\varepsilon(k) = O(k^{-1}) \mathfrak{E}^{(x_l,x_r)}(k)$.

□

Lemma 67. *There exists an $r > \sqrt{M_\gamma}$ and $M_\Psi > 0$ such that, for $k \in \mathbb{C}^+(r)$, $x \in \overline{\mathcal{D}}$, and $y \in \overline{\mathcal{D}}$,*

$$|\Psi(k,x,y)| \leq M_\Psi \mathfrak{E}^{(x_l,x_r)}(k) \qquad \text{and} \qquad \left| \frac{\Psi(k,x,y)}{\mathfrak{b}_0(k)} \right| \leq M_\Psi |k|^b \mathfrak{E}^{(x_l,x_r)}(k). \tag{5.1.26}$$

Therefore, $\Psi(k,x,y)$ and $\Psi(k,x,y)/\mathfrak{b}_0(k)$ are well-defined functions.

Proof. For Boundary Case:

1. If $(a:b)_{2,4} \neq 0$, from (2.3.7), we find $\Psi(k,x,y) = O(k^0) \mathfrak{E}^{(x_l,x_r)}(k)$. With $b = 0$, we have (5.1.26), using Lemma 66.
2. If $(a:b)_{2,4} = 0$ and $m_{\mathfrak{c}_0} \neq 0$, from (2.3.7), we find $\Psi(k,x,y) = O(k^{-1}) \mathfrak{E}^{(x_l,x_r)}(k)$. With $b = 0$, we have (5.1.26).

3. If $(a:b)_{2,4}=0$, $m_{\mathfrak{c}_0}=0$, $m_{\mathfrak{c}_1}=0$, and $(a:b)_{1,3}\neq 0$, from (2.3.7a), we have for $x_l<y<x<x_r$,

$$\Psi(k,x,y)=-\frac{4\beta(x_r)(a:b)_{1,2}}{k\sqrt{(\beta\mu)(x_l)}\sqrt{(\beta\mu)(x_r)}}\Xi(k)\mathcal{S}_0^{(y,x)}(k)+O(k^{-2})\mathfrak{E}^{(x_l,x_r)}(k), \tag{5.1.27a}$$

and similarly for $x_l<x<y<x_r$,

$$\Psi(k,x,y)=-\frac{4\beta(x_l)(a:b)_{3,4}}{\sqrt{(\beta\mu)(x_l)}\sqrt{(\beta\mu)(x_r)}}\Xi(k)\mathcal{S}_0^{(x,y)}(k)+O(k^{-2})\mathfrak{E}^{(x_l,x_r)}(k), \tag{5.1.27b}$$

This boundary case is regular if both $(a:b)_{1,2}=0$ and $(a:b)_{3,4}=0$ and irregular if either $(a:b)_{1,2}\neq 0$ or $(a:b)_{3,4}\neq 0$, see Remark 6. Lemma 66 then gives, for $x_l<y<x<x_r$,

$$\frac{\Psi(k,x,y)}{\mathfrak{b}_0(k)}=M_\Psi^{(-)}k\Xi(k)\mathcal{S}_0^{(y,x)}(k)+O(k^0)\mathfrak{E}^{(x_l,x_r)}(k), \tag{5.1.28a}$$

where $M_\Psi^{(-)}=4\beta(x_r)(a:b)_{1,2}/(m_{\mathfrak{s}}\sqrt{(\beta\mu)(x_l)}\sqrt{(\beta\mu)(x_r)})$, and similarly for $x_l<x<y<x_r$,

$$\frac{\Psi(k,x,y)}{\mathfrak{b}_0(k)}=M_\Psi^{(+)}k\Xi(k)\mathcal{S}_0^{(x,y)}(k)+O(k^0)\mathfrak{E}^{(x_l,x_r)}(k), \tag{5.1.28b}$$

with $M_\Psi^{(+)}=4\beta(x_l)(a:b)_{3,4}/(m_{\mathfrak{s}}\sqrt{(\beta\mu)(x_l)}\sqrt{(\beta\mu)(x_r)})$.

4. If $(a:b)_{2,4}=0$, $m_{\mathfrak{c}_0}=0$, $m_{\mathfrak{c}_1}\neq 0$, and $m_{\mathfrak{c}_1}\mathfrak{u}_+-8m_{\mathfrak{s}}\neq 0$, from (2.3.7), for $x_l<y<x<x_r$,

$$\begin{aligned}\Psi(k,x,y)=&\frac{2m_{\mathfrak{c}_1}}{k}\Xi(k)\left(\mathcal{S}_0^{(x_l,y)}(k)\mathcal{C}_0^{(x,x_r)}(k)-\mathcal{C}_0^{(x_l,y)}(k)\mathcal{S}_0^{(x,x_r)}(k)\right)\\&-\frac{4\beta(x_r)(a:b)_{1,2}}{k\sqrt{(\beta\mu)(x_l)}\sqrt{(\beta\mu)(x_r)}}\Xi(k)\mathcal{S}_0^{(y,x)}(k)+O(k^{-2})\mathfrak{E}^{(x_l,x_r)}(k).\end{aligned} \tag{5.1.29}$$

Similarly for $x_l<x<y<x_r$, we also reach $\Psi(k,x,y)=O(k^{-1})\mathfrak{E}^{(x_l,x_r)}(k)$. With Lemma 66, this gives (5.1.26).

□

Lemma 68. *There exists an $r>\sqrt{M_\gamma}$ and $M_{\mathcal{B}}>0$ such that for $k\in\mathbb{C}^+(r)$ and $x\in\overline{\mathcal{D}}$,*

$$|\mathcal{B}_m(k,x)|\le M_{\mathcal{B}}\mathfrak{E}^{(x_l,x_r)}(k) \qquad \text{and} \qquad \left|\frac{\mathcal{B}_m(k,x)}{\mathfrak{b}_0(k)}\right|\le M_{\mathcal{B}}|k|^{b+1}\mathfrak{E}^{(x_l,x_r)}(k). \tag{5.1.30}$$

Therefore, the functions $\mathcal{B}_m(k,x)$ and $\mathcal{B}_m(k,x)/\mathfrak{b}_0(k)$ are well defined for both regular *and* irregular *problems.*

Proof. Using Lemma 63 in (2.3.6c), we have

$$\mathcal{B}_{2-j}(k,x)=(-1)^j\frac{4\Xi(k)}{\sqrt{(\beta\mu)(x)}}\left\{\frac{a_{j2}\beta(x_r)}{\sqrt{(\beta\mu)(x_r)}}\mathcal{C}_0^{(x_l,x)}(k)+\frac{b_{j2}\beta(x_l)}{\sqrt{(\beta\mu)(x_l)}}\mathcal{C}_0^{(x,x_r)}(k)\right\}+O(k^{-1})\mathfrak{E}^{(x_l,x_r)}(k),\quad j=1,2. \tag{5.1.31}$$

Using

$$\Xi(k)\mathcal{C}_0^{(x_l,x)}(k)=\frac{1}{2}\mathcal{J}_0^{(x,x_r)}[1](k)\left(\mathcal{J}_0^{(x_l,x)}[2](k)+1\right)=\frac{1}{2}\mathcal{I}_0^{(x,x_r)}(k)+O(k^{-1})\mathfrak{E}^{(x_l,x_r)}(k), \tag{5.1.32}$$

and similarly for $\Xi(k)\mathcal{C}_0^{(x,x_r)}(k)$, where $\mathcal{I}_0^{(a,b)}(k)$ is defined in (4.1.23), then

$$\mathcal{B}_{2-j}(k,x)=(-1)^j\frac{2}{\sqrt{(\beta\mu)(x)}}\left\{\frac{a_{j2}\beta(x_r)}{\sqrt{(\beta\mu)(x_r)}}\mathcal{I}_0^{(x,x_r)}(k)+\frac{b_{j2}\beta(x_l)}{\sqrt{(\beta\mu)(x_l)}}\mathcal{I}_0^{(x_l,x)}(k)\right\}+O(k^{-1})\mathfrak{E}^{(x_l,x_r)}(k),\quad j=1,2. \tag{5.1.33}$$

For Boundary Case 1 or 2 and for the *irregular* boundary conditions, (5.1.30) follows from the above and Lemma 66. For the *regular* version of Boundary Case 3, we have $a_{ij}=0$ and $b_{ij}=0$ for all $i,j=1,2$, except for a_{11} and b_{21}, see Remark 6. Thus, using Lemma 66, we have (5.1.30) for the *regular* Boundary Case 3. □

Lemma 69. *There exists an $r > \sqrt{M_\gamma}$ and $M_\Phi > 0$ such that for $k \in \mathbb{C}^+(r)$ and $x \in \overline{\mathcal{D}}$,*

$$|\Phi_0(k,x)| \le M_\Phi \|q_0\|_{\mathcal{D}} \mathfrak{E}^{(x_l,x_r)}(k) \qquad \text{and} \qquad \left|\frac{\Phi_0(k,x)}{\mathfrak{b}_0(k)}\right| = o(k)\|q_0\|_{\mathcal{D}}\mathfrak{E}^{(x_l,x_r)}(k). \tag{5.1.34}$$

Therefore, $\Phi_0(k,x)$ and $\Phi_0(k,x)/\mathfrak{b}_0(k)$ are well-defined functions.

Proof. For the *regular* problems, using Lemma 67 in (4.1.28a), we find (5.1.34) with $M_\Phi = M_\Psi/(m_\alpha m_\beta m_\mathfrak{n})$. For the *irregular* version of Boundary Case 3, using (5.1.28) in (4.1.28a),

$$\frac{\Phi_0(k,x)}{\mathfrak{b}_0(k)} = M_\Psi^{(-)} \int_{x_l}^{x} \frac{k\Xi(k)\mathcal{S}_0^{(y,x)}(k)q_\alpha(y)}{\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,y)}}\,dy + M_\Psi^{(+)} \int_{x}^{x_r} \frac{k\Xi(k)\mathcal{S}_0^{(x,y)}(k)q_\alpha(y)}{\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,y)}}\,dy + O(k^0)\|q_0\|_{\mathcal{D}}\mathfrak{E}^{(x_l,x_r)}(k). \tag{5.1.35}$$

The Riemann-Lebesgue lemma [16] then gives that $\Phi_0(k,x)/\mathfrak{b}_0(k) = o(k)\|q_0\|_{\mathcal{D}}\mathfrak{E}^{(x_l,x_r)}(k)$. The same argument applies to Boundary Case 4. □

Lemma 70. *There exists an $r > \sqrt{M_\gamma}$ and $M_f > 0$ such that for $k \in \mathbb{C}^+(r)\backslash\Omega(r)$,* i.e., *the green region of Figure 4.1, for $x \in \overline{\mathcal{D}}$, and for $t \in [0,T]$,*

$$\left|\Phi_f(k,x,t)e^{-k^2 t}\right| = O(k^{-2})\mathfrak{E}^{(x_l,x_r)}(k) \qquad \text{and} \qquad \left|\frac{\Phi_f(k,x,t)e^{-k^2 t}}{\mathfrak{b}_0(k)}\right| = O(k^{b-2})\mathfrak{E}^{(x_l,x_r)}(k). \tag{5.1.36}$$

Therefore, $\Phi_f(k,x,t)$ and $\Phi_f(k,x,t)/\mathfrak{b}_0(k)$ are well-defined functions.

Proof. For $k \in \mathbb{C}^+(r)\backslash\Omega(r)$, *i.e.*, the green region of Figure 4.1, $|e^{-k^2(t-s)}| < 1$, and so, integrating (1.2.16) by parts, we find

$$\tilde{f}_\alpha(k^2,x,t) = \frac{f_\alpha(x,t)e^{k^2 t}}{k^2} - \frac{f_\alpha(x,0)}{k^2} - \frac{\mathcal{G}[f_{\alpha,t}](k^2,x,t)}{k^2}. \tag{5.1.37}$$

and by Assumption 3.1, we have

$$\int_{\mathcal{D}} \left|\tilde{f}_\alpha(k^2,x,t)e^{-k^2 t}\right| dx \le \frac{2\|f\|_{\mathcal{D}} + \|f_t\|_{\mathcal{D}}}{m_\alpha |k|^2}. \tag{5.1.38}$$

Using this and (5.1.26) in (4.1.28b), we reach (5.1.36) with $M_f = M_\Psi/(m_\alpha m_\beta m_\mathfrak{n})$. □

Lemma 71. *There exists an $r > \sqrt{M_\gamma}$ so that for $x \in \mathcal{D}$ and $t \in [0,T]$, $\mathcal{J}_n^{(a,b)}(k)$, $\Delta(k)$, $\Psi(k,x,y)$, and $\Phi_0(k,x)$ are analytic functions for $k \in \Omega_{\text{ext}}(r)$, and $\Phi_f(k,x,t)e^{-k^2 t}$ and $\mathcal{B}_m(k,x)e^{-k^2 t}$ are analytic functions for $k \in \Omega_{\text{ext}}(r)\backslash\Omega(r)$.*

Proof. The proof is identical to that of Lemma 28. □

Lemma 72. *For $k \in \Omega_{\text{ext}}(r)$ and integer $N > 0$,*

$$\frac{1}{\Delta(k)} = \frac{1}{\mathfrak{b}_0(k)}\left(\Delta_N^{-1}(k) + \delta_N(k)\right), \tag{5.1.39}$$

where $\delta_N(k) = O(k^{-N})$ for $k \in \Omega(r)$ and $\Delta_N^{-1}(k) = O(k^0)[\mathfrak{E}^{(x_l,x_r)}(k)]^N$ and is analytic for $k \in \Omega_{\text{ext}}(r)$.

Proof. From Lemma 66, we have that $\Delta(k) = \mathfrak{b}_0(k)(1+\varepsilon(k))$, where $\varepsilon(k) = O(k^{-1})\mathfrak{E}^{(x_l,x_r)}(k)$. We then write

$$\frac{1}{\Delta(k)} = \frac{1}{\mathfrak{b}_0(k)}\left[\sum_{j=0}^{N-1}(-1)^j[\varepsilon(k)]^j + (-1)^N\frac{[\varepsilon(k)]^N}{1+\varepsilon(k)}\right] = \frac{1}{\mathfrak{b}_0(k)}\left[\Delta_N^{-1}(k) + \delta_N(k)\right], \tag{5.1.40}$$

for any integer $N > 0$, where we define

$$\Delta_N^{-1}(k) = \sum_{j=0}^{N-1}(-1)^j[\varepsilon(k)]^j \qquad \text{and} \qquad \delta_N(k) = (-1)^N\frac{[\varepsilon(k)]^N}{1+\varepsilon(k)}. \tag{5.1.41}$$

Since $\varepsilon(k) = O(k^{-1})\mathfrak{E}^{(x_l,x_r)}(k)$ and $\varepsilon(k)$ is analytic (by Lemma 71) for $k \in \Omega_{\text{ext}}(r)$, $\Delta_N^{-1}(k) = O(k^0)[\mathfrak{E}^{(x_l,x_r)}(k)]^N$ and $\Delta_N^{-1}(k)$ is analytic for $k \in \Omega_{\text{ext}}(r)$. Since $\varepsilon(k) = O(k^{-1})$ for $k \in \Omega(r)$, $\delta_N(k) = O(k^{-N})$ for $k \in \Omega(r)$. □

Lemma 73. *There exists an $r > \sqrt{M_\gamma}$ such that for $x \in \overline{\mathcal{D}}$ and $t \in (0,T)$, $q_{\mathcal{B}_m}(x,t)$ (4.1.29c) can be written*

$$q_{\mathcal{B}_m}(x,t) = \frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}(r)} \frac{\mathcal{B}_m(k,x)}{\mathfrak{b}_0(k)}\mathfrak{F}_m^{(0)}(k)e^{-k^2t}\,dk + \frac{1}{2\pi}\int_{\partial\Omega(r)} \frac{\mathcal{B}_m(k,x)}{\mathfrak{b}_0(k)}\mathfrak{F}_m^{(1)}(k,t)e^{-k^2t}\,dk, \tag{5.1.42}$$

where

$$\mathfrak{F}_m^{(0)}(k) = \Delta_N^{-1}(k)\left(-\frac{f_m(0)}{k^2}+\frac{f_m'(0)}{k^4}\right), \tag{5.1.43a}$$

$$\mathfrak{F}_m^{(1)}(k,t) = \delta_N(k)\left(-\frac{f_m(0)}{k^2}+\frac{f_m'(0)}{k^4}\right) + \left(\Delta_N^{-1}(k)+\delta_N(k)\right)\frac{\mathcal{G}[f_m''](k^2,t)}{k^4}. \tag{5.1.43b}$$

With Assumption 3.3, the function $\mathfrak{F}_m^{(0)}(k)$ is $O(k^{-2})[\mathfrak{E}^{(x_l,x_r)}(k)]^N$ for $k \in \Omega_{\text{ext}}(r)$ and $\mathfrak{F}_m^{(1)}(k,t)$ is $O(k^{-\min\{N+2,6\}})$ for $k \in \Omega(r)$. With $N \geq 4$, the function $q_{\mathcal{B}_m}(x,t)$ is well defined.

