Active Phase for the Stochastic Sandpile on Z

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Abstract

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We prove that the critical value of the one-dimensional Stochastic Sandpile Model is less than one. This verifies a conjecture of Rolla and Sidoravicius.

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DEDICATION

To my lovely wife, Xuanxuan, and my beloved parents, Wenjin and Wei.

Chapter 1

INTRODUCTION

Sandpile models have a long history in the statistical mechanics literature as paradigms of self-organized criticality [3, 7, 13]. Of particular importance is the Stochastic Sandpile Model (SSM). This abelian variant of Manna's model [13] is widely believed to exhibit universality [6]. Since the SSM gained popularity in the mathematics community the prime challenge has been to prove that the critical density for the one-dimensional Stochastic Sandpile Model is less than one [17]. In this paper we prove this conjecture.

The SSM is an interacting particle system on $\mathbb{Z}$ defined as follows. At each time, the state of the system is given by a function $\eta_t : \mathbb{Z} \to \{0, 1, 2, \dots\}$, where $\eta_t(x)$ represents the number of particles at site x at time t. Sites x such that $\eta_t(x) \leqslant 1$ are considered stable while sites with $\eta_t(x) \geqslant 2$ are unstable. Unstable sites topple independently at exponential rate 1 and when an unstable site topples two particles from the site independently move to neighboring sites.

Criticality of the SSM is defined with respect to whether or not the system remains active. We say the stochastic sandpile *locally fixates* if for each x the function $t \mapsto \eta_t(x)$ is eventually constant and *stays active* otherwise. The main result of [17] about the SSM is that if the initial distribution ν is given by i.i.d. Poisson random variables with parameter μ, then there exists a critical value $\mu_c \in [1/4, 1]$ such that the system locally fixates almost surely if $\mu < \mu_c$ and stays active almost surely in $\mu > \mu_c$, see [17, Theorem 1]. The argument that $\mu_c \leqslant 1$ is essentially trivial and comes down to showing that for trivial reasons if $\mu > 1$ then the site 0 topples infinitely many times, while the argument that $\mu_c \geqslant 1/4$ is subtle. Since [17] improvements have been made on the lower bound, see [15], but the problem posed in [17, Section 7] of finding a non-trivial upper bound has remained open. Our main result is the following theorem, which gives the first non-trivial upper bound on μ_c.

Theorem 1.1. *For any independent starting configuration with multiple particles at some sites a.s., the critical value μ_c for the SSM is strictly less than 1. In fact, $\mu_c \leqslant 1 - e^{-2\times 10^5}$.*

In this paper we consider an infinite version of the SSM but there are also finite versions (e.g. driven dissipative and fixed energy [8,14]). Universality would imply that all the critical values and exponents should be the same across these models. Some sandpile models with deterministic toppling rules have been shown to not exhibit universality [9], while there is numerical evidence for universality in the SSM [6]. Our result should transfer to finite versions of this model [5].

The methods used in this paper are based on those recently developed to study Activated Random Walk (ARW). ARW is a related abelian network that is also believed to exhibit self-organized criticality and universality [11,17]. See [16] for a nice survey on ARW. While there are many similarities between the SSM and activated random walk, since [17] activated random walk was introduced as it was believed to be a more tractable model to study [4, 11]. For example, the analogue of Theorem 1.1 was established in [10] and, indeed, has recently been extended to higher dimensions [1,12]. There is also evidence that ARW exhibits universality [18]. Our approach in this paper is similar to the one taken in [10]. However the analysis is much more delicate for the SSM than what was required in previous papers on ARW. The added difficulties arise because in ARW particles move or sleep independently, while in the SSM the particle moves are correlated in pairs. This correlation makes the SSM much more difficult to rigorously analyze.

An essential component of this analysis is to use a half-toppling scheme, defined in Chapter 2. Half-topplings allow us to follow similar (albeit significantly more complicated) approaches to those that have been used to study ARW. Half-toppling schemes have also been used in the previous lower bound on the critical density for the SSM [17] and the recent improvement that proves $\mu_c \geqslant 1/2$ [15]. In the next chapter we introduce some notation, detail the half-toppling scheme and then outline the rest of the paper.

Chapter 2

SETUP AND OUTLINE

2.1 Sandpile dynamics

We start by defining the 'site-wise' representation for SSM. We identify configurations of particles as functions $\eta : \mathbb{Z} \to \mathbb{N}$, i.e. $\eta(x)$ counts the number of particles at site x. To run the dynamics on a finite interval $I \subset \mathbb{Z}$, every site $x \in I$ is assigned an infinite sequence of iid **instructions**, $(\xi^x_j : j \in \mathbb{N})$ each taking one of the two possible values $\xi_{-,x}$ or $\xi_{+,x}$ with equal probability, where $\xi_{-,x}$ and $\xi_{+,x}$ are operators on the space of particle configurations that act via

$$\xi_{-,x}(\eta)(y) = \begin{cases} \eta(y), & y \notin \{x, x-1\} \\ \eta(y) \pm 1, & y = x - \frac{1}{2} \mp \frac{1}{2} \end{cases}$$

and

$$\xi_{+,x}(\eta)(y) = \begin{cases} \eta(y), & y \notin \{x, x+1\} \\ \eta(y) \pm 1, & y = x + \frac{1}{2} \pm \frac{1}{2}. \end{cases}$$

In the usual *discrete-time* setup, the system evolves one step by choosing a site $x \in I$ with $\eta(x) \geqslant 2$, and applying the *two* stack instructions ξ^x_j, ξ^x_{j+1} where j is such that all instructions $\xi^x_{j'}$ for $j' < j$ have already been used, but ξ^x_j has not. In this version, particles always topple in pairs. For an initial particle configuration η and a sequence of sites $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k), \alpha_i \in I$ to be toppled, let

$$\xi_\alpha = \xi^{\alpha_k}_{j_{2k}} \circ \xi^{\alpha_k}_{j_{2k-1}} \circ \cdots \circ \xi^{\alpha_2}_{j_4} \circ \xi^{\alpha_2}_{j_3} \circ \xi^{\alpha_1}_{j_2} \circ \xi^{\alpha_1}_{j_1}$$

be the operator obtained by performing all topplings of α in order. The **odometer** of the pair (η, α) is the function which records the total number of times each site was toppled, i.e.

$$m_I(\eta, \alpha, x) := \#\{j : \alpha_j = x\}.$$

Such a sequence α is called **legal** for η if $\xi_{(\alpha_1,\ldots,\alpha_{\ell-1})}\eta(\alpha_\ell) \geqslant 2$ for every $\ell \in \{1,\ldots,k\}$.

The odometer function m_I is universal in the sense that if α, α' are any two legal toppling sequences for a configuration η on interval I such that $\xi_\alpha\eta$ and $\xi_{\alpha'}\eta$ have no sites with at least two particles, then α is a permutation of α' and thus $m_I(\eta,\alpha,\cdot) = m_I(\eta,\alpha',\cdot)$. For our argument, it is only necessary that the half-toppling version of the dynamics gives a lower bound on this odometer; see Lemma 2.2.

We need the following result from [17, Lemma 4] which relates the site-wise representation to the continuous-time process $(\eta_t)_{t\geqslant 0}$ in Chapter 1. Define

$$m(\eta, x) := \sup_{I,\alpha} m_I(\eta, \alpha, x)$$

where the supremum is over all finite intervals I and all legal toppling sequences α for the configuration η on I.

Lemma 2.1. *Suppose the initial state η_0 is a translation-invariant, ergodic distribution on $\mathbb{Z}$ with finite density $\mathbf{E}(\eta_0(0))$, then*

$$\mathbb{P}\left((\eta_t)_{t\geqslant 0} \text{ stays active}\right) = \mathbb{P}(m(\eta_0, 0) = \infty) \in \{0, 1\}.$$

The above lemma says that in order to establish the system staying active a.s., it suffices to show that some site is toppled infinitely many times with positive probability.

2.2 Half-topplings

To allow more freedom in our toppling procedure (and retain some independence between particles), we work with an equivalent version of the dynamics. Instead of toppling two particles at a time, we topple single particles, i.e. perform **half-topplings**. In the usual dynamics, the number of particles toppled at a given site is always even, so there are restrictions on which half-topplings are allowed. Namely, we must keep track of how many times each site has been half-toppled, and only allow another half-toppling if that number is odd, or if that site has at least two particles.

Thus, in the half-toppling scheme, a configuration of particles consists of two functions, $\eta : \mathbb{Z} \to \mathbb{N}$, the number of particles per site, and a parity sequence $\omega : \mathbb{Z} \to \{0, 1\}$. A site x is *legal* for a pair of configurations (η, ω) if either $\eta(x) \geqslant 2$ or $\eta(x) = \omega(x) = 1$. At each step of the half-toppling scheme, a legal site x is chosen for the current pair of configurations, and the first unused instruction ξ_j^x acts upon the pair by sending a single particle at x to a uniform random neighbor and adding 1 (mod 2) to $\omega(x)$. A half-toppling sequence of sites $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ is **legal** for (η, ω) if the chosen site is legal for every step, that is, α_ℓ is legal for

$$\xi_{j_{\ell-1}}^{\alpha_{\ell-1}} \circ \cdots \circ \xi_{j_2}^{\alpha_2} \circ \xi_{j_1}^{\alpha_1}(\eta, \omega)$$

for every $\ell \in \{1, \dots, k\}$.

One key fact about this version of the dynamics is that it provides a lower bound on the total odometer. Recall the odometer function m_I for the classical dynamics, and for any sequence α of half-topplings, let $m_I(\eta, \omega, \alpha, x)$ denote the corresponding half-toppling odometer function, after performing all half-topplings in α started from the configuration (η, ω). We have:

Lemma 2.2. *(Abelian lemma for half-topplings) Fix a particle configuration η on a finite interval I. Let α be any legal half-toppling sequence for $(\eta, \vec{0})$, and let $\overline{\alpha}$ be any legal toppling sequence for η (in the sense of Section 2.1) such that $\xi_{\overline{\alpha}}\eta$ has no site with at least two particles. Then for any $x \in I$,*

$$m_I(\eta, \vec{0}, \alpha, x) \leqslant 2m_I(\eta, \overline{\alpha}, x).$$

We omit the proof, which follows identically to that of [17, Lemma 6].

2.3 Outline

Our strategy is to alternate between two types of legal half-toppling sequences, taking place on a sequence of nested intervals in $\mathbb{Z}$.

The first of these is called the 'carpet/hole toppling procedure' and is essentially a renormalization scheme of the SSM, detailed in Chapter 3. The procedure starts from and remains in a very particular set of configurations, and runs until it partially stabilizes the interval – when there are no more legal topplings among a small class of marked ('free') particles. We will show that each iteration is well behaved – that many particles reach the boundary – with exponentially high probability. The main technical hurdle in analyzing the carpet/hole procedure is the single block estimate Lemma 4.3, which we deal with in Chapters 6, 7 and 8. In Chapter 4 we use the single block estimate to prove a version of the above statement, via an energy-entropy calculation similar to [10].

The carpet/hole procedure needs to start with relatively nice configurations. Thus when partial stabilization is achieved on an interval, before restarting the carpet/hole procedure on the larger interval, it is necessary to perform the other 'bootstrap' toppling procedure to restore the configuration. In Chapter 5 we analyze the alternation of both procedures and prove Theorem 1.1.

In the remaining chapters, we focus on the carpet/hole dynamics inside a single block. In Chapter 6, to exploit the independence among random walks, we introduce an auxiliary process that only reveals partial information about the parity configuration. In Chapter 7, we turn to the intricate correlation between particles in our toppling procedure, by carefully controlling the typical number of zeros in the auxiliary process. Finally, in Chapter 8 we combine these results and prove Lemma 4.3.

Finally, in Chapter 9 we illustrate the argument in Chapter 4 through the Activated Random Walk model, following the exposition in [12].

Chapter 3

CARPET/HOLE TOPPLING PROCEDURE

The carpet/hole toppling procedure divides the space into *blocks* as well as long *transit regions* between blocks. Each particle is designated as one of two types: there are *carpet* particles which do not move and occupy all sites except for one location per block called *hole*, and *free* particles which are toppled repeatedly and legally thanks to the carpet particles. At some point, a free particle may turn into a carpet particle and vice versa, but the total number of free particles will be preserved throughout. In view of Lemma 2.2, we will make sure the half-topplings performed during the procedure are always legal. After the definition, we state in Proposition 3.2 the main fact that will be used regarding the carpet/hole toppling procedure.

3.1 Valid configurations

Fix integers $a > 0$ and $K = a^4$. The procedure described in this section is well-defined for any positive integer a, although later in Proposition 3.2 and Lemma 4.3 we will need to choose a large enough a. The toppling procedure needs to start from relatively nice particle configurations on a finite interval, which we call valid configurations.

Definition 3.1. *For any $n \in \mathbb{N}$, a pair of particle configuration and parity sequence (η, ω) on $D_n := (-K + a, nK)$ is **valid** if*

$$\eta = \sum_{i=-K+a+1}^{nK-1} \delta_i - \sum_{u \in U} \delta_u + \sum_{i=0}^{n-1} (b_{i,0}\delta_{iK} + b_{i,1}\delta_{iK+a})$$

for some $b_{i,0}, b_{i,1} \geqslant 0$ and some $U \subset \cup_{i=0}^{n-1} [iK, iK + a]$ satisfying

$$|U \cap [iK, iK + a]| \leqslant 1, \quad \textit{for } i = 0, 1, \ldots, n-1.$$

The subinterval $[iK, iK + a]$ is called the ith **block** for $i = 0, 1, \dots, n-1$, and the complement of the blocks are called **transit regions**. In other words, a valid configuration has single particles everywhere, except for at most one empty spot per block and some additional particles at the boundary of every block.

Fix a valid configuration (η, ω). In order to perform the carpet/hole toppling procedure starting from (η, ω), we need to identify the special site *hole* in each block based on (η, ω), as well as classify the particles of η into several categories. Note that the definitions given in the current Section 3.1 only apply to the starting configuration (η, ω). In Section 3.2, we will provide rules regarding how the holes move and particles switch types during the procedure.

Every block has exactly one **hole**, uniquely determined by the pair (η, ω). For each $i = 0, \dots, n-1$, if block i has a position with no particle, then the empty site is the hole of block i. If block i has a particle in every position, the leftmost site x in that block with $\omega(x) = 1$ is declared the hole. If there is no such x, then declare $iK + a$ to be the position of the hole.

With the definition of the holes, we allocate the particles in η into several types. By definition, each site in D_n (containing both blocks and transit regions) that does not have a hole should have at least one particle. We declare one particle at every such site to be a **carpet** particle. (If there are multiple particles at a site, we simply pick an arbitrary one and declare it a carpet particle.) All the remaining particles in D_n are **free** particles.

In the configuration η, free particles are either in a hole or at an endpoint of a block. Free particles are further divided into two groups. For each $i = 0, \dots, n-1$, if block i has $\eta(x) \geqslant 1$ and $\omega(x) = 0$ for all $x \in [iK, iK + a]$, then the hole was defined to be at $iK + a$ and we declare an arbitrary (free) particle at $iK + a$ to be a **frozen** free particle. All the other free particles are **thawed**. There is always one special thawed particle, the **hot** particle, that actually gets toppled. We will explain the rule of determining the hot particle in Section 3.2.

3.2 Carpet/hole dynamics

We now formally define the carpet/hole toppling procedure inside a given interval $D_n = (-K+a, nK)$. Fix a valid configuration (η, ω). Suppose we've picked the starting locations of the holes and labelled the particles as 'carpet', 'free', 'frozen' and 'thawed' according to Section 3.1. The toppling procedure works as follows:

(L1) We follow a *leftmost priority policy* for choosing the hot particle, which implies Lemma 4.2 below. Find the leftmost block i that contains a thawed particle. If no such block exists, the procedure ends and we say the procedure reaches *partial stabilization*. Among the thawed particles in block i, we choose the one inside the hole to be the hot particle if there is one. Otherwise, the thawed particles are at the boundary iK or $iK+a$, and we declare an arbitrary thawed particle the hot one. This particular order of choice is to ensure (F2) below. Once we choose the hot particle, we proceed with (L2) or (L3) depending on whether block i contains a frozen particle.

(L2) Suppose block i does not contain a frozen particle. If the hot particle is at the hole but the hole is not the leftmost position y with parity value $\omega(y) = 1$, then go directly to (L2a) or (L2b). Otherwise, repeatedly topple the hot particle until it returns to the hole of block i, or hits either $(i-1)K+a$ or $(i+1)K$, which could be a boundary point of a neighboring block or a boundary point of the interval D_n. There are three cases:

(L2a) Suppose the hot particle returned to the hole of block i and there exists some position in the block i with odd parity. Find the leftmost position y in block i with parity value $\omega(y) = 1$. Declare the hot particle at the hole a carpet particle, move the hole in block i to position y, and declare the carpet particle at y a thawed and hot particle. Return to (L2).

(L2b) Suppose the hot particle returned to the hole of block i, but there is no site in block i with odd parity. Turn the hot particle at the hole into a carpet particle, move the hole to position $iK + a$, and declare the carpet particle at $iK + a$ a frozen free particle. Return to (L1).

(L2c) If the hot particle reached $(i-1)K + a$ or $(i+1)K$, declare it no longer hot but still free and thawed. Return to (L1).

(L3) If block i does contain a frozen particle, then every site in block i has a particle which is not the hot particle. Repeatedly topple the hot particle until it reaches $(i-1)K+a$ or $(i+1)K$. Then declare it no longer hot but still free and thawed. If block i now has the leftmost site y with parity value $\omega(y) = 1$, then turn the frozen free particle at $iK + a$ into a carpet particle, move the hole to y, and declare the carpet particle at x free and thawed. Return to (L1).

The block size a is large, so the typical occurrence is (L2a): a hot particle makes an excursion inside its block. Our aim is to show that step (L2b) occurs much *less* often than (L2c).

More broadly, what we will show is that the carpet/hole toppling procedure, as a renormalization scheme of the Stochastic Sandpile Model, 'converges' to the Activated Random Walk model (ARW) with a small sleep rate when the block size goes to infinity. Each block and every free particle in our procedure correspond to a single site and a normal particle in the ARW model respectively. Step (L2c) mimics an ARW particle moving to a neighboring site, whereas the rare (L2b) step represents an ARW particle becoming sleepy with a small sleep rate. If (L2b) does happen and a free particle becomes frozen, it can become thawed through (L3), just as a sleepy ARW particle gets reactivated by the presence of another particle.

Once we make the 'convergence' rigorous in certain sense, the analysis of a 1D ARW-like system with a small sleep rate yields to a classical energy-entropy argument [12]. The overall outcome of these is Proposition 3.2, stated at the end of Chapter 3.

The following properties hold trivially for the initial configuration (η, ω) and are preserved by the half-toppling sequence, which can be checked by induction.

(P1) Each block i has exactly one hole which is located at some site $x \in [iK, iK + a]$.

(P2) Whenever the hot particle is inside the hole of a block without a frozen particle, we always make sure this hole is located at the leftmost site y in that block with parity value $\omega(y) = 1$. If there is no site in block i with odd parity, then we make sure the hole is at $iK + a$ and contains a frozen particle.

(P3) There is exactly one carpet particle at every site in D_n except for the holes.

(P4) All free particles except the hot particle are in a hole, or at a site iK or $iK + a$ for some i.

(P5) We call a block which contains a frozen particle a **frozen block**. In a frozen block, both the frozen particle and the hole are at $iK + a$. Thus every site in this block has a particle that is not hot.

We also collect some facts that will be useful throughout the paper.

(F1) All half-topplings performed during this procedure are legal.

(F2) After the first free particle has been designated hot in block i, all free particles in block i except the hot particle (if there is one) are at iK or $iK + a$.

(F3) A free particle designated hot in a block with a frozen particle reaches a neighboring block before being declared not hot.

(F4) The number of free particles is conserved during the procedure.

(F5) When the procedure ends, all free particles are either frozen, with at most one frozen particle per block, or at the boundary points of D_n. Also, there is at most one particle (either carpet or frozen free particle) at every site inside D_n.

3.3 Bounds on frozen particles

We prove the following regarding the carpet/hole toppling procedure. Roughly speaking, it says that the dynamics is likely to sustain 'enough activity' so that at the end of the procedure, there are few frozen particles left inside every subinterval.

Let (η, ω) be a valid configuration on $D_n = (-K + a, nK)$. Run carpet/hole dynamics with initial condition (η, ω) until *partial stabilization* when there is no more thawed particles inside D_n. Denote its law by $\mathbb{P}_{(\eta,\omega)}$. For $0 \leqslant n_0 \leqslant n_1 \leqslant n-1$, define $\mathrm{Frozen}(n_0, n_1)$ to be the number of frozen free particles remaining in the blocks $n_0, n_0+1, \ldots, n_1$ at the end of the carpet/hole dynamics on D_n. Also let $m_n(\eta, \omega, x)$ be the (half-toppling) odometer function at site $x \in D_n$ resulting from the carpet/hole toppling procedure on D_n, and define

$$H_i := \{m_n(\eta, \omega, iK) \geqslant \beta n\},\ H(n_0, n_1) := H^c_{n_0-1} \bigcap \left(\bigcap_{i=n_0}^{n_1} H_i \right)$$

for some constant $\beta > 0$. The purpose of the event $H(n_0, n_1)$ is to ensure each block starting from a nice configuration and to facilitate a bootstrap argument later in the proof of Lemma 5.8. Write $H := H(0, n-1)$ and $\mathrm{Frozen} := \mathrm{Frozen}(0, n-1)$.

