Alternate Approaches to the Cup Product and Gerstenhaber Bracket on Hochschild Cohomology

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Abstract

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The Hochschild cohomology $HH^\bullet(A)$ of an algebra A is a derived invariant of the algebra which admits both a graded ring structure (called the cup product) and a compatible graded Lie algebra structure (called the Gerstenhaber bracket). The Lie structure is particularly important as it provides a means of addressing the deformation theory of the algebra A.

In this thesis we produce some new methods for analyzing the cup product and Gerstenhaber bracket on Hochschild cohomology. For the cup product we produce a number of new, and rather fundamental, relations between the theories of twisting cochains and Hochschild cohomology. In the case of a Koszul algebra A, our results imply that the Hochschild cohomology ring of A is a subquotient of the tensor product algebra $A \otimes A^!$ of A with its Koszul dual $A^!$.

We also investigate the Hochschild cohomology of smash product algebras $A*\mathcal{G}$. (Here A is an algebra equipped with an action of a Hopf algebra $\mathcal{G}$.) In this setting, we produce new methods for computing both the cup product and Gerstenhaber bracket. For the Gerstenhaber bracket in particular, we show that there is an intermediate cohomology $H^\bullet_{\mathrm{Int}}(A*\mathcal{G})$ which is a braided commutative algebra in the category of Yetter-Drinfeld modules over $\mathcal{G}$, admits a braided anti-commutative bracket $[,]_{\mathsf{YD}}$, and can be used to recover both the cup product and Gerstenhaber bracket on the standard Hochschild cohomology of $A*\mathcal{G}$.

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DEDICATION

To Aiyana Jones, Rekia Boyd, and Freddie Gray.

INTRODUCTION AND THE SECRETS WITHIN

0.1 An introduction to Hochschild cohomology and what it's good for

Let k be a field of arbitrary characteristic. The Hochschild cohomology of a (k-)algebra is a derived invariant $A \mapsto HH^\bullet(A)$ which produces the center in degree 0, classifies (outer) derivations on A in degree 1, infinitesimal deformations of A in degree 2, and obstructions to lifting deformations in degree 3. This cohomology was first introduced by Hochschild [30] and then popularized in the many works of Gerstenhaber, including the foundational papers [18, 19]. Formally, we can define the cohomology as the graded extension group

$$HH^\bullet(A) = \mathrm{Ext}^\bullet_{A\text{-bimod}}(A, A)$$

with its standard Yoneda product. The product on $HH^\bullet(A)$ is often referred to as the cup product. (It is shown that the cup product and Yoneda product agree at [9, Proposition 1.1].)

Hochschild cohomology also carries a, somewhat mysterious, graded Lie structure which is compatible with the cup product. The Hochschild cohomology, or more specifically the Hochschild cochain complex, along with its Lie structure controls the deformation theory of A (in the general sense of deformation theory via dg Lie algebras). This point is elaborated on in Section 0.3. The graded Lie bracket on Hochschild cohomology is called the Gerstenhaber bracket, and the compatibility condition between the cup product and Gerstenhaber bracket requires that for each $f \in HH^i(A)$ the operation $[f, -]$ is a graded degree $(i-1)$ derivation on the graded ring $HH^\bullet(A)$ (see Section 1.1.1).

One can also consider the Hochschild cohomology $HH^\bullet(A, M)$ with "coefficients" in a bimodule M. This cohomology is, again, given by bimodule extensions $HH^\bullet(A, M) =$

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$\mathrm{Ext}^{\bullet}_{A\text{-bimod}}(A, M)$, and when $M = B$ is an algebra extension of A the associated Hochschild cohomology has a canonical graded ring structure. The cohomology rings $HH^{\bullet}(A, B)$, when B is something other than A, can be of some use, and in fact will be a principle object of study in the latter two chapters of this dissertation. There is also an obvious notion of Hochschild homology, which is defined using Tor groups. In the case of a d-Calabi-Yau algebra A the Hochschild cohomology and homology are related by a (Van den Bergh) duality $HH^{\bullet}(A) \overset{\cong}{\to} HH_{d-\bullet}(A)$ [87].