Proof. With Assumption 3.3, integrating (2.2.7b) by parts yields

$$F_m(k^2,t) = -\frac{f_m(0)}{k^2} - \frac{f_m'(t)e^{k^2t}}{k^4} + \frac{f_m'(0)}{k^4} + \frac{\mathcal{G}[f_m''](k^2,t)}{k^4}. \tag{5.1.44}$$

By Lemmas 68, 71, and 72, $\mathcal{B}_m(k,x)/(k^4\Delta(k))$ is $O(k^{b-3})$ and analytic for $k \in \Omega(r)$. Therefore,

$$\int_{\partial\Omega(r)} \frac{\mathcal{B}_m(k,x)}{\Delta(k)}\frac{f_m'(t)}{k^4}\,dk = 0, \tag{5.1.45}$$

by Cauchy's theorem. Using (5.1.44) then in (4.1.29c) gives

$$q_{\mathcal{B}_m}(x,t) = \frac{1}{2\pi}\int_{\partial\Omega(r)} \frac{\mathcal{B}_m(k,x)}{\Delta(k)}\left[-\frac{f_m(0)}{k^2}+\frac{f_m'(0)}{k^4}+\frac{\mathcal{G}[f_m''](k^2,t)}{k^4}\right]e^{-k^2t}\,dk, \tag{5.1.46}$$

Using Lemma 72 in (5.1.46), we find

$$q_{\mathcal{B}_m}(x,t) = \frac{1}{2\pi}\int_{\partial\Omega(r)} \frac{\mathcal{B}_m(k,x)}{\mathfrak{b}_0(k)}\mathfrak{F}_m^{(0)}(k)e^{-k^2t}\,dk + \frac{1}{2\pi}\int_{\partial\Omega(r)} \frac{\mathcal{B}_m(k,x)}{\mathfrak{b}_0(k)}\mathfrak{F}_m^{(1)}(k,t)e^{-k^2t}\,dk, \tag{5.1.47}$$

where $\mathfrak{F}_m^{(0)}(k)$ and $\mathfrak{F}_m^{(1)}(k,t)$ are from (5.1.43). Using Assumption 3.3 and Lemma 72, the function $\mathfrak{F}_m^{(0)}(k)$ is analytic and $O(k^{-2})[\mathfrak{E}^{(x_l,x_r)}(k)]^N$ for $k \in \Omega_{\text{ext}}(r)$ and $\mathfrak{F}_m^{(1)}(k,t)$ is $O(k^{-\min\{N+2,6\}})$ for $k \in \Omega(r)$. Let θ_0 be defined as in Lemma 64 and define the contour $C_R = \{k \in \mathbb{C} : |k| = R \text{ and } \theta_0 < \theta < \pi/4 \text{ or } 3\pi/4 < \theta < \pi - \theta_0\}$, see Figure 4.1. We can deform the first integral in (5.1.47) from $\partial\Omega(r)$ to $\partial\Omega_{\text{ext}}(r)$, since (using symmetry)

$$\begin{aligned}\left|\int_{C_R} \frac{\mathcal{B}_m(k,x)}{\mathfrak{b}_0(k)}\mathfrak{F}_m^{(0)}(k)e^{-k^2t}\,dk\right| &\leq 2M_{\mathcal{B}}O(R^b)\int_0^{\frac{\pi}{4}} [\mathfrak{E}^{(x_l,x_r)}(k)]^{N+1}e^{-R^2\cos(2\theta)t}\,d\theta \\ &\leq 2M_{\mathcal{B}}O(R^b)\int_0^{\frac{\pi}{4}} e^{((N+1)M_{\mathcal{J}}R-\frac{4}{\pi}R^2t)(\frac{\pi}{4}-\theta)}\,d\theta \to 0,\end{aligned} \tag{5.1.48}$$

as $R \to \infty$. We then obtain (5.1.42). The first integrand in (5.1.42) is exponentially decaying (by e^{-k^2t}), and we choose $N \geq 4$ so that the second integrand is $O(k^{-4})$ and is absolutely integrable (as is its t-derivative and two x-derivatives). ☐

Lemma 74. *There exists an $r > \sqrt{M_\gamma}$ so that for $x \in \overline{\mathcal{D}}$ and $t \in (0,T)$, $q_0(x,t)$ (4.1.29a) can be written as*

$$q_0(x,t) = \frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}(r)} \frac{\Phi_0(k,x)}{\mathfrak{b}_0(k)}\Delta_N^{-1}(k)e^{-k^2t}\,dk + \frac{1}{2\pi}\int_{\partial\Omega(r)} \frac{\Phi_0(k,x)}{\mathfrak{b}_0(k)}\delta_N(k)e^{-k^2t}\,dk, \tag{5.1.49}$$

and is well defined.

Proof. Using Lemma 72 in (4.1.29a),

$$q_0(x,t) = \frac{1}{2\pi}\int_{\partial\Omega(r)} \frac{\Phi_0(k,x)}{\mathfrak{b}_0(k)}\Delta_N^{-1}(k)e^{-k^2t}\,dk + \frac{1}{2\pi}\int_{\partial\Omega(r)} \frac{\Phi_0(k,x)}{\mathfrak{b}_0(k)}\delta_N(k)e^{-k^2t}\,dk. \tag{5.1.50}$$

By Lemmas 69, 71, and 72, $\Phi_0(k,x)/\mathfrak{b}_0(k)$ and $\Delta_N^{-1}(k)$ are bounded, well defined, and analytic for $k \in \Omega_{\text{ext}}(r)$. Let C_R be defined as in Lemma 73, see Figure 4.1. We can deform the first integral in (5.1.50) from $\partial\Omega(r)$ to $\partial\Omega_{\text{ext}}(r)$, since (using symmetry)

$$\begin{aligned}\left|\int_{C_R} \frac{\Phi_0(k,x)}{\mathfrak{b}_0(k)}\Delta_N^{-1}(k)e^{-k^2t}\,dk\right| &\le o(R^2)\|q_0\|_{\mathcal{D}}\int_0^{\frac{\pi}{4}} e^{M_{\mathcal{J}}(N+1)(x_r-x_l)R(\frac{\pi}{4}-\theta)-R^2\cos(2\theta)t}\,d\theta \\ &\le o(R^2)\|q_0\|_{\mathcal{D}}\int_0^{\frac{\pi}{4}} e^{(M_{\mathcal{J}}(N+1)(x_r-x_l)R-\frac{4}{\pi}R^2t)(\frac{\pi}{4}-\theta)}\,d\theta \to 0,\end{aligned} \tag{5.1.51}$$

as $R \to \infty$. Thus, we have (5.1.49). The first integral in (5.1.49) is exponential decaying (by e^{-k^2t}), and we choose $N \ge 4$ so that the second integral is $O(k^{-4})$ and thus is absolutely integrable (along with its t-derivative and two x-derivatives). □

Lemma 75. *There exists an $r > \sqrt{M_\gamma}$ so that for $x \in \overline{\mathcal{D}}$ and $t \in (0,T)$, $q_f(x,t)$ (4.1.29b) can be written as*

$$q_f(x,t) = \frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}(r)} \frac{\Phi_{\mathfrak{f}}^{(0)}(k,x,t)}{\mathfrak{b}_0(k)}e^{-k^2t}\,dk + \frac{1}{2\pi}\int_{\partial\Omega(r)} \frac{\Phi_{\mathfrak{f}}^{(1)}(k,x,t)}{\mathfrak{b}_0(k)}e^{-k^2t}\,dk, \tag{5.1.52a}$$

where

$$\Phi_{\mathfrak{f}}^{(m)}(k,x,t) = \int_{\mathcal{D}} \frac{\Psi(k,x,y)\mathfrak{f}_\alpha^{(m)}(k^2,y,t)}{\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,y)}}\,dy, \qquad m = 0,1, \tag{5.1.52b}$$

and, for regular *problems,*

$$\mathfrak{f}_\alpha^{(0)}(k^2,x,t) = -\Delta_N^{-1}(k)\frac{f_\alpha(x,0)}{k^2}, \tag{5.1.53a}$$

$$\mathfrak{f}_\alpha^{(1)}(k^2,x,t) = -\delta_N(k)\frac{f_\alpha(x,0)}{k^2} - \left(\Delta_N^{-1}(k)+\delta_N(k)\right)\frac{\mathcal{G}[f_{\alpha,t}](k^2,x,t)}{k^2}. \tag{5.1.53b}$$

and for irregular *problems,*

$$\mathfrak{f}_\alpha^{(0)}(k^2,x,t) = \Delta_N^{-1}(k)\left(-\frac{f_\alpha(x,0)}{k^2} + \frac{f_{\alpha,t}(x,0)}{k^4}\right), \tag{5.1.54a}$$

$$\mathfrak{f}_\alpha^{(1)}(k^2,x,t) = \delta_N(k)\left(-\frac{f_\alpha(x,0)}{k^2} + \frac{f_{\alpha,t}(x,0)}{k^4}\right) + \left(\Delta_N^{-1}(k)+\delta_N(k)\right)\frac{\mathcal{G}[f_{\alpha,tt}](k^2,x,t)}{k^4}. \tag{5.1.54b}$$

Therefore, $q_f(x,t)$ is well defined for all three problems.

Proof. For *regular* problems, with Assumption 3.1, the first term of (5.1.37) in (4.1.29b) gives

$$\int_{\partial\Omega(r)} \frac{dk}{k^2\Delta(k)}\int_{\mathcal{D}} \frac{\Psi(k,x,y)f_\alpha(y,t)}{\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,y)}}\,dy = \int_{\mathcal{D}} dy\, f_\alpha(y,t)\int_{\partial\Omega(r)} \frac{\Psi(k,x,y)}{k^2\Delta(k)\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,y)}}\,dk = 0, \tag{5.1.55}$$

by Cauchy's theorem. Thus, when we insert (5.1.37) into (4.1.29b), we find

$$q_f(x,t) = \frac{1}{2\pi}\int_{\partial\Omega(r)} dk\,\frac{e^{-k^2t}}{\Delta(k)}\int_{\mathcal{D}} \frac{\Psi(k,x,y)}{\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,y)}}\left[-\frac{f_\alpha(y,0)}{k^2} - \frac{\mathcal{G}[f_{\alpha,t}](k^2,y,t)}{k^2}\right]dy. \tag{5.1.56}$$

Using (5.1.39) in the expression above, we obtain

$$q_f(x,t) = \frac{1}{2\pi}\int_{\partial\Omega(r)} \frac{\Phi_{\mathfrak{f}}^{(0)}(k,x,t)}{\mathfrak{b}_0(k)}e^{-k^2t}\,dk + \frac{1}{2\pi}\int_{\partial\Omega(r)} \frac{\Phi_{\mathfrak{f}}^{(1)}(k,x,t)}{\mathfrak{b}_0(k)}e^{-k^2t}\,dk, \tag{5.1.57}$$

where $\Phi_{\mathfrak{f}}^{(m)}(k,x,t)$ is defined in (5.1.52b). We deform the contour of the first integral from $\partial\Omega(r)$ to $\partial\Omega_{\text{ext}}(r)$ to reach (5.1.52). The deformation is justified by Lemmas 67, 71, and 72, similar to Lemma 73.

With Assumption 3.5, we integrate (5.1.37) by parts a second time to obtain

$$\tilde{f}_\alpha(k^2,x,t) = \frac{f_\alpha(x,t)e^{k^2t}}{k^2} - \frac{f_\alpha(x,0)}{k^2} - \frac{f_{\alpha,t}(x,t)e^{k^2t}}{k^4} + \frac{f_{\alpha,t}(x,0)}{k^4} + \frac{\mathcal{G}[f_{\alpha,tt}](k^2,x,t)}{k^4}. \tag{5.1.58}$$

When inserting this equation into (4.1.29b), we again integrate the first and third term to zero by Cauchy's theorem (and Jordan's lemma). We then obtain

$$q_f(x,t) = \frac{1}{2\pi}\int_{\partial\Omega(r)} dk\,\frac{e^{-k^2t}}{\Delta(k)}\int_{\mathcal{D}}\frac{\Psi(k,x,y)}{\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,y)}}\left[-\frac{f_\alpha(y,0)}{k^2} + \frac{f_{\alpha,t}(y,0)}{k^4} + \frac{\mathcal{G}[f_{\alpha,ss}](k^2,y,t)}{k^4}\right]dy, \tag{5.1.59}$$

which gives (5.1.57), but with $\mathfrak{f}_\alpha^{(m)}(k^2,x,t)$ defined in (5.1.54). A contour deformation then gives (5.1.52). □

Theorem 76. *There exists an $r > \sqrt{M_\gamma}$ such that the solution expressions (2.3.2) are well defined for $x \in \overline{\mathcal{D}}$ and $t \in (0,T)$.*

Proof. Combining Lemmas 73, 74, and 75, we obtain our result. □

5.2 The solution expressions solve the evolution equation

In this section, we will prove that the solution expression (2.3.2) solves the evolution equation (2.3.1a) in their respective domains. We have the same derivatives of the *accumulation functions* as in Lemma 33, the same summation identity of Lemma 35 (for the finite interval), the same definitions in Definition 36, and the same identities in Lemma 37 (for the finite interval). We now prove the bounds on these derivative functions.

Lemma 77. *There exists an $r > \sqrt{M_\gamma}$ and $M_\Psi > 0$ such that for $k \in \mathbb{C}^+(r)$, for $x \in \overline{\mathcal{D}}$, and for $y \in \overline{\mathcal{D}}$,*

$$|\overline{\Psi}(k,x,y)| \le M_\Psi|k|\mathfrak{E}^{(x_l,x_r)}(k) \qquad \textit{and} \qquad \left|\frac{\overline{\Psi}(k,x,y)}{\mathfrak{b}_0(k)}\right| \le M_\Psi|k|^{b+1}\mathfrak{E}^{(x_l,x_r)}(k). \tag{5.2.1}$$

Therefore, $\overline{\Psi}(k,x,y)$ and $\overline{\Psi}(k,x,y)/\mathfrak{b}_0(k)$ are well-defined functions.

Proof. The proof is identical to that of Lemma 67. Note that the M_Ψ's only differ by a factor of $M_\mathfrak{n}$ from Lemma 13. Without loss of generality, we can take them to be the same. □

Lemma 78. *For all $k \in \mathbb{C}^+(r)$ and $x \in \overline{\mathcal{D}}$*

$$\chi(k,x) = 2ik(\beta\mathfrak{n})(k,x)\Delta(k), \tag{5.2.2}$$

where $\chi(k,x)$ is defined in (4.2.18b).