Proposition 3.2. *For $\delta = \beta = 4 \times 10^{-4}$ and any $L^* > 0$, there exist $c_1 > 0$ and sufficiently large $K = e^{1.8\times 10^5}$ such that the following hold for large enough $n' = n_1 - n_0 + 1$. If (η, ω) is valid and the total number of particles $\sum_{x \in D_n} \eta(x) \leqslant L^* |D_n|$, then we have*

$$\mathbb{P}_{(\eta,\omega)}(\{\mathit{Frozen}(n_0, n_1) > \delta n'\} \cap H(n_0, n_1)) \leqslant \exp(-c_1 n').$$

In particular,

$$\mathbb{P}_{(\eta,\omega)}(\{\mathit{Frozen} > \delta n\} \cap H) \leqslant \exp(-c_1 n).$$

Proposition 3.2 will be proved in the next chapter. We will rely on Proposition 3.2 to do the analysis in Chapter 5.

Chapter 4

FILTRATIONS AND COARSE-GRAINED PARTICLE FLOWS

Fix any initial configuration (η, ω) on D_n. Consider one iteration of the carpet/hole dynamics on an interval $D_n = (-K + a, nK)$ with n blocks. Recall our aim is to prove Proposition 3.2. The 'single block estimate' Lemma 4.3 roughly says that the probabilities of a block being frozen at certain stopping times are small. To combine these estimates for different blocks, it is necessary to introduce some independence between the blocks, and enumerate the possible trajectories of the system. Chapter 4 is devoted to these goals, and then giving a proof of Proposition 3.2 by using Lemma 4.3. Though more technical, the argument in this chapter is essentially the same as the one in [10].

4.1 Decorated stacks and filtration

Whenever a free particle is designated as the hot particle in step (L1), we designate the block i it is in as the **hot block**. For each $y \in D_n$ in a transit region, we split the stack instructions for y into two separate stacks, $\xi^y = (\xi^{y,L}, \xi^{y,R})$. Any free particle toppled at site y uses instructions from the L stack if the nearest block to the left of y is the hot block, and from the R stack if the nearest block to the right of y is the hot block. Sites y in a block have just a single instruction stack, ξ^y. We work with the filtration $\mathcal{F}_i$ of all the stack instructions decorated by blocks $j \leqslant i$, i.e.

$$\mathcal{F}_i = \sigma[\{\xi^y : y \in (-K + a, iK + a]\} \cup \{\xi^{y,L} : y \in [iK + a, (i + 1)K)\}].$$

4.2 Coarse-grained particle flows and mass balance equations

To index possible 'trajectories' of the carpet/hole dynamics, we count the flow of particles between blocks. For each $i \in \{0, \dots, n-1\}$ and $s \geqslant 0$ let

$$\eta^+(s, i) = \eta + s\delta_{iK+a}$$

be the configuration η with s additional particles at the right edge of block i. We define the following 'coarse-grained counters':

Definition 4.1. *For each $j \in \{0, \dots, n-1\}$, run carpet/hole dynamics started from configuration $(\eta^+(s, j), \omega)$ until there are no legal topplings of free particles in $(-K + a, jK)$, the first j blocks. Define*

- $L_i^j(s)$ = *the total number of times during this toppling procedure that a free particle goes left using an instruction from the stack $\xi^{(i-1)K+a+1,R}$ for any $i \leq j$;*
- $F_i^j(s)$ = *the number of frozen particle in block i at the end of the toppling procedure (either 0 or 1 by (F5)) for any $i \leq j$;*
- $F_i := F_i^{n-1}(0)$ *and* $L_i := L_i^{n-1}(0)$*. Set* $L_n := 0$*.*

Note that $F_i^j(s), L_i^j(s) \in \mathcal{F}_j$. Our leftmost priority policy for choosing the hot particle guarantees the next lemma.

Lemma 4.2. *We have, for each $i \in \{0, \dots, n-1\}$,*

$$F_i = F_i^i(L_{i+1}) \quad \textit{and} \quad L_i = L_i^i(L_{i+1}).$$

See the corresponding lemma in [10] for a proof. This allows us to study the number of frozen free particles in block i at the end of the procedure by running the carpet/hole dynamics up to block i, plus some random number of extra particles input from the right.

To avoid dependence, we make a uniform argument over all possible values of the input from the right. A vector of integers $\boldsymbol{s} = (s_0, s_1, \cdots, s_n = 0)$ is said to satisfy the *mass balance equations* if

$$L_i^i(s_{i+1}) = s_i \text{ for } i = 0, \ldots, n-1.$$

Note that the *random* vector $(L_0, L_1, \ldots, L_n = 0)$ satisfies the mass balance equations by definition. In what follows, we will sum over all possible vectors $\boldsymbol{s}$ satisfying these equations.

4.3 Single block estimate

The following Lemma is an upper bound on the number of frozen particles F_i in a fixed block. Chapters 6, 7 and 8 are devoted to its proof.

Lemma 4.3. *For $\delta = 4 \times 10^{-4}$, there exist sufficiently large constants $K = e^{1.8\times 10^5}$ and c, c' such that $\log c' < \delta c$, and the following holds for any $i \leqslant n$. If the initial configuration (η, ω) on $D_n = (-K + a, nK)$ is valid, then*

$$\sup_{l \geqslant 0} \sum_{s \geqslant 2} \mathbf{E}_{(\eta,\omega)}[e^{cF_i^i(s)} 1\{L_i^i(s) = l\} | \mathcal{F}_{i-1}] < c'.$$

Additionally, for any $k \geq 0$,

$$\sup_{l \geqslant 0} \mathbb{P}_{(\eta,\omega)} \left(\sum_{s \geqslant 0} 1\{L_i^i(s) = l\} > k \,\middle|\, \mathcal{F}_{i-1} \right) \leqslant \theta^k$$

for some $\theta \in (0, 1)$.

Note the lower limit of the summation over s in the first statement. Since we only assume the starting configuration to be valid, we need at least two input particles to 'fix' the configuration – see the proof of Lemma 4.3 in Chapter 8.

In this section we use the single block estimate to prove Proposition 3.2.

Proof of Proposition 3.2. We start by treating the special case where $n_0 = 0$ and $n_1 = n-1$. Let η be any valid carpet, and recall the events H_i that the odometer of site iK is at least

βn and H that all H_i's occur (note that $H_{-1} = \emptyset$). Also let J_i be the event that for the first βn many times a hot particle is toppled at site iK, at least twice it takes a step left and then do an excursion to site $(i-1)K + a$ before returning to site iK. Note that $J_i \in \mathcal{F}_i$ and

$$\mathbb{P}(J_i^c \mid \mathcal{F}_{i-1}) \leqslant \beta n \left(1 - \frac{1}{2K}\right)^{\beta n - 1} \leqslant \exp(-c_2 n) \tag{4.4}$$

for some $c_2 > 0$ depending only on K and β.

To bound Frozen, the number of frozen free particles remaining in D_n after the carpet/hole dynamics, we use the fact that $H_i \subset \{L_i \geqslant 2\} \cup J_i^c$. Since $\text{Frozen} = \sum_i F_i$,

$$\begin{aligned}
\mathbf{E}[e^{c\text{Frozen}} 1_H] &\leqslant e^c \cdot \mathbf{E}\left[\prod_{i=0}^{n-2} e^{cF_i} 1_{H_{i+1}}\right] \\
&\leqslant e^c \cdot \mathbf{E}\left[\prod_{i=0}^{n-2} \left(e^{cF_i} 1\{L_{i+1} \geqslant 2\} + e^c 1_{J_{i+1}^c} 1\{L_{i+1} < 2\}\right)\right].
\end{aligned}$$

Recall there are at most $L^*|D_n|$ particles inside D_n at the beginning. Let $c_4 = L^*(K+1)$. Using Lemma 4.2, we may rewrite the expectation as

$$\mathbf{E}\left[\sum_{s_0=0}^{c_4 n} \sum_{s_1} \cdots \sum_{s_{n-1}} \prod_{i=0}^{n-2} \left(e^{cF_i^i(s_{i+1})} 1\{L_{i+1} \geqslant 2\} + e^c 1_{J_{i+1}^c} 1\{L_{i+1} < 2\}\right) 1\{L_i^i(s_{i+1}) = s_i\} 1\{L_i = s_i\}\right]$$

$$\leqslant \mathbf{E}\left[\sum_{s_0=0}^{c_4 n} \sum_{s_1} \cdots \sum_{s_{n-1}} \pi_{n-1}\right] = \mathcal{S}(n-1),$$

where

$$\pi_k := \prod_{i=0}^{k-1} \left(e^{cF_i^i(s_{i+1})} 1\{L_i^i(s_{i+1}) = s_i\} 1\{s_{i+1} \geqslant 2\} + e^c 1_{J_{i+1}^c} 1\{s_{i+1} < 2\}\right) \in \mathcal{F}_k$$

and

$$\mathcal{S}(k) := \mathbf{E}\left[\sum_{s_0=0}^{c_4 n} \sum_{s_1} \cdots \sum_{s_k} \pi_k\right].$$

We will inductively show that $\mathcal{S}(k) \leqslant (c_4 n + 1)(c' + 4e^{2c - c_2 \wedge c_3 n})^k$ for constants c_2, c_3, c_4. The base case $k = 0$ is trivial and the case $k = 1$ follows from Lemma 4.3 and the estimate (4.4). Suppose that the inequality is true for $k-2$ and $k-1$. Let

$$U_i(s_{i+1}, s_i) := e^{cF_i^i(s_{i+1})} 1\{L_i^i(s_{i+1}) = s_i\} \in \mathcal{F}_i \quad \text{and} \quad V_i := e^c 1_{J_i^c} \in \mathcal{F}_i.$$

Decomposing the last two sums depending on whether $S_{k-1} \geq 2$ and $S_k \geq 2$, we get

$$\begin{aligned}\mathcal{S}(k) = &\mathbf{E}\left[\sum_{s_0=0}^{c_4 n}\sum_{s_1}\cdots\sum_{s_{k-1}}\pi_{k-1}\cdot\sum_{s_k=0,1}V_k\right]\\ &+\mathbf{E}\left[\sum_{s_0=0}^{c_4 n}\sum_{s_1}\cdots\sum_{s_{k-2}}\pi_{k-2}\cdot\sum_{s_{k-1}\geqslant 2}U_{k-2}(s_{k-1},s_{k-2})\sum_{s_k\geqslant 2}U_{k-1}(s_k,s_{k-1})\right]\\ &+\mathbf{E}\left[\sum_{s_0=0}^{c_4 n}\sum_{s_1}\cdots\sum_{s_{k-2}}\pi_{k-2}\cdot\sum_{s_{k-1}=0,1}\sum_{s_k\geqslant 2}V_{k-1}U_{k-1}(s_k,s_{k-1})\right].\end{aligned}$$

By conditioning on $\mathcal{F}_{k-1}$ and using (4.4), the first sum is bounded above by $\mathcal{S}(k-1)\cdot 2e^{c-c_2 n}$. Similarly, the second sum is at most $\mathcal{S}(k-1)\cdot c'$ by conditioning on $\mathcal{F}_{k-2}$ and using the first part of Lemma 4.3. For the last sum, we have

$$\begin{aligned}&\mathbf{E}\left[\sum_{s_{k-1}=0,1}\sum_{s_k\geqslant 2}V_{k-1}U_{k-1}(s_k,s_{k-1})\middle|\mathcal{F}_{k-2}\right]\\ &\leqslant 2e^{2c}\mathbf{E}\left[\left(\sum_{s_k}1\{L_{k-1}^{k-1}(s_k)=s_{k-1}\}\right)\cdot 1_{J_{k-1}^c}\middle|\mathcal{F}_{k-2}\right].\end{aligned}$$

It follows from (4.4) and the second part of Lemma 4.3 that the third sum is upper bounded $\mathcal{S}(k-2)\cdot 2e^{2c-c_3 n}$ for some $c_3 > 0$. Combining all three bounds finishes the inductive step.

To bound Frozen, we apply Markov's inequality together with the bound on $\mathcal{S}(n-1)$ to get

$$\mathbb{P}(\{\text{Frozen} > \delta n\}\cap H) \leqslant (c_4 n + 1)\exp(c + (\log(c' + 4e^{2c-c_2\wedge c_3 n}) - \delta c)n).$$

By the assumption $\log c' < \delta c$, the event in question is exponentially unlikely in n. This completes the proof of the case $n_0 = 0$ and $n_1 = n-1$.

The proof of the general case is almost the same as the above case. The main difference is that when $n_0 \neq 0$, the counter L_{n_0} no longer enjoys the deterministic upper bound as $L_0 \leqslant c_4 n$. Instead it suffices to argue that $H_{n_0-1}^c$ and $L_{n_0} > 4\beta/(1-\theta)$ occur simultaneously with probability exponentially small in n. By the second part of Lemma 4.3, out of every two added particles at $(n_0-1)K + a$, with probability at least $1-\theta$ some particle reaches

$(n_0 - 2)K + a$ and thus visits $(n_0 - 1)K$. Then a standard concentration bound gives the claim. $\square$

Chapter 5

INDEPENDENT STARTING CONFIGURATIONS

In this chapter we shall use Proposition 3.2 to prove Theorem 1.1. The main challenge is that Proposition 3.2 requires a valid initial configuration, as well as sufficient activity in every block as stipulated in the event H, whereas in Theorem 1.1 we directly start from an independent configuration. We bridge the gap by giving an explicit toppling procedure, alternating between legal IDLA steps and the carpet/hole toppling procedure run on a nested sequence of intervals. The particles collected at the boundary of a smaller interval help restore a valid configuration and ensure sufficient activity on a larger interval, which, in turn, guarantees that enough particles reach the endpoints of the larger interval. We will show that this procedure runs forever with positive probability.

Recall that $K = a^4$ is the period of the blocks in the carpet/hole procedure. Throughout Chapter 5, we use the same $K = e^{1.8\times 10^5}$ from Proposition 3.2 and Lemma 4.3. We will prove the following re-statement of Theorem 1.1.

Theorem 5.1. *Let μ be a probability distribution supported on finitely many non-negative integers with mean $p \in (1 - \frac{1}{3K}, 1)$ and $\sum_{j\geqslant 2}\mu(j) > 0$. Let $\{X(i)\}_{i\in\mathbb{Z}}$ be i.i.d. random variables with distribution μ. Then the system with starting configuration $\{X(i)\}_{i\in\mathbb{Z}}$ stays active a.s.*

Proof of Theorem 1.1. Theorem 5.1 is a quantitative version of Theorem 1.1 with $\mu_c \leqslant 1 - \frac{1}{3K} \leqslant 1 - \exp(-2\times 10^5)$. It causes no loss of generality to assume that μ is supported on finitely many integers and $\mathbf{E}(\mu) < 1$. To see this, note that for any probability distribution μ on $\mathbb{N}$ satisfying $\mathbf{E}(\mu) > 1 - \frac{1}{3K}$ and $\sum_{j\geqslant 2}\mu(j) > 0$, one can find another distribution $\tilde{\mu}$ stochastically dominated by μ, such that $\tilde{\mu}$ is supported on finitely many integers, $\mathbf{E}(\tilde{\mu}) \in$

$(1 - \frac{1}{3K}, 1)$ and $\sum_{j \geqslant 2} \tilde{\mu}(j) > 0$. By monotonicity, if the stochastic sandpile with independent initial distributions $\tilde{\mu}$ does not fixate, then the model with distribution μ also does not fixate. □

Throughout the rest of this chapter, we assume μ and $\{X(i)\}_{i \in \mathbb{Z}}$ satisfy all the conditions in Theorem 5.1.

5.1 Partial stabilization on nested intervals

Define a sequence $\{\tilde{M}_i\}_{i \geqslant 0}$ with $\tilde{M}_{i+1} = \lfloor (\tilde{M}_i)(1+\gamma) \rfloor$ for some large $\tilde{M}_0$ with $\gamma = .02$. Let $M_i = (\tilde{M}_i + 1)K - a/2$. Consider a sequence of intervals of integers

$$\text{Interval}_i := \{-M_i, \ldots, M_i\},$$

which is a shifted version of the interval $D_{2\tilde{M}_i+1} = (-K+a, (2\tilde{M}_i+1)K)$ with $2\tilde{M}_i + 1$ many blocks. For $i \geqslant 1$, write

$$\text{Interval}_{\text{left}} = \{-M_i, \ldots, -M_{i-1} - 1\}$$

and

$$\text{Interval}_{\text{right}} = \{M_{i-1} + 1, \ldots, M_i\}.$$

Also define $S_i^+ := \sum_{j \in \text{Interval}_{\text{right}}} jX(j)$ and $S_i^- := \sum_{j \in \text{Interval}_{\text{left}}} jX(j)$. Then

$$\mathbf{E}(S_i^\pm) = \pm E(\mu)(M_i - M_{i-1})(M_i + M_{i-1} + 1)/2.$$

Finally, let $\text{Event}_{4,i}^\pm$ be the event that

$$|S_i^\pm - \mathbf{E}(S_i^\pm)| \leqslant .01\gamma \tilde{M}_i M_i.$$

The above notation Interval_i works for all $i \geqslant 0$. For $i = -1$, we use the convention that $\text{Interval}_{-1} := \{-a/2\}$. Here $-a/2$ is the left endpoint of the center block containing site zero. For $i = 0$, we write $\text{Interval}_{\text{left}} = \{-M_0, \ldots, -a/2-1\}$ and $\text{Interval}_{\text{right}} = \{-a/2+1, \ldots, M_0\}$.

We will inductively define partial stabilization procedures on the nested sequence of intervals $\{\text{Interval}_i\}_{i\geqslant 0}$ and the resulting configurations $\{Y_i(z)\}_{z\in\text{Interval}_i}$. Set $Y_{-1}(-a/2) = X(-a/2)$. Suppose $\{Y_{i-1}(z)\}_{z\in\text{Interval}_{i-1}}$ is defined, we may extend the definition $Y_{i-1}(z) := X(z)$ for all $z \notin \text{Interval}_{i-1}$, and then define $\{Y_i(z)\}_{z\in\text{Interval}_i}$ to be the configuration after the partial stabilization of Interval_i with initial configuration $\{Y_{i-1}(z)\}_{z\in\text{Interval}_i}$ and particles frozen when they get to the boundary of the interval. Let $u_i(z)$ be the site odometer at z in the partial stabilization of Interval_i from Y_{i-1}.

For $i \geqslant 1$, to go from Y_{i-1} to Y_i we do the partial stabilization in the following order:

- Run IDLA on the particles in the two intervals

  $$\text{Interval}_{\text{left}} \setminus \{-M_i\} \text{ and } \text{Interval}_{\text{right}} \setminus \{M_i\}$$

  and freeze particles at the boundaries $-M_i, -M_{i-1}, M_{i-1}$ or M_i. In other words, keep toppling every particle inside both intervals until it either becomes alone at its site or reaches the boundary.

- Run IDLA on the particles at $-M_{i-1}$ and M_{i-1}, freezing particles at $-M_i$ and M_i, until $\text{Interval}_{\text{left}} \setminus \{-M_i\}$ and $\text{Interval}_{\text{right}} \setminus \{M_i\}$ are completely filled or we run out of particles at $-M_{i-1}$ and M_{i-1}.

- If the configuration inside Interval_i becomes valid after the previous two steps, stabilize Interval_i according to the carpet/hole dynamics, freezing particles at $-M_i$ and M_i.

For $i = 0$, we perform a similar procedure by replacing the interval endpoints $-M_{i-1}$ and M_{i-1} in the first and second steps by $-a/2$.

Let $Z_{i,1}$, $Z_{i,2}$ and $Z_{i,3} = Y_i$ the configurations after each of these three steps at stage i respectively. Each of these configurations has at most one particle per site inside Interval_i except at $-M_{i-1}$ and M_{i-1} (or $-a/2$ when $i = 0$).

Let $L^* := \max\{j : \ \mu(j) > 0\} \geqslant 2$, and let Stage_{-1} be the event that

- every site in Interval_0 contains L^* particles initially,

which occurs with small but positive probability. Suppose Stage_{i-1} happens, we will define the event Stage_i inductively. Conditioned on Stage_{i-1}, we will observe the following typical behavior during the partial stabilization on Interval_i:

- After we run IDLA on the particles in

$$\text{Interval}_{\text{left}} \setminus \{-M_i\} \text{ and } \text{Interval}_{\text{right}} \setminus \{M_i\},$$

  the density of sites inside each of those intervals that is covered by a particle is between $1 - 1/(3K)$ and 1. We also expect that $\text{Event}_{4,i}^+ \cap \text{Event}_{4,i}^-$ occurs when $i \geqslant 1$. Call the intersection of these three events to be $\text{Event}_{1,i}$.

- Then we run IDLA on the particles at $-M_{i-1}$ and M_{i-1} (or $-a/2$ when $i = 0$) until $\text{Interval}_{\text{left}} \setminus \{-M_i\}$ and $\text{Interval}_{\text{right}} \setminus \{M_i\}$ are completely filled. We expect that there are at least $.2\tilde{M}_i$ particles left at both $-M_{i-1}$ and M_{i-1} (or $-a/2$ when $i = 0$) at the end of this step. Call this event $\text{Event}_{2,i}$.