The simplest example of Hochschild cohomology is provided by the famed Hochschild-Kostant-Rosenberg theorem [32]. They state that for a smooth affine scheme X, with global functions $k[X]$, we have an isomorphism

$$HH_{\bullet}(A) = \bigwedge\nolimits^{\bullet}_{k[X]} \Omega_X, \quad \text{where } \Omega_X \text{ is the module of global Kahler differentials}$$

and a cohomological version of the theorem provides an isomorphism

$$HH^{\bullet}(A) = \bigwedge\nolimits^{\bullet}_{k[X]} T_X, \quad \text{where } T_X \text{ is the module of global vector fields.}$$

Here the ring structure is, as the notation suggests, just the exterior algebra generated by T_X, and the Lie structure is induced by the Lie bracket on vector fields. As a slightly more interesting example, when X is complex affine space and G is a finite group acting on X, then the Hochschild cohomology of the skew group ring $k[X] * G$ is given by

$$HH^{\bullet}(k[X] * G) = \big(\oplus_{g \in G} \bigwedge\nolimits^{\bullet - \mathrm{codim}(X^g)}_{k[X^g]} T_{X^g} \big)^G. \tag{0.1.1}$$

Here X^g is the subscheme fixed by $g \in G$ [16]. The same formula holds when X is a symplectic manifold and G acts by symplectic automorphisms, and the result was originally provided in this setting [22, proof of Proposition 6.2].

Hochschild cohomology is related to a number of other invariants and important mathematical structures. For a rather deep example one can consider cyclic (co)homology, both the standard version HC and negative version HC^{-}. This (co)homology plays a principle role in Connes' theory of noncommutative geometry [13].

Cyclic (co)homology is strongly related to Hochschild (co)homology via Connes' exact sequences. In the negative instance for homology, this sequence appears as

$$\cdots \to HC_n^-(A) \to HH_n(A) \to HC_{n+1}^- \to HC_{n-1}^-(A) \to \cdots$$

[50, Section 5.1.3]. When A is Calabi-Yau the above sequence along with the duality between Hochschild cohomology and homology induces, by way of the algebraic structures on $HH^\bullet(A)$, a graded Lie structure on negative cyclic homology $HC_\bullet^-(A)$ [61, Proposition 26], [6, Theorem 10.2]. This graded Lie structure on negative cyclic homology is referred to as the string topology bracket and, according to [6], the corresponding dg Lie algebra controls the deformation theory of the given Calabi-Yau algebra.

For some slightly more straightforward relations there is, for example, a general isomorphism

$$HH^\bullet(A, \mathrm{Hom}_k(P, Q)) = \mathrm{Ext}_A(P, Q),$$

for arbitrary A-modules P and Q. This identity is one of graded rings when $P = Q$ (see Proposition 2.32). Whence all cohomology rings can be expressed as Hochschild cohomology rings. We also have a spectral sequence relating a canonical cohomology for Poisson algebras to the Hochschild cohomology of their quantizations [8]. As a final example, Keller has shown that the Hochschild cohomology of an algebra A is isomorphic, as a graded Lie algebra, to the Lie algebra Lie(DPic(A)) of its derived Picard group [41].

In recent years Hochschild cohomology has been extended beyond its original algebraic setting and we now have Hochschild cohomologies of schemes, dg categories, and general abelian categories [83, 89, 52, 43]. The dg extension allowed Keller to prove invariance of the B_∞-structure on Hochschild cohomology under Koszul duality [37], and Hochschild homology has emerged as a means of studying derived categories of non-affine schemes (e.g. [47]). We continue our discussion of Hochschild cohomology in a more pointed manner below.

0.2 More background and the specific contents of this thesis

The thesis is broken up into three chapters. The first two chapters are basically complete papers, the second of which was published in the Journal of Algebra [64]. The third chapter is also a self contained paper, but may not appear in print until some examples have been given. Most of the material of [64] has not been changed significantly, but we have made superficial changes in notation in order to maintain consistency with the material of the other chapters. Also, two examples have been included which were promised in the printed version, but only appear here.

0.2.1 More on the cup product and the contents of Chapter 1: twisting cochains and the cup product on Hochshcild cohomology for Koszul algebras

In some sense, the Lie bracket is the most important structure on Hochschild cohomology. Namely, it is the Lie bracket which one employs as a means of addressing the deformation theory of an algebra A. However, computation of the ring structure on $HH^\bullet(A)$ can be a very helpful step in the computation of the bracket. One first computes the product then extends the Lie bracket from the generators via the derivation identity

$$[f_0, (f_1 \dots f_n)] = \sum_i \pm f_1 \dots f_{i-1}[f_0, f_i]f_{i+1} \dots f_n.$$