Proof. The proof is identical to that of Lemma 39. □

Lemma 79. *For $j = 1,2$,*

$$\begin{aligned}\mathcal{B}_{2-j,x}(k,x) = -(-1)^j\frac{4k\mathfrak{n}(k,x)\Xi(k)}{\sqrt{(\beta\mathfrak{n})(k,x)}}\Bigg\{&\frac{\beta(x_r)}{\sqrt{(\beta\mathfrak{n})(k,x_r)}}\left[\frac{a_{j1}}{k\mathfrak{n}(k,x_l)}\sum_{n=0}^\infty(-1)^n\mathcal{C}_n^{(x_l,x)}(k) + a_{j2}\sum_{n=0}^\infty(-1)^n\mathcal{S}_n^{(x_l,x)}(k)\right]\\ &+\frac{\beta(x_l)}{\sqrt{(\beta\mathfrak{n})(k,x_l)}}\left[\frac{b_{j1}}{k\mathfrak{n}(k,x_r)}\sum_{n=0}^\infty\mathcal{C}_n^{(x,x_r)}(k) - b_{j2}\sum_{n=0}^\infty(-1)^n\mathcal{S}_n^{(x,x_r)}(k)\right]\Bigg\},\end{aligned} \tag{5.2.3a}$$

and there exists an $r > \sqrt{M_\gamma}$ and $M_\mathcal{B} > 0$ such that for $k \in \mathbb{C}^+(r)$ and $x \in \overline{\mathcal{D}}$,

$$|\mathcal{B}_{m,x}(k,x)| \le M_\mathcal{B}|k|\mathfrak{E}^{(x_l,x_r)}(k) \qquad \textit{and} \qquad \left|\frac{\mathcal{B}_{m,x}(k,x)}{\mathfrak{b}_0(k)}\right| \le M_\mathcal{B}|k|^{b+2}\mathfrak{E}^{(x_l,x_r)}(k), \tag{5.2.3b}$$

Therefore, the functions $\mathcal{B}_{m,x}(x,t)$ and $\mathcal{B}_{m,x}(x,t)/\mathfrak{b}_0(k)$ are well defined.

Proof. Lemma 33 and direct computation gives us (5.2.3a). The proof for (5.2.3b) is identical to the proof of Lemma 68 in Section 5.1. Note that, as in Lemma 77, the $M_{\mathcal{B}}$'s only differ by a factor of $M_{\mathfrak{n}}$ from Lemma 13. Without loss of generality, we can take them to be the same. ☐

Lemma 80. *We have*

$$\Phi_{0,x}(k,x) = \int_{\mathcal{D}} \frac{\overline{\Psi}(k,x,y)q_\alpha(y)}{\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,y)}}\,dy, \tag{5.2.4}$$

where $\Phi_0(k,x)$ is defined in (4.1.28a). For all three problems, there exists a $M_\Phi > 0$ such that

$$|\Phi_{0,x}(k,x)| \le M_\Phi |k| \|q_0\|_{\mathcal{D}} \mathfrak{E}^{(x_l,x_r)}(k) \qquad \text{and} \qquad \left|\frac{\Phi_{0,x}(k,x)}{\mathfrak{b}_0(k)}\right| \le M_\Phi |k|^2 \|q_0\|_{\mathcal{D}} \mathfrak{E}^{(x_l,x_r)}(k). \tag{5.2.5}$$

Therefore, $\Phi_{0,x}(k,x)$ and $\Phi_{0,x}(k,x)/\mathfrak{b}_0(k)$ are well defined.

Proof. We break the integral over $\mathcal{D}$ in (4.1.28a) into the two regions $y < x$ and $y > x$. Then we use the Leibniz integral rule to obtain

$$\Phi_{0,x}(k,x) = \frac{(\Psi(k,x,x^-) - \Psi(k,x,x^+))q_\alpha(x)}{(\beta\mathfrak{n})(k,x)} + \int_{\mathcal{D}} \frac{\overline{\Psi}(k,x,y)q_\alpha(y)}{\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,y)}}\,dy. \tag{5.2.6}$$

Since $\Psi(k,x,x^-) = \Psi(k,x,x^+)$ for both all three problems, we have (5.2.4). We obtain (5.2.5) from Lemma 77. Since the integrand in (5.2.4) is absolutely integrable, differentiation under the integral is allowed. ☐

Lemma 81. *For $k \in \mathbb{C}^+(r)$, $x \in \overline{\mathcal{D}}$, and $t \in (0,T)$, the x-derivative of $\Phi_{\mathfrak{f}}^{(m)}(k,x,t)$ (5.1.52b) is*

$$\Phi_{\mathfrak{f},x}^{(m)}(k,x,t) = \int_{\mathcal{D}} \frac{\overline{\Psi}(k,x,y)\mathfrak{f}_\alpha^{(m)}(k^2,y,t)}{\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,y)}}\,dy, \qquad m = 0,1, \tag{5.2.7}$$

and we choose $N \ge 4$ so that $\Phi_{\mathfrak{f},x}^{(1)}(k,x,t)/\mathfrak{b}_0(k) = O(k^{-4})\mathfrak{E}^{(x_l,x_r)}(k)$.

Proof. We break the integral over $\mathcal{D}$ in (5.1.52b) into the two regions $y < x$ and $y > x$. Then we use the Leibniz integral rule to obtain

$$\Phi_{\mathfrak{f},x}^{(m)}(k,x,t) = \frac{(\Psi(k,x,x^-) - \Psi(k,x,x^+))\,\mathfrak{f}_\alpha^{(m)}(k^2,x,t)}{\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,x)}} + \int_{\mathcal{D}} \frac{\overline{\Psi}(k,x,y)\mathfrak{f}_\alpha^{(m)}(k^2,y,t)}{\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,y)}}\,dy, \qquad m = 0,1. \tag{5.2.8}$$

Since $\Psi(k,x,x^-) = \Psi(k,x,x^+)$, we have (5.2.7). We find the bound $\Phi_{\mathfrak{f},x}^{(1)}(k,x,t)/\mathfrak{b}_0(k) = O(k^{-4})\mathfrak{E}^{(x_l,x_r)}(k)$ and other relevant bounds using (5.1.53), (5.1.54), and (5.2.1). ☐

Lemma 82. *For all $x \in \overline{\mathcal{D}}$ and $t \in (0,T)$,*

$$q_{\mathcal{B}_m,x}(x,t) = \frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}} \frac{\mathcal{B}_{m,x}(k,x)}{\mathfrak{b}_0(k)}\mathfrak{F}_m^{(0)}(k)e^{-k^2t}\,dk + \frac{1}{2\pi}\int_{\partial\Omega} \frac{\mathcal{B}_{m,x}(k,x)}{\mathfrak{b}_0(k)}\mathfrak{F}_m^{(1)}(k,t)e^{-k^2t}\,dk, \tag{5.2.9a}$$

$$q_{0,x}(x,t) = \frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}} \frac{\Phi_{0,x}(k,x)}{\mathfrak{b}_0(k)}\Delta_N^{-1}(k)e^{-k^2t}\,dk + \frac{1}{2\pi}\int_{\partial\Omega} \frac{\Phi_{0,x}(k,x)}{\mathfrak{b}_0(k)}\delta_N(k)e^{-k^2t}\,dk, \tag{5.2.9b}$$

$$q_{f,x}(x,t) = \frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}} \frac{\Phi_{\mathfrak{f},x}^{(0)}(k,x,t)}{\mathfrak{b}_0(k)}e^{-k^2t}\,dk + \frac{1}{2\pi}\int_{\partial\Omega} \frac{\Phi_{\mathfrak{f},x}^{(1)}(k,x,t)}{\mathfrak{b}_0(k)}e^{-k^2t}\,dk, \tag{5.2.9c}$$

are well defined, i.e., *we can differentiate under the integral sign.*

Proof. Differentiating (5.1.42), (5.1.49), and (5.1.52a) gives us (5.2.9). The first integrands in (5.2.9) are exponentially decaying for $t \in (0,T)$, and therefore are well defined for $x \in \overline{\mathcal{D}}$. The second integrands in (5.2.9) are absolutely convergent by Lemma 79 and (5.1.43b), by choosing $N \ge 4$ and by Lemmas 72, 80, and 81. ☐

Lemma 83. *For $k \in \Omega_{\text{ext}}$, $x, y \in \mathcal{D}$, $t \in (0,T)$, and $m = 0,1$,*

$$\tilde{\Psi}(k,x,y) = -\frac{k^2 + \gamma(x)}{\alpha(x)}\Psi(k,x,y) \qquad \text{and} \qquad (\beta\mathcal{B}_{m,x})_x(k,x) = -\frac{k^2 + \gamma(x)}{\alpha(x)}\mathcal{B}_m(k,x). \tag{5.2.10}$$

Proof. The computation is the same as in Lemma 44. □

Lemma 84. *For all three problems,*

$$\alpha(x)(\beta\Phi_{0,x})_x(k,x) = 2ik\Delta(k)q_0(x) - (k^2+\gamma(x))\Phi_0(k,x), \tag{5.2.11a}$$

$$\alpha(x)\big(\beta\Phi_{\mathfrak{f},x}^{(m)}\big)_x(k,x,t) = 2ik\Delta(k)\mathfrak{f}^{(m)}(k^2,x,t) - (k^2+\gamma(x))\Phi_{\mathfrak{f}}^{(m)}(k,x,t), \qquad m=0,1, \tag{5.2.11b}$$

where $\mathfrak{f}^{(m)}(k^2,x,t) = \alpha(x)\mathfrak{f}_\alpha^{(m)}(k^2,x,t)$.

Proof. The computation is the same as in Lemma 45. □

Lemma 85. *For* $x \in \mathcal{D}$ *and* $t \in (0,T)$, *the* t*-derivatives are*

$$q_{\mathcal{B}_m,t}(x,t) = -\frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}} \frac{k^2\mathcal{B}_m(k,x)}{\mathfrak{b}_0(k)}\mathfrak{F}_m^{(0)}(k)e^{-k^2t}\,dk - \frac{1}{2\pi}\int_{\partial\Omega} \frac{k^2\mathcal{B}_m(k,x)}{\mathfrak{b}_0(k)}\mathfrak{F}_m^{(1)}(k,t)e^{-k^2t}\,dk, \tag{5.2.12a}$$

$$q_{0,t}(x,t) = -\frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}} \frac{k^2\Phi_0(k,x)}{\mathfrak{b}_0(k)}\Delta_N^{-1}(k)e^{-k^2t}\,dk - \frac{1}{2\pi}\int_{\partial\Omega} \frac{k^2\Phi_0(k,x)}{\mathfrak{b}_0(k)}\delta_N(k)e^{-k^2t}\,dk, \tag{5.2.12b}$$

$$q_{f,t}(x,t) = -\frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}} \frac{k^2\Phi_{\mathfrak{f}}^{(0)}(k,x,t)}{\mathfrak{b}_0(k)}e^{-k^2t}\,dk - \frac{1}{2\pi}\int_{\partial\Omega} \frac{k^2\Phi_{\mathfrak{f}}^{(1)}(k,x,t)}{\mathfrak{b}_0(k)}e^{-k^2t}\,dk, \tag{5.2.12c}$$

and are well defined.

Proof. From (5.1.43b), we have $\mathfrak{F}_{m,t}^{(1)}(k^2,t)e^{-k^2t} = \big(\Delta_N^{-1}(k)+\delta_N(k)\big)\,f_m''(t)/k^4$, and

$$\int_{\partial\Omega} \frac{\mathcal{B}_m(k,x)}{\mathfrak{b}_0(k)}\mathfrak{F}_{m,t}^{(1)}(k,t)e^{-k^2t}\,dk = f_m''(t)\int_{\partial\Omega} \frac{\mathcal{B}_m(k,x)}{k^4\mathfrak{b}_0(k)}\big(\Delta_N^{-1}(k)+\delta_N(k)\big)\,dk = 0, \tag{5.2.13}$$

by Cauchy's theorem since the integrand is $O(k^{-2})$. Differentiating (5.1.42) with respect to t, we then obtain (5.2.12a). The first integrand in (5.2.12a) is exponentially decaying and the second integrand is $O(k^{-2})$ and so is absolutely integrable. Thus, the differentiation under the integral is justified and $q_{\mathcal{B}_m,t}(x,t)$ is well defined.

Differentiating (5.1.49) with respect to t gives (5.2.12b). Since the integrands are absolutely integrable, the differentiation under the integral is justified and $q_{0,t}(x,t)$ is well defined.

Since, from (5.1.53) and (5.1.54), $\mathfrak{f}_{\alpha,t}^{(0)}(k^2,x,t)$ and $\Phi_{\mathfrak{f},t}^{(0)}(k,x,t) = 0$. For *regular* problems, $\mathfrak{f}_{\alpha,t}^{(m)}(k^2,x,t)e^{-k^2t} = -\big(\Delta_N^{-1}(k)+\delta_N(k)\big)f_{\alpha,t}(x,t)/k^2$, and for *irregular* problems, $\mathfrak{f}_{\alpha,t}^{(m)}(k^2,x,t)e^{-k^2t} = \big(\Delta_N^{-1}(k)+\delta_N(k)\big)f_{\alpha,tt}(x,t)/k^4$. Then we have $k^2\Phi_{\mathfrak{f},t}^{(1)}(k,x,t)e^{-k^2t}/\mathfrak{b}_0(k) = O(k^{-2})\mathfrak{E}^{(x_l,x_r)}(k)$, so that the integral over $\partial\Omega$ is zero by Cauchy's theorem. Differentiating (5.1.52b) with respect to t, we then obtain (5.2.12c). Since the integrands of (5.2.12c) are absolutely integrable the differentiation under the integral is justified and $q_{f,t}(x,t)$ is well defined. □

Lemma 86. *For* $x \in \mathcal{D}$ *and* $t \in (0,T)$, *the derivatives*

$$\alpha(x)(\beta q_{0,x})_x(x,t) + \gamma(x)q_0(x,t) = q_{0,t}(x,t), \tag{5.2.14a}$$

$$\alpha(x)(\beta q_{f,x})_x(x,t) + \gamma(x)q_f(x,t) + f(x,t) = q_{f,t}(x,t), \tag{5.2.14b}$$

$$\alpha(x)(\beta q_{\mathcal{B}_m,x})_x(x,t) + \gamma(x)q_{\mathcal{B}_m}(x,t) = q_{\mathcal{B}_m,t}(x,t), \qquad m=0,1. \tag{5.2.14c}$$

are well defined, i.e., *we can differentiate under the integral sign.*

Proof. Direct differentiation of Lemma 82 yields

$$(\beta q_{\mathcal{B}_m,x})_x(x,t) = \frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}} \frac{(\beta\mathcal{B}_{m,x})_x(k,x)}{\mathfrak{b}_0(k)}\mathfrak{F}_m^{(0)}(k)e^{-k^2t}\,dk + \frac{1}{2\pi}\int_{\partial\Omega} \frac{(\beta\mathcal{B}_{m,x})_x(k,x)}{\mathfrak{b}_0(k)}\mathfrak{F}_m^{(1)}(k,t)e^{-k^2t}\,dk, \tag{5.2.15a}$$

$$(\beta q_{0,x})_x(x,t) = \frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}} \frac{(\beta\Phi_{0,x})_x(k,x)}{\mathfrak{b}_0(k)}\Delta_N^{-1}(k)e^{-k^2t}\,dk + \frac{1}{2\pi}\int_{\partial\Omega} \frac{(\beta\Phi_{0,x})_x(k,x)}{\mathfrak{b}_0(k)}\delta_N(k)e^{-k^2t}\,dk, \tag{5.2.15b}$$

$$(\beta q_{f,x})_x(x,t) = \frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}} \frac{\big(\beta\Phi_{\mathfrak{f},x}^{(0)}\big)_x(k,x,t)}{\mathfrak{b}_0(k)}e^{-k^2t}\,dk + \frac{1}{2\pi}\int_{\partial\Omega} \frac{\big(\beta\Phi_{\mathfrak{f},x}^{(1)}\big)_x(k,x,t)}{\mathfrak{b}_0(k)}e^{-k^2t}\,dk, \tag{5.2.15c}$$

Using Lemmas 83 and 85, we find (5.2.14c). Since

$$\int_{\partial\Omega_{\text{ext}}} \frac{k\Delta(k)\Delta_N^{-1}(k)}{\mathfrak{b}_0(k)} e^{-k^2 t}\,dk + \int_{\partial\Omega} \frac{k\Delta(k)\delta_N(k)}{\mathfrak{b}_0(k)} e^{-k^2 t}\,dk = \int_{\partial\Omega_{\text{ext}}} \frac{k\Delta(k)\big(\Delta_N^{-1}(k)+\delta_N(k)\big)}{\mathfrak{b}_0(k)} e^{-k^2 t} dk = 0, \quad (5.2.16)$$

then using Lemmas 84 and 85, we find (5.2.14a). using Lemmas 84 and 85, we find

$$\alpha(x)(\beta q_{f,x})_x(x,t) + \gamma(x) q_f(x,t) = q_{f,t}(x,t) + I_f(x,t), \quad (5.2.17)$$

where

$$I_f(x,t) = \frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}} \frac{2ik\Delta(k)\mathfrak{f}^{(0)}(k^2,x,t)}{\mathfrak{b}_0(k)} e^{-k^2 t}\,dk + \frac{1}{2\pi}\int_{\partial\Omega} \frac{2ik\Delta(k)\mathfrak{f}^{(1)}(k^2,x,t)}{\mathfrak{b}_0(k)} e^{-k^2 t}\,dk \quad (5.2.18)$$

For *regular* problems, we use (5.1.53) in $I_f(x,t)$ to find

$$I_f(x,t) = \frac{1}{i\pi}\int_{\partial\Omega_{\text{ext}}} \left(\frac{f(x,0)}{k} + \int_0^t f_s(x,s) e^{k^2 s}\,ds\right) e^{-k^2 t}\,dk = f(x,t), \quad (5.2.19)$$

with a contour integration as in Lemma 47. Equation (5.2.17) then gives (5.2.12). Since the integrands in (5.2.14) are absolutely integrable, the differentiation under the integral is justified. ☐

Theorem 87. *The solution expression* (2.3.2) *solves the evolution equation* (2.3.1a).

