Estimation and Control of Nonlinear Hybrid Systems and Nonaffine Control

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Abstract

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This dissertation consists of three parts which present new results in three areas of nonlinear estimation and control. The first part presents systematic methods to synthesize interval observers, which are set-based state estimators, for nonlinear switched systems and nonlinear impulsive systems. Interval observers are designed to ensure positivity and input-to-state (ISS) stability of the error dynamics. The observer gains and other interval observer parameters are synthesized by solving convex programming problems in the forms of linear programs and linear matrix inequalities (LMIs). In general, it is a difficult task to ensure both the positivity and stability of the error dynamics. To overcome this challenge, a common approach is to find a different set of coordinates where the interval observer can be synthesized to ensure positivity and stability more easily. Interval observers for hybrid systems usually require multiple coordinate transformations to be incorporated into the design. For example, in impulsive systems, two coordinate systems are required: one for the continuous part, and one for the jump part.

The second part of this dissertation is focused on estimation and control of nonlinear systems that are implemented on digital platforms. Traditionally, digital control systems are implemented in a periodic fashion. In contrast, self-triggered and event-triggered control/estimation is implemented in an aperiodic fashion by introducing feedback into the

sensing and actuation. The goal is to reduce the amount of sampling compared to periodically sampled controllers/estimators. Systematic methods to design self-triggered estimators and periodic event-triggered controllers for nonlinear systems are presented. These methods guarantee a reduction of sampling.

The final part proposes constructive methods to design controllers for Coulomb spacecraft formations. Coulomb formations are controlled by manipulating the charges of the the spacecraft in the formation, so, by Coulomb's law, the dynamics are nonlinear in the control.

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DEDICATION

To my grandparents Helmi and Harold.

Chapter 0
OVERVIEW OF THE DISSERTATION

This dissertation consists of three parts which present new results in three areas of nonlinear estimation and control.

The first part is focused on the synthesis of interval observers, which are set-based state estimators, for nonlinear hybrid systems. This part is based on the following publications:

- **A.M. Tahir**, X. Xu, and B. Açıkmeşe, "Synthesis of interval observers for polytopic systems and conic systems," *Conference on Decision and Control*, Nice, France, 2019.
- **A.M. Tahir** and B. Açıkmeşe, "ISS Interval Observers for Nonlinear Switched Systems under Constrained Switching," (under review).
- **A.M. Tahir**, X. Xu, and B. Açıkmeşe, "Interval Observer Synthesis for Polytopic Nonlinear Impulsive Systems," (in preparation).

The second part is focused on estimation and control of nonlinear systems with periodic event-triggered and self-triggered sampling. This part is based on the following publications:

- **A.M. Tahir**, X. Xu, and B. Açıkmeşe, "Self-triggered interval observers-based for Lipschitz nonlinear systems," *American Control Conference*, Philadelphia, PA, 2019.
- X. Xu, **A.M. Tahir**, and B. Açıkmeşe, "Stabilization of Nonlinear Systems Using Periodic Event-triggered Control," (under review).

- X. Xu, **A.M. Tahir** and B. Açıkmeşe, "Periodic event-triggered control design for incrementally conic nonlinear systems," *Conference on Decision and Control*, Miami, FL, 2018.

The third part is focused on the construction of stabilizing control laws for Coulomb spacecraft formations which have dynamics that are nonlinear in the input. This part is based on the following publication:

- **A.M. Tahir** and A. Narang-Siddarth, "Constructive nonlinear approach to Coulomb formation control," *AIAA Guidance, Navigation, and Control Conference*, Kissimmee, FL, 2018.

Part I

SET-BASED STATE ESTIMATION FOR NONLINEAR HYBRID SYSTEMS

Chapter 1
INTERVAL OBSERVERS

"As a rule, what is out of sight
disturbs men's minds more seriously
than what they see."

Gallic War
Gaius Julius Caesar

A set-based state estimator provides a compact set to which the state of a system is guaranteed to belong to at each instance of time. There are many different types of set-based state estimators that have been studied in the literature [23, 88, 135, 8]. The vast majority of the literature considers linear systems. There are many applications of set-based estimation such as cyber-security [46], fault detection and isolation [118, 25, 121], safety constraint satisfaction [69, §III.C], robust model-predictive output feedback control [97], and event-triggered output feedback control [35, §4].

An interval observer is a particular type of set-based state estimator which takes advantage of the theory of positive systems. Interval observers have received a lot of attention in the literature due to their effectiveness in providing set-based state estimation for uncertain, nonlinear, and hybrid systems. An interval observer provides an upper bound $\overline{x}$ and a lower bound $\underline{x}$ for the state x at each instance of time. The only information available to $\overline{x}$ and $\underline{x}$ is the output of the system and the bounds on the disturbance and measurement noise. Interval observers are designed by ensuring that the dynamics of the errors $\overline{e} = \overline{x} - x$ and $\underline{e} = x - \underline{x}$ with output feedback are positive systems and input-to-state (ISS) stable with respect to the disturbance and measurement noise.

A positive system is a dynamical system whose state remains elementwise nonnegative for all time if its initial condition is elementwise nonnegative and its input is elementwise nonnegative for all time. See Appendix A for a review of key concepts from positive systems theory that will be used throughout this part of the dissertation. The errors $\overline{e}$ and $\underline{e}$ are defined in such a way that if $\overline{e}$ and $\underline{e}$ are nonnegative vectors, then $\underline{x} \leq x \leq \overline{x}$[1]. Therefore, if the dynamics of $\overline{e}$ and $\underline{e}$ are positive systems, it implies that if $\overline{e}(0) \geq 0$ and $\underline{e}(0) \geq 0$, then $\overline{e}(t) \geq 0$ and $\underline{e}(t) \geq 0$ for all $t \geq 0$. Hence, if $\underline{x}(0) \leq x(0) \leq \overline{x}(0)$, then $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ for all $t \geq 0$. This means that at each instance of time t, an interval to which the state $x(t)$ belongs to is known.

ISS stability of the errors ensures that the errors remain bounded and converge to a bounded set, the size of which, roughly speaking, depends on the size of the disturbance and noise. Therefore, the size of the interval which contains the state depends on the size of the disturbance and noise. See Appendix B for definitions and sufficient conditions for ISS stability which will be used throughout this part.

It is a challenging task to ensure positivity and stability of the error dynamics simultaneously. Positive systems often occur in biological systems because they model phenomena such as population dynamics and concentrations of drugs where quantities are naturally nonnegative [70, 13]. Hence, many of the early results for interval observers were applied to biological systems [65]. Many systems of interest, however, are not intrinsically positive systems, and cannot be made stable and positive using output feedback. To overcome this challenge, the system can be expressed in a different coordinate frame where positivity and stability can be enforced more readily [99, 120].

In recent years, many researchers have been studying the problems of designing interval observers for hybrid systems. All of the available results are for linear hybrid systems: sampled linear systems [101, 51]; switched linear systems in continuous-time [80, 57] and

[1]The inequalities are understood to be elementwise.

discrete-time [48]; and linear impulsive systems [47]. The use of coordinate transformations is more difficult in hybrid systems. In switched systems, the error dynamics is a switched system and each mode of the error dynamics should be positive and stable; however, a single coordinate transformation that makes all of the modes of the error dynamics positive and stable is either difficult to find or does not exist. To overcome this, multiple coordinate transformations are used: one for each mode. In impulsive systems, positivity and stability should be enforced through the continuous flow and the jumps. Again, a single coordinate transformation cannot always be found to ensure positivity and stability of the continuous and jump parts, so multiple coordinate transformations can be incorporated: one for the continuous part and one for the jump part.

In the following chapters, interval observers will be synthesized via the solutions to convex optimization problems. Convex optimization problems can be solved numerically efficiently, and there are many software packages available that can solve them numerically such as CVX [66], YALMIP [95], and others. The convex programs will take the form of semidefinite programs [30]. In these semidefinite programs, ISS stability conditions are given in the form of linear matrix inequalities (LMIs) and positivity conditions are given in the form of linear inequalities. The synthesis of interval observers from the solutions to convex programs has been done extensively in the literature for linear systems [124, 53]; however, there are fewer of such synthesis results for nonlinear and hybrid systems.

Notation Throughout this part, the following notation will be used. $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_{\geq 0}$ denotes the set of nonnegative real numbers. $\mathbb{R}^n$ denotes the set of real vectors of dimension n, $\mathbb{R}^n_{\geq 0}$ denotes the set of elementwise nonnegative vectors of dimension n. $\mathbb{Z}_{\geq 0}$ denotes the set of nonnegative integers, and $\mathbb{Z}_{\geq 1}$ denotes the set of positive integers. $\|\cdot\|$ denotes the Euclidean norm. $|\cdot|$ denotes the element-wise absolute value operator. For a matrix $A \in \mathbb{R}^{n\times m}$, $A^{\oplus}$ is defined elementwise as $A^{\oplus}_{ij} = \max\{0, A_{ij}\}$ and $A^{\ominus} = A^{\oplus} - A$. $A \geq 0$ means that A is nonnegative, i.e., all the elements of A are nonnegative. For two matrices $A, B \in \mathbb{R}^{n\times m}$, $A \geq B \Leftrightarrow A - B \geq 0$. For a matrix $A \in \mathbb{R}^{n\times m}$, A^T denotes the transpose of A. For a square matrix A, $\mathrm{He}(A) = A + A^T$. For a symmetric matrix $P \in \mathbb{R}^{n\times n}$, $P \succ 0 (\succeq 0)$ means that P is positive definite (positive semi-definite). A matrix $A \in \mathbb{R}^{n\times n}$ is Metzler if all of its off-diagonal elements are nonnegative. Denote $\mathcal{M}_n$ as the set of $n \times n$ matrices that are Metzler. The $n \times n$ identity matrix is denoted by I_n. The vector of all ones of dimension n is denoted by $\mathbf{1}_n$. The Kronecker product of A with B is denoted $A \otimes B$. A matrix $\Theta \in \mathrm{Conv}\{\Theta_1, \ldots, \Theta_\nu\}$ means that there exist ν nonnegative scalars $\lambda_1, \ldots, \lambda_\nu \geq 0$ where $\sum_{i=1}^{\nu} \lambda_i = 1$ such that $\Theta = \sum_{i=1}^{\nu} \lambda_i \Theta_i$. A matrix $\Theta \in \mathrm{Cone}\{\Theta_1, \ldots, \Theta_\nu\}$ means that there exist ν nonnegative scalars $\lambda_1, \ldots, \lambda_\nu \geq 0$ such that $\Theta = \sum_{i=1}^{\nu} \lambda_i \Theta_i$.

1.1 Basic Ideas behind the Design of Interval Observers

Here, the basic ideas behind the design of interval observers are illustrated by the design of an interval observer for linear systems. A comprehensive survey can be found in [53]. These ideas will be extended to nonlinear systems and applied to nonlinear hybrid systems in the following chapters.

Consider a system with a state $x(t) \in \mathbb{R}^n$,

$$\dot{x}(t) = Ax(t) + w(t), \tag{1.1}$$

$$y(t) = Cx(t) + v(t). \tag{1.2}$$

A bounded disturbance $w(t)$ acts on the system. $y(t) \in \mathbb{R}^m$ is the output of the system which is corrupted by the bounded noise $v(t)$. It is assumed that $\overline{V} \in \mathbb{R}_{\geq 0}$ and $\underline{w}, \underline{w} \in \mathbb{R}^n$ are known such that

$$\underline{w} \leq w(t) \leq \overline{w}, \tag{1.3}$$

and

$$|v(t)| \leq \overline{V}\mathbf{1}_m, \tag{1.4}$$

for all $t \geq 0$, where $|v(t)|$ denotes the elementwise absolute value of $v(t)$.

The interval observer consists of two estimates $\underline{x}(t)$ and $\overline{x}(t)$. The dynamics of $\underline{x}(t)$ and $\overline{x}(t)$ are designed to use output feedback so that the following objectives hold:

(1) **Framer**: $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ for all $t \geq 0$ if the initial conditions satisfy $\underline{x}(0) \leq x(0) \leq \overline{x}(0)$, which is referred to in the literature as the *framer property* of an interval observer [98].

(2) **Input-to-State Stability**: The dynamics of the follow error is ISS

$$\varepsilon(t) = \begin{bmatrix} \overline{e}(t) \\ \underline{e}(t) \end{bmatrix} = \begin{bmatrix} \overline{x}(t) - x(t) \\ x(t) - \underline{x}(t) \end{bmatrix}. \tag{1.5}$$

Consider the following interval observer:

$$\dot{\overline{x}}(t) = A\overline{x}(t) + L(y(t) - C\overline{x}(t)) + \overline{w} + |L|\overline{V}\mathbf{1}_m, \tag{1.6}$$

$$\dot{\underline{x}}(t) = A\underline{x}(t) + L(y(t) - C\underline{x}(t)) + \underline{w} - |L|\overline{V}\mathbf{1}_m. \tag{1.7}$$

This structure is representative of many interval observers in the literature. It consists of two copies of the system with output feedback terms and terms which bound the measurement noise and disturbance [53]. Now consider the dynamics of the errors $\overline{e}$ and $\underline{e}$:

$$\dot{\overline{e}}(t) = (A - LC)\overline{e}(t) + \overline{d}(t), \tag{1.8}$$

$$\dot{\underline{e}}(t) = (A - LC)\underline{e}(t) + \underline{d}(t), \tag{1.9}$$

where $\overline{d}(t) = \overline{w} - w(t) + |L|\overline{V}\mathbf{1}_m - Lv(t)$, and $\underline{d}(t) = w(t) - \underline{w} + |L|\overline{V}\mathbf{1}_m + Lv(t)$. $\overline{e}$ and $\underline{e}$ are defined such that if $\overline{e} \geq 0$ and $\underline{e} \geq 0$, then $\underline{x} \leq x \leq \overline{x}$. The observer gain L should be designed such that (1.8) and (1.9) are positive systems so that if $\overline{e}(0) \geq 0$ and $\underline{e}(0) \geq 0$, then $\overline{e}(t) \geq 0$ and $\underline{e}(t) \geq 0$ for all $t \geq 0$. This implies that if $\underline{x}(0) \leq x(0) \leq \overline{x}(0)$, then $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ for all $t \geq 0$.

(1.8) is a positive system if and only if $\overline{d}(t) \geq 0$ and $A - LC$ is a Metzler matrix [59, 127]. A matrix is Metzler if and only if all of its off-diagonal elements are nonnegative. Since (1.3) and (1.4) hold, $\overline{d}(t) \geq 0$ for all $t \geq 0$. For $\overline{e}(t)$ to be ISS $A - LC$ must be Hurwitz . The same analysis applies to show positivity and ISS stability of $\underline{e}$.

In summary, the design of an interval observer for the linear system (1.1)-(1.2) boils down to finding an observer gain L such that $A - LC$ is Metzler and Hurwitz; however, this is not always possible. In this case, suppose there exists a matrix $L \in \mathbb{R}^{n\times m}$ and an invertible matrix $U \in \mathbb{R}^{n\times n}$ such that $U^{-1}(A-LC)U$ is Hurwitz and Metzler. L can be found such that $A - LC$ is Hurwitz but not necessarily Metzler, then the matrix U can be constructed using the Jordan normal form in the case where all the eigenvalues of $A - LC$ are real. Otherwise, [120, Lemma 1] can be used to satisfy Assumption 3[2].

Consider the transformation $z = U^{-1}x$. The continuous-time system (1.1)-(1.2) under consideration can be written in the new coordinate system as

$$\dot{z}(t) = U^{-1}AUz(t) + U^{-1}w(t),$$
$$y(t) = CUz(t) + v(t).$$

Consider the following interval observer

$$\dot{\overline{z}}(t) = U^{-1}AU\overline{z}(t) + U^{-1}L(y(t) - CU\overline{z}(t)) + (U^{-1})^{\oplus}\overline{w} - (U^{-1})^{\ominus}\underline{w} + |LU|\,\overline{V}\mathbf{1}_m,$$
$$\dot{\underline{z}}(t) = U^{-1}AU\underline{z}(t) + U^{-1}L(y(t) - CU\underline{z}(t)) + (U^{-1})^{\oplus}\underline{w} - (U^{-1})^{\ominus}\overline{w} - |LU|\,\overline{V}\mathbf{1}_m.$$

The dynamics of $\overline{e}_z(t) = \overline{z}(t) - z(t)$ and $\underline{e}_z(t) = z(t) - \underline{z}(t)$ are positive and ISS since $U^{-1}(A - LC)U$ is Hurwitz and Metzler. This means that if $\underline{z}(0) \leq z(0) \leq \overline{z}(0)$, then $\underline{z}(t) \leq z(t) \leq \overline{z}(t)$ and $\overline{e}_z(t), \underline{e}_z(t)$ remain bounded for all $t \geq 0$.

Consider the following lemma.

---

[2]Note that it is not always possible to find a static matrix U and observer gain L which guarantees that $U^{-1}(A - LC)U$ is Hurwitz and Metzler. A recent development for linear systems has been the use of time-varying changes of coordinates which are derived from the Jordan decomposition of a matrix with complex eigenvalues [98, 99, 41].

Lemma 1 ([120])**.** *If a vector* $x \in \mathbb{R}^n$ *is such that* $\underline{x} \le x \le \overline{x}$ *for* $\overline{x}, \underline{x} \in \mathbb{R}^n$, *then* $A^{\oplus}\underline{x} - A^{\ominus}\overline{x} \le Ax \le A^{\oplus}\overline{x} - A^{\ominus}\underline{x}$.

It follows from Lemma 1 that if $\underline{\xi}_0 \le x(0) \le \overline{\xi}_0$, and the interval observer is initialized by

$$\overline{z}(0) = (U^{-1})^{\oplus}\overline{\xi}_0 - (U^{-1})^{\ominus}\underline{\xi}_0,$$
$$\underline{z}(0) = (U^{-1})^{\oplus}\underline{\xi}_0 - (U^{-1})^{\ominus}\overline{\xi}_0,$$

then $\underline{x}(t) \le x(t) \le \overline{x}(t)$ for all $t \ge 0$ where

$$\overline{x}(t) = U^{\oplus}\overline{z}(t) - U^{\ominus}\underline{z}(t),$$
$$\underline{x}(t) = U^{\oplus}\underline{z}(t) - U^{\ominus}\overline{z}(t).$$

Clearly, $\overline{x}(t)$ and $\underline{x}(t)$ remain bounded if $\overline{z}(t)$ and $\underline{z}(t)$ remain bounded.

For linear discrete time systems, finding an interval observer boils down to finding an observer gain L such that $A - LC$ is both Schur and nonnegative. If both of these properties cannot be satisfied, a transformation matrix U can be found such that $U^{-1}(A - LC)U$ is both Schur and nonnegative (cf. [53, §III.B]). L can be found such that $A - LC$ is Schur but not necessarily nonnegative, then the matrix U can be constructed using the Jordan normal form in the case where all the eigenvalues of $A - LC$ are nonnegative and real. [120, Lemma 1] can be used in other cases by first solving an inverse eigenvalue problem to find a nonnegative matrix R that has the same set of eigenvalues as $A - LC$. More discussion on this process can be found in [52, §II.B] and the references therein.

1.2 Characterization of Incrementally Quadratic Nonlinearities

In the following chapters, interval observers will be synthesized for systems with nonlinearities which satisfy incremental quadratic constraints (δQCs). δQC nonlinearities have been studied for observer design [7, 5], observer-based control [155], and robust model-predictive control [6].

Given a function $p : \mathbb{R}^{n_q} \to \mathbb{R}^{n_p}$, a symmetric matrix $M \in \mathbb{R}^{(n_q+n_p)\times(n_q+n_p)}$ is called an *incremental multiplier matrix* for p if it satisfies the following δQC:

$$\begin{bmatrix} q-\tilde{q} \\ \delta p(q,\tilde{q}) \end{bmatrix}^T M \begin{bmatrix} q-\tilde{q} \\ \delta p(q,\tilde{q}) \end{bmatrix} \geq 0,\ \forall q,\tilde{q} \in \mathbb{R}^{n_q}. \tag{1.10}$$

Clearly, the incremental multiplier matrix is not unique. The multiplier matrices can be derived from the matrix representations of the nonlinearities (see [44, 7] for detail).

Lemma 2 (Lemma 4.6 in [44])**.** *Suppose that $p(q) : \mathbb{R}^{n_p} \to \mathbb{R}^{n_q}$ is a continuously differentiable function with derivative $\frac{\partial p}{\partial q}$ and let Ω be any closed convex set of real matrices such that $\frac{\partial p}{\partial q} \in \Omega$ for all $q \in \mathbb{R}^{n_q}$. Then, for every $q, \tilde{q} \in \mathbb{R}^{n_q}$, there is a matrix $\Theta \in \Omega$ such that*

$$p(q) - p(\tilde{q}) = \Theta(q - \tilde{q}). \tag{1.11}$$

This lemma is the mean-value theorem for vector valued functions which allows for nonlinear systems to be described as linear parameter-varying systems.

This part of the dissertation considers three types of nonlinearities: globally Lipschitz nonlinearities, polytopic nonlinearities, and conic nonlinearities. These nonlinearities satisfy δQCs which will be helpful when formulating convex programs to solve for observer gains [7].

1.2.1 Globally Lipschitz Nonlinearities

A nonlinear function $p(q) : \mathbb{R}^{n_q} \to \mathbb{R}^{n_p}$ is *globally Lipschitz* if there exists a constant $\gamma > 0$, which is referred to as the Lipschitz constant such that

$$\|p(q) - p(\tilde{q})\| \leq \gamma\|q - \tilde{q}\|,\ \forall q, \tilde{q} \in \mathbb{R}^{n_q}. \tag{1.12}$$

The condition (1.12) can be written as an incremental quadratic constraint with the following multiplier matrix

$$M = \lambda \begin{bmatrix} \gamma^2 I_{n_q} & 0 \\ 0 & -I_{n_p} \end{bmatrix} \tag{1.13}$$

for any $\lambda > 0$.

1.2.2 Polytopic Nonlinearties

A nonlinear function is referred to as a polytopic nonlinearty if its Jacobian belongs to a polytope.

Definition 1 ([44, 7]). *A continuously differentiable function $p(q) : \mathbb{R}^{n_q} \to \mathbb{R}^{n_p}$ is called a polytopic nonlinearity if for some matrices $\Theta_1, \ldots, \Theta_\nu \in \mathbb{R}^{n_p \times n_q}$,*

$$\frac{\partial p}{\partial q}(q) \in \Omega = Conv\{\Theta_1, \ldots, \Theta_\nu\} \tag{1.14}$$

for all $q \in \mathbb{R}^{n_q}$. The matrices $\Theta_1, \ldots, \Theta_\nu$ are the vertices of the polytope.

Polytopic nonlinearities satisfy incremental quadratic constraints. See [44, §6.4.1] and [7, §5.2.1] for more information.

Lemma 3 (c.f. [44, 7]). *If p is a polytopic nonlinearity with vertices $\Theta_1, \ldots, \Theta_\nu$, then it satisfies a δQC with the following multiplier matrix*

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix} \tag{1.15}$$

where M_{11}, M_{12}, M_{22} satisfy the following linear matrix relations for $k = 1, \ldots, \nu$:

$$\begin{cases} M_{11} + M_{12}\Theta_k + {\Theta_k}^T M_{12}^T + \Theta_k^T M_{22}\Theta_k \succeq 0, \\ M_{22} \preceq 0. \end{cases} \tag{1.16}$$

Clearly, all polytopic nonlinearities are also globally Lipschitz. So a Lipschitz constant γ can be found and a multiplier matrix in the form of (1.13) for some $\lambda > 0$ can be utilized; however, a multiplier matrix satisfying (1.16) is a better characterization of the polytopic nonlinearity, because it is less conservative as evidenced by the observer synthesis problem of incrementally quadratic systems [44, 7]. Moreover, the polytopic description is appealing because it provides a tight characterization of nonlinearities with bounded Jacobians. Consider the following example of a nonlinearity $p : \mathbb{R}^2 \to \mathbb{R}^2$,

$$p = \begin{bmatrix} \sin q_1 \\ 2\sin q_1 + \sin q_2 \end{bmatrix},$$

which is described by the polytope Ω where $v(\Omega) = 4$,

$$\Omega = \text{Conv}\left\{ \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ -2 & -1 \end{bmatrix} \right\}.$$

This is a better description than the Lipschitz description (i.e. $\|p(q) - p(\tilde{q})\| \leq 2\|q - \tilde{q}\|$) which implies that each element of the Jacobian can vary independently between -2 and 2. In contrast, the polytopic description accounts for the fact that the $(1, 2)$-th element of the Jacobian is always zero and that the $(2, 1)$-th element is always twice the $(1, 1)$-th element of the Jacobian of p.

1.2.3 Conic Nonlinearities

A nonlinear function is referred to as a conic nonlinearity if its Jacobian belongs to a cone.

Definition 2 ([44, 7])**.** *A continuously differentiable function $p(q) : \mathbb{R}^{n_q} \to \mathbb{R}^{n_p}$ is a conic nonlinearity if for some matrices $\Theta_1, \ldots, \Theta_\nu \in \mathbb{R}^{n_p \times n_q}$*

$$\frac{\partial p}{\partial q}(q) \in \Omega = Cone\{\Theta_1, \ldots, \Theta_\nu\} \tag{1.17}$$

for all $q \in \mathbb{R}^{n_q}$. The set of linearly independent matrices $\{\Theta_1, \ldots, \Theta_\nu\}$ is referred to as the conic basis of p.

Conic nonlinearities satisfy incremental quadratic constraints. See [44, §6.4.2] and [7, §5.2.2] for more information.

Lemma 4 (c.f. [44, 7]). *If p is a conic nonlinearity with conic basis $\{\Theta_1, \ldots, \Theta_\nu\}$, then it satisfies a δQC with the following multiplier matrix*

$$M = \begin{bmatrix} 0 & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix} \tag{1.18}$$

where M_{12}, M_{22} satisfy the following linear matrix relations $\forall k = 1, \ldots, \nu$

$$M_{12}\Theta_k + {\Theta_k}^T M_{12}^T \succeq 0, M_{22}\Theta_k = 0, M_{22} \preceq 0. \tag{1.19}$$

Chapter 2

SYNTHESIS OF INTERVAL OBSERVERS FOR NONLINEAR CONTINUOUS-TIME SYSTEMS

"Other planes lie beyond the reach of normal sense and common roads, but they are no less real than what we see or touch or feel."

Lost Wisdom, *Det Som Engang Var*
Burzum

The synthesis of interval observers from the solutions to convex programs has been done extensively in the literature for linear systems [124, 53]; however, there are fewer of such synthesis results for nonlinear systems. This chapter presents new results for synthesizing interval observers for two types of nonlinear systems. The first is detailed in §2.1 for polytopic systems, and the second is detailed in §2.2 for conic systems.

Observer design for systems with globally Lipschitz nonlinearities was first studied in Thau [146], which provided a procedure to check the stability of the error dynamics with a given observer gain from an LMI. In [119], the observer gain is found from the solution to the algebraic Riccatti equation. [122] presents necessary and sufficient conditions for the stability of the error dynamics from the frequency domain properties of the linear part of the error dynamics $A - LC$. [17] extends the results for globally Lipschitz nonlinear systems to systems which satisfy a motononocity condition. The authors present observers which have two output feedback terms: one which is the usual linear correction term, and the other is an output feedback term which is injected into the nonlinearity. In [7], observers for systems with nonlinearities which satisfy δQCs are synthesized via the solution to LMIs. δQC nonlinearities can be parameterized by incremental multiplier matrices and include a wide variety of nonlinearities including: globally Lipschitz nonlinearities, sector-bounded nonlinearities, monotonic nonlinearities, as well as polytopic nonlinearities and conic nonlinearities.

An approach to synthesizing observers for globally Lipschitz nonlinear systems is the LPV approach, where the nonlinear error dynamics is expressed as a linear parameter varying (LPV) system using the mean-value theorem [160, 116, 79]. Through a numerical investigation, the authors of [160] found that the LPV approach for observer synthesis is able to find observer gains for systems with much larger Lipschitz constants than the δQC multiplier method using a Lipschitz multiplier matrix (1.13).

The applications of nonlinear observers are numerous, including: slip-angle estimation for cars [81, 116], spacecraft formation control [90, 24], and spacecraft attitude estimation [113, 147]. Other state estimation techniques for continuous time nonlinear systems are: the extended Kalman filter [82], the unscented Kalman filter [133], moving horizon observers [104], adaptive observers [142], passivity-based observers [61], and high-gain observers [86].

Interval observers for globally Lipschitz nonlinear systems have been studied in [120, 105, 55]. In these papers, the error dynamics are made positive by enforcing the positivity of the linear and nonlinear parts separately. The positivity of the linear part is enforced by the selection of the observer gains, and the positivity of the nonlinear part is enforced by exploiting the mixed monotonicity of globally Lipschitz functions. This introduces coupling between the upper bound and lower bound dynamics. A function is mixed monotone if it can be decomposed into increasing and decreasing parts [156, 42]. In [105], the decomposition is explicitly constructed for globally Lipschitz nonlinearities, and in [120, 55] a mixed monotone decomposition is assumed to be known. Here, conditions for positivity of the proposed interval observers for polytopic systems, which are globally Lipschitz, are relaxed in the sense that the error dynamics can be positive even if either the linear or nonlinear part is not positive on its own. The observer gains and coupling terms are synthesized from convex programs to satisfy the proposed positivity conditions in addition to a stability condition.

For two of the convex programs proposed for polytopic systems, ISS stability is proven with quadratic ISS-Lyapunov functions, so the stability conditions are given in the form of LMIs. One exploits the incremental quadratic constraints (δQCs) that polytopic nonlinearities satisfy [7]. The other is derived using the LPV approach.Due to the positivity constraints, the LPV system that is equivalent to the nonlinear error dynamics also allows for the formulation of linear programs to synthesize the observer gains and coupling terms. In this case, ISS stability is proven with linear copositive ISS-Lyapunov functions [127]. This is appealing because linear programs have lesser computational complexity than semidefinite programs, so interval observers large systems can be handled more readily [127].

The use of coordinate transformations for interval observer design for nonlinear systems (which is also done in the literature [120, 55]) is also considered in this chapter. For polytopic systems, the observer gain and transformation matrix are assumed to be found, then the coupling matrices are synthesized from a linear program.

Few results exist in the literature for the synthesis of interval observers for nonlinear systems that are not globally Lipschitz. In [100], interval observers are designed for triangular systems which are not globally Lipschitz by exploiting the triangular structure; however, the interval observers described in [100] are open-loop with no output feedback. §2.2 considers systems with conic nonlinearities, which are not necessarily globally Lipschitz. Positivity conditions can be derived based on the conic bases of the nonlinearities and a nonlinear injection term. The stability conditions in the proposed convex programs for synthesis are in the form of LMIs that exploit the δQCs. The δQCs for conic nonlinearities hold globally and do not imply boundedness of the Jacobian, so the synthesized interval observers are globally stable.

2.1 Systems with Polytopic Nonlinearities

Consider the following system

$$\dot{x}(t) = Ax(t) + p(x(t)) + w(t), \tag{2.1}$$

$$y(t) = Cx(t) + v(t), \tag{2.2}$$

where the dynamics of the state $x \in \mathbb{R}^n$ has a linear part, a nonlinear part, and a bounded disturbance $w(t)$. The output $y \in \mathbb{R}^m$ is linear in the state and is corrupted by bounded noise $v(t)$. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n_p}$, and $C \in \mathbb{R}^{m\times n}$. $p : \mathbb{R}^n \to \mathbb{R}^n$ is a polytopic nonlinearity defined by the polytope $\Omega = \text{Conv}\{\Theta_1, \dots, \Theta_\nu\} \subset \mathbb{R}^{n\times n}$.

The disturbance $w(t)$ and measurement noise $v(t)$ are unknown, but they are bounded and the bounds on them are assumed to be known.

Assumption 1. *The disturbance belongs to a known interval:* $\underline{w} \leq w(t) \leq \overline{w}$ *for all* $t \geq 0$.

Assumption 2. *The measurement noise is bounded as follows:* $|v(t)| \leq \overline{V}\mathbf{1}_m$ *for all* $t \geq 0$, *and* $\overline{V} \geq 0$ *is known.*

Consider the following interval observer

$$\dot{\overline{x}}(t) = A\overline{x}(t) + \overline{p}(\overline{x}(t), \underline{x}(t)) + \overline{L}(y(t) - C\overline{x}(t)) + \overline{w} + \left|\overline{L}\right| \overline{V}\mathbf{1}_m, \tag{2.3}$$

$$\dot{\underline{x}}(t) = A\underline{x}(t) + \underline{p}(\underline{x}(t), \overline{x}(t)) + \underline{L}(y(t) - C\underline{x}(t)) + \underline{w} - \left|\underline{L}\right| \overline{V}\mathbf{1}_m, \tag{2.4}$$

which is a standard structure for an interval observer for nonlinear continuous-time systems. Define the lumped error,

$$\varepsilon(t) = \begin{bmatrix} \overline{e}(t) \\ \underline{e}(t) \end{bmatrix} = \begin{bmatrix} \overline{x}(t) - x(t) \\ x(t) - \underline{x}(t) \end{bmatrix}. \tag{2.5}$$

ε has the following dynamics

$$\dot{\varepsilon}(t) = \begin{bmatrix} A - \overline{L}C & 0_n \\ 0_n & A - \underline{L}C \end{bmatrix} \varepsilon(t) + \begin{bmatrix} \overline{p}(\overline{x}(t), \underline{x}(t)) - p(x(t)) \\ p(x(t)) - \underline{p}(\underline{x}(t), \overline{x}(t)) \end{bmatrix} + \Delta(t) \tag{2.6}$$

where

$$\Delta(t) = \begin{bmatrix} \overline{w} - w(t) + \left|\overline{L}\right| \overline{V}\mathbf{1}_m + \overline{L}v(t) \\ w(t) - \underline{w} + \left|\underline{L}\right| \overline{V}\mathbf{1}_m - \underline{L}v(t) \end{bmatrix}. \tag{2.7}$$

ε is defined by (2.31) in such a way that if it is a nonnegative vector then $\underline{x} \leq x \leq \overline{x}$. Therefore, the observer gains $\overline{L}, \underline{L}$ and nonlinear functions $\overline{p}$ and $\underline{p}$ should be designed such that (2.6) is a positive system.

There are many possibilities to construct the functions $\overline{p}$ and $\underline{p}$. A particular choice of structure is the following:

$$\overline{p}(\overline{x}, \underline{x}) = p(\overline{x}) + \overline{G}(\overline{x} - \underline{x}), \tag{2.8}$$

$$\underline{p}(\underline{x}, \overline{x}) = p(\underline{x}) + \underline{G}(\underline{x} - \overline{x}), \tag{2.9}$$

where $\overline{G}, \underline{G} \in \mathbb{R}^{n\times n}$. The impetus for this particular choice is that $\overline{G}$ and $\underline{G}$ can be synthesized from convex programs that ensures ISS stability and positivity of the interval observer.

Lemma 2 allows for the nonlinear error dynamics to be written as an LPV system.

Lemma 5. *The nonlinear error dynamics* (2.6) *where* $p : \mathbb{R}^n \to \mathbb{R}^n$ *is a polytopic nonlinearity defined by the polytope* $\Omega = Conv\{\Theta_1, \ldots, \Theta_\nu\}$ *and* $\overline{p}$, $\underline{p}$ *are defined by* (2.8), (2.9), *respectively, is equivalent to an LPV system*

$$\dot{\varepsilon}(t) = \mathcal{A}(\varrho(t))\varepsilon(t) + \Delta(t) \tag{2.10}$$

where $\varrho(t)$ *is a time-varying parameter that belongs to bounded set*

$$\varrho(t) \in \Xi = \left\{ \varrho \in \mathbb{R}^{\nu\times\nu}_{\geq 0} : \sum_{i=1}^{\nu}\sum_{j=1}^{\nu} \varrho_{ij} = 1 \right\},\ \forall t \geq 0, \tag{2.11}$$

and

$$\mathcal{A}(\varrho(t)) = \sum_{i=1}^{\nu}\sum_{j=1}^{\nu} \varrho_{ij}(t)\mathcal{A}_{ij}$$

where

$$\mathcal{A}_{ij} = \begin{bmatrix} A + \Theta_i - \overline{L}C + \overline{G} & \overline{G} \\ \underline{G} & A + \Theta_j - \underline{L}C + \underline{G} \end{bmatrix}. \tag{2.12}$$

Proof. By Lemma 2, there exist ν nonnegative scalars $\overline{\lambda}_1(t), \ldots, \overline{\lambda}_\nu(t)$ such that $\sum_{i=1}^{\nu} \overline{\lambda}_i(t) = 1$ for all $t \geq 0$, and

$$p(\overline{x}(t)) - p(x(t)) = \left(\sum_{i=1}^{\nu} \overline{\lambda}_i(t) \Theta_i \right) (\overline{x}(t) - x(t)).$$

Moreover, there exist ν nonnegative scalars $\underline{\lambda}_1(t), \ldots, \underline{\lambda}_\nu(t)$ such that $\sum_{i=1}^{\nu} \underline{\lambda}_i(t) = 1$ for all $t \geq 0$, and

$$p(x(t)) - p(\underline{x}(t)) = \left(\sum_{j=1}^{\nu} \underline{\lambda}_j(t) \Theta_j \right) (x(t) - \underline{x}(t)).$$

Define $\varrho_{ij}(t) = \overline{\lambda}_i(t) \cdot \underline{\lambda}_j(t)$ for all $(i, j) \in \{1, \ldots, \nu\}^2$ and $t \geq 0$. It is clear that $\varrho_{ij}(t)$ is nonnegative and $\sum_{i=1}^{\nu} \sum_{j=1}^{\nu} \varrho_{ij}(t) = 1$ for all $(i, j) \in \{1, \ldots, \nu\}^2$ and $t \geq 0$. Furthermore,

$$\begin{aligned} \left(\sum_{i=1}^{\nu} \sum_{j=1}^{\nu} \varrho_{ij}(t) \Theta_i \right) (\overline{x}(t) - x(t)) &= \left(\left(\sum_{i=1}^{\nu} \overline{\lambda}_i(t) \Theta_i \right) \left(\sum_{j=1}^{\nu} \underline{\lambda}_j(t) \right) \right) (\overline{x}(t) - x(t)) \\ &= p(\overline{x}(t)) - p(x(t)), \end{aligned}$$

and, by the same reasoning,

$$\left(\sum_{i=1}^{\nu} \sum_{j=1}^{\nu} \varrho_{ij}(t) \Theta_j \right) (x(t) - \underline{x}(t)) = p(x(t)) - p(\underline{x}(t)).$$

Therefore,

$$\begin{bmatrix} A - \overline{L}C & 0_n \\ 0_n & A - \underline{L}C \end{bmatrix} \varepsilon(t) + \begin{bmatrix} \overline{p}(\overline{x}(t), \underline{x}(t)) - p(x(t)) \\ p(x(t)) - \underline{p}(\underline{x}(t), \overline{x}(t)) \end{bmatrix} = \left(\sum_{i=1}^{\nu} \sum_{j=1}^{\nu} \varrho_{ij}(t) \mathcal{A}_{ij} \right) \varepsilon(t).$$

This concludes the proof. □

The LPV description of the nonlinear error dynamics elucidates conditions on the observer gains $\overline{L}, \underline{L}$ and coupling matrices $\overline{G}, \underline{G}$ for the nonlinear error dynamics (2.6) to be a positive system.

Proposition 1. *Consider the system* (2.1)-(2.2) *with a polytopic nonlinearity* $p : \mathbb{R}^n \to \mathbb{R}^n$ *that is defined by the polytope* $\Omega = Conv\{\Theta_1, \ldots, \Theta_\nu\}$ *and suppose assumptions 1 and 2 hold. Consider an interval observer* (2.3)-(2.4) *with* (2.8)-(2.9) *where* $\overline{L}, \underline{L}, \overline{G}$, *and* $\underline{G}$ *are such that*

$$\mathcal{A}_{ij} \in \mathcal{M}_{2n},\ \forall (i,j) \in \{1, \ldots, \nu\}^2, \tag{2.13}$$

where $\mathcal{A}_{ij}$ *is defined by* (2.12). *Then,* $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ *for all* $t \geq 0$ *provided* $\underline{x}(0) \leq x(0) \leq \overline{x}(0)$.

Proof. Since $\varepsilon(t) \geq 0$ for all $t \geq 0$ implies that $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ for all $t \geq 0$ it must be shown that the error dynamics (2.6) is a positive system. This is accomplished by showing that the equivalent LPV system (2.10) is a positive system.

$\mathcal{A}_{ij} \in \mathcal{M}_{2n}$ for all $(i,j) \in \{1, \ldots, \nu\}^2$ implies that $\mathcal{A}(\varrho) \in \mathcal{M}_{2n}$ for all $\varrho \in \Xi$ since any convex combination of Metzler matrices is Metzler. Moreover, by assumption 1

$$\overline{w} - w(t) \geq 0,\ \forall t \geq 0,$$
$$w(t) - \underline{w} \geq 0,\ \forall t \geq 0,$$

and, by assumption 2,

$$\left|\overline{L}\right| \overline{V} \mathbf{1}_m + \overline{L} v(t) \geq 0,\ \forall t \geq 0,$$
$$\left|\underline{L}\right| \overline{V} \mathbf{1}_m - \underline{L} v(t) \geq 0,\ \forall t \geq 0.$$

Hence, $\Delta(t) \geq 0$ for all $t \geq 0$. Therefore, by [13, Lemma VIII.1], (2.6) is a positive system, which implies that (2.6) with (2.8)-(2.9) is a positive system, which yields the result. □

In the literature, $\overline{p}$ and $\underline{p}$ are typically assumed to satisfy the following property:

$$\underline{p}(\underline{x}, \overline{x}) \leq p(x) \leq \overline{p}(\overline{x}, \underline{x}) \tag{2.14}$$

for all $\underline{x}, \overline{x}, x \in \mathbb{R}^n$ such that $\underline{x} \leq x \leq \overline{x}$. Examples of this construction in the literature are the following: [120, Theorem 4], [105, Property 4], and [55, Assumption 1].

If $A-\overline{L}C \in \mathcal{M}_n$ and $A-\underline{L}C \in \mathcal{M}_n$, assumptions 1 and 2 hold, and $\overline{p}, \underline{p}$ satisfy (2.14), then (2.6) is a positive system. Therefore, the positivity conditions can be ensured by ensuring positivity of the linear part and nonlinear parts separately.

(2.14) is satisfied when $p(x)$ is a mixed-monotone function on $\mathbb{R}^n$ and $\overline{p}$, $\underline{p}$ decomposition functions for $p(x)$. Mixed-monotone functions are functions that can be decomposed into increasing and decreasing parts and are defined as the following:

Definition 3 (cf. [42, 156])**.** *$f : \mathcal{X} \to \mathcal{Z}$ is mixed monotone on $\mathcal{X}$ if $\exists g : \mathcal{X} \times \mathcal{X} \to \mathcal{Z}$ satisfying the following:*

i. $g(x, x) = f(x), \forall x \in \mathcal{X}$,

ii. $x_1 \geq x_2 \implies g(x_1, y) \geq g(x_2, y), \forall x_1, x_2, y \in \mathcal{X}$,

iii. $y_1 \geq y_2 \implies g(x, y_2) \geq g(x, y_1), \forall x, y_1, y_2 \in \mathcal{X}$.

g is referred to as the decomposition function for f on $\mathcal{X}$.

The positivity condition (2.13) proposed in Proposition 1 is different from the conditions: $A - \overline{L}C \in \mathcal{M}_n$ and $A - \underline{L}C \in \mathcal{M}_n$, and $\overline{p}, \underline{p}$ satisfy (2.14), which are the typical conditions imposed on interval observers for nonlinear continuous-time systems. For (2.8)-(2.9) to satisfy (2.14), then $\overline{G}, \underline{G} \geq 0$ and the follow should hold to satisfy Definition 3:

$$\overline{G} + \Theta_i \geq 0, \ \forall i \in \{1, \ldots, \nu\},$$
$$\underline{G} + \Theta_i \geq 0, \ \forall i \in \{1, \ldots, \nu\}.$$

The condition (2.13) accounts for the interaction between the linear part and the nonlinear part, whereas the condition (2.14) does not. Consider a two-dimensional example with no disturbance or measurement noise with the following parameters:

$$A = \begin{bmatrix} 0 & -1 \\ -1 & -0.5 \end{bmatrix}, p(x) = \begin{bmatrix} 0 \\ 0.5\sin(x_1) \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Using Theorem 3, $\overline{L}, \underline{L}, \overline{G}$, and $\underline{G}$ are found for an interval observer (2.3)-(2.4) :

$$\overline{L} = \underline{L} = \begin{bmatrix} 2.0030 \\ -1.5000 \end{bmatrix}, \overline{G}, \underline{G} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The interval observer is positive and stable even though neither $A - \overline{L}C$ nor $A - \underline{L}C$ are Metzler and (2.14) does not hold. In fact, the does not exist an $L \in \mathbb{R}^2$ such that $A - LC$ is Metzler. This example demonstrates that the conditions that are derived from the LPV approach are less restrictive that the typical conditions imposed in the literature.

2.1.1 Synthesis from Linear Matrix Inequalities

The following theorems derive ISS-stability conditions using quadratic ISS-Lyapunov functions

$$V(\varepsilon) = \varepsilon^T P \varepsilon. \tag{2.15}$$

The following theorem uses the LPV approach to synthesize the gains and coupling matrices.

Theorem 1. *Consider the system* (2.1)-(2.2) *with a polytopic nonlinearity* $p : \mathbb{R}^n \to \mathbb{R}^n$ *that is defined by the polytope* $\Omega = Conv\{\Theta_1, \dots, \Theta_\nu\}$, *suppose assumptions 1 and 2 hold, and* $x(t)$ *remains bounded for all* $t \geq 0$. *Suppose there exist diagonal matrices* $\overline{P}, \underline{P} \in \mathbb{R}^{n\times n}$, *matrices* $\overline{Y}, \underline{Y} \in \mathbb{R}^{n\times m}$, *matrices* $\overline{W}, \underline{W} \in \mathbb{R}^{n\times n}$, *scalars* $\alpha > 0$ *and* $\gamma > 0$ *such that the* *following holds*

$$\begin{bmatrix} He(\mathcal{Q}_{ij}) + \alpha P & P \\ \star & -\gamma I_{2n} \end{bmatrix} \preceq 0, \ \forall (i,j) \in \{1, \ldots, \nu\}^2, \tag{2.16}$$

$$\mathcal{Q}_{ij} \in \mathcal{M}_{2n}, \ \forall (i,j) \in \{1, \ldots, \nu\}^2, \tag{2.17}$$

$$P \succ 0, \tag{2.18}$$

where

$$P = \begin{bmatrix} \overline{P} & 0_n \\ \star & \underline{P} \end{bmatrix}, \tag{2.19}$$

$$\mathcal{Q}_{ij} = \begin{bmatrix} \overline{P}(A + \Theta_i) - \overline{Y}C + \overline{W} & \overline{W} \\ \underline{W} & \underline{P}(A + \Theta_j) - \underline{Y}C + \underline{W} \end{bmatrix}.$$

Consider an interval observer (2.3)-(2.4) *with* (2.8)-(2.9) *where*

$$\overline{L} = \overline{P}^{-1}\overline{Y}, \underline{L} = \underline{P}^{-1}\underline{Y}, \overline{G} = \overline{P}^{-1}\overline{W}, \underline{G} = \underline{P}^{-1}\underline{W}.$$

The following hold:

i. If $\underline{x}(0) \leq x(0) \leq \overline{x}(0)$*, then* $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ *for all* $t \geq 0$*,*

ii. $\overline{x}(t)$ *and* $\underline{x}(t)$ *are bounded for all* $t \geq 0$*,*

iii. The ultimate bound on $\varepsilon(t)$*, defined in* (2.31)*, depends on* $\|\Delta\|_{[0,\infty)}$*.*

Proof. $\mathcal{Q}_{ij} = P\mathcal{A}_{ij}$, where $\mathcal{A}_{ij}$ is defined in (2.12), with the following variable substitutions

$$\overline{Y} = \overline{P}\,\overline{L}, \underline{Y} = \underline{P}\,\underline{L}, \overline{W} = \overline{P}\,\overline{G}, \underline{W} = \underline{P}\,\underline{G}.$$

Since P is positive definite and diagonal, P^{-1} exists and is diagonal. $\mathcal{A}_{ij} = P^{-1}\mathcal{Q}_{ij}$ is Metzler for all $(i,j) \in \{1, \ldots, \nu\}$ since the matrix product of a positive definite diagonal matrix and a Metzler matrix is Metzler. Hence, by Proposition 1, assertion i holds.

The candidate ISS-Lyapunov function (2.15) evolves as follows,

$$\begin{aligned}\langle \nabla V(\varepsilon(t)), \mathcal{A}(\varrho(t))\varepsilon(t) + \Delta(t)\rangle &= \begin{bmatrix} \varepsilon(t) \\ \Delta(t) \end{bmatrix}^T \begin{bmatrix} \mathrm{He}\left(P\mathcal{A}(\varrho(t))\right) & P \\ \star & 0_{2n} \end{bmatrix} \begin{bmatrix} \varepsilon(t) \\ \Delta(t) \end{bmatrix} \\ &= \begin{bmatrix} \varepsilon(t) \\ \Delta(t) \end{bmatrix}^T \begin{bmatrix} \mathrm{He}\left(P\left(\sum_{i=1}^{\nu}\sum_{j=1}^{\nu} \varrho_{ij}(t)\mathcal{A}_{ij}\right)\right) & P \\ \star & 0_{2n} \end{bmatrix} \begin{bmatrix} \varepsilon(t) \\ \Delta(t) \end{bmatrix}, \\ &= \begin{bmatrix} \varepsilon(t) \\ \Delta(t) \end{bmatrix}^T \begin{bmatrix} \mathrm{He}\left(\left(\sum_{i=1}^{\nu}\sum_{j=1}^{\nu} \varrho_{ij}(t)\mathcal{Q}_{ij}\right)\right) & P \\ \star & 0_{2n} \end{bmatrix} \begin{bmatrix} \varepsilon(t) \\ \Delta(t) \end{bmatrix}.\end{aligned}$$

When (2.16) is feasible,

$$\begin{aligned}\langle \nabla V(\varepsilon(t)), \mathcal{A}(\varrho(t))\varepsilon(t) + \Delta(t)\rangle &= \sum_{i=1}^{\nu}\sum_{j=1}^{\nu} \varrho_{ij}(t) \begin{bmatrix} \varepsilon(t) \\ \Delta(t) \end{bmatrix}^T \begin{bmatrix} \mathrm{He}\left(\mathcal{Q}_{ij}\right) & P \\ \star & 0_{2n} \end{bmatrix} \begin{bmatrix} \varepsilon(t) \\ \Delta(t) \end{bmatrix} \\ &\leq \sum_{i=1}^{\nu}\sum_{j=1}^{\nu} \varrho_{ij}(t) \begin{bmatrix} \varepsilon(t) \\ \Delta(t) \end{bmatrix}^T \begin{bmatrix} -\alpha P & 0_{2n} \\ \star & \gamma I_{2n} \end{bmatrix} \begin{bmatrix} \varepsilon(t) \\ \Delta(t) \end{bmatrix} \\ &= -\alpha V(\varepsilon(t)) + \gamma \|\Delta(t)\|^2.\end{aligned}$$

Therefore, $V(\varepsilon)$ is an ISS-Lyapunov function, and the error ε is ISS with respect to the disturbance Δ. Since $\Delta(t)$ is bounded for all $t \geq 0$, it follows from Definition 10 that $\varepsilon(t)$ remains bounded for all $t \geq 0$. Hence, $\overline{x}(t) = x(t) + \overline{e}(t)$ is bounded, and $\underline{x}(t) = x(t) - \underline{e}(t)$ is bounded. This proves assertion ii. Moreover, by the definition of ISS, the ultimate bound on $\varepsilon(t)$ depends on the size of $\|\Delta\|_{[0,\infty)}$. This proves assertion iii, and concludes the proof. □

The approach in Theorem 1 is referred to as the LPV approach since an equivalent LPV system to the nonlinear error dynamics is found and the observer gains and coupling matrices are constructed by analyzing the vertices of a polytope. The name 'LPV approach' comes from [160] which studied the Luenberger observer synthesis problem using this idea.

The constraint that P is a diagonal matrix is reasonable as the conditions (2.16)-(2.17) imply that $\mathcal{A}_{ij}$ is Hurwtiz and Metzler for all $i, j = 1, \dots, \nu$. A Metzler matrix is Hurwitz if and only if it admits a diagonal Lyapunov function [127].

The following theorem uses the incremental quadratic constraints that polytopic nonlinearities satisfy to synthesize the observer gains and coupling matrices.

Theorem 2. *Consider the system* (2.1)-(2.2) *with a polytopic nonlinearity* $p : \mathbb{R}^n \to \mathbb{R}^n$ *that is defined by the polytope* $\Omega = Conv\{\Theta_1, \dots, \Theta_\nu\}$*, suppose assumptions 1 and 2 hold, and* $x(t)$ *remains bounded for all* $t \geq 0$*. Suppose there exist diagonal matrices* $\overline{P}, \underline{P} \in \mathbb{R}^{n\times n}$*, matrices* $\overline{Y}, \underline{Y} \in \mathbb{R}^{n\times m}$*, matrices* $\overline{W}, \underline{W} \in \mathbb{R}^{n\times n}$*, scalars* $\alpha > 0$ *and* $\gamma > 0$ *such that*

$$\begin{bmatrix} He(\mathcal{Q}) + \alpha P + I_2 \otimes M_{11} & P + I_2 \otimes M_{12} & P \\ \star & I_2 \otimes M_{22} & 0_{2n} \\ \star & \star & -\gamma I_{2n} \end{bmatrix} \preceq 0 \tag{2.20}$$

and (2.17) *and* (2.18) *hold, where* $M_{11}, M_{12}, M_{22} \in \mathbb{R}^{n\times n}$ *satisfy* (1.16)*,* P *is defined in* (2.19)*, and*

$$\mathcal{Q} = \begin{bmatrix} \overline{P}A - \overline{Y}C + \overline{W} & \overline{W} \\ \underline{W} & \underline{P}A - \underline{Y}C + \underline{W} \end{bmatrix}.$$

Consider an interval observer (2.3)-(2.4) *with* (2.8)-(2.9) *where*

$$\overline{L} = \overline{P}^{-1}\overline{Y}, \underline{L} = \underline{P}^{-1}\underline{Y}, \overline{G} = \overline{P}^{-1}\overline{W}, \underline{G} = \underline{P}^{-1}\underline{W}.$$

The conclusions of Theorem 1 hold.

Proof. Assertion i follows from the same proof as Theorem 1.

Assertions ii-iii are proved by showing that the satisfaction of (2.20) and (2.18) imply that (2.15) is an ISS Lyapunov function for the nonlinear error dynamics (2.6). V evolves as follows

$$\langle \nabla V(\varepsilon(t)), \mathfrak{A}\varepsilon(t) + \delta p(t) + \Delta(t)\rangle = \begin{bmatrix} \varepsilon(t) \\ \delta p(t) \\ \Delta(t) \end{bmatrix}^T \begin{bmatrix} \text{He}(P\mathfrak{A}) & P & P \\ \star & 0_{2n} & 0_{2n} \\ \star & \star & 0_{2n} \end{bmatrix} \begin{bmatrix} \varepsilon(t) \\ \delta p(t) \\ \Delta(t) \end{bmatrix}$$

where

$$\mathfrak{A} = \begin{bmatrix} A - \overline{L}C + \overline{G} & \overline{G} \\ \underline{G} & A - \underline{L}C + \underline{G} \end{bmatrix}, \tag{2.21}$$

and

$$\delta p(t) = \begin{bmatrix} p(\overline{x}(t)) - p(x(t)) \\ p(x(t)) - p(\underline{x}(t)) \end{bmatrix}.$$

By Lemma 3, since p is a polytopic nonlinearity, it satisfies a δQC with a multiplier matrix

$$M = \begin{bmatrix} M_{11} & M_{12} \\ \star & M_{22} \end{bmatrix}$$

where $M_{11}, M_{12}, M_{22} \in \mathbb{R}^{n\times n}$ satisfy (1.16). Therefore, the following δQC holds

$$\begin{bmatrix} \varepsilon(t) \\ \delta p(t) \\ \Delta(t) \end{bmatrix}^T \begin{bmatrix} I_2 \otimes M_{11} & I_2 \otimes M_{12} & 0_{2n} \\ \star & I_2 \otimes M_{22} & 0_{2n} \\ \star & \star & 0_{2n} \end{bmatrix} \begin{bmatrix} \varepsilon(t) \\ \delta p(t) \\ \Delta(t) \end{bmatrix} \geq 0, \ \forall t \geq 0.$$

$\mathcal{Q} = P\mathcal{A}$, where $\mathcal{A}$ is defined in (2.21), with the following variable substitutions

$$\overline{Y} = \overline{P}\,\overline{L}, \underline{Y} = \underline{P}\,\underline{L}, \overline{W} = \overline{P}\,\overline{G}, \underline{W} = \underline{P}\,\underline{G}.$$

The satisfaction of (2.20) means that

$$\langle \nabla V(\varepsilon(t)), \mathfrak{A}\varepsilon(t) + \delta p(t) + \Delta(t)\rangle \leq -\alpha V(\varepsilon(t)) + \gamma\|\Delta(t)\|^2$$

so V is an ISS-Lyapunov function and, thus, assertions ii-iii hold, which concludes the proof.

