Harnack inequalities for nonlocal operators

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Abstract

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Harnack inequalities are a fundamental property in both probability theory and analysis. The scale-variant Harnack inequalities play an important role in studying various properties, such as the regularity of harmonic functions in probability and analysis. This thesis focuses on scale-invariant Harnack inequalities for a class of nonlocal operators and a class of weakly coupled nonlocal operators.

In Chapter 1, we show the scale-invariant elliptic Harnack inequality holds for a class of nonlocal operators $\mathcal{L}$, which are second order elliptic differential operators perturbed by nonlocal operators. We assume the existence of a conservative Hunt process $\{X_t, t \geq 0; \mathbb{P}^x, x \in \mathbb{R}^d\}$ corresponding to each operator $\mathcal{L}$ in that class, and establish the scale-invariant Harnack inequalities for nonnegative functions that are $\mathcal{L}$-harmonic. This is achieved by using Krylov estimate approach, where the comparison constant depends solely on the parameters of the class of the operators.

Moreover, unlike purely diffusive operators, for which Hölder regularity of bounded harmonic functions is a direct consequence of the Harnack inequality, we establish the Hölder regularity for bounded harmonic functions for nonlocal operators in this class using a probabilistic approach.Additionally, we demonstrate that the scale-invariant parabolic Harnack inequality holds for nonnegative $\mathcal{L}$-caloric functions and establish Hölder regularity for bounded $\mathcal{L}$-caloric functions.

In Chapter 2, utilizing the result from Chapter 1, we consider a system $\mathcal{G}$ of nonlocal

operators $\{\mathcal{L}_i, i = 1, ..m\}$, as discussed in Chapter 1, connected by an index switching process $\{\Lambda_i, i = 1, ..., m\}$ determined by its switching rate matrix Q. Such a system of operator whose coupling terms do not involve the derivatives of the unknown functions is called a *weakly coupled nonlocal system.* Weakly coupled systems are widely investigated in the field of physics, finance and engineering, etc. Through a piecing-together procedure, there exists a Hunt process $\Big((X_t, \Lambda_t), t \geq 0; \mathbb{P}^{(t,x)}, (t, x) \in [0, \infty) \times \mathbb{R}^d\Big)$ corresponding to the weakly coupled operator $\mathcal{G}$ wthin a certain class. Using the two-sided scale-invariant Green function estimates, we prove the scale-invariant Harnack inequalities for the weakly coupled nonlocal operators $\mathcal{G}$. Under the irreducibility assumption of the switching matrix, we further derive a full rank scale-invariant Harnack inequality for this class of weakly coupled operators.

The Appendix A serves as a complimentary part to Chapter 1. One of the essential intermediate components in Krylov's estimate approach presented in Chapter 1 is a lower bound of the hitting probability estimate. This result, along with other related important theorems, all ultimately relies on the equivalence between the Martingale problems and SDE for this class of nonlocal operators. However, there is limited literature available on this topic in English. Therefore, for both learning purposes and the reader's convenience, we summarize and provide the detail explanations of some existing results originally in French [36, 56], which are used in the proof of Proposition 1.3.1 of Chapter 1. In addition, the proof of the support theorem for diffusion processes (4.9) (orginally from [6]) and the Krylov's estimate for diffusion operators in Lemma 1.4.4, (originally from [2, 51]) have also been rewritten and clarified in details.

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DEDICATION

To my parents and my God Jehovah

Chapter 1

HARNACK INEQUALITIES FOR A CLASS OF NONLOCAL OPERATORS

1.1 Introduction

This project is concerned with the scale invariant Harnack inequality and parabolic Harnack inequality as well as a priori Hölder regularity for the following type non-symmetric non-local operators defined on $C_b^2(\mathbb{R}^d)$ by

$$\begin{aligned}\mathcal{L}u(x) &= \frac{1}{2}\sum_{i,j=1}^{d} a_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j}u(x) + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}u(x) \\ &\quad + \int_{\mathbb{R}^d\setminus\{0\}} \Big(u(x+z) - u(x) - \nabla u(x)\cdot z \mathbb{1}_{\{|z|\le 1\}}\Big)\, n(x,dz). \qquad (1.1)\end{aligned}$$

In probability theory, the above non-local operator corresponds to a semimartingale Markov process that has both diffusive and jumping motions. We assume that there is a Hunt process $\{X_t, t > 0; \mathbb{P}^x, x \in \mathbb{R}^d\}$ on $\mathbb{R}^d$ that solves the martingale problem of $(\mathcal{L}, C_b^2(\mathbb{R}^d))$; that is, for every $f \in C_b^2(\mathbb{R}^d)$,

$$M_t^f := f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)ds$$

is a martingale under $\mathbb{P}^x$ for every $x \in \mathbb{R}^d$ and $\mathbb{P}^x(X_0 = x) = 1$. The matrix $(a_{ij}(x))$ is the diffusion matrix for the process X, $b(x) = (b_1(x), \cdots, b_d(x))$ describes the drift, and kernel $n(x, dz)$ describes the jumping rate for the process to jump from x to $x + z$. A function u is said to be $\mathcal{L}$-harmonic in a domain D if $u(X_t)$ is a local martingale up to the first exit time τ_D by the process X exiting from D. Heuristically, u is $\mathcal{L}$-harmonic in D if $\mathcal{L}u = 0$ in D. But we are not going to establish this analytic characterization. See Chen [17] for the equivalent characterizations between probabilistic and analytic notions of harmonicity when the underlying process is symmetric with respect to some reference measure.

The classical elliptic Harnack inequality asserts that if u is a non-negative harmonic function in a ball $B(x_0, r) \subset \mathbb{R}^d$ in the sense that $\Delta u = 0$ in $B(x_0, r)$, then the values of u on $B(x_0, r/2)$ are universally comparable. This scale invariant property is an easy consequence of the Poisson representation of classical harmonic functions. When $d = 2$, the above inequality was obtained by Harnack in 1887. Scale invariant Harnack inequalities have many important consequences in analysis and PDEs. There is also a parabolic version of Harnack inequality for non-negative caloric functions (functions satisfying the parabolic equation $\frac{\partial}{\partial t}u(t,x) = \mathcal{L}u(t,x)$, called parabolic Harnack inequality. However, it takes much more challenging efforts to establish Harnack inequality for non-homogenous second order elliptic differential operators. The first general result of De Giorgi [35] and Nash [63] in late 1950 on a priori Hölder estimates of the solutions of partial differential equations served as a powerful stimulus in the development of the theory of elliptic and parabolic equations of the *divergence* form. Harnack inequalities have been extended to elliptic or parabolic partial differential equations by many authors. In particular, in his influential work [61, 62], Moser established Harnack inequalities for solutions of second order elliptic or parabolic partial differential equations of divergence form, and used them to give another proof of a prior Hölder regularity results of De Giorgi and Nash.

In 1979 and early 1980's, Krylov and Safonov made a breakthrough in establishing the Harnack's inequality and a priori Hölder estimates for the solutions of second order linear elliptic and parabolic differential equations of *nondivergence* form with measurable coefficients. The probabilistic arguments played a major role in their approach [51, 53, 67]. We refer the reader to [47] for an account of a history of classical results on Harnack inequalities.

While Moser, Krylov and Safonov established the Harnack inequality for second order elliptic differential operators, which corresponds to diffusion processes, there are some fruitful developments recently for a certain class of non-local operators, whose corresponding Markov processes have discontinuous sample paths; see [8, 9, 10, 20, 21, 22, 39] and the references therein. Some of these work are concerned with symmetric Markov processes and symmetric non-local operators. In [8, 9, 20, 21], the jump kernel is comparable to that of a radically symmetric stable process and mixed stable processes. In this project, we consider processes that have both diffusive and jump parts and with quite general jumping kernel $n(x, dz)$. Harnack inequality and Hölder regularity for harmonic functions with respect to non-local operator $\mathcal{L}$ of the form (1.1) has been studied in [39] under the assumption that $n(x, dz)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$ together with some comparability assumption. However the paper [39] contains a critical error in the proof of a key support theorem (see §1.4.2 for details); in addition the claimed Harnack inequality there is not scale invariant.

A closely related topic is a priori Hölder regularity. For recent development on priori Hölder estimates for non-local operators, we refer the reader to [23, 27] and the references therein.

Given positive constants Λ_1, Λ_2, Λ_3, we say a non-local operator $\mathcal{L}$ of (1.1) is in class $\mathcal{N}(\Lambda_1, \Lambda_2, \Lambda_3)$ if the following holds:

(a) (uniform ellipticity on diffusion matrix)

$$\Lambda_1|\xi|^2 \leq \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \leq \Lambda_1^{-1}|\xi|^2 \quad \text{for every } x, \xi \in \mathbb{R}^d, \tag{1.2}$$

(b) (bounded drift) $\|b\|_\infty := \sup_{x\in\mathbb{R}^d} |b(x)| \leq \Lambda_2$.

(c) (Lévy jumping kernel condition) $\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d\setminus\{0\}} (|z|^2 \wedge 1)n(x, dz) \leq \Lambda_3$

In this paper, we establish scale invariant elliptic and parabolic Harnack inequality as well as a priori Hölder regularity for non-local operator $\mathcal{L} \in \mathcal{N}(\Lambda_1, \Lambda_2, \Lambda_3)$. The following are two of the main results of this paper. The first one is on a priori Hölder estimates and the second one is on scale invariant Harnack inequality.

Theorem 1.1.1. *Suppose $\mathcal{L} \in \mathcal{N}(\Lambda_1, \Lambda_2, \Lambda_3)$ and that*

$$\psi(\delta) := \sup_{x\in\mathbb{R}^d} \int_{\{z:|z|\leq\delta\}} (|z|^2 \wedge 1)n(x, dz) \to 0 \quad \text{as } \delta \to 0. \tag{1.3}$$

There exist a constant $\widetilde{r}_1 > 0$ depending on $(d, \Lambda_1, \Lambda_2, \Lambda_3)$, and $\alpha \in (0, 1), c > 0$ that depend only on $(d, \Lambda_1, \Lambda_2, \Lambda_3)$ and the rate of ψ converging to 0 so that for any $x_0 \in \mathbb{R}^d$, $r \in (0, \widetilde{r}_0]$, and any bounded function u on $\mathbb{R}^d$ that is $\mathcal{L}$-harmonic in $B(x_0, 2r)$, one has

$$|u(x) - u(y)| \leq c\,\|u\|_\infty\,(|x - y|/r)^\alpha \quad \text{for any } x, y \in B(x_0, r), \tag{1.4}$$

where $\|u\|_\infty = \sup_{x\in\mathbb{R}^d} |u(x)|$.

Theorem 1.1.2. *Let* $\mathcal{L} \in \mathcal{N}(\Lambda_1, \Lambda_2, \Lambda_3)$ *satisfying* (1.3). *Suppose that for every* $r \in (0,1]$ *and* $x_0 \in \mathbb{R}^d$,

$$n(x_0, A - x_0) \le c_0 n(x, A - x) \quad \text{for } x \in B(x_0, r) \text{ and } A \subset B(x_0, 2r)^c. \tag{1.5}$$

Here $A - x := \{y - x : y \in A\}$. *Then there is a constant* $r_1 >$ *depending on* $(d, \Lambda_1, \Lambda_2, \Lambda_3, c_0)$, *and* $C > 0$ *depending only on* $(d, \Lambda_1, \Lambda_2, \Lambda_3, c_0)$ *as well as the rate of* ψ *of* (1.3) *converging to* 0 *so that for any* $x_0 \in \mathbb{R}^d$, $r \in (0, r_1]$, *and any nonnegative function* u *on* $\mathbb{R}^d$ *that is* $\mathcal{L}$*-harmonic in* $B(x_0, 2r)$,

$$u(x) \le C u(y) \qquad \text{for } x, y \in B(x_0, r).$$

Our approach is mainly probabilistic. A key to establish the above two results is a hitting probability estimate (Theorem 1.4.9). For this, we establish a support theorem (Theorem 1.4.1) and a Krylov estimate (Theorem 1.4.6) for non-local operator $\mathcal{L}$. These results are of interest in their own. However, before we can do these, we need to represent the semimartingale Markov process having infinitesimal generator $\mathcal{L} \in \mathcal{N}(\Lambda_1, \Lambda_2, \Lambda_3)$ as a weak solution of an SDE driven by Brownian motion and Poisson point processes. This part is known in literature but the references are not easily accessible. We carry out the details in the Appendix §A.1 of this paper.

Parabolic Harnack inequality for non-local operator $\mathcal{L}$ is studied in Section §1.7.

In this paper, we use := as a way of definition. For a Borel or Lebesgue measurable set $A \subset \mathbb{R}^k$, we use $|A|$ to denote its Lebesgue measure.

1.2 Preliminaries

We first start with a remark on condition (1.12).

Remark 1.2.1. **(i)** If we define

$$J(x, dy) := n(x, dy - x) \tag{2.1}$$

in the sense that $J(x, A) := n(x, A - x)$ for any $A \subset \mathbb{R}^d \setminus \{x\}$, then $J(x, dy)$ is the jumping rate for jumps from x to y by the process X. Condition (1.12) is equivalent to

$$J(x_0, A) \le c_0 J(x, A) \quad \text{for } x \in B(x_0, r) \text{ and } A \subset B(x_0, 2r)^c. \tag{2.2}$$

Condition (2.2) is satisfied if $J(x, dy) = j(x, y)\nu(dy)$ for some measure ν on $\mathbb{R}^d$ and $j(x, y)$ has the property that for any $x_0 \in \mathbb{R}^d$ and $r \in (0, 1]$,

$$j(x_0, y) \le c_0 j(x, y) \quad \text{for any } x \in B(x_0, r) \text{ and } y \in B(x_0, 2r)^c. \tag{2.3}$$

We say two functions f and g are comparable and denote as $f \asymp g$ if there are two positive constants c_1, c_2 so that $c_1 g \le f \le c_2 g$ in their common domains of definition. If $j(x, y) \asymp \phi(|y-x|)$, where ϕ is a decreasing function on $(0, \infty)$ that has the property that $\phi(r) \le c\phi(2r)$ for all $r > 0$, then $j(x, y)$ satisfies (2.2).

(ii) Note that

$$\int_{\mathbb{R}^d \setminus \{0\}} (1 \wedge |z|^2) n(x, dz) = \int_{\mathbb{R}^d} (1 \wedge |x - y|^2) J(x, dy)$$

When $J(x, dy) = j(x, y)\nu(dy)$, property (c) in the definition of $\mathcal{N}(\Lambda_1, \Lambda_2, \Lambda_3)$ is equivalent to

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} (1 \wedge |x - y|^2) j(x, y)\nu(dy) \le \Lambda_3 < \infty. \tag{2.4}$$

Typical examples for measure $\nu(dy)$ on $\mathbb{R}^d$ and function $j(x, y)$ having properties (1.3), (2.3) and (2.4) are

(a) Lebesgue measure on $\mathbb{R}^d$ and $j(x, y) \asymp |x - y|^{-(d+\alpha)}$ for some $0 < \alpha < 2$;

(b) k-dimensional Lebesgue measure on a finite union of k-dimensional affine subspaces of $\mathbb{R}^d$ with $k \in [1, d]$, and $j(x, y) \asymp |x - y|^{-(k+\alpha)}$ with $0 < \alpha < 2$;

(c) More generally, ν is a Borel measure on $\mathbb{R}^d$ so that there is some $n \in (0, d]$ and $c > 0$ so that

$$\nu(B(x, r)) \le c\, r^n \quad \text{for any } x \in \mathbb{R}^d \text{ and } r \in (0, 1], \tag{2.5}$$

and $\phi(|x - y|) \asymp |x - y|^{-(n+\alpha)}$ with $0 < \alpha < 2$. Any Ahlfors n-regular measure on an n-set in $\mathbb{R}^d$ (for example, Sierpinski gasket or carpet in $\mathbb{R}^d$) with $n \in (0, d]$ has property (1.9).

(iii) Note that for any $x \in B(x_0, r/2)$, $x_0 \in B(x, r/2)$ and $B(x, r) \subset B(x_0, 2r)$, so by (1.12) (with the role of x_0 and x interchanged and with $r/2$ in place of r),

$$n(x, B - x) \le c_0 n(x_0, B - x_0) \quad \text{for } x \in B(x_0, r/2) \text{ and } B \subset B(x, r)^c.$$

Now for any $y \in B(x_0, r)$, there is some $x \in B(x_0, r/2)$ so that $|y - x| < r/2$. Since $B(y, r) \subset B(x_0, 2r)$, we have from the above display with (y, x) and $r/2$ in place of (x, x_0) and $r/2$ there that for any $A \subset B(x_0, 2r)^c$,

$$n(y, A - y) \le c_0 n(x, A - x) \le c_0^2\, n(x_0, A - x_0). \tag{2.6}$$

In other words, by increasing the value of c_0 if needed, condition (1.12) is equivalent to

$$n(x, A - x) \le c_0 n(y, A - y) \quad \text{for } x, y \in B(x_0, r) \text{ and } A \subset B(x_0, 2r)^c. \tag{2.7}$$

Another way to put it, by increasing the value of c_0 if needed, condition (2.2) is equivalent to

$$J(x, A) \le c_0 J(y, A) \quad \text{for } x, y \in B(x_0, r) \text{ and } A \subset B(x_0, 2r)^c. \tag{2.8}$$

Throughout this paper, we assume that $X = \{X_t, t \ge 0; \mathbb{P}^x, x \in \mathbb{R}^d\}$ is a conservative Hunt process on $\mathbb{R}^d$ having infinitesimal generator $\mathcal{L} \in \mathcal{N}(\Lambda_1, \Lambda_2, \Lambda_3)$ in the sense that $\mathbb{P}^x(X_0 = x) = 1$, X has infinite lifetime, and for every $f \in C_c^2(\mathbb{R}^d)$,

$$f(X_t) - f(X_0) - \int_0^t \mathcal{L} f(X_s) ds$$

is a martingale under $\mathbb{P}^x$ for every $x \in \mathbb{R}^d$. In this paper, we say a Hunt process X corresponds to $\mathcal{L}$ if $X = \{X_t, t \ge 0; \mathbb{P}^x, x \in \mathbb{R}^d\}$ is a Hunt process having infinitesimal generator $\mathcal{L}$.

A real-valued function u on $\mathbb{R}^d$ is said to be $\mathcal{L}$-harmonic in an open subset $D \subset \mathbb{R}^d$ if for every ball $B(x_0, r)$ whose closure is inside D and for every $x \in B(x_0, r)$,

$$\mathbb{E}^x \left[|u(X_{\tau_{B(x_0,r)}})| \right] < \infty \quad \text{and} \quad u(x) = \mathbb{E}^x \left[u(X_{\tau_{B(x_0,r)}}) \right].$$

The following rough scaling property will be used several times in the paper.

Lemma 1.2.2. *Suppose that $X = \{X_t, t \geq 0; \mathbb{P}^x, x \in \mathbb{R}^d\}$ is a Hunt process having infinitesimal generator $\mathcal{L} \in \mathcal{N}(\Lambda_1, \Lambda_2, \Lambda_3)$. For $\lambda \in (0,1]$, define $Y_t = \lambda^{-1} X_{\lambda^2 t}$ and $\mathbb{P}^x_Y = \mathbb{P}^{\lambda x}$. Then $\{Y_t, t \geq 0; \mathbb{P}^x_Y, x \in \mathbb{R}^d\}$ is a Hunt process having infinitesimal generator*

$$\begin{aligned}\mathcal{L}^{(\lambda)} f(x) &= \frac{1}{2}\sum_{i,j=1}^d a_{i,j}(\lambda x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \nabla f(x)\cdot\left(\lambda b(\lambda x) - \int_{\mathbb{R}^d\setminus\{0\}} \lambda z \mathbb{1}_{\{\lambda<|z|\leq 1\}} n(\lambda x, dz)\right) \\ &\quad + \int_{\mathbb{R}^d\setminus\{0\}} \left(f(x+z) - f(x) - \nabla f(x)\cdot z\mathbb{1}_{|z|\leq 1}\right)\lambda^2 n(\lambda x, d(\lambda z))\end{aligned}$$

for $f \in C^2_c(\mathbb{R}^d)$. In particular, $\mathcal{L}^{(\lambda)} \in \mathcal{N}(\Lambda_1, \Lambda_2 + \Lambda_3, \Lambda_3)$ for every $\lambda \in (0,1]$.

Proof. For $f \in C^2_c(\mathbb{R}^d)$, define $f_\lambda(x) = f(x/\lambda)$. Denote by $\{\mathbb{P}^Y_t; t \geq 0\}$ the transition semigroup of Y. Then

$$P^Y_t f(x) = \mathbb{E}^x\left[f(Y_t)\right] = \mathbb{E}^{\lambda x}\left[f(\lambda^{-1}X_{\lambda^2 t})\right] = (P_{\lambda^2 t} f_\lambda)(\lambda x).$$

Thus the generator $\mathcal{L}^Y$ of Y is

$$\begin{aligned}\mathcal{L}^Y f(x) &= \lim_{t\to 0}\frac{P^Y_t f(x) - f(x)}{t} \\ &= \lim_{t\to 0}\frac{P_{\lambda^2 t} f_\lambda(\lambda x) - f_\lambda(\lambda x)}{t} = \lambda^2(\mathcal{L} f_\lambda)(\lambda x) \\ &= \frac{1}{2}\sum_{i,j=1}^d a_{i,j}(\lambda x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \lambda b(\lambda x)\cdot\nabla f(x) \\ &\quad + \int_{\mathbb{R}^d\setminus\{0\}}\left(f(x+z) - f(x) - z\mathbb{1}_{\{|z|\leq 1/\lambda\}}\cdot\nabla f(x)\right)\lambda^2 n(\lambda x, d(\lambda z)) \\ &= \frac{1}{2}\sum_{i,j=1}^d a_{i,j}(\lambda x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \nabla f(x)\left(\lambda b(\lambda x) - \int_{\mathbb{R}^d\setminus\{0\}}\lambda^2 z\mathbb{1}_{\{1<|z|\leq 1/\lambda\}} n(\lambda x, d(\lambda z))\right) \\ &\quad + \int_{\mathbb{R}^d\setminus\{0\}}\left(f(x+z) - f(x) - z\mathbb{1}_{\{|z|\leq 1\}}\cdot\nabla f(x)\right)\lambda^2 n(\lambda x, d(\lambda z)) \\ &= \frac{1}{2}\sum_{i,j=1}^d a_{i,j}(\lambda x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \nabla f(x)\left(\lambda b(\lambda x) - \int_{\mathbb{R}^d\setminus\{0\}}\lambda w\mathbb{1}_{\{\lambda<|w|\leq 1\}} n(\lambda x, dw)\right) \\ &\quad + \int_{\mathbb{R}^d\setminus\{0\}}\left(f(x+z) - f(x) - z\mathbb{1}_{\{|z|\leq 1\}}\cdot\nabla f(x)\right)\lambda^2 n(\lambda x, d(\lambda z)) \\ &= \mathcal{L}^{(\lambda)} f(x).\end{aligned}$$

Hence the process Y has diffusion matrix $a(\lambda x)$ which satisfies the same uniform ellipticity condition (1.2) as $a(x)$, drift coefficient

$$b_\lambda(x) := \lambda b(\lambda x) - \int_{\mathbb{R}^d\setminus\{0\}}\lambda w\mathbb{1}_{\{\lambda<|w|\leq 1\}} n(\lambda x, dw)$$

and the jumping measure $n_\lambda(x,dz) := \lambda^2 n(\lambda x, d(\lambda z))$. Clearly for $\lambda \in (0,1]$,

$$\|b_\lambda\|_\infty \le \lambda \|b\|_\infty + \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d\setminus\{0\}} w^2 \mathbb{1}_{\{\lambda<|w|\le 1\}} n(x,dw) \le \Lambda_2 + \Lambda_3$$

and

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d\setminus\{0\}} (1\wedge |z|^2) n_\lambda(x,dz) = \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d\setminus\{0\}} (\lambda^2 \wedge |w|^2) n(\lambda x, dw) \le \Lambda_3.$$

This shows that $\mathcal{L}^{(\lambda)} \in \mathcal{N}(\Lambda_1, \Lambda_2+\Lambda_3, \Lambda_3)$ for every $\lambda \in (0,1]$. □

Lemma 1.2.3. *There are $r_0 \in (0,1/4]$ and $c_1 > 0$ that depend only on $(d, \Lambda_1, \Lambda_2, \Lambda_3)$ so that for every Hunt process X which corresponds to the generator $\mathcal{L} \in \mathcal{N}(\Lambda_1,\Lambda_2,\Lambda_3)$ and for every $x_0 \in \mathbb{R}^d$ and $r \in (0, r_0]$,*

$$\mathbb{E}^x\left[\tau_{B(x_0,r)}\right] \le c_1 r^2 \quad \text{for every } x \in B(x_0,r).$$

Here $\tau_{B(x_0,r)} := \inf\{t \ge 0 : X_t \notin B(x_0,r)\}$ is the first exit time from $B(x_0,r)$ for the process X.

Proof. Fix a smooth non-decreasing function φ on $[0,\infty)$ so that $\varphi(r) = r$ for $r \in (0,3/2]$ and $\varphi(r) = 4$ for $r \ge 4$. For each $y_0 \in \mathbb{R}^d$, let $f(x) := \varphi(|x-y_0|^2)$.

Note that $M_t^f := f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)ds$ is a martingale under $\mathbb{P}^x$ for every $x \in \mathbb{R}^d$. Take $R_0 := \frac{\Lambda_1}{4(\Lambda_2+\Lambda_3)} \wedge \frac{1}{2}$. Then

$$\mathbb{E}^{y_0}\left[f(X_{t\wedge\tau_{B(y_0,R_0)}})\right] = \mathbb{E}^{y_0}\left[\int_0^{t\wedge\tau_{B(y_0,R_0)}} \mathcal{L}f(X_s)ds\right]. \tag{2.9}$$

Note that $\partial_{ij}f(x) = 2\delta_{ij}$ and so $\frac{1}{2}\sum_{1\le i,j\le d} a_{ij}(x)\partial_{ij}^2 f(x) \ge \Lambda_1 d$ for $x \in B(y_0,3/2)$, while $|\nabla f(x)| \le 2R_0 d$ for $x \in B(y_0,R_0)$. Hence

$$\frac{1}{2}\sum_{1\le i,j\le d} a_{ij}(x)\partial_{ij}^2 f(x) + b(x)\cdot\nabla f(x) \ge \Lambda_1 d - 2R_0 d\|b\|_\infty \ge \Lambda_1 d/2, \quad x \in B(y_0,R_0), \tag{2.10}$$

for any $\|b\|_\infty \le \Lambda_2+\Lambda_3$. By the convexity of f in $B(y_0,3/2)$, we have

$$f(x+z) - f(x) - \nabla f(x)\cdot z \ge 0 \quad \text{for } x \in B(y_0,R_0) \text{ and } |z| \le 1.$$

By the definition of $f(x)$, we have $f(x) \le 1/4$ for $x \in B(y_0,R_0) \subset B(y_0,1/2)$ and $f(y) \ge 1/4$ for $y \in B(y_0,1/2)^c$. Hence we have

$$f(x+z) - f(x) \ge 0 \quad \text{for } x \in B(y_0,R_0) \text{ and } |z| \ge 1.$$

It follows from the last two displays that

$$\int_{\mathbb{R}^d\setminus\{0\}} \Big(f(x+z) - f(x) - \nabla f(x)\cdot z \mathbb{1}_{\{|z|\le 1\}} \Big)\, n(x,dz) \ge 0 \quad \text{for } x\in B(y_0,R_0). \tag{2.11}$$

This together with (1.8), implies that $\mathcal{L}f(x) \ge \Lambda_1 d/2$ on $B(y_0,R_0)$. We now have by (1.6) that

$$\frac{\Lambda_1 d}{2}\mathbb{E}^{y_0}\left[t\wedge \tau_{B(y_0,R_0)}\right] \le \mathbb{E}^{y_0}\left[f(X_{t\wedge \tau_{B(y_0,R_0)}})\right].$$

Taking $t\to\infty$ yields

$$\mathbb{E}^{y_0}\left[\tau_{B(y_0,R_0)}\right] \le \frac{2}{\Lambda_1 d}\mathbb{E}^{y_0}\left[f(X_{\tau_{B(y_0,R_0)}})\right] \le \frac{8}{\Lambda_1 d} \quad \text{for every } y_0\in\mathbb{R}^d.$$

Define $r_0 = R_0/2$. Then for every $x_0\in\mathbb{R}^d$ and $r\in(0,r_0]$,

$$\mathbb{E}^{x}\left[\tau_{B(x_0,r_0)}\right] \le \mathbb{E}^{x}\left[\tau_{B(x,2r_0)}\right] \le \frac{8}{\Lambda_1 d} \quad \text{for every } x\in B(x_0,r_0). \tag{2.12}$$

The above holds for any $\mathcal{L}$ that satisfying (1.2) and for any b with $\|b\|_\infty \le \Lambda_2+\Lambda_3$.

Now for any Hunt process with infinitesimal generator $\mathcal{L}\in\mathcal{N}(\Lambda_1,\Lambda_2,\Lambda_3)$, clearly (2.12) holds. For any $r\in(0,r_0]$, let $\lambda = r/r_0$ and define $Y_t = \lambda^{-1}X_{\lambda^2 t}$. Note that since $\lambda\in(0,1]$, the generator $\mathcal{L}^Y$ of Y is in $\mathcal{N}(\Lambda_1,\Lambda_2+\Lambda_3,\Lambda_3)$ by Lemma 1.2.2. Thus the estimate (2.12) holds for Y. For every $x_0\in\mathbb{R}^d$, note that

$$\tau_{B(x_0,r)} = \lambda^2 \tau^Y_{B(x_0/\lambda, r_0)}$$

and so for every $x\in B(x_0,r)$,

$$\mathbb{E}^{x}\left[\tau_{B(x_0,r)}\right] = \mathbb{E}^{x/\lambda}\left[\lambda^2\tau^Y_{B(x_0/\lambda,r_0)}\right] \le \frac{8}{\Lambda_1 d}\,\frac{r^2}{r_0^2} =: c_1 r^2.$$

This proves the lemma. □

Corollary 1.2.4. *Assume that the condition of Lemma 1.2.3 holds and let* $r_0\in(0,1/4]$ *and* $c_1>0$ *be the constants in Lemma 1.2.3. For every* $x_0\in\mathbb{R}^d$ *and* $r\in(0,r_0]$,

$$\mathbb{E}^{x}\left[\tau^2_{B(x_0,r)}\right] \le 2c_1^2 r^4 \quad \textit{for } x\in B(x_0,r).$$

Proof. By Lemma 1.2.3 and the Markov property of X, we have for every $x\in B(x_0,r)$,

$$\mathbb{E}^{x}\left[\tau^2_{B(x_0,r)}\right] \;=\; 2\mathbb{E}^{x}\left[\int_0^{\tau_{B(x_0,r)}} \mathbb{1}_{B(x_0,r)}(X_s)\left(\int_s^{\tau_{B(x_0,r)}} \mathbb{1}_{B(x_0,r)}(X_u)du\right)ds\right]$$

$$
\begin{aligned}
&= 2\mathbb{E}^x \left[\int_0^{\tau_{B(x_0,r)}} \mathbb{E}^{X_s} \left[\tau_{B(x_0,r)} \right] ds \right] \\
&\le 2c_1 r^2 \mathbb{E}^x \left[\tau_{B(x_0,r)} \right] \le 2c_1^2 r^4.
\end{aligned}
$$

□

Remark 1.2.5. *Sometimes it is more convenient to use cubes instead of balls. For $x_0 \in \mathbb{R}^d$ and $r > 0$, denote by $\mathcal{Q}(x_0, r)$ the open cube in $\mathbb{R}^d$ centered at x_0 with side length r. Clearly,*

$$
\mathcal{Q}(x_0, 2r/\sqrt{d}) \subset B(x_0, r) \subset \mathcal{Q}(x_0, 2r) \tag{2.13}
$$

and so

$$
\tau_{\mathcal{Q}(x_0,2r/\sqrt{d})} \le \tau_{B(x_0,r)} \le \tau_{\mathcal{Q}(x_0,2r)}. \tag{2.14}
$$

Lemma 1.2.6. *Suppose that $X = (X_t, t \ge 0; \mathbb{P}^x, x \in \mathbb{R}^d\}$ is a Hunt process having infinitesimal generator $\mathcal{L} \in \mathcal{N}(\Lambda_1, \Lambda_2, \Lambda_3)$. For any $D > 0$, there is a positive constant c_2 that depends only on $(d, \Lambda_1, \Lambda_2, \Lambda_3, \gamma)$ so that for every $x_0 \in \mathbb{R}^d$ and $r \in (0, 1]$,*

$$
\mathbb{E}^x \left[\tau_{B(x_0,r)} \right] \ge \mathbb{E}^x \left[\tau_{B(x_0,r)} \wedge (\gamma r^2) \right] \ge c_2 r^2 \quad \textit{for every } x \in B(x_0, r/2).
$$

Proof. Let $\varphi(s)$ be a smooth nondecreasing function on $[0,\infty)$ so that $0 \le \varphi \le 6$, $0 \le \varphi'(s) \le 2$, $\varphi(s) = s$ for $s \in [0,4]$, and $\varphi(s) = 6$ when $s \ge 5$. Fix $x_0 \in \mathbb{R}^d$ and set $f(x) := \varphi(|x-x_0|^2)$. Note that $|\nabla f(x)| \le 2|x-x_0| \le 2$ and $\partial_{ij} f(x) = 2\delta_{ij}$ for $x \in B(x_0,1)$. Consequently, $|\frac{1}{2}\sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2 f(x)}{\partial x_i \partial x_j}| \le d/\Lambda_1$, and $|\sum_{i,j=1}^d b_i(x) \frac{\partial f(x)}{\partial x_i}| \le 2\Lambda_2 d$ on $B(x_0,1)$. Note also for any $x \in \mathbb{R}^d$,

$$
\begin{aligned}
& |\int_{\mathbb{R}^d} (f(x+z) - f(x) - \nabla f(x) \cdot \mathbb{1}_{\{|z| \le 1\}} z) n(x, dz)| \\
\le\ & \int_{|z| \le 1} |f(x+z) - f(x) - \nabla f(x) \cdot \mathbb{1}_{\{|z| \le 1\}} z| n(x, dz) \\
& + \int_{|z|>1} |f(x+z) - f(x)| n(x, dz) \\
\le\ & \frac{1}{2} \int_{|z| \le 1} |z|^2 \|D^2 f\|_\infty n(x, dz) + 12 \int_{|z|>1} n(x, dz) \\
\le\ & \max\{\|D^2 f\|_\infty / 2, 12\} \Lambda_3 \le 12\Lambda_3
\end{aligned}
$$

Thus

$$
|\mathcal{L} f(x)| \le \frac{d}{\Lambda_1} + 2\Lambda_2 d + 12\Lambda_3 =: a_1 \quad \text{for every } x \in B(x_0, 1). \tag{2.15}
$$

Let $r \in (0,1]$ and $x_0 \in \mathbb{R}^d$. For every $x \in B(x_0, r/2)$,

$$\begin{aligned} \mathbb{E}^x \left[f(X_{t\wedge\tau_{B(x_0,r)}}) \right] &= f(x) + \mathbb{E}^x \left[\int_0^{t\wedge\tau_{B(x_0,r)}} \mathcal{L}f(X_s)ds \right] \\ &\le r^2/4 + a_1 \mathbb{E}^x \left[t \wedge \tau_{B(x_0,r)} \right]. \end{aligned} \tag{2.16}$$

Let $c_1 > 0$ and $r_0 \in (0, 1/4]$ be the constants from Lemma 1.2.3 and define $\gamma_0 = 2c_1$. By Lemma 1.2.3, for every $x_0 \in \mathbb{R}^d$ and $r \in (0, r_0]$,

$$\mathbb{P}^x(\tau_{B(x_0,r)} \ge \gamma_0 r^2) \le \frac{\mathbb{E}^x[\tau_{B(x_0,r)}]}{\gamma_0 r^2} \le \frac{c_1 r^2}{\gamma_0 r^2} = 1/2 \quad \text{for every } x \in B(x_0, r). \tag{2.17}$$

Consequently,

$$\mathbb{E}^x \left[f(X_{(\gamma_0 r^2)\wedge\tau_{B(x_0,r)}}) \right] \ge \mathbb{E}^x \left[f(X_{\tau_{B(x_0,r)}}); \tau_{B(x_0,r)} < \gamma_0 r^2 \right] \ge r^2\, \mathbb{P}^x(\tau_{B(x_0,r)} < \gamma_0 r^2) > r^2/2. \tag{2.18}$$

Taking $t = \gamma_0 r^2$ in (2.16) yields that for every $x_0 \in \mathbb{R}^d$ and $r \in (0, r_0]$,

$$\mathbb{E}^x \left[(\gamma_0 r^2) \wedge \tau_{B(x_0,r)} \right] > r^2/(4a_1) \quad \text{for every } x \in B(x_0, r/2).$$

When $r \in (r_0, 1]$, we clearly have for every $x \in B(x_0, r/2)$,

$$\mathbb{E}^x \left[(\gamma_0 r^2) \wedge \tau_{B(x_0,r)} \right] \ge \mathbb{E}^x \left[(\gamma_0 (r_0/2)^2) \wedge \tau_{B(x,r_0/2)} \right] > r_0^2/(16a_1) \ge r_0^2 r^2/(16a_1). \tag{2.19}$$

For $\gamma \in (0, \gamma_0)$, by (2.19),

$$\mathbb{E}^x \left[\tau_{B(x_0,r)} \wedge (\gamma r^2) \right] \ge (\gamma/\gamma_0) \mathbb{E}^x \left[\tau_{B(x_0,r)} \wedge (\gamma_0 r^2) \right] \ge \frac{\gamma r_0^2}{16 a_1 \gamma_0} r^2 \quad \text{for } x \in B(x_0, r/2).$$

This established the desired inequality by taking $c_2 = \gamma r_0^2/(16a_1\gamma_0)$. □

Corollary 1.2.7. *Under the condition of Lemma 1.2.6, for any $\gamma > 0$, there exists $c_3 > 0$ depending on $(d, \Lambda_1, \Lambda_2, \Lambda_3, \gamma)$ so that for every $x_0 \in \mathbb{R}^d, r \in (0, 2]$,*

$$\mathbb{E}^x[\tau_{Q(x_0,2r)}] \ge \mathbb{E}^x[\tau_{Q(x_0,2r)} \wedge (\gamma r^2)] \ge c_3 r^2 \quad \textit{for } x \in Q(x_0, r).$$

Proof. Note that for any $x \in Q(x_0, r), B(x, r/8) \subset Q(x_0, 2r)$,then by Lemma 1.2.6,

$$\mathbb{E}^x[\tau_{Q(x_0,2r)} \wedge (\gamma r^2)] \ge \mathbb{E}^x[\tau_{B(x,r/8)} \wedge (\gamma r^2)] \ge c_3 r^2.$$

□

Next, we need a lemma that will be used later to prove Harnack inequality. It is the only place requiring condition (1.12). But we need a special case of Proposition **??** in the Appendix beforehand.

Proposition 1.2.8. *If A and B are two Borel sets with positive distance apart, then for each x,*

$$\sum_{s\le t} \mathbb{1}_{(X_{s-}\in A, X_s\in B)} - \int_0^t \int_{B-X_s} \mathbb{1}_A(X_s) n(X_s, dz) ds \tag{2.20}$$

is a $\mathbb{P}^x$-martingale.

Proof. Following the approach of the proof in [19], since $\{X_t, t\ge 0; \mathbb{P}^x, x\in\mathbb{R}^d\}$ is a Hunt process having infinitesimal generator $\mathcal{L}$. That is for any $f\in C_c^\infty(\mathbb{R}^d)$,

$$M_t^f := f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)ds \text{ is a martingale under } \mathbb{P}^x. \tag{2.21}$$

In particular, it implies $X_t = (X_t^1, ..., X_t^d)$ is a semi-martingale.By Ito's formula, for any $f\in C_c^\infty(\mathbb{R}^d)$,we have

$$f(X_t) - f(X_0) = \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(X_{s^-})dX_s^i + \sum_{0<s\le t}\eta_s(f) + \frac{1}{2}A_t(f), \tag{2.22}$$

where

$$\eta_s(f) = f(X_s) - f(X_{s^-}) - \sum_{i=1}^d \frac{\partial f}{\partial x_i}(X_{s^-})(X_s - X_{s^-}) \tag{2.23}$$

and

$$A_t(f) = \sum_{i,j=1}^d \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X_{s^-})d\langle (X^i)^c, (X^j)^c\rangle_s. \tag{2.24}$$

Let A and B be two bounded closed Borel sets with positive distance apart.Let $f\in C_c^\infty(\mathbb{R}^d)$ with $f=0$ on A and $f=1$ on B. Then it is clear that $N_t^f := \int_0^t \mathbb{1}_A(X_{s^-})dM_s^f$ is a martingale. Then by (2.21), (2.22),(2.23),and (2.24), we have

$$\begin{aligned} N_t^f : &= \sum_{i=1}^d \int_0^t \mathbb{1}_A(X_{s^-})\frac{\partial f}{\partial x_i}(X_{s^-})dX_s^i + \sum_{0<s\le t}\mathbb{1}_A(X_{s^-})[(f(X_s) - f(X_{s^-}) - \sum_{i=1}^d \frac{\partial f}{\partial x_i}(X_{s^-})(X_s - X_{s^-})] \\ &+ \frac{1}{2}\int_0^t \mathbb{1}_A(X_{s^-})\frac{\partial^2 f}{\partial x_i \partial x_j}(X_{s^-})d\langle (X^i)^c, (X^j)^c\rangle_s - \int_0^t \mathbb{1}_A(X_{s^-})\mathcal{L}f(X_{s^-})ds \end{aligned}$$

$$= \sum_{0<s\le t} \mathbb{1}_A(X_{s^-}) f(X_s) - \int_0^t \mathbb{1}_A(X_{s^-}) \mathcal{L} f(X_{s^-}) ds$$
$$= \sum_{0<s\le t} \mathbb{1}_A(X_{s^-}) f(X_s) - \int_0^t \mathbb{1}_A(X_{s^-}) \int_{\mathbb{R}^d\setminus\{0\}} f(X_{s^-}+z) n(X_{s^-}, dz) ds, \tag{2.25}$$

where the second inequality is because $\mathbb{1}_A(x)f(x) = 0$.

By taking a sequence of functions $f_n \in C_c^\infty(\mathbb{R}^d)$ with $f_n = 0$ on A,$f_n = 1$ on B and $f_n \downarrow \mathbb{1}_B$, for any $x \in \mathbb{R}^d$, and the fact that $X_{s^-} \neq X_s$ for only countably many values of s,we have

$$\sum_{0<s\le t} \mathbb{1}_A(X_{s^-}) \mathbb{1}_B(X_s) - \int_0^t \mathbb{1}_A(X_s) \int_{B-X_s} n(X_s, dz) ds$$

is a martingale under $\mathbb{P}^x$.

Since any Borel set can be approximated by bounded closed set [13, Theorem1.1], then by taking a sequence of bounded closed set $\{A_n\}_{n\ge1} \uparrow A$, and $\{B_n\}_{n\ge1} \uparrow B$, and montone convergence theorem, (2.20) holds.

Remark 1.2.9. *By taking limits on a sequence of Borel sets which have positive distance apart,* (2.20) *holds for two disjoint Borel sets A and B.*

Proposition 1.2.10. *Suppose that $X = (X_t, t \ge 0; \mathbb{P}^x, x \in \mathbb{R}^d\}$ is a Hunt process having infinitesimal generator $\mathcal{L} \in \mathcal{N}(\Lambda_1, \Lambda_2, \Lambda_3)$. Assume condition* (1.12) *holds. There is a constant $c_4 > 0$ that depends only on $d, \Lambda_1, \Lambda_2, \Lambda_3$ and c_0 such that for each $x_0 \in \mathbb{R}^d$, $r \in (0, r_0]$ and every nonnegative measurable bounded function φ supported in $B(x_0, 2r)^c$,*

$$\mathbb{E}^x \left[\varphi(X_{\tau_{B(x_0,r)}})\right] \le c_4 \mathbb{E}^y \left[\varphi(X_{\tau_{B(x_0,r)}})\right] \quad \text{for } x, y \in B(x_0, r/2),$$

where $r_0 \in (0, 1/4)$ is the constant in Lemma 1.2.3.

Proof. The proof is quite straightforward using the Lévy system of X. Since any nonnegative bounded measurable function is a limit of a linear combination of the indicator functions, it suffices to consider $\varphi(x) = \mathbb{1}_F(x)$, where F is a closed set contained in $B(x_0, 2r)^c$. By the Lévy system formula Proposition 1.2.8 with $A = B(x_0, r)$ and $B = F$, and Lemma 1.2.6 , for each $y \in B(x_0, r/2)$,

$$\begin{aligned}
\mathbb{P}^y\left(X_{\tau_{B(x_0,r)}} \in F\right) &= \mathbb{E}^y \sum_{s\le \tau_{B(x_0,r)}} \mathbb{1}_{(X_{s-}\neq X_s, X_s\in F)} \\
&= \mathbb{E}^y \int_0^{\tau_{B(x_0,r)}} n(X_s, F-X_s)ds \\
&\geq c_0^{-1} n(x_0, F-x_0)\,\mathbb{E}^y\left[\tau_{B(x_0,r)}\right] \\
&\geq c_0^{-1} c_3 r^2 n(x_0, F-x_0).
\end{aligned}$$

Similarly, we have by (2.6), Lemma 1.2.3 and (2.13) that for any $x \in B(x_0, r/2)$,

$$\begin{aligned}
\mathbb{P}^x\left(X_{\tau_{B(x_0,r)}} \in F\right) &= \mathbb{E}^x \int_0^{\tau_{B(x_0,r))}} n(X_s, F-X_s)ds \\
&\leq c_0^2\, n(x_0, F-x_0)\mathbb{E}^x\left[\tau_{B(x_0,r}\right] \\
&\leq c_0^2 c_1 r^2\, n(x_0, F-x_0).
\end{aligned}$$

This proves that for any closed set $F \subset B(x_0, 2r)^c$,

$$\mathbb{P}^x\left(X_{\tau_{B(x_0,r)}} \in F\right) \leq c_4\,\mathbb{P}^y\left(X_{\tau_{B(x_0,r)}} \in F\right) \quad \text{for every } x, y \in B(x_0, r/2).$$

with $c_4 = c_0^3 c_1/c_3$. So the lemma holds. □

To study the hitting probability for the associated process, we need an SDE representation for the solution of the martingale problem for $\mathcal{L}$.

1.3 Martingale problems and SDEs

Let $\{X_t, t \geq 0; \mathbb{P}^x, x \in \mathbb{R}^d\}$ be a Hunt process having infinitesimal generator $\mathcal{L} \in \mathcal{N}(\Lambda_1, \Lambda_2, \Lambda_3)$. Define

$$Y_t = \begin{cases} \Delta X_t & \text{if } \Delta X_t \neq 0 \\ 0. & \text{otherwise} \end{cases}$$

Since $X = \{X_t, t \geq 0; \mathbb{P}^x, x \in \mathbb{R}^d\}$ is a Hunt process, we know that Y is a quasi-left continuous and σ-finite point process, and Y is adapted with respect to the right-continuous filtration $\{\mathcal{F}_t, t \geq 0\}$ generated by the process X. According to Proposition A.1.17, and Theorem A.1.26 in the Appendix, we know that there exists a diffusive σ-finite measure λ on $\mathbb{R}^d \setminus \{0\}$ such that

$$n(x, \mathbb{R}^d \setminus \{0\}) \leq \lambda(\mathbb{R}^d \setminus \{0\}) \quad \text{for every } x \in \mathbb{R}^d, \tag{3.1}$$

and a $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^d \setminus \{0\})$- measurable function $F(x,z)$ on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{0\}$ such that

$$Y_s = F(X_{s^-}, \widehat{Y}_s), \tag{3.2}$$

where $\widehat{Y}$ is a Poisson point process with the intensity measure $ds \times \lambda(dx)$.

From Theorem A.1.26, for any $A \in \mathbb{R}^d$, then

$$\int_{\mathbb{R}^d \setminus \{0\}} \mathbb{1}_A(F(x,z))\lambda(dz) = \int_{\mathbb{R}^d \setminus \{0\}} \mathbb{1}_A(z) n(x,dz).$$

Since $\mathcal{L} \in \mathcal{N}(\Lambda_1, \Lambda_2, \Lambda_3)$, by Proposition A.1.17, we know that the measure λ also satisfies

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d \setminus \{0\}} (|F(x,z)|^2 \wedge 1)\lambda(dz) \leq \Lambda_3. \tag{3.3}$$

Define the random measure μ on $(0,\infty) \times \mathbb{R}^d$ such that

$$\mu([0,t] \times A) = \sum_{s \leq t} \mathbb{1}_{\{\Delta X_s \in F(X_{s-}, A)\}}. \tag{3.4}$$

So from (3.2), μ is a Poisson random measure under $\mathbb{P}^x$ with the intensity measure $\nu([0,t] \times A) = t\lambda(A)$.

The following theorem relates the existence of the solution $\{X_t, t \geq 0; \mathbb{P}^x\}$ to the martingale problem with respect to our nonlocal operator $\mathcal{L}$ to the existence of the solution of the SDE (3.6).This theorem is very important in proving the support theorem and an Krylov's inequality in Section 2.5. The outline of the proof is stated in [56, Section II, Theorem 10]. We also give a very detailed proof in the Appendix §A.1.

Proposition 1.3.1. *Suppose that $X = \{X_t, t \geq 0; \mathbb{P}^x\}$ is the solution to the martingale problem starting from x for the non-local operator $\mathcal{L} \in \mathcal{N}(\Lambda_1, \Lambda_2, \Lambda_3)$ with the jump kernel $n(x,dz)$.*

Then there exist a $d \times d$ matrix-valued function $\sigma(x)$, an $\mathbb{R}^d$-valued function $b(x)$, an independent probability space $(\widehat{\Omega}, \widehat{\mathscr{F}}, \widehat{\mathscr{F}_t})$, a d-dimensional Brownian motion W, a $\mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^d \setminus \{0\})$-measurable function $F(x,z)$, a Poisson point process $\widehat{Y}$, independent of W, with the intensity measure $dt \times \lambda(dx)$, where λ is a diffusive σ-finite measure satisfying (3.1) and

$$\int_{\mathbb{R}^d \setminus \{0\}} f(z) n(x,dz) = \int_{\mathbb{R}^d \setminus \{0\}} f(F(x,z))\lambda(dz) \text{ for any measurable function } f, \tag{3.5}$$

so that X has the same distribution as the solution to the following SDE:

$$\begin{aligned} dX_t &= \sigma(X_t)dW_t + b(X_t)dt + \int_{\{|F(X_{t-},z)|\le 1\}} F(X_{t-},z)(\mu-\nu)(dt,dz) \\ &\quad + \int_{\{|F(X_{t-},z)|>1\}} F(X_{t-},z)\mu(dt,dz), \\ X_0 &= x, \end{aligned} \tag{3.6}$$

where μ is the Poisson random measure on $[0,\infty)\times\mathbb{R}^d$ satisfying (3.4) with intensity measure $dt\times\nu(dx)$.

Proof. See the proof of Theorem A.1.33 in the Appendix .

1.4 Support theorem for diffusion processes with jumps

In this section, we establish support theorem for jump diffusion X having infinitesimal generator $\mathcal{L}\in\mathcal{N}(\Lambda_1,\Lambda_2,\Lambda_3)$ of (1.1). By (3.6), the process X satisfies the following SDE

$$\begin{aligned} dX_t &= \sigma(X_t)dW_t + b(X_t)dt + \int_{\{|F(X_{t-},z)|\le 1\}} F(X_{t-},z)(\mu-\nu)(dz,dt) \\ &\quad + \int_{\{|F(X_{t-},z)|>1\}} F(X_{t-},z)\mu(dz,dt), \end{aligned} \tag{4.1}$$

where W is a d-dimensional Brownian motion and $\sigma(x)$ is a $d\times d$ symmetric matrix-valued function satisfying $(\sigma\sigma^T)_{ij}=a_{ij}$. As $\mathcal{L}\in\mathcal{N}(\Lambda_1,\Lambda_2,\Lambda_3)$, σ is bounded and uniformly elliptic.

1.4.1 Meyer's construction

We first review Meyer's construction on adding/removing jumps.

For $\delta\in(0,1)$, define

$$n_\delta(x,dz)=\mathbb{1}_{\{|z|\le\delta\}}n(x,dz) \quad\text{and}\quad \lambda_\delta(x)=n(x,\{z:|z|>\delta\}). \tag{4.2}$$

Note that

$$\|\lambda_\delta\|_\infty := \sup_{x\in\mathbb{R}^d}\lambda_\delta(x)\le\delta^{-2}\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}(|z|^2\wedge 1)n(x,dz)\le\Lambda_3/\delta^2. \tag{4.3}$$

For fixed $\delta \in (0,1)$, let $\bar{X} = \{\bar{X}_t; t \geq 0; \bar{\mathbb{P}}^x, x \in \mathbb{R}^d\}$ be the jump diffusion having generator $\mathcal{L}^{(\delta)}$ which is the same as $\mathcal{L}$ but with $n_\delta(x,dz)$ in place of $n(x,dz)$. By (3.6), the process $\bar{X}$ is a weak solution to the following SDE:

$$d\bar{X}_t = \sigma(\bar{X}_t)dW_t + b(\bar{X}_t)dt + \int_{\{z:|F(\bar{X}_{t-},z)|\leq\delta\}} F(\bar{X}_{t-},z)(\mu-\nu)(dz,dt). \tag{4.4}$$

We can construct X from $\bar{X}$ through the following procedure. Let U_1 be an exponential random variable with parameter 1 that is independent of $(\bar{X},W)$. Run a copy of $\bar{X}$ starting from $x_0 \in \mathbb{R}^d$. Define

$$T_1 = \inf\left\{t > 0 : \int_0^t \lambda_\delta(\bar{X}_s)ds > U_1\right\}.$$

Define

$$Y_t = \begin{cases} \bar{X}_t & \text{for } t \in (0,T_1), \\ \bar{X}_{T_1-} + \xi & \text{for } t = T_1, \end{cases}$$

where $\xi \in \mathbb{R}^d$ is a random point chosen according to the probability measure

$$\frac{\mathbb{1}_{\{z:|z|>\delta\}} n(X_{T_1-},dz)}{\lambda_\delta(X_{T_1})}.$$

Let U_2 be another exponential random variable with parameter 1 which is independent of $(\bar{X},W,U_1)$. Run another copy of $\bar{X}$ but starting from Y_{T_1}, call it $\bar{X}'$, till time

$$T_2 = \inf\left\{t > 0 : \int_0^t \lambda_\delta(\bar{X}'_s)ds > U_2\right\},$$

where U_2 is another exponential random variable with parameter 1 that is independent of $\{T_t, t \in [0,T]\}$ and $\bar{X}'$. Define

$$Y_t = \begin{cases} \bar{X}'_{t-T_1} & \text{for } t \in [T_1, T_1+T_2), \\ \bar{X}'_{T_1-} + \xi' & \text{for } t = T_1+T_2, \end{cases}$$

where $\xi' \in \mathbb{R}^d$ is a random point chosen according to the probability measure

$$\frac{\mathbb{1}_{\{z:|z|>\delta\}} n(X'_{T_2-},dz)}{\lambda_\delta(X'_{T_2})}.$$

This defines a process Y on $[0,T_1+T_2]$. Repeating this procedure we get a process Y on $[0,\infty)$, as λ_δ is bounded and so $\sum_{k=1}^\infty T_k = \infty$. It is easy to see that Y has the same distribution as X in (4.1). See [65] for details. We have by [5, Lemma 2.6] that for any $A \in \sigma\{\bar{X}(t) : t \in [0,1]\}$,

$$\mathbb{P}^x(\{X_s = \bar{X}_s \text{ for all } 0 \le s \le 1\} \cap A) \ge e^{-\|\lambda_\delta\|_\infty} \bar{\mathbb{P}}^x(A). \tag{4.5}$$

1.4.2 The Support theorem

The following result has been claimed in [39, Theorem 4.2] but its proof contains a critical error when doing lower bound estimate on the probability $\mathbb{Q}(\sup_{t \le t_0} |D_t| \ge \varepsilon)$ in the last line of Page 33 there. We present a different approach here.

Theorem 1.4.1. *Suppose $\mathcal{L} \in \mathcal{N}(\Lambda_1, \Lambda_2, \Lambda_3)$ satisfying condition (1.3) and X is a Hunt process that solves the corresponding martingale problem for $(\mathcal{L}, C_b^2(\mathbb{R}^d))$. For any $\gamma \in (0,1]$ and $a > 0$, there exists a non-decreasing positive function $\Phi^{(\gamma,a)}$ on $(0,\infty)$ that depends on $d, \Lambda_1, \Lambda_2, \Lambda_3$, γ, a and the rate of the function ψ in (1.3) converging to 0 so that for any Lipschitz continuous function with $\phi : [0,\gamma] \to \mathbb{R}^d$, $\sup_{t \in [0,\gamma]} |\phi'(t)| \le a$ for a.e. $t \in [0,\gamma]$ and $\phi(0) = x_0$,*

$$\mathbb{P}^{x_0}\left(\sup_{t \in (0,\gamma]} |X_t - \phi(t)| < \varepsilon \right) \ge \Phi^{(\gamma,a)}(\varepsilon). \tag{4.6}$$

Moreover, the function $\Phi^{(\gamma,a)}$ is decreasing in $a \in (0,\infty)$ and increasing in γ.

Proof. (i) Given $\varepsilon \in (0,1/2)$, we first consider process $\bar{X}$ in (4.4) with suitably chosen $\delta = \delta(\varepsilon) \in (0,1)$ whose value will be specified below. For $t \in (0,\gamma]$, define

$$M_t := \exp\left(\int_0^t \langle \sigma^{-1}(\bar{X}_{s-})(\phi'(s) - b(\bar{X}_s)), dW_s \rangle - \frac{1}{2} \int_0^t |\sigma^{-1}(\bar{X}_{s-})(\phi'(s) - b(\bar{X}_s))|^2 ds \right).$$

Since

$$\begin{aligned}
& \bar{\mathbb{E}}^x \left[M_t^2 \right] \\
= \ & \bar{\mathbb{E}}^x \Big[\exp\left(\int_0^t 2\langle \sigma^{-1}(\bar{X}_{s-})(\phi'(s) - b(\bar{X}_s)), d\widetilde{W}_s \rangle - \frac{1}{2} \int_0^t |2\sigma^{-1}(\bar{X}_{s-})(\phi'(s) - b(\bar{X}_s))|^2 ds \right) \\
& \qquad \times \exp\left(\int_0^t |\sigma^{-1}(\bar{X}_{s-})(\phi'(s) - b(\bar{X}_s))|^2 ds \right) \Big] \\
\le \ & \exp\left(t\Lambda_1^{-1} \left(\Lambda_2 + |\phi'(s)|_{L^\infty([0,t])} \right)^2 \right) \le \exp\left(\gamma \Lambda_1^{-1} (\Lambda_2 + a)^2 \right),
\end{aligned} \tag{4.7}$$

$\{M_t, t \geq 0\}$ is a square-integrable martingale under $\bar{\mathbb{P}}^x$ for every $x \in \mathbb{R}^d$. So it defines a family of probability measures $\{\mathbb{Q}^x, x \in \mathbb{R}^d\}$ by

$$\frac{d\mathbb{Q}^x}{d\bar{\mathbb{P}}^x}\Big|_{\mathcal{F}_t} = M_t, \quad t \in [0, \gamma].$$

By Girsanov's theorem,

$$\widehat{W}_t := W_t + \int_0^t \sigma^{-1}(\bar{X}_s)(b(\bar{X}_s) - \phi'(s))ds, \quad t \in [0, \gamma]$$

is a d-dimensional Brownian motion under each $\mathbb{Q}^x$. Hence

$$\bar{X}_t - \phi(t) - \int_0^t \int_{\{|z:F(\bar{X}_{s-},z)|\leq\delta\}} F(\bar{X}_{s-}, z)(\mu - \nu)(dz, ds) = \bar{X}_0 - \phi(0) + \int_0^t \sigma(\bar{X}_s)d\widehat{W_s}$$

is an $\{\mathcal{F}_t\}$-martingale under each $\mathbb{Q}^x$. Under $\mathbb{Q}^{x_0}$,

$$\bar{X}_t - \phi(t) = \int_0^t \sigma(\bar{X}_s)d\widehat{W_s} + \int_0^t \int_{\{z:|F(\bar{X}_{s-},z)|\leq\delta\}} F(\bar{X}_{s-}, z)(\mu - \nu)(dz, ds). \tag{4.8}$$

We next estimate $\mathbb{Q}^{x_0}\left(\sup_{t\in[0,\gamma]} |\bar{X}_t - \phi(t)| < \varepsilon\right)$. Note that

$$\begin{aligned}
&\mathbb{Q}^{x_0}\left(\sup_{t\in[0,\gamma]} |\bar{X}_t - \phi(t)| < \varepsilon\right) \\
\geq\ &\mathbb{Q}^{x_0}\left(\sup_{t\in[0,\gamma]} \Big|\int_0^t \sigma(\bar{X}_s)d\widehat{W_s}\Big| < \varepsilon/2 \text{ and } \sup_{t\in[0,\gamma]} \Big|\int_0^t \int_{\{z:|F(\bar{X}_{s-},z)|\leq\delta\}} F(\bar{X}_{s-}, z)(\mu - \nu)(dz, ds)\Big| < \varepsilon/2\right) \\
\geq\ &\mathbb{Q}^{x_0}\left(\sup_{t\in[0,\gamma]} \Big|\int_0^t \sigma(\bar{X}_s)d\widehat{W_s}\Big| < \varepsilon/2\right) \\
&-\mathbb{Q}^{x_0}\left(\sup_{t\in[0,\gamma]} \Big|\int_0^t \int_{\{z:|F(\bar{X}_{s-},z)|\leq\delta\}} F(\bar{X}_{s-}, z)(\mu - \nu)(dz, ds)\Big| \geq \varepsilon/2\right).
\end{aligned}$$

Since each component of $t \mapsto \int_0^t \sigma(\bar{X}_s)d\widehat{W_s}$ is a continuous martingale, using the idea in the proof of Theorem I.8.3 in [6], it can be shown that there is a positive increasing function $\phi_1(\varepsilon)$ on $(0, \infty)$ that depends only on (d, Λ_1) so that

$$\mathbb{Q}^{x_0}\left(\sup_{t\in[0,\gamma]} \Big|\int_0^t \sigma(\bar{X}_s)d\widehat{W_s}\Big| < \varepsilon/2\right) \geq \phi_1(\varepsilon/\sqrt{\gamma}). \tag{4.9}$$

We put its proof in the Appendix §A.2.

Next, by Doob's maximal inequality,

$$\begin{aligned}
&\mathbb{Q}^{x_0}\left(\sup_{t\in[0,\gamma]}\Big|\int_0^t\int_{\{z:|F(\bar X_{s-},z)|\le\delta\}}F(\bar X_{s-},z)(\mu-\nu)(dz,ds)\Big|\ge\varepsilon/2\right)\\
\le\;& 4\varepsilon^{-2}\,\mathbb{E}^{\mathbb{Q}^{x_0}}\left[\left(\int_0^\gamma\int_{\{z:|F(\bar X_{s-},z)|\le\delta\}}F(\bar X_{s-},z)(\mu-\nu)(dz,ds)\right)^2\right]\\
=\;& 4\varepsilon^{-2}\,\mathbb{E}^{\mathbb{Q}^{x_0}}\left[\int_0^\gamma\int_{\{z:|F(\bar X_{s-},z)|\le\delta\}}F(\bar X_{s-},z)^2\nu(dz,ds)\right]\\
=\;& 4\varepsilon^{-2}\,\mathbb{E}^{\mathbb{Q}^{x_0}}\left[\int_0^\gamma\int_{\{z:|z|\le\delta\}}|z|^2 n(\bar X_s,dz)ds\right]\\
\le\;& 4\gamma\varepsilon^{-2}\sup_{x\in\mathbb{R}^d}\int_{\{z:|z|\le\delta\}}|z|^2 n(x,dz)=4\gamma\varepsilon^{-2}\psi(\delta).
\end{aligned}$$

Choose $\delta=\delta(\gamma,\varepsilon)\in(0,1)$ so that $\psi(\delta)\le\varepsilon^2\phi_1(\varepsilon/\sqrt{\gamma})/(8\gamma)$. Since both ϕ_1 and ψ are increasing functions, the above $\delta(\gamma,\varepsilon)$ can be chosen in such a way that it is increasing in $\varepsilon>0$ and decreasing in γ. Then we have from the above

$$\mathbb{Q}^{x_0}\left(\sup_{t\in[0,\gamma]}|\bar X_t-\phi(t)|<\varepsilon\right)\ge\phi_1(\varepsilon/\sqrt{\gamma})/2.$$

As $d\mathbb{Q}^x=M_1 d\bar{\mathbb{P}}^{x_0}$ on $\mathcal{F}_1$, we have by Cauchy-Schwarz inequality and (4.7),

$$\begin{aligned}
\phi_1(\varepsilon)/2 \;\le\;& \mathbb{Q}^{x_0}\left(\sup_{t\in[0,\gamma]}|\bar X_t-\phi(t)|<\varepsilon\right)\\
\le\;& \left(\mathbb{E}^{\bar{\mathbb{P}}^{x_0}}\left[M_\gamma^2\right]\right)^{1/2}\bar{\mathbb{P}}^{x_0}\left(\sup_{t\in[0,\gamma]}|\bar X_t-\phi(t)|<\varepsilon\right)^{1/2}\\
\le\;& \exp\left(\Lambda_1^{-1}(\Lambda_2+a)^2/2\right)\bar{\mathbb{P}}^{x_0}\left(\sup_{t\in[0,\gamma]}|\bar X_t-\phi(t)|<\varepsilon\right)^{1/2}.
\end{aligned}$$

Consequently,

$$\bar{\mathbb{P}}^{x_0}\left(\sup_{t\in[0,\gamma]}|\bar X_t-\phi(t)|<\varepsilon\right)\ge\exp\left(-\Lambda_1^{-1}(\Lambda_2+a)^2\right)\phi_1^2(\varepsilon/\sqrt{\gamma})/4 \tag{4.10}$$

(ii) Now by (4.3) and (4.5),

$$\begin{aligned}
\mathbb{P}^{x_0}\left(\sup_{t\in[0,\gamma]}|X_t-\phi(t)|<\varepsilon\right) \;\ge\;& e^{-\Lambda_3/\delta(\gamma,\varepsilon)^2}\,\bar{\mathbb{P}}^{x_0}\left(\sup_{t\in[0,\gamma]}|\bar X_t-\phi(t)|<\varepsilon\right)\\
\ge\;& e^{-\Lambda_3/\delta(\gamma,\varepsilon)^2}\,e^{-(\Lambda_2+a)^2/\Lambda_1}\phi_1^2(\varepsilon/\sqrt{\gamma})/4;
\end{aligned}$$

that is, the estimate (4.6) holds for

$$\Phi^{(\gamma,a)}(\varepsilon):=e^{-\Lambda_3/\delta(\gamma,\varepsilon)^2}\,e^{-(\Lambda_2+a)^2/\Lambda_1}\phi_1^2(\varepsilon/\sqrt{\gamma})/4.$$

Clearly, $\Phi^{(\gamma,a)}(\varepsilon)$ is a positive increasing function of ε on $(0,\infty)$, and $\Phi^{(\gamma,a)}$ is decreasing in $a>0$ and increasing in $\gamma\in(0,1]$. This completes the proof of the theorem. □

Corollary 1.4.2. *Suppose that $\mathcal{L}\in\mathcal{N}(\Lambda_1,\Lambda_2,\Lambda_3)$ and that condition (1.3) holds. Let $X=\{X_t,t\geq 0;\mathbb{P}^x,x\in\mathbb{R}^d\}$ be the strong Markov process that solves the martingale problem $(\mathcal{L},C_b^2(\mathbb{R}^d))$. For any $\delta\in(0,1]$, there is a non-decreasing positive function Φ_δ on $(0,\infty)$ that depends on $d,\Lambda_1,\Lambda_2,\Lambda_3$, a and the rate of the function ψ in (1.3) converging to 0 so that for any $x_0\in\mathbb{R}^d$, $0<h<r\leq 1$, $y,z\in B(x_0,r)$ with $B(y,h)\cup B(z,h)\subset B(x_0,r)$,*

$$\mathbb{P}^y(T_{B(z,h)}<\tau_{B(x_0,r)}<\delta r^2)\geq\Phi_\delta(h/r).$$

Moreover, the function Φ_δ is non-decreasing in $\delta>0$. In fact, we can take $\Phi_\delta:=\Phi^{(\delta,\,2/\delta)}$, where $\Phi^{(\gamma,a)}$ is the positive function in Theorem 1.4.1.

Proof. It suffice to consider the case when $r=1$. Let $\{X_t,t\geq 0;\mathbb{P}^x,x\in\mathbb{R}^d\}$ be any strong Markov process associated with the generator $\mathcal{L}\in\mathcal{N}(\Lambda_1,\Lambda_2+\Lambda_3,\Lambda_3)$.

Let ϕ be the linear function taking values in $\mathbb{R}^d$ defined on $[0,\delta]$ so that $\phi(0)=y$ and $\phi(\delta)=z$. Given $\sup_{t\in[0,\delta]}|X_t-\phi(t)|<h$, X_t would never exit $B(x_0,1)$ by time δ. Note that $|\phi'(t)|=|y-z|\leq 2r/\delta\leq 2/\delta$ for $t\in(0,\delta)$. By (4.6),

$$\mathbb{P}^y\left(T_{B(z,h)}<\tau_{B(x_0,1)}<\delta\right)\geq\mathbb{P}^y\left(\sup_{t\in[0,\delta]}|X_t-\phi(t)|<h\right)\geq\Phi_\delta(h).$$

□

Theorem 1.4.3. *Suppose that $\mathcal{L}\in\mathcal{N}(\Lambda_1,\Lambda_2,\Lambda_3)$ and the condition (1.3) holds. Let $X=\{X_t,t\geq 0;\mathbb{P}^x,x\in\mathbb{R}^d\}$ be the Hunt process that solves the martingale problem $(\mathcal{L},C_b^2(\mathbb{R}^d))$. For any $\delta\in(0,1]$, there is a non-decreasing positive function Φ_δ on $(0,\infty)$ that depends on $d,\Lambda_1,\Lambda_2,\Lambda_3$, a and the rate of the function ψ in (1.3) converging to 0 so that for any $x_0\in\mathbb{R}^d$, $0<h<r\leq 1$, $y,z\in Q(x_0,r)$ with $\overline{Q(y,h)}\cup\overline{Q(z,h)}\subset Q(x_0,r)$,*

$$\mathbb{P}^y\left(T_{Q(z,h)}\circ\theta_{\delta r^2}<\delta r^2 \text{ and } \tau_{Q(x_0,r)}>2\delta r^2\right)\geq\Phi_\delta(h/2r).$$

Moreover, the function Φ_δ is non-decreasing in $\delta>0$. In fact, we can take $\Phi_\delta:=\Phi^{(2\delta,\,\sqrt{d}/\delta)}$, where $\Phi^{(\gamma,a)}$ is the positive function in Theorem 1.4.1.

Proof. In view of Lemma 1.2.2, it suffices to prove this theorem for $r = 1$. For $\delta \in (0, 1]$, define $\Phi_\delta := \Phi^{(2\delta, \sqrt{d}/\delta)}$, where $\Phi^{(\gamma,a)}$ is the positive function in Theorem 1.4.1. Let ϕ be the piecewise linear function taking values in $\mathbb{R}^d$ defined on $[0, 2\delta]$ so that $\phi(s) = y$ for $s \in [0, \delta]$ and $\phi(s) = y + (s - \delta)(z - y)/\delta$ for $r \in [\delta, 2\delta]$. Note that ϕ is continuous on $[0, 2\delta]$ with $|\phi'(s)| \leq |y - z|/\delta \leq \sqrt{d}/\delta$ for $s \in [0, 2\delta] \setminus \{\delta\}$.