- If the typical events occur up to this point, by definition the configuration inside Interval_i should be valid and thus we may carry out the carpet/hole toppling procedure. After we stabilize according to the carpet/hole dynamics, there will be less than $\delta(2\tilde{M}_i + 1) = .0004(2\tilde{M}_i + 1)$ blocks without a hole in Interval_i. At both $-M_i$ and M_i there will be at least $\tilde{M}_i/4$ particles. Moreover, the odometer $u_i(z)$ at the left endpoint $z = -a/2 + m'K$ of every block $m' \in \{-\tilde{M}_i, \ldots, \tilde{M}_i\}$ during this carpet/hole procedure is at least $\beta(2\tilde{M}_i + 1) = .0004(2\tilde{M}_i + 1)$. Call this event $\text{Event}_{3,i}$.

When all these three events happen, we call the procedure successful at stage i and define inductively

$$\text{Stage}_i := \text{Stage}_{i-1} \cap \text{Event}_{1,i} \cap \text{Event}_{2,i} \cap \text{Event}_{3,i}.$$

We will show that the event $\text{Stage}_\infty := \cap_{i\geqslant -1}\text{Stage}_i$ happens with positive probability. Once this is proved, we can show that the odometer lower bound in the definition of $\text{Event}_{3,i}$ implies the system stays active almost surely, thus proving Theorem 5.1.

The goal of the rest of Chapter 5 is to show that $\mathbb{P}(\text{Stage}_\infty) > 0$. The first two steps of each stage are straightforward IDLA processes. We will give lower bounds on the probabilities of $\text{Event}_{1,i}$ and $\text{Event}_{2,i}$ in Lemmata 5.2 and 5.3. The stabilization in the third step is more involved and most of the work in this chapter will come in bounding the probability of $\text{Event}_{3,i}$. Other than Proposition 3.2, we will need the 'center of mass' calculation in Lemmata 5.4–5.6 that guarantees enough particles reaching both endpoints. In the proof of Lemma 5.8 we will also carry out a bootstrap argument which, starting from the blocks $\pm\tilde{M}_{i-1}$, proves the odometer of every block is high. Since the definition and analysis of stage 0 are slightly different from those of stages $i \geqslant 1$, we only treat stage $i \geqslant 1$ in all lemmata of Chapter 5 but discuss the modifications for stage 0 at the end of the chapter.

5.2 IDLA steps

We start by bounding the probability that the first step is successful.

Lemma 5.2. *Let $X(i)$ be i.i.d. with distribution μ satisfying the conditions in Theorem 5.1. There exists $c > 0$ such that for M_i large enough,*

$$\mathbb{P}\left(\mathit{Event}_{1,i}\right) = \mathbb{P}\left(\mathit{Event}_{1,i} \mid \mathit{Stage}_{i-1}\right) > 1 - e^{-cM_i}.$$

Proof. First note that $\text{Event}_{1,i}$ and Stage_{i-1} are independent as they were defined on disjoint sets of independent random variables. This justifies the equality.

Next, notice that S_i^+ and S_i^- are the sums of less than M_i independent random variables each bounded by CM_i for some C, as μ is finitely supported. Since M_i and $\tilde{M}_i$ differ by a constant, standard concentration bounds give that

$$\mathbb{P}\left(\text{Event}_{4,i}^+ \cap \text{Event}_{4,i}^-\right) > 1 - e^{-cM_i}$$

for some $c > 0$.

Recall $\{Z_{i,1}(j)\}_{j\in\{M_{i-1},\ldots,M_i\}}$ is the sequence generated by running IDLA on $\{X(j)\}_{j\in\{M_{i-1},\ldots,M_i\}}$ with particles frozen on the boundaries. If $Z_{i,1}(M_{i-1}) = k$, then there exists $y \geqslant 0$ such that

$$\sum_{M_{i-1}}^{M_{i-1}+y} X(j) \geqslant y + k.$$

The random variables $\{X(i)\}$ have mean less than 1 and are bounded and i.i.d. So the probability of the previous inequality is decreasing exponentially in k and y. Summing up over all y gives that the probability that $Z_{i,1}(M_{i-1}) = k$ is exponentially small in k.

If

$$\sum_{M_{i-1}+1}^{M_i-1} Z_{i,1}(j) \leqslant (1 - 1/(3K))(M_i - M_{i-1} - 1),$$

then one of the following events must be true:

1. $\sum_{M_{i-1}+1}^{M_i-1} X_i(j) \leqslant (1/2)(\mathbf{E}(\mu) + 1 - 1/(3K))(M_i - M_{i-1} - 1)$;

2. there exists $k \geqslant (1/4)(\mathbf{E}(\mu) - (1 - 1/(3K)))(M_i - M_{i-1} - 1)$ and $y \geqslant 0$ such that $\sum_{M_{i-1}}^{M_{i-1}+y} X(j) = y + k$;

3. there exists $k \geqslant (1/4)(\mathbf{E}(\mu) - (1 - 1/(3K)))(M_i - M_{i-1} - 1)$ and $y \geqslant 0$ such that $\sum_{M_i-y}^{M_i} X(j) = y + k$.

By standard estimates on sums of independent bounded random variables, the probability of the first event is decreasing exponentially in M_i. The probabilities of the second and third events are decreasing exponentially in M_i by the argument earlier in the lemma. This completes the proof for $\text{Interval}_{\text{right}}$. The proof for $\text{Interval}_{\text{left}}$ is similar. □

Next we condition on $\text{Stage}_{i-1} \cap \text{Event}_{1,i}$ and bound the probability that the second step is successful.

Lemma 5.3. *For $\gamma = .02$, there exists $c > 0$ such that for M_i large enough,*

$$\mathbb{P}({Event_{2,i}}^C \mid Stage_{i-1} \cap Event_{1,i}) < e^{-cM_i}.$$

Proof. By the definition of $\text{Event}_{1,i}$ and Stage_{i-1} respectively,

$$\#\{j \in \text{Interval}_{\text{right}} \setminus \{M_i\} \ : \ Z_{i,1}(j) = 0\} < (\tilde{M}_i - \tilde{M}_{i-1})/3,$$

and

$$\#\{j \in \{M_{i-1} - (M_i - M_{i-1}), \ldots, M_{i-1}\} \ : \ Z_{i,1}(j) = 0\} \leqslant \tilde{M}_i - \tilde{M}_{i-1},$$

since $Z_{i,1}(j) = Y_{i-1}(j)$ for $j \in \text{Interval}_{i-1}$ and the configuration Y_{i-1} remains valid after the carpet/hole dynamics of stage $i-1$.

If $\text{Event}_{2,i}$ fails, then there are less than $.2\tilde{M}_i \leqslant .2(1+\gamma)\tilde{M}_{i-1}$ particles at M_{i-1} (or $-M_{i-1}$ when $i = 0$) when the second step ends. We only treat the case where the event is violated at M_{i-1} because the other case would be similar. On the event Stage_{i-1} we have at least $.25\tilde{M}_{i-1}$ particles initially at M_{i-1}, so on $\text{Stage}_{i-1} \cap \text{Event}_{1,i} \cap {\text{Event}_{2,i}}^C$, the number of particles released from M_{i-1} during the second step must be at least

$$\begin{aligned}
.25\tilde{M}_{i-1} - .2(1+\gamma)\tilde{M}_{i-1} &> .045\tilde{M}_{i-1} \\
&> 2\gamma\tilde{M}_{i-1} \\
&= 2((1+\gamma)\tilde{M}_{i-1} - \tilde{M}_{i-1}) \\
&\geqslant 2(\tilde{M}_i - \tilde{M}_{i-1})
\end{aligned}$$

for $\gamma = 0.02$. After $\tilde{M}_i - \tilde{M}_{i-1}$ particles have settled to the left of M_{i-1}, every site in $\{M_{i-1} - (M_i - M_{i-1}), \ldots, M_{i-1}\}$ has one particle. Thus the nearest vacancy to the right of M_{i-1} (if it exists) is at least as close as the nearest vacancy to the left of M_{i-1}, and the probability that each subsequent particle settles to the right of M_{i-1} is at least $1/2$. The probability that it takes at least $(\tilde{M}_i - \tilde{M}_{i-1})$ particles to get less than $(\tilde{M}_i - \tilde{M}_{i-1})/3$ of them settling to the right of M_{i-1} is exponentially unlikely in $\tilde{M}_i - \tilde{M}_{i-1}$ and thus in M_i. □

5.3 *Center of mass*

In this section, we carry out the 'center of mass' calculation, which is useful for bounding the probability that the third step is successful. Define

$$N_i := \sum_{j \in \text{Interval}_i} X(j) - [(2M_i + 1) - (2\tilde{M}_i + 1)].$$

This is the number of excess particles in Interval_i above the level of one hole per block.

Lemma 5.4. *If $\textit{Stage}_{i-1} \cap \textit{Event}_{1,i}$ occurs, then*

$$\begin{aligned} \sum_{j \in \textit{Interval}_i} j Z_{i-1,3}(j) \;\; &\geqslant \;\; -(2\tilde{M}_{i-1} + 1)(a/2) + .25\tilde{M}_{i-1}M_{i-1} \\ &\quad -(N_{i-1} - .25\tilde{M}_{i-1})M_{i-1} - .02\gamma\tilde{M}_i M_i. \end{aligned}$$

Proof. We break this sum up into sums over particles in Interval_{i-1}, $\text{Interval}_{\text{left}}$ and $\text{Interval}_{\text{right}}$. For the particles in Interval_{i-1}, we further partition them into three groups. The first group consists of all the carpet particles in $\text{Interval}_{i-1} \setminus \{-M_{i-1}, M_{i-1}\}$, with one particle at every location except for the holes. The second group is all the particles at M_{i-1}. The third group consists of all the remaining particles, which are all of the free particles in $\text{Interval}_{i-1} \setminus \{-M_{i-1}, M_{i-1}\}$ plus all the particles at $-M_{i-1}$.

The absolute value of the sum over the first group is at most

$$(2\tilde{M}_{i-1} + 1)(a/2).$$

As Stage_{i-1} occurs, we get that the sum of the locations over the second group is at least

$$.25\tilde{M}_{i-1}M_{i-1}.$$

in absolute value. By (F4), the number of particles in the third group is N_{i-1} minus the number of particles in the second group, which is at least $N_{i-1} - .25\tilde{M}_{i-1}$. Thus the sum over the third group is at least

$$(N_{i-1} - .25\tilde{M}_{i-1})(-M_{i-1}).$$

For the particles in $\text{Interval}_{\text{left}}$ and $\text{Interval}_{\text{right}}$, since $\text{Event}_{1,i}$ occurs, so does $\text{Event}^+_{4,i} \cap \text{Event}^-_{4,i}$. By the definition of those events, we get that the absolute value of the both sums combined is at most $.02\gamma\tilde{M}_i M_i$. Combining these estimates proves the lemma. □

Let $\daleth_i$ be the event that there are at most $\delta(2\tilde{M}_i + 1)$ blocks without a hole at the end of stage i. The next lemma says that if the majority of free particles exit through the endpoints, the proportion of the particles leaving from each side cannot become more unbalanced without causing a significant shift in the center of mass.

Lemma 5.5. *Let $\gamma = .02$. The following holds for $\delta = 4 \times 10^{-4}$ and sufficiently large M_i. On the event*

$$Stage_{i-1} \cap Event_{1,i} \cap Event_{2,i} \cap \daleth_i \cap \{Z_{i,3}(M_i) < .25\tilde{M}_i\},$$

we have

$$D_i := \sum_{j \in Interval_i} jZ_{i-1,3}(j) - jZ_{i,3}(j) > (\gamma/10K)M_i^2.$$

Proof. First we show that under the same hypotheses,

$$\begin{aligned}\sum_{j\in\text{Interval}_i} jZ_{i,3}(j) &< (2\tilde{M}_i+1)(a/2) + (.25\tilde{M}_i + \delta(2\tilde{M}_i+1))M_i \\ &\quad - \Big(N_{i-1} + (4/3)(\tilde{M}_i - \tilde{M}_{i-1}) - .25\tilde{M}_i - \delta(2\tilde{M}_i+1)\Big)M_i.\end{aligned}$$

The proof of this fact proceeds in a similar manner to Lemma 5.4. We decompose the particles into three groups: the carpet particles in $\text{Interval}_i \setminus \{-\tilde{M}_i, \tilde{M}_i\}$, the free particles in $\text{Interval}_i \setminus \{-\tilde{M}_i, \tilde{M}_i\}$ plus the particles at M_i, and the particles at $-M_i$. The sum of the locations of the carpet particles is at most

$$(2\tilde{M}_i + 1)(a/2).$$

Since $\daleth_i \cap \{Z_{i,3}(M_i) < .25\tilde{M}_i\}$ occurred, the sum of the locations of the second group is at most

$$(.25\tilde{M}_i + \delta(2\tilde{M}_i + 1))M_i.$$

By $\text{Event}_{1,i}$ we have $N_i \geqslant N_{i-1} + 2(2/3)(\tilde{M}_i - \tilde{M}_{i-1})$, so there are at least

$$N_{i-1} + (4/3)(\tilde{M}_i - \tilde{M}_{i-1}) - (.25\tilde{M}_i + \delta(2\tilde{M}_i + 1))$$

many particles at $-M_i$, each contributing $-M_i$ to the sum. Putting these estimates together gives the desired upper bound.

Combining this result with Lemma 5.4, we get

$$\begin{aligned}
&\sum_{j \in \text{Interval}_i} jZ_{i-1,3}(j) - jZ_{i,3}(j) \\
&\quad > \quad (4/3)(\tilde{M}_i - \tilde{M}_{i-1})M_i - .5(\tilde{M}_i M_i - \tilde{M}_{i-1}M_{i-1}) + N_{i-1}(M_i - M_{i-1}) \\
&\qquad\quad -2\delta(2\tilde{M}_i + 1)M_i - .02\gamma\tilde{M}_i M_i - (2\tilde{M}_i + 1)a \\
&\quad > \quad (1.3 - 1.1 + 0 - .08 - .02)\gamma\tilde{M}_i M_i \\
&\quad > \quad (\gamma/10K)M_i^2,
\end{aligned}$$

for $\delta = 0.0004 = 0.02\gamma$ and M_i sufficiently large. □

Lastly, we show that with high probability there is not enough time for the center of mass to reach a displacement of the size implied by Lemma 5.5. Recall the notation D_i from the last lemma.

Lemma 5.6. *There exists $c > 0$ such that for all large enough M_i,*

$$\mathbb{P}(|D_i| > M_i^{1.6} \mid \mathit{Stage}_{i-1}) \leqslant e^{-cM_i^{0.1}}.$$

Before proving Lemma 5.6, we also state the following variant of it which will be useful in another instance of 'center of mass' argument. For $m'_0, m'_1 \in \mathbb{Z}$ such that $-\tilde{M}_i \leqslant m'_0 \leqslant m'_1 \leqslant \tilde{M}_i$, let $\hat{m}_0 := (m_0 - 1)K + a/2$ and $\hat{m}_1 := (m_1 + 1)K - a/2$, and define $\rho_{m'_0,m'_1}(j) : \text{Interval}_i \to \mathbb{Z}$ to be the piecewise linear function $\rho_{m'_0,m'_1}(j) := j$ when $j \in \{\hat{m}_0, \ldots, \hat{m}_1\}$ and $\rho_{m'_0,m'_1}(j) := \hat{m}_0$ (resp. $\hat{m}_1$) when $j < \hat{m}_0$ (resp. $j > \hat{m}_1$). Finally, we define

$$D'_{i,m'_0,m'_1} := \sum_{j \in \text{Interval}_i} \rho_{m'_0,m'_1}(j)Z_{i,2}(j) - \rho_{m'_0,m'_1}(j)Z_{i,3}(j).$$

Different from D_i, the above definition records the change of a restricted version of center of mass in the *third* step of stage i. Also, recall that $u_i(z)$ denotes the odometer at $z \in \mathrm{Interval}_i$ during the third step of stage i.

Lemma 5.7. *For $-\tilde{M}_i \leqslant m_0' \leqslant m_1' \leqslant \tilde{M}_i$, there exists $c > 0$ such that for all large enough M_i,*

$$\mathbb{P}(|D'_{i,m_0',m_1'}| > M_i^{1.6} + \max\{u_i(\hat{m}_0), u_i(\hat{m}_1)\} \mid \mathit{Stage}_{i-1} \cap \mathit{Event}_{1,i} \cap \mathit{Event}_{2,i}) \leqslant e^{-cM_i^{0.1}}.$$

Proof of Lemmata 5.6 and 5.7. We start with Lemma 5.7. If the inequality in question occurs, then either (1) the carpet/hole dynamics from $Z_{i,2}$ to $Z_{i,3}$ takes more than $M_i^{3.1}$ many topplings, or (2) the restricted center of mass, i.e. the sum of $\rho_{m_0',m_1'}$ function values of all particles' locations, moves by at least $M_i^{1.6} + \max\{u_i(\hat{m}_0), u_i(\hat{m}_1)\}$ during the carpet/hole dynamics, which undergoes at most $M_i^{3.1}$ topplings. It suffices to bound the probabilities of both of them.

On one hand, since μ is finitely supported, there is a deterministic upper bound on the number of particles inside $\mathrm{Interval}_i$ that is linear in M_i. So if the system takes at least $M_i^{3.1}$ steps, then there exists one particle that moved $CM_i^{2.1}$ many times for some $C > 0$. As particles are frozen at $\pm M_i$, it must have done so without hitting these two points. For each particle, this has probability at most $e^{-cM_i^{0.1}}$ for some $c > 0$. By a union bound over all particles in $\mathrm{Interval}_i$, the first event has probability at most $e^{-cM_i^{0.1}}$ for some $c > 0$ and large enough M_i.

For the second event, let τ be the total number of topplings taken during the third step of stage i, and let $\mathcal{L}(t) \in \mathrm{Interval}_i$ and $\sigma(t) \in \{-1, +1\}$ be the starting location and direction of the tth toppling respectively. We may write

$$\begin{aligned} D'_{i,m_0',m_1'} = &\sum_{t=1}^{\tau} \sigma(t) 1\{\hat{m}_0 < \mathcal{L}(t) < \hat{m}_1\} \\ &+ \sum_{t=1}^{\tau} 1\{\mathcal{L}(t) = \hat{m}_0, \sigma(t) = 1\} - \sum_{t=1}^{\tau} 1\{\mathcal{L}(t) = \hat{m}_1, \sigma(t) = -1\}. \end{aligned}$$

The first sum with the upper limit τ replaced by some time variable $\tau' = 0, \ldots, \tau$ is a martingale with bounded differences. So by applying the Azuma's inequality and a union bound over times, the probability that $\tau \leqslant M_i^{3.1}$ and the first sum exceeds $M_i^{1.6}$ is at most $e^{-cM_i^{0.1}}$ for some $c > 0$ and large enough M_i. Since both sums in the second line above are at most $u_i(\hat{m}_0)$ and $u_i(\hat{m}_1)$ respectively, we get the desired bound on the second event. This completes the proof of Lemma 5.7.

For Lemma 5.6, note that by an almost identical argument, one could state and prove an analog of Lemma 5.7 that applies to the partial stabilization of the whole stage i (from $Z_{i-1,3}$ to $Z_{i,3}$) instead of just the third step ($Z_{i,2}$ to $Z_{i,3}$). So Lemma 5.6 is simply a special case of this analog where $m_0' = -\tilde{M}_i$ and $m_1' = \tilde{M}_i$. Note that the odometer term in the inequality disappears because no toppling starts from the boundary. □

5.4 Carpet/hole step

Finally, we bound the probability that the third step is successful.

Lemma 5.8. *Let $\gamma = .02$. There exist some $\alpha, c > 0$ such that for $\delta = \beta = 4 \times 10^{-4}$ and sufficiently large K, M_i, we have*

$$\mathbb{P}(\mathit{Event}_{3,i}{}^C \mid \mathit{Stage}_{i-1} \cap \mathit{Event}_{1,i} \cap \mathit{Event}_{2,i}) < e^{-cM_i^\alpha}.$$

Proof. We pick constants γ, δ, β, K and M_i as stated so that Proposition 3.2 and all previous lemmata in Chapter 5 are true. If $\text{Event}_{3,i}{}^C$ occurs, then one of these three events during the third step of stage i must happen:

1. the odometer $u_i(-a/2 + m'K)$ at the left endpoint of some block m' is less than $\beta(2\tilde{M}_i + 1)$;

2. the odometer $u_i(-a/2+m'K) \geqslant \beta(2\tilde{M}_i+1)$ for every block m' but Frozen $> \delta(2\tilde{M}_i+1)$;

3. Frozen $\leqslant \delta(2\tilde{M}_i + 1)$ but there are less than $.25\tilde{M}_i$ particles at either $-M_i$ or M_i.

The second event is exponentially unlikely in M_i by the second statement of Proposition 3.2. For the third event which we denote as $\mathcal{T}$, the 'center of mass' calculation, Lemmata 5.5 and 5.6, as well as its symmetric version shows that $\mathbb{P}(\text{Event}_{1,i} \cap \text{Event}_{2,i} \cap \mathcal{T} \mid \text{Stage}_i)$ is exponentially small in $M_i^{0.1}$. So by Lemmata 5.2 and 5.3 we get the desired bound on $\mathbb{P}(\mathcal{T} \mid \text{Stage}_i \cap \text{Event}_{1,i} \cap \text{Event}_{2,i})$. In the remainder of the proof, we focus on the bootstrap argument which bounds the first event.