Proof. Since $q(x,t) = q_0(x,t) + q_f(x,t) + q_{\mathcal{B}_0}(x,t) + q_{\mathcal{B}_1}(x,t)$, (5.2.14) gives the result. ☐

5.3 The solution expressions satisfy the boundary values

We keep the same definitions from Definition 49 and Lemmas 50–52 still apply.

Lemma 88. *For* $\ell = 1,2,\ m = 0,1,$ *for* $k \in \Omega_{\text{ext}}$*, and for* $t \in [0,T]$*,*

$$\mathcal{Q}_{\mathcal{B}_m}^{(\ell)}(t) = f_m(t)\delta_{\ell-1,m}, \qquad \mathcal{Q}_0^{(\ell)}(t) = 0, \qquad \text{and} \qquad \mathcal{Q}_f^{(\ell)}(t) = 0, \quad (5.3.1)$$

where $\delta_{\ell-1,m}$ *is the Kronecker delta.*

Proof. From Lemmas 73, 74, 75, and 82, $Q_{\mathcal{B}_m}^{(\ell)}(t)$ (4.3.5b), $\mathcal{Q}_0^{(\ell)}(t)$ (4.3.6b), and $\mathcal{Q}_f^{(\ell)}(t)$ (4.3.7b) are well defined functions. Inserting (5.1.49) and (5.2.9b) in (4.3.6b), and inserting (5.1.52) and (5.2.9c) in (4.3.7b), we obtain we find

$$\mathcal{Q}_0^{(\ell)}(t) = \frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}} \frac{\mathcal{P}_0^{(\ell)}(k)}{\mathfrak{b}_0(k)}\Delta_N^{-1}(k) e^{-k^2 t}\,dk + \frac{1}{2\pi}\int_{\partial\Omega} \frac{\mathcal{P}_0^{(\ell)}(k)}{\mathfrak{b}_0(k)}\delta_N(k) e^{-k^2 t}\,dk, \qquad \ell = 1,2, \quad (5.3.2a)$$

$$Q_f^{(\ell)}(t) = \frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}} \frac{\mathcal{P}_f^{(\ell,0)}(k,t)}{\mathfrak{b}_0(k)} e^{-k^2 t}\,dk + \frac{1}{2\pi}\int_{\partial\Omega} \frac{\mathcal{P}_f^{(\ell,1)}(k,t)}{\mathfrak{b}_0(k)} e^{-k^2 t}\,dk, \qquad \ell = 1,2, \quad (5.3.2b)$$

which, using Lemma 52, gives the second two equations in (5.3.1). Using (5.1.42) and (5.2.9a) in (4.3.5b), we find

$$\mathcal{Q}_{\mathcal{B}_m}^{(\ell)}(t) = \frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}(r)} \frac{\mathfrak{B}_m^{(\ell)}(k)}{\mathfrak{b}_0(k)}\mathfrak{F}_m^{(0)}(k) e^{-k^2 t}\,dk + \frac{1}{2\pi}\int_{\partial\Omega(r)} \frac{\mathfrak{B}_m^{(\ell)}(k)}{\mathfrak{b}_0(k)}\mathfrak{F}_m^{(1)}(k,t) e^{-k^2 t}\,dk, \qquad \ell = 1,2,\ m = 0,1, \quad (5.3.3)$$

which, using Lemma 51 and (5.1.43), we obtain

$$\mathcal{Q}_{\mathcal{B}_m}^{(\ell)}(t) = -\frac{\delta_{\ell-1,m}}{i\pi}\int_{\partial\Omega_{\text{ext}}} \left(\frac{f_m(0)}{k} - \frac{f_m'(0)}{k^3} - \frac{1}{k^3}\int_0^t f_m''(s) e^{k^2 s}\,ds\right) e^{-k^2 t}\,dk, \qquad \ell = 1,2,\ m = 0,1. \quad (5.3.4)$$

Integrating the s-integral by parts, we have

$$\mathcal{Q}_{\mathcal{B}_m}^{(\ell)}(t) = -\frac{\delta_{\ell-1,m}}{i\pi}\int_{\partial\Omega_{\text{ext}}} \left(\frac{f_m(0)}{k} - \frac{f_m'(t) e^{k^2 t}}{k^3} + \frac{1}{k}\int_0^t f_m'(s) e^{k^2 s}\,ds\right) e^{-k^2 t}\,dk, \qquad \ell = 1,2,\ m = 0,1. \quad (5.3.5)$$

The integral over the middle term in the integrand is zero by Cauchy's theorem. The remaining integrand is $O(k^{-3})$, and so we can deform down to the real axis, and since the integrand is odd, the principal value is zero. Therefore, we only have the residue contribution at the origin. This gives

$$\mathcal{Q}^{(\ell)}_{\mathcal{B}_m}(t) = \delta_{\ell-1,m}\mathrm{Res}\left(\left(\frac{f_m(0)}{k} - \frac{f_m'(0)}{k^3} - \frac{1}{k^3}\int_0^t f_m''(s)e^{k^2 s}\,ds\right)e^{-k^2 t};\ k=0\right) = f_m(t)\delta_{\ell-1,m}. \tag{5.3.6}$$

□

Theorem 89. *The boundary conditions for all three problems are satisfied.*

Proof. From Lemma 88, we have

$$\begin{aligned} a_{\ell 1}q(x_l,t) + a_{\ell 2}q_x(x_l,t) + b_{\ell 1}q(x_r,t) + b_{\ell 2}q_x(x_r,t) &= \mathcal{Q}^{(\ell)}_0(t) + \mathcal{Q}^{(\ell)}_f(t) + \mathcal{Q}^{(\ell)}_{\mathcal{B}_0}(t) + \mathcal{Q}^{(\ell)}_{\mathcal{B}_1}(t) \\ &= f_0(t)\tilde{\delta}_{\ell-1,0} + f_1(t)\tilde{\delta}_{\ell-1,1}. \end{aligned} \tag{5.3.7}$$

□

5.4 The solution expressions satisfy the initial condition

Definition 90. *For this section we define a function $\mathcal{I}(k,y)$ to be any function consisting of a linear combination of or integral over e^{ikm_j} (with absolutely summable or integrable coefficients) for any nonzero functions m_j, independent of k, with $\arg(m_j) \in [-\pi/4, \pi/4]$. In this way, we may write*

$$\mathcal{I}(k,y) = \sum_{j=1}^{J}\int_{\mathcal{D}^{n_j}} c_j(k,y,\boldsymbol{\zeta}_{n_j})e^{ikm_j(y,\boldsymbol{\zeta}_{n_j})}\,d\boldsymbol{\zeta}_{n_j}, \tag{5.4.1}$$

for $k \in \mathbb{C}^+$ and $y \in \mathcal{D}$, where $c_j : \mathbb{C}\times\mathcal{D}\times\mathcal{D}^{n_j} \to \mathbb{C}$ and $m_j : \mathcal{D}\times\mathcal{D}^{n_j} \to \mathbb{C}$. We require c_j to be absolutely integrable in y and $\boldsymbol{\zeta}_n$ and analytic at $k=\infty$, i.e., *there exists a convergent Laurent series. We require $\arg(m_j) \in [-\pi/4, \pi/4]$ and $m_j \neq 0$ for almost all $y \in \mathcal{D}$ and $\boldsymbol{\zeta}_n \in \mathcal{D}^n$. We allow c_j and m_j to depend on x and we allow $n_j = 0$, in which case, we simply do not have an integral over $\boldsymbol{\zeta}_{n_j} \in \mathcal{D}^{n_j}$.*

This acts like an absorbing element similar to the $o(\cdot)$, $O(\cdot)$ notation. In this way, we have $c\mathcal{I}(k,y) = \mathcal{I}(k,y)$, $\mathcal{I}(k,y)+\mathcal{I}(k,y) = \mathcal{I}(k,y)$, $\mathcal{I}(k,y)\mathcal{I}(k,y) = \mathcal{I}(k,y)$, etc. *We will define $\mathcal{I}(y)$ to be any bounded, integrable function of y (independent of k, but may depend on x) and $\mathcal{I}(0)$ to be any constant (independent of y and k, but may depend on x). We also define $M_a(b) = \int_a^b \mu(\xi)\,d\xi$ and*

$$\mathcal{I}^{(a,b)}_0(k) = \exp\left(\int_a^b ik\mu(\xi)\,d\xi\right) = e^{ikM_a(b)}, \tag{5.4.2}$$

so that, from (4.1.10b) and (4.1.15), if $\sigma > 0$, $\mathcal{J}^{(a,b)}_0[\sigma](k) = \mathcal{I}^{(a,b)}_0(\sigma k)\big(1+O(k^{-1})\big) = \mathcal{I}(k,y)$. Note that $\Xi(k) = \mathcal{J}^{(x_l,x_r)}_0[1](k) \to 0$ exponentially fast, so that $\mathcal{I}^{(x_l,x_r)}_0(k) \to 0$ exponentially fast as well.

Lemma 91. *For $n \geq 0$, we have*

$$\mathcal{J}^{(a(y),b(y))}_n[\sigma_{p,n}](k) = \frac{\mathcal{I}(y)+\mathcal{I}(k,y)}{k^{\lfloor\frac{n+1}{2}\rfloor}}, \tag{5.4.3a}$$

and if a and b are independent of y, we have

$$\mathcal{J}^{(a,b)}_n[\sigma_{p,n}](k) = \frac{\mathcal{I}(0)+\mathcal{I}(k,y)}{k^{\lfloor\frac{n+1}{2}\rfloor}}. \tag{5.4.3b}$$

Proof. From (4.1.10b) and (4.1.16), we have (5.4.3) for $n=0$. From (5.1.17), we find (5.4.3) for $n=1$. Assuming (5.4.3) holds for $m = 0,1,\ldots,n-1$, using it in (5.1.18) shows that (5.4.3) holds for all $n \geq 0$ by induction. □

Lemma 92. *We have*

$$\mathcal{J}_1^{(a,b)}[\sigma_{p,1}](k) = \frac{1}{4\lambda_{1,1}ik}\left(\mathfrak{u}(b)\mathcal{I}_0^{(a,b)}(\sigma_{0,1}k) - \mathfrak{u}(a)\mathcal{I}_0^{(a,b)}(\sigma_{1,1}k)\right) - \frac{1}{4\lambda_{1,1}ik}\int_a^b \mathfrak{u}'(y)\mathcal{I}_0^{(a,y)}(\sigma_{0,1}k)\mathcal{I}_0^{(y,b)}(\sigma_{1,1}k)\,dy + \frac{\mathcal{I}(y)+\mathcal{I}(k,y)}{k^2}. \tag{5.4.4a}$$

and if $\sigma_{0,2} = \sigma_{2,2}$,

$$\mathcal{J}_2^{(a,b)}[\sigma_{p,2}](k) = \frac{m_{\text{int}}(a,b)}{16\lambda_{1,2}ik}\mathcal{I}_0^{(a,b)}(\sigma_{0,2}k) + \frac{\mathcal{I}(y)+\mathcal{I}(k,y)}{k^2}, \tag{5.4.4b}$$

where

$$m_{\text{int}}(a,b) = \int_a^b \mu(y)\mathfrak{u}(y)^2\,dy = \int_a^b \frac{1}{\mu(y)}\left(\frac{\beta'(y)}{\beta(y)} - \frac{\alpha'(y)}{\alpha(y)}\right)^2 dy. \tag{5.4.4c}$$

Under Assumption 2.7,

$$\mathcal{J}_1^{(a,b)}[\sigma_{p,1}](k) = \frac{1}{4\lambda_{1,1}ik}\left(\mathfrak{u}(b)\mathcal{J}_0^{(a,b)}[\sigma_{0,1}](k) - \mathfrak{u}(a)\mathcal{J}_0^{(a,b)}[\sigma_{1,1}](k)\right) - \frac{1}{(2\lambda_{1,1}ik)^2}\left(\frac{\mathfrak{u}'(b)}{\mu(b)}\mathcal{I}_0^{(a,b)}(\sigma_{0,1}k) - \frac{\mathfrak{u}'(a)}{\mu(a)}\mathcal{I}_0^{(a,b)}(\sigma_{1,1}k)\right) + \frac{1}{(2\lambda_{1,1}ik)^2}\int_a^b \left(\frac{\mathfrak{u}'(y)}{\mu(y)}\right)'\mathcal{I}_0^{(a,y)}(\sigma_{0,1}k)\mathcal{I}_0^{(y,b)}(\sigma_{1,1}k)\,dy + \frac{\mathcal{I}(y)+\mathcal{I}(k,y)}{k^3}. \tag{5.4.5a}$$

and if $\sigma_{0,2} = \sigma_{2,2}$,

$$\mathcal{J}_2^{(a,b)}[\sigma_{p,2}](k) = \frac{m_{\text{int}}(a,b)}{16\lambda_{1,2}ik}\mathcal{J}_0^{(a,b)}[\sigma_{0,2}](k) - \frac{\mathfrak{u}^2(b)+\mathfrak{u}^2(a)}{2(4\lambda_{1,2}ik)^2}\mathcal{I}_0^{(a,b)}(\sigma_{0,2}k) + \frac{\mathfrak{u}(a)\mathfrak{u}(b)}{(4\lambda_{1,2}ik)^2}\mathcal{I}_0^{(a,b)}(\sigma_{1,2}k) + \frac{\mathcal{I}(y)+\mathcal{I}(k,y)}{k^3}. \tag{5.4.5b}$$

and if $\sigma_{0,3} = \sigma_{2,3}$ *and* $\sigma_{1,3} = \sigma_{3,3}$,

$$\mathcal{J}_3^{(a,b)}[\sigma_{p,3}](k) = \frac{\mathfrak{u}(b)m_{\text{int}}(a,b)}{(8\lambda_{1,3}ik)^2}\mathcal{I}_0^{(a,b)}(\sigma_{0,3}k) + \frac{\mathfrak{u}(a)m_{\text{int}}(a,b)}{(8\lambda_{1,3}ik)^2}\mathcal{I}_0^{(a,b)}(\sigma_{1,3}k) + \frac{\mathcal{I}(y)+\mathcal{I}(k,y)}{k^3}. \tag{5.4.5c}$$