Aside from its applications to computations of the Gerstenhaber bracket, the cup product has also been of independent interest. For example, the Hochschild cohomology ring replaces group cohomology in Snashall and Solberg's extension of the theory of support varieties to the situation of arbitrary finite dimensional algebras [80]. Snashall, Solberg, and coauthors have given a number of computations of Hochschild cohomology rings in relation to this theory, see for example [25, 26, 81]. Some additional computations of the cup product can be found at [9, 20, 29, 73]. The Hochschild cohomology ring also generally provides a rather informative derived invariant which seems to be much more amenable to study than the entire Gerstenhaber (or, further, B_∞) structure.[1]

In this chapter the Hochschild cohomology is approached at multiple levels. We give a general result relating twisting cochains to the cup product on Hochschild cohomology. The most well known appearance of twisting cochains in an algebraic context relates a (filtered) Koszul algebra to its Koszul dual in a concrete expression of Koszul duality (see [40] and Section 1.4). While I will not give a concrete definition of (filtered) Koszul algebras here, important examples include quantum polynomial rings, Sklyanin algebras, Universal enveloping algebras, Clifford algebras, and Weyl algebras. In the event that our algebra is filtered as opposed to graded, as is the case for the latter three examples, the role of the Koszul dual will be played by a dg algebra, or curved dg algebra. This filtered Koszul duality was first proposed by Positselski, and is repeated here in Sections 1.2.3 and 1.2.4.

We apply our general results to the Koszul context to deduce a very specific approach to the Hochschild cohomology of Koszul rings. Namely, we show that the cup product on the Hochschild cohomology of a Koszul ring is specified entirely by that of the Koszul ring itself and its Koszul dual. Finally, we illustrate our main result in the Koszul case by computing the Hochschild cohomology ring of the universal enveloping algebra of the Heisenberg Lie algebra. We also follow a specific example related to quantum polynomial rings throughout the chapter. We enumerate some of our findings here.

Theorem I (Theorem 1.32). *Let A be any dg algebra, C be any curved dg coalgebra, and $\pi : C \to A$ be any acyclic twisting cochain. Then we have an identification of graded rings*

$$H^\bullet(\mathrm{Hom}_k^\pi(C, A)) = HH^\bullet(A),$$

where $\mathrm{Hom}_k^\pi(C, A)$ denotes the twisted hom complex and $HH^\bullet(A)$ is the Hochschild cohomology of A with the cup product.

An acyclic twisting cochain is a degree 1 k-linear map π from C to A which satisfies the Maurer-Cartan equation, and the corresponding twisted hom complex is the standard set of

---

[1] As evidence that the cup product is in fact more accessible than the Gerstenhaber bracket one may consider the much larger number of algebras for which the cup product is understood, in comparison to those for which the Lie bracket is understood.

graded homs

$$\mathrm{Hom}_k^\pi(C,A)=\oplus_i\left\{\begin{array}{c}\text{homogeneous degree } i\ k\text{-linear maps}\\ f:C\to A\end{array}\right\}$$

with the convolution product and altered differential

$$d_{\mathrm{Hom}^\pi}(f)=d_Af-(-1)^{|f|}fd_C-[\pi,f].$$

With this product and differential, the twisted hom complex $\mathrm{Hom}_k^\pi(C,A)$ becomes a dg algebra and its cohomology therefore inherits a natural graded ring structure. As a corollary of the main theorem we get

Theorem II (Corollary 1.33)**.** *Let $A=k\langle x_1,\dots,x_n\rangle/(R)$ be a Koszul algebra (with R the k-space of homogeneous degree 2 relations) and $A^!=\langle\lambda_1,\dots,\lambda_n\rangle/(R^\perp)$ be its Koszul dual. Let e be the degree 1 element $e=\sum_i x_i\otimes\lambda_i$ in the tensor product algebra $A\otimes A^!$. Then*

1. *the graded commutator operation $[e,-]:A\otimes A^!\to A\otimes A^!$ is a square zero degree 1 derivation on the algebra $A\otimes A^!$, where we grade by the degree on $A^!$,*
2. *the cohomology of the resulting dg algebra admits an identification $H^\bullet(A\otimes A^!)=HH^\bullet(A)$.*

In the statement of the above theorem the λ_i are dual to x_i so that $k\{\lambda_1,\dots,\lambda_n\}=(k\{x_1,\dots,x_n\})^*$.