As we are conditioning on $\text{Stage}_{i-1} \cap \text{Event}_{1,i} \cap \text{Event}_{2,i}$, there are a large number (at least $.2\tilde{M}_i$) of initial particles at $-M_{i-1}$ right before the third step of stage i. By the second statement of Lemma 4.3, with probability exponentially close to one, the odometer $u_i(-M_{i-1}-a)$ at the left endpoint of the block containing $-M_{i-1}$ must be at least $\beta(2\tilde{M}_i+1)$. So if there is some block whose left endpoint odometer is less than $\beta(2\tilde{M}_i + 1)$, then there exists some largest interval $I' = \{m'_0, \ldots, m'_1\}$ such that $-\tilde{M}_{i-1} \in I'$ and $u_i(-a/2 + m'K) \geqslant \beta(2\tilde{M}_i + 1)$ for any block $m' \in I'$.

First, we claim it is exponentially unlikely in M_i that $\text{Frozen}(m'_0, m'_1) > \delta(2\tilde{M}_i + 1)$. Indeed, by Proposition 3.2 for any interval $I'' = (m''_0, m''_1)$ satisfying $|I''| \geqslant \delta(2\tilde{M}_i + 1)$, the probability that $m''_0 = m'_0$ and $m''_1 = m'_1$ but $\text{Frozen}(m''_0, m''_1) > \delta(2\tilde{M}_i + 1)$ is exponentially small in M_i. The inequality $\text{Frozen}(m''_0, m''_1) < \delta(2\tilde{M}_i + 1)$ holds trivially for other intervals $I'' = (m''_0, m''_1)$ such that $|I''| < \delta(2\tilde{M}_i + 1)$, so taking a union bound over all such intervals proves the claim.

Next, we show that with exponentially high probability either $m'_0 = -\tilde{M}_i$ or $m'_1 = \tilde{M}_i$. Suppose not, then by definition both $u_i(-a/2 + (m'_0 - 1)K)$ and $u_i(-a/2 + (m'_1 + 1)K)$ are less than $\beta(2\tilde{M}_i + 1)$. So with the notation $\hat{m}_0$ and $\hat{m}_1$ from Lemma 5.7, the number of free particles that stayed inside $(\hat{m}_0, \hat{m}_1)$ before the third step of stage i but become at or to the right of $\hat{m}_1$ afterwards is at most $\beta(2\tilde{M}_i + 1) + 1$ by (F5). The same also holds for those becoming at or to the left of $\hat{m}_0$. Since there are at least $.2\tilde{M}_i$ particles inside $(\hat{m}_0, \hat{m}_1)$ before the third step, at least $.2\tilde{M}_i - 2\beta(2\tilde{M}_i + 1) - 2 \geqslant .19\tilde{M}_i$ particles remain frozen in this interval at the end of the carpet/hole procedure. This is exponentially unlikely in M_i by the above claim.

Thus we may assume, without loss of generality, that $m'_0 = -\tilde{M}_i$, as the case $m'_1 = \tilde{M}_i$ would follow from a symmetric argument. By the claim above we may also assume that $\text{Frozen}(m'_0, m'_1) \leqslant \delta(2\tilde{M}_i + 1)$ which happens with exponentially high probability. Our next goal is to show that if $m'_0 = -\tilde{M}_i$ and $\text{Frozen}(m'_0, m'_1) \leqslant \delta(2\tilde{M}_i + 1)$, then we have $m'_1 = \tilde{M}_i$ with exponentially high probability. This would give the bound on the first event, thus proving Lemma 5.8.

From now on we suppose that $m'_0 = -\tilde{M}_i$ and $\text{Frozen}(m'_0, m'_1) \leqslant \delta(2\tilde{M}_i+1)$ but $m'_1 < \tilde{M}_i$. We will show that this is exponentially unlikely in $M_i^{0.1}$ by carrying out another 'center of mass' calculation. Recall the notations from Lemma 5.7. By assumption, we have $\hat{m}_0 = -M_i$. Let $V'_{i,2}$ (resp. $V'_{i,3}$) be the set of locations of the carpet particles of $Z_{i,2}$ (resp. $Z_{i,3}$) inside $(\hat{m}_0, \hat{m}_1)$. Similar to Lemma 5.5,

$$\left| \sum_{j \in \text{Interval}_i} \rho_{m'_0,m'_1}(j)\delta_{V'_{i,2}}(j) - \rho_{m'_0,m'_1}(j)\delta_{V'_{i,3}}(j) \right| \leqslant (m'_1 - m'_0 + 1)a.$$

For the rest of the particles, we consider a new kind of sum with each location shifted by $-\hat{m}_0$ for simplicity. Let $R_{i,2}$ (resp. $R_{i,3}$) be the number of particles of $Z_{i,2}$ (resp. $Z_{i,3}$) in Interval_i at or to the right of $\hat{m}_1$. On one hand, as there are at least $.2\tilde{M}_i$ particles at $-M_{i-1}$ initially, we get

$$\sum_{j \in \text{Interval}_i} (\rho_{m'_0,m'_1}(j) - \hat{m}_0)(Z_{i,2}(j) - \delta_{V'_{i,2}}(j)) \geqslant (.2\tilde{M}_i - 1)(M_i - M_{i-1}) + R_{i,2}(\hat{m}_1 - \hat{m}_0).$$

On the other hand, by assumption we have

$$\sum_{j \in \text{Interval}_i} (\rho_{m'_0,m'_1}(j) - \hat{m}_0)(Z_{i,3}(j) - \delta_{V'_{i,3}}(j)) \leqslant (R_{i,3} + \delta(2\tilde{M}_i + 1))(\hat{m}_1 - \hat{m}_0).$$

By the definition of (m'_0, m'_1), we have $u_i(\hat{m}_1) < \beta(2\tilde{M}_i + 1)$, which implies $R_{i,3} - R_{i,2} < \beta(2\tilde{M}_i + 1) + 1$ by (F5). Using this fact and the inequality $\hat{m}_1 - \hat{m}_0 < 2M_i$, we combine the above estimates into

$$\begin{aligned} D'_{i,m'_0,m'_1} &> (.2\tilde{M}_i - 1)(M_i - M_{i-1}) - 2(\beta + \delta)(2\tilde{M}_i + 1)M_i - 2\tilde{M}_i a \\ &> (.19 - .16)\gamma\tilde{M}_i M_i \end{aligned}$$

$$> \ (\gamma/40K)M_i^2,$$

for $\beta = \delta = 0.02\gamma$ and M_i sufficiently large.

Lastly, since $u_i(\hat{m}_0) = 0$ and $u_i(\hat{m}_1) < \beta(2\tilde{M}_i + 1)$, Lemma 5.7 and a union bound over all intervals $I'' = (m_0'', m_1'')$ imply the above event occurs with exponentially small probability in $M_i^{0.1}$. This completes the 'center of mass' argument and thus the proof of Lemma 5.8. □

Proof of Theorem 5.1. As mentioned above, we've only proved Lemmata 5.2–5.8 for stage $i \geqslant 1$. We briefly discuss the counterparts and proofs for stage $i = 0$ before putting everything together. Recall the different definitions of $\mathrm{Interval}_{i-1}$, $\mathrm{Interval}_{\mathrm{left}}$, $\mathrm{Interval}_{\mathrm{right}}$, Stage_{i-1}, $\mathrm{Event}_{1,i}$ and $\mathrm{Event}_{2,i}$ when $i = 0$. If Stage_{-1} occurs, i.e. there are exactly $L^* \geqslant 2$ particles at every location in $\mathrm{Interval}_0$ before stage 0, then $\mathrm{Event}_{1,i}$ occurs almost surely as there will be exactly one particle at every site of $\mathrm{Interval}_{\mathrm{left}} \setminus \{-M_0\}$ and $\mathrm{Interval}_{\mathrm{right}} \setminus \{M_0\}$. Also, the second step of stage 0 will be trivial, so by a concentration bound on the IDLA particles in the first step, there will be at least $0.49(L^* - 1)(2M_0)$ particles at $-a/2$ with probability exponentially close to one. This shows that Lemmata 5.2 and 5.3 hold for $i = 0$.

For Lemma 5.4, in fact on the event Stage_{-1} we have $\sum_{j \in \mathrm{Interval}_0} jX(j) = 0$. In the proof of Lemma 5.5, by using the fact $N_0 \geqslant (L^* - 1)(2M_i + 1)$ and the counterpart of Lemma 5.4, we get $D_i > M_i^2$. Lemmata 5.6 and 5.7 work for all $i \geqslant 0$. Finally, essentially the same proof works for Lemma 5.8 by using the above counterparts of Lemmata 5.4 and 5.5 and replacing $-M_{i-1}$ by $-a/2$. In other words, Lemma 5.8 also holds for $i = 0$.

From the definition of Stage_i and Lemmata 5.2, 5.3, and 5.8 we get

$$\begin{aligned}
&\mathbb{P}({\mathrm{Stage}_i}^C \mid \mathrm{Stage}_{i-1}) \\
&\quad = \mathbb{P}({\mathrm{Event}_{1,i}}^C \mid \mathrm{Stage}_{i-1}) \\
&\qquad + \mathbb{P}({\mathrm{Event}_{2,i}}^C \mid \mathrm{Stage}_{i-1} \cap \mathrm{Event}_{1,i}) \\
&\qquad + \mathbb{P}({\mathrm{Event}_{3,i}}^C \mid \mathrm{Stage}_{i-1} \cap \mathrm{Event}_{1,i} \cap \mathrm{Event}_{2,i}) \\
&\quad < e^{-cM_i} + e^{-cM_i} + e^{-cM_i^\alpha}
\end{aligned}$$

for some $c, \alpha > 0$ and sufficiently large M_i. We have that Stage_{-1} happens with small but positive probability and M_i grows exponentially, so there exists a large enough M_0 such that

$$\mathbb{P}(\cap_{i \geqslant -1} \text{Stage}_i) > 0.$$

Thus by the definition of $\text{Event}_{3,i}$ the odometer at site $z = -a/2$ is infinite with positive probability. By Lemmata 2.1 and 2.2 and (F1), this implies

$$\mathbb{P}(\text{stochastic sandpile stays active}) = 1.$$

□

Chapter 6
CARPET PROCESSES

We now turn to analyzing the dynamics within a single block, which involves keeping track of the parity configuration as it evolves during the carpet/hole procedure. At a high level, our aim is to show that the parity configuration typically has many 1's. Chapters 6 and 7 are devoted to analyzing an auxiliary version of the particle dynamics, which allow us to unravel the complex combinatorics of our half-toppling procedure, by retaining some independence throughout the process. In particular, Chapter 6 provides a formulation of such independence, while Chapter 7 controls the possible impact of the remaining correlation. The outcomes are Lemmata 6.8 and 7.1. Finally, these two lemmata are used in Chapter 8 to prove Lemma 4.3 by bounding the probability that a block becomes frozen.

6.1 Carpet process and excursions

We focus on the dynamics happening inside a single block, with the hot particle at the hole initially. In Section 6.1, we shall recall some relevant aspects of the dynamics from Chapter 3 and provide the notations that will be used throughout Chapters 6, 7 and 8.

Recall that the block is made up of a string of a carpet particles, except for a single empty site – the hole – and some parity configuration, $\tilde{\omega} \in \tilde{\Omega} = \{0,1\}^{[a]}$. For the remainder of the analysis we shift coordinates so that the leftmost point of the block is position 0. For any such $\tilde{\omega}$, denote by $L(\tilde{\omega})$ the location of the leftmost 1 of $\tilde{\omega}$,

$$L(\tilde{\omega}) := \inf\{i : \tilde{\omega}(i) = 1\},$$

which is the position of hole when the hot particle is inside the hole. If $\tilde{\omega}$ is identically zero then we write $L(\tilde{\omega}) = a + 1$. Let $Q : [\mathrm{length}(Q)] \to \mathbb{Z}$ denote the path taken by the hot particle starting at position $L(\tilde{\omega})$ and ending on the first return to that site, where length(Q) is the number of steps taken in the path Q and $Q(u)$ is the position of Q at step u. Define Local_Q, the local time of Q, by

$$\text{Local}_Q(i) := \#\{u \in [\text{length}(Q) - 1] : \ Q(u) = i\}.$$

Then define Parity_Q, the parity of the local time of Q, by

$$\text{Parity}_Q(i) := \text{Local}_Q(i) \bmod 2.$$

Recall from Chapter 3 that our carpet process inside a single block is a Markov chain $\tilde{\omega}^t$, one step of which is defined as follows: Let $L^t = L(\tilde{\omega}^t) \in [0, a]$.

- (Do an excursion) The hot particle at L^t performs a random walk Q^t on $\mathbb{Z}$ that starts at L^t and ends on its first return to L^t.

- (Update the parity configuration) For each $i \in [a]$, set

$$\tilde{\omega}^{t+1}(i) = \tilde{\omega}^t(i) + \text{Parity}_{Q^t}(i) \bmod 2.$$

We will be only interested in the carpet process $\tilde{\omega}^t$ until the first time the block gets frozen. In Lemma 8.4, we will deal with what happens thereafter by using the renewal property of carpet processes.

To describe the random walk path Q^t, we introduce three types of random walk paths, with a right (resp. left) version for each kind, namely: for $L \in [0, a]$,

(Type 1) a simple random walk started from L and conditioned to hit K (resp. $-K + a$) without returning to L;

(Type 2) a simple random walk started at a (resp. 0) and conditioned to hit L without reaching K (resp. $-K + a$);

(Type 3) a right (resp. left) type 1 walk followed by a corresponding version of type 2 walk.

Most of the time, Q^t is a simple random walk excursion on $\mathbb{Z}$ starting and ending at L^t. However, when the hot particle reaches a neighboring block, by (F2) it respawns from one of the boundary points: Q^t could also be

1. *'long excursion'*: an emission to the left (resp. right) from L^t followed by a re-arrival from the left (resp. right) boundary, i.e. a path of type 3;

2. *'double-sided path'*: an emission to the left (resp. right) followed by a re-arrival from the right (resp. left) boundary, a union of type 1 and type 2 paths on opposite sides of L^t;

3. *'failed re-arrival'*: after the emission, the newly designated hot particle fails to arrive at L^t but makes another emission to a neighboring block.

Among the three corner cases, the 'long excursion' is very similar to a random walk excursion in terms of their change of the parities inside the block. In the following, we will simply refer to a long excursion as an *'excursion'*.

6.2 Random walk parities

We will need the following lemma regarding the parity function for a single random walk excursion. Note that the lemma no longer holds at $i = \text{Extreme}$ for excursions to the right, or at $i = L-1$ for those to the left – the parity in such case is determined by the conditioning.

Lemma 6.1. *Let Q be a SRW excursion on $\mathbb{Z}$ starting and ending at L. Let*

$$\mathit{Range} = \{i : \exists\, u \text{ with } Q(u) = i\}.$$

Let Extreme be its other extreme value of Range (besides L). If Extreme > L, then for any $i \in Range \setminus \{L, Extreme\}$,

$$\frac{1}{6} < \mathbb{P}\bigg(Parity_Q(i) = 0 \mid \{Parity_Q(j')\}_{j' \in [L,i)}, Extreme \bigg) < \frac{5}{6}.$$

Instead if Extreme < L, then for any $i \in Range \setminus \{L-1, L\}$,

$$\frac{1}{6} < \mathbb{P}\bigg(Parity_Q(i) = 0 \mid \{Parity_Q(j')\}_{j' \in [Extreme,i)}, Extreme \bigg) < \frac{5}{6}.$$

Proof. We only treat the case where the excursion goes to the right from L so that $L \leqslant Q(u) \leqslant \text{Extreme}$ for all $0 \leqslant u \leqslant \text{length}(Q)$, as the proof of the other case would be similar. Fix i, a sequence of parities $(x_j)_{j=L}^{i-1}$, and an extreme value $E > i$. For a deterministic excursion q away from L, we let $\tau_0(q) = \tau_0'(q) := 0$ and recursively define

$$\tau_j(q) := \inf\{t > \tau_{j-1}'(q) : q(t) = i\}$$

and

$$\tau_j'(q) := \inf\{t > \tau_j(q) : q(t) \in \{i-1, i+2\}\}.$$

Note that between $\tau_j(q)$ and $\tau_j'(q)$, the excursion q might bounces back and forth between i and $i+1$.

We will count the number of visits to i in these intervals conditioning on the entire rest of the path. Let $B = \cup_j(\tau_j(q), \tau_j'(q))$ and define $\Gamma(0) = 0$ and $\Gamma(j) = \inf\{j > \Gamma(j-1) : j \notin B\}$. We then define $Reduce_q(j) = q(\Gamma(j))$, which is the result of removing the bounces of q back and forth between i and $i+1$. Let $\tilde{q}$ be any deterministic excursion away from L such that $\text{Parity}_{\tilde{q}}(j) = x_j$ for $L \leqslant j \leqslant i-1$ and $max(\tilde{q}) = E$. Let $\hat{q} = Reduce_{\tilde{q}}$ and $\tilde{k} = \max\{k : \tau_k'(\tilde{q}) < \infty\}$. Consider $\tilde{Q}$ being distributed like Q conditioned on $Reduce_Q = \hat{q}$. For $1 \leqslant j \leqslant \tilde{k}$, let N_j be the number of visits of $\tilde{Q}$ to i in $[\tau_j(\tilde{Q}), \tau_j'(\tilde{Q}))$.

When $E \geqslant i+2$, a direct computation shows that $N_1, \dots, N_{\tilde{k}}$ are independent Geometric(3/4) random variables. Let

$$N = \sum_{j=1}^{\tilde{k}} \mathbf{1}\{N_j \text{ is odd}\}.$$

Then N has a Binomial(4/5) distribution and a direct computation shows that $1/5 \leqslant \mathbb{P}(N \text{ is even}) \leqslant 4/5$. Since $\text{Parity}_{\tilde{Q}}(i) = \text{Parity}(N)$, the result follows by unwinding the conditioning.

When $E = i+1$, we always have $q(\tau'_j(q)) = i-1$ and there must exist some $k_0 \in [1, \tilde{k}]$ with $N_{k_0} > 1$. So conditioned on each N_k being either equal to or greater than one with at least one inequality $N_{k_0} > 1$ for some k_0, we have $N_1, \ldots, N_{\tilde{k}}$ distributed either as 1 almost surely or as Geometric(3/4) random variables. The result then follows from a similar argument. □

In Section 6.1, we introduced the random walk paths of Type 1–3, with a right (resp. left) version for each kind. We also require a similar result for these paths.

Lemma 6.2. *Define Extreme as in Lemma 6.1. For a right version walk Q of any type and any $i \in [L+1, a] \setminus \{Extreme\}$,*

$$\frac{1}{6} < \mathbb{P}\Big(Parity_Q(i) = 0 \mid \{Parity_Q(j')\}_{j' \in [L,i)}, Extreme\Big) < \frac{5}{6}.$$

For a left version walk Q of any type and any $i \in [0, L-2]$,

$$\frac{1}{6} < \mathbb{P}\Big(Parity_Q(i) = 0 \mid \{Parity_Q(j')\}_{j' \in [-K+a,i)}, Extreme\Big) < \frac{5}{6}.$$

Proof. The proof follows similarly to that of Lemma 6.1. □

We shall give a description of the cases where Lemma 6.1 or 6.2 fails when applied to the random walk path Q^t defined in 6.1. Following the comment above Lemma 6.1, we consider the event that the random variable $Z \in \{0,1\}$ is *uniquely determined* by the σ-algebra $\mathcal{R}$, that is,

$$\mathbb{P}(Z = 1 \mid \mathcal{R}) \in \{0,1\}.$$

Let $\mathcal{T}(Q)$ denote whether the random walk path Q is an excursion, a double-sided path, or a failed re-arrival. Let

$$\text{Range}(Q) := \{i : \exists\, u \text{ with } Q(u) = i\}.$$

Lemma 6.3. *Consider the σ-algebra $\mathcal{R}_{t,i}$ generated by*

$$\mathcal{T}(Q^t), L^t, Range(Q^t) \text{ and } \{Parity_{Q^t}(j')\}_{j'<i}.$$

Suppose Q^t is not a failed re-arrival. Almost surely, the random variable $Parity_{Q^t}(i)$, for some $i \in (Range(Q^t) \setminus \{L^t\}) \cap [0, a]$, is uniquely determined by $\mathcal{R}_{t,i}$ if and only if one of the following is true:

- *Q^t is an excursion to the left or a double-sided path, and $i = L^t - 1$;*

- *Q^t is an excursion to the right or a double-sided path, and $i = max(Range(Q^t))$.*

Otherwise, a.s.

$$\mathbb{P}(Parity_{Q^t}(i) = 1 \mid \mathcal{R}_{t,i}) \in (1/6, 5/6).$$

Proof. This is a consequence of the discussion in this section. □

6.3 Auxiliary carpet process ω^t

We start with the following heuristic. Suppose we had the following two facts:

1. there exist $\epsilon, D > 0$ such that if $L(\tilde{\omega}^t) > \epsilon a$ then

$$\mathbf{E}\left(L(\tilde{\omega}^{t+1}) - L(\tilde{\omega}^t) \mid \tilde{\omega}^t\right) < -D;$$

2. there exists $G > 0$ such that for all a and all $\tilde{\omega}^t$,

$$\mathbf{E}\left(L(\tilde{\omega}^{t+1}) - L(\tilde{\omega}^t) \mid \tilde{\omega}^t\right) < G.$$

Since the process $\tilde{\omega}^t$ is a Markov process, the above two facts, together with some regularity conditions, would be sufficient to show that the expected value of the smallest t with $L(\tilde{\omega}^t) = a + 1$ is growing exponentially in a, that is, the block does not get frozen for a long time. Unfortunately neither of the facts above is true.