Given $\sup_{t\in[0,2\delta]} |X_t - \phi(t)| < h/2$, X_t would never exit $\mathcal{Q}(x_0, r)$ before time 2δ and it will visit $\mathcal{Q}(z, h)$ during the time interval $(\delta, 2\delta)$, Thus by (4.6),

$$\mathbb{P}^y \left(T_{\mathcal{Q}(z,h)} \circ \theta_\delta < \delta \text{ and } \tau_{\mathcal{Q}(x_0,1)} > 2\delta \right) \geq \mathbb{P}^y \left(\sup_{t\in[0,2\delta]} |X_t - \phi(t)| < h/2 \right) \geq \Phi_\delta(h/2).$$

□

1.4.3 Krylov's inequality for non-local operator

Define a molifier function

$$p(t, x) = \begin{cases} C \exp\left(\frac{1}{|t|^2-1}\right) \exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |t| < 1, |x| < 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $C > 0$ is a normalized constant so that $\int_{-\infty}^{\infty} \int_{\mathbb{R}^d} \zeta(t, x)dxdt = 1$. Given a function $G(t, x) \in L^{d+1}(\mathbb{R}^{d+1})$, define for $\varepsilon > 0$,

$$G^\varepsilon(t, x) := \varepsilon^{-(d+1)} \int_0^\infty \int_{\mathbb{R}^d} \zeta((t - s)/\varepsilon, (x - y)/\varepsilon)G(t, y)dyds.$$

It is known that $z^\varepsilon(t, x)$ is infinitely diferentiable in t and x, and $z^\varepsilon \to z$ a.e. as $\varepsilon \to 0$.

Lemma 1.4.4. *Suppse that $r > 0$ and $f(t, x) \geq 0$ is in $L^{d+1}(\mathbb{R}\times\mathbb{R}^d)$ such that $f = 0$ outside $(0, \infty) \times B(x_0, r)$. Then there exists a bounded non-positive function $G(t, x)$ on $\mathbb{R} \times \mathbb{R}^d$ that vanishes on $(-\infty, 0) \times \mathbb{R}^d$ and has the following properties for small $\varepsilon > 0$.*

(i) *$G(t, x)$ is convex in $x \in B(x_0, 2r)$ for every fixed $t > 0$ and decreasing in t for every fixed $x \in \mathbb{R}^d$;*

(ii) *If $A = (a^{ij})$ is a nonnegative definite symmetric matrix, then there exists a constant $c_1 > 0$ depending only on d such that for every $(t, x) \in (0, \infty) \times B(x_0, r)$,*

$$\sum_{i,j=1}^d a^{ij} \frac{\partial^2}{\partial x_i \partial x_j} G^\varepsilon(t, x) - \frac{\partial}{\partial t} G^\varepsilon(t, x) \geq c_1 (\det A)^{1/(d+1)} f^\varepsilon(t, x), (0, \infty) \times B(x_0, r);$$

(iii) *For any vector b in $\mathbb{R}^d$ and any $c_0 > 0$ s.t.* $\sup_{x\in B(x_0,r)} |b(x)| \le c_0\, r/2$,

$$\sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i} G^{\varepsilon}(t,x) \ge c_0\, G^{\varepsilon}(t,x) \quad \text{for } (t,x) \in (0,\infty)\times B(x_0,r),$$

where $|b(x)|$ is the Euclidean norm of $b(x)$.

(iv) *There is a constant $c_2 = (d+1)(2r)^d/\omega_d$ such that for every $(t,x) \in (0,\infty)\times \mathbb{R}^d$,*

$$|G(t,x)|^{d+1} \le c_2 \Big| \int_0^\infty \int_{B(x_0,r)} f(s,y)^{d+1} dyds \Big|.$$

The above result was given in [51] but its proof is quite sketchy. For reader's convenience, we give a detailed proof in the Appendix §A.3 of this paper.

Theorem 1.4.5. *Suppose $\{X_t, t \ge 0; \mathbb{P}^x, x \in \mathbb{R}^d\}$ is a Hunt process having infinitesimal generator $\mathcal{L} \in \mathcal{N}(\Lambda_1,\Lambda_2,\Lambda_3)$. For every $\lambda > 0$, there is a positive constant c_3 depending only on $(d,\Lambda_1,\Lambda_2,\Lambda_3,\lambda)$ such that for every $x_0 \in \mathbb{R}^d$ and $f \in L^{d+1}(\mathbb{R}^+ \times B(x_0,1/4))$,*

$$\mathbb{E}^x \int_0^{\tau_{B(x_0,1/4)}} e^{-\lambda s} |f(s,X_s)| ds \le c_3 \|f\|_{L^{d+1}(\mathbb{R}^+\times B(x_0,1/4))}$$

for every $x \in B(x_0,1/4)$.

Proof. This proof follows a similar idea as that for [1, Theorem 1.1]. For any function $f \in L^{d+1}(\mathbb{R}^+ \times B(x_0,1/4))$ and integer $n \ge 1$, define

$$g_n(t,x) = \begin{cases} |f(n-t,x)| & \text{when } (t,x) \in (0,n)\times B(x_0,1/4), \\ 0 & \text{elsewhere,} \end{cases} \tag{4.11}$$

which is a non-negative function in $L^{d+1}(\mathbb{R}^{d+1})$. Let

$$r_0 := 1 \vee (2\sqrt{d}\Lambda_2/c_0) \tag{4.12}$$

where c_0 is the constant in Lemma 1.4.4(iii). By Lemma 1.4.4, there exists a non-positive bounded function $G_n(t,x)$ on $(-\infty,\infty)\times\mathbb{R}^d$ such that $G_n(t,x) = 0$ for $t < 0$ that has the properties stated in Lemma 1.4.4 with $B(x_0,r_0)$ in place of $B(x_0,r)$ there. For simplicity, denote $\tau_{B(x_0,1/4)}$ as τ. By (3.6), applying Ito's formula to $e^{-\lambda s} G_n^{\varepsilon}(n-s,X_s)$ yields

$$\mathbb{E}^x \left[e^{-\lambda(\tau\wedge n)} G_n^{\varepsilon}(n-\tau\wedge n, X_{\tau\wedge n}) \right] - G_n^{\varepsilon}(n,x)$$

$$
\begin{aligned}
= & \ \mathbb{E}^x\Big[\int_0^{\tau\wedge n} e^{-\lambda s}\Big(\sum_{i,j=1}^d \frac{1}{2}a_{ij}\frac{\partial^2 G_n^\varepsilon}{\partial x_i\partial x_j}(n-s,X_{s^-}) + \sum_{i=1}^d b_i\frac{\partial G_n^\varepsilon}{\partial x_i}(n-s,X_{s^-}) - \lambda G_n^\varepsilon(n-s,X_{s-}) \\
& -\frac{\partial G_n^\varepsilon}{\partial s}(n-s,X_{s^-})\Big)ds + \int_0^{\tau\wedge n}\int_{|F(X_{s-},z)|\le 1} e^{-\lambda s}\Big(G_n^\varepsilon(n-s,X_{s^-}+F(X_{s^-},z)) - G_n^\varepsilon(n-s,X_{s^-}) \\
& \quad -\nabla_x G_n^\varepsilon(n-s,X_{s^-})\cdot F(X_{s^-},z)\Big)\nu(dzds) \\
& +\int_0^{\tau\wedge n}\int_{|F(X_{s-},z)|>1} e^{-\lambda s}(G_n^\varepsilon(n-s,X_{s-}+F(X_{s-},z)) - G_n^\varepsilon(n-s,X_{s-}))\mu(dzds)\Big] \\
= & \ \mathbb{E}^x\Big[\int_0^{\tau\wedge n} e^{-\lambda s}\Big(\frac{1}{2}\sum_{i,j=1}^d a_{ij}\frac{\partial^2 G_n^\varepsilon}{\partial x_i\partial x_j}(n-s,X_{s-}) + \sum_{i=1}^d b_i\frac{\partial G_n^\varepsilon}{\partial x_i}(n-s,X_{s-}) - \lambda G_n^\varepsilon(n-s,X_{s-}) \\
& -\frac{\partial G_n^\varepsilon}{\partial s}(n-s,X_{s-}) + \int_{|z|\le 1}(G_n^\varepsilon(n-s,X_{s^-}+z) - G_n^\varepsilon(n-s,X_{s^-}) \\
& -\nabla_x G_n^\varepsilon(n-s,X_s)\cdot z)n(X_{s^-},dz) + \int_{|z|>1}(G_n^\varepsilon(n-s,X_{s^-}+z) - G_n^\varepsilon(n-s,X_{s^-}))n(X_s,dz)\Big)ds\Big],
\end{aligned} \tag{4.13}
$$

where the second equality is due to (3.5).

For $s\in(0,\tau\wedge n]$, $X_{s-}\in B(x_0,1/4)$. So by Lemma 1.4.4(ii),

$$
\frac{1}{2}\sum_{i,j=1}^d a_{ij}\frac{\partial^2 G_n^\varepsilon}{\partial x_i\partial x_j}(n-s,X_{s^-}) - \frac{\partial G_n^\varepsilon}{\partial t}(n-s,X_{s^-}) \ge c_1(\det A)^{1/(d+1)}g_n^\varepsilon(n-s,X_{s^-}).
$$

Noting G_n is bounded and non-positive on $\mathbb{R}\times\mathbb{R}^d$, this together with (4.13) implies that

$$
\begin{aligned}
& -G_n^\varepsilon(n,x) \\
& \mathbb{E}^x\left[e^{-\lambda(\tau\wedge n)}G_n^\varepsilon(n-\tau\wedge n, X_{\tau\wedge n})\right] - G_n^\varepsilon(n,x) \\
\ge & \ \mathbb{E}^x\Big[\int_0^{\tau\wedge n} e^{-\lambda s}\Big(c_1(\det A)^{1/(d+1)}g_n^\varepsilon(n-s,X_{s^-}) + \sum_{i=1}^d b_i(n-s,X_s)\frac{\partial G_n^\varepsilon}{\partial x_i}(n-s,X_{s^-}) \\
& +\int_{|z|\le 1}(G_n^\varepsilon(n-s,X_{s^-}+z) - G_n^\varepsilon(n-s,X_{s^-}) - \nabla_x G_n^\varepsilon(n-s,X_{s^-})\cdot z)n(X_{s^-},dz) \\
& +\int_{|z|>1}(G_n^\varepsilon(n-s,X_{s^-}+z) - G_n^\varepsilon(n-s,X_{s^-}))n(X_{s^-},dz)\Big)ds\Big].
\end{aligned} \tag{4.14}
$$

Since by Lemma 1.4.4, G_n is convex in $B(x_0,2r_0)$, so G_n^ε is convex in x in $B(x_0,3/2)$ for any $\varepsilon\in(0,1/4)$. Thus for every $x\in B(x_0,1/4)$, $z\in\mathbb{R}^d$ with $|z|\le 1$ and $t>0$,

$$
G_n^\varepsilon(t,x+z) - G_n^\varepsilon(t,x) - \nabla_x G_n^\varepsilon(t,x)\cdot z \ge 0. \tag{4.15}
$$

On the other hand, since $r_0 \geq 1/4$, by (4.11) and Lemma 1.4.4(iv),

$$\sup_{(t,x)\in\mathbb{R}^{d+1}} |G_n^\varepsilon(t,x)| \leq \sup_{(t,x)\in\mathbb{R}^{d+1}} |G_n(t,x)| \leq c_2\|g_n\|_{L^{d+1}(\mathbb{R}\times B(x_0,r_0))} = c_2\|f\|_{L^{d+1}((0,n)\times B(x_0,1/4))}. \tag{4.16}$$

Hence

$$\begin{aligned} & \int_0^{\tau\wedge n} e^{-\lambda s} \int_{|z|>1} \left|G_n^\varepsilon(n-s, X_{s^-}+z) - G_n^\varepsilon(n-s, X_{s^-})\right| n(X_{s^-}, dz)ds \\ \leq\ & 2\|G_n^\varepsilon\|_\infty \int_0^\infty e^{-\lambda s} \int_{|z|>1} n(X_{s^-}, dz)ds \\ \leq\ & 2c_2\|f\|_{L^{d+1}((0,n)\times B(x_0,1/4))}\lambda^{-1}\Lambda_3. \end{aligned} \tag{4.17}$$

By (1.2) and (4.12), $|b(x)| \leq \sqrt{d}\Lambda_2 \leq c_0 r_0/2$, for any $x \in B(x_0, r_0)$, then by Lemma 1.4.4(iii),

$$\sum_{i=1}^d b_i \frac{\partial G_n^\varepsilon}{\partial x_i}(t,x) \geq c_0 G_n^\varepsilon(t,x) \quad \text{for any } (t,x) \in \mathbb{R}^+ \times B(x_0, r_0). \tag{4.18}$$

Thus by (4.14)-(4.18) and the fact that $\det A \geq \Lambda_1^d$, it yields

$$\begin{aligned} & c_2\|f\|_{L^{d+1}((0,n)\times B(x_0,1/4))} \\ \geq\ & \mathbb{E}^x\Big[\int_0^{\tau\wedge n} e^{-\lambda s}\Big(c_1\Lambda_1^{d/(d+1)} g_n^\varepsilon(n-s, X_{s^-}) + c_0 G_n^\varepsilon(n-s, X_{s^-})\Big)ds\Big] \\ & -2c_2\|f\|_{L^{d+1}((0,n)\times B(x_0,1/4))}\lambda^{-1}\Lambda_3 \\ \geq\ & c_1\Lambda_1^{d/(d+1)}\mathbb{E}^x\Big[\int_0^{\tau\wedge n} e^{-\lambda s}\ g_n^\varepsilon(n-s, X_{s-})ds\Big] - c_2\lambda^{-1}(c_0+2\Lambda_3)\|f\|_{L^{d+1}((0,n)\times B(x_0,1/4))}. \end{aligned}$$

Thus we get

$$\mathbb{E}^x\Big[\int_0^{\tau\wedge n} e^{-\lambda s}\ g_n^\varepsilon(n-s, X_{s^-})ds\Big] \leq c_3\|f\|_{L^{d+1}((0,n)\times B(x_0,1/4))}, \tag{4.19}$$

where

$$c_3 = \frac{c_2(1+\lambda^{-1}(c_0+2\Lambda_3))}{c_1\Lambda_1^{d/(d+1)}}.$$

Passing $\varepsilon \to 0$ in (4.19) yields

$$\begin{aligned} \mathbb{E}^x\Big[\int_0^{\tau\wedge n} e^{-\lambda s}|f(s,X_s)|ds\Big] &= \mathbb{E}^x\Big[\int_0^{\tau\wedge n} e^{-\lambda s} g_n(n-s, X_s)ds\Big] \\ &= \mathbb{E}^x\Big[\int_0^{\tau\wedge n} e^{-\lambda s} g_n(n-s, X_{s^-})ds\Big] \\ &\leq c_3\|f\|_{L^{d+1}((0,n)\times B(x_0,1/4))}. \end{aligned}$$

Taking $n \to \infty$ gives

$$\mathbb{E}^x\Big[\int_0^\tau e^{-\lambda s}|f(s,X_s)|ds\Big] \le c_3\|f\|_{L^{d+1}(\mathbb{R}^+\times B(x_0,1/4))}.$$

This completes the proof of the theorem. $\square$

Theorem 1.4.6. *Suppose $\{X_t, t\ge 0; \mathbb{P}^x, x\in\mathbb{R}^d\}$ is a Hunt process having infinitesimal generator $\mathcal{L}\in\mathcal{N}(\Lambda_1,\Lambda_2,\Lambda_3)$. For every $\lambda>0$, there exists a constant $c_4>0$ that depends only on $(d,\Lambda_1,\Lambda_2,\Lambda_3,\lambda)$ such that for any $f\in L^{d+1}(\mathbb{R}^+\times\mathbb{R}^d)$ and $x\in\mathbb{R}^d$,*

$$\mathbb{E}^x\int_0^\infty e^{-\lambda s}|f(s,X_s)|ds \le c_4\|f\|_{L^{d+1}(\mathbb{R}^+\times\mathbb{R}^d)}.$$

Proof. The proof is similar to that of [6, Theorem V.5.3]. Define $S_0 := 0$, $S_1 := \inf\{t>0 : |X_t - X_0| > 1/4\}$, and

$$S_{i+1} := S_i + S_1\circ\theta_{S_i} \quad \text{for } i\ge 1.$$

Clearly S_k is a stopping time for every $k\ge 1$.

Fix a smooth non-decreasing function φ on $[0,\infty)$ so that $\varphi(r)=r$ for $r\in[0,4]$ and $\varphi(r)=6$ for $r\ge 5$. Fix $x\in\mathbb{R}^d$, let $f(y)=\varphi(|y-x|^2)$, which is a bounded smooth function with bounded derivatives on $\mathbb{R}^d$. Hence $|\mathcal{L}f(x)|$ is bounded by a constant c that depends only on $(d,\Lambda_1,\Lambda_2,\Lambda_3)$. Note that $f(x)=0$ and $f(X_{t\wedge\tau_{B(x,1/4)}})\ge 1/16$ on $\{\tau_{B(x,1/4)}\le t\}$. Thus we have for every $t>0$,

$$\frac{1}{16}\mathbb{P}^x(\tau_{B(x,1/4)}\le t) \le \mathbb{E}^x\left[f(X_{t\wedge\tau_{B(x,1/4)}})\right] = \mathbb{E}^x\left[\int_0^{t\wedge\tau_{B(x,1/4)}}\mathcal{L}f(X_s)ds\right] \le ct.$$

Take $t_0 = 1/(32c)$. Then $\mathbb{P}^x(\tau_{B(x,1/4)}\le t_0)\le\frac{1}{2}$. Thus

$$\begin{aligned}\mathbb{E}^x\left[e^{-\lambda S_1}\right] &\le \mathbb{P}^x(S_1\le t_0) + e^{-\lambda t_0}\mathbb{P}^x(S_1>t_0)\\ &= (1-e^{-\lambda t_0})\mathbb{P}^x(S_1\le t_0) + e^{-\lambda t_0}\mathbb{P}^x(S_1>t_0)\\ &\le \frac{1}{2}+\frac{1}{2}e^{-\lambda t_0} < 1.\end{aligned}$$

Let $\rho = (1+e^{-\lambda t_0})/2$, which depends only on $(d,\Lambda_1,\Lambda_2,\Lambda_3)$. We have just shown that $\mathbb{E}^x\left[e^{-\lambda S_1}\right]\le\rho$ for every $x\in\mathbb{R}^d$. By the strong Markov property of X, for $i\ge 1$,

$$\mathbb{E}^x\left[e^{-\lambda S_{i+1}}\right] = \mathbb{E}^x\left[e^{-\lambda S_i}\mathbb{E}^x\left[e^{-\lambda S_1\circ\theta_{S_i}}|\mathcal{F}_{S_i}\right]\right] = \mathbb{E}^x\left[e^{-\lambda S_i}\mathbb{E}^{X_{S_i}}\left[e^{-\lambda S_1}\right]\right] \le \rho\mathbb{E}^x\left[e^{-\lambda S_i}\right].$$

Thus by induction, we have $\mathbb{E}^x e^{-\lambda S_i} \le \rho^i$ for every $i \ge 1$. Consequently, by Theorem 1.4.5 and the strong Markov property of X,

$$\begin{aligned}\mathbb{E}^x \int_{S_i}^{S_{i+1}} e^{-\lambda s}|f(s,X_s)|ds &= \mathbb{E}^x \left[e^{-\lambda S_i}\, \mathbb{E}^{X_{S_i}} \left[\int_0^{S_1} e^{-\lambda t}|f(s,X_s)|ds \right]\right] \\ &\le c_3 \|f\|_{L^{d+1}(\mathbb{R}^+\times\mathbb{R}^d)}\, \mathbb{E}^x e^{-\lambda S_i} \\ &\le c_3 \rho^i \, \|f\|_{L^{d+1}(\mathbb{R}^+\times\mathbb{R}^d)}.\end{aligned}$$

Hence

$$\begin{aligned}\mathbb{E}^x \int_0^\infty e^{-\lambda s}|f(s,X_s)|ds &= \sum_{i=0}^\infty \mathbb{E}^x \int_{S_i}^{S_{i+1}} e^{-\lambda s}|f(s,X_s)|ds \\ &\le c_3 \sum_{i=0}^\infty \rho^i \, \|f\|_{L^{d+1}(\mathbb{R}^+\times\mathbb{R}^d)} = \frac{c_3}{1-\rho} \|f\|_{L^{d+1}(\mathbb{R}^+\times\mathbb{R}^d)}.\end{aligned}$$

This establishes the theorem. □

1.4.4 Hitting probability estimates

Proposition 1.4.7. *Suppose that $\mathcal{L} \in \mathcal{N}(\Lambda_1, \Lambda_2, \Lambda_3)$. There exist constant $C_0 > 0$ and $\varepsilon > 0$ that depends only on $(d, \Lambda_1, \Lambda_2, \Lambda_3)$ so that for every $x_0 \in \mathbb{R}^d$, $r \in (0, \widetilde{r}_0]$ and any $A \subset \mathcal{Q}(x_0, r)$ with $|\mathcal{Q}(x_0,r) \setminus A| \le \varepsilon r^d$,*

$$\mathbb{P}^x(T_A < \tau_{\mathcal{Q}(x_0,r)}) \ge C_0 \quad \text{for every } x \in \overline{\mathcal{Q}(x_0, r/2)},$$

where $\widetilde{r}_0 = ((2/\sqrt{d}) \wedge 1) r_0$ and r_0 is the constant in Lemma 1.2.3.

Proof. In view of the scaling property of Lemma 1.2.2, it suffices to prove the proposition for any $\mathcal{L} \in \mathcal{N}(\Lambda_1, \Lambda_2 + \Lambda_3, \Lambda_3)$ and for $r = \widetilde{r}_0$. Let $x_0 \in \mathbb{R}^d$. For simplicity denote $\tau_{\mathcal{Q}(x_0, \widetilde{r}_0)}$ by τ. By Corollary 1.2.4, Corollary 1.2.7, and Remark 1.2.5, there are positive constants c_1 and c_2 that depend only on $(d, \Lambda_1, \Lambda_2, \Lambda_3)$ so that

$$\mathbb{E}^x[\tau] \ge c_1 \widetilde{r}_0 \ \text{ for } x \in \mathcal{Q}(x_0, \widetilde{r}_0/2) \quad \text{and} \quad \mathbb{E}^x[\tau^2] \le c_2 \widetilde{r}_0^2 \ \text{ for } x \in \mathcal{Q}(x_0, \widetilde{r}_0),$$

where c_1, c_2 depends only on $(d, \Lambda_1, \Lambda_2, \Lambda_3)$.

Take $t_0 = 4c_2\widetilde{r}_0/(c_1 \wedge c_2 \wedge c_3)$. Then by Theorem 1.4.6,

$$\begin{aligned}
\mathbb{E}^x \int_0^\tau \mathbb{1}_{\mathcal{Q}(x_0,\widetilde{r}_0)\setminus A}(X_s)ds &\leq \mathbb{E}^x \int_0^{t_0\wedge\tau} \mathbb{1}_{\mathcal{Q}(x_0,\widetilde{r}_0)\setminus A}(X_s)ds + \mathbb{E}^x\left[\tau; \tau \geq t_0\right] \\
&\leq e^{t_0}\mathbb{E}^x \int_0^\infty e^{-s}\mathbb{1}_{[0,t_0]\times(\mathcal{Q}(x_0,\widetilde{r}_0)\setminus A)}(s, X_s)ds + \mathbb{E}^x[\tau^2]/t_0 \\
&\leq c_3 e^{t_0}\|\mathbb{1}_{[0,t_0]\times(\mathcal{Q}(x_0,\widetilde{r}_0)\setminus A)}\|_{L^{d+1}(\mathbb{R}^{d+1})} + c_2\widetilde{r}_0^2/t_0 \\
&\leq c_3 e^{4c_2\widetilde{r}_0/(c_1\wedge c_2\wedge c_3)}(t_0\,|\mathcal{Q}(x_0,\widetilde{r}_0)\setminus A|)^{1/(d+1)} + (c_1\wedge c_2\wedge c_3)\widetilde{r}_0/4,
\end{aligned}$$

where $c_3 > 0$ is a constant that depends only on d, Λ_1, Λ_2 and Λ_3. Let

$$\varepsilon := ((c_1\wedge c_2\wedge c_3)/4\widetilde{r}_0 c_2)((c_1\wedge c_2\wedge c_3)(4c_3)^{-1}e^{-t_0})^{d+1} < (1/4)^{d+1}.$$

Then for any $A \subset \mathcal{Q}(x_0,\widetilde{r}_0)$ with $|\mathcal{Q}(x_0,\widetilde{r}_0)\setminus A| \leq \varepsilon\widetilde{r}_0^d$, we have

$$\mathbb{E}^x \int_0^\tau \mathbb{1}_{\mathcal{Q}(x_0,\widetilde{r}_0)\setminus A}(X_s)ds < (c_1\wedge c_2\wedge c_3)\widetilde{r}_0/2 \quad \text{for every } x \in \mathcal{Q}(x_0,\widetilde{r}_0/2).$$

On the other hand, for $x \in \mathcal{Q}(x_0,\widetilde{r}_0/2)$,

$$\begin{aligned}
(c_1\wedge c_2\wedge c_3)\widetilde{r}_0 \leq \mathbb{E}^x\tau &= \mathbb{E}^x \int_0^\tau \mathbb{1}_A(X_s)ds + \mathbb{E}^x \int_0^\tau \mathbb{1}_{\mathcal{Q}(x_0,\widetilde{r}_0)\setminus A}(X_s)ds \\
&< \mathbb{E}^x\left[\tau; T_A < \tau\right] + (c_1\wedge c_2\wedge c_3)\widetilde{r}_0/2 \\
&\leq \Big(\mathbb{E}^x[\tau^2]\,\mathbb{P}^x(T_A < \tau)\Big)^{1/2} + (c_1\wedge c_2\wedge c_3)\widetilde{r}_0/2 \\
&\leq \Big(c_2\widetilde{r}_0^2\,\mathbb{P}^x(T_A < \tau)\Big)^{1/2} + (c_1\wedge c_2\wedge c_3)\widetilde{r}_0/2.
\end{aligned}$$

Hence $\mathbb{P}^x(T_A < \tau) > (c_1\wedge c_2\wedge c_3)^2/(4c_2)$ for every $x \in \mathcal{Q}(x_0,\widetilde{r}_0/2)$. □

We need the following proposition for the lower bound estimate of the hitting probability. Given an open cube $\mathcal{Q}$, let $\overline{\mathcal{Q}}$ denote its closure, and let $\widehat{\mathcal{Q}}$ be the closed cube that has the same center as $\mathcal{Q}$ but with 3 times its side length. The following is [6, Proposition V.7.2] (with a slight modification in the statement that our cubes $\mathcal{Q}$ are open cubes instead of closed cubes).

Proposition 1.4.8. *Let $q \in (0,1), x_0 \in \mathbb{R}^d$ and $r \in (0,1]$. For any measurable $A \subset \mathcal{Q}(x_0,r)$ with $|A| \leq q|\mathcal{Q}(x_0,r)|$, there exists a countable family $\mathcal{R}$ of pairwise disjoint open cubes $\mathcal{Q}$ in $\mathcal{Q}(x_0,r)$ so that $|A\cap\mathcal{Q}| \geq q|\mathcal{Q}|$ for each $\mathcal{Q} \in \mathcal{R}$, and $|A| \leq q|D\cap\mathcal{Q}(x_0,r)|$ with $D := \cup_{\mathcal{Q}\in\mathcal{R}}\overline{\widehat{\mathcal{Q}}}$.*

Theorem 1.4.9. *Let $\mathcal{L} \in \mathcal{N}(\Lambda_1, \Lambda_2, \Lambda_3)$ and $X = \{X_t, t \geq 0; \mathbb{P}^x, x \in \mathbb{R}^d\}$ be a Hunt process that solves the martingale problem for $(\mathcal{L}, C_b^2(\mathbb{R}^d))$. There exists a non-decreasing strictly positive function ϕ on $(0,1]$ that depend only on $(d, \Lambda_1, \Lambda_2, \Lambda_3)$ and the rate of the function ψ in (1.3) converging to 0 so that for any $r \in (0, \widetilde{r}_0]$, $x_0 \in \mathbb{R}^d$, and $A \subset Q(x_0, r)$,*

$$\mathbb{P}^x(T_A < \tau_{Q(x_0,r)}) \geq \phi(|A|/r^d) \quad \text{for } x \in \overline{Q(x_0, r/2)},$$

where $\widetilde{r}_0$ is the constant in Proposition 1.4.7.

Proof. Our proof follows the idea of [6, Theorem V.7.4].

Define a function ϕ on $[0,1]$ by

$$\begin{aligned}\phi(\varepsilon) \quad = \quad \inf\Big\{\mathbb{P}^y(T_A < \tau_{Q(x_0,r)}) : \mathcal{L} \in \mathcal{N}(\Lambda_1, \Lambda_2, \Lambda_3), r \in (0, \widetilde{r}_0], y \in \overline{Q(x_0, r/2)}, \\ A \subset Q(x_0, r) \text{ with } |A| \geq \varepsilon |Q(x_0, r)|\Big\}.\end{aligned}$$

Without loss of generality we can assume $d = 2$. Clearly ϕ is non-decreasing. By Proposition 1.4.7, $\phi(\varepsilon) > 0$ when ε is sufficiently close to 1. Define $q_0 = \inf\{\varepsilon : \phi(\varepsilon) > 0\}$. We claim that $q_0 = 0$, and will prove this by contradiction.

Suppose that $q_0 > 0$. Take $q \in (q_0, \frac{\sqrt{1+8q_0}-1}{2}) \subset (q_0, 1)$, which is always possible since $0 < q_0 < 1$. Since $\eta := (q + q^2)/2 < q_0 < q$, we have $\phi(\eta) = 0$ and $\phi(q) > 0$. Let

$$\beta := 1 - ((3+q)/4)^{1/d} < 1/2 \quad \text{and} \quad \rho = \Phi_1(\frac{\beta}{18} \wedge \frac{1}{8}), \tag{4.20}$$

where Φ_1 is the strictly positive non-decreasing function Φ_δ on $(0, \infty)$ in Corollary 1.4.2 with $\delta = 1$.

Note that $\phi(\frac{q_0+\eta}{2}) = 0$ as $\frac{q_0+\eta}{2} < q_0$. By the definition of ϕ, there exist some $\mathcal{L}_0 \in \mathcal{N}(\Lambda_1, \Lambda_2, \Lambda_3)$, some $x_0 \in \mathbb{R}^d$, $r_0 \in (0, \widetilde{r}_0]$, $A \subset Q(x_0, r_0)$ with $|A|/|Q(x_0, r_0)| \geq (q_0 + \eta)/2$, and some $x_1 \in \overline{Q(x_0, r_0/2)}$ so that

$$\mathbb{P}^{x_1}(T_A < \tau_{Q(x_0,r_0)}) < \phi((q_0 + \eta)/2) + \rho\phi(q)^2 = \rho\phi(q)^2, \tag{4.21}$$

where $\{X_t, t \geq 0; \mathbb{P}^x, x \in \mathbb{R}\}$ is the Hunt process associated with the infinitesimal generator $\mathcal{L}_0$. We claim that $|A|/|Q(x_0, r_0)| < q$. This is because if $|A|/|Q(x_0, r_0)| \geq q$, we would then have

$$\rho\phi(q)^2 > \mathbb{P}^{x_1}(T_A < \tau_{Q(x_0,r_0)}) \geq \phi(q)$$

and so $\rho\phi(q) > 1$, which is impossible as $\phi(q) \le 1$ and $\rho < 1$.

As $|A| < q|\mathcal{Q}(x_0, r_0)|$, by Proposition 1.4.8, there is a countable family $\mathcal{R}$ of disjoint open cubes in $\mathcal{Q}(x_0, r_0)$ so that $|A \cap \mathcal{Q}| \ge q|\mathcal{Q}|$ for each $\mathcal{Q} \in \mathcal{R}$, and $|A| \le q|D \cap \mathcal{Q}(x_0, r)|$ with $D := \cup_{\mathcal{Q}\in\mathcal{R}} \widehat{\mathcal{Q}}$. Enumerate $\mathcal{R}$ by $\{\mathcal{Q}_k; k \ge 1\}$.

As

$$\lim_{n\to\infty} |\cup_{k=1}^n \widehat{\mathcal{Q}}_k \cap \mathcal{Q}(x_0, r_0)| = |D \cap \mathcal{Q}(x_0, r_0)| \ge \frac{|A|}{q} \ge \frac{q_0 + \eta}{2q} |\mathcal{Q}(x_0, r_0)| > \frac{1+q}{2} |\mathcal{Q}(x_0, r_0)|,$$

there is some $N \ge 1$ so that

$$|\cup_{k=1}^N \widehat{\mathcal{Q}}_k \cap \mathcal{Q}(x_0, r_0)| > \frac{1+3q}{4} |\mathcal{Q}(x_0, r_0)|.$$

Let $D_1 := \cup_{k=1}^N \widehat{\mathcal{Q}}_k \cap \mathcal{Q}(x_0, (1-\beta)r_0)$. Then

$$\begin{aligned} |D_1| &\ge |\cup_{k=1}^N \widehat{\mathcal{Q}}_k \cap \mathcal{Q}(x_0, r_0)| - |\mathcal{Q}(x_0, r_0) \setminus \mathcal{Q}(x_0, (1-\beta)r_0)| \\ &\ge \left(\frac{1+3q}{4} - \frac{1-q}{4} \right) |\mathcal{Q}(x_0, r_0)| \\ &= q\, |\mathcal{Q}(x_0, r_0)|. \end{aligned}$$

Thus by the definition of ϕ,

$$\mathbb{P}^x(T_{D_1} < \tau_{\mathcal{Q}(x_0, r_0)}) \ge \phi(q) \quad \text{for every } x \in \overline{\mathcal{Q}(x_0, r_0/2)}. \tag{4.22}$$

We claim that

$$\mathbb{P}^y(T_A < \tau_{\mathcal{Q}(x_0, r_0)}) \ge \rho\, \phi(q) \quad \text{for every } y \in \overline{D}_1. \tag{4.23}$$

For $y \in \overline{D}_1$, there is some $1 \le k \le N$ so that $y \in \overline{\widehat{\mathcal{Q}}_k}$ and $\mathcal{Q}(y, \beta r_0) \subset \mathcal{Q}(x_0, r_0)$. Denote $\mathcal{Q}_k$ by $\mathcal{Q}(y_0, r)$ and let $\mathcal{Q}^* = \mathcal{Q}(y_0, r/2)$. If $r \ge \beta r_0/9$, we have by Corollary 1.4.2 with $\delta = 1$,

$$\mathbb{P}^y(T_{\mathcal{Q}^*} < \tau_{\mathcal{Q}(x_0, r_0)}) \ge \Phi_1(\beta/18) \ge \rho.$$

When $r < \beta r_0/9$, $\mathcal{Q}(y_0, r/2) \subset \mathcal{Q}(y, 4r) \subset \mathcal{Q}(x_0, r_0)$ as $|y - y_0|_\infty \le 3r/2 < \beta r_0/6$. Again by Corollary 1.4.2 with $\delta = 1$,

$$\mathbb{P}^y(T_{\mathcal{Q}^*} < \tau_{\mathcal{Q}(x_0, r_0)}) \ge \mathbb{P}^y(T_{\mathcal{Q}^*} < \tau_{\mathcal{Q}(y, 4r)}) \ge \Phi_1(1/8) \ge \rho.$$

Since $|A \cap \mathcal{Q}| \ge q\, |\mathcal{Q}|$, in both cases we have by the definition of $\phi(q)$ that

$$\mathbb{P}^y(T_A < \tau_{\mathcal{Q}(x_0, r_0)}) \quad \ge \quad \mathbb{E}^y \left[\mathbb{P}^{X_{T_{\mathcal{Q}_k^*}}} (T_{A\cap\mathcal{Q}} < \tau_{\mathcal{Q}});\ T_{\mathcal{Q}^*} < \tau_{\mathcal{Q}(x_0, r_0)}) \right]$$

$$\geq \quad \phi(q)\mathbb{P}^y(T_{\mathcal{Q}^*} < \tau_{\mathcal{Q}(x_0,r_0)}) \ \geq \ \rho\,\phi(q).$$

This establishes the claim (4.23).

It follows from (4.22) and (4.23) that for every $x \in \overline{\mathcal{Q}(x_0, r_0/2)}$,

$$\begin{aligned}
\mathbb{P}^x(T_A < \tau_{\mathcal{Q}(x_0,r_0)}) \quad &\geq \quad \mathbb{P}^x(T_{D_1} < T_A < \tau_{\mathcal{Q}(x_0,r_0)}) \\
&\geq \quad \mathbb{E}^x\left[\mathbb{P}^{X_{T_{D_1}}}(T_A < \tau_{\mathcal{Q}(x_0,r_0)}); T_{D_1} < \tau_{\mathcal{Q}(x_0,r_0)}\right] \\
&\geq \quad \rho\phi(q)\mathbb{P}^{x_1}(T_{D_1} < \tau_{\mathcal{Q}(x_0,r_0)}) \geq \rho\phi(q)^2.
\end{aligned}$$

This contradiction with (4.21) shows that $q_0 = 0$. □

Remark 1.4.10. Our proof follows that of the corresponding result [6, Theorem V.7.4] for diffusion processes. However, the proof in [6] has a gap on line 15 of p.118 t as one does not have a control on the size of $|R_i^*|$ relative to $|\mathcal{Q}(0,1)|$. But it can be fixed using the argument above by considering the length of R_i in two cases.)

Corollary 1.4.11. *There exists a non-decreasing function $\varphi : (0,1] \mapsto (0,1]$ depending only on $(d, \Lambda_1, \Lambda_2, \Lambda_3)$ and the rate of the function ψ in (1.3) converging to 0 such that for any $x_0 \in \mathbb{R}^d, r \in (0, \widetilde{r}_1]$, if $A \subset B(x_0, r)$, with $|A| > 0$, for any $x \in B(x_0, r/2)$,*

$$\mathbb{P}^x(T_A \leq \tau_{B(x_0,r)}) \geq \varphi(|A|/|B(x_0, r)|). \tag{4.24}$$

where $\widetilde{r}_1 = \widetilde{r}_0/2$, and $\widetilde{r}_0$ is the constant in Proposition 1.4.7.

Proof. Our proof is motivated by that for [39, Corollary 4.9] and fixes some ambiguous claims in [39, lines -10 and -6 on p.36]. Without loss of generality, we may and do assume that $x_0 = 0$. For notational convenience, we assume dimension $d = 2$ but the proof works for any dimension $d \geq 2$. Let $\lambda = |A|/|B(0, r)|$. Take $\varepsilon = \varepsilon(\lambda) \in (0, 1/4)$ so that $(1-(1-2\varepsilon)^2) = \lambda/2$. Clearly, $\varepsilon(\lambda)$ is an increasing function in $\lambda \in (0, 1]$. It follows that

$$|A \cap B(0, (1 - 2\varepsilon)r)|/|B(0, r)| \geq \lambda/2.$$

Let $k = k(\lambda) \in \mathbb{N}$ be sufficiently large so that for any cube

$$R_{i,j} := [(i-1)2^{-k}r, i2^{-k}r] \times [(j-1)2^{-k}r, j2^{-k}r]$$

with $R_{i,j} \cap B(0,(1-2\varepsilon)r) \neq \emptyset$ for $1-2^k \leq i,j \leq 2^k$, we have $R_{i,j} \subset B(0,(1-\varepsilon)r)$. We can choose $k(\lambda)$ in such a way that it is a decreasing function in ε and hence a decreasing function in λ. Let

$$\mathcal{C} := \{R_{i,j} : |R_{i,j} \cap B(0,(1-2\varepsilon)r) \cap A| > 0\}$$

and $M = M(\lambda)$ be the total number of elements in $\mathcal{C}$. Clearly, $M \leq 2^{2(k+1)}$. Denote by $R^*_{i,j}$ the closed cube that has the same center as $R_{i,j}$ with side length half as long. As

$$\sum_{R_{i,j} \in \mathcal{C}} |R_{i,j} \cap A| \geq |B(0,(1-2\varepsilon)r) \cap A| \geq \lambda |B(0,r)|/2 = |A|/2, \tag{4.25}$$

there is some $R_{i,j} \in \mathcal{C}$ so that

$$|R_{i,j} \cap A| \geq |A|/(2M). \tag{4.26}$$

Since $|R_{i,j}| = (2^{-k}r)^2$ and $M \leq 2^{2(k+1)}$, we have

$$|R_{i.j} \cap A|/|R_{i,j}| \geq |A|/(8r^2) \geq |A|/(3\pi r^2) \geq \lambda/6.$$

Let ϕ be the non-decreasing strictly positive function on $(0,1]$ in Theorem 1.4.9. We then have by Theorem 1.4.9,

$$\mathbb{P}^x(T_{R_{i,j}\cap A} < \tau_{R_{i,j}}) \geq \phi(|R_{i,j} \cap A|/|R_{i.j}|) \geq \phi(\lambda/6) \quad \text{for } x \in R^*_{i,j}. \tag{4.27}$$

Let y_0 be the center of the cube $R_{i,j}$. Then $B(y_0, r/(4k)) \subset R^*_{i,j}$. For any $x \in B(0,r/2)$, by the strong Markov property, (4.27) and Corollary 1.4.2,

$$\begin{aligned}
\mathbb{P}^x(T_A < \tau_{B(0,r)}) &\geq \mathbb{P}^x\left(T_{R^*_{i,j}} < \tau_{B(0,r)} \text{ and } T_{R_{i,j}\cap A} \circ \theta_{T_{R^*_{i,j}}} < \tau_{R_{i,j}} \circ \theta_{T_{R^*_{i,j}}}\right) \\
&\geq \mathbb{E}^x\left[\mathbb{P}^{T_{R^*_{i,j}}}\left(T_{R_{i,j}\cap A} < T_{R_{i,j}}\right); T_{R^*_{i,j}} < \tau_{B(0,r)}\right] \\
&\geq \phi(\lambda/6)\, \mathbb{P}^x(T_{R^*_{i,j}} < \tau_{B(0,r)}) \\
&\geq \phi(\lambda/6)\, \mathbb{P}^x(T_{B(y_0,r/(4k))} < \tau_{B(0,r)}) \\
&\geq \phi(\lambda/6)\, \Phi_1(1/(4k)),
\end{aligned}$$

where Φ_1 is the non-decreasing strictly positive function on $(0,\infty)$ in Corollary 1.4.2 with $\delta = 1$. This completes the proof of this corollary by taking

$$\varphi(\lambda) := \phi(\lambda/6)\, \Phi_1(1/(4k)) = \phi(\lambda/8)\, \Phi_1(1/(4k(\lambda))),$$

which is a strictly positive non-decreasing function on $(0,1]$. $\square$

1.5 Hölder regularity

Proof of Theorem 1.1.1. Given Corollary 1.4.11 on a lower bound estimate of the hitting probability, the idea of the proof is similar to those in literature, see, e.g., [39, Theorem 5.1] or [20, Theorem 4.14] on Hölder regularity for parabolic functions. Due to the general form of jumping kernel $n(x, dz)$ in this paper, for reader's convenience, we spell out the details here.

Fix $x_0 \in \mathbb{R}^d$ and $r \in (0, \widetilde{r}_0]$,where $\widetilde{r}_0$ is the constant in Proposition 1.4.7. Suppose that u is a bounded function on $\mathbb{R}^d$ that is $\mathcal{L}$-harmonic in $B(x_0, 2r)$. Without loss of generality, we assume $\|u\|_\infty = 1/2$.

Take $z_1 \in B(x_0, r)$ and set $r_1 = r/2$. Clearly, $B(z_1, r_1) \subset B(x_0, 3r/2)$. Recall that φ is the non-decreasing function in Corollary 1.4.11,

Let

$$a := \sqrt{1 - \tfrac{1}{4}\varphi(1/3)} \vee (1/\sqrt{2}) \quad \text{ and } \quad \rho := \left(\frac{\varphi(1/3)}{40 c_1 \Lambda_3} \wedge \frac{1}{4}\right)^{1/2} \sqrt{a}, \tag{5.1}$$

where $c_1 > 1$ is the constant in Lemma 1.2.3. Note that $1/\sqrt{2} < a < 1$ and $0 < \rho < 1/2$. For $n \geq 2$, define $r_n = \rho^{n-1} r_1$. For simplicity, denote $B(z_1, r_n)$ by B_n and τ_{B_n} by τ_n.

Let $M_n := \sup_{x \in B_n} u(x)$ and $m_n := \inf_{x \in B_n} u(x)$. We claim that

$$M_n - m_n \leq a^{n-2} \quad \text{ for all } n \geq 1. \tag{5.2}$$

Clearly, $M_2 - m_2 \leq M_1 - m_1 \leq 1$ so (5.2) holds for $n = 1, 2$. Suppose $M_i - m_i \leq a^{i-2}$ is true for $i = 2, \cdots, n \geq 2$, we want to show it holds for $n + 1$. For this, define

$$A_n = \{x \in B_n : u(x) \leq (M_n + m_n)/2\}.$$

We may assume $|A_n|/|B_n| \geq 1/2$. Otherwise we consider $M_n - u$ instead of u. Let A be a compact subset of A_n such that $|A|/|B_n| \geq 1/3$. By Corollary 1.4.11,

$$\mathbb{P}^x(T_A < \tau_n) \geq \varphi(1/3) \quad \text{ for } x \in B_{n+1}. \tag{5.3}$$

For any given $\varepsilon > 0$, there are $y, z \in B_{n+1}$ so that $u(y) < m_{n+1} + \varepsilon$ and $u(z) > M_{n+1} - \varepsilon$. As $\mathcal{L} \in \mathcal{N}(\Lambda_1, \Lambda_2, \Lambda_3)$, by Lemma 1.2.3, for $1 \leq i \leq n - 1$, it yields

$$\sup_{x \in B_n} \mathbb{P}^x(X_{\tau_n} \notin B_{n-i})$$

$$
\begin{aligned}
&\le \sup_{x\in B_n} \mathbb{E}^x \int_0^{\tau_n} n(X_s, B_{n-i}-X_s)ds \\
&\le \sup_{x\in B_n} \mathbb{E}^x \int_0^{\tau_n} n(X_s, \{h: |h|\ge r_{n-i}-r_n\})ds \\
&\le \sup_{x\in B_n} \mathbb{E}^x \int_0^{\tau_n} (r_{n-i}-r_n)^{-2} \int_{\mathbb{R}^d} (|h|^2\wedge 1) n(X_s, dh)ds \\
&\le (r_{n-i}-r_n)^{-2}\Lambda_3 \sup_{x\in B_n} \mathbb{E}^x \tau_n \\
&\le c_1\Lambda_3\, r_n^2\, (r_{n-i}-r_n)^{-2} \\
&\le c_1\Lambda_3\, \rho^{2i}/(1-\rho^i)^2. \qquad (5.4)
\end{aligned}
$$

Note that $\rho^2 \le a/4 < 1/4$. We have by (5.3) and (5.4) that for $n\ge 2$,

$$
\begin{aligned}
&\quad M_{n+1}-m_{n+1}-2\varepsilon \\
&< u(z)-u(y) = \mathbb{E}^z[u(X_{T_A\wedge\tau_n})-u(y)] \\
&= \mathbb{E}^z[u(X_{T_A})-u(y); T_A<\tau_n] + \mathbb{E}^z[u(X_{\tau_n})-u(y); T_A\ge\tau_n, X_{\tau_n}\in B_{n-1}] \\
&\quad + \sum_{i=2}^{n-1} \mathbb{E}^z[u(X_{T_A})-u(y); T_A\ge\tau_n, X_{\tau_n}\in B_{n-i}\setminus B_{n+1-i}] \\
&\quad + \mathbb{E}^z[u(X_{\tau_n})-u(y); T_A\ge\tau_n, X_{\tau_n}\notin B_1] \\
&\le ((M_n+m_n)/2-m_n)\,\mathbb{P}^z(T_A<\tau_n) + (M_{n-1}-m_{n-1})\,\mathbb{P}^z(T_A\ge\tau_n) \\
&\quad + \sum_{i=2}^{n-1}(M_{n-i}-m_{n-i})\,\mathbb{P}^z(X_{\tau_n}\notin B_{n+1-i}) + \mathbb{P}^z(X_{\tau_n}\notin B_1) \\
&\le \frac{1}{2}a^{n-2}\mathbb{P}^z(T_A<\tau_n) + a^{n-3}(1-\mathbb{P}^z(T_A<\tau_n)) \\
&\quad + \sum_{i=2}^{n-1} a^{n-i-2}c_1\Lambda_3\rho^{2(i-1)}/(1-\rho^{i-1})^2 + c_1\Lambda_3\rho^{2(n-1)}/(1-\rho^{n-1})^2 \\
&\le a^{n-3}\,(1-(1-a/2)\mathbb{P}^z(T_A<\tau_n)) + \frac{c_1\Lambda_3 a^{n-3}}{(1-\rho)^2}\sum_{i=2}^{n-1}(\rho^2/a)^{i-1} + \frac{c_1\Lambda_3\rho^{2(n-1)}}{(1-\rho)^2} \\
&\le a^{n-3}\,(1-(1-a/2)\varphi(1/3)) + 4c_1\Lambda_3 a^{n-3}(\rho^2/a)\sum_{k=0}^{n-3}2^{-k} + 4c_1\Lambda_3\rho^{2(n-1)} \\
&\le a^{n-3}\left(1-\tfrac{1}{2}\varphi(1/3)\right) + 8c_1\Lambda_3 a^{n-3}(\rho^2/a) + 4c_1\Lambda_3\rho^{2(n-1)} \\
&\le a^{n-1}\left(\frac{1-\frac{1}{2}\varphi(1/3)}{a^2} + 8c_1\Lambda_3\left((2\rho^2/a)+\tfrac{1}{2}(\rho^2/a)^{n-1}\right)\right) \\
&\le a^{n-1}\left(\frac{1-\frac{1}{2}\varphi(1/3)}{a^2} + \frac{10c_1\Lambda_3\,\rho^2}{a}\right) \\
&\le a^{n-1}, \qquad (5.5)
\end{aligned}
$$

where the last inequality is due to the definition (5.1) of a and ρ. Since the above holds for any $\varepsilon > 0$, passing $\varepsilon \to 0$ shows that (5.2) holds for $n+1$ and hence, by induction, (5.2) holds for all $n \geq 1$.

Given any $z_2 \in B(z_1, r_1)$, let $k \geq 0$ be the largest non-negative integer so that $|z_2 - z_1| \leq \rho^k r/2$, that is, $k \geq 0$ is the largest integer not exceeding $\log_\rho(2|z_1 - z_2|/r)$. Then $z_2 \in B(z_1, r_{k+1})$ and so

$$|u(z_2) - u(z_1)| \leq a^{k-2} \leq a^{\log_\rho(2|z_1 - z_2|/r) - 3} = \frac{1}{a^3} \left(\frac{2|z_2 - z_1|}{r} \right)^{(\log a)/\log \rho}.$$

Since $1/\sqrt{2} < a < 1$ and $0 < \rho < 1/2$, then $\log a / \log \rho \in (0, 1/2)$. This establishes (1.4) with $\alpha := \log a / \log \rho \in (0, 1/2)$ and $C = 2a^{-3}$, both depend only on $\Lambda_1, \Lambda_2, \Lambda_3, d$ and the rate of the function ψ in (1.3) converging to 0. □

1.6 Elliptic Harnack inequality

Lemma 1.6.1. *Suppose $\mathcal{L} \in \mathcal{N}(\Lambda_1, \Lambda_2, \Lambda_3)$ and condition* (1.3) *holds. For any bounded function $h(x)$ on $\mathbb{R}^d$ that is $\mathcal{L}$-harmonic in an open subset D of $\mathbb{R}^d$, $s \mapsto h(X_{s \wedge \tau_D})$ is right continuous $\mathbb{P}^x$-a.s. for any $x \in D$, where $\{X_t, t \geq 0; \mathbb{P}^x, x \in \mathbb{R}^d\}$ is the Hunt process associated with the operator $\mathcal{L}$.*

Proof. By the definition of $\mathcal{L}$-harmonicity, for any relatively compact open subset D_1 of D, $h(x) = \mathbb{E}^x[h(X_{\tau_{D_1}})]$ for every $x \in D_1$.By the strong Markov property of X, for every stopping time S of $\{\mathcal{F}_s, s \geq 0\}$,

$$h(X_{S \wedge \tau_{D_1}}) = \mathbb{E}^x[h(X_{\tau_{D_1}}) | \mathcal{F}_{S \wedge \tau_{D_1}}] \quad \mathbb{P}^x - a.s.$$

Here the martingle $s \to \mathbb{E}^x[h(X_{\tau_{D_1}}) | \mathcal{F}_{S \wedge \tau_{D_1}}]$ is taken to be its right continuous version. As $s \to X_s$ càdlàg and h is Borel-measurable, then $s \to h(X_s \wedge \tau_{D_1})$ is optional. Then by Optional Section Theorem [42, Theorem 4.10], we have

$$\mathbb{P}^x[h(X_{s \wedge \tau_{D_1}}) = \mathbb{E}^x(h(X_{\tau_{D_1}}) | \mathcal{F}_{s \wedge \tau_{D_1}}) \text{ for all } s \geq 0] = 1.$$

That is to say that $s \to h(X_{s \wedge \tau_{D_1}})$ is right continuous $\mathbb{P}^x$-a.s., for every $x \in D_1$. Since this holds for every relatively compact open subset D_1 of D, then the lemma is proved.

The right continuity of $s \mapsto h(X_{s\wedge\tau_D})$ is a consequence of the right continuity of the Hunt process $t \mapsto X_t$. □

Proof of Theorem 1.1.2. Given Corollary 1.4.11 on hitting probability estimate, the proof follows the nowadays standard ideas in literature; see, for example, that of [20, Proposition 4.3] which is for parabolic Harnack inequality. The elliptic Harnack inequality case is simpler.

Denote $r_1 = 2\widetilde{r}_1$. Let $r \in (0, \widetilde{r}_1]$, $x_0 \in \mathbb{R}^d$ and u be any nonnegative function on $\mathbb{R}^d$ that is $\mathcal{L}$-harmonic in $B(x_0, 2r)$, where $\widetilde{r}_1$ is the constant in Corollary 1.4.11. By shrinking the value of r a little bit, we may assume without loss of generality that $u(x) = \mathbb{E}^x\left[u(X_{\tau_{B(x_0,2r)}})\right]$. As u is the increasing limit of bounded harmonic functions $u_n(x) := \mathbb{E}^x\left[(u\wedge n)(X_{\tau_{B(x_0,2r)}})\right]$, it suffices to establish the Harnack inequality for bounded harmonic functions in $B(x_0, 2r)$. On the other hand, since X is assumed to be conservative, constant functions are harmonic. By considering $u + \varepsilon$ with $\varepsilon > 0$ and then passing $\varepsilon \to 0$, it suffices to establish Harnack inequality for positive bounded harmonic function u with $\inf_{\mathbb{R}^d} u > 0$. Taking a constant multiple of such u if needed, we may further assume that $\inf_{B(x_0,r)} u = 1/2$. Thus there is $y_0 \in B(x_0, r)$ so that $u(y_0) < 1$.

Our aim is to prove u is bounded above on $B(x_0, r)$ by a universal constant that depends only on $(\Lambda_1, \Lambda_2, \Lambda_3, c_0, d)$ and the rate of the function ψ of (1.3) converging to 0, where c_0 is the constant in (1.12).

Let $\varphi : (0, 1] \to (0, 1]$ be the non-decreasing function in Corollary 1.4.11. Taking φ to be $\varphi(r) \wedge r$ if needed, we may assume that $\lim_{r\to 0+} \varphi(r) = 0$.

From the support theorem and a scaling argument, we will prove by induction that for any $k \in N$, there exists a $\delta_0 \in (0, 1)$ so that for every $R \in (0, 1]$,

$$\mathbb{P}^y\left(T_{B(z,4^{-k}R)} < \tau_{B(x_0,R)}\right) \geq \delta_0^k \quad \text{for } y, z \in B(x_0, 3R/4). \tag{6.1}$$

Indeed, by Corollary 1.4.2 with $\delta = 1$, for $y, z \in B(x_0, 3R/4)$,

$$\mathbb{P}^y\left(T_{B(z,R/4)} < \tau_{B(x_0,R)}\right) \geq \Phi_1(1/4) =: \delta_0.$$

Suppose that $\mathbb{P}^y\left(T_{B(z,4^{-k}R)} < \tau_{B(x_0,R)}\right) \geq \delta_0^k$ holds for $k = n$.

Since $B(z,(4^{-n}+4^{-(n+1)})R)\cup B(z,4^{-(n+1)}R)\subset B(z,4^{-n+1}R)$, by the strong Markov property of X and Corollary 1.4.2,

$$\begin{aligned}
& \mathbb{P}^y(T_{B(z,4^{-k}R)}<\tau_{B(x_0,R)}) \\
\geq\ & \mathbb{E}^y.\left[\mathbb{P}^{X_{T_{B(z,4^{-k+1}R)}}}(T_{B(z,4^{-k}R)}<\tau_{B(z,4^{-k+2}R)});T_{B(z,4^{-k+1}R)}<\tau_{B(x_0,R)}\right] \\
\geq\ & \mathbb{E}^y\left[\Phi_1(1/16);T_{B(z,4^{-k+1}R)}<\tau_{B(x_0,R)}\right] \\
\geq\ & \delta_0\cdot\delta_0^{k-1}=\delta_0^k.
\end{aligned}$$

Therefore, by Corollary 1.4.11, (6.1) implies that for any $R\in(0,\widetilde{r}_1]$, $k\in N$, any $y,z\in B(x_0,3R/4)$ and $A\subset B(y,4^{-k}R)$,

$$\begin{aligned}
\mathbb{P}^z(T_A<\tau_{B(x_0,R)}) &\geq \mathbb{E}^z\left[\mathbb{P}^{X_{T_{B(y,4^{-(k+1)}R)}}}\left(T_A<\tau_{B(y,4^{-k}R)}\right);T_{B(y,4^{-(k+1)}R)}<\tau_{B(x_0,R)}\right] \\
&\geq \varphi(|A|/|B(y,4^{-k}R)|)\mathbb{P}^z\left(T_{B(y,4^{-(k+1)}R)}<\tau_{B(x_0,R)}\right) \\
&\geq \varphi(|A|/|B(y,4^{-k}R)|)\delta_0^{k+1}. \qquad (6.2)
\end{aligned}$$

Set

$$\eta=\frac{\varphi(1/2^{d+1})}{4}\wedge\frac{c_4}{2},\quad \xi=\eta/c_4\quad\text{and}\quad \gamma_0=\frac{\varphi(1/2^{d+1})}{4(1-\varphi(1/2^{d+1}))}. \qquad (6.3)$$

where $c_4>0$ is the constant in Proposition 1.2.10. For any real number a, denote by $\lfloor a\rfloor$ the largest integer not exceeding a. By decreasing the value of $\delta_0\in(0,1)$ in (6.1) if needed, we may assume that $0<\delta_0<\frac{1}{1+\gamma_0/2}$.Let m be the smallest nonnegative integer so that

$$1/(4^{\log_{\delta_0}1/(1+\gamma_0/2)}-1)\leq 4^{m\log_{\delta_0}1/(1+\gamma_0/2)}. \qquad (6.4)$$

As

$$\lim_{k\to\infty}(1+\gamma_0)^k\delta_0^{\lfloor(k+m)\log_{\delta_0}1/(1+\gamma_0/2)\rfloor}\to\infty,$$

Take

$$a_0:=\inf_{k\geq 1}(1+\gamma_0)^k\delta_0^{\lfloor(k+m)\log_{\delta_0}1/(1+\gamma_0/2)\rfloor}\leq 1+\gamma_0.$$

Let

$$K=\frac{2(1+\gamma_0)}{a_0\xi\delta_0^2\ \varphi(\frac{1}{4}\cdot(\frac{1}{8})^d)} \qquad (6.5)$$

Note that $0<\ \eta<1/4$, $0<\xi<1/2$ Then by and $K>2/\xi\geq 4$. For simplicity, set

$$r_k:=r/4^{(k+m)\log_{\delta_0}1/(1+\gamma_0/2)}\text{ for }k\geq 1.$$

Note by (6.4), $\sum_{k=1}^{\infty} r_k < r$, and for each $k \geq 1$,

$$4^{-n_k-1} r \leq r_k \leq 4^{-n_k} r, \quad \text{where } n_k := \lfloor (k+m) \log_{\delta_0} 1/(1+\gamma_0/2) \rfloor. \tag{6.6}$$

We will show that $u \leq K$ on $B(x_0, r)$. Suppose not. Then there exists a point $z_1 \in B(x_0, r)$ with $u(z_1) > K$. We claim that in this case

$$\begin{aligned} &\text{there is a sequence of points } \{z_j; j \geq 1\} \subset B(x_0, 2r) \text{ so that} \\ &z_j \in B(z_{j-1}, r_{j-1}) \text{ having } u(z_j) \geq K(1+\gamma_0)^{j-1}, \end{aligned} \tag{6.7}$$

with the convention that $z_0 := x_0$ and $r_0 := r$.

We will establish the existence of such a sequence $\{z_j, j \geq 1\}$ by induction. By assumption, we have $z_1 \in B(x_0, r) = B(z_0, r_0)$ with $u(z_1) \geq K$. Suppose we have z_j for $j = 1, \cdots, k$ for some $k \geq 1$ with the mentioned properties in (6.7). We proceed to find z_{k+1}. Let

$$A = \left\{ y \in B(z_k, r_k/4) : u(y) \geq \xi K (1+\gamma_0)^{k-1} \right\}.$$

We claim that

$$|A| < |B(z_k, r_k/4)|/2. \tag{6.8}$$

If not, $|A| \geq |B(z_k, r_k/4)|/2$. Let A' be any compact subset of A so that $|A'| > |B(z_k, r_k/4)|/4$. By (6.2), for every $x \in B(x_0, r)$,

$$\mathbb{P}^x(T_{A'} < \tau_{B(x_0,2r)}) \quad \geq \quad \delta_0^{n_k+2}\, \varphi(|A'|/|B(z_k, r_k/4)|) \geq \delta_0^{n_k+2} \varphi(\frac{1}{4} \cdot (\frac{1}{8})^d).$$

Here n_k is the integer defined in (6.6) and the last inequality above is due to

$$\frac{|A'|}{|B(z_k, 4^{-n_k-1}2r)|} = \frac{|A'|}{|B(z_k, r_k/4)|} \frac{|B(z_k, r_k/4)|}{|B(z_k, 4^{-n_k-1}2r)|} \geq \frac{1}{4} \cdot (\frac{1}{8})^d$$

This together with the harmonicity of u and right continuity of $t \mapsto u(X_t)$ gives

$$1 > u(y_0) \geq \mathbb{E}^{y_0}[u(X_{T_{A'}}); T_{A'} < \tau_{B(x_0,2r)}] \geq \xi K (1+\gamma_0)^{k-1} \delta_0^{n_k+2} \varphi(\frac{1}{4} \cdot (\frac{1}{8})^d) > 2.$$

This contradiction proves the claim (6.8). Consequently, there is a compact subset $E \subset B(z_k, r_k/4) \setminus A$ having

$$|E| > |B(z_k, r_k/4)|/2 \tag{6.9}$$

such that

$$u(y) < \xi K(1+\gamma_0)^{k-1} \text{ for every } y \in E. \tag{6.10}$$

We claim that

$$\mathbb{E}^{z_k}\left[u(X_{\tau_{B(z_k,r_k/2)}}); X_{\tau_{B(z_k,r_k/2)}} \notin B(z_k,r_k)\right] \le \eta K(1+\gamma_0)^{k-1}. \tag{6.11}$$

If not, $\mathbb{E}^{z_k}\left[u(X_{\tau_{B(z_k,r_k/2)}}); X_{\tau_{B(z_k,r_k/2)}} \notin B(z_k,r_k)\right] > \eta K(1+\gamma_0)^{k-1}$. By Proposition 1.2.10, for every $y \in B(z_k, r_k/4)$,

$$\begin{aligned} u(y) = \mathbb{E}^y[u(X_{\tau_{B(z_k,r_k/2)}})] &\ge \mathbb{E}^y\left[u(X_{\tau_{B(z_k,r_k/2)}}); X_{\tau_{B(z_k,r_k/2)}} \notin B(z_k,r_k)\right] \\ &\ge c_4^{-1}\mathbb{E}^{z_k}\left[u(X_{\tau_{B(z_k,r_k/2)}}); X_{\tau_{B(z_k,r_k/2)}} \notin B(z_k,r_k)\right] \\ &> c_4^{-1}\eta\, K(1+\gamma_0)^{k-1} \ge \xi K(1+\gamma_0)^{k-1}. \end{aligned}$$

This contradiction with (6.8) establishes the claim (6.11).

Let $M_k = \sup_{x\in B(z_k,r_k)} u(x)$. Since $t \mapsto u(X_{t\wedge\tau_{B(z_k,r_k)}})$ is a bounded martingale, by (6.11),

$$\begin{aligned} u(z_k) &= \mathbb{E}^{z_k}\left[u(X_{T_E\wedge\tau_{B(z_k,r_k/2)}})\right] \\ &= \mathbb{E}^{z_k}\left[u(X_{T_E}); T_E < \tau_{B(z_k,r_k/2)}\right] \\ &\quad +\mathbb{E}^{z_k}\left[u(X_{\tau_{B(z_k,r_k/2)}}); \tau_{B(z_k,r_k/2)} < T_E; X_{\tau_{B(z_k,r_k/2)}} \in B(z_k,r_k)\right] \\ &\quad +\mathbb{E}^{z_k}\left[u(X_{\tau_{B(z_k,r_k/2)}}); \tau_{B(z_k,r_k/2)} < T_E; X_{\tau_{B(z_k,r_k/2)}} \notin B(z_k,r_k)\right]. \end{aligned} \tag{6.12}$$

By (6.10), Lemma 1.6.1,we know that

$$\mathbb{E}^{z_k}[u(X_{T_E}); T_E < \tau_{B(z_k,r_k/2)}] < \xi K(1+\gamma_0)^{k-1}\,\mathbb{P}^{z_k}(T_E < \tau_{B(z_k,r_k/2)}) \tag{6.13}$$

Then by (6.11),(6.12),(6.13),

$$\begin{aligned} u(z_k) &< \xi K(1+\gamma_0)^{k-1}\,\mathbb{P}^{z_k}(T_E < \tau_{B(z_k,r_k/2)}) + M_k\mathbb{P}^{z_k}(\tau_{B(z_k,r_k/2)} < T_E) + \eta K(1+\gamma_0)^{k-1} \\ &= \xi K(1+\gamma_0)^{k-1}\,\mathbb{P}^{z_k}(T_E < \tau_{B(z_k,r_k/2)}) + M_k\left(1-\mathbb{P}^{z_k}(T_E < \tau_{B(z_k,r_k/2)})\right) + \eta K(1+\gamma_0)^{k-1}. \end{aligned}$$

As $u(z_k) \ge K(1+\gamma_0)^{k-1}$, we conclude from the above that

$$\frac{M_k}{K(1+\gamma_0)^{k-1}} > \frac{1-\eta-\xi\mathbb{P}^{z_k}(T_E < \tau_{B(z_k,r_k/2)})}{1-\mathbb{P}^{z_k}(T_E < \tau_{B(z_k,r_k/2)})}$$

$$= \quad 1 + \frac{(1-\xi)\mathbb{P}^{z_k}(T_E < \tau_{B(z_k, r_k/2)}) - \eta}{1 - \mathbb{P}^{z_k}(T_E < \tau_{B(z_k, r_k/2)})}.$$

Note that by Corollary 1.4.11,

$$\mathbb{P}^{z_k}(T_E < \tau_{B(z_k, r_k/2)}) \geq \varphi(|E|/|B(z_k, r_k/2)|) = \varphi(1/2^{d+1}).$$

Recalling the definition of ξ, η and γ_0 from (6.3) and (6.5), we have from the above two displays that

$$\frac{M_k}{K(1+\gamma_0)^{k-1}} > 1 + \frac{(1-\xi)\varphi(1/2^{d+1}) - \eta}{1 - \varphi(1/2^{d+1})} \geq 1 + \gamma_0.$$

Consequently, $M_k > K(1+\gamma_0)^k$ and so there is $z_{k+1} \in B(z_k, r_k)$ with $u(z_{k+1}) \geq K(1+\gamma_0)^k$.

Thus by induction, we have a sequence $\{z_j; j \geq 1\}$ that has the property (6.7). But this contradicts to the assumption that u is bounded on $B(x_0, 2r)$.

Hence $u(x) \leq K$ must holds for any $x \in B(x_0, r)$.

Hence

$$u(x) \leq C_1 u(y) \text{ for any } x, y \in B(x_0, r),$$

where C_1 depends only on $d, \Lambda_1, \Lambda_2, \Lambda_3, c_4$, the rate of ψ converging to 0, and independent of the radius r.

□

1.7 Scale-invariant Parabolic Harnack inequality(PHI)

Let $X = \{X_t, t \geq 0; \mathbb{P}^x, x \in \mathbb{R}^d\}$ be a Hunt process on $\mathbb{R}^d$ that solves the martingale problem of $(\mathcal{L}, C_b^2(\mathbb{R}^d))$. Denote by $Z_t := (V_t, X_t)$ the corresponding space-time process, where $V_t = V_0 - t$. The law of $Z = \{Z_t, t \geq 0\}$ starting from $Z_0 = (t_0, x)$ is denoted by $\mathbf{P}^{(t_0,x)}$. The minimal augmented filtration generated by Z will be denoted as $\{\mathscr{F}_t, t \geq 0\}$. A measurable function u defined on $\mathbb{R}^+ \times \mathbb{R}^d$ is said to be $\mathcal{L}$-*caloric* or $\mathcal{L}$-*parabolic* in an open set $D \subset \mathbb{R}^+ \times \mathbb{R}^d$ if for every relatively compact open subset U of D,

$$\mathbf{E}^{(t_0,x)}[|u(Z_{\tau_U})|] < \infty \quad \text{and} \quad u(t_0, x) = \mathbf{E}^{(t_0,x)}[u(Z_{\tau_U})] \qquad \text{for every } (t_0, x) \in U.$$

Here $\tau_U = \inf\{s \geq 0 : X_s \notin U\}$. Clearly, if $u(t,x)$ is a function defined on $\mathbb{R}^+ \times \mathbb{R}^d$ and is $C_b^{1,2}$-smooth and satisfies $\frac{\partial u}{\partial t} = \mathcal{L}u$ in an open set $D \subset \mathbb{R}^+ \times \mathbb{R}^d$, then u is $\mathcal{L}$-parabolic in D.

Note that under $\mathbf{P}^{(t_0,x)}$,

$$\tau_{(t_0-r^2,t_0)\times\mathcal{Q}(x_0,r)} = \inf\{t \geq 0 : Z_t \notin (t_0 - r^2, t_0) \times \mathcal{Q}(x_0, r)\} = \tau_{\mathcal{Q}(x_0,r)} \wedge r^2.$$

Theorem 1.7.1. *Let $\mathcal{L} \in \mathcal{N}(\Lambda_1, \Lambda_2, \Lambda_3)$ satisfying conditions* (1.3) *and* (1.12)*. There exists a constant $C > 0$ depending on $(d, \Lambda_1, \Lambda_2, \Lambda_3, c_0)$ and the rate of ψ in* (1.3) *converging to* 0 *and a constant $\widetilde{r}_1 > 0$ so that for any $x_0 \in \mathbb{R}^d, r \in (0, \widetilde{r}_1)$, $t_0 \geq r^2$, and any bounded nonnegative function u that is $\mathcal{L}$-parabolic with in $Q = (t_0 - r^2, t_0) \times B(x_0, r)$, we have*

$$\sup_{(s,x)\in Q_-} u(s,x) \leq C \inf_{(t,y)\in Q_+} u(t,y), \tag{7.1}$$

where

$$Q_- = (t_0 - 3r^2/4, t_0 - r^2/2) \times B(x_0, r/2), Q_+ = (t_0 - 3r^2/8, t_0 - r^2/8) \times B(x_0, r/2),$$

and $\widetilde{r}_1$ is the constant in Proposition 1.4.11.

The following follows immediately from Corollary 1.4.2.

Proposition 1.7.2. *Suppose that $\mathcal{L} \in \mathcal{N}(\Lambda_1, \Lambda_2, \Lambda_3)$ satisfying* (1.3)*. Let $\{X_t, t \geq 0; \mathbb{P}^x, x \in \mathbb{R}^d\}$ be the Hunt process having $\mathcal{L}$ as its generator, and denote by $\mathbf{P}^{(t_0,y)}$ the law of the space-time process $\{Z_t = (V_t, X_t), t \geq 0\}$ of X starting at (t_0, y). For every $\delta \in (0, 1]$, there is a non-decreasing positive function Φ_δ on $(0, 1)$ that depends on $d, \Lambda_1, \Lambda_2, \Lambda_3, \delta$ and the rate of the function ψ in* (1.3)*converging to* 0 *so that for any $r \in (0, 1], t_0 \geq r^2, x_0 \in \mathbb{R}^d$, $t_1 \in [t_0 - r^2, t_0 - \delta r^2]$ and for any $h \in (0, r)$ with $\mathcal{Q}(y, h) \cup \mathcal{Q}(z, h) \subset \mathcal{Q}(x_0, r)$,*

$$\mathbf{P}^{(t_0,y)}(T_{(t_1,t_1+\delta r^2)\times\mathcal{Q}(z,h)} < \tau_{(t_1,t_0)\times\mathcal{Q}(x_0,r)}) \geq \Phi_\delta(h/2r), \tag{7.2}$$

where Φ_δ is defined in Corollary 1.4.2.