□

In the case that $0_n \in \Omega$, if (2.13) holds, then $\mathcal{A} \in \mathcal{M}_n$ where $\mathcal{A}$ is defined in (2.21). If (2.20) holds, then there exists a symmetric matrix $P \in \mathbb{R}^{2n\times 2n}$ such that $P \succ 0$ and

$$\begin{bmatrix} \mathrm{He}(P\mathfrak{A}) & P & P \\ \star & 0_{2n} & 0_{2n} \\ \star & \star & 0_{2n} \end{bmatrix} + \tilde{M} \preceq 0 \tag{2.22}$$

where

$$\tilde{M} = \begin{bmatrix} I_2 \otimes M_{11} & I_2 \otimes M_{12} & 0_{2n} \\ \star & I_2 \otimes M_{22} & 0_{2n} \\ \star & \star & -\gamma I_{2n} \end{bmatrix}.$$

By [125, Theorem 1], (2.22) is equivalent to the following frequency domain condition

$$\begin{bmatrix} (\iota\omega I_{2n} - A)^{-1}B \\ I_{4n} \end{bmatrix}^{*} \tilde{M} \begin{bmatrix} (\iota\omega I_{2n} - A)^{-1}B \\ I_{4n} \end{bmatrix} \geq 0, \ \forall \omega \in [0, \infty) \tag{2.23}$$

where $B = \begin{bmatrix} I_{2n} & I_{2n} \end{bmatrix}$. By [126, Theorem 1], the frequency domain condition (2.23) is equivalent to (2.22) holding with P being a diagonal positive definite matrix. Hence, using the KYP lemma for positive systems shows that the restriction of P to being a diagonal matrix is a reasonable restriction.

2.1.2 Synthesis from Linear Programs

One advantage of positive systems is that they can admit *linear copositve Lyapunov functions* which take the following form

$$V(\varepsilon) = r^T \varepsilon \tag{2.24}$$

where $r \in \mathbb{R}^{2n}$ is a strictly positive vector.

Theorem 3. *Consider the system* (2.1)-(2.2) *with a polytopic nonlinearity* $p : \mathbb{R}^n \to \mathbb{R}^n$ *that is defined by the polytope* $\Omega = Conv\{\Theta_1, \ldots, \Theta_\nu\}$*, suppose assumptions 1 and 2 hold, and* $x(t)$ *remains bounded for all* $t \geq 0$*. Suppose there exist diagonal matrices* $\overline{P}, \underline{P} \in \mathbb{R}^{n \times n}$*, matrices* $\overline{Y}, \underline{Y} \in \mathbb{R}^{n \times m}$*, matrices* $\overline{W}, \underline{W} \in \mathbb{R}^{n \times n}$*, scalars* $\alpha > 0$ *and* $\gamma > 0$ *such that the* *following holds*

$$\mathbf{1}_{2m}^T \mathcal{Q}_{ij} < 0,\ \forall (i,j) \in \{1,\ldots,\nu\}^2, \tag{2.25}$$

$$\mathbf{1}_{2m}^T P \leq \gamma \mathbf{1}_{2m}^T, \tag{2.26}$$

$$\mathcal{Q}_{ij} \in \mathcal{M}_{2n},\ \forall (i,j) \in \{1,\ldots,\nu\}^2, \tag{2.27}$$

$$P \succ 0, \tag{2.28}$$

where

$$P = \begin{bmatrix} \overline{P} & 0_n \\ \star & \underline{P} \end{bmatrix},$$

$$\mathcal{Q}_{ij} = \begin{bmatrix} \overline{P}(A+\Theta_i) - \overline{Y}C + \overline{W} & \overline{W} \\ \underline{W} & \underline{P}(A+\Theta_j) - \underline{Y}C + \underline{W} \end{bmatrix}.$$

Consider an interval observer (2.3)-(2.4) *with* (2.8)-(2.9) *where*

$$\overline{L} = \overline{P}^{-1}\overline{Y}, \underline{L} = \underline{P}^{-1}\underline{Y}, \overline{G} = \overline{P}^{-1}\overline{W}, \underline{G} = \underline{P}^{-1}\underline{W}.$$

The conclusions of Theorem 1 hold.

Proof. Consider the ISS Lyapunov function

$$V(\varepsilon) = \mathbf{1}_{2m}^T P \varepsilon.$$

The satisfaction of (2.27) for diagonal positive definite P implies that $\varepsilon(t) \geq 0$ for all $t \geq 0$, $V(\varepsilon(t)) > 0$ for all $t \geq 0$. So V satisfies the conditions be a Lyapunov function. V evolves as follows:

$$\langle \nabla V(\varepsilon(t)), \mathcal{A}(\varrho(t))\varepsilon(t) + \Delta(t)\rangle = \mathbf{1}_{2m}^T P \mathcal{A}(\varrho(t))\varepsilon(t) + \mathbf{1}_{2m}^T P \Delta(t).$$

The satisfaction of (2.25)-(2.26) imply that

$$\langle \nabla V(\varepsilon(t)), \mathcal{A}(\varrho(t))\varepsilon(t) + \Delta(t)\rangle \leq \gamma \mathbf{1}_{2m}^T \Delta(t).$$

So V is an ISS Lyapunov function, and the results follow. □

Theorem 3 proposes a linear program to synthesize the gains and coupling matrices. This is appealing because linear programs have lesser computational complexity than semidefinite programs, so interval observers for large systems can be handled more readily [127].

2.1.3 Synthesis with Coordinate Transformation

Whilst it is shown that positivity and stability of the error dynamics is possible in some cases where neither $A - \overline{L}C$ nor $A - \underline{L}C$ is Metzler, the dynamics of the system can be expressed in a new set of coordinates where positivity and stability of the linear part can be enforced easily [120]. The change of coordinates is linear and driven by the linear part of the error dynamics. The process begins from the following assumption:

Assumption 3. *There exists a matrix $L \in \mathbb{R}^{n\times m}$ and an invertible matrix $U \in \mathbb{R}^{n\times n}$ such that $U^{-1}(A - LC)U$ is Hurwitz and Metzler.*

L can be found such that $A - LC$ is Hurwitz but not necessarily Metzler, then the matrix U can be constructed using the Jordan normal form in the case where all the eigenvalues of $A - LC$ are real. Otherwise, [120, Lemma 1] can be used to satisfy Assumption 3.

Consider the transformation $z = U^{-1}x$ for some invertible matrix $U \in \mathbb{R}^{n\times n}$. The nonlinear continuous-time system (2.1)-(2.2) under consideration can be written in the new coordinate system as

$$\dot{z}(t) = U^{-1}AUz(t) + U^{-1}p(Uz(t)) + U^{-1}w(t), \tag{2.29}$$

$$y(t) = CUz(t) + v(t). \tag{2.30}$$

Consider the following interval observer which uses the coordinate transformation U and associated observer gain L

$$\begin{aligned}\dot{\overline{z}}(t) &= U^{-1}AU\overline{z}(t) + U^{-1}p(U\overline{z}(t)) + U^{-1}L(y(t) - CU\overline{z}(t)) \\ &\quad + (U^{-1})^{\oplus}\overline{w} - (U^{-1})^{\ominus}\underline{w} + \overline{G}(\overline{z}(t) - \underline{z}(t)) + |LU|\,\overline{V}\mathbf{1}_m, \\ \dot{\underline{z}}(t) &= U^{-1}AU\underline{z}(t) + U^{-1}p(U\underline{z}(t)) + U^{-1}L(y(t) - CU\underline{z}(t)) \\ &\quad + (U^{-1})^{\oplus}\underline{w} - (U^{-1})^{\ominus}\overline{w} + \underline{G}(\underline{z}(t) - \overline{z}(t)) - |LU|\,\overline{V}\mathbf{1}_m.\end{aligned}$$

The lumped error

$$\varepsilon_z(t) = \begin{bmatrix} \overline{z}(t) - z(t) \\ z(t) - \underline{z}(t) \end{bmatrix} \tag{2.31}$$

has the following dynamics

$$\begin{aligned}\dot{\varepsilon}_z(t) &= \begin{bmatrix} U^{-1}(A - LC)U + \overline{G} & \overline{G} \\ \underline{G} & U^{-1}(A - LC)U + \overline{G} \end{bmatrix} \varepsilon_z(t) \\ &+ \begin{bmatrix} U^{-1}(p(U\overline{z}(t)) - p(Uz(t))) \\ U^{-1}(p(Uz(t)) - p(U\underline{z}(t))) \end{bmatrix} + \Delta(t)\end{aligned} \tag{2.32}$$

where

$$\Delta(t) = \begin{bmatrix} (U^{-1})^{\oplus}\overline{w} - (U^{-1})^{\ominus}\underline{w} - U^{-1}w(t) + |LU|\,\overline{V}\mathbf{1}_m + LUv(t) \\ U^{-1}w(t) - (U^{-1})^{\oplus}\underline{w} + (U^{-1})^{\ominus}\overline{w} + |LU|\,\overline{V}\mathbf{1}_m - LUv(t) \end{bmatrix}. \tag{2.33}$$

$\Delta(t) \geq 0$ for all $t \geq 0$ by Lemma 1 and assumptions 1 and 2. From the LPV approach, if $\overline{G}, \underline{G}$ satisfy:

$$\mathfrak{A}_{U_{ij}} = \begin{bmatrix} U^{-1}(A + \Theta_i - LC)U + \overline{G} & \overline{G} \\ \underline{G} & U^{-1}(A + \Theta_j - LC)U + \underline{G} \end{bmatrix} \in \mathcal{M}_{2n}\ \forall (i,j) \in \{1, \dots, \nu\}^2,$$

then if $\varepsilon_z(0) \geq 0$, then $\varepsilon(t) \geq 0$ for all $t \geq 0$. This means that if $\underline{z}(0) \leq z(0) \leq \overline{z}(0)$, then $\underline{z}(t) \leq z(t) \leq \overline{z}(t)$. It then follows from Lemma 1 that if $\underline{\xi}_0 \leq x(0) \leq \overline{\xi}_0$, and the interval observer is initialized by

$$\overline{z}(0) = (U^{-1})^{\oplus}\overline{\xi}_0 - (U^{-1})^{\ominus}\underline{\xi}_0,$$
$$\underline{z}(0) = (U^{-1})^{\oplus}\underline{\xi}_0 - (U^{-1})^{\ominus}\overline{\xi}_0,$$

then $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ for all $t \geq 0$ where

$$\overline{x}(t) = U^{\oplus}\overline{z}(t) - U^{\ominus}\underline{z}(t),$$
$$\underline{x}(t) = U^{\oplus}\underline{z}(t) - U^{\ominus}\overline{z}(t).$$

If, in addition, ε_z is ISS stable, then $\varepsilon_z(t)$ is bounded for all $t \geq 0$. Therefore, it follows that

$$\varepsilon(t) = \begin{bmatrix} U^{\oplus} & U^{\ominus} \\ U^{\ominus} & U^{\oplus} \end{bmatrix} \varepsilon_z(t)$$

is bounded for all $t \geq 0$. If $x(t)$ is bounded for all $t \geq 0$, then $\overline{x}(t) = x(t) + \overline{e}(t)$ and $\underline{x}(t) = x(t) - \underline{e}(t)$ are bounded for all $t \geq 0$.

For ISS stability, it is necessary that $\mathfrak{A}_{U_{ij}}$ is both Metzler and Hurwitz for all $(i, j) \in \{1, \ldots, \nu\}^2$. Recall that a Metzler and Hurwitz matrix has strictly negative diagonal elements [108]. While it is not always guaranteed that the given coordinate transformation U and observer gain L will yield a positive and ISS stable interval observer, $U^{-1}(A-LC)U$, which is both Metzler and Hurwitz, supplies diagonal elements that are strictly negative, thus making it more likely that positivity and ISS stability can hold. The matrices $\overline{G}$ and $\underline{G}$ are coupling terms which must be nonnegative and help cancel out the negative off-diagonal elements from the nonlinearity.

It is assumed that U and L have been found to satisfy assumption 3, the matrices $\overline{G}$ and $\underline{G}$ still need to be found and ISS stability of ε_z needs to be verified. Convex programs can be formulated in the same spirit as Theorems 1- 3 to find $\overline{G}$ and $\underline{G}$ and the ISS Lyapunov function for ISS stability. For example, based on Theorem 3, the matrices $\overline{G}$ and $\underline{G}$ can be synthesized by solving the following linear program for diagonal positive definite matrices $\overline{P}, \underline{P} \in \mathbb{R}^{n\times n}$ and matrices $\overline{W}, \underline{W} \in \mathbb{R}^{n\times n}$:

$$\begin{cases} \mathbf{1}_{2m}^T \mathcal{A}_{U_{ij}} < 0, \\ \mathbf{1}_{2m}^T \overline{P} \leq \gamma \mathbf{1}_n, \ \mathbf{1}_{2m}^T \underline{P} \leq \gamma \mathbf{1}_n, \quad \forall (i,j) \in \{1,\ldots,\nu\}^2, \\ \mathcal{A}_{U_{ij}} \in \mathcal{M}_{2n}, \end{cases} \tag{2.34}$$

where

$$\mathcal{A}_{U_{ij}} = \begin{bmatrix} \overline{P}U^{-1}(A+\Theta_i - LC)U + \overline{W} & \overline{W} \\ \underline{W} & \underline{P}U^{-1}(A+\Theta_j - LC)U + \underline{W} \end{bmatrix}.$$

Then, $\overline{G} = \overline{P}^{-1}\overline{W}$ and $\underline{G} = \underline{P}^{-1}\underline{W}$.

If the linear program (2.34) is infeasible, a different coordinate transformation U and observer gain L should then be tried. It is difficult to pose a problem which solves for L and U to satisfy assumption 3 in addition to the rest of the constraints in (2.34) due to the nonconvexity introduced by the product of U^{-1}, U, and L.

2.1.4 Numerical Example

Consider the following enzyme kinetics model, which is referred to as the Goodwin model of enzyme kinetics [39]:

$$\dot{x}_1(t) = -x_1(t) + \frac{360}{43 + x_3^{10}(t)} + w(t), \tag{2.35}$$

$$\dot{x}_2(t) = x_1(t) - 0.6x_2(t), \tag{2.36}$$

$$\dot{x}_3(t) = x_2(t) - 0.3x_2(t), \tag{2.37}$$

$$y(t) = x_1(t) + v(t). \tag{2.38}$$

The disturbance $w(t)$ is bounded by

$$0 \leq w(t) \leq 3,$$

for all $t \geq 0$. Moreover, the measurement noise $v(t)$ is bounded by

$$|v(t)| \leq 0.1,$$

for all $t \geq 0$. The nonlinearity

$$p(x) = \begin{bmatrix} \dfrac{360}{43 + x_3^{10}} \\ 0 \\ 0 \end{bmatrix},$$

is a polytopic nonlinearity which is described by a polytope with the following vertices:

$$\Theta_1 = \begin{bmatrix} 0 & 0 & 14.514 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \Theta_2 = -\Theta_1.$$

By solving the convex program proposed in Theorem 1, the following observer gains and coupling matrices for the interval observer (2.3)-(2.4) with (2.8)-(2.9) are found:

$$\overline{L} = \underline{L} = \begin{bmatrix} 34.7691 \\ 1 \\ 0 \end{bmatrix}, \overline{G} = \underline{G} = \begin{bmatrix} 0 & 0 & 14.514 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

A simulation of the interval observer is given in Figure 2.1. The upper bound $\overline{x}(t)$ (magenta) and lower bound $\underline{x}(t)$ (blue) remain close to the state $x(t)$ (black). The ordering $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ is also maintained for all $t \geq 0$, which can be seen by the fact that none of the lines cross over each other.

2.2 Systems with Conic Nonlinearities

Consider the following system

$$\dot{x} = Ax + B_p p(q), \tag{2.39}$$

$$y = Cx, q = C_q x, \tag{2.40}$$

where $p : \mathbb{R}^{n_q} \to \mathbb{R}^{n_p}$ is a conic nonlinearity defined by a cone Ω with a conic basis $\{\Theta_1, \dots, \Theta_\nu\}$, $A \in \mathbb{R}^{n\times n}, C \in \mathbb{R}^{m\times n}, B_p \in \mathbb{R}^{n\times n_p}$, and $C_q \in \mathbb{R}^{n_q\times n}$.

Consider the following interval observer for the system (2.39)-(2.40):

$$\dot{\overline{x}} = A\overline{x} + B_p p(\overline{q}) + \overline{L}(y - C\overline{x}), \tag{2.41}$$

$$\dot{\underline{x}} = A\underline{x} + B_p p(\underline{q}) + \underline{L}(y - C\underline{x}), \tag{2.42}$$

$$\overline{q} = C_q\overline{x} + \overline{L}_n(C\overline{x} - y), \underline{q} = C_q\underline{x} + \underline{L}_n(C\underline{x} - y). \tag{2.43}$$

The nonlinear injection gains $\overline{L}_n$ and $\underline{L}_n$ are to be designed along with $\overline{L}$ and $\underline{L}$ for positivity and stability of the following error dynamics:

$$\dot{\overline{e}} = (A - \overline{L}C)\overline{e} + B_p\delta p(\overline{q}, q), \tag{2.44}$$

$$\dot{\underline{e}} = (A - \underline{L}C)\underline{e} + B_p\delta p(q, \underline{q}). \tag{2.45}$$

By Lemma 2, for all $q, \overline{q} \in \mathbb{R}^{n_q}$, there exists a $\Theta \in \Omega$ such that $\delta p(\overline{q}, q) = \Theta(\overline{q} - q) = \Theta(C_q + \overline{L}_n C)\overline{e}$. Therefore, (2.44) is positive if $A - \overline{L}C + B_p\Theta(C_q + \overline{L}_n C) \in \mathcal{M}_n$ for all $\Theta \in \Omega$, and (2.45) is positive if $A - \underline{L}C + B_p\Theta(C_q + \underline{L}_n C) \in \mathcal{M}_n$ for all $\Theta \in \Omega$. Therefore, the observer gains $\overline{L}, \underline{L}, \overline{L}_n$, and $\underline{L}_n$ should be designed such that $A - \overline{L}C \in \mathcal{M}_n$, $A - \underline{L}C \in \mathcal{M}_n$,

$$B_p\Theta_i(C_q + \overline{L}_n C) \in \mathcal{M}_n, \tag{2.46}$$

$$B_p\Theta_i(C_q + \underline{L}_n C) \in \mathcal{M}_n, \tag{2.47}$$

for all $i = 1, \dots, \nu$. The exponential stability of $\overline{e} = 0$ and $\underline{e} = 0$ follows from the following linear matrix inequalities:

$$\begin{bmatrix} \mathrm{He}(\overline{P}(A-\overline{L}C))+\alpha\overline{P} & \overline{P}B_p \\ \star & 0 \end{bmatrix} + \overline{M} \preceq 0, \tag{2.48}$$

$$\begin{bmatrix} \mathrm{He}(\underline{P}(A-\underline{L}C))+\alpha\underline{P} & \underline{P}B_p \\ \star & 0 \end{bmatrix} + \underline{M} \preceq 0, \tag{2.49}$$

where $\overline{P}, \underline{P} \succ 0$, $\alpha > 0$, $\overline{M} = \overline{\Upsilon}^T M \overline{\Upsilon}$, $\underline{M} = \underline{\Upsilon}^T M \underline{\Upsilon}$,

$$\overline{\Upsilon} = \begin{bmatrix} C_q + \overline{L}_n C & 0 \\ 0 & I_{n_p \times n_p} \end{bmatrix}, \underline{\Upsilon} = \begin{bmatrix} C_q + \underline{L}_n C & 0 \\ 0 & I_{n_p \times n_p} \end{bmatrix},$$

and M is a multiplier matrix for the conic nonlinearity p.

By [125, Theorem 1], (2.48) is equivalent to the the following frequency domain condition on $A - \overline{L}C$:

$$\begin{bmatrix} \left(\iota\omega I - (A-\overline{L}C)\right)^{-1} B_p \\ I \end{bmatrix}^* \overline{M} \begin{bmatrix} \left(\iota\omega I - (A-\overline{L}C)\right)^{-1} B_p \\ I \end{bmatrix} \preceq 0, \tag{2.50}$$

for all $\omega \in \mathbb{R} \cup \{\infty\}$. Since $A - \overline{L}C \in \mathcal{M}_n$, by [126, Theorem 1], (2.50) is equivalent to (2.48) where $\overline{P}$ is a positive definite diagonal matrix. Similarly, (2.49) implies a frequency domain condition on $A - \underline{L}C$ which, since $A - \underline{L}C \in \mathcal{M}_n$, is then equivalent to (2.49) where $\underline{P}$ is a positive definite diagonal matrix. So the Kalman-Yakubovich-Popov (KYP) Lemma for positive systems shows that restricting $\overline{P}$ and $\underline{P}$ to be diagonal is a reasonable restriction. This restriction will be helpful in formulating a convex program to find the observer gains $\overline{L}, \underline{L}$ for positivity and stability.

Synthesizing the observer gains follows a two step process. First, find $\overline{L}_n$ and $\underline{L}_n$ to satisfy (2.46) and (2.47). Then, solve a convex program for $\overline{L}$ and $\underline{L}$.

Theorem 4. *Consider the system* (2.39)-(2.40) *where p is a conic nonlinearity with conic basis $\{\Theta_1, \ldots, \Theta_\nu\}$ and a multiplier matrix M. Suppose $\overline{L}_n$, and $\underline{L}_n$ are such that* (2.46) *and* (2.47) *hold, and there exis positive definite diagonal matrices* $\overline{P}, \underline{P} \in \mathbb{R}^{n\times n}$*, matrices* $\overline{Y}, \underline{Y} \in \mathbb{R}^{n\times m}$ *and* $\alpha > 0$ *such that* $\overline{P}A - \overline{Y}C \in \mathcal{M}_n$*,* $\underline{P}A - \underline{Y}C \in \mathcal{M}_n$*,*

$$\begin{bmatrix} He(\overline{P}A - \overline{Y}C) + \alpha\overline{P} & \overline{P}B_p \\ \star & 0 \end{bmatrix} + \overline{M} \preceq 0, \tag{2.51}$$

$$\begin{bmatrix} He(\underline{P}A - \underline{Y}C) + \alpha\underline{P} & \underline{P}B_p \\ \star & 0 \end{bmatrix} + \underline{M} \preceq 0. \tag{2.52}$$

Consider an interval observer (2.41)-(2.43) *with* $\overline{L} = \overline{P}^{-1}\overline{Y}$ *and* $\underline{L} = \underline{P}^{-1}\underline{Y}$*. If* $\underline{x}(0) \leq x(0) \leq \overline{x}(0)$*, then* $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ *for all* $t \geq 0$*. Moreover,* $\overline{e} = 0$ *and* $\underline{e} = 0$ *are exponentially stable.*

Theorem 4 provides a systematic way to synthesis observer gains which are positive and globally stable. It is possible to use a local Lipschitz constant to synthesize the observer gains $\overline{L}$ and $\underline{L}$ by using a multiplier matrix M from the Lipschitz constant, i.e. (1.13); however, the synthesized interval observers would be only be guaranteed to be stable when the initial conditions satisfy the Lipschitz condition. Thus, synthesizing an interval observer using a local Lipschitz constant limits how large the initial condition uncertainty can be while guaranteeing stability. On the other hand, using the conic multiplier matrix allows for the synthesized interval observers to be stable regardless of how large the initial condition uncertainty is.

2.2.1 Numerical Example

Consider the following system:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -x_1 - x_2 - x_2^3, \\ y = x_1. \end{cases} \tag{2.53}$$

The nonlinearity $p(q) = -q^3$ where $q = x_2$ is a monotically decreasing function which is a conic nonlinearity whose Jacobian $\frac{\partial p}{\partial q} = -3q^2 \in \Omega = \text{Cone}\left\{\begin{bmatrix}-1\end{bmatrix}\right\}$ for all $q \in \mathbb{R}$. Using Theorem 4, the following observer gains for an interval observer (2.41)-(2.43) are found:

$$\overline{L} = \underline{L} = \begin{bmatrix} 2.0691 \\ -1.4709 \end{bmatrix}, \ \overline{L}_n = \underline{L}_n = 0.$$

Figure 2.2 shows an example simulation. The upper bound $\overline{x}(t)$ (magenta) and lower bound $\underline{x}(t)$ (blue) approach the state $x(t)$ (black). The ordering $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ is also maintained for all $t \geq 0$, which can be seen by the fact that none of the lines cross over each other. In the simulation, $\|p(\overline{q}(0)) - p(q(0))\| \leq \gamma^2 \|\overline{q}(0) - q(0)\|$, where $\gamma^2 = 11^2$. The convex program proposed in Theorem 4 would be infeasible if, instead of using the multiplier matrix for the conic nonlinearity, the multiplier matrix (1.13) is used with this value of γ^2. In fact, the convex program is infeasible for any $\gamma^2 \geq 0.3$, so the observers that can be synthesized using the local Lipschitz constant would only be guaranteed to be stable when $\overline{x}(0)$ and $\underline{x}(0)$ are very close to $x(0)$.

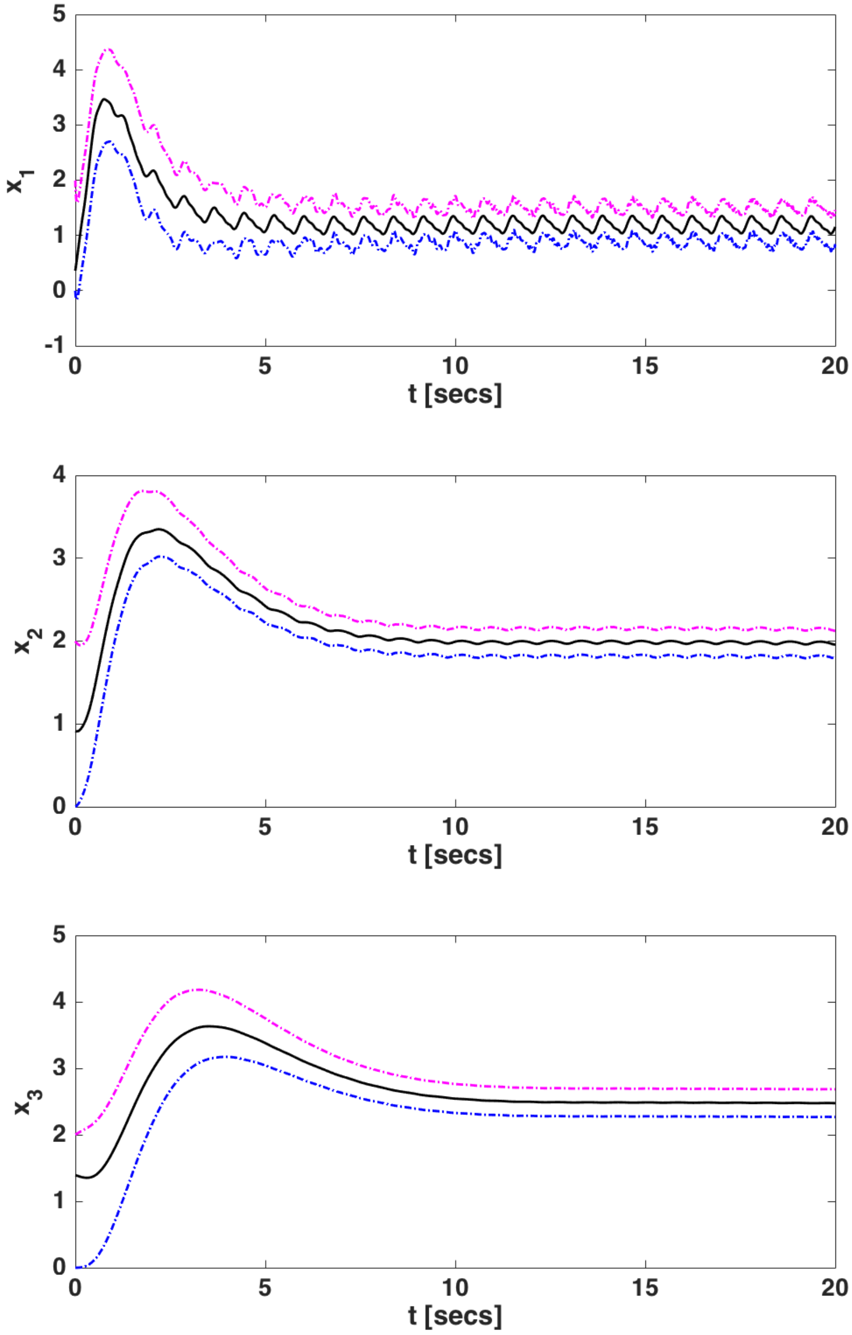

Figure 2.1: Interval observer simulation for Goodwin's model of enzyme kinetics.

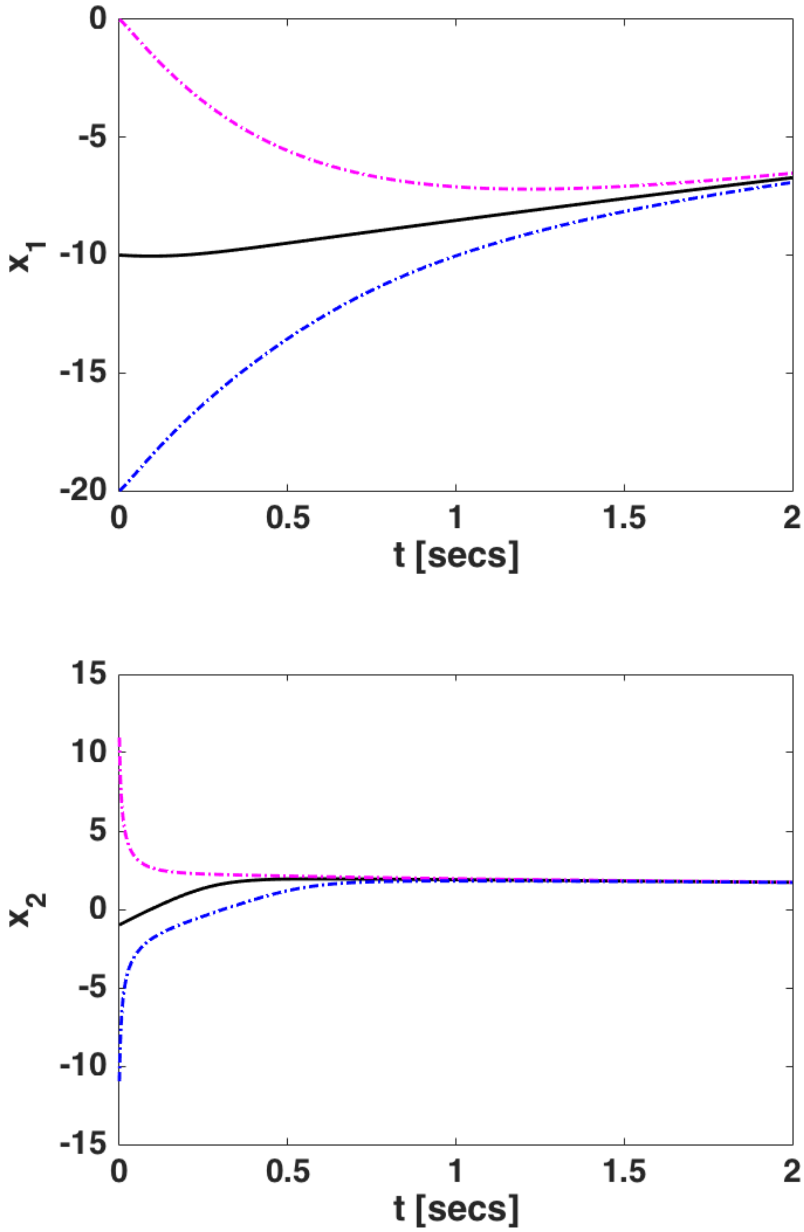

Figure 2.2: A simulation of an interval observer for the conic example.

Chapter 3

SYNTHESIS OF INTERVAL OBSERVERS FOR NONLINEAR DISCRETE-TIME SYSTEMS

> "Time is not a thing, thus nothing which is, and yet it remains constant in its passing away without being something temporal like the beings in time."
>
> *Time and Being*
>
> Martin Heidegger

Compared to the continuous-time case, less attention has been given towards the design of observers for nonlinear discrete-time systems. One motivation to study the observer design for nonlinear discrete-time systems is that nonlinear systems are often discretized. Moreover, systems with sampled output can be analyzed as discrete-time systems [18, 77, 1]. LMI-based techniques for Luenberger observer design have been considered in [158, 20, 79, 1, 159] for discrete-time systems with Lipschitz nonlinearities. [79, 159] use the LPV approach. More recently, [161] is an extension of [7] to nonlinear discrete time systems with nonlinearities which satisfy incremental quadratic constraints which includes interesting cases such as one-sided Lipschitz nonlinear systems [2]. There are many practical applications of nonlinear observers in discrete-time that have been studied in the literature: stealthy attack detection [157], fault detection [114], spacecraft attitude estimation [10], and estimation for biological systems [29]. Other state estimation techniques for nonlinear discrete-time systems include: the extended Kalman filter [129], interval analysis [88], and moving horizon estimators [128].

Interval observers for linear discrete-time systems have been widely studied in the literature [52, 153, 54, 102], but not much attention has been given to nonlinear discrete-time systems. [102, §III] provides sufficient conditions for interval observers for nonlinear discrete-time systems but no systematic procedure to synthesize them. Moreover, the nonlinear example in that paper is an asymptotically stable one-dimensional system with no output. This chapter proposes a systematic procedure to synthesize interval observers for discrete-time systems with polytopic nonlinearities.

3.1 Problem Setup

Consider the following discrete time nonlinear system

$$x_{k+1} = Ax_k + p(x_k) + w_k, \tag{3.1}$$

$$y_k = Cx_k + v_k. \tag{3.2}$$

where $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{m\times n}$, and $p(x_k) : \mathbb{R}^n \to \mathbb{R}^n$ is a polytopic nonlinearity with a polytope defined by ν vertices $\Theta_1, \ldots, \Theta_\nu \in \mathbb{R}^{n\times n}$. w_k is a disturbance, and v_k is the measurement noise. The disturbance and noise are unknown, but they are assumed to belong to known compact sets:

Assumption 4. *The disturbance belongs to a known interval:* $\underline{w} \le w_k \le \overline{w}$ *for all* $k \in \mathbb{Z}_{\ge 0}$.

Assumption 5. *The measurement noise is bounded as follows:* $|v_k| \le V\mathbf{1}_m$ *for all* $k \in \mathbb{Z}_{\ge 0}$.

The interval observer consists of two estimates $\underline{x}_k$ and $\overline{x}_k$. The dynamics of $\underline{x}_k$ and $\overline{x}_k$ are designed to use output feedback so that the following objectives hold:

(1) **Framer**: $\underline{x}_k \le x_k \le \overline{x}_k$ for all $k \in \mathbb{Z}_{\ge 0}$ if the initial conditions satisfy $\underline{x}_0 \le x_0 \le \overline{x}_0$.

(2) **Input-to-State Stability**: The dynamics of the follow error is ISS

$$\varepsilon_k = \begin{bmatrix} \overline{e}_k \\ \underline{e}_k \end{bmatrix} = \begin{bmatrix} \overline{x}_k - x_k \\ x_k - \underline{x}_k \end{bmatrix} \tag{3.3}$$

3.2 LPV Approach to Interval Observer Synthesis

Consider the following dynamics of a candidate interval observer for (3.1)-(3.2) which is inspired by the structure in [153]:

$$\overline{\zeta}_{k+1} = \overline{T}A\overline{x}_k + \overline{L}(y_k - C\overline{x}_k) + \overline{T}p(\overline{x}_k) + \overline{G}(\overline{x}_k - \underline{x}_k), \tag{3.4}$$

$$\overline{x}_k = \overline{\zeta}_k + \overline{N}y_k + \overline{\Delta}, \tag{3.5}$$

$$\underline{\zeta}_{k+1} = \underline{T}A\underline{x}_k + \underline{L}(y_k - C\underline{x}_k) + \underline{T}p(\underline{x}_k) + \underline{G}(\underline{x}_k - \overline{x}_k), \tag{3.6}$$

$$\underline{x}_k = \underline{\zeta}_k + \underline{N}y_k + \underline{\Delta}, \tag{3.7}$$

where $\overline{T}, \underline{T}, \overline{G}, \underline{G} \in \mathbb{R}^{n\times n}$, $\overline{L}, \underline{L}, \overline{N}, \underline{N} \in \mathbb{R}^{n\times m}$. $\overline{\Delta}$ and $\underline{\Delta}$ are the following:

$$\overline{\Delta} = \overline{T}^{\oplus}\underline{w} - \overline{T}^{\ominus}\overline{w} + \left|\overline{L}\right| V\mathbf{1}_m + \left|\overline{N}\right| V\mathbf{1}_m, \tag{3.8}$$

$$\underline{\Delta} = \underline{T}^{\oplus}\overline{w} - \underline{T}^{\ominus}\underline{w} - \left|\underline{L}\right| V\mathbf{1}_m - \left|\underline{N}\right| V\mathbf{1}_m. \tag{3.9}$$

Lastly, $\overline{T}, \underline{T}, \overline{N}, \underline{N}$ are constrained to satisfy the following algebraic conditions,

$$\overline{T} + \overline{N}C = I_n, \tag{3.10}$$

$$\underline{T} + \underline{N}C = I_n. \tag{3.11}$$

Define the errors

$$\overline{e}_k = \overline{x}_k - x_k \tag{3.12}$$

$$\underline{e}_k = x_k - \underline{x}_k. \tag{3.13}$$

Now compute the dynamics of the errors, starting with $\overline{e}_{k+1}$. By (3.10),

$$\begin{aligned}\overline{e}_{k+1} &= \overline{\zeta}_{k+1} + \overline{N}y_{k+1} + \overline{\Delta} - (\overline{T} + \overline{N}C)x_{k+1} \\ &= \overline{\zeta}_{k+1} + \overline{N}Cx_{k+1} + \overline{N}v_{k+1} + \overline{\Delta} - \overline{T}x_{k+1} - \overline{N}Cx_{k+1} \\ &= \overline{T}A\overline{x}_k + \overline{L}(y_k - C\overline{x}_k) + \overline{T}p(\overline{x}_k) + \overline{G}(\overline{x}_k - \underline{x}_k) - \overline{T}(Ax_k + p(x_k) + w_k) + \overline{N}v_{k+1} + \overline{\Delta} \\ &= (\overline{T}A - \overline{L}C + \overline{G})\overline{e}_k + \overline{G}\underline{e}_k + \overline{T}(p(\overline{x}_k) - p(x_k)) + \overline{d}_k. \end{aligned} \tag{3.14}$$

where $\overline{d}_k = \overline{L}v_k - \overline{T}w_k + \overline{N}v_{k+1} + \overline{\Delta}$. Similarly,

$$\underline{e}_{k+1} = (\underline{T}A - \underline{L}C + \underline{G})\underline{e}_k + \underline{G}\overline{e}_k + \underline{T}(p(x_k) - p(\underline{x}_k)) + \underline{d}_k. \tag{3.15}$$

where $\underline{d}_k = \underline{T}w_k - \underline{L}v_k - \underline{N}v_{k+1} - \underline{\Delta}$. Define

$$d_k = \begin{bmatrix} \overline{d}_k \\ \underline{d}_k \end{bmatrix}, \delta p(x_k, \overline{x}_k, \underline{x}_k) = \begin{bmatrix} \overline{T}\left(p(\overline{x}_k) - p(x_k)\right) \\ \underline{T}\left(p(x_k) - p(\underline{x}_k)\right) \end{bmatrix}.$$

The nonlinear error dynamics can be written compactly as

$$\varepsilon_{k+1} = \begin{bmatrix} \overline{T}A - \overline{L}C + \overline{G} & \overline{G} \\ \underline{G} & \underline{T}A - \underline{L}C + \underline{G} \end{bmatrix} \varepsilon_k + \delta p(x_k, \overline{x}_k, \underline{x}_k) + d_k. \tag{3.16}$$

$\overline{e}$ and $\underline{e}$ are defined in (3.12)-(3.13) in such a way that if they are both nonnegative vectors then $\underline{x} \leq x \leq \overline{x}$. Therefore, the observer gains $\overline{L}$, $\underline{L}$, and the other matrices $\overline{T}, \underline{T}, \overline{N}, \underline{N}, \overline{G}$ and $\underline{G}$ must be designed such that (3.16) is a positive system. Moreover, $\overline{x}$ and $\underline{x}$ should remain close to x. This is accomplished by also designing the gains and other matrices such that (3.16) is input-to-state stable (ISS) with respect to the bounded disturbance signal $d = \{d_k\}_{k \in \mathbb{Z}_{\geq 0}}$.

By Lemma 2, the nonlinear system (3.16) is equivalent to an LPV system

$$\varepsilon_{k+1} = \mathcal{A}\left(\varrho(k)\right)\varepsilon_k + d_k. \tag{3.17}$$

where $\varrho(k)$ is a time-varying parameter which belongs to the bounded set

$$\varrho(k) \in \Xi = \left\{ \varrho \in \mathbb{R}_{\geq 0}^{\nu \times \nu} : \sum_{i=1}^{\nu} \sum_{j=1}^{\nu} \varrho_{ij} = 1 \right\}, \ \forall k \in \mathbb{Z}_{\geq 0},$$

and

$$\mathcal{A}(\varrho(k)) = \sum_{i=1}^{\nu} \sum_{j=1}^{\nu} \varrho_{ij}(k)\mathcal{A}_{ij},$$

where

$$\mathcal{A}_{ij} = \begin{bmatrix} \overline{T}(A+\Theta_i) - \overline{L}C + \overline{G} & \overline{G} \\ \underline{G} & \underline{T}(A+\Theta_j) - \underline{L}C + \underline{G} \end{bmatrix}. \tag{3.18}$$

The following proposition gives sufficient conditions for the error dynamics (3.16) to be a positive system.

Proposition 2. *Consider the system* (3.1)-(3.2) *where* $p(x_k) : \mathbb{R}^n \to \mathbb{R}^n$ *is a polytopic nonlinearity with* $\Omega = \{\Theta_1, \ldots, \Theta_\nu\}$ *and the observer* (3.4)-(3.11) *where assumptions 4 and 5 hold and*

$$\mathcal{A}_{ij} \geq 0,\ \forall (i,j) \in \{1, \ldots, \nu\}^2 \tag{3.19}$$

where $\mathcal{A}_{ij}$ *is defined in* (3.18)*. Then, if* $\underline{x}_0 \leq x_0 \leq \overline{x}_0$*, then* $\underline{x}_k \leq x_k \leq \overline{x}_k$ *for all* $k \in \mathbb{Z}_{\geq 0}$*.*

Proof. First, it must be shown that d_k is a nonnegative vector for all $k \in \mathbb{Z}_{\geq 0}$. Consider the following,

$$\overline{d}_k = \overline{L}v_k + \left|\overline{L}\right| V\mathbf{1}_m - \overline{T}w_k + \overline{T}^{\oplus}\underline{w} - \overline{T}^{\ominus}\overline{w} + \overline{N}v_{k+1} + \left|\overline{N}\right| V\mathbf{1}_m,$$

$-\overline{T}w_k + \overline{T}^{\oplus}\underline{w} - \overline{T}^{\ominus}\overline{w} \geq 0$ for all $k \in \mathbb{Z}_{\geq 0}$ by Assumption 4 and Lemma 1. It follows from Assumption 5 that $\overline{L}v_k + \left|\overline{L}\right| V\mathbf{1}_m \geq 0$ and $\overline{N}v_{k+1} + \left|\overline{N}\right| V\mathbf{1}_m \geq 0$ for all $k \in \mathbb{Z}_{\geq 0}$. Therefore, $\overline{d}_k \geq 0$ for all $k \in \mathbb{Z}_{\geq 0}$. By the same reasoning, $\underline{d}_k \geq 0$ for all $k \in \mathbb{Z}_{\geq 0}$. Hence, $d_k \geq 0$ for all $k \in \mathbb{Z}_{\geq 0}$.

By (3.19), $\mathcal{A}_{TLG}(\varrho) \geq 0$ for all $\varrho \in \Xi$. Therefore, if $\varepsilon_0 \geq 0$, then $\varepsilon_k \geq 0$ for all $k \in \mathbb{Z}_{\geq 0}$. This implies, by the definitions of ε_k, $\overline{e}$, and $\underline{e}$, that if $\underline{x}_0 \leq x_0 \leq \overline{x}_0$, then $\underline{x}_k \leq x_k \leq \overline{x}_k$ for all $k \in \mathbb{Z}_{\geq 0}$. □

Remark 1 (Sufficiency of the Positivity Conditions)**.** *It is well known that a discrete-time linear system* $x_{k+1} = Ax_k$ *is positive if and only if* A *is an nonnegative matrix. A linear-*

time varying discrete-time linear system $x_{k+1} = A(k)x_k$ *is positive if and only if the state-transition matrix*

$$\phi(t,s) = \begin{cases} I_n, & t = s, \\ A(t-1)A(t-2)\dots A(s), & t > s. \end{cases}$$

is nonnegative for all $t \geq s \geq 0$ *where* $t, s \in \mathbb{Z}_{\geq 0}$ *[87]. Since the LPV system* (3.17) *is generated from a nonlinear system* (3.16)*, its state-transition matrix is not known ahead of time. Hence,* (3.19) *is a sufficient condition that can be enforced through the selection of gains and other matrices.*

The use of the matrices $\overline{T}, \underline{T}, \overline{N}$, and $\underline{N}$ with the algebraic constraints (3.10)-(3.11) was first proposed by [153] to synthesize interval observers for discrete-time linear systems. In a traditional interval observer design for linear discrete-time systems, the observer gains $\overline{L}$ and $\underline{L}$ are solved for such that $A - \overline{L}C$ and $A - \underline{L}C$ are both Schur and nonnegative, which are, typically, difficult conditions to enforce simultaneously. [153] proposes the addition of $\overline{T}, \underline{T}, \overline{N}$, and $\underline{N}$ so that the conditions become that $\overline{T}A - \overline{L}C$ and $\underline{T}A - \underline{L}C$ are both nonnegative and Schur stable. The extra degrees of freedom help ensure the positivity and stability conditions. This can be seen as an alternative to expressing the interval observer in a different coordinate system. Coordinate transformations are discussed later in §3.2.1.

The extra variables help ensure positivity and stability in the nonlinear case considered in this paper; however, it is not always possible to find a $\overline{T}, \underline{T}, \overline{L}, \underline{L}$ such that $\overline{T}(A+\Theta_i) - \overline{L}C$ and $\underline{T}(A+\Theta_i) - \underline{L}C$ are nonnegative for all $i \in \{1, \dots, \nu\}$. Hence, the proposed interval observer (3.4)-(3.11) introduces coupling between the $\overline{x}$ and $\underline{x}$ dynamics through the matrices $\overline{G}$ and $\underline{G}$. This coupling is also inspired by the literature on the nonlinear interval observer problem in continuous-time which is discussed in §2. (3.19) implies that $\overline{G}$ and $\underline{G}$ must be nonnegative and $\overline{T}(A+\Theta_i) - \overline{L}C + \overline{G}$ and $\underline{T}(A+\Theta_i) - \underline{L}C + \underline{G}$ must also be nonnegative for all $i \in \{1, \dots, \nu\}$. Therefore, (3.19) can be satisfied even if $\overline{T}(A+\Theta_i) - \overline{L}C$ and $\underline{T}(A+\Theta_i) - \underline{L}C$

have some negative elements. Since $\overline{T}$ and $\underline{T}$ are not known *a priori*, $\overline{G}$ and $\underline{G}$ will be solved for in the proposed semidefinite program at the same time $\overline{T}$ and $\underline{T}$ are solved for.

The observer gains and other matrices can be synthesized via a solution to a semidefinite program to meet the requirements of Proposition 2 and guarantee ISS of the errors with respect to the disturbances. Using the LPV description of the nonlinear discrete-time system (3.17), parameter-dependent ISS-Lyapunov functions of the form [43]

$$V(\varepsilon, \varrho) = \varepsilon^T P(\varrho)\varepsilon \tag{3.20}$$

are used. $P(\rho) = \sum_{i=1}^{\nu}\sum_{j=1}^{\nu}\rho_{ij}P_{ij}$, for positive definite matrices $P_{ij} \in \mathbb{R}^{2n\times 2n}$, $\forall(i,j) \in \{1,\ldots,\nu\}^2$. $P(\rho)$ satisfies $0 \prec \underline{\kappa}I_{2n} \preceq P(\rho) \preceq \overline{\kappa}I_{2n}$ for a sufficiently small $\underline{\kappa} > 0$ and $\overline{\kappa} = \sum_{i=1}^{\nu}\sum_{j=1}^{\nu}\lambda_{\max}(P_{ij})$.

Theorem 5. *Consider the system* (3.1)-(3.2) *where* $p(x_k) : \mathbb{R}^n \to \mathbb{R}^n$ *is a polytopic nonlinearity with* $\Omega = \{\Theta_1,\ldots,\Theta_\nu\}$ *and* x_k *remains bounded for all* $k \in \mathbb{Z}_{\geq 0}$. *Suppose assumptions 4 and 5 hold and there exist* ν^2 $2n\times 2n$ *matrices* $P_{ij} = P_{ij}^T \succ 0$, $\forall(i,j) \in \{1,\ldots,\nu\}^2$, *M-Matrices* $\overline{J}, \underline{J} \in \mathbb{R}^{n\times n}$, *matrices* $\overline{Y}, \underline{Y} \in \mathbb{R}^{n\times m}$, $\overline{W}, \underline{W} \in \mathbb{R}^{n\times n}$, $\overline{F}, \underline{F} \in \mathbb{R}^{n\times(n+m)}$ *and* $\gamma > 0$ *such that the following holds*

$$\begin{bmatrix} -P_{ij} & \Phi_{ij} & 0_{2n} \\ \star & P_{gh} - J - J^T & J \\ \star & \star & -\gamma I_{2n} \end{bmatrix} \prec 0,\ \forall(g,h,i,j) \in \{1,\ldots,\nu\}^4 \tag{3.21}$$

$$\Phi_{ij} \geq 0,\ \forall(i,j) \in \{1,\ldots,\nu\}^2 \tag{3.22}$$

where

$$J = \begin{bmatrix} \overline{J} & 0 \\ 0 & \underline{J} \end{bmatrix},\ \Phi_{ij} = \begin{bmatrix} (\overline{J}\chi^\dagger + \overline{F}\Psi)\alpha_1(A+\Theta_i) - \overline{Y}C + \overline{W} & \overline{W} \\ \underline{W} & (\underline{J}\chi^\dagger + \underline{F}\Psi)\alpha_1(A+\Theta_j) - \underline{Y}C + \underline{W} \end{bmatrix},$$

$$\chi = \begin{bmatrix} I_n \\ C \end{bmatrix}, \Psi = I_{n+m} - \chi\chi^\dagger, \alpha_1 = \begin{bmatrix} I_n \\ 0_{m\times n} \end{bmatrix}, \alpha_2 = \begin{bmatrix} I_m \\ 0_{n\times m} \end{bmatrix}$$

With the observer dynamics (3.4)-(3.11) *and*

$$\overline{L} = \overline{J}^{-1}\overline{Y}, \underline{L} = \underline{J}^{-1}\underline{Y}, \overline{S} = \overline{J}^{-1}\overline{F}, \underline{S} = \underline{J}^{-1}\underline{F}, \overline{G} = \overline{J}^{-1}\overline{W}, \underline{G} = \underline{J}^{-1}\underline{W},$$
$$\overline{T} = \chi^{\dagger}\alpha_1 + \overline{S}\Psi\alpha_1,\ \underline{T} = \chi^{\dagger}\alpha_1 + \underline{S}\Psi\alpha_1, \overline{N} = \chi^{\dagger}\alpha_2 + \overline{S}\Psi\alpha_2,\ \underline{N} = \chi^{\dagger}\alpha_2 + \underline{S}\Psi\alpha_2,$$

then the following hold:

i. If $\underline{x}_0 \le x_0 \le \overline{x}_0$ *then* $\underline{x}_k \le x_k \le \overline{x}_k$ *for all* $k \in \mathbb{Z}_{\ge 0}$*,*

ii. $\overline{x}_k, \underline{x}_k$ *remain bounded, and the ultimate bound of* ε *depends on* $\|d\|_{\sup}$.

Proof. Consider the candidate ISS-Lyapunov function (3.20) for the LPV error dynamics (3.17) which evolves as follows

$$V(\varepsilon_{k+1}, \varrho(k+1)) - V(\varepsilon_k, \varrho(k)) =$$
$$\begin{bmatrix} \varepsilon_k \\ d_k \end{bmatrix}^T \begin{bmatrix} \mathcal{A}^T(\varrho(k))P(\varrho(k+1))\mathcal{A}(\varrho(k)) - P(\varrho(k)) & \mathcal{A}^T(\varrho(k))P(\varrho(k+1)) \\ \star & P(\varrho(k+1)) \end{bmatrix} \begin{bmatrix} \varepsilon_k \\ d_k \end{bmatrix}.$$

V is an ISS-Lyapunov function, and, thus, the error system is ISS if $V(\varepsilon_{k+1}, \varrho(k+1)) - V(\varepsilon_k, \varrho(k)) < \gamma d_k^T d_k$ for all $\varrho(k), \varrho(k+1) \in \Xi$ [83], which leads to the following set of LMIs

$$\begin{bmatrix} \mathcal{A}^T(\varrho(k))P(\varrho(k+1))\mathcal{A}(\varrho(k)) - P(\varrho(k)) & \mathcal{A}^T(\varrho(k))P(\varrho(k+1)) \\ \star & P(\varrho(k+1)) - \gamma I_{2n} \end{bmatrix} \prec 0 \tag{3.23}$$

for all $\varrho(k), \varrho(k+1) \in \Xi$. By the Schur complement [30, pp. 7-8], (3.23) is equivalent to

$$\begin{bmatrix} -P(\varrho_k) & \mathcal{A}^T(\varrho_k)P(\varrho_{k+1}) & P(\varrho_{k+1})\mathcal{A}(\varrho_k) \\ \star & P(\varrho_{k+1}) - \gamma I_{2n} & 0_{2n} \\ \star & \star & -P(\varrho_{k+1}) \end{bmatrix} \prec 0 \tag{3.24}$$

for all $\varrho(k), \varrho(k+1) \in \Xi$. When the following holds

$$Q_{ghij} = \begin{bmatrix} -P_{ij} & \mathcal{A}_{ij}^T P_{gh} & P_{gh}\mathcal{A}_{ij} \\ \star & P_{gh} - \gamma I_{2n} & 0_{2n} \\ \star & \star & -P_{gh} \end{bmatrix} \prec 0,\ \forall (g,h,i,j) \in \{1,\ldots,\nu\}^4$$

then (3.24) holds since LHS(3.24)= $\sum_{i=1}^{\nu}\sum_{j=1}^{\nu}\varrho_{gh}\left(\sum_{g=1}^{\nu}\sum_{h=1}^{\nu}\varrho_{ij}Q_{ghij}\right)$ where $\varrho(k) = [\varrho_{ij}] \in \Xi$ and $\varrho(k+1) = [\varrho_{gh}] \in \Xi$. By the Schur complement, $Q_{ghij} \prec 0$ is equivalent to

$$R_{ghij} = \begin{bmatrix} I_{2n} & 0_{2n} \\ \mathcal{A}_{ij} & I_{2n} \\ 0_{2n} & I_{2n} \end{bmatrix}^T \begin{bmatrix} -P_{ij} & 0_{2n} & 0_{2n} \\ 0_{2n} & P_{gh} & 0_{2n} \\ 0_{2n} & 0_{2n} & -\gamma I_{2n} \end{bmatrix} \begin{bmatrix} I_{2n} & 0_{2n} \\ \mathcal{A}_{ij} & I_{2n} \\ 0_{2n} & I_{2n} \end{bmatrix} \prec 0,\ \forall (g,h,i,j) \in \{1,\ldots,\nu\}^4$$

By the Projection Lemma, $R_{ghij} \prec 0$ is equivalent to

$$\begin{bmatrix} -P_{ij} & X\mathcal{A}_{ij} & 0_{2n} \\ \star & P_{gh} - X - X^T & X \\ \star & \star & -\gamma I_{2n} \end{bmatrix} \prec 0 \tag{3.25}$$

for an arbitrary matrix $X \in \mathbb{R}^{2n\times 2n}$. (3.21) is (3.25) with the following variable substitutions

$$\overline{T} = \chi^{\dagger}\alpha_1 + \overline{S}\Psi\alpha_1,\ \underline{T} = \chi^{\dagger}\alpha_1 + \underline{S}\Psi\alpha_1$$
$$X = J, \overline{Y} = \overline{J}\,\overline{L}, \underline{Y} = \underline{J}\,\underline{Y}, \overline{F} = \overline{J}\,\overline{S}, \underline{F} = \underline{J}\,\underline{S}, \overline{W} = \overline{J}\,\overline{G}, \underline{W} = \underline{J}\,\underline{G}.$$

J is invertible since $P_{gh} \prec J + J^T$ for (3.21) to be feasible and P_{gh} are positive definite. (3.10)-(3.11) are satisfied. Therefore, ε is ISS. Hence, there exists a $\mathcal{KL}$ function β and a $\mathcal{K}$ function ρ such that $|\varepsilon_k| \leq \beta(|\varepsilon_0, k) + \rho(\|d\|_{\sup})$. Therefore, ε_k is bounded for all $k \in \mathbb{Z}_{\geq 0}$ and the ultimate bound on $|\varepsilon_k|$ depends on $\rho(\|d\|_{\sup})$ which is 'small' when $\|d\|_{\sup}$ is 'small.' Since x_k is bounded, $\overline{x}_k = x_k + \overline{e}_k$ and $\underline{x}_k = x_k - \underline{e}_k$ remain bounded.