To make this heuristic rigorous we will introduce an auxiliary carpet process $\{\omega^t\}_{t\geqslant 0}$ and a related filtration $\{\mathbf{Reveal}^t\}_{t\geqslant 0}$, where we do not reveal the full information about the state of $\tilde{\omega}^t$. It turns out that

$$\mathbf{E}\left(L(\omega^{t+1}) - L(\omega^t) \mid \mathbf{Reveal}^t\right)$$

will have a more uniform drift, which allows us to make the counterparts of the above heuristic rigorous.

We inductively define an increasing family of σ-algebras $(\mathbf{Reveal}^t)_{t\geqslant 0}$ and the auxiliary carpet process $(\omega^t)_{t\geqslant 0}$. Recall the notations from Section 6.1 and Lemma 6.3. Every σ-algebra $\mathbf{Reveal}^t$ will be generated by $\{\mathcal{R}_{s,\,i(s,t)}\}_{s<t}$ for some random $i(s,t)$. At each time t, the process ω^t takes values in the space consisting of all

$$\omega \in \Omega \subset \{0, 1, ?\}^{[a]}$$

such that $\omega(j) = 0$ for $j < L(\omega)$, where

$$L(\omega) := \inf\{i : \omega(i) = 1\}.$$

In particular, if ω contains no 1, then $\omega = (0, 0, \ldots, 0)$. The variable ω^t will be defined as a function of the paths $\{Q^s\}_{s<t}$ and measurable in $\mathbf{Reveal}^t$.

To keep track of the information in $\mathbf{Reveal}^t$, we also define a family of 0/1-valued random variables, $\mathrm{Hidden}(s, i, t)$, that are indexed by triples

$$(s, i, t) \in \mathbb{Z}_{\geqslant 0} \times [a] \times \mathbb{Z}_{>0} \text{ with } s < t.$$

Our definition will ensure that a.s. $\mathrm{Hidden}(s, i, t) = 1$ if and only if $\mathrm{Parity}_{Q^s}(i)$ is not uniquely determined by $\mathbf{Reveal}^t$, that is,

$$\mathbb{P}\left(\mathrm{Parity}_{Q^s}(i) = 1 \mid \mathbf{Reveal}^t\right) \notin \{0, 1\}.$$

Set $\omega^0 = \tilde{\omega}^0$ and $\mathbf{Reveal}^0 = \sigma(\tilde{\omega}^0)$. Suppose we have defined the σ-algebra $\mathbf{Reveal}^t$, the state ω^t and the family of random variables $\mathrm{Hidden}(\cdot, \cdot, t)$ after the t-th random walk path Q^{t-1}. We start by including $\mathbf{Reveal}^t$ and $\mathcal{T}(Q^t)$ in $\mathbf{Reveal}^{t+1}$, i.e. all the previously revealed plus whether Q^t is an excursion, a double-sided path, or a failed re-arrival. For any $s < t$, set

$$\text{Hidden}(s, i, t+1) = 0 \text{ if } \text{Hidden}(s, i, t) = 0.$$

If Q^t is an excursion or a double-sided path, then ω^{t+1}, $\mathbf{Reveal}^{t+1}$ and $\text{Hidden}(\cdot, \cdot, t+1)$ are further defined via the three-step procedure:

1. Refresh the range of Q^t (other than L^t) to get $\omega_*^{t+1} \in \{0, 1, ?\}$[a]:

   If $\max(\text{Range}(Q^t)) \neq L^t + 1$ and $\min(\text{Range}(Q^t)) \neq L^t - 1$, then

$$\omega_*^{t+1}(i) = \begin{cases} ?, & i \in \text{Range}(Q^t) \setminus \{L^t\} \\ 0, & i = L^t \\ \omega^t(i), & \text{otherwise.} \end{cases}$$

   Include the random variable $\text{Range}(Q^t)$ in $\mathbf{Reveal}^{t+1}$ and set $\text{Hidden}(t, i, t+1) = 0$ for all $i \in \text{Range}(Q^t)^c \cup \{L^t\}$.

   If $\max(\text{Range}(Q^t)) = L^t + 1$, then we define ω_*^{t+1} similarly but change the symbol at $L^t + 1$ to

$$\omega_*^{t+1}(L^t + 1) = \begin{cases} ?, & \text{if } \omega^t(L^t + 1) = ? \\ (\omega^t(L^t + 1) + 1) \bmod 2, & \text{otherwise.} \end{cases}$$

   Additionally, set $\text{Hidden}(t, L^t + 1, t + 1) = 0$. If $\min(\text{Range}(Q^t)) = L^t - 1$, then we do the similar change at $L^t - 1$.

2. Find the leftmost one to get $\omega^{t+1} \in \Omega$:

$$\omega^{t+1}(i) = \begin{cases} 0, & i < L^{t+1} \\ 1, & i = L^{t+1} \\ \omega_*^{t+1}(i), & i \geqslant L^{t+1} + 2. \end{cases}$$

We leave the definition of $\omega^{t+1}(L^{t+1}+1)$ to the last step. Include in $\mathbf{Reveal}^{t+1}$ all the random variables $\text{Parity}_{Q^s}(i)$ for $s \leqslant t$ and $i \leqslant L^{t+1}$ which have not been in $\mathbf{Reveal}^t$. Set $\text{Hidden}(s,i,t) = 0$ for any such s, i.

3. Inspect the bit at $L^{t+1}+1$ to see if it is uniquely determined by $\mathbf{Reveal}^{t+1}$. More specifically, for any $s \leqslant t$ such that $\text{Hidden}(s, L^{t+1}+1, t+1)$ has not been set to zero, set $\text{Hidden}(s, L^{t+1}+1, t+1) = 0$ if one of the two cases from Lemma 6.3 holds with Q^s and $i = L^{t+1}+1$. If after this, we have $\text{Hidden}(s, L^{t+1}+1, t+1) = 0$ for all $s \leqslant t$, then we define $\omega^{t+1}(L^{t+1}+1) = \tilde{\omega}^{t+1}(L^{t+1}+1)$; otherwise, let $\omega^{t+1}(L^{t+1}+1) = ?$. Finally, we set all the undefined $\text{Hidden}(\cdot,\cdot,t+1)$ to one.

If Q^t is a failed re-arrival, then we simply leave ω^{t+1}, $\mathbf{Reveal}^{t+1}$ and $\text{Hidden}(\cdot,\cdot,t+1)$ undefined. This completes the inductive definition of all three items.

The last definition implies that all results about the auxiliary carpet process ω^t, including but not limited to the key Lemmata 6.8, 7.1 and 8.3 below, only apply until the first occurrence of a failed re-arrival. However, since the auxiliary process $\omega^0 = \tilde{\omega}^0$ may start from an arbitrary configuration, we could restart the definition of $\{\omega^{T+t}\}_{t\geqslant 0}$ with $\omega^T = \tilde{\omega}^T$, at some stopping time T close enough to the time in question. This is done in Lemma 8.4.

6.4 Properties of the process ω^t

We now state several consequences of these definitions before proving Lemma 6.8, which summarizes our progress in Chapter 6. For $s < t$, let

$$i_h(s,t) = \inf\{i : \text{Hidden}(s,i,t) = 1\}.$$

Also let

$$\text{Hidden}_L(s,t) := \{i : \text{Hidden}(s,i,t) = 1\} \cap (-\infty, L^t)$$

and

$$\text{Hidden}_R(s,t) := \{i : \text{Hidden}(s,i,t) = 1\} \cap (L^t, \infty).$$

Lemma 6.4. *The following are true.*

1. $\omega^t(i) = \tilde{\omega}^t(i)$ *if* $\omega^t(i) \neq ?$.

2. $L(\omega^t) = L(\tilde{\omega}^t)$.

3. $\omega^t(i) = ?$ *if and only if there exists some* $s < t$ *such that* $\mathit{Hidden}(s, i, t) = 1$.

Proof. All items follow from the definition of ω^t_* and ω^t by induction on t. □

Lemma 6.5. *The following are true.*

1. ω^t *is measurable in* $\boldsymbol{Reveal}^t$.

2. $\boldsymbol{Reveal}^t$ *is generated by* $\{\mathcal{R}_{t,\, i_h(s,t)}\}_{s<t}$ *where* $R_{t,i}$ *is defined in Lemma 6.3.*

3. *For any* $s < t$*, both* $\mathit{Hidden}_L(s, t)$ *and* $\mathit{Hidden}_R(s, t)$ *are either empty or a connected set of size at least two.*

Proof. The proof of each item goes by induction. In particular, for item (1) and (2) we use Lemma 6.3 in the 'inspect' step and trivially in the two corner cases of the 'refresh' step. Item (3) is guaranteed by the definition of Hidden, both from the two corner cases in the 'refresh' step and from the cases of Lemma 6.3 in the 'inspect' step. □

For the purpose of Lemma 6.8, it is more convenient to study the intermediate state ω^t_* defined in the 'refresh' step of the procedure. We consider the corresponding $\mathbf{Reveal}^t_*$ and $\mathrm{Hidden}_*(\cdot, \cdot, t)$ associated with ω^t_*. Let $\mathbf{Reveal}^{t+1}_*$ (resp. $\mathrm{Hidden}_*(\cdot, \cdot, t+1)$) be $\mathbf{Reveal}^{t+1}$ (resp. $\mathrm{Hidden}(\cdot, \cdot, t+1)$) at the end of the 'refresh' step without performing the changes in the subsequent two steps. All undefined $\mathrm{Hidden}_*(\cdot, \cdot, t+1)$ are set to one. We also define $i_{h,*}(s, t)$, $\mathrm{Hidden}_{L,*}(s, t)$ and $\mathrm{Hidden}_{R,*}(s, t)$ by replacing $\mathrm{Hidden}(s, i, t)$ in the original definitions with $\mathrm{Hidden}_*(s, i, t)$. The next lemma follows as a by-product of the inductive proof of the last two lemmata.

Lemma 6.6. *Except for Lemma 6.4 (2), Lemmata 6.4 and 6.5 still hold if we replace ω^t, $\boldsymbol{Reveal}^t$ and $\mathrm{Hidden}(\cdot,\cdot,t)$ with the corresponding ω^t_*, $\boldsymbol{Reveal}^t_*$ and $\mathrm{Hidden}_*(\cdot,\cdot,t)$.* □

Lemma 6.7. *If $\omega^t_*(i) = ?$, then a.s.*

$$\mathbb{P}\left(\tilde{\omega}^t(i) = 1 \mid \boldsymbol{Reveal}^t_*, \{\mathcal{R}_{s,i-1}\}_{s<t}\right) \in (1/6, 5/6).$$

Proof. In the following, when we refer to the items in Lemmata 6.4 and 6.5, we actually mean their counterparts in Lemma 6.6. Since $\omega^t_*(i) = ?$, by Lemma 6.4 (3) there exists some $s_0 < t$ with $\mathrm{Hidden}_*(s_0, i, t) = 1$. By Lemma 6.5 (3), we must have

$$\{j, j+1\} \subseteq \mathrm{Hidden}_{L,*}(s_0, t) \cup \mathrm{Hidden}_{R,*}(s_0, t)$$

holds for either (1) $j = i$, or (2) $j = i-1$ with $i = \max(\mathrm{Hidden}_{L,*}(s_0, t))$ or $i = \max(\mathrm{Hidden}_{R,*}(s_0, t))$.

In the first case, neither of the events in Lemma 6.3 occurs with Q^{s_0} at i. So by combining Lemma 6.5 (2), the fact that $i_{h,*}(s_0, t) \leqslant i$, independence and Lemma 6.3, we obtain that a.s.

$$\mathbb{P}\left(\mathrm{Parity}_{Q^{s_0}}(i) = 1 \mid \mathbf{Reveal}^t_*, \{\mathcal{R}_{s,i}\}_{s<t}\right) \in (1/6, 5/6).$$

Since $\tilde{\omega}^t(i) = \tilde{\omega}^0(i) + \sum_{s<t} \mathrm{Parity}_{Q^s}(i)$, this proves Lemma 6.7 in the first case.

For the second case, the display from the first case can be shown similarly if we replace both i's with $i-1$'s. The assumption also implies $\mathrm{Parity}_{Q^{s_0}}(i)$ is uniquely determined by $\mathcal{R}_{s_0,i}$, so a.s.

$$\mathbb{P}\left(\mathrm{Parity}_{Q^{s_0}}(i) = 1 \mid \mathbf{Reveal}^t_*, \{\mathcal{R}_{s,i-1}\}_{s<t}\right) \in (1/6, 5/6).$$

Lemma 6.7 then follows. □

For $\omega \in \{0, 1, ?\}^{[a]}$, define

$$N(\omega) := \#\{i \in [0, a] : \omega(i) \neq 0\}.$$

One key part of our analysis is to show that when $L(\omega^t)$ is large, $N(\omega^t)$ behaves like a random walk biased to increase, see Lemma 8.1. Lemma 6.8 says that $N(\omega^t)$ is unlikely to decrease dramatically by erasing many ?s.

Lemma 6.8. *For any $t, k \in \mathbb{N}$, a.s.*

$$\mathbb{P}\left(N(\omega_*^{t+1}) - N(\omega^{t+1}) \geqslant k \mid \boldsymbol{Reveal}_*^t\right) \leqslant (5/6)^{(k-1)/2}.$$

Proof. Let i_n be the location of the nth leftmost ? in ω_*^{t+1} (if it exists). Since we might lose one more ? in the 'inspect' step, the above conditional probability is at most

$$\mathbb{P}\left(\tilde{\omega}(i_1) = 0, \tilde{\omega}(i_2) = 0, \ldots, \tilde{\omega}(i_{k-1}) = 0 \mid \mathbf{Reveal}_*^t\right).$$

The above expression is at most $(5/6)^{\lceil (k-1)/2 \rceil}$ by repeated conditioning and use of Lemma 6.7 at every other i_n. This proves the desired estimate. □

Chapter 7

ZEROS OF AUXILIARY CARPET PROCESS

With Lemma 6.8, we will show that when $L(\omega^t)$ is large, the number of nonzero symbols $N(\omega^t)$ has a bias to increase. However, this does not rule out the possibility that $N(\omega^t)$ becomes small when $L(\omega^t)$ is small. The main goal of Chapter 7 is to show that with a nice initial configuration, this is unlikely to occur for an exponentially long time.

To state Lemma 7.1, the main result of this chapter, we fix a few notations. First we introduce several special states in the space Ω, defined in Section 6.3. Let

$$\text{Base} = 01??\cdots??$$

$$\text{Exit} = 0000\cdots00.$$

The 'Base' state starts with 01 followed by $a-1$ many ?, whereas the 'Exit' state is the all-zero state which occurs exactly when the block becomes frozen. Define

$$\tau_{\text{Exit}} := \inf\{t \geqslant 0 : \omega^t = \text{Exit}\}.$$

For $\epsilon = 1/200$, define an 'ϵ-Base' state to be any configuration of the form

$$00\cdots01??\cdots??$$

starting with *at most* ϵa many 0's from the left. In particular, the Base state is an ϵ-Base state.

Since ω^t is not a Markov chain with respect to its natural filtration, we consider the carpet processes $\{\tilde{\omega}^t\}_{t\geqslant t_0}$ and $\{\omega^t\}_{t\geqslant t_0}$ that start from some negative t_0 instead of zero, and use the shorthand notation $\overline{\mathbb{P}}_{\omega_0}(A) \leqslant x$ to mean that

$$\sup_{R\in\mathbf{Reveal}^0} \mathbb{P}\left(A \mid \omega^0 = \omega_0, R\right) \leqslant x.$$

Write $\underline{\mathbb{P}}$ similarly for the infimum. In fact, all results in Chapter 7 are true in a stronger sense where the supremum/infimum is over all $R \in \sigma(\{Q^s\}_{s<0})$, but the above notation has the advantage of working for both Chapters 7 and 8. We also write $\overline{\mathbb{P}}_{\epsilon\text{-Base}}$ in the case that the statement holds for any initial ϵ-Base state ω_0.

Lemma 7.1. *Consider the event*

$$B = \left\{L(\omega^t) \leqslant \epsilon a \text{ and } N(\omega^t) \leqslant \epsilon a \text{ for some } 0 \leqslant t < \tau_{Exit}\right\}.$$

For $\epsilon = 1/200$ *and sufficiently large* a*,*

$$\overline{\mathbb{P}}_{\epsilon\text{-}Base}(B) \leqslant \exp(-a/100).$$

The rest of Chapter 7 is devoted to the proof of Lemma 7.1. In order to produce so many zeros in ω^t, the auxiliary carpet process needs to follow certain scheme – it has to work from right to left and generate the zeros without refreshing the pre-existing zeros to its right. The proof then goes by identifying such schemes and bounding each scheme's probability, via estimates mimicking those for birth-death chains.

7.1 *History of zeros in* ω^t

We start by studying how zeros are generated in the auxiliary carpet process. Write

$$\text{Visited}(t, i) = 1\{i \in \text{Range}(Q^t)\}.$$

Recall that the process ω^t starts from some negative time t_0. For any time $t \geqslant 0$, define

$$\text{Right Zeros}^t := \{i > L(\omega^t) : \ \omega^t(i) = 0\}.$$

For any time $t \geqslant 0$ and $i \in [L(\omega^t), a]$, define

$$\text{Last}^t(i) := \sup\{t' \in [0, t) : \text{Visited}(t', i) = 1\} \vee -1,$$

which is the last *nonnegative* time t' before t when i gets visited by a random walk path.

Lemma 7.2. *For any s and i with $L(\omega^{s+1}) < i \leq L(\omega^s)$, we have Visited$(s,i) = 1$*

Proof. Q^s cannot be an excursion to the right, since necessarily $L(\omega^{s+1}) > L(\omega^s)$ in that case. If Q^s is a double-side path, then $\text{Visited}(s,i) = 1$ for all i. So assume Q^s is an excursion to the left. Then $L(\omega^{s+1}) \geqslant \min(\text{Range}(Q^s))$ and thus $\min(\text{Range}(Q^s)) < i \leq L(\omega^s)$. This implies $\text{Visited}(s,i) = 1$. □

Lemma 7.3. *For any $t \geqslant 0$, $i \in$ Right Zeros^t and $t' \in [\text{Last}^t(i) + 1, t]$,*

$$L(\omega^{t'}) < i.$$

Proof. Suppose that $L(\omega^{t'}) \geqslant i$. Since $i \in \text{Right Zeros}^t$, we have $L(\omega^t) < i \leq L(\omega^{t'})$. Thus there exists some s such that $t' \leqslant s < t$ with $L(\omega^{s+1}) < i \leq L(\omega^s)$. But by Lemma 7.2 we must have a time $s \in (\text{Last}^t(i), t)$ with $\text{Visited}(s,i) = 1$, which contradicts the definition of $\text{Last}^t(i)$. Therefore $L(\omega^{t'}) < i$. □

Lemma 7.4. *Suppose $L(\omega^s) \notin \{i, i+1\} \subseteq$ Range(Q^s), and $L(\omega^{t'}) < i$ for any $t' \in [s+1, t]$, then*

$$\omega^t(i) = \omega^t(i+1) = ?.$$

Proof. Recall the definition of the procedure in Section 6.3. By the first assumption, $\text{Hidden}(s, i', s+1)$ has not been set to zero at the end of the 'refresh' step for $i' \in \{i, i+1\}$. By the second assumption, $\text{Hidden}(s, i', t')$ will remain to be one for $i' \in \{i, i+1\}$ and $t' \in [s+1, t]$. So $\text{Hidden}(s, i, t) = \text{Hidden}(s, i+1, t) = 1$ and the lemma follows by Lemma 6.4 (3). □

Lemma 7.5. *Let $i_{\max} := \max\{i > L(\omega^t) : \omega^t(i) = 0\}$. We have*

1. *the function Last$^t(i)$ is decreasing on the set $[L(\omega^t), a]$;*

2. *for any $i, j \in$ Right Zeros^t with $j < i$ and Last$^t(j) =$ Last$^t(i) \neq$ Last$^t(i_{\max})$, we have $i = j+1$ and at time Last$^t(i)$, there was an excursion to its left $Q^s = Q^{\text{Last}^t(i)}$ starting from i such that Visited$(s, i-2) = 1$ and Parity$_{Q^s}(i-1) = 0$;*

3. *there do not exist three distinct values* $i, j, k \in$ *Right Zeros*^{t} *with* $\mathit{Last}^t(i) = \mathit{Last}^t(j) = \mathit{Last}^t(k) \neq \mathit{Last}^t(i_{\max})$.

4. *if* ω^0 *is an* ϵ*-Base state, then there do not exist four distinct values in Right Zeros*^{t} *with the same function value* $\mathit{Last}^t(\cdot)$.

Proof. Since the random walk always starts from the leftmost one, a straightforward induction on the time t proves item (1). Item (3) follows directly from item (2), so it remains to show items (2) and (4).

For item (2), write $s = \mathrm{Last}^t(j) = \mathrm{Last}^t(i)$. By item (1), we have $s > \mathrm{Last}^t(i_{\max}) \geqslant -1$, so $s \geqslant 0$ and $\mathrm{Visited}(s, j) = \mathrm{Visited}(s, i) = 1$. Thus $\{j, j+1\} \subseteq \mathrm{Range}(Q^s)$. We claim that $L(\omega^s) \in \{j, j+1\}$. Suppose not, then Lemmata 7.3 and 7.4 combined imply that $\omega^t(j) = ?$, which is a contradiction. This proves the claim.