Proposition 1.7.3. *Suppose that $\mathcal{L} \in \mathcal{N}(\Lambda_1, \Lambda_2, \Lambda_3)$ satisfying* (1.3)*. Let $\{X_t, t \geq 0; \mathbb{P}^x, x \in \mathbb{R}^d\}$ be the Hunt process having $\mathcal{L}$ as its generator, and denote by $\mathbf{P}^{(t_0,y)}$ the law of the space-time process $\{Z_t = (V_t, X_t), t \geq 0\}$ of X starting at (t_0, y). For every $\delta \in (0, 1]$, there is a non-decreasing positive function Φ_δ on $(0, 1)$ that depends on $d, \Lambda_1, \Lambda_2, \Lambda_3, \delta$ and the* *rate of the function ψ in* (1.3)*converging to* 0 *so that for any* $r \in (0,1], t_0 \geq r^2, x_0 \in \mathbb{R}^d$, $t_1 \in [t_0 - r^2, t_0 - \delta r^2]$ *and for any* $h \in (0,r)$ *with* $B(y,h) \cup B(z,h) \subset B(x_0,r)$,

$$\mathbf{P}^{(t_0,y)}(T_{(t_1,t_1+\delta r^2)\times B(z,h)} < \tau_{(t_1,t_0)\times B(x_0,r)}) \geq \Phi_\delta(h/r), \tag{7.3}$$

where Φ_δ *is defined in Corollary 1.4.2.*

Proposition 1.7.4. *Suppose that* $\mathcal{L} \in \mathcal{N}(\Lambda_1, \Lambda_2, \Lambda_3)$ *satisfying* (1.3)*. There exist constants* $C_0 > 0$ *and* $\varepsilon \in (0,1)$ *that depend only on* $(d, \Lambda_1, \Lambda_2, \Lambda_3)$ *so that for any* $r \in (0, \widetilde{r}_0]$, $t_0 \geq r^2, x_0 \in \mathbb{R}^d$ *and any set* $A \subset (t_0 - r^2, t_0) \times \mathcal{Q}(x_0, r)$ *with* $|(t_0 - r^2, t_0) \times \mathcal{Q}(x_0,r) \setminus A| < \varepsilon r^{d+2}$,

$$\mathbf{P}^{(t_0,x)}\left(T_A < \tau_{(t_0-r^2,t_0)\times \mathcal{Q}(x_0,r)}\right) \geq C_0 \quad \text{for } x \in \mathcal{Q}(x_0, r/2),$$

where $\widetilde{r}_0$ *is the constant in Proposition 1.4.7.*

Proof. We apply the similar method for showing Proposition 1.4.7 to the space-time process $Z_t = (V_t, X_t)$. For simplicity, let $\tau := \tau_{(t_0-r^2,t_0)\times \mathcal{Q}(x_0,r)} = \tau^X_{\mathcal{Q}(x_0,r)} \wedge r^2$. In view of the scaling property in Lemma 1.2.2, it suffices to consider the case when $r = \widetilde{r}_0$.

By Remark 1.2.5 and Corollary 1.2.7, there are constants c_2 and c_3 that depend only on $(d, \Lambda_1, \Lambda_2, \Lambda_3)$ so that for any $x_0 \in \mathbb{R}^d$,

$$\mathbf{E}^{(t_0,x)}[\tau] \geq c_2 \quad \text{and} \quad \mathbf{E}^{(t_0,x)}[\tau^2] \leq c_3 \quad \text{for } x \in \mathcal{Q}(x_0, \widetilde{r}_0/2).$$

Let $t_1 = 4c_3/(c_2 \wedge c_4)$, where $c_4 = c_4(d, \Lambda_1, \Lambda_2, \Lambda_3)$ is the constant in Theorem 1.4.6. Note that

$$\mathbf{E}^x\left[\tau - (\tau \wedge t_1)\right] \leq \mathbf{E}^x[\tau; \tau \geq t_1] \leq \mathbf{E}^x[\tau^2]/t_1 \leq (c_2 \wedge c_4)/4.$$

Hence by Theorem 1.4.6,

$$\begin{aligned}
& \mathbf{E}^{(t_0,x)} \int_0^\tau \mathbb{1}_{A^c}(Z_s)ds \\
\leq\ & \mathbf{E}^{(t_0,x)} \int_0^{t_1\wedge\tau} \mathbb{1}_{(t_0-\tilde{r}_0^2,t_0)\times B(x_0,\tilde{r}_0)\setminus A}(Z_s)ds + \mathbf{E}^{(t_0,x)}\left[\tau - (\tau\wedge t_1)\right] \\
\leq\ & e^{t_1}\mathbb{E}^x \int_0^\infty e^{-s}\mathbb{1}_{(t_0-\tilde{r}_0^2,t_0)\times B(x_0,\tilde{r}_0)\setminus A}(t_0 - s, X_s)ds + (c_2\wedge c_4)/4 \\
\leq\ & c_3 e^{t_1}\|\mathbb{1}_{(t_0-\tilde{r}_0^2,t_0)\times \mathcal{Q}(x_0,\tilde{r}_0)\setminus A}\|_{L^{d+1}(\mathbb{R}^{d+1})} + (c_2\wedge c_4)/4
\end{aligned}$$

$$\leq \quad c_3 e^{t_1} \varepsilon^{1/(d+1)} + (c_2 \wedge c_4)/4$$

Taking $\varepsilon > 0$ so that $c_3 e^{t_1} \varepsilon^{1/(d+1)} = (c_2 \wedge c_4)/4$, we have

$$\mathbf{E}^{(t_0,x)} \int_0^\tau \mathbb{1}_{A^c}(Z_s)ds \leq (c_2 \wedge c_4)/2 \quad \text{for } x \in \mathcal{Q}(x_0, \widetilde{r}_0/2).$$

On the other hand, since

$$\begin{aligned} c_2 \leq \mathbf{E}^{(t_0,x)}[\tau] &= \mathbf{E}^{(t_0,x)}[\tau; T_A < \tau] + \mathbf{E}^{(t_0,x)} \int_0^\tau \mathbb{1}_{A^c}(Z_s)ds \\ &\leq \left(\mathbf{E}^{(t_0,x)}[\tau^2]\, \mathbf{P}^{(t_0,x)}(T_A < \tau)\right)^{1/2} + (c_2 \wedge c_4)/2 \\ &\leq (c_3 \mathbf{P}^{(t_0,x)}(T_A < \tau))^{1/2} + (c_2 \wedge c_4)/2, \end{aligned}$$

then for any $x \in \mathcal{Q}(x_0, \widetilde{r}_0/2)$, it yields that

$$\mathbf{P}^{(t_0,x)}(T_A < \tau) \geq (c_2 \wedge c_4)^2/(4c_3).$$

This proves the theorem with $C_0 := (c_2 \wedge c_4)^2/(4c_3)$ and $\varepsilon = ((c_2 \wedge c_4)/(4c_3 e^{t_1}))^{d+1}$. □

Now we need the following lemma for proving the lower bound estimate of the hitting probability in Theorem 1.7.6. It is the parabolic version of Proposition 1.4.8, which is called the "crawling of ink spots” lemma given in [53], which we modify a bit according to our needs.

For any $q, \alpha, r \in (0, 1], t_0 \geq r^2, x_0 \in \mathbb{R}^d$, let $\widetilde{Q}$ be an open box in $(t_0 - r^2, t_0) \times \mathcal{Q}(x_0, r)$ of the form

$$(t_1 - r_1^2, t_1) \times \mathcal{Q}(x_1, r_1), \text{ for some } (t_1, x_1) \in (t_0 - r^2, t_0) \times \mathcal{Q}(x_0, r). \tag{7.4}$$

For each such set $\widetilde{Q}$, set

$$\widetilde{Q}^1 := \Big((t_1 - 4r_1^2, t_1 + 3r_1^2) \times \mathcal{Q}(x_1, 3r_1)\Big) \cap \Big((t_0 - r^2, t_0) \times \mathcal{Q}(x_0, r)\Big), \tag{7.5}$$

$$\widetilde{Q}^2 := \Big\{(s, x) : t_1 + r_1^2 \leq s \leq t_1 + r_1^2 + 5r_1^2/\alpha,\ x \in \mathcal{Q}(x_1, 3r_1) \cap \mathcal{Q}(x_0, r)\Big\}. \tag{7.6}$$

Namely, The set $\widetilde{Q}^1$ is formed by intersecting $(t_0 - r^2, t_0) \times \mathcal{Q}(x_0, r)$ with a box assembled from smaller boxes arranged like bricks, each congruent to the reference box $\widetilde{Q}$ (up to boundaries). Specifically, that box is constructed with seven layers of these smaller boxes along the t-direction. Each layer consists of a central box congruent to $\widetilde{Q}$, surrounded on all sides and corners by additional boxes, all congruent to $\widetilde{Q}$ in each spatial direction, forming a unified layer. The set $\widetilde{Q}^2$ is constructed along the positive t-direction of $\widetilde{Q}$, keeping the same dimensions in each spatial direction as $\widetilde{Q}^1$, but with a different length in the t-direction.

Lemma 1.7.5. *Let $q, \alpha, r \in (0,1), t_0 \geq r^2, x_0 \in \mathbb{R}^d$ and A be any measurable set in $(t_0 - r^2, t_0) \times \mathcal{Q}(x_0, r)$ with $|A| \leq q|(t_0 - r^2, t_0) \times \mathcal{Q}(x_0, r)|$. Then there exists a countable collection $\mathcal{B}$ of disjoint open sets $\widetilde{Q}$ of the form (7.4) such that $|A \cap \widetilde{Q}| \geq q|\widetilde{Q}|$ for each $\widetilde{Q} \in \mathcal{B}$. Let $\mathcal{B}^1 = \cup_{\widetilde{Q} \in \mathcal{B}} \widetilde{Q}^1$, $\mathcal{B}^2 = \cup_{\widetilde{Q} \in \mathcal{B}} \widetilde{Q}^2$, where $\widetilde{Q}^1, \widetilde{Q}^2$ are of the form as (7.5),(7.6), respectively, then $|A \setminus \mathcal{B}^1| = 0$ and $|A| \leq q|\mathcal{B}^1| \leq q(1+\alpha)|\mathcal{B}^2|$.*

Proof. Without loss of generality, we assume that the space dimension $d = 2$, $t_0 = 0, x_0 = 0, r = 1$. Higher dimensions only differ in notation.

The proof of the assertion $|\mathcal{B}^1| \leq (1+\alpha)|\mathcal{B}^2|$ follows the idea of the proof of Lemma 2.3 in [53]. We spell the details here . Since $|\mathcal{B}^1| \leq |\mathcal{B}^1 \cup \mathcal{B}^2|$, it suffices to show that

$$|\mathcal{B}^1 \cup \mathcal{B}^2| \leq (1+\alpha)|\mathcal{B}^2|. \tag{7.7}$$

Define the x_1-section of $\widetilde{Q}^1$ as $\widetilde{Q}^1_{x_1} = \{x_1 : (x_1, x_2) \in \widetilde{Q}^1\}$. Note that by (7.5), (7.6), $\widetilde{Q}^1_{x_1} = \widetilde{Q}^2_{x_1}$, $\widetilde{Q}^1_{x_2} = \widetilde{Q}^2_{x_2}$ for each $\widetilde{Q} \in \mathcal{B}$. Then

$$|\{x : (t,x) \in \mathcal{B}^1 \cup \mathcal{B}^2)\}| = |\{x : (t,x) \in \mathcal{B}^2\}|.$$

Therefore, to show (7.7), it suffices to prove

$$|\{t : (t,x) \in \mathcal{B}^1 \cup \mathcal{B}^2\}| \leq (1+\alpha)|\{t : (t,x) \in \mathcal{B}^2\}|. \tag{7.8}$$

Fix $\widetilde{Q} \in \mathcal{B}$, let $A(\widetilde{Q})$ be the length of the t-section of $\widetilde{Q}^1 \cup \widetilde{Q}^2$, i.e, $A(\widetilde{Q}) = |(\widetilde{Q}^1 \cup \widetilde{Q}^2)_t|$, and $B(\widetilde{Q}) = |(\widetilde{Q}^2)_t|$. Then by (7.5), (7.6),we have

$$A(\widetilde{Q}) \leq (1+\alpha)B(\widetilde{Q}). \tag{7.9}$$

By (7.9) and applying Lemma 2.2 in [53] with $(t_1, t_2) = (-\infty, \infty), B = \cup_{\widetilde{Q}^2 \in \mathcal{B}^2}\{t : (t,x) \in \widetilde{Q}^2\}$, $g(I)$ be the function that stretches an interval $I = [a,b]$ to the left by a factor of $(1+\alpha)$, starting from the right endpoint b, (7.8) holds.

To prove the first part and the assertion $|A \setminus \mathcal{B}^1| = 0$ and $|A| \leq q|\mathcal{B}^1|$, we follow the idea of the proof of [6, Proposition V.7.2] and [53, Lemma 2.1] with more details provided and slight modifications according to our construction.

Divide the set $(-1,0) \times Q(0,1)$ with the hyperplane $t = -1/4, t = -1/2, , t = -3/4, x = 0, y = 0$ into 2^4 congruent boxes whose interiors are denoted as $\widetilde{Q}(j_1), j_1 = 1, ..., 2^4$. Each $\widetilde{Q}(j_1)$ is congruent to $\{(t,x) : 0 < t < 1/4, 0 < x^1, x^2 < 1/2\}$. For $j_1 = 1, 2..., 16$, if $\widetilde{Q}_{j_1}$ satisfies $|\widetilde{Q}_{j_1} \cap A| \geq q|\widetilde{Q}_{j_1}|$, then that $\widetilde{Q}_{j_1}$ belongs to the collection $\mathcal{B}$. If not, we split $\widetilde{Q}_{j_1}$ into 2^4 equal sub-boxes whose interiors are denoted as $\widetilde{Q}_{j_2}, j_2 = 1, ..., 2^4$, with each $\widetilde{Q}_{j_2}$ is congruent to $\{(t,x); 0 < t < 1/16, 0 < x^1, x^2 < 1/4\}$ and repeat the same procedure... For any $n \in \mathbb{N}$, let $\widehat{\mathcal{Q}}_n$ be the set of the open boxes congruent to $\{(t,x); 0 < t < 1/2^{n+1}, 0 < x^1, x^2 < 1/2^n\}$ and each element $\widetilde{Q}_{j_n}$ in $\widehat{\mathcal{Q}}_n$ satisfies $|\widetilde{Q}_{j_n} \cap A| \geq q|\widetilde{Q}_{j_n}|$.Then $\mathcal{B} = \cup_{n\geq 1} \widehat{\mathcal{Q}}_n$ and thus $\mathcal{B}$ is countable.

It is obvious that $\widehat{\mathcal{Q}}_n \cap \widehat{\mathcal{Q}}_m = \emptyset$, for $m \neq n$ and $|A \cap \widetilde{Q}_j| \geq q|\widetilde{Q}_j|$ for each $\widetilde{Q}_j \in \mathcal{B}$.

Let A^d be the collection of points of density of A.That is, for each $z \in A^d$, $\lim_{r\to 0} \frac{|B(z,r)\cap A|}{|B(z,r)|} = 1$. Then by Lebesgue density theorem,

$$|A \setminus A^d| = 0. \tag{7.10}$$

Since $q \in (0,1)$, we claim that

$$\text{For any point } z \in A^d, z \in \cup_{\widetilde{Q}_j \in \mathcal{B}} \overline{\widetilde{Q}_j}. \tag{7.11}$$

Suppose not. Then there exists an $r_0 > 0$ such that $B(z, r_0) \cap \widetilde{Q}_j = \emptyset$ for any $\widetilde{Q}_j \in \mathcal{B}$. Since z is a point of density of A, then $\lim_{r\to 0} |B(z,r) \cap A||/|B(z,r)| = 1$. Hence there exists an $\varepsilon \in (\sqrt[d]{q}, 1)$ and $\widetilde{r}_0(q,\varepsilon) > 0$ such that

$$|B(z,r) \cap A|/|B(z,r)| \geq q/(\varepsilon^d) > q$$

for any $r \in (0, \widetilde{r}_0]$. Choose $k(q,\varepsilon,r_0) \in \mathbb{N}$ sufficiently large so that there are cubes

$$R_{ij} = [i/2^k, (i+1)/2^k] \times [j/2^k, (j+1)/2^k], \quad 0 \leq i,j \leq (2^k - 1).$$

such that

$$B(z, \varepsilon(r_0 \wedge \widetilde{r}_1)) \subset \mathcal{C}_1 = \{R_{ij} \subset B(z, r_0 \wedge \widetilde{r}_1) : 0 \leq i,j \leq (2^k - 1)\}. \tag{7.12}$$

Let $M_1 = |\mathcal{C}_1|$. Set

$$\mathcal{C}_2 = \{R_{ij} \subset B(z, r_0 \wedge \widetilde{r}_1) : R_{ij} \cap A \cap B(z, \varepsilon(r_0 \wedge \widetilde{r}_1)) \neq \emptyset\}.$$

Since z is a point of density of A, By (7.12), $\mathcal{C}_2 \neq \emptyset$ and $|\mathcal{C}_2| \leq M_1$. Then

$$\begin{aligned} \sum_{m=1}^{|C_2|} |R_{ij}^m \cap A \cap B(z, \varepsilon(r_0 \wedge \widetilde{r}_1))| &\geq |A \cap B(z, \varepsilon(r_0 \wedge \widetilde{r}_1))| \\ &\geq (q/(\varepsilon^d))|B(z, \varepsilon(r_0 \wedge \widetilde{r}_1))| \\ &\geq q|B(z, r_0 \wedge \widetilde{r}_1)| \\ &\geq qM|R_{ij}|. \end{aligned}$$

Therefore, there exists at least one cube R_{ij} in $\mathcal{C}_2$ satisfying $|R_{ij}^m \cap A| \geq q|R_{ij}^m|$. Obviously, the interior $(R_{ij}^m)^\circ$ of such R_{ij}^m is an element in $\mathcal{B}$. Therefore, by contradiction, (7.11) is proved.

Since $|\cup_{\widetilde{Q}_j \in \mathcal{B}} \partial\widetilde{Q}_j| \leq \sum_{\widetilde{Q}_j \in \mathcal{B}} |\partial\widetilde{Q}_j| = 0$, then $|(\cup_{\widetilde{Q}_j \in \mathcal{B}} \overline{\widetilde{Q}_j}) \setminus \mathcal{B}^1| = |\mathcal{B} \setminus \mathcal{B}^1| = 0$. Then by (7.10), (7.11),

$$|A \setminus \mathcal{B}^1| = |A^d \setminus \mathcal{B}^1| = 0. \tag{7.13}$$

To show $|A| \leq q|\mathcal{B}^1|$, we first represent $\mathcal{B}^1$, up to a set of measure 0, as the union of boxes of the form $(t_1 - r_1^2, t_1) \times \mathcal{Q}(x_1, r_1)$, where $(t_1, x_1) \in (-1, 0) \times \mathcal{Q}(0, 1)$.

Divide $(-1, 0) \times \mathcal{Q}(0, 1)$ with the hyperplane $t = -1/4, t = -1/2, , t = -3/4, x = 0, y = 0$, up to a set of measure of 0, into 2^4 congruent disjoint open boxes $\widetilde{Q}(j_1)$, each is which is congruent to $\{(t, x) : 0 < t < 1/4, 0 < x^1, x^2 < 1/2\}$. For $j_1 = 1, 2..., 16$, if $\widetilde{Q}(j_1) \subset \mathcal{B}^1$, then $\widetilde{Q}(j_1) \in \widehat{Q}_1$; Otherwise, divide $\widetilde{Q}(j_1)$,up to a set of measure of 0, into 2^4 equal open sub-boxes, each of which is congruent to $\{(t, x) : 0 < t < 1/16, 0 < x^1, x^2 < 1/4\}$, and repeat the above procedure to create the set $\widehat{Q}_2$... Eventually, for any $n \in \mathbb{N}$, $\widehat{Q}_n$ the set in which each element $\widetilde{Q}(j_n)$ satisfies $\widetilde{Q}(j_n) \subset \mathcal{B}^1$ and is congruent to $\{(t, x); 0 < t < 1/2^{n+1}, 0 < x^1, x^2 < 1/2^n\}$. Also, it is easy to see that $\widehat{Q}_n \cap \widehat{Q}_m = \emptyset$, whenever $m \neq n$.

Therefore,

$$|\mathcal{B}^1| = \sum_{n \geq 1} \sum_{\widetilde{Q}(j_n) \in \widehat{Q}_n} |\widetilde{Q}(j_n)| \tag{7.14}$$

and

$$|A \cap \mathcal{B}^1| = \sum_{n \geq 1} \sum_{\widetilde{Q}(j_n) \in \widehat{Q}_n} |A \cap \widetilde{Q}(j_n)|. \tag{7.15}$$

If every $\widetilde{Q}(j_1)$ is in $\mathcal{B}^1$, there is nothing to prove. Otherwise, there exists some $\widetilde{Q}(j_1) \notin \mathcal{B}^1$.Under this assumption, we are going to show that for every $n \geq 1$, each $\widetilde{Q}(j_n) \in \widehat{Q}_n$,

$$|A \cap \widetilde{Q}(j_n)| < q|\widetilde{Q}(j_n)|. \tag{7.16}$$

Suppose not. Then there exists some $\widetilde{Q}(j_n)$ in some $\widehat{Q}_n$ for some $n \in N$ such that

$$|A \cap \widetilde{Q}(j_n)| \geq q|\widetilde{Q}(j_n)|.$$

Since $\widetilde{Q}(j_n) \subset \mathcal{B}^1$, then $\widetilde{Q}(j_n)$ is contained in some $\widetilde{Q}(j_{n-1}) \subset \mathcal{B}^1$.However, since $\widehat{Q}_n$ is the collection of boxes $\widetilde{Q}(j_n)$ whose $\widetilde{Q}(j_{n-1}) \not\subset \mathcal{B}^1$. Therefore this contradiction proves (7.16) and thus by (7.13),(7.14),(7.15) (7.16), we have $|A| \leq q|\mathcal{B}^1|$. □

Theorem 1.7.6. *Suppose $\mathcal{L} \in \mathcal{N}(\Lambda_1, \Lambda_2, \Lambda_3)$ satisfying* (1.3) *and X is a Hunt process that solves the martingale problem for $(\mathcal{L}, C_b^2(\mathbb{R}^d))$. Denote by $\mathbf{P}^{(t_0,x)}$ the law of the space-time process $Z_t = (V_0 - t, X_t)$ starting from (t_0, x). There exists a non-decreasing function ϕ on $[0,1]$, which depends only on $(d, \Lambda_1, \Lambda_2, \Lambda_3)$ and the rate of the function ψ in (1.3) converging to* 0 *so that for any $r \in (0, \widetilde{r}_0], x_0 \in \mathbb{R}^d, y \in \overline{Q(x_0, r/2)}$, and any measurable set $A \subset (t_0 - r^2, t_0) \times Q(x_0, r)$with $|A| > 0$,*

$$\mathbf{P}^{(t_0,y)}(T_A \leq \tau_{(t_0 - r^2, t_0) \times Q(x_0, r)}) \geq \phi(|A|/r^{d+2}). \tag{7.17}$$

where $\widetilde{r}_0$ is the constant in Proposition 1.4.7.

Proof. By the similar idea as that for Theorem 1.4.9, and inspired by the idea in the proof of Theorem 1 in [52], define

$$\begin{aligned} \phi(\varepsilon) \quad = \quad \inf \Big\{ & \mathbf{P}^{(t_0,y)}(T_A < \tau_{(t_0 - r^2, t_0) \times Q(x_0, r)} : \mathcal{L} \in \mathcal{N}(\Lambda_1, \Lambda_2, \Lambda_3),\ x_0 \in \mathbb{R}^d, r \in (0, \widetilde{r}_0], \\ & t_0 \geq r^2, y \in \overline{Q(x_0, r/2)}, A \subset (t_0 - r^2, t_0) \times Q(x_0, r), |A| \geq \varepsilon r^{d+2} \Big\}. \end{aligned}$$

Without loss of generality, we assume that the space dimension $d = 2$, and the proof works for higher dimensions.

Obviously, ϕ is non-decreasing. By Proposition 1.7.4, there exists an $\varepsilon_0 \in (0,1)$ so that $\phi(\varepsilon) > 0$ for every $\varepsilon \in [\varepsilon_0, 1]$.

Define $q_0 = \inf\{\varepsilon > 0 : \phi(\varepsilon) > 0\}$ and we want to prove that $q_0 = 0$ by contradiction.

Were $q_0 > 0$, as $q_0 < 1$, take $q \in (q_0, \frac{1}{4}(q_0 + \sqrt{q_0^2 + 8q_0}) \subset (q_0, 1)$, then $q(2q - q_0) < q_0$. Choose $\alpha \in (0, (\frac{q_0}{q(2q-q_0)} - 1) \wedge 1)$ and let $\beta := (2q - q_0)q(1 + \alpha)$. Then $\beta < q_0 < 2q - q_0$, and thus

$$\phi(\beta) = 0. \tag{7.18}$$

Set

$$\begin{aligned} &\beta_1 := 1 - (1 - (q - q_0)/4)^{1/2}, \quad \kappa := \left(\frac{1-q}{1-(q+q_0)/2}\right)^{1/4}, \\ &\gamma_1 = \frac{1}{2a_1} \wedge 1 \wedge [8(1/\kappa^2 - 1)], \quad \rho = \Phi_1(\frac{\beta}{72} \wedge \frac{1}{32} \wedge \frac{\kappa}{8}\sqrt{\frac{\alpha(q-q_0)}{5+\alpha}}). \end{aligned} \tag{7.19}$$

where a_1 is defined in (2.15). Note that $0 < \gamma_1 \leq 1$ and $0 < \beta_1, \kappa, \rho < 1$.

As $\phi(\beta) = 0$, by the definition of q_0, there exists some $\mathcal{L}_0 \in \mathcal{N}(\Lambda_1, \Lambda_2, \Lambda_3)$, some $r_0 \in (0, \tilde{r}_0], x_0 \in \mathbb{R}^d, t_0 \geq r_0^2, y \in \overline{Q(x_0, r_0/2)}$, and $A \subset \bar{Q} := (t_0 - r_0^2, t_0) \times Q(x_0, r_0)$ with $|A|/|\bar{Q}| \geq \beta$, such that

$$\mathbf{P}^{(t_0,y)}(T_A < \tau_{\bar{Q}}) < \rho\phi(q)\phi(\frac{q+q_0}{2})/4,$$

where $\{Z_t, t \geq 0; \mathbf{P}^x, x \in \mathbb{R}^d\}$ is the space-time process of X having the infinitesimal generator $\mathcal{L}_0$. We claim that $|A| < q|\bar{Q}|$. Indeed, otherwise, were $|A|/|\bar{Q}| \geq q$, by the definition of ϕ,

$$\rho\phi(q)\phi(\frac{q+q_0}{2})/4 > \mathbf{P}^{(t_0,y)}(T_A < \tau_{\bar{Q}}) \geq \phi(q),$$

which is impossible as $\rho, \phi(\frac{q+q_0}{2}) < 1$. Next by breaking the proof into two cases, we want to show that

$$\mathbf{P}^{(t_0,y)}(T_A < \tau_{\bar{Q}}) \geq \rho\phi(q)\phi(\frac{q+q_0}{2})/4 \text{ for every } y \in \overline{Q(x_0, r/2)}. \tag{7.20}$$

(i) If $|\mathcal{B}^2 \setminus \bar{Q}| < (q - q_0)|\bar{Q}|$, as $|A| < q|\bar{Q}|$, by Lemma 1.7.5, there exists a collection $\mathcal{B}$ of disjoint open boxes $\widetilde{Q}$'s in $\bar{Q}$ of the form (7.4) such that $|A \cap \widetilde{Q}| > q|\widetilde{Q}|$ for each

$\widetilde{Q} \in \mathcal{B}$ and $|A| \leq q(1+\alpha)|\mathcal{B}^2|$ with $\mathcal{B}^2 = \cup_{\widetilde{Q}\in\mathcal{B}}\widetilde{Q}^2$, where $\widetilde{Q}^2$ is of the form in (7.6). Since $|A| \geq \beta|\bar{Q}|$, then

$$|\mathcal{B}^2 \cap \bar{Q}| \quad = \quad |\mathcal{B}^2| - |\mathcal{B}^2 \setminus \bar{Q}| > |A|/(q(1+\alpha)) - (q-q_0)|\bar{Q}| = q|\bar{Q}|.$$

Enumerate $\mathcal{B}$, as

$$\lim_{n\to\infty} | \cup_{i=1}^n \widetilde{Q}_i^2 \cap \bar{Q}| \geq q|\bar{Q}|,$$

there exists an $N \in \mathbb{N}$ such that $| \cup_{i=1}^N \widetilde{Q}_i^2 \cap \bar{Q}| \geq ((3q+q_0)/4)|\bar{Q}|$.
Let $\Gamma := (\cup_{i=1}^N \widetilde{Q}_i^2) \cap ((t_0 - r_0^2, t_0) \times \mathcal{Q}(x_0,(1-\beta_1)r_0))$. Then

$$\Gamma \quad \geq \quad | \cup_{i=1}^N \widetilde{Q}_i^2 \cap \bar{Q}| - |(t_0 - r_0^2, t_0) \times (\mathcal{Q}(x_0, r_0) \setminus \mathcal{Q}(x_0,(1-\beta_1)r_0))| \geq ((q+q_0)/2)|\bar{Q}|,$$

and therefore by the definition of ϕ,

$$\mathbf{P}^{(t_0,x)}(T_\Gamma < \tau_{\bar{Q}}) \geq \phi(\frac{q+q_0}{2}) \text{ for every } \overline{x \in \mathcal{Q}(x_0, r/2)}. \tag{7.21}$$

We claim that

$$\mathbf{P}^{(t,z)}(T_A < \tau_{\bar{Q}}) \geq \rho/4 \text{ for every } (t,z) \in \overline{\Gamma}.$$

For $(t,z) \in \overline{\Gamma}$, $(t,z) \in \overline{\widetilde{Q}_k^2}$ for some $1 \leq k \leq N$. Denote $\widetilde{Q}_k$ as $(t_1 - r_1^2, t_1) \times \mathcal{Q}(x_1, r_1)$ and by (7.6), $t_1 + \gamma_1 r_1^2/8 < t$. let $\Theta := (t_1, t_1 + 2\gamma_1(r_1/4)^2) \times \mathcal{Q}(x_1, r_1/4)$.
If $r_1 \geq \beta_1 r_0/9$, since $\Theta_2 := (t_1, t_1 + 2\gamma_1\beta_1^2 r_0^2/36^2) \times \mathcal{Q}(x_1, \beta_1 r_0/36) \subset \Theta$, and by Proposition 1.7.2, Φ_δ is non-decreasing in δ, then with $\delta = 1$,

$$\mathbf{P}^{(t,z)}(T_{\Theta_1} < \tau_{(t_1,t)\times\mathcal{Q}(0,r_0)}) \geq \mathbf{P}^{(t,z)}(T_{\Theta_2} < \tau_{(t_1,t)\times\mathcal{Q}(x_0,r_0)}) \geq \Phi_1(\beta_1/72) \geq \rho. \tag{7.22}$$

If $r_1 < \beta_1 r_0/9$, $\mathcal{Q}(z, r_1/4) \cup \mathcal{Q}(x_1, r_1) \subset \mathcal{Q}(x_1, 4r_1)$ as $|z - x_1|_\infty \leq 3r_1/2 \leq \beta_1 r/6$. Then with $\delta = 1$,

$$\mathbf{P}^{(t,z)}(T_\Theta < \tau_{(t_1 - r_1^2, t)\times\mathcal{Q}(x_1, 4r_1)}) > \Phi_1(1/32) \geq \rho, \tag{7.23}$$

Apply the same smooth non-decreasing test function φ on $[0,\infty)$ as in Lemma 1.2.6, with x_1 in place of x_0 in $f(y)$, i.e., $f(y) = \varphi(|y-x_1|^2)$. Given $d = 2$, since $\mathcal{Q}(x_1, r_1/4) \leq B(x_1, 1)$, then similar as (2.15),

$$|\mathcal{L}f(x)| \leq a_1 \text{ for every } \widetilde{x} \in \mathcal{Q}(x_1, r_1/4), \tag{7.24}$$

where a_1 is the constant in (2.15), depending only on $(d, \Lambda_1, \Lambda_2, \Lambda_3)$.

As $f(X_{t_2 \wedge \tau_{Q(x_1, r_1/2)}}) \le r_1^2/8$ on $\{\tau_{Q(x_1,r_1/2)} \le t_2\}$, by Lemma 1.2.3, (2.13), (2.16), taking $t_2 = \gamma_1 r_1^2/8$, one has

$$\begin{aligned}(r_1^2/8)\mathbb{P}^{\tilde{x}}(\tau_{Q(x_1,r_1/2)} \le \gamma_1 r_1/8) &\le \mathbb{E}^{\tilde{x}}\left[f(X_{t_2 \wedge \tau_{Q(x_1,r_1/2)}})\right] \\ &= f(x) + \mathbb{E}^{\tilde{x}}\left[\int_0^{t_2 \wedge \tau_{Q(x_1,r_1/2)}} \mathcal{L}f(X_s)ds\right] \\ &\le r_1^2/32 + a_1\gamma_1 r_1^2/8,\end{aligned}$$

and it follows that

$$\mathbb{P}^{\tilde{x}}(\tau_{Q(x_1,r_1/2)} > 2\gamma_1(r_1/4)^2) = 1/4 \text{ for every } \widetilde{x} \in \mathcal{Q}(x_1, r_1/4). \tag{7.25}$$

Let $U_1 = \{t_1\} \times \mathcal{Q}(x_1, r_1/2)$.

Given $\{T_\Theta < \tau_{\bar{Q}}\}$, if the process X starts from a point in $\mathcal{Q}(x_1, r_1/4)$, and takes more than $\gamma_1 r_1^2/8$ amount of time to exit $\mathcal{Q}(x_1, r_1/2)$,where $\gamma_1 \le 1$, then the process Z will definitely hit U_1 before exiting $\bar{Q}$. Therefore, by (7.22),(7.23),(7.25),

$$\mathbf{P}^{(t,z)}(T_{U_1} < \tau_{\bar{Q}}) \ge \mathbf{E}^{(t,z)}[\mathbb{P}^{X_{T_{\mathcal{Q}(x_1,r_1/4)}}}(\tau_{\mathcal{Q}(x_1,r_1/2)} > \gamma_1 r_1^2/8); T_\Theta < \tau_{\bar{Q}}] \ge \rho/4,$$

and it follows that

$$\begin{aligned}\mathbf{P}^{(t,z)}(T_A < \tau_{\bar{Q}}) &\ge \mathbf{E}^{(t,z)}[\mathbf{P}^{Z(T_{U_1})}(T_A < \tau_{\bar{Q}}); T_{U_1} < \tau_{\bar{Q}}] \\ &\ge \mathbf{E}^{(t,z)}[\mathbf{P}^{Z(T_{U_1})}(T_{A\cap\widetilde{Q}} < \tau_{\widetilde{Q}}); T_{U_1} < \tau_{\bar{Q}}] \ge \rho\phi(q)/4.\end{aligned} \tag{7.26}$$

Finally, by (7.21) and (7.26), for every $y \in \overline{\mathcal{Q}(x_0, r/2)}$,

$$\begin{aligned}\mathbf{P}^{(t_0,y)}(T_A < \tau_{\bar{Q}}) &\ge \mathbf{E}^{(t_0,y)}[\mathbf{P}^{Z_{T_\Gamma}}(T_A < \tau_{\bar{Q}}); T_\Gamma < \tau_{\bar{Q}}] \\ &\ge \rho\phi(q)\phi(\frac{q+q_0}{2})/4.\end{aligned} \tag{7.27}$$

(ii) If $|\mathcal{B}^2 \backslash \bar{Q}| \ge (q - q_0)|\bar{Q}|$, then by the definition of $\mathcal{B}^2$, there must exists a $\widetilde{Q}^2 \in \mathcal{B}^2$ intersecting with the hyperplane $t = t_0 + (q - q_0)r_0^2$. Recall the form of $\widetilde{Q}^2$ in (7.6), we have $t_2 + 5r_2^2/\alpha + r_2^2 > t_0 + (q - q_0)r_0^2$, i.e., $r_2 > r_0\sqrt{\frac{\alpha(q-q_0)}{5+\alpha}}$.

Let $\widetilde{D} = (t_2 - r_2^2, t_2 + (\kappa^2 - 1)r_2^2) \times \mathcal{Q}(x_2, \kappa r_2)$ and

$$\Theta_2 = (t_2 + (\kappa^2 - 1)r_2^2, t_2 + (\kappa^2 - 1 + \gamma_1\kappa^2/8)r_2^2) \times \mathcal{Q}(x_2, \kappa r_2/4).$$

By the definition of γ_1 in (7.19), $t_2 + (\kappa^2 + \gamma_1\kappa^2/8 - 1)r_2^2 \le t_2$. As $\kappa < 1$, for any $y \in \overline{Q(x_0, r_0/2)}$, $Q(y, \kappa r_2/4) \cup Q(x_2, \kappa r_2/4) \subset Q(x_0, r_0)$. Hence, again by Proposition 1.7.2 with $\delta = 1$,

$$\mathbf{P}^{(t_0,y)}(T_{\Theta_2} < \tau_{\bar{Q}}) \ge \Phi_1(\frac{\kappa}{8}\sqrt{\frac{\alpha(q-q_0)}{5+\alpha}}) \ge \rho, \tag{7.28}$$

In addition, by the same argument in (7.25), for any $\bar{x} \in Q(x_2, \kappa r_2/4)$,

$$\mathbb{P}^{\bar{x}}(\tau_{Q(x_2,\kappa r_2/2)} \ge \gamma_1\kappa^2 r_2^2/8) = 1/4. \tag{7.29}$$

Let $U_2 = \{t_2 + (\kappa^2 - 1)r_2^2\} \times Q(x_2, \kappa r_2/2)$. By (7.28) and (7.29),

$$\mathbf{P}^{(t_0,y)}(T_{U_2} < \tau_{\bar{Q}}) \ge \rho/4. \tag{7.30}$$

As $|\widetilde{D}| = \kappa^4|\widetilde{Q}|$, by the definition of κ, one has

$$|A \cap \widetilde{D}| \quad \ge \quad |A \cap \widetilde{Q}| - |\widetilde{Q} \setminus \widetilde{D}| \ge (q - (1-\kappa^4))|\widetilde{Q}| = \frac{q_0 + q}{2}|\widetilde{D}|. \tag{7.31}$$

Then by the definition of ϕ, (7.30) and (7.31),

$$\begin{aligned} \mathbf{P}^{(t_0,y)}(T_A < \tau_{\bar{Q}}) \quad &\ge \quad \mathbf{E}^{(t_0,y)}[\mathbf{P}^{Z(\tau_{U_2})}(T_{A\cap\widetilde{D}} < \tau_{\widetilde{D}}); T_{U_2} < \tau_{\bar{Q}}] \\ &\ge \quad \rho\phi(\frac{q_0 + q}{2})/4. \end{aligned} \tag{7.32}$$

Finally, combining (7.27) and (7.32), one has (7.20), which implies that $\phi(\beta) \neq 0$. This contradiction with (7.18) proves that $q_0 = 0$. □

Corollary 1.7.7. *There exists a non-decreasing function $\varphi : (0,1] \mapsto (0,1/4]$ depending only on $(d, \Lambda_1, \Lambda_2, \Lambda_3)$ and the rate of the function ψ in* (1.3) *converging to* 0 *such that for any $r \in (0, \widetilde{r}_1]$, $x_0 \in \mathbb{R}^d, x \in B(x_0, r/2)$, and any measurable $A \subset (t_0 - r^2, t_0) \times B(x_0, r)$with $|A| > 0$,*

$$\mathbf{P}^{(t_0,x)}(T_A \le \tau_{(t_0-r^2,t_0)\times B(x_0,r)}) \ge \varphi(|A|/|(t_0 - r^2, t_0) \times B(x_0, r)|). \tag{7.33}$$

where $\widetilde{r}_1$ is the constant in Corollary 1.4.11.

Proof. We will follow the similar idea as the proof of Corollary 1.4.11. Fix $r \in (0, \widetilde{r}_1)$, and $t_0 \geq r^2$. Without loss of generality, we assume $x_0 = 0$ and for the notional convenience, we assume the space dimension $d = 2$, and the proof works for higher dimensions.

Denote $\bar{Q} := (t_0 - r^2, t_0) \times B(0, r)$ and let $\lambda = |A|/|\bar{Q}|$. Take $\varepsilon = \varepsilon(\lambda) \in (0, 1/2)$ so that $(1 - (1 - 2\varepsilon)^4) = \lambda/2$. It is clear that $\varepsilon(\lambda)$ is an increasing function in $\lambda \in (0, 1]$. Define $m(\varepsilon) := (1 - (1 - \varepsilon)^2)/2$. It follows that

$$|A \cap (t_0 - r^2 + m(2\varepsilon)r^2, t_0 - m(2\varepsilon)r^2) \times B(0, (1 - 2\varepsilon)r)|/|\bar{Q}| \geq \lambda/2.$$

Let $k = k(\lambda) \in \mathbb{N}$ be sufficiently large so that $k \geq 1/(2\sqrt{\varepsilon^2 - 2\varepsilon})$ and for any box $H_{lij} \in (t_0 - r^2 + m(\varepsilon)r^2, t_0 - m(\varepsilon)r^2) \times B(0, (1 - \varepsilon)r)$ of the form

$$H_{lij} := [t_l - r^2/k^2, t_l] \times R_{ij} = [t_0 - (l+1)r^2/k^2, t_0 - lr^2/k^2] \times [ir/k, (i+1)/k] \times [jr/k, (j+1)r/k],$$

with $0 \leq l \leq k^2 - 1, -k \leq i, j \leq k - 1$, we have $|H_{lij} \cap (t_0 - r^2 + m(2\varepsilon)r^2, t_0 - m(2\varepsilon)r^2) \times B(0, (1 - 2\varepsilon)r)| \neq \emptyset$.We can choose $k(\lambda)$ in such a way that it is a decreasing function in ε and hence a decreasing function in λ. Let

$$\mathcal{C} = \{H_{lij} : |H_{lij} \cap (t_0 - r^2 + m(2\varepsilon)r^2, t_0 - m(2\varepsilon)r^2) \times B(0, (1 - 2\varepsilon)r) \cap A| \neq \emptyset\}$$

and $M = M(\lambda)$ be the total number of elements in $\mathcal{C}$. Clearly, M is an increasing function in k and thus a decreasing function in λ, and $|M| \leq k^2 \cdot (2k)^2$. Denote by R^*_{ij} the closed concentric cube as R_{ij} but with side length half as long. As

$$\sum_{H_{lij} \in \mathcal{C}} |H_{lij} \cap A| \geq |(t_0 - r^2 + m(2\varepsilon)r^2, t_0 - m(2\varepsilon)r^2) \times B(0, (1 - 2\varepsilon)r) \cap A| \geq \lambda|\bar{Q}|/2 = |A|/2,$$

there must exist some $H_{lij} \in \mathcal{C}$ such that

$$|H_{lij} \cap A| \geq |A|/(2M).$$

Since $|H_{lij}| = r^4/k^4$ and $M \leq 4k^4$,then

$$|H_{lij} \cap A|/|H_{lij}| \geq |A|/(8r^4) \geq \lambda|\bar{Q}|/(2|(t_0 - r^2, t_0) \times Q(0, 2r)|) \geq \lambda/2.$$

Let ϕ be the non-decreasing strictly positive function on $(0, 1]$ in Theorem 7.17. We have by Theorem 7.17,

$$\mathbf{P}^{(t_l, x)}(T_{H_{lij} \cap A} < \tau_{H_{lij}}) \geq \phi(\lambda/2) \quad \text{for } x \in R^*_{ij}. \tag{7.34}$$

Let y_0 be the center of R_{ij} and $\gamma_1 = 1/2a_2 \wedge 1$, where a_2 is the constant in (7.24). Define $\bar{H}_{lij} = (t_l, t_l + 2\gamma_1 (r/4k)^2) \times B(y_0, r/8k)$. Since $k \geq 1/(2\sqrt{\varepsilon^2 - 2\varepsilon})$ and $t_l < t_0 - m(\varepsilon) r^2$, then $t_l + 2\gamma_1 (r/4k)^2 < t_0$. For any $y \in B(0, r/2)$, we have $B(y, r/8k) \subset B(0, r)$. Then by Proposition 1.7.3 with $\delta = 1$,

$$\mathbf{P}^{(t_0,y)}(T_{\bar{H}_{lij}} < \tau_{\bar{Q}}) \geq \Phi_1(1/8k). \tag{7.35}$$

Also, by the similar argument in (7.25),

$$\mathbb{P}^{\bar{x}}(\tau_{B(y_0,r/4k)} \geq 2\gamma_1 (r/4k)^2) \geq 1/4 \quad \text{for } \bar{x} \in B(y_0, r/8k). \tag{7.36}$$

Let $U = \{t_l\} \times B(y_0, r/4k)$. Then by (7.34), (7.35), (7.36), we have

$$\begin{aligned}
\mathbf{P}^{(t_0,y)}(T_A \leq \tau_{\bar{Q}}) &\geq \mathbf{E}^{(t_0,y)}[\mathbf{P}^{Z_{T_U}}(T_A < \tau_{\bar{Q}}); T_U \leq \tau_{\bar{Q}}] \\
&\geq \mathbf{E}^{(t_0,y)}[\mathbf{P}^{Z_{T_U}}(T_{A \cap H_{lij}} < \tau_{H_{lij}}); T_U \leq \tau_{\bar{Q}}] \\
&\geq \phi(\lambda/2) \mathbf{E}^{(t_0,y)}[\mathbb{P}^{X_{T_{B(y_0,r/8k)}}}(\tau_{B(y_0,r/4k)} \geq \gamma_1 r^2/(32k^2)); T_{\bar{H}_{lij}} < \tau_{\bar{Q}}] \\
&\geq \phi(\lambda/2) \Phi_1(1/8k)/4,
\end{aligned}$$

where Φ_1 is the non-decreasing strictly positive function on $(0,\infty)$ in Proposition 1.7.3 with $\delta = 1$. This completes the proof of this corollary by taking

$$\varphi(\lambda) := \phi(\lambda/2)\, \Phi_1(1/(8k))/4 = \phi(\lambda/8)\, \Phi_1(1/(8k(\lambda)))/4,$$

which is a strictly positive non-decreasing function on $(0, 1]$. □

Next, we are going to show the Hölder property of $\mathcal{L}$-parabolic function u.

Theorem 1.7.8. *Suppose $\mathcal{L} \in \mathcal{N}(\Lambda_1, \Lambda_2, \Lambda_3)$ satisfying* (1.3)*. There exist constants $C > 0$ and $\beta \in (0, 1)$ that depend only on $d, \Lambda_1, \Lambda_2, \Lambda_3$ and the rate of ψ converging to 0 so that for any $x_0 \in \mathbb{R}^d$, $r \in (0, \widetilde{r}_0]$, and any bounded function u parabolic with respect to $\mathcal{L}$ in $(t_0 - r^2, t_0) \times B(x_0, r)$, one has*

$$|u(s,x) - u(t,y)| \leq C \, \|u\|_\infty \, r^{-\beta} (|t-s|^{1/2} + |x-y|)^\beta \quad \textit{for any } (s,x), (t,y) \in (t_0 - r^2/4, t_0) \times B(x_0, r/2), \tag{7.37}$$

where $\widetilde{r}_0$ is the constant in Proposition 1.4.7, and $\|u\|_\infty = \sup_{(t,x) \in (0,\infty) \times \mathbb{R}^d} |u(t,x)|$.

Proof. Fix $x_0 \in \mathbb{R}^d$ and $r \in (0, \widetilde{r}_0]$. Suppose that u is a function defined in $(0,\infty) \times \mathbb{R}^d$ that is $\mathcal{L}$-parabolic in $(t_0 - r^2, t_0) \times B(x_0, r)$. Since u is bounded, then without loss of generality, we assume that $\|u\|_\infty = 1/2$.

Take $(t,x) \in (t_0 - r^2/4, t_0) \times B(x_0, r/2)$ and $r_1 = r/2$, so that $(t - r_1^2, t) \times B(x, r_1) \subset (t_0 - r^2, t_0) \times B(x_0, r)$. Let φ be the non-decreasing function defined in Corollary 1.7.7. Set

$$a := \sqrt{1 - \varphi(1/3)/4} \vee (1/\sqrt{2}), \text{ and } \rho := (\frac{\varphi(1/3)}{40 c_1 \Lambda_3} \wedge \frac{1}{4})^{1/2} \sqrt{a},$$

where $c_1 > 1$ is the constant in Lemma 2.3. Note that $1/\sqrt{2} \le a < 1$ and $0 < \rho < 1/2$. For $n \ge 2$, let $r_n = \rho^{n-1} r_1$, and for simplicity, denote $(t - r_n^2, t) \times B(x, r_n)$ as Q_n, τ_{Q_n}as τ_n. Let $M_n := \sup_{(s,y)\in Q_n} u(s,y), m_n := \inf_{(s,y)\in Q_n} u(s,y)$.

We claim that

$$M_n - m_n \le a^{n-2}, \text{ for each } n \ge 1.$$

Obviously, $M_2 - m_2 \le M_1 - m_1 \le 1$. For $n \ge 3$, define

$$A_n = \{(s,y) \in Q_n : u(s,y) \le (M_n + m_n)/2\}.$$

We claim that $|A_n/Q_n| \ge 1/2$. Otherwise, we consider $M_n - u$ instead of M_n. Let A be a compact subset of A_n such that $|A/Q_n| \ge 1/3$. By Corollary 1.7.7, we have

$$\mathbf{P}^{(t,\bar{x})}(T_A < \tau_n) \ge \varphi(1/3) \quad \text{for } \bar{x} \in B(x, r_{n+1}). \tag{7.38}$$

For any $\varepsilon > 0$, there are $z_1, z_2 \in Q_{n+1}$ so that $u(z_1) \ge M_{n+1} - \varepsilon, u(z_2) \le m_{n+1} + \varepsilon$.

As $\mathcal{L} \in \mathcal{N}(\Lambda_1, \Lambda_2, \Lambda_3)$, by (5.4), we have that for $1 \le i \le n-1$,

$$\sup_{(s,y)\in Q_n} \mathbf{P}^{(s,y)}(Z_{\tau_n} \notin Q_{n-i}) \quad \le \quad c_1 \Lambda_3 \rho^{2i}/(1-\rho^i)^2. \tag{7.39}$$

Note that $\rho^2 \le a/4 < 1/4$. Following the same calculation as in (5.5), we have by (7.38) and (7.39) that for $n \ge 2$,

$$\begin{aligned} & M_{n+1} - m_{n+1} - 2\varepsilon \\ < \quad & u(z_1) - u(z_2) \\ = \quad & \mathbf{E}^{(t,x)}[u(Z_{T_A \wedge \tau_n}) - u(z_2)] \end{aligned}$$

$$
\begin{aligned}
&= \mathbf{E}^{(t,x)}[u(Z_{T_A}) - u(z_2); T_A < \tau_n] + \mathbb{E}^{(t,x)}[u(Z_{\tau_n}) - u(z_2); T_A \geq \tau_n, Z_{\tau_n} \in Q_{n-1}] \\
&\quad + \sum_{i=2}^{n-1} \mathbf{E}^{(t,x)}[u(Z_{\tau_n}) - u(z_2); T_A \geq \tau_n, Z_{\tau_n} \in Q_{n-i} \setminus Q_{n+1-i}] \\
&\quad + \mathbf{E}^{(t,x)}[u(Z_{\tau_n}) - u(z_2); T_A \geq \tau_n, Z_{\tau_n} \notin Q_1] \\
&\leq ((M_n + m_n)/2 - m_n)\mathbf{P}^{(t,x)}(T_A < \tau_n) + (M_{n-1} - m_{n-1})\mathbf{P}^{(t,x)}(T_A \geq \tau_n) \\
&\quad + \sum_{i=2}^{n-1} (M_{n-i} - m_{n-i})\mathbf{P}^{(t,x)}(Z_{\tau_n} \notin Q_{n+1-i}) + \mathbf{P}^{(t,x)}(Z_{\tau_n} \notin Q_1) \\
&\leq \frac{1}{2} a^{n-2}\mathbf{P}^{(t,x)}(T_A < \tau_n) + a^{n-3}(1 - \mathbf{P}^{(t,x)}(T_A < \tau_n)) \\
&\quad + \sum_{i=2}^{n-1} a^{n-i-2} c_1 \Lambda_3 \rho^{2(i-1)}/(1-\rho^{i-1})^2 + c_1\Lambda_3 \rho^{2(n-1)}/(1-\rho^{n-1})^2 \\
&\leq a^{n-3}\left(1 - (1 - a/2)\mathbf{P}^{(t,x)}(T_A < \tau_n)\right) + \frac{c_1\Lambda_3 a^{n-3}}{(1-\rho)^2} \sum_{i=2}^{n-1} (\rho^2/a)^{i-1} + \frac{c_1\Lambda_3\rho^{2(n-1)}}{(1-\rho)^2} \\
&\leq a^{n-3}\left(1 - (1 - a/2)\varphi(1/3)\right) + 4c_1\Lambda_3 a^{n-3}(\rho^2/a) \sum_{k=0}^{n-3} 2^{-k} + 4c_1\Lambda_3\rho^{2(n-1)} \\
&\leq a^{n-3}\left(1 - \tfrac{1}{2}\varphi(1/3)\right) + 8c_1\Lambda_3 a^{n-3}(\rho^2/a) + 4c_1\Lambda_3\rho^{2(n-1)} \\
&\leq a^{n-1}\left(\frac{1 - \frac{1}{2}\varphi(1/3)}{a^2} + 8c_1\Lambda_3\left((2\rho^2/a) + \tfrac{1}{2}(\rho^2/a)^{n-1}\right)\right) \\
&\leq a^{n-1}\left(\frac{1 - \frac{1}{2}\varphi(1/3)}{a^2} + \frac{10c_1\Lambda_3\,\rho^2}{a}\right) \\
&\leq a^{n-1}.
\end{aligned}
$$

Sending $\varepsilon \to 0$, we have $M_{n+1} - m_{n+1} \leq a^{n-1}$.

Given $z_1 = (t,x), z_2 = (s,y) \in (t_0 - r^2, t_0) \times B(x_0, r/2)$,with $s \leq t$, let k be the largest nonnegative integer not exceeding $\log_\rho(|z_2 - z_1|/r)$. Then $z_2 \in (t - r_{k+1}^2, t) \times B(x, r_{k+1})$, and $|z_2 - z_1| := (|t-s|)^{1/2} + |x - y| \leq \rho^k r$. Therefore,

$$
|u(z_2) - u(z_1)| \leq a^{k-1} \leq \frac{1}{a} a^{\log_\rho(|z_2 - z_1|/r)} \leq \frac{1}{a}(|z_2 - z_1|/r)^{\log a/\log\rho}.
$$

Since $1/\sqrt{2} < a < 1$ and $0 < \rho < 1/2$, so $\log a/\log\rho \in (0, 1/2)$, then we complete the proof by taking $\beta := \log a/\log\rho \in (0,1/2)$ and $C = 2a^{-1}$, which both depend on $d, \Lambda_1, \Lambda_2, \Lambda_3$ and the rate of the function ψ converging to 0. □

Proposition 1.7.9. *Suppose* $\mathcal{L} \in \mathcal{N}(\Lambda_1, \Lambda_2, \Lambda_3)$ *satisfying* (1.3) *and* X *is a Hunt process that solves the martingale problem for* $(\mathcal{L}, C_b^2(\mathbb{R}^d))$*. Denote by* $\mathbf{P}^{(t_0,x)}$ *the law of the space-time process* $Z_t = (V_0 - t, X_t)$ *starting from* (t_0, x)*. Then for any* $x_0 \in \mathbb{R}^d, r \in (0, r_0], t_0 \geq r^2$*, there exists* $c_5 > 0$ *that depends only on* $(d, \Lambda_1, \Lambda_2, \Lambda_3, c_0)$ *such that for any bounded non-negative function* h *supported on* $[0, t_0] \times B(x_0, 2r)^c$ *that is* $\mathcal{L}$*-parabolic in* $(t_0 - r^2, t_0) \times B(x_0, r)$*, one has*

$$\mathbf{E}^{(t_0,x)}[h(Z_{\tau_r})] \leq c_5 \mathbf{E}^{(t_0,y)}[h(Z_{\tau_r})] \quad \text{for } x, y \in B(x_0, r/2),$$

where $\tau_r = \tau_{(t_0 - r^2, t_0) \times B(x_0, r)} = \inf\{t \geq 0 : X_t \notin B(x_0, r)\} \wedge r^2 = \tau_{B(x_0,r)} \wedge r^2$*,* c_0 *is the constant in* (1.12)*, and* r_0 *is the constant in Lemma 1.2.3.*

Proof. It just suffices to prove for $h(t, x) = \mathbb{1}_{[0,t_0] \times F}(t, x)$, where F is a closed set in $B(x_0, 2r)^c$. By Lemma 1.2.6, Proposition 1.2.8, for any $x \in B(x_0, r/2)$, we have

$$\begin{aligned} \mathbf{E}^{(t_0,y)}[\mathbb{1}_{[0,t_0] \times F}(t_0 - \tau_r, X_{\tau_r})] &= \mathbb{P}^y[X_{\tau_r} \in F] \\ &= \mathbb{E}^y[\sum_{s \leq \tau_r} \mathbb{1}_{(X_{s-} \neq X_s, X_s \in F)}] \\ &= \mathbb{E}^y[\int_0^{\tau_r} n(X_s, F - X_s) ds] \\ &\geq c_0^{-1} n(x_0, F - x_0)\, \mathbb{E}^y[\tau_r] \\ &\geq c_0^{-1} n(x_0, F - x_0)\, c_2 r^2. \end{aligned} \tag{7.40}$$

Similarly,by Lemma 1.2.3, for any $x \in B(x_0, r)$, we have

$$\begin{aligned} \mathbf{E}^{(t_0,x)}[\mathbb{1}_{[0,t_0] \times F}(t_0 - \tau_r, X_{\tau_r})] &= \mathbb{E}^x[\int_0^{\tau_r} n(X_s, F - X_s) ds] \\ &\leq c_0^2\, n(x_0, F - x_0) c_1 r^2. \end{aligned} \tag{7.41}$$

Hence, let $c_5 = c_0^3 c_2 / c_1$, and by (7.40),(7.41), it yields that

$$\mathbf{E}^{(t_0,x)}[\mathbb{1}_{[0,t_0] \times F}(Z_{\tau_r})] \leq c_5 \mathbf{E}^{(t_0,y)}[\mathbb{1}_{[0,t_0] \times F}(Z_{\tau_r})], \text{ for } x, y \in B(x_0, r/2).$$

□

Proposition 1.7.10. *Suppose* $\mathcal{L} \in \mathcal{N}(\Lambda_1, \Lambda_2, \Lambda_3)$ *with condition* (1.3) *holds.For any bounded function* $u(t, x)$ *on* $(0, \infty) \times \mathbb{R}^d$ *that is* $\mathcal{L}$*-parabolic in an open subset* D *of* $(0, \infty) \times \mathbb{R}^d$*, then*

$s \mapsto u(Z_{s\wedge\tau_D})$ is right continuous $\mathbf{P}^{(t,x)}$-a.s. for any $(t,x) \in D$,where $\{Z_s : s \geq 0\}$ is the space-time process of X.

Proof. This follows the same proof as Lemma 1.6.1. □

Proof of Theorem 1.7.1. For $r \in (0, \widetilde{r}_1)$ and u be any nonnegative function on $(0,\infty) \times \mathbb{R}^d$ that is $\mathcal{L}$-parabolic in $(t_0 - r^2, t_0) \times B(x_0, r)$. Similar to the beginning of the argument in Theorem 1.1.2, it suffices to establish the parabolic Harnack inequality for bounded nonnegative functions u on $(0,\infty) \times \mathbb{R}^d$ that is $\mathcal{L}$-parabolic in $(t_0 - r^2, t_0) \times B(x_0, r)$ with $\inf_{(0,\infty)\times\mathbb{R}^d} u > 0$.Taking a constant multiple of u if needed, we may further assume

$$\inf_{Q_+} u = 1/2, \tag{7.42}$$

where $Q_+ := (t_0 - 3r^2/8, t_0 - r^2/8) \times B(x_0, r/2)$.

Then there exists a point $(t,x) \in Q_+$ such that $u(t,x) < 1$. Let φ be the function in Corollary 1.7.7, and take $\varphi(r) \wedge r$ if needed, we way assume that $\lim_{r\to 0^+} \varphi(r) = 0$.

Our goal is to show that u is bounded above on Q_- by an universal constant which depend only on $(d, c_0, \Lambda_1, \Lambda_2, \Lambda_3)$ and the rate of the function ψ in (1.12) converging to 0 and we will do it by contradiction.

Let

$$\gamma_1 = 1/((2a_1) \vee 2), \text{where } a_1 \text{ is the constant in (2.15)},$$

and denote $\tau_r = \inf\{t > 0 : Z_t \notin (t_0 - r^2, t) \times B(x_0, r)\}$.

We claim that for any $r \in (0, \widetilde{r}_1), y, z \in B(x_0, 3r/4)$, any $t - r^2/8 \geq t_1 \geq t_2 \geq ... \geq t_k \geq t_{k+1} \geq ... \geq t_0 - r^2, k \in N$, we have

$$\mathbf{P}^{(t,y)}(T_{(t_k, t_k+2\gamma_1 4^{-2(k+1)}r^2)\times B(z,4^{-(k+1)}r)} < \tau_r) = \delta_0^{k+1}. \tag{7.43}$$

When $k = 1$, by Proposition 1.7.3 with $\delta = 1$,

$$\mathbf{P}^{(t,y)}(T_{(t_1,t_1+2\gamma_1(r/16)^2)\times B(z,4^{-2}r)} < \tau_r) \geq \Phi_1(1/16) \geq \Phi_1(1/16)^2 =: \delta_0^2.$$

Since $B(z, r/4) \cup B(y, r/4) \subset B(x_0, r)$, let $T_1 = (t_2 + 2\gamma_1(4^{-3}r)^2, t_2 + 2\gamma_1(4^{-3}r)^2 + \gamma_1 r^2/16) \times B(z, r/4)$, and by the strong Markov property of Z and Proposition 1.7.3, we have

$$\mathbf{P}^{(t,y)}(T_{(t_2,t_2+2\gamma_1(4^{-3}r)^2)\times B(z,r/64)}) < \tau_r)$$

$$\begin{aligned}
&\geq \mathbf{E}^{(t,y)}[\mathbf{P}^{Z_{T_1}}(T_{(t_2,t_2+2\gamma_1(4^{-3}r)^2)\times B(z,r/64)} < \tau_{(t_0-r^2,t)\times B(z,r/4)}); T_1 < \tau_r] \\
&\geq \Phi_1(1/16)\mathbf{P}^{(t,y)}[T_1 < \tau_r] \geq \delta_0^3. \qquad (7.44)
\end{aligned}$$

Suppose that for $k \geq 3$, $\mathbf{P}^{(t,y)}(T_{(t_{k-1},t_{k-1}+2\gamma_1 4^{-2k}r^2)\times B(z,4^{-k}r)} < \tau_{(t_0-r^2,t)\times B(x_0,r)}) \geq \delta_0^k$ holds. Let $T_2 = T_{(t_k+2\gamma_1 4^{-2(k+1)}r^2, t_k+2\gamma_1 4^{-2(k+1)}r^2+\gamma_1 4^{-2k}r^2)\times B(z,4^{-k}r)}$. Since $B(z, 4^{-(k+1)}r + 4^{-k}r) \subset B(z, 4^{-(k-1)}r)$, then

$$\begin{aligned}
&\mathbf{P}^{(t,y)}(T_{(t_k,t_k+2\gamma_1 4^{-2(k+1)}r^2)\times B(z,4^{-(k+1)}r)} < \tau_r) \\
\geq\ &\mathbf{E}^{(t,y)}[\mathbf{P}^{Z_{T_2}}(T_{(t_k,t_k+2\gamma_1 4^{-2(k+1)}r^2)\times B(z,4^{-(k+1)}r)}) < \tau_{(t_k,t_k+2\gamma_1 4^{-2(k+1)}r^2+\gamma_1 4^{-2k}r^2)\times B(z,4^{-(k-1)}r)}; \\
&T_2 < \tau_r] \\
\geq\ &\mathbf{E}^{(t,y)}[\Phi_1(1/16); T_2 < \tau_r] \\
\geq\ &\delta_0 \cdot \delta_0^k = \delta_0^{k+1}.
\end{aligned}$$

For $k \in \mathbb{N}$, define $\widetilde{U}_k := \{t_k\} \times B(z, 4^{-k}r/2)$ for $z \in B(x_0, 3r/4)$, $0 \leq r \leq \widetilde{r}_1$.

Then applying the same test function φ in Lemma 1.2.6, but with $(z, B(x_0, 1))$ in place of $(x_1, B(z, 4^{-k}r/2))$ in $f(y)$ and (2.15), and then by the same argument as in (7.24), (7.25), we have

$$\mathbb{P}^{\tilde{x}}[\tau_{B(z,4^{-k}r/2)} \geq 2\gamma_1 4^{-2(k+1)}r^2] \geq 1/4 \text{ for any } \widetilde{x} \in B(z, 4^{-(k+1)}r). \qquad (7.45)$$

Then by (7.43),(7.45), it follows that, for any $y, z \in B(x_0, 3r/4)$,

$$\begin{aligned}
&\mathbf{P}^{(t,y)}[T_{\widetilde{U}_k} \leq \tau_r] \\
\geq\ &\mathbf{E}^{(t,y)}[\mathbb{P}^{X_{T_{B(z,4^{-(k+1)}r/2)}}}(\tau_{B(z,4^{-k}r/2)} \geq 2\gamma_1 4^{-2(k+1)}r^2); \\
&T_{(t_k,t_k+2\gamma_1 4^{-2(k+1)}r^2)\times B(z,4^{-(k+1)}r)} < \tau_r] \\
\geq\ &\delta_0^{k+1}/4. \qquad (7.46)
\end{aligned}$$

Hence by the strong Markov property of Z, (7.46) and Corollary 1.7.7, for any $A \subset (t_k - 4^{-2k}r^2, t_k) \times B(z, 4^{-k}r), y, z \in B(x_0, 3r/4), k \in \mathbb{N}$,

$$\begin{aligned}
&\mathbf{P}^{(t,y)}(T_A < \tau_r) \\
\geq\ &\mathbf{E}^{(t,y)}[\mathbf{P}^{Z_{T_{\tilde{U}_k}}}(T_A < \tau_{(t_k-4^{-2k}r^2,t_k)\times B(z,4^{-k}r)}); T_{\widetilde{U}_k} < \tau_r]
\end{aligned}$$

$$\geq \quad \varphi(|A|/|(t_k - 4^{-2k}r^2, t_k) \times B(z, 4^{-k}r)|)\delta_0^{k+1}/4. \tag{7.47}$$

Set

$$\eta_0 = \frac{\varphi(1/2)}{4} \wedge \frac{c_5}{2}, \quad \xi_0 = \eta_0/c_5 \quad \text{and} \quad \gamma_0 = \frac{\varphi(1/2)}{4(1-\varphi(1/2))}. \tag{7.48}$$

where $c_5 \geq 1$ is the constant in Proposition 1.7.9. For any real number a, denote by $\lfloor a \rfloor$ the largest integer not exceeding a. By decreasing the value of $\delta_0 \in (0,1)$ in (7.43) if needed, we may assume that $0 < \delta_0 < \frac{1}{1+\gamma_0/2}$. Let m be the smallest nonnegative integer so that

$$1/(4^{\log_{\delta_0} 1/(1+\gamma_0/2)} - 1) \leq 4^{m \log_{\delta_0} 1/(1+\gamma_0/2)}. \tag{7.49}$$

As

$$\lim_{k\to\infty} (1+\gamma_0)^k \delta_0^{\lfloor (k+m) \log_{\delta_0} 1/(1+\gamma_0/2) \rfloor} \to \infty,$$

take

$$a_0 := \inf_{k\geq 1} (1+\gamma_0)^k \delta_0^{\lfloor (k+m) \log_{\delta_0} 1/(1+\gamma_0/2) \rfloor} \leq 1+\gamma_0,$$

and let

$$K = \frac{5(1+\gamma_0)}{a_0 \xi_0 \delta_0 \varphi(\frac{1}{4} \cdot (\frac{1}{16})^{d+2})}. \tag{7.50}$$

Note that $0 < \eta_0 < 1/4$, $0 < \xi_0 < 1/2$, and thus $K > 5/\xi_0 > 10$. Let $r_k = r/(2 \cdot 4^{(k+m) \log_{\delta_0} 1/(1+\gamma_0/2)})$ for $k \geq 1$ and by (7.49), $\sum_{k=1}^{\infty} r_k \leq r/2$, and $\sum_{k\geq 1} r_k^2 < r^2/4$. In addition, for each $k \in \mathbb{N}$,

$$4^{-n_k - 1} r/2 \leq r_k \leq 4^{-n_k} r/2, \quad \text{where } n_k := \lfloor (k+m) \log_{\delta_0} 1/(1+\gamma_0/2) \rfloor. \tag{7.51}$$

We will show that $u \leq K$ on Q_-. Suppose not. Then there exists a point $(s_1, y_1) \in Q_- = (t_0 - 3r^2/4, t_0 - r^2/2) \times B(x_0, r/2)$ with $u(s_1, y_1) > K$. We claim that in this case

there is a sequence of points $\{(s_k, y_k); k \geq 1\} \subset (t_0 - r^2, t_0) \times B(x_0, r)$ so that

$(s_k, y_k) \in (s_{k-1} - r_{k-1}^2, s_{k-1}) \times B(y_{k-1}, r_{k-1})$ having

$u(s_k, y_k) \geq K(1+\gamma_0)^{k-1}$, (7.52)

with the convention that $s_0 = t_0 - r^2/2, y_0 = x_0, r_0 = r/2$. Let $U_k = \{s_k\} \times B(y_k, r_k/4)$. Since by (7.51),

$$4^{-n_k - 3} r \leq r_k/8 \leq 4^{-n_k - 2} r.$$

Notice that $t - s_k \geq r^2/8$ for every $k \in \mathbb{N}$. Then by (7.43) with$(s_k, y_k, n_k + 3)$ in place of $(t_k, z, k+1)$, we have

$$\begin{aligned} & \mathbf{P}^{(t,x)}(T_{(s_k, s_k+2\gamma_1(r_k/8)^2)\times B(y_k, r_k/8)} < \tau_r) \\ \geq \ & \mathbf{P}^{(t,x)}(T_{(s_k, s_k+2\gamma_1 4^{-2n_k-6}r)\times B(y_k, 4^{-n_k-3}r)} < \tau_r) \\ \geq \ & \delta_0^{n_k+3}, \end{aligned} \tag{7.53}$$

Then with $(s_k, y_k, r_k/4)$ in place of $(t_k, z, 4^{-k}r/2)$ in (7.46), by (7.45), (7.53), it follows that

$$\begin{aligned} \mathbf{P}^{(t,x)}(T_{U_k} < \tau_r) \ \geq \ & \mathbf{E}^{(t,x)}[\mathbb{P}^{X_{\tau_{B(y_k,r_k/8)}}}(\tau_{B(y_k,r_k/4)} > \gamma_1 r_k^2/32); T_{(s_k, s_k+\gamma_1 r_k^2/32)\times B(y_k, r_k/8)} < \tau_r] \\ \geq \ & \delta_0^{n_k+3}/4. \end{aligned}$$

If $u \geq \xi_0 K(1+\gamma_0)^{k-1}$ on U_k, then by (7.50),

$$\begin{aligned} 1 \ > \ & u(t,x) = \mathbf{E}^{(t,x)}[u(Z_{T_{U_k}\wedge\tau_r})] \geq \xi_0 K(1+\gamma_0)^{k-1}\mathbf{P}^{(t,x)}(T_{U_k} < \tau_r) \\ \geq \ & \xi_0 K(1+\gamma_0)^{k-1}\delta_0^{n_k+3}/4 > 2. \end{aligned}$$

By contradiction, there is at least one point on U_k taking values less than $\xi_0 K(1+\gamma_0)^{k-1}$.