Since J is also an M-matrix, $J^{-1} \geq 0$. Therefore, $\mathcal{A}_{ij} = J^{-1}\Phi_{ij} \geq 0$. Hence, (3.19) is satisfied and, by Proposition 2, if $\underline{x}_0 \leq x_0 \leq \overline{x}_0$, then $\underline{x}_k \leq x_k \leq \overline{x}_k$ for all $k \in \mathbb{Z}_{\geq 0}$. □

Some remarks need to be made about the proposed interval observer (3.4)-(3.11) and the approach for synthesis in Theorem 5.

Remark 2 (The ISS-Lyapunov Function)**.** *The LPV approach has been used to design Luenberger observers for Lipschitz nonlinear systems in continuous-time with quadratic Lyapunov* *functions [116, 160]. [160] shows through a numerical investigation that the LPV approach can handle much larger Lipschitz constants than other common LMI methods for synthesizing Luenberger observers.*

The LPV approach allows for parameter-varying Lyapunov functions to be utilized in the discrete-time problem [43]. Parameter-varying Lyapunov functions are less conservative than quadratic Lyapunov functions since there are more solution variables in the proposed semidefinite program and the special case of a quadratic Lyapunov function can be obtained by enforcing $P = P^T \succ 0$ *and* $P_{ij} = P$ *for all* $(i, j) \in \{1, \ldots, \nu\}^2$ *(c.f. the discussion in [117, §6.1]).*

It is well known that a nonnegative matrix A *is Schur if and only if there exists an* $n \times n$ *diagonal matrix* $P \succ 0$ *such that* $A^T PA - P \prec 0$ *[127, Proposition 1]. What is more, it is shown in [87] that a linear time-varying positive system in discrete-time is stable if and only if it admits a diagonal quadratic (time-varying) Lyapunov function; however, since the projection lemma adds in the extra variable* J *which 'separates' the dynamics from the* P_{ij}*'s in the LMI, the restriction of the* P_{ij}*'s to be diagonal does not need to be explicitly made in* (3.21).

Remark 3 (Ultimate Bound of the Interval Estimate)**.** *Theorem 5 ensures that the system* (3.16) *is ISS with respect to* d*. So the ultimate bound on the errors and, hence, the interval width depends on* $\|d\|_{\sup}$*.* $\|d\|_{\sup}$ *is influenced by the width of the disturbance interval* $\|\overline{w} - \underline{w}\|$*, the size of* V*, the norm of the matrices* $\overline{T}, \underline{T}, \overline{L}, \underline{L}, \overline{N}$*, and* $\underline{N}$*, and the parameter* γ*. It is a nontrivial problem to be able to find the smallest possible interval width for given disturbance and noise bounds since it is difficult to directly minimize the norms of* $\overline{T}, \underline{T}, \overline{L}, \underline{L}, \overline{N}$*, and* $\underline{N}$ *in the proposed semidefinite program in Theorem 5.*

When there is no disturbance or measurement noise and $\overline{w}, \underline{w}, V$ *are set to zero in assumptions 4 and 5, then, since* $d_k = 0$ *uniformly, it follows from Definition 12 that* $\overline{x}_k$ *and* $\underline{x}_k$ *asymptotically approach* x_k*.*

Remark 4 (Degeneration to the Case of Linear Systems)**.** *If the system* (3.1)-(3.2) *is a linear system, then* $p(x) = 0$ *for all* $x \in \mathbb{R}^n$*, so the results presented in Proposition 2 and Theorem 5 can be applied by setting* $\Omega = \{0_n\}$*. Then, the necessary and sufficient condition based on* (3.19) *for positivity is that* $\overline{G}, \underline{G}, \overline{T}A - \overline{L}C + \overline{G}$*, and* $\underline{T}A - \underline{L}C + \underline{G}$ *are nonnegative matrices. This is different from the positivity conditions in [153], since it is possible for* (3.19) *to be satisfied when* $\overline{T}A - \overline{L}C$ *and* $\underline{T}A - \underline{L}C$ *have negative elements.*

The LMI conditions for stability proposed in [153] are based on the Schur complement, whereas the LMI condition (3.21) *in this paper is based on the projection lemma, which adds the extra variable* J*. Therefore, there are more degrees of freedom to ensure the stability condition, which can help reduce conservatism.*

By enforcing $\overline{T} = \underline{T} = I_n$ and $\overline{N} = \underline{N} = 0_{n\times m}$, the interval observer (3.4)-(3.11) mimics the structure used in many papers on interval observers for nonlinear continuous time systems (i.e. (2.3)-(2.4)). In this case, the observer gains $\overline{L}, \underline{L}$ and matrices $\overline{G}$ and $\underline{G}$ can be synthesized from the following corollary to Theorem 5. The proof follows the same steps as the proof of Theorem 5.

Corollary 1. *Consider the system* (3.1)-(3.2) *where* $p(x_k) : \mathbb{R}^n \to \mathbb{R}^n$ *is a polytopic nonlinearity with* $\Omega = \{\Theta_1, \ldots, \Theta_\nu\}$ *and* x_k *remains bounded for all* $k \in \mathbb{Z}_{\geq 0}$*. Suppose assumptions 4 and 5 hold and there exist* ν^2 $2n \times 2n$ *matrices* $P_{ij} = P_{ij}^T \succ 0,\ \forall (i,j) \in \{1,\ldots,\nu\}^2$*, M-Matrices* $\overline{J}, \underline{J} \in \mathbb{R}^{n\times n}$*, matrices* $\overline{Y}, \underline{Y} \in \mathbb{R}^{n\times m}$*,* $\overline{W}, \underline{W} \in \mathbb{R}^{n\times n}$*, and* $\gamma > 0$ *such that* (3.21)-(3.22) *hold, where*

$$J = \begin{bmatrix} \overline{J} & 0 \\ 0 & \underline{J} \end{bmatrix},\ \Phi_{ij} = \begin{bmatrix} \overline{J}(A+\Theta_i) - \overline{Y}C + \overline{W} & \overline{W} \\ \underline{W} & \underline{J}(A+\Theta_j) - \underline{Y}C + \underline{W} \end{bmatrix}.$$

With the observer dynamics (3.4)-(3.11) *where* $\overline{T} = \underline{T} = I_n$ *and* $\overline{N} = \underline{N} = 0_{n\times m}$ *and*

$$\overline{L} = \overline{J}^{-1}\overline{Y}, \underline{L} = \underline{J}^{-1}\underline{Y}, \overline{G} = \overline{J}^{-1}\overline{W}, \underline{G} = \underline{J}^{-1}\underline{W},$$

the conclusions of Theorem 5 hold.

3.2.1 Synthesis with Coordinate Transformation

Similarly to the continuous-time case, interval observers for discrete-time systems can be synthesized with the use of coordinate transformations. Consider the following assumption.

Assumption 6. *There exists a matrix $L \in \mathbb{R}^{n\times m}$ and an invertible matrix $U \in \mathbb{R}^{n\times n}$ such that $U^{-1}(A - LC)U$ is Schur and nonnegative.*

L can be found such that $A-LC$ is Schur but not necessarily nonnegative, then the matrix U can be constructed using the Jordan normal form in the case where all the eigenvalues of $A - LC$ are nonnegative and real. [120, Lemma 1] can be used in other cases by first solving an inverse eigenvalue problem to find a nonnegative matrix R that has the same set of eigenvalues as $A - LC$. More discussion on this process can be found in [52, §II.B] and the references therein.

Consider the following interval observer dynamics:

$$\overline{z}_{k+1} = U^{-1}AU\overline{z}_k + U^{-1}p(U\overline{z}_k) + \overline{G}_z(\overline{z}_k - \underline{z}_k) + U^{-1}L(y_k - CU\overline{z}_k) + \overline{\Delta}_z, \tag{3.26}$$

$$\underline{z}_{k+1} = U^{-1}AU\underline{z}_k + U^{-1}p(U\underline{z}_k) + \underline{G}_z(\underline{z}_k - \overline{z}_k) + U^{-1}L(y_k - CU\underline{z}_k) + \underline{\Delta}_z, \tag{3.27}$$

where

$$\overline{\Delta}_z = (U^{-1})^{\oplus}\overline{w} - (U^{-1})^{\ominus}\underline{w} + |U^{-1}L|V\mathbf{1}_m, \tag{3.28}$$

$$\underline{\Delta}_z = (U^{-1})^{\ominus}\overline{w} - (U^{-1})^{\oplus}\underline{w} - |U^{-1}L|V\mathbf{1}_m. \tag{3.29}$$

By expressing the system in a new coordinate system $z = U^{-1}x$, where the linear part is both stable and positive, it can be easier to ensure the positivity of the nonlinear part by careful choice of $\overline{G}_z$ and $\underline{G}_z$, which can be synthesized through the following theorem that also verifies ISS stability of the errors $\overline{e}_{z_k} = \overline{z}_k - z_k$ and $\underline{e}_{z_k} = z_k - \underline{z}_k$ where $z_k = U^{-1}x_k$.

Theorem 6. *Consider the system* (3.1)-(3.2) *where $p(x_k) : \mathbb{R}^n \to \mathbb{R}^n$ is a polytopic nonlinearity with $\Omega = \{\Theta_1, \ldots, \Theta_\nu\}$ and x_k remains bounded for all $k \in \mathbb{Z}_{\geq 0}$. Suppose Assumptions 4 and 5 hold and there exists a matrix $L \in \mathbb{R}^{n\times m}$ and an invertible matrix*

$U \in \mathbb{R}^{n\times n}$ such that Assumption 6 holds. Further, suppose there exist ν^2 $2n \times 2n$ matrices $P_{ij} = P_{ij}^T \succ 0$, $\forall (i,j) \in \{1,\ldots,\nu\}^2$, M-Matrices $\overline{J}, \underline{J} \in \mathbb{R}^{n\times n}$, matrices $\overline{W}_z, \underline{W}_z \in \mathbb{R}^{n\times n}$, and $\gamma > 0$ such that (3.21)-(3.22) *hold, where*

$$J = \begin{bmatrix} \overline{J} & 0 \\ 0 & \underline{J} \end{bmatrix}, \; \Phi_{ij} = \begin{bmatrix} \overline{J}U^{-1}(A - LC + \Theta_i)U + \overline{W}_z & \overline{W}_z \\ \underline{W}_z & \underline{J}U^{-1}(A - LC + \Theta_j)U + \underline{W}_z \end{bmatrix}.$$

If $\underline{\xi}_0 \leq x_0 \leq \overline{\xi}_0$ and the interval observer (3.26)-(3.29) *where $\overline{G}_z = \overline{J}^{-1}\overline{W}_z$ and $\underline{G}_z = \underline{J}^{-1}\underline{W}_z$ is initialized as the following*

$$\overline{z}_0 = (U^{-1})^{\oplus}\overline{\xi}_0 - (U^{-1})^{\ominus}\underline{\xi}_0, \tag{3.30}$$

$$\underline{z}_0 = (U^{-1})^{\oplus}\underline{\xi}_0 - (U^{-1})^{\ominus}\overline{\xi}_0, \tag{3.31}$$

then $\underline{x}_k \leq x_k \leq \overline{x}_k$ for all $k \in \mathbb{Z}_{\geq 0}$ where

$$\overline{x}_k = U^{\oplus}\overline{z}_k - U^{\ominus}\underline{z}_k,$$

$$\underline{x}_k = U^{\oplus}\underline{z}_k - U^{\ominus}\overline{z}_k,$$

and $\overline{x}_k, \underline{x}_k$ remain bounded for all $k \in \mathbb{Z}_{\geq 0}$.

Proof. (3.30)-(3.31) ensures that, by Lemma 1, the initial condition of (3.26)-(3.29) satisfies $\underline{z}_0 \leq z_0 \leq \overline{z}_0$ where $z_0 = U^{-1}x_0$. The same arguments from Theorem 5 apply to show positivity and ISS of the errors $\overline{e}_{z_k}$ and $\underline{e}_{z_k}$. Therefore, $\underline{z}_k \leq z_k \leq \overline{z}_k$ and $\underline{z}_k, \overline{z}_k$ remain bounded for all $k \in \mathbb{Z}_{\geq 0}$. Since $\underline{z}_k, \overline{z}_k$ remain bounded, it follows that $\overline{x}_k, \underline{x}_k$ remain bounded for all $k \in \mathbb{Z}_{\geq 0}$. Lastly, it follows from Lemma 1 that $\underline{x}_k \leq x_k \leq \overline{x}_k$ for all $k \in \mathbb{Z}_{\geq 0}$. □

3.3 Numerical Example

Consider the following system:

$$x_1[k+1] = -x_2[k] + 0.02e^{-(x_1[k])^2}, \tag{3.32}$$

$$x_2[k+1] = x_1[k] + 0.199x_2[k], \tag{3.33}$$

$$x_3[k+1] = x_1[k] - 5.7 - 0.02e^{-(x_3[k])^2} + w[k], \tag{3.34}$$

$$y[k] = x_1[k] + v[k]. \tag{3.35}$$

In [20], a Luenberger observer was designed for this system. The nonlinearity

$$p(x) = \begin{bmatrix} 0.02e^{-{x_1}^2} \\ 0 \\ -0.02e^{-{x_3}^2} \end{bmatrix}$$

can be described by a polytope with the following vertices:

$$\Theta_1 = \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \Theta_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}, \Theta_3 = -\Theta_1, \Theta_4 = -\Theta_2.$$

Using Theorem 5, the following interval observer matrices are found:

$$\overline{L} = \underline{L} = \begin{bmatrix} -0.0002 \\ 0.9878 \\ 0.9998 \end{bmatrix}, \overline{G} = \underline{G} = \begin{bmatrix} 0.0991 & 0.0933 & 0.0414 \\ 0.1016 & 0.0883 & 0.0420 \\ 0.0939 & 0.0867 & 0.3042 \end{bmatrix},$$

$$\overline{T} = \underline{T} = \begin{bmatrix} -0.0101 & 0 & 0 \\ 0.1712 & 1.0000 & 0 \\ -0.0120 & 0 & 1.0000 \end{bmatrix}, \overline{N} = \underline{N} = \begin{bmatrix} 1.0101 \\ -0.1712 \\ 0.0120 \end{bmatrix}.$$

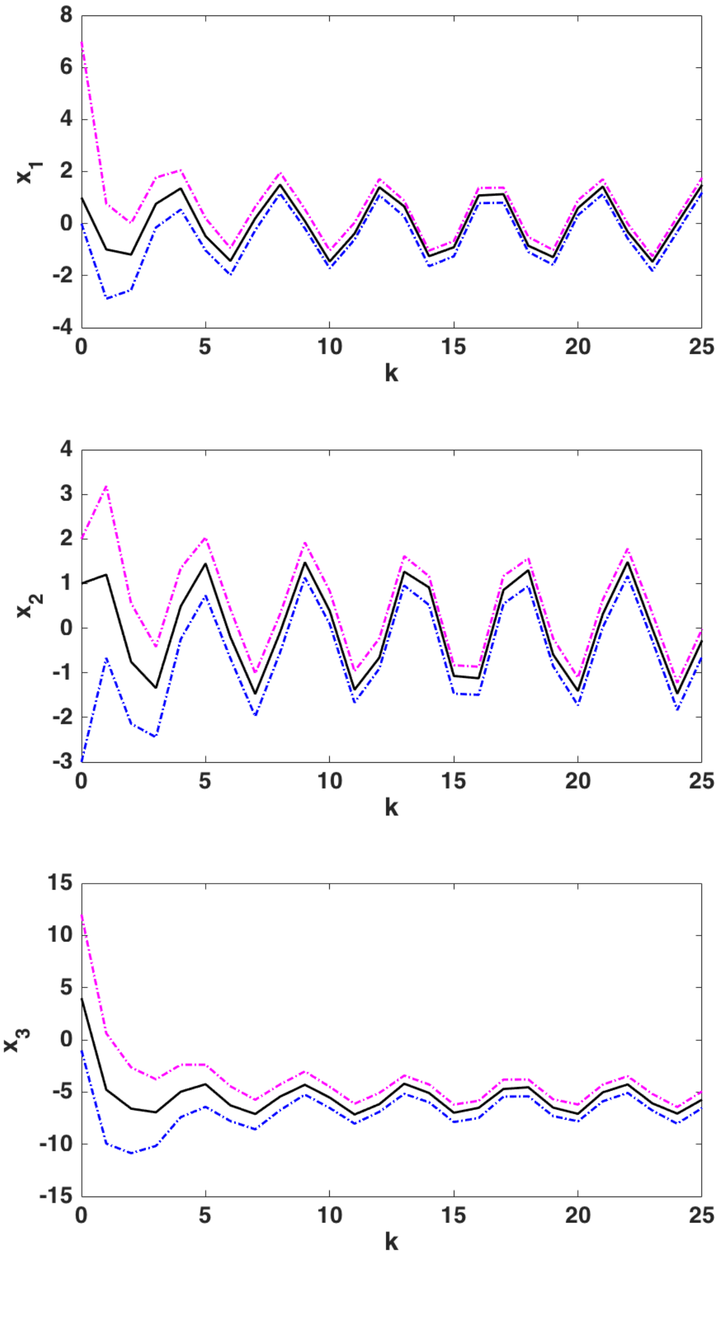

Figure 3.1: Interval observer simulation for a nonlinear discrete-time example.

Chapter 4
SYNTHESIS OF INTERVAL OBSERVERS FOR NONLINEAR SWITCHED SYSTEMS

"Never walk away from home ahead of your axe and sword. You can't feel a battle in your bones or foresee a fight."

*Háv*amál

This chapter is focused on synthesizing interval observers for switched systems with globally Lipschitz nonlinearities under the assumption that the switching signal is known at each instance of time. Luenberger observers are synthesized for switched linear systems in [9], where the stability of the switched error dynamics follows from a common Lyapunov function. When the switches are state-dependent, a jump at the switching detection times can be introduced into the observer to improve the state-estimation scheme [115, 19]. Observers for switched systems with unknown inputs was studied in [21] where coordinate transformations are used. The use of different coordinate transformations for different modes introduces jumps at the switching instances.

Observability of switched linear systems has been studied in [145], where the subsystems that make up the switched systems are not necessarily observable. Few results exist on the synthesis of observers for nonlinear switched systems. Observability of switched nonlinear systems was studied in and an observer is proposed in [138]. Norm-observers (or norm-estimators), which provide an upper bound on the norm of a switched nonlinear system are constructed in [106, 75] based on an ISS property.

In [57], interval observers are designed for switched linear systems, where ISS stability of the switched error dynamics is shown using common ISS-Lyapunov functions. The authors also introduced the use of multiple coordinate transformations. In [80], ISS stability is proven by using multiple quadratic ISS-Lyapunov functions under average dwell time constraints where all modes of the switched error dynamics are ISS on their own.

In designing interval observers for switched systems, the switching signal can be leveraged for easier synthesis compared to the non-switched cases. For example, in general, it is not always possible for all modes of the switched error dynamics to be made both positive and ISS. If the modes which cannot be made positive and ISS are just made positive, which can always be done, and their activation time is properly constrained, then the switched error dynamics are positive and ISS [106].

It is difficult to incorporate coordinate transformations when designing interval observers for switched systems since it is not always possible to find a single coordinate frame where all of the modes of the switched nonlinear error dynamics are positive and ISS. To overcome this, each mode is associated with its own coordinate system. To ensure positivity is preserved when changing the coordinate system, impulses are applied at the switching instances. Therefore, the error dynamics becomes a *switched impulsive system*. The impulses are open-loop and tend to be destabilizing, so the switches (and therefore impulses) should not occur too frequently to ensure ISS of the switched impulsive error dynamics (cf. [76]).

Interval observers are synthesized for nonlinear switched systems, where the interval observer parameters are found from the solutions to semidefinite programs. For the case where interval observers are designed in the original coordinate system, unstable modes in the switched error dynamics are allowed since not every mode can be made both positive and stable on its own. ISS stability is guaranteed when the unstable modes are not active for too long. When coordinate transformations are used, an average dwell time constraint is derived usingg multiple Lyapunov functions which guarantees ISS of the switched impulsive error dynamics.

4.1 Problem Setup

Consider a family of systems

$$\begin{cases} \dot{x}(t) &= A_i x(t) + B_i p_i(q(t)) + w(t), \\ q(t) &= E_i x(t), \\ y(t) &= C_i x(t) + v(t). \end{cases} \tag{4.1}$$

for all $i \in \mathcal{P}$, where $\mathcal{P} = \{1, \ldots, \mathcal{N}\}$ is the index set. For each $i \in \mathcal{P}$, $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times n_p}$, $E_i \in \mathbb{R}^{n_q\times n}$, $C_i \in \mathbb{R}^{n\times m}$, and the nonlinearities $p_i : \mathbb{R}^{n_q} \to \mathbb{R}^{n_p}$ are globally Lipschitz with Lipschitz constants $\gamma_i > 0$.

Consider the following *switched system*:

$$\begin{cases} \dot{x}(t) &= A_{\sigma(t)} x(t) + B_{\sigma(t)} p_{\sigma(t)}(q(t)) + w(t), \\ q(t) &= E_{\sigma(t)} x(t), \\ y(t) &= C_{\sigma(t)} x(t) + v(t), \end{cases} \tag{4.2}$$

which is generated from: the family of systems (4.1), an initial condition $x(0)$, and a switching signal $\sigma(\cdot)$. The function $\sigma : \mathbb{R}_{\geq 0} \to \mathcal{P}$ is a piecewise constant, right continuous function which, for each $t \geq 0$, specifies the index of the active mode. Let $\{t_0, t_1, t_2, t_3, \ldots\}$ be a strictly increasing sequence denoting the switching times and $t_0 = 0$. It is assumed that the active mode is known at each $t \geq 0$.

The switching signal σ is said to satisfy an average dwell time constraint with *average dwell time* $\tau_a > 0$ if there exist $N_0, \tau_a > 0$ such that

$$N(t,s) \leq \frac{t-s}{\tau_a} + N_0 \tag{4.3}$$

for all $t \geq s \geq 0$, where $N(t,s)$ is the number of switches that occur on the semi-open interval $(s,t]$.

The disturbance $w(t)$ is unknown, but it is assumed to belong to a known compact interval:

Assumption 7. *There exists* $\overline{w}, \underline{w} \in \mathbb{R}^n$ *that are known and are such that* $\underline{w} \leq w(t) \leq \overline{w}$ *for all* $t \geq 0$.

Moreover, the measurement noise $v(t)$ is unknown, but is assumed to belong to a known compact set:

Assumption 8. *There exists* $\overline{V} \in \mathbb{R}_{\geq 0}$ *which is known and is such that* $|v(t)| \leq \overline{V}\mathbf{1}_m$ *for all* $t \geq 0$.

The interval observer consists of two estimates $\underline{x}(t)$ and $\overline{x}(t)$. The dynamics of $\underline{x}(t)$ and $\overline{x}(t)$ are designed to use output feedback so that the following objectives hold:

(1) **Framer**: $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ for all $t \geq 0$ if the initial conditions satisfy $\underline{x}(0) \leq x(0) \leq \overline{x}(0)$.

(2) **Input-to-State Stability**: Under switching constraints, the dynamics of the follow error is ISS

$$\varepsilon(t) = \begin{bmatrix} \overline{e}(t) \\ \underline{e}(t) \end{bmatrix} = \begin{bmatrix} \overline{x}(t) - x(t) \\ x(t) - \underline{x}(t) \end{bmatrix}. \tag{4.4}$$

4.2 Interval Observer Synthesis under Constrained Switching

Consider the following interval observer for the switched system (4.2):

$$\begin{cases} \dot{\overline{x}}(t) & = A_{\sigma(t)}\overline{x}(t) + \overline{L}_{\sigma(t)}(y(t) - C_{\sigma(t)}\overline{x}(t)) \\ & \quad + B_{\sigma(t)}p_{\sigma(t)}(\overline{q}(t)) + \overline{G}_{\sigma(t)}(\overline{x}(t) - \underline{x}(t)) + \overline{\Delta}_{\sigma(t)}, \\ \dot{\underline{x}}(t) & = A_{\sigma(t)}\underline{x}(t) + \underline{L}_{\sigma(t)}(y(t) - C_{\sigma(t)}\underline{x}(t)) \\ & \quad + B_{\sigma(t)}p_{\sigma(t)}(\underline{q}(t)) + \underline{G}_{\sigma(t)}(\underline{x}(t) - \overline{x}(t)) + \underline{\Delta}_{\sigma(t)}, \\ \overline{q}(t) & = E_{\sigma(t)}\overline{x}(t), \ \overline{q}(t) = E_{\sigma(t)}\underline{x}(t), \end{cases} \tag{4.5}$$

where

$$\overline{\Delta}_i = \overline{w} + |\overline{L}_i|\overline{V}\mathbf{1}_m,\ \underline{\Delta}_i = \underline{w} - |\underline{L}_i|\overline{V}\mathbf{1}_m$$

for all $i \in \mathcal{P}$. The observer gains $\overline{L}_1, \ldots, \overline{L}_{\mathcal{N}}, \underline{L}_1, \ldots, \underline{L}_{\mathcal{N}} \in \mathbb{R}^{n\times m}$ and coupling matrices $\overline{G}_1, \ldots, \overline{G}_{\mathcal{N}}, \underline{G}_1, \ldots, \underline{G}_{\mathcal{N}} \in \mathbb{R}^{n\times n}$ are to be found to satisfy the two objectives of the interval observer. To this end, the error ε dynamics are expressed as follows:

$$\dot{\varepsilon}(t) = \mathfrak{A}_{\sigma(t)}\varepsilon(t) + \mathfrak{B}_{\sigma(t)}\Delta p_{\sigma(t)}(\overline{q}(t), q(t), \underline{q}(t)) + d(t), \tag{4.6}$$

which is a switched system, where

$$\mathfrak{A}_i = \begin{bmatrix} A_i - \overline{L}_i C_i + \overline{G}_i & \overline{G}_i \\ \underline{G}_i & A_i - \underline{L}_i C_i + \underline{G}_i \end{bmatrix},$$

$$\mathfrak{B}_i = I_2 \otimes B_i,$$

$$\Delta p_i(\overline{q}, q, \underline{q}) = \begin{bmatrix} p_i(\overline{q}) - p_i(q) \\ p_i(q) - p_i(\underline{q}) \end{bmatrix},$$

for all $i \in \mathcal{P}$, and

$$d(t) = \begin{bmatrix} \overline{\Delta}_{\sigma(t)} - \overline{L}_{\sigma(t)}v(t) - w(t) \\ w(t) + \underline{L}_{\sigma(t)}v(t) - \underline{\Delta}_{\sigma(t)} \end{bmatrix},$$

for all $t \geq 0$.

4.2.1 Sufficient Conditions for the Framer Property

To show positivity of the switched error dynamics (4.6), it is sufficient to show that, for each $i \in \mathcal{P}$, the modes

$$\dot{\varepsilon}(t) = \mathfrak{A}_i\varepsilon(t) + \mathfrak{B}_i\Delta p_i(\overline{q}(t), q(t), \underline{q}(t)) + d(t), \tag{4.7}$$

which generate the switched system (4.6) are positive systems.

By assumption 7, $\overline{w} - w(t) \geq 0$ and $w(t) - \underline{w} \geq 0$ for all $t \geq 0$. By assumption 8, $|\overline{L}_i|\overline{V}\mathbf{1}_m - \overline{L}_i v(t) \geq 0$ and $\underline{L}_i v(t) + |\underline{L}_i|\overline{V}\mathbf{1}_m \geq 0$ for all $i \in \mathcal{P}$ and $t \geq 0$. Therefore,

$$d(t) \geq 0, \ \forall t \geq 0. \tag{4.8}$$

By [160, Lemma 7], for every $\overline{q}, q \in \mathbb{R}^{n_q}$ there exists a matrix $\overline{\Theta}_i \in \mathbb{R}^{n_p \times n_q}$ such that $p_i(\overline{q}) - p_i(q) = \overline{\Theta}_i(\overline{q} - q)$ where $-\gamma_i \mathbf{1}_{n_p \times n_q} \leq \overline{\Theta}_i \leq \gamma_i \mathbf{1}_{n_p \times n_q}$. Similarly, for every $q, \underline{q} \in \mathbb{R}^{n_q}$, there exists a matrix $\underline{\Theta}_i \in \mathbb{R}^{n_p \times n_q}$ such that $p_i(q) - p_i(\underline{q}) = \underline{\Theta}_i(q - \underline{q})$ where $-\gamma_i \mathbf{1}_{n_p \times n_q} \leq \underline{\Theta}_i \leq \gamma_i \mathbf{1}_{n_p \times n_q}$. Hence, along the same lines as [160, §3.2], (4.7) can be rewritten as a linear time-varying (LTV) system

$$\dot{\varepsilon}(t) = (\mathfrak{A}_i + \mathfrak{B}_i \Theta(t) \mathfrak{E}_i)\, \varepsilon(t) + d(t), \tag{4.9}$$

where $\mathfrak{E}_i = I_2 \otimes E_i$ and $\Theta(t)$ is a time-varying parameter that belongs to a bounded set:

$$-\gamma_i I_2 \otimes \mathbf{1}_{n_p \times n_q} \leq \Theta(t) \leq \gamma_i I_2 \otimes \mathbf{1}_{n_p \times n_q}, \tag{4.10}$$

for all $t \geq 0$. Showing that LTV system (4.9) is a positive system implies that the nonlinear error dynamics (4.7) is also a positive system. By [13, Lemma VIII.1], (4.9) is a positive system if (4.8) holds, and $\mathfrak{A}_i + \mathfrak{B}_i \Theta(t) \mathfrak{E}_i \in \mathcal{M}_{2n}$ for all $t \geq 0$. The following lemma will be used to translate this condition on positivity into a condition which can be easily enforced in a semidefinite program.

Lemma 6. *Consider matrices $A \in \mathbb{R}^{k \times l}$, $\underline{B}, \overline{B} \in \mathbb{R}^{l \times m}$, and $C \in \mathbb{R}^{m \times n}$ such that $\underline{B} \leq \overline{B}$. The following hold*

$$ABC \geq A^{\oplus}\underline{B}C^{\oplus} - A^{\ominus}\overline{B}C^{\oplus} - A^{\oplus}\overline{B}C^{\ominus} + A^{\ominus}\underline{B}C^{\ominus}, \tag{4.11}$$

$$ABC \leq A^{\oplus}\overline{B}C^{\oplus} - A^{\ominus}\underline{B}C^{\oplus} - A^{\oplus}\underline{B}C^{\ominus} + A^{\ominus}\overline{B}C^{\ominus}, \tag{4.12}$$

for any $B \in \mathbb{R}^{l \times m}$ such that $\underline{B} \leq B \leq \overline{B}$.

Proof. Consider the following decomposition $A = A^{\oplus} - A^{\ominus}$. $A^{\oplus}\overline{B} - A^{\ominus}\underline{B} - AB = A^{\oplus}(\overline{B} - B) + A^{\ominus}(B - \underline{B})$. By assumption, $\overline{B} - B \geq 0$ and $B - \underline{B} \geq 0$. Moreover, $A^{\oplus} \geq 0$ and $A^{\ominus} \geq 0$. $A^{\oplus}(\overline{B} - B) - A^{\ominus}(B - \underline{B}) \geq 0$. In addition, $AB - A^{\oplus}\underline{B} + A^{\ominus}\overline{B} = A^{\oplus}(B - \underline{B}) + A^{\ominus}(\overline{B} - B) \geq 0$ by the same reasoning.

Now consider the decomposition $C = C^{\oplus} - C^{\ominus}$. $ABC - (\underline{A}B^{\oplus} - \overline{A}B^{\ominus})C^{\oplus} + (\overline{A}B^{\oplus} - \underline{A}B^{\ominus})C^{\ominus} = (AB - \underline{A}B^{\oplus} - \overline{A}B^{\ominus})C^{\oplus} + (\overline{A}B^{\oplus} - \underline{A}B^{\ominus} - AB)C^{\ominus}$ which is nonnegative since $C^{\oplus}$ and $C^{\ominus}$ are nonnegative and it has been show that $\underline{A}B^{\oplus} - \overline{A}B^{\ominus} \leq AB \leq \overline{A}B^{\oplus} - \underline{A}B^{\ominus}$. Hence, $ABC \geq (\underline{A}B^{\oplus} - \overline{A}B^{\ominus})C^{\oplus} + (\overline{A}B^{\oplus} - \underline{A}B^{\ominus})C^{\ominus}$ which yields (4.11). The inequality (4.12) follows from the same reasoning. □

By Lemma 6, the following holds:

$$\mathfrak{B}_i\Theta(t)\mathfrak{E}_i \geq -\gamma_i(\mathfrak{B}_i^{\oplus} + \mathfrak{B}_i^{\ominus})(I_2 \otimes \mathbf{1}_{n_p \times n_q})(\mathfrak{E}_i^{\oplus} + \mathfrak{E}_i^{\ominus}), \tag{4.13}$$

for all $t \geq 0$. Therefore, if the following holds:

$$\mathfrak{A}_i - \gamma_i(\mathfrak{B}_i^{\oplus} + \mathfrak{B}_i^{\ominus})(I_2 \otimes \mathbf{1}_{n_p \times n_q})(\mathfrak{E}_i^{\oplus} + \mathfrak{E}_i^{\ominus}) \in \mathcal{M}_{2n}, \tag{4.14}$$

then $\mathfrak{A}_i + \mathfrak{B}_i\Theta(t)\mathfrak{E}_i \in \mathcal{M}_{2n}$ for all $t \geq 0$. This is a consequence of the fact that, for two matrices J and Q, if $J \in \mathcal{M}_n$ and $Q \geq J$, then Q must be Metzler.

4.2.2 Sufficient Conditions for ISS

Consider the following set of positive definite functions:

$$V_i(\varepsilon) = \varepsilon^T P_i \varepsilon, \tag{4.15}$$

where $P_i \in \mathbb{R}^{2n \times 2n}$ and $P_i = P_i^T \succ 0$ for all $i \in \mathcal{P}$. Suppose each of these functions satisfy Lyapunov-like conditions:

$$\begin{aligned} &\|\varepsilon\| \geq \varphi(\|d\|) \\ \implies &\left\langle \nabla V_i(\varepsilon), \mathfrak{A}_i\varepsilon + \mathfrak{B}_i\Delta p_i(\overline{q}, q, \underline{q}) + d \right\rangle \leq -\lambda_i V_i(\varepsilon), \end{aligned} \tag{4.16}$$

where $\varphi \in \mathcal{K}_\infty$ and $\lambda_i \in \{\lambda_s, -\lambda_u\}$ for two scalars $\lambda_s, \lambda_u > 0$. Let $\mathcal{P}_s = \{i \in \mathcal{P} : \lambda_i = \lambda_s\}$ and $\mathcal{P}_u = \mathcal{P} \setminus \mathcal{P}_s$. That is, $\mathcal{P}_s$ is the set of modes which are ISS on their own, and $\mathcal{P}_u$ is the set of modes which are not ISS on their own.

Let $T^u(t,s)$ denote the total activation time of modes in $\mathcal{P}_u$ during the semi-open time interval $[s,t)$ where $0 \le s \le t$. If the activation time of the modes in $\mathcal{P}_u$ is limited by

$$T^u(t,s) \le T_0 + \rho(t-s),\ \forall t \ge s \ge 0, \tag{4.17}$$

for some positive constants ρ, T_0 such that

$$\rho < \frac{\lambda_s}{\lambda_s + \lambda_u}, \tag{4.18}$$

and σ has an average dwell time τ_a which satisfies

$$\tau_a > \frac{\ln \mu}{\lambda_s(1-\rho) - \lambda_u \rho}, \tag{4.19}$$

for some $\mu \ge 1$ such that

$$P_i \preceq \mu P_j, \tag{4.20}$$

for all $(i,j) \in \mathcal{P} \times \mathcal{P}$, then (4.6) is ISS with respect to d by [106, Theorem 2].

Using the $\mathcal{S}$-procedure [30], there exists $\varphi \in \mathcal{K}_\infty$ such that (4.16) holds if the following LMI holds:

$$\begin{bmatrix} \mathrm{He}\,(P_i \mathfrak{A}_i) + \lambda_i P_i + s_i \mathfrak{E}_i^T \mathfrak{E}_i & P_i \mathfrak{B}_i & P_i \\ \star & -\frac{s_i}{\gamma_i^2} I_{2n_p} & 0_{2n_p \times 2n} \\ \star & \star & -\varrho^2 I_{2n} \end{bmatrix} \prec 0, \tag{4.21}$$

for scalars $s_i, \varrho > 0$ and $\lambda_i \in \{\lambda_s, -\lambda_u\}$.

4.2.3 Synthesis

Now a semidefinite program is proposed that finds the observer gains and coupling matrices such that (4.14) and (4.21) are satisfied for all $i \in \mathcal{P}$. This guarantees that (4.6) is a positive system. Moreover, the dwell time constraints on σ that guarantee ISS of (4.6) are found from λ_s and λ_u and the positive definite matrices $P_1, \ldots, P_{\mathcal{N}}$.

The proposed semidefinite program in Theorem 7 restricts P_i to be a positive definite diagonal matrix for all $i \in \mathcal{P}$. This restriction is useful because for any two matrices J and Q where $J \in \mathbb{R}^{n\times n}$ is a positive definite diagonal matrix and $Q \in \mathcal{M}_n$, $JQ \in \mathcal{M}_n$, and, therefore, it is used in many results for the synthesis of positive systems [127, 144]. The Kalman-Yakubovich-Popov (KYP) lemma states that the satisfaction of the LMI (4.21) holds if and only if a certain frequency domain condition on $\mathfrak{A}_i$ holds [125]. Since the condition (4.14) implies that $\mathfrak{A}_i \in \mathcal{M}_{2n}$, the KYP lemma for positive systems guarantees that the frequency condition on $\mathfrak{A}_i$ holds if and only if the LMI (4.21) holds where P_i is a positive definite diagonal matrix [126, 40]. Therefore, the restriction of P_i to be positive definite diagonal is reasonable.

Theorem 7. *Consider a switched system* (4.2) *with a switching signal* σ *that is generated from the family of systems* (4.1) *where the nonlinearities* $p_i : \mathbb{R}^{n_q} \to \mathbb{R}^{n_p}$ *are globally Lipschitz with Lipschitz constants* $\gamma_i > 0$ *for all* $i \in \mathcal{P}$. *Let assumptions 7 and 8 hold. Suppose there exist scalars* $\lambda_s, \lambda_u, \varrho > 0$, *positive definite diagonal matrices* $\overline{P}_i, \underline{P}_i \in \mathbb{R}^{n\times n}$, *matrices* $\overline{Y}_i, \underline{Y}_i \in \mathbb{R}^{n\times m}$, *matrices* $\overline{W}_i, \underline{W}_i \in \mathbb{R}^{n\times n}$, *positive scalars* $s_i > 0$, *and scalars* $\lambda_i \in \mathbb{R}$ *such that the following is satisfied:*

$$\begin{bmatrix} He(\mathcal{Q}_i) + \lambda_i P_i + s_i \mathfrak{E}_i^T \mathfrak{E}_i & P_i \mathfrak{B}_i & P_i \\ \star & -\dfrac{s_i}{\gamma_i^2} I_{2n_p} & 0_{2n_p \times 2n} \\ \star & \star & -\varrho^2 I_{2n} \end{bmatrix} \prec 0, \tag{4.22}$$

$$\mathcal{Q}_i - \gamma_i P_i (\mathfrak{B}_i^{\oplus} + \mathfrak{B}_i^{\ominus})(I_2 \otimes \mathbf{1}_{n_p \times n_q})(\mathfrak{E}_i^{\oplus} + \mathfrak{E}_i^{\ominus}) \in \mathcal{M}_{2n},$$

and $\lambda_i \in \{\lambda_s, -\lambda_u\}$*, for all* $i \in \mathcal{P}$*, where*

$$P_i = \begin{bmatrix} \overline{P}_i & 0_n \\ \star & \underline{P}_i \end{bmatrix}, \tag{4.23}$$
$$\mathcal{Q}_i = \begin{bmatrix} \overline{P}_i A_i - \overline{Y}_i C_i + \overline{W}_i & \overline{W}_i \\ \underline{W}_i & \underline{P}_i A_i - \underline{Y}_i C_i + \underline{W}_i \end{bmatrix}.$$

Let $\mathcal{P}_s = \{i \in \mathcal{P} : \lambda_i = \lambda_s\}$ *and* $\mathcal{P}_u = \mathcal{P} \setminus \mathcal{P}_s$*. An interval observer* (4.5) *with*

$$\overline{L}_i = \overline{P}_i^{-1}\overline{Y}_i, \underline{L}_i = \underline{P}_i^{-1}\underline{Y}_i, \overline{G}_i = \overline{P}_i^{-1}\overline{W}_i, \underline{G}_i = \underline{P}_i^{-1}\underline{W}_i,$$

for all $i \in \mathcal{P}$ *satisfies the following:*

i. If $\underline{x}(0) \leq x(0) \leq \overline{x}(0)$*, then* $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ *for all* $t \geq 0$*.*

ii. If $x(t)$ *remains bounds for all* $t \geq 0$ *and* σ *has an average dwell time* τ_a *which satisfies* (4.19)-(4.20) *and there exist* ρ, T_0 *such that* (4.17)-(4.18) *hold, then* $\overline{x}(t)$ *and* $\underline{x}(t)$ *remain bounded for all* $t \geq 0$*.*

Proof. For each $i \in \mathcal{P}$, $\mathcal{Q}_i = P_i\mathfrak{A}_i$ with the following variable substitutions: $\overline{Y}_i = \overline{P}_i\overline{L}_i, \underline{Y}_i = \underline{P}_i\underline{L}_i, \overline{W}_i = \overline{P}_i\overline{G}_i, \underline{W}_i = \underline{P}_i\underline{G}_i$. Since P_i is positive definite diagonal, P_i^{-1} exists and is also positive definite diagonal. Therefore,

$$\begin{aligned} &P_i^{-1}\left(\mathcal{Q}_i - \gamma_i P_i(\mathfrak{B}_i^{\oplus} + \mathfrak{B}_i^{\ominus})(I_2 \otimes \mathbf{1}_{n_p \times n_q})(\mathfrak{E}_i^{\oplus} + \mathfrak{E}_i^{\ominus})\right) \\ &= \mathfrak{A}_i - \gamma_i(\mathfrak{B}_i^{\oplus} + \mathfrak{B}_i^{\ominus})(I_2 \otimes \mathbf{1}_{n_p \times n_q})(\mathfrak{E}_i^{\oplus} + \mathfrak{E}_i^{\ominus}) \in \mathcal{M}_{2n}, \end{aligned}$$

for all $i \in \mathcal{P}$ because the product of a positive definite diagonal matrix and a Metzler matrix is Metzler. Since, in addition, assumptions 7 and 8 hold, the error dynamics (4.6) is a positive system. This proves assertion i.

The LMI (4.22) is the LMI (4.21) by substituting in the variable $\mathcal{Q}_i$. The satisfaction of this LMI means that the functions (4.15) satisfy (4.16). By [106, Theorem 2], the conditions on the activation times of the unstable systems (4.17)-(4.18) and average dwell time of the switching signal (4.19)-(4.20) imply that (4.6) is ISS. Hence $\varepsilon(t)$ remains bounded for all $t \geq 0$. If $x(t)$ remains bounded for all $t \geq 0$, then $\overline{x}(t) = \overline{e}(t) + x(t)$ and $\underline{x}(t) = x(t) - \underline{e}(t)$ remain bounded for all $t \geq 0$. This proves assertion ii. □

Theorem 7 provides a systematic way to find the observer gains, coupling matrices, and dwell time constraints on the switching signal to guarantee positivity and ISS stability of the nonlinear switched error dynamics. Note that (4.22) is nonconvex in λ_i; however, a gridded search for the λ_i's can be performed.

4.3 Interval Observer Synthesis with Coordinate Transformations

The result in Theorem 7 limits the activation time of the modes which cannot be made positive and ISS in the original coordinate frame. In many cases, the restriction on the activation times is impracticable. To overcome this, suppose a coordinate transformation can be found for each mode such that the following assumption holds.

Assumption 9. *For each $i \in \mathcal{P}$, there exist an observer gain $L_i \in \mathbb{R}^{n\times m}$ and invertible matrix $U_i \in \mathbb{R}^{n\times n}$ such that*

$$\aleph_i = U_i^{-1}(A_i - L_iC_i)U_i \tag{4.24}$$

is both Hurwitz and Metzler.

Let $z(t)$ be the coordinates of the active system:

$$z(t) = U_{\sigma(t_k)}^{-1}x(t), \tag{4.25}$$

for all $t \in [t_k, t_{k+1})$ and $k \in \mathbb{Z}_{\geq 0}$.

The dynamics of the switched system (4.2) can be expressed using z defined in (4.25) as a switched impulsive system. The continuous part of the dynamics is the following:

$$\begin{cases} \dot{z}(t) &= U_{\sigma(t)}^{-1}A_{\sigma(t)}U_{\sigma(t)}z(t) \\ &\quad +U_{\sigma(t)}^{-1}B_{\sigma(t)}p_{\sigma(t)}(q(t)) + U_{\sigma(t)}^{-1}w(t), \\ q(t) &= E_{\sigma(t)}U_{\sigma(t)}z(t), \\ y(t) &= C_{\sigma(t)}U_{\sigma(t)}z(t) + v(t), \end{cases}$$

for all $t \in [t_k, t_{k+1})$ and $k \in \mathbb{Z}_{\geq 0}$. At the switching times, the change in coordinates introduces an impulse

$$z(t_k) = \mathfrak{U}_{\sigma(t_{k-1})\to\sigma(t_k)}z(t_k^-),$$

for all $k \in \mathbb{Z}_{\geq 1}$, where

$$\mathfrak{U}_{i\to j} = U_j^{-1}U_i, \tag{4.26}$$

for all $(i, j) \in \mathcal{P} \times \mathcal{P}$.

Consider an interval observer that is also a switched impulsive system. The following is the continuous part of the dynamics:

$$\begin{cases} \dot{\overline{z}}(t) &= U_{\sigma(t)}^{-1}A_{\sigma(t)}U_{\sigma(t)}\overline{z}(t) + U_{\sigma(t)}^{-1}p_{\sigma(t)}(\overline{q}(t)) \\ &\quad +U_{\sigma(t)}^{-1}L_{\sigma(t)}(y(t) - C_{\sigma(t)}U_{\sigma(t)}\overline{z}(t)) \\ &\quad +\overline{G}_{\sigma(t)}(\overline{z}(t) - \underline{z}(t)) + \overline{\Delta}_{\sigma(t)}, \\ \dot{\underline{z}}(t) &= U_{\sigma(t)}^{-1}A_{\sigma(t)}U_{\sigma(t)}\underline{z}(t) + U_{\sigma(t)}^{-1}p_{\sigma(t)}(\underline{q}(t)) \\ &\quad +U_{\sigma(t)}^{-1}L_{\sigma(t)}(y(t) - C_{\sigma(t)}U_{\sigma(t)}\underline{z}(t)) \\ &\quad +\underline{G}_{\sigma(t)}(\underline{z}(t) - \overline{z}(t)) + \underline{\Delta}_{\sigma(t)}, \\ \overline{q}(t) &= E_{\sigma(t)}U_{\sigma(t)}\overline{z}(t),\ \underline{q}(t) = E_{\sigma(t)}U_{\sigma(t)}\underline{z}(t), \end{cases} \tag{4.27}$$

for all $t \in [t_k, t_{k+1})$ and $k \in \mathbb{Z}_{\geq 0}$ where the observer gains $L_1, \ldots, L_{\mathcal{N}}$ come from assumption 9 and

$$\overline{\Delta_i} = U_i^{\oplus}\overline{w} - U_i^{\ominus}\underline{w} + |U_i^{-1}L_i|\overline{V}\mathbf{1}_m,$$
$$\underline{\Delta_i} = U_i^{\oplus}\underline{w} - U_i^{\ominus}\overline{w} + |U_i^{-1}L_i|\overline{V}\mathbf{1}_m,$$

for all $i \in \mathcal{P}$. The jump part is the following:

$$\begin{cases} \overline{z}(t_k) & = \mathfrak{U}^{\oplus}_{\sigma(t_{k-1})\to\sigma(t_k)}\overline{z}(t_k^-) - \mathfrak{U}^{\ominus}_{\sigma(t_{k-1})\to\sigma(t_k)}\underline{z}(t_k^-), \\ \underline{z}(t_k) & = \mathfrak{U}^{\oplus}_{\sigma(t_{k-1})\to\sigma(t_k)}\underline{z}(t_k^-) - \mathfrak{U}^{\ominus}_{\sigma(t_{k-1})\to\sigma(t_k)}\overline{z}(t_k^-), \end{cases} \tag{4.28}$$

for all $k \in \mathbb{Z}_{\geq 1}$.

Since the observer gains and coordinate transformations are assumed to have been found to satisfy assumption 9, the coupling matrices $\overline{G}_1, \ldots, \overline{G}_{\mathcal{N}}, \underline{G}_1, \ldots, \underline{G}_{\mathcal{N}} \in \mathbb{R}^{n\times n}$ are to be synthesized from a semidefinite program, and dwell-time conditions on σ are to be determined to guarantee the positivity and ISS stability of the following error

$$\varepsilon_z(t) = \begin{bmatrix} \overline{z}(t) - z(t) \\ z(t) - \underline{z}(t) \end{bmatrix}. \tag{4.29}$$

ε_z evolves according to a switched impulsive system with a continuous part

$$\dot{\varepsilon}_z(t) = \tilde{\mathfrak{A}}_{\sigma(t)}\varepsilon_z(t) + \Delta\tilde{p}_{\sigma(t)}(\overline{q}(t), q(t), \underline{q}(t)) + \tilde{d}(t) \tag{4.30}$$

for all $t \in [t_k, t_{k+1})$ and $k \in \mathbb{Z}_{\geq 0}$, where

$$\tilde{\mathfrak{A}}_i = \begin{bmatrix} \aleph_i + \overline{G}_i & \overline{G}_i \\ \underline{G}_i & \aleph_i + \underline{G}_i \end{bmatrix},$$
$$\tilde{\mathfrak{B}}_i = I_2 \otimes \left(U_i^{-1}B_i\right)$$
$$\Delta\tilde{p}_i(\overline{q}, q, \underline{q}) = \begin{bmatrix} U_i^{-1}\left(p_i(\overline{q}) - p_i(q)\right) \\ U_i^{-1}\left(p_i(q) - p_i(\underline{q})\right) \end{bmatrix},$$

$\aleph_i$ is defined in (4.24), for all $i \in \mathcal{P}$, and

$$\tilde{d}(t) = \begin{bmatrix} \overline{\Delta}_{\sigma(t)} - U_{\sigma(t)}^{-1}\overline{L}_{\sigma(t)}v(t) - U_{\sigma(t)}^{-1}w(t) \\ U_{\sigma(t)}^{-1}w(t) + U_{\sigma(t)}^{-1}\underline{L}_{\sigma(t)}v(t) - \underline{\Delta}_{\sigma(t)} \end{bmatrix},$$

for all $t \geq 0$. The jump part of the error dynamics is the following:

$$\varepsilon_z(t_k) = \mathfrak{J}_{\sigma(t_{k-1})\to\sigma(t_k)}\varepsilon_z(t_k^-) \tag{4.31}$$

for all $k \in \mathbb{Z}_{\geq 1}$, where

$$\mathfrak{J}_{i\to j} = \begin{bmatrix} \mathfrak{U}_{i\to j}^{\oplus} & \mathfrak{U}_{i\to j}^{\ominus} \\ \mathfrak{U}_{i\to j}^{\ominus} & \mathfrak{U}_{i\to j}^{\oplus} \end{bmatrix}, \tag{4.32}$$

and $\mathfrak{U}_{i\to j}$ is defined in (4.26), for all $(i, j) \in \mathcal{P} \times \mathcal{P}$.

4.3.1 Sufficient Conditions for the Framer Property

Showing that the switched impulsive error dynamics (4.30)-(4.31) is a positive system implies that if $\underline{z}(0) \leq z(0) \leq \overline{z}(0)$, then $\underline{z}(t) \leq z(t) \leq \overline{z}(t)$ for all $t \geq 0$. For every $t \geq 0$, if $\underline{z}(t) \leq z(t) \leq \overline{z}(t)$, then, by Lemma 1 and (4.25), $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ where

$$\underline{x}(t) = \left(U_{\sigma(t)}\right)^{\oplus}\underline{z}(t) - \left(U_{\sigma(t)}\right)^{\ominus}\overline{z}(t), \tag{4.33}$$

$$\overline{x}(t) = \left(U_{\sigma(t)}\right)^{\oplus}\overline{z}(t) - \left(U_{\sigma(t)}\right)^{\ominus}\underline{z}(t). \tag{4.34}$$

If the initial condition of the switched system (4.2) satisfies $\underline{\xi}_0 \leq x(0) \leq \overline{\xi}_0$, for $\overline{\xi}_0, \underline{\xi}_0 \in \mathbb{R}^n$, then, by Lemma 1 and (4.25), $\underline{z}(0) \leq z(0) \leq \overline{z}(0)$ where

$$\underline{z}(0) = \left(U_{\sigma(0)}^{-1}\right)^{\oplus}\underline{\xi}_0 - \left(U_{\sigma(0)}^{-1}\right)^{\ominus}\overline{\xi}_0, \tag{4.35}$$

$$\overline{z}(0) = \left(U_{\sigma(0)}^{-1}\right)^{\oplus}\overline{\xi}_0 - \left(U_{\sigma(0)}^{-1}\right)^{\ominus}\underline{\xi}_0. \tag{4.36}$$

To show that (4.30)-(4.31) is a positive system, it is sufficient to show that positivity is preserved over the continuous parts and over the jumps. This amounts to showing that

$$\dot{\varepsilon}_z(t) = \tilde{\mathfrak{A}}_i\varepsilon_z(t) + \tilde{\mathfrak{B}}_i\Delta\tilde{p}_i(\overline{q}(t), q(t), \underline{q}(t)) + \tilde{d}(t) \tag{4.37}$$

is a positive system for all $i \in \mathcal{P}$, and

$$\varepsilon_z(t_k) = \mathfrak{J}_{i\to j}\varepsilon_z(t_k^-) \tag{4.38}$$

is a positive system for all $(i,j) \in \mathcal{P}\times\mathcal{P}$.

By assumption 7 and Lemma 1, $U_i^{\oplus}\overline{w} - U_i^{\ominus}\underline{w} - U_i^{-1}w(t) \geq 0$ and $U_i^{-1}w(t) - U_i^{\oplus}\underline{w} - U_i^{\ominus}\overline{w} \geq 0$ for all $t \geq 0$. By assumption 8, for each $i \in \mathcal{P}$, $|U_i^{-1}L_i|\overline{V}\mathbf{1}_m - U_i^{-1}L_iv(t)$ and $U_i^{-1}L_iv(t) - |U_i^{-1}L_i|\overline{V}\mathbf{1}_m$ for all $t \geq 0$. Therefore, $\tilde{d}(t) \geq 0$ for all $t \geq 0$.

Using arguments similar to §4.2.1, the following conditions on $U_i, L_i, \overline{G}_i$, and $\underline{G}_i$ should hold:

$$\tilde{\mathfrak{A}}_i - \gamma_i(\tilde{\mathfrak{B}}_i^{\oplus} + \tilde{\mathfrak{B}}_i^{\ominus})(I_2 \otimes \mathbf{1}_{n_p\times n_q})(\tilde{\mathfrak{E}}_i^{\oplus} + \tilde{\mathfrak{E}}_i^{\ominus}) \in \mathcal{M}_{2n}, \tag{4.39}$$

where $\tilde{\mathfrak{E}}_i = I_2 \times (E_iU_i)$, for all $i \in \mathcal{P}$, for positivity to hold during the continuous parts.

By the definitions of the operators $(\cdot)^{\oplus}$ and $(\cdot)^{\ominus}$, $\mathfrak{J}_{i\to j} \geq 0$ for all $(i,j) \in \mathcal{P}\times\mathcal{P}$. Hence, positivity is preserved over the jumps.

4.3.2 Sufficient Conditions for ISS

ISS will be shown using a family of positive definite functions (4.15) for all $i \in \mathcal{P}$ and a switched Lyapunov function $V_{\sigma(t)}(\varepsilon_z(t))$. Note that there exist $\underline{\alpha}, \overline{\alpha} > 0$ such that

$$\underline{\alpha}I_{2n} \preceq P_i \preceq \overline{\alpha}I_{2n} \tag{4.40}$$

for all $i \in \mathcal{P}$.

Suppose for some $\lambda_c > 0$, the following holds:

$$\begin{bmatrix} \mathrm{He}\left(P_i\tilde{\mathfrak{A}}_i\right) + \lambda_c P_i + s_i\tilde{\mathfrak{E}}_i^T\tilde{\mathfrak{E}}_i & P_i\tilde{\mathfrak{B}}_i & P_i \\ \star & -\dfrac{s_i}{\gamma_i^2}I_{2n_p} & 0_{2n_p\times 2n} \\ \star & \star & -\varrho^2 I_{2n} \end{bmatrix} \prec 0, \tag{4.41}$$

for some positive scalar $s_i > 0$, for all $i \in \mathcal{P}$. Moreover, suppose for some $\lambda_d \in \mathbb{R}$, the following holds

$$\mathfrak{J}_{j\to i}^T P_i \mathfrak{J}_{j\to i} \preceq e^{-\lambda_d} P_j, \tag{4.42}$$

for all $(i, j) \in \mathcal{P} \times \mathcal{P}$.