We consider all the possible cases of Q^s satisfying $L(\omega^s) \in \{j, j+1\}$. First, Q^s cannot be a double-sided path: otherwise, we would have $\mathrm{Last}^t(i) = \mathrm{Last}^t(i_{\max})$. Secondly, Q^s cannot be an excursion to the right: otherwise, from $L(\omega^s) \in \{j, j+1\}$ we must have $L(\omega^{s+1}) > j$, which contradicts Lemma 7.3. Lastly, if Q^s is an excursion to the left, then we must have $L(\omega^s) = j+1$ to guarantee $\mathrm{Visited}(s, j+1) = 1$. This implies $i = j+1$. Since no random walk after time s visits j, it follows that $\mathrm{Parity}_{Q^s}(j) = \tilde{\omega}^s(j) = \omega^s(j) = 0$, which in turn implies that $\mathrm{Visited}(s, i-2) = 1$. This proves item (2).

For item (4), we first show that for any $s \geqslant 0$, $|\text{Right Zeros}^t(s)| \leqslant 3$, where

$$\text{Right Zeros}^t(s) := \{i \in \text{Right Zeros}^t : \mathrm{Last}^t(i) = s\}.$$

Suppose not, then out of any four such elements we can find $i \in \text{Right Zeros}^t(s)$ such that $L(\omega^s) \notin \{i, i+1\} \subseteq \mathrm{Range}(Q^s)$, which leads to a similar contradiction to that in the proof of item (2). This proves the bound for $s \geqslant 0$.

It remains to check the case where $s = -1$. We show that $|\text{Right Zeros}^t(-1)| \leqslant 1$ again by contradiction. Suppose not and there exist $i, j \in \text{Right Zeros}^t(-1)$ with $i < j$. Since $\mathrm{Last}^t(j) = -1$, by item (1) we have $j > L(\omega^0)$ and thus $\omega^0(j) = ?$ due to the definition of an ϵ-Base state. Moreover, by Lemma 7.3 we get $L(\omega^{t'}) < i$ for any $t' \in [0,t]$, so it follows from the definition of ω^t that $\omega^{t'}(j) = ?$ for any $t' \in [0,t]$. This contradicts with the fact that $\omega^t(j) = 0$, which proves item (4). □

We introduce some more notation. For $p < a$ write $[p,a] = [p,a] \cap \mathbb{Z}$. We call a sequence $x \in \{0, *, ?/1\}^{[p,a]}$ **good** if for all j such that $x(j) = *$ we have $x(j+1) = 0$. The good sequences capture different ways in which zeros may be generated in ω^t, with adjacent $*$ and 0 representing a pair of zeros generated as described in Lemma 7.5 (2).

Given a good x and $p < a$, we partition the zeros in x as follows:

$$\mathrm{Set}_0 := \{j \in [p,a] : x(j) = 0\} = \mathrm{Set}_1 \cup \mathrm{Set}_2 \cup \mathrm{Set}_3 \cup \{\tilde{M}(x)\},$$

where

$$\begin{aligned} \tilde{M}(x) &:= \max\{j : \ x(j) = 0\} \\ \tilde{N}(x) &:= \max\{j < \tilde{M}(x) : \ x(j) = 0\} \end{aligned}$$

and

$$\begin{aligned} \mathrm{Set}_1 &:= \{j \in [p,a] : x(j) = 0, x(j+1) \neq ?/1 \text{ and } x(j-1) \neq *\} \ \setminus \{\tilde{N}(x)\} \\ \mathrm{Set}_2 &:= \{j \in [p,a] : x(j) = 0, x(j+1) \neq ?/1 \text{ and } x(j-1) = *\} \ \setminus \{\tilde{N}(x)\} \\ \mathrm{Set}_3 &:= \{j \in [p,a] : x(j) = 0, x(j+1) = ?/1\} \cup \{\tilde{N}(x)\} \setminus \{\tilde{M}(x)\}. \end{aligned}$$

For $j \in \mathrm{Set}_0 \setminus \{\tilde{M}(x)\}$, let $k(j)$ be such that

$$j + k(j) = \inf\{j' > j : j' \in \mathrm{Set}_0\}$$

and let $r(j)$ be such that

$$j + r(j) = \inf\{j' > j : x(j') = 0 \text{ or } *\},$$

except when $j = \tilde{N}(x)$, we have $r(\tilde{N}(x)) = \tilde{M}(x) + 1 - \tilde{N}(x)$ instead. Note that in a good sequence, for $j \neq \tilde{N}(x)$ we always have $r(j) = k(j)$ or $k(j) - 1$.

Finally, for $r \geqslant 2$ divide Set_3 into pieces

$$\mathrm{Set}_{3,r} := \mathrm{Set}_3 \cap \{j : \ r(j) = r\}.$$

7.2 Counters, stopping times and events

In this section, we outline the proof of Lemma 7.1. Suppose we're given a good sequence x. To bound the probability of the right-to-left dynamics generating x, we will inductively define a counter $y^s(i)$ at every vertex $i \in \mathrm{Set}_0$ starting from $\tilde{M}(x)$. The goal of the counters is twofold. On one hand, given a sequence $z \in \mathbb{N}^{\mathrm{Set}_0}$ the counters can be used to identify a sequence of stopping times $\tilde{T}(i)$ for every $i \in \mathrm{Set}_0$. We will tailor our definitions so that $\tilde{T}(i) = \mathrm{Last}^t(i) + 1$ with the right choice of z (in most cases), see Lemma 7.8. On the other hand, the counters also give us bounds on the probability of the key events $A_{x,z}$, see Lemma 7.7.

We start with the definition of $y^s(i)$ and $\tilde{T}(i)$ at $i = \tilde{M}(x)$.

- For $i = \tilde{M}(x)$, the counter $y^0(i)$ starts at zero. For any $s \geqslant 0$, the counter $y^{s+1}(i) = y^s(i) + 1$ increases by one if at time s the random walk starts from $L(\omega^s) \geqslant \tilde{M}(x)$ and either $s = 0$ or the previous random walk path Q^{s-1} was from $L(\omega^{s-1}) < \tilde{M}(x)$; otherwise, the counter stays put and we have $y^{s+1}(i) = y^s(i)$. Given $z(i)$, define the stopping time $\tilde{T}(\tilde{M}(x))$ to be the smallest s such that $y^s(i) = z(i)$ and the next path starts at $L(\omega^s) < \tilde{M}(x)$ if such s exists; in this case, we say $\tilde{T}(\tilde{M}(x))$ is well-defined.

In order to define other counters we will work inductively. Suppose that the stopping time $\tilde{T}(i + k(i))$ is well-defined, we shall define $y^s(i)$ for all $s \geqslant \tilde{T}(i + k(i))$.

- For $i \in \mathrm{Set}_1 \cup \mathrm{Set}_2$, then initially at $s = \tilde{T}(i+k(i))$, we set the counter $y^s(i) = \tilde{\omega}^s(i) - 1$. For any $s \geqslant \tilde{T}(i + k(i))$, let

  $$y^{s+1}(i) = y^s(i) + \mathrm{Local}_{Q^s}(i).$$

  In words, the counter increases by one every time a particle moves from i.

- For $i \in \mathrm{Set}_3$, then the counter $y^{\tilde{T}(i+k(i))}(i)$ starts at zero and for $s \geqslant \tilde{T}(i + k(i))$, we have

$$y^{s+1}(i) = \begin{cases} y^s(i) + 1, & L(\omega^s) \in [i, i + r(i)) \\ y^s(i) + \text{Local}_{Q^s}(i), & L(\omega^s) < i. \end{cases}$$

In words, the counter increases by one every time a random walk path starts at a location belonging to $[i, i + r(i))$ or every time a particle moves from i in a path that starts to the left of i.

Given $z(i)$, let $\tilde{T}(i)$ be the smallest s such that $y^s(i) = z(i)$ if such s exists; in this case, we say $\tilde{T}(i)$ is well-defined. This completes the definition of the counter $y^s(i)$ and stopping time $\tilde{T}(i)$ at every $i \in \text{Set}_0$.

Finally, we define the key events $A_{x,z}$ in the analysis of counters. For a good sequence x and $z \in \mathbb{N}^{\text{Set}_0}$, define $A_{x,z}$ to be the event that

1. the stopping times $\tilde{T}(i)$ are well-defined for all $i \in \text{Set}_0$;
2. $\tilde{T}(\tilde{M}(x)) < \tau_{\text{Exit}}$;
3. for any $i \in \text{Set}_0 \setminus \{\tilde{M}(x)\}$, none of Q^s visits $i + r(i)$ during $\tilde{T}(i + k(i)) \leq s < \tilde{T}(i)$;
4. for $i \in \text{Set}_2$, just before $\tilde{T}(i)$ the particle made an excursion to the left $Q^s = Q^{\tilde{T}(i)-1}$ starting from i where $\text{Visited}(s, i-2) = 1$ and $\text{Parity}_{Q^s}(i-1) = 0$.

Lemma 7.6. *If the event B in Lemma 7.1 occurs, then there exists*

- $p \leqslant \epsilon a$,
- *a good* $x \in \{0, *, ?/1\}^{[p,a]}$ *and*
- $z \in \mathbb{N}^{Set_0}$

such that

- *the event $A_{x,z}$ occurs and*
- *all the $z(i)$'s are odd for $i \in Set_1 \cup Set_2$.*

Lemma 7.7. *Fix any $p \leqslant a$. Fix any good sequence $x \in \{0, *, ?/1\}^{[p,a]}$. Fix any sequence $z \in \mathbb{N}^{Set_0}$. We have*

$$\overline{\mathbb{P}}_{\epsilon\text{-}Base}(A_{x,z}) \leqslant \left(1 - p_{\frac{1}{2}}^{-\tilde{M}(x)+a+1}\right)^{z(\tilde{M}(x))} \prod_{i \in Set_1} \left(\frac{1}{2}\right)^{z(i)} \prod_{i \in Set_2} p_{\frac{1}{6}} \left(\frac{1}{2}\right)^{z(i)-1} \prod_{r>1} \prod_{i \in Set_{3,r}} \left(1 - \frac{1}{2r}\right)^{z(i)},$$

where $p_{\frac{1}{2}} = \frac{1}{2} - \frac{1}{a^4}$ and $p_{\frac{1}{6}} = \frac{1}{6} + \frac{1}{a^4}$.

In Section 7.3, we will first prove the above two lemmata, and then combine them to prove Lemma 7.1 using a union bound.

7.3 Analysis of zero generation in ω^t

We start by proving Lemma 7.6. Assume the event B from Lemma 7.1 occurs, that is, $L(\omega^t) \leqslant \epsilon a$ and $N(\omega^t) \leqslant \epsilon a$ for some $0 \leqslant t < \tau_{\text{Exit}}$. We define the corresponding p, x and z as follows. Recall $i_{\max} = \max\{i > L(\omega^t) : \omega^t(i) = 0\}$.

- Let $p := L(\omega^t) \leqslant \epsilon a$.
- Set $\tilde{M} := \min\{i \in \text{Right Zeros}^t : \text{Last}^t(i) = \text{Last}^t(i_{\max})\}$.
- Define a sequence x in $\{0, *, ?/1\}^{[p,a]}$ by first setting $x(\tilde{M}) := 0$ and $x(i) := ?/1$ for $i > \tilde{M}$. For $i < \tilde{M}$, let

$$x(i) := \begin{cases} ?/1 & \text{if } \omega^t(i) = ?/1, \\ * & \text{if } \omega^t(i) = \omega^t(i+1) = 0 \text{ and } \text{Last}^t(i) = \text{Last}^t(i+1), \\ 0 & \text{otherwise.} \end{cases}$$

  Note that x is good due to Lemma 7.5 (3).

- For $i = \tilde{M}(x)$ let $z(\tilde{M}(x)) := y^t(\tilde{M}(x))$. Note that $\tilde{T}(\tilde{M}(x))$ is well-defined by our definition of $z(\tilde{M}(x))$ and the fact that $L(\omega^t) < \tilde{M}(x)$. To define $z(i)$ for $i \in \text{Set}_0 \setminus \{\tilde{M}(x)\}$ we will work inductively. Suppose that $z(i + k(i))$ is given and $\tilde{T}(i + k(i))$ is well-defined. Then we let $z(i) := y^t(i)$. Again $\tilde{T}(i)$ is well-defined by our choice of $z(i)$. This completes the definition of z.

Lemma 7.8. *Let p, x and z be defined as above. For $i = \tilde{M}(x)$, we have $Last^t(i+1)+1 \leq \tilde{T}(i) \leq Last^t(i)+1$. For any $i \in Set_0 \setminus \{\tilde{M}(x)\}$, we have $\tilde{T}(i) = Last^t(i)+1$.*

Proof. We shall prove Lemma 7.8 by induction on i. We start with the base case where $i = \tilde{M}(x)$. If $z(i) = y^t(i) = 0$, then $\tilde{T}(i) = 0$ and the upper bound on $\tilde{T}(i)$ is trivial. If $y^t(i) > 0$, then by the definition of $\tilde{T}(\tilde{M}(x))$ we have $L(\omega^{\tilde{T}(i)}) < i \leq L(\omega^{\tilde{T}(i)-1})$. Lemma 7.2 implies $\text{Visited}(\tilde{T}(i)-1, i) = 1$ and thus $\tilde{T}(i) - 1 \leq \text{Last}^t(i)$. This proves the upper bound.

To show $\tilde{T}(i) \geqslant \text{Last}^t(i+1)+1$ for $i = \tilde{M}(x)$, we argue by contradiction. Suppose for some $s \geqslant \tilde{T}(\tilde{M}(x))$, the path Q^s visits $\tilde{M}(x)+1$. Since $s \geqslant \tilde{T}(\tilde{M}(x))$, by definition $L(\omega^{t'}) < \tilde{M}(x)$ for any $t' \in [s, t]$, so the conditions of Lemma 7.4 are satisfied. This implies $\omega^t(\tilde{M}(x)) = ?$, which is a contradiction. This completes the proof of the base case.

Now suppose $\tilde{T}(i + k(i)) \leq \text{Last}^t(i + k(i)) + 1$ holds for some $i \in \text{Set}_0 \setminus \{\tilde{M}(x)\}$, we will prove $\tilde{T}(i) = \text{Last}^t(i) + 1$. First, note that $\text{Last}^t(i) \neq \text{Last}^t(i + k(i))$; otherwise, Lemma 7.5 (2) would imply $k(i) = 1$ and $x(i) = *$, which contradicts $i \in \text{Set}_0$. So by Lemma 7.5 (1) and the induction hypothesis we have $\text{Last}^t(i) > \text{Last}^t(i + k(i)) \geqslant \tilde{T}(i + k(i)) - 1$. Thus $\text{Last}^t(i) \geqslant \tilde{T}(i + k(i))$ and it makes sense to talk about $y^{\text{Last}^t(i)}$.

In order to prove $\tilde{T}(i) = \text{Last}^t(i) + 1$, it suffices to check

$$y^{\text{Last}^t(i)+1} > y^{\text{Last}^t(i)} \tag{7.9}$$

and for any $t' \in [\text{Last}^t(i) + 1, t)$,

$$y^{t'+1}(i) = y^{t'}(i). \tag{7.10}$$

Since $Q^{\text{Last}^t(i)}$ visits i, the inequality (7.9) follows from the definition of counter directly if $i \in \text{Set}_1 \cup \text{Set}_2$ or $i \in \text{Set}_3$ and $L(\omega^{\text{Last}^t(i)}) < i + r(i)$. The remaining case that $i \in \text{Set}_3$

and $L(\omega^{\mathrm{Last}^t(i)}) \geqslant i + r(i)$ cannot happen due to the fact that $\mathrm{Last}^t(i) > \mathrm{Last}^t(i + r(i))$ and Lemma 7.3. For (7.10), $Q^{t'}$ does not visit i for $t' \in [\mathrm{Last}^t(i) + 1, t)$, so it follows that $y^{t'+1}(i) = y^{t'}(i)$ for any such t'. □

Proof of Lemma 7.6. We finish the proof by checking all requirements on x and z. We've checked that x is a good sequence.

We check that $z(i) = y^t(i) \geqslant 0$ for all $i \in \mathrm{Set}_1 \cup \mathrm{Set}_2$. In the proof of Lemma 7.8, we've shown $\mathrm{Last}^t(i) \geqslant \tilde{T}(i + k(i))$, so $y^t(i) \geqslant y^{\mathrm{Last}^t(i)+1} \geqslant y^{\mathrm{Last}^t(i)} + 1 \geqslant y^{\tilde{T}(i+k(i))} + 1 \geqslant 0$.

To see why $z(i) = y^t(i)$ is odd for $i \in \mathrm{Set}_1 \cup \mathrm{Set}_2$, note that by definition $y^{\tilde{T}(i+1)}(i) = \tilde{\omega}^{\tilde{T}(i+1)}(i) - 1$. Also $y^t(i) - y^{\tilde{T}(i+1)}(i)$ has the same parity as $\tilde{\omega}^t(i) - \tilde{\omega}^{\tilde{T}(i+1)}(i)$. Combining these with $\tilde{\omega}^t(i) = \omega^t(i) = 0$ proves $y^t(i)$ is odd.

Finally, we show $A_{x,z}$ occurs by checking every item of its definition: we've checked item (1) in the definition of z above; item (2) holds because $t < \tau_{\mathrm{Exit}}$; item (3) follows from the lower bound on $\tilde{T}(i)$ in Lemma 7.8; item 4 follows from Lemma 7.5 (2) and Lemma 7.8. □

Next we prove Lemma 7.7.

Proof of Lemma 7.7. We will estimate the probability inductively. We start with $p = \tilde{M}(x)$ and then progressively lower it until we get the full result.

For $p = \tilde{M}(x)$, $x|_{[\tilde{M}(x),a]}$ and $z|_{[\tilde{M}(x),a]}$, define the j-th iteration of the counter $y^s(\tilde{M}(x))$ to be the set of paths Q^s such that $y^{s+1}(\tilde{M}(x)) = j$ and $L(\omega^s) \geqslant \tilde{M}(x)$, for any $j \in [1, z(\tilde{M}(x)]$. For each iteration of the counter, we may sample an infinite sequence of paths and reveal as many of them as needed. In each iteration we cannot have the first $a - \tilde{M}(x) + 1$ paths all being excursions to their right; otherwise, the leftmost one in the configuration would exceed a, which contradicts the definition of $A_{x,z}$ item 2. Since the probability of any path reaching a neighboring block is at most $2/a^4$ (see the proof of Lemma 8.4), the probability of each path being an excursion to the right (excluding the double-sided path and the failed re-arrival) is at least $1/2 - 1/a^4$. Thus we get the following upper bound on the probability of $A_{x,z}$

$$\left(1 - (1/2 - 1/a^4)^{-\tilde{M}(x)+a+1}\right)^{z(\tilde{M}(x))}.$$

Suppose we have established the upper bound $UB_{x,z,p}$ for p, $x|_{[p,a]}$ and $z|_{[p,a]}$. Let $p' = \max\{j < p : x(j) = 0\}$. We will establish the bound $UB_{x,z,p''}$ for p'', $x|_{[p'',a]}$ and $z|_{[p'',a]}$, where $p'' = p'$ if $p' \in \mathrm{Set}_1 \cup \mathrm{Set}_3$ and $p'' = p' - 1$ if $p' \in \mathrm{Set}_2$.

If $p' \in \mathrm{Set}_1$, i.e. $r(p') = 1$ and $x(p'-1) \neq *$, then by the definition of $A_{x,z}$ item 3 we know that every movement of a particle from p' in the time interval $[\tilde{T}(p), \tilde{T}(p'))$ must be to the left. By the definition of the counter, there are

$$y^{\tilde{T}(p')} - y^{\tilde{T}(p)} = z(p') - (\tilde{\omega}^s(i) - 1) \geqslant z(p')$$

many such movements. Thus we get an upper bound of

$$UB_{x,z,p}\, 2^{-z(p')}.$$

If $p' \in \mathrm{Set}_2$, i.e. $r(p') = 1$ and $x(p'-1) = *$, then the same analysis as in the last case $p' \in \mathrm{Set}_1$ implies that there are at least $z(p')$ many movements from p' during $[\tilde{T}(p), \tilde{T}(p'))$, all of which are to the left. Moreover, by the definition of $A_{x,z}$ item 4, at the last such movement from p' the particle makes an excursion to the left (including a long excursion to the left) that visits $p'-1$ an even number of times. This happens with probability at most $1/6 + 1/a^4$. In fact, for a *usual* random walk excursion starting from zero (whose range could exceed $\pm K$), the probability of going to the left and visiting site -1 even times is exactly $1/6$. The estimate then follows from a simple union bound by considering whether the path reaches a neighboring block. Therefore, in this case we obtain a slightly improved bound of

$$UB_{x,z,p}\,(1/6 + 1/a^4)(1/2)^{z(p')-1}.$$

If $p' \in \mathrm{Set}_{3,r}$, i.e. $r(p') = r$ with $r \geqslant 2$, then by the definition of $A_{x,z}$ item 3, we know that every time a random walk path starts at a location in $[p', p'+r)$ during $[\tilde{T}(p), \tilde{T}(p'))$, the first step either moves to the left or moves to the right without hitting $p'+r$. This has probability at most $1 - 1/2r$. Also, every time during $[\tilde{T}(p), \tilde{T}(p'))$ the particles moves from p' in a path that starts to the left of p', the particle either moves to the left or moves to the right and does not hit $p'+r$ before returning to p', which also has probability at most

$1 - 1/2r$. Combining these with the definition of the counter, we get the bound

$$UB_{x,z,p}(1 - 1/2r)^{z(p')}.$$

Putting these together gives us the lemma. □

Finally, we complete the argument by proving Lemma 7.1.