Next define $\tau_{r_k/2} = \tau_{B(y_k, r_k/2)} \wedge (r_k^2/4) = \inf\{t > 0 : Z_t \notin (s_k - r_k^2/4, s_k) \times B(y_k, r_k/2)\}$ and we claim that

$$\mathbf{E}^{(s_k,y_k)}[u(Z_{\tau_{r_k/2}}) : X_{\tau_{r_k/2}} \notin B(y_k, r_k)] \leq \eta_0 K(1+\gamma_0)^{k-1}. \tag{7.54}$$

Indeed, otherwise, by Proposition 1.7.9, we would have for every $z \in B(y_k, r_k/4)$,

$$\begin{aligned} u(s_k, z) \ \geq \ & \mathbf{E}^{(s_k,y)}[u(Z_{\tau_{r_k/2}}) : X_{\tau_{r_k/2}} \notin B(y_k, r_k)] \\ \geq \ & c_5^{-1}\mathbf{E}^{(s_k,y_k)}[u(Z_{\tau_{r_k/2}}) : X_{\tau_{r_k/2}} \notin B(y_k, r_k)] \\ \geq \ & c_5^{-1}\eta_0 K(1+\gamma_0)^{k-1} \\ \geq \ & \xi_0 K(1+\gamma_0)^{k-1}, \end{aligned}$$

contradicting with the fact that there is at least one point on U_k with $u < \xi_0 K(1+\gamma_0)^{k-1}$.

Define $A := \{(\bar{t}, \bar{y}) \in (s_k - r_k^2/4, s_k) \times B(y_k, r_k/2) : u(\bar{t}, \bar{y}) \geq \xi_0 K(1+\gamma_0)^{k-1}\}$, and claim that

$$|A| < |(s_k - r_k^2/4, s_k) \times B(y_k, r_k/2)|/2.$$

If not, let $\tilde{A}$ be a compact subset of A such that $|\tilde{A}| > |(s_k - r_k^2/4, s_k) \times B(y_k, r_k/2)|/4$.As $4^{-n_k-2}r \leq r_k/2 \leq 4^{-n_k-1}r$, then

$$\begin{aligned} & \frac{|\tilde{A}|}{|(s_k - 4^{-2n_k}r^2, s_k) \times B(y_k, 4^{-n_k}r)|} \\ = \quad & \frac{|\tilde{A}|}{|(s_k - r_k^2/4, s_k) \times B(y_k, r_k/2)|} \cdot \frac{|(s_k - r_k^2/4, s_k) \times B(y_k, r_k/2)|}{|(s_k - 4^{-2n_k}r^2, s_k) \times B(y_k, 4^{-n_k}r)|} \\ \geq \quad & (1/16)^{d+2}/4. \end{aligned}$$

Then by (7.47), with (s_k, y_k, n_k) in place of (t_k, z, k),

$$\mathbf{P}^{(t,x)}(T_{\tilde{A}} < \tau_r) \geq \varphi((1/16)^{d+2}/4)\delta_0^{n_k+1}/4. \tag{7.55}$$

Therefore, by the definition of parabolic functions and (7.47), (7.50),

$$\begin{aligned} 1 \geq u(t,x) \quad & \geq \quad \mathbf{E}^{(t,x)}[u(Z_{T_{\tilde{A}}}) : T_{\tilde{A}} < \tau_r] \\ & \geq \quad \xi_0 K(1+\gamma_0)^{k-1}\mathbf{P}^{(t,x)}(T_{\tilde{A}} < \tau_r) \\ & \geq \quad \xi_0 K(1+\gamma_0)^{k-1}\varphi(\frac{|\tilde{A}|}{|(s_k - 4^{-2n_k}r^2, s_k) \times B(y_k, 4^{-n_k}r)|})\delta_0^{n_k+1} \\ & \geq \quad \xi_0 K(1+\gamma_0)^{k-1}\varphi((1/16)^{d+2}/4)\delta_0^{n_k+1}/4 > 5/4. \end{aligned}$$

So by contradiction, $|A| \leq |(s_k - r_k^2/4, s_k) \times B(y_k, r_k/2)|/2$, and there is a compact subset $E \subset (s_k - r_k^2/4, s_k) \times B(y_k, r_k/2) \setminus A$ such that

$$|E|/|(s_k - r_k^2/4, s_k) \times B(y_k, r_k/2)| \geq 1/2, \tag{7.56}$$

with

$$u(t, \widetilde{y}) < \xi_0 K(1+\gamma_0)^{k-1} \text{ for every } (t, \widetilde{y}) \in E. \tag{7.57}$$

Let $M_k = \sup_{(\tilde{s},\tilde{y}) \in (s_k - r_k^2, s_k) \times B(y_k, r_k)} u(\tilde{s}, \tilde{y})$.Since $t \mapsto u(Z_{t \wedge \tau_{(s_k - r_k^2, s_k) \times B(y_k, r_k)}})$ is a bounded martingale, then by (7.54),

$$\begin{aligned} u(s_k, y_k) \quad & = \quad \mathbf{E}^{(s_k,y_k)}[u(Z_{T_E \wedge \tau_{r_k/2}}] \\ & = \quad \mathbf{E}^{(s_k,y_k)}[u(Z_{T_E}); T_E \leq \tau_{r_k/2}] \end{aligned}$$

$$+\mathbf{E}^{(s_k,y_k)}[u(Z_{\tau_{r_k/2}});\tau_{r_k/2}<T_E,X_{\tau_{r_k/2}}\in B(y_k,r_k)]$$
$$+\mathbf{E}^{(s_k,y_k)}[u(Z_{\tau_{r_k/2}});\tau_{r_k/2}<T_E,X_{\tau_{r_k/2}}\notin B(y_k,r_k)] \tag{7.58}$$

By (7.57), Lemma 1.7.10, we know that

$$\mathbf{E}^{(s_k,y_k)}[u(Z_{T_E});T_E\le\tau_{r_k/2}]\le\xi_0K(1+\gamma_0)^{k-1}\,\mathbf{P}^{(s_k,y_k)}(T_E<\tau_{B(z_k,r_k/2)}). \tag{7.59}$$

Therefore, by (7.54), (7.58), (7.59),

$$\begin{aligned} u(s_k,y_k) &< \xi_0K(1+\gamma_0)^{k-1}\,\mathbf{P}^{(s_k,y_k)}(T_E<\tau_{B(z_k,r_k/2)})+M_k\mathbf{P}^{(s_k,y_k)}(\tau_{B(z_k,r_k/2)}<T_E)+\eta_0K(1+\gamma_0)^{k-1} \\ &= \xi_0K(1+\gamma_0)^{k-1}\,\mathbf{P}^{(s_k,y_k)}(T_E<\tau_{B(z_k,r_k/2)})+M_k\left(1-\mathbf{P}^{(s_k,y_k)}(T_E<\tau_{B(z_k,r_k/2)})\right) \\ &\quad +\eta_0K(1+\gamma_0)^{k-1}. \end{aligned}$$

As $u(s_k,y_k)\ge K(1+\gamma_0)^{k-1}$, we conclude from the above that

$$M_k/((1+\gamma_0)^{k-1}K)\ge 1+\frac{(1-\xi_0)\mathbf{P}^{(s_k,y_k)}(T_E\le\tau_{r_k/2})-\eta_0}{1-\mathbf{P}^{(s_k,y_k)}(T_E\le\tau_{r_k/2})}.$$

Notice that by Corollary 1.7.7 and (7.56),

$$\mathbf{P}^{(s_k,y_k)}(T_E\le\tau_{r_k/2})\ge\varphi(1/2).$$

Hence by the definition of ξ_0,η_0, and $\gamma_0$in (7.48), we have from the above two displays that $M_k/((1+\gamma_0)^{k-1}K)\ge 1+\gamma_0$.

Thus there exists a point $(s_{k+1},y_{k+1})\in(s_k-r_k^2,s_k)\times B(y_k,r_k)$ such that $u(s_{k+1},y_{k+1})\ge K(1+\gamma_0)^k$. Thus by induction, there exists a sequence $\{(s_j,y_j);j\ge 1\}$ that has the property (7.52), contradicting with the assumption that u is bounded on $(t_0-r^2,t_0)\times B(x_0,r)$.Therefore, $u\le K$ in Q_-. Then together with (7.42), we complete the proof by setting $C=2K$ in (7.1), which depends only on $(d,\Lambda_1,\Lambda_2,\Lambda_3,c_0)$ and the rate of the function ψ in (1.3) converging to 0, and independent of x_0,t_0 and r.

□

Chapter 2

HARNACK INEQUALITIES FOR WEAKLY COUPLED NONLOCAL SYSTEMS

2.1 Introduction

Let $\mathcal{M} = \{1, ..., m\}$, for $m \geq 2$. Suppose that $Q(x) = (q_{ij})_{m\times m}(x)$ is an $m \times m$ matrix valued function on $\mathbb{R}^d$ such that for a.e. $x \in \mathbb{R}^d$,

$$q_{ij}(x) \geq 0 \text{ for } i \neq j, \quad \text{ and } \sum_{j=1}^{m} q_{ij}(x) \leq 0 \text{ for each } i \in \mathcal{M}.$$

For any function

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} : \mathbb{R}^d \mapsto \mathbb{R}^m,$$

define

$$Q(x)u(x, \cdot)(i) := \sum_{j\in\mathcal{M}} q_{ij}(x)u(x, j), \quad (x, i) \in \mathbb{R}^d \times \mathcal{M}. \tag{1.1}$$

Consider the following weakly coupled operator $\mathcal{G}$ such that for each $i \in \mathcal{M}, u(\cdot, i) \in C_b^2(\mathbb{R}^d)$,

$$\mathcal{G}u(x, i) := \mathcal{L}_i u(x, i) + Q(x)u(x, \cdot)(i), \quad (x, i) \in \mathbb{R}^d \times \mathcal{M} \tag{1.2}$$

with

$$\mathcal{L}_i u(x, i) = \mathcal{L}_i^{b_1} u(x, i) + \mathcal{S}^{b_2} u(x, i), \tag{1.3}$$

where

$$\mathcal{L}_i^{b_1} u(x, i) := \frac{1}{2}\sum_{k,l=1}^{d} a_{kl}(x, i)\frac{\partial^2}{\partial x_k \partial x_l}u(x, i) + \sum_{k=1}^{d} b_1^k(x, i)\frac{\partial}{\partial x_k}u(x, i); \tag{1.4}$$

$$\mathcal{S}^{b_2} u(x, i) := \int_{\mathbb{R}^d\setminus\{0\}} (u(x + z, i) - u(x, i) - \nabla u(x, i) \cdot \mathbb{1}_{|z|\leq 1} z) b_2(x, z, i) j_i(z) dz. \tag{1.5}$$

Here $a_{kl}(x, i) = a_{lk}(x, i)$ and there are positive constants $\Theta_1, \Theta_2, \Theta_3, \Theta_4, c, c_1$ and $\gamma \in (0, 1)$ such that $(a_{kl}(x, i))_{1\leq k,l\leq m}, (b_1^k(x, i))_{1\leq k\leq m}, b_2(x, z, i), j_i(z)$ satisfy the following conditions:

(a) (Uniform ellipticity and Hölder continuity on diffusion matrix)

$$\Theta_1|\zeta|^2 \le \sum_{k,l=1}^{d} a_{kl}(x,i)\zeta_k\zeta_l \le \Theta_1^{-1}|\zeta|^2, \quad \text{for } \zeta \in \mathbb{R}^d; \tag{1.6}$$

$$|a_{kl}(x_1,i) - a_{kl}(x_2,i)| \le c|x_1 - x_2|^{\gamma}, \text{ for } (x_1,i),(x_2,i) \in \mathbb{R}^d \times \mathcal{M}, \text{ some } \gamma \in (0,1). \tag{1.7}$$

(b) (Bounded drift)

$$\|b_1\|_\infty := \sup_{(x,i)\in\mathbb{R}^d\times\mathcal{M}} |b_1(x,i)| \le \Theta_2. \tag{1.8}$$

(c) (Lévy jumping kernel condition and nonnegative local boundedness)

$$\|b_2(x,z,i)\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d\setminus\{0\}\times\mathcal{M})} \le \Theta_3. \tag{1.9}$$

$$\int_{\mathbb{R}^d\setminus\{0\}} (1\wedge|z|^2)j_i(z)dz < \Theta_4, \tag{1.10}$$

where each $j_i(z)$ is a nonnegative locally bounded function on $\mathbb{R}^d\setminus\{0\}$ such that there exists a constant $\beta \in (1,2)$ and $c_1 > 0$ satisfying

$$j_i(z) \le \frac{c_1}{|z|^{d+\beta}} \text{ for } z \in B(0,1)\setminus\{0\}, i \in \mathcal{M}. \tag{1.11}$$

In matrix form,

$$\mathcal{G}u = \begin{pmatrix} \mathcal{L}_1 & 0 & \dots & 0 \\ 0 & \mathcal{L}_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{L}_m \end{pmatrix} u + Qu. \tag{1.12}$$

Such operators with no derivatives appearing in the coupling terms are called *weakly coupled* operators. There are studies on potential theory for weakly coupled elliptic systems where $\mathcal{L}_i$ in (1.3) are all differential operators. In 1989, Skorohord in [70] introduced a Markov process $\{(X_t,\Lambda_t), t \ge 0; \mathbb{P}^{(x,i)}, (x,i) \in (\mathbb{R}^d\times\mathcal{M})\}$in a descriptive manner under the condition that $\sum_{j\in\mathcal{M}} q_{ij}(x) = 0$ a.e. in $\mathbb{R}^d$ having $\mathcal{G}$ as its infinitesimal generator (1.2). In [28], Chen and Zhao established the existence of switched diffusion processes corresponding to the weakly coupled elliptic operators in divergence form with measurable coefficients such that the diffusion matrix is uniformly elliptic, and the coefficients of the first order term and each term in the switching rate matrix Q belongs to a Kato class. They also showed the solvability of the Cauchy problems for weakly coupled elliptic systems via a Dirichlet space approach, i.e. $\partial u/\partial t = \mathcal{G}u$ and a strong positivity result for the solutions. In [28], the regularity condition on coefficient a^k, b^k, and q_{kl} are the following :

$$a_{kl}(\cdot, i) \in W^{1,2}(D) \text{ for each i}, \sum_{k=1}^{d}(\partial b_k(\cdot, i))/\partial x_k) \text{ is bounded from below for each } i,$$
$$\text{and } \sum_{i=1}^{m} q_{ij} \text{ is bounded from above for each } j,$$

where D is a bounded domain. In [29], Chen and Zhao generalized the previous result to weakly coupled elliptic system (i.e. $\mathcal{G}u = 0$) whose coefficients are only measurable such that the diffusion matrix is uniformly elliptic with lower-order terms in a Kato class on bounded domains. They also gave a probabilistic representation of the solution and proved a strong positivity result for that problem via a probabilistic approach. Later, the corresponding Harnack inequality for weakly coupled elliptic systems and the full Harnack inequality under the irreducibility assumption of Q and Hölder continuous coefficients are established by an analytic method in [30]. The Harnack inequality improves the result of F. Mandras [58], in which the Harnack inequality takes the form :

$$\sum_{i=1}^{m} \sup_{x\in D} u_i(x) \leq C \sum_{i=1}^{m} \inf_{x\in D} u_i(x),$$

where D is bounded. The approach in [30] is based on the representation theorem obtained in [29] and the estimates of Green functions and harmonic measures of the operators $\mathcal{L}_k + q_{kk}$ in small balls. Stimulated by the ongoing research on nonlocal operators which correspond to various jump processes, see [6, 9, 14, 20, 71], as well as the surgent needs of applications of switched Markov processes (X, Λ) to the fields of engeering and finance [41, 69, 59, 46], the scale-invariant Harnack inequality for $\mathcal{L}_i$ of the form (1.3) with uniformly elliptic diffusion matrix, bounded drift and more general jump kernel assumption has been studied in [24], and the weakly coupled nonlocal system of (1.2) has also been studied, for example, in [15]. In [15], the non scale-invariant Harnack inequality for each level and the full rank Harnack inequality for the operator (1.2) satisfying (1.6),(1.8),(1.10), the additional continuity assumption on $(a_{kl}(x))$ and $(b_k(x))$, and the relative comparison of the jump intensity have been established. The purpose of this paper is to establish the scale-invariant Harnack inequality for the weakly coupled system. In this paper we assume that the dimension $d \geq 2$ and our goal is to establish the scale-invariant Harnack inequality for harmonic functions with respect to the operator $\mathcal{G}$ in (1.2). We say that the non-local operator $\mathcal{L}$ of the form (1.3) belongs to **class** $\mathcal{N}(\Theta_1, \Theta_2, \Theta_3, \Theta_4, \gamma, \beta, c_1)$ if the following (1.6)-(1.11) hold and an operator $\mathcal{G}$ belongs to **class** $\mathcal{N}(c_0, m, \gamma, \beta, \Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, c_1, \vartheta)$ if (1.6)-(1.11) hold with the additional conditions (1.14) and (1.15) for the switching rate matrix Q:

$$\text{There exists a constant matrix } Q^0 = (q^0_{ij}) \text{ and a positive constant } 0 < c_0 < 1$$

$$\text{such that } c_0 q^0_{ij} \leq q_{ij}(x) \leq q^0_{ij} \leq \Theta_5, i \neq j \ a.e. \text{ on } \mathbb{R}^d \tag{1.13}$$

$$\text{and } -q_{ii}(x) \leq -q^0_{ii} \leq \Theta_5 \ a.e. \text{ on } \mathbb{R}^d. \tag{1.14}$$

Define $J^{b_2}(x, y, i) = b_2(x, y - x, i) j_i(y - x)$, for each $x, y \in \mathbb{R}^d, i \in \mathcal{M}$. We say that the jumping kernel $J^{b_2}(x, x+z, i), i \in \mathcal{M}$ satisfies the **UJS condition** if there exists a constant $\vartheta = \vartheta(d, \beta, b_2) > 0$ so that for a.e. $x \in \mathbb{R}^d, z \neq 0$, every $i \in \mathcal{M}$,

$$J^{b_2}(x, x+z, i) \leq \frac{\vartheta}{|B(x, r)|} \int_{B(x,r)} J^{b_2}(u, x+z, i) du \text{ for every } 0 < r < |z|/2. \tag{1.15}$$

Here $|B(x_0, r)|$ denotes the Lebesgue measure of the ball $B(x_0, r)$, which is $\omega_d r^d$ with ω_d being the volume of the unit ball in $\mathbb{R}^d$. For the typical examples of the nonlocal operators whose jump kernel J^{b_2} satisfy the UJS condition, we refer the readers to Remark 1.6 in [26].

Given a switching rate matrix Q, we denote the collection of n-step path from state i to j by

$$\begin{aligned} \Psi(n; i, j) : \;&=\; \{(l_0, l_1, ..., l_n) : l_0 = i, l_n = j, l_i \in \mathcal{M}, l_i \neq l_{i+1}, \{q_{l_i l_{i+1}}(x) > 0\} \\ &\quad \text{has a positive Lebsgue measure on } \mathbb{R}^d, \text{ for } i = 0, 1, ..., n-1\}. \end{aligned} \tag{1.16}$$

By (1.13),$\Psi^0(n; i, j)$, which is defined correspondingly for Q^0, satisfies that $\Psi^0(n; i, j) = \Psi(n; i, j)$ for any $i, j \in \mathcal{M}, n \in \mathbb{N}$.

Definition 2.1.1. *We say that the operator $\mathcal{G}$ or its associated matrix-valued function Q is* **irreducible** *in $\mathbb{R}^d$ if for any distinguished $k, l \in \mathcal{M}$, there exist $l_0, ..., l_n$ in $\mathcal{M}$ with $l_{i-1} \neq l_i$ for $1 \leq i \leq n$, $l_0 = k$ and $l_n = l$ such that $q^0_{l_{i-1} l_i} \neq 0$.*

Definition 2.1.2. *We say that the operator $\mathcal{G}$ or its associated matrix-valued function Q is* **strictly irreducible** *if $q^0_{kl} > 0$ for any $k, l \in \mathcal{M}$.*

And we denote

$$q_0 = \min\{q^0_{ij} : i, j \in \mathcal{M}, i \neq j\}. \tag{1.17}$$

For any $\mathcal{G} \in \mathcal{N}(c_0, m, \gamma, \beta, \Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, c_1, \vartheta)$, we can show that there is a unique Hunt process $Y = \{(X_t, \Lambda_t), t \geq 0; \mathbb{P}^{(x,i)}, (x, i) \in (\mathbb{R}^d \times \mathcal{M})\}$ solving the martingale problem for $(\mathcal{G}, C^2_b(\mathbb{R}^d \times \mathcal{M}))$ starting at (x, i) : that is, for every function $u(x, i) : \mathbb{R}^d \times \mathcal{M} \mapsto \mathbb{R}$, such that $u(\cdot, i) \in C^2_b(\mathbb{R}^d)$ for every $i \in \mathcal{M}$,

$$M^u_t := u(X_t, \Lambda_t) - u(X_0, \Lambda_0) - \int_0^t \mathcal{G}u(X_s, \Lambda_s)ds$$

is a $\mathbb{P}^{(x,i)}$-local maringale for each $(x, i) \in \mathbb{R}^d \times \mathcal{M}$ and $\mathbb{P}^{(x,i)}(X_0 = x, \Lambda_0 = i) = 1$, where X is an $\mathbb{R}^d-$valued process, and Λ is a switching process taking values in $\mathcal{M}$. In this case, we say that the Hunt process $Y = \{(X_t, \Lambda_t), t \geq 0; \mathbb{P}^{(x,i)}, (x, i) \in (\mathbb{R}^d \times \mathcal{M})\}$ **has the generator $\mathcal{G}$ or corresponds to the operator $\mathcal{G}$**.

The process Y can be constructed in the way described in [30, Remark 2.2]: Under the assumption (1.6)-(1.11), by Theorem 5.2 in [49], there is a unique conservative strong Markov process $\bar{X} = \{\bar{X}_t, t \geq 0; \mathbb{P}^x, x \in \mathbb{R}^d\}$ whose distribution with $\mathbb{P}^x(X_0 = 0) = 1$ solves the martingale problem for $(\mathcal{L}, C^2_b(\mathbb{R}^d))$, where $\mathcal{L} \in \mathcal{N}(\Theta_1, \Theta_2, \Theta_3, \Theta_4, \gamma, \beta, c_1)$. By the assumption (1.9), we know that $b(x, z)j(z)dz \leq \Theta_3 j(z)dz$ for every $x \in \mathbb{R}^d$, then by [49, Theorem 2.1, Theorem 2.2], the process $\bar{X}$ is quasi-left continuous under $\mathbb{P}^x$. Fix $i_0 \in \mathcal{M}$ and let $(X^{i_0}, \mathbb{P}^{(x_0,i_0)})$ be the Hunt process corresponding to the infinitesimal generator $\mathcal{L}_{i_0}$ starting from (x_0, i_0).We run the subprocess $\widetilde{X}^{i_0}$ of X^{i_0} that is killed with the rate $-q_{i_0 i_0}(x)$. Note that the subprocess $\widetilde{X}^{i_0}$ has the infinitesimal generator $\mathcal{L}_{i_0} + q_{i_0 i_0}$. At the lifetime τ_1, the killed subprocess $\widetilde{X}^{i_0}$ is killed with probability $1 + \sum_{j \in \mathcal{M}} q_{i_0 j}(X^{i_0}_{\tau_1-})/q_{i_0 i_0}(X^{i_0}_{\tau_1-})$ and jumps to plane $j \neq i$ with probability $-q_{i_0 j}(X^{i_0}_{\tau_1-})/q_{i_0 i_0}(X^{i_0}_{\tau_1-})$ and then starting from $X^{i_0}(\tau_1-)$, we run an independent copy of a subprocess $\widetilde{X}^j$ of X^j with the killing rate $-q_{jj}(x)$. Iterating this procedure, the resulting process $Y = ((X_t, \Lambda_t), t \geq 0; \mathbb{P}^{(x,i_0)})$ is a Hunt proces with lifetime ζ by [43, 60]. For each $x \in \mathbb{R}^d$, we say that $Q(x)$ is *Markovian* if $\sum_{j \in \mathcal{M}} q_{ij}(x) = 0$ a.e. on $\mathbb{R}^d$ for every $i \in \mathcal{M}$, and *sub-Markovian* if $\sum_{j \in \mathcal{M}} q_{ij}(x) \leq 0$ a.e.

on $\mathbb{R}^d$ for every $i \in \mathcal{M}$. When $Q(x)$ is *Markovian*, the lifetime $\zeta = \infty$, and when $Q(x)$ is *sub-Markovian*, $\zeta < \infty$ with positive probability. We use the convention $(X_t, \Lambda_t) = \partial$ for $t \geq \zeta$, where ∂ is a cemetery point and any function is extended to ∂ by taking value zero. It is easy to check that $((X, \Lambda), \mathbb{P}^{(x_0,i_0)})$ solves the maringale problem for $(\mathcal{G}, C_b^2(\mathbb{R}^d \times \mathcal{M}))$, where we denote by $C_b^2(\mathbb{R}^d \times \mathcal{M})$ the class of functions u defined on $\mathbb{R}^d \times \mathcal{M}$ such that $u(\cdot, i)$ is in $C_b^2(\mathbb{R}^d)$ for each $i \in \mathcal{M}$. This way of constructing a switched diffusion process by patching together the pre-switching process $\widetilde{X}^i$ for the operator $\mathcal{L}_i + q_{ii}$ with its switching distribution has also been utilized in [29, Page 296].

In this paper, we always assume that D is a bounded and connected open set in $\mathbb{R}^d$. We say that a Borel measurable function $u : D \times \mathcal{M} \mapsto \mathbb{R}^d$ is **$\mathcal{G}$-harmonic** in $D \times \mathcal{M}$ if for any relatively compact open subset V of $D \times \mathcal{M}$,

$$\mathbb{E}^{(x,i)}[u(X_{\tau_V}, \Lambda_{\tau_V})] < \infty, \text{ and } u(x,i) = \mathbb{E}^{(x,i)}[u(X_{\tau_V}, \Lambda_{\tau_V})]$$

for every $(x, i) \in V$, where $\tau_V := \inf\{t \geq 0 : (X_t, \Lambda_t) \notin V\}$ is the first exit time from the set V. Heuristically, u is $\mathcal{G}$-harmonic in V if $\mathcal{G}u = 0$ in V. But we are not going to establish this analytic characterization. See [17, 57] for the equivalent characterizations between probabilistic and analytic notions of harmonicity under some suitable conditions.

Our first main result is on the local Hölder regularity for bounded $\mathcal{G}$-harmonic functions.

Theorem 2.1.3. *Let $\mathcal{G} \in \mathcal{N}(c_0, m, \gamma, \beta, \Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, c_1, \vartheta)$. There exist constants $\widetilde{r}_0 = \widetilde{r}_0(d, \Theta_1, \Theta_2, \Theta_3, \Theta_4) \in (0, 1/4)$, $\alpha_1 = \alpha_1(d, \beta, \Theta_1, \Theta_2, \Theta_3, \Theta_4) \in (0, 1)$ and $C_1 = C_1(d, \beta, \Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, m) > 0$ such that for any $x_0 \in \mathbb{R}^d, r \in (0, \widetilde{r}_0)$, any bounded function u defined in $\mathbb{R}^d$ that is harmonic with respect to $\mathcal{G}$ in $B(x_0, r)$,*

$$\|u(x) - u(y)\| \leq C_1 \|u\|_\infty \left(\frac{\|x - y\|}{r}\right)^{\alpha_1}, \quad \text{for any } x, y \in B(x_0, r/2).$$

Here $\|\cdot\|$ is the supremum norm of a vector.

The next one is on the scale-invariant Harnack inequality for nonnegative $\mathcal{G}$-harmonic functions.

For any nonnegative function u defined in $\mathbb{R}^d$ that is $\mathcal{G}$-harmonic in $B(x_0, r)$, $x_0 \in \mathbb{R}^d, r > 0$, define

$$h(x,k) := \mathbb{E}^{(x,k)}[u(X_{\tau_{B(x_0,r/2)}}, \Lambda_{\tau_{B(x_0,r/2)}}); \tau_{B(x_0,r/2)} < \tau_1], \quad (x,k) \in B(x_0, r/2), \tag{1.18}$$

where

$$\tau_1 = \inf\{t > 0 : \Lambda_t \neq \Lambda_0\}, \tag{1.19}$$

is the first switching time for (X, Λ).

Theorem 2.1.4. *Let $\mathcal{G} \in \mathcal{N}(c_0, m, \gamma, \beta, \Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, c_1, \vartheta)$ with Q satisfying* (1.13)*. Then there exist constants $0 \leq \widetilde{r}_1 < 1$ and $C_2, C_3 > 0$, which depends only on $(d, m, c_0, \beta, \gamma, \Theta_1, \Theta_2, \Theta_3, \Theta_4, \vartheta, Q^0)$, such that for any $x_0 \in \mathbb{R}^d, r \in (0, \widetilde{r}_1]$, any nonnegative function u defined on $\mathbb{R}^d \times \mathcal{M}$ that is $\mathcal{G}$-harmonic on $B(x_0, r) \times \mathcal{M}$, we have for any $x, y \in B(x_0, r/8), k \in \mathcal{M}$,*

$$C_2\Big(h(y,k) + \sum_{l \in E(k)} r^{m_{kl}} h(y,l)\Big) \leq u(x,k) \leq C_3\Big(h(y,k) + \sum_{l \in E(k)} r^{m_{kl}} h(y,l)\Big),$$

where $h(y,l)$ is defined in (1.18)*, $E(k) = \{l \in \mathcal{M} \setminus \{k\} : \Psi(n,k,l) \neq \emptyset,$ for some $n \in \mathbb{N}\}$ and m_{kl} is the smallest integer n such that $\Psi^0(n,k,l) \neq \emptyset$.*

In particular, for each $k \in \mathcal{M}$

$$u(x,k) \leq Cu(y,k), \text{ for any } x, y \in B(x_0, r/8),$$

where $C = C_3/C_2$.

Under the assumption of strict irreducibility for $\mathcal{G}$, we have the following scale-invariant full rank Harnack inequality for nonnegative $\mathcal{G}$-harmonic functions.

Theorem 2.1.5. *(Full Rank Harnack Inequality) Let $\mathcal{G} \in \mathcal{N}(c_0, m, \gamma, \beta, \Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, c_1, \vartheta)$ with Q satisfying* (1.13) *and $\mathcal{G}$ is strictly irreducible. Then there exist constants $\widetilde{C}_1, \widetilde{C}_3 > 0$ and $0 \leq \widetilde{r}_1 < 1$ depending on $(d, \beta, \gamma, \Theta_1, \Theta_2, \Theta_3, \Theta_4, \vartheta, m, c_0, q_0, Q^0)$, such that for any $r \in (0, \widetilde{r}_1]$, any nonnegative function u defined on $\mathbb{R}^d \times \mathcal{M}$ that is $\mathcal{G}$-harmonic on $B(x_0, r) \times \mathcal{M}$, we have*

$$u(x,k) \leq \widetilde{C}_1(u(y,l) + h(y,k)) \quad \text{for any } (x,k), (y,l) \text{ in } B(x_0, r/8) \times \mathcal{M}.$$

In particular,

$$u(x,k) \leq \widetilde{C}_3 r^{-2} u(y,l) \quad \text{for any } (x,k), (y,l) \text{ in } B(x_0, r/8) \times \mathcal{M} \text{ with } k \neq j.$$

Here $h(y,k)$is defined in (1.18)*, and $\widetilde{r}_1, q_0$ are the constants in Theorem 2.1.4 and* (1.17) *respecitvely.*

The rest of the paper is organized as follows. In Section 2, we derive the two-sided scale invariant Green function estimate and the Martin integral representation formula for non-negative harmonic functions with respect to the operator $\mathcal{L}_i + q_{ii}$, as well as the probabilistic representation formula for nonnegative $\mathcal{G}$-harmonic functions in term of harmonic functions for $\mathcal{L}_i + q_{ii}, i \in \mathcal{M}$ in small balls. The proof of the Hölder regularity of bounded $\mathcal{G}$-harmonic functions is given in Section 3. The proof of Theorem 2.1.4 is given in Section 4 and the proof for Theorem 2.1.5 is shown in Section 5.

2.2 Preliminaries

A Borel measurable function f is said to be in **Kato class** $\mathbb{K}_d$ if and only if for each ball B in $\mathbb{R}^d$:

$$\lim_{r\to 0}[\sup_{x\in B}\int_{|y-x|\le r}|f(y)|g(y-x)dy] = 0, \tag{2.1}$$

where

$$g(x) = \begin{cases} -\ln|x| \text{ when } d=2; \\ |x|^{2-d} \text{ when } d\ge 3. \end{cases} \tag{2.2}$$

Remark 2.2.1. *The boundedness assumption of q_{kk} in* (1.13) *implies that each $q_{kk} \in \mathbb{K}_d, k \in \mathcal{M}$.*

The following rough scaling property of the infinitesimal generator identifies the class of the operators for which the scale-invariant Harnack inequalities will be established in this paper.

Lemma 2.2.2. *Suppose that $\{(X_t,\Lambda_t), t\ge 0; \mathbb{P}^{(x,i)}, (x,i)\in\mathbb{R}^d\times\mathcal{M}\}$ is a Hunt process having the infinitesimal generator $\mathcal{G}\in\mathcal{N}(c_0,m,\gamma,\beta,\Theta_1,\Theta_2,\Theta_3,\Theta_4,\Theta_5,c_1,\vartheta)$. For any $\lambda\in(0,1]$, define $Y_t=\lambda^{-1}X_{\lambda^2 t}$, $\Gamma_t=\Lambda_{\lambda^2 t}$, and $\mathbf{P}^{(x,i)}=\mathbb{P}^{(\lambda x,i)}$. Then $\{(Y_t,\Gamma_t), t\ge 0; \mathbf{P}^{(x,i)}, (x,i)\in\mathbb{R}^d\times\mathcal{M}\}$ is a Hunt process corresponding to the infinitesimal generator*

$$\mathcal{G}^{(\lambda)}f(x,i) = \mathcal{L}_i^{(\lambda)}f(x,i) + Q^{(\lambda)}(x)f(x,\cdot)(i), \quad \text{for } (x,i)\in\mathbb{R}^d\times\mathcal{M},$$

where $f(\cdot,i)\in C_b^2(\mathbb{R}^d), i\in\mathcal{M}$,

$$\begin{aligned}\mathcal{L}_i^{(\lambda)}f(x,i) &= \frac{1}{2}\sum_{k,l=1}^{d} a_{k,l}(\lambda x,i)\frac{\partial^2 f}{\partial x_k\partial x_l}(x,i)\\ &+\Big(\lambda b_1(\lambda x,i)-\int_{\mathbb{R}^d\setminus\{0\}} z\mathbb{1}_{\{1<|z|\le 1/\lambda\}}\lambda^{d+2}b_2(\lambda x,\lambda z,i)j_0(\lambda z)dz\Big)\cdot\nabla f(x,i)\\ &+\int_{\mathbb{R}^d\setminus\{0\}}\Big(f(x+z,i)-f(x,i)-z\mathbb{1}_{\{|z|\le 1\}}\cdot\nabla f(x,i)\Big)\lambda^{d+2}b_2(\lambda x,\lambda z,i)j_i(\lambda z)dz\\ &+\lambda^2 Q(\lambda x)f(x,\cdot)(i), \end{aligned} \tag{2.3}$$

and

$$q_{ij}^{(\lambda)}(x)=\lambda^2 q_{ij}(\lambda x),\ \textit{for every } i,j\in\mathcal{M}.$$

In particular, $\mathcal{G}^{(\lambda)}\in\mathcal{N}(\Theta_1,\Theta_2+\Theta_3\Theta_4,\Theta_3,\Theta_4,\Theta_5,\vartheta,\beta,\gamma,c_1)$ *with* $Q^{0,(\lambda)}=\lambda^2Q^0$ *in* (1.13) *for every* $\lambda\in(0,1]$.

Proof. For any $f\in C_b^2(\mathbb{R}^d,\mathcal{M})$, define $f_\lambda(x,i)=f(x/\lambda,i)$. Denote by $\mathbf{P}^{(x,i)}$ the transition semigroup of (Y,Γ).Then

$$\mathbf{P}_tf(x,i)=\mathbf{E}^{(x,i)}[f(Y_t,\Gamma_t)]=\mathbb{E}^{(\lambda x,i)}f_\lambda[(X_{\lambda^2t},\Lambda_{\lambda^2t})]=\mathbb{P}_{\lambda^2t}^{(\lambda x,i)}f_\lambda(\lambda x,i).$$

Then

$$\begin{aligned}&\mathcal{G}^{(\lambda)}f(x,i)\\ =\ &\lim_{t\to0}\frac{\mathbf{E}^{(x,i)}[f(Y_t,\Gamma_t)]-f(x,i)]}{t}\\ =\ &\lim_{t\to0}\frac{\mathbb{E}^{(\lambda x,i)}[f_\lambda(X_{\lambda^2t},\Lambda_{\lambda^2t})]-f_\lambda(x,i)]}{\lambda^2t}\lambda^2=\lambda^2\mathcal{G}f_\lambda(\lambda x,i).\\ =\ &\frac{1}{2}\sum_{i,j=1}^{d}a_{i,j}(\lambda x,i)\frac{\partial^2 f}{\partial x_i\partial x_j}(x,i)+\Big(\lambda b_1(\lambda x,i)-\int_{\mathbb{R}^d\setminus\{0\}}z\mathbb{1}_{\{1<|z|\le1/\lambda\}}\lambda^{d+2}b_2(\lambda x,\lambda z,i)j_0(\lambda z)dz)\Big)\cdot\nabla f(x,\\ &+\int_{\mathbb{R}^d\setminus\{0\}}\Big(f(x+z,i)-f(x,i)-z\mathbb{1}_{\{|z|\le1\}}\cdot\nabla f(x,i)\Big)\lambda^{d+2}b_2(\lambda x,\lambda z,i)j_i(\lambda z)dz+\lambda^2Q(\lambda x)f(x,\cdot)(i)\\ =\ &\mathcal{L}^{(\lambda)}f(x,i)+Q^{(\lambda)}f(x,i). \end{aligned} \tag{2.}$$

Hence the process (Y,Γ) has diffusion matrix $a(\lambda x)$ which satisfies the same uniform ellipticity and Hölder continuity condition (1.6) as $a(x)$, the drift coefficient vector

$$|b_1^\lambda(x,i)|:\ =\ |\lambda b_1(\lambda x,i)-\lambda\int_{\mathbb{R}^d\setminus\{0\}}w\mathbb{1}_{\{\lambda\le|w|\le1\}}b_2(\lambda x,w,i)j_0(w)d(w)|$$

$$
\begin{aligned}
&\leq \quad \Theta_2+\Theta_3\int_{\mathbb{R}^d\setminus\{0\}}|w|^2\mathbb{1}_{\{\lambda\leq|w|\leq1\}}\frac{c_1}{|w|^{d+\beta}}dw \\
&\leq \quad \Theta_2+\Theta_3\Theta_4 \qquad (2.5)
\end{aligned}
$$

and by(1.15), the jumping kernel $J_\lambda^{b_2}(x,x+z):=\lambda^2 b_2(\lambda x,\lambda z)j_0(\lambda z)$ satisfies

$$
\begin{aligned}
J_\lambda^{b_2}(x,x+z) &= \lambda^2 b_2(\lambda x,\lambda z)j_0(\lambda z) \\
&\leq \frac{\vartheta}{|B(\lambda x,\lambda r)|}\int_{B(\lambda x,\lambda r)}\lambda^2 b_2(\lambda u,\lambda(x+z))j_0(\lambda z)d(\lambda u) \\
&= \frac{\lambda^d\vartheta}{|B(\lambda x,\lambda r)|}\int_{B(x,r)}J_\lambda^{b_2}(u,x+z)du \\
&= \frac{\vartheta}{|B(x,r)|}\int_{B(x,r)}J_\lambda^{b_2}(u,x+z)du, \qquad (2.6)
\end{aligned}
$$

which satisfies the UJS condition with the same constant ϑ. Clearly for $\lambda\in(0,1]$,

$$\|b_2^\lambda\|_\infty:=\lambda^2\|b_2(\lambda x,\lambda z,i)\|_\infty\leq\Theta_3,$$

and

$$j_i^\lambda(z):=j_i(\lambda z)\leq\frac{c_1}{|\lambda z|^{d+\beta}},\qquad \text{for } |z|\leq 1/\lambda.$$

with

$$\int_{\mathbb{R}^d\setminus\{0\}}(1\wedge|z|^2)j_i^\lambda(z)dz=\int_{\mathbb{R}^d\setminus 0}(\lambda^2\wedge|w|^2)j_i(w)dw\leq\Theta_4.$$

Also, by (1.13) and (1.17), since

$$c_0\lambda^2 q_0\leq c_0\lambda^2 q_{ij}^0\leq q_{ij}^{(\lambda)}(x)\leq\lambda^2 q_{ij}^0\leq\lambda^2\Theta_5\leq\Theta_5\quad a.e. \text{ in } \mathbb{R}^d.$$

Then $Q^{0,(\lambda)}=\lambda^2Q^0$ with $c_0^{(\lambda)}=c_0$. This shows that $\mathcal{G}^{(\lambda)}\in\mathcal{N}(\Theta_1,\Theta_2+\Theta_3\Theta_4,\Theta_3,\Theta_4,\Theta_5,\vartheta,\beta,\gamma)$ with $Q^{0,(\lambda)}=\lambda^2Q^0$ for every $\lambda\in(0,1]$.

□

Suppose that $\{(X_t,\Lambda_t),t\geq0;\mathbb{P}^{(x_0,i)},(x_0,i)\in\mathbb{R}^d\times\mathcal{M}\}$ is a Hunt process having the infinitesimal generator $\mathcal{G}\in\mathcal{N}(\Theta_1,\Theta_2,\Theta_3,\Theta_4,\Theta_5,\gamma,\beta,c_1)$ starting from (x_0,i), define

$$X_t^0=\begin{cases}X_t, & t<\tau_1;\\ \partial, & t\geq\tau_1;\end{cases}$$

where τ_1 is defined in (1.19) and ∂ is a cemetery point. Then by Lemma 3.7 in [75],

$$\{X_t^0, t \le \tau_1; \mathbb{P}^{(x_0,i)}, x \in \mathbb{R}^d\} \text{ is a Hunt process corresponding to the operator } \mathcal{L}_i + q_{ii} \text{ starting from } x. \tag{2.7}$$

For any $\varphi \ge 0, \alpha \ge 0$, define

$$\begin{aligned} \widetilde{G}_\alpha^i \varphi(x) : &= \mathbb{E}^{(x,i)}[\int_0^\infty e^{-\alpha s}\varphi(X_s^0)ds] \\ &= \mathbb{E}^{(x,i)}[\int_0^{\tau_1} e^{-\alpha s}\varphi(X_s)ds], \end{aligned} \tag{2.8}$$

Proposition 2.2.3. *Suppose that* $\{(X_t, \Lambda_t), t \ge 0; \mathbb{P}^{(x,i)}, x \in \mathbb{R}^d, i \in \mathcal{M}\}$ *is the Hunt process having the infinitesimal generator* $\mathcal{G} \in \mathcal{N}(c_0, m, \gamma, \beta, \Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, c_1, \vartheta)$ *starting from* (x, i)*. Then for every nonnegative Borel measurable function* φ *defined on* $\mathbb{R}^d$*,* $x \in \mathbb{R}^d$*, any* $\alpha \ge 0$*,*

$$\mathbb{E}^{(x,i)}[e^{-\alpha\tau_1}\varphi(X_{\tau_1-})] = \widetilde{G}_\alpha^i(-q_{ii}\varphi)(x),$$

where τ_1 *is defined in* (1.19)*, and* $\widetilde{G}_\alpha^i$ *is defined in* (2.8)*.*

Proof. Let $\{Z_t^i, t \le \xi; \mathbf{P}^x, x \in \mathbb{R}^d\}$ be the Hunt process corresponding to the operator $(\mathcal{L}_i, C_b^2(\mathbb{R}^d))$. Define

$$\xi = \inf\{t > 0 : -\int_0^t q_{ii}(Z_s^i)ds > \eta\}, \tag{2.9}$$

where η is an exponential random variable with mean 1. Let the probability measure $\mathbb{P}^x$ on Ω determined by

$$\mathbb{E}^x[f(Z_t^i)] = \mathbf{E}^x[f(Z_t^i); t < \xi] = \mathbf{E}^x[e_q(t)f(Z_t^i)], \quad t \ge 0, \tag{2.10}$$

for every $f \in C_b^2(\mathbb{R}^d)$, where $e_q(t) := \exp(\int_0^t q_{ii}(Z_s^i)ds)$. Then by Ito's formula,

$$\{Z_t^i, t \le \xi; \mathbb{P}^x, x \in \mathbb{R}^d\} \text{ is a Hunt process corresponding to the operator } \mathcal{L}_i + q_{ii} \text{ starting from } x, \text{ whose lifetime is } \xi. \tag{2.11}$$

Then by Theorem 5.2 [49], (2.7), (2.11), $\mathbb{E}^{(x,i)}[\mathbb{1}_{\{0<t\le\tau_1\}}f(X_t^0)g(\tau_1)] = \mathbb{E}^x[\mathbb{1}_{\{0<t\le\xi\}}f(Z_t^i)g(\xi)]$ for any $f, g \in C_b^2(\mathbb{R}^d)$.Then for any $\varphi \ge 0, x \in \mathbb{R}^d$,

$$\mathbb{E}^{(x,i)}[e^{-\alpha\tau_1}\varphi(X_{\tau_1^-})] = \mathbb{E}^{(x,i)}[e^{-\alpha\tau_1}\varphi(X^0_{\tau_1^-})] = \mathbb{E}^x[e^{-\alpha\xi}\varphi(Z^i_{\xi-})]$$

and

$$\widetilde{G}^i_\alpha\varphi(x) = \mathbb{E}^{(x,i)}[\int_0^\infty e^{-\alpha t}\varphi(X^0_t)dt] = \mathbb{E}^x[\int_0^\infty e^{-\alpha t}\varphi(Z^i_t)dt]. \tag{2.12}$$

Let $\mathcal{F}_t = \sigma\{Z^i_s; 0 \le s \le t\}$.Then by (2.9),(2.10),(2.12), and (61.2) on [68, p.286] (by putting $m_t = e^{\int_0^t q_{ii}(Z^i_s)ds}\mathbb{1}_{\{t<\xi\}}$ there and noticing that $\mathbf{P}^x - a.e.Z_{\xi^-} = \partial$) that

$$\begin{aligned}
\mathbb{E}^x[e^{-\alpha\xi}\varphi(Z^i_{\xi-})] &= \mathbf{E}^x[\int_0^\infty e^{-\alpha t}\varphi(Z^i_t)(-dm_t)] \\
&= \mathbf{E}^x[\int_0^\infty e^{-\alpha t}\varphi(Z^i_t)(-q_{ii}(Z^i_t))\exp(\int_0^t q_{ii}(Z^i_s)ds))dt] \\
&= \mathbb{E}^x[\int_0^\infty e^{-\alpha t}(-q_{ii}\varphi)(Z^i_t)dt] \\
&= \widetilde{G}^i_\alpha(-q_{ii}\varphi)(x).
\end{aligned}$$

□

Remark 2.2.4. *By* (2.12)*, when $\alpha = 0$, we know that $\widetilde{G}^i$ is the Green operator of $\mathcal{L}_i + q_{ii}$ in $\mathbb{R}^d$ with zero Dirichlet boundary condition.*

For any nonnegative Borel measurable function ϕ defined on $\mathbb{R}^d \times \mathcal{M}$, define

$$u(x,i) := \mathbb{E}^{(x,i)}[\phi(X_{\tau_D}, \Lambda_{\tau_D})] \text{ for any } (x,i) \in D \times \mathcal{M}. \tag{2.13}$$

□

Proposition 2.2.5. *Let D be a bounded and connected open set in $\mathbb{R}^d$. For any nonnegative Borel measurable function ϕ defined on $\mathbb{R}^d \times \mathcal{M}$, the function u defined in* (2.13) *satisfies*

$$u(x,i) = h(x,i) + \sum_{j=1, j\neq i}^m \widetilde{G}^i_D(q_{ij}u(\cdot,j))(x), \text{ for every } (x,i) \in D \times \mathcal{M}, \tag{2.14}$$

where

$$h(x,i) = \mathbb{E}^{(x,i)}[\phi(X_{\tau_D}, \Lambda_{\tau_D}); \tau_D < \tau_1], \tag{2.15}$$

$\tau_D = \inf\{t > 0 : (X_t, \Lambda_t) \notin D \times \mathcal{M}\}$ and $\widetilde{G}^i_D$ is the Green operator with respect to the operator $\mathcal{L}_i + q_{ii}$ on D with zero Dirichlet boundary condition.

Proof. First consider any nonnegative function $\phi \in C^\infty_c(\mathbb{R}^d \times \mathcal{M})$, where $C^\infty_c(\mathbb{R}^d \times \mathcal{M})$ is the space of functions f such that $f(\cdot, i) \in C^\infty_c(\mathbb{R}^d)$ for each $i \in \mathcal{M}$. Then by (2.15),

$$u(x,i) = \mathbb{E}^{(x,i)}[\phi(X_{\tau_D}, \Lambda_{\tau_D}); \tau_D < \tau_1] + \mathbb{E}^{(x,i)}[\phi(X_{\tau_D}, \Lambda_{\tau_D}); \tau_D > \tau_1] + \mathbb{E}^{(x,i)}[\phi(X_{\tau_D}, \Lambda_{\tau_D}); \tau_D = \tau_1]$$

$$= \quad h(x,i) + I + II; \tag{2.16}$$

Then by the strong Markov property of (X, Λ), Lemma 3.6 in [75], and Proposition 2.2.3,

$$\begin{aligned}
I &= \mathbb{E}^{(x,i)}[\mathbb{E}^{(X_{\tau_1},\Lambda_{\tau_1})}[\phi(X_{\tau_D},\Lambda_{\tau_D})];\tau_D > \tau_1] \\
&= \mathbb{E}^{(x,i)}[u(X_{\tau_1},\Lambda_{\tau_1});\tau_D > \tau_1] \\
&= \sum_{j=1,j\neq i}^{m} \mathbb{E}^{(x,i)}[\mathbb{1}_{\{\tau_1<\tau_D\}} u(X_{\tau_1-}, j)(-\frac{q_{ij}}{q_{ii}}(X_{\tau_1-}))] \\
&= \sum_{j=1,j\neq i}^{m} \mathbb{E}^{(x,i)}[\int_0^{\tau_1} \mathbb{1}_{\{\tau_1<\tau_D\}} u(X_s, j)\frac{q_{ij}}{q_{ii}}(X_s) q_{ii}(X_s) ds)] \\
&= \sum_{j=1,j\neq i}^{m} \mathbb{E}^{(x,i)}[\int_0^{\tau_D\wedge\tau_1} \mathbb{1}_{\{\tau_1<\tau_D\}} u(X_s, j)\frac{q_{ij}}{q_{ii}}(X_s) q_{ii}(X_s) ds)] \\
&= \sum_{j=1,j\neq i}^{m} \mathbb{E}^{(x,i)}[\int_0^{\tau_D} u(X^0_s, j)\frac{q_{ij}}{q_{ii}}(X^0_s) q_{ii}(X^0_s) ds)] \\
&= \sum_{j=1,j\neq i}^{m} \widetilde{G}^i_D(q_{ij}u(\cdot,j))(x). \qquad (2.17)
\end{aligned}$$

By the definition of τ_1 in (1.19), τ_1 is $\mathcal{F}_{\tau_1^-}$ measurable, then again by Lemma 3.6 in [75], and Proposition 2.2.3,

$$\begin{aligned}
II &= \mathbb{E}^{(x,i)}[\mathbb{1}_{D^c}(X_{\tau_1})\mathbb{1}_{\{\tau_D=\tau_1\}}\phi(X_{\tau_1},\Lambda_{\tau_1})] \\
&= \sum_{j=1,j\neq i}^{m} \mathbb{E}^{(x,i)}[\mathbb{1}_{D^c}(X_{\tau_D-})\mathbb{1}_{\{\tau_D=\tau_1\}}\phi(X_{\tau_1^-}, j)\frac{q_{ij}}{q_{ii}}(X_{\tau_1^-})] \\
&= 0. \qquad (2.18)
\end{aligned}$$

Then by (2.16),(2.17),(2.18),

$$u(x,i) \quad = \quad h(x,i) + \sum_{j=1,j\neq i}^{m} \widetilde{G}^i_D(q_{ij}u(\cdot,j))(x).$$

Finally, by monotone convergence theorem, (2.14) holds for any nonnegative Borel measurable function ϕ defined on $\mathbb{R}^d \times \mathcal{M}$ harmonic with respect to $\mathcal{G}$ in D.

□

Recall that an open set D in $\mathbb{R}^d$ (when $d \geq 2$) is said to be $C^{1,1}$ if there exist a localization radius $R_0 > 0$ and a constant $\Xi_0 > 0$ such that for every $Q \in \partial D$, there exists a $C^{1,1}$ function $\phi = \phi_Q : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\phi(0) = \nabla\phi(0) = 0$, $\|\phi\|_\infty \leq \Xi_0, |\nabla\phi(x) - \nabla\phi(y)| \leq \Xi_0|x - y|$, and an orthonormal coordinate system $CS_Q : y = (y_1, ...y_{d-1}, y_d) =: (\widetilde{y}, y_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$ with its origin at Q such that

$$B(Q, R_0) \cap D = \{y = (\widetilde{y}, y_d) \in B(0, R_0) \text{ in } CS_Q : y_d > \phi(\widetilde{y})\}.$$

The pair (R_0, Ξ_0) is called the characteristics of the $C^{1,1}$ open set D.

For any $\mathcal{L} \in \mathcal{N}(\Theta_1, \Theta_2, \Theta_3, \Theta_4, \gamma, \beta, c_1)$, let $(X_t, t \geq 0; \mathbb{P}^x, x \in \mathbb{R}^d)$ be the process having the infinitesimal generator $\mathcal{L}$ and denote X^D as the subprocess of X in D. That is, $X_t^D(\omega) = X_t(\omega)$ if $t < \tau_D(\omega)$ and $X^D(\omega) = \partial$ if $t \geq \tau_D(\omega)$, where ∂ is a cemtery state. If $p(t, x, y)$ is the transition density of the process X. Then

$$p_D(t, x, y) = p(t, x, y) - \mathbb{E}^x[p(t - \tau_D, X_{\tau_D}, y), \tau_D < t]$$

is the transition density of X^D. Let **Green function** of $\mathcal{L}$ in a bounded $C^{1,1}$ domain D. That is, $G_D(x, y) = \int_0^{\tau_D} p_D(t, x, y)dt$ for $x, y \in D$. Then for any $\lambda \in (0, 1)$, we denote the Green function for $\mathcal{L}$ in λD as $G_{\lambda D}(x, y)$, for $x, y \in \lambda D$ and the the Green function for $\mathcal{L}^{(\lambda)}$ of the form (2.3) in D as $G_D^{(\lambda)}(x, y)$, for $x, y \in D$.

Let $G_{\lambda D}^{\Delta}$ be the Green function for Δ in λD. Notice that by Brownian scaling,

$$G_{\lambda D}^{\Delta}(x, y) = \lambda^{2-d} G_D^{\Delta}(x/\lambda, y/\lambda), \text{ for every } x, y \in \lambda D, x \neq y.$$

For the details of how to transform the relationship between $G_D^{(\lambda)}(x/\lambda, y/\lambda)$ and $G_D^{\Delta}(x/\lambda, y/\lambda)$ to the relationship between $G_{\lambda D}(x, y)$ and $G_{\lambda D}^{\Delta}(x, y), x, y \in \lambda D$, the reader can refer to [26]. Next we quote a series of theorems for the sharpe two-sided scale-invariant Green function estimates from [26], which helps us obtain scale-invariant properties for the Green function of the killed operator $\mathcal{L}+q$ as a preparation to derive the Martin integral representation formula for nonnegative harmonic functions with respect to $\mathcal{L}+q$, where $q < 0$ and $\|q\|_\infty = \Theta_5$.

The following lemmas on the scale-invariant comparison inequalities of the Green functions for the Laplacian operator Δ in small balls can be founded in [26, (2.19)] for $d \geq 3$ and in [79, Lemma 3] for $d = 2$ respectively.

Lemma 2.2.6. *There exists a constant $K_1 = K_1(d) > 1$ for $d \geq 3$ such that for any $x_0 \in \mathbb{R}^d, r > 0$, such that for any $x \neq y \in B(x_0, r), x \neq y$,*

$$\frac{K_1^{-1}}{|x - y|^{d-2}} \Big(1 \wedge \frac{\delta_{B(x_0,r)}(x)}{|x - y|}\Big)\Big(1 \wedge \frac{\delta_{B(x_0,r)}(y)}{|x - y|}\Big) \leq G_{B(x_0,r)}^{\Delta}(x, y)$$

$$\leq \quad \frac{K_1}{|x-y|^{d-2}}\Big(1\wedge \frac{\delta_{B(x_0,r)}(x)}{|x-y|}\Big)\Big(1\wedge \frac{\delta_{B(x_0,r)}(y)}{|x-y|}\Big), \tag{2.19}$$

where $\delta_{B(x_0,r)}(y)$ is the Euclidean distance between the point y to $B(x_0,y)^c$.

Lemma 2.2.7. *When $d=2$, for any $x_0\in\mathbb{R}^d, r\in(0,1)$, any $x,y\in B(x_0,r)$, we have*

$$\frac{1}{2\pi}\ln\Big(1+\frac{\delta_{B(x_0,r)}(x)\delta_{B(x_0,r)}(y)}{|x-y|^2}\big)\leq G^{\Delta}_{B(x_0,r)}(x,y)\Big)\leq \frac{1}{2\pi}\ln\Big(1+4\frac{\delta_{B(x_0,r)}(x)\delta_{B(x_0,r)}(y)}{|x-y|^2}\Big), \tag{2.20}$$

We also quote the two-sided scale-invariant Green function estimate for each $\mathcal{L}$ from [26, Lemma 2.17].

Lemma 2.2.8. *Let $\mathcal{L}\in\mathcal{N}(\Theta_1,\Theta_2,\Theta_3,\Theta_4,\gamma,\beta,c_1)$ and D be a bounded $C^{1,1}$ domain with characteristics (R_0,Ξ_0). There exist positive constants $\delta_1\in(0,1)$depending on $(d,\beta,\gamma,\Theta_1,\Theta_2,\Theta_3,\Theta_4,R_0,\Xi_0,diam(D))$ and $K_0=K_0(d,\Theta_1,\gamma)>1$ such that for any $\lambda\in(0,\delta_1)$ and bounded functions b_2 with $\|b_2\|_\infty\leq\Theta_3\vee(\Theta_2+\Theta_3\Theta_4)$,*

$$K_0^{-1}G^{\Delta}_{\lambda D}(x,y)\leq G_{\lambda D}(x,y)\leq K_0 G^{\Delta}_{\lambda D}(x,y)\quad \textit{for } x,y\in\lambda D, \tag{2.21}$$

□

Let $\mathcal{L}\in\mathcal{N}(\Theta_1,\Theta_2,\Theta_3,\Theta_4,\gamma,\beta,c_1)$. Fix $x'\in D$ and let

$$M_D(x,z)=\frac{G_D(x,z)}{G_D(x',z)},\ \text{for } x\in D, y\in D\setminus\{x,x'\} \tag{2.22}$$

be the **Martin kernel** of $\mathcal{L}$ for $x\in D, z\in D\setminus\{x,x'\}$. The *Martin boundary* for $\mathcal{L}$ in D is defined to be the set $\partial_M D=D^*\backslash D$, where D^* is the smallest compact set for which $M_D(x,z)$ is continuous in z in the extended sense(See [7, Theorem 7.1] for detailed explanation for this definition). The *minimal boundary* $\partial_M^{min}D$ for $\mathcal{L}$ in D is the collection of all points $z\in\partial_M D$ so that $x\mapsto M_D(x,y)$ is a positive *minimal $\mathcal{L}$-harmonic function* in D in the sense that if $h\geq 0$ is $\mathcal{L}$-harmonic in D and $h(x)\leq M_D(x,y)$ on D, then $h(x)=cM_D(x,y)$ for some constant $0<c\leq 1$.

By Theorem 3.6 in [26] which is quoted below, we can identify the Euclidean boundary as its Martin boundary with respect to the operator $\mathcal{L}$ of the form (1.3) when D is a bounded $C^{1,1}$ domain.

Theorem 2.2.9. *Suppose that D is a bounded $C^{1,1}$ domain with characteristics (R_0, Ξ_0). Then there exists a positive constant $\delta_1 > 0$ such that that for any $\lambda \in (0, \delta_1)$, the Martin boundary and minimal Martin boundary λD with respect to $\mathcal{L}$ of the form* (1.3) *can be both identified with its Euclidean boundary $\partial(\lambda D)$, where δ_1 is the constant in Lemma 2.2.8.*

Moreover, the Martin integral representation formula for nonnegative harmonic function with respect to $\mathcal{L}$ is obtained in [26, Theorem 3.8].

Theorem 2.2.10. *Let D be a bounded $C^{1,1}$ domain and $\lambda < \delta_1$. If h is a nonnegative harmonic function in λD with respect to $\mathcal{L} \in \mathcal{N}(\Theta_1, \Theta_2, \Theta_3, \Theta_4, \gamma, \beta, c_1)$, then there exists a unique measure μ_h on $\partial(\lambda D)$ such that*

$$h(x) = \int_{\partial(\lambda D)} M_{\lambda D}(x,z)h(z)\mu_h(dz) + \int_{\overline{\lambda D}^c} \int_{\lambda D} G_{\lambda D}(x,y)J^{b_2}(y,z)dyh(z)dz, \qquad (2.23)$$

where δ_1 is the constant in Lemma 2.2.8, and J^{b_2} is the jump kernel defined in (1.15)*.*

Suppose that $\mathcal{L} \in \mathcal{N}(\Theta_1, \Theta_2, \Theta_3, \Theta_4, \gamma, \beta, c_1)$ and $q < 0$ such that $\|q\|_\infty = \Theta_5$. Let $\widetilde{G}_{B(x_0,r)}(\cdot,\cdot)$ be the **Green function** for the operator $\mathcal{L} + q$ in the ball $B(x_0, r)$. We are going to prove the existence and a comparison inequality for $\widetilde{G}_{B(x_0,r)}(\cdot,\cdot)$. First, we quote the $3G$-lemma from [30, Lemma 2.1], which can also be found in [31, 78].

Lemma 2.2.11. *(3G-Lemma) For any ball $B \in \mathbb{R}^d$,*

$$\frac{G_B^{\Delta}(x,y)G_B^{\Delta}(y,z)}{G_B^{\Delta}(x,z)} \le C_G(g(x-y) + g(y-z)) \qquad (2.24)$$

for every $x, y, z \in B$, where the constant C_G depends only on d, and the function g is defined in (2.2)*.*

Proof. The proof can be found in [31]. □

For any $\mathcal{L} \in \mathcal{N}(\Theta_1, \Theta_2, \Theta_3, \Theta_4, \beta, \gamma, c_1)$, let $\{\bar{X}_t, t \ge 0; \mathbb{P}^x, x \in \mathbb{R}^d\}$ be the Hunt process corresponding to the operator $\mathcal{L}$. Define

$$\tau_D = \inf\{t \ge 0; \bar{X}_t \notin D\}.$$

Similar as the h-conditioned Brownian motion defined in [78, Page 131, 132], for any positive Borel-measurable function h, $M_t := h(\bar{X}_t)/h(\bar{X}_0)$ is a multiplicative functional $M = \{M_t : 0 \le t < \infty\}$ of $\bar{X}$ such that $\mathbb{E}^x[M_t] \le 1$ for every $t \ge 0$. For any $\mathcal{F}_t$-measurable function $\Phi \ge 0$, where $\mathcal{F}_t = \sigma\{\bar{X}_s : s \le t\}, t > 0$, define

$$\mathbb{E}_h^x[\Phi; t < \tau_D] := (h(x))^{-1}\mathbb{E}^x[\Phi \cdot h(\bar{X}_t); t < \tau_D]; \quad x \in D. \tag{2.25}$$

and $\mathbb{E}_h^x[\Phi; t < \tau_D]$ is reduced to be a probability distribution function in B when Φ is of the form $\mathbb{1}_{\{X_t \in B\}}$, $B \in \mathcal{D}$, where $\mathcal{D}$ is a Borel σ-field of D.

When D is a bounded $C^{1,1}$ domain, since the Green function $G_D(\cdot, v)$ for the operator $\mathcal{L}$ is harmonic with respect to $\mathcal{L}$ in $D \setminus \{v\}$, then substituting h with $G_D(\cdot, v)$ in (2.25), we can define the $G_D(\cdot, v)$-*conditioned Markov process* whose state space is $D \setminus \{v\} \cup \{\partial\}$, and the associated probability and expectation are denoted as $\mathbb{P}_v^x$ and $\mathbb{E}_v^x$ respectively. By the definition of Martin kernel in (2.22) and by [26, Lemma 3.2, Lemma 3.4], $M_D(\cdot, z) = \lim_{y \in D, y \to z} \frac{G_D(x,y)}{G_D(x',y)}$ exists for any $x \in D, z \in \partial D$, and it is harmonic with respect to $\mathcal{L}$ in D. Then we can similarly define the $M_D(\cdot, z)$-*conditioned Markov process*, whose associated probability and expectation are denoted as $\mathbb{P}_z^x$ and $\mathbb{E}_z^x$ respectively.

Definition 2.2.12. *Let $\mathcal{L} \in \mathcal{N}(\Theta_1, \Theta_2, \Theta_3, \Theta_4, \beta, \gamma, c_1)$ and $(\bar{X}_t, t \ge 0; \mathbb{P}^x, x \in \mathbb{R}^d)$ be the Hunt process corresponding to $\mathcal{L}$. We say that a set $A \subset \mathbb{R}^d$ is* **nearly Borel** *if for every $x \in \mathbb{R}^d$, there exist Borel subsets B and B' in $\mathcal{B}(\mathbb{R}^d)$ such that $B \subset A \subset B'$ and*

$$\mathbb{P}^x(\bar{X}_t \in B' \setminus B \text{ for some } t > 0) = 0.$$

The collection of nearly Borel sets is a σ-field, which we denote as $\mathcal{B}^n(\mathbb{R}^d)$. We say that a set $A \subset \mathbb{R}^d$ is **polar** *if there exists a set $D \in \mathcal{B}^n(\mathbb{R}^d)$ such that $A \subset D$ and $\mathbb{P}^x(T_D < \infty) = 0$ for every $x \in \mathbb{R}^d$.*

Notice that the definition of nearly Borel set depends on the process $\bar{X}$ and roughly speaking, a set is nearly Borel if the process $\bar{X}$ cannot distinguish it from a Borel set. Also, it is easy to see that $\mathcal{B}(\mathbb{R}^d) \subset \mathcal{B}^n(\mathbb{R}^d)$. We will omit further discussions and the reader can refer to [11, Page 60] for more details on nearly Borel sets.

Lemma 2.2.13. *Let $\mathcal{L} \in \mathcal{N}(\Theta_1, \Theta_2, \Theta_3, \Theta_4, \beta, \gamma, c_1)$ and $(\bar{X}_t, t \ge 0; \mathbb{P}^x, x \in \mathbb{R}^d)$ be the Hunt process corresponding to the operator $\mathcal{L}$. Then any singleton set is a polar set.*

Proof. Fix a point $v \in \mathbb{R}^d$. By Lemma 2.2.8 with $D = B(v,1)$, there exists constants $\varepsilon_1 = \varepsilon_1(d,\Theta_1,\Theta_2,\Theta_3,\Theta_4,\beta,\gamma)$ and $K_0 = (d,\Theta_1,\gamma)$ such that for any $r \in (0,\varepsilon_1)$, (2.21) holds with $\lambda = r$. Since (2.21) is independent of the center point rv, then the Green function $G_{B(v,r)} = G_{r(B(v,1)-v)+v}$ of $\bar{X}$ in $B(v,r)$ satisfies

$$K_0^{-1} G^{\Delta}_{B(v,r)}(x,y) \le G_{B(v,r)}(x,y) \le K_0 G^{\Delta}_{B(v,r)}(x,y), \text{ for } x \in B(v,r), y \in B(v,r)\setminus\{x\}. \tag{2.26}$$

Define $\sigma_{B(v,r)} = \inf\{t > 0 : \bar{X}_t \in B(v,r)\}$ and $\tau_{B(v,r)} = \inf\{t > 0 : \bar{X}_t \notin B(v,r)\}$. Notice that for any $x \in \mathbb{R}^d$,

$$\begin{aligned} U(x,\{v\}) : &= \int_0^\infty \mathbb{P}^x_s(\{v\})ds \\ &= \mathbb{E}^x[\mathbb{E}^{\bar{X}_{\sigma_{B(v,r)}}}[\int_0^{\tau_{B(v,r)}\circ\theta_{\sigma_{B(v,r)}}} \mathbb{1}_{\{v\}}(\bar{X}_s)ds]] + \mathbb{E}^x[\int_{\tau_{B(v,r)}}^\infty \mathbb{1}_{\{v\}}(\bar{X}_s)ds] \\ &= \mathbb{E}^x[\int_{\{v\}} G_{B(v,r)}(\bar{X}_{\sigma_{B(v,r)}},y)dy] + \mathbb{E}^x[U(\bar{X}_{\tau_{B(v,r)}},\{v\})]. \end{aligned} \tag{2.27}$$

Since $\mathbb{P}_t U(x,\{v\}) = \int_0^\infty \mathbb{P}^x_{t+s}(\{v\})ds \to U(x,\{v\})$ as $t \to 0$. Then $U(\cdot,\{v\})$ is an excessive function on $\mathbb{R}^d$.

Denote by $p(t,x,y)$ the transition density function of $\bar{X}_t$. By (2.1) and (2.6) in [26], the transition density of $\bar{X}_t$ for $t > 0$, there exists constants $\bar{c}_k, k = 1,2,3$ depending on $(d,\beta,\gamma,\Theta_1,\Theta_2,\Theta_3)$ such that

$$\bar{c}_1^{-1} e^{-c_1 t} p_0(t,\bar{c}_2 x,\bar{c}_2 y) \le p(t,x,y) \le \bar{c}_1 e^{\bar{c}_1 t} p_1(t,\bar{c}_3 x,\bar{c}_3 y) + t\|J_1\|_\infty,$$

where $p_0(t,x,y)$ and $p_1(t,x,y)$ are the transition density functions for Δ and $\Delta^{\beta/2}$ respectively, and $J_1 = \int_{\mathbb{R}^d} \mathbb{1}_{\{|z|\ge 1\}} b_2(x,z) j_0(z)dz$.

Then for any $x \in \mathbb{R}^d$, $\mathbb{P}^x_t(\{v\}) = \int_{\{v\}} p(t,x,y)dy = 0$ for each $t > 0$, and thus $U(x,\{v\}) = 0$ for every $x \in \mathbb{R}^d$. Also, by (2.19),(2.20),(2.26), when $x = v$ in (2.27),

$$U(v,\{v\}) \ge \int_{\{v\}} G_{B(v,r)}(v,y)dy \ge K_0{}^{-1} \int_{\{v\}} G^{\Delta}_{B(v,r)}(v,y)dy = \infty.$$

Then by [11, Proposition 3.14], $\{v\}$ is a polar set. □

Let $\mathcal{F}_{\tau^-}$ be the $\sigma-$field of events of Y strictly prior to the stopping time τ. That is, $\mathcal{F}_{\tau^-}$ is $\sigma-$field generated by $\mathcal{F}_0$ and the sets $A \cap \{\tau > t\}$ for $A \in \mathcal{F}_t$ and $t > 0$.

In the following Lemma 2.2.14-Theorem 2.2.21, unless we mentioned in particular, we will assume that $\mathcal{L} \in \mathcal{N}(\Theta_1, \Theta_2, \Theta_3, \Theta_4, \gamma, \beta, c_1)$ and $(\bar{X}_t, t \geq 0; \mathbb{P}^x, x \in \mathbb{R}^d)$ is the Hunt process corresponding to $\mathcal{L}$ and $\widetilde{X}$ is the subprocess of $\bar{X}$ with killing rate $-q$, such that $\|q\|_\infty = \Theta_5$.