Proof. Using Definition 90 in (5.1.17), we find (5.4.4a). With Assumption 2.7 and Lemma 14, integrating (4.1.10) for $n = 1$ by parts gives

$$\mathcal{J}_1^{(a,b)}[\sigma_{p,1}](k) = \frac{1}{2\lambda_{1,1}ik}\left(\mathfrak{v}(k,b)\exp\left(\sigma_{0,1}\int_a^b ik\mathfrak{n}(k,\xi)\,d\xi\right) - \mathfrak{v}(k,a)\exp\left(\sigma_{1,1}\int_a^b ik\mathfrak{n}(k,\xi)\,d\xi\right)\right) - \frac{1}{2\lambda_{1,1}ik}\int_a^b \mathfrak{v}'(k,y)\exp\left(\sigma_{0,1}\int_a^y + \sigma_{1,1}\int_y^b ik\mathfrak{n}(k,\xi)\,d\xi\right)dy, \tag{5.4.6}$$

which reduces down to

$$\mathcal{J}_1^{(a,b)}[\sigma_{p,1}](k) = \frac{1}{4\lambda_{1,1}ik}\left(\mathfrak{u}(b)\exp\left(\sigma_{0,1}\int_a^b ik\mathfrak{n}(k,\xi)\,d\xi\right) - \mathfrak{u}(a)\exp\left(\sigma_{1,1}\int_a^b ik\mathfrak{n}(k,\xi)\,d\xi\right)\right) - \frac{1}{(2\lambda_{1,1}ik)^2}\int_a^b \frac{\mathfrak{u}'(y)}{\mu(y)}\lambda_{1,1}ik\mathfrak{n}(k,y)\exp\left(\sigma_{0,1}\int_a^y + \sigma_{1,1}\int_y^b ik\mathfrak{n}(k,\xi)\,d\xi\right)dy + \frac{\mathcal{I}(y)+\mathcal{I}(k,y)}{k^3}. \tag{5.4.7}$$

Integrating by parts again, we arrive at (5.4.5a). For $n = 2$, with Lemma 95, (5.1.18) reduces down to

$$\mathcal{J}_2^{(a,b)}[\sigma_{p,2}](k) = -\frac{1}{16\lambda_{2,2}ik}\int_a^b \mathfrak{u}^2(y)\mu(y)\exp\left(\sigma_{0,2}\int_a^y + \sigma_{2,2}\int_y^b ik\mu(\xi)\,d\xi\right)dy + \frac{\mathcal{I}(y)+\mathcal{I}(k,y)}{k^2}. \tag{5.4.8}$$

If $\sigma_{0,2} = \sigma_{2,2}$, we find (5.4.4b). Similar to (5.1.18), under Assumption 2.7, we integrate (4.1.10a) by parts to obtain the recurrence relation

$$\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k) = \frac{\mathfrak{v}(k,b)}{2\lambda_{n,n}ik}\mathcal{J}_{n-1}^{(a,b)}[\sigma_{p,n}](k) - \frac{1}{4\lambda_{n,n}ik}\int_a^b \mathfrak{v}^2(k,y)\mathfrak{n}(k,y)\exp\left(\sigma_{n,n}\int_y^b ik\mathfrak{n}(k,\xi)\,d\xi\right)\mathcal{J}_{n-2}^{(a,y)}[\sigma_{p,n}](k)\,dy$$
$$- \frac{1}{2\lambda_{n,n}ik}\int_a^b \mathfrak{v}'(k,y)\exp\left(\sigma_{n,n}\int_y^b ik\mathfrak{n}(k,\xi)\,d\xi\right)\mathcal{J}_{n-1}^{(a,y)}[\sigma_{p,n}](k)\,dy. \tag{5.4.9}$$

For $n = 2$, if $\sigma_{0,2} = \sigma_{2,2}$, then $\lambda_{2,2} = -\lambda_{1,2}$ and this reduces down to

$$\mathcal{J}_2^{(a,b)}[\sigma_{p,2}](k) = -\frac{\mathfrak{u}(b)}{4\lambda_{1,2}ik}\mathcal{J}_1^{(a,b)}[\sigma_{p,2}](k) + \frac{m_{\text{int}}(a,b)}{16\lambda_{1,2}ik}\mathcal{J}_0^{(a,b)}[\sigma_{0,2}](k)$$
$$+ \frac{1}{4\lambda_{1,2}ik}\int_a^b \mathfrak{u}'(y)\mathcal{J}_0^{(y,b)}[\sigma_{2,2}](k)\mathcal{J}_1^{(a,y)}[\sigma_{p,2}](k)\,dy + \frac{\mathcal{I}(y)+\mathcal{I}(k,y)}{k^3}, \tag{5.4.10}$$

and using (5.4.5a),

$$\mathcal{J}_2^{(a,b)}[\sigma_{p,2}](k) = \frac{m_{\text{int}}(a,b)}{16\lambda_{1,2}ik}\mathcal{J}_0^{(a,b)}[\sigma_{0,2}](k) - \frac{\mathfrak{u}^2(b)+\mathfrak{u}^2(a)}{2(4\lambda_{1,2}ik)^2}\mathcal{I}_0^{(a,b)}(\sigma_{0,2}k) + \frac{\mathfrak{u}(a)\mathfrak{u}(b)}{(4\lambda_{1,2}ik)^2}\mathcal{I}_0^{(a,b)}(\sigma_{1,2}k)$$
$$- \frac{\mathfrak{u}(a)}{(4\lambda_{1,2}ik)^2}\int_a^b \mathfrak{u}'(y)\exp\left(\sigma_{1,2}\int_a^y + \sigma_{2,2}\int_y^b ik\mathfrak{n}(k,\xi)\,d\xi\right)dy + \frac{\mathcal{I}(y)+\mathcal{I}(k,y)}{k^3}. \tag{5.4.11}$$

Integrating the final integral by parts yields (5.4.5b). Under Assumption 2.7, for $n = 3$, using (5.4.5a) and (5.4.5b), if $\sigma_{0,3} = \sigma_{2,3}$ and $\sigma_{1,3} = \sigma_{3,3}$, then $\lambda_{2,3} = -\lambda_{1,3}$ and $\lambda_{3,3} = \lambda_{1,3}$, and (5.4.9) reduces to

$$\mathcal{J}_3^{(a,b)}[\sigma_{p,3}](k) = \frac{\mathfrak{u}(b)m_{\text{int}}(a,b)}{(8\lambda_{1,3}ik)^2}\mathcal{I}_0^{(a,b)}(\sigma_{0,3}k) + \frac{\mathfrak{u}(a)m_{\text{int}}(a,b)}{(8\lambda_{1,3}ik)^2}\mathcal{I}_0^{(a,b)}(\sigma_{1,3}k) + \frac{\mathcal{I}(y)+\mathcal{I}(k,y)}{k^3}$$
$$- \frac{1}{(8\lambda_{1,3}ik)^2}\int_a^b \left(\mathfrak{u}^3(y)\mu(y) + m_{\text{int}}(a,y)\mathfrak{u}'(y)\right)\exp\left(\sigma_{0,3}\int_a^y + \sigma_{1,3}\int_y^b ik\mu(\xi)\,d\xi\right)dy. \tag{5.4.12}$$

Another integration by parts gives (5.4.5c). □

Corollary 93. *We have*

$$\mathcal{C}_1^{(a,b)}(k) = \frac{\mathfrak{u}_+(a,b)}{16ik}\left(\mathcal{I}_0^{(a,b)}(2k) - 1\right) - \frac{1}{16ik}\int_a^b \mathfrak{u}'(y)\left(\mathcal{I}_0^{(a,y)}(2k) - \mathcal{I}_0^{(y,b)}(2k)\right)dy + \frac{\mathcal{I}(y)+\mathcal{I}(k,y)}{k^2}, \tag{5.4.13a}$$

$$\mathcal{S}_1^{(a,b)}(k) = -\frac{\mathfrak{u}_-(a,b)}{16k}\left(\mathcal{I}_0^{(a,b)}(2k) + 1\right) + \frac{1}{16k}\int_a^b \mathfrak{u}'(y)\left(\mathcal{I}_0^{(a,y)}(2k) + \mathcal{I}_0^{(y,b)}(2k)\right)dy + \frac{\mathcal{I}(y)+\mathcal{I}(k,y)}{k^2}, \tag{5.4.13b}$$

$$\mathcal{C}_2^{(a,b)}(k) = \frac{m_{\text{int}}(a,b)}{64ik}\left(\mathcal{I}_0^{(a,b)}(2k) - 1\right) + \frac{\mathcal{I}(y)+\mathcal{I}(k,y)}{k^2}, \tag{5.4.13c}$$

$$\mathcal{S}_2^{(a,b)}(k) = -\frac{m_{\text{int}}(a,b)}{64k}\left(\mathcal{I}_0^{(a,b)}(2k) + 1\right) + \frac{\mathcal{I}(y)+\mathcal{I}(k,y)}{k^2}, \tag{5.4.13d}$$

and under Assumption 2.7,

$$\mathcal{C}_1^{(a,b)}(k) = \frac{\mathfrak{u}_+(a,b)}{16ik}\left(\mathcal{J}_0^{(a,b)}[2](k) - 1\right) + \frac{1}{32k^2}\left(\frac{\mathfrak{u}'(b)}{\mu(b)} - \frac{\mathfrak{u}'(a)}{\mu(a)}\right)\left(\mathcal{I}_0^{(a,b)}(2k) + 1\right)$$
$$- \frac{1}{32k^2}\int_a^b \left(\frac{\mathfrak{u}'(y)}{\mu(y)}\right)'\left(\mathcal{I}_0^{(a,y)}(2k) + \mathcal{I}_0^{(y,b)}(2k)\right)dy + \frac{\mathcal{I}(y)+\mathcal{I}(k,y)}{k^3}, \tag{5.4.14a}$$

$$\mathcal{S}_1^{(a,b)}(k) = -\frac{\mathfrak{u}_-(a,b)}{16k}\left(\mathcal{J}_0^{(a,b)}[2](k) + 1\right) + \frac{1}{32ik^2}\left(\frac{\mathfrak{u}'(b)}{\mu(b)} + \frac{\mathfrak{u}'(a)}{\mu(a)}\right)\left(\mathcal{I}_0^{(a,b)}(2k) - 1\right)$$
$$- \frac{1}{32ik^2}\int_a^b \left(\frac{\mathfrak{u}'(y)}{\mu(y)}\right)'\left(\mathcal{I}_0^{(a,y)}(2k) - \mathcal{I}_0^{(y,b)}(2k)\right)dy + \frac{\mathcal{I}(y)+\mathcal{I}(k,y)}{k^3}, \tag{5.4.14b}$$

$$\mathcal{C}_2^{(a,b)}(k) = \frac{m_{\text{int}}(a,b)}{64ik}\left(\mathcal{J}_0^{(a,b)}[2](k)-1\right) + \frac{\mathfrak{u}_-^2(a,b)}{(16k)^2}\left(\mathcal{I}_0^{(a,b)}(2k)+1\right) + \frac{\mathcal{I}(y)+\mathcal{I}(k,y)}{k^3}, \tag{5.4.14c}$$

$$\mathcal{S}_2^{(a,b)}(k) = -\frac{m_{\text{int}}(a,b)}{64k}\left(\mathcal{J}_0^{(a,b)}[2](k)+1\right) + \frac{\mathfrak{u}_+^2(a,b)}{i(16k)^2}\left(\mathcal{I}_0^{(a,b)}(2k)-1\right) + \frac{\mathcal{I}(y)+\mathcal{I}(k,y)}{k^3}, \tag{5.4.14d}$$

$$\mathcal{C}_3^{(a,b)}[\sigma_{p,3}](k) = -\frac{\mathfrak{u}_+(a,b)m_{\text{int}}(a,b)}{(32k)^2}\left(\mathcal{I}_0^{(a,b)}(2k)+1\right) + \frac{\mathcal{I}(y)+\mathcal{I}(k,y)}{k^3}, \tag{5.4.14e}$$

$$\mathcal{S}_3^{(a,b)}[\sigma_{p,3}](k) = -\frac{\mathfrak{u}_-(a,b)m_{\text{int}}(a,b)}{i(32k)^2}\left(\mathcal{I}_0^{(a,b)}(2k)-1\right) + \frac{\mathcal{I}(y)+\mathcal{I}(k,y)}{k^3}. \tag{5.4.14f}$$

Proof. Using (5.4.4) in (4.1.12), we find (5.4.13). Similarly, using (5.4.5) in (4.1.12), we find (5.4.14). □

Remark 94. *Note that if a, b are independent of y the $\mathcal{I}(y)$'s appearing in Lemma 91 and Corollary 93 are $\mathcal{I}(0)$'s.*

Lemma 95. *For almost every $y \in \mathcal{D}$*

$$\lim_{t\to 0^+}\int_{\partial\Omega_{\text{ext}}} \frac{\mathcal{I}(k,y)}{k}e^{-k^2t}\,dk = 0. \tag{5.4.15}$$

Proof. From Definition 90, we have

$$\int_{\partial\Omega_{\text{ext}}} \frac{\mathcal{I}(k,y)}{k}e^{-k^2t}\,dk = \sum_{j=1}^{J}\int_{\mathcal{D}^n}\int_{\partial\Omega_{\text{ext}}} \frac{c_j(k,y,\boldsymbol{\zeta}_n)}{k}e^{ikm_j(y,\boldsymbol{\zeta}_n)-k^2t}\,dk\,d\boldsymbol{\zeta}_n. \tag{5.4.16}$$

Since c_j is analytic at $k=\infty$, we can write it as a Laurent series with coefficients denoted $c_j^{(\ell)}(y,\boldsymbol{\zeta}_n)$, so that

$$\int_{\partial\Omega_{\text{ext}}} \frac{\mathcal{I}(k,y)}{k}e^{-k^2t}\,dk = \sum_{j=1}^{J}\sum_{\ell=1}^{\infty}\int_{\mathcal{D}^n} c_j^{(\ell)}(y,\boldsymbol{\zeta}_n)\int_{\partial\Omega_{\text{ext}}} \frac{e^{ikm_j(y,\boldsymbol{\zeta}_n)-k^2t}}{k^\ell}\,dk\,d\boldsymbol{\zeta}_n. \tag{5.4.17}$$

For the $O(k^{-2})$ terms, we deform up to $\partial\Omega$ and since the integrals are absolutely convergent, we use the DCT and Cauchy's theorem to conclude that

$$\sum_{\ell=2}^{\infty} c_j^{(\ell)}(y,\boldsymbol{\zeta}_n)\int_{\partial\Omega_{\text{ext}}} \frac{e^{ikm_j(y,\boldsymbol{\zeta}_n)-k^2t}}{k^\ell}\,dk \to 0, \tag{5.4.18}$$

as $t \to 0^+$. For the $O(k^{-1})$ term, we deform down to the real axis, indenting the contour around the possible singularity at $k=0$, denoting the contour as $\Gamma_{\mathbb{R}}$. We obtain

$$\int_{\Gamma_{\mathbb{R}}} \frac{e^{ikm_j(y,\boldsymbol{\zeta}_n)-k^2t}}{k}\,dk = \int_{\Gamma_{\mathbb{R}}} \frac{\cos(km_j(y,\boldsymbol{\zeta}_n))e^{-k^2t}}{k}\,dk + i\int_{-\infty}^{\infty} \frac{\sin(km_j(y,\boldsymbol{\zeta}_n))e^{-k^2t}}{k}\,dk. \tag{5.4.19}$$

The first integrand is odd and we only have a contribution from the residue theorem. The second integrand is even, but is analytic at $k=0$. Integrating these, we find

$$\int_{\Gamma_{\mathbb{R}}} \frac{e^{ikm_j(y,\boldsymbol{\zeta}_n)-k^2t}}{k}\,dk = -i\pi\,\text{erfc}\left(\frac{m_j(y,\boldsymbol{\zeta}_n)}{2\sqrt{t}}\right) \to 0, \tag{5.4.20}$$

as $t \to 0^+$ [8], since $m_j \in [-\pi/4, \pi/4]$ for almost all $x, y \in \mathcal{D}$ and $\boldsymbol{\zeta}_n \in \mathcal{D}^n$. □