Corollary 7.11. *If B occurs and p, x are defined as in Lemma 7.6, then $x(p) = ?/1$ and*

$$\tilde{M}(x) \geqslant a - N(\omega^t) - 1 \geqslant a(1 - \epsilon) - 1, \tag{7.12}$$

$$|Set_1| + 2|Set_2| + 2|Set_3| \geqslant a - L(\omega^t) - N(\omega^t) - 2 \geqslant a(1 - 2\epsilon) - 2, \tag{7.13}$$

$$|Set_3| \leqslant \sum\nolimits_r (r-1)|Set_{3,r}| \leq N(\omega^t) \leq a\epsilon, \tag{7.14}$$

$$|Set_{?/1}| := |\{j \in [p, a] : x(j) = ?/1\}| \leqslant N(\omega^t) + 2 \leqslant \epsilon a + 2. \tag{7.15}$$

Proof. For (7.12), there are only $?/1$'s to the right of $\tilde{M}(x)$: at most $N(\omega^t) - 1$ of them correspond to ?'s and 1's in ω^t, and at most two of them come from zeros in ω^t by the choice of $\tilde{M}$ in the definition of x and Lemma 7.5 (1)(4). For (7.14), notice the convention that $\tilde{N}(x) \in \mathrm{Set}_3$ and $r(\tilde{N}(x)) = \tilde{M}(x) + 1 - \tilde{N}(x)$. □

Proof of Lemma 7.1. By Lemma 7.6, it suffices to bound

$$\sum_{x \text{ good}} \sum_{z} \mathbb{P}(A_{x,z}),$$

where the sum is taken over all good x's satisfying the bounds in Corollary 7.11 and all $z \in \mathbb{N}^{\mathrm{Set}_0}$ satisfying the parity constraint in Lemma 7.6.

By (7.12), the leading term of the product in Lemma 7.7 is at most

$$(1 - 2.1^{-2-a\epsilon})^{z(\tilde{M}(x))}$$

for sufficiently large a. Summing up over all possible values of $z(\tilde{M}(x))$ gives us

$$\frac{1}{1 - (1 - 2.1^{-2-a\epsilon})} \leqslant 5 \cdot 2.1^{a\epsilon}. \tag{7.16}$$

From (7.13) and (7.14), we get

$$|\mathrm{Set}_1| \geqslant a(1-4\epsilon) - 2|\mathrm{Set}_2| - 2. \tag{7.17}$$

Also choose a large enough such that

$$p_{\frac{1}{6}} = 1/6 + 1/a^4 \leqslant 0.334 \cdot 1/2. \tag{7.18}$$

Finally, from (7.14) we have

$$\prod_r (2r)^{|\mathrm{Set}_{3,r}|} \leqslant 2^{\sum_r r|\mathrm{Set}_{3,r}|} \leqslant 2^{\sum_r 2(r-1)|\mathrm{Set}_{3,r}|} \leqslant 2^{2a\epsilon}. \tag{7.19}$$

Fix x and write $m := |\mathrm{Set}_2|$. By Lemma 7.7 and the distributive property, we have

$$\begin{aligned}
&\sum_z \mathbb{P}(A_{x,z})\\
&\leqslant \quad 5\cdot 2.1^{a\epsilon} \prod_{i\in\mathrm{Set}_1}\left(\sum_{z\geqslant 1} 2^{1-2z}\right)\prod_{i\in\mathrm{Set}_2}\left(\sum_{z\geqslant 1} 0.334\cdot 2^{1-2z}\right)\prod_r \prod_{i\in\mathrm{Set}_{3,r}}\left(\sum_{z\geqslant 0}\left(1-\frac{1}{2r}\right)^z\right)\\
&\leqslant \quad 5\cdot 2.1^{a\epsilon}(2/3)^{a(1-4\epsilon)-2m-2}(2/9)^m 1.002^m \prod_r (2r)^{|\mathrm{Set}_{3,r}|}\\
&\leqslant \quad 15\cdot 2.1^{a\epsilon}(2/3)^{a(1-4\epsilon)}.501^m 2^{2a\epsilon}\\
&\leqslant \quad 15\cdot 50^{\epsilon a}(2/3)^a.501^m,
\end{aligned}$$

where we've used the fact that all $z(i)$'s are odd for $i \in \mathrm{Set}_1 \cup \mathrm{Set}_2$ and equations (7.16), (7.17), (7.18) and (7.19).

Note that the bound we just developed only depends on $m = |\mathrm{Set}_2|$. For $m \in \mathbb{N}$ let

$$\mathcal{W}(m) := \{x : x \text{ is good, } x(p) = ?/1, |\mathrm{Set}_2| = m \text{ and } |\mathrm{Set}_{?/1}| \leqslant \epsilon a + 2\}.$$

Write

$$S_{n,k} = \sum_{i=0}^{k} \binom{n}{i}.$$

Also note that by the binomial theorem, for $a \geqslant 2m$ and $\lambda \geqslant 0$ we have

$$\binom{a-m}{m}(1/2+.001)^m \lambda^{a-2m} \leq (\lambda + 1/2 + .001)^{a-m}. \tag{7.20}$$

Then by (7.15) we have

$$\begin{aligned}
\sum_{x \text{ good}} \sum_z \mathbb{P}(A_{x,z}) &= \sum_m \sum_{x \in \mathcal{W}(m)} \sum_z \mathbb{P}(A_{x,z}) \\
&\leqslant \sum_m a S_{a,\epsilon a+1} \binom{a-m}{m} 2^{\epsilon a} \cdot 15 \cdot 50^{\epsilon a} (2/3)^a .501^m \\
&= 10^{2\epsilon a} S_{a,\epsilon a} \cdot 15 a^2 (2/3)^a \sum_m \binom{a-m}{m} .501^m \\
&\leqslant 10^{2\epsilon a} S_{a,\epsilon a} \cdot 15 a^2 (2/3)^a \sum_m (7/4 + .001)^{a-m} (4/5)^{a-2m} \\
&\leqslant 10^{2\epsilon a} S_{a,\epsilon a} \cdot 150 a^2 (14/15 + .001)^a,
\end{aligned}$$

where in the second last inequality, we use (7.20) by picking a suitable $\lambda = 5/4$.

We complete the proof of Lemma 7.1 by taking a small enough ϵ. Indeed, recall the Chernoff bound

$$\log S_{a,\epsilon a} \leqslant H(\epsilon) a$$

where $H(\cdot)$ is the binary entropy function $H(x) = -x \log x - (1-x)\log(1-x)$. For $\epsilon = 1/200$,

$$\log(14/15 + .001) + 2\epsilon \log 10 + H(\epsilon) < -\frac{1}{100},$$

so the above probability is upper bounded by $\exp(-a/100)$ for all a sufficiently large. $\square$

Chapter 8

SINGLE BLOCK ESTIMATE

Using the results of the previous chapters, which show that the carpet process ω is well behaved in terms of the number of ?'s and 1s, we return to the main task of bounding the probability that a particle is frozen and proving Lemma 4.3.

Define

$$A = \{L(\omega) \leqslant \epsilon a \text{ and } N(\omega) > \epsilon a\} \subset \Omega.$$

Recall

$$\text{Base} = 01?? \cdots ??$$

$$\text{Exit} = 0000 \cdots 00$$

$$\tau_{\text{Exit}} = \inf\{t \geqslant 0 : \omega^t = \text{Exit}\}$$

as well as the definition of an ϵ-Base state.

Our first lemma is obtained by analyzing the bias in the process $N(\omega^t)$. It says that starting with a large enough $N(\omega_0)$, the process is much more likely to return to a state $L(\omega) \leqslant \epsilon a$ than become frozen.

Lemma 8.1. *For any state $\omega_0 \in A$ and sufficiently large a,*

$$\overline{\mathbb{P}}_{\omega_0}(\tau_{Exit} < a^3 \wedge \inf\{t > 0 : L(\omega^t) \leqslant \epsilon a\}) \leqslant e^{-\frac{1}{3}10^{-6}\sqrt{a}}.$$

Proof. Consider the value of

$$\Delta_t N := N(\omega^{t+1}) - N(\omega^t) = (N(\omega_*^{t+1}) - N(\omega^t)) + (N(\omega^{t+1}) - N(\omega_*^{t+1})).$$

Let $\ell(Q^t) := L^t - \min(\text{Range}(Q^t))$ be the maximum distance reached by the random walk path Q^t to its left. By the definition of the 'refresh' step, the first term

$$N(\omega_*^{t+1}) - N(\omega^t) \geqslant (\ell(Q^t) \wedge L^t) - 2.$$

When $L^t > \epsilon a$ and $\ell(Q^t) > 0$, this is at least $(\ell(Q^t) \wedge \epsilon a) - 2$; otherwise, we have the lower bound -2. Using Lemma 6.8, the second term

$$N(\omega^{t+1}) - N(\omega_*^{t+1}) \succ -\text{Geo}(p) - 1,$$

where $p = 1 - \sqrt{5/6}$ and $\succ$ is stochastic domination.

Combining both estimates, we obtain that for $t < \inf\{s > 0 : L(\omega^s) \leqslant \epsilon a\}$, $N(\omega^t) - N(\omega^0)$ stochastically dominates a sum of the form

$$S_t = \sum_{i=1}^{B_{t-1}} Z_i - \sum_{i=1}^{t} (Y_i + 3),$$

where $B_{t-1} \sim \text{Binomial}(t-1, 1/2)$, $Y_i \sim \text{Geo}(p)$ are iid, and Z_i are iid positive variables with tail $\mathbb{P}(Z = q) \geqslant \frac{1}{2q^2}$ for $q \in [1, \epsilon a]$. Our aim is to show that for $t < a^3$, $S_t \geqslant -\epsilon a$ with high probability, which implies that with the same probability, for $t < a^3 \wedge \inf\{s > 0 : L(\omega^s) \leqslant \epsilon a\}$, $N(\omega^t) > 0$, i.e. τ_{Exit} has not occurred.

To prove this, first observe that throwing out the positive sum in S_t and using a tail bound for a sum of Geometric random variables,

$$\mathbb{P}(S_{\epsilon a/100} < -\epsilon a/2) \leqslant \mathbb{P}(\sum_{i=1}^{\epsilon a/100} Y_i > \epsilon a/3) \leqslant \exp(-\epsilon a/100).$$

To see that $N(\omega^t)$ stays away from 0 at later times, we carry out the following concentration bound for the sum of Z_i. Observe that by standard tail bounds for Binomials (e.g. a Chernoff bound), for any $q = 1, 2, \ldots, a^{1/4}$ and any $t > \epsilon a/100$,

$$\begin{aligned}\mathbb{P}\left(\#\{i \leqslant B_{t-1} : Z_i = q\} \leqslant \frac{1}{12} t q^{-2}\right) &\leqslant \mathbb{P}(\text{Bin}(t-1, \frac{1}{2} \cdot \frac{1}{2} q^{-2}) \leqslant \frac{1}{12} t q^{-2}) \\ &\leqslant \exp(-t q^{-2}/72) \\ &\leqslant \exp(-\epsilon \sqrt{a}/7200).\end{aligned}$$

By a union bound over all such q, none of these events occurs with probability at least $1 - \exp(-\frac{\sqrt{a}}{2*10^6})$ if a is sufficiently large, and if none of them occurs, we have

$$\sum_{i=1}^{B_{t-1}} Z_i \geqslant \sum_{q=1}^{a^{1/4}} \frac{1}{12} t q^{-1} \geqslant \frac{1}{50} t \log a.$$

For the negative part of S_t, again using a similar tail bound for a sum of Geometrics, for any $t > \epsilon a/100$,

$$\mathbb{P}\left(\sum_{i=1}^{t} (Y_i + 3) \geqslant t(3 + 10p^{-1})\right) \leqslant \exp(-a/10^4).$$

Combining these bounds, since we can take sufficiently large $a \geqslant e^{6000}$ so that $\frac{1}{50}\log(a) > 3 + 10p^{-1}$, we obtain for any $t > \epsilon a/100$,

$$\mathbb{P}(S_t < -\epsilon a) \leqslant \exp(-\frac{1}{2} 10^{-6} \sqrt{a}).$$

Taking a union bound over all values $t \leqslant a^3$ gives the result.

□

The next lemma gives a lower bound on the frequency at which the carpet process ω^t refreshes by returning to the Base state.

Lemma 8.2. *The following hold for a sufficiently large. If ω_0 satisfies $L(\omega_0) \leqslant a$,*

$$\underline{\mathbb{P}}_{\omega_0}(\omega^1 = \mathit{Base} \text{ or } \omega^2 = \mathit{Base}) \geqslant \frac{1}{600a^2}.$$

Additionally,

$$\overline{\mathbb{P}}_{\omega_0}(\tau_{\mathit{Base}} > a^3) \leqslant \exp(-a/2000),$$

where $\tau_{\mathit{Base}} = \inf\{t > 0 : \omega^t = \mathit{Base}\}$.

Proof. The second claim follows immediately by repeated application of the first claim, bounding the probability of hitting the Base state independently for every successive pair of times:

$$\overline{\mathbb{P}}_{\omega_0}(\tau_{\text{Base}} > a^3) \leqslant (1 - \frac{1}{600a^2})^{a^3/2} \leqslant \exp(-a/2000).$$

One way to return to the base state is by first taking an excursion to the left beyond 0 and not matching the parity at 1 – this makes $L(\omega^1) = 1$ – and then, at the next step, taking an excursion to the right beyond a and not matching the parity at 2. Recall that the probability for a simple random walk excursion to reach distance at least ℓ' is $\frac{1}{\ell'}$, so the probability for an excursion to reach distance $\ell' \in [a, a^2)$ is at least $\frac{1}{2a}$. Note that by the choice of K, an excursion that reaches maximum distance $\ell' \in [a, a^2)$ visits every point in the block but does not reach any neighboring block. Thus if $L(\omega_0) \geqslant 2$, then by Lemma 6.1

$$\underline{\mathbb{P}}_{\omega_0}(\omega^2 = \text{Base}) \geqslant \left(\frac{1}{2} \cdot \frac{1}{2a} \cdot \frac{1}{6}\right)^2 \geqslant \frac{1}{600a^2}.$$

If $L(\omega_0) = 1$, only the second step is necessary. □

Now we are ready to state Lemma 8.3 which is the culmination of the previous two chapters. It shows that in a typical scenario, the carpet process keeps refreshing itself by returning to the Base rather than become frozen.

Lemma 8.3. *For a sufficiently large,*

$$\overline{\mathbb{P}}_{\epsilon\text{-}Base}(\tau_{Exit} < \tau_{Base}) \leqslant e^{-\frac{1}{4}10^{-6}\sqrt{a}}$$

Proof. By Lemma 8.2, the process hits the Base in time at most a^3 with exponentially high probability. So it suffices to bound the event that $\tau_{\text{Exit}} < a^3$.

If $\tau_{\text{Exit}} < a^3$ occurs, then either

1. there exists a time $t < \tau_{\text{Exit}}$ with $L(\omega^t) \leqslant \epsilon a$ and $N(\omega^t) \leqslant \epsilon a$ or

2. there exists a time $t < \tau_{\text{Exit}} < a^3$ with $\omega^t \in A$ and $L(\omega^s) > \epsilon a$ for any s such that $t < s < \tau_{\text{Exit}}$.

By Lemma 7.1 the first event occurs with exponentially small probability in a. By Lemma 8.1 and a union bound over all $t < a^3$, the probability of the second event is exponentially small in $\sqrt{a}$. Combining these estimates, we obtain that the process reaches the Exit after a^3 steps with exponentially high probability. □

8.1 *Proof of Lemma 4.3*

Following [10], we first re-parameterize the process by the number of *attempted emissions* from block i, instead of the number of additional input particles from the right. Each attempted emission, as a portion of the carpet/hole toppling procedure described in Section 3.2, is defined as the evolution until the hot particle reaches a new block or until the hot particle becomes frozen.

Denote by $\text{Left}(k)$ and $F(k)$ the number of particles emitted to the left from block i and the number of frozen particles in block i after the kth attempted emission respectively (a slight abuse of notation). Let $e(0) = 0$ and $e(k)$ denote the number of steps taken by the auxiliary carpet process ω^t until the completion of the kth attempted emission. Note that $e(k)$ counts steps of the chain ω^t, not individual topplings in the carpet/hole procedure.

The following lemma gives a conclusion in the same spirit as Lemma 8.3, but for a process not necessarily started from the Base or ϵ-Base state.

Lemma 8.4. *The following holds for a sufficiently large. If the initial carpet (η, ω) is valid, for any $k \geqslant 2$, almost surely,*

$$\mathbb{P}_{(\eta,\omega)}(F(k) = 1|\mathcal{F}_{i-1}) \leqslant 8a^{-1}.$$

Proof. We consider three cases.

Case I: $F(k-2) = 0$ and $F(k-1) = 0$. In this case, the carpet process keeps performing random walk excursions from the hole, until one such excursion reaches a neighboring block. Recall that the probability that a random walk excursion reaches maximum distance at least l is $\frac{1}{l}$, and the distance from any point in block i to a neighboring block is at most K and at least $K/2$. Thus each excursion of the hot particle in block i has probability at least a^{-4} and at most $2a^{-4}$ of reaching a neighboring block, uniformly over the initial position of the excursion (inside block i). Elementary estimates give that for $k \geqslant 2$,

$$\overline{\mathbb{P}}_{\omega_0}(e(k-1) - e(k-2) > a^3) \leqslant 1 - (1 - 2a^{-4})^{a^3} \leq 2a^{-1} \tag{8.5}$$

and

$$\overline{\mathbb{P}}_{\omega_0}(e(k) - e(k-2) < 2a^5) \leqslant (1 - a^{-4})^{2a^5} + 2a(1 - a^{-4})^{2a^5-1}$$
$$\leq a^{-1}.$$

Note that the probability of the $k-1$th emission being followed by a failed re-arrival is at most $a/K = 1/a^3$. Following the comment in Section 6.3, we consider the auxiliary carpet process started at $e(k-2)$, and assume $Q^{e(k-1)-1}$ is not a failed re-arrival so that all results since Chapter 6 apply until time $e(k)$.

By Lemma 8.2, for any $k \geqslant 2$ we have

$$\overline{\mathbb{P}}_{\omega_0}(\omega^t \neq \text{Base},\ t \in [e(k-2), e(k-2) + a^3]) \leqslant \exp(-a/2000).$$

By Lemma 8.3 and the previous line, since there are at most $2a^5$ returns to the Base state up to time $2a^5$, for any $k \geqslant 2$,

$$\overline{\mathbb{P}}_{\omega_0}(\omega^t = \text{Exit for some } t \in [e(k-2) + a^3, e(k-2) + 2a^5]) \leqslant 2a^5 \exp(-\frac{1}{4}10^{-6}\sqrt{a}).$$

Combining all these estimates and taking a sufficiently large gives an upper bound $4a^{-1}$ in the first case.

Case II: $F(k-2) = 1$. By (F3), we have $e(k-1) = e(k-2) + 1$, cf. (8.5), and in the $(k-1)$th attempted emission, we topple the hot particle started from 0 or a until it reaches a neighboring block. Since we've chosen $K = a^4$, the probability that the hot particle reaches a neighboring block without visiting the entire block $[0, a]$ is at most $a/K = 1/a^3$: it must arrive at one endpoint of $[0, a]$ first, and then reach a neighboring block without touching the other endpoint. By Lemma 6.2 for type 1 paths,

$$\overline{\mathbb{P}}_{\omega_0}(\omega^{e(k-1)} \text{ is not } \epsilon\text{-Base} \mid [0, a] \subset \text{Range}(Q^{e(k-1)-1})) \leqslant (5/6)^{\epsilon a} \leqslant \exp(-a/1200).$$

Moreover, a similar argument using an upper tail bound on $e(k) - e(k-1)$ and Lemma 8.3 as in Case I shows that

$$\overline{\mathbb{P}}_{\omega_0}(F(k) = 1 \mid \omega^{e(k-1)} \text{ is } \epsilon\text{-Base}) \leqslant a^5 \exp\frac{1}{4}10^{-6}\sqrt{a}.$$

Combining all these, we obtain an upper bound $2a^{-1}$ for sufficiently large a in the second case.

Case III: $F(k-1) = 1$. By (F3) and Lemma 6.2, with probability at least $1 - 2/a^3$ we have $\omega^{e(k)}$ is not an all-zero configuration and thus $F(k) = 0$.