Lemma 2.2.14. *There exists a constant* $r_5 = r_5(d, \beta, \gamma, \Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5) < 1$ *such that for any* $r \in (0, r_5), x_0 \in \mathbb{R}^d$*, we have*

$$1/2 \leq \mathbb{E}_v^x[\exp(\int_0^{\tau_{B(x_0,r)\setminus\{v\}}} |q|(\bar{X}_s)ds)] \leq 2, \ \textit{for } v \in B(x_0, r), x \in B(x_0, r) \setminus \{v\}, \quad (2.28)$$

and

$$1/2 \leq \mathbb{E}_z^x[\exp(\int_0^{\tau_B} |q|(\bar{X}_s)ds)] \leq 2, \ \textit{for } x \in B(x_0, r), z \in \partial B(x_0, r). \quad (2.29)$$

where $\tau_{B(x_0,r)\setminus\{v\}} = \inf\{t > 0 : \bar{X}_t \notin B(x_0, r) \setminus \{v\}\}$*,* $\tau_B = \inf\{t > 0 : \bar{X}_t \notin B(x_0, r)\}$*, and* $\mathbb{P}_v^x, \mathbb{P}_z^x$ *are defined in* (2.25)*.*

Proof. Fix $x_0 \in \mathbb{R}^d$. By Lemma 2.2.8 with $D = B(x_0, 1)$, then there exists a constant $\varepsilon_2 = \varepsilon_2(d, \beta, \gamma, \Theta_1, \Theta_2, \Theta_3, \Theta_4) > 0$ and $K_0 = K_0(d, \gamma, \Theta_1) > 0$ such that for any $r \in (0, \varepsilon_2)$, satisfies (2.21) with $\lambda = r$. Since $B(x_0, r) = r(D - x_0) + x_0$, then by denoting $B = B(x_0, r)$, the Green function $G_B(\cdot, \cdot)$ of $\bar{X}$ in B satisfies

$$K_0^{-1} G_B^\Delta(x, y) \leq G_B(x, y) \leq K_0 G_B^\Delta(x, y), \text{ for } x \in B(x_0, r), y \in B(x_0, r) \setminus \{x\}.$$

Then for any $x \in B$, $v \in B \setminus \{x\}$, by Lemma 2.2.13, and the definition of the polar set and the fact that $\{v\} \in \mathcal{B}^n(\mathbb{R}^d)$, $\tau_{B\setminus\{v\}} = \tau_B \quad a.s.$

Then by the definition of $\mathbb{E}_v^x$, (2.21),

$$\begin{aligned} \mathbb{E}_v^x[\int_0^{\tau_{B\setminus\{v\}}} |q|(\bar{X}_s)ds] &= (G_B(x, v))^{-1}\mathbb{E}^x[\int_0^{\tau_B} G_B(\bar{X}_s, v)|q|(\bar{X}_s)ds] \\ &= \int_B \frac{G_B(x, y)G_B(y, v)}{G_B(x, v)}|q|(y)dy \\ &\leq K_0^3 \int_B \frac{G_B^\Delta(x, y)G_B^\Delta(y, v)}{G_B^\Delta(x, v)}|q|(y)dy. \quad (2.30) \end{aligned}$$

Then by Lemma 2.2.11, there exists a constant $r_5 \in (0, \varepsilon_2)$ depending on (d, K_0, Θ_5) such that for any $r \in (0, r_5)$,

$$K_0^3 \int_B \frac{G_B^\Delta(x, y)G_B^\Delta(y, v)}{G_B^\Delta(x, v)}|q|(y)dy \leq 1/2, \quad (2.31)$$

so that by Jenson's inequality, we have for any $r \in (0, r_5), x \in B(x_0, r), v \in B(x_0, r) \setminus \{x\}$,

$$\mathbb{E}_v^x[\exp(\int_0^{\tau_{B\setminus\{v\}}} q(\bar{X}_s)ds)] \geq \exp(\mathbb{E}_v^x[-\int_0^{\tau_{B\setminus\{v\}}} |q|(\bar{X}_s)ds]) \geq e^{1/2} > 1/2.$$

Then by (2.30),(2.31) and Khas'minskii's Lemma in [31, Lemma 3.7], we have

$$\begin{aligned}\mathbb{E}_v^x[\exp(\int_0^{\tau_{B\setminus\{v\}}} q(\bar{X}_s)ds)] &\leq \mathbb{E}_v^x[\exp(\int_0^{\tau_{B\setminus\{v\}}} |q|(\bar{X}_s)ds)] \\ &\leq \frac{1}{1-\sup_{x\in B(x_0,r)} \mathbb{E}_v^x[\int_0^{\tau_{B\setminus\{v\}}} |q|(\bar{X}_s)ds]} \leq 2.\end{aligned}$$

For any $x_0 \in \mathbb{R}^d$, by [26, Lemma 3.2] with $D = B(x_0, 1)$ and the fact $B(x_0, r) = r(D - x_0) + x_0$ for any $r \in (0, \varepsilon_2)$, where ε_2 is the constant in the above, denoting $B = B(x_0, r)$, for any $x \in B$, $z \in \partial B$, $M_B(x, z) := \lim_{y\in B, y\to z} G_B(x, y)/G_B(x', y)$ exists.

Then by the definition of $\mathbb{E}_z^x$, and dominated convergence theorem,

$$\begin{aligned}\mathbb{E}_z^x[\int_0^{\tau_B} |q|(\bar{X}_s)ds] &= (M_B(x,y))^{-1}\mathbb{E}^x[\int_0^{\tau_B} M_B(\bar{X}_s, z)|q|(\bar{X}_s)ds] \\ &= \int_B \frac{G_B(x,v)M_B(v,z)}{M_B(x,z)}|q|(v)dv \\ &= \int_B \lim_{y\in B, y\to z} \frac{G_B(x,v)G_B(v,y)}{G_B(x,y)}|q|(v)dv \\ &\leq \lim_{y\in B, y\to z} \int_B \frac{G_B(x,v)G_B(v,y)}{G_B(x,y)}|q|(v)dv. \qquad (2.32)\end{aligned}$$

Following the above similar argument, by (2.32), Lemma 2.2.8 with $D = B(x_0, 1)$ and $\lambda = r$, Lemma 2.2.11, Jenson's inequality and Khas'minskii's Lemma in [78, Lemma 3.7], for any $r \in (0, r_5)$,

$$1/2 \leq \mathbb{E}_z^x[\exp(\int_0^{\tau_B} q(\bar{X}_s)ds)] \leq 2, \text{ for } x \in B(x_0, r), z \in \partial B(x_0, r). \qquad (2.33)$$

□

Next we will establish the lemma on the scale-invariant estimate for the Green function $\widetilde{G}_{B(x_0,r)}$ of $\mathcal{L} + q$ in a ball $B(x_0, r)$ for any $r \in (0, r_5)$.

Lemma 2.2.15. *There exists a constant $C_0 = C_0(d, \gamma, \Theta_1) > 1$ such that for any $x_0 \in \mathbb{R}^d$, $r \in (0, r_5)$, the Green function $\widetilde{G}_{B(x_0,r)}(\cdot, \cdot)$ for $\mathcal{L} + q$ in $B(x_0, r)$ exists and satisfies that*

$$C_0^{-1} G^{\Delta}_{B(x_0,r)}(x,y) \leq \widetilde{G}_{B(x_0,r)}(x,y) \leq C_0 G^{\Delta}_{B(x_0,r)}(x,y), \quad x \in B(x_0,r), y \in B(x_0,r) \setminus \{x\}, \qquad (2.34)$$

where r_5 is the constant in Lemma 2.2.14 and $C_0 = 2K_0$ with K_0 being the constant in Lemma 2.2.8.

Proof. Fix $x_0 \in \mathbb{R}^d$.Then by Lemma 3.5 in [16] with $G(x,y) = G_{B(x_0,r)}(x,y)$ and $\zeta^y = \tau_{B(x_0,r)\setminus\{y\}}$,

$$\widetilde{G}_{B(x_0,r)}(x,y) \quad = \quad G_{B(x_0,r)}(x,y)\mathbb{E}^x_y[\exp(\int_0^{\tau_{B(x_0,r)\setminus\{y\}}} |q|(\bar{X}_s)ds)],$$

where $G_{B(x_0,r)}(\cdot,\cdot)$ is the Green function for $\mathcal{L}$ in $B(x_0,r)$. Then the conclusion follows by Lemma 2.2.8 and (2.28). □

Corollary 2.2.16. *For any $\mathcal{G} \in \mathcal{N}(c_0, m, \gamma, \beta, \Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, c_1, \vartheta)$, let $((X_t, \Lambda_t), t \geq 0; \mathbb{P}^{(x,i)}, (x,i) \in \mathbb{R}^d \times \mathcal{M})$ be the solution solving the martingale problem $(\mathcal{G}, C_b^2(\mathbb{R}^d \times \mathcal{M}))$. Then there exists a constant $c_1 = c_1(d,\gamma,\Theta_1) > 0$ and $c_2 = c_2(\varepsilon, d, \gamma, \Theta_1) > 0$ such that for any $x_0 \in \mathbb{R}^d, r \in (0,1), \varepsilon \in (0,1), (x,i) \in B(x_0,(1-\varepsilon)r) \times \mathcal{M}$,*

$$c_2 r^2 \leq \mathbb{E}^{(x,i)}[\tau_{B(x_0,r)} \wedge \tau_1] \leq c_1 r^2.$$

Proof. Let $\{(X,\Lambda); \mathbb{P}^{(x,i)}(x,i) \in \mathbb{R}^d \times \mathcal{M}\}$ be the strong Markov process corresponding to the operator $\mathcal{G} \in \mathcal{N}(c_0, m, \gamma, \beta, \Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, c_1, \vartheta)$. And let $\tau^B_{B(x_0,r)}$ be the exit time of the ball $B(x_0,r)$ for a Brownian motion B starting from x. Recall that τ_1 is defined in (1.19). Then by (2.34), there exist a constant $c_2 > 0$ depending only on $(\varepsilon, d, \gamma, \Theta_1)$ such that

$$\begin{aligned} \mathbb{E}^{(x,i)}[\tau_{B(x_0,r)} \wedge \tau_1] &= \int_{\mathbb{R}^d} \widetilde{G}^i_{B(x_0,r)}(x,y)dy \\ &\geq \int_{\mathbb{R}^d} C_0^{-1} G^{\Delta}_{B(x_0,r)}(x,y)dy \\ &= C_0^{-1}\mathbb{E}^x[\tau^B_{B(x_0,r)}] \\ &\geq c_2 r^2. \end{aligned} \tag{2.35}$$

Similarly, by (2.34), there exists a constant $\widetilde{c}_1 > 0$ depending only on (d,γ,Θ_1), for any $(x,i) \in B(x_0,r) \times \mathcal{M}$,

$$\begin{aligned} \mathbb{E}^{(x,i)}[\tau_{B(x_0,r)} \wedge \tau_1] &\leq C_0\mathbb{E}^x[\tau^B_{B(x_0,r)}] \\ &\leq \widetilde{c}_1 r^2. \end{aligned} \tag{2.36}$$

□

Then similar like [26, Lemma 3.3], by the Lévy system formula, we have the following lemma.

Lemma 2.2.17. *For any $x_0 \in \mathbb{R}^d, r \in (0, r_5)$, any non-negative function h defined on $\mathbb{R}^d$, we have for every $x \in B(x_0, r)$,*

$$\mathbb{E}^x[h(\widetilde{X}_{\tau_{B(x_0,r)}}), \widetilde{X}_{\tau_{B(x_0,r)}-} \neq \widetilde{X}_{\tau_{B(x_0,r)}}] = \int_{\overline{B(x_0,r)}^c} \int_{B(x_0,r)} \widetilde{G}_{B(x_0,r)}(x,y) J^{b_2}(y,u) dy h(u) du, \tag{2.37}$$

where $J^{b_2}(y,u) = b_2(y, u-y) j_0(y-u)$ and r_5 is the constant in Lemma 2.2.15.

□

Denote by $M^{\Delta}_{B(x_0,r)}(x,z)$ the Martin kernel of the Laplacian operator Δ in $B(x_0,r)$, $r \in (0, r_5), x \in B(x_0, r), z \in \partial B(x_0, r)$. In the following, we will obtain the existence and the scale-invariant estimate for the Martin kernel $\widetilde{M}_{B(x_0,r)}(\cdot,\cdot)$ for $\mathcal{L}+q$ in $B(x_0,r)$ for any $r \in (0, r_5)$.

Lemma 2.2.18. *For any $x_0 \in \mathbb{R}^d$, $r \in (0, r_5)$, any $x \in B(x_0,r), z \in \partial B(x_0,r)$, $\widetilde{M}_{B(x_0,r)}(x,z) = \lim_{y\to z} \frac{\widetilde{M}_{B(x_0,r)}(x,y)}{\widetilde{M}_{B(x_0,r)}(x',y)}$ exists and satisfies that*

$$C_0^{-2} M^{\Delta}_{B(x_0,r)}(x,z) \leq \widetilde{M}_{B(x_0,r)}(x,z) \leq C_0^2 M^{\Delta}_{B(x_0,r)}(x,z), \quad x \in B(x_0,r), z \in \partial B(x_0,r), \tag{2.38}$$

where r_5 and C_0 are the constants in Lemma 2.2.15.

Proof. The proof of existence of follows by the definition of $\mathbb{E}^x_y$ in (2.25) and [26, Lemma 3.1, Lemma 3.2]. And the inequality (2.38) follows by Lemma 2.2.15. □

Following the proof of [25, Lemma 3.4], we can obtain the following lemma on the harmonicity for the killed Martin kernel $\widetilde{M}_{B(x_0,r)}(\cdot,z)$ with respect to $\mathcal{L}+q$ in small balls.

Lemma 2.2.19. *For any $x_0 \in \mathbb{R}^d, r \in (0, r_5)$, for any $z \in \partial B(x_0,r)$, $\widetilde{M}_{B(x_0,r)}(\cdot,z)$ is harmonic with respect to $\mathcal{L}+q$ in $B(x_0,r)$.*

Similar like [26, Lemma 3.6], we obtain the following theorem.

Theorem 2.2.20. *For any $x_0 \in \mathbb{R}^d, r \in (0, r_5)$, the Martin boundary and minimal Martin boundary of $B(x_0,r)$ for $\mathcal{L}+q$ can all be identified with its Euclidean boundary $\partial B(x_0,r)$.*

Proof. The proof follows by Lemma 2.2.18 and [18, Theorem 4.4]. □

Hence, following the similar arguement in [26, Theorem 3.8], by Lemma 2.2.15, Lemma 2.2.17 and Theorem 2.2.20, we have the following theorem.

Theorem 2.2.21. *For $x_0 \in \mathbb{R}^d, r \in (0, r_5)$, if $\widetilde{h}$ is a nonnegative function defined in $\mathbb{R}^d$ that is harmonic with with respect to $\mathcal{L} + q$ in $B(x_0, r)$, there exists a unique measure $\mu_{\widetilde{h}}$ on $\partial B(x_0, r)$ such that for any $x \in B(x_0, r)$,*

$$\widetilde{h}(x) = \int_{\partial B(x_0,r)} \widetilde{M}_{B(x_0,r)}(x, z)\widetilde{h}(z)\mu_{\widetilde{h}}(dz) + \int_{\overline{B(x_0,r)}^c} \int_{B(x_0,r)} \widetilde{G}_{B(x_0,r)}(x, y)J^{b_2}(y, z)dy\widetilde{h}(z)dz. \tag{2.39}$$

Next, given an operator $\mathcal{G} \in \mathcal{N}(c_0, m, \gamma, \beta, \Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, c_1, \vartheta)$, for any ball in $\mathbb{R}^d$ with $r(B) \le r_5$, define $\widetilde{G}_B(\cdot, \cdot)$ on $B \times B$ and another $m \times m$ matrix function $\widetilde{Q} : \mathbb{R}^d \mapsto \mathbb{R}^{m\times m}$ by

$$\widetilde{G}_B(\cdot, \cdot) = \begin{pmatrix} \widetilde{G}^1_B(\cdot, \cdot) & 0 & \dots & 0 \\ 0 & \widetilde{G}^2_B(\cdot, \cdot) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \widetilde{G}^m_B(\cdot, \cdot) \end{pmatrix}, \qquad \widetilde{Q} = \begin{pmatrix} 0 & q_{12} & \cdots & q_{1m} \\ q_{21} & 0 & \cdots & q_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ q_{m1} & q_{m2} & \cdots & 0 \end{pmatrix}, \tag{2.40}$$

where $\widetilde{G}^k_B(\cdot, \cdot)$ is the Green function of $\mathcal{L}_k + q_{kk}$ on B, and r_5 is the constant in Lemma 2.2.14. Then $\widetilde{G}_B(\cdot, \cdot)$ is the Green function of the non-coupled operator $\widetilde{\mathcal{L}}$,where

$$\widetilde{\mathcal{L}} = \begin{pmatrix} \mathcal{L}_1 + q_{11} & 0 & \dots & 0 \\ 0 & \mathcal{L}_2 + q_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{L}_m + q_{mm} \end{pmatrix}. \tag{2.41}$$

By abusing the notation, let us denote by $\widetilde{G}_B$ the Green operator of $\widetilde{\mathcal{L}}$ and $\widetilde{G}^k_B$ the Green operator with respect to $\mathcal{L}_k + q_{kk}$ on B with zero Dirichlet boundary condition.

To develop a representation of u in terms of $\{h(\cdot, i)\}_{i\in\mathcal{M}}$ in (2.15), we need the following lemma on the boundedness of the operator $\widetilde{G}_B\widetilde{Q}$.

Lemma 2.2.22. *Let $\mathcal{G} \in \mathcal{N}(c_0, m, \gamma, \beta, \Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, c_1, \vartheta)$.Then there exists a $\delta_3 = \delta_3(d, m, \beta, \gamma, \Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5) > 0$ such that for any $x_0 \in \mathbb{R}^d, r \in (0, \delta_3]$, $\widetilde{G}_{B(x_0,r)}\widetilde{Q}$ is a bounded operator on $((B(\overline{B(x_0, r)} \times \mathcal{M})), \|\cdot\|_\infty)$ with*

$$\|\widetilde{G}_{B(x_0,r)}\widetilde{Q}\|_\infty < 1/4. \tag{2.42}$$

Proof. Fix $x_0 \in \mathbb{R}^d$. When $d \geq 3$, By Lemma 2.2.6, for any $x_0 \in \mathbb{R}^d, r > 0$,

$$\sup_{x \in B(x_0,r)} \int_{B(x_0,r)} G^{\Delta}_{B(x_0,r)}(x, y)dy \leq \omega_d K_1 r^2, \tag{2.43}$$

where ω_d is the volume of d-dimensional unit sphere.

When $d = 2$, since $\ln(1+z) \leq 1+z$ for $z \geq 0$, then by Lemma 2.2.7, for any $x_0 \in \mathbb{R}^d, r \in (0, 1)$,

$$\sup_{x \in B(x_0,r)} \int_{B(x_0,r)} G^{\Delta}_{B(x_0,r)}(x, y)dy \leq (\omega_d r^2)(1 + 4\ln 2)/(2\pi) \leq \omega_d K_1 r^2, \tag{2.44}$$

Let

$$\delta_3 = (1/\sqrt{4(m-1)\omega_d C_0(C_G + K_1)\Theta_5}) \wedge (r_5/2), \tag{2.45}$$

where C_G, r_5 and C_0 are the constants in Lemma 2.2.11, Lemma 2.2.14 and Lemma 2.2.15 respectively.Then for any $r \in (0, \delta_3]$, $u \in (B(B(x_0, r) \times \mathcal{M}))^m$, it follows that for any $k, l \in \mathcal{M}$, by (2.14),(2.34), (2.43),(2.44),

$$\begin{aligned}
\left|[\widetilde{G}^k_{B(x_0,r)}\widetilde{Q}u]_k(x)\right| &= \Big|\sum_{l=1, l\neq k}^{l=m} \int_{B(x_0,r)} \widetilde{G}^k_{B(x_0,r)}(x, y)q_{kl}(y)u(y, l)dy\Big| \\
&\leq C_0\Theta_5(m-1)\|u\|_\infty \int_{B(x_0,r)} G^{\Delta}_{B(x_0,r)}(x, y)dy < \|u\|_\infty/4.
\end{aligned}$$

Therefore, for any $r \in (0, \delta_3)$, $\widetilde{G}_{B(x_0,r)}\widetilde{Q}$ is a bounded operator from $((\mathcal{B}(\overline{B}(x_0, r) \times \mathcal{M}))^m$ to itself such that $\|\widetilde{G}_{B(x_0,r)}\widetilde{Q}\|_\infty < 1/4$. Here $\mathcal{B}(\overline{B}(x_0, r) \times \mathcal{M})$ is the space of bounded functions defined on $\overline{B}(x_0, r) \times \mathcal{M}$.

□

Proposition 2.2.23. *Let $\mathcal{G} \in \mathcal{N}(c_0, m, \gamma, \beta, \Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, c_1, \vartheta)$ and $((X_t, \Lambda_t), t \geq 0; \mathbb{P}^{(x,i)}, x \in \mathbb{R}^d \times \mathcal{M}))$ be the Hunt process corresponding to the operator $\mathcal{G}$. For any function*

u defined in (2.13) with $D = B(x_0, r) \times \mathcal{M}$, where $x_0 \in \mathbb{R}^d, r > 0$, we have for any integer $K \in \mathbb{N}$, any $(x, i) \in B(x_0, r) \times \mathcal{M}$,

$$\begin{aligned} u(x,i) &= h(x,i) + \sum_{k=1}^{K} \sum_{\substack{l_1,\ldots,l_k=1,\\ l_1\neq i, l_2\neq l_1,\ldots,l_k\neq l_{k-1}}}^{m} \widetilde{G}^i_{B(x_0,r)}(q_{il_1}(\widetilde{G}^{l_1}_{B(x_0,r)} q_{l_1l_2}(\ldots(\widetilde{G}^{l_{k-1}}_{B(x_0,r)} q_{l_{k-1}l_k} h(\cdot, l_k)\ldots)))(x) \\ &+ \sum_{\substack{l_1,\ldots,l_K=1,\\ l_1\neq i, l_2\neq l_1,\ldots,l_K\neq l_{K-1}}}^{m} \widetilde{G}^i_{B(x_0,r)}(q_{il_1}(\widetilde{G}^{l_1}_{B(x_0,r)} q_{l_1l_2}(\ldots(\widetilde{G}^{l_{K-1}}_{B(x_0,r)} q_{l_{K-1}l_K} u(\cdot, l_K))\ldots)))(x), \end{aligned} \tag{2.46}$$

where $h(x, i)$ is defined in (2.15).

When ϕ in (2.13) is bounded, for any $x_0 \in \mathbb{R}^d, r \in (0, \delta_3]$, we have for any $(x, i) \in B(x_0, r) \times \mathcal{M}$,

$$u(x,i) = h(x,i) + \sum_{k=1}^{\infty} \sum_{\substack{l_1,\ldots,l_k=1,\\ l_1\neq i, l_2\neq l_1,\ldots,l_k\neq l_{k-1}}}^{m} \widetilde{G}^i_{B(x_0,r)}(q_{il_1}(\widetilde{G}^{l_1}_{B(x_0,r)} q_{l_1l_2}(\ldots(\widetilde{G}^{l_{k-1}}_{B(x_0,r)} q_{l_{k-1}l_k} h(\cdot, l_k))\ldots)))(x), \tag{2.47}$$

and we have

$$u(x,i) = h(x,i) + \sum_{n=1}^{\infty} \sum_{l\in\mathcal{M}} \sum_{(i,l_1,\ldots,l_n)\in\Psi(n;i,l)} \widetilde{G}^i_B(q_{il_1}(\widetilde{G}^{l_1}_B q_{l_1l_2}(\ldots(\widetilde{G}^{l_{n-1}}_B q_{l_{n-1}l_n} h(\cdot, l_n))\ldots)))(x), \tag{2.48}$$

if we represent (2.47) in terms of valid paths in each $\Psi(n; i, l)$. Here δ_3 is the constant in Lemma 2.2.22 and $\Psi(n; i, l)$ is defined in (1.16).

Proof. Define $\tau_2 = \inf\{t > \tau_1 : \Lambda_t \neq \Lambda_{\tau_1}\}$, where τ_1 is defined in (1.19). Then by Proposition 2.2.5, and the strong Markov property of (X, Λ), we have for any $(x, i) \in B(x_0, r) \times \mathcal{M}$,

$$\begin{aligned} u(x,i) &= h(x,i) + \sum_{j=1,j\neq i}^{m} \widetilde{G}^i_{B(x_0,r)}(q_{ij} u(\cdot, j))(x) \\ &= h(x,i) + \sum_{j=1,j\neq i}^{m} \mathbb{E}^{(x,i)}[\int_0^{\tau_{B(x_0r)}\wedge\tau_1} q_{ij} u(X_s, j) ds] \\ &= h(x,i) + \sum_{j=1,j\neq i}^{m} \mathbb{E}^{(x,i)}[\int_0^{\tau_{B(x_0r)}\wedge\tau_1} q_{ij}(h_j(X_s) + \sum_{l=1,l\neq j}^{m} (\widetilde{G}^j_{B(x_0,r)} q_{jl} u(\cdot, l))(X_s)) ds] \end{aligned}$$

$$= \quad h(x,i) + \sum_{j=1,j\neq i}^{m} \widetilde{G}^i_{B(x_0,r)}(q_{ij}h(\cdot,j))(x) + \sum_{j,l=1,j\neq l,j\neq i}^{m} \widetilde{G}^i_{B(x_0,r)}(q_{ij}(\widetilde{G}^j_{B(x_0,r)}q_{jl}u(\cdot,l)))(x).$$

Iterating (2.14) for $K-1$ times as above, we obtain (2.46). Then by (2.40), if we represent (2.46) in terms of vectors, we have for any $x \in B(x_0,r)$,

$$u(x) = h(x) + \sum_{k=1}^{K}(\widetilde{G}_{B(x_0,r)}\widetilde{Q})^k h(x) + (\widetilde{G}_{B(x_0,r)}\widetilde{Q})^K u(x), \tag{2.49}$$

where $u(x) = [u(x,1),...,u(x,m)]^T$, and $h(x) = [h(x,1),...,h(x,i)]^T$.

If ϕ is bounded, then by (2.13), u is bounded. Then by Lemma 2.2.22, for any $x_0 \in \mathbb{R}^d, r \in (0,\delta_3]$, any bounded and nonnegative function u on $B(x_0,r)$,

$$\lim_{K\to\infty}(\widetilde{G}_{B(x_0,r)}\widetilde{Q})^K u(x) = \vec{0},$$

where $\vec{0}$ represents the zero vector of dimension m. Therefore,

$$u(x) = h(x) + \sum_{k=1}^{\infty}(\widetilde{G}_{B(x_0,r)}\widetilde{Q})^k h(x), \quad x \in B(x_0,r),$$

i.e., for any $(x,i) \in B(x_0,r)\times\mathcal{M}$,

$$u(x,i) = h(x,i) + \sum_{k=1}^{\infty} \sum_{\substack{l_1,...,l_n=1, \\ l_1\neq i, l_2\neq l_1,...l_{k-1}\neq l_k}}^{m} \widetilde{G}^i_B(q_{il_1}(\widetilde{G}^{l_1}_B q_{l_1l_2}(...(\widetilde{G}^{l_{k-1}}_B q_{l_{k-1}l_n}h(\cdot,l_k))...)))(x).$$

If we represent it in terms of valid paths,

$$u(x,i) = h(x,i) + \sum_{n=1}^{\infty}\sum_{l\in\mathcal{M}} \sum_{(i,l_1,...,l_n)\in\Psi(n;i,l)}^{m} \widetilde{G}^i_B(q_{il_1}(\widetilde{G}^{l_1}_B q_{l_1l_2}(...(\widetilde{G}^{l_{n-1}}_B q_{l_{n-1}l_n}h(\cdot,l_n))...)))(x).$$

□

2.3 Hölder regularity

Proof of Theorem 2.1.3.

For any $x_0 \in \mathbb{R}^d, r \in (0,\widetilde{r}_0)$, denote $B = B(x_0, 2r/3)$, where $\widetilde{r}_0$ will be determined later. For any $i \in \mathcal{M}$, let $\{X^i_s, s\geq 0; \mathbb{P}^x, x \in \mathbb{R}^d\}$ be the Hunt process corresponding to the operator $\mathcal{L}_i$ starting from x. Define $\tau^i_B = \inf\{t > 0 : X^i_t \notin B\}$.

First we claim that for any $x \in B$,

$$h_i(x) = \widetilde{u}_i(x) + G_B^i(q_{ii}h_i)(x),$$

where $\widetilde{u}_i(x) = \mathbb{E}^x[u(X^i_{\tau^i_B}, i)]$, G^i_B is the Green operator for $\mathcal{L}_i$ with zero Dirichlet boundary condition. By the definition of harmonic functions with respect to $\mathcal{L}_i + q_{ii}$,

$$\begin{aligned}
h_i(x) &= \mathbb{E}^x[\exp(\int_0^{\tau^i_B} q_{ii}(X^i_s)ds)u(X^i_{\tau^i_B}, i)] \\
&= \widetilde{u}_i(x) + \mathbb{E}^x[(\exp(\int_0^{\tau^i_B} q_{ii}(X^i_s)ds) - 1)u(X^i_{\tau^i_B}, i)] \\
&= \widetilde{u}_i(x) + \mathbb{E}^x[\int_0^{\tau^i_B}(\exp(\int_s^{\tau^i_B} q_{ii}(X^i_t)dt))q_{ii}(X^i_s)u(X^i_{\tau^i_B}, i)ds] \\
&= \widetilde{u}_i(x) + \mathbb{E}^x[\int_0^{\tau^i_B} q_{ii}(X^i_s)\mathbb{E}^{X^i_s}[\exp(\int_0^{\tau^i_B} q_{ii}(X^i_t)dt)u(X^i_{\tau^i_B}, i)]ds] \\
&= \widetilde{u}_i(x) + \mathbb{E}^x[\int_0^{\tau^i_B} q_{ii}(X^i_s)h_i(X^i_s)ds] \\
&= \widetilde{u}_i(x) + \int_B G^i_B(x,y)q_{ii}(y)h_i(y)dy. \\
&= \widetilde{u}_i(x) + G^i_B q_{ii} h_i(x), \qquad (3.1)
\end{aligned}$$

Notice that by Lemma 2.36, for any $x \in B(x_0, r/2), r \in (0, r_0)$,

$$|\int_B G^i_B(x,y)q_{ii}(y)h_i(y)dy| \leq 4\widetilde{c}_1\Theta_5\|u\|_\infty r^2/9, \qquad (3.2)$$

where r_0 is the constant in Lemma 2.36. We will get the similar result when x is replaced by y.

Since by Theorem 1.1.1, there exists a constant $C^i_3 = C^i_3$, $r_4 \in (0, 1/4)$ and $\alpha \in (0,1)$ which both depend on$(d, \Theta_1, \Theta_2, \Theta_3, \Theta_4, \beta)$ such that for any $x_0 \in \mathbb{R}^d, r \in (0, r_4)$, any bounded function $\widetilde{u}_i$ defined in $\mathbb{R}^d$ that is harmonic with respect to $\mathcal{L}_i$ in $B(x_0, r)$,

$$(\widetilde{u}_i(x) - \widetilde{u}_i(y))| \leq C^i_3\|u_i\|_\infty(\frac{\|x-y\|}{r})^\alpha, \text{for } x, y \in B(x_0, r/2). \qquad (3.3)$$

Let $\widetilde{r}_0 = r_0 \wedge r_4$. Therefore by (3.1), (3.2), (3.3), for any $r \in (0, \widetilde{r}_0) \subset (0, 1/4), x, y \in B(x_0, r/2)$,

$$\begin{aligned}
|h_i(x) - h_i(y)| &\leq |\widetilde{u}_i(x) - \widetilde{u}_i(y)| + |G^i_B q_{ii} h_i(x)| + |G^i_B q_{ii} h_i(y)| \\
&\leq C^i_3\|u\|_\infty(\frac{|x-y|}{r})^\alpha + 8\widetilde{c}_1 r^2\Theta_5\|u\|_\infty/9. \qquad (3.4)
\end{aligned}$$

Set $r = |x-y|^{1/2}$, we have

$$\begin{aligned}
|h_i(x)-h_i(y)| &\le |\widetilde{u}_i(x)-\widetilde{u}_i(y)| + |G^i_B q_{ii} h_i(x)| + |G^i_B q_{ii} h_i(y)| \\
&\le C^i_3 \|u\|_\infty |x-y|^{\alpha/2} + 8\widetilde{c}_1 |x-y| \Theta_5 \|u\|_\infty /9 \\
&\le (C^i_3 + 8\Theta_5 \widetilde{c}_1 \widetilde{r}_0^{1-\alpha/2}/9)\|u\|_\infty |x-y|^{\alpha/2}. \qquad (3.5)
\end{aligned}$$

For any $x_0 \in \mathbb{R}^d, r \in (0, \widetilde{r}_0)$, denote $\widetilde{B} = B(x_0, 2r/3)$. Define $\tau_{\widetilde{B}} = \inf\{t > 0 : (X_t, \Lambda_t) \notin \widetilde{B} \times \mathcal{M}\}$. By Lemma 2.36, for any $x \in B(x_0, r/2)$, we estimate each term in the second part of (2.14) such that

$$|\widetilde{G}^i_{\widetilde{B}}(q_{ik} u(\cdot,k))(x)| \le 4\widetilde{c}_1 r^2 \Theta_5 \|u\|_\infty/9, \qquad (3.6)$$

Therefore, by (2.14), (3.5), (3.6), for any $r \in (0, \widetilde{r}_0)$, $x, y \in B(x_0, r/2)$, we have

$$\begin{aligned}
\|u(x)-u(y)\| &= \sup_{i\in\mathcal{M}} |u(x,i)-u(y,i)| \\
&\le \sup_{i\in\mathcal{M}} (|h_i(x)-h_i(y)| + \sum_{k=1,k\neq i}^{m} (|G^i_{\widetilde{B}}(q_{ik}u(\cdot,k))(x)| + |G^i_{\widetilde{B}}(q_{ik}u(\cdot,k))(y)|) \\
&\le (\max_{i\in\mathcal{M}} C^i_3 + 8\Theta_5 \widetilde{c}_1 \widetilde{r}_0^{1-\alpha/2}/9)\|u\|_\infty |x-y|^{\alpha/2} + 8(m-1)\widetilde{c}_1 \Theta_5 \|u\|_\infty r^2/9.
\end{aligned}$$

Set $r = |x-y|^{1/2}$, $\alpha_1 = \alpha/2$. We have for any $r \in (0, \widetilde{r}_0), x, y \in B(x_0, r/2)$,

$$\begin{aligned}
\|u(x)-u(y)\| &\le (\max_{i\in\mathcal{M}} C^i_3 + 8\Theta_5 \widetilde{c}_1 \widetilde{r}_0^{1-\alpha/2}/9)\|u\|_\infty |x-y|^{\alpha/2} + 8(m-1)\widetilde{c}_1 \Theta_5 \|u\|_\infty |x-y|/9 \\
&\le (\max_{i\in\mathcal{M}} C^i_3 + 8m\widetilde{c}_1 \widetilde{r}_0^{1-\alpha/2}/9)\Theta_5 \|u\|_\infty |x-y|^{\alpha/2} := C_1 \|u\|_\infty |x-y|^{\alpha/2}. \qquad (3.7)
\end{aligned}$$

2.4 Scale-invariant Harnack inequality for each level

Proposition 2.4.1. *Suppose that the operator $\mathcal{L} \in \mathcal{N}(\Theta_1, \Theta_2, \Theta_3, \Theta_4, \gamma, \beta, c_1)$, and $\{\bar{X}_t, t \ge 0; \mathbb{P}^x, x \in \mathbb{R}^d\}$ is the Hunt process corresponding to the operator $\mathcal{L}$ starting at x. Let $\widetilde{X}$ be the subprocess of $\bar{X}$ with killing rate $-q$ such that $\|q\|_\infty = \Theta_5$. There exists a constant $\widetilde{C}_2 = \widetilde{C}_2(d, \beta, \gamma, \Theta_1, \vartheta) > 0$ such that for any $x_0 \in \mathbb{R}^d, r \in (0, \delta_3]$, any nonnegative functions h defined in $\mathbb{R}^d$ that is harmonic with respect to $\mathcal{L} + q$ in $B(x_0, r)$,*

$$h(y_1) \le C_2 h(y_2), \quad \textit{for any } y_1, y_2 \in B(x_0, r/4). \qquad (4.1)$$

Here δ_3 is the constant in Lemma 2.2.22.

Proof. For simplicity, we just give a proof for the $d \geq 3$, the case for $d = 2$ is similar by using Lemma 2.2.7 instead of Lemma 2.2.6. This follows the same idea of [26, Theorem 1.7]. We spell out the details here for the reader's convenience.

For any $x_0 \in \mathbb{R}^d, r \in (0, \delta_3]$,where δ_3 is the constant in Lemma 2.2.22, let $\tau_{B(x_0,r)} = \inf\{t > 0 : \bar{X}_t \notin B(x_0, r)\}$. By shrinking the value of r a little bit, we may assume without loss of generality that $h(x) = \mathbb{E}^x\big[h(\widetilde{X}_{\tau_{B(x_0,r)}})\big], x \in B(x_0, r)$. As h is the increasing limit of bounded harmonic functions $h_n(x) := \mathbb{E}^x\big[(h \wedge n)(\widetilde{X}_{\tau_{B(x_0,r)}})\big]$, it suffices to establish the Harnack inequality for bounded nonnegative harmonic functions h in $B(x_0, r)$.

Denote $B_r = B(x_0, r)$. Then by Theorem 2.2.21, there exists a unique measure μ_h on ∂B_r, such that for any for any $x \in B(x_0, r)$,

$$
\begin{aligned}
h(x) &= \int_{\partial B_r} \widetilde{M}_{B_r}(x,z)h(z)\mu_h(dz) + \int_{\overline{B_r}^c} \int_{B_r} \widetilde{G}_{B_r}(x,y)J^{b_2}(y,z)dyh(z)dz \\
&= \int_{\partial B_r} \widetilde{M}_{B_r}(x,z)h(z)\mu_h(dz) + \int_{\overline{B_r}^c} \int_{B_r \setminus B_{r/2}} \widetilde{G}_{B_r}(x,y)J^{b_2}(y,z)dyh(z)dz \\
&\quad + \int_{\overline{B_r}^c} \int_{B_{r/2}} \widetilde{G}_{B_r}(x,y)J^{b_2}(y,z)dyh(z)dz. \\
&= h_1(x) + h_2(x) + h_3(x). \qquad (4.2)
\end{aligned}
$$

Since for any $z \in \partial B_r$, $M^{\Delta}_{B_r}(\cdot, z)$ is harmonic with respect to Δ in $B(x_0, r)$, then there exists a constant $C_2 = C_2(d) > 0$ such that for any $x, x' \in B(x_0, r/2)$,

$$M^{\Delta}_{B_r}(x,z) \leq C_2 M^{\Delta}_{B_r}(x',z). \qquad (4.3)$$

Therefore, by Lemma 2.2.18,

$$h_1(x) \leq C_0^4 C_2 h_1(x') \text{ for any } x, x' \in B(x_0, r/2). \qquad (4.4)$$

Also, by Lemma 2.2.6, there exists a constant $C_3 = C_3(d) > 0$ such that

$$G^{\Delta}_{B_r}(x,y) \leq C_3 G^{\Delta}_{B_r}(x',y) \text{ for any } x, x' \in B(x_0, r/4), y \in B_r \setminus B_{r/2}. \qquad (4.5)$$

Therefore, by Lemma 2.2.15, (4.5), $h_2(x) \leq C_0^2 C_3 h_2(x')$ for any $x, x' \in B(x_0, r/4)$.
Next, on one hand, by the UJS condition (1.15) for J^{b_2}, (2.43), Lemma 2.2.15, for any $x \in B(x_0, r/2)$,

$$h_3(x) = \int_{\overline{B_r}^c} \int_{B_{r/2}} \widetilde{G}_{B_r}(x,y)J^{b_2}(y,z)dyh(z)dz$$

$$
\begin{aligned}
&\leq C_0\vartheta\int_{\overline{B_r}^c}\int_{B_{r/2}} G^{\Delta}_{B_r}(x,y)\frac{1}{|B(y,r/4)|}\int_{B(y,r/4)} J^{b_2}(u,z)dudyh(z)dz \\
&\leq 4^dC_0\vartheta/(\omega_d r^d)\int_{\overline{B_r}^c}(\int_{B_{r/2}} G^{\Delta}_{B_r}(x,y)dy)\int_{B_{3r/4}} J^{b_2}(u,z)duh(z)dz \\
&\leq c_3C_0K_1\vartheta r^{2-d}\int_{\overline{B_r}^c}\int_{B_{3r/4}} J^{b_2}(u,z)duh(z)dz, \qquad (4.6)
\end{aligned}
$$

where ω_d and c_3 are constants depending only on d, and therefore $c_4C_0K_1\vartheta/\omega_d$ depends on $(d,\beta,\gamma,\Theta_1,\vartheta)$, and independent of r.

On the other hand, by Lemma 2.2.6, Lemma 2.2.15, for any $x'\in B(x_0,r/2)$,

$$
\begin{aligned}
h(x') &\geq \mathbb{E}^{x'}[h(\widetilde{X}_{\tau_{B_{7r/8}}}); \widetilde{X}_{\tau_{B_{7r/8}}}\in\overline{B_r}^c] \\
&\geq C_0^{-1}\int_{\overline{B_r}^c}\Big(\int_{B_{3r/4}} G^{\Delta}_{B_{7r/8}}(x',y)J^{b_2}(y,z)dy\Big)h(z)dz \\
&\geq c_4(C_0K_1)^{-1}r^{2-d}\int_{\overline{B_r}^c}\int_{B_{3r/4}} J^{b_2}(y,z)dyh(z)dz, \qquad (4.7)
\end{aligned}
$$

where $c_4(C_0K_1)^{-1}$ depends only on (d,γ,Θ_1) and independent of r.

Therefore by (4.2), (4.3),(4.5),(4.6),(4.7), for any $x,x'\in B(x_0,r/4)$,

$$
h_1(x)+h_2(x)+h_3(x)\leq (C_2+C_3+c_3C_0^2K_1^2\vartheta/c_4)h(x') := \widetilde{C}_2h(x'),
$$

where $\widetilde{C}_2$ depends only on $(d,\beta,\gamma,\Theta_1,\vartheta)$.

□

Proof of Theorem 2.1.4:

Fix $x_0\in\mathbb{R}^d, r\in(0,\widetilde{r}_1]$, with $\widetilde{r}_1=r_0\wedge\delta_3$, where r_0 and δ_3 are the constants in Lemma 2.36 and Lemma 2.2.22 respectively. Notice that $\widetilde{r}_1<r_5<1$ by Lemma 2.2.14 and (2.45).

Similar to the beginning of the argument in Proposition 2.4.1, it suffices to establish the Harnack inequality for bounded nonnegative functions u defined in $\mathbb{R}^d$ that is $\mathcal{G}$-harmonic in $B(x_0,r)$. For any $k\in\mathcal{M}$, let $\widetilde{X}^k$ be the Hunt process corresponding to the operator $\mathcal{L}_k+q_{kk}$ for $\mathcal{L}_k\in\mathcal{N}(\Theta_1,\Theta_2,\Theta_3,\Theta_4,\gamma,\beta,c_1)$.

For any $x_0\in\mathbb{R}^d, r\in(0,\widetilde{r}_1]$, denoting $B=B(x_0,r/2)$, by the definition of $h(x,k)$ and τ_1 in (1.18) and (1.19), (2.7), $h(x,k)$ is harmonic with respect to $\mathcal{L}_k+q_{kk}$ in B, i.e,

$$
h(x,k)=\mathbb{E}^{(x,k)}[u(\widetilde{X}^k_{\tau_B},k)] \text{ for } x\in B.
$$

Then by Theorem 2.2.21, there exists a unique measure μ_u^k on ∂B, such that for any $x \in B$,

$$h(x,k) = \int_{\overline{B}^c} \int_B \widetilde{G}_B^k(x,y) J^{b_2,k}(y,z) dy u(z,k) dz + \int_{\partial B} \widetilde{M}_B^k(x,z) u(z,k) \mu_u^k(dz).$$

Hence by Proposition 2.4.1, there exists a constant $\widetilde{C}_2 = \widetilde{C}_2(d,\beta,\gamma,\Theta_1,\vartheta) > 0$ such that for any $x, x' \in B(x_0, r/8)$,

$$h(x,k) \le \widetilde{C}_2 h(x',k). \tag{4.8}$$

For any $C > 0, x \in B, k, l \in \mathcal{M}, r \in (0, \widetilde{r}_1]$, define

$$\begin{aligned} F_{kl}^I(C;x,z,r) :&= \sum_{n=1}^{\infty} \sum_{\Psi^0(n;k,l)} C^n q^0_{kl_1} q^0_{l_1 l_2} ... q^0_{l_{n-1} l} \int_{B^n} G_B^\Delta(x,y_1) G_B^\Delta(y_1,y_2) ... G_B^\Delta(y_{n-1},y_n) (\int_B \\ &\quad C G_B^\Delta(y_n,y) J^{b_2,l}(y,z) \mathbb{1}_{\overline{B}^c}(z) dy) dy_1 dy_2 ... dy_n, \text{ for } z \in \overline{B}^c, \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} F_{kl}^{II}(C;x,z,r) :&= \sum_{n=1}^{\infty} \sum_{\Psi^0(n;k,l)} C^n q^0_{kl_1} q^0_{l_1 l_2} ... q^0_{l_{n-1} l} \int_{B^n} G_B^\Delta(x,y_1) G_B^\Delta(y_1,y_2) ... G_B^\Delta(y_{n-1},y_n) \\ &\quad (C^2 M_B^\Delta(y_n,z) \mathbb{1}_{\partial B}(z)) dy_1 dy_2 ... dy_n, \text{ for } z \in \partial B. \end{aligned} \tag{4.10}$$

For each $k \in \mathcal{M}$, recall that

$$E(k) = \{l \in \mathcal{M} \setminus k : \Psi^0(n,k,l) \ne \emptyset, n \in \mathbb{N}\}. \tag{4.11}$$

Then by (1.13), Lemma 2.2.15, Lemma 2.2.18, Theorem 2.2.21, (2.48), (4.9),(4.10),(4.11), for any $(x,k) \in B \times \mathcal{M}$, it yields that

$$\begin{aligned} & h(x,k) + \sum_{l \in E(k) \cup \{k\}} (\int_{\overline{B}^c} F_{kl}^I(c_0 C_0^{-1}; x,z,r) u(z,l) dz + \int_{\partial B} F_{kl}^{II}(c_0 C_0^{-1}; x,z,r) u(z,l) \mu_u^l(dz)) \\ \le \; & u(x,k) \le h(x,k) + \sum_{l \in E(k) \cup \{k\}} (\int_{\overline{B}^c} F_{kl}^I(C_0; x,z,r) u(z,l) dz + \int_{\partial B} F_{kl}^{II}(C_0; x,z,r) u(z,l) \mu_u^l(dz)). \end{aligned} \tag{4.12}$$

By the 3G-Lemma (Lemma 2.2.11), for any $x, y \in B$,

$$\begin{aligned} \int_B G_B^\Delta(x,y_1) G_B^\Delta(y_1,y) dy_1 &\le C_G G_B^\Delta(x,y) (\sup_{x \in B} \int_B |x-y_1|^{2-d} dy_1 + \sup_{y \in B} \int_B |y-y_1|^{2-d} dy_1) \\ &\le C_G \omega_d r^2 G_B^\Delta(x,y) =: C_g r^2 G_B^\Delta(x,y), \end{aligned} \tag{4.13}$$

Then by Lemma 2.2.11, (4.13), for any $r \in (0, \widetilde{r}_1], x \in B, z \in \overline{B}^c$, we have

$$\begin{aligned} F^I_{kl}(C_0; x, z, r) &\le \sum_{n=1}^{\infty} \sum_{\Psi^0(n;k,l)} (C_g C_0)^n q^0_{kl_1} q^0_{l_1 l_2} ... q^0_{l_{n-1}l} r^{2n} \Big(\int_B C_0 G^{\Delta}_B(x,y) J^{b_2,l}(y,z) \mathbb{1}_{\overline{B}^c}(z) dy\Big) \\ &\le \sum_{n=1}^{\infty} (C_g C_0 r^2)^n \sum_{\Psi^0(n;k,l)} q^0_{kl_1} q^0_{l_1 l_2} ... q^0_{l_{n-1}l} \Big(\int_B C_0 G^{\Delta}_B(x,y) J^{b_2,l}(y,z) \mathbb{1}_{\overline{B}^c}(z) dy\Big). \end{aligned} \tag{4.14}$$

By Lemma 2.2.6, for any $x, y \in B(x_0, 3r/8), B = B(x_0, r/2)$,

$$G^{\Delta}_B(x,y) \ge K_1^{-1}(3r/4)^{2-d}/36, \tag{4.15}$$

and for any $x \in B(x_0, r/8), y \in B(x_0, 3r/8)$,

$$G^{\Delta}_B(x,y) \ge 3K_1^{-1}(r/2)^{2-d}/16 \ge K_1^{-1}(3r/4)^{2-d}/36. \tag{4.16}$$

Then by (4.15), (4.16), for any $k, l \in \mathcal{M}, r \in (0, \widetilde{r}_1], x \in B, z \in \overline{B}^c$, we have

$$\begin{aligned} & F^I_{kl}(c_0 C_0^{-1}; x, z, r) \\ \ge\ & \sum_{n=1}^{\infty} \sum_{\Psi^0(n;k,l)} c_0^n C_0^{-n} q^0_{kl_1} q^0_{l_1 l_2} ... q^0_{l_{n-1}l} \int_{B(x_0,3r/8)^{n-1}} G^{\Delta}_B(x,y_1) ... \int_{B(x_0,r/8)} G^{\Delta}_B(y_{n-1},y_n) \Big(\int_B C_0^{-1} G^{\Delta}_B(y_n,y) \\ & J^{b_2,l}(y,z) \mathbb{1}_{\overline{B}^c}(z) dy\Big) dy_n dy_{n-1} ... dy_1 \\ \ge\ & \sum_{n=1}^{\infty} \sum_{\Psi^0(n;k,l)} c_0^n C_0^{-n} q^0_{kl_1} q^0_{l_1 l_2} ... q^0_{l_{n-1}l} \Big(K_1^{-1} \int_{B(x_0,3r/8)} ((3r/4)^{2-d}/36) dy_1\Big)^{n-1} \int_{B(x_0,r/8)} (K_1^{-1}(3r/4)^{2-d}/36) \\ & \Big(\int_B C_0^{-1} G^{\Delta}_B(y_n,y) J^{b_2,l}(y,z) \mathbb{1}_{\overline{B}^c}(z) dy\Big) dy_n \\ \ge\ & \sum_{n=1}^{\infty} \sum_{\Psi^0(n;k,l)} c_0^n C_0^{-n} q^0_{kl_1} q^0_{l_1 l_2} ... q^0_{l_{n-1}l} \big((36K_1)^{-1}(3r/4)^{2-d} \cdot \omega_d (r/8)^d\big)^n \Big(\inf_{y_n \in B(x_0,r/8)} \int_B C_0^{-1} G^{\Delta}_B(y_n,y) \\ & J^{b_2,l}(y,z) \mathbb{1}_{\overline{B}^c}(z) dy\Big) \\ \ge\ & \sum_{n=1}^{\infty} (c_0 K_d r^2/(K_1 C_0))^n \sum_{\Psi^0(n;k,l)} q^0_{kl_1} q^0_{l_1 l_2} ... q^0_{l_{n-1}l} \Big(\inf_{y_n \in B(x_0,r/8)} \int_B C_0^{-1} G^{\Delta}_B(y_n,y) J^{b_2,l}(y,z) \mathbb{1}_{\overline{B}^c}(z) dy\Big), \end{aligned} \tag{4.17}$$

where $K_d = K_d(d)$ is some constant such that $K_d \le C_g$ in (4.13).

For any $s \ge 0, k, l \in \mathcal{M}$, define

$$H_{kl}(s) := \sum_{n=1}^{\infty} a_n(kl) s^n, \tag{4.18}$$

where $a_n(kl) = \sum_{\Psi^0(n;k,l)} q^0_{kl_1} q^0_{l_1 l_2} ... q^0_{l_{n-1}l}$. Since each $a_n \le \Theta_5^n (m-1)^n$, then by the choice of δ_3 in (2.45), for any $r \in (0, \widetilde{r}_1] \subset (0, \delta_3)$,

$$H_{kl}(s) \le \sum_{n=1}^{\infty} ((m-1)\Theta_5 s)^n < 2 \text{ for any } s \in (0, C_g C_0 \widetilde{r}_1^2]. \tag{4.19}$$

Then by (4.14), (4.18), for any $r \in (0, \widetilde{r}_1]$, $x \in B, z \in \overline{B}^c, k, l \in \mathcal{M}$,

$$F^I_{kl}(C_0; x, z, r) \quad \le \quad H_{kl}(C_g C_0 r^2)(C_0 \int_B G^\Delta_B(x,y) J^{b_2,l}(y,z) \mathbb{1}_{\overline{B}^c}(z) dy). \tag{4.20}$$

Similarly, by (2.22), (4.10),(4.14), (4.20), for any $r \in (0, \widetilde{r}_1], x \in B, z \in \partial B, k, l \in \mathcal{M}$,

$$F^{II}_{kl}(C_0; x, z, r) \quad \le \quad H_{kl}(C_g C_0 r^2)(C_0^2 M^\Delta_B(x,z) \mathbb{1}_{\partial B}(z)). \tag{4.21}$$

Also, by (4.17), (4.18), for any $r \in (0, \widetilde{r}_1]$, $x \in B, z \in \overline{B}^c, k, l \in \mathcal{M}$,

$$F^I_{kl}(c_0 C_0^{-1}; x, z, r) \quad \ge \quad H_{kl}(c_0 K_d r^2/(K_1 C_0))(\inf_{\widetilde{y} \in B(x_0, r/8)} \int_B C_0^{-1} G^\Delta_B(\widetilde{y}, y) J^{b_2,l}(y,z) \mathbb{1}_{\overline{B}^c}(z) dy) \tag{4.22}$$

Similarly, by (4.10), (4.17) (4.22), for any $r \in (0, \widetilde{r}_1], x \in B, z \in \partial B, k \in \mathcal{M}, l \in E(k)$,

$$F^{II}_{kl}(c_0 C_0^{-1}; x, z, r) \ge H_{kl}(c_0 K_d r^2/(K_1 C_0))(C_0^{-2} \inf_{\widetilde{y} \in B(x_0, r/4)} M^\Delta_B(\widetilde{y}, z) \mathbb{1}_{\partial B}(z)). \tag{4.23}$$

Recall that $m_{kl} := \inf\{n \ge 0 : \Psi^0(n,k,l) \ne \emptyset\}$, the smallest integer n such that one can go from level k to level l in n steps. By (4.19) , for any $s \in (0, C_g C_0 \widetilde{r}_1^2]$,

$$a_{m_{kl}} s^{m_{kl}} \le H_{kl}(s) = a_{m_{kl}} s^{m_{kl}} (1 + o(s)) \le 3 a_{m_{kl}} s^{m_{kl}}, \tag{4.24}$$

where $o(s)$ stands for some function of s converging to 0 as s approaches 0.

Hence by (4.8),(4.12),(4.19), (4.20),(4.21), (4.24), Lemma 2.2.15, Lemma 2.2.18, Theorem 2.2.21, for any $x, x' \in B(x_0, r/8), k \in \mathcal{M}$,

$$\begin{aligned} u(x,k) \quad &\le \quad h(x,k) + \sum_{l \in E(k) \cup \{k\}} H_{kl}(C_g C_0 r^2)(C_0 \int_{\overline{B}^c} \int_B G^\Delta_B(x,y) J^{b_2,l}(y,z) u(z,l) dy dz \\ &\qquad + \int_{\partial B} C_0^2 M^\Delta_B(x,z) u(z,l) \mu^l_u(dz)) \\ &\le \quad (1 + 3 a_{m_{kl}} (C_g C_0 r^2)^{m_{kl}} C_0^4) h(x,k) + \sum_{l \in E(k)} 3 a_{m_{kl}} (C_g C_0 r^2)^{m_{kl}} C_0^4 h(x,l) \end{aligned}$$

$$\leq \widetilde{C}_2(1+3a_{m_{kl}}(C_gC_0r^2)^{m_{kl}}C_0^4)h(x',k)+\sum_{l\in E(k)}3\widetilde{C}_2C_0^4a_{m_{kl}}(C_gC_0)^{m_{kl}}r^{2m_{kl}}h(x',l). \tag{4.25}$$

Let $C_3 = \max_{k\in\mathcal{M}}(\sum_{l\in E(k)\cup\{k\}}3\widetilde{C}_2C_0^4a_{m_{kl}}(C_gC_0)^{m_{kl}}+\widetilde{C}_2)$. Since each m_{kl} is finite, then C_3 is finite. By (4.25), for any $x, x' \in B(x_0, r/8), k \in \mathcal{M}$,

$$u(x,k)\leq C_3(h(x',k)+\sum_{l\in E(k)}r^{2m_{kl}}h(x',l)). \tag{4.26}$$

Since by (4.8) and Theorem 2.2.21, for any $k \in \mathcal{M}$, $x' \in B(x_0, r/8)$,

$$\begin{aligned}
&\widetilde{C}_2\inf_{\widetilde{y}\in B(x_0,r/8)}\Big(\int_{\overline{B}^c}\int_B\widetilde{G}_B^k(\widetilde{y},y)J^{b_2,k}(y,z)dyu(z,k)dz+\int_{\partial B}\widetilde{M}_B^k(\tilde{y},z)u(z,k)\mu_u^k(dz)\Big),\\
\geq\ &\int_{\overline{B}^c}\int_B\widetilde{G}_B^k(x',y)J^{b_2,k}(y,z)dyu(z,k)dz+\int_{\partial B}\widetilde{M}_B^k(x',z)u(z,k)\mu_u^k(dz),
\end{aligned} \tag{4.27}$$

then by (4.8), (4.22), (4.23),(4.24),(4.27), Lemma 2.2.15, Lemma 2.2.18, Theorem 2.2.21, for any $x, x' \in B(x_0, r/8), k \in \mathcal{M}$,

$$\begin{aligned}
u(x,k)\ &\geq\ h(x,k)+\sum_{l\in E(k)\cup\{k\}}H_{kl}(c_0K_dr^2/(K_1C_0))(C_0^{-1}\int_{\overline{B}^c}\int_B G_B^{\Delta}(x,y)J^{b_2,l}(y,z)u(z,l)dydz\\
&\quad+\int_{\partial B}C_0^{-2}M_B^{\Delta}(x,z)u(z,l)\mu_u^l(dz))\\
&\geq\ h(x,k)+\sum_{l\in E(k)}a_{m_{kl}}(c_0K_dr^2/(K_1C_0))^{m_{kl}}C_0^{-4}h(x,l)\\
&\geq\ h(x,k)+\sum_{l\in E(k)}a_{m_{kl}}(c_0K_d/(K_1C_0))^{m_{kl}}r^{2m_{kl}}C_0^{-4}h(x,l)\\
&\geq\ \widetilde{C}_2h(x',k)+\sum_{l\in E(k)}(\widetilde{C}_2C_0^4)^{-1}a_{m_{kl}}(c_0K_d/(K_1C_0))^{m_{kl}}r^{2m_{kl}}h(x',l).
\end{aligned} \tag{4.28}$$

Let $C_2 = \min\{(\widetilde{C}_2C_0^4)^{-1}a_{m_{kl}}(c_0K_d/(K_1C_0))^{m_{kl}}, l \in E(k), k \in \mathcal{M}\} \wedge (1/\widetilde{C}_2)$, which is also finite. Then by (4.28), for any $x, x' \in B(x_0, r/8), k \in \mathcal{M}$,

$$u(x,k)\geq C_2(h(x',k)+\sum_{l\in E(k)}r^{2m_{kl}}h(x',l)). \tag{4.29}$$

Here both C_2 and C_3 depend on $(d, m, c_0, \beta, \gamma, \Theta_1, \Theta_2, \Theta_3, \Theta_4, \vartheta, Q^0)$, and independent of r.

In particular, by (4.26),(4.29), for any $x_0 \in \mathbb{R}^d, r \in (0, \widetilde{r}_1], x, x' \in B(x_0, r/8), k \in \mathcal{M}$,

$$u(x,k)\leq (C_3/C_2)C_2(h(x',k)+\sum_{l\in E(k)}r^{2m_{kl}}h(x',l))\leq (C_3/C_2)u(x',k)=:Cu(x',k). \tag{4.30}$$

□

2.5 Scale-invariant Full Rank Harnack inequality

Proof of Theorem 2.1.5.

Fix $x_0 \in \mathbb{R}^d$ and $r \in (0, \widetilde{r}_1)$,where $\widetilde{r}_1$ is the constant in Theorem 2.1.4. Similar to the beginning of the argument in Proposition 2.4.1, it suffices to establish the Harnack inequality for bounded nonnegative functions u defined in $\mathbb{R}^d$ that is $\mathcal{G}$-harmonic in $B(x_0, r)$. Since Q is strictly irreducible, then by (1.13), (1.17), we know that

$$0 < q_0 \le q^0_{kl}, \text{ for any } k, l \in \mathcal{M}, k \neq l. \tag{5.1}$$

Also, in (4.26) (4.28), $E(k) = \mathcal{M}$ for each $k \in \mathcal{M}$, such that

$$m_{kl} = 1, a_{m_{kl}} = q^0_{kl} \text{ for each } l \in \mathcal{M} \setminus \{k\}, \text{ and } m_{kk} = 2. \tag{5.2}$$

Then by (5.1),(5.2), for any $x, x' \in B(x_0, r/8), k \in \mathcal{M}$, we have

$$u(x,k) \ge (\widetilde{C}_2 C_0^4)^{-1} c_0 q_0 K_d r^2/(K_1 C_0) \sum_{l \in \mathcal{M}\setminus\{k\}} h(x', l) := K_3 r^2 \sum_{l \in \mathcal{M}\setminus\{k\}} h(x', l), \tag{5.3}$$

where $K_3 = (\widetilde{C}_2 C_0^4)^{-1} c_0 q_0 K_d r^2/(K_1 C_0)$. Since $\widetilde{r}_1 < 1$ by Theorem 2.1.4, then by (4.26), (5.1),(5.2), for any $r \in (0, \widetilde{r}_1], x, x' \in B(x_0, r/8), k \in \mathcal{M}$, we have

$$u(x,k) \quad \le \quad C_3 h(x', k) + C_3 r^2 \sum_{l \in \mathcal{M}\setminus\{k\}} h(x', l). \tag{5.4}$$

Hence by Theorem 2.1.4, (5.3), (5.4), for any $r \in (0, \widetilde{r}_1], x, x' \in B(x_0, r/8)$, any $k, i \in \mathcal{M}, k \neq i$,

$$\begin{aligned} u(x,k) \quad &\asymp \quad u(x', k) \le C_3 r^2 \sum_{l \neq k,i} h(x', l) + C_3 r^2 h(x', i) + C_3 h(x', k) \\ &\le \quad (C_3/K_3) K_3 r^2 \sum_{l \neq k,i} h(x', l) + C_3 u(x', i) + C_3 h(x', k) \\ &\le \quad (C_3/K_3 + 2C_3)(u(x', i) + h(x', k)) = \widetilde{C}_1 (u(x', i) + h(x', k)), \end{aligned} \tag{5.5}$$

where $\widetilde{C}_1 := C_3/K_3 + 2C_3 + C$ depending only on $(d, \beta, \gamma, m, c_0, q_0, \Theta_1, \Theta_2, \Theta_3, \Theta_4, \vartheta, Q^0)$ and independent of r.

Furthermore, by (5.3),(5.4), for any $r \in (0, \widetilde{r}_1], x, x' \in B(x_0, r/8)$, any $k, i \in \mathcal{M}, k \neq i$,

$$u(x,k) \quad \le \quad C_3(1 + \widetilde{r}_1^2) \sum_{l \in \mathcal{M}} h(x', l) \le C_3(1 + \widetilde{r}_1^2)(r^{-2} h(x', i) + \sum_{l \in \mathcal{M}\setminus\{i\}} h(x', l))$$

$$
\begin{aligned}
&\leq C_3(1+\tilde{r}_1^2)(r^{-2}u(x',i)+(1/K_3)r^{-2}u(x',i)) \leq C_3(1+\tilde{r}_1^2)(1+1/K_3)r^{-2}u(x',i) \\
&\leq \widetilde{C}_3(1+\tilde{r}_1^2)r^{-2}u(x',i).
\end{aligned}
$$

□

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Appendix A

APPENDIX TO CHAPTER 1

A.1 Martingale problems and SDEs

Given a nonlocal operator $\mathcal{L} \in \mathcal{N}(\Lambda_1, \Lambda_2, \Lambda_3)$, let us first consider in what conditions the solution $(X_t, t \geq 0; \mathbb{P}^x)$ of the martingale problem for $(\mathcal{L}, C^2(\mathbb{R}^d))$ can be transferred into the weak solution of an SDE.

Let us first consider a Lusin space U(a space that is homeomorphic to a Borel subset of a compact metric space) with its Borel σ-algebra $\mathscr{U}$. Let $\widehat{U} = U \cup \{\Delta\}$, where Δ is the trivial point, with the extended σ-algebra $\widehat{\mathscr{U}}$ constructed as $\widehat{\mathscr{U}} = \{A \cup B : A \in \mathscr{U}, B \subset \{\Delta\}\}$.

Definition A.1.1. *Consider a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and a $\widehat{\mathscr{U}}$-measurable process $\{Y_s\}_{s\geq 0}$ on $\widehat{U}$ is a* **point process** *if for almost every $\omega \in \Omega$, $D_\omega = \{t \geq 0 : Y_t(\omega) \neq \Delta\}$ is countable.*

Definition A.1.2. *We say that a point process $Y = (Y_s)$ is* ***discrete*** *if for almost every $\omega \in \Omega$, the set $D_\omega = \{t \geq 0 : Y_t(\omega) \neq \Delta\}$ has no accumulation point. A point process Y is said to be* **σ-discrete** *if there exists a sequence of increasing sets $\{U_n\}_{n\in\mathbb{N}} \subset \mathscr{U}$ such that $U = \cup_{n\in\mathbb{N}} U_n$, and for almost every $\omega \in \Omega$, $D_\omega^n = \{t \geq 0 : Y_t(\omega) \in U_n\}$ has no accumulation point for each $n \in \mathbb{N}$.*

In the case of σ-discrete point processes, denote $D_\omega = \cup_{n\in\mathbb{N}} D_\omega^n$.

Next, define a random counting measure ξ^Y on$(\mathbb{R}^+ \times U, \mathscr{B}(\mathbb{R}^+) \otimes \mathscr{U})$ associated with the σ-discrete point process Y as

$$\xi^Y(\omega, \cdot) := \sum_{s\geq 0} \delta_{(s, Y_s(\omega))} \tag{1.1}$$

with $\xi^Y(\omega, \{0\} \times U) = 0$. For any $B \in \mathscr{U}$, denote $\xi_t^Y(B) := \xi^Y(\cdot, [0,t] \times B)$ for each $t > 0$ and $\xi_0^Y(B) = 0$, as **the counting process with respect to the set** B. Also, for any positive $\mathscr{F} \otimes \mathscr{B}(\mathbb{R}^+) \otimes \mathscr{U}$-measurable function Z defined on $\Omega \times [0, \infty) \times U$, denote

$$\xi_t^Y(Z)(\omega) := \sum_{0<s\in D_\omega} Z(\omega, s, Y_s(\omega))\mathbb{1}_{(0,t]}(s)\text{for } t > 0 \text{ and } \quad \xi^Y(Z)(\omega) := \lim_{t\to\infty} \xi_t^Y(Z)(\omega) \quad (1.2)$$

for almost every $\omega \in \Omega$ with$\xi_0^Y(Z) = 0$.

The probability space $(\Omega, \mathscr{F}, \mathbb{P})$ will be equipped with a nondecreasing family of σ-algebra $\{\mathscr{F}_s\}_{s\geq 0}$ satisfying the usual conditions: that is, they are right-continuous and augmented with all $\mathbb{P}$-negligible sets of $\mathscr{F}_\infty$.

A σ-discrete point process Y is said to be *adapted* if Y_0 is $\mathcal{F}_0$-measurable, and for any $B \in \mathscr{U}$, any U_n defined in Definition A.1.2, $\xi_s^Y(B \cap U_n)$ is $\mathscr{F}_s$-measurable for every $s \geq 0$.

The σ-field on $\Omega \times \mathbb{R}^+$, generated by all càdlàg adapted processes, is called *optional* σ-algebra, denoted by $\mathscr{O}$. The σ-field on $\Omega \times \mathbb{R}^+$, generated by all left-continuous adapted processes or the following types of sets: $A \times \{0\}$, where $A \in \mathscr{F}_0$ and $B \times (s, t]$, where $B \in \mathscr{F}_s, 0 \leq s < t$, is called *predictable* σ-algebra, denoted by $\mathscr{P}$. A stochastic process is said to be *predictable* (resp. *optional*) if it is $\mathscr{P}$(resp. $\mathscr{O}$)-measurable.

Hence if a point process Y is adapted and càdlàg, it is therefore optional.

Definition A.1.3. *[44, 1.22, 1.30 in Chapter I]A random set D is called* **thin** *if it is of the form $D = \cup_{m\in\mathbb{N}}[T_m]$, where $\{T_m\}_{m\geq 1}$ is a sequence of stopping times and $[T_m]$ is the graph of the map $T_m : \Omega \to [0, \infty]$ to the set $(\Omega, [0, \infty))$; Moreover, if the sequence $\{T_m\}_{m\geq 1}$ satisfies $[T_n] \cap [T_m] = \emptyset$ for all $m \neq n$, it is called an* **exhausting sequence** *for D.*

Theorem A.1.4. *If ξ is a random counting measure on $\Omega \times [0, \infty) \times U$, then there exists a thin random set $D \subset \Omega \times [0, \infty)$ and a U-valued optional process Y such that*

$$\xi(\omega, \cdot) = \sum_{s\geq 0} \mathbb{1}_D(\omega, s)\delta_{\{s, Y_s(\omega)\}}.$$

Proof. The proof can be found in the proof of Proposition 1.14 of Chapter II of [44].

Definition A.1.5. *Following Part B in the Introduction of [36],we say that a point process Y is* **σ-finite** *if there exists a strictly positive $\mathscr{P} \otimes \mathscr{U}$-measurable function H defined on $\Omega \times [0, \infty) \times U$ such that $\mathbb{E}[\xi^Y(H)] < \infty$, where ξ^Y is the random counting measure with respect to the point process Y.*

Theorem A.1.6. *A σ-discrete point process Y defined on $\hat{U}$ is σ-finite.*

Proof. We construct H as follows. By the definition of σ-discreteness in Definition A.1.2, there exists a partition of disjoint sets in U such that $\{U_n\}_{n\in\mathbb{N}} \subset \mathscr{U}, U = U_{n\geq 1}U_n$, and $D^n_\omega = \{s > 0 : Y_s \in U_n \text{ and } Y_s \neq \Delta\}$ is has no accumulation points for almost every $\omega \in \Omega$. Fix $n \in \mathbb{N}$. Let $S_{n,0} = 0$, and define $S_{n,k}(\omega) = \inf\{t > S_{n,k-1} : Y_t \neq \Delta, Y_t \in U_n\}, k \in \mathbb{N}$.

Since D^n_ω is discrete, then for almost every $\omega \in \Omega$, if $D^n_\omega \setminus \{0\} = \emptyset$, set $S_{n,1}(\omega) = \inf \emptyset = \infty$; If $D^n_\omega \setminus \{0\} \neq \emptyset$, then $D^n_\omega \setminus \{0\}$ must have an infimum which is strictly positve or infinity. Because, otherwise, 0 would be the accumulation point for D^n_ω. Hence $S_{n,1} > 0$ a.s.

By induction, we find the sequence $\{S_{n,k}\}_{k\geq 0}$ of stopping times, which exhausts the set D^n_ω for almost every ω.

For each $n \in \mathbb{N}$, let $A_0 = \Omega \times \{0\}$, and $A_{n,k} = (S_{n,k-1}, S_{n,k}], k \in \mathbb{N}$. Hence A_0 and $A_{n,k}$'s are all $\mathscr{P}$-measurable, and $\cup_{k\geq 0} A_{n,k} = (0, \infty)$, and for almost every $\omega \in \Omega$,

$$\xi(\mathbb{1}_{A_{n,k}(\omega)\times U_n}) = \sum_{0<s\in D^n_\omega} \mathbb{1}_{A_{n,k}\times U_n}(s, Y_s) \leq 1,$$

i.e., Y jumps at most once in $A_{n,k}(\omega) \times U_n$ almost surely. Let

$$H(\omega, s, u) := \mathbb{1}_{A_0(\omega)} + \sum_{n\geq 1} \frac{1}{2^n} \sum_{k\geq 0} \frac{1}{2^k} \mathbb{1}_{A_{n,k}(\omega)\times U_n}(s, u)$$

Hence H is $\mathscr{P} \times \mathscr{U}$-measurable, strictly postive almost surely, and

$$\mathbb{E}[\xi H] = \mathbb{E}[\sum_{s\in D_\omega} H(\omega, s, Y_s)] \leq 1 + \sum_{n\geq 1} \frac{1}{2^n} \sum_{k\geq 0} \frac{1}{2^k} \leq 5.$$

Hence the process Y is σ-finite. □

Definition A.1.7. (i) An adapted càdlàg process X is said to be *quasi-left continuous* if for any increasing sequence $\{T_n\}_{n\in\mathbb{N}}$ of stopping times with limit T, we have $\lim_{n\to\infty} X_{T_n} = X_T$ a.s. on $\{T < \infty\}$.