Lemma 96. *For $\varepsilon(k)$ in Lemma 66, we may write*

$$\varepsilon(k) = \frac{\mathcal{I}(0)+\mathcal{I}(k,y)}{k} + O(k^{-2})\mathfrak{E}^{(x_l,x_r)}(k). \tag{5.4.21a}$$

and more specifically, for irregular problems, *we write*

$$\varepsilon(k) = \frac{\mathcal{I}(0)}{ik} + \frac{1}{k}\int_{x_l}^{x_r}\mathcal{I}(\zeta)\left(\mathcal{I}_0^{(x_l,\zeta)}(2k)+\mathcal{I}_0^{(\zeta,x_r)}(2k)\right)d\zeta + \frac{\mathcal{I}(0)+\mathcal{I}(k,y)}{k^2} + O(k^{-3})\mathfrak{E}^{(x_l,x_r)}(k). \tag{5.4.21b}$$

Proof. Using Lemma 91 in (5.1.22), (5.1.23), and (5.1.24) gives (5.4.21a). Using Lemma 91 and Corollary 93 in (5.1.24), we find

$$\varepsilon(k) = -\frac{4\mathfrak{u}_- + m_{\text{int}}}{32ik} + \frac{1}{8ik}\int_{x_l}^{x_r} \mathfrak{u}'(\zeta)\left(\mathcal{I}_0^{(x_l,\zeta)}(2k) + \mathcal{I}_0^{(\zeta,x_r)}(2k\right)d\zeta + \frac{\mathcal{I}(0)+\mathcal{I}(k,y)}{k^2} + O(k^{-3})\mathfrak{E}^{(x_l,x_r)}(k). \quad (5.4.22)$$

which gives (5.4.21b). Using Lemma 91 and $\mathfrak{c}_0(k) = \mathcal{I}(0)/k^3 + O(k^{-5})$ in (5.1.25), we find

$$\varepsilon(k) = \frac{8k^2}{m_{\mathfrak{c}_1}\mathfrak{u}_+ - 8m_{\mathfrak{s}}}\left(\frac{\mathcal{I}(0)}{k^3} - \frac{m_{\mathfrak{c}_1}\mathfrak{u}_+}{8k^2} - \frac{2im_{\mathfrak{c}_1}}{k}\sum_{\substack{n=1\\ n \text{ odd}}}^{3} \mathcal{C}_n^{(x_l,x_r)}(k) + \frac{2im_{\mathfrak{s}}}{k^2}\sum_{n=1}^{2} \mathcal{S}_n^{(x_l,x_r)}(k)\right)$$
$$+ \frac{\mathcal{I}(0)+\mathcal{I}(k,y)}{k^2} + O(k^{-3})\mathfrak{E}^{(x_l,x_r)}(k), \quad (5.4.23)$$

and using Corollary 93,

$$\varepsilon(k) = \frac{\mathcal{I}(0)}{k} + \frac{\mathcal{I}(0)}{k}\int_a^b \left(\frac{\mathfrak{u}'(\zeta)}{\mu(\zeta)}\right)'\left[\mathcal{I}_0^{(a,\zeta)}(2k) + \mathcal{I}_0^{(\zeta,b)}(2k)\right]d\zeta + \frac{\mathcal{I}(0)+\mathcal{I}(k,y)}{k^2} + O(k^{-3})\mathfrak{E}^{(x_l,x_r)}(k), \quad (5.4.24)$$

which also gives (5.4.21b). ☐

Lemma 97. *As $t \to 0^+$, with $m = 2-j$ for $m = 0,1$ (and $j = 1,2$), we have*

$$q_{\mathcal{B}_m}(x,t) = \frac{f_m(0)}{2\sqrt{\pi t}\sqrt{(\beta\mu)(x)}}\left(B_{j-}e^{-\frac{1}{4t}M_{x_l}^2(x)} + B_{j+}e^{-\frac{1}{4t}M_x^2(x_r)}\right) + o(t^0), \quad (5.4.25)$$

where $b_{j\pm} = 0$ for regular problems *and for* irregular problems

$$B_{j-} = \frac{16(-1)^{j+1}\beta(x_l)b_{j2}}{(m_{\mathfrak{c}_1}\mathfrak{u}_+ - 8m_{\mathfrak{s}})\sqrt{(\beta\mu)(x_l)}} \qquad \text{and} \qquad B_{j+} = \frac{16(-1)^{j+1}\beta(x_r)a_{j2}}{(m_{\mathfrak{c}_1}\mathfrak{u}_+ - 8m_{\mathfrak{s}})\sqrt{(\beta\mu)(x_r)}}, \quad (5.4.26)$$

setting $m_{\mathfrak{c}_1}\mathfrak{u}_+ = 0$ for Boundary Case 3.

Proof. Consider $q_{\mathcal{B}_m}(x,t)$ given in (5.1.42). The second integrand is $O(k^{-2})$ and is therefore absolutely integrable. We may then use the DCT and Cauchy's theorem to conclude that the second integral goes to zero in the limit as $t \to 0^+$. Using that $\Xi(k) = \mathcal{J}_0^{(x_l,x)}[1](k)\mathcal{J}_0^{(x,x_r)}[1](k)$, we pull out the leading order of $\mathcal{B}_{2-j}(k)$, using Lemma 91, and write (2.3.6c) as

$$\mathcal{B}_{2-j}(k,x) = \frac{4(-1)^j}{\sqrt{(\beta\mu)(x)}}\left\{\frac{a_{j2}\beta(x_r)}{\sqrt{(\beta\mu)(x_r)}}\mathcal{J}_0^{(x,x_r)}[1](k)\mathcal{C}_0^{(x_l,x)}(k) + \frac{b_{j2}\beta(x_l)}{\sqrt{(\beta\mu)(x_l)}}\mathcal{J}_0^{(x_l,x)}[1](k)\mathcal{C}_0^{(x,x_r)}(k)\right\} + \frac{\mathcal{I}(k,y)}{k}.$$
$$+ O(k^{-2})\mathfrak{E}^{(x_l,x_r)}(k). \quad (5.4.27)$$

Using Lemma 92 and absorbing the $\Xi(k)$ terms into the $O(k^{-2})\mathfrak{E}^{(x_l,x_r)}(k)$ terms,

$$\mathcal{B}_{2-j}(k,x) = \frac{2(-1)^j}{\sqrt{(\beta\mu)(x)}}\left\{\frac{a_{j2}\beta(x_r)}{\sqrt{(\beta\mu)(x_r)}}\mathcal{I}_0^{(x,x_r)}(k) + \frac{b_{j2}\beta(x_l)}{\sqrt{(\beta\mu)(x_l)}}\mathcal{I}_0^{(x_l,x)}(k)\right\} + \frac{\mathcal{I}(k,y)}{k} + O(k^{-2})\mathfrak{E}^{(x_l,x_r)}(k). \quad (5.4.28)$$

From (5.1.43) and Lemma 96,

$$\frac{\mathfrak{F}_m^{(0)}(k)}{\mathfrak{b}_0(k)} = -\frac{f_m(0)}{\mathfrak{b}_0(k)k^2} + \frac{\mathcal{I}(0)+\mathcal{I}(k,y)}{k} + O(k^{-2})\mathfrak{E}^{(x_l,x_r)}(k), \quad (5.4.29)$$

so that

$$\frac{\mathcal{B}_{2-j}(k,x)}{\mathfrak{b}_0(k)}\mathfrak{F}_{2-j}^{(0)}(k) = \frac{f_{2-j}(0)}{\sqrt{(\beta\mu)(x)}}\left(B_{j-}\mathcal{I}_0^{(x_l,x)}(k) + B_{j+}\mathcal{I}_0^{(x,x_r)}(k)\right) + \frac{\mathcal{I}(k,y)}{k} + O(k^{-2})\mathfrak{E}^{(x_l,x_r)}(k), \quad (5.4.30)$$

where $B_{j\pm}$ are given in (5.4.26). We insert this into (5.1.42). The $O(k^{-2})\mathfrak{E}^{(x_l,x_r)}(k)$ terms can be deformed up to $\partial\Omega$ and we can use the DCT [8] and Cauchy's theorem to conclude it goes to zero as $t \to 0$. We use Lemma 95 for the $\mathcal{I}(k,y)/k$ term. Then

$$\int_{\partial\Omega_{\mathrm{ext}}} \frac{\mathcal{B}_{2-j}(k,x)}{\mathfrak{b}_0(k)} \mathfrak{F}_{2-j}^{(0)}(k) e^{-k^2 t}\, dk = \frac{f_{2-j}(0)}{\sqrt{(\beta\mu)(x)}} \int_{\partial\Omega_{\mathrm{ext}}} \left(B_{j-}\mathcal{I}_0^{(x_l,x)}(k) + B_{j+}\mathcal{I}_0^{(x,x_r)}(k) \right) e^{-k^2 t}\, dk + o(t^0). \quad (5.4.31)$$

Deforming down the real axis, letting $M_x(y) = \int_x^y \mu(\xi)\, d\xi$, and integrating, we find (5.4.25). □

Lemma 98. *For* regular problems*,*

$$\frac{\Psi(k,x,y)}{\mathfrak{b}_0(k)} \Delta_N^{-1}(k) = \mathcal{I}_0^{(x,y)}(\operatorname{sgn}(y-x)k) + \mathcal{I}(0)\mathcal{I}_0^{(x_l,x)}(k)\mathcal{I}_0^{(x_l,y)}(k) + \mathcal{I}(0)\mathcal{I}_0^{(x,x_r)}(k)\mathcal{I}_0^{(y,x_r)}(k)$$
$$+ \mathcal{I}(y)\mathcal{I}_0^{(x_l,\min(x,y))}(k)\mathcal{I}_0^{(\max(x,y),x_r)}(k) + \frac{\mathcal{I}(k,y)}{k} + O(k^{-2})\mathfrak{E}^{(x_l,x_r)}(k), \quad (5.4.32a)$$

and for irregular problems*,*

$$\frac{\Psi(k,x,y)}{\mathfrak{b}_0(k)} \Delta_N^{-1}(k) = \mathcal{I}_0^{(x,y)}(\operatorname{sgn}(y-x)k) + \mathcal{I}(y)\mathcal{I}_0^{(x_l,x)}(k)\mathcal{I}_0^{(x_l,y)}(k) + \mathcal{I}(y)\mathcal{I}_0^{(x,x_r)}(k)\mathcal{I}_0^{(y,x_r)}(k) + \frac{\mathcal{I}(k,y)}{k} + O(k^{-2})\mathfrak{E}^{(x_l,x_r)}(k)$$
$$- \frac{8m_{\mathfrak{c}_1} ik}{m_{\mathfrak{c}_1}\mathfrak{u}_+ - 8m_{\mathfrak{s}}} \left(\mathcal{I}_0^{(x_l,x)}(k)\mathcal{I}_0^{(x_l,y)}(k) - \mathcal{I}_0^{(x,x_r)}(k)\mathcal{I}_0^{(y,x_r)}(k) \right)$$
$$+ \mathcal{I}_0^{(x_l,\min(x,y))}(k)\mathcal{I}_0^{(\max(x,y),x_r)}(k) \Bigg\{ c(y-x)ik + \mathcal{I}(y)$$
$$+ \mathcal{I}(y) \int_{\min(x,y)}^{\max(x,y)} \mathcal{I}(\zeta) \left(\mathcal{I}_0^{(\min(x,y),\zeta)}(2k) + \mathcal{I}_0^{(\zeta,\max(x,y))}(2k) \right) d\zeta$$
$$+ \mathcal{I}(y) \int_{x_l}^{x_r} \mathcal{I}(\zeta) \left(\mathcal{I}_0^{(x_l,\zeta)}(2k) + \mathcal{I}_0^{(\zeta,x_r)}(2k) \right) d\zeta \Bigg\}$$
$$+ \mathcal{I}(0) \left(\mathcal{I}_0^{(x_l,x)}(k)\mathcal{I}_0^{(x_l,y)}(k) - \mathcal{I}_0^{(x,x_r)}(k)\mathcal{I}_0^{(y,x_r)}(k) \right) \int_{x_l}^{x_r} \mathcal{I}(\zeta) \left(\mathcal{I}_0^{(x_l,\zeta)}(2k) + \mathcal{I}_0^{(\zeta,x_r)}(2k) \right) d\zeta, \quad (5.4.32b)$$

where

$$c(s) = -\frac{16}{(m_{\mathfrak{c}_1}\mathfrak{u}_+ - 8m_{\mathfrak{s}})\sqrt{(\beta\mu)(x_l)}\sqrt{(\beta\mu)(x_r)}} \begin{cases} \beta(x_r)(a:b)_{1,2}, & \text{if } s<0, \\ \beta(x_l)(a:b)_{3,4}, & \text{if } s>0. \end{cases} \quad (5.4.32c)$$

Proof. For Boundary Case:

1. if $(a:b)_{2,4} \neq 0$, using Lemma 91 in (2.3.7), we find, for $x_l < y < x < x_r$,

$$\frac{\Psi(k,x,y)}{\mathfrak{b}_0(k)} = 4\mathcal{J}_0^{(y,x)}[1](k)\mathfrak{C}_0^{(x_l,y)}(k)\mathfrak{C}_\ell^{(x,x_r)}(k) + \frac{\mathcal{I}(k,y)}{k}, \quad (5.4.33)$$

   and similarly, for $x_l < x < y < x_r$. Expanding the cosines with Lemma 92, we find

$$\frac{\Psi(k,x,y)}{\mathfrak{b}_0(k)} = \mathcal{I}_0^{(x_l,x)}(k)\mathcal{I}_0^{(x_l,y)}(k) + \mathcal{I}_0^{(x,x_r)}(k)\mathcal{I}_0^{(y,x_r)}(k) + \mathcal{I}_0^{(x,y)}(\operatorname{sgn}(y-x)k) + \frac{\mathcal{I}(k,y)}{k} + O(k^{-2})\mathfrak{E}^{(x_l,x_r)}(k), \quad (5.4.34)$$

   which, using Lemma 96, gives (5.4.32).

2. if $(a:b)_{2,4} = 0$ and $m_{\mathfrak{c}_0} \neq 0$, then using Lemma 91 in (2.3.7), for $x_l < y < x < x_r$,

$$\frac{\Psi(k,x,y)}{\mathfrak{b}_0(k)} = -\frac{m_{\mathfrak{c}_1}}{m_{\mathfrak{c}_0}} \left(\mathcal{I}_0^{(x_l,x)}(k)\mathcal{I}_0^{(x_l,y)}(k) - \mathcal{I}_0^{(x,x_r)}(k)\mathcal{I}_0^{(y,x_r)}(k) \right) + \mathcal{I}_0^{(x,y)}(\operatorname{sgn}(y-x)k)$$
$$- \frac{2\beta(x_r)(a:b)_{1,2}}{m_{\mathfrak{c}_0}\sqrt{(\beta\mu)(x_l)}\sqrt{(\beta\mu)(x_r)}} \mathcal{I}_0^{(x_l,y)}(k)\mathcal{I}_0^{(x,x_r)}(k) + \frac{\mathcal{I}(k,y)}{k} + O(k^{-2})\mathfrak{E}^{(x_l,x_r)}(k), \quad (5.4.35)$$

   and similarly, for $x_l < x < y < x_r$. Using Lemma 96, this gives (5.4.32).