Taking a union bound over the three cases completes the proof. $\square$

Proof of Lemma 4.3. We use Lemma 8.4 to prove the single block estimate. We will focus on the first statement because the second one will be shown in (8.7) as an intermediate step. Fix an $l \geqslant 0$, and block $i \in \{0, \dots, n-1\}$. Our aim is to uniformly bound the expression

$$\mathbf{E}\left[\sum_{s\geqslant 2} e^{cF_i^i(s)} 1\{L_i^i(s) = l\} \middle| \mathcal{F}_{i-1}\right].$$

Note that each particle added at site $iK + a$ causes at least one attempted emission in block i (possibly more if additional particles arrive from the left of block i as a result). Thus the sum is upper bounded by

$$\widetilde{\mathbf{E}}\left[\sum_{k\geqslant 2} e^{cF(k)} 1\{\mathrm{Left}(k) = l\}\right],$$

where we use $\widetilde{\mathbf{E}}$ for the conditional expectation w.r.t. $\mathcal{F}_{i-1}$. Set

$$\tau_l = \inf\{k \geqslant 0 : \mathrm{Left}(k) = l\},$$

and re-write the latter sum (over $k \geqslant 2$) as

$$\widetilde{\mathbf{E}}\left[\sum_{k=\tau_l\vee 2}^{\tau_{l+1}-1} e^{cF(k)}\right] = \widetilde{\mathbf{E}}\left[\sum_{k\geqslant 0} 1\{\tau_{l+1} - (\tau_l \vee 2) > k\} e^{cF((\tau_l\vee 2)+k)}\right]. \tag{8.6}$$

We claim that for any $k, l \geqslant 0$,

$$\widetilde{\mathbb{P}}(\tau_{l+1} - \tau_l \geqslant k) \leqslant (2/3)^{\lfloor k/2 \rfloor} \tag{8.7}$$

and

$$\widetilde{\mathbf{E}}[e^{cF((\tau_l\vee 2)+k)}] \leqslant 1 + 100e^c a^{-1}(k + \log(a)). \tag{8.8}$$

For (8.7), note that either the $k+1$st attempted emission is a successful emission, or it is frozen in which case the $k+2$nd attempted emission is emitted to a neighboring block with probability 1 by (F3). By gambler's ruin, the probability to be emitted to the right in either case is at most $\frac{K}{2K-a} \leqslant 2/3$. Thus

$$\widetilde{\mathbb{P}}(\mathrm{Left}(k+2) = \mathrm{Left}(k)) \leqslant 2/3.$$

This proves the first claim.

For (8.8), when $l = 0$, we have $\tau_l = 0$ and the claim follows directly from Lemma 8.4. For $l \geqslant 2$, note that $\tau_l \geqslant \tau_{l-2} + 2$. By (8.7),

$$\widetilde{\mathbb{P}}(\tau_l - \tau_{l-2} > k') \leqslant 2(\frac{2}{3})^{\lfloor k'/4 \rfloor}$$

Moreover, Lemma 8.4 implies that

$$\widetilde{\mathbb{P}}(F(j) = 1 \text{ for some } j \in [\tau_{l-2} + 2, \tau_{l-2} + k' + k]) \leqslant 8(k + k')a^{-1}.$$

Combining these two estimates into a union bound by setting $k' = \lceil 12\log(a) \rceil$ yields the result. When $l = 1$, a similar argument works by replacing τ_{l-2} with 0. This proves (8.8).

Now we turn back to the task of bounding the sum (8.6). Putting the above bounds (8.7) and (8.8) together, we get

$$\widetilde{\mathbf{E}}\left[1\{\tau_{l+1} - (\tau_l \vee 2) > k\} e^{cF((\tau_l \vee 2)+k)}\right] \leqslant \min\{1 + 100e^c a^{-1}(k + \log(a)), (2/3)^{\lfloor k/2 \rfloor} e^c\}.$$

Splitting the sum (8.6) at $k = \lceil 6\log(a) \rceil$, and using both bounds, the original sum is bounded by

$$c' = 7\log(a) + 2500e^c \log^2(a) a^{-1}$$

Finally, we pick $a = e^{1.1c}$ and large enough $c \geqslant 4 \times 10^4$ so that the condition $\delta c = 0.0004c > \log c'$ holds, which proves Lemma 4.3 for large enough a.

It remains to obtain the explicit values for a and K. Note that throughout Chapters 7 and 8, we've taken a sufficiently large multiple times. Two sharpest constraints, however, are $a \geqslant e^{6000}$ in the proof of Lemma 8.1 to achieve the bias of the random walk, and $a \geqslant e^{4.4\times10^4}$

above for Lemma 4.3 to work under a very small $\delta = .0004$. Thus, we conclude that Lemma 4.3 holds for $a \geqslant e^{4.4\times 10^4}$ and $K = a^4 \geqslant e^{1.76\times 10^5}$. □

Chapter 9

ACTIVATED RANDOM WALK ON $\mathbb{Z}^2$

We study the dynamics of the simple symmetric Activated Random Walk (ARW) on the square lattice $\mathbb{Z}^2$, with sleep rate λ and initial density ζ. Initially all particles are active. For every sleep rate λ, the system exhibits the so-called *absorbing-state phase transition*: when the initial density of particles is below a critical value $\zeta_c(\lambda)$, the system fixates; above $\zeta_c(\lambda)$, the system stays active a.s. It is known that $\zeta_c(\lambda) \leq 1$ for all λ.

Theorem 9.1. *There is a constant $C > 0$ such that for all $\lambda < 1$,*

$$\zeta_c(\lambda) \leq \frac{C}{\ln(1/\lambda)}.$$

In other words, the system stays active a.s. if $\zeta > \frac{C}{\ln(1/\lambda)}$.

Corollary 9.2. *For small enough λ, the critical density $\zeta_c(\lambda) < 1$.*

Corollary 9.3. *The critical density $\zeta_c(\lambda) \to 0$ as $\lambda \to 0$.*

In one dimension, similar results as Corollary 9.2 and 9.3 were first proved in the breakthrough work of [4]. The approach from [4] was later reformulated in [16] as based on two major ingredients: a mass balance equation between blocks and a single-block estimate. More recently, this reformulation has been adopted successfully in [2, 10] to find the right order of asymptotics in Corollary 9.3 and extend Corollary 9.2 for all $\lambda > 0$ respectively. All those results rely crucially on the nice property of $\mathbb{Z}$ that any point in the line separates it into two disjoint parts, in order to obtain some independence structure between blocks. However, this topological fact breaks down in $\mathbb{Z}^2$, making the block arguments rather unwieldy in two dimensions. In this note, we take a step back and consider the mass balance equations between *sites* instead of blocks. Our main goal is to show that a similar line of arguments still works in two dimensions and yields non-trivial results.

In order to establish Theorem 9.1, it suffices to prove the following quantitative estimate for the *finite-volume dynamics*, see [16, Theorem 2.11]. For any positive integer N, let $\mathcal{B}_N := \{-N, \dots, N\}^2$ be the finite box in $\mathbb{Z}^2$ and let $|\mathcal{B}_N|$ be the cardinality of $\mathcal{B}_N$. Consider the ARW dynamics on $\mathcal{B}_N$ where the walks are killed upon leaving $\mathcal{B}_N$. Without loss of generality, we only consider the initial configurations with at most $|\mathcal{B}_N|$ particles in B_N. The finite-volume dynamics eventually stabilizes, when all particles that remain in $\mathcal{B}_N$ are sleeping. We call $S(\mathcal{B}_N)$ the number of sleeping particles in $\mathcal{B}_N$ after stabilization.

Theorem 9.4. *There exist positive constants C and c, such that for any integer N and any initial configuration in $\mathcal{B}_N$,*

$$\mathbb{P}\left(S(\mathcal{B}_N) \geq \frac{C}{\ln(1/\lambda)}|\mathcal{B}_N|\right) \leq e^{-cN^2}.$$

9.1 Modified site-wise representation

Formally, the configuration of the system at time $t \geq 0$ is given by $\eta_t \in \{0, \mathfrak{s}, 1, 2, \dots\}^{\mathbb{Z}^2}$, where $\mathfrak{s}$ represents a sleeping particle. Write $x \sim y$ if x is a neighbor of y. For $x \sim y$, the *movement* transition $\mathfrak{t}_{x,y}$ is defined by

$$\mathfrak{t}_{x,y}(\eta)(z) = \begin{cases} \eta(y) + 1, & z = y, \\ \eta(x) - 1, & z = x, \\ \eta(z), & \text{otherwise}, \end{cases}$$

where we use the convention $\mathfrak{s} + 1 = 2$.

Due to the Abelian property, one may consider the site-wise representation of ARW. We will, however, use a slight modification of the usual site-wise representation. Every site $x \in \mathbb{Z}^2$ is associated with one stack of i.i.d. movement instructions $(\xi_k^x)_{k \in \mathbb{N}^+}$, together with another independent stack of i.i.d. geometric random variables $(g_k^x)_{k \in \mathbb{N}}$. More specifically, each movement ξ_k^x has the distribution of $\mathfrak{t}_{x,y_k}$, where y_k is a uniformly random neighbor of x, while each geometric g_k^x encodes the number of sleep instructions between consecutive movements ξ_k^x and ξ_{k+1}^x, thus taking value in $\{0, 1, 2, \dots\}$ with success probability $1/(1+\lambda)$.

For $x \sim y$, let $n_{x,y}(m)$ be the number of times that $\mathfrak{t}_{x,y}$ appears in the first m movement instructions $(\xi_k^x)_{k=1}^m$. Also let $\chi_x(m) := 1\{g_m^x > 0\}$ be the Bernoulli random variable with success probability $\lambda/(1+\lambda)$.

9.2 Mass balance equation

In this section we focus on the finite-volume dynamics and use the mass balance equation to prove Theorem 9.4.

We start by introducing three *random* fields $\mathbf{M} = (M_x) \in \mathbb{N}^{\mathcal{B}_N}$, $\mathbf{S} = (S_x) \in \{0,1\}^{\mathcal{B}_N}$ and $\boldsymbol{\Phi} = (\Phi_x) \in \mathbb{N}^{\partial \mathcal{B}_N}$. First, define the *activity odometer field* M_x which counts the number of movement instructions used at x until stabilization. Note that in M_x, we do *not* count any sleep instruction. Secondly, let S_x be the indicator random variable of there being a sleeping particle at x after stabilization. Thirdly, define the *exit measure* Φ_x to be the number of particles killed upon entering a site $x \in \partial B_N$. Here, for any $A \subseteq \mathbb{Z}^2$, we define the boundary set $\partial A := \{x \in A^c; \text{there exists } y \in A \text{ such that } y \sim x\}$.

Now suppose we're given *deterministic* fields $\mathbf{m} = (m_x) \in \mathbb{N}^{\mathcal{B}_N}$, $\mathbf{s} = (s_x) \in \{0,1\}^{\mathcal{B}_N}$ and $\boldsymbol{\phi} = (\phi_x) \in \mathbb{N}^{\partial \mathcal{B}_N}$. A tuple of fields $(\mathbf{m}, \mathbf{s}, \boldsymbol{\phi})$ is said to satisfy the *mass balance equation* if

(i) the mass balance equation holds at every $x \in \mathcal{B}_N$:

$$\eta_0(x) + \sum_{y \sim x} n_{y,x}(m_y) = m_x + s_x, \tag{9.5}$$

where η_0 denotes the initial configuration;

(ii) the *boundary condition* is met at every $x \in \partial \mathcal{B}_N$:

$$n_{R(x),x}(m_{R(x)}) = \phi_x, \tag{9.6}$$

where $R(x)$ is the unique neighbor of x contained in $\mathcal{B}_N$.

Note that the tuple of random fields $(\mathbf{M}, \mathbf{S}, \mathbf{\Phi})$ defined above always satisfies the mass balance equation.

To prove Theorem 9.4, we rely on the following entropy bound on all possible field configurations satisfying the mass balance equation. Denote by $\mathcal{M}(x, \mathbf{m}, \mathbf{s}, \boldsymbol{\phi})$ the event where equation (9.5) holds at $x \in \mathcal{B}_N$; for $x \in \partial\mathcal{B}_N$, we use the same notation for the event where equation (9.6) holds at x. Also note that $\|\mathbf{\Phi}\|_\infty \leq |\eta_0|$, where $|\eta_0| := \sum_{x\in\mathcal{B}_N} \eta_0(x)$.

Lemma 9.7. *There exists $c_1 > 0$ such that for any $N \in \mathbb{N}^+$ and initial configuration η_0 in $\mathcal{B}_N$ with $|\eta_0| \leq |B_N|$,*

$$\sum_{\mathbf{m},\mathbf{s},\boldsymbol{\phi}} \mathbb{P}\left(\bigcap_{x\in\mathcal{B}_N\cup\partial\mathcal{B}_N} \mathcal{M}(x, \mathbf{m}, \mathbf{s}, \boldsymbol{\phi}) \right) \leq e^{c_1|\mathcal{B}_N|}, \tag{9.8}$$

where the summation is over $\|\boldsymbol{\phi}\|_\infty \leq |\eta_0|$ and arbitrary $\mathbf{m}, \mathbf{s}$.

The proof of Lemma 9.7 will be given in the next section.

Proof of Theorem 9.4. Recall from Theorem 9.4 that $S(\mathcal{B}_N)$ represents the number of sleeping particles in $\mathcal{B}_N$ after stabilization, so we have $S(\mathcal{B}_N) = \sum_{x\in\mathcal{B}_N} S_x$. Note also that

$$S_x \leq \chi_x(M_x),$$

where $\chi_x(m)$ is defined in Section 9.1. Thus in order to prove Theorem 9.4, it suffices to upper bound $\mathbb{P}(\mathcal{A}(\mathbf{M}))$, where $\mathbf{M}$ is the odometer field, and for a fixed $\mathbf{m}$, we use $\mathcal{A}(\mathbf{m})$ to denote the event that

$$\sum_{x\in\mathcal{B}_N} \chi_x(m_x) \geq \frac{C}{\ln(1/\lambda)} |\mathcal{B}_N|.$$

By decomposing over the fields we get $\mathbb{P}(\mathcal{A}(\mathbf{M}))$ is equal to

$$\sum_{\mathbf{m},\mathbf{s},\boldsymbol{\phi}} \mathbb{P}\left(\mathcal{A}(\mathbf{m}) \bigcap \{(\mathbf{m}, \mathbf{s}, \boldsymbol{\phi}) = (\mathbf{M}, \mathbf{S}, \mathbf{\Phi})\} \bigcap \bigcap_{x\in\mathcal{B}_N\cup\partial\mathcal{B}_N} \mathcal{M}(x, \mathbf{m}, \mathbf{s}, \boldsymbol{\phi}) \right)$$
$$\leq \sum_{\mathbf{m},\mathbf{s},\boldsymbol{\phi}} \mathbb{P}\left(\mathcal{A}(\mathbf{m}) \bigcap \bigcap_{x\in\mathcal{B}_N\cup\partial\mathcal{B}_N} \mathcal{M}(x, \mathbf{m}, \mathbf{s}, \boldsymbol{\phi}) \right)$$

$$\leq \sum_{\mathbf{m},\mathbf{s},\boldsymbol{\phi}} \mathbb{P}\left(\mathcal{A}(\mathbf{m})\right) \mathbb{P}\left(\bigcap_{x \in \mathcal{B}_N \cup \partial \mathcal{B}_N} \mathcal{M}(x, \mathbf{m}, \mathbf{s}, \boldsymbol{\phi}) \right), \tag{9.9}$$

where in the last inequality we use the independence between $\chi_x(m)$'s and $n_{x,y}(m)$'s. For any fixed $\mathbf{m}$, the probability of $\mathcal{A}(\mathbf{m})$ is controlled via Chernoff bound by

$$e^{-D_{\mathrm{KL}}\left(\frac{C}{\ln(1/\lambda)} \middle\| \frac{\lambda}{1+\lambda}\right) |\mathcal{B}_N|} \tag{9.10}$$

where

$$D_{\mathrm{KL}}(p_1 \| p_2) = p_1 \ln \frac{p_1}{p_2} + (1-p_1) \ln \left(\frac{1-p_1}{1-p_2} \right)$$

is the Kullback-Leibler divergence between Bernoulli distributed random variables with parameters p_1 and p_2 respectively. One can check that the divergence term in (9.10) grows at least linearly in C uniformly for all $\lambda \in (0,1)$. This fact, combined with (9.9), (9.10) and Lemma 9.7 completes the proof of Theorem 9.4. □

9.3 Entropy bound

In this section we prove the entropy bound in Lemma 9.7. Partition $\mathcal{B}_N$ into a set of cycles $\mathcal{B}_N = \coprod_{n=0}^N C_n$, where each $C_n := \{x \in \mathbb{Z}^2; \|x\|_\infty = N - n\}$. Set $C_{-1} := \partial \mathcal{B}_N$. Recall from Section 9.1 that there is a stack of movement instruction $(\xi_k^x)_{k \in \mathbb{N}^+}$ at every $x \in \mathcal{B}_N$. Consider the filtration $(\mathcal{F}_n)_{n=-1}^N$, where $\mathcal{F}_{-1}$ is the trivial σ-field and for $n \geq 0$, each $\mathcal{F}_n$ is the σ-field generated by the instructions $\{\xi_k^x; k \in \mathbb{N}^+, x \in C_i, 0 \leq i \leq n\}$ in the outermost $n+1$ cycles.

Lemma 9.11. *Fix $N, \eta_0, \mathbf{s}$ and $\boldsymbol{\phi}$. For every $n = 0, \ldots, N$, a.s.,*

$$\mathbf{E}\left[\sum_{\mathbf{m}_n} \prod_{x \in C_{n-1}} 1_{\mathcal{M}(x,\mathbf{m},\mathbf{s},\boldsymbol{\phi})} \middle| \mathcal{F}_{n-1} \right] \leq 4^{|C_n|}, \tag{9.12}$$

where $\mathbf{m}_n := (m_x)_{x \in C_n}$ is the restriction of $\mathbf{m}$ to C_n, and the summation is over all $\mathbf{m}_n \in \mathbb{N}^{C_n}$.

Proof. Define the backward neighbor function B that maps each $y \in C_n$ to a neighbor of y in C_{n-1}; whenever there are multiple such neighbors, just pick one arbitrarily. Note that

B is a bijection from C_n to $B(C_n)$. Rather than working with (9.12), it suffices to give an upper bound of a slightly larger expression

$$\mathbf{E}\Big[\sum_{\mathbf{m}_n}\prod_{y\in C_n} 1_{\mathcal{M}(B(y),\mathbf{m},\mathbf{s},\boldsymbol{\phi})}\Big|\mathcal{F}_{n-1}\Big]. \tag{9.13}$$

By either the mass balance equation (9.5) when $n \geq 1$ or the boundary condition (9.6) when $n = 0$, the event $\mathcal{M}(B(y), \mathbf{m}, \mathbf{s}, \boldsymbol{\phi})$ above depends on $\mathbf{m}_n$ only through m_y. Thus we may rewrite (9.13) as

$$\mathbf{E}\Big[\prod_{y\in C_n}\sum_{m_y} 1_{\mathcal{M}(B(y),\mathbf{m},\mathbf{s},\boldsymbol{\phi})}\Big|\mathcal{F}_{n-1}\Big]. \tag{9.14}$$

Now note that all randomness defining $\mathcal{M}(B(y), \mathbf{m}, \mathbf{s}, \boldsymbol{\phi})$, except for $n_{y,B(y)}(m_y)$, are measurable with respect to $\mathcal{F}_{n-1}$. Therefore, the sums over m_y in (9.14), conditioned on $\mathcal{F}_{n-1}$, are i.i.d. geometric random variables with parameter $1/4$. This implies that both (9.14) and (9.13) are equal to $4^{|C_n|}$, thus proving (9.12). □

Proof of Lemma 9.7. We will inductively show that for $0 \leq n \leq N$,

$$\mathbf{E}\Big[\sum_{\mathbf{m}_0}\sum_{\mathbf{m}_1}\cdots\sum_{\mathbf{m}_n}\prod_{x\in A_{n-1}} 1_{\mathcal{M}(x,\mathbf{m},\mathbf{s},\boldsymbol{\phi})}\Big] \leq 4^{|A_n\setminus A_{-1}|}, \tag{9.15}$$

where $A_n := \bigcup_{i=-1}^{n} C_i$. Assuming (9.15) for $n = N$, it follows that the left-hand side of (9.8) is bounded above by $4^{|\mathcal{B}_N|} \cdot 2^{|\mathcal{B}_N|} \cdot |\mathcal{B}_N|^{|\partial\mathcal{B}_N|}$. This would imply Lemma 9.7 by choosing c_1 large enough.

It remains to prove (9.15) by induction. The base case of $n = 0$ is nothing but Lemma 9.11 when $n = 0$. Suppose the inequality is true for $n - 1$. By conditioning on $\mathcal{F}_{n-1}$ and using Lemma 9.11, we get

$$\begin{aligned}
&\mathbf{E}\Big[\sum_{\mathbf{m}_0}\sum_{\mathbf{m}_1}\cdots\sum_{\mathbf{m}_n}\prod_{x\in A_{n-1}} 1_{\mathcal{M}(x,\mathbf{m},\mathbf{s},\boldsymbol{\phi})}\Big] \\
&\quad = \mathbf{E}\Big[\sum_{\mathbf{m}_0}\sum_{\mathbf{m}_1}\cdots\sum_{\mathbf{m}_{n-1}}\prod_{x\in A_{n-2}} 1_{\mathcal{M}(x,\mathbf{m},\mathbf{s},\boldsymbol{\phi})}\times \\
&\qquad\qquad \mathbf{E}\Big[\sum_{\mathbf{m}_n}\prod_{x\in C_{n-1}} 1_{\mathcal{M}(x,\mathbf{m},\mathbf{s},\boldsymbol{\phi})}\Big|\mathcal{F}_{n-1}\Big]\Big]
\end{aligned}$$

$$\leq 4^{|C_n|} \cdot \mathbf{E}\Big[\sum_{\mathbf{m}_0} \sum_{\mathbf{m}_1} \cdots \sum_{\mathbf{m}_{n-1}} \prod_{x \in A_{n-2}} 1_{\mathcal{M}(x,\mathbf{m},\mathbf{s},\boldsymbol{\phi})} \Big]$$

$$\leq 4^{|C_n|} \cdot 4^{|A_{n-1} \setminus A_{-1}|} = 4^{|A_n \setminus A_{-1}|}.$$

□

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