It is generally understood that Koszul rings are a class of rings for which the Homological algebra is extremely tangible and, at times, computable. For example, we have canonical resolutions of modules over a Koszul ring via the Koszul dual [85], one can easily check the global dimension by considering the Koszul dual, we know that a Koszul ring is Artin-Schelter regular (and hence twisted Calabi-Yau) exactly when the Koszul dual is Frobenius [79], etc. Each of these properties is expressed as a concrete relationship between a Koszul ring and its Koszul dual. Theorem II allows us to understand the Hochschild cohomology ring as yet another expression of this general philosophy of Koszul duality.

We would be remiss at this point to not mention the work of Buchweitz, Green, Snashall, and Solberg [9]. In the paper [9] the authors relate the multiplication on the Hochschild cohomology of a Koszul (path) algebra A to a comultiplicative structure on a special collection of right ideals in A. One manner in which our results differ is that the methods presented here are essentially basis free–the element e does not depend on choice of basis–while those of [9] seem to depend heavily on choices of bases and a concrete analysis of a certain coalgebra related to the Koszul ring. Obviously, our work also extends naturally to give results in a vastly broader framework.

In Chapter 2 we give analogs of Theorems I and II for the Hochschild cohomology $HH^\bullet(A,M) = \mathrm{Ext}_{A\text{-bimod}}(A,M)$ with coefficients is a (dg) bimodule M. In the case that $M = A\otimes A$, we show that our complex calculates the bimodule structure on $HH^\bullet(A, A\otimes A)$, and can therefore be used to compute the Nakayama automorphism on twisted Calabi-Yau algebras. Corollary 1.30, in particular, provides a generalization of methods given by Van den Bergh for computing the Nakayama automorphism in the case of a Koszul ring [85, Theorem 9.1].

0.2.2 Chapter 2: spectral sequences for the cohomology rings of a smash product

Fix a algebra A and a Hopf algebra $\mathcal{G}$ acting on A. We assume that the antipode on $\mathcal{G}$ is bijective, and that the action of $\mathcal{G}$ on A gives it the structure of a $\mathcal{G}$-module algebra [62, Definition 4.1.1]. We can then form the smash product algebra $A * \mathcal{G}$, which is the vector space $A \otimes \mathcal{G}$ with multiplication

$$(a\otimes\gamma)(a'\otimes\gamma') := \sum_i a(\gamma_{i_1}\cdot a')\otimes\gamma_{i_2}\gamma',$$

where $\Delta(\gamma) = \sum_i \gamma_{i_1}\otimes\gamma_{i_2}$. In the case that $\mathcal{G}$ is the group algebra of a group G we use the notation $A * G$ as a shorthand for the smash product $A * kG$.

Smash products have appeared in a number of different contexts in the literature. Under certain conditions, the smash product can serve as a replacement for the invariant algebra $A^{\mathcal{G}}$ [62, Section 4.5] [1] [11, Proposition 5.2]. Smash products have also appeared as a means of untwisting, or unbraiding, certain twisted structures. For example, one can untwists a twisted Calabi-Yau algebra with a smash product [72, 24], or unbraid a braided Hopf algebra [3, Section 1.5]. In a more classical context, smash products have played an integral role in a classification program proposed by Andruskiewitsch and Schneider, which began at [2]. There is also a geometric interpretation of the smash product which motivates the work of Chapter 3 and is discussed in Section 0.2.3 below.

In this chapter we equip some known spectral sequences, which converge to the Hochschild cohomology and Ext groups of a smash product, with multiplicative structures. These spectral sequences, with their new multiplicative structures, can then be used to compute the products on these cohomologies. Specifically, we provide spectral sequences which converge to the Hochschild cohomology $HH^\bullet(A*\mathcal{G}, B)$, along with the cup product, and the extension algebra $\mathrm{Ext}_{A*\mathcal{G}\text{-mod}}(M, M)$, along with the standard Yoneda product. Here we allow M to be any (left) $A*\mathcal{G}$-module and B to be any algebra extension of the smash product, i.e. any algebra equipped with an algebra map $A*\mathcal{G} \to B$.

In the theorem below, by a multiplicative spectral sequence we mean a spectral sequence $E = (E_r)$ equipped with bigraded products $E_r \otimes E_r \to E_r$ which are compatible with the differentials and structural isomorphism $E_{r+1} \cong H^\bullet(E_r)$. (One can refer to Section 2.6 for a more precise definition.) We say that a multiplicative spectral sequence converges to a graded algebra H if H carries an additional filtration and there is an isomorphism of bigraded algebras $E_\infty \cong \mathrm{grH}$. One of our main result is the following.