In general, since the jumps are open-loop, they will tend to be destabilizing so $\lambda_d < 0$ to satisfy (4.42). When (4.41) is satisfied for $\lambda_c > 0$ then (4.37) is ISS for all $i \in \mathcal{P}$. (4.41) is satisfied with $\lambda_c > 0$ only if $\tilde{\mathfrak{A}}_i$ is Hurwitz for all $i \in \mathcal{P}$. This shows that the coordinate transformations and observer gains introduced in assumption 9 permit an increase the allowable activation times of the modes which would be unstable in the original coordinate system; however, the cost of using the different coordinate frames is that the switching must not occur too frequently due to the destabilizing impulses.

The following proposition will show that ISS of (4.30)-(4.31) can be guaranteed when σ has a dwell time

$$\tau_a = \frac{-\lambda_d}{\lambda_c - \eta} \tag{4.43}$$

for any $\eta \in (0, \lambda_c)$ when $\lambda_d < 0$.

Proposition 3. *Consider the switched impulsive error dynamics* (4.30)-(4.31)*, and suppose there exists* $P_i = P_i^T \succ 0$ *for all* $i \in \mathcal{P}$ *such that* (4.40)-(4.42) *hold for scalars* $\lambda_c, \underline{\alpha}, \overline{\alpha} > 0$ *and* $\lambda_d < 0$*. If* σ *satisfies an average dwell time constraint* (4.3) *with average dwell time* (4.43) *for* $\eta \in (0, \lambda_c)$ *and* $N_0 > 0$*, then* ε_z *is ISS with respect to* $\tilde{d}$*.*

Proof. The LMI (4.41) implies that there exists $\varphi \in \mathcal{K}_\infty$ such that

$$\begin{aligned} &\|\varepsilon_z\|^2 \geq \varphi(\|\tilde{d}\|^2) \\ \implies &\left\langle \nabla V_i(\varepsilon_z), \tilde{\mathfrak{A}}_i \varepsilon_z + \tilde{\mathfrak{B}}_i \Delta\tilde{p}_i(\overline{q}, q, \underline{q}) + \tilde{d} \right\rangle \leq -\lambda_c V_i(\varepsilon_z), \end{aligned} \tag{4.44}$$

for all $i \in \mathcal{P}$. The LMI (4.42) implies that

$$V_i(\mathfrak{J}_{j\to i}\varepsilon_z) \le e^{-\lambda_d} V_j(\varepsilon_z) \tag{4.45}$$

for all $(i,j) \in \mathcal{P} \times \mathcal{P}$.

By right-continuity of ε_z and $\tilde{d}$, there exists a sequence of times $0 = \hat{t}_0 \le \check{t}_1 < \hat{t}_1 < \check{t}_2 < \hat{t}_2 < \dots$ such that the following holds

$$\|\varepsilon_z(t)\|^2 \ge \varphi(\|\tilde{d}(t)\|^2_{[0,t]}) \;\; \forall t \in [\hat{t}_k, \check{t}_{k+1}),\; k \in \mathbb{Z}_{\ge 0}, \tag{4.46}$$

$$\|\varepsilon_z(t)\|^2 \le \varphi(\|\tilde{d}(t)\|^2_{[0,t]}) \;\; \forall t \in [\check{t}_k, \hat{t}_k),\; k \in \mathbb{Z}_{\ge 1}. \tag{4.47}$$

Suppose that $\check{t}_1 > 0$. For any $t \in [t_k, t_{k+1}) \cap [0, \check{t}_1]$, the following holds from (4.44),

$$V_{\sigma(t_k)}(\varepsilon_z(t)) \le e^{-\lambda_c(t-t_k)} V_{\sigma(t_k)}(\varepsilon(t_k)). \tag{4.48}$$

Therefore,

$$V_{\sigma(t_{k+1}^-)}(\varepsilon_z(t_{k+1}^-)) \le e^{-\lambda_c(t_{k+1}-t_k)} V_{\sigma(t_k)}(\varepsilon(t_k)).$$

By (4.45),

$$V_{\sigma(t_{k+1})}(\varepsilon_z(t_{k+1})) \le e^{-\lambda_d-\lambda_c(t_{k+1}-t_k)} V_{\sigma(t_k)}(\varepsilon_z(t_k)). \tag{4.49}$$

By iterating the same reasoning that led to (4.49) over all switching instances in $[0, \check{t}_1]$, it follows that

$$V_{\sigma(t)}(\varepsilon_z(t)) \le e^{-\lambda_d N(t,0)-\lambda_c t} V_{\sigma(0)}(\varepsilon_z(0)), \tag{4.50}$$

for all $t \in [0, \check{t}_1]$. If σ satisfies (4.3) with average dwell time (4.43), then

$$V_{\sigma(t)}(\varepsilon_z(t)) \le e^{-\eta t} e^{N_0} V_{\sigma(0)}(\varepsilon_z(0)), \tag{4.51}$$

for all $t \in [0, \check{t}_1]$. By (4.47), $V_{\sigma(t)}(\varepsilon_z(t)) \le \overline{\alpha}\varphi(\|\tilde{d}(t)\|^2_{[0,t]})$ for all $t \in [\check{t}_k, \hat{t}_k)$. If $\hat{t}_k$ is an switching instance then, by (4.45),

$$V_{\sigma(\hat{t}_k)}(\varepsilon_z(\hat{t}_k)) \le e^{-\lambda_d} \overline{\alpha}\varphi(\|\tilde{d}(t)\|^2_{[0,t]}).$$

Therefore, the following holds

$$V_{\sigma(t)}(\varepsilon_z(t)) \leq e^{-\lambda_d}\overline{\alpha}\varphi(\|\tilde{d}(t)\|^2_{[0,t]}),$$

for all $t \in [\check{t}_k, \hat{t}_k]$ since $e^{-\lambda_d} > 1$.

Using the same reasoning that led to (4.50),

$$V_{\sigma(t)}(\varepsilon_z(t)) \leq e^{-\lambda_d N(t,\hat{t}_k)-\lambda_c(t-\hat{t}_k)}V_{\sigma(\hat{t}_k)}(\varepsilon_z(\hat{t}_k)),$$

for all $t \in [\hat{t}_k, \check{t}_{k+1})$. Using the dwell time constraint, this means that

$$\begin{aligned} V_{\sigma(t)}(\varepsilon_z(t)) &\leq e^{-\eta(t-\hat{t}_k)}e^{N_0}V_{\sigma(\hat{t}_k)}(\varepsilon_z(\hat{t}_k)) \\ &\leq e^{N_0}V_{\sigma(\hat{t}_k)}(\varepsilon_z(\hat{t}_k)) \leq e^{N_0}e^{-\lambda_d}\overline{\alpha}\varphi(\|\tilde{d}(t)\|^2_{[0,t]}). \end{aligned} \tag{4.52}$$

Combining (4.51) and (4.52), the following holds

$$\begin{aligned} V_{\sigma(t)}(\varepsilon_z(t)) \leq \max\Big\{&e^{-\eta t}e^{N_0}V_{\sigma(0)}(\varepsilon_z(0)), \\ &e^{N_0}e^{-\lambda_d}\overline{\alpha}\varphi(\|\tilde{d}(t)\|^2_{[0,t]})\Big\} \end{aligned}$$

for all $t \geq 0$. By the standard arguments, ε_z is ISS. □

4.3.3 Synthesis

The following theorem finds the coupling matrices to guarantee positivity and proscribes the average dwell time constraints that ensure ISS of the switched impulsive error dynamics.

Theorem 8. *Consider a switched system* (4.2) *with a switching signal* σ *that is generated from the family of systems* (4.1) *where the nonlinearities* $p_i : \mathbb{R}^{n_q} \to \mathbb{R}^{n_p}$ *are globally Lipschitz with Lipschitz constants* $\gamma_i > 0$ *for all* $i \in \mathcal{P}$*. Let assumptions 7 and 8 hold. Suppose there exist scalars* $\lambda_c, \varrho > 0$*, scalar* $\lambda_d < 0$*, positive definite diagonal matrices* $\overline{P}_i, \underline{P}_i \in \mathbb{R}^{n\times n}$*, and matrices* $\overline{W}_i, \underline{W}_i \in \mathbb{R}^{n\times n}$*, positive scalars* $s_i > 0$ *such that*

$$\begin{bmatrix} He\left(\tilde{\mathcal{Q}}_i\right) + \lambda_c P_i + s_i \tilde{\mathfrak{E}}_i^T \tilde{\mathfrak{E}}_i & P_i \tilde{\mathfrak{B}}_i & P_i \\ \star & -\frac{s_i}{\gamma_i^2} I_{2n_p} & 0_{2n_p \times 2n} \\ \star & \star & -\varrho^2 I_{2n} \end{bmatrix} \prec 0, \tag{4.53}$$

$$\mathcal{Q}_i - \gamma_i P_i (\tilde{\mathfrak{B}}_i^{\oplus} + \tilde{\mathfrak{B}}_i^{\ominus})(I_2 \otimes \mathbf{1}_{n_p \times n_q})(\tilde{\mathfrak{E}}_i^{\oplus} + \tilde{\mathfrak{E}}_i^{\ominus}) \in \mathcal{M}_{2n}, \tag{4.54}$$

for all $i \in \mathcal{P}$ *and* (4.47) *holds for all* $(i, j) \in \mathcal{P} \times \mathcal{P}$*, where* P_i *is defined in* (4.23)*,* L_i *and* U_i *satisfy assumption 9, and*

$$\tilde{\mathcal{Q}}_i = \begin{bmatrix} \overline{P}_i \aleph_i + \overline{W}_i & \overline{W}_i \\ \underline{W}_i & \underline{P}_i \aleph_i + \underline{W}_i \end{bmatrix}.$$

An interval observer (4.27)*-*(4.28) *with* $\overline{G}_i = \overline{P}_i^{-1} \overline{W}_i, \underline{G}_i = \underline{P}_i^{-1} \underline{W}_i$*, for all* $i \in \mathcal{P}$ *satisfies the following:*

i. If $\underline{\xi}_0 \le x(0) \le \overline{\xi}_0$ *for* $\overline{\xi}_0, \underline{\xi}_0 \in \mathbb{R}^n$*, and* (4.27)*-*(4.28) *is initialized by* (4.35)*-*(4.36)*, then* $\underline{x}(t) \le x(t) \le \overline{x}(t)$ *for all* $t \ge 0$ *where* $\underline{x}(t)$ *and* $\overline{x}(t)$ *are* (4.33) *and* (4.34)*, respectively.*

ii. If $x(t)$ *remains bounded for all* $t \ge 0$ *and* σ *satisfies an average dwell time constraint* (4.3) *with average dwell time* (4.43) *for* $\eta \in (0, \lambda_c)$ *and* $N_0 > 0$*, then* $\overline{x}(t)$ *and* $\underline{x}(t)$ *remain bounded for all* $t \ge 0$*.*

Proof. For each $i \in \mathcal{P}$, $\tilde{\mathcal{Q}}_i = P_i \tilde{\mathfrak{A}}_i$ with the following variable substitutions: $\overline{W}_i = \overline{P}_i \overline{G}_i, \underline{W}_i = \underline{P}_i \underline{G}_i$. Since P_i is positive definite diagonal, P_i^{-1} exists and is also positive definite diagonal. Hence, the satisfaction of (4.54) implies that (4.39) holds. By the arguments of §4.3.1, the satisfaction of (4.39) and assumptions 7 and 8 imply that assertion i holds.

Satisfaction of the LMI (4.53) implies that (4.41) holds. Therefore, by Proposition 3, with (4.41) and (4.47) holding for λ_c and λ_d, (4.30)-(4.31) is ISS under the dwell time constraints. Therefore, $\varepsilon_z(t)$ is bounded for all $t \geq 0$. From (4.33) and (4.34),

$$\varepsilon(t) = \begin{bmatrix} (U_{\sigma(t)})^{\oplus} & (U_{\sigma(t)})^{\ominus} \\ (U_{\sigma(t)})^{\ominus} & (U_{\sigma(t)})^{\oplus} \end{bmatrix} \varepsilon_z(t).$$

Therefore, $\varepsilon(t)$ remains bounded for all $t \geq 0$. If $x(t)$ remains bounded for all $t \geq 0$, $\overline{x}(t)$ and $\underline{x}(t)$ remain bounded for all $t \geq 0$. This proves assertion ii. □

4.4 Numerical Examples

4.4.1 Example with Unstable Mode

Consider an example of a pendulum with position x_1 and velocity x_2 which is subject to a disturbance $w(t) \in \mathbb{R}^2$ which satisfies assumption 7 where

$$\overline{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \underline{w} = -\overline{w}.$$

The output switches between position output $y = x_1 + v(t)$ and velocity output $y = x_2 + v(t)$ which are both corrupted by noise $v(t)$ that satisfies assumption 8 with $\overline{V} = 0.1$. The family of systems (4.1) is described by the following:

$$A_1 = A_2 = \begin{bmatrix} 0 & 1 \\ 0 & -0.5 \end{bmatrix}, B_1 = B_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$
$$E_1 = E_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, p_1(q) = p_2(q) = 0.5 \sin q,$$
$$C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, C_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

Note that the pair (A_2, C_2) is unobservable. Therefore, there does not exist a matrix $L \in \mathbb{R}^2$ such that $A_2 - LC_2$ is Hurwitz. Therefore, assumption 9 cannot hold, and Theorem 8 cannot be used to synthesize interval observers. Theorem 7 can be used since it allows for unstable interval observer modes and guarantees boundedness of $\overline{x}(t)$ and $\underline{x}(t)$ if the switching signal σ of the switched system (4.2) satisfies the dwell time constraints proscribed in assertion ii of Theorem 7. Using Theorem 7 the following interval observer matrices are found for $\lambda_1 = \lambda_s = 0.7$ and $\lambda_2 = -\lambda_u = -0.2$:

$$\overline{G}_1 = \underline{G}_1 = \begin{bmatrix} 6.07 & 0.33 \\ 0.22 & 0.01 \end{bmatrix}, \overline{G}_2 = \underline{G}_2 = \begin{bmatrix} 0.00 & 0.12 \\ 0.95 & 6.32 \end{bmatrix},$$
$$\overline{P}_1 = \underline{P}_1 = \begin{bmatrix} 2.71 & 0 \\ 0 & 6.67 \end{bmatrix}, \overline{L}_1 = \underline{L}_1 = \begin{bmatrix} 43.49 \\ -0.49 \end{bmatrix},$$
$$\overline{P}_2 = \underline{P}_2 = \begin{bmatrix} 13.84 & 0 \\ 0 & 2.60 \end{bmatrix}, \overline{L}_2 = \underline{L}_2 = \begin{bmatrix} 1.00 \\ 34.06 \end{bmatrix},$$

and $\varrho^2 = 121.45$. Figure 4.1 shows an example simulation of the interval observer (4.5). $\overline{x}(t)$ and $\underline{x}(t)$ remain bounded since the switching signal limits the activation time of the unstable subsystem $i = 2$ by (4.17) with $T_0 = 1$ and $\rho = 0.5$ which satisfies (4.18). Moreover, it is clear that $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ for all $t \geq 0$ since the magenta, black, and blue lines do not cross over each other.

4.4.2 Example with Multiple Coordinate Transformations

Now consider an example with two modes where the family of systems (4.1) is described by the following:

$$A_1 = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, B_1 = B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$
$$E_1 = \begin{bmatrix} 1 & -0.5 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, p_1(q) = p_2(q) = 0.2 \sin q,$$
$$C_1 = \begin{bmatrix} 1 & 0.5 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}.$$

Note that there does not exist a matrix $L \in \mathbb{R}^2$ such that $A_1 - LC_1$ is both Metzler and Hurwtiz. Suppose there is no measurement noise, but there is a disturbance $w(t) \in \mathbb{R}^2$

satisfies assumption 7 where

$$\overline{w} = \begin{bmatrix} 0.005 \\ 0.05 \end{bmatrix}, \underline{w} = -\overline{w}.$$

The following satisfy assumption 9

$$U_1 = \begin{bmatrix} 2.41 & -0.59 \\ 1 & 1 \end{bmatrix}, L_1 = \begin{bmatrix} 6.01 \\ 1.81 \end{bmatrix},$$
$$U_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, L_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Using Theorem 8 the following interval observer matrices are found for $\lambda_c = 0.07$ and $\lambda_d = -0.8$:

$$\overline{G}_1 = \underline{G}_1 = \begin{bmatrix} 1.09 & 0.37 \\ 0.28 & 0.01 \end{bmatrix}, \overline{G}_2 = \underline{G}_2 = \begin{bmatrix} 0 & 0 \\ 0.2 & 0 \end{bmatrix},$$
$$\overline{P}_1 = \underline{P}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2.12 \end{bmatrix}, \overline{P}_2 = \underline{P}_2 = \begin{bmatrix} 4.34 & 0 \\ 0 & 4.41 \end{bmatrix},$$

and $\varrho^2 = 48.11$. Figure 4.2 shows an example simulation of the interval observer (4.27)-(4.28). $\overline{x}(t)$ and $\underline{x}(t)$ remain bounded since σ satisfies the dwell time constraint (4.3) with average dwell time (4.43) where $\eta = 0.01$ and $N_0 = 3$. The first impulse is the most pronounced since the error at that point is large. The impulses are smaller as time moves forward and the error decreases.

4.5 Concluding Remarks and Suggestions for Future Work

Interval observers for nonlinear switched systems were synthesized, and dwell time conditions were proscribed using multiple Lyapunov functions. When using multiple coordinate transformations, one associate with each mode, an impulse must be applied at the switching instances to ensure positivity when changing coordinate frames. These impulses tend to be destabilizing, so they should not occur too frequently to ensure boundedness of the interval.

Future work involves synthesizing interval observers for nonlinear switched systems where multiple coordinate transformations are used and not all the continuous modes of the switched impulsive error dynamics are stable. Another direction for future work is in the design of interval observers for nonlinear switched discrete time systems with multiple Lyapunov functions. For the case where no coordinate transformations are made, [91, Proposition 1] can be used as it is a discrete-time equivalent of [106, Theorem 2]. Interval observer-based output feedback control and interval observers with unknown switching signal are two other interesting areas to explore.

Further research should be devoted to applications. For example, an interesting application area is the state estimation of Unmanned Aerial Systems (UAS) which intermittently fly in GPS-denied environments [96, 93]. When the GPS signal is denied, the system is undetectable. Theorem 7 can give an indication of how long an aircraft can fly in a GPS-denied environment before needing to switch to other, possibly more ‘expensive,’ sources of position measurements so that the interval remains bounded.

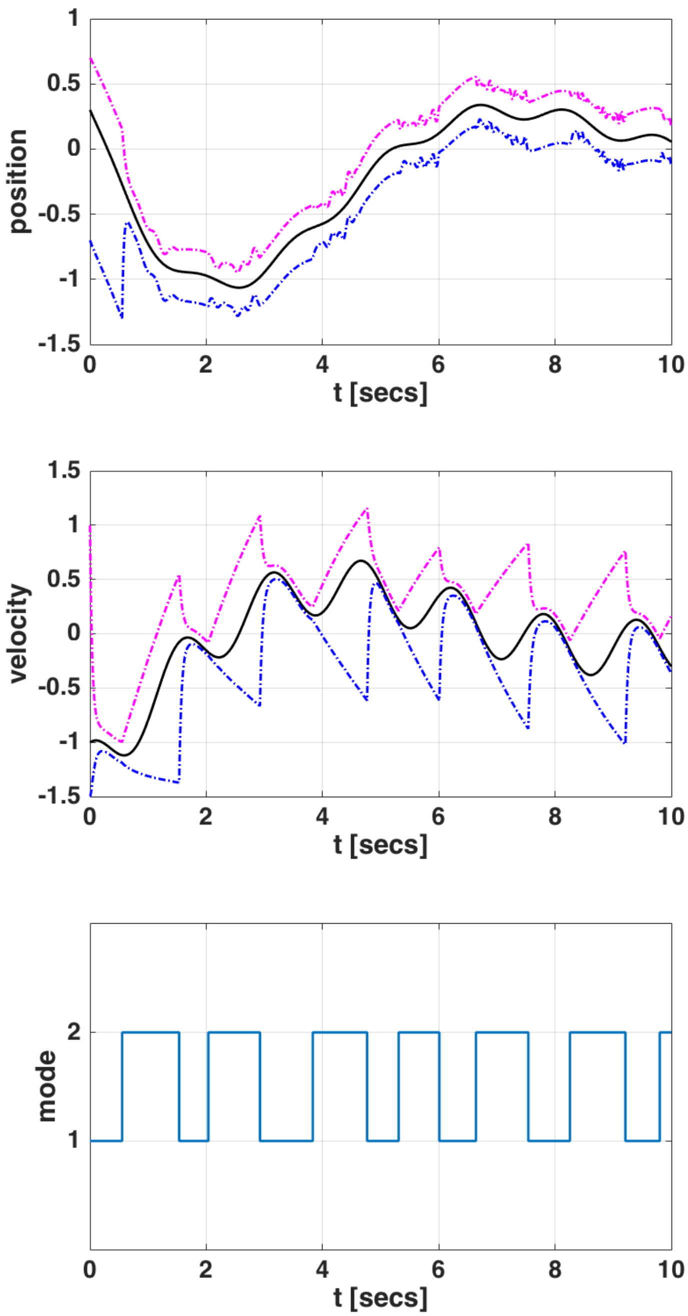

Figure 4.1: A simulation of an interval observer for a switched system with an unstable error dynamics mode.

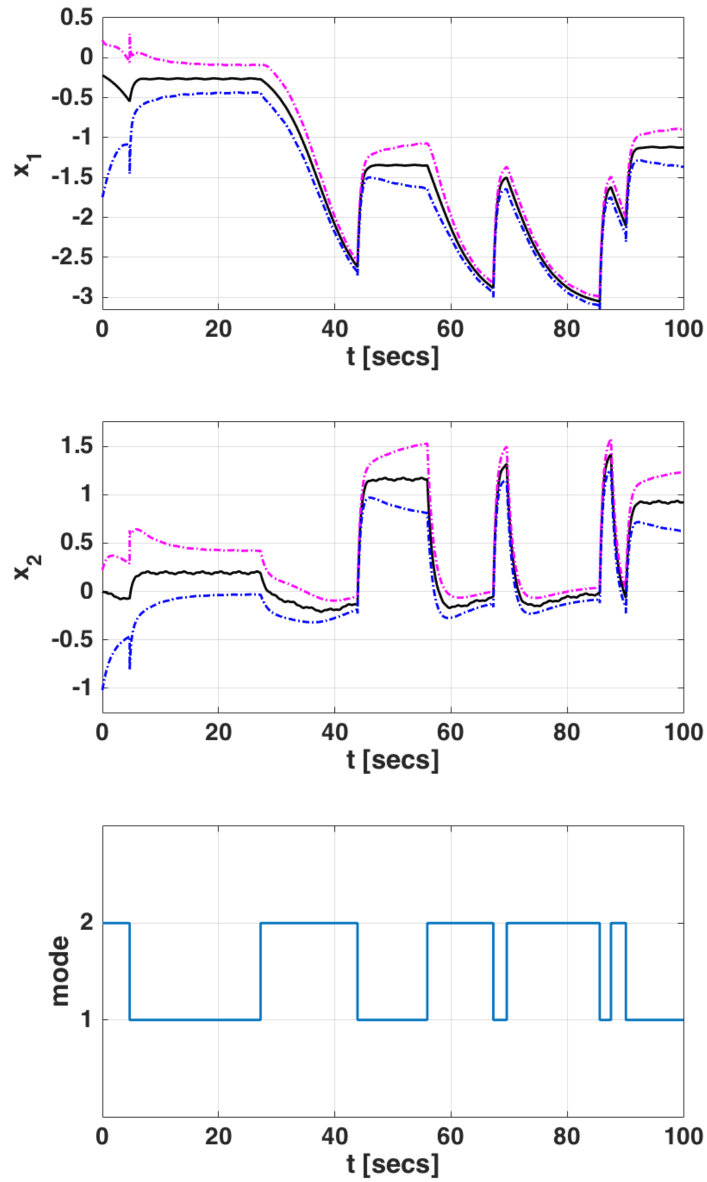

Figure 4.2: A simulation of an interval observer for switched system using coordinate transformations.

Chapter 5

SYNTHESIS OF INTERVAL OBSERVERS FOR NONLINEAR IMPULSIVE SYSTEMS

> "Take time to deliberate, but when the time for action has arrived, stop thinking and go."
>
> NAPOLEON BONAPARTE

Impulsive systems consist of: a continuous part, which evolves according to an ordinary differential equation; a jump part, which evolves according to a difference equation; and a sequence of impulse times. This chapter focuses on the synthesis of interval observers for nonlinear impulsive systems, where the continuous and jump dynamics have polytopic nonlinearities. Luenberger observers for linear impulsive systems have been studied in [22, 103], but there are no known results on observer design for nonlinear hybrid systems.

The structure of jump part of the proposed interval observers is partly inspired by the results of [153] which addressed the problem of interval observer synthesis for linear discrete-time systems by introducing extra variables which satisfy a certain algebraic condition that help ensure positivity and stability. The advantage of this structure is that the extra variables can be solved for in the proposed semidefinite programs and the need to change coordinate systems can be eliminated.

In impulsive systems, it is difficult to find a single coordinate transformation that ensures the stability and positivity of the continuous and jump parts [47]. There are two approaches proposed for the use of coordinate transformations. The first is to find a single transformation that ensures positivity and stability of the continuous dynamics, then the extra variables from [153] are synthesized to help ensure positivity and stability of the jump part of the dynamics without relying on a second coordinate transformation for the jump part. The second approach uses two different coordinate transformations: one for the continuous part and the other for the jump part.

In impulsive systems, the conditions for ISS of the errors can be relaxed compared to those of purely continuous-time or purely discrete-time systems. That is, either, but not both, the continuous or jump parts could be unstable and ISS can be guaranteed if appropriate dwell-time conditions are met [22]. So the ISS stability of the errors depends on the impulse sequence. Roughly speaking, if the jump part is destabilizing, then the impulsive error dynamics could still be ISS if the impulses do not occur too frequently. *Vice versa* if the continuous part is destabilizing, then the jumps should happen frequently enough to guarantee ISS. The stability of impulsive systems can be analyzed by finding an equivalent discrete-time system. This approach is straightforward for linear systems [33]; however, it is difficult to discretize nonlinear systems reliably, particularly when the time between impulses varies.

5.1 Problem Setup

Consider the following nonlinear impulsive system with linear output:

$$\dot{x}(t) = A_c x(t) + p_c(x(t)) + w(t),\ t \neq t_k, k \in \mathbb{Z}_{\geq 1}, \tag{5.1}$$

$$x(t_k) = A_d x(t_k^-) + p_d(x(t_k^-)) + w(t_k^-),\ k \in \mathbb{Z}_{\geq 1}, \tag{5.2}$$

$$y(t) = Cx(t) + v(t), \tag{5.3}$$

where $\{t_1, t_2, \dots\}$ is strictly increasing sequence in $(0, \infty)$ of the impulse times. Denote $N(t, s)$ as the number of impulse times t_k in the semi-open interval $(s, t]$ for $0 \leq s < t$. The impulse sequence is assumed to not have a finite accumulation point, so a finite number of impulses occur in a finite amount of time. The state $x(t) \in \mathbb{R}^n$ evolves according to the differential equation (5.1), which is referred to as the *continuous part* of the impulsive system, between consecutive impulses. The difference equation (5.2), which is referred to as the *jump part* of the impulsive system, describes the instantaneous change of the state at the impulse times.

The nonlinearity $p_c : \mathbb{R}^n \to \mathbb{R}^n$ is a polytopic nonlinearity defined by ν_c vertices $\Theta_1 \ldots, \Theta_{\nu_c} \in \mathbb{R}^{n\times n}$, and $p_d : \mathbb{R}^n \to \mathbb{R}^n$ is a polytopic nonlinearity defined by ν_d vertices $\Upsilon_1 \ldots, \Upsilon_{\nu_d} \in \mathbb{R}^{n\times n}$. The linear parts of the dynamics are defined by the matrices $A_c, A_d \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{m\times n}$. A bounded disturbance $w(t) \in \mathbb{R}^n$ acts on the impulsive system. It is assumed that $\overline{w}_c, \underline{w}_c \in \mathbb{R}^n$ are known such that

$$\underline{w}_c \leq w(t) \leq \overline{w}_c,\ t \neq t_k, k \in \mathbb{Z}_{\geq 1}, \tag{5.4}$$

and $\overline{w}_d, \underline{w}_d \in \mathbb{R}^n$ are known such that

$$\underline{w}_d \leq w(t_k^-) \leq \overline{w}_d,\ k \in \mathbb{Z}_{\geq 1}. \tag{5.5}$$

The output $y(t) \in \mathbb{R}^m$ is linear in the state and is corrupted by measurement noise $v(t) \in \mathbb{R}^m$. It is assumed that $\overline{V} \in \mathbb{R}_{\geq 0}$ is known such that

$$|v(t)| \leq \overline{V}\mathbf{1}_m, \tag{5.6}$$

for all $t \geq 0$, where $|v(t)|$ denotes the elementwise absolute value of $v(t)$.

Interval observers consist of two coupled impulsive systems with states $\underline{x}(t), \overline{x}(t) \in \mathbb{R}^n$ and the error is defined as

$$\varepsilon(t) = \begin{bmatrix} \overline{e}(t) \\ \underline{e}(t) \end{bmatrix} = \begin{bmatrix} \overline{x}(t) - x(t) \\ x(t) - \underline{x}(t) \end{bmatrix}. \tag{5.7}$$

The dynamics of $\underline{x}(t)$ and $\overline{x}(t)$ are designed to use output feedback such that the error ε satisfies the following properties:

(1) **Positivity**: If $\varepsilon(0) \geq 0$, then $\varepsilon(t) \geq 0$ for all $t \geq 0$. This implies that if $\underline{x}(0) \leq x(0) \leq \overline{x}(0)$, then $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ for all $t \geq 0$.

(2) **Input-to-State Stability**: Under constraints on the impulse sequence, the error ε is ISS with respect to w, v. This implies that $\varepsilon(t)$ remains bounded for all $t \geq 0$.

5.2 Synthesis in Original Coordinates

Consider the following interval observer for (5.1)-(5.3):

$$\begin{cases} \dot{\overline{x}}(t) = A_c\overline{x}(t) + \overline{L}_c(y(t) - C\overline{x}(t)) + p_c(\overline{x}(t)) + \overline{G}_c(\overline{x}(t) - \underline{x}(t)) + \overline{\Delta}_c, \\ \dot{\underline{x}}(t) = A_c\underline{x}(t) + \underline{L}_c(y(t) - C\underline{x}(t)) + p_c(\underline{x}(t)) + \underline{G}_c(\underline{x}(t) - \overline{x}(t)) + \underline{\Delta}_c, \end{cases} \quad t \neq t_k, k \in \mathbb{Z}_{\geq 1}, \tag{5.8}$$

$$\begin{cases} \overline{\zeta}(t_k) = \overline{T}A_d\overline{x}(t_k^-) + \overline{L}_d(y(t_k^-) - C\overline{x}(t_k^-)) + \overline{T}p_d(\overline{x}(t_k^-)) + \overline{G}_d(\overline{x}(t_k^-) - \underline{x}(t_k^-)), \\ \overline{x}(t_k) = \overline{\zeta}(t_k) + \overline{N}y(t_k) + \overline{\Delta}_d, \\ \underline{\zeta}(t_k) = \underline{T}A_d\underline{x}(t_k^-) + \underline{L}_d(y(t_k^-) - C\underline{x}(t_k^-)) + \underline{T}p_d(\underline{x}(t_k^-)) + \underline{G}_d(\underline{x}(t_k^-) - \overline{x}(t_k^-)), \\ \underline{x}(t_k) = \underline{\zeta}(t_k) + \underline{N}y(t_k) + \underline{\Delta}_d, \end{cases} \quad k \in \mathbb{Z}_{\geq 1}, \tag{5.9}$$

where,

$$\overline{\Delta}_c = \overline{w}_c + \left|\overline{L}_c\right| \overline{V}\mathbf{1}_m, \tag{5.10}$$

$$\underline{\Delta}_c = \underline{w}_c - \left|\underline{L}_c\right| \overline{V}\mathbf{1}_m, \tag{5.11}$$

$$\overline{\Delta}_d = \overline{T}^{\oplus}\overline{w}_d - \overline{T}^{\ominus}\underline{w}_d + \left|\overline{L}_d\right| \overline{V}\mathbf{1}_m + \left|\overline{N}\right| \overline{V}\mathbf{1}_m, \tag{5.12}$$

$$\underline{\Delta}_d = \underline{T}^{\oplus}\underline{w}_d - \underline{T}^{\ominus}\overline{w}_d - \left|\underline{L}_d\right| \overline{V}\mathbf{1}_m - \left|\underline{N}\right| \overline{V}\mathbf{1}_m, \tag{5.13}$$

and $\overline{T}, \overline{N}, \underline{T}, \underline{N}$ satisfy the following algebraic constraints:

$$\overline{T} + \overline{N}C = I_n, \tag{5.14}$$

$$\underline{T} + \underline{N}C = I_n. \tag{5.15}$$

The dynamics of the error ε defined in (5.7) is an impulsive system. The continuous part of the error dynamics evolve according to the following differential equation:

$$\dot{\varepsilon}(t) = \begin{bmatrix} A_c - \overline{L}_c C + \overline{G}_c & \overline{G}_c \\ \underline{G}_c & A_c - \underline{L}_c C + \underline{G}_c \end{bmatrix} \varepsilon(t)$$
$$+ \begin{bmatrix} p_c(\overline{x}(t)) - p_c(x(t)) \\ p_c(x(t)) - p_c(\underline{x}(t)) \end{bmatrix} + d_c(t),\ t \neq t_k, k \in \mathbb{Z}_{\geq 1} \tag{5.16}$$

where

$$d_c(t) = \begin{bmatrix} \overline{\Delta}_c - w(t) - \overline{L}_c v(t) \\ w(t) + \underline{L}_c v(t) - \underline{\Delta}_c \end{bmatrix}.$$

At all impulse times t_k, the error $\overline{e}(t_k)$ is the following:

$$\begin{aligned}\overline{e}(t_k) &= \overline{\zeta}(t_k) + \overline{N}y(t_k) + \overline{\Delta}_d - x(t_k). \\ &= \overline{\zeta}(t_k) + \overline{N}y(t_k) + \overline{\Delta}_d - (\overline{T} + \overline{N}C)x(t_k),\end{aligned}$$

by (5.14). From (5.3), $\overline{N}y(t_k) = \overline{N}Cx(t_k) + \overline{N}v(t_k)$, therefore,

$$\overline{e}(t_k) = \overline{\zeta}(t_k) + \overline{N}v(t_k) + \overline{\Delta}_d - \overline{T}x(t_k).$$

Hence,

$$\begin{aligned}\overline{e}(t_k) &= (\overline{T}A_d - \overline{L}C + \overline{G}_d)\overline{e}(t_k^-) + \overline{G}_d\underline{e}(t_k^-) \\ &+ \overline{T}(p(\overline{x}(t_k^-) - p(x(t_k^-)) + \overline{\Delta}_d + \overline{L}_d v(t_k^-) - \overline{T}w(t_k^-) + \overline{N}v(t_k).\end{aligned}$$

Using similar reasoning, the jump part of the impulsive dynamics of $\underline{e}$ can be found, which leads to the following:

$$\varepsilon(t_k) = \begin{bmatrix} \overline{T}A_d - \overline{L}_d C + \overline{G}_d & \overline{G}_d \\ \underline{G}_d & A_d - \underline{L}_d C + \underline{G}_d \end{bmatrix} \varepsilon(t_k^-)$$
$$+ \begin{bmatrix} \overline{T}(p_d(\overline{x}(t_k^-)) - p_d(x(t_k^-)) \\ \underline{T}(p_d(x(t_k^-)) - p_d(\underline{x}(t_k^-))) \end{bmatrix} + d_d(k), \tag{5.17}$$

where

$$d_d(k) = \begin{bmatrix} \overline{\Delta}_d + \overline{L}_d v(t_k^-) - \overline{T} w(t_k^-) + \overline{N} v(t_k) \\ \underline{T} w(t_k^-) - \underline{L} v(t_k^-) - \underline{N} v(t_k) - \underline{\Delta}_d \end{bmatrix}.$$

The matrices $\overline{L}_c, \underline{L}_c, \overline{L}_d, \underline{L}_d, \overline{N}, \underline{N} \in \mathbb{R}^{n\times m}$ and $\overline{T}, \underline{T}, \overline{G}_c, \underline{G}_c, \overline{G}_d, \underline{G}_d \in \mathbb{R}^{n\times n}$ are to be found to satisfy the two objectives of the interval observer. The following theorem proposes a semidefinite program from which positivity of the interval observer is guaranteed. Moreover, the constraints on the impulse sequence that guarantee ISS stability are given.

Theorem 9. *Consider the impulsive system (5.1)-(5.3) with impulse sequence* $\{t_1, t_2, \dots\}$ *where* $p_c : \mathbb{R}^n \to \mathbb{R}^n$ *is a polytopic nonlinearity defined by* ν_c *vertices* $\Theta_1 \dots, \Theta_{\nu_c} \in \mathbb{R}^{n\times n}$*, and* $p_d : \mathbb{R}^n \to \mathbb{R}^n$ *is a polytopic nonlinearity defined by* ν_d *vertices* $\Upsilon_1 \dots, \Upsilon_{\nu_d} \in \mathbb{R}^{n\times n}$*. Suppose* $x(t)$ *remains bounded for all* $t \geq 0$*,* $\overline{w}_c, \underline{w}_c \in \mathbb{R}^n$ *satisfy* (5.4)*,* $\overline{w}_d, \underline{w}_d \in \mathbb{R}^n$ *satisfy* (5.5)*, and* $\overline{V} \in \mathbb{R}_{\geq 0}$ *satisfies* (5.6)*. Suppose there exist* $n \times n$ *diagonal matrices* $\overline{P}, \underline{P} \succ 0$*, M-matrices* $\overline{J}, \underline{J} \in \mathbb{R}^{n\times n}$*, matrices* $\overline{Y}_c, \underline{Y}_c, \overline{Y}_d, \underline{Y}_d \in \mathbb{R}^{n\times m}$*, matrices* $\overline{F}, \underline{F} \in \mathbb{R}^{n\times(n+m)}$*, matrices* $\overline{W}_c, \underline{W}_c, \overline{W}_d, \underline{W}_d \in \mathbb{R}^{n\times n}$*, a scalar* $\gamma > 0$*, and scalars* $\alpha_c, \alpha_d \in \mathbb{R}$*,* $\alpha_d \neq 0$ *such that*

$$\begin{bmatrix} He(\mathcal{Q}_{ij}) + \alpha_c P & P \\ \star & -\gamma I_{2n} \end{bmatrix} \preceq 0, \tag{5.18}$$

$$\mathcal{Q}_{ij} \in \mathcal{M}_{2n}, \tag{5.19}$$

for all $i, j = 1, \dots, \nu_c$*, and*

$$\begin{bmatrix} -e^{-\alpha_d} P & \Phi_{ij} & 0_{2n} \\ \star & P - J - J^T & J \\ \star & \star & -\gamma I_{2n} \end{bmatrix} \preceq 0, \tag{5.20}$$

$$\Phi_{ij} \geq 0, \tag{5.21}$$

for all $i, j = 1, \ldots, \nu_d$, *where*

$$J = \begin{bmatrix} \overline{J} & 0_n \\ 0_n & \underline{J} \end{bmatrix}, P = \begin{bmatrix} \overline{P} & 0_n \\ 0_n & \underline{P} \end{bmatrix} \tag{5.22}$$

$$\mathcal{Q}_{ij} = \begin{bmatrix} \overline{P}(A_c + \Theta_{c_i}) - \overline{Y}_c C + \overline{W}_c & \overline{W}_c \\ \underline{W}_c & \underline{P}(A_c + \Theta_{c_j}) - \underline{Y}_c C + \underline{W}_c \end{bmatrix} \tag{5.23}$$

$$\Phi_{ij} = \begin{bmatrix} (\overline{J}\chi^\dagger + \overline{F}\Psi)\alpha_1(A_d + \Theta_{d_i}) - \overline{Y}_d C + \overline{W}_d & \overline{W}_d \\ \underline{W}_d & (\underline{J}\chi^\dagger + \underline{F}\Psi)\alpha_1(A_d + \Theta_{d_j}) - \underline{Y}_d C + \underline{W}_d \end{bmatrix},$$

$$\chi = \begin{bmatrix} I_n \\ C \end{bmatrix}, \Psi = I_{n+m} - \chi\chi^\dagger, \alpha_1 = \begin{bmatrix} I_n \\ 0_{m\times n} \end{bmatrix}, \alpha_2 = \begin{bmatrix} I_m \\ 0_{n\times m} \end{bmatrix}. \tag{5.24}$$

An interval observer (5.8)-(5.13) *where*

$$\overline{L}_c = \overline{P}^{-1}\overline{Y}_c, \underline{L}_c = \underline{P}^{-1}\underline{Y}_c, \overline{L}_d = \overline{J}^{-1}\overline{Y}_d, \underline{L}_d = \underline{J}^{-1}\underline{Y}_d, \tag{5.25}$$

$$\overline{G}_c = \overline{P}^{-1}\overline{W}_c, \underline{G}_c = \underline{P}^{-1}\underline{W}_c, \overline{G}_d = \overline{J}^{-1}\overline{W}_d, \underline{G}_d = \underline{J}^{-1}\underline{W}_d, \tag{5.26}$$

$$\overline{S} = \overline{J}^{-1}\overline{F}, \underline{S} = \underline{J}^{-1}\underline{F},$$

$$\overline{T} = \chi^\dagger\alpha_1 + \overline{S}\Psi\alpha_1, \ , \overline{N} = \chi^\dagger\alpha_2 + \overline{S}\Psi\alpha_2, \tag{5.27}$$

$$\underline{T} = \chi^\dagger\alpha_1 + \underline{S}\Psi\alpha_1, \ \underline{N} = \chi^\dagger\alpha_2 + \underline{S}\Psi\alpha_2, \tag{5.28}$$

satisfies the following:

i. If $\underline{x}(0) \leq x(0) \leq \overline{x}(0)$*, then* $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ *for all* $t \geq 0$.

ii. If there exist constants $\mu, \lambda > 0$ *such that the impulse sequence satisfies,*

$$-\alpha_d N(t, s) - (\alpha_c - \lambda)(t - s) \leq \mu, \ \forall t \geq s \geq 0, \tag{5.29}$$

then $\overline{x}(t)$ *and* $\underline{x}(t)$ *remain bounded for all* $t \geq 0$.

Proof. The positivity of the impulsive error dynamics follows by showing that the continuous and jump parts of the error dynamics are positive systems on their own. Positivity of the continuous part implies that if $\varepsilon(t_k) \geq 0$, then $\varepsilon(t) \geq 0$ for all $t \in [t_k, t_{k+1})$, $k \in \mathbb{Z}_{\geq 0}$. Positivity of the jump part implies that if $\varepsilon(t_k^-) \geq 0$, then $\varepsilon(t_k) \geq 0$ for all $k \in \mathbb{Z}_{\geq 1}$. Clearly, this implies that if $\varepsilon(0) \geq 0$, the $\varepsilon(t) \geq 0$ for all $t \geq 0$.

Positivity of Continuous Part By Lemma 2, the continuous part of the impulsive error dynamics (5.16) can be rewritten as the following LPV system:

$$\dot{\varepsilon}(t) = \mathcal{A}_c(\varrho_c(t))\varepsilon(t) + d_c(t), \tag{5.30}$$

where ϱ_c belongs to a compact set:

$$\varrho_c(t) \in \Xi_c = \left\{ \varrho_c \in \mathbb{R}_{\geq 0}^{\nu_c \times \nu_c} : \sum_{i=1}^{\nu_c} \sum_{j=1}^{\nu_c} \varrho_{c_{ij}} = 1 \right\}, \ \forall t \geq 0,$$

and

$$\mathcal{A}_c(\varrho_c(t)) = \sum_{i=1}^{\nu_1} \sum_{j=1}^{\nu_1} \varrho_{c_{ij}}(t) \mathcal{A}_{c_{ij}},$$

where $\mathcal{A}_{c_{ij}}$ is defined as

$$\mathcal{A}_{c_{ij}} = \begin{bmatrix} A_c + \Theta_i - \overline{L}_c C + \overline{G}_c & \overline{G}_c \\ \underline{G}_c & A_c + \Theta_j - \underline{L}_c C + \underline{G}_c \end{bmatrix}, \tag{5.31}$$

for all $i, j = 1, \ldots, \nu_c$. The matrix $\mathcal{Q}_{ij} = P\mathcal{A}_{c_{ij}}$ with the following variable substitutions:

$$\overline{Y}_c = \overline{P}\,\overline{L}_c, \underline{Y}_c = \underline{P}\,\underline{L}_c, \overline{W}_c = \overline{P}\,\overline{G}_c, \underline{W}_c = \underline{P}\,\underline{G}_c.$$

Since P is positive definite and diagonal, P^{-1} exists and is positive definite and diagonal. Hence, $\mathcal{A}_{c_{ij}} \in \mathcal{M}_{2n}$ for all $i, j = 1, \ldots, \nu_c$ because $\mathcal{A}_{c_{ij}} = P^{-1}\mathcal{Q}_{ij}$ and the the product of a Metzler matrix and a positive definite diagonal matrix is Metzler. Therefore, $\mathcal{A}_c(\varrho_c) \in \mathcal{M}_{2n}$ for all $\varrho_c \in \Xi_c$ since a convex combination of Metzler matrices is Metzler. Since (5.4) holds, $\overline{w}_c - w(t) \geq 0$ and $w(t) - \underline{w}_c \geq 0$ for all $t \neq t_k$, $k \in \mathbb{Z}_{\geq 1}$. Since (5.6) holds, $-\overline{L}_c v(t) + |\overline{L}_c|\overline{V}\mathbf{1}_m \geq 0$ and $\underline{L}_c v(t) + |\underline{L}_c|\overline{V}\mathbf{1}_m \geq 0$ for all $t \neq t_k$, $k \in \mathbb{Z}_{\geq 1}$. Hence, $d_c(t) \geq 0$ for all $t \neq t_k$, $k \in \mathbb{Z}_{\geq 1}$. By [13, Lemma VIII.1], (5.30) is a positive continuous-time system. Therefore, if $\varepsilon(t_k) \geq 0$, then $\varepsilon(t) \geq 0$ for all $t \in [t_k, t_{k+1})$, $k \in \mathbb{Z}_{\geq 0}$.

Positivity of Jump Part The definitions of $\overline{T}$ and $\overline{N}$ in (5.27) ensure that (5.14) holds. This follows from the fact that the general solution to the algebraic equation $\overline{T} + \overline{N}C = I_n$ is

$$\begin{bmatrix} \overline{T} & \overline{N} \end{bmatrix} = \chi^\dagger + S\Psi,$$

for any matrix $S \in \mathbb{R}^{n\times(n+m)}$. Similarly, the definitions of $\underline{T}$ and $\underline{N}$ in (5.28) ensures that (5.15) holds, so the jump part of the error dynamics is (5.17). By Lemma 2, (5.17) can be rewritten as the following LPV system:

$$\varepsilon(t_k) = \mathcal{A}_d(\varrho_d(t_k^-))\varepsilon(t_k^-) + d_d(k), \tag{5.32}$$

where ϱ_d belongs to a compact set:

$$\varrho_d(t_k^-) \in \Xi_d = \left\{ \varrho_d \in \mathbb{R}_{\geq 0}^{\nu_d\times\nu_d} : \sum_{i=1}^{\nu_d}\sum_{j=1}^{\nu_d} \varrho_{d_{ij}} = 1 \right\},\ \forall k \in \mathbb{Z}_{\geq 1},$$

and

$$\mathcal{A}_d(\varrho_d(t_k^-)) = \sum_{i=1}^{\nu_d}\sum_{j=1}^{\nu_d} \varrho_{d_{ij}}(t_k^-)\mathcal{A}_{d_{ij}},$$

where $\mathcal{A}_{d_{ij}}$ is defined as follows:

$$\mathcal{A}_{d_{ij}} = \begin{bmatrix} \overline{T}(A_d + \Upsilon_i) - \overline{L}_d C + \overline{G}_d & \overline{G}_d \\ \underline{G}_d & \underline{T}(A_d + \Upsilon_j) - \underline{L}_d C + \underline{G}_d \end{bmatrix}, \tag{5.33}$$

for all $i, j = 1, \ldots, \nu_d$. The matrix $\Phi_{ij} = J\mathcal{A}_{d_{ij}}$ with the following variable substitutions:

$$\overline{T} = \chi^\dagger\alpha_1 + \overline{S}\Psi\alpha_1,\ \underline{T} = \chi^\dagger\alpha_1 + \underline{S}\Psi\alpha_1 \tag{5.34}$$

$$\overline{Y} = \overline{J}\,\overline{L}, \underline{Y} = \underline{J}\,\underline{Y}, \overline{F} = \overline{J}\,\overline{S}, \underline{F} = \underline{J}\,\underline{S}, \overline{W} = \overline{J}\,\overline{G}, \underline{W} = \underline{J}\,\underline{G}. \tag{5.35}$$

The satisfaction of the LMI (5.20) implies that J must be invertible. Since $\overline{J}, \underline{J}$ are M-matrices, J is an M-matrix, so J^{-1} is a nonnegative matrix [67, Theorem 10.3.2]. Therefore, the satisfaction of (5.21) implies that $\mathcal{A}_{d_{ij}} = J^{-1}\Phi_{ij} \geq 0$ for all $i, j = 1, \ldots, \nu_d$. This implies that $\mathcal{A}_d(\varrho_d) \geq 0$ for all $\varrho_d \in \Xi_d$. Since (5.5) holds, $\overline{T}^{\oplus}\overline{w}_d - \overline{T}^{\ominus}\underline{w}_d - Tw(t_k^-) \geq 0$ for all $k \in \mathbb{Z}_{\geq 1}$ by Lemma 1. Similarly, $Tw(t_k^-) - \underline{T}^{\oplus}\underline{w}_d + \underline{T}^{\ominus}\overline{w}_d \geq 0$ for all $k \in \mathbb{Z}_{\geq 1}$. Moreover, since (5.6) holds, $\left|\overline{L}_d\right|\overline{V}\mathbf{1}_m + \left|\overline{N}\right|\overline{V}\mathbf{1}_m + \overline{L}_d v(t_k^-) + \overline{N}v(t_k) \geq 0$ and $|\underline{L}_d|\,\overline{V}\mathbf{1}_m + |\underline{N}|\,\overline{V}\mathbf{1}_m - \underline{L}v(t_k^-) - \underline{N}v(t_k) \geq 0$ for all $k \in \mathbb{Z}_{\geq 1}$. Therefore, $d_d(k) \geq 0$ for all $k \in \mathbb{Z}_{\geq 1}$. Hence, if $\varepsilon(t_k^-) \geq 0$, then $\varepsilon(t_k) \geq 0$ for all $k \in \mathbb{Z}_{\geq 1}$.

Input-to-State Stability Consider the ISS Lyapunov function $V(\varepsilon) = \varepsilon^T P \varepsilon$. The satisfaction of the LMI (5.18) implies that

$$\langle \nabla V(\varepsilon), \mathcal{A}_c(\varrho_c)\varepsilon + d_c \rangle \leq -\alpha_c V(\varepsilon) + \gamma d_c^T d_c,\ \forall \varrho_c \in \Xi_c. \tag{5.36}$$

From the variable substitutions (5.34)-(5.35), the LMI condition (5.20) implies that

$$\begin{bmatrix} -e^{-\alpha_d}P & J\mathcal{A}_{d_{ij}} & 0_{2n} \\ \star & P - J - J^T & J \\ \star & \star & -\gamma I_{2n} \end{bmatrix} \preceq 0. \tag{5.37}$$

By the Projection Lemma [117], (5.37) is equivalent to the following LMI.

$$\begin{bmatrix} \mathcal{A}_{d_{ij}}^T P \mathcal{A}_{d_{ij}} - e^{-\alpha_d}P & \mathcal{A}_{d_{ij}}^T P \\ \star & P - \gamma I_{2n} \end{bmatrix} \preceq 0. \tag{5.38}$$

Since (5.38) holds for all $i, j = 1, \ldots, \nu_d$, then

$$V(\mathcal{A}_d(\varrho_d)\varepsilon + d_d) \leq e^{-\alpha_d}V(\varepsilon) + \gamma d_d^T d_d,\ \forall \varrho_d \in \Xi_d. \tag{5.39}$$

By [76, Theorem 1], since the ISS Lyapunov function satisfies (5.36)-(5.39), the impulsive error dynamics are ISS if the impulse sequence satisfies (5.29). Therefore, $\overline{e}(t)$ and $\underline{e}(t)$ are bounded for all $t \geq 0$. Since $x(t)$ is bounded for all $t \geq 0$, then $\overline{x}(t) = x(t) + \overline{e}(t)$ and $\underline{x}(t) = x(t) - \underline{e}(t)$ are bounded for all $t \geq 0$. □

Note that (5.18) and (5.21) are not convex because of the bilinearity introduced with the multiplication of α_c and P as well as $e^{-\alpha_d}$ and P; however, since α_c and α_d are scalar quantities, a gridded search for a feasible set of α_c and α_d can easily be performed. If $\alpha_c, \alpha_d > 0$, then the impulsive error dynamics is ISS for any impulsive sequence since, in this case, μ and λ can be always be found that satisfies (5.29). On the other hand, preservation of ISS in impulsive systems is dependent on the sequence of the impulses if either, but not both, the continuous or the impulsive parts is destabilizing (i.e. $\alpha_c\alpha_d < 0$) [76]. In this case, the boundedness of $\overline{x}(t)$ and $\underline{x}(t)$ will depend on the impulse sequence. Roughly speaking, if $\alpha_c > 0$ but $\alpha_d < 0$, then boundedness of $\overline{x}(t)$ and $\underline{x}(t)$ can be guaranteed if the impulses do not occur too frequently. *Vice versa*, if $\alpha_c < 0$ but $\alpha_d > 0$, the impulses must occur frequently enough to guarantee boundedness of $\overline{x}(t)$ and $\underline{x}(t)$.

The restriction of P to be a diagonal matrix is used frequently in positive systems synthesis [127, 50, 40]. This is a reasonable restriction since it is shown in [87] that a linear time-varying positive system is stable if and only if it admits a diagonal quadratic Lyapunov function in the continuous-time case [87, Theorem 4.2] and discrete-time case [87, Theorem 4.1].

The bound of the interval width $\|\overline{x}(t) - \underline{x}(t)\|$ as $t \to \infty$ (the ultimate bound) depends on γ and $\max\left\{\|d_c\|_{[0,\infty)}, \|d_d\|_{[0,\infty)}\right\}$ which depends on the size of the disturbance and noise, the gains $\overline{L}_c, \underline{L}_c, \overline{L}_d, \underline{L}_d$, and the matrices $\overline{T}, \underline{T}, \overline{N}, \underline{N}$. It is a nontrivial problem to find the smallest possible interval width for given disturbance and noise bounds. When there is no disturbance or measurement noise and $\overline{w}_c, \underline{w}_c, \overline{w}_d, \underline{w}_d$, and $\overline{V}$ are set to zero, then $\overline{x}(t)$ and $\underline{x}(t)$ asymptotically approach $x(t)$.

The use of the matrices $\overline{T}, \underline{T}, \overline{N}$, and $\underline{N}$ with the algebraic constraints (5.14)-(5.15) is inspired by [153] which synthesizes interval observers for discrete-time linear systems. In a traditional interval observer design for linear discrete-time systems (cf. [52, 102]), the observer gains $\overline{L}$ and $\underline{L}$ are solved for such that $A - \overline{L}C$ and $A - \underline{L}C$ are both Schur and nonnegative, which are, typically, difficult conditions to enforce simultaneously. To overcome this, coordinate transformations could be used. As an alternative to using coordinate transformations, [153] proposes the addition of $\overline{T}, \underline{T}, \overline{N}$, and $\underline{N}$ to require that $\overline{T}A - \overline{L}C$ and $\underline{T}A - \underline{L}C$ are both nonnegative and Schur stable.

When $\overline{T} = \underline{T} = I_n$ and $\overline{N} = \underline{N} = 0_{n\times m}$, the interval observer (5.8)-(5.13) has a more conventional structure in that it consists of a copies of the original system with output feedback and coupling terms. Moreover, with this structure, only the output measurement immediately before the impulse $y(t_k^-)$ is used. An interval observer with this structure can be synthesized from Theorem 9 by replacing Φ_{ij} in (5.24) with

$$\Phi_{ij} = \begin{bmatrix} \overline{J}(A_d + \Theta_{d_i}) - \overline{Y}_d C + \overline{W}_d & \overline{W}_d \\ \underline{W}_d & \underline{J}(A_d + \Theta_{d_j}) - \underline{Y}_d C + \underline{W}_d \end{bmatrix},$$

and replacing (5.27)-(5.28) with $\overline{T} = \underline{T} = I_n$ and $\overline{N} = \underline{N} = 0_{n\times m}$.