3. if $(a:b)_{2,4}=0$, $m_{\mathfrak{c}_0}=0$, $m_{\mathfrak{c}_1}=0$, and $(a:b)_{1,3}\neq 0$, then using Lemma 91 in (2.3.7), for $x_l<y<x<x_r$,

$$\frac{\Psi(k,x,y)}{\mathfrak{b}_0(k)}=-4\mathcal{I}_0^{(y,x)}(k)\mathcal{S}_0^{(x_l,y)}(k)\mathcal{S}_0^{(x,x_r)}(k)+\frac{4\beta(x_r)(a:b)_{1,2}k}{m_{\mathfrak{s}}\sqrt{(\beta\mu)(x_l)}\sqrt{(\beta\mu)(x_r)}}\Xi(k)\sum_{n=0}^{2}\mathcal{S}_n^{(y,x)}(k)+\frac{\mathcal{I}(k,y)}{k},\quad (5.4.36)$$

Expanding using Lemma 92,

$$\begin{aligned}\frac{\Psi(k,x,y)}{\mathfrak{b}_0(k)}&=-\mathcal{I}_0^{(x_l,x)}(k)\mathcal{I}_0^{(x_l,y)}(k)-\mathcal{I}_0^{(x,x_r)}(k)\mathcal{I}_0^{(y,x_r)}(k)+\mathcal{I}_0^{(y,x)}(k)+\frac{\mathcal{I}(k,y)}{k}+O(k^{-2})\mathfrak{E}^{(x_l,x_r)}(k)\\&\quad+c(y-x)\mathcal{I}_0^{(x_l,\min(x,y))}(k)\mathcal{I}_0^{(\max(x,y),x_r)}(k)\Bigg(ik+\mathcal{I}(y)\\&\qquad+\mathcal{I}(y)\int_{\min(x,y)}^{\max(x,y)}\mathcal{I}(\zeta)\left(\mathcal{I}_0^{(\min(x,y),\zeta)}(2k)+\mathcal{I}_0^{(\zeta,\max(x,y))}(2k)\right)d\zeta\Bigg),\quad (5.4.37)\end{aligned}$$

and similarly for $x_l<x<y<x_r$. Using Lemma 96, this gives

$$\begin{aligned}\frac{\Psi(k,x,y)}{\mathfrak{b}_0(k)}\Delta_N^{-1}(k)&=-\mathcal{I}_0^{(x_l,x)}(k)\mathcal{I}_0^{(x_l,y)}(k)-\mathcal{I}_0^{(x,x_r)}(k)\mathcal{I}_0^{(y,x_r)}(k)+\mathcal{I}_0^{(y,x)}(k)+\frac{\mathcal{I}(k,y)}{k}+O(k^{-2})\mathfrak{E}^{(x_l,x_r)}(k)\\&\quad+\mathcal{I}_0^{(x_l,\min(x,y))}(k)\mathcal{I}_0^{(\max(x,y),x_r)}(k)\Bigg\{c(y-x)ik+\mathcal{I}(y)\\&\qquad+\mathcal{I}(y)\int_{\min(x,y)}^{\max(x,y)}\mathcal{I}(\zeta)\left(\mathcal{I}_0^{(\min(x,y),\zeta)}(2k)+\mathcal{I}_0^{(\zeta,\max(x,y))}(2k)\right)d\zeta\\&\qquad+\mathcal{I}(y)\int_{x_l}^{x_r}\mathcal{I}(\zeta)\left(\mathcal{I}_0^{(x_l,\zeta)}(2k)+\mathcal{I}_0^{(\zeta,x_r)}(2k)\right)d\zeta\Bigg\},\quad (5.4.38)\end{aligned}$$

which yields (5.4.32).

4. if $(a:b)_{2,4}=0$, $m_{\mathfrak{c}_0}=0$, $m_{\mathfrak{c}_1}\neq 0$, and $m_{\mathfrak{c}_1}\mathfrak{u}_+-8m_{\mathfrak{s}}\neq 0$, then using Definition 90 and Lemma 92 in (2.3.7),

$$\begin{aligned}\frac{\Psi(k,x,y)}{\mathfrak{b}_0(k)}&=\mathcal{I}_0^{(x,y)}(\operatorname{sgn}(y-x)k)+\mathcal{I}(y)\mathcal{I}_0^{(x_l,x)}(k)\mathcal{I}_0^{(x_l,y)}(k)+\mathcal{I}(y)\mathcal{I}_0^{(x,x_r)}(k)\mathcal{I}_0^{(y,x_r)}(k)\\&\quad-\frac{8m_{\mathfrak{c}_1}ik}{m_{\mathfrak{c}_1}\mathfrak{u}_+-8m_{\mathfrak{s}}}\left(\mathcal{I}_0^{(x_l,x)}(k)\mathcal{I}_0^{(x_l,y)}(k)-\mathcal{I}_0^{(x,x_r)}(k)\mathcal{I}_0^{(y,x_r)}(k)\right)\\&\quad+c(y-x)(ik+\mathcal{I}(y))\mathcal{I}_0^{(x_l,\min(x,y))}(k)\mathcal{I}_0^{(\max(x,y),x_r)}(k)+\frac{\mathcal{I}(k,y)}{k}+O(k^{-2})\mathfrak{E}^{(x_l,x_r)}(k),\quad (5.4.39)\end{aligned}$$

where $c(s)$ is given in (5.4.32c). Using Lemma 96,

$$\begin{aligned}\frac{\Psi(k,x,y)}{\mathfrak{b}_0(k)}\Delta_N^{-1}(k)&=\mathcal{I}_0^{(x,y)}(\operatorname{sgn}(y-x)k)+\mathcal{I}(y)\mathcal{I}_0^{(x_l,x)}(k)\mathcal{I}_0^{(x_l,y)}(k)+\mathcal{I}(y)\mathcal{I}_0^{(x,x_r)}(k)\mathcal{I}_0^{(y,x_r)}(k)\\&\quad-\frac{8m_{\mathfrak{c}_1}ik}{m_{\mathfrak{c}_1}\mathfrak{u}_+-8m_{\mathfrak{s}}}\left(\mathcal{I}_0^{(x_l,x)}(k)\mathcal{I}_0^{(x_l,y)}(k)-\mathcal{I}_0^{(x,x_r)}(k)\mathcal{I}_0^{(y,x_r)}(k)\right)\\&\quad+c(y-x)(ik+\mathcal{I}(y))\mathcal{I}_0^{(x_l,\min(x,y))}(k)\mathcal{I}_0^{(\max(x,y),x_r)}(k)+\frac{\mathcal{I}(k,y)}{k}+O(k^{-2})\mathfrak{E}^{(x_l,x_r)}(k)\\&\quad+\mathcal{I}(0)\left(\mathcal{I}_0^{(x_l,x)}(k)\mathcal{I}_0^{(x_l,y)}(k)-\mathcal{I}_0^{(x,x_r)}(k)\mathcal{I}_0^{(y,x_r)}(k)\right)\int_{x_l}^{x_r}\mathcal{I}(\zeta)\left(\mathcal{I}_0^{(x_l,\zeta)}(2k)+\mathcal{I}_0^{(\zeta,x_r)}(2k)\right)d\zeta\\&\quad+\mathcal{I}(y)\mathcal{I}_0^{(x_l,\min(x,y))}(k)\mathcal{I}_0^{(\max(x,y),x_r)}(k)\int_{x_l}^{x_r}\mathcal{I}(\zeta)\left(\mathcal{I}_0^{(x_l,\zeta)}(2k)+\mathcal{I}_0^{(\zeta,x_r)}(2k)\right)d\zeta,\quad (5.4.40)\end{aligned}$$

which gives (5.4.32).

□

Lemma 99. *We have $q_f(x,t)\to 0$, as $t\to 0^+$.*

Proof. Consider $q_f(x,t)$ given in (5.1.52a). From Lemmas 67 and 75, the second integrand is $O(k^{-2})$ and is therefore absolutely integrable. We may then use the DCT and Cauchy's theorem to conclude that the second integral goes to zero in the limit as $t \to 0^+$. Similarly, if the problem is *regular*, the integrand of the first integral in (5.1.52a) is $O(k^{-2})\mathfrak{E}^{(x_l,x_r)}(k)$, so that it may be deformed back up to $\partial\Omega$ and we can use the DCT to Cauchy's theorem to conclude that it also goes to zero as $t \to 0^+$. For the *irregular problems*, from (5.1.52a), we have

$$q_f(x,t) = -\frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}(r)} \frac{e^{-k^2 t}}{\mathfrak{b}_0(k)} \int_{\mathcal{D}} \frac{\Psi(k,x,y)\Delta_N^{-1}(k) f_\alpha(y,0)}{k^2\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,y)}}\, dy\, dk + o(t^0). \tag{5.4.41}$$

Since this is absolutely integrable, we may switch the order of integration, so that

$$q_f(x,t) = -\frac{1}{2\pi}\int_{\mathcal{D}} \frac{f_\alpha(y,0)}{\sqrt{(\beta\mu)(x)}\sqrt{(\beta\mu)(y)}} \int_{\partial\Omega_{\text{ext}}(r)} \frac{\Psi(k,x,y)}{k^2\mathfrak{b}_0(k)}\Delta_N^{-1}(k) e^{-k^2 t}\, dk\, dy + o(t^0). \tag{5.4.42}$$

Using Lemma 98, we have

$$\frac{\Psi(k,x,y)}{k^2\mathfrak{b}_0(k)}\Delta_N^{-1}(k) = \frac{\mathcal{I}(k,x,y)}{k} + O(k^{-2})\mathfrak{E}^{(x_l,x_r)}(k),$$

so that from Lemma 95, we have that $q_f(x,t) \to 0$ as $t \to 0^+$. ☐

Lemma 100. *Define* $\mathcal{X}(y) = \sum_j M_{a_j(y)}(b_j(y))$ *(where* $M_a(b) = \int_a^b \mu(\xi)\, d\xi$*),* $\chi(y) = \mathcal{X}'(y)/\mu(y)$*, and*

$$I_0[\eta](x,t) = \frac{1}{2\pi}\int_{\mathcal{D}} \eta(y)\mu(y) \int_{\partial\Omega_{\text{ext}}} \left(\prod_j \mathcal{I}_0^{(a_j(y),b_j(y))}(k)\right) e^{-k^2 t}\, dk\, dy, \tag{5.4.43a}$$

$$I_1[\eta](x,t) = \frac{1}{2\pi}\int_{\mathcal{D}} \eta(y)c(y-x)\mu(y) \int_{\partial\Omega} ik\left(\prod_j \mathcal{I}_0^{(a_j(y),b_j(y))}(k)\right) e^{-k^2 t}\, dk\, dy. \tag{5.4.43b}$$

We assume $\eta \in L^1(\mathcal{D})$*,* $\mathcal{X}(y)$ *is continuous for* $y \in \mathcal{D}$*, and*

$$\chi(y) = \begin{cases} \chi_+, & \text{if } y > x, \\ \chi_-, & \text{if } y < x, \end{cases} \qquad \text{and} \qquad c(y-x) = \begin{cases} c_+, & \text{if } y > x, \\ c_-, & \text{if } y < x, \end{cases} \tag{5.4.44}$$

for two nonzero constants $\chi_\pm$ *and two constants* $c_\pm$*.*

- *If* $\arg(\mathcal{X}(y)) \in [-\pi/4, \pi/4]$ *for all* $y \in \mathcal{D}$*, then* $I_0[\eta](x,t) \to 0$ *as* $t \to 0^+$*.*
- *For almost all* $x \in \mathcal{D}$*, if* $\mathcal{X}(x) = 0$*,* $\chi_- = \chi_+ = 1$*,* $\arg(\mathcal{X}(y)) \in [-\pi/4, \pi/4]$ *for* $y > x$*, and* $\arg(\mathcal{X}(y)) \in [3\pi/4, 5\pi/4]$ *for* $y < x$*, then* $I_0[\eta](x,t) \to \eta(x)$ *as* $t \to 0^+$*.*
- *If* $\arg(\mathcal{X}(y)) \in [-\pi/4, \pi/4]$ *for all* $y \in \mathcal{D}$*, then*

$$I_1[\eta](x,t) = \frac{1}{2\sqrt{\pi t}}\left(\frac{c_-}{\chi_-} - \frac{c_+}{\chi_+}\right)\eta(x)e^{-\frac{1}{4t}\mathcal{X}^2(x)} - \frac{c_-}{2\chi_-\sqrt{\pi t}}\eta(x_l)e^{-\frac{1}{4t}\mathcal{X}^2(x_l)} + \frac{c_+}{2\chi_+\sqrt{\pi t}}\eta(x_r)e^{-\frac{1}{4t}\mathcal{X}^2(x_r)} + o(t^0). \tag{5.4.45}$$

Proof. Integrating the k-integral of (5.4.43), we find

$$I_0[\eta](x,t) = \frac{1}{2\sqrt{\pi t}}\int_{\mathcal{D}} \eta(y)\mu(y)e^{-\frac{1}{4t}\mathcal{X}^2(y)}\, dy, \tag{5.4.46a}$$

$$I_1[\eta](x,t) = -\frac{1}{4\sqrt{\pi t^3}}\int_{\mathcal{D}} \eta(y)c(x-y)\mu(y)\mathcal{X}(y)e^{-\frac{1}{4t}\mathcal{X}^2(y)}\, dy. \tag{5.4.46b}$$

For a fixed $x \in \mathcal{D}$, if $\eta(x)$ is finite, using that $\eta \in L^1(\mathcal{D})$, it follow that for any $t > 0$, there exists a $\varphi \in \mathrm{AC}^1(\mathcal{D})$ [16], so that

$$\int_{\mathcal{D}} |\eta(y) - \varphi(y)|\, dy < t^2 \qquad \text{and} \qquad |\varphi(x) - \eta(x)| < t^2. \tag{5.4.47}$$

Using this, we have $I_j[\eta](x,t) = I_j[\eta-\varphi](x,t) + I_j[\varphi](x,t)$ for both $j=0,1$. Since $\|\mathcal{X}\|_\infty \le \|\mu\|_{\mathcal{D}} \le M_{\mathfrak{n}}(x_r - x_l)$,

$$|I_0[\eta-\varphi](x,t)| \le \frac{M_{\mathfrak{n}}}{2\sqrt{\pi t}} \int_{\mathcal{D}} |\eta(y)-\varphi(y)|\,dy < \frac{M_{\mathfrak{n}}}{2\sqrt{\pi}}\sqrt{t^3} \to 0, \tag{5.4.48a}$$

$$|I_1[\eta-\varphi](x,t)| \le \frac{\max(|c_-|,|c_+|)M_{\mathfrak{n}}^2(x_r-x_l)}{4\sqrt{\pi}} \int_{\mathcal{D}} |\eta(y)-\varphi(y)|\,dy < \frac{\max(|c_-|,|c_+|)M_{\mathfrak{n}}^2(x_r-x_l)}{4\sqrt{\pi}}\sqrt{t} \to 0, \tag{5.4.48b}$$

as $t\to 0^+$. We break (5.4.46) into two regions $y<x$ and $y>x$,

$$I_0[\varphi](x,t) = \frac{1}{2\sqrt{\pi t}}\left(\frac{1}{\chi_-}\int_{y<x} + \frac{1}{\chi_+}\int_{y>x}\right)\varphi(y)\chi(y)\mu(y)e^{-\frac{1}{4t}\mathcal{X}^2(y)}\,dy, \tag{5.4.49a}$$

$$I_1[\varphi](x,t) = -\frac{1}{4\sqrt{\pi t^3}}\left(\frac{c_-}{\chi_-}\int_{y<x} + \frac{c_+}{\chi_+}\int_{y>x}\right)\varphi(y)\chi(y)\mu(y)\mathcal{X}(y)e^{-\frac{1}{4t}\mathcal{X}^2(y)}\,dy. \tag{5.4.49b}$$