(ii) [44, 2.7, 2.20 of Chapter I] A *predictable time* is a mapping $S : \Omega \mapsto \bar{\mathbb{R}}_+$ such that the interval $[0, S)$ is $\mathscr{P}$-measurable, and a stopping time is called **totally inaccessible** if $\mathbb{P}(T = S < \infty) = 0$ for all predictable times S.

(iii) We call a process $A = (A_t)$ *an increasing process* if it is adapted, and its path is right continuous and nondecreasing $\mathbb{P}$-a.s. with $A_0 = 0$.

(iv) An increasing process $A = (A_t)$ is said to be *integrable* if $\mathbb{E}[A_\infty] < \infty$ and *locally integrable* if there is an increasing sequence of stopping times $\{T_n; n \geq 1\}$ with $\lim_{n\to\infty} T_n = \infty$ $\mathbb{P}$-a.s. and $\mathbb{E}[A_{T_n}] < \infty$ for every $n \geq 0$.

(v) A random set $D \subset \Omega \times [0,\infty)$ is called *evanescent* if $\mathbb{P}(\{\omega \in \Omega : (\omega, t) \in D$ for some $t \geq 0\}) = 0$.

(vi) For any locally integrable increasing process A, there is a predictable process, called the *compensator* A^p of A, which is unique up to an evanescent set, such that $A - A^p$ is a local martingale. A^p is called the *predictable compensator* of A, or the *dual predictable projection* of A. (See [44, Theorem 3.17 of Chapter I]). For any random counting measure μ on $\mathbb{R}^+ \times U$, the *dual predictable projection/compansator* μ^p of a random measure is defined in the similar idea. (See [44, Theorem 1.8 of Chapter II]).

If Y is an adapted σ-finite point process, then for any $\mathscr{P} \times \mathscr{U}$-measurable positive function Z on $\Omega \times [0,\infty) \times U$, $\{\xi_t^Y(Z)\}_{t\geq 0}$ is adapted, where ξ_t^Y is defined in (1.2). Indeed, if $A \in \mathscr{F}_s$, $0 < s \leq v$, $B \in \mathscr{U}$, the function

$$Z = \mathbb{1}_A \times \mathbb{1}_{(s,v]\times B}$$

is $\mathscr{P} \times \mathscr{U}$-measurable, and $\xi_t^Y(Z) = \mathbb{1}_A[\xi^Y_{t\wedge v}(B) - \xi^y_{t\wedge s}(B)]$ is adapted. By monotone class theorem, for any $C \in \mathscr{P} \times \mathscr{U}$, $\{\xi_t^Y(\mathbb{1}_C)\}_{t\geq 0}$ is adapted. As Y is σ-finite, by definition, there exists a strictly positive $\mathscr{P} \times \mathscr{U}$-measurable function H such that $\mathbb{E}[\xi^Y(H)] < \infty$. The set

$$\begin{aligned} \mathbf{D} &= \{(\omega, t) : Y_t \neq \Delta\} \\ &= \{(\omega, 0) : Y_0 \neq \Delta\} \cup \{(\omega, t) : \xi^Y_{t-}(H)(\omega) \neq \xi^Y_t(H)(\omega), t > 0\} \end{aligned} \tag{1.3}$$

is the union of the graph of a countable sequence of stopping times. Indeed, since for each $n \in \mathbb{N}$, define $A_n = \{t \in [0,\infty), \Delta\xi^Y(H) \in [1/2^{n+1}, 1/2^n)\}$ and let $A_0 = \{t \in [0,\infty), \Delta\xi^Y(H) \geq 1/2\}$. Since $\mathbb{E}[\xi^Y(H)] < \infty$, then $\xi^Y(H) < \infty$ a.s. Since ξ_t^H is strictly increasing in time a.s., then there must be at most finitely many elements in A_n for each $n \in \mathbb{N} \cup \{0\}$ almost surely.(Otherwise, $\mathbb{P}(\xi^Y(H) = \sum_{\{\Delta\xi^Y(H)\neq 0\}} \Delta\xi^Y(H) = \infty) > 0$). Then for each $n \in \mathbb{N} \cup \{0\}$, define recursively $T^n_{k+1} := \{t > T^n_k, t \in A_n\}$. And it is not hard to see that $\mathbf{D} = \cup_{\{\mathbb{N}\cup 0\}} \cup_{k=1}^{\infty} [T^n_k]$, which is the countably many union of graph of stopping times.

Definition A.1.8. *We say that an adapted σ-finite point process Y is* **quasi-left continuous** *if $\mathbf{D}\cap[0,\infty)$ is the union of the graph of a sequence of totally inaccessible stopping times.*

Theorem A.1.9. *Let X be an adapted càdlàg process. The following three statements are equivalent to each other:*

(a) X is quasi-left continuous;

(b) There exists a sequence of totally inaccessible stopping times that exhausts the times of jumps of X;

(c) For any increasing sequence $\{T_n\}_{n\geq 1}$ of stopping times with limit T, we have

$$\lim_{n\to\infty} X_{T_n} = X_T \quad a.s. \text{ on } \{T < \infty\}.$$

Proof. The proof can be found in Theorem 2.26 in Chapter II of [44].

Theorem A.1.10. *Suppose that X is a locally integrable non-decreasing process. Its dual predictable projection X^p is continuous a.s. if and only if X is quasi-left continuous.*

Proof. By the definition of dual predictable projection, $X - X^p$ is a martingale starting from 0 at $t = 0$. Since if Y is a local martingale, then by Lemma 2.27 in [44], $\mathbb{E}[Y_T|\mathscr{F}_{T-}] = Y_{T-}$ on $\{T < \infty\}$ for every predictable times T. Also, for any predictable time T,

$$\mathbb{E}[X_T - X^p_T|\mathscr{F}_{T-}] = X_{T^-} - X^p_{T^-} \quad a.s. \text{ on } \{T < \infty\},$$

i.e.

$$\mathbb{E}[\Delta X_T - \Delta X^p_T|\mathscr{F}_{T-}] = 0 \quad a.s. \text{ on } \{T < \infty\}.$$

Since X^p is predictable, then by Lemma 25.3(i) in [55], $X^p_T\mathbb{1}_{\{T<\infty\}}$ is $\mathscr{F}_{T-}$-measurable, and therefore

$$\Delta X^p_T = \mathbb{E}[\Delta X_T\mathbb{1}_{\{T<\infty\}}|\mathscr{F}_{T-}] \quad a.s. \text{ on } \{T<\infty\}.$$

Since X is quasi-left continuous, then $X_T = X_{T-} \quad a.s.$ on $\{T<\infty\}$, then

$$\Delta X^p_T = 0 \quad a.s. \text{ on } \{T<\infty\}. \tag{1.4}$$

Since for any predictable process Y, by Theorem 3.33 in [42], there is a sequence of strictly positive predictable stopping times $\{T_k\}_{k\geq 1}$ so that $\{(\omega,t) : t>0, Y_t(\omega)\neq Y_{t-}(\omega)\}\subset \cup_{k\geq 1}[T_k]$, then by (1.4), for each T_k, there exists a set $A_k\subset\Omega$ such that

$$A_k \subset \pi_1([T_k]\cap\{(\omega,t) : t>0, X^p_t(\omega)\neq X^p_{t-}(\omega)\}),$$

where $[T_k]$ is defined in Definition A.1.3, and π_1 is the projection map on the first coordinate). By (1.4), we have $\mathbb{P}(A_k)=0$ and $\Delta X^p_{T_k}(\omega)=0$ for any $\omega\in A^c_k$. Then on the set $\Omega\setminus\cup_{k\geq 1}A_k$, $\mathbb{P}(\cap_{k\geq 1}A^c_k) = 1-\mathbb{P}(\cup_{k\geq 1}A_k)\geq 1-\sum_{k\geq 1}\mathbb{P}(A_k) = 1$, and for every $\omega\in\Omega\setminus\cup_{k\geq 1}A_k$, $X^p_t(\omega)=X^p_{t-}(\omega)$ for all $t\geq 0$. Hence, X^p is continuous a.s.

Conversely, if X^p is a.s. continuous and predictable, then for any predictable time T,

$$\Delta X^p_T = 0 \quad a.s. \text{ on } \{T<\infty\}.$$

Hence

$$\mathbb{E}[\Delta X_T\mathbb{1}_{\{T<\infty\}}] = \mathbb{E}[\Delta X^p_T] = 0. \quad a.s.\{T<\infty\}.$$

Since X is increasing, then ΔX_T only takes nonnegative values on $\{T<\infty\}$. Hence $\Delta X_T = 0 \quad a.s.$ on $\{T<\infty\}$. Hence the jumps of the process X only happens at totally inaccessible times, i.e., X is quasi-left continuous. □

For any σ-finite positive diffusive measure ν on a Lusin space $(U,\mathscr{U})$, there exists a Poisson point process Y whose associated random measure $\mu^Y(\omega,\cdot) := \sum_{s\geq 0}\delta_{(s,Y_s(\omega))}$ on $(\mathbb{R}^+\times U, \mathscr{B}(\mathbb{R}^+)\otimes\mathscr{U})$ satisfies that

(i) For any disjoint sets $B_1, B_2, ..., B_n$ on $\mathscr{B}(\mathbb{R}^+)\otimes\mathscr{U}$, $\mu^Y(B_i)$'s are independently distributed;

(ii) $\mu^Y(B)$ is Poisson distributed with *intensity measure* $\nu(du)ds$ for every set $B \in \mathscr{B}(\mathbb{R}^+) \times \mathscr{U}$.

The following theorem provides a criteria for determining whether a point process Y is a Poisson point process.

Theorem A.1.11. *A necessary and sufficient condition for a σ-finite point process Y on a Lusin space U to be a Poisson point process is that there exists a σ-finite positive measure ν on $\mathscr{U}$ such that for any $B \in \mathscr{U}$ with $\nu(B) < \infty$, the associated random measure μ^Y on $([0,\infty) \times U, \mathscr{B}([0,\infty)) \otimes \mathscr{U})$satisfies that $\mu_t^Y(B) - t\nu(B)$ is an $\{\mathscr{F}_t\}$-martingale for each $t > 0$, where $\mu_t^Y(B) := \mu^Y(\cdot, [0,t) \times B)$.*

Proof. The idea of the proof can be found in Theorem 3 in [36], and we willl provide a more detailed proof here. The proof of the necessity is trivial, which is by the defintion of the dual predictable projection of the Poisson random measure. To show the sufficiency, it suffices to check that for any disjoint set $A_i = (s_i, t_i) \times B_i$ with $\nu(B_i) < \infty, i \in \{1, 2, ..., n\}$, $\mu^Y(\cdot, A_i)$'s are independent and each follows Poisson distribution with parameter $\nu^Y(B_i)(t_i - s_i)$.

Given $\theta_1, \theta_2, ..., \theta_n \in \mathbb{R}$, define $\varphi(s,u) : \mathbb{R}^+ \times U \to \mathbb{R}$ such that $\varphi(s,u) := \sum_{j=1}^n \theta_j \mathbb{1}_{A_j}(s,u)$, where $A_j's$ are disjoint. Notice that $\mathbb{E}[\exp(i\mu_0^Y(\varphi))] = 0$, and for each $t > 0$,

$$\begin{aligned}\mathbb{E}[\exp(i\mu_t^Y(\varphi))] &= 1 + \mathbb{E}[\sum_{0<s\le t}(\exp(i\mu_s^Y(\varphi)) - \exp(i\mu_{s-}^Y(\varphi))] \\ &= 1 + \mathbb{E}[\sum_{0<s\le t}\exp(i\mu_{s-}^Y(\varphi))\Big(\exp(i[\mu_s^Y(\varphi) - \mu_{s-}^Y(\varphi)]) - 1\Big)]. \quad (1.5)\end{aligned}$$

Notice that

$$\exp(i[\mu_s^Y(\varphi) - \mu_{s-}^Y(\varphi)]) = \exp(i\delta_{(s,Y_s)}(\varphi)) = \exp i[\sum_{j=1}^n \theta_j \mathbb{1}_{A_j}(s,Y_s)] = \Pi_{j=1}^n \exp(i\theta_j \mathbb{1}_{A_j}(s,Y_s)).$$

Therefore, by (1.5), and the disjointness of A_j's, $1 \le j \le n$,

$$\begin{aligned}\mathbb{E}[\exp(i\mu_t^Y(\varphi))] &= 1 + \mathbb{E}[\sum_{0<s\le t}\exp(i\mu_{s-}^Y(\varphi))(\sum_{j=1}^n \exp(i\theta_j \mathbb{1}_{A_j}(s,Y_s)) - 1)] \\ &= 1 + \mathbb{E}[\sum_{0<s\le t}\exp(i\mu_{s-}^Y(\varphi))(\sum_{j=1}^n \exp(i\theta_j \mathbb{1}_{A_j}(s,Y_s)) - \sum_{j=1}^n \mathbb{1}_{A_j}(s,Y_s)]\end{aligned}$$

$$\begin{aligned} &= 1+\mathbb{E}[\sum_{0<s\le t}\exp(i\mu^Y_{s^-}(\varphi))(\sum_{j=1}^n(\exp(i\theta_j)-1)\mathbb{1}_{A_j}(s,Y_s))] \\ &= 1+\sum_{j=1}^n(\exp(i\theta_j)-1)\mathbb{E}[\sum_{0<s\le t}\exp(i\mu^Y_{s^-}(\varphi))\mathbb{1}_{A_j}(s,Y_s)]. \end{aligned}$$

Since $\mu_t^Y(\mathbb{1}_{A_j}(s,Y_s))-\nu(B_j)[t_i\wedge t-s_j\wedge t]=\sum_{0<s\le t}\mathbb{1}_{A_j}(s,Y_s)-\int_0^t\nu(B_j)\mathbb{1}_{(s_j,t_j)}(s)ds$ is an $\mathscr{F}_t$-martingale for each $t>0$, and $\mu^Y_{s^-}(\varphi)$ is $\mathscr{F}_s$-measurable, then

$$\begin{aligned} \mathbb{E}[\exp(i\mu_t^Y(\varphi))] &= 1+\sum_{j=1}^n(\exp i\theta_j-1)\mathbb{E}[\int_0^t\exp(i\mu^Y_{s^-}(\varphi))\mathbb{1}_{(s_j,t_j)}(s)\nu(B_j)ds] \\ &= 1+\sum_{j=1}^n(\exp i\theta_j-1)\nu(B_j)\int_0^t\mathbb{E}[\exp(i\mu^Y_s(\varphi))]\mathbb{1}_{(s_j,t_j)}(s)ds. \end{aligned}$$

Then

$$\begin{aligned} \mathbb{E}[\exp(i\mu_t^Y(\varphi))] &= \exp[\sum_{j=1}^n(\exp i\theta_j-1)\nu(B_j)(t_j\wedge t-s_j\wedge t))] \\ &= \sum_{j=1}^n\exp[(\exp i\theta_j-1)\nu(B_j)(t_j\wedge t-s_j\wedge t). \end{aligned}$$

That is to say that the characteristic function of the random variables $(\mu_t^Y(A_1),...,\mu_t^Y(A_n))$ are the products of each characteristic function, and thus these $\mu_t^Y(A_j)'s$ are independent and each $\mu_t^Y(A_j)$ follows Poisson distribution with intensity measure $(t_j\wedge t-s_j\wedge t)\nu(B_j)$. □

Consider any increasing process A. It uniquely determines a Radon measure μ^ω on $[0,\infty)$ with $\mu^\omega(\{0\})=0$ and $\mu^\omega(a,b]=A_b(\omega)-A_a(\omega)$ for any $b>a\ge 0,\omega\in\Omega$. We denote this measure by $dA_t(\omega)$.

Theorem A.1.12. *Suppose that A and C are two increasing locally integrable processes defined on the same probability space $(\Omega,\mathscr{F},\mathbb{P})$ so that for almost every ω, $dA_t(\omega)\ll dC_t(\omega)$ on $\mathbb{R}^+$. Then their corresponding dual predictable projections are increasing processes satisfying $dA_t^p(\omega)\ll dC_t^p(\omega)$ on $[0,\infty)$ a.s..*

Proof. The locally integrable increasing processes A and C define two σ-finite measures μ_A and μ_C on $\mathcal{B}(\mathbb{R}^+)\times\mathcal{F}_\infty$ by

$$\mu_A(F)=\mathbb{E}\int_{[0,\infty)}1_F(s,\omega)dA_s(\omega)\quad\text{and}\quad\mu_C(F)=\mathbb{E}\int_{[0,\infty)}1_F(s,\omega)dC_s(\omega)$$

for any $F \in \mathcal{B}(\mathbb{R}^+) \times \mathcal{F}_\infty$. Conversely, a σ-finite measure μ on $\mathcal{B}(\mathbb{R}^+) \times \mathcal{F}_\infty$ is induced by an increasing process if and only if for each $t \geq 0$,

$$Q_t(\Gamma) := \mu([0,t] \times \Gamma), \quad \Gamma \in \mathcal{F}_\infty,$$

is a σ-finite measure absolutely continuous with respect to $\mathbb{P}$; Moreover, by Theorem 5.11 in [42], such increasing process is unique.

The dual predictable projections A^p and C^p of A and C are the unique predictable increasing processes so that for any non-negative bounded measurable process H,

$$\int_{\mathbb{R}^+ \times \Omega} H d\mu_A = \mathbb{E} \int_{[0,\infty)} {}^p H_s(\omega) dA^p_s(\omega) \quad \text{and} \quad \int_{\mathbb{R}^+ \times \Omega} H d\mu_C = \mathbb{E} \int_{[0,\infty)} {}^p H_s(\omega) dC^p_s(\omega),$$

where ${}^pH$ is the predictable projection of H; see [42, Theorem 5.20]. Clearly, $\mu_A = \mu_{A^p}$ and $\mu_C = \mu_{C^p}$ on $\mathscr{P}$. Since $dA_t(\omega) \ll dC_t(\omega)$ on $\mathbb{R}^+$ for a.s. $\omega \in \Omega$, we have $\mu_A \ll \mu_C$ on $\mathcal{B}(\mathbb{R}^+) \times \mathcal{F}_\infty$ and consequently, on $\mathscr{P}$. Let K be the Radon-Nikodym derivative of μ_{C^p} with respect to μ_{A^p} on $\mathscr{P}$, which is a non-negative predictable process. For each integer $n \geq 1$, let $K^{(n)} := K \wedge n$. Define $\widetilde{A}_t(\omega) = \int_0^t K_s(\omega) dC^p_s(\omega)$ and $\widetilde{A}^{(n)}_t(\omega) = \int_0^t K^{(n)}_s(\omega) dC^p_s(\omega)$. Observe that $\widetilde{A}$ and $\widetilde{A}^{(n)}$ are predictable increasing processes, and $\widetilde{A}^{(n)}$ increases to $\widetilde{A}$ by monotone convergence theorem. For any non-negative bounded measurable process H,

$$\begin{aligned}
\int_{\mathbb{R}^+ \times \Omega} H d\mu_{\widetilde{A}} &= \int_{\mathbb{R}^+ \times \Omega} {}^p H_s(\omega) d\mu_{\widetilde{A}} = \int_{\mathbb{R}^+ \times \Omega} {}^p H_s(\omega) K_s(\omega) d\mu_{C^p} \\
&= \int_{\mathbb{R}^+ \times \Omega} {}^p H_s(\omega) d\mu_{A^p} = \int_{\mathbb{R}^+ \times \Omega} H_s(\omega) d\mu_{A^p}.
\end{aligned}$$

By the uniqueness between the increasing process and σ-finite measure on $\mathcal{B}(\mathbb{R}^+) \times \mathcal{F}_\infty$ mentioned earlier, the above shows that $A^p_t = \widetilde{A}_t = \int_0^t K_s(\omega) dC^p_s$, proving $dA^p_t \ll dC^p_t$. □

Next, we will quote [40, Corollary 3.2] to prove Theorem A.1.15. Before proceeding, we introduce some definitions relevant to Theorem A.1.14.

Definition A.1.13. *A kernel N from a measurable space $(E, \mathscr{E})$ to another measurable Lusin space $(S, \mathscr{S})$ is a function $N(x, A)$ for each $x \in E$ and $A \in \mathscr{S}$ such that*

- *$N(\cdot, A)$ is $\mathscr{E}$-measurable for each $A \in \mathscr{S}$;*

- *$N(x, \cdot)$ is a measure on $\mathscr{S}$ for each $x \in E$.*

We say that the kernel N is bounded if $\sup\{N(x,S) : x \in E\} < \infty$.

Let $\mathscr{H} \subset \mathscr{E}$ be closed under countable unions and be hereditary in the sense that if $N \in \mathscr{H}$, and $B \subset N, B \in \mathscr{E}$, then $B \in \mathscr{H}$. We assume that $E \notin \mathscr{H}$. We call the sets in $\mathscr{H}$ *negligible* and $\mathscr{H}$ the defining class of negligible sets. A property $p(x)$ depending on $x \in E$ is said to hold **a.e.** $\mathscr{H}$ if the set of x for which $p(x)$ does not hold in an element of $\mathscr{H}$.

Theorem A.1.14. *Let S be a Lusin space and $\mathscr{S}$ be the Borel σ-algebra. Define S^* as the space of bounded Borel-measurable function on S.Suppose that $\mathscr{H} \subset \mathscr{E}$ is the defining class of negligible sets. Let $(E, \mathscr{E})$ be a measurable space with a σ-finite measure μ and define E^* as the Banach space of bounded real-valued $\mathscr{E}$-measurable functions on E under the supremum norm. Let $T : S^* \to E^*$ be an almost positive and almost linear map (i.e, for every $f, g \in S^*, \alpha, \beta \in \mathbb{R}$, $T(\alpha f + \beta g) = \alpha Tf + \beta Tg$ and for every $f \in S^*, f \geq 0, Tf \geq 0$ a.e. in $\mathscr{H}$) satisfying that for any nonnegative increasing sequence of functions $\{f_n\}_{n\geq 1}$ in S^* such that $f_n \uparrow f$ for some $f \in S^*$, $Tf_n \uparrow Tf$ a.e. in $\mathscr{H}$.*

Then there exists a bounded kernel N from $(E, \mathscr{E})$ to $(S, \mathscr{S})$ such that $Tf = Nf$ a.e. in $\mathscr{H}$ for each $f \in S^*$.

Proof. See Corollary 3.2 in [40]. □

The following theorem is to show the existence of Lévy system for the adapted σ-finite point process Y on a Lusin space. The idea of the proof is from [36, Theorem 3], and we will provide more details here.

Theorem A.1.15. *Let Y be an adapted σ-finite point process on a Lusin space U.Then there exists a strictly increasing predictable process A and a σ-finite kernel $N(\omega, s, \cdot)$ from $(\Omega \times [0, \infty), \mathscr{P})$ to $(U, \mathscr{U})$, such that for every nonnegative $\mathscr{U}$-measurable function f, the associated random counting process $\xi^Y_t(f) = \sum_{\{s\in\mathbf{D},0<s\leq t\}} f(Y_s)$ has a dual predictable projection $\int_0^t N(\omega, s, f)dA_s$, which is an increasing process. The pair (N, A) is called the* **Lévy system** *of the point process Y.*

Proof. Since $\{Y_s\}_{s\geq 0}$ is σ-finite, then there exists a strictly positive $\mathscr{P} \otimes \mathscr{U}$-measurable process H such that $\mathbb{E}[\xi^Y(H)] < \infty$, and the jumping times are defined in $\mathbf{D} \cap (0, \infty]$.

Since $\{\xi_t^Y(H)\}_{t\geq 0}$ is an integrable increasing process, then by Theorem A.1.10, the dual predictable projection $\bar{A}_t$ of $\xi_t^Y(H)$ is a.s. continuous.

For every $\mathscr{U}$-measurable bounded positive function f on U, define

$$\xi_t^Y(fH) := \sum_{0<s\in\mathbf{D}} f(Y_s)H(\omega,s,Y_s)\mathbb{1}_{[0,t]}(s), \text{ and } \xi^Y(fH) := \lim_{t\to\infty}\xi_t^Y(fH)$$

with $\xi_0^Y(fH) = 0$. Denote $(\xi_t^Y(fH))^p$ as the dual predictable projection of $\xi_t^Y(fH)$.

Let

$$\mathcal{C} = \{C \in \mathscr{F}_\infty \times \mathcal{B}([0,\infty)), \mathbb{P}\times m(C) = 0\},$$

where m is the Lebesgue measure on $[0,\infty)$. Then $\mathcal{C}$ is closed under countable unions and if $B \subset C$ for some $C \in \mathcal{C}$ and $B \subset \mathscr{F}_\infty \times \mathcal{B}([0,\infty))$, then $B \in \mathcal{C}$. Hence $\mathcal{C}$ can be taken as the defining negligible class $\mathscr{H}$ of negligible sets in Theorem A.1.14.

Since $\xi_t^Y(H)$ and $\xi_t^Y(fH)$ are both increasing processes, then by the definition of the dual predictable projection, $\bar{A}$ and $(\xi^Y(fH))^p$ are both increasing processes. Since $d\xi_t^Y(fH) \ll d\xi_t^Y(H)$ a.s., then by Theorem A.1.12, $d(\xi_t^Y(fH))^p \ll d\bar{A}_t$ a.s.. Then there exists a $\mathscr{P}$-measurable Radon-Nikodym derivative (Tf) defined on $(\Omega\times[0,\infty)$, nonnegative a.e. on $(\Omega\times[0,\infty)$ such that

$$d(\xi_t^Y(fH))^p = (Tf)(\omega,s)d\bar{A}_t \quad \mathbb{P}\times m - a.e. \tag{1.6}$$

Note that for each

$$f \in F = \{\text{equivalent class of bounded and positive } (U,\mathscr{U}) - \text{measurable functions}\},$$

$$Tf \in E = \{\text{equivalence class of } (\Omega\times[0,\infty),\mathscr{P})-\text{measurable functions, i.e. modulo } \mathbb{P}\times m\text{-null sets}).$$

In addition, it is easy to see that $\|Tf\| \leq \|f\|$, $T\mathbb{1}_{id} = \mathbb{1}_{id}$ and T is almost positive and almost linear bounded map (with respect to the the measure $\mathbb{P}\times m$. Here "$\|\cdot\|$" stands for the essential sup norm on $\Omega\times[0,\infty)$.

Also for any $0 \leq \{f_n\} \uparrow f \in F$, $Tf_n \uparrow Tf$ $\mathbb{P}\times m$-a.e. By Theorem A.1.14, there exists a bounded kernel $N_1(\omega,s,du)$ from $\Omega\times[0,\infty)$ to $(U,\mathscr{U})$, such that for almost every (ω,s), every bounded and positive $\mathscr{U}$-measurable function f,

$$(Tf)(\omega,s) = N_1(\omega,s,f) \quad \mathbb{P}\times m - a.e. \tag{1.7}$$

where $N_1(\omega,s,f) = \int_U f(u)N_1(\omega,s,du)$. Since $\|Tf\| \leq \|f\|$, $N_1(\omega,s,U) \leq 1$ for almost every (ω,s).

Let $D = \{(\omega,s)\in\Omega\times[0,\infty) : (Tf)(\omega,s) \neq N_1(\omega,s,f)\}, D_\omega = \{s\in[0,\infty) : (\omega,s)\in D\}$. Since D is a null set with respect to $d\mathbb{P}\otimes ds$, by Fubini's theorem, $m(D_\omega) = 0 \quad \mathbb{P}$-.a.s. Hence, by (1.6),(1.7),

$$(\xi_t^Y(fH))^p = \int_0^t \int_U f(u)N_1(\omega,s,du)d\bar{A}_s \text{ for every } t\geq 0 \quad \mathbb{P}-a.s. \tag{1.8}$$

In addition, for any positive $\mathscr{U}$-measurable function f, there exists a sequence of bounded functions $\{f_n\}_{n\in\mathbb{N}}$, where $f_n := f\mathbb{1}_{\{f\leq n\}}$, such that $f_n \uparrow f$ a.e. on U.Therefore, by monotone convergence theorem, (1.8) holds for every positive $\mathscr{U}$-measurable function f.

Similarly, for any positive $\mathscr{P}\otimes\mathscr{U}$-measurable function $Z(\omega,s,u)$, we have

$$(\xi_t^Y(ZH))^p = \int_0^t \int_U Z(\omega,s,u)N_1(\omega,s,du)d\bar{A}_s \text{ for every } t\geq 0 \quad a.s.$$

Let $A_t = \int_0^t e^{-s}ds + \bar{A}_t$.Then A is a.s. strictly increasing with respect to time. Thus, $d\bar{A}_s(\omega) \ll dA_s(\omega)$ a.s. So there exists a $\mathbb{P}\times m$-a.e. nonnegative Radon-Nikodym derivative $\varphi(\omega,s)$ such that $\|\varphi\|\leq 1$ and

$$d\bar{A}_s = \varphi(\omega,s)dA_s \quad \mathbb{P}\times m - a.e.. \tag{1.9}$$

By Fubini theorem, we have

$$\bar{A}_t = \int_0^t \varphi(\omega,s)dA_s \text{ for every } t\geq 0 \quad a.s. \tag{1.10}$$

Let

$$N(\omega,s,\mathbb{1}_U(\cdot)) := \varphi(\omega,s)N_1(\omega,s,\frac{\mathbb{1}_U(\cdot)}{H(\omega,s,\cdot)}) \text{ for every } (\omega,s). \tag{1.11}$$

For each $(\omega,s)\in\Omega\times[0,\infty), n\in\mathbb{N}$, define $E_n(\omega,s) = \{u\in U : H(\omega,s,u)\geq 1/n\}$. Then $U = \cup_{n\geq 1}E_n(\omega,s)$. By the fact that $N_1(\omega,s,U)\leq 1$, and $\varphi(\omega,s)\leq 1$ for almost every (ω,s), we have

$$N(\omega,s,\mathbb{1}_{\{E_n\}}) = \int_{E_n} \frac{\varphi(\omega,s)}{H(\omega,s,u)} N_1(\omega,s,du) \leq n$$

for every (ω,s). Hence $N(\omega,s,\cdot)$ is a σ-finite measure on U for every (ω,s).

Hence, by (1.8),(1.10) and (1.11), the dual predictable projection of $\xi_t^Y f$ is

$$(\xi_t^Y f)^p = \int_0^t N_1(\omega, s, \frac{f}{H(\omega, s, \cdot)})d\bar{A}_s = \int_0^t \int_U f(u)N(\omega, s, du)dA_s \text{ for every } t \geq 0 \quad a.s.,$$

and (N, A) is the pair required in the Lévy system of the σ-finite point process Y.

Similarly, for any positive $\mathscr{P} \otimes \mathscr{U}$-measurable function Z, the dual predictable projection of $\xi_t^Y(Z) = \sum_{s \in \mathbf{D}} \mathbb{1}_{[0,t]}(s)(Z)(\omega, s, Y_s)$ is

$$(\xi_t^Y(Z))^p = \int_0^t N(\omega, s, Z)dA_s = \int_0^t \int_U Z(\omega, s, u)N(\omega, s, du)dA_s \text{ for every } t \geq 0 \quad a.s.$$

□

Remark A.1.16. *(i) From the last few lines of the proof, we note that the pairs in the Lévy system is not unique.*

(ii) One can also refer to Theorem 1.9 in Chapter II of [44], which proves the existence of the pair of an integrable increasing process A and a kernel K for an optional $\mathscr{P} \times \widehat{\mathscr{U}}$-measurable σ-finite random measure μ on $[0, \infty) \times U$.

(iii) For Lévy system for discontinuous Hunt processes, one can also refer to S.Watanbe[76] and Benveniste and Jacod [12]. However, this theorem is about the existence of a Lévy system for an adapted σ-finite quasi-left continuous point process, with no underlying Markov processes involved.

(iv) Notice that in our project, the process $\{X_t, t \geq 0; \mathbb{P}^x, x \in \mathbb{R}^d\}$ corresponding to each $\mathcal{L}$ in $\mathcal{N}(\Lambda_1, \Lambda_2, \Lambda_3)$ is a Hunt process. Let Y_s be the process on $\mathbb{R}^d$ defined as

$$Y_s := \begin{cases} X_s - X_{s^-} & \text{if } X_{s^-} \neq X_s, \\ 0 & \text{if } X_{s^-} = X_s. \end{cases} \tag{1.12}$$

Define $\widetilde{\mathbf{D}} := \{(\omega, s) : s > 0, X_s \neq X_{s^-}\}$. By Proposition 1.2.8 and [44], we know that Lévy system for Y is $(n(x, dz), t)$, where $n(x, dz)$ is our jumping measure, $x \in \mathbb{R}^d, z \in \mathbb{R}^d \setminus \{0\}$. And by the jump measure condition 1.1, $n(x, dz)$ is σ-finite on $\mathbb{R}^d \setminus \{0\}$ for every $x \in \mathbb{R}^d$. Indeed, for each $x \in \mathbb{R}^d, n \in \mathbb{N}$, define $E_n(x) = \{z \in \mathbb{R}^d \setminus \{0\} : |z| \in$

$[1/2^n, 1/2^{n-1})\}$, and $E_0(x) = \{z \in \mathbb{R}^d \setminus \{0\} : |z| \geq 1\}$. Then $|E_n(x)| \leq \Lambda_3 2^{2n}$ for each $n \in \mathbb{N}$, and $|E_0(x)| \leq \Lambda_3$.

Then by Theorem A.1.15, for any positive $\mathscr{B}(\mathbb{R}^d \setminus \{0\})$-measurable function f, if we define the associated random counting measure $\xi_t(\cdot,\cdot) := \sum_{\{s\in\widetilde{\mathbf{D}},0<s\leq t\}} \delta_{(X_{s-},Y_s)}$ on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{0\}$, then the dual predictable projection for $\xi_t(f) = \sum_{\{s\in\widetilde{\mathbf{D}},s\leq t\}} f(Y_s) = \sum_{\{s\in\widetilde{\mathbf{D}},s\leq t\}} f(X_s - X_{s-})$ is $\int_0^t \int_{\mathbb{R}^d\setminus\{0\}} f(z)n(X_{s-}, dz)ds$.

The following theorem is to represent the kernel $N(x, du)$ with $N(x,\cdot)$ defined on a Lusin space U and is a σ-finite measure on $\mathscr{U}$ for each $x \in E$ in terms of some diffusive measure under some conditions. We follows the idea in Theorem 6 in [36], and we give a detailed proof here.

Theorem A.1.17. *Let $(E, \mathscr{E})$ and $(U, \mathscr{U})$be two measurable spaces where U is Lusin. Let N be a kernel from E to U such that $N(x, .)$ is a σ-finite measure on $\mathscr{U}$.*

Then for any σ-finite positive diffusive measure $\bar{\nu}$ on U such that

$$N(x,U) \leq \bar{\nu}(U) \text{ for each } x \in E, \tag{1.13}$$

there exists a $\mathscr{E} \otimes \mathscr{U}$-measurable function $\bar{F}^{\bar{\nu}}(x,u) : E \times U \mapsto U \cup \{\Delta\}$ such that for any $\mathscr{U}$-measurable positive function f,

$$N(x,f) = \int_U f(\bar{F}^{\bar{\nu}}(x,u))\bar{\nu}(du), \tag{1.14}$$

where $\bar{F}^{\bar{\nu}}$ depends on the measure $\bar{\nu}$.

Proof. First, notice that any σ-finite diffusive measure m such that $\bar{\nu}(U) = \infty$ always satisfies (1.13).

Next, By the definition of Lusin space U, there exists a continuous bi-measurable bijection ψ from $(U, \mathscr{U})$ to $([0,1], \mathscr{B}([0,1])$. Define $\psi(\Delta) = \infty$.

Fix a diffusive measure $\bar{\nu}$ satisfying (1.13). For $x \in E$, let $\bar{N}$ and $\bar{\bar{\nu}}$ be measures on $[0,1]$ such that $\bar{N}(x, \psi(C)) := N(x,C)$ and $\bar{\bar{\nu}}(\psi(C)) := \bar{\nu}(C)$ for each $C \in \mathscr{U}$.

Claim:There exists a function $\tau : E \times [0,1] \mapsto [0,1] \cup \{\infty\}$ such that for each $C \in \mathscr{U}$, $\bar{N}(x, \psi(C)) = \bar{\bar{\nu}}(\{\tau(x,t) : \tau(x,t) \in [0,t]\})$, i.e.,

$$N(x,C) = \bar{\nu}(\{u : \bar{F}^{\bar{\nu}}(x,u) \in C\}),$$

where $\bar{F}^{\bar{\nu}}(x,u) = \psi^{-1}(\tau(x,\psi^{-1}(u)))$.

W.L.O.G, it suffices to first consider $U = [0,1]$.

1. Suppose $\bar{N}(x,U)$ is finite for each $x \in E$. For each $x \in E$, let $k(x,U) := \bar{\bar{\nu}}(U)$ and denote $H(x,t) = k(x,[0,t]), G(x,t) = \bar{N}(x,[0,t])$ for $t \in [0,1]$. For each $(x,u) \in E \times [0,1]$, define

$$\tau(x,u) := \begin{cases} \inf\{s \in [0,1] : \ G(x,s) \geq H(x,u)\} & \text{if } H(x,u) \leq G(x,1), \\ \infty & \text{otherwise.} \end{cases}$$

Since $\bar{m}$ is diffusive, then $H(x,t)$ is continuous and nondecreasing for $0 \leq t \leq 1$. Since $G(x,t)$ is right-continuous and nondecreasing for $0 \leq t \leq 1$, then $\tau(x,u)$ is nondecreasing and right-continuous in u. Since $\{\tau(x,u) > t\} = \{G(x,t) \geq H(x,u)\}$ is $\mathscr{E} \otimes \mathscr{U}$-measurable, then $\tau(x,u)$ is $\mathscr{E} \otimes \mathscr{U}$-measurable. Then for every $0 \leq t \leq 1$,

$$\begin{aligned} \int_0^1 \bar{\bar{\nu}}(du)\mathbb{1}_{\{\tau(x,u)\leq t\}} &= \int_0^1 k(x,du)\mathbb{1}_{\{\tau(x,u)\leq t\}} \\ &= \int_0^1 k(x,du)\mathbb{1}_{\{G(x,t)\geq \bar{F}^{\bar{\nu}}(x,u)\}} \\ &= G(x,t) = \bar{N}(x,[0,t]). \end{aligned}$$

Fix $x \in E$. Let

$$\mathscr{C} := \{[0,t] \subset [0,1], 0 \leq t \leq 1\}$$

and

$$\mathscr{D} := \{C \in \mathscr{B}[0,1] : \bar{N}(x,C) = \int_0^1 \mathbb{1}_{\{\tau(x,u)\in C\}}\bar{\bar{\nu}}(du)\}.$$

Then $\mathscr{C}$ is a π-system, and $\mathscr{D}$ is a λ-system such that $\mathscr{C} \subset \mathscr{D}$. By monotone class theorem, $\mathscr{B}[0,1] = \sigma(\mathscr{C}) \subset \mathscr{D}$.

2. If $\bar{N}(x,\cdot)$ is σ-finite on U for each $x \in E$, then there exists a countable partition of disjoint sets $\{A_k(x)\}_{k\in N} \subset [0,1]$ such that $\bar{N}(x,A_k(x)) < \infty$.

   Since $\bar{\bar{\nu}}$ is σ-finite, then there exists a partition $\{E_i\}_{i\geq 1}$of $[0,1]$ such that $\bar{\bar{\nu}}(E_i) < \infty$ for each $i \in \mathbb{N}$. Then if there exists some E_i such that $\bar{\bar{\nu}}(E_i) \geq \bar{N}(x,A_1)$, then since $\bar{m}$ is diffusive, then there exists a set $B_1(x) \subset E_i$ such that $\bar{m}(B_1(x)) = \bar{N}(x, A_1(x))$.Otherwise, since $\bar{N}(x,U) \leq \bar{\bar{\nu}}(U)$, then there must be some integer $\bar{\nu} \geq 2$ such that

$$\sum_{i=1}^{m-1} \bar{\bar{\nu}}(E_i) < \bar{N}(x, A_1(x)) \leq \sum_{i=1}^{m-1} \bar{\bar{\nu}}(E_i).$$

Then by the diffusive property of $\bar{m}$, there exists an $\widetilde{E}_m \subset E_m$ such that $B_1(x) = \cup_{i=1}^{m-1} E_i \cup \widetilde{E}_m$ satisfying $\bar{\bar{\nu}}(B_1(x)) = \bar{N}(x, A_1)$.

Therefore, $\bar{N}(x, A_1^c) \leq \bar{\bar{\nu}}(B_1^c(x))$. Repeat the above procedure. Hence, by induction, for each $x \in E$, there exists a countable partition of disjoint sets $\{B_k(x)\}_{k \in N}$ such that $\bar{\bar{\nu}}(B_k(x)) = \bar{N}(x, A_k)$ for each $k \in \mathbb{N}$, and from (1), for each k, there exists a function $\tau_k(x,u) : B_k(x) \mapsto A_k \cup \{\infty\}$ such that for every $C \in \mathscr{B}(0,1)$, $\bar{N}(x, \mathbb{1}_{A_k \cap C}) = \int \mathbb{1}_{B_k(x)}(u) \mathbb{1}_{\tau_k(x,u) \in \{C \cap A_k\}} \bar{\bar{\nu}}(du)$. Therefore, by letting

$$\tau(x,u) := \begin{cases} \tau_k(x,u) & \text{if } u \in B_k(x), k \in \mathbb{N} \\ \infty & \text{if } u \notin \cup_k B_k(x), \end{cases}$$

we have $\bar{N}(x, \mathbb{1}_C) = \int \mathbb{1}_{\{\tau(x,u) \in C\}} \bar{\bar{\nu}}(du)$.

Define $\bar{F}^{\bar{\nu}}(x,u) := \psi^{-1}(\tau(x, \psi(u)))$ for each $u \in U$.Then $\bar{F}^{\bar{\nu}}$ is a function from $E \times U$ to $U \cup \{\Delta\}$, such that for each $C \in \mathscr{U}$,

$$N(x, \mathbb{1}_C) = \bar{N}(x, \mathbb{1}_{\psi(C)}) = \int \mathbb{1}_{\{\tau(x,u) \in \psi(C)\}} \bar{\bar{\nu}}(du) = \int \mathbb{1}_{\{\bar{F}^{\bar{\nu}}(x,u) \in C\}} \bar{\nu}(du),$$

where $\bar{F}^{\bar{\nu}}$ depends on the mesure m.Therefore, for any positive $\mathscr{U}$-measurable function f,

$$N(x, f) = \int_U f(\bar{F}^{\bar{\nu}}(x,u)) \bar{\nu}(du).$$

$\square$

Remark A.1.18. *The jump process Y defined in (1.12) has a Lévy system $(n(x,dz), t)$, where $n(x,dz)$ is the jump kernel for X, which are σ-finite on $\mathscr{B}(\mathbb{R}^d \setminus \{0\})$ for each $x \in \mathbb{R}^d$. Also notice that $X_{s^-} : (\omega, s) \mapsto \mathbb{R}^d$ is function valued on $\mathbb{R}^d$. Then applying Theorem A.1.17 with $E = \mathbb{R}^d, U = \mathbb{R}^d \setminus \{0\}$, for any σ-finite diffusive measure m on $\mathbb{R}^d \setminus \{0\}$ such* *that* $n(x,\mathbb{R}^d\setminus\{0\})\le\bar\nu(\mathbb{R}^d\setminus\{0\})$, *there exists a* $\mathscr{E}\otimes\mathscr{U}$*-measurable function* $F^{\bar\nu}(x,u):E\times U\mapsto U\cup\{\Delta\}$ *such that for any positive* $\mathscr{B}(\mathbb{R}^d\setminus\{0\})$*-measurable function* f,

$$n(X_{s^-}(\omega),f)=\int_{\mathbb{R}^d\setminus\{0\}}f(F^{\bar\nu}(X_{s^-}(\omega),u))\bar\nu(du)\text{for any }(\omega,s)\in\Omega\times[0,\infty),\tag{1.15}$$

where $F^{\bar\nu}$ *is depends on the measure* $\bar\nu$.

Now we quote a Doob's theorem from [32, 4.44] to prove Theorem A.1.22. As usual, we introduce some concepts that are used in Doob's theorem.

Definition A.1.19. *A* σ*-algebra* $\mathscr{G}$ *on* Ω *is said to be a separable if it is generated by some sequence* $\{H_n\}_{n\ge1}$ *of subsets of* Ω. *Then let* $\mathcal{F}_n=\sigma(H_1,...,H_n)$, *we obtain a filtration* $(\mathcal{F}_n)$ *such that*

$$\mathcal{F}_\infty=\lim_{n\to\infty}\mathcal{F}_n=\vee_n\mathcal{F}_n=\mathscr{G}.$$

For example, if $\Omega=[0,1]$, its Borel σ-algebra $\mathscr{G}$ is separable: Let $P_n=([0,1/2^n],(1/2^n,2/2^n],(2/2^n,3/2^n],...,(1-1/2{,}1])$ and $\mathcal{F}_n=\sigma(P_n)$ for each $n\in\mathbb{N}$.Then the partition P_n on $[0,1]$ is more and more refined as n increases. So $\mathscr{G}$ is a separable σ-algebra if and only if it is generated by a sequence $\{P_n\}_{n\ge1}$ of finite partitions of $[0,1]$.

Since any Lusin space U is homeomorphic to $[0,1]$, then there is a bijection is $\phi:U\to[0,1]$. For each $n\in\mathbb{N}$, let the partition $\bar P_n$ on U be $(\phi^{-1}[0,1/2^n],\phi^{-1}(1/2^n,2/2^n],...,\phi^{-1}(1-1/2^n,1])$. and set $\mathcal{F}_n=\sigma(\bar P_n)$. Then we have $\mathscr{U}=\vee_n\mathcal{F}_n$. Hence the Borel σ-algebra $\mathscr{U}$ is separable.

The following is Doob's theorem.

Theorem A.1.20. *Let* S *be a set and* $\mathscr{G}$ *a separable* σ*-algebra on it. Let* $(E,\mathscr{E})$ *be an arbitrary measurable space. Let* $P,\mathbb{Q}$ *be two a bounded kernels both from* $(E,\mathscr{E})$ *to* $(S,\mathscr{G})$. *Suppose that, for each* x *in* E, *the measure* $H\to\mathbb{Q}(x,H)$ *is absolutely continuous with respect to the measure* $H\to P(x,H)$. *Then there exists a positive* $\mathscr{E}\otimes\mathscr{G}$*-function* $Z(x,\omega)$ *such that*

$$\mathbb{Q}(x,H)=\int_H Z(x,\omega)P(x,d\omega),x\in E,H\in\mathscr{G}.$$

Remark A.1.21. *The Doob's theorem in [32, 4.44] states the condition when* P *is a probability kernel, and we modify it as a bounded kernel for our purpose. The proof follows the same line in [32, 4.44].*

Theorem A.1.22. *Let $(E,\mathscr{E})$ and $(U,\mathscr{U})$ be two measurable spaces where U is Lusin. Let N be a kernel from E to U such that $N(x,.)$ is a σ-finite measure on $\mathscr{U}$ for each $x \in E$. Then there exists a family of $\mathscr{E}\otimes\mathscr{U}$-measurable sub-Markovian kernel $\widehat{Q}(x,u,\cdot)$ from $(E\times U)$ to $\mathscr{U}$, such that for every σ-finite diffusive measure $\bar{\nu}$ satisfying (1.13), every positive $\mathscr{U}$-measurable function f, and every $A \in \mathscr{U}$, we have*

$$\int_U \mathbb{1}_A(u) f(F^{\bar{\nu}}(x,u))\bar{\nu}(du) = \int_U \widehat{Q}(x,u,A) f(u) N(x,du).$$

Moreover, when $N(x,dz)$ is the jump kernel for the jump process Y of a Hunt process X defined in (1.12), we have the dual predictable projection of $\sum_{0<s\le t} \mathbb{1}_{\{s\in\widetilde{\mathbf{D}}\}}\widehat{Q}(X_{s^-},Y_s,A)$ satisfies that

$$[\sum_{0<s\le t} \mathbb{1}_{\{s\in\widetilde{\mathbf{D}}\}}\widehat{Q}(X_{s^-},Y_s,A)]^p = \int_0^t \int_{\mathbb{R}^d\setminus\{0\}} \mathbb{1}_A(u)\mathbb{1}_{\{F^{\bar{\nu}}(X_{s^-},u)\neq 0\}}\bar{\nu}(du)ds \quad a.s.,$$

where $F^{\bar{\nu}}$ is the function defined in Remark A.1.18, $\widetilde{\mathbf{D}} := \{(\omega,s) : s>0, X_s \neq X_{s^-}\}$.

Proof. This proof is based on the idea of Theorem 8 in [36], and we modify it for our purpose and add more details. Since $N(x,dz)$ is σ-finite on U for each $x\in E$, then for each $x \in E$, there exists a sequence of disjoint sets $\{E_n(x)\}_{\{n\ge 1\}} \subset \mathscr{U}$ and $U = \cup_{n\ge1} E_n(x)$, such that $N(x,E_n(x)) < \infty$ for each $x \in E, n\in N$. Define $h(x,u) := \sum_{n\ge1}\frac{1}{2^n}\mathbb{1}_{E_n(x)}(u)$, for each $(x,u)\in E\times U$, then $h(x,u)$ is a strictly positive function on $E\times U$, and

$$\int_U h(x,u)N(x,du) < \infty \text{ for every } x \in E. \tag{1.16}$$

Fix $A, C \in \mathscr{U}$, and consider the following two $\mathscr{E}$-measurable kernels from $E\times U$ to $\mathscr{U}$:

$$\begin{aligned} \mu_A^1(x,C) &= \int_U \mathbb{1}_A(u) h(x,F^{\bar{\nu}}(x,u))\mathbb{1}_C(F^{\bar{\nu}}(x,u))\bar{\nu}(du) \\ &= \int_A h(x,F^{\bar{\nu}}(x,u))\mathbb{1}_C(F^{\bar{\nu}}(x,u))\bar{\nu}(du) \end{aligned}$$

and

$$\mu^2(x,C) = \int_U h(x,F^{\bar{\nu}}(x,u))\mathbb{1}_C(F^{\bar{\nu}}(x,u))\bar{\nu}(du).$$

Then for every $x\in E$, $\mu_A^1(x,C)$ and $\mu^2(x,C)$ are two measures on $\mathscr{U}$ such that

$$\mu_A^1(x,dv) \ll \mu^2(x,dv).$$

Also, $\mathscr{U}$ is a separable σ-algebra on U.Then by Doob's theorem A.1.20, there exists a Radon-Nikodym derivative $\varphi_A(x,v)$ defined on $E\times U$ such that $\varphi_A(x,v)\leq 1$ and

$$\mu^1_A(x,dv)=\varphi_A(x,v)\mu^2(x,u,dv) \text{ for every } (x,v).$$

Since for fixed $A\in\mathscr{U}$, both $\mu^1_A(x,dv)$ and $\mu^2(x,dv)$ are both $\mathscr{E}\otimes\mathscr{U}$-measurable, then $\varphi_A(x,v)$ is $\mathscr{E}\otimes\mathscr{U}$-measurable. Define $\widehat{Q}(x,u,A)=\varphi_A(x,F^{\bar{\nu}}(x,u),A)\leq 1$ for each $(x,u)\in E\times U$ and $A\in\mathscr{U}$, which means that $\widehat{Q}(x,u,A)$ is a sub-Markovian kernel from $E\times U$ to $\mathscr{U}$. Since $\bar{F}^m$ is $\mathscr{E}\otimes\mathscr{U}$-measurable, then for fixed $A\in\mathscr{U}$, $\widehat{Q}(\cdot,\cdot,A)$ is $\mathscr{E}\otimes\mathscr{U}$-measurable. Then for every positive $\mathscr{U}$-measurable function f, $A\in\mathscr{U}$, $x\in E$,

$$\begin{aligned}
\int_U h(x,v)f(v)\bar{\nu}(dv) &= \mu^1_A(x,f)\\
&= \int_U \varphi_A(x,F^{\bar{\nu}}(x,u),A)h(x,F^{\bar{\nu}}(x,u))f(F^{\bar{\nu}}(x,u))\bar{\nu}(du)\\
&= \mu^2(x,f\widehat{Q}(x,\cdot,A))\\
&= \int_U \widehat{Q}(x,u,A)h(x,F^{\bar{\nu}}(x,u))f(F^{\bar{\nu}}(x,u))\bar{\nu}(du). \qquad (1.17)
\end{aligned}$$

Since h is strictly positive, divide $h(x,u)$ (where on the left hand side, it is of the form $h(x,F^{\bar{\nu}}(x,u))$ from both sides of (1.17).Then for every $x\in E$, $A\in\mathscr{U}$, positive $\mathscr{U}$-measurable function f, by Theorem A.1.17,

$$\int_U \mathbb{1}_A(u)f(F^{\bar{\nu}}(x,u))\bar{\nu}(du)=\int_U \widehat{Q}(x,u,A)f(u)N(x,du). \qquad (1.18)$$

Similarly, replacing f with any $\mathscr{E}\otimes\mathscr{U}$-measurable positive function $Z(x,u)$ in (1.18),

$$\int_U \mathbb{1}_A(u)Z(x,F^{\bar{\nu}}(\omega,s,u))\bar{\nu}(du)=\int_U \widehat{Q}(x,u,A)Z(x,u)N(x,du).$$

Fix $A\in\mathscr{U}$, $\widehat{Q}(\cdot,\cdot,A)$ is a positive $\mathscr{E}\otimes\mathscr{U}$-measurable.

When $N(x,du)$ is the jump kernel for the jump process Y defined in (1.12), with X_{s^-} is $\mathscr{P}$-measurable, Y is quasi-left continuous, then by Remark A.1.16, the dual predictable projection of $\xi_t(\widehat{Q}(X_{s^-},Y_s,A))=\sum_{\{0<s\leq t,s\in\widetilde{\mathbf{D}}\}}\widehat{Q}(X_{s^-},Y_s,A)$ is

$$[\sum_{\{0<s\leq t,s\in\widetilde{\mathbf{D}}\}}\widehat{Q}(X_{s^-},Y_s,A)]^p=\int_0^t\int_{\mathbb{R}^d\setminus\{0\}}\widehat{Q}(X_{s^-},u,A)N(X_{s^-},du)ds.$$

Therefore, by (1.18),

$$
\begin{aligned}
\Big(\sum_{\{0<s\leq t, s\in\widetilde{\mathbf{D}}\}} Q(X_{s^-}, Y_s, A)\Big)^p &= \int_0^t \int_{\mathbb{R}^d\setminus\{0\}} \widehat{Q}(X_{s^-}, u, A) N(X_{s^-}, du) ds \\
&= \int_0^t \int_{\mathbb{R}^d\setminus\{0\}} \mathbb{1}_A(u) \mathbb{1}_{\{F^{\bar{\nu}}(X_{s^-}, u)\neq 0\}} \bar{\nu}(du) ds.
\end{aligned}
$$

That the indicator $\mathbb{1}_{\{F^{\bar{\nu}}(X_{s^-},u)\neq 0\}}$ appears follows from the fact that (1.18) is only valid for any positive $\mathscr{U}$-measurable function on U. Here $F^{\bar{\nu}}$ is the function defined in Remark A.1.18(iv). □

Remark A.1.23. *For any σ-finite quasi-left continuous Point process Y defined in a Lusin space U, whose Lévy system is $(N(\omega, s, du), t)$ as shown in Theorem A.1.15, apply Theorem A.1.22 with $E = \Omega \times \mathbb{R}^+$, we also have a $\mathscr{P} \times \mathscr{U}$-measurable sub-Markovian kernel $Q(\omega, s, u, A)$ such that for every $Q(\omega, s, u, A)$ from $(\Omega \times \mathbb{R}^+ \times U)$ to $\mathscr{U}$, such that for every σ-finite diffusive measure m satisfying* (1.13)*, every positive $\mathscr{U}$-measurable function f, and every $A \in \mathscr{U}$, we have*

$$
\int_U \mathbb{1}_A(u) f(\bar{F}^{\bar{\nu}}(\omega, s, u)) \bar{\nu}(du) = \int_U Q(\omega, s, u, A) f(u) N(\omega, s, du),
$$

where $\bar{F}^{\bar{\nu}}$ is the function in Theorem A.1.17.

Next, we follow the idea of Lemma 9-Theorem 12 in [36] for σ-finite quasi-left continuous Point process Y on a Lusin space U, whose Lévy system is $(N(\omega, s, du), t)$, to construct a new point process $\bar{Y}$ on U based on the sub-markovian kernel $Q(\omega, s, u, A)$, and prove that $\bar{Y}$ is a Poisson point process. After we give the details of the proof in each theorem, we apply the result to our jump process on $\mathbb{R}^d \setminus \{0\}$ defined in (1.12).

Denote a point on the space $\widehat{U}^{\mathbb{R}^+}$ as $\tilde{u}$ and let $\tilde{u}_t$ be the coordinate of $\tilde{u}$, taking values on $\widehat{U}$. By letting

$$
Q(\omega, s, u, \{\Delta\}) = 1 - Q(\omega, s, u, U), \tag{1.19}
$$

we extend the sub-Markov kernel $Q(\omega, s, u, \cdot)$ to be Markovian from $(\Omega, \mathbb{R}^+) \times U)$ to $\widehat{U}$.

Next define a new space $\bar{\Omega} = \Omega \times \widehat{U}^{\mathbb{R}^+}$ with the induced σ-algebra $\bar{\mathscr{G}} = \mathscr{F} \otimes \widehat{\mathscr{U}}^{\otimes\mathbb{R}^+}$.

For each $A_i \in \widehat{\mathscr{U}}$, let $\bar{Q}$ be a new kernel from $(\Omega \times \mathbb{R}^+ \times \widehat{U})$ to $\widehat{\mathscr{U}}$ such that

$$\bar{Q}(\omega, t_i, Y_{t_i}, A_i) := \begin{cases} Q(\omega, t_i, Y_{t_i}, A_i) & \text{if } Y_{t_i}(\omega) \neq \Delta, \\ \delta_{A_i}(\Delta) & \text{otherwise.} \end{cases} \tag{1.20}$$

Hence,

$$\bar{Q}_\omega(\widetilde{u}_{t_1} \in A_1, ..., \widetilde{u}_{t_n} \in A_n) = \Pi_{i=1}^n \bar{Q}(\omega, t_i, Y_{t_i}, A_i)$$

defines a transition kernel from $(\Omega, \mathscr{F})$ to $(\widehat{U}^{\mathbb{R}^+}, \widehat{\mathscr{U}}^{\otimes \mathbb{R}^+})$. On the probability space $(\bar{\Omega}, \bar{\mathscr{G}})$, we define a probability $\bar{\mathbb{P}}$ such that $d\bar{\mathbb{P}} = d\mathbb{P}(\omega)\bar{Q}_\omega(\cdot)$ and the σ-algebra $\bar{\mathscr{G}}$, which is complete with respect to $\bar{\mathbb{P}}$.

For each $\omega \in \Omega$, let $D_\omega = \{t > 0 : Y_t \neq \Delta\}$. Then we define a new point process $\bar{Y}$ on $\bar{\Omega}$ as follows:

$$\bar{Y}_t(\omega, \widetilde{u}) := \begin{cases} Y_0(\omega) & \text{if } t = 0, \\ \widetilde{u}_t & \text{if } t \in D_\omega \text{ and } t > 0, \\ \Delta & \text{if } t \notin D_\omega \text{ and } t > 0, \end{cases} \tag{1.21}$$

It is not hard to see that if Y is σ-discrete, so is $\bar{Y}$. Let $\bar{\xi}$ be the random counting measure $\bar{\xi}_t(\omega, \widetilde{u})$ associated with $\bar{Y}$ on $(\mathbb{R}^+ \times U, \mathscr{B}(\mathbb{R}^+) \times \mathscr{U})$ as

$$\bar{\xi}_t(\omega, \cdot) = \sum_{\{s \in D_\omega, 0 < s \leq t\}} \delta_{\{s, \bar{Y}_s(\omega)\}}.$$

To show that $\bar{Y}$ is a Possion point process, we first need to define the filtration $\{\bar{\mathscr{G}}_t\}_{t \geq 0}$, each of which is a sub σ-algebra of $\bar{\mathscr{G}}$ and that $\bar{Y}$ is adapted with. Let $(\bar{\mathscr{G}}_t)$ be the augmented filtration of $\bar{Y}$ by including all the $\bar{\mathbb{P}}$-negligible sets.

Lemma A.1.24. *The probability $\bar{Q}_\omega$ induces a transition kernel from $(\Omega, \mathscr{F}_t)$ to $(\widehat{U}^{(0,t]}, \widehat{\mathscr{U}}^{(0,t]})$ for each $t > 0$.*

Proof. First, $\bar{Q}_\omega(\widehat{U}^{(0,t]}) = \Pi_{\{0 < s \leq t, s \in \mathbf{D}_\omega\}} \bar{Q}(\omega, s, Y_s, \widehat{U}) = 1$.

For each $B \in \widehat{\mathscr{U}}^{(0,t]}$, denote $B_\tau := \pi_\tau(B)$, where π_τ is the projection map on the τ-th component such that $0 < \tau \leq t$. Then we have

$$\bar{\mathbb{E}}^{\bar{Q}_\omega}[\bar{\xi}_t(B)] = \Pi_{\{0 < s \leq t, s \in \mathbf{D}_\omega\}} \bar{Q}(\omega, s, Y_s, B_s).$$

Let $\widehat{\mathbf{D}}$ is a countable union of graph of quasi-left continuous jump times, i.e, $\widehat{\mathbf{D}} = \{Y_t(\omega) \neq \Delta\} = \{T_1, T_2, ..., T_k, ...\}$.Then

$$\Pi_{\{0<s\leq t, s\in\widehat{\mathbf{D}}\}}\bar{Q}(\omega, s, Y_s, \pi_s(B)) = \Pi_k \mathbb{1}_{\{T_k\leq t\}}\bar{Q}(\omega, T_k, Y_{T_k}, B_{T_k}).$$

Since T_k is $\mathscr{F}_{T_k}$-measurable, $B_{T_k} = \pi_{T_k}(B)$ is $\mathscr{F}_{T_k}$-measurable. Since the point process Y is quasi-left continuous, then Y is progressively measurable. Therefore, Y_{T_k} is thus $\mathscr{F}_{T_k}$-measurable.

Also, fix $A_i \in \mathscr{U}$ in the definition of $\bar{Q}$ in (1.20), then $\bar{Q}$ is $\mathscr{P}\otimes\widehat{\mathscr{U}}$-measurable. Hence for every $k \in \mathbb{N}$, $\bar{Q}(\omega, T_k, Y_{T_k}, B_{T_k})$ is $\mathscr{F}_{T_k}$-measurable. Since $\mathscr{F}_{T_k} \subset \mathscr{F}_t$ on $\{T_k \leq t\}$. Therefore, $\Pi_{\{0<s\leq t, s\in\widehat{\mathbf{D}}\}}\bar{Q}(\omega, T_k, Y_{T_k}, B_{T_k})$ is $\mathscr{F}_t$-measurable.

On the other hand, fix $\omega \in \Omega$, $\bar{Q}_\omega(\widehat{U}^{(0,t]}) = 1$. And for any set $B \in \widehat{U}^{(0,t]}$,

$$\bar{Q}_\omega((\bar{Y}_s)_{0<s\leq t} \in B) = \Pi_{\{0\leq s\leq t, s\in D_\omega\}}\bar{Q}(\omega, s, Y_s, B_s) = \Pi_{\{0\leq T_k\leq t\}}\bar{Q}(\omega, T_k, Y_{T_k}, B_{T_k}).$$

For any sequence of sets $\{B_j\}_{j\in\mathbb{N}} \subset \widehat{U}$, such that $B_j \cap B_i = \emptyset$, for any $i, j \in \mathbb{N}$, by the definition of product measure,

$$\bar{Q}_\omega(\cup_{j\in\mathbb{N}}B_j) = \sum_{j=1}^{\infty}\Pi_{\{0\leq T_k\leq t\}}\bar{Q}(\omega, T_k, Y_{T_k}, B_{j,T_k}) = \sum_{j=1}^{\infty}\bar{Q}_\omega(B_j) \leq 1,$$

and when $B_j = \hat{U}$ for every $j \in \mathbb{N}$, then

$$\bar{Q}_\omega(\cup_{j\in\mathbb{N}}\hat{U}) = \sum_{j=1}^{\infty}\bar{Q}_\omega(\hat{U}) = 1.$$

Hence, for each $\omega \in \Omega$, $\bar{Q}_\omega$ is probability measure in $\widehat{\mathscr{U}^{(0,t]}}$.

□

The following theorem is about the Lévy system of the point process $\bar{Y}$.

Theorem A.1.25. *Let H be the $\mathscr{P}\otimes\mathscr{U}$-measurable strictly positive function on $(\Omega, [0,\infty, U)$ such that $\mathbb{E}(\xi^Y(H)] < \infty$, where ξ^Y_t is the random counting measure associated with the original Point process Y. Then for any $B \in U$, the dual predictable projection of $\bar{\mu}_t(B) := \sum_{0<s\leq t} H(\omega, s, Y_s(\omega))\mathbb{1}_{\{\bar{Y}_s(\omega,u)\in B\}}$ with respect to $\bar{\mathscr{G}}_t$ is*

$$\bar{\nu}_t(B) = \int_0^t ds \int \mathbb{1}_{\{u\in B\}}H(\omega, s, \bar{F}^{\bar{\nu}}(\omega, s, u))\bar{\nu}(du),$$

where $\bar{\nu}$ is denoted as the intensity measure of $\bar{\mu}$ on U, and F is function corresponding to $\bar{\nu}$ in Theorem A.1.17 with $E = \Omega \times \mathbb{R}^+$.

Moreover, the random counting measure $\bar{\xi}_t(B) = \sum_{0<s\le t} \delta_{\{\bar{Y}_s(\omega,u)\in B\}}$ has a dual predictable projection with respect to the filtration $\bar{\mathscr{G}_t}$ with

$$\int_0^t ds \int_U \mathbb{1}_B(u) \mathbb{1}_{\{\bar{F}^{\bar{\nu}}(\omega,s,u)\neq\Delta\}} \bar{\nu}(du), \tag{1.22}$$

which indicates that the point process $\bar{Y}$ is quasi-left continuous with Lévy system $(\mathbb{1}_{\{\bar{F}^{\bar{\nu}}(\omega,s,u)\neq\Delta\}}\bar{\nu}, t)$.

Proof. By the definition of dual predictable projection, it suffices to show that $(\bar{\mu}_t(B) - m_t(B))_{t\ge 0}$ is a martingale with respect to the filtration $\{\bar{\mathscr{G}_t}\}_{t\ge0}$.

By Theorem A.1.24, we have shown that $\bar{\mu}_t(B) - \bar{\mu}_s(B)$ is independent of $\bar{\mathscr{G}_s}$ under the law of $\bar{Q}_\omega$. Hence for any $A \in \mathscr{F}_s$, by the definition of $\bar{Y}(\omega, \tilde{u}) \in B$, and (1.19),

$$\begin{aligned}
\bar{\mathbb{E}}[\mathbb{1}_A(\bar{\mu}_t(B) - \bar{\mu}_s(B))] \quad &= \bar{\mathbb{E}}[\bar{\mathbb{E}}^{\bar{Q}_\omega}[\mathbb{1}_A(\bar{\mu}_t(B) - \bar{\mu}_s(B))]] \\
&= \mathbb{E}[\bar{\mathbb{E}}^{\bar{Q}_\omega}(\mathbb{1}_A)\bar{\mathbb{E}}^{\bar{Q}_\omega}[\bar{\mu}_t(B) - \bar{\mu}_s(B)]] \\
&= \mathbb{E}[\bar{\mathbb{E}}^{\bar{Q}_\omega}(\mathbb{1}_A) \sum_{\{s<v\le t, v\in\widehat{\mathbf{D}}\}} H(\omega, v, Y_v)\bar{Q}(\omega, v, Y_v, B)] \\
&= \mathbb{E}[\bar{\mathbb{E}}^{\bar{Q}_\omega}(\mathbb{1}_A) \sum_{\{s<v\le t, v\in\widehat{\mathbf{D}}\}} H(\omega, v, Y_v)Q(\omega, v, Y_v, B)].
\end{aligned}$$

Since Lévy system of Y is (N, t) and by Theorem A.1.22 with $E = \Omega \times [0, \infty)$,

$$\begin{aligned}
&\mathbb{E}[\bar{\mathbb{E}}^{\bar{Q}_\omega}(\mathbb{1}_A) \sum_{\{s<v\le t, s\in\mathbf{D}\}} H(\omega, v, Y_v)Q(\omega, v, Y_v, B)] \\
&= \mathbb{E}[\bar{\mathbb{E}}^{Q_\omega}(\mathbb{1}_A) \int_s^t dv \int_U N(\omega, s, du)H(\omega, s, u)Q(\omega, s, u, B)] \\
&= \mathbb{E}[\bar{\mathbb{E}}^{\bar{Q}_\omega}(\mathbb{1}_A) \int_s^t dv \int_U H(\omega, s, \bar{F}^{\bar{\nu}}(\omega, s, u))\mathbb{1}_{\{u\in B\}}\bar{\nu}(du)] \\
&= \mathbb{E}[\bar{\mathbb{E}}^{\bar{Q}_\omega}(\mathbb{1}_A)(\bar{\nu}_t(B) - \bar{\nu}_s(B))] \\
&= \bar{\mathbb{E}}[\mathbb{1}_A(\bar{\nu}_t(B) - \bar{\nu}_s(B))].
\end{aligned} \tag{1.23}$$

The second part of the theorem follows by dividing $H(\omega, v, Y_v)$ on the left-hand side) from both sides of (1.23).

□

To obtain a Poisson point process $\widehat{Y}$ whose Lévy system is $(\bar{\nu}(du), t)$, and the above characteristic measure misses the part $\mathbb{1}_{\{\bar{F}^{\bar{\nu}}(\omega,s,u)=\Delta\}}\bar{\nu}(du)$, which corresponds to the scenario when the original point process $Y_s(\omega) = \Delta$. So we complement this part by introducing an auxiliary probability space and a Poisson point process with Lévy system $(\mathbb{1}_{\{\bar{F}^{\bar{\nu}}(\omega,s,u)=\Delta\}}\bar{\nu}, t)$ on that probability space to construct a Poisson point process whose characteristic measure is $\bar{\nu}$.

Define a Poisson point process Z on the probability space $(W, \mathscr{A}, \mathscr{A}_t, \widetilde{P})$, whose characteristic measure is m. Let Π_t be the associated Poisson random measure.
Denote $\widehat{\Omega}$ for the space $\bar{\Omega} \times W$, $\widehat{\mathscr{G}}$ for $\bar{\mathscr{G}} \times \mathscr{A}$, $\widehat{\mathbb{P}}$ for the probability on $(\widehat{\Omega}, \widehat{\mathscr{G}})$ defined by $\bar{\mathbb{P}} \times \widetilde{\mathbb{P}}$, and $\widehat{\mathscr{F}_t}$ for the increasing complete right-continuous filtration constructed by the pair $\bar{\mathscr{G}_t} \times \mathscr{A}_t$.

Theorem A.1.26. *Consider the above defined space $(\widehat{\Omega}, \widehat{\mathscr{F}}, \widehat{\mathscr{F}}_t, \widehat{\mathscr{P}})$. The process $\widehat{Y}_t(\omega, \widetilde{u}, \tilde{\omega})$ is defined as follows :*

$$\widehat{Y}_t(\omega, \widetilde{u}, \tilde{\omega}) := \begin{cases} \bar{Y}_t(\omega, \widetilde{u}) \ \text{if } Y_t(\omega) \neq \Delta \text{ and } Z_t(\tilde{\omega}) = \Delta; \\ Z_t(\tilde{\omega}) \quad \text{if } Y_t(\omega) = \Delta \text{ and } Z_t(\tilde{\omega}) \neq \Delta \text{ and } \bar{F}^{\bar{\nu}}(\omega, t, Z_t(\tilde{\omega})) = \Delta; \\ \Delta \qquad \text{otherwise,} \end{cases} \tag{1.24}$$

is a Poisson point process with characteristic measure m.

Proof. The idea of the proof is the same as in [36], we just elaborate it here.
Let $C = \{(\omega, \tilde{u}, \tilde{\omega}) : \text{if there exists } t > 0 \text{ s.t. } Z_t(\tilde{\omega}) \neq \Delta \text{ and } Y_t(\omega) \neq \Delta\}$. Given $B \in \mathscr{U}$ such that $m(B) < \infty$, let $T_1, ..., T_n, ...$ be the successive jumping times of the process $\{\Pi_t(B)\}_{t\geq 0}$.