5.3 Synthesis with Coordinate Transformations

5.3.1 Synthesis with One Coordinate Transformation

In some cases, the semidefinite program in Theorem 9 is only feasible if $\alpha_c < 0$ because there does not exist an observer gain $L_c \in \mathbb{R}^{n\times m}$ such that $A_c - L_c C$ is both Metzler and Hurwtiz [120]. If the impulse sequence requires α_c to be positive for (5.29) to hold, a coordinate transformation can be used by finding an invertible matrix $U \in \mathbb{R}^{n\times n}$ and an observer gain $L_c \in \mathbb{R}^{n\times m}$ such that $U^{-1}(A_c - L_c C)U$ is Hurwtiz and Metzler. With the transformation $z = U^{-1}x$. The impulsive system (5.1)-(5.3) can be recast with the coordinate transformation as

$$\dot{z}(t) = U^{-1}A_c U z(t) + U^{-1}p_c(Uz(t)) + U^{-1}w(t),\ t \neq t_k, k \in \mathbb{Z}_{\geq 1}, \tag{5.40}$$

$$z(t_k) = U^{-1}A_d U z(t_k^-) + U^{-1}p_d(Uz(t_k^-)) + U^{-1}w(t_k^-),\ k \in \mathbb{Z}_{\geq 1}, \tag{5.41}$$

$$y(t) = CUz(t) + v(t). \tag{5.42}$$

Consider the following interval observer which is modified from (5.8)-(5.13) to account for the use of coordinate transformation U and the observer gain L_c,

$$\begin{cases} \dot{\overline{z}}(t) = U^{-1}A_cU\overline{z}(t) + U^{-1}p_c(U\overline{z}) \\ +U^{-1}L_c(y(t) - CU\overline{z}(t)) + \overline{G}_{z_c}(\overline{z}(t) - \underline{z}(t)) + \overline{\Delta}_{z_c}, \\ \dot{\underline{z}}(t) = U^{-1}A_cU\underline{z}(t) + U^{-1}p_c(U\underline{z}) \\ +U^{-1}L_c(y(t) - CU\underline{z}(t)) + \overline{G}_{z_c}(\underline{z}(t) - \overline{z}(t)) + \underline{\Delta}_{z_c}, \end{cases} \quad t \neq t_k, k \in \mathbb{Z}_{\geq 1}, \tag{5.43}$$

$$\begin{cases} \overline{\zeta}_z(t_k) = \overline{T}U^{-1}A_dU\overline{z}(t_k^-) + \overline{L}_d(y(t_k^-) - CU\overline{z}(t_k^-)) \\ +\overline{T}U^{-1}p_d(U\overline{z}(t_k^-)) + \overline{G}_{z_d}(\overline{z}(t_k^-) - \underline{z}(t_k^-)), \\ \overline{z}(t_k) = \overline{\zeta}_z(t_k) + \overline{N}y(t_k) + \overline{\Delta}_{z_d}, \\ \underline{\zeta}_z(t_k) = \underline{T}U^{-1}A_dU\underline{z}(t_k^-) + \underline{L}_d(y(t_k^-) - CU\underline{z}(t_k^-)) \\ +\underline{T}U^{-1}p_d(U\underline{z}(t_k^-)) + \underline{G}_{z_d}(\underline{z}(t_k^-) - \overline{z}(t_k^-)), \\ \underline{z}(t_k) = \underline{\zeta}_z(t_k) + \underline{N}y(t_k) + \underline{\Delta}_{z_d}, \end{cases} \quad k \in \mathbb{Z}_{\geq 1}. \tag{5.44}$$

where

$$\overline{\Delta}_{z_c} = (U^{-1})^{\oplus}\overline{w}_c - (U^{-1})^{\ominus}\underline{w}_c + |U^{-1}L_c|\overline{V}\mathbf{1}_m, \tag{5.45}$$

$$\underline{\Delta}_{z_c} = (U^{-1})^{\oplus}\underline{w}_c - (U^{-1})^{\ominus}\overline{w}_c - |U^{-1}L_c|\overline{V}\mathbf{1}_m, \tag{5.46}$$

$$\overline{\Delta}_{z_d} = (\overline{T}U^{-1})^{\oplus}\overline{w}_d - (\overline{T}U^{-1})^{\ominus}\underline{w}_d + |\overline{L}_d|\overline{V}\mathbf{1}_m + |\overline{N}|\overline{V}\mathbf{1}_m, \tag{5.47}$$

$$\underline{\Delta}_{z_d} = (\underline{T}U^{-1})^{\oplus}\underline{w}_d - (\underline{T}U^{-1})^{\ominus}\overline{w}_d - |\underline{L}_d|\overline{V}\mathbf{1}_m - |\underline{N}|\overline{V}\mathbf{1}_m. \tag{5.48}$$

The synthesis process is a two-step process where L_c and U are found, then the matrices $\overline{G}_{z_c}, \underline{G}_{z_c}, \overline{L}_d, \underline{L}_d, \overline{G}_{z_d}, \underline{G}_{z_d}, \overline{T}, \underline{T}, \overline{N}$, and $\underline{N}$ are synthesized from a semidefinite program to ensure positivity and ISS stability of the error dynamics under dwell-time constraints.

Theorem 10. *Consider the impulsive system (5.1)-(5.3) with impulse sequence* $\{t_1, t_2, \dots\}$ *where* $p_c : \mathbb{R}^n \to \mathbb{R}^n$ *is a polytopic nonlinearity defined by* ν_c *vertices* $\Theta_1 \dots, \Theta_{\nu_c} \in \mathbb{R}^{n\times n}$*, and* $p_d : \mathbb{R}^n \to \mathbb{R}^n$ *is a polytopic nonlinearity defined by* ν_d *vertices* $\Upsilon_1 \dots, \Upsilon_{\nu_d} \in \mathbb{R}^{n\times n}$*. Suppose* $x(t)$ *remains bounded for all* $t \geq 0$, $\overline{w}_c, \underline{w}_c \in \mathbb{R}^n$ *satisfy* (5.4), $\overline{w}_d, \underline{w}_d \in \mathbb{R}^n$ *satisfy* (5.5), *and* $\overline{V} \in \mathbb{R}_{\geq 0}$ *satisfies* (5.6).

Suppose an observer gain $L_c \in \mathbb{R}^{n\times m}$ *and an invertible matrix* $U \in \mathbb{R}^{n\times n}$ *are given. Moreover, suppose there exist* $n \times n$ *diagonal matrices* $\overline{P}, \underline{P} \prec 0$*, M-matrices* $\overline{J}, \underline{J} \in \mathbb{R}^{n\times n}$*, matrices* $\overline{Y}_d, \underline{Y}_d \in \mathbb{R}^{n\times m}$*, matrices* $\overline{F}, \underline{F} \in \mathbb{R}^{n\times(n+m)}$*, matrices* $\overline{W}_{z_c}, \underline{W}_{z_c}, \overline{W}_{z_d}, \underline{W}_{z_d} \in \mathbb{R}^{n\times n}$*, a scalar* $\gamma > 0$*, and scalars* $\alpha_c, \alpha_d \in \mathbb{R}$*,* $\alpha_d \neq 0$ *such that such that* (5.18)-(5.19) *hold for all* $i, j = 1, \ldots, \nu_c$ *and* (5.20)-(5.21) *hold for all* $i, j = 1, \ldots, \nu_d$ *where* P *and* J *are defined in* (5.22)*,*

$$
\begin{aligned}
\mathcal{Q}_{ij} &= \begin{bmatrix} \overline{P}U^{-1}(A_c - L_cC + \Theta_i)U + \overline{W}_{z_c} & \overline{W}_{z_c} \\ \underline{W}_{z_c} & \underline{P}U^{-1}(A_c - L_cC + \Theta_j)U + \underline{W}_{z_c} \end{bmatrix} \\
\Phi_{ij} &= \begin{bmatrix} (\overline{J}\chi^{\dagger} + \overline{F}\Psi)\alpha_1 U^{-1}(A_d + \Upsilon_i)U - \overline{Y}_d CU + \overline{W}_d & \overline{W}_d \\ \underline{W}_d & (\underline{J}\chi^{\dagger} + \underline{F}\Psi)\alpha_1 U^{-1}(A_d + \Upsilon_j)U - \underline{Y}_d CU + \underline{W}_d \end{bmatrix}, \\
\chi &= \begin{bmatrix} I_n \\ CU \end{bmatrix}, \Psi = I_{n+m} - \chi\chi^{\dagger}, \alpha_1 = \begin{bmatrix} I_n \\ 0_{m\times n} \end{bmatrix}, \alpha_2 = \begin{bmatrix} I_m \\ 0_{n\times m} \end{bmatrix}.
\end{aligned}
$$

An interval observer (5.43)-(5.48) *where*

$$
\begin{aligned}
&\overline{L}_d = \overline{J}^{-1}\overline{Y}_d, \underline{L}_d = \underline{J}^{-1}\underline{Y}_d, \\
&\overline{G}_{z_c} = \overline{P}^{-1}\overline{W}_{z_c}, \underline{G}_{z_c} = \underline{P}^{-1}\underline{W}_{z_c}, \overline{G}_{z_d} = \overline{J}^{-1}\overline{W}_{z_d}, \underline{G}_{z_d} = \underline{J}^{-1}\underline{W}_{z_d}, \\
&\overline{S} = \overline{J}^{-1}\overline{F}, \underline{S} = \underline{J}^{-1}\underline{F}, \\
&\overline{T} = \chi^{\dagger}\alpha_1 + \overline{S}\Psi\alpha_1,\ \underline{T} = \chi^{\dagger}\alpha_1 + \underline{S}\Psi\alpha_1, \overline{N} = \chi^{\dagger}\alpha_2 + \overline{S}\Psi\alpha_2,\ \underline{N} = \chi^{\dagger}\alpha_2 + \underline{S}\Psi\alpha_2,
\end{aligned} \tag{5.49}
$$

satisfies the following:

i. If $\underline{\xi}_0 \leq x(0) \leq \overline{\xi}_0$*, and the interval observer is initialized by*

$$\overline{z}(0) = (U^{-1})^{\oplus}\overline{\xi}_0 - (U^{-1})^{\ominus}\underline{\xi}_0, \tag{5.50}$$

$$\underline{z}(0) = (U^{-1})^{\oplus}\underline{\xi}_0 - (U^{-1})^{\ominus}\overline{\xi}_0, \tag{5.51}$$

then $\underline{x}(t) \le x(t) \le \overline{x}(t)$ *for all* $t \ge 0$ *where*

$$\overline{x}(t) = U^{\oplus}\overline{z}(t) - U^{\ominus}\underline{z}(t), \tag{5.52}$$

$$\underline{x}(t) = U^{\oplus}\underline{z}(t) - U^{\ominus}\overline{z}(t). \tag{5.53}$$

ii. If there exist constants $\mu, \lambda > 0$ *such that the impulse sequence satisfies* (5.29)*, then* $\overline{x}(t)$ *and* $\underline{x}(t)$ *remain bounded for all* $t \ge 0$*.*

Proof. Firstly, positivity of the following error

$$\varepsilon_z(t) = \begin{bmatrix} \overline{e}_z(t) \\ \underline{e}_z(t) \end{bmatrix} = \begin{bmatrix} \overline{z}(t) - z(t) \\ z(t) - \underline{z}(t) \end{bmatrix}. \tag{5.54}$$

will be shown. The dynamics of ε_z is an impulsive system with continuous part:

$$\begin{aligned} \dot{\varepsilon}_z(t) = & \begin{bmatrix} U^{-1}(A_c - L_cC)U + \overline{G}_{z_c} & \overline{G}_{z_c} \\ \underline{G}_{z_c} & U^{-1}(A_c - L_cC)U + \underline{G}_{z_c} \end{bmatrix} \varepsilon_z(t) \\ & + \begin{bmatrix} U^{-1}(p_c(U\overline{z}(t)) - p_c(Uz(t))) \\ U^{-1}(p_c(Uz(t)) - p_c(U\underline{z}(t))) \end{bmatrix} + d_{z_c}(t),\ t \ne t_k, k \in \mathbb{Z}_{\ge 1}, \end{aligned} \tag{5.55}$$

where

$$d_{z_c}(t) = \begin{bmatrix} \overline{\Delta}_{z_c} - U^{-1}w(t) - U^{-1}L_cv(t) \\ U^{-1}w(t) + U^{-1}L_cv(t) - \underline{\Delta}_{z_c} \end{bmatrix}.$$

Since (5.4) holds, $(U^{-1})^{\oplus}\overline{w}_c - (U^{-1})^{\ominus}\underline{w}_c - U^{-1}w(t) \ge 0$ and $U^{-1}w(t) - (U^{-1})^{\oplus}\underline{w}_c + (U^{-1})^{\ominus}\overline{w}_c \ge 0$ for all for all $t \ne t_k$, $k \in \mathbb{Z}_{\ge 1}$. Since (5.6) holds, $-U^{-1}\overline{L}_cv(t) + |U^{-1}L_c|\overline{V}\mathbf{1}_m \ge 0$ and $U^{-1}\overline{L}_cv(t) + |U^{-1}L_c|\overline{V}\mathbf{1}_m \ge 0$ for all $t \ne t_k$, $k \in \mathbb{Z}_{\ge 1}$. Hence, $d_c(t) \ge 0$ for all $t \ne t_k$, $k \in \mathbb{Z}_{\ge 1}$. By the same arguments as the proof of Theorem 9, $d_c(t) \ge 0$ and the satisfaction of (5.19) with P diagonal positive definite, implies that if $\varepsilon(t_k) \ge 0$, then $\varepsilon(t) \ge 0$ for all $t \in [t_k, t_{k+1})$, $k \in \mathbb{Z}_{\ge 0}$.

The definitions of $\overline{T}, \underline{T}, \overline{N}$ and $\underline{N}$ in (5.49) ensure that $\overline{T}+\overline{N}CU = I_n$ and $\underline{T}+\underline{N}CU = I_n$. Therefore, the jump part of the dynamics of ε_z is the following:

$$\varepsilon_z(t_k) = \begin{bmatrix} \overline{T}U^{-1}A_dU - \overline{L}_dCU + \overline{G}_{z_d} & \overline{G}_{z_d} \\ \underline{G}_{z_d} & \underline{T}U^{-1}A_cU - \underline{L}_dCU + \underline{G}_{z_d} \end{bmatrix} \varepsilon_z(t_k^-) \tag{5.56}$$
$$+ \begin{bmatrix} \overline{T}U^{-1}(p_d(U\overline{z}(t_k^-)) - p_d(Uz(t_k^-)) \\ \underline{T}U^{-1}(p_d(Uz(t_k^-)) - p_d(U\underline{z}(t_k^-)) \end{bmatrix} + d_{z_d}(k),$$

for all $k \in \mathbb{Z}_{\geq 1}$, where

$$d_{z_d}(k) = \begin{bmatrix} \overline{\Delta}_{z_d} - \overline{T}U^{-1}w(t_k^-) - \overline{L}_dv(t_k^-) + \overline{N}v(t_k) \\ \underline{T}U^{-1}w(t_k^-) + \underline{L}_dv(t_k^-) - \underline{N}v(t_k) - \underline{\Delta}_{z_d} \end{bmatrix}.$$

Since (5.5) holds, $(\overline{T}U^{-1})^{\oplus}\overline{w}_d - (\overline{T}U^{-1})^{\ominus}\underline{w}_d - \overline{T}U^{-1}w(t_k^-) \geq 0$ for all $k \in \mathbb{Z}_{\geq 1}$ by Lemma 1. Similarly, $\underline{T}U^{-1}w(t_k^-) - (\underline{T}U^{-1})^{\oplus}\underline{w}_d - (\underline{T}U^{-1})^{\ominus}\overline{w}_d \geq 0$ for all $k \in \mathbb{Z}_{\geq 1}$. Moreover, since (5.6) holds, $\left|\overline{L}_d\right| \overline{V}\mathbf{1}_m + \left|\overline{N}\right| \overline{V}\mathbf{1}_m + \overline{L}_dv(t_k^-) + \overline{N}v(t_k) \geq 0$ and $|\underline{L}_d| \overline{V}\mathbf{1}_m + |\underline{N}| \overline{V}\mathbf{1}_m - \underline{L}v(t_k^-) - \underline{N}v(t_k) \geq 0$ for all $k \in \mathbb{Z}_{\geq 1}$. Therefore, $d_d(k) \geq 0$ for all $k \in \mathbb{Z}_{\geq 1}$. By the same arguments as the proof of Theorem 9, $d_d(k) \geq 0$ and the satisfaction of (5.21) with J being an M-matrix implies that if $\varepsilon(t_k^-) \geq 0$, then $\varepsilon(t_k) \geq 0$ for all $k \in \mathbb{Z}_{\geq 1}$.

Therefore, if $\underline{z}(0) \leq z(0) \leq \overline{z}(0)$, then $\underline{z}(t) \leq z(t) \leq \overline{z}(t)$ for all $t \geq 0$. Since $\underline{\xi}_0 \leq x(0) \leq \overline{\xi}_0$, then, by Lemma 1, $\underline{z}(0) \leq z(0) \leq \overline{z}(0)$ when $\underline{z}(0)$ and $\overline{z}(0)$ are defined by (5.50) and (5.51), respectively. Since $x(t) = Uz(t)$ for all $t \geq 0$, it then follows from Lemma 1 that $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ for all $t \geq 0$ when $\overline{x}(t)$ and $\underline{x}(t)$ are defined by (5.52) and (5.53), respectively.

With the ISS Lyapunov function $V(\varepsilon_z) = \varepsilon_z^T P \varepsilon_z$ the satisfaction of the LMIs (5.18) and (5.20) along with the existence of $\lambda, \mu > 0$ such that the impulse sequence satisfies (5.29) implies that ε_z is ISS stable. Therefore, $\varepsilon_z(t)$ is bounded for all $t \geq 0$. Therefore, it follows that

$$\varepsilon(t) = \begin{bmatrix} U^{\oplus} & U^{\ominus} \\ U^{\ominus} & U^{\oplus} \end{bmatrix} \varepsilon_z(t)$$

is bounded for all $t \geq 0$. Since $x(t)$ is bounded for all $t \geq 0$, then $\overline{x}(t) = x(t) + \overline{e}(t)$ and $\underline{x}(t) = x(t) - \underline{e}(t)$ are bounded for all $t \geq 0$. □

While the goal of the coordinate transformation is to make it easier to ensure ISS stability of the continuous part of the error dynamics when there does not exist an L_c such that $A_c - L_cC$ is both Hurwitz and Metzler; however, even though $U^{-1}(A_c - L_cC)U$ is Hurwitz, this does not necessarily mean that the semidefinite program in Theorem 10 will be feasible with $\alpha_c > 0$ since $U^{-1}(A_c - L_cC)U$ being Hurwitz is a necessary but insufficient condition for the ISS stability of the continuous part of the nonlinear error dynamics [122]. To ensure positivity and ISS stability of the jump part in the new coordinate system, the additional matrices $\overline{T}, \underline{T}, \overline{N}$, and $\underline{N}$ are incorporated. This can avoid the need for a second transformation for the jump part.

5.3.2 Synthesis with Two Coordinate Transformations

In this subsection, the use of two transformations, one for the continuous part, and one for the jump part is derived for nonlinear systems. The use of two coordinate transformations is useful in situations where both α_c and α_d must be positive because the impulse sequence is unknown ahead of time. Since it is difficult to find a single invertible matrix U where there are two observer gains $L_c, L_d \in \mathbb{R}^{n\times m}$ such that $U^{-1}(A_c - L_cC)U$ is Metzler and Hurwtiz and $U^{-1}(A_d - L_dC)U$ is Schur and nonnegative, it is sufficient to find $L_c, L_d \in \mathbb{R}^{n\times m}$ and invertible matrices $U_c, U_d \in \mathbb{R}^{n\times n}$ such that $U_c^{-1}(A_c - L_cC)U_c$ is Metzler and Hurwtiz and $U_d^{-1}(A_d - L_dC)U_d$ is Schur and nonnegative. The continuous part of the impulsive system (5.1), can be written in the coordinates $z = U_c^{-1}x$, and the jump part of the impulsive system (5.2), can be written in the coordinates $\xi = U_d^{-1}x$.

Consider the following interval observer which uses two transformations U_c and U_d and their corresponding observer gains L_c and L_d.

$$\begin{cases} \dot{\overline{z}}(t) = U_c^{-1}A_cU_c\overline{z}(t) + U_c^{-1}p_c(U_c\overline{z}) + U_c^{-1}L_c(y(t) - CU_c\overline{z}(t)) \\ +\overline{G}_{z_c}(\overline{z}(t) - \underline{z}(t)) + \overline{\Delta}_{z_c}, \\ \dot{\underline{z}}(t) = U^{-1}A_cU\underline{z}(t) + U_c^{-1}p_c(U\underline{z}) + U^{-1}L_c(y(t) - CU_c\underline{z}(t)) \\ +\overline{G}_{z_c}(\underline{z}(t) - \overline{z}(t)) + \underline{\Delta}_{z_c}, \end{cases} \quad t \neq t_k, k \in \mathbb{Z}_{\geq 1}, \tag{5.57}$$

$$\begin{cases} \overline{\xi}(t_k^-) = (U_d^{-1}U_c)^{\oplus}\overline{z}(t_k^-) - (U_d^{-1}U_c)^{\ominus}\underline{z}(t_k^-), \\ \underline{\xi}(t_k^-) = -(U_d^{-1}U_c)^{\ominus}\overline{z}(t_k^-) + (U_d^{-1}U_c)^{\oplus}\underline{z}(t_k^-), \\ \overline{\xi}(t_k) = U_d^{-1}A_dU_d\overline{\xi}(t_k^-) + U_d^{-1}L_d(y(t_k^-) - CU_d\overline{\xi}(t_k^-)) \\ +U_d^{-1}p_d(U_d\overline{\xi}(t_k^-)) + \overline{G}_{\xi_d}(\overline{\xi}(t_k^-) - \underline{\xi}(t_k^-)) + \overline{\Delta}_{\xi_d}, \\ \underline{\xi}(t_k) = U_d^{-1}A_dU_d\underline{\xi}(t_k^-) + U_d^{-1}L_d(y(t_k^-) - CU_d\underline{\xi}(t_k^-)) \\ +U_d^{-1}p_d(U_d\underline{\xi}(t_k^-)) + \underline{G}_{\xi_d}(\underline{\xi}(t_k^-) - \overline{\xi}(t_k^-)) + \underline{\Delta}_{\xi_d}, \\ \overline{z}(t_k^-) = (U_c^{-1}U_d)^{\oplus}\overline{\xi}(t_k^-) - (U_c^{-1}U_d)^{\ominus}\underline{\xi}(t_k^-), \\ \underline{z}(t_k^-) = -(U_c^{-1}U_d)^{\ominus}\overline{\xi}(t_k^-) + (U_c^{-1}U_d)^{\oplus}\underline{\xi}(t_k^-), \end{cases} \quad k \in \mathbb{Z}_{\geq 1}, \tag{5.58}$$

where $\overline{\Delta}_{z_c}$ and $\underline{\Delta}_{z_c}$ are defined in (5.45) and (5.46), respectively, and

$$\overline{\Delta}_{\xi_d} = (U_d^{-1})^{\oplus}\overline{w}_d - (U_d^{-1})^{\ominus}\underline{w}_d + |U_d^{-1}L_d|\overline{V}\mathbf{1}_m, \tag{5.59}$$

$$\underline{\Delta}_{\xi_d} = (U_d^{-1})^{\oplus}\underline{w}_d - (U_d^{-1})^{\ominus}\overline{w}_d - |U_d^{-1}L_d|\overline{V}\mathbf{1}_m. \tag{5.60}$$

The following theorem provides sufficient conditions for positivity and boundedness of the interval observer (5.57)-(5.60) with (5.45)-(5.46).

Theorem 11. *Consider the impulsive system (5.1)-(5.3) with impulse sequence* $\{t_1, t_2, \dots\}$ *where* $p_c : \mathbb{R}^n \to \mathbb{R}^n$ *is a polytopic nonlinearity defined by* ν_c *vertices* $\Theta_1 \dots, \Theta_{\nu_c} \in \mathbb{R}^{n\times n}$*, and* $p_d : \mathbb{R}^n \to \mathbb{R}^n$ *is a polytopic nonlinearity defined by* ν_d *vertices* $\Upsilon_1 \dots, \Upsilon_{\nu_d} \in \mathbb{R}^{n\times n}$*. Suppose* $x(t)$ *remains bounded for all* $t \geq 0$, $\overline{w}_c, \underline{w}_c \in \mathbb{R}^n$ *satisfy* (5.4), $\overline{w}_d, \underline{w}_d \in \mathbb{R}^n$ *satisfy* (5.5), *and* $\overline{V} \in \mathbb{R}_{\geq 0}$ *satisfies* (5.6).

Suppose there exist matrices $P \in \mathbb{R}^{2n\times 2n}$ *where* $P = P^T \succ 0$ *such that:*

$$\begin{bmatrix} He(P\mathcal{A}_{c_{ij}}) + \alpha_c P & P \\ \star & -\gamma I_{2n} \end{bmatrix} \preceq 0, \tag{5.61}$$

$$\mathcal{A}_{c_{ij}} \in \mathcal{M}_{2n}, \tag{5.62}$$

for all $i, j = 1, \dots, \nu_c$,

$$\begin{bmatrix} \mathcal{U}_{cd}^T \mathcal{A}_{d_{ij}}^T \mathcal{U}_{dc}^T P \mathcal{U}_{dc} \mathcal{A}_{d_{ij}} \mathcal{U}_{cd} - e^{-\alpha_d} P & \mathcal{A}_{d_{ij}}^T P \\ \star & P - \gamma I_{2n} \end{bmatrix} \preceq 0, \tag{5.63}$$

$$\mathcal{A}_{d_{ij}} \geq 0, \tag{5.64}$$

for all $i, j = 1, \dots, \nu_d$, *where*

$$\mathcal{A}_{c_{ij}} = \begin{bmatrix} U_c^{-1}(A_c + \Theta_i - L_c C)U_c + \overline{G}_{z_c} & \overline{G}_{z_c} \\ \underline{G}_{z_c} & U_c^{-1}(A_c + \Theta_j - L_c C)U_c + \underline{G}_{z_c} \end{bmatrix}, \tag{5.65}$$

$$\mathcal{A}_{d_{ij}} = \begin{bmatrix} U_d^{-1}(A_d + \Upsilon_i - L_d C)U_d + \overline{G}_{z_d} & \overline{G}_{z_d} \\ \underline{G}_{z_d} & U_d^{-1}(A_d + \Upsilon_j - L_d C)U_c + \underline{G}_{z_d} \end{bmatrix}, \tag{5.66}$$

$$\mathcal{U}_{dc} = \begin{bmatrix} (U_d^{-1}U_c)^{\oplus} & (U_d^{-1}U_c)^{\ominus} \\ (U_d^{-1}U_c)^{\ominus} & (U_d^{-1}U_c)^{\oplus} \end{bmatrix}, \mathcal{U}_{cd} = \begin{bmatrix} (U_c^{-1}U_d)^{\oplus} & (U_c^{-1}U_d)^{\ominus} \\ (U_c^{-1}U_d)^{\ominus} & (U_c^{-1}U_d)^{\oplus} \end{bmatrix}. \tag{5.67}$$

The interval observer (5.57)-(5.60) *with* (5.45)-(5.46) *satisfies the following:*

i. If $\underline{\xi}_0 \leq x(0) \leq \overline{\xi}_0$ *and the interval observer is intialized by* (5.50)-(5.51), *then* $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ *for all* $t \geq 0$, *where* $\overline{x}(t)$ *and* $\underline{x}(t)$ *are defined by* (5.52) *and* (5.53), *respectively.*

ii. If there exist constants $\mu, \lambda > 0$ such that the impulse sequence satisfies (5.29)*, then $\overline{x}(t)$ and $\underline{x}(t)$ remain bounded for all $t \geq 0$.*

Proof. Define the following:

$$\varepsilon_z(t) = \begin{bmatrix} \overline{e}_z(t) \\ \underline{e}_z(t) \end{bmatrix} = \begin{bmatrix} \overline{z}(t) - z(t) \\ z(t) - \underline{z}(t) \end{bmatrix}, \ \varepsilon_\xi(t) = \begin{bmatrix} \overline{\xi}(t) - \xi(t) \\ \xi(t) - \underline{\xi}(t) \end{bmatrix}.$$

The positivity of the interval observer follows by showing that if $\varepsilon_z(0) \geq 0$, then $\varepsilon_z(t) \geq 0$ for all $t \geq 0$. The dynamics of ε_z is an impulsive system with continuous part:

$$\dot{\varepsilon}_z(t) = \begin{bmatrix} U_c^{-1}(A_c - L_cC)U_c + \overline{G}_{z_c} & \overline{G}_{z_c} \\ \underline{G}_{z_c} & U_c^{-1}(A_c - L_cC)U_c + \underline{G}_{z_c} \end{bmatrix} \varepsilon_z(t)$$
$$+ \begin{bmatrix} U_c^{-1}(p_c(U_c\overline{z}(t)) - p_c(Uz(t)) \\ U_c^{-1}(p_c(U_cz(t)) - p_c(U_c\underline{z}(t)) \end{bmatrix} + d_{z_c}(t),$$

The same argument as the proof of Theorem 10 apply to show that the satisfaction of (5.62) implies that if $\varepsilon_z(t_k) \geq 0$, then $\varepsilon_z(t) \geq 0$ for all $t \in [t_k, t_{k+1})$, $k \in \mathbb{Z}_{\geq 0}$. Therefore, if $\underline{z}(t_k) \leq z(t_k) \leq \overline{z}(t_k)$, then $\underline{z}(t) \leq z(t) \leq \overline{z}(t)$ for all $t \in [t_k, t_{k+1})$, $k \in \mathbb{Z}_{\geq 0}$.

Now consider the jump part of the dynamics of ε_z:

$$\varepsilon_\xi(t_k^-) = \mathcal{U}_{dc}\varepsilon_z(t_k^-), \tag{5.68}$$

$$\varepsilon_\xi(t_k) = \begin{bmatrix} U_d^{-1}(A_d - L_dC)U_d + \overline{G}_{z_d} & \overline{G}_{z_d} \\ \underline{G}_{z_d} & U_d^{-1}(A_d - L_dC)U_c + \underline{G}_{z_d} \end{bmatrix} \varepsilon_\xi(t_k^-)$$
$$+ \begin{bmatrix} U_d^{-1}(p_d(U_d\overline{\xi}(t_k^-) - p_d(U_d\xi(t_k^-)) \\ U_d^{-1}(p_d(U_d\xi(t_k^-) - p_d(U_d\underline{\xi}(t_k^-)) \end{bmatrix} + d_{\xi_d}(k) \tag{5.69}$$

$$\varepsilon_z(t_k) = \mathcal{U}_{cd}\varepsilon_\xi(t_k). \tag{5.70}$$

If $\varepsilon_z(t_k^-) \geq 0$, then from (5.68), $\varepsilon_\xi(t_k^-) \geq 0$ since $\mathcal{U}_{dc} \geq 0$ by the definition of the operators $(\cdot)^\oplus$ and $(\cdot)^\ominus$. Therefore, $\underline{\xi}(t_k^-) \leq \xi(t_k^-) = U_d^{-1}x(t_k^-) \leq \overline{\xi}(t_k^-)$. By Lemma 2, (5.69) can be written as an LPV system:

$$\varepsilon_\xi(t_k) = \mathcal{A}_{\xi_d}(\varrho_{\xi_d}(t_k^-))\varepsilon_\xi(t_k^-) + d_{\xi_d}(k) \tag{5.71}$$

where ϱ_{ξ_d} belongs to a compact set

$$\varrho_{\xi_d}(t_k^-) \in \Xi_d = \left\{ \varrho_{\xi_d} \in \mathbb{R}_{\geq 0}^{\nu_d \times \nu_d} : \sum_{i=1}^{\nu_d} \sum_{j=1}^{\nu_d} \varrho_{\xi_{d_{ij}}} = 1 \right\}, \ \forall k \in \mathbb{Z}_{\geq 1},$$

and $\mathcal{A}_{\xi_d}\varrho_{\xi_d}((t_k^-)) = \sum_{i=1}^{\nu_d} \sum_{i=1}^{\nu_d} \varrho_{\xi_{d_{ij}}} \mathcal{A}_{d_{ij}}$ where $\mathcal{A}_{d_{ij}}$ is defined in (5.66). Therefore, if (5.64) holds, then $\varepsilon_\xi(t_k) \geq 0$, so $\underline{\xi}(t_k) \leq \xi(t_k) = U_d^{-1}x(t_k) \leq \overline{\xi}(t_k)$. Using (5.70), $\varepsilon_z(t_k) \geq 0$ since $\mathcal{U}_{cd} \geq 0$, so $\underline{z}(t_k) \leq z(t_k) \leq \overline{z}(t_k)$.

With the ISS Lyapunov function $V(\varepsilon_z) = \varepsilon_z^T P \varepsilon_z$ the satisfaction of the LMIs (5.61) and (5.63) along with the existence of $\lambda, \mu > 0$ such that the impulse sequence satisfies (5.29) implies that ε_z is ISS stable. Therefore, $\varepsilon_z(t)$ is bounded for all $t \geq 0$. Therefore, it follows that $\varepsilon(t)$ is bounded for all $t \geq 0$. Since $x(t)$ is bounded for all $t \geq 0$, then $\overline{x}(t) = x(t) + \overline{e}(t)$ and $\underline{x}(t) = x(t) - \underline{e}(t)$ are bounded for all $t \geq 0$. □

5.3.3 Coordinate Transformations Driven by the Jump Part

A special case is when measurements are only made when the jumps are detected (i.e. $y(t_k^-)$) and no measurements are available during the continuous part. This case is studied in Luenberger observer design for linear impulsive systems in [22, §IV] and [103]. Moreover, there are applications which exhibit this case, for example, in blood glucose monitoring for diabetes patients [131, 130]. In this case, the matrices $\overline{T}, \underline{T}, \overline{N}$, and $\underline{N}$ cannot be incorporated in the design because, to use these matrices, the jump part uses the measurement of the output immediately before the impulse with a correction immediately after.

To simplify the discussion, consider that the jump part is linear and that there are no disturbances nor measurement noise.

$$\begin{aligned}\dot{x}(t) &= A_c x(t) + p_c(x(t)),\ t \neq t_k, k \in \mathbb{Z}_{\geq 1},\\ x(t_k) &= A_d x(t_k^-),\ k \in \mathbb{Z}_{\geq 1},\\ y(t_k^-) &= Cx(t_k^-),\ k \in \mathbb{Z}_{\geq 1}.\end{aligned}$$

Suppose there does not exist an $L \in \mathbb{R}^{n\times m}$ such that $A_d - LC$ is both Schur and nonnegative, but there exists an $L \in \mathbb{R}^{n\times m}$ and an invertible matrix U such that $U^{-1}(A - LC)U$ is Schur and nonnegative. Consider the following interval observer dynamics,

$$\begin{aligned}\dot{\overline{z}}(t) &= U^{-1}A_c U\overline{z}(t) + U^{-1}p_c(U\overline{z}(t)) + \overline{G}_{z_c}(\overline{z}(t) - \underline{z}(t)),\ t \neq t_k, k \in \mathbb{Z}_{\geq 1},\\ \dot{\underline{z}}(t) &= U^{-1}A_c U\underline{z}(t) + U^{-1}p_c(U\underline{z}(t)) + \underline{G}_{z_c}(\underline{z}(t) - \overline{z}(t)),\ t \neq t_k, k \in \mathbb{Z}_{\geq 1},\\ \overline{z}(t_k) &= U^{-1}A_c U\overline{z}(t_k^-) + U^{-1}L(y(t_k^-) - CU\overline{z}(t_k^-),\ k \in \mathbb{Z}_{\geq 1},\\ \underline{z}(t_k) &= U^{-1}A_c U\underline{z}(t_k^-) + U^{-1}L(y(t_k^-) - CU\underline{z}(t_k^-),\ k \in \mathbb{Z}_{\geq 1}.\end{aligned}$$

The jump part of the error dynamics is positive because $U^{-1}(A - LC)U$ is nonnegative, and the positivity of the continuous part of the error dynamics can always be enforced through the choices of $\overline{G}_{z_c}$ and $\underline{G}_{z_c}$. Therefore, if $\overline{z}, \underline{z}$ are initialized as

$$\begin{aligned}\overline{z}(0) &= (U^{-1})^{\oplus}\overline{\xi}_0 - (U^{-1})^{\ominus}\underline{\xi}_0,\\ \underline{z}(0) &= (U^{-1})^{\oplus}\underline{\xi}_0 - (U^{-1})^{\ominus}\overline{\xi}_0,,\end{aligned}$$

then $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ for all $t \geq 0$ where

$$\begin{aligned}\overline{x}(t) &= U^{\oplus}\overline{z}(t) - U^{\ominus}\underline{z}(t),\\ \underline{x}(t) &= U^{\oplus}\underline{z}(t) - U^{\ominus}\overline{z}(t).\end{aligned}$$

Asymptotic stability of $\overline{x}(t)$ and $\underline{x}(t)$ follows if the impulses happen frequently enough because the jump part is stable and it is likely that the continuous part is unstable since no output feedback is available in between impulses.

5.4 Numerical Examples

5.4.1 Synthesis in Original Coordinates with Dwell-Time Constraints

Consider an academic example of a two state impulsive system with the following system matrices:

$$A_c = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, A_d = \begin{bmatrix} 0.99 & 0.2 \\ -0.1 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & -1 \end{bmatrix}.$$

The continuous part has a nonlinearity

$$p_c(x) = \begin{bmatrix} 0.5e^{-x_2^2} \\ 0.5\cos x_1 \end{bmatrix},$$

which is a polytopic nonlinearity that is described by four vertices:

$$\Theta_1 = \begin{bmatrix} 0 & 0.4289 \\ 0.5 & 0 \end{bmatrix},\ \Theta_2 = -\Theta_1,\ \Theta_3 = \begin{bmatrix} 0 & 0.4289 \\ -0.5 & 0 \end{bmatrix},\ \Theta_4 = -\Theta_3.$$

The jump part has a nonlinearity

$$p_d(x) = \begin{bmatrix} 0 \\ \dfrac{x_2}{1+x_2^2} \end{bmatrix},$$

which is a polytopic nonlinearity that is described by two vertices:

$$\Upsilon_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ \Upsilon_2 = \begin{bmatrix} 0 & 0 \\ 0 & -0.125 \end{bmatrix}.$$

Theorem 9 is used to find the following matrices for the interval observer (5.8)-(5.13) where $\alpha_c = 0.1$ and $\alpha_d = -0.05$:

$$\overline{L}_c = \underline{L}_c = \begin{bmatrix} 1.8071 \\ -63.7541 \end{bmatrix}, \ \overline{G}_c = \underline{G}_c = \begin{bmatrix} 0.0488 & 0.0479 \\ 1.8582 & 1.8219 \end{bmatrix},$$
$$\overline{L}_d = \underline{L}_d = \begin{bmatrix} 0.0725 \\ 0.0726 \end{bmatrix}, \ \overline{G}_d = \underline{G}_d = \begin{bmatrix} 0.0013 & 0.0012 \\ 0.0021 & 0.0020 \end{bmatrix},$$
$$\overline{T} = \underline{T} = \begin{bmatrix} 0.1733 & 0.8267 \\ 0.1810 & 0.8190 \end{bmatrix}, \ \overline{N} = \underline{N} = \begin{bmatrix} 0.8267 \\ -0.1810 \end{bmatrix}.$$

The disturbance satisfies (5.4) and (5.5) where

$$\overline{w}_c = \overline{w}_d = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \underline{w}_c = \underline{w}_d = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Moreover, the measurement noise satisfies (5.6) with $\overline{V} = 0.1$.

A simulation is shown in Figure 5.1 where the impulse sequence satisfies (5.29), so the interval observer states $\overline{x}(t)$ (magenta) and $\underline{x}(t)$ (blue) remain bounded, since $x(t)$ (black) remains bounded. Moreover, the ordering $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ for all $t \geq 0$.

5.4.2 Synthesis with Coordinate Transformation

Consider the following model of a pendulum with impacts (cf. [132, Example 6.1]) where x_1 is the position of the pendulum, and x_2 is the velocity of the pendulum. The system matrices are the following

$$A_c = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, A_d = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \end{bmatrix}. \tag{5.72}$$

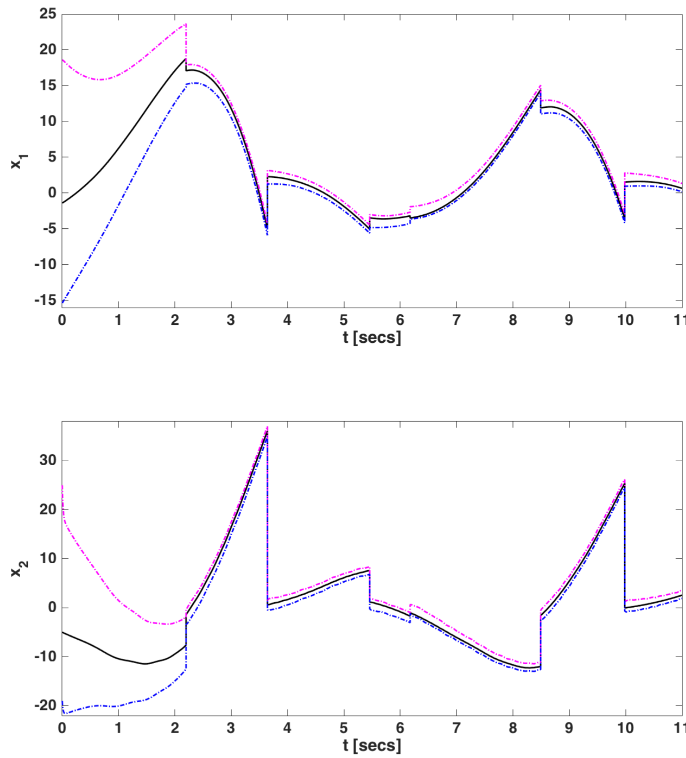

Figure 5.1: Simulation of an interval observer for a nonlinear impulsive system with average dwell-time constraints and no coordinate transformation.

The jump part is linear, and the continuous part has the following nonlinearity:

$$p_c(x) = \begin{bmatrix} 0 \\ -\sin x_1 \end{bmatrix}, \tag{5.73}$$

which is a polytopic nonlinearity described by two vertices:

$$\Theta_1 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \Theta_2 = -\Theta_1. \tag{5.74}$$

The jumps occur when the pendulum hits the surface. For this example, the surface is located at $x_1 = -0.1$:

$$t_{k+1} = \inf\{t > t_k : x_1 = -0.1, x_2 < 0, t_0 = 0\}.$$

Once the pendulum hits the surface, its velocity changes direction but maintains the same magnitude with a small disturbance that decreases the velocity, so the disturbance satisfies (5.5) with

$$\overline{w}_d = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \underline{w}_d = \begin{bmatrix} 0 \\ -0.02 \end{bmatrix}.$$

Between impacts, the disturbance acts to slow down the pendulum due to air resistance or friction, and satisfies (5.4) with

$$\overline{w}_c = \begin{bmatrix} 0 \\ 0.02 \end{bmatrix}, \ \underline{w}_c = \begin{bmatrix} 0 \\ -0.02 \end{bmatrix}.$$

There does not exist an $L_c \in \mathbb{R}^2$ such that $A_c - L_c C$ is both Hurwitz and Metzler, so the semidefinite program in Theorem 9 can only be feasible for negative values of α_c. Since the velocity of the pendulum decreases over time due to air resistance, friction, and the losses during the impacts, the pendulum eventually will not have enough energy to swing back up towards the surface, so the impacts do not occur past a certain time. Therefore, for the interval observer to remain bounded, a coordinate transformation is made so that α_c is positive for the semidefinite program in Theorem 10. Consider the following observer gain and transformation matrix:

$$L_c = \begin{bmatrix} 0.0114 \\ 4.8804 \end{bmatrix}, U = \begin{bmatrix} -0.2837 & -0.7139 \\ 1 & 1 \end{bmatrix}.$$

$U^{-1}(A_c - L_c C)U$ is Metzler and Hurwtiz. U is constructed from the Jordan normal form of $A_c - L_c C$.

Using the observer (5.43)-(5.48) and solving the semidefinite program in Theorem 10 yields the following

$$\overline{L}_d = \begin{bmatrix} 0.3364 \\ -0.8771 \end{bmatrix}, \underline{L}_d = \begin{bmatrix} 0.3334 \\ -0.8295 \end{bmatrix}, \overline{G}_{z_c} = \begin{bmatrix} 0.2659 & 1.3370 \\ 0.3133 & 0.0420 \end{bmatrix}, \underline{G}_{z_c} = \begin{bmatrix} 0.2524 & 1.3483 \\ 0.3139 & 0.0390 \end{bmatrix},$$
$$\overline{G}_{z_d} = \begin{bmatrix} 0.0480 & 0.0393 \\ 0.0571 & 0.0464 \end{bmatrix}, \underline{G}_{z_d} = \begin{bmatrix} 0.0788 & 0.0562 \\ 0.0834 & 0.0594 \end{bmatrix}, \overline{T} = \begin{bmatrix} -0.4578 & -0.5822 \\ 1.0529 & 1.4205 \end{bmatrix},$$
$$\underline{T} = \begin{bmatrix} -0.4350 & -0.5731 \\ 1.0587 & 1.4228 \end{bmatrix}, \overline{N} = \begin{bmatrix} 2.0352 \\ -1.4700 \end{bmatrix}, \underline{N} = \begin{bmatrix} 2.0035 \\ -1.4780 \end{bmatrix},$$

with $\alpha_c = 0.1$ and $\alpha_d = 0.1$. Since α_c and α_d are positive, boundedness of the errors is guaranteed regardless of the impulse sequence. Note that there does not exist an $L_d \in \mathbb{R}^2$ such that $U^{-1}A_dU - L_dCU$ is both Schur and nonnegative; however, $\overline{T}U^{-1}A_dU - \overline{L}_dCU + \overline{G}_{z_d}$ and $\underline{T}U^{-1}A_dU - \underline{L}_dCU + \underline{G}_{z_d}$ are both Schur and nonnegative. A simulation is shown in Figure 5.2. $\overline{x}(t)$ (magenta) and $\underline{x}(t)$ (blue) remain close to $x(t)$ (black) whilst also maintaining the ordering $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ for all $t \geq 0$.

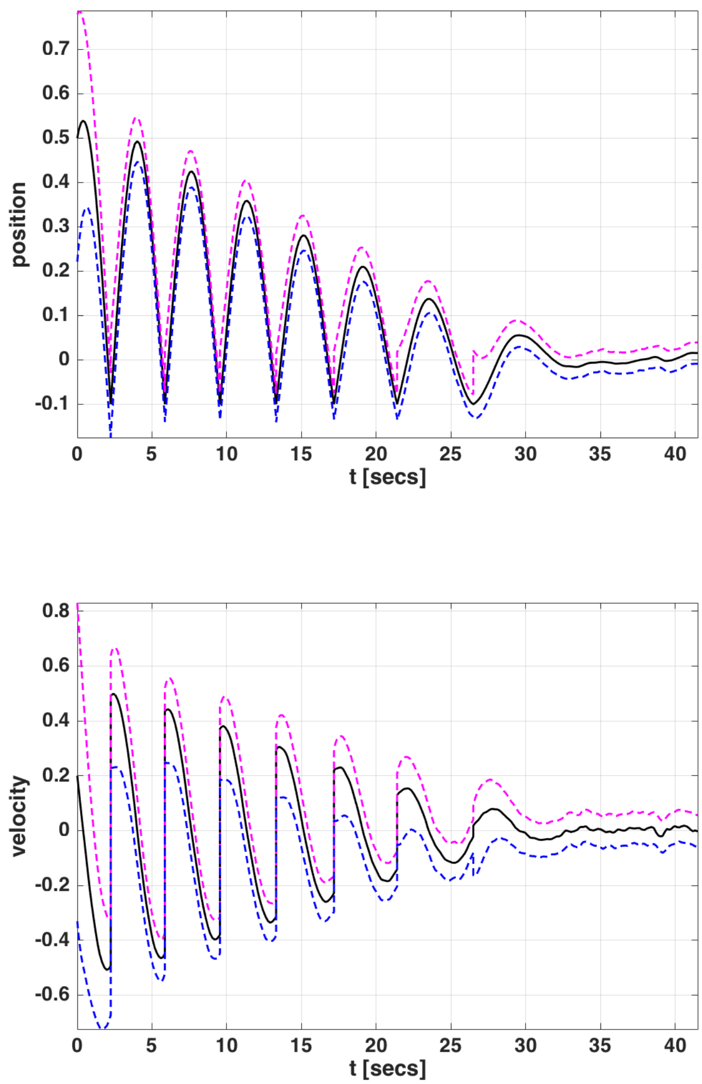

Figure 5.2: Interval observer simulation for a pendulum with impacts using a coordinate transformation.

Part II

RESOURCE AWARE ESTIMATION AND CONTROL OF NONLINEAR SYSTEMS

Chapter 6

SELF-TRIGGERED INTERVAL OBSERVERS FOR LIPSCHITZ NONLINEAR SYSTEMS

"If the gods have determined about me and about the things which must happen to me, they have determined well, for it is not easy even to imagine a deity without forethought."

Meditations

Marcus Aurelius

For many engineering applications such as systems that are implemented over a wireless network, the output is measured discretely at times $\{t_k\}_{k=1}^{\infty}$. Frequently, measurements are assumed to occur periodically (i.e. $t_k = kh$ for $h > 0$ being a desired sampling time) or the time in between samples lies within a compact interval (i.e. $t_{k+1} - t_k \in [T_1, T_2]$ for $0 < T_1 \leq T_2$) [60, 77, 64]. Self-triggered observers are inspired by self-triggered controllers where the feedback control is implemented by sampling the system and applying a constant control signal until the next sampling instance, which is determined as a function of the state [73, 14, 148]. A self-triggered observer uses the state estimate and output at a given sampling time to determine when the next measurement should be made. The goal of self-triggered control and observers is to reduce the amount of times signals are sent over the network. In [11], self-triggered output feedback controllers for linear systems are designed by taking advantage of the cascade interconnection structure between the controller and the observer; in [89] self-triggered output feedback controllers for linear systems are designed using a model predictive control framework; in [12], self-triggered observers are designed for nonlinear systems in triangular form.

The interval observer framework is used for self-triggered observer design. In an interval observer, it is assumed that it is known *a priori* that the initial condition lies in a compact interval defined by an upper bound $\overline{x}(0)$ and a lower bound $\underline{x}(0)$. The upper and lower errors are asymptotically stabilized to zero whilst ensuring that the upper bound remains an upper bound and the lower bound remains a lower bound [105, 99, 55]. Interval observers have been designed for systems with sampled measurements for linear systems [101, 51]. However, a systematic way to construct self-triggered interval observers for linear and Lipschitz nonlinear systems is still lacking. The utility of interval observers in constructing observers with self-triggered measurements comes from the fact that the state always remains within a known compact set. Therefore, a bound on the errors is always known.

In this chapter, the use of the interval observer framework to design observers with self-triggered measurements is proposed. Moreover, convex programs to construct interval observers for linear and Lipschitz nonlinear systems are given. The process begins with determining necessary conditions for periodically sampled interval observers of linear systems and proposes two convex programs to solve for the Lyapunov function and observer gain. This result is built on to derive an interval observer for linear systems with self-triggered measurements by ensuring that the proposed Lyapunov function decreases sufficiently throughout the jumps while also maintaining the positivity of the upper and lower errors. The design of sampled interval observers for Lipschitz systems begins by constructing a linear framer system which bounds the Lipschtiz system. Thus, the tools derived for linear systems can be incorporated into the design of periodically sampled and self-triggered interval observers for Lipschitz nonlinear systems.

Notation. $\mathbb{R} := (-\infty, \infty)$, $\mathbb{R}_{\geq 0} := [0, \infty)$, $\mathbb{Z}_{\geq 0} := \{0, 1, 2, \dots\}$. For a vector $x \in \mathbb{R}^n$, $x \geq 0$ denotes that the elements of x are nonnegative. For a matrix M, $M \geq 0$ denotes that the elements of M are nonnegative, and M is referred to as a nonnegative matrix. $\|\cdot\|$ denotes the Euclidean norm. A function $f(x) : \mathbb{R}^n \to \mathbb{R}^n$ is globally Lipschitz if $\exists \gamma > 0$ such that $\|f(x) - f(y)\| \leq \gamma \|x - y\|, \forall x, y \in \mathbb{R}^n$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class $\mathcal{K}_\infty$ function if $\alpha(0) = 0$, it is continuous, strictly increasing, and unbounded. ∇f is the gradient of a C^1 function f. A matrix M is Metzler if all of its off-diagonal elements are nonnegative. A matrix M is an M-matrix if all of its off-diagonal elements are nonpositive and all of its diagonal elements are positive. A matrix S is Stieltjes if it is a symmetric positive definite M-matrix. For a symmetric matrix S, $S \succ 0$ denotes that S is positive definite, $S \succeq 0$ denotes S is positive semidefinite. $\lambda_i(M)$ denotes the ith eigenvalue of the matrix M; $\lambda_{min}(M)$, $\lambda_{max}(M)$ denote the minimum and maximum eigenvalues, respectively. $e^{(\cdot)}$ denotes the matrix exponential for matrix arguments, and the exponential for scaler arguments. RHS(i) refers to the right-hand side of equation (i). The notation $\star$ denotes entries whose values follow from symmetry.

6.1 Hybrid Systems Background

This section recalls definitions for the hybrid systems framework in [64]. A set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is a compact hybrid time domain if $E = \cup_{j=0}^{J}([t_j, t_{j+1}], j)$ for finite sequence of times $0 = t_0 \leq t_1 \leq \cdots \leq t_{J+1}$ where $J \in \mathbb{Z}_{\geq 0}$. E is a hybrid time domain if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, \ldots, J\})$ is a compact hybrid time domain. A hybrid arc is a function ϕ defined on a hybrid time domain such that $\phi(., j)$ is locally absolutely continuous for each j. A hybrid arc $\phi : \text{dom}\phi \to \mathbb{R}^n$ is a solution to the following hybrid system

$$\dot{x} \in F(x), x \in \mathcal{C}; \quad x^+ \in G(x), x \in \mathcal{D} \tag{6.1}$$

where $\mathcal{C}$ is the flow set and $\mathcal{D}$ is the jump set, if the following is true:

i. $\phi(0, 0) \in \mathcal{C} \cup \mathcal{D}$.

ii. For all $j \in \mathbb{Z}_{\geq 0}$ and almost all $t \in \mathbb{R}_{\geq 0}$ such that for $(t, j) \in \mathrm{dom}\phi, \phi(t, j) \in \mathcal{C}$ and $\dot{\phi}(t, j) \in F(\phi(t, j))$.

iii. For all $(t, j) \in \mathrm{dom}\phi$ such that for all $(t, j+1) \in \mathrm{dom}\phi, \phi(t, j) \in \mathcal{D}, \phi(t, j+1) \in G(\phi(t, j))$.

A solution to the hybrid system is maximal if it cannot be extended, and complete if its domain is unbounded.

Let $\mathcal{A} \subset \mathbb{R}^n$ be a closed set. $\|\cdot\|_{\mathcal{A}} := \inf_{a \in \mathcal{A}} \|x - a\|$. $\mathcal{A}$ is said to be uniformly globally pre-asymptotically stable if it is uniformly globally stable and uniformly globally pre-attractive (Definition 3.6 in [64]). If, in addition, the maximal solutions are complete, then $\mathcal{A}$ is globally asymptotically stable. Note that the maximal solutions are complete for all the systems analyzed in this chapter. $\mathcal{A}$ is said to be globally exponentially stable if there exists positive numbers k, ϱ such that for each solution ϕ satisfies

$$\|\phi(t, j)\|_{\mathcal{A}} \leq k e^{-\varrho(t+j)} \|\phi(0, 0)\|_{\mathcal{A}}, \forall (t, j) \in \mathrm{dom}\phi.$$

For a hybrid system (6.1), stability can be determined using Lyapunov theory [64]. It is clear that pre-asymptotic stability is guaranteed if there exists a candidate Lyapunov function which is strictly decreasing during the flow and the jumps; however, here, Lyapunov functions will be constructed which are nonstrictly decreasing during the flow but strictly decreasing during the jumps, so it is necessary to ensure that the jumps occur often enough to guarantee stability. The following result will be used.

Lemma 7 (Proposition 3.24 in [64])**.** *Consider the hybrid system (6.1). Let $\mathcal{A} \subset \mathbb{R}^n$ be a closed set. Suppose that V is a candidate Lyapunov function which satisfies the following:*

$$\alpha_1(\|x\|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(\|x\|_{\mathcal{A}}), \forall x \in \mathcal{C} \cup \mathcal{D} \cup G(\mathcal{D})$$
$$\langle \nabla V(x), f \rangle \leq 0, \forall x \in \mathcal{C}, f \in F(x)$$
$$V(x^+) - V(x) \leq -\rho(\|x\|_{\mathcal{A}}), \forall x \in \mathcal{D}, x^+ \in G(x)$$

for $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$*, and positive definite function* $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$*. If, for each* $r > 0$*, there exists* $\gamma_r \in \mathcal{K}_\infty$*,* $N_r \geq 0$ *such that for every solution* ϕ *to (6.1),* $\|\phi(0,0)\|_{\mathcal{A}} \in (0, r]$*,* $(t, j) \in dom\phi$*,* $t + j \geq T$ *imply* $j \geq \gamma_r(T) - N_r$*, then* $\mathcal{A}$ *is uniformly globally pre-asymptotically stable.*

6.2 Sampled Interval Observer Design for Linear Systems

Consider a linear system with state $x \in \mathbb{R}^n$ and output measurements $y \in \mathbb{R}^m$ sampled at times t_k, $t_0 = 0$ as follows:

$$\dot{x}(t) = Ax(t) + Bu(t),\ \forall t \in \mathbb{R}_{\geq 0}, \tag{6.2}$$

$$y(t_k) = Cx(t_k),\ \forall k \in \mathbb{Z}_{\geq 0}. \tag{6.3}$$

It is assumed that the input is known. An interval observer is designed such that the continuous dynamics mimic the system during flow (in between measurements), and the estimator updates via an impulse when a measurement is made at t_k:

$$\dot{\overline{x}}(t) = A\overline{x}(t) + Bu(t),\ t \in [t_k, t_{k+1}), \tag{6.4}$$

$$\dot{\underline{x}}(t) = A\underline{x}(t) + Bu(t),\ t \in [t_k, t_{k+1}), \tag{6.5}$$

$$\overline{x}(t_k^+) = \overline{x}(t_k) + L(y(t_k) - C\overline{x}(t_k)), \tag{6.6}$$

$$\underline{x}(t_k^+) = \underline{x}(t_k) + L(y(t_k) - C\underline{x}(t_k)). \tag{6.7}$$

This system is augmented with a clock variable in a fashion similar to [60]:

$$\dot{\tau}(t) = -1, t \in [t_k, t_{k+1}); \quad \tau(t_k^+) = T_k,$$

where $T_k = t_{k+1} - t_k > 0$ is the kth inter-sampling time. The set of all inter-sampling times is denoted T_{inter}, and the largest inter-sampling time in T_{inter} is denoted as h_{max}. This system can be written with the formalism of [64] as:

$$\mathcal{H} = \begin{cases} \left.\begin{aligned} \dot{x} &= Ax + Bu \\ \dot{\overline{x}} &= A\overline{x} + Bu \\ \dot{\underline{x}} &= A\underline{x} + Bu \\ \dot{\tau} &= -1 \end{aligned}\right\} & (x, \overline{x}, \underline{x}, \tau) \in \mathcal{C} \\ \left.\begin{aligned} x^+ &= x \\ \overline{x}^+ &= \overline{x} + L(y - C\overline{x}) \\ \underline{x}^+ &= \underline{x} + L(y - C\underline{x}) \\ \tau^+ &\in T_{inter} \end{aligned}\right\} & (x, \overline{x}, \underline{x}, \tau) \in \mathcal{D} \end{cases}$$

where $\mathcal{C} = \{(x, \overline{x}, \underline{x}, \tau) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times [0, h_{max}]\}$ and $\mathcal{D} = \{(x, \overline{x}, \underline{x}, \tau) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \{0\}\}$.