Theorem III (Corollary 2.30). *For any algebra extension B of the smash product $A*\mathcal{G}$, there are two multiplicative spectral sequences*

$$E_2 = \mathrm{Ext}_{\mathcal{G}\text{-mod}}(k, HH^\bullet(A, B)) \Rightarrow HH^\bullet(A*\mathcal{G}, B)$$

and

$${}'E_1 = \mathrm{Ext}_{\mathcal{G}\text{-mod}}(k, \mathrm{RHom}_{A\text{-bimod}}(A, B)) \Rightarrow HH^\bullet(A*\mathcal{G}, B)$$

which converge to the Hochschild cohomology as an algebra.

To be clear, we mean that there is some $\mathcal{G}$-module algebra structure on $HH^\bullet(A,B)$ and that the second term E_2 is the bigraded algebra $\mathrm{Ext}_{\mathcal{G}\text{-mod}}(k, HH^\bullet(A,B))$. Similarly, for the term ${}'E_1$, we mean there is some particular model for $\mathrm{RHom}_{A\text{-bimod}}(A,B))$ which is a $\mathcal{G}$-module (dg) algebra and that ${}'E_1$ is the given Ext algebra. We also provide a version of the above theorem for the cohomology rings $\mathrm{Ext}_{A*\mathcal{G}\text{-mod}}(M,M)$.

Theorem IV (Corollary 2.38)**.** *For any $A*\mathcal{G}$-module M, there are two multiplicative spectral sequences*

$$\bar{E}_2 = \mathrm{Ext}_{\mathcal{G}\text{-mod}}(k, \mathrm{Ext}_{A\text{-mod}}(M,M)) \Rightarrow \mathrm{Ext}_{A*\mathcal{G}\text{-mod}}(M,M)$$

and

$${}'\bar{E}_1 = \mathrm{Ext}_{\mathcal{G}\text{-mod}}(k, \mathrm{RHom}_{A\text{-mod}}(M,M)) \Rightarrow \mathrm{Ext}_{A*\mathcal{G}\text{-mod}}(M,M)$$

which converge to $\mathrm{Ext}_{A*\mathcal{G}\text{-mod}}(M,M)$ *as an algebra.*

In the text it is shown that all four of the above spectral sequences exist as explicit isomorphism at the level of cochains. Let us explain what is meant by this statement in the case of Hochschild cohomology.

Let B be an algebra extension of $A*\mathcal{G}$, as in Theorem III. For a free A-bimodule resolution $K \to A$, equipped with a $\mathcal{G}$-action satisfying certain natural conditions, and any resolution $L \to k$ of the trivial $\mathcal{G}$-module, we produce a dg algebra structure on the double complex

$$\mathrm{Hom}_{\mathcal{G}\text{-mod}}(L, \mathrm{Hom}_{A\text{-bimod}}(K,B)). \tag{0.2.1}$$

From this data we also produce a $A*\mathcal{G}$-bimodule resolution $\mathcal{K}$ of $A*\mathcal{G}$, and dg algebra structure on the associated complex $\mathrm{Hom}_{A*\mathcal{G}\text{-bimod}}(\mathcal{K},B)$. The dg algebra structure is chosen so that the homology of $\mathrm{Hom}_{A*\mathcal{G}\text{-bimod}}(\mathcal{K},B)$ is the Hochschild cohomology $HH^\bullet(A*\mathcal{G},B)$ with the cup product.

In Theorem 2.27, which can be seen as a lifting of Theorem III to the level of cochains, we show that there is an explicit isomorphism of dg algebras

$$\mathrm{Hom}_{A*\mathcal{G}\text{-bimod}}(\mathcal{K},B) \overset{\cong}{\to} \mathrm{Hom}_{\mathcal{G}\text{-mod}}(L, \mathrm{Hom}_{A\text{-bimod}}(K,B)).$$

It follows then that the Hochschild cohomology ring of the smash product can be computed as the homology of the double complex $\mathrm{Hom}_{\mathcal{G}\text{-mod}}(L, \mathrm{Hom}_{A\text{-bimod}}(K, B))$. We get Theorem III as an easy corollary of this fact.

The full power of Theorem 2.27 is employed to compute some examples. For both examples we assume that k is characteristic 0. Let $k[X]$ denote the algebra of global functions on an affine scheme X, and $T_{poly}(X)$ denote the Gerstenhaber algebra of polyvector fields on X. In the first example we provide the following result.