Fix $\omega \in \Omega$, recall that $D_\omega = \{t > 0 : Y_t \neq \Delta\}$, which is is countable, and notice that $Y_{T_n(\tilde{\omega})}(\omega) \neq \Delta \Rightarrow T_n \in D_\omega$.Since T_n follows Gamma distribution, then by the definition of $\bar{Y}$ in(1.21), $\widetilde{\mathbb{P}}(T_n(\tilde{\omega}) \in D_\omega) = 0$. In addition, since $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are independent, then $\widehat{\mathbb{P}}(Y_{T_n(\tilde{\omega})}(\omega) \neq \Delta) = \mathbb{P}[\widetilde{\mathbb{P}}(T_n \in D_\omega)] = 0$.

Therefore,

$$\widehat{\mathbb{P}}(C) = 0, \tag{1.25}$$

i.e., $\widehat{\mathbb{P}}$ almost surely, the processes Z_t and Y_t won't jump at the same time. Recall that $\{t > 0 : \bar{Y}_t(\omega, \widetilde{u}) \neq \Delta\} \subset D_\omega \quad \mathbb{P}-$a.s., then $\widehat{\mathbb{P}}$ almost surely, the processes Z_t and $\bar{Y}_t$ won't jump at the same time as well.

Next, denote $\widehat{\xi}$ as the associated random counting measure of $\widehat{Y}$. By Theorem A.1.11, to show that $\widehat{Y}$ is a Poisson point process, it suffices to show that $(\widehat{\xi}_t(B) - t\bar{\nu}(B))$ is a $(\widehat{\mathscr{F}_t})$-martingale.

By the definition of $\widehat{Y}_t$,

$$\widehat{\xi}_t(B) = \sum_{0<s\leq t} \mathbb{1}_{\{\bar{Y}_s \in B\}} \mathbb{1}_{\{Z_s=\Delta\}} + \sum_{0<s\leq t} \mathbb{1}_{\{\bar{Y}_s=\Delta\}} \mathbb{1}_{\{Z_s(\tilde{\omega})\in B\}} \mathbb{1}_{\{\bar{F}^{\bar{\nu}}(\omega,s,Z_t(\tilde{\omega}))=\Delta)\}}.$$

We know that

$$\widehat{\mathbb{E}}(\mathbb{1}_{\{Z_s=\Delta\}} \mathbb{1}_{\{\bar{Y}_s\in B\}}) = \bar{\mathbb{P}}(\bar{Y}_s \in B)\widetilde{\mathbb{P}}(Z_s(\tilde{\omega}) = \Delta) = \bar{\mathbb{P}}(\bar{Y}_s \in B), \tag{1.26}$$

where the first equality is due to the independence of $\bar{Y}$ and Z, and the second equality is because each jumping time of the Poisson point process $\{Z_s\}_{s\geq 0}$ follows continuous probability distribution, which leads to $\widetilde{\mathbb{P}}(\widetilde{T}_k = s) = 0 \Leftrightarrow \widetilde{\mathbb{P}}(Z_s(\tilde{\omega}) = \Delta) = 1$. Hence

$$\widehat{\mathbb{E}}(\mathbb{1}_{\{\bar{Y}_s=\Delta\}} \mathbb{1}_{\{Z_s(\tilde{\omega})\in B\}}) = \bar{\mathbb{P}}(\bar{Y}_s = \Delta)\widetilde{\mathbb{P}}(Z_s(\tilde{\omega}) \in B), \tag{1.27}$$

Since

$$\bar{\mathbb{P}}(Y_s \neq \Delta)\widetilde{\mathbb{P}}(Z_s(\tilde{\omega}) \in B) = 0.$$

then

$$\bar{\mathbb{P}}(\bar{Y}_s = \Delta)\widetilde{\mathbb{P}}(Z_s(\tilde{\omega}) \in B) = \widetilde{\mathbb{P}}(Z_s(\tilde{\omega}) \in B). \tag{1.28}$$

By (1.26), (1.27) and (1.28),

$$\widehat{\xi}_t(B) = \bar{\xi}_t(B) + \sum_{0<s\leq t} \mathbb{1}_{\{\bar{F}^{\bar{\nu}}(\omega,s,Z_s(\tilde{\omega}))=\Delta\}} \mathbb{1}_{\{Z_s(\tilde{\omega})\in B\}},$$

where $\bar{\xi}$ is the random counting measure associated with $\bar{Y}$. Recall that by (1.22),

$$(\bar{\xi}_t(B))^p = \int_0^t \int_U \mathbb{1}_{\{u\in B\}} \mathbb{1}_{\{\bar{F}^{\bar{\nu}}(\omega,s,u)\neq\Delta\}} \bar{\nu}(du)ds, \tag{1.29}$$

Also, given the Lévy system of the Poisson point process $\{Z_s(\tilde{\omega})\}_{s\geq 0}$ is (m, t), we have

$$(\sum_{0<s\leq t} \mathbb{1}_{\{\bar{F}^{\bar{\nu}}(\omega,s,Z_s(\tilde{\omega}))=\Delta\}} \mathbb{1}_{\{Z_s(\tilde{\omega})\in B\}})^p = \int_0^t ds \int_U \mathbb{1}_{\{u\in B\}} \mathbb{1}_{\{\bar{F}^{\bar{\nu}}(\omega,s,u)=\Delta\}} \bar{\nu}(du). \tag{1.30}$$

Therefore, by (1.29), (1.30), $(\widehat{\xi}_t(B) - \int_0^t ds \int_U \mathbb{1}_{\{u\in B\}}\bar{\nu}(du))$ is a $(\widehat{\mathscr{F}_t})$-martingale for any $B \in \mathscr{U}$ s.t. $\bar{\nu}(du)(B) < \infty$, and thus $\widehat{Y}$ is a Poisson point process whose characteristic measure is m.

Theorem A.1.27. *The point process Y with Lévy system (N, t) is the image of the Poisson point process $\widehat{Y}$ under the function $\bar{F}^{\bar{\nu}}$ such that*

$$Y_t = \bar{F}^{\bar{\nu}}(\omega, t, \widehat{Y}(\omega, \widetilde{u}, \tilde{\omega})) \quad \widehat{\mathbb{P}} - a.s. \tag{1.31}$$

Proof. Since

$$\{(\omega, \widetilde{u}, \tilde{\omega}) : Y_t = \bar{F}^{\bar{\nu}}(\omega, t, \widehat{Y}_t), Y_t \neq \Delta)\} = \{(\omega, \widetilde{u}, \tilde{\omega}) : Y_t = \bar{F}^{\bar{\nu}}(\omega, t, \bar{Y}_t), Y_t \neq \Delta\},$$

then it suffices to show that $\{(\omega, \widetilde{u}, \tilde{\omega}) : Y_t \neq \bar{F}^{\bar{\nu}}(\omega, t, \widehat{Y}(\omega, \widetilde{u}, \tilde{\omega}))\}$ is $\widehat{\mathbb{P}}$-negligible.
Notice that

$$\begin{aligned}
&\{(\omega, \widetilde{u}, \tilde{\omega}) : \exists\, t \text{ such that } \bar{F}^{\bar{\nu}}(\omega, t, \widehat{Y}_t) \neq Y_t, Y_t \neq \Delta\}.\\
&= \{(\omega, \widetilde{u}, \tilde{\omega}) : \exists\, t \text{ such that } \bar{F}^{\bar{\nu}}(\omega, t, \widehat{Y}_t) = \bar{F}^{\bar{\nu}}(\omega, t, \bar{Y}_t), \bar{F}^{\bar{\nu}}(\omega, t, \widehat{Y}_t) \neq Y_t, Y_t \neq \Delta\}\\
&= \{(\omega, \widetilde{u}, \tilde{\omega}) : \exists\, t \text{ such that } \bar{F}^{\bar{\nu}}(\omega, t, \bar{Y}_t) \neq \widehat{Y}_t, t \in D_\omega\}\\
&= \emptyset \quad \widehat{\mathbb{P}} - a.s.
\end{aligned} \tag{1.32}$$

In addition,

$$\begin{aligned}
&\widehat{\mathbb{E}}[\sum_{0<s\le t} \mathbb{1}_{\{\bar{F}^{\bar{\nu}}(\omega,s,\widehat{Y}_s(\omega))\neq\Delta\}}\mathbb{1}_{\{\bar{F}^{\bar{\nu}}(\omega,s,\widehat{Y}_s(\omega))\neq Y_s(\omega)\}}]\\
&= \widehat{\mathbb{E}}[\sum_{0<s\le t} \mathbb{1}_{\{\bar{F}^{\bar{\nu}}(\omega,s,\bar{Y}_s(\omega))\neq\Delta\}}\mathbb{1}_{\{\bar{F}^{\bar{\nu}}(\omega,s,\bar{Y}_s(\omega))\neq Y_s(\omega)\}}]\\
&= \widehat{\mathbb{E}}[\sum_{0<s\le t} Q(\omega, s, Y_s, \mathbb{1}_{\{\bar{F}^{\bar{\nu}}(\omega,s,\bar{Y}_s(\omega))\neq\Delta\}}\mathbb{1}_{\bar{F}^{\bar{\nu}}(\omega,s,\bar{Y}_s(\omega))\neq Y_s(\omega)\}})]\\
&= \mathbb{E}[\int_0^t \int_U \bar{\nu}(du)\mathbb{1}_{\{\bar{F}^{\bar{\nu}}(\omega,s,u)\neq\Delta\}}\mathbb{1}_{\{\bar{F}^{\bar{\nu}}(\omega,s,u)\neq n(\omega,s,u)\}}] = 0 \text{ for all } t \ge 0.
\end{aligned} \tag{1.33}$$

The last two equalities is due to Theorem A.1.25.
Hence the increasing process $(\sum_{0<s\le t} \mathbb{1}_{\{\bar{F}^{\bar{\nu}}(\omega,s,\widehat{Y}_s(\omega))\neq\Delta\}}\mathbb{1}_{\{\bar{F}^{\bar{\nu}}(\omega,s,\widehat{Y}_s(\omega))\neq Y_s(\omega)\}})$ is evanescent.
Therefore,

$$\{(\omega, \widetilde{u}, \tilde{\omega}) : \exists\, t \text{ such that } Y_t(\omega) \neq \bar{F}^{\bar{\nu}}(\omega, t, \widehat{Y}_t(\omega, \widetilde{u}, \tilde{\omega})), Y_t \neq \Delta\} \tag{1.34}$$

is a null set under $\widehat{\mathbb{P}}$. Therefore, by(1.32) and (1.33),

$$\{(\omega, \widetilde{u}, \tilde{\omega}) : \exists\, t \text{ such that } Y_t(\omega) \neq \bar{F}^{\bar{\nu}}(\omega, t, \widehat{Y}_t(\omega, \widetilde{u}, \tilde{\omega}))\}$$

is a $\widehat{\mathbb{P}}$-negligible set.

Remark A.1.28. *Apply the same extension and definition in (1.19),(1.20) to $\widehat{Q}(X_{s-}, Y_s, \cdot)$ that is obtained in Theorem A.1.22, for the jump process Y on $\mathbb{R}^d \setminus \{0\}$ defined in* (1.12). *Then $\widehat{Q}(X_{s-}, u, \{0\}) = 1 - \widehat{Q}(X_{s-}, u, \mathbb{R}^d \setminus \{0\})$. And for each $A_i \in \mathbb{R}$, let $\bar{\widehat{Q}}$ be a new kernel from $(\mathbb{R}^d \times \mathbb{R}^d)$ to $\mathbb{R}$ such that*

$$\bar{\widehat{Q}}(X_{t_i-}(\omega), Y_{t_i}, A_i) := \begin{cases} \widehat{Q}(X_{t_i-}, Y_{t_i}, A_i) & \textit{if } X_{t_i-}(\omega) \neq X_{t_i(\omega)}, \\ \delta_{A_i}(0) & \textit{otherwise.} \end{cases} \tag{1.35}$$

Apply the argument in Theorem A.1.24,Theorem A.1.26, Theorem A.1.27, and we can obtain a Poisson process $\widehat{Y}$ such that $Y_t = F^{\bar{\nu}}(X_{t-}, \widehat{Y}_t)$ $\quad \widehat{\mathbb{P}} - a.s.$, such that

$$\widehat{Y}_t(\omega, \widetilde{u}, \tilde{\omega}) := \begin{cases} \bar{Y}_t(\omega, \widetilde{u}) \ \textit{if } X_{t-}(\omega) \neq X_t \ \textit{and } Z_t(\tilde{\omega}) = 0; \\ Z_t(\tilde{\omega}) \quad \textit{if } X_{t-}(\omega) = X_t \ \textit{and } Z_t(\tilde{\omega}) \neq 0 \ \textit{and } F^{\bar{\nu}}(X_{t-}, Z_t(\tilde{\omega})) = 0; \\ 0 \quad \textit{otherwise.} \end{cases}$$

□

The following is to rewrite the proof of Theorem 13 in [36] in details.

Let M be a quasi-left continuous, square-integrable martingale on $\mathbb{R}^d$, and define:

$$Y_t := \begin{cases} \Delta M_t & \text{if } \Delta M_t \neq 0; \\ 0 & \text{otherwise.} \end{cases} \tag{1.36}$$

Since M square integrable, then $\mathbb{E}[\sum_{s \le t}(\Delta M_s)^2] < \infty$ for every $t > 0$, then Y is σ-finite. Hence the process Y is a adapted σ-finite quasi-left continuous point process.

Assume that the Lévy system of Y is $(N(\omega, s, du), t)$,we want to show that the compensated sum of the totally inaccessible jumps of the square integrable martingale M is a stochastic integral with respect to a Poisson martingale, where $N(\omega, s, du)$ is a kernel from $(\Omega \times [0, \infty))$ to $\mathscr{B}(\mathbb{R}^d \setminus \{0\})$.

Theorem A.1.29. *Let M be a square-integrable and quasi-left continuous martingale defined on the probability space $(\Omega, \mathscr{F}_t, \mathscr{F}, \mathbb{P})$, taking values on $\mathbb{R}^d$. The associated jump process ΔM is absolutely has a Lévy system (N, t).Then there exists two probability spaces $(\widetilde{\Omega}, \widetilde{\mathscr{F}}, \widetilde{\mathscr{F}_t})$, $(\widehat{\Omega}, \widehat{\mathscr{F}}, \widehat{\mathscr{F}_t})$, a d-dimensional Brownian Motion W, a Poisson point process $\widehat{Y}$ with σ-finite intensity measure $\bar{\nu}$, independent of W, a predictable $d \times d$ matrix process $\{\sigma(\omega, t)\}_{t>0}$, a $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}^d \setminus \{0\})$-measurable function $F(\omega, s, u)$ defined on $\Omega \times \mathbb{R}^+ \times \mathbb{R}^d \setminus \{0\}$ satisfying :*

$$\mathbb{E} \int_0^t \langle \theta, \sigma\sigma^*(\omega, s)\theta \rangle ds < \infty \text{ for every } \theta \in \mathbb{R}^d \setminus \{0\},$$

and

$$\mathbb{E} \int_0^t ds \int_{\mathbb{R}^d \setminus \{0\}} (\bar{F}^{\bar{\nu}})^2(\omega, s, u) \bar{\nu}(du) < \infty,$$

such that

$$M_t = M_0 + \int_0^t \sigma(\omega, s) dB_s + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} \bar{F}^{\bar{\nu}}(\omega, s, u)(\mu - \nu)(du, ds),$$

where μ is the Poisson random measure associated with the Poisson point process $\widehat{Y}$ on $(\mathbb{R}^+ \times \mathbb{R}^d \setminus \{0\}, \mathscr{B}(\mathbb{R}^+) \times \mathscr{B}(\mathbb{R}^{d \setminus \{0\}}))$, and ν is its intensity measure.

Proof. For every square-integrable martingale, it could be uniquely decomposed into

$$M_t = M_t^c + M_t^d,$$

where M^c is the continuous part and M^d is the discontinuous part of M.

Let $\{Y_t\}_{t \geq 0}$ be a point process defined in (1.36). Then for any set $A \in \mathscr{B}(\mathbb{R}^d)$, define the associated counting measure γ^Y on $(\mathbb{R}^+ \times \mathbb{R}^d \setminus \{0\}, \mathscr{B}(\mathbb{R}^+) \times \mathscr{B}(\mathbb{R}^d \setminus \{0\}))$ as

$$\gamma^Y((0, t] \times A) = \sum_{s>0} \delta_{\{(s, Y_s) \in ((0,t] \times A)\}}.$$

Since M is square-integrable, i.e., for every $t > 0$,

$$\mathbb{E}[\sum_{0<s\leq t} (M_s - M_{s^-})^2] \leq \mathbb{E}[M_t^2] < \infty, \tag{1.37}$$

then the measure γ is σ-finite. By Theorem A.1.26 and A.1.27, there exists a probability space $(\widehat{\Omega}, \widehat{\mathscr{F}}, \widehat{\mathscr{F}}_t, \widehat{\mathbb{P}})$, a Poisson point process $\widehat{Y}$ with intensity measure $\bar{\nu}$ and a $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}^d \setminus$

$\{0\})$- measurable process $\bar{F}^{\bar{\nu}}(\omega, s, u)$ such that

$$Y_t = \bar{F}^{\bar{\nu}}(\omega, t, \widehat{Y}_t) \quad \widehat{\mathbb{P}}\text{- a.s.,} \tag{1.38}$$

and

$$\begin{aligned} & \widehat{\mathbb{E}}[\int_0^t ds \int_{\mathbb{R}^d} (\bar{F}^{\bar{\nu}})^2(\omega, s, u)\bar{\nu}(du)] = \mathbb{E}[\int_0^t ds \int_{\mathbb{R}^d} (\bar{F}^{\bar{\nu}})^2(\omega, s, u)\bar{\nu}(du)] \\ = \; & \mathbb{E}[\sum_{0<s\le t} (Y_s)^2] < \infty. \end{aligned}$$

For any $A \in \mathscr{B}(\mathbb{R}^d)$, let $\mu((0,t] \times A) = \sum_{s>0} \mathbb{1}_{\{(s,\widehat{Y}_s)\in(0,t]\times A\}}$. By (1.38),

$$\mu((0,t] \times A) := \sum_{0<s\le t} \mathbb{1}_{\{\bar{F}^{\bar{\nu}}(\omega,s,\widehat{Y}_s)\in \bar{F}^{\bar{\nu}}(\omega,s,A)\}}, \tag{1.39}$$

and μ is a Poisson measure on $\mathbb{R}^+ \times \mathbb{R}^d$ with intensity measure ν which satisfies

$$\nu((0,t] \times A) = \mathbb{E}[\mu((0,t] \times A)] = t\bar{\nu}(A). \tag{1.40}$$

Then by (1.31), (1.39), $\gamma((0,t) \times \bar{F}^{\bar{\nu}}(\omega, s, A)) = \mu((0,t] \times A) \qquad \mathbb{P} - a.s.$

Define a process $M^{d,p}$ as $M_0^{d,p} = 0$ and for each $t \ge 0$,

$$\begin{aligned} M_t^{d,p} &= \sum_{0<s\le t} (\Delta M_s)\mathbb{1}_{\{\frac{1}{p}<|\Delta M_s|\le p\}} - \int_0^t \int_{\mathbb{R}^d} \bar{F}^{\bar{\nu}}(\omega, s, u)\mathbb{1}_{\{\frac{1}{p}<|\bar{F}^{\bar{\nu}}(\omega,s,u)|\le p\}}\bar{\nu}(du)ds. \\ &= \sum_{0<s\le t} \bar{F}^{\bar{\nu}}(\omega, s, \widehat{Y}_s)\mathbb{1}_{\{\frac{1}{p}<|\Delta M_s|\le p\}} - \int_0^t \int_{\mathbb{R}^d} \bar{F}^{\bar{\nu}}(\omega, s, u)\mathbb{1}_{\{\frac{1}{p}<|\bar{F}^{\bar{\nu}}(\omega,s,u)|\le p\}}\bar{\nu}(du)ds \end{aligned}$$

Since $\bar{F}^{\bar{\nu}}$ is a positive function, by (1.37), for every $p > 0$,

$$\mathbb{E}[|M_t^{d,p}|^2] \le \mathbb{E}[\sum_{0<s\le t} |\Delta M_s|^2\mathbb{1}_{\{\frac{1}{p}<|\Delta M_s|\le p\}}] \le \mathbb{E}[\sum_{0<s\le t} |M_s - M_{s-}|^2] \le \mathbb{E}[[|M_t|^2] < \infty.$$

Therefore,

$$\lim_{p\to\infty} \mathbb{E}[|M_t^{d,p}|^2] \le \mathbb{E}[|M_t|^2] < \infty,$$

i.e. $M_t^{d,p}$ converges uniformly in $\mathcal{L}^2(\Omega)$ as $p \to \infty$.

Denote M_t^d as the $\mathcal{L}^2$-limit, $t \ge 0$.Then there exists a subsequence $\{M_t^{d,p_k}\}_{k\in N}$ of $\{M_t^{d,p}\}_{p\in N}$ for each $t \ge 0$ such that $M_t^{d,p_k} \to M_t^d$, $\mathbb{P}$-a.s. such that $|M_t^d| < \infty, \mathbb{P}$-a.s., as $p_k \to \infty$.

Then by dominated convergence theorem,

$$\mathbb{E}[M_t^d|\mathscr{F}_s] = \mathbb{E}[\lim_{p_k\to\infty} M_t^{d,p_k}|\mathscr{F}_s] = \lim_{p_k\to\infty} \mathbb{E}[M_t^{d,p_k}|\mathscr{F}_s] = \lim_{p_k\to\infty} M_s^{d,p_k} = M_s^d.$$

Therefore, M_t^d is also a martingale with respect to $\{\mathscr{F}_t\}$, and by (1.39) and (1.40),

$$M_t^d = \sum_{0<s\le t} \bar{F}^{\bar{\nu}}(\omega, s, \widehat{Y}_s) - \int_0^t ds \int_{\mathbb{R}^d\setminus\{0\}} \bar{F}^{\bar{\nu}}(\omega, s, u)\bar{\nu}(du) = \int_0^t \int_{\mathbb{R}^d\setminus\{0\}} \bar{F}^{\bar{\nu}}(\omega, s, u)(\mu-\nu)(du, ds).$$

Hence $M_t^c := M_t - M_t^d$ is a continuous square-ntegrable martingale starting with $M_0^c = 0$.

By Theorem 3.4 in [66, Chapter V], for any continuous martingale starting from 0, such that $d\langle M_t^c, M_t^c\rangle \ll dt$, there exists an auxiliary probability space $(\widetilde{\Omega}, \widetilde{\mathscr{F}}, \widetilde{\mathbb{P}})$, and by defining

$$\Omega' = \Omega \times \widetilde{\Omega}, \mathscr{F}' = \mathscr{F} \otimes \widetilde{\mathscr{F}}, \mathbb{P}' = \mathbb{P} \otimes \widetilde{\mathbb{P}},$$

there exists a d-dimensional $\{\mathscr{F}_t'\}_{t\ge 0}$-measurable d-dimensional Brownian motion W and a $\mathscr{P}$-measurable matrix process $\{\sigma(\omega, s)\}_{s>0}$ such that

$$M_t^c = M_0 + \int_0^t \sigma(\omega, s) dW_s \quad \mathbb{P} - a.s.$$

Finally we will show that W_t and $\widehat{Y}_t$ are independent under $\mathbb{P}$. Since by definition and the uniqueness of M^c and M^d, then $\mathbb{E}[M_t^c M_t^d] = 0, t \ge 0$. Then for a d-dimensional vector function $\phi(s) = \sum_{j=1}^d u_j \mathbb{1}_{(t_j, t_{j+1}]}$, where those $(t_j, t_{j+1}]$'s are disjoint.

Let

$$U_t := \exp\{i \int_0^t \langle \phi(s), dW_s\rangle + \frac{1}{2}\int_0^t |\phi(s)|^2 ds\}.$$

Then (U_t) is an adapted continuous martingale starting from $U_0 = 1$.

Similarly, define $\mu_t := \mu((0,t] \times \mathbb{R}^d) = \sum_{s>0} \mathbb{1}_{\{(s,\widehat{Y}_s)\in(0,t]\times\mathbb{R}^d\}}$, and a d-dimensional vector function $\varphi(s) = \sum_{j=1}^d v_j \mathbb{1}_{\Gamma_j} \times \mathbb{1}_{(s_j, s_{j+1}]}$, where $\Gamma_j \in \mathscr{B}(\mathbb{R}^d \setminus \{0\})$ and those $\Gamma_j \times (s_j, s_{j+1}]$'s are disjoint,

$$\begin{aligned}
& \exp\{i \int_0^t \int_{\mathbb{R}^d} \varphi\mu(ds, du)\} \\
= \;& \exp\{i\mu_t(\varphi)\} \\
= \;& \sum_{0<s\le t} \exp\{i\mu_s(\varphi) - i\mu_{s^-}(\varphi)\} \\
= \;& 1 + \sum_{0<s\le t} \exp\{i\mu_{s^-}(\varphi)\}[\exp\{i\varphi(s, \widehat{Y}_s)\} - 1]\mathbb{1}_{\{\widehat{Y}_s \ne 0\}}
\end{aligned}$$

Notice that $\widehat{\mathbb{P}}$-almost surely, $\{s \le t : \widehat{Y}_s \ne 0\}$ is countable.

Let $K_t := U_t \exp\{i\mu_t(\varphi)\}$, then

$$\mathbb{E}(K_t) = \mathbb{E}\left[U_t(\exp\{\mu_{s^-}(\varphi)\}(\exp\{i\varphi(s, \widehat{Y}_s)\} - 1)\right]$$

$$
\begin{aligned}
&= \mathbb{E}\Big[1 + \sum_{0<s\le t} [U_{s^-} \exp\{i\mu_{s^-}(\varphi)\}(\exp\{i\varphi(s,\widehat{Y}_s)\} - 1)] \\
&\quad +(U_s - U_{s^-}) \exp\{i\mu_{s^-}(\varphi)\}(\exp\{i\varphi(s,\widehat{Y}_s)\} - 1)]\Big] \\
&= \mathbb{E}\Big[1 + \sum_{0<s\le t} [U_{s^-} \exp\{i\mu_{s^-}(\varphi)\}(\exp\{i\varphi(s,\widehat{Y}_s)\} - 1)]\Big] \\
&= 1 + \mathbb{E}\int_0^t K_s \int_{\mathbb{R}^d\setminus\{0\}} (\exp\{i\varphi(s,u)\} - 1)\bar{\nu}(du)ds,
\end{aligned}
$$

where the last second equality is due to the predictability of U_s.

Therefore,

$$
\mathbb{E}(K_t) = 1 \cdot \exp(\int_0^t ds \int_{\mathbb{R}^d} (\exp\{i\varphi(s,u)\} - 1)\bar{\nu}(du)),
$$

i.e.,

$$
\begin{aligned}
&\mathbb{E}\left[\exp\left(i\int_0^t \langle\phi(s), dW_s\rangle\right)\exp\left(i\int_0^t \varphi(s,u)\mu_t(ds,du)\right)\right] \\
&= \mathbb{E}\left[\exp\left(i\int_0^t \langle\phi(s), dW_s\rangle\right)\right]\mathbb{E}\left[\exp\left(i\int_0^t \exp\{i\varphi(s,u)\mu_t(ds,du)\right)\right].
\end{aligned}
$$

Therefore W_t and $\widehat{Y}_t$ are independent under $\mathbb{P}$.

Definition A.1.30. *A set $M \subset C_b(S)$ is* **separating** *if whenever $P, Q \in \mathscr{P}(S)$ and*

$$
\int f dP = \int f dQ, \quad \forall f \in M,
$$

we have $P = Q$.

The following result is known, see, e.g., [37, Theorem 3.12 of Chapter 4].

Theorem A.1.31. *Let S be a separable space. Let $\mathcal{A} \subset C_b(S) \times B(S)$, i.e., the graph of $\mathcal{A}$, $(f, \mathcal{A}f) \subset C_b(S) \times B(S)$, and suppose that $\mathscr{D}(\mathcal{A})$ is separating. Let X be a solution of the martingale problem for $\mathcal{A}$ with respect to $\{\mathscr{F}_t\}_{t\ge 0}$, having sample paths in $\mathcal{B}_S[0,\infty)$. Let $\tau_1 \le \tau_2 \le \ldots$ be a sequence of $\{\mathscr{F}_t\}$-stopping times and $\tau := \lim_{n\to\infty} \tau_n$. Then*

$$
\mathbb{P}\left(\lim_{n\to\infty} X_{\tau_n} = X_\tau, \tau < \infty\right) = \mathbb{P}(\tau < \infty).
$$

In particular, $\mathbb{P}(X_t = X_{t^-}) = 1$ for each $t > 0$.

Proof. For any $f \in \mathscr{D}(\mathcal{A})$, since X is a solution of the martingale problem for $\mathcal{A}$, for every $t \geq 0$,

$$f(X_{\tau_n \wedge t}) - \int_0^{\tau_n \wedge t} \mathcal{A}f(X_s)ds = \mathbb{E}\left[f(X_{\tau \wedge t}) - \int_0^{\tau \wedge t} \mathcal{A}f(X_s)ds \Big| \mathcal{F}_{\tau_n \wedge t}\right].$$

Hence with $\mathcal{G} := \sigma\{\mathcal{F}_{\tau_n}, n \in \mathbb{N}\}$, we have

$$\begin{aligned} f(\lim_{n\to\infty} X_{\tau_n \wedge t}) &= \lim_{n\to\infty} f(X_{\tau_n \wedge t}) = \lim_{n\to\infty} \mathbb{E}\left[f(X_{\tau \wedge t}) - \int_{\tau_n \wedge t}^{\tau \wedge t} \mathcal{A}f(X_s)ds \Big| \mathcal{F}_{\tau_n}\right] \\ &= \mathbb{E}[f(X_{\tau \wedge t})|\mathcal{G}], \end{aligned} \tag{1.41}$$

After sending $t \to \infty$, we have

$$f(\lim_{n\to\infty} X_{\tau_n}) = \mathbb{E}[f(X_\tau)|\mathcal{G}] \quad \text{on } \{\tau < \infty\} \tag{1.42}$$

In particular, $\mathbb{E}\left[\mathbb{1}_{\{\tau<\infty\}} f(\lim_{n\to\infty} X_{\tau_n})\right] = \mathbb{E}\left[\mathbb{1}_{\{\tau<\infty\}} f(X_\tau)\right]$ for every $f \in \mathscr{D}(\mathcal{A})$. Since $\mathscr{D}(\mathcal{A})$ is separating, it follows that $\lim_{n\to\infty} X_{\tau_n}$ and X_τ have the same distribution on $\{\tau < \infty\}$. Consequently,

$$\begin{aligned} & \mathbb{E}\left[\mathbb{1}_{\{\tau<\infty\}}(f(\lim_{n\to\infty} X_{\tau_n}) - f(X_\tau))^2\right] \\ = \ & \mathbb{E}\left[\mathbb{1}_{\{\tau<\infty\}}\left(f(\lim_{n\to\infty} X_{\tau_n})^2 - 2f(\lim_{n\to\infty} X_{\tau_n})f(X_\tau)) + f(X_\tau)^2\right)\right] \\ = \ & \mathbb{E}\left[\mathbb{1}_{\{\tau<\infty\}}\left(f(\lim_{n\to\infty} X_{\tau_n})^2 - 2\mathbb{E}[f(\lim_{n\to\infty} X_{\tau_n})f(X_\tau))|\mathcal{G}] + f(X_\tau)^2\right)\right] \\ = \ & \mathbb{E}\left[\mathbb{1}_{\{\tau<\infty\}} f(X_\tau)^2\right] - \mathbb{E}\left[\mathbb{1}_{\{\tau<\infty\}} f(\lim_{n\to\infty} X_{\tau_n})^2\right] \\ = \ & 0. \end{aligned}$$

Hence $f(\lim_{n\to\infty} X_{\tau_n}) = f(X_\tau)$ $\mathbb{P}$-a.s. on $\{\tau < \infty\}$ for every $f \in \mathscr{D}(\mathcal{A}) \subset C_b(S)$. We claim that $\lim_{n\to\infty} X_{\tau_n} = X_\tau$ $\mathbb{P}$-a.s. on $\{\tau < \infty\}$. Suppose not, then there is a Borel set $C \subset S$ so that $\mathbb{1}_{\{\tau<\infty\}}\mathbb{1}_C(\lim_{n\to\infty} X_{\tau_n}) \neq \mathbb{1}_{\{\tau<\infty\}}\mathbb{1}_C(X_\tau)$ with positive probability. Let

$$\Omega_1 := \left\{\tau < \infty, \ \lim_{n\to\infty} X_{\tau_n} \in C \text{ and } X_\tau \in C^c\right\}$$

and

$$\Omega_2 := \left\{\tau < \infty, \ X_\tau \in C \text{ and } \lim_{n\to\infty} X_{\tau_n} \in C^c\right\}.$$

Then either $\mathbb{P}(\Omega_1) > 0$ or $\mathbb{P}(\Omega_2) > 0$. Without loss of generality, assume that $\mathbb{P}(\Omega_1) > 0$. Denote by μ_1 and μ_2 the probability distribution of the random variables $\lim_{n\to\infty} X_{\tau_n}$

and X_τ restricted to the probability space $(\Omega_1, \mathbb{P}(\Omega_1)^{-1}\mathbb{P}|_{\Omega_1})$. Clearly $\mu_1 \neq \mu_2$ as μ_1 concentrates on C while μ_2 concentrates on C^c. On the other hand, for every $f \in \mathscr{D}(\mathcal{A})$, $f(\lim_{n\to\infty} X_{\tau_n}) = f(X_\tau)$ $\mathbb{P}$-a.s. on Ω_1, and so

$$\int_S f(x)\mu_1(dx) = \mathbb{P}(\Omega_1)^{-1}\mathbb{E}\left[\mathbb{1}_{\Omega_1} f(\lim_{n\to\infty} X_{\tau_n})\right] = \mathbb{P}(\Omega_1)^{-1}\mathbb{E}\left[\mathbb{1}_{\Omega_1} f(X_\tau)\right] = \int_S f(x)\mu_2(dx).$$

As $\mathscr{D}(\mathcal{A})$ is separating, the above yields that $\mu_1 = \mu_2$, which is a contradiction. This proves the claim that $\lim_{n\to\infty} X_{\tau_n} = X_\tau$ $\mathbb{P}$-a.s. on $\{\tau < \infty\}$. □

The following gives a sufficient condition for a set $M \subset C_b(S)$ to be separating.

Theorem A.1.32. *Suppose that (S, d) is a complete and separable metric space, and $M \subset C_b(S)$ is an algebra. If M separates points in S, then M is separating.*

Proof. See Theorem 4.5 of Chapter 3 in[37]. □

Proposition A.1.33. *Suppose that there is a solution $\{X_t, t \geq 0; \mathbb{P}^x\}$ to the martingale problem starting from x relative to the operator $\mathcal{L} \in \mathcal{N}(\Lambda_1, \Lambda_2, \Lambda_3)$, where $n(x, dz)$ is the jumping kernel of X. There exists a $d \times d$ matrix σ, two probability space $(\widehat{\Omega}, \widehat{\mathscr{F}}, \widehat{\mathscr{F}_t}), (\widetilde{\Omega}, \widetilde{\mathscr{F}}, \widetilde{\mathscr{F}_t})$, a $\mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^d \setminus \{0\})$-measurable function $F(x, u)$, a Brownian motion W_t, a Poisson point process $\widehat{Y}_t$ with σ-finite intensity measure λ satisfying (3.1) , independent of W_t, satisfying the property that*

$$\int_{\mathbb{R}^d} f(u)n(x, du) = \int_{\mathbb{R}^d} f(F(x, u))\lambda(du) \text{ for each Borel measurable function } f,$$

so that $\{X_t, t \geq 0\}$ satisfies the following SDE $\mathbb{P}^x$-a.s.:

$$\begin{aligned} dX_t &= \sigma(X_t)dW_t + b(X_t)dt + \int_{\{|F(X_{t-},z)|\leq 1\}} F(X_{t-}, z)(\mu - \nu)(dz, dt) \\ &\quad + \int_{\{|F(X_{t-},z)|>1\}} F(X_{t-}, z)\mu(dz, dt) \\ X_0 &= x, \end{aligned} \tag{1.43}$$

where μ is the Poisson random measure satisfying (3.4) with the intensity measure ν.

Proof. The idea follows the proof for Theorem II_{10} in [56]. Since $\{\mathbb{P}^x, x \in \mathbb{R}^d; X_t, t > 0\}$ is a solution to the martingale problem associated with $\mathcal{L} \in \mathcal{N}(\Lambda_1, \Lambda_2, \Lambda_3)$ starting from x.

Then

$$\widetilde{M}_t = f(X_t) - f(x) - \int_0^t \mathcal{L}f(X_s)ds \tag{1.44}$$

is a martingale for any $f \in C^2(\mathbb{R}^d)$.

Notice that for any $f \in C_c^2(\mathbb{R}^d)$, $\mathcal{L}f \in C_c(\mathbb{R}^d) \subset B(\mathbb{R}^d)$ for any $\mathcal{L} \in \mathcal{N}(\Lambda_1, \Lambda_2, \Lambda_3)$.

Since $C_c^2(\mathbb{R}^d)$ is an algebra and separates points, then by Theorem A.1.32, $C_c^2(\mathbb{R}^d)$ is separating. As $\{X_t\}_{t\geq 0}$ is càdlàg, the solution $\{X_t\}_{t\geq 0}$ is quasi-left continuous by Theorem A.1.31.

Next, for $f(x) \in C^2(\mathbb{R}^d)$, $f(x)\mathbb{1}_{\{|h|>\varepsilon\}}$,$f(x+h)\mathbb{1}_{\{|h|>\varepsilon\}}$ and $\langle h, \nabla f(x)\mathbb{1}_{\{|h|>\varepsilon\}}\rangle$ are both nonnegative, and

$$\mathbb{E}^x[\int_0^t \int_{\mathbb{R}^d} h\cdot\nabla f(X_{s^-})\mathbb{1}_{\{\varepsilon<|h|\leq 1\}} n(X_{s^-}, dh)ds] = \mathbb{E}^x[\sum_{0<s\leq t}\langle \nabla f(X_{s^-}), \Delta X_s\rangle \mathbb{1}_{\{\varepsilon<|\Delta X_s|\leq 1\}})].$$

Then

$$\begin{aligned} &\mathbb{E}^x[\int_0^t \int_{\mathbb{R}^d}(f(X_{s^-}+h) - f(X_{s^-}))\mathbb{1}_{\{|h|>\varepsilon\}} - h\cdot\nabla f(X_{s^-})\mathbb{1}_{\{\varepsilon<|h|\leq 1\}} n(X_s, dh)ds] \\ = \;& \mathbb{E}^x[\sum_{0<s\leq t}(f(X_s) - f(X_{s^-})\mathbb{1}_{\{|h|>\varepsilon\}} - \langle \nabla f(X_{s^-}), \Delta X_s\rangle \mathbb{1}_{\{\varepsilon<|\Delta X_s|\leq 1\}})]. \end{aligned}$$

Sending $\varepsilon \to 0$, it yields

$$\begin{aligned} &\mathbb{E}^x[\int_0^t \int_{\mathbb{R}^d}(f(X_{s^-}+h) - f(X_{s^-})) - h\cdot\nabla f(X_{s^-})\mathbb{1}_{\{|h|\leq 1\}} n(X_s, dh)ds] \\ = \;& \mathbb{E}^x[\sum_{0<s\leq t}(f(X_s) - f(X_{s^-}) - \langle \nabla f(X_{s^-}), \Delta X_s\rangle \mathbb{1}_{\{||\Delta X_s|\leq 1\}})]. \end{aligned} \tag{1.45}$$

Therefore, by (1.44) and (1.45),

$$\begin{aligned} H_t^f : \;=\; & f(X_t) - f(x) - \int_0^t \textstyle\sum_{i,j=1}^d a_{ij}(X_{s^-})\dfrac{\partial^2 f(X_{s^-})}{\partial x_i \partial x_j} + \displaystyle\sum_{i=1}^d b_i(X_{s^-})\frac{\partial f(X_{s^-})}{\partial x_i}ds \\ & - \sum_{0<s\leq t}\Big((f(X_s) - f(X_{s^-})) - \langle \nabla f(X_{s^-}), \Delta X_s\rangle \mathbb{1}_{\{|\Delta X_s|\leq 1\}}\Big) \end{aligned}$$

is also a martingale under $\mathbb{P}^x$. Let $f = \langle \theta, x\rangle$, where θ is an arbrary vector in $\mathbb{R}^d$, then $\frac{\partial^2 f(x)}{\partial x_i \partial x_j} = 0, i, j = 1, .., d$, and

$$H_t^f \;=\; \langle \theta, X_t - x\rangle - \int_0^t \sum_{i=1}^d \theta^i b_i(X_{s^-})ds - \sum_{0<s\leq t}(\langle \theta, \Delta X_s\rangle - \sum_{i=1}^d \theta^i \Delta X_s^i \mathbb{1}_{\{|\Delta X_s|\leq 1\}})$$

$$= \quad \langle \theta, X_t - x\rangle - \int_0^t \sum_{i=1}^d \theta^i b_i(X_{s^-})ds - \sum_{0<s\le t} (\langle \theta, \Delta X_s\rangle \mathbb{1}_{\{|\Delta X_s|>1\}})$$

is a martingale under $\mathbb{P}^x$. Specifically, letting $\langle \theta_i, M_t\rangle = M_t^i, i = 1, ..., d$, where

$$M_t = X_t - x - \int_0^t b(X_{s^-})ds - \sum_{0<s\le t} \Delta X_s \mathbb{1}_{\{|\Delta X_s|>1\}}, \tag{1.46}$$

each component of M_t is a martingale, and then M_t is a martingale in $\mathbb{R}^d$.

Since X is a Hunt process, then the jumping times of X is quasi-left continuous. Let $\bar{A}(x)$ denote the diffusion matrix (a_{ij}). For any $\theta \in \mathbb{R}^d$, $(\langle \theta, M_t\rangle)$ is a martingale, then by Lemma 9 of [56], Lévy jump kernel assumption (1.3), for each $t \ge 0$,

$$\mathbb{E}^x[\langle \theta, M_t\rangle^2] \quad = \int_0^t \langle \theta, \bar{A}(X_{s^-})\theta\rangle ds + \int_0^t \int_{\mathbb{R}^d} \mathbb{1}_{\{|h|\le 1\}} \langle \theta, h\rangle^2 n(X_{s^-}, dh)ds < \infty,$$

then $\langle \theta, M_t\rangle$ is square-integrable, whose quadratic variation is absolutely continuous with respect to time.

Since each M_t^i is square integrable such that $[M^i, M^i]_t = \langle M^i, M^i\rangle_t + \sum_{0<s\le t} (\Delta M_s^i)^2 = \int_0^t a_{ii}(X_{s^-})ds + \int_0^t \int_{\mathbb{R}^d} \mathbb{1}_{\{|h|\le 1\}} |h_i|^2 n(X_{s^-}, dh)ds < \infty$, then $d[M^i, M^i]_t \ll dt$, i.e., M is a square-integrable martingale vector in $\mathbb{R}^d$ with each component of $[M, M]_t$ is absolutely continuous with respect to time.

Moreover, by Theorem A.1.17, Theorem A.1.27, there exists a probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathcal{F}_t})$, a Poisson point process $\widehat{Y}$ with characteristic measure m, and a $\mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^d \setminus \{0\})$-measurable function $F(x, u)$ such that

$$\Delta X_t = F(X_{t^-}, \widehat{Y}_t). \tag{1.47}$$

Let μ be the Poisson random measure associated with the process $\widehat{Y}$ defined on $([0, \infty) \times \mathbb{R}^d \setminus \{0\}, \mathscr{B}([0, \infty)) \times \mathscr{B}(\mathbb{R}^d \setminus \{0\})$ with intensity measure ν. Then $\nu((0, t] \times A) = t\lambda(A)$ for any $A \in \mathscr{B}(\mathbb{R}^d \setminus \{0\})$. Then the dual predictable projection of $\sum_{0<s\le t} \Delta X_s \mathbb{1}_{\{|\Delta X_s|>1\}}$ satisfies

$$\begin{aligned}(\sum_{0<s\le t} \Delta X_s \mathbb{1}_{\{|\Delta X_s|>1\}})^p \quad &= \quad \int_0^t \int_{|h|>1} h n(X_{s^-}, dh)ds \\ &= \quad \int_0^t \int_{|F(X_{s^-}, u)|>1} F(X_{s^-}, u)\nu(duds)\end{aligned}$$

$$= \int_0^t \int_{|F(X_{s^-},u)|>1} F(X_{s^-},u)\lambda(du)ds < \infty, \qquad (1.48)$$

where the last inequality is due to the Lévy jump kernel assumption (1.3).

Since $\Delta M_t = \Delta X_t \mathbb{1}_{\{|\Delta X_t| \le 1\}}$, then by Theorem A.1.29, there exists a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathcal{F}_t})$, a d-dimensional Brownian motion W, independent of the Poisson point process $\widehat{Y}$ satisfying

$$M_t = \int_0^t \sigma(X_{s^-})dW_s + \int_0^t \int_{\{|F(X_{s^-},u)|\le 1\}} F(X_{s^-},u)(\mu - \nu)(du, ds), \qquad (1.49)$$

where μ is the Poisson random measure associated with the process $\widehat{Y}$ defined on $([0,\infty) \times \mathbb{R}^d \setminus \{0\}, \mathscr{B}([0,\infty)) \times \mathscr{B}(\mathbb{R}^d \setminus \{0\}))$. Then by (1.46)-(1.49),

$$\begin{aligned} X_t &= x_0 + \int_0^t \sigma(X_s)dW_s + \int_0^t b(X_s)ds + \int_0^t \int_{|F(X_{s^-},u)|\le 1} F(X_{s^-},u)(\mu - \nu)(du, ds) \\ &\quad + \int_0^t \int_{|F(X_{s^-},u)|>1} F(X_{s^-},u)\mu(du, ds), \end{aligned}$$

with $X_0 = x$.

A.2 Proof of Property (4.9)

Indeed, define $Y_t := \int_0^t \sigma(\bar{X}_s) d\widehat{W_s}$. We divide it into cases of $d = 1$ and $d \ge 2$.

(i) When $d = 1$, Y is a 1-dimensional continuous martingale with

$$\langle Y \rangle_t = \int_0^t |\sigma(\bar{X}_{s^-})|^2 ds = \int_0^t a_{11}(\bar{X}_{s^-})^2 ds.$$

Note that $\Lambda_1 t \le \langle Y \rangle_t \le \Lambda_1^{-1} t$. By defining $D_t = \inf\{u : \langle Y \rangle_u > t\}$, Y_{D_t} is a one-dimensional Brownian motion, which we denote as $\bar{B}_s$. Hence,

$$\mathbb{Q}^{x_0}\left(\sup_{s\le 1} |Y_s| \le \varepsilon/2\right) \ge \mathbb{Q}^{x_0}\left(\sup_{s\le 1/\Lambda_1} |\bar{B}_s| \le \varepsilon/2\right) = \mathbb{Q}^{x_0}\left(\sup_{s\le 1} |\bar{B}_s| \le \varepsilon\sqrt{\Lambda_1}/2\right)$$

Let $\varphi(r) := \mathbb{Q}^{x_0}(\sup_{s\le 1} |\bar{B}_s| \le r)$, which is an increasing function in r. We have

$$\mathbb{Q}^{x_0}\left(\sup_{s\le 1} |Y_s| \le \varepsilon/2\right) \le \varphi(\varepsilon\sqrt{\Lambda_1}/2).$$

(ii) When $d \geq 2$, Y is a d-dimensional process with each coordinate process being a continuous martingale. In general, we can not use time change to turn Y into a d-dimensional Brownian motion. We consider a one-dimensional semimartingale $V(t) = |Y_t - y_0|^2$ instead, where $y = (\varepsilon, 0, .., 0)$. Note that $V_0 = |y_0|^2 = \varepsilon^2$. Define

$$\tau := \inf\{t > 0 : |Y_t| > \varepsilon/2\} = \inf\left\{t > 0 : V_t \notin [\varepsilon^2/4, 9\varepsilon^2/4]\right\}.$$

Let B is a one-dimensional Brownian motion independent of Y, and set

$$\bar{V}_t = V_{t\wedge\tau} + \varepsilon(B_{t\vee\tau} - B_\tau).$$

By Ito's formula applied to V_t,

$$\bar{V}_t = \bar{V}_0 + 2\int_0^{t\wedge\tau} \langle (Y_s - y)\sigma(\bar{X}_s), d\widehat{W_s}\rangle + \varepsilon(B_{t\vee\tau} - B_\tau) + \int_0^{t\wedge\tau} \mathrm{Tr}A(\bar{X}_s)ds,$$

where $\mathrm{Tr}A$ denotes the trace of the matrix A. So $\bar{V}_t - \bar{V}_0$ is a continuous semimartingale with martingale part

$$\bar{M}_t := 2\int_0^{t\wedge\tau} \langle (Y_s - y)\sigma(\bar{X}_s), d\widehat{W_s}\rangle + \varepsilon B_{t\vee\tau} - \varepsilon B_\tau$$

As $\langle \bar{M}\rangle_t = 4\int_0^{t\wedge\tau} |(Y_s - y)\sigma(\bar{X}_s)|^2 ds + \varepsilon(t\vee\tau - \tau)$, we have

$$c_1 := \varepsilon^2 \wedge (\Lambda_1\varepsilon^2) \leq d\langle \bar{M}\rangle_t/dt \leq \varepsilon^2 \vee (9\varepsilon^2/\Lambda_1) := c_2. \tag{2.1}$$

Let $A_t := \int_0^{t\wedge\tau} \mathrm{Tr}A(\bar{X}_s)ds$ and observe that

$$\mathrm{Tr}A(\bar{X}_t) \leq d/\Lambda_1 := c_3. \tag{2.2}$$

With $D_t := \inf\{u : \langle \bar{M}\rangle_u > t\}$, $\widetilde{W}_t := \bar{M}_{D_t}$ is a one-dimensional Brownian motion. Due to (2.1), D_t is continuous and strictly increasing. In fact, D_t is differentiable almost everywhere.

Let $Z_t := \bar{V}_{D_t} = \bar{V}_0 + \widetilde{W}_t + A_{D_t}$. Note that $Z_0 = \varepsilon^2$ and $A_{D_t} = \int_0^t e_s ds$ with

$$e_s = \mathbb{1}_{\{D_s\leq\tau\}}\,\mathrm{tr}A(\bar{X}_{D_s})\frac{dD_s}{ds} \leq c_3/c_1.$$

Define

$$\widetilde{M}_t := \exp\left(-\int_0^t e_s d\widehat{W}_s - \frac{1}{2}\int_0^t e_s^2 ds\right).$$

By a similar argument as that for (4.7), we have

$$\mathbb{E}^{\mathbb{Q}^{x_0}}\left[\widetilde{M}_t^2\right] \leq \exp(t(c_3/c_1)^2). \tag{2.3}$$

It therefore defines another family of probability measure $\{\widetilde{\mathbb{Q}}^x : x \in \mathbb{R}^d\}$ by

$$\frac{d\widetilde{\mathbb{Q}}^x}{d\mathbb{Q}^x}\Big|_{\mathcal{F}_t} = \widetilde{M}_t, \quad t \geq 0,$$

and $Z_t - Z_0$ is a one-dimensional standard Brownian motion under each $\widetilde{\mathbb{Q}}^x$. Let

$$t_0 := c_2. \tag{2.4}$$

Then

$$\widetilde{\mathbb{Q}}^{x_0}\left(\sup_{s\leq t_0}|Z_s - Z_0| \leq 3\varepsilon^2/4\right) = \varphi(3\varepsilon^2/4\sqrt{t_0}).$$

By Cauchy-Schwartz inequality, (2.3) and (2.4),

$$\begin{aligned}\left(\widetilde{\mathbb{Q}}^{x_0}\Big(\sup_{s\leq t_0}|Z_s - Z_0| \leq 3\varepsilon^2/4\Big)\right)^2 &\leq \mathbb{E}^{\mathbb{Q}^{x_0}}\left[\widetilde{M}_{t_0}^2\right]\mathbb{Q}^{x_0}\Big(\sup_{s\leq t_0}|Z_s - Z_0| \leq 3\varepsilon^2/4\Big)\\ &\leq \exp\left(c_2c_3^2/c_1^2\right)\mathbb{Q}^{x_0}\Big(\sup_{s\leq t_0}|Z_s - Z_0| \leq 3\varepsilon^2/4\Big).\end{aligned}$$

Consequently,

$$\mathbb{Q}^{x_0}\left(\sup_{s\leq t_0}|Z_s - Z_0| \leq 3\varepsilon^2/4\right) \geq \exp\left(-c_2c_3^2/c_1^2\right)\varphi^2(3\varepsilon^2/4\sqrt{t_0}).$$

By the definition of D_{t_0},

$$\begin{aligned}&\mathbb{Q}^{x_0}\left(\sup_{s\leq 1}|\bar{V}_s - \bar{V}_0| \leq 3\varepsilon^2/4\right) = \mathbb{Q}^{x_0}\left(\sup_{D_s\leq 1}|\bar{V}_{D_s} - \bar{V}_0| \leq 3\varepsilon^2/4\right)\\ \geq\ &\mathbb{Q}^{x_0}\left(\sup_{s\leq\langle\bar{M}\rangle_1}|\bar{V}_{D_s} - \bar{V}_0| \leq 3\varepsilon^2/4\right) \geq \mathbb{Q}^{x_0}\left(\sup_{s\leq t_0}|Z_s - Z_0| \leq 3\varepsilon^2/4\right).\end{aligned}$$

As

$$\left\{\sup_{s\leq 1}|Y_s| \leq \varepsilon/2\right\} = \left\{\varepsilon/2 \leq \sup_{s\leq 1}|Y_s - y_0| \leq 3\varepsilon/2\right\} \supset \left\{\sup_{s\leq 1}|V_s - V_0| \leq 3\varepsilon^2/4\right\},$$

we have

$$\mathbb{Q}^{x_0}\Big(\sup_{s\leq 1}|Y_s| \leq \varepsilon/2\Big) \geq \mathbb{Q}^{x_0}\Big(\sup_{s\leq 1}|V_s - V_0| \leq 3\varepsilon^2/4\Big)$$

$$= \quad \mathbb{Q}^{x_0}\Big(\sup_{s\leq 1}|\bar{V}_s - \bar{V}_0| \leq 3\varepsilon^2/4\Big)$$
$$= \quad \exp\Big(-c_2c_3^2/c_1^2\Big)\ \varphi^2(3\varepsilon^2/4\sqrt{t_0}).$$

This establishes the claim (4.9) with

$$\phi_1(\varepsilon) := \begin{cases} \varphi(\varepsilon\sqrt{\Lambda_1}/2) & \text{if } d = 1, \\ \exp\left(-c_2c_3^2/c_1^2\right)\ \varphi^2(3\varepsilon^2/4\sqrt{t_0}) & \text{if } d \geq 2. \end{cases}$$

A.3 Proof of Lemma 1.4.4

This part is mainly referred from [2], [3] [4], [50] and [51].

Definition A.3.1. *[3, Section 2.1] Let M be a set in $\mathbb{R}^d$. A hyperplane H_α is called* ***supporting*** *to the set M if $H_\alpha \cap M \neq \emptyset$ and the whole set M lies on one side of H_α.*

Definition A.3.2. *[3, Section 9.2] We introduce a new d-dimensional space $E^d = \{p : p = (p_1, p_2, ..., p_d)\}$ called* ***gradient space*** *with the canonical scalar product*

$$(p, q) = \sum_{i=1}^{d} p_i q_i$$

and denote by $|p| = (p,p)^{1/2}$ the length of any vector $p \in E^d$. Obviously, E^d is a d-dimensional Euclidean space.To abuse the notation, we will use $\mathbb{R}^d$ when referring to gradient space.

Definition A.3.3. *Let $z(x)$ be a convex function defined on a set D, S_z be the graph of the function $z(x)$, H_α as an arbitrary supporting hyperplane $z(x)$ passing through (x_0, z_0),where $z_0 = z(x_0)$. Then the equation*

$$z - z_0 = \langle p^0, x - x_0\rangle = p_1^0(x_1 - x_0^1) + ... + p_d^0(x_d - x_0^d)$$

is the equation of H_α.The point $p^0 = (p_1^0, p_2^0, ..., p_d^0) \in S_z \cap H_\alpha$. The point $p^0 = (p_1^0, ..., p_d^0) \in \mathbb{R}^d$ is called ***the normal image of the supporting hyperplane*** *H_α and is denoted by*

$$p^0 = \nu_z(H_\alpha).$$

The set

$$\nu_z(x_0) = \cup_\alpha \nu_z(H_\alpha)$$

is called the ***the normal image of the point*** x_0 ***with respect to the function*** z***.****(More precisely,* $\nu_z(x_0)$ *is the normal image of the point* x_0 *with respect to the function* $z(x)$*).*

For any set $\Gamma \subset D$*, the set*

$$\nu_z(\Gamma) = \cup_{x_0 \in \Gamma} \nu_z(x_0)$$

is ***the normal image of*** Γ ***with respect to the function*** z*.*

Notice that $\nu_z(\Gamma)$ is a subset of the gradient space $\mathbb{R}^d$. The mapping ν_z, which maps the set $\Gamma \subset D$ to the set $\nu_z(\Gamma) \subset \mathbb{R}^d$, is called **normal**. By [3, Section 9.4], for any Borel subset Γ of D, the set $\nu_z(\Gamma)$ is Lebesgue measurable in the gradient space $\mathbb{R}^d$.

Denote $\omega(z, \Gamma)$ as **the normal volume of** $\nu_z(\Gamma)$, i.e., the Lebesgue measure of $\nu_z(\Gamma)$, $\omega(z, \Gamma) = |\nu_z(\Gamma)|$.

Definition A.3.4. *A convex body* F *in* $\mathbb{R}^{d+1}$ *is called an* $(n+I)$***-convex solid polyhedron*** *if* F *is the intersection of a finite number of closed halfspaces. If the solid polyhedron* F *is a bounded set in* $\mathbb{R}^{d+1}$*, then its boundary of is called a* ***closed convex polyhedron*** *or a* ***polyhedron surface****.*

It is not hard to see when the graph S_z of the function z is a polyhedral surface, there are only finitely many points $\{x_1, ..., x_n\} \in \Gamma$ whose $\omega(z, x_i)$are non-zero.(See for[50] more details.)

Since **Alexandrov's theorem** says that a convex function is twice differentiable almost everywhere, then for any convex function z defined on a set D, and a Borel subset $\Gamma \subset D$, by the change of the coordinates, the normal volume $\omega(z, \Gamma)$ is

$$\omega(z, \Gamma) := \int_{\nu_z(\Gamma)} dp = \int_{\Gamma} \det(z_{x_i x_j}(x)) dx, \tag{3.1}$$

where $(z_{x_i x_j}(x))$ is the Hessian matrix of $z(x)$ at the point x and is nonnegative semidefinite. Hence by (3.1),

$$\begin{aligned} &\omega(z, \cdot) \text{ is a } \sigma\text{-finite measure on Borel sets of } D, \text{ and absolutely} \\ &\text{continuous with respect to the Lebesgue measure on } \mathbb{R}^d. \end{aligned} \tag{3.2}$$

Lemma A.3.5. *Let a sequence of convex functions $\{z^n\}_{n\geq 1}$ defined on a domain D such that $\{z^n\}_{n\geq 1}$ converges to the convex function z for almost every $x \in D$. Then*

$$\limsup_{n\to\infty} \omega(z^n, F) \leq \omega(z, F)$$

for every closed subset F of D, where $\omega(z^n, F)$ is defined in (3.1)*.*

Proof. See Lemma 9.2 on page 119 of [3].

Next let us prove the weak convergence of the measure $\omega(z, \cdot)$.

Lemma A.3.6. *Let a sequence of convex functions $\{z^n\}_{n\geq 1}$ defined on a domain D such that $\{z^n\}_{n\geq 1}$ converges to the convex function z for almost every $x \in D$, Then $\omega(z^n, D)$ converges weakly to $\omega(z, D)$ in D, i.e.*

$$\lim_{n\to\infty} \int_D g(x)\omega(z^n, dx) = \int_D g(x)\omega(z, dx).$$

for any function $g \in C_b(D)$.

Proof. It suffices to show when $g = \mathbb{1}_A$ where A is a Borel set of D.

Without loss of generality, we assume that $\omega(z, D) < \infty$. Because if $\omega(z, D)$ is σ-finite, there exists a countable sequence of sets $\{E_n\}_{n\geq 1} \subset D$ such that $\cup_{n\geq 1} E_n = D$ and $\omega(z, E_n) < \infty$ for each $E_n \subset D$.

Since $z^n \to z$ pointwise on D as $n \to \infty$, and $\{z^n\}_{n\geq 1}$ and z are both convex functions, then $\det z^n_{x_i x_j}(x) \geq 0, \det z_{x_i x_j}(x) \geq 0$ a.e. on D and $\det z^n_{x_i x_j}(x) \to \det z_{x_i x_j}(x)$ a.e. on D as $n \to \infty$, $1 \leq i, j \leq d$.

Then for any Borel set A in D, by monotone convergence theorem,

$$\lim_{n\to\infty} \omega(z^n, A) = \lim_{n\to\infty} \int_D \mathbb{1}_A \det z^n_{x_i x_j} dx = \int_D \mathbb{1}_A \det z_{x_i x_j} dx = \omega(z, A) \tag{3.3}$$

□

Now we are going to prove the following main lemma.

Lemma A.3.7 (Lemma 1 in [51])**.** *Let D be a bounded, convex and open subset of $\mathbb{R}^d$ and z_1, z_2 be two nonpositive convex functions defined on $\bar{D}$ such that $z_1 = z_2 = 0$ on*

∂D.Suppose that there exists a constant $K > 0$ such that $|z_i(x) - z_i(y)| \leq K|x-y|$ for every $x, y \in \bar{D}, i = 1, 2$. Define

$$z(t) := tz_2(x) + (1-t)z_1(x), \quad t \in [0,1]. \tag{3.4}$$

Then for every $t \in [0,1]$,

$$\frac{d}{dt}\int_D z(t)\omega(z(t), dx) = (d+1)\int_D z_t(t)\omega(z(t), dx), \tag{3.5}$$

$$\frac{d}{dt}\int_D (z_2 - z_1)\omega(z(t), dx) \leq 0. \tag{3.6}$$

Proof. We follow the idea of proof of Lemma 1 in [51]. However, we provide with more necessary details.

Take a sequence of bounded open and convex domains $\{D_n\}_{n\geq 1}$ such that each boundary ∂D_n is infinitely differentiable, and $D_1 \subset D_2 \subset ...$ with $D = \cup_n D_n$.

For each $i = 1, 2$, define a sequence of infinitely differentiable convex functions $\{z_i^n\}_{n\geq 1}$ such that $z_i^n|_{\partial D_n} = 0$ and $z_i^n(x) \to z_i(x)$ for any $x \in \bar{D}$ as $n \to \infty$. For each $n \in \mathbb{N}$, define

$$z^n(t) = tz_2^n(x) + (1-t)z_1^n(x), \quad t \in [0,1]. \tag{3.7}$$

Since each z_i is convex on its bounded domain $\bar{D}$, then by (3.1), $\omega(z_i, \bar{D}) < \infty$.Then by definition of (3.4), (3.7), $\omega(z(t), \bar{D}) < \infty$, and $\omega(z^n(t), \bar{D}) < \infty$.

For any sequence of bounded continuous functions $\{f^n\}_{n\geq 1}$ on defined $\bar{D}$ such that $f^n \to f$ uniformly on $\bar{D}$, by dominated convergence theorem and Lemma A.3.6,

$$\begin{aligned}
& |\lim_{n\to\infty}\int_{\bar{D}} f^n(x)\omega(z^n(t), dx) - \int_{\bar{D}} f(x)\omega(z(t), dx)| \\
= & |\lim_{n\to\infty}\int_{\bar{D}} f^n(x)\det(z^n_{x_ix_j}(t))dx - \int_{\bar{D}} f(x)\omega(z(t), dx)| \\
\leq & |\lim_{n\to\infty}\int_{\bar{D}} (f^n(x) - f(x))\det(z^n_{x_ix_j}(t))dx| + |\lim_{n\to\infty}\int_{\bar{D}} f(x)\det(z^n_{x_ix_j}(t))dx - \int_{\bar{D}} f(x)\omega(z(t), dx)| \\
\leq & \lim_{n\to\infty}\int_{\bar{D}} |f^n(x) - f(x)|\sup_{n\geq 1}\det(z^n_{x_ix_j}(t))dx + 0 \\
= & 0 + 0 = 0,
\end{aligned}$$

i.e.,

$$\lim_{n\to\infty}\int_{\bar{D}} f^n(x)\det(z^n_{x_ix_j}(t))dx = \int_{\bar{D}} f(x)\omega(z(t), dx). \tag{3.8}$$

Therefore, for every $t \in [0,1]$, by (3.8),

$$\lim_{n\to\infty}\int_D z^n(t)\det(z^n_{x_ix_j}(t))dx = \int_D z(t)\omega(z(t),dx).$$

Furthermore, by (3.7), $z_t(t) = z_2^n - z_1^n$, then

$$\begin{aligned}\frac{d}{dt}\int_D z_t(t)\det(z^n_{x_ix_j}(t))dx &= \lim_{n\to\infty}\frac{d}{dt}\int_D (z_2^n - z_1^n)\det(z^n_{x_ix_j}(t))dx \\ &= \lim_{n\to\infty}\sum_{i,j=1}^{n}\int_D z_t^n(t) z^n_{x_ix_jt}(t)\frac{\partial}{\partial z^n_{x_iz_j}(t)}\det(z^n_{x_ix_j}(t))dx.\end{aligned} \tag{3.9}$$

Let $D_j = \{x_j : (x_1,...x_d)\in D\}, 1\le j\le d$. Since $z_t^n(t) = 0$ on ∂D, and $\partial D_j \subset \partial D$, then $z_t^n(t) = 0$ on ∂D_j. Integrate (3.9) by part with respect to x_j, by the fact that

$$\sum_{j=1}^{d}\frac{\partial}{\partial x_j}\frac{\partial}{\partial z^n_{x_ix_j}(t)}\det(z^n_{x_ix_j}(t)) = 0, \text{for every } i\in\{1,..,d\}, \tag{3.10}$$

we obtain

$$\begin{aligned}&\sum_{i,j=1}^{d}\int_D z_t^n(t)\frac{\partial}{\partial z^n_{x_iz_j}(t)}\det(z^n_{x_ix_j}(t))\frac{\partial}{\partial x_j}z^n_{x_it}(t)dx_1...dx_{j-1}dx_{j+1}...dx_d dx_j \\ =\ &\sum_{i,j=1}^{d}\int_{D_1}\int_{D_2}...\int_{D_d} z_t^n(t)\frac{\partial}{\partial z^n_{x_iz_j}(t)}\det(z^n_{x_ix_j}(t))dx_d...dx_{j+1}dx_{j-1}...dx_1 z^n_{x_it}(t)|_{\partial D_j} \\ &-\sum_{i,j=1}^{d}\int_D z^n_{x_it}(t)\frac{\partial}{\partial x_j}(z_t^n(t)\frac{\partial}{\partial z^n_{x_ix_j}(t)}\det(z^n_{x_ix_j}(t)))dx \\ =\ &-\sum_{i,j=1}^{d}\int_D z^n_{x_it}(t)z^n_{x_jt}(t)\frac{\partial}{\partial z^n_{x_ix_j}(t)}\det(z^n_{x_ix_j}(t))dx - \sum_{i=1}^{d}\int_D z_t^n(t)z^n_{x_it}(t)\sum_{j=1}^{d}\frac{\partial}{\partial x_j}\frac{\partial}{\partial z^n_{x_ix_j}(t)}\det(z^n_{x_ix_j}(t))dx \\ =\ &-\sum_{i,j=1}^{d}\int_D z^n_{x_it}(t)z^n_{x_jt}(t)\frac{\partial}{\partial z^n_{x_ix_j}(t)}\det(z^n_{x_ix_j}(t))dx.\end{aligned} \tag{3.11}$$

For each $n\in N$, denote the matrix $A_n := (a^n_{ij}) = (\frac{\partial}{\partial z^n_{x_ix_j}(t)}\det(z^n_{x_ix_j}(t))$, and $C_n := (c^n_{ij}) = (z^n_{x_ix_j}(t))$.

Then $a^n_{ij} = (Adj(C_n))_{ij}$, and $A_n\cdot C_n = Adj(C_n)\cdot C_n = (\det C_n)I$, where $Adj(C_n)$ is called adjugate matrix of C_n, i.e, the transpose of the cofactor matrix of C_n.

Since z_n is convex, then $\det(C_n)\ge 0$.

Therefore

(i) If $\det C_n = 0$, A_n is the zero matrix, and hence (3.6) obviously holds;

(ii) If $\det(C_n) > 0$, then C_n is invertible and positive semi-definite, and $A_n = Adj(C_n) = \det(C_n)C_n^{-1}$. Since C_n is positive semi-definite, then for each vector $\xi \in \mathbb{R}^d$, $\xi^T A_n \xi = \xi^T \det(C_n) C_n^{-1}\xi = \det(C_n)\xi^T C_n^{-1}\xi > 0$. Hence for vector $\xi^T = (z_{x_i t})_{1\le i\le d}$,

$$\xi^T A_n \xi = \sum_{i,j=1}^{d} z^n_{x_i t}(t) z^n_{x_j t}(t) a^n_{ij} = \sum_{i,j=1}^{d} z^n_{x_i t}(t) z^n_{x_j t}(t) \frac{\partial}{\partial z^n_{x_i x_j}(t)} \det(z^n_{x_i x_j}(t)) \ge 0.$$

Hence,

$$\begin{aligned}
\frac{d}{dt}\int_D z_t(t)\omega(z(t),dx) &= \frac{d}{dt}\lim_{n\to\infty}\int_D z^n_t(t)\det(z^n_{x_ix_j}(t))dx \\
&= \lim_{n\to\infty}\frac{d}{dt}\int_D z^n_t(t)\det(z^n_{x_ix_j}(t))dx \\
&= \lim_{n\to\infty}\sum_{i,j=1}^{d}\int_D z^n_t(t)\frac{\partial}{\partial t}(\det z^n_{x_ix_j}(t))dx \\
&= -\sum_{i,j=1}^{d}\lim_{n\to\infty}\int_D z^n_{x_jt}(t)z^n_{x_it}(t)\frac{\partial}{\partial z^n_{x_ix_j}(t)}\det(z^n_{x_ix_j}(t))dx \\
&\le 0.
\end{aligned}$$

which proves (3.6).