It is assumed that the initial condition of the system lies within an interval $\underline{x}(0,0) \leq x(0,0) \leq \overline{x}(0,0)$. One of the primary functions of the interval observer is to maintain the ordering for all time and across all jumps, so

$$\underline{x}(t,j) \leq x(t,j) \leq \overline{x}(t,j), \forall (t,j) \in \text{dom}\xi \tag{6.8}$$

where ξ is a solution to $\mathcal{H}$. Furthermore, define the lower and upper errors as

$$\underline{e} := x - \underline{x}, \quad \overline{e} := \overline{x} - x. \tag{6.9}$$

The errors have the following dynamics

$$\mathcal{H}_e = \begin{cases} \left.\begin{aligned} \dot{\overline{e}} &= A\overline{e} \\ \dot{\underline{e}} &= A\underline{e} \\ \dot{\tau} &= -1 \end{aligned}\right\} & (\overline{e}, \underline{e}, \tau) \in \mathcal{C} \\ \left.\begin{aligned} \overline{e}^+ &= (I - LC)\overline{e} \\ \underline{e}^+ &= (I - LC)\underline{e} \\ \tau^+ &\in T_{inter} \end{aligned}\right\} & (\overline{e}, \underline{e}, \tau) \in \mathcal{D} \end{cases}$$

where $\mathcal{C} = \{(\overline{e}, \underline{e}, \tau) \in \mathbb{R}^n_{\geq 0} \times \mathbb{R}^n_{\geq 0} \times [0, h_{max}]\}$ and $\mathcal{D} = \{(\overline{e}, \underline{e}, \tau) \in \mathbb{R}^n_{\geq 0} \times \mathbb{R}^n_{\geq 0} \times \{0\}\}$. For a solution ζ to $\mathcal{H}_e$, domξ = domζ when they have the same clock dynamics.

Clearly $\overline{e}, \underline{e} \geq 0$ implies $\underline{x} \leq x \leq \overline{x}$, so the positivity of the errors must be shown to guarantee (6.8). In addition, L and the inter-sampling times T_k shall be designed to ensure that the set $\mathcal{A} := \{(\underline{e}, \overline{e}, \tau) \in \mathbb{R}^n \times \mathbb{R}^n \times [0, h_{max}] : (\underline{e}, \overline{e}) = (0, 0)\}$ is globally asymptotically stable.

6.2.1 Periodically Sampled Measurements

If the measurements are sampled periodically, $T_k = h$ for all $k \in \mathbb{Z}_{\geq 0}$ where $h > 0$ is a desired sampling time. The following lemma gives the conditions for the stability and positivity of the interval observer.

Lemma 8. *Suppose A is Metzler, there exists $L \in \mathbb{R}^{n \times m}$ such that $I - LC \geq 0$, and there exists a $P \in \mathbb{R}^{n \times n}$ where $P = P^T \succ 0$ such that for a desired sampling time $h > 0$,*

$$(I - LC)^T e^{A^T h} P e^{Ah} (I - LC) - P + \beta I \prec 0 \tag{6.10}$$

for some $\beta > 0$. Then the closed set $\mathcal{A} := \{(\underline{e}, \overline{e}, \tau) \in \mathbb{R}^n \times \mathbb{R}^n \times [0, h] : (\underline{e}, \overline{e}) = (0, 0)\}$ is globally exponentially stable for $\mathcal{H}_e$. Moreover, if $\underline{x}(0, 0) \leq x(0, 0) \leq \overline{x}(0, 0)$ then $\underline{x}(t, j) \leq x(t, j) \leq \overline{x}(t, j)$ for all $(t, j) \in$ domζ.

Proof. To prove the result both positivity and stability need to be shown. Positivity of the flow is ensured if and only if A is Metzler (c.f. Theorem 2 in [59]), and positivity of the jumps follows is ensured if and only if $I - LC \geq 0$. By [60, Theorem 1], (6.10) ensures that the set $\mathcal{A}$ is globally exponentially stable. □

The following is a convex program which can be solved numerically to find L and P to satisfy the positivity and stability conditions of Lemma 8. This convex program relies on M-matrices which are invertible and their inverses are nonnegative matrices (see pp. 161 in [67]).

Proposition 4. *Suppose that there exist* $G \in \mathbb{R}^{n\times n}$ *which is an M-matrix,* $P \in \mathbb{R}^{n\times n}$ *where* $P = P^T \succ 0$*, and* $J \in \mathbb{R}^{n\times m}$ *such that* $G - JC \geq 0$ *and*

$$\begin{bmatrix} -G - G^T & G - JC & e^{A^T h}P \\ \star & -P + \beta I & 0 \\ \star & \star & -P + \beta I \end{bmatrix} \prec 0. \tag{6.11}$$

Then $I - LC \geq 0$ *and* (6.10) *holds with* $L = G^{-1}J$*.*

Proof. With the variable transformation $J = GL$, feasibility of (6.11) implies feasibility of (6.10) by Proposition 1 in [60]. Since G is an M-matrix, $G^{-1} \geq 0$. Therefore, $G^{-1}(G - JC) = I - LC \geq 0$. □

Another convex program is shown in the following which is derived by enforcing the commutativity of A and P (which ensures that e^{Ah} and P also commute) and using the Schur complement. The commutativity property can be useful for designing self-triggered functions.

Proposition 5. *Suppose that there exist* $P \in \mathbb{R}^{n\times n}$ *which is Stieltjes, and* $Y \in \mathbb{R}^{n\times m}$ *such that* $P - YC \geq 0, PA = AP$*, and*

$$\begin{bmatrix} P & e^{Ah}(P - YC) \\ \star & P - \beta I \end{bmatrix} \succ 0.$$

Then $I - LC \geq 0$ *and* (6.10) *holds with* $L = P^{-1}Y$*.*

6.2.2 Self-triggered Measurements

The self-triggering formulation is based on the periodically sampled interval observer given in the preceding subsection. The self-triggering mechanism is embedded in the clock dynamics $T_k = h + \tau_s(x_d(t_k))$ where $x_d = \overline{x} - \underline{x}$ for $\tau_s(x_d(t_k)) \geq 0$, which guarantees that the uniform minimum inter-sampling time is lower bounded by h, which, in turn, guarantees that Zeno behaviour is not exhibited [73].

The self-triggered system can be written as a hybrid system $\mathcal{H}_{ds}$ as follows:

$$\mathcal{H}_{ds} = \begin{cases} \left.\begin{aligned} \dot{x}_d &= Ax_d \\ \dot{\tau} &= -1 \end{aligned}\right\} & (x_d, \tau) \in \mathcal{C} \\ \left.\begin{aligned} x_d^+ &= (I - LC)x_d \\ \tau^+ &= h + \tau_s(x_d) \end{aligned}\right\} & (x_d, \tau) \in \mathcal{D} \end{cases}$$

the flow set is $\mathcal{C} = \left\{(x_d, \tau) \in \mathbb{R}^n_{\geq 0} \times [0, h_{max}]\right\}$ and the jump set is $\mathcal{D} = \left\{(x_d, \tau) \in \mathbb{R}^n_{\geq 0} \times \{0\}\right\}$. Notice that $x_d = \overline{e} + \underline{e}$. When A is Metzler and $I - LC \geq 0$, showing that $x_d \to 0$ ensures that $\overline{e}$ and $\underline{e}$ are positive and $\overline{e}$, $\underline{e} \to 0$ as $t + j \to \infty$ [55]. The following theorem designs self-triggering functions to guarantee the stability and positivity of $\mathcal{H}_{ds}$.

Theorem 12. *Consider the linear system in* (6.2), (6.3) *where A is Metzler. Suppose that there exist P, L such that* (6.10) *is satisfied and $I - LC \geq 0$ for a given sampling time $h > 0$, and rank$(I - LC) = n$. If $\underline{x}(0,0) \leq x(0,0) \leq \overline{x}(0,0)$, then $\underline{x}(t,j) \leq x(t,j) \leq \overline{x}(t,j)$ for all $(t,j) \in dom\eta_{ds}$, and with the following self-triggering function*

$$\tau_s(x_d) = \max\left\{0, \frac{1}{\lambda_M}\log\left(\frac{x_d^T\,(P - \beta I)\,x_d}{\lambda_{max}(P)x_d^T Q_1 x_d}\right)\right\} \tag{6.12}$$

where $\lambda_M = \max_{i\in[1,\dots,n]}\{|\lambda_i(A + A^T)|\}$ and $Q_1 = (I - LC)^T e^{A^T h} e^{Ah}(I - LC)$, the set $\mathcal{A}_d = \{(x_d, \tau) \in \mathbb{R}^n_{\geq 0} \times [0, h_{max}] : x_d = 0\}$ is globally asymptotically stable for $\mathcal{H}_{ds}$, which ensures that $\mathcal{A} = \{(\underline{e}, \overline{e}, \tau) \in \mathbb{R}^n \times \mathbb{R}^n \times [0, h_{max}] : (\underline{e}, \overline{e}) = (0, 0)\}$ is globally asymptotically stable. Particularly, if A and P commute, then the conclusion holds with the following triggering function:

$$\tau_s(x_d) = \frac{1}{\lambda_M}\log\left(\frac{x_d^T\,(P - \beta I)\,x_d}{x_d^T Q_2 x_d}\right) \tag{6.13}$$

where $Q_2 = (I - LC)^T e^{A^T h} P e^{Ah}(I - LC)$.

Proof. Positivity follows from the same arguments as Lemma 8. To show stability, consider the Lyapunov function $V(x_d, \tau) = x_d^T e^{A^T\tau} P e^{A\tau} x_d$ which remains constant during the flows. During the jumps, the Lyapunov function changes as

$$V(x_d^+, h + \tau_s(x_d)) - V(x_d, 0) =$$
$$x_d^T \left((I - LC)^T e^{A^T(h+\tau_s)} P e^{A(h+\tau_s)} (I - LC) - P \right) x_d.$$

Now consider the first part of the term,

$$x_d^T \left((I - LC)^T e^{A^T(h+\tau_s)} P e^{A(h+\tau_s)} (I - LC) \right) x_d$$
$$\leq \lambda_{max}(P) x_d^T \left((I - LC)^T e^{A^T h} e^{(A^T+A)\tau_s} e^{Ah} (I - LC) \right) x_d$$
$$\leq e^{\lambda_M \tau_s} x_d^T Q_1 x_d.$$

τ_s in (6.12) ensures $e^{\lambda_M \tau_s} x_d^T Q_1 x_d \leq x_d^T (P - \beta I) x_d$. Thus, $V(x_d^+, h + \tau_s(x_d)) - V(x_d, 0) \leq -\beta x_d^T x_d$ during the jump.

In the case that $AP = PA$,

$$x_d^T \left((I - LC)^T e^{A^T(h+\tau_s)} P e^{A(h+\tau_s)} (I - LC) \right) x_d =$$
$$x_d^T \left((I - LC)^T e^{A^T h} e^{(A^T+A)\tau_s} P e^{Ah} (I - LC) \right) x_d. \tag{6.14}$$

Therefore, $RHS(6.14) \leq e^{\lambda_M \tau_s} x_d^T Q_2 x_d$. τ_s in (6.13) ensures that $V(x_d^+, h+\tau_s(x_d)) - V(x_d, 0) \leq -\beta x_d^T x_d$ during the jump. Note that $\tau_s(x_d) \geq 0$ in (6.13) by virtue of (6.10) for all $x_d \in \mathbb{R}^n$. Since $\text{rank}(I - LC) = n$,

$$h_{max} = h + \frac{1}{\lambda_M} \log \left(\frac{\lambda_{max}(P - \beta I)}{\lambda_{min}(Q_2)} \right) < \infty.$$

Therefore, $\alpha_1 \|x_d\|_{\mathcal{A}_d} \leq V(x_d, \tau) \leq \alpha_2 \|x_d\|_{\mathcal{A}_d}$ with the constants $\alpha_1 = \min_{s \in [h, h_{max}]} \lambda_{min}(e^{A^T s} P e^{As})$ and $\alpha_2 = \max_{s \in [h, h_{max}]} \lambda_{max}(e^{A^T s} P e^{As})$. For all $(t, j) \in \text{dom}\eta_{ds}$, $t \leq (j + 1) h_{max}$. Therefore, for $t + j \geq T$, $j \geq \frac{T}{h_{max}+1} - \frac{h_{max}}{h_{max}+1}$. By Lemma 7, $\mathcal{A}_d$ is asymptotically stable. Therefore, since $x_d = \bar{e} + \underline{e} \to 0$, then $\bar{e}$, $\underline{e} \to 0$ as $t + j \to \infty$. The asymptotic stability of $\mathcal{A}_d$ and positivity of the errors ensures the asymptotic stability of $\mathcal{A}$ [55]. □

The self-triggering functions $\tau_s(x_d)$ are designed to be implementable in real time and ensure that the Lyapunov function decreases sufficiently during each jump.

Remark 5. *If A is not Metzler, then a coordinate transformation can often be found such that it is Metzler in a new set of coordinates [99, 55]. Then the results of Lemma 8 and Theorem 12 can be used. Alternatively, a framer system can be incorporated (see Remark 6).*

6.3 Sampled Interval Observer Design for Lipschitz Nonlinear Systems

Consider a nonlinear system with state $x \in \mathbb{R}^n$ and output measurements $y \in \mathbb{R}^m$ sampled at times t_k, $t_0 = 0$:

$$\dot{x}(t) = Ax(t) + \Psi(x(t)) + Bu(t),\ \forall t \geq 0, \tag{6.15}$$

$$y(t_k) = Cx(t_k),\ \forall k \in \mathbb{Z}_{\geq 0}. \tag{6.16}$$

where Ψ is a globally Lipschitz nonlinear function.

A linear framer system bounds the nonlinear system by a linear system. This enables the incorporation of the tools for linear systems that were developed in the previous section for designing sampled interval observers of Lipschitz nonlinear systems. The linear framer system is constructed using the following results of [105]:

Lemma 9 (Properties 3-5 in [105])**.** *Suppose that $\psi(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is a globally Lipschitz function. Then there exists a function $\Psi(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$\Psi(x, x) = \psi(x) \tag{6.17}$$

and for $\underline{x} \leq x \leq \overline{x}$,

$$\Psi(\underline{x}, \overline{x}) \leq \Psi(x, x) \leq \Psi(\overline{x}, \underline{x}). \tag{6.18}$$

Moreover, there exists nonnegative matrices N_i such that

$$\Psi(\overline{x}, \underline{x}) - \Psi(x, x) \leq N_1\overline{e} + N_2\underline{e}, \tag{6.19}$$

$$\Psi(x, x) - \Psi(\underline{x}, \overline{x}) \leq N_3\overline{e} + N_4\underline{e}, \tag{6.20}$$

for $\underline{x} \leq x \leq \overline{x}$ where $\overline{e}$ and $\underline{e}$ are defined in (6.9).

Now consider the following interval observer for (6.15), (6.16) where F_1 and F_2 are non-negative matrices:

$$\begin{aligned}
\dot{\overline{x}}(t) =& A\overline{x}(t) + Bu(t) + \Psi(\overline{x}(t), \underline{x}(t)) \\
&+ F_1(\overline{x}(t) - \underline{x}(t)), t \in [t_k, t_{k+1}), \\
\dot{\underline{x}}(t) =& A\underline{x}(t) + Bu(t) + \Psi(\underline{x}(t), \overline{x}(t)) \\
&- F_2(\overline{x}(t) - \underline{x}(t)), t \in [t_k, t_{k+1}), \\
\overline{x}(t_k^+) =& \overline{x}(t_k) + L_1(y(t_k) - C\overline{x}(t_k)), \\
\underline{x}(t_k^+) =& \underline{x}(t_k) + L_2(y(t_k) - C\underline{x}(t_k)).
\end{aligned}$$

The error dynamics are given as follows:

$$\mathcal{H}_{ne} = \begin{cases} \left.\begin{aligned} \dot{\eta} = \hat{A}\eta + \Phi(x, \overline{x}, \underline{x}) \\ \dot{\tau} = -1 \end{aligned}\right\} & (\eta, \tau) \in \mathcal{C} \\ \left.\begin{aligned} \eta^+ = A_d\eta \\ \tau^+ \in T_{inter} \end{aligned}\right\} & (\eta, \tau) \in \mathcal{D} \end{cases}$$

where $\eta = \begin{bmatrix} \overline{e}^T & \underline{e}^T \end{bmatrix}^T$,

$$\Phi(x, \overline{x}, \underline{x}) = \begin{bmatrix} \Psi^T(\overline{x}, \underline{x}) - \psi^T(x) & \psi^T(x) - \Psi^T(\underline{x}, \overline{x}) \end{bmatrix}^T$$

$$\hat{A} = \begin{bmatrix} A + F_1 & F_1 \\ F_2 & A + F_2 \end{bmatrix}, A_d = \begin{bmatrix} I - L_1C & 0 \\ 0 & I - L_2C \end{bmatrix}$$

and the flow set is $\mathcal{C} = \left\{(\eta, \tau) \in \mathbb{R}^{2n}_{\geq 0} \times [0, h_{max}]\right\}$, the jump set is $\mathcal{D} = \left\{(\eta, \tau) \in \mathbb{R}^{2n}_{\geq 0} \times \{0\}\right\}$.

Define the following system which will be a linear framer system for $\mathcal{H}_{ne}$:

$$\mathcal{H}_f = \begin{cases} \left.\begin{aligned} \dot{\tilde{\eta}} = \tilde{A}\tilde{\eta} \\ \dot{\tau} = -1 \end{aligned}\right\} & (\tilde{\eta}, \tau) \in \mathcal{C} \\ \left.\begin{aligned} \tilde{\eta}^+ = A_d\tilde{\eta} \\ \tau^+ = h \end{aligned}\right\} & (\tilde{\eta}, \tau) \in \mathcal{D} \end{cases}$$

where $\tilde{A} = \hat{A} + N$ for

$$N = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_4 \end{bmatrix},$$

and the flow set is $\mathcal{C} = \big\{(\tilde{\eta}, \tau) \in \mathbb{R}^{2n}_{\geq 0} \times [0, h]\big\}$, the jump set is $\mathcal{D} = \big\{(\tilde{\eta}, \tau) \in \mathbb{R}^{2n}_{\geq 0} \times \{0\}\big\}$.

Remark 6. *The results in this section can be used for linear systems* (6.2)-(6.3) *where A is not Metzler as an alternative to using coordinate transformations. F_1, F_2 should be chosen to make $A + F_1$ and $A + F_2$ Metzler, and $N = 0$.*

Remark 7. *While F_1 and F_2 are used to help relax the condition that A is Metzler, coordinate transformations can also be used to make A Metzler for the nonlinear system* (6.15)-(6.16) *(c.f. [55]).*

6.3.1 Periodically Sampled Measurements

For a periodically sampled system with sampling time $h > 0$, $T_k = h$. Stability and positivity of the error dynamics $\mathcal{H}_{ne}$ is shown.

Lemma 10. *Suppose $A + F_1$ and $A + F_2$ are Metzler, $I - L_1C \geq 0$, $I - L_2C \geq 0$, and there exists a matrix $P = P^T \succ 0$ such that*

$$A_d^T e^{\tilde{A}^T h} P e^{\tilde{A}h} A_d - P + \beta I \prec 0 \tag{6.21}$$

for some $\beta > 0$. If $\underline{x}(0,0) \leq x(0,0) \leq \overline{x}(0,0)$, then $\underline{x}(t,j) \leq x(t,j) \leq \overline{x}(t,j)$ for all $(t,j) \in dom\eta$ where η is a solution to $\mathcal{H}_{ne}$, and $\mathcal{A} = \{(\eta, \tau) \in \mathbb{R}^{2n}_{\geq 0} \times [0,h] : \eta = 0\}$ is globally exponentially stable for $\mathcal{H}_{ne}$.

Proof. Positivity of η follows from (6.18), the choice of F_1 and F_2 that make $A+F_1$ and $A+F_2$ Metzler, and $I - L_1C \geq 0$, $I - L_2C \geq 0$. Defining $\varepsilon := \tilde{\eta} - \eta$. If $\tilde{\eta}(0,0) \geq \eta(0,0) \geq 0$, therefore, $\varepsilon(0,0) \geq 0$. ε evolves during the flow according to $\dot{\varepsilon} = \hat{A}\varepsilon + N\tilde{\eta} - \Phi(x, \overline{x}, \underline{x})$. N is nonnegative, therefore, $N\tilde{\eta} \geq N\eta$. By (6.19)-(6.20), $N\tilde{\eta} - \Phi(x, \overline{x}, \underline{x}) \geq 0$. Therefore, since $\hat{A}$ is Metzler, ε remains positive during the flows. Positivity is also preserved during the jumps since $A_d \geq 0$. Therefore, if $\tilde{\eta}(0,0) \geq \eta(0,0) \geq 0$, then, $\tilde{\eta}(t,j) \geq \eta(t,j) \geq 0$ for all $(t,j) \in \text{dom}\tilde{\eta}$, and $\mathcal{H}_f$ is a linear framer system for $\mathcal{H}_{ne}$. Note that $\text{dom}\eta = \text{dom}\tilde{\eta}$ since the flows and jumps of both the framer system and the error dynamics coincide. By [60, Theorem 1], feasibility of the LMI (6.21) ensures that for some $k, \varrho > 0$, $\|\eta(t,j)\|_{\mathcal{A}} \leq \|\tilde{\eta}(t,j)\|_{\mathcal{A}_f} \leq ke^{-\varrho(t+j)}\|\tilde{\eta}(0,0)\|_{\mathcal{A}_f}$ where $\mathcal{A}_f = \left\{(\tilde{\eta}, \tau) \in \mathbb{R}^{2n}_{\geq 0} \times [0, h] : \tilde{\eta} = 0\right\}$. $\square$

The gains L_1, L_2, and the Lyapunov matrix P which satisfy the conditions of Lemma 10 can be solved for by a convex program using the same techniques as Proposition 4.

Proposition 6. *Suppose that there exist $G_1, G_2 \in \mathbb{R}^{n\times n}$ that are M-matrices, $J_1, J_2 \in \mathbb{R}^{n\times m}$, and $P \in \mathbb{R}^{2n\times 2n}$ where $P = P^T \succ 0$ such that $G - J\tilde{C} \geq 0$ and*

$$G = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix}, J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}, \tilde{C} = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix},$$

$$\begin{bmatrix} -G - G^T & G - J\tilde{C} & e^{\tilde{A}^T h}P \\ \star & -P + \beta I & 0 \\ \star & \star & -P + \beta I \end{bmatrix} \prec 0.$$

Then $A_d \geq 0$ and (6.21) *holds with $L_1 = G_1^{-1}J_1$ and $L_2 = G_2^{-1}J_2$.*

6.3.2 Self-triggered Measurements

In this case, $\mathcal{H}_f$ and $\mathcal{H}_{ne}$ have the clock reset $\tau^+ = h + \tau_s(\tilde{\eta})$. A self-triggered observer for the Lipschitz nonlinear system is derived by showing asymptotic stability of the linear framer system with self-triggered measurements, which guarantees the asymptotic stability of the errors.

Theorem 13. *Consider the nonlinear system* (6.15), (6.16). *Suppose $A + F_1$ and $A + F_2$ are Metzler and that there exist P, L_1, L_2 such that* (6.21) *holds and $I - L_1C \geq 0$, $I - L_2C \geq 0$*

for a given sampling time $h > 0$, and rank$(I - L_1C) =$ rank$(I - L_2C) = n$. If $\underline{x}(0,0) \leq x(0,0) \leq \overline{x}(0,0)$, then $\underline{x}(t,j) \leq x(t,j) \leq \overline{x}(t,j)$ for all $(t,j) \in \text{dom}\eta$ where η is a solution to $\mathcal{H}_{ne}$. Let $\tilde{\eta}$ be a solution to $\mathcal{H}_f$ and its initial condition is such that

$$\tilde{\eta}(0,0) \geq \begin{bmatrix} \overline{x}(0,0) - \underline{x}(0,0) \\ \overline{x}(0,0) - \underline{x}(0,0) \end{bmatrix}. \tag{6.22}$$

With the following self-triggering function

$$\tau_s(\tilde{\eta}) = \max\left\{0, \frac{1}{\lambda_M}\log\left(\frac{\tilde{\eta}^T(P - \beta I)\tilde{\eta}}{\lambda_{max}(P)\tilde{\eta}^T Q_1 \tilde{\eta}}\right)\right\} \tag{6.23}$$

where $\lambda_M = \max_{i\in[1,\ldots,n]}\{|\lambda_i(\tilde{A} + \tilde{A}^T)|\}$, and $Q_1 = A_d^T e^{\tilde{A}^T h} e^{\tilde{A}h} A_d$, the set $\mathcal{A} = \{(\eta,\tau) \in \mathbb{R}^{2n}_{\geq 0} \times [0, h_{max}] : \eta = 0\}$ is globally asymptotically stable. Particularly, if A and P commute, then the conclusion holds with the following triggering function:

$$\tau_s(\tilde{\eta}) = \frac{1}{\lambda_M}\log\left(\frac{\tilde{\eta}^T(P - \beta I)\tilde{\eta}}{\tilde{\eta}^T A_d^T e^{\tilde{A}^T h} P e^{\tilde{A}h} A_d \tilde{\eta}}\right). \tag{6.24}$$

Proof. From (6.22) and the arguments of the proof of Lemma 10, $0 \leq \eta(t,j) \leq \tilde{\eta}(t,j)$ for all $(t,j) \in \text{dom}\tilde{\eta}$. The same arguments from Theorem 12 follow to determine the self-triggering functions (6.23) and (6.24), which guarantee the asymptotic stability of $\mathcal{A}$. □

6.4 Numerical Simulations

6.4.1 Linear Example

Consider an example of a linear system (6.2)-(6.3) where

$$A = \begin{bmatrix} 0 & 1 \\ 1 & -0.4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Let $h = 0.05$ be the sampling time. By using Proposition 5, the observer gain

$$L = \begin{bmatrix} 0.2432 \\ 0 \end{bmatrix}$$

and Lyapunov matrix

$$P = \begin{bmatrix} 5.7998 & 0 \\ 0 & 5.7998 \end{bmatrix},$$

are found. Since A and P commute, the triggering function (6.13) can be used. The errors $\overline{e}$ and $\underline{e}$ from a simulation of the periodically sampled interval observer are shown in Figure 6.1. The errors for the self-triggered observer are shown in Figure 6.2 from the same initial condition. Within the first second of the simulations, the output is measured 5 times in the self-triggered case (see Fig. 6.2), and 20 times in the periodically sampled case (see Fig. 6.1).

Note that

$$I - LC = \begin{bmatrix} 0.7568 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, $\overline{e}_2$ and $\underline{e}_2$ do not change throughout the jumps; however, they change during the flow because the eigenvalues of $e^{Ah}(I - LC)$ are strictly in the unit circle (c.f. Remark 3 in [60]).

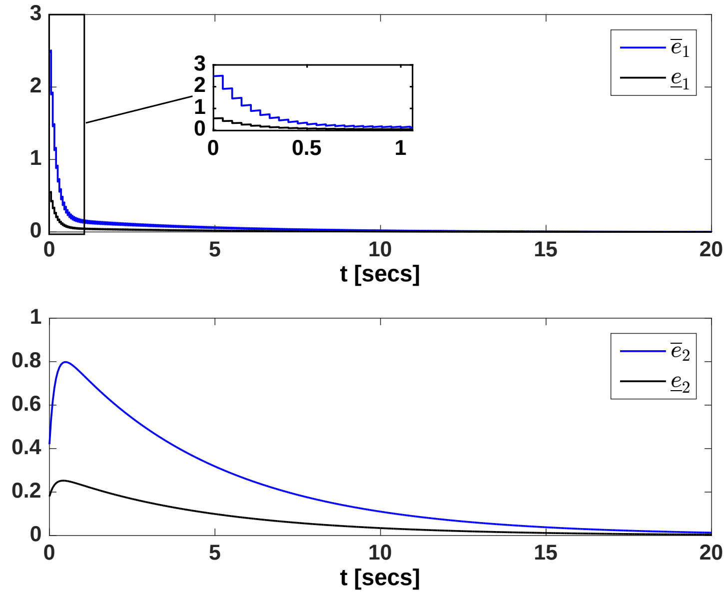

Figure 6.1: Periodically sampled interval observer for the linear example.

6.4.2 Nonlinear Example

Consider the following model for a single-link robot arm:

$$\dot{x}_1 = x_2, \ \dot{x}_2 = -\alpha \sin(x_1) - \nu x_2, \ y = x_1,$$

where $\alpha = 0.5$, $\nu = 0.4$ and $h = 0.1$. The nonlinear function from Lemma 9 is $\Psi(x_a, x_b) = \alpha x_a - \alpha x_b - \alpha \sin(x_b)$, and

$$N_1 = \begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix}, N_2 = \begin{bmatrix} 0 & 0 \\ 2\alpha & 0 \end{bmatrix},$$
$$N_3 = \begin{bmatrix} 0 & 0 \\ 2\alpha & 0 \end{bmatrix}, N_4 = \begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix}.$$

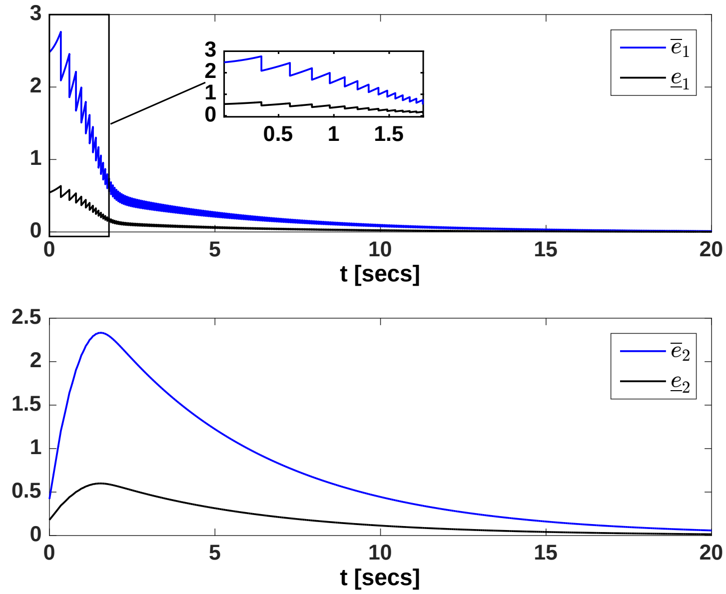

Figure 6.2: Self-triggered interval observer for the linear example.

F_1, F_2 are chosen to be zero since the linear part is already Metzler. Using Proposition 6, $L_1 = L_2 = \begin{bmatrix} 0.9303 & -0.0089 \end{bmatrix}^T$.

$$P = \begin{bmatrix} 23.6535 & -2.1083 & -0.7145 & -1.0730 \\ -2.1083 & 72.7132 & -1.0730 & 1.8223 \\ -0.7145 & -1.0730 & 23.6535 & -2.1083 \\ -1.0730 & 1.8223 & -2.1083 & 72.7132 \end{bmatrix}.$$

The M-matrices are

$$G_1 = G_2 = \begin{bmatrix} 76.5774 & 0 \\ -1.0967 & 71.6495 \end{bmatrix}.$$

Figure 3 shows a simulation of the periodically sampled interval observer and Figure 4 shows the self-triggered observer using the self-triggering function (6.23) from the same initial conditions. Within the first two seconds of the simulations, the output is measured 14 times in the self-triggered case, and 20 times in the periodically sampled case.

Note that

$$I - L_1 C = I - L_2 C = \begin{bmatrix} 0.0697 & 0 \\ 0.0089 & 1.0000 \end{bmatrix}.$$

Therefore, the jumps of $\overline{x}_2$ and $\underline{x}_2$ are not are pronounced as those of $\overline{x}_1$ and $\underline{x}_1$.

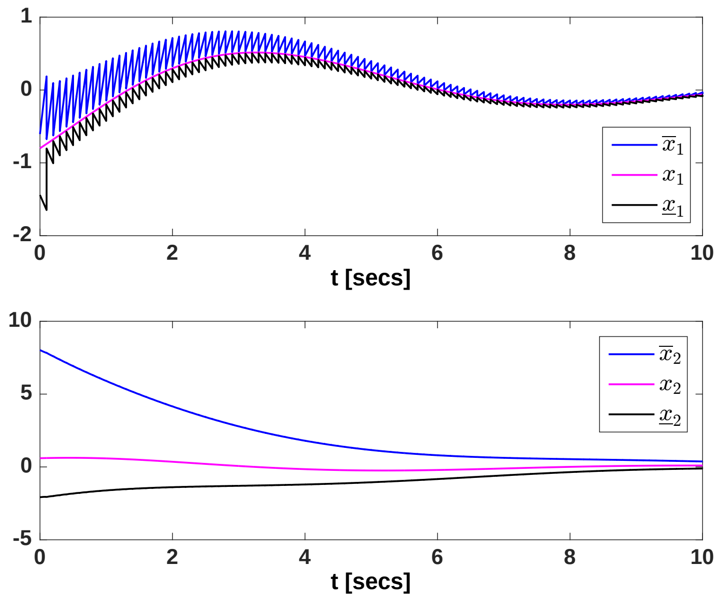

Figure 6.3: Periodically sampled interval observer for the nonlinear example.

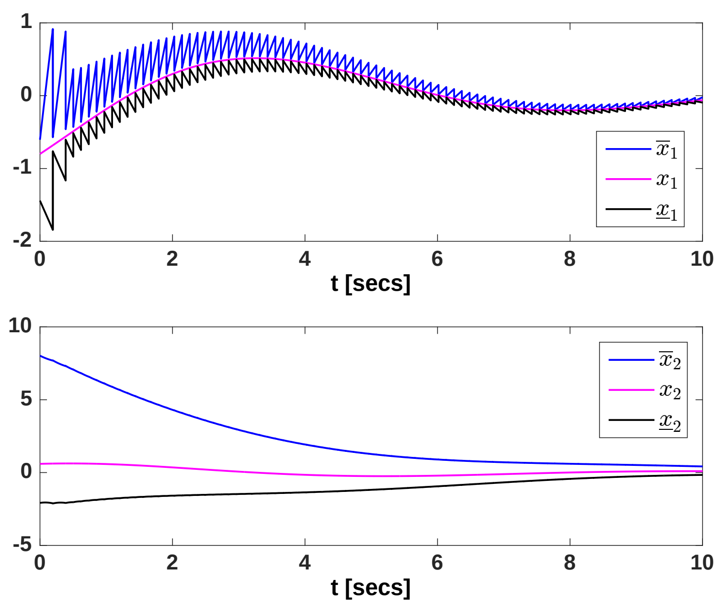

Figure 6.4: Self-triggered interval observer for the nonlinear example.

Chapter 7

STABILIZATION OF NONLINEAR SYSTEMS USING PERIODIC EVENT-TRIGGERED CONTROL

"Often is there regret for saying too much, and seldom regret for saying too little."

The Saga of Hrafnkel Frey's Gothi

Digital control systems are traditionally executed in a time-triggered fashion where the sensors and actuators are accessed periodically. In contrast, event-triggered control (ETC) executes the sensing and actuation only when certain triggering rules are satisfied; this can be seen as adding feedback to the sensing and actuation processes (see a recent survey paper [73] and references therein). The ETC paradigm is designed to avoid unnecessary waste of communication/computation resources by reducing the number of sensing/actuation executions, while still guaranteeing a desirable closed-loop performance [143, 139, 152, 62, 27, 49, 4, 3]; this shows potential in applications of systems with limited resources such as networked control systems and embedded control systems.

Since the triggering condition of ETC has to be monitored continuously, it is difficult to implement ETC in digital platforms directly. To overcome this problem, periodic event-triggered control (PETC) was proposed [72, 71, 74]. By evaluating the triggering conditions periodically and deciding whether to update the sensing/actuation at each sampling time, PETC inherits both the benefits of ETC and sampled-data control, and can be implemented on a standard digital platform. Furthermore, Zeno phenomenon is avoided since the sampling period is a lower bound for the minimum inter-execution time. Although ETC for discrete-time models can be considered as PETC (e.g., see [71, 56]), the inter-sample behavior of the original continuous-time systems are not captured in the discrete-time analysis. PETC design for (continuous-time) linear systems was investigated in [72] where three analysis approaches were presented. However, PETC design for nonlinear systems is difficult because of an intrinsic difficulty: the discrete-time dynamics of a nonlinear system can not be exactly known from its continuous-time dynamics [58, 28, 150, 15]. There are few existing papers on PETC design for nonlinear systems: [150] studied state feedback PETC design for undisturbed nonlinear systems using the hybrid system approach and proved globally asymptotically stability of the closed-loop system; [151] studied output feedback PETC design for disturbed nonlinear systems and proved input-to-state stability of the closed-loop system; [28] investigated state feedback PETC design for nonlinear systems by redesigning the event function of an existing continuous ETC system using overapproximation techniques, such that the control performance guarantees for the continuous ETC system are preserved; [58] studied output feedback PETC design for Lipschitz systems using impulsive observers and proved practical stabilization of the resulting system. In spite of these interesting results, many PETC design problems for nonlinear systems are largely open and deserve to be further explored. For example, asynchronous sensing and actuation PETC mechanisms have not been explored, and systematic methods to determine the sampling period and triggering functions are still rare.

This chapter investigates the input-to-state stabilization of disturbed nonlinear systems using PETC design. Configurations of state feedback and observer-based output feedback are considered separately. The impulsive system approach is used to formulate and analyze the closed-loop system. A systematic way is given to design the sampling period and the triggering functions. For PETC design of incrementally quadratic nonlinear system sufficient conditions in the form of LMIs are given. *Notation.* Denote the set of real, non-negative real and non-negative integer numbers by $\mathbb{R}$, $\mathbb{R}_{\geq 0}$ and $\mathbb{Z}_{\geq 0}$, respectively. Denote the 2-norm by

$\|\cdot\|$. Given a non-empty and closed set $\mathcal{A}$, the point-to-set distance from x to $\mathcal{A}$ is denoted by $\|x\|_{\mathcal{A}} = \inf_{y\in\mathcal{A}} \|y - x\|$. Denote the identity matrix of size $n \times n$ by I_n. Denote the zero matrix of size $n_1 \times n_2$ by $\mathbf{0}_{n_1\times n_2}$ and the zero vector of size n by $\mathbf{0}_n$; the subscripts will be omitted when clear from context. Denote the block diagonal matrix by $diag\{M_1, ..., M_n\}$ where $M_1, ..., M_n$ are matrices in the diagonal block. For symmetric matrices, $*$ stands for entries whose values follow from symmetry. A signal $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is called left-continuous if $\lim_{s\to t^-} x(s) = x(t)$ for all $t > 0$. "$\forall x$ $a.e.$" means for every $x \in \mathbb{R}^{n_x}$ except for a set of zero Lebesgue-measure in $\mathbb{R}^{n_x}$.

7.1 Problem Setup

Fig.7.1 (a) shows the configuration of implementing the state feedback PETC. The plant is a nonlinear system given as

$$\dot{x}(t) = f(x(t), u(t), w(t)) \tag{7.1}$$

where $x \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^{n_u}$ is the control input, $w \in \mathbb{R}^{n_w}$ is the disturbance, f is a locally Lipschitz continuous function. The state feedback controller is given as $u(t) = k(x(t))$ where $k(\cdot)$ is a continuous function. Assume $u(t)$ is designed such that the solution to the system $\dot{x}(t) = f(x(t), k(x(t)), w(t))$ exists for all time and all initial conditions, and the closed-loop system is input-to-state stable (ISS) with respect to (w.r.t.) w.

Denote the sampling period to be $h > 0$, and define the sampling times as $t_k := kh$ for any $k \in \mathbb{Z}_{\geq 0}$. With the event-triggering mechanism (ETM), the state of the plant, $x(t)$, is sampled at each sampling time t_k. The input to the controller, $\tilde{x}_c(t)$, is updated only when the event-triggering condition for the state is satisfied. Specifically, $\tilde{x}_c(t)$ is a left-continuous, piecewise constant signal that is defined for $t \in (t_k, t_{k+1}]$ as

$$\tilde{x}_c(t) = \begin{cases} x(t_k), & \text{if } \Gamma_x(x(t_k), e(t_k)) \geq 0, \\ \tilde{x}_c(t_k), & \text{if } \Gamma_x(x(t_k), e(t_k)) < 0, \end{cases} \tag{7.2}$$

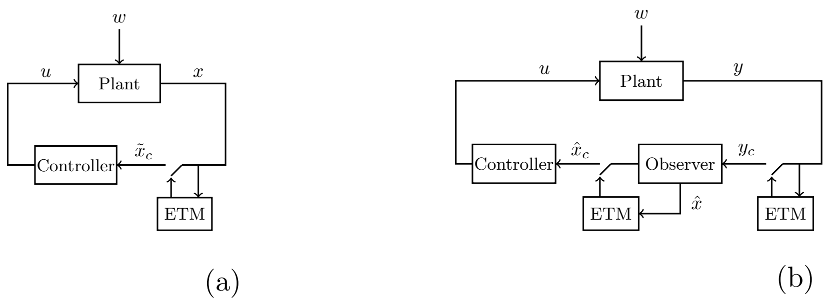

Figure 7.1: (a) Configuration of the state feedback PETC (b) Configuration of the observer-based output feedback PETC

where $e(t) = x(t) - \tilde{x}_c(t)$ and $\Gamma_x(x(t), e(t))$ is the triggering function that will be determined later. The triggering times $t_0^x, t_1^x, t_2^x, \ldots$ are given by $t_0^x = 0$ and $t_{k+1}^x = \min_{i \in \mathbb{Z}_{\geq 0}}\{ih \mid ih > t_k^x, \Gamma_x(x(ih), e(ih)) \geq 0\}$. The control input to the plant, $u(t)$, is given as

$$u(t) = k(\tilde{x}_c(t)). \tag{7.3}$$

Fig.7.1 (b) shows the configuration of implementing the observer-based output feedback PETC, where ETMs exist in both the sensing and actuation channels. The plant is given in (7.1) and the output is

$$y(t) = g(x(t)) \tag{7.4}$$

where $y \in \mathbb{R}^{n_y}$ and g is a continuous function. The observer is

$$\dot{\hat{x}} = \varphi(\hat{x}, u, y) \tag{7.5}$$

where $\hat{x} \in \mathbb{R}^{n_x}$, φ is a continuously differentiable function, and the observer-based controller is given as $u(t) = k(\hat{x}(t))$ where $k(\cdot)$ is a continuous function. Assume that $\varphi(\cdot)$ and $k(\cdot)$ are designed for (7.1) and (7.4) such that without ETMs, the solution to the closed-loop system exists for all time and all initial conditions, $\hat{x}$ asymptotically converges to x when $w = 0$, and the system (7.1) implementing the controller $u(t)$ is ISS w.r.t. w. When the ETMs are implemented, the output of the plant, $y(t)$, is sampled at each sampling time t_k. The input to the observer, $y_c(t)$, is updated only when the event-triggering condition for the output is satisfied. Specifically, $y_c(t)$ is a left-continuous, piecewise constant signal that is defined for $t \in (t_k, t_{k+1}]$ as

$$y_c(t) = \begin{cases} y(t_k), & \text{if } \Gamma_y(y(t_k), y_e(t_k)) \geq 0, \\ y_c(t_k), & \text{if } \Gamma_y(y(t_k), y_e(t_k)) < 0, \end{cases} \tag{7.6}$$

where $y_e(t) = y_c(t) - y(t)$ and $\Gamma_y(y(t), y_e(t))$ is the triggering function of the output that will be determined later.

The triggering times $t_0^y, t_1^y, t_2^y, \dots$ are given by $t_0^y = 0$ and $t_{k+1}^y = \min_{i \in \mathbb{Z}_{\geq 0}}\{ih \mid ih > t_k^y, \Gamma_y(y(ih), y_e(ih)) \geq 0\}$. Under ETMs, the observer (7.5) becomes

$$\dot{\hat{x}} = \varphi(\hat{x}, u, y_c). \tag{7.7}$$

The input to the plant, $u(t)$, is updated only when the event-triggering condition for the input is satisfied. Specifically, define a left-continuous, piecewise constant signal $\hat{x}_c(t)$ for $t \in (t_k, t_{k+1}]$ as

$$\hat{x}_c(t) = \begin{cases} \hat{x}(t_k), & \text{if } \Gamma_u(\hat{x}(t_k), x_e(t_k)) \geq 0, \\ \hat{x}_c(t_k), & \text{if } \Gamma_u(\hat{x}(t_k), x_e(t_k)) < 0, \end{cases} \tag{7.8}$$

where $x_e(t) = \hat{x}_c(t) - \hat{x}(t)$ and $\Gamma_u(\hat{x}(t), x_e(t))$ is the triggering function of the input that will be determined later. The triggering times $t_0^u, t_1^u, t_2^u, \dots$ are given by $t_0^u = 0$ and $t_{k+1}^u = \min_{i \in \mathbb{Z}_{\geq 0}}\{ih \mid ih > t_k^u, \Gamma_u(\hat{x}(ih), x_e(ih)) \geq 0\}$.

Then the control input to the plant, $u(t)$, is given as

$$u(t) = k(\hat{x}_c(t)). \tag{7.9}$$

Systems that are implemented with ETMs are impulsive systems, which evolve continuously based on ODEs most of the time and exhibit impulses at some instances. Clearly, for systems implemented with ETMs, the impulses happen when the triggering conditions are met. Inspired by [76], the input-to-state stability of impulsive systems w.r.t. a given set is defined below.

Definition 4. *Consider the following impulsive system*

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)), t \in (T_i, T_{i+1}], \\ x^+(t) = g(x(t), u(t)), t = T_i, \end{cases} \tag{7.10}$$

where f is locally Lipschitz, $i \in \mathbb{Z}_{\geq 0}$, $\{T_0, T_1, T_2, \dots\}$ is a sequence of impulsive times with $T_0 < T_1 < \dots$, the state $x(t) \in \mathbb{R}^n$ is absolutely continuous between impulses, $u(t) \in \mathbb{R}^m$ is a locally bounded Lebesgue-measurable input, and $x^+(t) := \lim_{s \to t^+} x(s)$. Given a time sequence $\{T_i\}$, the impulsive system (7.10) *is ISS w.r.t. a given non-empty and closed set $\mathcal{A}$ if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$, such that for every initial condition $x(T_0)$ and every admissible input u, the solution to* (7.10) *exists globally and satisfies*

$$\|x(t)\|_{\mathcal{A}} \leq \beta(\|x(T_0)\|_{\mathcal{A}}, t - T_0) + \gamma(\|u\|_{[T_0, T]}) \tag{7.11}$$

where $\|\cdot\|_I$ denotes the supremum norm on an interval I. The impulsive system (7.10) *is* uniformly ISS *w.r.t. $\mathcal{A}$ over a given class $\mathcal{S}$ of admissible sequences of impulse times if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ that are independent of the choice of the time sequence, such that* (7.11) *holds for every time sequence in $\mathcal{S}$.*

In the following, the closed-loop system implemented with ETMs is called *uniformly ISS*, or just ISS for short, w.r.t. a given (non-empty and closed) set $\mathcal{A}$, if it is uniformly ISS over all impulsive times generated by the periodic event-triggering mechanisms. It should be noted that the impulsive times generated by the periodic event-triggering mechanisms have no accumulation point (i.e. Zeno phenomenon is avoided) since the inter-execution times are lower bounded by the sampling period.

The PETC design problems that will be investigated are the following:

1. Given the configuration in Fig.7.1 (a), design the sampling period h *and the triggering function* $\Gamma_x(x, e)$ *such that the resulting closed-loop system is ISS w.r.t.* w*;*

2. Given the configuration in Fig.7.1 (b), design the sampling period h *and the triggering functions* $\Gamma_y(y, y_e)$ *and* $\Gamma_u(\hat{x}, x_e)$ *such that the resulting closed-loop system is ISS w.r.t.* w*.*

7.2 Input-to-state Stabilization Using PETC

In this section, the input-to-state stabilization of disturbed nonlinear systems will be investigated. One key technique that will be used is from the emulation approach, which has been widely used to analyze the stability property of a system under sampling [92, 111]. Computation of the *maximum allowable sampling period* in the emulation approach was investigated in [4, 37, 112], which will be also used later to determine the sampling period h in the PETC design.

Lemma 11. *[37] Let* $\phi : [0, \tilde{\mathcal{T}}] \to \mathbb{R}$ *be the solution of the following ODE:*

$$\dot{\phi} = -2\mu\phi - \gamma(\phi^2 + 1) \tag{7.12}$$

with $\phi(0) = \lambda^{-1}, 0 < \lambda < 1,\ \mu > 0, \gamma > 0,$ *and*

$$\tilde{\mathcal{T}}(\mu, \gamma, \lambda) = \begin{cases} \frac{1}{\mu r} arctan\left(\frac{r(1-\lambda)}{2\frac{\lambda}{1+\lambda}(\frac{\gamma}{\mu}-1)+1+\lambda}\right), & \gamma > \mu, \\ \frac{1}{\mu}\frac{1-\lambda}{1+\lambda}, & \gamma = \mu, \\ \frac{1}{\mu r} arctanh\left(\frac{r(1-\lambda)}{2\frac{\lambda}{1+\lambda}(\frac{\gamma}{\mu}-1)+1+\lambda}\right), & \gamma < \mu, \end{cases} \tag{7.13}$$

$$r = \sqrt{\left|\left(\frac{\gamma}{\mu}\right)^2 - 1\right|}. \tag{7.14}$$

Then, $\phi(\tau) \in [\lambda, \lambda^{-1}]$ *for all* $\tau \in [0, \tilde{\mathcal{T}}]$*, and* $\phi(\tilde{\mathcal{T}}) = \lambda$*.*

With r given in (7.14), define $\mathcal{T}(\mu,\gamma)$ as

$$\mathcal{T}(\mu,\gamma)=\begin{cases}\frac{1}{\mu r}arctan(r), & \gamma>\mu,\\ \frac{1}{\mu}, & \gamma=\mu,\\ \frac{1}{\mu r}arctanh(r), & \gamma<\mu.\end{cases} \tag{7.15}$$

Remark 8. *Clearly, $\mathcal{T}(\mu,\gamma)$ and $\tilde{\mathcal{T}}(\mu,\gamma,\lambda)$ are both positive, and $\mathcal{T}(\mu,\gamma)=\tilde{\mathcal{T}}(\mu,\gamma,0)$. Furthermore, for fixed μ,γ, $\tilde{\mathcal{T}}(\mu,\gamma,\cdot)$ is a strictly decreasing function, and $\tilde{\mathcal{T}}(\mu,\gamma,\lambda)\to 0$ as $\lambda\to 1$.*

7.2.1 State Feedback PETC Design

Consider the configuration in Fig.7.1 (a) where the plant is (7.1) and the state feedback controller is (7.3). Define $\tau\in\mathbb{R}_{\geq 0}$ as the clock variable and

$$x_s(t)=\begin{pmatrix}x(t)\\ e(t)\\ \tau(t)\end{pmatrix}, x_s^+=\begin{pmatrix}x(t^+)\\ e(t^+)\\ \tau(t^+)\end{pmatrix}$$

. The closed-loop system with the ETM in Fig.7.1 (a) is expressed as an impulsive model as follows:

$$\dot{x}_s=F_s(x,e,w):=\begin{pmatrix}\tilde{f}_s(x,e,w)\\ \tilde{f}_s(x,e,w)\\ 1\end{pmatrix}, \quad t\in(t_k,t_{k+1}], \tag{7.16}$$

$$x_s^+=G_s(x,e):=\begin{pmatrix}x\\ g_s(x,e)\\ 0\end{pmatrix}, \qquad t=t_k, \tag{7.17}$$

where

$$\tilde{f}_s(x, e, w) = f(x, k(x - e), w),$$
$$g_s(x, e) = \begin{cases} 0, & \text{if } \Gamma_x(x, e) \geq 0, \\ e, & \text{if } \Gamma_x(x, e) < 0. \end{cases}$$

Theorem 14. *Consider the configuration shown in Fig.7.1 (a) where the plant is* (7.1) *and the controller is* (7.3)*. Suppose that there exist positive numbers* μ, γ, α, d*, and a differentiable, positive definite, radially unbounded function* $V_1(x) : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$ *such that* $\forall x_s$ *a.e.,* $\forall w$*,*

$$\nabla V(x_s) F_s(x, e, w) \leq -\alpha V(x_s) + d\|w\|^2 \tag{7.18}$$

where $V(x_s) = V_1(x) + V_2(e, \tau)$*,* $V_2(e, \tau) = \phi(\tau) e^\top e$*,* ϕ *is the solution of ODE* (7.12)*. Choose positive numbers* α_0, s, h, λ *satisfying* $\alpha_0 < \alpha$*,* $\lambda < 1$ *and*

$$\frac{\ln(1+s)}{\alpha_0} < h < \mathcal{T}(\mu, \gamma), \tag{7.19}$$
$$h = \tilde{\mathcal{T}}(\mu, \gamma, \lambda), \tag{7.20}$$
$$(1+s)\lambda^2 < 1, \tag{7.21}$$

where $\tilde{\mathcal{T}}(\mu, \gamma, \lambda)$ *and* $\mathcal{T}(\mu, \gamma)$ *are defined in* (7.13) *and* (7.15)*. Let the initial condition of* ϕ *be* $\phi(0) = \lambda^{-1}$*. If the triggering function is chosen as*

$$\Gamma_x(x, e) = (\lambda^{-1} - (1+s)\lambda)\|e\|^2 - sV_1(x), \tag{7.22}$$

then the closed-loop system (7.16)-(7.17) *is ISS w.r.t. the set* $\{(x, e, \tau) | (x, e) = (0, 0)\}$*.*

Proof. By Lemma 11, $\phi(\tau) \in [\lambda, \lambda^{-1}]$ for any $\tau \in [0, h]$, and $\phi(h) = \lambda$. Because V_1 and V_2 are both positive definite, the function V is positive definite w.r.t. x and e (i.e., $V(x_s) \geq 0$ for any $x, e \in \mathbb{R}^{n_x}$, and $V(x_s) = 0$ when $x = e = 0$, $V(x_s) \neq 0$ otherwise). Furthermore, $V(x_s)$ is differentiable and radially unbounded for any $x, e \in \mathbb{R}^{n_x}$.

During the continuous dynamics when $t \in (t_k, t_{k+1}]$, inequality (7.18) implies

$$V(x_s(t)) \geq \frac{d}{\alpha - \alpha_0}\|w(t)\|^2$$
$$\Rightarrow \dot{V}(x_s(t)) \leq -\alpha_0 V(x_s(t)),\ \forall t \in (t_k, t_{k+1}]\ a.e. \tag{7.23}$$

where $\dot{V}(x_s)$ is the derivative of V along (7.16).

At the impulse time when $t = t_k$, there are two cases. Note that $(1+s)\lambda^2 < 1$ implies $\lambda^{-1} - (1+s)\lambda > 0$.

i. If $\Gamma_x(x,e) < 0$, the triggering condition is not met. Since $\Gamma_x(x,e) < 0$ implies $\lambda^{-1}\|e\|^2 < (1+s)\lambda\|e\|^2 + sV_1(x)$, it holds that $V(x_s^+) = V_1(x) + \lambda^{-1}\|e\|^2 < (1+s)V(x_s)$.

ii. If $\Gamma_x(x,e) \geq 0$, the triggering condition is met. Then from (7.17) and since $e(t_k^+) = 0$, it holds that $V(x_s^+) = V_1(x) \leq V(x_s)$.