Integrating (5.4.49a) by parts once and (5.4.49b) by parts twice, we obtain

$$\begin{aligned} I_0[\varphi](x,t) &= \frac{1}{2}\left(\frac{1}{\chi_-}-\frac{1}{\chi_+}\right)\varphi(x)\mathrm{erf}\left(\frac{\mathcal{X}(x)}{2\sqrt{t}}\right) + \frac{1}{2\chi_+}\varphi(x_r)\mathrm{erf}\left(\frac{\mathcal{X}(x_r)}{2\sqrt{t}}\right) - \frac{1}{2\chi_-}\varphi(x_l)\mathrm{erf}\left(\frac{\mathcal{X}(x_l)}{2\sqrt{t}}\right) \\ &\quad - \frac{1}{2}\left(\frac{1}{\chi_-}\int_{y<x} + \frac{1}{\chi_+}\int_{y>x}\right)\varphi'(y)\mathrm{erf}\left(\frac{\mathcal{X}(y)}{2\sqrt{t}}\right)dy, \end{aligned} \tag{5.4.50a}$$

$$\begin{aligned} I_1[\varphi](x,t) &= \frac{1}{2\sqrt{\pi t}}\left(\frac{c_-}{\chi_-}-\frac{c_+}{\chi_+}\right)\varphi(x)e^{-\frac{1}{4t}\mathcal{X}^2(x)} + \frac{c_+}{2\chi_+\sqrt{\pi t}}\varphi(x_r)e^{-\frac{1}{4t}\mathcal{X}^2(x_r)} - \frac{c_-}{2\chi_-\sqrt{\pi t}}\varphi(x_l)e^{-\frac{1}{4t}\mathcal{X}^2(x_l)} \\ &\quad - \frac{1}{2}\left(\frac{c_-}{\chi_-^2}-\frac{c_+}{\chi_+^2}\right)\frac{\varphi'(x)}{\mu(x)}\mathrm{erf}\left(\frac{\mathcal{X}(x)}{2\sqrt{t}}\right) - \frac{c_+}{2\chi_+^2}\frac{\varphi'(x_r)}{\mu(x_r)}\mathrm{erf}\left(\frac{\mathcal{X}(x_r)}{2\sqrt{t}}\right) + \frac{c_-}{2\chi_-^2}\frac{\varphi'(x_l)}{\mu(x_l)}\mathrm{erf}\left(\frac{\mathcal{X}(x_l)}{2\sqrt{t}}\right) \\ &\quad + \frac{1}{2}\left(\frac{c_-}{\chi_-^2}\int_{y<x} + \frac{c_+}{\chi_+^2}\int_{y>x}\right)\left(\frac{\varphi'(y)}{\mu(y)}\right)'\mathrm{erf}\left(\frac{\mathcal{X}(y)}{2\sqrt{t}}\right)dy. \end{aligned} \tag{5.4.50b}$$

If $\arg(\mathcal{X}(y)) \in [-\pi/4,\pi/4]$ for all $y\in\mathcal{D}$, then as $t\to 0^+$,

$$I_0[\varphi](x,t) \to \frac{1}{2\chi_-}(\varphi(x)-\varphi(x_l)) + \frac{1}{2\chi_+}(\varphi(x_r)-\varphi(x)) - \frac{1}{2\chi_-}\int_{y<x}\varphi'(y)\,dy - \frac{1}{2\chi_+}\int_{y>x}\varphi'(y)\,dy = 0, \tag{5.4.51a}$$

and similarly,

$$I_1[\varphi](x,t) = \frac{1}{2\sqrt{\pi t}}\left(\frac{c_-}{\chi_-}-\frac{c_+}{\chi_+}\right)\varphi(x)e^{-\frac{1}{4t}\mathcal{X}^2(x)} - \frac{c_-}{2\chi_-\sqrt{\pi t}}\varphi(x_l)e^{-\frac{1}{4t}\mathcal{X}^2(x_l)} + \frac{c_+}{2\chi_+\sqrt{\pi t}}\varphi(x_r)e^{-\frac{1}{4t}\mathcal{X}^2(x_r)} + o(t^0). \tag{5.4.51b}$$

Therefore, we have $I_0[\eta](x,t) = I_0[\eta-\varphi](x,t) + I_0[\varphi](x,t) \to 0$, as $t\to 0^+$ and (5.4.51b) becomes (5.4.45). If $\mathcal{X}(x)=0$, $\chi_-=\chi_+=1$, $\arg(\mathcal{X}(y))\in[-\pi/4,\pi/4]$ for $y>x$ and $\arg(\mathcal{X}(y))\in[3\pi/4,5\pi/4]$ for $y<x$, then, as $t\to 0^+$,

$$I_0[\varphi](x,t) = \frac{1}{2\chi_+}\varphi(x_r) + \frac{1}{2\chi_-}\varphi(x_l) + \frac{1}{2\chi_-}\int_{y<x}\varphi'(y)\,dy - \frac{1}{2\chi_+}\int_{y>x}\varphi'(y)\,dy = \frac{1}{2}\left(\frac{1}{\chi_-}+\frac{1}{\chi_+}\right)\varphi(x) = \varphi(x). \tag{5.4.52}$$

Therefore $I_0[\eta](x,t) = I_0[\eta-\varphi](x,t) + I_0[\varphi](x,t) \to \varphi(x) \to \eta(x)$, as $t\to 0^+$. Since $\eta\in L^1(\mathcal{D})$, it is finite for almost all $x\in\mathcal{D}$ and therefore the result holds for almost all $x\in\mathcal{D}$. □

Lemma 101. *If $q_0\in L^1(\mathcal{D})$, then for almost all $x\in\mathcal{D}$,*

$$q_0(x,t) = q_0(x) + \frac{1}{2\sqrt{\pi t}\sqrt{(\beta\mu)(x)}}\left(Q_-e^{-\frac{1}{4t}M_{x_l}^2(x)} + Q_+e^{-\frac{1}{4t}M_x^2(x_r)}\right) + o(t^0), \tag{5.4.53}$$

where $Q_\pm = 0$ for regular problems *and for* irregular problems*,*

$$Q_+ = -\frac{16\beta(x_r)}{(m_{\mathfrak{c}_1}\mathfrak{u}_+ - 8m_{\mathfrak{s}})\sqrt{(\beta\mu)(x_r)}}\big((a:b)_{2,3}q_0(x_r) - (a:b)_{1,2}q_0(x_l)\big), \tag{5.4.54a}$$

$$Q_- = \frac{16\beta(x_l)}{(m_{\mathfrak{c}_1}\mathfrak{u}_+ - 8m_{\mathfrak{s}})\sqrt{(\beta\mu)(x_l)}}\big((a:b)_{1,4}q_0(x_l) + (a:b)_{3,4}q_0(x_r)\big). \tag{5.4.54b}$$

Proof. The second integrand in (5.1.49) is absolutely integrable and so by the DCT and Cauchy's theorem it has the limit zero as $t \to 0^+$. For the first integral, we write

$$q_0(x,t) = \frac{1}{2\pi} \int_{\partial\Omega_{\text{ext}}} \Delta_N^{-1}(k) e^{-k^2 t} \int_{\mathcal{D}} \frac{\Psi(k,x,y)}{\mathfrak{b}_0(k)} \frac{q_\alpha(y)}{\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,y)}} \, dy \, dk + o(t^0). \tag{5.4.55}$$

For *regular problems*, using the asymptotics $1/\sqrt{(\beta\mathfrak{n})(k,x)} = 1/\sqrt{(\beta\mu)(x)} + O(k^{-2})$ and $\Psi(k,x,y)/\mathfrak{b}_0(k)\Delta_N^{-1}(k) = O(k^0)\mathfrak{E}^{(x_l,x_r)}(k)$ (see Lemma 67), and the Fubini-Tonelli theorem, we may swap the order of integration to obtain

$$q_0(x,t) = \frac{1}{2\pi\sqrt{(\beta\mu)(x)}} \int_{\mathcal{D}} \frac{q_\alpha(y)}{\sqrt{(\beta\mu)(y)}} \int_{\partial\Omega_{\text{ext}}} \frac{\Psi(k,x,y)}{\mathfrak{b}_0(k)} \Delta_N^{-1}(k) e^{-k^2 t} \, dk \, dy + o(t^0). \tag{5.4.56}$$

We use Lemma 98 in (5.4.56). The $O(k^{-2})\mathfrak{E}^{(x_l,x_r)}(k)$ terms go to zero as $t \to 0^+$ as above, by the DCT and Cauchy's theorem. The $\mathcal{I}(k,y)/k$ term goes to zero by Lemma 95. For the remaining terms, we apply Lemma 100. The $\mathcal{I}_0^{(x_l,x)}(k)\mathcal{I}_0^{(x_l,y)}(k)$, $\mathcal{I}_0^{(x,x_r)}(k)\mathcal{I}_0^{(y,x_r)}(k)$, and $\mathcal{I}_0^{(x_l,\min(x,y))}(k)\mathcal{I}_0^{(\max(x,y),x_r)}(k)$ terms have $\arg(\mathcal{X}(y)) = \arg(\mu(y)) \in [-\pi/4, \pi/4]$ so that those terms go to zero in the limit $t \to 0^+$. The final integral over $\mathcal{I}_0^{(y,x)}(\operatorname{sgn}(x-y)k)$ has $\mathcal{X}(y) = M_x(y)$, so that $\mathcal{X}(x) = 0$, $\chi(y) = 1$, $\arg(\mathcal{X}(y)) \in [-\pi/4, \pi/4]$ for $y > x$, and $\arg(\mathcal{X}(y)) \in [3\pi/4, 5\pi/4]$, so that (with $\eta(y) = q_\alpha(y)/(\mu(y)\sqrt{(\beta\mu)(y)}) \in L^1(\mathcal{D})$)

$$q_0(x,t) \to \frac{q_\alpha(x)}{\mu(x)\sqrt{(\beta\mu)(x)}\sqrt{(\beta\mu)(x)}} = q_0(x), \qquad \text{as } t \to 0^+. \tag{5.4.57}$$

Similarly, for *irregular problems*, we use Lemma 98 and the asymptotics $1/\sqrt{(\beta\mathfrak{n})(k,x)} = 1/\sqrt{(\beta\mu)(x)} + O(k^{-2})$ in (5.4.55). The $O(k^{-2})\mathfrak{E}^{(x_l,x_r)}(k)$ and $\mathcal{I}(k,y)/k$ terms go to zero as above. Using Lemma 100 as above, we reduce this to

$$\begin{aligned} q_0(x,t) = q_0(x) &- \frac{1}{2\pi\sqrt{(\beta\mu)(x)}} \frac{8m_{\mathfrak{c}_1}}{m_{\mathfrak{c}_1}\mathfrak{u}_+ - 8m_{\mathfrak{s}}} \int_{\mathcal{D}} \frac{q_\alpha(y)}{\mu(y)\sqrt{(\beta\mu)(y)}} \mu(y) \int_{\partial\Omega_{\text{ext}}} ik\mathcal{I}_0^{(x_l,x)}(k)\mathcal{I}_0^{(x_l,y)}(k) e^{-k^2 t} \, dk \, dy \\ &+ \frac{1}{2\pi\sqrt{(\beta\mu)(x)}} \frac{8m_{\mathfrak{c}_1}}{m_{\mathfrak{c}_1}\mathfrak{u}_+ - 8m_{\mathfrak{s}}} \int_{\mathcal{D}} \frac{q_\alpha(y)}{\mu(y)\sqrt{(\beta\mu)(y)}} \mu(y) \int_{\partial\Omega_{\text{ext}}} ik\mathcal{I}_0^{(x,x_r)}(k)\mathcal{I}_0^{(y,x_r)}(k) e^{-k^2 t} \, dk \, dy \\ &+ \frac{1}{2\pi\sqrt{(\beta\mu)(x)}} \int_{\mathcal{D}} \frac{q_\alpha(y)}{\mu(y)\sqrt{(\beta\mu)(y)}} c(y-x)\mu(y) \int_{\partial\Omega_{\text{ext}}} ik\mathcal{I}_0^{(x_l,\min(x,y))}(k)\mathcal{I}_0^{(\max(x,y),x_r)}(k) e^{-k^2 t} dk \, dy + o(t^0). \end{aligned}$$

Using Lemma 100 again, for the first integral, $\chi_- = \chi_+ = 1$ and $c_- = c_+ = 1$, for the second integral, $\chi_- = -1 = \chi_+$ and $c_- = c_+ = 1$, and for the last integral, $\chi_- = 1$, $\chi_+ = -1$, and $c_\pm$ are given in (5.4.32c). Using that $\arg(M_{x_l}(x_r)) \in (-\pi/4, \pi/4)$ so that those terms decay exponentially, this becomes

$$\begin{aligned} q_0(x,t) = q_0(x) &+ \frac{1}{2\sqrt{\pi t}\sqrt{(\beta\mu)(x)}} \frac{8m_{\mathfrak{c}_1}}{m_{\mathfrak{c}_1}\mathfrak{u}_+ - 8m_{\mathfrak{s}}} \left(\frac{q_\alpha(x_l)}{\mu(x_l)\sqrt{(\beta\mu)(x_l)}} e^{-\frac{1}{4t}M_{x_l}^2(x)} - \frac{q_\alpha(x_r)}{\mu(x_r)\sqrt{(\beta\mu)(x_r)}} e^{-\frac{1}{4t}M_x^2(x_r)} \right) \\ &+ \frac{1}{2\sqrt{\pi t}\sqrt{(\beta\mu)(x)}} \left(-\frac{c_- q_\alpha(x_l)}{\mu(x_l)\sqrt{(\beta\mu)(x_l)}} e^{-\frac{1}{4t}M_x^2(x_r)} - \frac{c_+ q_\alpha(x_r)}{\mu(x_r)\sqrt{(\beta\mu)(x_r)}} e^{-\frac{1}{4t}M_{x_l}^2(x)} \right) + o(t^0). \end{aligned}$$

Using that for *irregular problems*, $m_{\mathfrak{c}_1} = 2(a:b)_{1,4}/\mu(x_l) = 2(a:b)_{2,3}/\mu(x_r)$ and (5.4.32c), we find (5.4.53). ☐

Theorem 102. *If $q_0 \in L^1(\mathcal{D})$, for almost all $x \in \mathcal{D}$, for* regular problems*, $q(x,t) \to q_0(x)$, and for* irregular problems*, if the* compatibility conditions (1.2.8) *are satisfied, then $q(x,t) \to q_0(x)$.*

Proof. For *regular problems*, this follows from Lemmas 97, 99, and 101 and since $q(x,t) = q_0(x,t) + q_f(x,t) + q_{\mathcal{B}_0}(x,t) + q_{\mathcal{B}_1}(x,t)$. For *irregular problems*, combining Lemmas 97 and 101,

$$\begin{aligned} q_0(x,t) + q_{\mathcal{B}_0}(x,t) + q_{\mathcal{B}_1}(x,t) = q_0(x) + \frac{1}{2\sqrt{\pi t}\sqrt{(\beta\mu)(x)}} \Big(&\big(Q_- + f_0(0)B_{2-} + f_1(0)B_{1-}\big) e^{-\frac{1}{4t}M_{x_l}^2(x)} \\ &+ \big(Q_+ + f_0(0)B_{2+} + f_1(0)B_{1+}\big) e^{-\frac{1}{4t}M_x^2(x_r)} \Big) + o(t^0). \end{aligned} \tag{5.4.58}$$

Using (5.4.26) and (5.4.54)

$$Q_- + f_0(0)B_{2-} + f_1(0)B_{1-} = \frac{16\beta(x_l)}{(m_{\mathfrak{c}_1}\mathfrak{u}_+ - 8m_{\mathfrak{s}})\sqrt{(\beta\mu)(x_l)}}\big((a:b)_{1,4}q_0(x_l) + (a:b)_{3,4}q_0(x_r) - b_{22}f_0(0) + b_{12}f_1(0)\big), \tag{5.4.59a}$$

$$Q_+ + f_0(0)B_{2+} + f_1(0)B_{1+} = \frac{16\beta(x_r)}{(m_{\mathfrak{c}_1}\mathfrak{u}_+ - 8m_{\mathfrak{s}})\sqrt{(\beta\mu)(x_r)}}\big((a:b)_{1,2}q_0(x_l) - (a:b)_{2,3}q_0(x_r) - a_{22}f_0(0) + a_{12}f_1(0)\big). \tag{5.4.59b}$$

If the *compatibility conditions* (1.2.8) are satisfied, then

$$Q_- + f_0(0)B_{2-} + f_1(0)B_{1-} = 0, \tag{5.4.60a}$$

$$Q_+ + f_0(0)B_{2+} + f_1(0)B_{1+} = 0, \tag{5.4.60b}$$

and therefore, $q_0(x,t) + q_{\mathcal{B}_0}(x,t) + q_{\mathcal{B}_1}(x,t) = q_0(x) + o(t^0)$. □

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