To show (3.5), integrate by parts for the right hand side with respect to x_j, by the fact that $z^n_t(t) = 0$ on ∂D and (3.10), similar like (3.11), it yields that

$$\begin{aligned}
&\sum_{i,j=1}^{d}\int_D z^n(t) z^n_{x_ix_jt}(t)\frac{\partial}{\partial z^n_{x_ix_j}(t)}\det(z^n_{x_ix_j}(t))dx \\
=\ &\sum_{i,j=1}^{d}\int_D z^n(t)\frac{\partial}{\partial z^n_{x_ix_j}(t)}\det(z^n_{x_ix_j}(t))dx_1...dz^n_{x_it}(t)...dx_n \\
=\ &-\sum_{i=1}^{d}\int_D z^n_{x_it}(t)\Big[\sum_{j=1}^{d}\frac{\partial}{\partial x_j}\Big(z^n(t)\frac{\partial}{\partial z^n_{x_ix_j}(t)}\det(z^n_{x_ix_j}(t))\Big)\Big]dx \\
=\ &-\sum_{i=1}^{d}\int_D z^n_{x_it}(t)\Big[z^n(t)\Big(\sum_{j=1}^{d}\frac{\partial}{\partial x_j}\frac{\partial}{\partial z^n_{x_ix_j}(t)}\det(z^n_{x_ix_j}(t))\Big)\Big]dx - \sum_{i=1}^{d}\int_D z^n_{x_it}(t)\Big[\sum_{j=1}^{d} z^n_{x_j}(t)\frac{\partial}{\partial z^n_{x_ix_j}(t)}\det(z^n_{x_ix_j}(t))\Big]dx \\
=\ &-\sum_{i=1}^{d}\int_D z^n_{x_it}(t)\Big[\sum_{j=1}^{d} z^n_{x_j}(t)\frac{\partial}{\partial z^n_{x_ix_j}(t)}\det(z^n_{x_ix_j}(t))\Big]dx. \qquad (3.12)
\end{aligned}$$

Next, by doing integration by parts for $\sum_{i=1}^d \int_D z^n_{x_i t}(t)[\sum_{j=1}^d z^n_{x_j}(t)\frac{\partial}{\partial z^n_{x_i x_j}(t)} \det(z^n_{x_i x_j}(t))]dx$ with respect to x_i. By the fact that $z^n_t = 0$ on ∂D and (3.10) we have

$$\begin{aligned}
& -\sum_{i=1}^d \int_D z^n_{x_i t}\Big[\sum_{j=1}^d z^n_{x_j}(t)\frac{\partial}{\partial z^n_{x_i x_j}(t)} \det(z^n_{x_i x_j}(t))\Big]dx \\
= & -\sum_{i=1}^d \int_D \Big[\sum_{j=1}^d z^n_{x_j}(t)\frac{\partial}{\partial z^n_{x_i x_j}(t)} \det(z^n_{x_i x_j}(t))\Big]dx_1...dz^n_t...dx_n \\
= & \sum_{i=1}^d \int_D z^n_t(t)\frac{\partial}{\partial x_i}\Big(\sum_{j=1}^d z^n_{x_j}(t)\frac{\partial}{\partial z^n_{x_i x_j}(t)} \det(z^n_{x_i x_j}(t))\Big)dx \\
= & \int_D z^n_t(t)\sum_{i,j=1}^d \Big(z^n_{x_i x_j}(t)\frac{\partial}{\partial z^n_{x_i x_j}(t)} \det(z^n_{x_i x_j}(t))\Big)dx \\
& + \int_D z^n_t(t)\sum_{j=1}^d z^n_{x_j}(t)\sum_{i=1}^d \frac{\partial}{\partial x_i}\frac{\partial}{\partial z^n_{x_j x_i}(t)} \det(z^n_{x_i x_j}(t))dx. \\
= & \int_D z^n_t(t)\sum_{i,j=1}^d \Big(z^n_{x_i x_j}(t)\frac{\partial}{\partial z^n_{x_i x_j}(t)} \det(z^n_{x_i x_j}(t))\Big)dx. \qquad (3.13)
\end{aligned}$$

By the Euler's theorem for homogeneous functions, i.e.,

$$\sum_{i,j=1}^d (z^n_{x_i x_j}(t)\frac{\partial}{\partial z^n_{x_i x_j}(t)} \det(z^n_{x_i x_j}(t))) = d\det(z^n_{x_i x_j}(t)). \qquad (3.14)$$

Therefore, by (3.12), (3.13),(3.14),

$$\int_D z^n(t)\frac{\partial}{\partial t}(\det z^n_{x_i x_j}(t))dx = d\int_D z^n_t(t)\det(z^n_{x_i x_j}(t))dx. \qquad (3.15)$$

Hence, by (3.15),

$$\begin{aligned}
\frac{d}{dt}\int_D z^n(t)\det(z^n_{x_i x_j}(t))dx &= \int_D z^n_t(t)\det(z^n_{x_i x_j}(t))dx + \int_D z^n(t)\frac{\partial}{\partial t}\det(z^n_{x_i x_j}(t))dx \\
&= (d+1)\int_D z^n_t(t)\det(z^n_{x_i x_j}(t))dx, \qquad (3.16)
\end{aligned}$$

and thus

$$\begin{aligned}
\frac{d}{dt}\int_D z(t)\omega(z(t),dx) &= \frac{d}{dt}\lim_{n\to\infty}\int_D z^n(t)\det(z^n_{x_i x_j}(t))dx \\
&= \lim_{n\to\infty}(d+1)\int_D z^n_t(t)\det(z^n_{x_i x_j}(t))dx \\
&= (d+1)\int_D z_t(t)\omega(z(t),dx).
\end{aligned}$$

□

For the two nonpositive convex function z_1, z_2 defined in Lemma A.3.7, for any $x \in \bar{D}$, define

$$\mu(dx) = (z_1(x) - z_2(x))\omega(z_2, dx). \tag{3.17}$$

Corollary A.3.8. *With the assumption of Lemma A.3.7 for the two functions z_1, z_2, z_1, z_2 satisfy that*

$$-\int_D z_2\omega(z_2, dx) \leq -\int_D z_1\omega(z_1, dx) + (d+1)\mu(D). \tag{3.18}$$

Proof. From Equation (3.6), we know that for any $t \in [0,1]$,

$$\int_D (z_2 - z_1)\omega(z(t), dx) \geq \int_D (z_2 - z_1)\omega(z(1), dx) = -\mu(D).$$

Then integrating the above with respect to t, we have

$$\begin{aligned} & -\mu(D) \leq \int_0^1 dt \int_D z_t(t)\omega(z(t), dx) \leq \frac{1}{d+1}\int_D z(t)\omega(z(t), dx)|_{t=0}^{t=1} \\ = & \frac{1}{d+1}(\int_D z(1)\omega(z(1), dx) - \int_D z(0)\omega(z(0), dx) \\ = & \frac{1}{d+1}(\int_D z_2\omega(z_2, dx) - \int_D z_1\omega(z_1, dx)), \end{aligned}$$

where the second last equality is due to (3.5). □

Corollary A.3.9. *With the assumption in Lemma A.3.7, there exist a constant $r > 0$ such that for any $x \in \bar{D}$,*

$$|z_i(x)|^{d+1} \leq -\omega_d^{-1} r^d \int_D z_i\omega(z_i, dx), \quad i = 1, 2. \tag{3.19}$$

Proof. We will show the case for $i = 1$, the result for $i = 2$ follows the exact same procedures. Given the surface of z_1 is polyhedral, so there are only finitely many vertices on the surface. Consider an arbitrary vertex x_1. Let $z_1^{'}$ denote the function whose graph is the surface of the cone with vertex $(x_1, z_1(x_1))$ with the base D. Obviously,

$$z_1(x) \leq z_1^{'}(x) \quad \text{for any } x \in D.$$

Then by (3.17), $\mu(D) < 0$, and by (3.18),

$$-z_1(x_1)\omega(z_1^{'}, D) = -\int_D z_1^{'}\omega(z_1^{'}, D) \le -\int_D z_1\omega(z_1, D). \tag{3.20}$$

Given D is bounded, there exists a $r > 0$ such that $D \subset B_r(0)$.Then for any $p \in \nu_{z_1^{'}}(D)$,

$$p(x - x_1) + z_1^{'}(x_1) \le 0 \quad \text{for any } x \in B_r(0).$$

Hence for any $p \in \nu_{z_1^{'}}(D)$,

$$\sup_{x\in B_r(0)} p(x - x_1) + z_1^{'}(x_1) \le 0.$$

Hence

$$r|p| - px_1 + z_1^{'}(x_1) \le 0,$$

i.e.

$$r|p| - px_1 + z_1(x_1) \le 0, \quad \Rightarrow |p + \frac{z_1(x_1)x_1}{r^2 - |x_1|^2}| \le \frac{|z_1(x_1)|r}{r^2 - |x_1|^2},$$

which means that $\nu_{z_1^{'}}(x_1)$ contains a ball with radius $\frac{|z_1(x_1)|r}{r^2-|x_1|^2}$. Therefore,

$$\omega_d \frac{|z_1(x_1)|^d r^d}{(r^2 - |x_1|^2)^d} \le \omega(z_1^{'}, x_1),$$

where ω_d is the volume of the unit sphere. Then combined with (3.20), it yields

$$\omega_d \frac{|z_1(x_1)|^{d+1}}{r^d} \le -z_1^{'}(x_1)\omega(z_1^{'}, x_1) = -\int_D z_1(x)\omega(z_1, dx). \tag{3.21}$$

Since the vertex x_1 is arbitrary, then for any $x \in \bar{D}$,

$$|z_1(x)|^{d+1} \le \max_{x\in\bar{D}} |z_1(x)|^{d+1} \le -\omega_d^{-1} r^d \int_D z_1(x)\omega(z_1, dx).$$

□

Lemma A.3.10. *Let $D \subset \mathbb{R}^d$ be a bounded polyhedral convex open set, and μ be a measure with nonzero mass on $\{x_1, ..., x_m\} \subset D$. Suppose that $z_1(x)$ is a convex and continuous function defined on $\bar{D}$ such that $z_1|_{\partial D} = 0$ and its graph is a polyhedral surface with vertices whose projections onto D belong to $\{x_1, ..., x_m\}$. Then there exists a convex and continuous* *function* $z_2(x)$ *on* $\bar{D}$ *such that* $z_2 \leq z_1$ *and* $z_2|_{\partial D} = 0$*. Moreover, the graph of* z_2 *is a convex polyhedral surface with vertices whose projections onto* D *belong to* $\{x_1, ..., x_m\}$*. Meanwhile,*

$$\nu_{z_1}(D) \subset \nu_{z_2}(D), \tag{3.22}$$

and the function z_2 *also solves the equations*

$$(z_1(x_i) - z_2(x_i))\omega(z_2, x_i) = \mu(x_i), \quad i = 1, ..., m.$$

Proof. The proof of (3.22) is Lemma 10.2 in [3], and the rest of the proof follows the similar idea as Theorem 47 in [4] and Theorem 3 in [2],which we will show in detail in Subsection A.4.

Remark A.3.11. 1. *If* $\mu(D) = \sum_{i=1}^{m} \mu(x_i)$*, the by Lemma A.3.8 and A.3.9, it also yields that*

$$max_{x\in D}|z_2(x)|^{d+1} \leq \int_D |z_1|\omega(z_1, dx) + \sum_{i=1}^{m} \mu(x_i).$$

We will apply this result to prove the following theorem.

2. *According to [51, Lemma 2], it is claimed that Lemma A.3.10 is also true if we change the equation into*

$$(z_1(x_i) - z_2(x_i))\omega(z_2, x_i) \leq \mu(x_i), \quad i = 1, ..., m.$$

Theorem A.3.12. *Let* D *be a convex, open and bounded set in* $\mathbb{R}^d$ *such that* $D \subset B(0, r)$ *for some* $r \geq 0$*. Let the measures* $\mu_1, \mu_2, ...$*concentrated in a closed set* $U \subset D$*.Then there exists a sequence of functions* $\{z_i\}_{i\geq 0}$ *that are convex and continuous on* $\bar{D}$*, such that* $0 = z_0 \geq z_1 \geq z_2 \geq ...$ *on* D*,* $z_i|_{\partial D} = 0$ *for each* $i \in \mathbb{N}$*, and for every* $x \in D$*,*

$$\mu_i(dx) = (z_{i-1}(x) - z_i(x))\omega(z_i, dx), \quad i = 1, 2, ... \tag{3.23}$$

In addition, for any $x \in D$*,*

$$|z_j(x)|^{d+1} \leq -r^d/\omega_d \int_D z_j\omega(z_j, dx) \leq (d + 1)r^d/\omega_d \sum_{i=1}^{j} \mu_i(D), \quad j = 1, 2, ... \tag{3.24}$$

where ω_d *is the volume of the unit d-dimensional ball.*

Proof. Let $\{D^n\}_{n\geq 1}$ be a sequence of polyhedral convex sets such that

$$U \subset D^1 \subset ... \subset D^n \subset D^{n+1}, \text{ and } D = \cup_{n\in N} D^n. \tag{3.25}$$

Now fix an arbitrary $i \in \mathbb{N}$. Construct a sequence of measures $\{\mu_i^n\}_{n\geq 1}$ such that each μ_i^n concentrated in a set $U^n \subset U$, where each U^n consists of a finite number of points $\{x_1^n, x_2^n, ..., x_{m_i}^n\}$ so that $U^1 \subset U^2 \subset U^3 \subset ... \subset U$, and for each $i \in \mathbb{N}$,

$$\mu_i^n \to \mu_i \text{ weakly as } n \to \infty. \tag{3.26}$$

Fix $n \in \mathbb{N}$. Given $z_0^n = 0$ on $\bar{D}$, applying Lemma A.3.10 repeatedly, then for each $n \in \mathbb{N}$, there exists a sequence of convex functions $\{z_i^n\}_{i\geq 1}$ whose graphs are polyhedral surfaces with vertices whose projections onto C^n is a subset of D^n, such that $0 = z_0^n \geq z_1^n \geq ...$ on D^n, each $z_i^n|_{\partial D^n} = 0$, and z_i is the solution of the equation

$$\mu_i^n(dx) = (z_{i-1}^n(x) - z_i^n(x))\omega(z_i^n, dx), \quad i = 1, 2, ... \tag{3.27}$$

Therefore, by Corollary A.3.8,

$$-\int_{D^n} z_i^n \omega(z_i^n, dx) \leq -\int_{D^n} z_{i-1}^n \omega(z_{i-1}^n, dx) + (d+1)\mu_i^n(D^n), \quad i = 1, 2, ...$$

Adding the above inequalities from $i = 1$ to j, by (3.19), it yields

$$|z_j^n(x)|^{d+1} \leq -\frac{r^d}{\omega_d}\int_{D^n} z_j^n \omega(z_j^n, dx) \leq \frac{(d+1)r^d}{\omega_d}\sum_{i=1}^{j} \mu_i^n(D). \tag{3.28}$$

For each fixed $j \in \mathbb{N}$, $\sum_{i=1}^{j} \mu_i^n(D)$ is a finite number, then $\{z_j^n\}n \geq 1$ is uniformly bounded on D. Therefore, there exists a subsequence of $\{z_j^{n_k}\}_{k\geq 1}$ of $\{z_j^n\}_{n\geq 1}$ converging uniformly on D. Denote z_j as the limit.

Since the graph of z_j^n is a polyhedral surface, then by the definition of convexity, each z_j is convex on $\bar{D}$, and for any $x, y \in D$, there exists a constant $K_j > 0$ such that

$$|z_j^n(x) - z_j^n(y)| \leq K_j|x - y|, \text{for each } n \in \mathbb{N}. \tag{3.29}$$

Taking the limits of n_k on both sides of (3.27),(3.28), (3.29), combined with the fact of (3.26), we have (3.23), (3.24), $z_j|_{\partial C} = 0$ for each $j \in \mathbb{N}$, and for any $x, y \in \bar{D}$,

$$|z_j(x) - z_j(y)| \leq K_j|x - y|, \quad j = 1, 2, ...$$

Lemma A.3.13. *Let z be a convex function whose graph is a polyhedral surface. There exists a constant $\varepsilon > 0$ depending only on the dimension d such that for any $r > 0$,*

$$\int_{\partial B_r(0)} \frac{d}{dn_r} z(y) d\sigma_r \geq \varepsilon \sqrt[d]{\omega(z, B_{r/2}(0))} r^{d-1},$$

where n_r is the outward normal direction of the surface of the sphere $B_r(0)$.

Proof. The proof can be found in Lemma 1 of [50], but since it looks sketchy, we will write a detailed proof as follows.

W.L.O.G, let us assume $r = 4$. Once we prove the case when $r = 4$, we can apply it to the case of the radius $r/4$, and it holds for any other values of $r > 0$. Then it suffices to show

$$\int_{\partial B_4(0)} \frac{d}{dn_r} z(y) d\sigma_r \geq \varepsilon \sqrt[d]{\omega(z, B_2(0))} 4^{d-1},$$

Also, since the graph of z is a polyhedral surface and by shifting $z(x)$ by a constant vector affine functions, i.e., $px+b$, $z(x)+px+b$ is still convex and this does not change the targeted inequality. Then W.L.O.G, we can assume that $z(0) = 0$ andd $z(x) \geq 0$ for all $x \in \mathbb{R}^d$.

In addition, if $f(x) := p(x - x_0) + z(x_0)$ is a supporting hyperplane of z for some $x_0 \in B_2(0)$, then $p(x - x_0) + z(x_0) \leq z(x)$ for any $x \in \mathbb{R}^d$, and thus

$$\begin{aligned} [z] : &= \sup_{x_1, x_2 \in B_3(0)} \frac{|z(x_1) - z(x_2)|}{|x_1 - x_2|} \\ &\geq \sup_{|x - x_0| \leq 1} \frac{|z(x) - z(x_0)|}{|x - x_0|} \\ &\geq \sup_{|x - x_0| \leq 1} \frac{|p(x - x_0)|}{|x - x_0|} =: |p|, \end{aligned}$$

where the last line is by the definition of the norm of the bounded linear functional. Hence, $\nu_z(x) \subset B_{[z]}(0)$ for $x \in B_2(0)$, which means $\omega(z, B_2(0)) \subset \omega_d [z]^d$. Hence, if we can show

$$\varepsilon[z] \leq \int_{\partial B_4(0)} \frac{d}{dn_r} z(x) d\sigma_r, \tag{3.30}$$

for some positive constant ε, which depends only on dimension d, and for all the function z whose graph is a convex polyhedron such that $z(0) = 0$ and $z(x) \geq 0$ for all x, we are done with the proof.

Suppose that such a constant does not exist, i.e, then for any convex function z whose graph is a polyhedral surface such that $[z] = 1$ with $z(0) = 0$ and $z(x) \geq 0$ for all x, for any $\varepsilon > 0$,

$$\int_{\partial B_4(0)} \frac{d}{dn_r} z(x) d\sigma_r \leq \varepsilon[z] = \varepsilon.$$

Then there exists a sequence of such $\{z_k\}_{k\geq 1}$whose graphs are polyhedral surfaces such that with $z(0) = 0$ and $z(x) \geq 0$ for all x, and for each z_k,

$$\int_{\partial B_4(0)} \frac{d}{dn_r} z_k(x) d\sigma_r \leq [z_k]/k.$$

Since the multiplying z_k a constant factor, does not change the inequality, then we can assume that $[z_k] = 1$ for each $k \geq 1$, so that

$$\int_{\partial B_4(0)} \frac{d}{dn_r} z_k(x) d\sigma_r \leq [z_k]/k = 1/k. \tag{3.31}$$

Fix $x \in \partial B_4(0)$, then for any $t \in [0, 1]$, by the linearity of z_k and the definition of normal mappings $\frac{d}{dn_r} z_k(x)$,

$$0 \leq (1-t)z_k(x) = z_k(x) - z_k(tx) \leq \frac{d}{dn_r} z_k(x) \cdot (x - tx) = (1-t)x \cdot \frac{d}{dn_r} z_k(x). \tag{3.32}$$

Notice that, $x = 4x'$, where $x' = \frac{x}{|x|} \in \partial B_1(0)$, which is also the unit outward normal vector of $\partial B_4(0)$ at the point x. Then for any $k \in N$, $t \in [0, 1]$, by(3.31), (3.32),

$$\begin{aligned} 0 &\leq \int_{\partial B_4(0)} z_k(x) d\sigma_r - \int_{\partial B_4(0)} z_k(tx) d\sigma_r \\ &\leq \int_{\partial B_4(0)} (1-t)x \cdot \frac{d}{dn_r} z_k(x) d\sigma_r \\ &= \int_{\partial B_4(0)} (1-t) \frac{d}{dn_r} z_k(rx') \cdot 4x' d\sigma_r \\ &\leq \frac{4(1-t)}{k}. \end{aligned} \tag{3.33}$$

By the fact that $[z_k] = 1$ and $z(0) = 0$, $z_k(x)$ is uniformly bounded and equi-continuous on $B_3(0)$. Hence by Ascoli-Asali theorem, there exists a subsequence $\{z_{k_m}\}_{m\geq 1}$ converging uniformly on $B_3(0)$. Denote this limit function by z. Given $\lim_{k\to\infty} 4(1-t)/k = 0$, then by (3.33), $\lim_{k\to\infty} \int_{\partial B_4(0)} z_k(x) d\sigma_r$ exists, and by DCT and the fact $z(0) = 0$,

$$\lim_{k\to\infty} \int_{\partial B_4(0)} z_k(x) d\sigma_r = \lim_{k\to\infty} \int_{\partial B_4(0)} z_k(0) d\sigma_r$$

$$
\begin{aligned}
&= \lim_{k\to\infty}\int_{\partial B_4(0)} z_k(0)d\sigma_r \\
&= \lim_{m\to\infty}\int_{\partial B_4(0)} z_{k_m}(0)d\sigma_r \\
&= \int_{\partial B_4(0)} z(0)d\sigma_r \\
&= \omega_d 4^{d-1} z(0) = 0. \qquad (3.34)
\end{aligned}
$$

By the definition of polyhedral surface, for each $k \in \mathbb{N}$, define $z_k(x) := \min\{p_j^k x + b_j^k, j = 1, ..., r_k\}$ for some $r_k \in \mathbb{N}$, and there exists some subset $I_k \subset \{1, ..., r_k\}$, such that the hyperplane $y_j^k(x) = p_j^k x + b_j^k, j \in I_k$ intersects with the graph of $z_k(x)$ on the domain of$B_3(0)$. when $j = 1, ..., s_k$. For $j \in \{1, ..., r_k\} \setminus I_k$, $y_j^k(x)$ has no intersection with $z_k(x)$ on $B_3(0)$. Then $z_k(x) = \min\{p_j^k x + b_j^k, j \in I_k\}$ for $x \in B_3(0)$, and hence for any $x, y \in B_3(0)$,

$$|z_k(x) - z_k(y)| \le \max\{|p_j^k(x-y)|, j \in I_k\} \le |x-y| \max\{|p_j^k|, j \in I_k\}.$$

Since $[z_k] = 1$, then $\max\{|p_j^k|, j \in I_k\} \ge 1$ for each $k \in \mathbb{N}$.
W.O.L.G, we assume $|p_1^k| \ge 1$ and $p_1^k x_1^k + b_1^k = z_k(x_1^k)$, for some $x_1^k \in B_3(0)$ for each $k \in \mathbb{N}$. Then for every $x \in \mathbb{R}^d$,

$$z_k(x) \ge p_1^k(x - x_1^k) + z_k(x_1^k) \ge p_1^k(x - x_1^k), \quad k \in \mathbb{N}.$$

Then

$$\int_{\partial B_4(0)} z_k(x) d\sigma_r \ge \int_{\partial B_4(0)} [p_1^k(x - x_1^k)] \vee 0 \; d\sigma_r.$$

Since $x_1^k \in B_3(0)$, then

$$
\begin{aligned}
[p_1^k(x - x_1^k)] \vee 0 &\ge [|p_1^k|(\frac{p_1^k}{|p_1^k|}x - p_1^k x_1^k)] \vee 0 \\
&\ge [|p_1^k|\frac{p_1^k}{|p_1^k|}x - |p_1^k||x_1^k|] \vee 0 \\
&= [|p_1^k|\frac{p_1^k}{|p_1^k|}x - 3|p_1^k|] \vee 0 \\
&= [|p_1^k|(\frac{p_1^k}{|p_1^k|}x - 3)] \vee 0.
\end{aligned}
$$

Notice that $[\frac{p_1^k}{|p_1^k|}x-3]\vee 0 \geq 0$ for any $x \in \partial B_4(0)$ and when $\vec{p_1^k} \parallel \vec{x}$, it yields that $p_1^k x/|p_1^k| = 4$. Therefore,

$$\begin{aligned}\lim_{k\to\infty}\int_{\partial B_4(0)} z_k(x)d\sigma_r &\geq \int_{\partial B_4(0)} (|p_1^k|(\frac{p_1^k}{|p_1^k|}x-3)\vee 0)\ d\sigma_r \\ &= \int_{\partial B_4(0)} (|p_1^k|(\frac{p_1^k}{|p_1^k|}4-3)\vee 0)\ d\sigma_r > 0.\end{aligned}$$

which contradicts with (3.34). Therefore we prove (3.30). □

Now let us define $\mathcal{L}z := \frac{1}{2}\sum_{i,j=1}^d a_{ij}z_{x_ix_j}$ and denote $\det A$ as the determinant of the matrix $A = (a_{ij})$.

Lemma A.3.14. *Let S be a convex open set of $\mathbb{R}^d$ and $z(x)$ be a convex function defined on S. The measure $\omega(z,dx)$ can be decomposed into an absolutely continuous measure $\omega^a(z,dx)$, and a singular measure $\omega^s(z,dx)$ with respect to Lebesgue measure and the Radon-Nikodym derivative of $\omega^a(z,dx)$ is denoted as $(g(x))^d$, where $g(x) \geq 0$ and d is the dimension of the domain. Then for any nonnegative $\xi(x) \in C_c^\infty(\bar{S})$ and for large values of $|x|$,*

$$\beta\int_S \xi\sqrt[d]{\det A}g dx \leq \int_S z\mathcal{L}\xi dx, \tag{3.35}$$

where $\beta > 0$ and depends only on the dimension d. In other words, $\beta\sqrt[d]{\det A}g \leq \mathcal{L}z$, holds in the weak sense.

Proof. The first part can be obtained by applying the Lebesuge-Radon-Nikodym Theorem, since by (3.2), $\omega(z,\cdot)$ is a σ-finite nonnegative measure on S.

To see the inequality, first

Claim: there exists a sequence of convex functions whose graphs are polyhedra, such that $z^n \to z$ pointwise on S.

By the definition of polyhedron, any polyhedral surface can be represented by a system of affine functions: $\tilde{z}(x) = min\{m_ix + b_i, i = 1,...,j\}$, where $N = max_{i=1,...j}\{|m_i|,|b_i|\}$, and j are some finite integer numbers. Since for every convex function $z(x)$, there exists a set $A \in \mathbb{R}^{d\times d}$ such that $z(x) = \sup_{(a,b)\in U} ax + b$, and for every set U, there always exists a set $D \in \mathbb{Q}^{d\times d}$ dense in U, i.e., for every element $(a,b) \in U$, there exists a sequence of elements $(a_m^a, b_m^b) \in B$ such that $a_m^a \to a, b_m^b \to b$ as $m \to \infty$, then $z(x) = \sup_{(\tilde{a},\tilde{b})\in D} \tilde{a}x + \tilde{b}$. Since D is countable, then by denote $l^k(x) = \tilde{a}_k x + \tilde{b}_k$, for each $(\tilde{a}_k, \tilde{b}_k) \in D$, and let $z^n(x) = \sup_{k\leq n} l^k(x)$ for each $n \in \mathbb{N}$, then it yields $z^n(x) \to z(x)$ as $n \to \infty$.

Recall that $\mathcal{L}z := \frac{1}{2}\sum_{i,j=1}^d a_{ij} z_{x_i x_j}$. W.L.O.G, assume that $\det A \neq 0$, for the case of $\det A = 0$ is trivial.

For $t > 0$,let $\sigma = (A)^{1/2}$ and for any $h \in C(\mathbb{R}^d)$,

$$p(t,x,y) = (2\pi t)^{-d/2}(\det\sigma)^{-1}\exp[-|\sigma^{-1}(x-y)|^2/2t],$$

$$T_t^\sigma h(x) := \int_{\mathbb{R}^d} h(y)p(t,x,y)dy, e(t) := t^{-d+1}\int_t^\infty r^{d-1}e^{-r^2}dr.$$

Since convex function is continuous, then z is continuous on S, and extend the sequence of $\{z^n\}_{n\geq1}$, and z to the domain of $\mathbb{R}^d$, so that $\{z^n\}_{n\geq1}, z$ are all continuous on $\mathbb{R}^d$ and $z^n \to z$ pointwise on $\mathbb{R}^d$. Then it suffices to show the case when $x = 0$ since the other cases hold by the translation of coordinates. Let $\widetilde{y} = \sigma^{-1}y/\sqrt{2t}$, then by the change of coordinates, $d\widetilde{y} = (2t)^{-d/2}(\det\sigma)^{-1}dy$. Then changing into d-dimensional polar coordinates, it yields

$$\begin{aligned}
& T_t^\sigma z^n(0) - z^n(0) \\
= \; & (2\pi t)^{-d/2}(\det\sigma)^{-1}\int_{\mathbb{R}^d}[z^n(y) - z^n(0)]e^{-|\sigma^{-1}y|^2/2t}dy \\
= \; & (2\pi t)^{-d/2}(\det\sigma)^{-1}\int_{\mathbb{R}^d}[z^n(\sqrt{2t}\sigma\widetilde{y}) - z^n(0)]e^{-|\widetilde{y}|^2}(2t)^{d/2}\det\sigma d\widetilde{y} \\
= \; & \pi^{-d/2}\int_{\mathbb{R}^d}[z^n(\sqrt{2t}\sigma\widetilde{y}) - z^n(0)]e^{-|\widetilde{y}|^2}d\widetilde{y} \\
= \; & \pi^{-d/2}\int_0^\infty r^{d-1}e^{-r^2}\int_{\partial B_1(0)}[z^n(\sqrt{2t}\sigma r\omega) - z^n(0)]d\sigma_1(\omega)dr \\
= \; & \pi^{-d/2}\int_0^\infty r^{d-1}e^{-r^2}\int_{\partial B_1(0)}[\int_0^r \frac{d}{ds}z^n(\sqrt{2t}\sigma s\omega)ds]||df(\omega)||d\sigma_1(\omega)dr \qquad (3.36)
\end{aligned}$$

Let $f(\omega) := \sqrt{2t}\sigma\omega$, where ω is a vector in $\mathbb{R}^d$, then df is called **the differential of f at** $p \in \mathbb{R}^d$, which is the linear transformation from the tangent space $T_p(\mathbb{R}^d)$ to $T_{f(p)}(\mathbb{R}^d)$, i.e.,

$$df_p : T_p(\mathbb{R}^d) \mapsto T_{f(p)}(\mathbb{R}^d).$$

If we let $\{v_i, i = 1, ..., d\}$ be the basis of the tangent space $T_p(\mathbb{R}^d)$, then $\{f(v_i), i = 1, ..., d\}$ be the linear basis of the tangent space $T_{f(p)}(\mathbb{R}^d)$. By Proposition 3.38 in [73] , it is shown that

$$\|df_p(\omega)\| = \|\frac{f(v_1)\times ... \times f(v_d)}{v_1 \times ... \times v_d}\| = (2t)^{d/2}\det\sigma.$$

Hence, (3.36) equals

$$
\begin{aligned}
&= \pi^{-d/2}\int_0^\infty r^{d-1}e^{-r^2}\int_0^r[\int_{\partial B_1(0)}\frac{d}{ds}z^n(\sqrt{2t}\sigma s\omega)(2t)^{d/2}\det\sigma d\sigma_1(\omega)]dsdr \\
&= \pi^{-d/2}\int_0^\infty r^{d-1}e^{-r^2}\int_0^r[\int_{\partial B_1(0)}\frac{d}{dn_s}z^n(\sqrt{2t}\sigma s\omega)(2t)^{d/2}(\det A)^{1/2}d\sigma_1(\omega)]dsdr \\
&= \pi^{-d/2}\int_0^\infty r^{d-1}e^{-r^2}\int_0^r[\int_{\partial B_s(0)}\frac{d}{dn_s}z^n(\sqrt{2t}\sigma\widetilde{\omega})(2t)^{d/2}(\det A)^{1/2}s^{-d+1}d\sigma_s(\widetilde{\omega})]dsdr \\
&= \pi^{-d/2}\int_0^\infty r^{d-1}e^{-r^2}\int_0^r s^{-d+1}(\det A)^{1/2d}\sqrt{2t}[\int_{\partial B_{\sqrt{2t\det A}s}(0)}\frac{d}{dn_1}z^n(\sigma\widetilde{\omega})d\sigma_{\sqrt{2t\det A}s}(\widetilde{\omega})]dsdr.
\end{aligned}
\tag{3.37}
$$

Denote $k=\sqrt{2t(\det A)^{1/d}}$. Thus, combined with Lemma A.3.13, (3.37) becomes

$$
\begin{aligned}
&= \pi^{-d/2}\int_0^\infty k(s^{-d+1}\int_s^\infty r^{d-1}e^{-r^2}dr)(\int_{\partial B_{ks}(0)}\frac{d}{dn_1}z^n(\sigma\widetilde{\omega})d\sigma_{ks})ds \\
&= \pi^{-d/2}\varepsilon\int_0^\infty ke(s)\sqrt[d]{\omega(z^n,B_{ks/2}(0))}s^{d-1}ds.
\end{aligned}
$$

Therefore, for any $x\in\mathbb{R}^d$,

$$T_t^\sigma z^n(x)\geq z^n(x)+\pi^{-d/2}\varepsilon\int_0^\infty ke(s)\sqrt[d]{\omega(z^n,B_{ks/2}(x))}s^{d-1}ds$$

Taking lim inf from both sides, it yields

$$\liminf_{n\to\infty}T_t^\sigma z^n(x) \geq \liminf_{n\to\infty}z^n(x)+\pi^{-d/2}\varepsilon\liminf_{n\to\infty}\int_0^\infty ke(s)\sqrt[d]{\omega(z^n,B_{ks/2}(x))}s^{d-1}ds \tag{3.38}$$

Given any convex function $z$on $\mathbb{R}^d$,by Lebesgue-Radon-Nikodym theom on Page 91 of [38], and (3.2), $\omega(z,dx)$ can be decomposed into two parts $\omega^a(z,dx)$ and $\omega^s(z,dx)$ such that $\omega^a(z,dx) << \omega(z,dx), \omega^s(z,dx)\perp\omega(z,dx)$, and there exists a nonegative Lebesgue integrable function f on$\mathbb{R}^d$ such that $\omega^a(z,dx)=f(x)^d dx$.Let $g=f^{1/d}$.Then

$$
\begin{aligned}
&\liminf_{n\to\infty}\int_0^\infty ke(s)\sqrt[d]{\omega(z^n,B_{ks/2}(x))}s^{d-1}ds \\
\geq &\int_0^\infty ke(s)\liminf_{n\to\infty}\sqrt[d]{\omega(z^n,B_{ks/2}(x))}s^{d-1}ds \\
= &\int_0^\infty ke(s)\sqrt[d]{\liminf_{n\to\infty}\int_{B_{ks/2}(x)}\omega(z^n,dy)}s^{d-1}ds \\
= &\int_0^\infty ke(s)\sqrt[d]{\liminf_{n\to\infty}\int_{B_{ks/2}(x)}\omega^a(z^n,dy)}s^{d-1}ds
\end{aligned}
$$

$$= \int_0^\infty ke(s)\sqrt[d]{\int_{B_{ks/2}(x)}(g(y))^d dy s^{d-1}}ds, \tag{3.39}$$

where the second last equality holds due to the weak convergence of $\omega(z^n, dx) \to \omega(z, dx)$. By letting $\frac{k}{2}\widehat{y} = y - x$, it yields $dy = (\frac{k}{2})^d d\widehat{y}$, and thus

$$\begin{aligned}\int_0^\infty ke(s)\sqrt[d]{\int_{\sigma B_{ks/2}(x)}(g(y))^d dy s^{d-1}}ds &= \int_0^\infty k\sqrt[d]{\int_{B_s(0)}(g(x+\frac{k}{2}\widehat{y}))^d(\frac{k}{2})^d d\widehat{y}s^{d-1}}ds \\ &= \int_0^\infty e(s)\frac{k^2}{2}\sqrt[d]{\int_{B_s(0)}(g(x+\frac{k}{2}\widehat{y}))^d d\widehat{y}s^{d-1}}ds.\end{aligned} \tag{3.40}$$

By the fact that $(\int_{B_s(0)} d\widehat{y})^{1-\frac{1}{d}} = (\omega_d)^{1-\frac{1}{d}}s^{d(1-\frac{1}{d})}$, and Hölder's inequality,

$$\begin{aligned}&\int_0^\infty \frac{k^2}{2}e(s)(\int_{B_s(0)}(g(x+\frac{k}{2}\widehat{y}))^d d\widehat{y})^{1/d}(\int_{B_s} d\widehat{y})^{1-\frac{1}{d}}ds \\ = &\int_0^\infty \frac{k^2}{2}e(s)(\int_{B_s(0)}(g(x+\frac{k}{2}\widehat{y}))^d d\widehat{y})^{1/d}(\omega_d)^{1-\frac{1}{d}}s^{d(1-\frac{1}{d})}ds \\ \geq &\int_0^\infty \frac{k^2}{2}e(s)\int_{B_s(0)} g(x+\frac{k}{2}\widehat{y})d\widehat{y}ds,\end{aligned} \tag{3.41}$$

where ω_d is the volume of the unit d-dim sphere. Hence, by (3.40), (3.41),

$$\int_0^\infty \frac{k^2}{2}e(s)\sqrt[d]{\int_{B_s(0)} g^d(x+\frac{k}{2}\widehat{y})d\widehat{y}s^{d-1}}ds \geq (\omega_d)^{\frac{1}{d}-1}\int_0^\infty \frac{k^2}{2}e(s)\int_{B_s(0)} g(x+\frac{k}{2}\widehat{y})d\widehat{y}ds. \tag{3.42}$$

Therefore, by (3.38),(3.39),(3.40),(3.42),

$$T_t^\sigma z(x) \geq z(x) + \pi^{-d/2}\varepsilon(\omega_d)^{\frac{1}{d}-1}t\sqrt[d]{\det A}\int_0^\infty e(s)\int_{B_s(0)} g(x+\frac{\sqrt{2t}\sqrt[d]{\det\sigma}}{2}\widehat{y})d\widehat{y}ds.$$

i.e.,

$$\frac{T_t^\sigma z(x) - z(x)}{t} \geq N\sqrt[d]{\det A}\int_0^\infty e(s)\int_{B_s(0)} g(x+\frac{\sqrt{2t}\sqrt[d]{\det\sigma}}{2}\widehat{y})d\widehat{y}ds, \tag{3.43}$$

where $N = \pi^{-d/2}\varepsilon(\omega_d)^{\frac{1}{d}-1}$, where ε only depends on dimension d. Multiplying the inequality by ξ and intergration with respect to x,

$$\int_S \xi(x)\frac{T_t^\sigma z(x) - z(x)}{t}dx \geq N\sqrt[d]{\det A}\int_S \xi(x)[\int_0^\infty e(s)\int_{B_s(0)} g(x+\frac{\sqrt{2t}\sqrt[d]{\det\sigma}}{2}\widehat{y})d\widehat{y}ds]dx.$$

Since by Fubini's theorem,

$$\begin{aligned}\int_S \xi(x)T_t^\sigma z(x)dx &= \int_S \xi(x)\int_S p(t,x,y)z(y)dydx = \int_S z(y)\int_S p(t,x,y)\xi(x)dxdy \\ &= \int_S z(x)\int_S p(t,x,y)\xi(y)dydx = \int_S z(x)T_t^\sigma \xi(x)dx,\end{aligned}$$

which yields

$$\int_S \xi(x)\frac{T_t^\sigma z(x)-z(x)}{t}dx = \int_S z(x)\frac{T_t^\sigma \xi(x)-\xi(x)}{t}dx. \tag{3.44}$$

Hence, by (3.43) and (3.44),

$$\int_S z(x)\frac{T_t^\sigma \xi(x)-\xi(x)}{t}dx \geq N\sqrt[d]{\det A}\int_S \xi(x)[\int_0^\infty e(s)\int_{B_s(0)} g(x+\frac{\sqrt{2t}\sqrt[d]{\det\sigma}}{2}\widehat{y})d\widehat{y}ds]dx.$$

Since ξ is infinitely differentiable, then $\xi\in\mathcal{D}(\mathcal{L})$, i.e., $\lim_{t\to 0}\frac{T_t^\sigma\xi(x)-\xi(x)}{t}=\mathcal{L}\xi(x)$ for every $x\in\mathbb{R}^d$.

On the other hand, since $z(x)T_t^\sigma\xi(x)$ is bounded for every $x\in S$, then by DCT,

$$\lim_{t\to 0}\int_S z(x)\frac{T_t^\sigma \xi(x)-\xi(x)}{t}dx = \int_S z(x)\mathcal{L}\xi(x)dx.$$

Letting $\widehat{x}=x+k\widehat{y}$, i.e, $x=\widehat{x}-k\widehat{y}$, by Fubini's theorem,

$$\begin{aligned}&N\sqrt[d]{\det A}\int_S \xi(x)[\int_0^\infty e(s)\int_{B_s(0)} g(x+\frac{\sqrt{2t}\sqrt[d]{\det\sigma}}{2}\widehat{y})d\widehat{y}ds]dx \\ =\ & N\sqrt[d]{\det A}\int_0^\infty e(s)\int_{B_s(0)}\int_S \xi(x)g(x+\frac{\sqrt{2t}\sqrt[d]{\det\sigma}}{2}\widehat{y})dxd\widehat{y}ds \\ =\ & N\sqrt[d]{\det A}\int_0^\infty e(s)\int_{B_s(0)}\int_S \xi(\widetilde{x}-\frac{\sqrt{2t}\sqrt[d]{\det\sigma}}{2}\widehat{y})g(\widehat{x})d\widehat{x}d\widehat{y}ds\end{aligned}$$

Notice that when $t\to 0$,$k\to 0$, and thus $\widehat{x}\to x$ and $\liminf_{t\to 0}\xi(\widehat{x}-\frac{\sqrt{2t}\sqrt[d]{\det\sigma}}{2}\widehat{y})=\xi(x)$. Then by Fatou's lemma,for every $x\in\mathbb{R}^d$,

$$\begin{aligned}&\liminf_{t\to 0}\sqrt[d]{\det A}\int_0^\infty e(s)\int_{B_s(0)}\int_S \xi(\widehat{x}-\frac{\sqrt{2t}\sqrt[d]{\det\sigma}}{2}\widehat{y})g(\widehat{x})d\widehat{x}d\widehat{y}ds \\ \geq\ & \sqrt[d]{\det A}\int_0^\infty e(s)\int_{B_s(0)}\int_S \xi(x)g(x)dxd\widehat{y}ds\end{aligned}$$

$$= \quad \beta \sqrt[d]{\det A} \int_S \xi(x) g(x) dx$$

where $\beta = 2N\omega_d \int_0^\infty e(s) s^d ds$, a constant which only depends on d. Therefore, we prove that for any $x \in S$, any $\xi \in C_c^\infty(S)$ with $\xi = 0$ on ∂S,

$$\beta \sqrt[d]{\det A} \int_S \xi(x) g(x) dx \leq \int_S z(x) \mathcal{L}\xi(x) dx,$$

i.e.,

$$\beta \sqrt[d]{\det A} g \leq \mathcal{L} z,$$

holds in the weak sense.

Our main theorem is the following theorem.

Theorem A.3.15. *Suppse that $r > 0$ and $f(t,x) \geq 0$ is in $L^{d+1}((-\infty,\infty) \times \mathbb{R}^d)$ such that $f = 0$ outside $(0,\infty) \times B(x_0, r)$. Then there exists a bounded non-positive function $G(t,x)$ on $(-\infty,\infty) \times \mathbb{R}^d$ that vanishes on $(-\infty, 0) \times \mathbb{R}^d$and has the following properties:*

(i) *$G(t,x)$ is convex in $x \in B(x_0, 2r)$ for every fixed $t \in (0,\infty)$ and decreasing in t for every fixed $x \in \mathbb{R}^d$;*

(ii) *If $A = (a^{ij})$ is a symmetric nonnegative-definite matrix, then for $\varepsilon > 0$ small enough,*

$$N(d)(\det A)^{1/(d+1)} f^\varepsilon(t,x) \leq -\frac{\partial G^\varepsilon}{\partial t}(t,x) + \sum_{i,j=1}^d a^{ij} G^\varepsilon_{x_i x_j}(t,x)$$

for every $(t,x) \in [0,\infty) \times B(x_0, r)$ where $N(d) > 0$ depending only on d;

(iii) *If a vector b in $\mathbb{R}^d$ and a constant $c > 0$ satisfies $|b| \leq cr/2$,then*

$$\sum_{i=1}^d b_i G^\varepsilon_{x_i}(t,x) - cG^\varepsilon(t,x) \geq 0,$$

for every $(t,x) \in [0,\infty) \times B(x_0, r)$;

(iv) *For every $(t,x) \in (-\infty,\infty) \times \mathbb{R}^d$,*

$$|G(t,x)|^{d+1} \leq \frac{(d+1)(2r)^d}{\omega_d} \int_0^\infty \int_{B(x_0,r)} f^{d+1}(s,y) dy ds,$$

where ω_d is the volume of the unit sphere.

Proof. W.L.O.G. we can assume $x_0 = 0$. For the case when $x_0 \neq 0$, the following proof holds by first shifting the coordinates x_0 to the origin. Let $n = 1, 2, ...$ and $k = 0, 1, 2, ...$ and define

$$t_{nk} = 2^{-n}k \text{ and } f_{nk}(x) = 2^n \int_{t_{nk}}^{t_{n,k+1}} f(s,x)ds. \tag{3.45}$$

Fix n, by Theorem A.3.12, there exists a sequence of convex functions $\{G^{nk}(x)\}_{k\geq 0}$ defined on $B(0,2r)$ such that for every $x \in B(0,2r)$,

$$0 = G^{n0}(x) \geq G^{n1}(x) \geq ... \text{ and } G^{nk}(x) = 0 \text{ on } \partial B(0,2r) \text{ for every } k \in \mathbb{N} \tag{3.46}$$

and

$$2^{-n}(f_{nk}(x))^{d+1}dx = (G^{nk}(x) - G^{n,k+1}(x))\omega(G^{n,k+1}, dx). \tag{3.47}$$

Next, for each $n \in \mathbb{N}$ and construct the function $G^n(x,t)$ such that for any $x \in B(0,2r)$,

$$\begin{aligned} &G^n(t_{nk}, x) = G^{nk}(x), G^n(t_{n,k+1}, x) = G^{n,k+1}(x) \text{for each } k \in \mathbb{N} \\ &\text{and } G^n(t,x) \text{ is linear in } t \text{ for } t \in [t_{nk}, t_{n,k+1}]. \end{aligned} \tag{3.48}$$

Therefore, by (3.46),$G^n(t,x)$ is decreasing in t on $(0,\infty)$, and by (3.24), and for any $t \in (0,\infty)$ we have $t \in [t_{nk}, t_{n,k+1}]$ for some $k \in \mathbb{N}$, and

$$\begin{aligned} & (2r)^{-d}\omega_d|G^n(t,x)| \\ \leq\ & (2r)^{-d}\omega_d|G^{nk}(t_{nk},x)| \\ \leq\ & (d+1)\sum_{i=0}^{k-1}\int_{B(0,r)} 2^{-n}(f_{ni}(y))^{d+1}dy \\ \leq\ & (d+1)\sum_{i=0}^{k-1}\int_{B(0,r)} 2^{-n}(2^n \int_{t_{n,i}}^{t_{n,i+1}} f(y,s)ds)^{d+1}dy \\ \leq\ & (d+1)\sum_{i=0}^{k-1}\int_{B(0,r)} 2^{-n}(\int_{t_{n,i}}^{t_{n,i+1}} 2^{n(1+1/d)}ds)^d \int_{t_{ni}}^{t_{n,i+1}} f^{d+1}(s,y)dy \\ \leq\ & (d+1)\sum_{i=0}^{k-1}\int_{B(0,r)}\int_{t_{ni}}^{t_{n,i+1}} f(s,y)^{d+1}dyds \\ =\ & (d+1)\int_{(0,t_{n,k})\times B(0,r)} (f(s,y))^{d+1}dyds \\ \leq\ & (d+1)\int_{(0,\infty)\times B(0,r)} (f(s,y))^{d+1}dyds, \end{aligned} \tag{3.49}$$

where the second last inequality is by Hölder inequality. Notice that the upper bound in the last line is independent of t, which it prove (iv).

By (3.49), the sequence of functions $\{G^n(t,x)\}_{n\geq 1}$ are uniformly bounded in $(0,\infty)\times B(0,2r)$. Then there exists a subsequence of functions

$$\{G^{n_{j1}}(t_1,x)\}_{j1\geq 1}\subset\{G^n(t,x)\}_{n\geq 1} \text{ converges uniformly in } t_{11}\times B(0,2r).$$

Since$\{G^{n_{j1}}\}_{j1\geq 1}$ are bounded sequence, then there exists a subsequence

$$\{G^{n_{j2}}(t_2,x)\}_{j2\geq 1}\subset\{G^n(t,x)\}_{n\geq 1} \text{ converges uniformly in } t_{21}\times B(0,2r).$$

Repeat this process for each $t=2^{-n}k, k,n\in\mathbb{N}$.

Finally, the diagonal sequence

$$\{G^{n_{j1}}(t_{11},x),G^{n_{j2}}(t_{21},x),G^{n_{j3}}(t_{22},x),...\} \text{ converges uniformly in } x\in B(0,2r).$$

Denote the sequence $s_m=\{n_{j1},n_{j2},n_{j3},n_{j4}...\}$. Let

$$G(t,x)=\lim_{s_m\to\infty}G^{s_m}(t,x), \text{ for each } t=k/2^n$$

and let

$$G(t,x):=\lim_{2^{-n}k\downarrow t}G(2^{-n}k,x) \text{ for each non-dyadic } t.$$

Then $G(t,x)$ is a function decreasing in $t\in(0,\infty)$.

Thus we proved (i) and (iv).

Next we will show (iii). For $\varepsilon\in(0,r/2)$, consider

$$G^{\varepsilon}(t,x):=\varepsilon^{-(d+1)}\int_{\mathbb{R}^d}\zeta(t/\varepsilon,(x-y)/\varepsilon)G(t,y)dy$$

and

$$f^{\varepsilon}(t,x):=\varepsilon^{-(d+1)}\int_{\mathbb{R}^d}\zeta(t/\varepsilon,(x-y)/\varepsilon)f(t,y)dy,$$

where ζ is the mollifier defined in Subsection 1.4.3.

Since $G(t,x)\leq 0$ on $(0,\infty)\times B(0,2r)$ and $G(x,t)$ is convex in x on $B(0,2r)$, then $G^{\varepsilon}(t,x)$ is non-positive on $(0,\infty)\times B(0,2r)$ and convex in x on $B(0,3r/2)$.

Let $F(t_0,x)=p(x-x_0)+G^\varepsilon(t_0,x_0)$ be the tangent plane to the graph of $G^\varepsilon(t_0,x)$ at $x_0\in B(0,r)$. Notice that for any $y=x_0+pr/(2|p|)$,

$$y\in B(0,3r/2) \text{ and } \langle p, y-x_0\rangle = |p|r/2, \tag{3.50}$$

Then by (3.50) and the definition of supporting hyperplane of a convex function,

$$r|p|/2+G^\varepsilon(t_0,x_0)\le \sup_{|x|\le 3r/2}(p(x-x_0)+G^\varepsilon(t_0,x_0))\le G^\varepsilon(t_0,x)\le 0$$

for each $x\in B(0,3r/2)$.

Notice that $p=\nabla_x G^\varepsilon(x_0,t_0)$, then for any vector $b\in\mathbb{R}^d$ and constant $c>0$ such that $|b|\le r/2c$, we have

$$\sum_{i=1}^d b_i G^\varepsilon_{x_i}-cG^\varepsilon\ge -|b||p|-cG^\varepsilon\ge(-r|p|/2-G^\varepsilon)c\ge 0.$$

To show (ii), first by the first part of Lemma A.3.14, for each $\omega(G^{nk},dx)$, there exists a Randon-Nikodym derivative, in the form of $(g_{nk}(x))^d$, with respect to Lebesgue measure dx, such that

$$\omega^a(G^{nk},dx)=(g_{nk}(x))^d dx. \tag{3.51}$$

Let

$$a=(2^n(G^{nk}(x)-G^{n,k+1}(x)))^{d+1}, b=(\beta/d)^d\det A(g_{n,k+1}(x))^d,$$

where $A=(a_{ij})$ in the operator $\mathcal{L}$ and β is the parameter in Lemma A.3.14.

Then by the inequality $(d+1)(ab)^{1/(d+1)}\le a+db^{1/d}$,where d is the dimension, and (3.47), and (3.51), for every $x\in B(0,2r)$,

$$\begin{aligned}
&(d+1)\sqrt[d+1]{\det A}(\frac{\beta}{d})^{d/(d+1)}f_{nk}(x)\\
=\;&(d+1)\sqrt[d+1]{\det A}(\frac{\beta}{d})^{d/(d+1)}\Big((g_{n,k+1}(x))^d\cdot a\Big)^{1/d+1}\\
\le\;&2^n(G^{nk}(x)-G^{n,k+1}(x))+\beta g_{n,k+1}\sqrt[d]{\det A},
\end{aligned}\tag{3.52}$$

Let

$$\chi^\varepsilon(x)=1/(\varepsilon^d)\chi(x/\varepsilon),$$

where $\chi(x)$ be another mollifier function on $x \in \mathbb{R}^d$, and $\varepsilon < r$. By (3.52) and Lemma A.3.14, for every $x \in B(0,r)$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} \alpha \sqrt[d+1]{\det A} f_{nk}(x-y)\chi^\varepsilon(y)dy \\
\leq\ & \int_{\mathbb{R}^d} \Big(2^n \chi^\varepsilon(y)(G^{nk}(x-y) - G^{n,k+1}(x-y)) + (\mathcal{L}\chi^\varepsilon(y))G^{n,k+1}(x-y)\Big)dy,
\end{aligned} \tag{3.53}$$

where $\alpha = (d+1)(\beta/d)^{d/(d+1)}$.

Take two numbers $t_1 = 2^{-m}k_1$ and $t_2 = 2^{-m}k_2$, where $k_1 < k_2$, and choose $n > m$. Then by (3.45),

$$\sum_{k=2^{n-m}k_1}^{2^{n-m}k_2-1} f_{nk}(x) = 2^n \int_{t_1}^{t_2} f(x,s)ds, \tag{3.54}$$

and by (3.48),

$$\sum_{k=2^{n-m}k_1}^{2^{n-m}k_2-1} [2^n(G^{nk}(x) - G^{n,k+1}(x)) = 2^n(G^n(t_1,x) - G^n(t_2,x)), \tag{3.55}$$

and

$$\begin{aligned}
\sum_{k=2^{n-m}k_1}^{2^{n-m}k_2-1} G^{n,k+1} &= 2^n \sum_{k=2^{n-m}k_1}^{2^{n-m}k_2-1} 2^{-n}\frac{1}{2}(G^n(t_{nk},x) \\
&\quad +G^n(t_{n,k+1},x)) + 1/2(G^n(t_2,x) - G^n(t_1,x)) \\
&= 2^n \int_{t_1}^{t_2} G^n(s,x)ds + 1/2(G^n(t_2,x) - G^n(t_1,x)).
\end{aligned} \tag{3.56}$$

Hence, by (3.53)-(3.56), for each $x \in B(0,r)$,

$$\begin{aligned}
&\alpha \sqrt[d+1]{\det A} \int_{t_1}^{t_2} \int_{\mathbb{R}^d} f(s,x-y)\chi^\varepsilon(y)dyds \\
\leq\ & \int_{\mathbb{R}^d} G^n(t_1,x-y)\chi(y)dy - \int_{\mathbb{R}^d} G^n(t_2,x-y)\chi^\varepsilon(y)dy \\
&+ \int_{\mathbb{R}^d}\int_{t_1}^{t_2} G^n(s,x-y)\mathcal{L}\chi^\varepsilon(y)dsdy + \frac{1}{2^{n+1}}\int_{\mathbb{R}^d}(G^n(t_2,x-y) - G^n(t_1,x-y))\mathcal{L}\chi^\varepsilon(y)dy.
\end{aligned}$$

Notice that $|\mathcal{L}\chi^\varepsilon(y)| \leq 1$ and $G^n(t_2,x), G^n(t_1,x)$are bounded by (iv). then sending $n \to \infty$, by DCT,

$$\alpha \sqrt[d+1]{\det A} \int_{t_1}^{t_2} \int_{\mathbb{R}^d} f(s,x-y)\chi^\varepsilon(y)dyds$$

$$\leq \int_{\mathbb{R}^d} G(t_1, x-y)\chi^\varepsilon(y)dy - \int_{\mathbb{R}^d} G(t_2, x-y)\chi^\varepsilon(y)dy + \int_{\mathbb{R}^d} \mathcal{L}\int_{t_1}^{t_2} G(s, x-y)\chi^\varepsilon(y)dsdy. \tag{3.57}$$

Let $\chi^\varepsilon(t) = \chi(|t|)/\varepsilon$ be the mollifier function on $t \in (0,\infty)$,where $\varepsilon \leq r$, then by (3.57),

$$\alpha \sqrt[d+1]{\det A}\int_{t_1}^{t_2} f^\varepsilon(s,x)ds = \alpha \sqrt[d+1]{\det A}\int_{t_1}^{t_2}\int_{\mathbb{R}^d}\int_{-\infty}^{\infty} f(s,x-y)\chi^\varepsilon(y)\chi^\varepsilon(t-s)dtdyds$$
$$\leq G^\varepsilon(t_1,x) - G^\varepsilon(t_2,x) + \int_{t_1}^{t_2} \mathcal{L}G^\varepsilon(s,x)ds.$$

Then the inequality in (ii) follows immediately.

A.4 Proof of Lemma A.3.10

Instead of proving Lemma A.3.10, we will show a more general theorem– Theorem 3 in [2], which we will summarize the details of the theorem here. From now on, the function z mentioned in the rest of this subsection, will by default, always means a convex function z. Let $D \in \mathbb{R}^d$ and $z : D \to \mathbb{R}$ be a differentiable function and let

$$p := \nabla z = (z_{x_1}, ..., z_{x_d})$$

as the gradient of $z(x)$. Let $f(p,z,x)$ be a function satisfying the following conditions:

Condition A.4.1. *The function $f(p,z,x)$ is well-defined for all p,z and every $x \in D$; Furthermore, it is always nonnegative and may take the infinite value.*

Condition A.4.2. *In every closed bounded domain R of the variables p,z and x, there exists a summable function $f_0(p)$ such that for every $(p,z,x) \in R$,*

$$f(p,z,x) \leq f_0(p).$$

Condition A.4.3. *There exist a number $z_0 \in \mathbb{R}$ and a function $f_1(p) \geq 0$ such that*

$$\int_{\mathbb{R}^d} f_1(p)dp > 0$$

and for every $x \in D$ and $z < z_0$,

$$f(p,z,x) \geq f_1(p).$$

Define

$$A(f) = \sup \int_{\mathbb{R}^d} f_1(p)dp, \tag{4.1}$$

where the supremum is taken over all possible z_0 and over all possible functions $f_1(p)$.

Condition A.4.4. *For almost every p, the function $f(p, z, x)$ is continuous in z and x.*

For a function f satisfying the above four conditions, $p \to (x(p), z(p))$ is just the inverse of our normal mapping. The function $x(p)$ and $z(p)$ are not unique for those points p for which the supporting plane of direction p touches the surface at more than one point. However, we claim that **the set of such points p has zero measure.**

Proof. It is a claim in (9.8) in §9.4 of [3].We will give a proof by contradiction for the case when $d = 1$.For $d \geq 2$, it borrows the idea of Lemma 5.2 in [3] and need some modification of it.

Assume there exists a set $U \in \mathbb{R}^d$ with $|U| \neq 0$ such that for any $p \in U$, the supporting hyperplane with the direction of p touches the convex surface at at least two points $(x_1(p), z_1), (x_2(p), z_2)$ such that $\nabla z|_{x=x_1(p)} = \nabla z|_{x=x_2(p)} = p$. Given the convexity of the function $z(x)$, we know z is differentiable almost everywhere [74, Theorem 25.5], and the gradient p is monotone, i.e., for any $x, y \in \mathbb{R}^d$ where it is differentiable,

$$(y - x) \cdot (\nabla z(x) - \nabla z(y)) \geq 0. \tag{4.2}$$

Hence for any point $\bar{x}(p) = tx_1(p) + (1-t)x_2(p), 0 \leq t \leq 1$, if $\bar{x}(p)$ is differentiable, then by (4.2),

$$(x_1(p) - \bar{x}(p)) \cdot (p - \nabla z(\bar{x}(p))) \geq 0 \text{ and } (x_2(p) - \bar{x}(p)) \cdot (p - \nabla z(\bar{x}(p))) \geq 0,$$

which lead to $(1-t)(x_1(p)-x_2(p)) \cdot (p-\nabla z(\bar{x}(p))) \geq 0$ and $t(x_1(p)-x_2(p)) \cdot (p-\nabla z(\bar{x}(p))) \geq 0$.Hence

$$\nabla z(\bar{x}(p)) = p. \tag{4.3}$$

When $d = 1$, let $(x_1(p), x_2(p))$ be the maximal interval whose gradient is p. Then by (4.3), for different $p \neq q \in U$, $(x_1(p), x_2(p)) \cap (x_1(q), x_2(q)) = \emptyset$. Choose an arbitrary rational number lying in each interval. Since there are at most countably many rational numbers, then there are at most countably disjoint such open intervals. Hence U is countable.

When $d \geq 2$, denote $l(x_1(p), x_2(p))$ be the maximal line segment whose gradient is gradient p whose endpoints are $(x_1(p), z_1)$ and $(x_2(p), z_2)$. Then the rest of the proof will be discussed by considering if the line segments can be represented by how many numbers of nonzero orthonormal unit vectors out of d orthonormal ones. The reader can refer to Lemma 5.2 in [3] for more details. □

Therefore, the function $f(p, z(p), x(p))$ is uniquely defined for almost all p's in the gradient plane.

Define

$$\omega_f(M; S) = \int_{\nu_z(M)} f(p, z(p), x(p))dp,$$

where M is a Borel set of D and $\chi_z(M)$ is the gradient space of function z defined on the set M.We call it **the relative curvature of S relative to the plane of the domain**.

Remark A.4.5. *According to [2], in geometry, instead of saying the function $z(x)$, we say "the surface S given by the equation $z = z(x)$", then the paper denotes the relative curvature of convex function z as $\omega_f(M; S)$.Just to clarify, when $f = 1$, $\omega_1(M; S)$ means the normal volume of z on M, as denoted by $\omega(z, M)$ in Subsection A.3.*

In particular, if $f(p) = (1+p)^{-\frac{n+1}{2}}$, then $\omega_f(M; S)$ is **the area of the spherical image (integral curvature)** of the set on the surface S_z whose projection is M.

Let $D \in \mathbb{R}^d$ be a convex domain and D be its boundary. Not every $(n-1)$-dimensional surface projectable onto D can be the boundary of a convex surface S projectable onto D. (**The border of the surface** S is simply the set of its limit points not belonging to S.)

Definition A.4.6. *We say a surface L that could be the boundary of some other surface S is called* ***an admissible surface or an admissible border.*** *(For more details and examples of an admissible border, please refer to Section 3 of [2]).*

Theorem A.4.7. *Every convex surfaces S projectable onto a domain D, having a common border L are uniformly bounded if they all obey the inequalities*

$$\omega_f(D; S) \leq C < A(f), \tag{4.4}$$

where C is a constant, $A(f)$ is defined in (4.1).

Proof. The proof can also be found in Theorem 1 of [2].By the definition of the supremum $A(f)$, for a given $C < A(f)$, there clearly exists a $z_0 \in \mathbb{R}$ and a function $f_1(p)$ such that

$$f(p, z, x) \geq f_1(p) \geq 0 \quad \text{for every } z < z_0, x \in D \tag{4.5}$$

and

$$\int_{\mathbb{R}^d} f_1(p)dp \geq C.$$

We can assume that z_0 is such that the common border L of the surfaces S lies above the plane $z = z_0$. Then the plane cuts off a"cap" $\bar{S}$ from every "large" surface. It suffices to demonstrate that the "caps" are bounded.

Let $\bar{D}$ denote the projection of the cap $\bar{S}$. Then, clearly

$$\omega_f(\bar{D}, \bar{S}) \leq \omega_f(D; S). \tag{4.6}$$

By the definition of ω_f and by (4.5),

$$\omega_f(\bar{D}; \bar{S}) = \int_{\nu_z(\bar{D})} f(p, z, x)dp \geq \int_{\nu_z(\bar{D})} f_1(p)dp, \tag{4.7}$$

where $\nu_z(\bar{D})$ is the normal image of the cap $\bar{S}$. Since the cap $\bar{S}$ grows with the surface S.Also, $\omega_f(\cdot; S)$, then $\omega_f(\bar{D}; \bar{S}) \uparrow \infty$.

$$\int_{\nu_z(\bar{D})} f_1(p)dp > C. \tag{4.8}$$

Then, by (4.6), (4.7) (4.8), we obtain $\omega_f(D; S)$<C, contradicting with (4.4). Hence all the surfaces are uniformly bounded. □

By (3.2), $\omega_f(D; S)$ is an additive set function on the gradient space, then the next theorem is about the weak convergence of the relative curvature when the sequence of convex surfaces $\{S_m\}_{m\geq 1}$ converge to S.

Before proving the main theorem of this subsection, we first introduce another definition.

Definition A.4.8. *If a convex surface S_L, whose border is L, has zero area of its spherical image projectable onto D, we say that* ***the surface*** S_L ***is spanned over*** L ***from below.****In this case, S_L is one and the only supporting plane for each point on the surface S_L.*

Theorem A.4.9. *If the sequence of convex surfaces $\{S_m\}_{m\geq 1}$ projectable onto a given domain D converge to a surface S, then their relative curvatures $\omega_f(D; S_m)$ converge weakly to $\omega_f(D; S)$.*

Proof. The proof can be found in Theorem 2 of [2].

Theorem A.4.10. *Let D be a polyhedral convex domain in $\mathbb{R}^d$ and L an admissible $(n-1)$-dimensional polyhedral surface projectable onto the boundary of D. Let $\mu(M)$ be a set function defined on D such that μ has nonzero mass only at finitely many points $\{x_1, ..., x_m\} \in D$. Then if*

$$\sum_{i=1}^{m} \mu(x_i) < A(f), \tag{4.9}$$

then there exists a convex polyhedral surface S such that for $i = 1, .., m$,

$$\omega_f(x_i; S) = \mu_i, \tag{4.10}$$

where S is projected onto D and L as its border.

Proof. The proof is the same as in Theorem 3 of [2]. We will summarize it here.

Let us span the surface S_L on L from below. Clearly, S_L is a polyhedral surface with no vertices and L as its border. By the definition of the surface on L spanned from below, we know that it satisfies the requirement that it has no vertices except those are projectable into the points $x_1, ..., x_m \in D$.

Also, $\omega_f(x_i; S_L) = 0, i = 1, ..., m$, which satisfies (4.9).

Hence it shows there exists a polyhedral surface satisfying the border and vertices requirement and (4.9).

Next, consider all polyhedral surfaces, denoted by $\{S_j, j \in I\}$, whose borders are L, and have no vertices other than those that are projectable into the points $x_1, .., x_m$ such that for each $j \in I$,

$$\omega_f(x_i; S_j) \leq \mu_i, i = 1, ..., m. \tag{4.11}$$

By (4.9) and Theorem A.4.7, they are uniformly bounded, i.e., for all the functions $z_j(x), j \in I$, each of whose graph is associated with a polyhedral surface S_j, there exists a positive $M > 0$, such that $\sup_{x \in D, j \in I} |z_j(x)| < M$. Then there must exist a surface $S^0 \in \{S_j, j \in I\}$ whose function z^0 satisfies that $\sum_{i=1}^m z(x_i)$ is the minimal.

We claim that S^0 is the desired surface satisfying (4.10), which will be proved by contradiction. Assume it is not, then it means for some point x_k,

$$\omega_f(x_k; S^0) < \mu_k. \tag{4.12}$$

Let A_k be a point on S^0 whose projection on D is x_k.Notice that A_k might not be a vertex of S^1. Move A_k downwards vertically through a small distance $A_k^{'} A_k$. Construct the convex hull of the point $A_K^{'}$, the vertices of S^0 and a border L. Then it is a polyhedral surface S^1 with border L, with the vertices whose projection belongs to the set $\{x_1, ..., x_m\}$.

Next we will show that the surface S^1 satisfies (4.10).

Denote $A_i^{'}$'s are the points on S^1 whose projections are x_i's on D. If $A_i^{'}$ is a vertex of S^1 different from $A_k^{'}$, then from the construction of S^1, it is clear that $A_i^{'}$ is the vertex A_i of the surface S^0 and by

$$\nu_{S^1}(x_i) \subset \nu_{S^0}(x_i),$$

so by the definition of $\omega_f(x_i; S^1)$, since z_i and x_i are the same in S^0 and S^1, then

$$\omega_f(x_i; S^1) \leq \omega_f(x_i; S^0),$$

and thus by (4.11),

$$\omega_f(x_i; S^0) \leq \mu_i.$$

For the point $A_i^{'}$ which is not the vertex of S^1, which means $\omega_f(x_i; S^0) = 0 < \nu_i$. Finally, we consider $A_k^{'}$, the most shifted vertex, by the definition of relative curvature,

$$\omega_f(x_k; S^1) = \int_{\nu_{S^1}(x_k)} f(p, z_k^{\prime}, x_k) dp,$$

where (x_k, z'_k) is the point on the surface S^1. Let us show that $\omega_f(x_k; S^1)$ varies continuously along $A'_k A_k$ shifted continuously from the initial position A_k. For a polyhedral surface, the normal image $\nu_z(x_k)$ of A'_k is a convex polyhedron in the gradient plane, and it varies continuously as the point A'_k varies continuously.

In addition, by Condition A.4.4, the function $f(p, z'_k, x_k)$ is continuous in z'_k and x_k for almost all p. Hence, as the point A'_k varies continuously along $A'_k A_k$, the function $f(p, z'_k, x_k)$ varies continuously for all p. And by Condition A.4.2, the function $f(p, z'_k, x_k) \leq f_0(p)$ on bounded domain of (p, z'_k, x_k). Therefore, by dominated convergence theorem, $\omega_f(x_i; S^1)$, and therefore $\omega_f(x_k; S^1)$ really depends continuously on the position of A'_k.Consequently, with a sufficiently small shift $A'_k A_k$, since by (3.22), $\nu_{S^1}(x_k) \subset \nu_{S^0}(x_k)$, then $\omega_f(x_k; S^1) < \omega_f(x_k; S^0)$ for a very small amount.

Hence, by (4.12),

$$\omega_f(x_k; S^1) < \mu_k.$$

SInce S^1 satisfies (4.10) and $\sum_{j=1}^m z'_j < \sum_{j=1}^m z_j$, which contradicts with the definition of S^0. Hence S^0 satisfies (4.10).

Remark A.4.11. *1. From the proof, it is clear to see that the surface S^0 does not only satisfy $\sum_{j=1}^m z_j$ is the minimal among all the functions whose surfaces are $\{S_j, j \in I\}$, but also satisfies each coordinate z_j attains the minimum respectively .*

2. In [2], there is a more generalized theorem which extends the above theorem by proving the existence of a convex surface on a convex domain , instead of a polyhedral domain, with the same border requirement and satisfying (4.9). Please refer to Theorem 4 in [2] if the readers are interested in this area.

Proof of Lemma A.3.10. To show Lemma A.3.10, notice that each $z_1(x_i)$ is a known value. Let $f(p, z_2, x) = z_1(x) - z_2$which is well-defined for every $x \in D$ and z_2. Notice that the boundary L of S_{z_2} is $z_2|_D = 0$.

Next, given $z_1|_{\partial D} = 0$ and z_1 is convex, then $z_1 \leq 0$ on $\bar{D}$. Notice that $z_2 = z_1$ is a solution of

$$(z_1 - z_2)\omega(z_2, x_i) \leq \mu_i \text{ for every } i = 1, .., m. \tag{4.13}$$

Also, given the assumption that $z_2 < z_1$ if such z_2 exists, $z_1(x) - z_2 \geq 0$ and it is defined for every $x \in \bar{D}$ and every p in the gradient plane, which satisfies Condition A.4.1.Notice that the steeper the polyhedral surface of z_2 is, the larger the value of $\sup_{x \in D} z_1(x) - z_2$ and the value of $\max_{i \in \{1,\ldots,m\}} |\nu_{z_2}(x_i)|$ become.
Moreover, $f(p.z_2, x)$ satisfies Condition A.4.2 since given any compact set D of (p, z_2, x), there exists a bounded function $f_0 = \sup_D(z_1(x) - z_2)$, such that $f(p.z_2, x) \leq f_0$ on D.
Since the graph of z_1 are polyhedron, then $z_1(x)$ are continuous on $\bar{D}$,then $f(p, z_2, x) = z_1(x) - z_2$ is continuous in x and z_2, which satisfies Condition A.4.4.

To check if $f(p, z_2, x) = z_1(x) - z_2$ satisfies Condition A.4.3, we first notice that t,$\{z_1(x), x \in \bar{D}\}$ is bounded, and the minimum value of z_1 lies either on the vertices or on ∂D, which is deterministic. Hence, for any constant $z_0 < \min_{x \in \bar{D}} z_1(x)$, choose $f_1(p) = \min_{x \in \bar{D}} z_1(x) - z_0 > 0$, then

$$\int_{\mathbb{R}^d} f_1(p)dp = \infty$$

and for any $z_2 < z_0$, and every $x \in \bar{D}$,

$$z_1(x) - z_2 \geq f_1(p),$$

which proves that $f(p, z_2, x) = z_1(x) - z_2$ satisfies Condition A.4.3. In addition, given

$$\int_D z_1(x_i) - z_2(x_i)\omega(z_2, x_i)dx \leq \sum_{i=1}^{m} \mu(x_i) < \infty,$$

it satisfies (4.9). Hence, by Theorem A.4.10, there exists a convex function z_2 such that $z_2 \leq z_1$ on $\bar{D}$, whose graph is a polyhedral surface, satisfying the requirements for the border and vertices.