In summary, at the impulse time when $t = t_k$,

$$V(x_s^+) \leq (1+s)V(x_s) = e^{\ln(1+s)}V(x_s). \tag{7.24}$$

Then a bound for $V(x_s(t))$ can be shown using (7.23) and (7.24) as follows. Clearly, there exists a sequence of times $t_0 := \hat{t}_0 \leq \check{t}_1 < \hat{t}_1 < \check{t}_2 < \hat{t}_2 \dots$ such that

$$V(x_s(t)) \geq \frac{d}{\alpha - \alpha_0}\|w\|^2_{[t_0,t]},\ \forall t \in (\hat{t}_j, \check{t}_{j+1}], \tag{7.25}$$
$$V(x_s(t)) \leq \frac{d}{\alpha - \alpha_0}\|w\|^2_{[t_0,t]},\ \forall t \in (\check{t}_i, \hat{t}_i], \tag{7.26}$$

where $j = 0, 1, 2, \dots$ and $i = 1, 2, \dots$. Now consider the case when the first interval $(t_0, \check{t}_1]$ is non-empty, i.e., $\check{t}_1 > t_0$. If $\check{t}_1 < \infty$, then between any two consecutive impulses $t_{k-1}, t_k \in (t_0, \check{t}_1]$, from (7.23) and (7.25), it follows that $\dot{V}(x_s(t)) \leq -\alpha_0 V(x_s(t))$, $\forall t \in (t_{k-1}, t_k]$ a.e., which implies that $V(x_s(t_k)) \leq e^{-\alpha_0 h}V(x_s(t_{k-1}))$. From (7.24), it follows that $V(x_s(t_k^+)) \leq$

$e^{\ln(1+s)}V(x_s(t_k))$. Therefore, for any $t \in (t_0, \check{t}_1]$, it holds that

$$\begin{aligned} V(x_s(t)) &\leq e^{\ln(1+s)\frac{t-t_0}{h}} e^{-\alpha_0(t-t_0)} V(x_s(t_0)) \\ &= e^{\frac{\ln(1+s)-\alpha_0 h}{h}(t-t_0)} V(x_s(t_0)). \end{aligned} \tag{7.27}$$

If $\check{t}_1 = \infty$, then it is easy to see that (7.27) holds for any $t \in (t_0, \infty)$. Note that $\frac{\ln(1+s)-\alpha_0 h}{h} < 0$ by the choice of h in (7.19). Next, consider the case when $t \geq \check{t}_1$. For any subinterval $(\check{t}_i, \hat{t}_i], i \geq 1$ where $\hat{t}_i < \infty$, inequality (7.26) holds. If $\hat{t}_i$ is not an impulse time, then (7.26) holds for $t = \hat{t}_i$. If $\hat{t}_i$ is an impulse time, then (7.24) implies that

$$V(x_s(\hat{t}_i^+)) \leq e^{\ln(1+s)} \frac{d}{\alpha - \alpha_0} \|w\|^2_{[t_0, \hat{t}_i]}. \tag{7.28}$$

In either case, inequality (7.28) holds. For any subinterval $(\check{t}_i, \hat{t}_i]$, $i \geq 1$, where $\hat{t}_i = \infty$, it is easy to see that inequality (7.28) also holds. In summary, (7.28) holds for any subinterval $(\check{t}_i, \hat{t}_i], i \geq 1$.

For any subinterval $(\hat{t}_j, \check{t}_{j+1}], j \geq 1$, using the same argument that derives (7.27), the following inequality holds for any $t \in (\hat{t}_j, \check{t}_{j+1}]$:

$$\begin{aligned} V(x_s(t)) &\leq e^{\frac{\ln(1+s)-\alpha_0 h}{h}(t-\hat{t}_i)} V(x_s(\hat{t}_i)) \\ &\leq e^{\ln(1+s)} \frac{d}{\alpha - \alpha_0} \|w\|^2_{[t_0, \hat{t}_j]}. \end{aligned} \tag{7.29}$$

Combing (7.27), (7.28) and (7.29), it holds that

$$V(x_s(t)) \leq \max\{e^{\frac{\ln(1+s)-\alpha_0 h}{h}(t-t_0)} V(x_s(t_0)), e^{\ln(1+s)} \frac{d}{\alpha - \alpha_0} \|w\|^2_{[t_0, t]}\}$$

, $\forall t \geq t_0$. Since $e^{\frac{\ln(1+s)-\alpha_0 h}{h}(t-t_0)}$ is a strictly decreasing function for $t \geq t_0$, and V is positive definite and radially unbounded for any $x, e \in \mathbb{R}^{n_x}$, by the standard argument for ISS it can be concluded that (7.11) holds with the set $\mathcal{A} := \{(x, e, \tau) | (x, e) = (0, 0)\}$. □

Remark 9. *The procedure of choosing parameters in Theorem 14 is the following: find* μ, γ, α, d *satisfying* (7.18) *at first, then choose* α_0, h, s *satisfying* (7.19), *and finally choose*

λ by (7.20)-(7.21)*. The sampling period h is related to the system dynamics through μ, γ,* (7.19) *and* (7.20)*.*

Whenever μ, γ, α satisfying (7.18) *are found, α_0, s, h, λ satisfying* (7.19)-(7.21) *always exist. Specifically, since* $\ln(1+s) \to 0$ *as* $s \to 0^+$*, there always exist α_0, s, h satisfying* (7.19)*. Because $\mathcal{T}(\mu,\gamma)$ and $\tilde{\mathcal{T}}(\mu,\gamma,0)$ have the properties stated in Remark 8, there always exists λ satisfying* (7.20)*. If* (7.21) *does not hold with such s and λ, then it is always possible to find a smaller s such that* (7.21) *holds, while still guaranteeing that* (7.19) *holds. Therefore, α_0, s, h, λ can always be found. On the other hand, the choices of s, h, α_0 will affect the triggering frequencies of the ETM (i.e., the chance $\Gamma_x(x,e) \geq 0$) by changing $\Gamma_x(x,e)$. For instance, if h is chosen, which implies λ is fixed, then a smaller s will render the triggering condition* (7.22) *easier to be met, which will tend to increase the triggering frequency, while a larger s will tend to decrease the triggering frequency; if s is fixed, then a smaller h will results in a larger λ, which will tend to decrease the triggering frequency, while a larger h will tend to increase the triggering frequency.*

Remark 10. *A set of Lyapunov-based sufficient conditions for the input-to-state stability of impulsive systems were given in [76, 45]. Particularly, when the continuous dynamics are exponentially ISS but the impulses are destabilizing, it was shown in [76] that the impulsive system is uniformly ISS if some average dwell-time condition is satisfied, which was relaxed to be a generalized average dwell-time condition in [45]. These important results rely on the existence of (exponential) ISS-Lyapunov functions. Part of the proof of Theorem 14 is inspired by Theorem 1 in [76], and the sampling period h in PETC design is a lower bound for the dwell-time. The dwell-time constraint imposed in* (7.19) *is related to the triggering condition* (7.22) *through the variable s as s gives the largest allowable increase of the Lyapunov function throughout the jump, as can be seen in* (7.24)*; so the triggering condition is designed to ensure that if the Lyapunov function does increase during the jumps the decrease in the Lyapunov function during the flows is enough to counteract that increase. Hence, s*

cannot be taken to be arbitrarily large due to the upper bound imposed on the sampling time $\mathcal{T}(\mu, \gamma)$ as well as the decay rate of the Lyapunov function during the flows α.

7.2.2 Output Feedback PETC Design

Consider the configuration in Fig.7.1 (b) where the plant is (7.1), the output is (7.4), the observer is (7.7) and the observer-based output feedback controller is (7.9). Define

$$\hat{e}(t) = x(t) - \hat{x}(t), \xi(t) = \begin{pmatrix} x(t) \\ \hat{e}(t) \end{pmatrix}, \eta(t) = \begin{pmatrix} y_e(t) \\ x_e(t) \end{pmatrix}.$$

Define τ as a clock variable, and

$$x_o(t) = \begin{pmatrix} \xi(t) \\ \eta(t) \\ \tau(t) \end{pmatrix}, x_o^+ = \begin{pmatrix} \xi(h^+) \\ \eta(h^+) \\ \tau(h^+) \end{pmatrix}.$$

Then the closed-loop system with ETMs in Fig.7.1 (b) is expressed as an impulsive model as follows:

$$\dot{x}_o = F_o(\xi, \eta, w) := \begin{pmatrix} \tilde{f}_o^1(\xi, \eta, w) \\ \tilde{f}_o^2(\xi, \eta, w) \\ 1 \end{pmatrix}, \ t \in (t_k, t_{k+1}], \tag{7.30}$$

$$x_o^+ = G_o(\xi, \eta) := \begin{pmatrix} \xi \\ g_o(\xi, \eta) \\ 0 \end{pmatrix}, \qquad t = t_k, \tag{7.31}$$

where

$$\tilde{f}_o^1(\xi,\eta,w) = \begin{pmatrix} f(x,k(\hat{x}_c),w) \\ f(x,k(\hat{x}_c),w) - \varphi(\hat{x},k(\hat{x}_c),w) \end{pmatrix},$$
$$\tilde{f}_o^2(\xi,\eta,w) = \begin{pmatrix} \nabla g(x) \cdot f(x,k(\hat{x}_c),w) \\ -f(x,k(\hat{x}_c),w) + \varphi(\hat{x},k(\hat{x}_c),w) \end{pmatrix},$$
$$g_o(\xi,\eta) = \begin{pmatrix} g_o^1(\xi,\eta) \\ g_o^2(\xi,\eta) \end{pmatrix},$$
$$g_o^1(\xi,\eta) = \begin{cases} 0, \text{ if } \Gamma_y(y,y_e) \geq 0, \\ y_e, \text{if } \Gamma_y(y,y_e) < 0, \end{cases}$$
$$g_o^2(\xi,\eta) = \begin{cases} 0, \text{ if } \Gamma_u(\hat{x},x_e) \geq 0, \\ x_e, \text{if } \Gamma_u(\hat{x},x_e) < 0. \end{cases}$$

Theorem 15. *Consider the configuration shown in Fig.7.1 (b) where the plant, output, observer and controller are given by* (7.1)*,* (7.4)*,* (7.7) *and* (7.9)*, respectively. Suppose that there exist positive numbers $\mu_1, \mu_2, \gamma_1, \gamma_2$, c_1, c_2, α, d, and differentiable, positive definite, radially unbounded functions $V_1(\xi) : \mathbb{R}^{2n_x} \to \mathbb{R}_{\geq 0}$, $V_3(y) : \mathbb{R}^{n_y} \to \mathbb{R}_{\geq 0}$ and $V_4(\hat{x}) : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$ such that $\forall x_o$ a.e., $\forall w$,*

$$\nabla V(x_o) F_o(\xi,\eta,w) \leq -\alpha V(x_o) + d\|w\|^2, \tag{7.32}$$
$$c_1 V_3(g(x)) + c_2 V_4(\hat{x}) \leq V_1(\xi), \tag{7.33}$$

where $V(x_o) = V_1(\xi) + V_2(\eta,\tau)$, $V_2(\eta,\tau) = c_1\phi_1 y_e^\top y_e + c_2\phi_2 x_e^\top x_e$, and $\phi_i (i = 1,2)$ is the solution of ODE $\dot{\phi}_i = -2\mu_i\phi_i - \gamma_i(\phi_i^2+1)$. Choose positive numbers $\alpha_0, s, h, \lambda_1, \lambda_2$ satisfying

$\alpha_0 < \alpha$, $\lambda_1 < 1$, $\lambda_2 < 1$, and

$$\frac{\ln(1+s)}{\alpha_0} < h < \min\{\mathcal{T}(\mu_1,\gamma_1), \mathcal{T}(\mu_2,\gamma_2)\}, \tag{7.34}$$

$$h = \tilde{\mathcal{T}}(\mu_1,\gamma_1,\lambda_1), \quad h = \tilde{\mathcal{T}}(\mu_2,\gamma_2,\lambda_2), \tag{7.35}$$

$$(1+s)\lambda_1^2 < 1, \quad (1+s)\lambda_2^2 < 1, \tag{7.36}$$

where $\tilde{\mathcal{T}}(\mu,\gamma,\lambda)$ and $\mathcal{T}(\mu,\gamma)$ are defined in (7.13) *and* (7.15)*. Let the initial condition of ϕ_i be $\phi_i(0) = \lambda_i^{-1}$ for $i = 1, 2$. If the triggering functions are chosen as*

$$\Gamma_y(y, y_e) = (\lambda_1^{-1} - (1+s)\lambda_1)\|y_e\|^2 - sV_3(y), \tag{7.37}$$

$$\Gamma_u(\hat{x}, x_e) = (\lambda_2^{-1} - (1+s)\lambda_2)\|x_e\|^2 - sV_4(\hat{x}), \tag{7.38}$$

then the closed-loop system (7.30)-(7.31) *is ISS w.r.t. the set $\{(x,\hat{e},\tau)|(x,\hat{e}) = (0,0)\}$.*

Proof. By Lemma 11, $\phi_i(\tau) \in [\lambda_i, \lambda_i^{-1}]$ for any $\tau \in [0,h]$, and $\phi_i(h) = \lambda_i$, $i = 1, 2$. Because V_1, V_2 are both positive definite, the function V is positive definite w.r.t. ξ and η. (i.e., $V(x_o) \geq 0$ for any $\xi \in \mathbb{R}^{2n_x}, \eta \in \mathbb{R}^{n_x+n_y}$, and $V(x_o) = 0$ when $\xi = \eta = 0$, $V(x_o) \neq 0$ otherwise). Furthermore, $V(x_o)$ is differentiable and radially unbounded for any ξ, η.

During the continuous dynamics when $t \in (t_k, t_{k+1}]$, the inequality (7.32) holds. Hence,

$$\begin{aligned} & V(x_o(t)) \geq \frac{d}{\alpha - \alpha_0}\|w(t)\|^2 \\ \Rightarrow\ & \dot{V}(x_o(t)) \leq -\alpha_0 V(x_o(t)), \ \forall t \in (t_k, t_{k+1}] \ a.e. \end{aligned} \tag{7.39}$$

where $\dot{V}(x_o)$ is the derivative of V along (7.30).

At the impulse time when $t = t_k$, there are four cases regarding satisfaction of the input and output triggering conditions. Note that $(1+s)\lambda_i^2 < 1$ implies $\lambda_i^{-1} - (1+s)\lambda_i > 0$, for $i = 1, 2$.

i. If $\Gamma_y(y, y_c) < 0$ and $\Gamma_u(\hat{x}, x_e) < 0$, the output and input triggering conditions are not met. Since $\Gamma_y(y, y_c) < 0$, $\lambda_1^{-1}\|y_e\|^2 < (1+s)\lambda_1\|y_e\|^2 + sV_3(y)$; since $\Gamma_u(\hat{x}, x_e) < 0$, $\lambda_2^{-1}\|x_e\|^2 <$

$(1+s)\lambda_2\|x_e\|^2 + sV_4(\hat{x})$. Therefore, $V(x_o^+) = V_1(\xi) + c_1\lambda_1^{-1}\|y_e\|^2 + c_2\lambda_2^{-1}\|x_e\|^2 < V_1(\xi) + c_1(1+s)\lambda_1\|y_e\|^2 + c_1 sV_3(y) + c_2(1+s)\lambda_2\|x_e\|^2 + c_2 sV_4(\hat{x}) \leq (1+s)V(x_o)$.

ii. If $\Gamma_y(y, y_e) < 0$ and $\Gamma_u(\hat{x}, x_e) \geq 0$, then $V(x_o^+) = V_1(\xi) + c_1\lambda_1^{-1}\|y_e\|^2 < V_1(\xi) + c_1(1+s)\lambda_1\|y_e\|^2 + c_1 sV_3(y) \leq (1+s)V(x_o)$.

iii. If $\Gamma_y(y, y_e) \geq 0$ and $\Gamma_u(\hat{x}, x_e) < 0$, then $V(x_o^+) = V_1(\xi) + c_2\lambda_2^{-1}\|x_e\|^2 < V_1(\xi) + c_2(1+s)\lambda_2\|x_e\|^2 + c_2 sV_4(\hat{x}) \leq (1+s)V(x_o)$.

iv. If $\Gamma_y(y, y_e) \geq 0$ and $\Gamma_u(\hat{x}, x_e) \geq 0$, then $V(x_o^+) = V_1(\xi) \leq V(x_o)$.

In summary, at the impulse time when $t = t_k$,

$$V(x_o^+) \leq (1+s)V(x_o) = e^{\ln(1+s)}V(x_o). \tag{7.40}$$

From (7.39) and (7.40), the same argument as in the proof of Theorem 14 can be used to show that $V(x_o(t)) \leq \max\{e^{\frac{\ln(1+s)-\alpha_0 h}{h}(t-t_0)}V(x_o(t_0)), e^{\ln(1+s)}\frac{d}{\alpha-\alpha_0}\|w\|^2_{[t_0,t]}\}$, $\forall t \geq t_0$. Since $e^{\frac{\ln(1+s)-\alpha_0 h}{h}(t-t_0)}$ is a strictly decreasing function for $t \geq t_0$, and V is positive definite and radially unbounded for any ξ, η, by the standard argument for ISS, it can be concluded that the closed-loop system (7.30)-(7.31) is ISS w.r.t. the set $\{(\xi, \eta, \tau)|(\xi, \eta) = (0, 0)\}$, and therefore, it is ISS w.r.t. the set $\{(x, \hat{e}, \tau)|(x, \hat{e}) = (0, 0)\}$. □

Remark 11. *Discussion similar to that in Remark 9 also holds for the output feedback case. In particular, when parameters $\mu_1, \mu_2, \gamma_1, \gamma_2, c_1, c_2, \alpha, d$ are found, numbers $\alpha_0, s, h, \lambda_1, \lambda_2$ satisfying* (7.34)-(7.36) *always exist.*

7.3 PETC for Incrementally Quadratic Systems

Functions $V_i (i = 1, 2, 3, 4)$ that satisfy the conditions of Theorem 14 or Theorem 15 are not easy to find in general. In this section, those sufficient conditions will be presented in the form of LMIs, which can be solved by convex program solvers, for incrementally quadratic nonlinear systems. Therefore, the sampling period and the triggering functions can be obtained constructively and systematically for a general class of nonlinear systems.

Suppose that the plant in Fig.7.1 (a) and Fig.7.1 (b) is an incrementally quadratic nonlinear system given as

$$\begin{cases} \dot{x} = Ax + Bu + Ep(q) + E_w w, \\ q = C_q x, \end{cases} \tag{7.41}$$

where $x \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^{n_u}$ is the control input, $p : \mathbb{R}^{n_q} \to \mathbb{R}^{n_p}$ is a function representing the known nonlinearity, $w \in \mathbb{R}^{n_w}$ is the unknown external disturbance, and $A \in \mathbb{R}^{n_x \times n_x}, B \in \mathbb{R}^{n_x \times n_u}, C_q \in \mathbb{R}^{n_q \times n_x}, E \in \mathbb{R}^{n_x \times n_p}, E_w \in \mathbb{R}^{n_x \times n_w}$ are constant matrices with proper sizes. The characterization of p is based on the incremental multiplier matrix defined below.

Definition 5. *Given a function $p : \mathbb{R}^{n_q} \to \mathbb{R}^{n_p}$, a symmetric matrix $M \in \mathbb{R}^{(n_q+n_p)\times(n_q+n_p)}$ is called an* incremental multiplier matrix *[7, 44] for p if it satisfies the following* incremental quadratic constraint*:*

$$\begin{pmatrix} \delta q \\ \delta p \end{pmatrix}^\top M \begin{pmatrix} \delta q \\ \delta p \end{pmatrix} \geq 0, \quad \forall q_1, q_2 \in \mathbb{R}^{n_q}, \tag{7.42}$$

where $\delta q = q_2 - q_1$, $\delta p = p(q_2) - p(q_1)$.

Given a nonlinearity p, its incremental multiplier matrix that satisfies (7.42) is not unique. Particularly, if M is an incremental multiplier matrix for p, then λM is also a incremental multiplier matrix for p for any $\lambda > 0$. Assume that p satisfies $p(0_{n_q}) = 0_{n_p}$ in the following, which implies that $\begin{pmatrix} q \\ p \end{pmatrix}^\top M \begin{pmatrix} q \\ p \end{pmatrix} \geq 0, \ \forall q \in \mathbb{R}^q$.

7.3.1 State Feedback PETC Design For Incrementally Quadratic Nonlinear Systems

Consider the configuration in Fig.7.1 (a) where the full-state information is available. Suppose that the plant is given as (7.41)-(7.42) and the controller is $u = K_1 x + K_2 p(C_q x)$ where

$K_1 \in \mathbb{R}^{n_u \times n_x}, K_2 \in \mathbb{R}^{n_u \times n_p}$. In the following, matrices K_1, K_2 are assumed to be chosen such that the closed-loop system in Fig.7.1 (a) without ETM is ISS (e.g., by using the results of [155]). With ETMs in Fig.7.1 (a), the control input to the plant is given as

$$u(t) = K_1 \tilde{x}_c(t) + K_2 p(C_q \tilde{x}_c(t)) \tag{7.43}$$

where $\tilde{x}_c$ is defined in (7.2). The closed-loop system in Fig.7.1 (a) is expressed in the form of (7.16)-(7.17) with $\tilde{f}_s(x, e, w) = (A + BK_1)x - BK_1 e + (E + BK_2)p + BK_2 \delta\tilde{p} + E_w w$, where $\delta\tilde{p} = p(q + \delta\tilde{q}) - p(q)$, $\delta\tilde{q} = -C_q e$.

Theorem 16. *Consider the configuration in Fig.7.1 (a) where the plant is* (7.41)-(7.42) *and the control input is* (7.43). *Given $\alpha > 0$, suppose that there exist positive numbers μ, γ, d, non-negative numbers σ_1, σ_2, matrix $P \in \mathbb{R}^{n_x \times n_x}$ where $P = P^\top \succ 0$, such that the following LMI holds:*

$$\begin{pmatrix} \Psi & -PBK_1 & P(E+BK_2) & PBK_2 & (A+BK_1)^\top & PE_w \\ * & -\gamma I & 0 & 0 & -(BK_1)^\top + (\frac{\alpha}{2} - \mu) I & 0 \\ * & * & 0 & 0 & (E+BK_2)^\top & 0 \\ * & * & * & 0 & (BK_2)^\top & 0 \\ * & * & * & * & -\gamma I & E_w \\ * & * & * & * & * & -dI \end{pmatrix}$$
$$+\sigma_1 S_1^\top M S_1 + \sigma_2 S_2^\top M S_2 \preceq 0, \tag{7.44}$$

where Ψ, S_1, S_2 are given as

$$\begin{cases} \Psi = P(A + BK_1) + (A + BK_1)^\top P + \alpha P, \\ S_1 = \begin{pmatrix} C_q, 0_{n_q \times (2n_x + 2n_p + n_w)} \\ 0_{n_p \times 2n_x}, I_{n_p}, 0_{n_p \times (n_x + n_p + n_w)} \end{pmatrix}, \\ S_2 = \begin{pmatrix} 0_{n_q \times n_x}, -C_q, 0_{n_q \times (n_x + 2n_p + n_w)} \\ 0_{n_p \times (2n_x + n_p)}, I_{n_p}, 0_{n_p \times (n_x + n_w)} \end{pmatrix}. \end{cases} \tag{7.45}$$

Choose positive numbers α_0, s, h, λ satisfying $\alpha_0 < \alpha$, $\lambda < 1$ and $\frac{log(1+s)}{\alpha_0} < h < \mathcal{T}(\mu,\gamma)$, $h = \tilde{\mathcal{T}}(\mu,\gamma,\lambda)$, $(1+s)\lambda^2 < 1$. If the triggering function is chosen as $\Gamma_x(x,e) = (\lambda^{-1} - (1+s)\lambda)\|e\|^2 - sx^\top Px$, then the closed-loop system in Fig.7.1 (a) is ISS w.r.t. the set $\{(x,e,\tau)|(x,e)=(0,0)\}$.

Theorem 16 shows that h and $\Gamma_x(x,e)$ can be constructed for the state feedback case by solving LMI (7.44).

7.3.2 Observer-based Output Feedback PETC Design For Incrementally Quadratic Nonlinear Systems

Consider the configuration in Fig.7.1 (b) where the measured output information is available. The plant is (7.41)-(7.42) and the output is $y = Cx$ where $y \in \mathbb{R}^{n_y}$ and $C \in \mathbb{R}^{n_y \times n_x}$. Suppose that the observer is

$$\begin{cases} \dot{\hat{x}} = A\hat{x}+Bu+Ep(\hat{q}+L_1(\hat{y}-y))+L_2(\hat{y}-y), \\ \hat{y} = C\hat{x}, \\ \hat{q} = C_q\hat{x}, \end{cases} \tag{7.46}$$

with $L_1 \in \mathbb{R}^{n_q \times n_y}$, $L_2 \in \mathbb{R}^{n_x \times n_y}$, and the controller is

$$u(t) = K_1\hat{x}(t) + K_2 p(C_q\hat{x}(t)). \tag{7.47}$$

In the following, matrices L_1, L_2, K_1, K_2 are assumed to be chosen such that the closed-loop system in Fig.7.1 (b) without ETMs is ISS (e.g., by using the results of [155]). With ETMs in Fig.7.1 (b), the observer becomes

$$\begin{cases} \dot{\hat{x}} = A\hat{x}+Bu+Ep(\hat{q}+L_1(\hat{y}-y_c))+L_2(\hat{y}-y_c), \\ \hat{y} = C\hat{x}, \\ \hat{q} = C_q\hat{x}. \end{cases} \tag{7.48}$$

The observer-based controller now becomes

$$u(t) = K_1\hat{x}_c(t) + K_2 p(C_q\hat{x}_c(t)). \tag{7.49}$$

Then the closed-loop system in Fig.7.1 (b) is expressed in the form of (7.30)-(7.31) with

$$\tilde{f}_o^1(\xi,\eta,w) = A_1\xi + A_2\eta + H_1 p + H_2\delta\check{p} + H_3\delta\hat{p} + H_4 w,$$
$$\tilde{f}_o^2(\xi,\eta,w) = A_3\xi + A_4\eta + H_5 p + H_6\delta\check{p} + H_7\delta\hat{p} + H_8 w,$$

where $\delta\hat{p} = p(q+\delta\hat{q}) - p(q)$, $\delta\hat{q} = C_q(x_e - \hat{e})$, $\delta\check{p} = p(q+\delta\check{q}) - p(q)$, $\delta\check{q} = -(C_q + L_1 C)\hat{e} - L_1 y_e$, and

$$A_1 = \begin{pmatrix} A+BK_1 & -BK_1 \\ 0 & A+L_2C \end{pmatrix}, A_4 = \begin{pmatrix} 0 & -CBK_1 \\ L_2 & -BK_1 \end{pmatrix},$$
$$A_2 = \begin{pmatrix} 0 & BK_1 \\ L_2 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} -C(A+BK_1) & CBK_1 \\ -(A+BK_1) & A+BK_1+L_2C \end{pmatrix},$$
$$H_1 = \begin{pmatrix} E+BK_2 \\ 0 \end{pmatrix}, H_2 = \begin{pmatrix} 0 \\ -E \end{pmatrix}, H_3 = \begin{pmatrix} BK_2 \\ 0 \end{pmatrix},$$
$$H_4 = \begin{pmatrix} E_w \\ E_w \end{pmatrix}, H_5 = \begin{pmatrix} -C(E+BK_2) \\ -(E+BK_2) \end{pmatrix}, H_6 = \begin{pmatrix} 0 \\ -E \end{pmatrix},$$
$$H_7 = \begin{pmatrix} -CBK_2 \\ -BK_2 \end{pmatrix}, H_8 = \begin{pmatrix} -CE_w \\ 0 \end{pmatrix}.$$

Theorem 17. *Consider the configuration in Fig.7.1 (b) where the plant is* (7.41)-(7.42)*, the output is* $y = Cx$*, the observer is* (7.48)*, and the control input is* (7.49)*. Given* $\alpha > 0$*, suppose that there exist positive numbers* $\mu_1, \mu_2, a_1, a_2, b_1, b_2, d, \sigma_1, \sigma_2, \sigma_3$*, and matrix* $P \in \mathbb{R}^{2n_x \times 2n_x}$*,*

$P = P^\top \succ 0$, such that the following LMI holds:

$$\begin{pmatrix} PA_1 + A_1^\top P & PA_2 & PH_1 & PH_2 & A_3^\top \\ * & R_1 & 0 & 0 & (A_4 + R_3)^\top \\ * & * & 0 & 0 & H_3^\top \\ * & * & * & 0 & H_4^\top \\ * & * & * & * & R_2 \end{pmatrix} + \sigma_0 S_0^\top P_0 S_0 + \sigma_1 S_1^\top M S_1 + \sigma_2 S_2^\top M S_2 \preceq 0 \tag{7.50}$$

where $R_1, R_2, R_3, S_1, S_2, S_3$ are given as the following:

$$\begin{cases} R_1 = \begin{pmatrix} -a_1 I_{n_y} & 0 \\ 0 & -a_2 I_{n_x} \end{pmatrix}, \; R_2 = \begin{pmatrix} -b_1 I_{n_y} & 0 \\ 0 & -b_2 I_{n_x} \end{pmatrix}, \; R_3 = \begin{pmatrix} -\mu_1 I_{n_y} & 0 \\ 0 & -\mu_2 I_{n_x} \end{pmatrix}, \\ S_1 = \begin{pmatrix} C_q, 0_{n_q \times (3n_x + 2n_y + 3n_p + n_w)} \\ 0_{n_p \times (3n_x + n_y)}, I_{n_p}, 0_{n_p \times (n_x + n_y + 2n_p + n_w)} \end{pmatrix}, \\ S_2 = \begin{pmatrix} 0_{n_q \times n_x}, -(C_q + L_1 C), -L_1, 0_{n_q \times (2n_x + n_y + 3n_p + n_w)}, \\ 0_{n_p \times (3n_x + n_y + n_p)}, I_{n_p}, 0_{n_p \times (n_x + n_y + n_p + n_w)} \end{pmatrix}, \\ S_3 = \begin{pmatrix} 0_{n_q \times n_x}, -C_q, 0_{n_q \times n_y}, C_q, 0_{n_q \times (n_x + n_y + 3n_p + n_w)} \\ 0_{n_p \times (3n_x + n_y + 2n_p)}, I_{n_p}, 0_{n_p \times (n_x + n_y + n_w)} \end{pmatrix}. \end{cases} \tag{7.51}$$

Suppose that there exist matrices $P_1 \in \mathbb{R}^{n_y \times n_y}, P_1 = P_1^\top \succ 0$, $P_2 \in \mathbb{R}^{n_x \times n_x}, P_2 = P_2^\top \succ 0$, such that the following LMI (7.52) *holds:*

$$\begin{pmatrix} c_1 C^\top P_1 C & 0 \\ 0 & c_2 P_2 \end{pmatrix} \preceq \begin{pmatrix} I_{n_x} & I_{n_x} \\ 0 & -I_{n_x} \end{pmatrix} P \begin{pmatrix} I_{n_x} & 0 \\ I_{n_x} & -I_{n_x} \end{pmatrix} \tag{7.52}$$

where $c_1 = \sqrt{a_1/b_1}$, $c_2 = \sqrt{a_2/b_2}$. Choose positive numbers $\alpha_0, s, h, \lambda_1, \lambda_2$ satisfying $\alpha_0 < \alpha$, $\lambda_1 < 1$, $\lambda_2 < 1$, and $\frac{log(1+s)}{\alpha_0} < h < \min\{\mathcal{T}(\mu_1, \gamma_1), \mathcal{T}(\mu_2, \gamma_2)\}$, $h = \tilde{\mathcal{T}}(\mu_1, \gamma_1, \lambda_1)$,$h =$

$\tilde{\mathcal{T}}(\mu_2,\gamma_2,\lambda_2)$, $(1+s)\lambda_1^2<1$,$(1+s)\lambda_2^2<1$, *where* $\gamma_1=\sqrt{a_1b_1}$, $\gamma_2=\sqrt{a_2b_2}$. *If the triggering functions are chosen as*

$$\Gamma_y(y,y_e)=(\lambda_1^{-1}-(1+s)\lambda_1)\|y_e\|^2-sy^\top P_1y,$$

$$\Gamma_u(\hat{x},x_e)=(\lambda_2^{-1}-(1+s)\lambda_2)\|x_e\|^2-s\hat{x}^\top P_2\hat{x},$$

then the closed-loop system in Fig.7.1 (b) is ISS w.r.t. the set $\{(x,\hat{e},\tau)|(x,\hat{e})=(0,0)\}$.

Proof. Define $V(x_o)=V_1(\xi)+V_2(\eta,\tau)$ where $V_1(x)=\xi^\top P\xi$, $V_2(\eta,\tau)=c_1\phi_1y_e^\top y_e+c_2\phi_2x_e^\top x_e$, x_o is defined in Subsec. 7.2.2, and ϕ_i is the solution of ODE $\dot{\phi}_i=-2\mu_i\phi_i-\gamma_i(\phi_i^2+1)$ with the initial condition $\phi_i(0)=\lambda_i^{-1}$, for $i=1,2$. Define $V_3(y)=y^\top P_1y$ and $V_4(\hat{x})=\hat{x}^\top P_2\hat{x}$. It is easy to see that if (7.32) and (7.33) hold during the flow (i.e., when $t\in(t_k,t_{k+1}]$), then all the conditions of Theorem 15 hold with Γ_u,Γ_y given in (7.37)-(7.38), and the conclusion follows immediately. Define $\varrho=\begin{pmatrix}\varrho_y\\ \varrho_x\end{pmatrix}:=\begin{pmatrix}c_1\phi_1y_e\\ c_2\phi_2x_e\end{pmatrix}$ and $\zeta=(\xi^\top,\eta^\top,p^\top,\delta\check{p}^\top,\delta\hat{p},\varrho^\top,w^\top)^\top$. Clearly, $\varrho=Q\eta$, which implies that $V_2(\eta,\tau)=\eta^\top\varrho$. During the flow (7.30), $\langle\nabla V(x_o),F_o(\xi,\eta,w)\rangle=\frac{\partial V_1}{\partial\xi}\tilde{f}_o^1(\xi,\eta,w)+\frac{\partial V_2}{\partial\eta}\tilde{f}_o^2(\xi,\eta,w)+\eta^\top\frac{\partial Q}{\partial\tau}\eta=2\xi^\top P(A_1\xi+A_2\eta+H_1p+H_2\delta\check{p}+H_3\delta\hat{p}+H_4w)+2\varrho^\top(A_3\xi+A_4\eta+H_5p+H_6\delta\check{p}+H_7\delta\hat{p}+H_8w)+\eta^\top R_1\eta+\varrho^\top R_2\varrho+2\eta^\top R_3\varrho$. Noting that $\begin{pmatrix}q\\ p\end{pmatrix}=S_1\zeta$, $\begin{pmatrix}\delta\check{q}\\ \delta\check{p}\end{pmatrix}=S_2\zeta$, $\begin{pmatrix}\delta\hat{q}\\ \delta\hat{p}\end{pmatrix}=S_3\zeta$, it hold that $\sigma_1\zeta^\top S_1^\top MS_1\zeta\geq0$, $\sigma_2\zeta^\top S_2^\top MS_2\zeta\geq0$, $\sigma_3\zeta^\top S_3^\top MS_3\zeta\geq0$. Multiplying the left-hand side and the right-hand side of (7.50) by $\zeta^\top$ and ζ, respectively, it follows that $2\xi^\top P(A_1\xi+A_2\eta+H_1p+H_2\delta\check{p}+H_3\delta\hat{p}+H_4w)+2\varrho^\top(A_3\xi+A_4\eta+H_5p+H_6\delta\check{p}+H_7\delta\hat{p}+H_8w)+\eta^\top R_1\eta+\varrho^\top R_2\varrho+2\eta^\top R_3\varrho+\alpha\xi^\top P\xi+\alpha\eta^\top\varrho-d\|w\|^2+\sigma_1\zeta^\top S_1^\top MS_1\zeta+\sigma_2\zeta^\top S_2^\top MS_2\zeta+\sigma_3\zeta^\top S_3^\top MS_3\zeta\leq0$. Therefore, it is easy to obtain that $\langle\nabla V(x_o),F_o(\xi,\eta,w)\rangle\leq-\alpha(\xi^\top P\xi+\eta^\top\varrho)+d\|w\|^2=-\alpha V(x_o)+d\|w\|^2$. Therefore, (7.32) holds during the flow. Since $\xi=\begin{pmatrix}I_{n_x}&0\\ I_{n_x}&-I_{n_x}\end{pmatrix}\begin{pmatrix}x\\ \hat{x}\end{pmatrix}$, multiplying $\begin{pmatrix}x\\ \hat{x}\end{pmatrix}^\top$ and its transpose to the left-hand side and the right-hand side of (7.52), respectively, it follows that $c_1 x^\top C^\top P_1 C x + c_2 \hat{x}^\top P_2 \hat{x} \leq \xi^\top P \xi$, which is equivalent to $c_1 y^\top P_1 y + c_2 \hat{x}^\top P_2 \hat{x} \leq \xi^\top P \xi$. Therefore, (7.52) implies that (7.33) holds with $V_3 = y^\top P_1 y, V_4 = \hat{x}^\top P_2 \hat{x}$. □

7.3.3 Special Case: Linear Control Systems

PETC design for continuous-time linear systems was investigated in [72]. By letting $E = 0$, dynamics of (7.41) becomes a linear system $\dot{x} = Ax + Bu + E_w w$, for which results in preceding subsections can be applied directly.

For the configuration in Fig.7.1 (a), suppose that the state feedback controller implemented with ETM is $u(t) = K\tilde{x}_c(t)$ where $K \in \mathbb{R}^{n_u \times n_x}$ and $\tilde{x}_c$ is defined in (7.2). Then the conditions of Theorem 16 becomes finding positive numbers μ, γ, d, and a matrix $P = P^\top \succ 0$ such that

$$\begin{pmatrix} \Psi & -PBK & (A+BK)^\top & PE_w \\ * & -\gamma I & -(BK)^\top + (\frac{\alpha}{2} - \mu)I & 0 \\ * & * & -\gamma I & E_w \\ * & * & * & -dI \end{pmatrix} \preceq 0$$

where $\Psi = P(A+BK) + (A+BK)^\top P + \alpha P$ For the configuration in Fig.7.1 (b), suppose that the output is $y = Cx$ with $C \in \mathbb{R}^{n_y \times n_x}$, the observer is $\dot{\hat{x}} = A\hat{x} + Bu + L(C\hat{x} - y_c)$ where $L \in \mathbb{R}^{n_x \times n_y}$, and the controller is $u(t) = K\hat{x}_c(t)$ where $\hat{x}_c$ is defined in (7.8). Then the conditions in Theorem (17) becomes finding positive numbers $\mu_1, \mu_2, a_1, a_2, b_1, b_2, d$, non-negative numbers $\sigma_1, \sigma_2, \sigma_3$, and a matrix $P = P^\top \succ 0$ such that

$$\begin{pmatrix} \Psi & PA_2 & A_3^\top & PH_4 \\ * & R_1 & A_4^\top + R_3^\top + \frac{\alpha}{2} I & 0 \\ * & * & R_2 & H_8 \\ * & * & * & -dI \end{pmatrix} \preceq 0$$

where $\Psi = PA_1 + A_1^\top P + \alpha P$, $A_i (i = 1, 2, 3, 4)$ and H_4, H_8 are given in the preceding subsection, $R_i (i = 1, 2, 3)$ are given in (7.51).

7.4 Numerical Example

Consider the following dynamical model of the single-link robot arm given in [4]:

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -\sin(x_1) + u + w,\\ y &= x_1.\end{aligned}$$

The system can be written in the form of (7.41) with

$$A=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, B=\begin{pmatrix}0\\ 1\end{pmatrix}, C=(1,0), E=\begin{pmatrix}0\\ -1\end{pmatrix}, E_w=\begin{pmatrix}0\\ 1\end{pmatrix},$$

$q = x_1,\ C_q = (1,0),\ p(q) = \sin(q)$. The nonlinearity p is globally Lipschitz and satisfies the incremental quadratic constraint (7.42) with the following multiplier matrix:

$$M = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

Consider the configuration in Fig.7.1 (b). Assume that the continuous-time observer is (7.46) and the control input is (7.47). By the results of [155], the gains

$$\begin{aligned}K_1 &= \begin{pmatrix}-7.3936 & -3.9937\end{pmatrix},\ K_2 = 1,\\ L_1 &= -1, L_2 = \begin{pmatrix}-5.1294\\ -18.0352\end{pmatrix}\end{aligned}$$

are chosen. By letting $\alpha = 1.1$, the LMI (7.50) in Theorem 17 is solved, which yields the values of $a_1, a_2, b_1, b_2, \mu_1, \mu_2, d, \gamma_1, \gamma_2, c_1, c_2$, from which $\mathcal{T}(\mu_1,\gamma_1) = 0.0751, \mathcal{T}(\mu_2,\gamma_2) = 0.0639$. The LMI (7.52) is solved to obtain the following matrices

$$P_1 = 0.1462,\ P_2 = \begin{pmatrix}0.6307 & 0.1195\\ 0.1195 & 0.1434\end{pmatrix}.$$

Choose $h = 0.02, s = 0.02, \lambda_1 = 0.627, \lambda_2 = 0.575, \alpha_0 = 1$. From Theorem 17, the triggering functions are chosen as follows:

$$\Gamma_y(y, y_e) = 0.9554\|y_e\|^2 - 0.02y^\top P_1 y,$$
$$\Gamma_u(\hat{x}, x_e) = 1.1526\|x_e\|^2 - 0.02\hat{x}^\top P_2 \hat{x}.$$

Figure 7.2 shows a simulation of the derived observer-based periodic event-triggered controller where the initial states are $x_1(0) = -0.2, x_2(0) = 0.6, \hat{x}_1(0) = -0.3, \hat{x}_2(0) = 0.7$, and the disturbance is generated randomly and satisfies $\|w\|_\infty \leq 0.05$. It can be seen that $x_1, x_2, \hat{e}$ all eventually go to a neighbourhood of the origin in the presence of disturbances.

Now the decrease in the amount of triggering of the periodic event-triggered controller against a time-triggered implementation is investigated. For different choices of sampling time h, 100 simulations from random initial conditions are performed, and the average frequency of triggering of the observer and controller (f^y_{avg}, f^u_{avg}, respectively) are computed. Table 7.1 shows the average triggering frequencies of the observer and controller for different choices of h. It is clear that f^y_{avg} and f^u_{avg} increase when h increases.

h	$0.005s$	$0.01s$	$0.015s$	$0.02s$
f^y_{avg}	22.3%	40.0%	51.2%	59.9%
f^u_{avg}	16.1%	26.1%	32.1%	38.0%

Table 7.1: Average triggering frequencies f^y_{avg}, f^u_{avg} based on 100 simulations.

In addition to the triggering frequencies, T^y_{avg} and T^u_{avg} which are the average inter-execution times for the output and the input ETMs during the simulations, respectively, are computed. There is no clear trend to suggest what the best choice of sampling time h

would be to decrease triggering overall; however, there is clearly an improvement against time-triggered implementations.

h	$0.005s$	$0.01s$	$0.015s$	$0.02s$
T^y_{avg}	$0.028s$	$0.022s$	$0.021s$	$0.023s$
T^u_{avg}	$0.037s$	$0.033s$	$0.032s$	$0.034s$

Table 7.2: Average inter-execution times T^y_{avg}, T^u_{avg} based on 100 simulations.

7.5 Conclusions and Suggestions for Future Work

The periodic event-triggered control design for nonlinear systems subject to disturbances was investigated. The periodic event-triggered control was modeled as an impulsive system. Sufficient conditions were proposed to ensure the resulting closed-loop system is input-to-state stable using either a state feedback or an observer-based output feedback controller. LMI-based sufficient conditions for the PETC design of incrementally quadratic nonlinear systems were also proposed. For all the cases considered, the sampling period and the triggering functions were given explicitly. Future research will be devoted towards decreasing the amount of triggering further and using dynamic event-triggering functions (cf. [26, 63]).

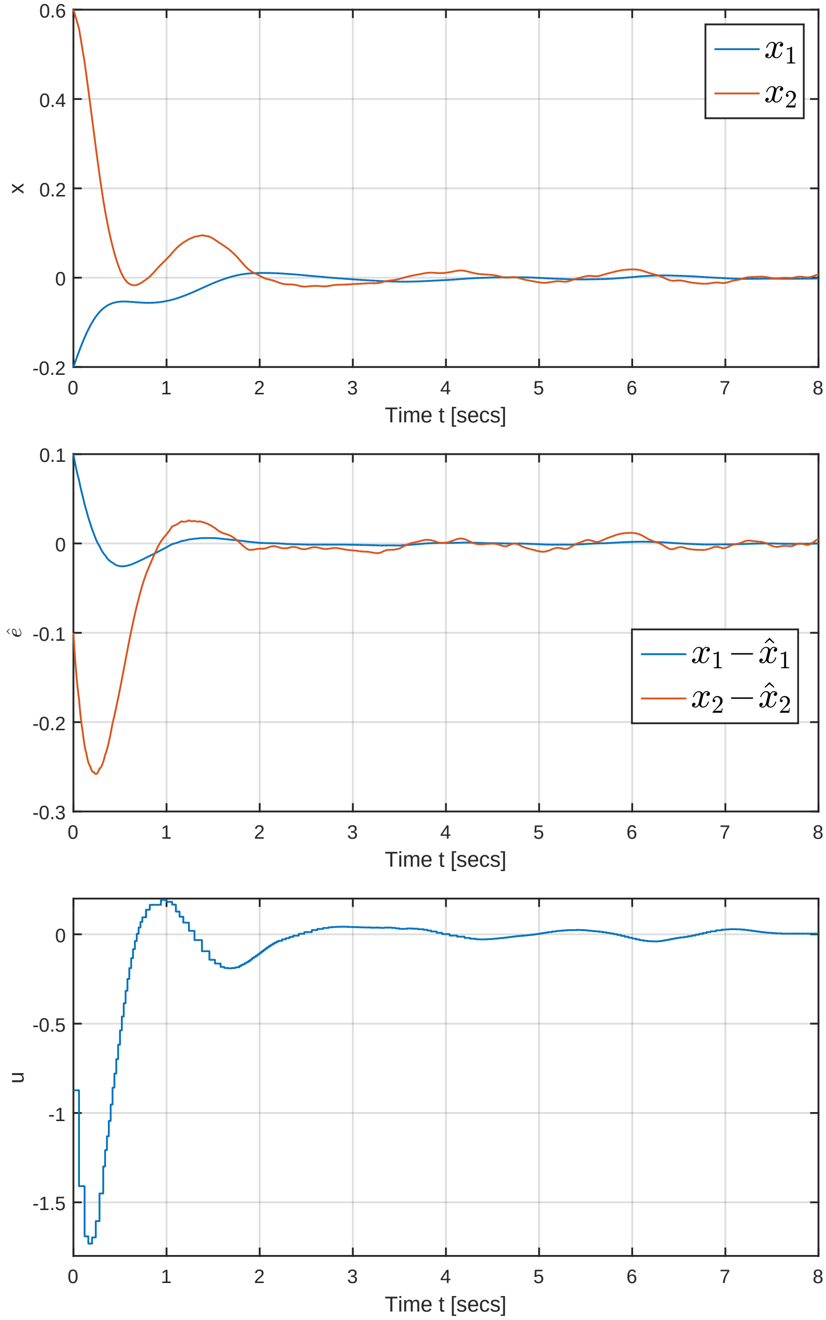

Figure 7.2: Trajectories of $x, \hat{x}, u$ in the simulation of single-link robot arm.

Part III

NONAFFINE CONTROL OF COULOMB SPACECRAFT FORMATIONS

Chapter 8

CONSTRUCTIVE NONLINEAR CONTROL

> "A career in flying was like climbing one of those ancient Babylonian pyramids made up of a dizzy progression of steps and ledges, a ziggurat, a pyramid extraordinarily high and steep; and the idea was to prove at every foot of the way up that pyramid that you were one of the elected and anointed ones who had *the right stuff*"
>
> *The Right Stuff*
>
> Tom Wolfe

This chapter reviews important concepts and proposes new results which will be used in the following chapter to derive controllers for Coulomb formations.

Notation The Lie derivative of a function V with respect to the vector field f is denoted as $L_f V$. The standard notation $\mathrm{ad}_f^k g$ is used for the Lie bracket.

8.1 Passivity

8.1.1 Definition

Consider a system, Σ_0,

$$\begin{aligned}\dot{x} &= f(x,u)\\ y &= h(x,u)\end{aligned} \tag{8.1}$$

where $x \in \mathbb{R}^n$, and $y, u \in \mathbb{R}^m$. Define dissipativity and passivity of Σ_0 [154],

Definition 6 (Dissipative System). Σ_0 *is dissipative if* $\exists V : \mathbb{R}^n \to \mathbb{R}$ *such that* $V(x) > 0$, $V(0) = 0$ *that satisfies the inequality,*

$$V(x) - V(x_0) \le \int_0^t w(y(s), u(s))ds, \;\; \forall t \ge 0, x \in \mathbb{R}^n \tag{8.2}$$

where V is referred to as a storage function*, and the integrand of the RHS,* $w(y,u)$*, is referred to as the* supply rate*.*

Definition 7 (Passive System). Σ_0 *is passive if it is dissipative with the supply rate* $w(u,y) = y^T u$*.*

Consider an example of a resistor inductor capacitor (RLC) circuit which has a voltage input u and a current output y. The power supplied to the RLC circuit is the product of the voltage input and the current output, uy. The storage function V is the energy stored in the inductor and the capacitor. If the inequality (8.2) holds, then the difference between the energy supplied and the energy stored is the energy that is dissipated by the resistor. If the inequality is an equality, then the energy supplied is equal to the energy that is stored, and the system is referred to as *lossless*.

8.1.2 Stability of Passive Systems

In an unforced passive system ($u = 0$) with a positive definite storage function, the origin is stable. This can be seen from the differential form of the passivity inequality,

$$\dot{V} \leq y^T u = 0 \tag{8.3}$$

From Hill and Moylan [78], the necessary condition for asymptotic stabilization using output feedback in the form,

$$u = -\phi(y) \tag{8.4}$$

where ϕ is a sector nonlinearity such that $y^T \phi(y) \geq 0$ for all $y \in \mathbb{R}^m$ is that the system must be zero-state detectible, which is defined as follows,

Definition 8 (Zero-State Detectability)**.** *Σ_0 is zero-state detectable if in an unforced system ($u = 0$) where $h(x(t), 0) = 0$ for all $t \geq 0$, $x(t) \to 0$ as $t \to \infty$.*

The closely related concept of zero-state observability is defined as such,

Definition 9 (Zero-State Observability)**.** *Σ_0 is zero-state observable if in an unforced system ($u = 0$) where $h(x(t), 0) = 0$ for all $t \geq 0$, $x(t) = 0$ for all $t \geq 0$.*

The following proposition relates the vector fields and storage function of the system to its detectability,

Proposition 7 (3.4 in Byrnes et al. [36])**.** *If Σ_0 is passive with a proper C^r, $r \geq 1$, storage function $V(x)$, then Σ is zero-state detectable if $S \cap \Omega = \{0\}$. Σ_0 is zero-state observable if it is lossless and $S = \{0\}$. Where,*

$$S = \left\{ x \in X \,\middle|\, L_f^k L_\tau V(x) = 0 \;\forall \tau \in \Delta, 0 \leq k \leq r \right\} \tag{8.5}$$

$$\Omega = \left\{ x \in X \,\middle|\, L_f^k V(x) = 0 \;\forall 1 \leq k \leq r \right\} \tag{8.6}$$

Δ is the distribution defined in Equation (8.7),

$$\Delta = span \left\{ ad_f^k g_i \,|0 \leq k \leq n-1, 1 \leq i \leq m \right\} \tag{8.7}$$

8.1.3 *Feedback Equivalence*

As system that is not passive can potentially be made passive by using state feedback. Consider the following theorem from Byrnes et al.[36],

Theorem 18 (4.7 in Byrnes et al. [36])**.** *If $x = 0$ is a regular point for Σ, then Σ is locally feedback equivalent to a passive system with a C^2 storage function V if and only if Σ has relative degree $\{1, \ldots, 1\}$ and is weakly minimum phase.*

The proof is available in Byrnes et al. [36]. What follows is the construction of the feedback law for feedback equivalence if the equations are expressed in a special case of the normal form[1] based on the derivation from Sepulchre et al. [136],

$$\begin{aligned} \dot{z} &= q(z) + p(z,\eta)\eta \\ \dot{\eta} &= a(z,\eta) + b(z,\eta)u \\ y &= \eta \end{aligned} \tag{8.8}$$

Since the system is weakly minimum phase, $\exists$ a positive definite function W for the zero dynamics[2] $\dot{z} = q(z)$ such that $L_{q(z)}W \leq 0$. So,

$$\dot{W}(z) = L_{q(z)}W + L_{p(z,\eta)}W\eta \leq L_{p(z,\eta)}W\eta \tag{8.9}$$

Now consider the positive definite storage function,

$$V(z,\eta) = W(z) + \frac{1}{2}\eta^T\eta \tag{8.10}$$

Now take the time derivative,

$$\begin{aligned} \dot{V}(z,\eta) &= \dot{W}(z) + \eta^T\dot{\eta} = L_{q(z)}W + L_{p(z,\eta)}W\eta + \eta^T\left(a(z,\eta) + b(z,\eta)u\right) \\ &\leq L_{p(z,\eta)}W\eta + \eta^T\left(a(z,\eta) + b(z,\eta)u\right) \end{aligned} \tag{8.11}$$

[1]The procedure is similar in Byrnes et al.[36] for general control-affine systems but with a couple extra coordinate transformations.

[2]If the system is zero state detectable, $z = 0$ is asymptotically stable for the zero dynamics $\dot{z} = q(z)$.

use the control input,

$$u(\eta, z) = b^{-1}(z,\eta)(-a(z,\eta) - (L_{p(z,\eta)}W)^T + v) \tag{8.12}$$

Note that $b^{-1}(z,\eta)$ exists by the the relative degree property. By a routine calculation,

$$\dot{V}(z,\eta) \leq \eta^T v = y^T v \tag{8.13}$$

so the system is passive from $v \to y$ by feedback. Notice that in constructing the feedback term, the storage function is also constructed. Theorem 18 has two conditions on the system; namely, that the relative degree must be $\{1,\ldots,1\}$ and that the system must be weakly minimum phase. These are necessary conditions for a system to be passive [36, 136], and those properties are invariant under state-feedback [107]; therefore, they are necessary to make the system passive by feedback. Moreover, for control-affine systems they are sufficient conditions by construction.

For asymptotic stabilization of the origin, a passive system to be zero-state detectible. The detectability of the system can be changed using state feedback. Consider the following proposition for systems in a particular case of the normal form (8.8) that is of interest in later problems,

Proposition 8. *Consider a system in the form,*

$$\begin{aligned}\dot{z} &= \eta\\ \dot{\eta} &= a(z,\eta) + u\\ y &= \eta\end{aligned} \tag{8.14}$$

that is passive from $u \to y$ with a storage function $\bar{V}(z,\eta) = \bar{V}_z(z) + \bar{V}_\eta(\eta)$ where $\bar{V}_z(z)$ and $\bar{V}_\eta(\eta)$ are C^r, $r \geq 2$, positive definite functions such that $\bar{V}_\eta(0) = 0$ and $\bar{V}_z(0) = 0$. Moreover, $\bar{V}_\eta(\eta)$ is a convex function. $a(z,0) = 0$ for all z. Let $P(z)$ be a C^2 convex positive definite potential function with $P(0) = 0$. With the input,

$$u = -\frac{\partial P^T}{\partial z}(z) + v \tag{8.15}$$

the resulting system is passive from $v \to y$ *with storage function* $V(z,\eta) = \bar{V}(z,\eta) + P(z)$ *that is also zero-state observable.*

Proof. First, show that the resulting system is passive with storage function V by taking a time derivative of V,

$$\dot{V}(z,\eta) = \dot{\bar{V}}(z,\eta) + \dot{P}(z) \tag{8.16}$$

$$\leq u^T y + \frac{\partial P}{\partial z}(z)\dot{z} \tag{8.17}$$

$$= -\frac{\partial P}{\partial z}(z)y + v^T y + \frac{\partial P}{\partial z}(z)\dot{z} = v^T y \tag{8.18}$$

Writing the system in control-affine form,

$$\begin{bmatrix} \dot{z} \\ \dot{\eta} \end{bmatrix} = f(z,\eta) + g(z,\eta)v \tag{8.19}$$

Now the vector fields of the system are,

$$f(z,\eta) = \begin{bmatrix} \eta \\ a(z,\eta) - \frac{\partial P}{\partial z}^T(z) \end{bmatrix} \tag{8.20}$$

and,

$$g(z,\eta) = \begin{bmatrix} 0_{m\times m} \\ I_{m\times m} \end{bmatrix} \tag{8.21}$$

Notice that the only solution for $f(z,\eta) = 0$ is $(z,\eta) = (0,0)$, so using the definition of Ω from Proposition 7, $0 \in \Omega$ since,

$$L_f^k V = \frac{\partial}{\partial(z,\eta)}\left(\frac{\partial}{\partial(z,\eta)}\left(\dots\left(\frac{\partial V(z,\eta)}{\partial(z,\eta)}\right)f(z,\eta)\right)f(z,\eta)\right)f(z,\eta) \tag{8.22}$$

The outer multiplication of $f(z,\eta)$ implies that $(0,0)$ is always a solution to $L_f^k V = 0$. Now consider,

$$L_g V = \frac{\partial V(z,\eta)}{\partial(z,\eta)}g(z,\eta) = \frac{\partial V(z,\eta)}{\partial\eta} = \frac{\partial\bar{V}(z,\eta)}{\partial\eta} + \frac{\partial P(z)}{\partial\eta} = \frac{\partial\bar{V}_\eta(\eta)}{\partial\eta} \tag{8.23}$$

So it is clear that $(0,0)$ is a solution to $L_gV = 0$. Moreover, η must be equal to 0 for $L_gV = 0$ by the convexity of $\bar{V}_\eta(\eta)$. Using (8.22) and (8.23), $(0,0)$ is a solution to $L_{[f,g]}V = L_fL_gV - L_gL_fV = 0$, and, by recursion, that solution is retained for higher Lie brackets in Δ, so $0 \in S$. Now consider,

$$L_fL_gV = \frac{\partial}{\partial(z,\eta)}\left(\frac{\partial \bar{V}_\eta(\eta)}{\partial \eta}\right) f(z,\eta) \tag{8.24}$$

$$= \frac{\partial^2 \bar{V}_\eta(\eta)}{\partial \eta^2}\left(a(z,\eta) - \frac{\partial P^T}{\partial z}(z)\right) \tag{8.25}$$

The only solution that satisfies both $L_gV = 0$ and $L_fL_gV = 0$ is $(0,0)$ by convexity of P. Therefore, $S = \{0\}$ and the system is zero-state observable. □

Arçak [16] uses a similar formulation for a passivity-based approach to a group coordination problem where the potential function is designed to steer the agents to a desired formation under the assumption that all the agents have strictly passive dynamics.

The construction is reminiscent of a gradient-descent algorithm and relies on the convexity of part of the storage function. Notice that the feedback term (8.12) which makes systems passive will guarantee the convexity condition in Proposition 8 because the η-part of the storage function is quadratic and, hence, convex. In fact, by choosing $W(z)$ in (8.12) to be a convex function, zero-state observability along with passivity can be guaranteed as is shown in the following Lemma which will be used for control synthesis later on.

Lemma 12. *Consider a system in the form,*

$$\begin{aligned}\dot{z} &= \eta\\ \dot{\eta} &= a(z,\eta) + b(z,\eta)u\\ y &= \eta\end{aligned} \tag{8.26}$$

where $b(z,\eta)$ is invertible. By applying the feedback (8.12), the system becomes passive from $v \to y$ with storage function $V(z,\eta) = W(z) + \frac{1}{2}\eta^T\eta$. If $W(z)$ is chosen to be a C^2 convex function, then the resulting system is zero-state observable.

Proof. By applying (8.12) the system becomes

$$\begin{aligned}\dot{z} &= \eta \\ \dot{\eta} &= -\frac{\partial W}{\partial z}^T(z) + v \\ y &= \eta\end{aligned} \tag{8.27}$$

and it follows from the proof of Proposition 8 that this system is zero-state observable if $W(z)$ is convex. □

Note that convexity of $W(z)$ is not the only condition that guarantees zero-state observability. For example, for storage functions of the form $W(z) = \log(1 + z^T z)$ also work. Since the zero dynamics are,

$$\dot{z} = 0 \tag{8.28}$$

it is easy to generate many Lyapunov functions $W(z)$ to guarantee zero-state observability for asymptotic stabilization.

8.1.4 Relative Degree of Nonaffine Systems

Consider a nonaffine system, Σ_1 with pure state output,

$$\begin{aligned}\dot{x} &= f(x) + \sum_{i=1}^{m-1}\sum_{j=i+1}^{m} g_{ij}(x)u_i u_j \\ y &= h(x)\end{aligned} \tag{8.29}$$

Theorem 19. Σ_1 *cannot be made passive by feedback in a neighbourhood of* $(x, u) = (0, 0)$.

Proof. Take a time derivative of the output,

$$\dot{y} = L_f h(x) + \sum_{i=1}^{m-1}\sum_{j=i+1}^{m} L_{g_{ij}} h(x) u_i u_j \tag{8.30}$$

Take the kth element of $\dot{y}$ and find the derivative,

$$\frac{\partial \dot{y}}{\partial u_k} = \sum_{i=1, i\neq k}^{m} L_{g_{ik}} h(x) u_i \tag{8.31}$$

$\partial \dot{y}/\partial u_k(0,0) = 0$, so the relative degree is higher than one which means that the system cannot be rendered passive by feedback. □

Let $u_m(x) = \alpha(x)$, then the system Σ_1 would be,

$$\begin{aligned} \dot{x} &= f(x) + \alpha(x) \sum_{i=1}^{m-1} g_{im}(x) u_i + \sum_{i=1}^{m-2} \sum_{j=i+1}^{m-1} g_{ij}(x) u_i u_j \\ y &= h(x) \end{aligned} \tag{8.32}$$

Which can be written compactly as,

$$\begin{aligned} \dot{x} &= f(x) + g_a(x) u + g_{na}(x) r(u) u \\ y &= h(x) \end{aligned} \tag{8.33}$$

where u now is equal to $[u_1, u_2, \ldots, u_{m-1}]^T$. If $\alpha(x)$ is designed such that $L_{g_a}V$ has rank $m-1$ in a neighbourhood of $x = 0$ for a positive definite storage function $V(x)$, then the system would have relative degree $\{1, \ldots, 1\}$, and therefore, could be feedback equivalent to a passive system if it is also weakly minimum phase.

The addition of an affine part means that, for example, the results of Lin [94] could be applied for stabilization if $L_f V \leq 0$. Lin [94] constructs smooth inputs that can be applied to the system (8.32) such that the contribution of the affine vector fields is larger than the the contribution of the nonaffine vector fields and, under a detectability condition, asymptotically stabilizes the origin.

8.2 Centre Manifold Theory

Centre manifold theory is a model reduction tool for systems in the form,

$$\begin{aligned}\dot{\Xi} &= A\Xi + f(\Xi, \Upsilon)\\ \dot{\Upsilon} &= B\Upsilon + g(\Xi, \Upsilon)\end{aligned} \tag{8.34}$$

where the eigenvalues of A have zero real parts, and all the eigenvalues of B have negative real parts. Moreover, $f(0,0), f'(0,0), g(0,0), g'(0,0) = 0$, where f' and g' are the Jacobian matrices of the functions f and g, respectively. Two results from Carr [38] will be used. The first is the conditions for the existence of a centre manifold, and the second is the conditions for stability of the origin when there is a centre manifold.

Theorem 20 (1.1 in Carr[38])**.** *There exists a centre manifold for (8.34),* $\Upsilon = H(\Xi)$ *(*$H(0) = 0$*),* $||\Xi|| < \delta$*, where* H *is* C^2*.*

The flow on the centre manifold is governed by,

$$\dot{\mu} = A\mu + f(\mu, H(\mu)) \tag{8.35}$$

If the flow on the centre manifold is stable, then the entire system is stable,

Theorem 21 (1.2 in Carr [38])**.** *(a) Suppose the zero solution of (8.35) is stable (asymptotically stable) (unstable). Then the zero solution of (8.34) is stable (asymptotically stable) (unstable).*

(b) Suppose the zero solution of (8.35) is stable. Let the initial conditions $(\Xi(0), \Upsilon(0))$ *be sufficiently small. There exists a solution* $\mu(t)$ *of (8.35) such that as* $t \to \infty$*,*

$$\begin{aligned}\Xi(t) &= \mu(t) + \mathcal{O}(e^{-\gamma t})\\ \Upsilon(t) &= H(\mu(t)) + \mathcal{O}(e^{-\gamma t})\end{aligned} \tag{8.36}$$

for $\gamma > 0$.

Chapter 9

COULOMB FORMATION CONTROL

> "Sometimes, flying feels too godlike to be attained by man. Sometimes, the world from above seems too beautiful, too wonderful, too distant for human eyes to see."
>
> *The Spirit of St. Louis*
> CHARLES LINDBERGH

Coulomb's law governs the electrostatic force that acts between charged particles. The magnitude of the force is proportional to the product of the charges, inversely proportional to the square of the distance between the particles, and acts along the line of sight between them. Coulomb formation control is a concept for the control of spacecraft formations which seeks to take advantage of the fact that spacecraft in high orbits accumulate charge and, therefore, exert electrostatic forces, herein referred to as Coulomb forces, on each other. Moreover, it is possible to actively control the charge of a spacecraft by emitting a current from it. So, by changing the charge of the spacecraft, the forces acting on them can be changed and, therefore, be used to control the formation. Active charge control requires vastly less propellant than a solely thruster-based formation control system. Current formation flying missions are constrained by strict propellant requirements, which makes Coulomb formation control a key enabling technology for the future of spacecraft formation flight [134].

There are two major challenges associated with Coulomb formation control. The first of which is the nonaffine dynamics. For two spacecraft Coulomb spacecraft formations the equations of motion have the general form,

$$\dot{x} = f(x) + g(x)u_1u_2 \tag{9.1}$$

which is a nonlinear system that is *biaffine* in u_1 and u_2, the charges of the first and second spacecraft, respectively.

For a two spacecraft formation, the product of the two charges $u = u_1u_2$ can be chosen as the control input for (9.1) to make the system affine in u. Then u_1 and u_2 can be determined from u; however, for formations that have $N > 2$ spacecraft, the equations will have the form,

$$\dot{x} = f(x) + \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} g_{ij}(x)u_iu_j \tag{9.2}$$

Using the same trick would not work as the products of charges cannot be chosen arbitrarily. Consider a three spacecraft example where the equations will have the form,

$$\dot{x} = f(x) + g_1(x)u_1u_2 + g_2(x)u_1u_3 + g_3(x)u_2u_3 \tag{9.3}$$

Now choose the control input as the products of the charges,

$$v = \begin{bmatrix} v_{12} \\ v_{13} \\ v_{23} \end{bmatrix} = \begin{bmatrix} u_1u_2 \\ u_1u_3 \\ u_2u_3 \end{bmatrix} \tag{9.4}$$

and substitute into (9.3) so it becomes affine in v. A realizable control law v must satisfy the following constraint,

$$v_{12}v_{13}v_{23} = (u_1u_2u_3)^2 \geq 0 \tag{9.5}$$

Wang and Schaub [149] derived a switching controller based on the multiple Lyapunov function approach from Branicky [31] where the switching scheme designs inputs to satisfy the constraint (9.5) and ensure that the uncontrolled Lyapunov functions are Lyapunov-like to ensure stability.

In this chapter, smooth feedback control laws for Coulomb formations of three collinear spacecraft where the control inputs u_1, u_2 and u_3 are constructed directly; hence yielding a realizable control without explicitly enforcing the constraint (9.5) through a sequential design methodology where the inputs are designed one at a time.

The methodology begins with a derivation of a control scheme for collinear two spacecraft formations using feedback equivalence. It is shown in Lemma 12 that the feedback law which makes a wide variety of systems (including collinear two Coulomb spacecraft formation dynamics) passive also makes the system zero-state observable by appropriate choice of storage function to guarantee asymptotic stabilization by output feedback.

The second challenge with Coulomb formation control is that the Coulomb actuation only occurs along the line of sight between spacecraft. Therefore, two spacecraft Coulomb formations are *underactuated* in $\mathbb{R}^2$. Natarajan and Schaub proposed that the Coulomb actuation can be augmented by using environmental forces such as gravity gradient in orbit [109] or using thrusters [110]. Their analyses use linearized dynamics for small perturbations. Here, the dynamics are not linearized, and a nonlinear formulation is constructed building off of the control scheme for two collinear spacecraft formations; the thruster input is chosen to be a proportional-derivation (PD) feedback controller similar to Natarajan and Schaub [110]. Asymptotic stability is shown by centre manifold theory.

9.1 Collinear Two Spacecraft Formations

9.1.1 Relative Dynamics

Consider an example of a collinear two spacecraft formation with the goemetry shown in Figure 9.1.

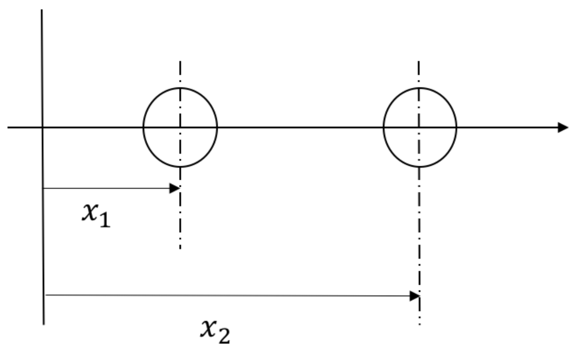

Figure 9.1: Geometry of a collinear two spacecraft formation.

Let u_1 be the charge of the first spacecraft and u_2 be the charge of the second spacecraft. The equations of motion of this system would be, using Coulomb's law,

$$\begin{aligned} \ddot{x}_1 &= \frac{1}{m_1}\frac{1}{4\pi\varepsilon_0}\frac{x_1 - x_2}{||x_1 - x_2||^3}u_1 u_2 \\ \ddot{x}_2 &= \frac{1}{m_2}\frac{1}{4\pi\varepsilon_0}\frac{x_2 - x_1}{||x_1 - x_2||^3}u_1 u_2 \end{aligned} \tag{9.6}$$

where m_1 and m_2 are the masses of the spacecraft. ε_0 is the permitivity of free space. If $m_1 = m_2$, then a constant k_c can be defined,

$$k_c = \frac{1}{m_1}\frac{1}{4\pi\varepsilon_0} \tag{9.7}$$

Define the relative coordinates,

$$\tilde{\xi}_1 = x_1 - x_2$$
$$\tilde{\xi}_2 = x_2 - x_1 \tag{9.8}$$

The relative velocities are similarly defined,

$$\nu_1 = \dot{\tilde{\xi}}_1 = \dot{x}_1 - \dot{x}_2$$
$$\nu_2 = \dot{\tilde{\xi}}_2 = \dot{x}_2 - \dot{x}_1 \tag{9.9}$$

So, the equations of motion in terms of the relative position and velocity are,

$$\dot{\tilde{\xi}}_1 = \nu_1$$
$$\dot{\tilde{\xi}}_2 = \nu_2$$
$$\dot{\nu}_1 = 2k_c \frac{\tilde{\xi}_1}{||\tilde{\xi}_1||^3} u_1 u_2$$
$$\dot{\nu}_2 = 2k_c \frac{\tilde{\xi}_2}{||\tilde{\xi}_2||^3} u_1 u_2 \tag{9.10}$$

Notice that,

$$\tilde{\xi}_1 = -\tilde{\xi}_2 = \tilde{\xi} \tag{9.11}$$

and

$$\nu_1 = -\nu_2 = \nu \tag{9.12}$$

so the equations can be reduced to,

$$\dot{\tilde{\xi}} = \nu$$
$$\dot{\nu} = 2k_c \frac{\tilde{\xi}}{||\tilde{\xi}||^3} u_1 u_2 \tag{9.13}$$

Let ξ^* be the desired relative distance between the two spacecraft, and define,

$$\xi = \tilde{\xi} - \xi^* \tag{9.14}$$

so,

$$\begin{aligned} \dot{\xi} &= \nu \\ \dot{\nu} &= 2k_c \frac{\xi + \xi^*}{||\xi + \xi^*||^3} u_1 u_2 \end{aligned} \tag{9.15}$$

a controller which designs the charges u_1 and u_2 to drive $\xi, \nu \to 0$ would achieve a formation with the desired relative distance ξ^*. Since there are only two spacecraft, the variable substitution $u = u_1 u_2$ can be made

$$\begin{aligned} \dot{\xi} &= \nu \\ \dot{\nu} &= 2k_c \frac{\xi + \xi^*}{||\xi + \xi^*||^3} u. \end{aligned} \tag{9.16}$$

It is assumed that both spacecraft have the same charging capabilities, so, once u is determined, u_1 and u_2 are chosen to have the same magnitude,

$$u_1 = \sqrt{|u|} \tag{9.17}$$

$$u_2 = \operatorname{sgn}(u) \sqrt{|u|} \tag{9.18}$$

9.1.2 Controller Design

The dynamics described in (9.16) are in the special normal form (8.8) if the output is chosen to be,

$$y = \nu \tag{9.19}$$

If $\xi^* \neq 0$, the system can be made passive at $(\xi, \nu) = (0, 0)$. The input which makes the system passive is the following,

$$u = \frac{1}{2k_c} \frac{||\xi + \xi^*||^3}{\xi + \xi^*} (-\xi + v) \tag{9.20}$$

so,

$$\dot{\xi} = \nu$$
$$\dot{\nu} = -\xi + v$$
$$y = \nu \tag{9.21}$$

Which is passive from $v \to y$ with storage function $V(\xi, \nu) = \frac{1}{2}(\xi^2 + \nu^2)$, and it is zero state observable by Lemma 12, so feedback term,

$$v = -\phi(y) \tag{9.22}$$

asymptotically stabilizes the origin if $\phi(y)y > 0$ for all $y \neq 0$. A simulation is shown in Figure 9.2.

9.2 Two Spacecraft Formations in Two Dimensions with Thrusters

The Coulomb force only acts along the line of sight between the two spacecraft, so in a scenario where the two spacecraft can move in $\mathbb{R}^2$ a thruster system can be used to augment the Coulomb actuation. Say that the thrust is constrained to act in the y-direction. The geometry is shown in Figure 9.3 and the dynamics will look like (9.23).

$$\dot{\xi}_x = \nu_x$$
$$\dot{\xi}_y = \nu_y$$
$$\dot{\nu}_x = u_{C_x}$$
$$\dot{\nu}_y = u_{T_y} + u_{C_y} \tag{9.23}$$

where $u_{C_{(.)}}$ is the input due to the Coulomb force, which is the product of charges of the two spacecraft in the formation; $u_{T_{(.)}}$ is the thruster input acceleration. The states of (9.23) are

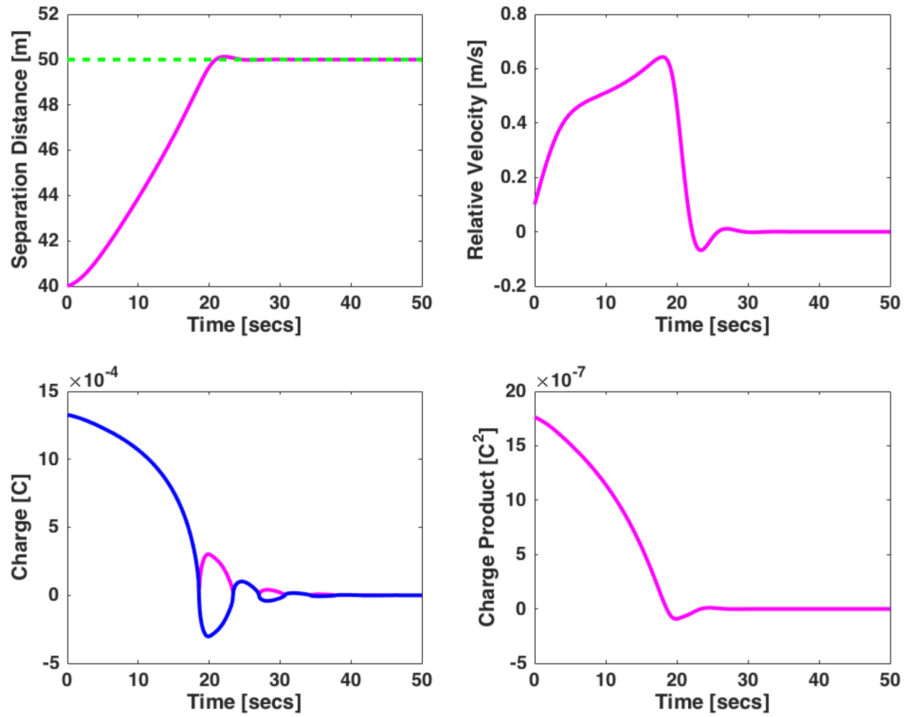

Figure 9.2: A simulation of a collinear two spacecraft formation using solely the Coulomb forces between the spacecraft.

defined similarly to the two-spacecraft formation,

$$\xi_x = \tilde{\xi}_x - \xi_x^* \tag{9.24}$$

$$\xi_y = \tilde{\xi}_y - \xi_y^* \tag{9.25}$$

where $\tilde{\xi}_x$ and $\tilde{\xi}_y$ are the relative distances between the two spacecraft in the x and y directions, respectively (shown in Figure 9.3), and ξ_x^*, ξ_y^* are the desired distances in the$x$ and y directions, respectively. The controller is designed tol drive the system from $(\tilde{\xi}_{x_0}, \tilde{\xi}_{y_0}, \tilde{\nu}_{x_0}, \tilde{\nu}_{y_0}) \rightarrow (\xi_x^*, 0, 0, 0)$.

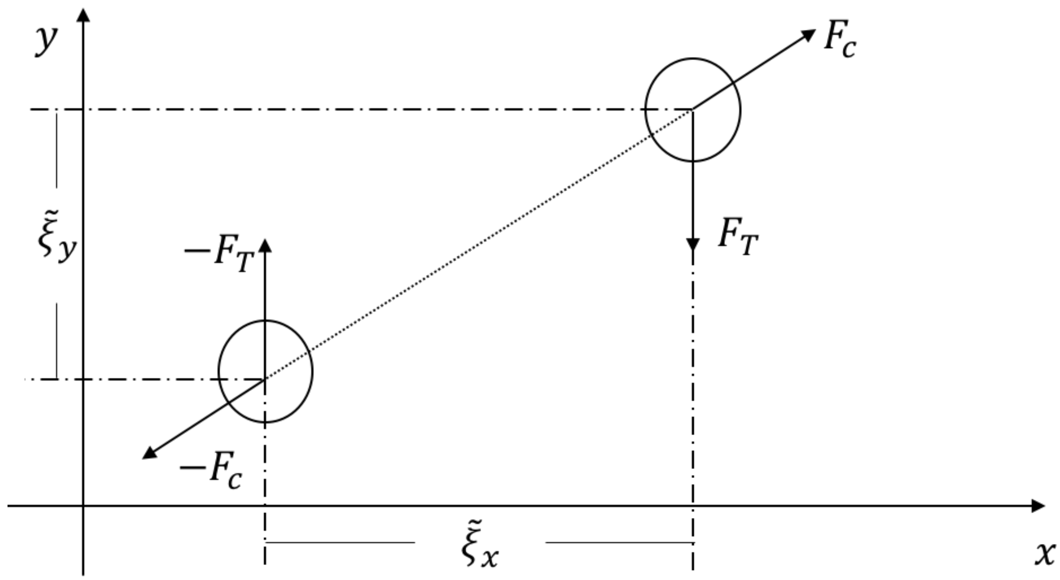

Figure 9.3: Schematic of the two spacecraft formation in two dimensions. The thrust is constrained to act solely in the y-direction for each spacecraft, and the Coulomb force between the two charged spacecraft will act along the line of sight between the two spacecraft.

If the thruster input is chosen to be a linear PD controller, the dynamics from (9.23) would look like,

$$\begin{aligned}
\dot{\xi}_x &= \nu_x \\
\dot{\xi}_y &= \nu_y \\
\dot{\nu}_x &= k_c \frac{\xi_x + \xi_x^*}{\left((\xi_x + \xi_x^*)^2 + \xi_y^2\right)^{\frac{3}{2}}} u \\
\dot{\nu}_y &= -a\xi_y - b\nu_y + k_c \frac{\xi_y}{\left((\xi_x + \xi_y^*)^2 + \xi_y^2\right)^{\frac{3}{2}}} u
\end{aligned} \tag{9.26}$$

where $a, b > 0$. Now use the following controller for the x dynamics based on the collinear two spacecraft scenario derived earlier where $W(\xi_x) = \frac{1}{2}\xi_x^2 + \frac{1}{4}\xi_x^4$,

$$u = \frac{1}{k_c} \frac{\left((\xi_x + \xi_x^*)^2 + \xi_y^2\right)^{\frac{3}{2}}}{\xi_x + \xi_x^*} \left(-\xi_x - \xi_x^3 - \frac{\nu_x}{1+\xi_x^2} - \nu_x^3\right) \tag{9.27}$$

This yields the following equations,

$$\begin{aligned}
\dot{\xi}_x &= \nu_x \\
\dot{\nu}_x &= -\xi_x - \xi_x^3 - \frac{\nu_x}{1+\xi_x^2} - \nu_x^3 \\
\dot{\xi}_y &= \nu_y \\
\dot{\nu}_y &= -a\xi_y - b\nu_y + \frac{\xi_y}{\xi_x + \xi_x^*}\left(-\xi_x - \xi_x^3 - \frac{\nu_x}{1+\xi_x^2} - \nu_x^3\right)
\end{aligned} \tag{9.28}$$

By Theorem 20, there exists a centre manifold,

$$\begin{bmatrix} \xi_y \\ \nu_y \end{bmatrix} = H(\xi_x, \nu_x) \tag{9.29}$$

and, by construction, the equilibrium $(\xi_x, \nu_x) = (0, 0)$ is asymptotically stable, so the equilibrium $(\xi_x, \xi_y, \nu_x, \nu_y) = (0, 0, 0, 0)$ is asymptotically stable by Theorem 21. Note that ξ_y shows up in the control (9.27); since ξ_x and ν_x belong to a compact set and H is continuous, H maps to a compact (therefore bounded) set, so the control is bounded so long as no new singularities are introduced.

A simulation of the derived controller is shown in Figure 9.4.

9.3 Collinear Three Spacecraft Formations

Consider a three spacecraft formation with the geometry shown in Figure 9.5.

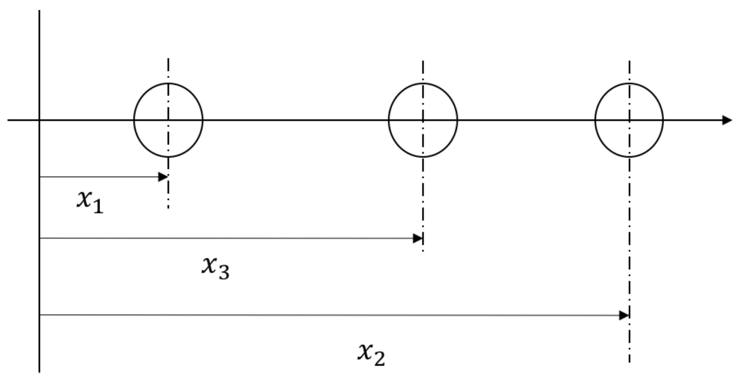

Figure 9.5: Geometry of a collinear three spacecraft formation.

Similarly to the two spacecraft example, define the relative coordinates,

$$\tilde{\xi}_1 = x_1 - x_2$$
$$\tilde{\xi}_2 = x_1 - x_3$$
$$\tilde{\xi}_3 = x_2 - x_3 = \tilde{\xi}_2 - \tilde{\xi}_1 \tag{9.30}$$

Now let,

$$\xi_1 = \tilde{\xi}_1 - \xi_1^*$$
$$\xi_2 = \tilde{\xi}_2 - \xi_2^*$$
$$\xi_3 = \tilde{\xi}_2 - \tilde{\xi}_1 - \xi_2^* + \xi_1^* \tag{9.31}$$

The equations of motion can be found using the same procedure as the collinear two spacecraft example,

$$
\begin{aligned}
\dot{\xi}_1 &= \nu_1 \\
\dot{\xi}_2 &= \nu_2 \\
\dot{\nu}_1 &= 2k_c \frac{\xi_1 + \xi_1^*}{||\xi_1 + \xi_1^*||^3} u_1 u_2 + k_c \frac{\xi_2 + \xi_2^*}{||\xi_2 + \xi_2^*||^3} u_1 u_3 - k_c \frac{\xi_2 - \xi_1 + \xi_2^* - \xi_1^*}{||\xi_2 - \xi_1 + \xi_2^* - \xi_1^*||^3} u_2 u_3 \\
\dot{\nu}_2 &= k_c \frac{\xi_1 + \xi_1^*}{||\xi_1 + \xi_1^*||^3} u_1 u_2 + 2k_c \frac{\xi_2 + \xi_2^*}{||\xi_2 + \xi_2^*||^3} u_1 u_3 + k_c \frac{\xi_2 - \xi_1 + \xi_2^* - \xi_1^*}{||\xi_2 + \xi_1 + \xi_2^* - \xi_1^*||^3} u_2 u_3
\end{aligned}
\tag{9.32}
$$

Let,

$$
u_3 = \beta(\xi_2 + \xi_2^*)(\xi_2 - \xi_1 + \xi_2^* - \xi_1^*) \tag{9.33}
$$

for a scalar $\beta > 0$. This designates the middle spacecraft as the leader spacecraft as is discussed in Section 8.1.8.1.4.

The equations of motion are now,

$$
\begin{aligned}
\dot{\xi}_1 &= \nu_1 \\
\dot{\xi}_2 &= \nu_2 \\
\dot{\nu}_1 &= \beta k_c \frac{(\xi_2 + \xi_2^*)^2(\xi_2 - \xi_1 + \xi_2^* - \xi_1^*)}{||\xi_2 + \xi_2^*||^3} u_1 - \beta k_c \frac{(\xi_2 + \xi_2^*)(\xi_2 - \xi_1 + \xi_2^* - \xi_1^*)^2}{||\xi_2 - \xi_1 + \xi_2^* - \xi_1^*||^3} u_2 + 2k_c \frac{\xi_1 + \xi_1^*}{||\xi_1 + \xi_1^*||^3} u_1 u_2 \\
\dot{\nu}_2 &= 2\beta k_c \frac{(\xi_2 + \xi_2^*)^2(\xi_2 - \xi_1 + \xi_2^* - \xi_1^*)}{||\xi_2 + \xi_2^*||^3} u_1 + \beta k_c \frac{(\xi_2 + \xi_2^*)(\xi_2 - \xi_1 + \xi_2^* - \xi_1^*)^2}{||\xi_2 - \xi_1 + \xi_2^* - \xi_1^*||^3} u_2 + k_c \frac{\xi_1 + \xi_1^*}{||\xi_1 + \xi_1^*||^3} u_1 u_2
\end{aligned}
\tag{9.34}
$$

More compactly,

$$
\begin{aligned}
\dot{\xi}_1 &= \nu_1 \\
\dot{\xi}_2 &= \nu_2 \\
\dot{\nu}_1 &= g_1(x)u_1 - g_2(x)u_2 + 2g_3(x)u_1u_2 \\
\dot{\nu}_2 &= 2g_1(x)u_1 + g_2(x)u_2 + g_3(x)u_1u_2
\end{aligned}
\tag{9.35}
$$

where $x = (\xi_1, \xi_2, \nu_1, \nu_2)$. Scale the input,

$$\begin{aligned} u_1 &= \frac{v_1}{g_1(x)} \\ u_2 &= \frac{v_2}{g_2(x)} \end{aligned} \tag{9.36}$$

So,

$$\begin{aligned} \dot{\xi}_1 &= \nu_1 \\ \dot{\xi}_2 &= \nu_2 \\ \dot{\nu}_1 &= v_1 - v_2 + 2\frac{g_3(x)}{g_1(x)g_2(x)}v_1 v_2 \\ \dot{\nu}_2 &= 2v_1 + v_2 + \frac{g_3(x)}{g_1(x)g_2(x)}v_1 v_2 \end{aligned} \tag{9.37}$$

Even more compactly,

$$\begin{aligned} \dot{\xi}_1 &= \nu_1 \\ \dot{\xi}_2 &= \nu_2 \\ \dot{\nu}_1 &= v_1 - v_2 + 2\tilde{g}_3(x)v_1 v_2 \\ \dot{\nu}_2 &= 2v_1 + v_2 + \tilde{g}_3(x)v_1 v_2 \end{aligned} \tag{9.38}$$

Choose v_2 as a PD controller,

$$v_2 = -a\nu_2 - b\xi_2 \tag{9.39}$$

where $a, b > 0$. Then choose v_1 to be nonlinear control based on the previous example,

$$v_1 = \frac{1}{1 + 2\tilde{g}_3(x)v_2}\left(v_2 - \xi_1 - \xi_1^3 - \frac{\nu_1}{1+\xi_1^2} - \nu_1^3\right) \tag{9.40}$$

Then the equations of motion become almost exactly the same as (9.28). There exists a centre manifold,

$$\begin{bmatrix} \xi_2 \\ \nu_2 \end{bmatrix} = H(\xi_1, \nu_1) \tag{9.41}$$

and the origin is asymptotically stable by the nonlinear construction. An example simulation is shown in Figure 9.6.

Unlike the example of the one dimensional thrust, there is an extra singularity that potentially is introduced in the feedback passivation term (9.40). This corresponds to a scenario where the force acting on the first spacecraft by the leader and by the second spacecraft cancel out each other. This is unlikely to occur since the leader is expected to have a higher magnitude of charge and the leader is chosen to be in between the two other spacecraft. In larger formations, this potentially could be more problematic because of the dependence of the Coulomb force on the inverse square of distance.

9.4 Concluding Remarks and Suggestions for Future Work

The dynamics for collinear two spacecraft Coulomb formations have a form such that, by choosing a convex storage function, the input that makes the system passive also makes the system zero-state observable by Lemma 12 which guarantees asymptotic stability by output feedback.

The three Coulomb spacecraft design problem is approached in a sequential manner. The charge of the middle spacecraft is chosen arbitrarily, partially motivated by a relative degree argument, to give the equations of motion an affine part. After scaling, the input for one of the spacecraft is chosen to be a PD controller without needing to know the value of the input of the third spacecraft. Based on that linear input, a centre manifold exists and the input of the third spacecraft can be determined as a nonlinear control law based on the derived control for collinear two spacecraft Coulomb formations. The input of the third spacecraft accounts for the choices of the inputs of the first and second spacecraft that were made earlier. This sequential design procedure simplifies the problem and yields smooth (and realizable) control that possibly can be used to design control for larger formations.

There are many possible directions for future research in Coulomb formation control.

One interesting direction is to extend the results of §9.2 to cases where the thrusters are impulsive thrusters. In this case, the sequence of impulses and the feedback controller need to be determined to stabilize the formation. Another interesting direction is to consider the problem of Coulomb force allocation [84]. Given a configuration of Coulomb spacecraft, is a desired Coulomb force realizable and what should the charges of each spacecraft be? This can be done by solving systems of bilinear equations [85] and analyzing the sign-solvability of systems of equations to ensure that the desired Coulomb force can be realised [34]. The effects of the space environment in geosynchronous orbits [68, 137] on controller design should also be studied.

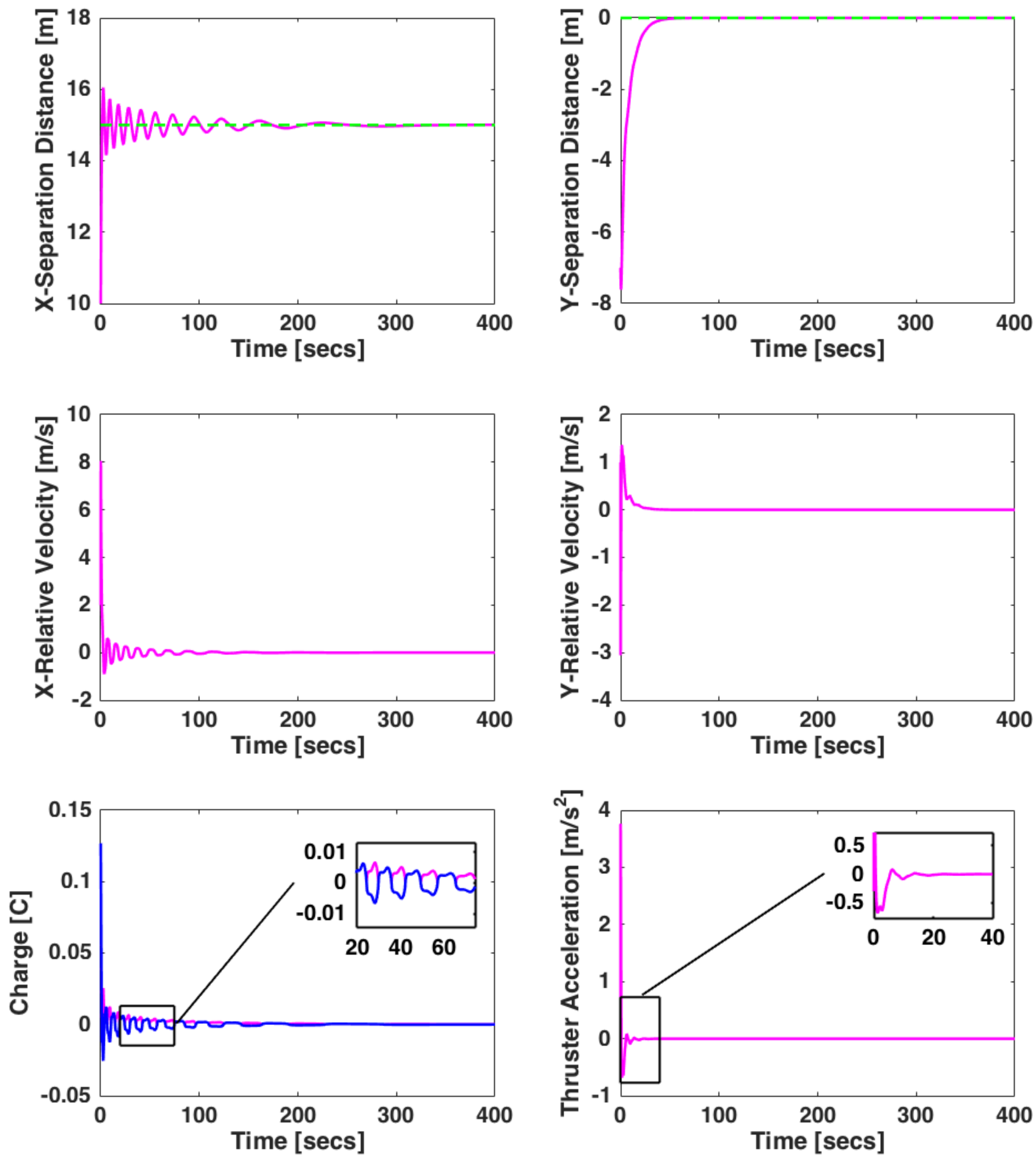

Figure 9.4: A simulation of the Coulomb-augmented system in two dimensions where the thruster is constrained to providing thrust in the y direction only.

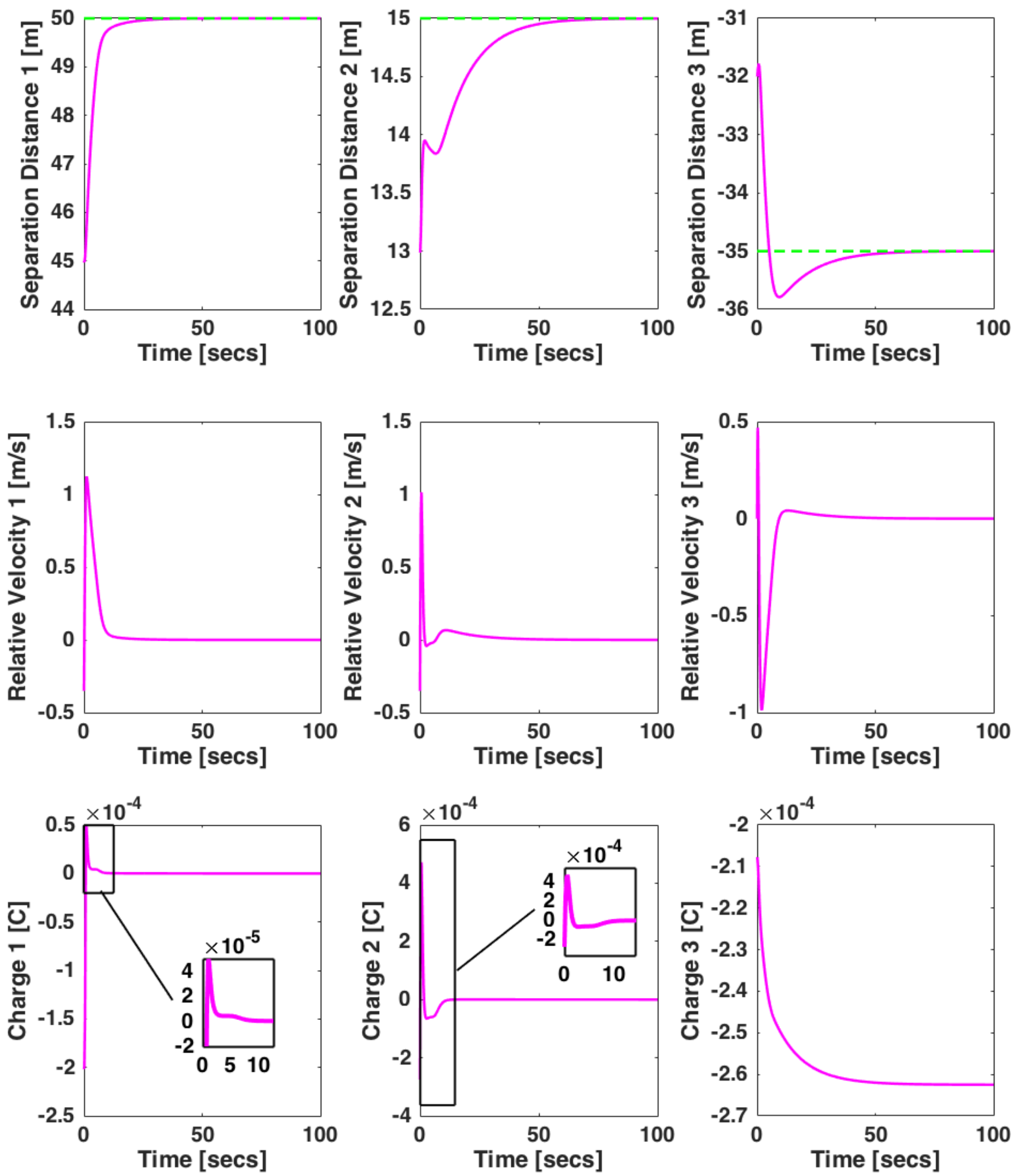

Figure 9.6: A simulation of the three spacecraft formation in one dimension using solely the Coulomb forces between the spacecraft.

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Appendix A
POSITIVE SYSTEMS

A *positive system* is a dynamical system whose state remains elementwise nonnegative for all time if its initial condition is elementwise nonnegative and its input is elementwise nonnegative for all time. This appendix reviews several key concepts about positive systems which are used to synthesize interval observers in Part I of this dissertation. For more details about positive systems, the reader is referred to the books [59, 140] and recent survey paper [127]. Note that this appendix uses the same notation as Part 1.

A.1 Continuous-Time Systems

A continuous-time system with state $x(t)$ and nonnegative input $u(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}^m_{\geq 0}$ is a positive system if $x(t) \geq 0$ for all $t \geq 0$ provided that the initial condition satisfies $x(0) \geq 0$.

A.1.1 Linear Time-Invariant Systems

The following linear time-invariant system:

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{A.1}$$

with state $x \in \mathbb{R}^n$ and nonnegative input $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m_{\geq 0}$, is a positive system if and only if $A \in \mathbb{R}^{n \times n}$ is a Metzler matrix and $B \in \mathbb{R}^{n \times m}$ is elementwise nonnegative (cf. [59, Theorem 2] and [127]). A matrix is Metzler if and only if all of its off-diagonal elements are nonnegative, i.e. $A_{ij} \geq 0$, for all $i \neq j$.

Asymptotic stability of the origin of the unforced system (A.1) follows if A is also Hurwitz, meaning that all of its eigenvalues have strictly negative real parts. A necessary, but not sufficient, condition for a Metzler matrix to be Hurwitz is that the diagonal elements must be strictly negative [108]. There are characteristics of matrices which are both Metzler and Hurwitz that are summarized in the following proposition from [127] that can be exploited for synthesizing positive controllers and observers.

Proposition 9 (Proposition 2 in [127]). *For a Metzler matrix $A \in \mathbb{R}^{n\times n}$, the following are equivalent:*

i. A is Hurwtiz;

ii. There exists a vector $\xi > 0$ such that $A\xi < 0$;

iii. There exists a vector $r > 0$ such that $r^T A < 0$;

iv. There exists a diagonal matrix $P \succ 0$ such that $PA + A^T P \prec 0$;

v. $-A^{-1}$ exists and has nonnegative entries.

Condition iv in Proposition 9 means that a Metzler matrix is Hurwitz if and only if it admits diagonal quadratic Lyapunov function $V(x) = x^T Px$. This is useful in synthesizing positive and stable systems because for any two matrices J and Q where $J \in \mathbb{R}^{n\times n}$ is a positive definite diagonal matrix and $Q \in \mathcal{M}_n$, $JQ \in \mathcal{M}_n$ [127, 50, 40].

Condition iii in Proposition 9 allows for a different kind of Lyapunov function known as the *linear copositive* Lyapunov function which takes the following form:

$$V(x) = r^T x \tag{A.2}$$

for $r > 0$. $V > 0$ for all $x \geq 0$. Since $r^T A < 0$, then $\dot{V} < 0$ for every positive x. Therefore, condition iii corresponds to saying that a Metzler matrix is Hurwitz if and only if it admits a linear copositive Lyapunov function. This condition allows for synthesizing positive and stable systems by solving linear programs [127, 124, 32, 123].

A.1.2 Linear Time-Varying Systems

The following linear time-varying system:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \tag{A.3}$$

is a positive system if $A(t)$ is Metzler for all $t \geq 0$, i.e. $A_{ij}(t) \geq 0$ for all $i \neq j$, and $B(t)$ is nonnegative for all $t \geq 0$ [13, Lemma VIII.1]. The converse holds if $A(\cdot)$ and $B(\cdot)$ are also continuous [13, Lemma VIII.1].

It is shown in [87, Theorem 4.2] that an unforced system (A.3) where $A(\cdot)$ is piecewise continuous and uniformly bounded is asymptotically stable if and only if it admits a quadratic Lyapunov function $V(x,t) = x^T P(t)x$, where $P(t)$ is differentiable and diagonal for all $t \geq 0$.

A.2 Discrete-Time Systems

A discrete-time system with state x_k and nonnegative input $u_k : \mathbb{Z}_{\geq 0} \to \mathbb{R}^m_{\geq 0}$ is a positive system if $x_k \geq 0$ for all $k \in \mathbb{Z}_{\geq 0}$ provided that the initial condition satisfies $x_0 \geq 0$.

The following linear time-invariant system:

$$x_{k+1} = Ax_k + Bu_k, \tag{A.4}$$

with state $x \in \mathbb{R}^n$ and nonnegative input $u : \mathbb{Z}_{\geq 0} \to \mathbb{R}^m_{\geq 0}$, is a positive system if and only if $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are nonnegative matrices (cf. [59, Theorem 2] and [127]).

Asymptotic stability of the origin of the unforced system (A.4) follows if A is also Schur, meaning that all of its eigenvalues have modulus strictly less than one. Similarly to the continuous-time case, the characteristics of matrices that are both nonnegative and Schur can be exploited to be able to synthesize positive and stable systems.

Proposition 10 (Proposition 1 in [127]). *For a nonnegative matrix $A \in \mathbb{R}^{n\times n}$, the following are equivalent:*

i. A is Schur;

ii. There exists a vector $\xi > 0$ such that $A\xi < \xi$;

iii. There exists a vector $r > 0$ such that $r^T A < r^T$;

iv. There exists a diagonal matrix $P \succ 0$ such that $A^T PA - P \prec 0$;

v. $(I_n - A)^{-1}$ exists and has nonnegative entries.

Condition iv in Proposition 10 means that a nonnegative matrix is Schur if and only if it admits diagonal quadratic Lyapunov function. Moreover, condition iii corresponds to saying that a nonnegative matrix is Schur if and only if it admits a linear copositive Lyapunov function which allows for synthesizing positive and stable systems by solving linear programs.

The following linear-time varying system in discrete-time:

$$x_{k+1} = A(k)x_k, \tag{A.5}$$

is positive if and only if its state-transition matrix

$$\phi(t,s) = \begin{cases} I_n, & t = s, \\ A(t-1)A(t-2)\dots A(s), & t > s. \end{cases}$$

is nonnegative for all $t \geq s \geq 0$ where $t, s \in \mathbb{Z}_{\geq 0}$ [87]. Clearly, the condition that $A(k)$ is nonnegative for all $k \in \mathbb{Z}_{\geq 0}$ is a sufficient condition for positivity of (A.5).

Similarly to the continuous time case, the linear time varying system (A.5) where $A : \mathbb{Z}_{\geq 0} \to \mathbb{R}^{n\times n}$ is uniformly bounded is asymptotically stable if and only if it admits a quadratic Lyapunov function $V(x,k) = x^T P(k)x$, where $P(k)$ is diagonal for all $k \in \mathbb{Z}_{\geq 0}$ [87, Theorem 4.1].

Appendix B

INPUT-TO-STATE STABILITY

A system with an input is input-to-state stable (ISS) if, regardless of its initial condition, it eventually converges to a neighbourhood of the origin, and, roughly speaking, that neighbourhood is ‘small’ if the input signal is ‘small.’

To define ISS, the definitions of $\mathcal{K}$ and $\mathcal{KL}$ functions must be recalled. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a $\mathcal{K}$-function if it is continuous, strictly increasing and $\gamma(0) = 0$; it is a $\mathcal{K}_\infty$-function if it is a $\mathcal{K}$-function and also $\gamma(s) \to \infty$ as $s \to \infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a $\mathcal{KL}$-function if, for each fixed $t \geq 0$, the function $\beta(\cdot, t)$ is a $\mathcal{K}$-function, and for each fixed $s \geq 0$, the function $\beta(s, \cdot)$ is decreasing and $\beta(s, t) \to 0$ as $t \to \infty$.

For a function $\phi : \mathbb{Z}_{\geq 0} \to \mathbb{R}^n$, $\|\phi\|_{\sup} = \sup\{\|\phi(k)\| : \forall k \in \mathbb{Z}_{\geq 0}\}$. For a function $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, $\|\psi\|_J$ denotes the (essential) supremum norm on an interval $J \subseteq \mathbb{R}_{\geq 0}$.

B.1 Nonlinear Continuous-Time Systems

Consider the following nonlinear continuous-tie system with a state $x \in \mathbb{R}^n$ and input $u \in \mathbb{R}^m$:

$$\dot{x}(t) = f(x(t), u(t)). \tag{B.1}$$

Let $x(\cdot, x_0, u)$ denote a trajectory of (B.1) starting from initial state x_0 and with a Lesbegue-measurable essentially bounded input u. ISS for the continuous-time system (B.1) is defined as follows.

Definition 10 (Definition 2.1 in [141]). (B.1) *is input-to-state stable if there exists a $\mathcal{KL}$ function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and a $\mathcal{K}$ function γ such that, for each bounded input u and* *each* $x_0 \in \mathbb{R}^n$*, it holds that*

$$\|x(t, x_0, u)\| \leq \beta(\|x_0\|, t) + \gamma(\|u\|_{[0,t]}) \tag{B.2}$$

for each $t \geq 0$.

(B.2) implies that the solution to (B.1) is ultimately bounded. The ultimate bound depends on the 'size' of the input, i.e. $\gamma(\|u\|_{[0,t]})$, which is larger when $\|u\|_{[0,t]}$ is larger. It is clear also that when $u = 0$ uniformly, then $x = 0$ is globally asymptotically stable.

Definition 11 (Definition 2.2 in [141])**.** *A continuous function* $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *is called an ISS-Lyapunov function if the following hold:*

i. There exist $\mathcal{K}_\infty$*-functions* α_1, α_2*, such that*

$$\alpha_1(\|\xi\|) \leq V(\xi) \leq \alpha_2(\|\xi\|),\ \forall \xi \in \mathbb{R}^n. \tag{B.3}$$

ii. There exist a $\mathcal{K}_\infty$*-function* α_3 *and a* $\mathcal{K}$*-function* χ*, such that*

$$\langle \nabla V(\xi), f(\xi, \mu) \rangle \leq -\alpha_3(\|\xi\|),$$

for all $\xi \in \mathbb{R}^n$ *and* $\mu \in \mathbb{R}^m$ *such that* $\|\xi\| \geq \chi(\|\mu\|)$.

The existence of an ISS-Lyapunov function is a sufficient condition for (B.1) to be ISS.

Lemma 13. *If* (B.1) *admits an ISS-Lyapunov function, then it is ISS.*

Note that, a smooth function V where there exists $\alpha_1, \ldots, \alpha_4 \in \mathcal{K}_\infty$ such that (B.3) and

$$\langle \nabla V(\xi), f(\xi, \mu) \rangle \leq -\alpha_3(\|\xi\|) + \alpha_4(\|\mu\|) \tag{B.4}$$

hold is also an ISS-Lyapunov function since clearly (B.4) implies property ii in Definition 11. This is the so-called "dissipation" characterization of ISS [141].

B.2 Nonlinear Discrete-Time Systems

Consider the following nonlinear discrete-time system with a state $x \in \mathbb{R}^n$ and input $u \in \mathbb{R}^m$:

$$x_{k+1} = f(x_k, u_k),\ \forall k \in \mathbb{Z}_{\geq 0}. \tag{B.5}$$

Let $x(\cdot, x_0, u)$ denote a trajectory starting from initial state x_0 with a bounded input u. ISS for the discrete-time system (B.5) is defined as follows.

Definition 12 (Definition 3.1 in [83])**.** (B.5) *is input-to-state stable if there exists a $\mathcal{KL}$ function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and a $\mathcal{K}$ function γ such that, for each bounded input u and each $x_0 \in \mathbb{R}^n$, it holds that*

$$\|x(k, x_0, u)\| \leq \beta(\|x_0\|, k) + \gamma(\|u\|_{\text{sup}})$$

for each $k \in \mathbb{Z}_{\geq 0}$.

To show that a nonlinear discrete-time system is ISS an ISS-Lyapunov function is constructed. An ISS-Lyapunov function for (B.5) is defined below.

Definition 13 (Definition 3.2 in [83])**.** *A continuous function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called an ISS-Lyapunov function if the following hold:*

i. There exist $\mathcal{K}_\infty$-functions α_1, α_2, such that

$$\alpha_1(\|\xi\|) \leq V(\xi) \leq \alpha_2(\|\xi\|),\ \forall \xi \in \mathbb{R}^n.$$

ii. There exist a $\mathcal{K}_\infty$-function α_3 and a $\mathcal{K}$-function σ, such that

$$V(f(\xi, \mu)) - V(\xi) \leq -\alpha_3(\|\xi\|) + \sigma(\|\mu\|),\ \forall \xi \in \mathbb{R}^n, \mu \in \mathbb{R}^m.$$

The following lemma states that the existence of an ISS-Lyapunov function for (B.5) implies that it is ISS.

Lemma 14 (Lemma 3.5 in [83])**.** *If* (B.5) *admits a continuous ISS-Lyapunov function, then it is ISS.*

B.3 Nonlinear Impulsive Systems

Consider the following nonlinear impulsive system with a state $x \in \mathbb{R}^n$ and input $u \in \mathbb{R}^m$:

$$\dot{x}(t) = f(x(t), u(t)),\ t \geq 0, t \neq t_k, \tag{B.6}$$

$$x(t) = g(x(t^-), u(t^-)),\ t = t_k, \tag{B.7}$$

where $\{t_1, t_2, \ldots, \}$ is a strictly increasing sequence of impulse times in $(0, \infty)$. $x(t, x_0, u)$ is a trajectory starting from x_0 with input u.

Definition 14 ([76])**.** (B.6)-*(B.7) is input-to-state stable if there exists a $\mathcal{KL}$ function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and a $\mathcal{K}$ function γ such that, for each locally bounded, Lesbegue-measurable input u and each $x(0) \in \mathbb{R}^n$, it holds that*

$$\|x(t, x(0), u)\| \leq \beta(\|x(0)\|, t) + \gamma(\|u\|_{[0,t]})$$

for each $t \geq 0$.

Definition 15. *A function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a candidate exponential ISS-Lyapunov function for* (B.6)-*(B.7) with rate coefficients $c, d \in \mathbb{R}$ if V is locally Lipschitz, positive definie, radially unbounded, and satisfies*

$$\langle \nabla V, f(x, u) \rangle \leq -cV(x) + \chi(\|u\|),\ \forall x \in \mathbb{R}^n\, a.e., \forall u \in \mathbb{R}^m,$$

$$V(g(x, u)) \leq e^{-d} V(x) + \chi(\|w\|),\ \forall x \in \mathbb{R}^n, \forall u \in \mathbb{R}^m.$$

for some function $\chi \in \mathcal{K}_\infty$.

Once a candidate exponential ISS-Lyapunov function has been identified, a class of impulse time sequences which guarantee ISS of the impulsive system can be identified from the rate coefficients c and d [76, 45].

Theorem 22 (Theorem 1 in [76]). *Let V be a candidate exponential ISS-Lyapunov function for* (B.6)-*(B.7) with rate coeficients $c, d \in \mathbb{R}$ with $d \neq 0$. For arbitrary constants $\mu, \lambda > 0$, let $\mathcal{S}[\mu, \lambda]$ denote the class of impulse time sequences $\{t_k\}$ satisfying*

$$-dN(t,s) - (c-\lambda)(t-s) \leq \mu, \; \forall t \geq s \geq 0. \tag{B.8}$$

Then (B.6)-*(B.7) is uniformly ISS over $\mathcal{S}[\mu, \lambda]$.*

If $c > 0$ and $d > 0$, then the system is ISS for any impulse time sequence since, in this case, μ and λ can be always be found that satisfies (B.8). On the other hand, preservation of ISS in impulsive systems is dependent on the sequence of the impulses if either, but not both, the continuous or the impulsive parts is destabilizing (i.e. $cd < 0$) [76, 45]. Generally speaking, if $c > 0$ but $d < 0$, then ISS can be guaranteed if the impulses do not occur too frequently. *Vice versa*, if $c < 0$ but $d > 0$, the impulses must occur frequently enough to guarantee ISS.

Although an impulsive system consists of a continuous part (B.6) and a jump part (B.7), (B.6) being ISS on its own, à la Definition 10, and (B.7) being ISS on its own, à la Definition 12, is not sufficient for the impulsive system to be ISS. They must also share a common ISS Lyapunov function.