Theorem V. *Let $\mathbb{Z}^n$ act on affine space $\mathbf{A}^n = \mathbf{A}^n_k$ by translation, and $\mathbf{T}^n$ denote the k-torus. Then we have an identification of Gerstenhaber algebras*

$$HH^\bullet(k[\mathbf{A}^n] * \mathbb{Z}) = (T_{poly}(\mathbf{A}^n))^{\mathbb{Z}}$$

and a graded algebra isomorphism $k[\mathbf{T}^n] \otimes HH^\bullet(k[\mathbf{A}^n] * \mathbb{Z}) \overset{\cong}{\to} T_{poly}(\mathbf{T}^n)$.

The theorem is established in the discussion following Theorem 2.48. The second example requires some explanation. We take q to be a nonzero scalar in k which is not a root of unity. Let $k_q[x, y]$ denote the skew polynomial ring in 2-variables,

$$k_q[x, y] = \frac{k\langle x, y\rangle}{(yx - qxy)}.$$

This algebra is twisted Calabi-Yau. If we let $\mathbb{Z} = \langle \phi \rangle$ act on $k_q[x, y]$ by the automorphism $\phi : x \mapsto q^{-1}x$, $y \mapsto qy$, then, according to [72, Proposition 7.3] and [24], the smash product $k_q[x, y] * \mathbb{Z}$ will be Calabi-Yau. By way of Theorem 2.27, we can provide the following computation.

Theorem VI (Theorem 2.55)**.** *Let λ, ε, ξ_i, ζ, and η_i be a variables of respective degrees* 0, 1, 1, 2, *and* 2. *Then there is an isomorphism of graded algebras*

$$HH^\bullet(k_q[x, y] * \mathbb{Z}) \cong \frac{k[\varepsilon, \lambda, \xi_1, \xi_2, \zeta]}{(\lambda\zeta - \xi_1\xi_2, \xi_i\zeta, \zeta^2)} \times_{k[\varepsilon]} \frac{k[\varepsilon, \eta_i : i \in \mathbb{Z} - \{-1\}]}{(\eta_i^2, \eta_i\eta_j)}.$$

Furthermore, there is a natural embedding of graded algebras $HH^\bullet(k_q[x, y]) \to HH^\bullet(k_q[x, y] * \mathbb{Z})$ identifying $HH^\bullet(k_q[x, y])$ with the subalgebra generated by ξ_1, ξ_2, and η_0.[2]

---

[2] The relations $\xi_i\zeta$ were missing from the presentation given in [64].

In the statement of the above theorem $k[X_1, \ldots, X_n]$ denotes the free graded commutative algebra on graded generators X_i, $k[X_1, \ldots, X_n] = k\langle X_1, \ldots, X_n\rangle/(X_iX_j - (-1)^{|X_i||X_j|}X_jX_i)$.

Relation to the work of Stefan, Guichardet, and others

Theorems III and IV can be seen as a refinement of results of Stefan and Guichardet given in [82] and [28] respectively. However, both Stefan and Guichardet work with classes of algebras that are slightly different than general smash products. Guichardet provides spectral sequences

$$\mathrm{Ext}_{kG\text{-mod}}(k, HH^\bullet(A, M)) \Rightarrow HH^\bullet(A *_\alpha G, M)$$

for crossed product algebras $A *_\alpha G$, where G is a group, while Stefan provides spectral sequences

$$\mathrm{Ext}_{\mathcal{G}\text{-mod}}(k, HH^\bullet(A, M)) \Rightarrow HH^\bullet(E, M)$$

for Hopf Galois extensions $A \to E$. Guccione and Guccione extend the results of Guichardet to allow for crossed products with arbitrary Hopf algebras in [27]. None of these spectral sequences carry any multiplicative structures. For definitions of these different classes of algebras one can see [62]. Let us only mention that there are strict containments

$$\{\text{Smash Products}\} \subsetneq \{\text{Crossed Products}\} \subsetneq \{\text{Hopf Galois Extensions}\}.$$

So these more limited results (taken together) do apply to larger classes of algebras.

Let us mention here that Guccione and Guccione also provide spectral sequences for Ext groups in [27], by way of a standard relation [88, Lemma 9.1.9]. Also, some results involving multiplicative structures are given by Sanada, in a rather constrained setting, in [74]. Further analysis of the situation can be found in [4].

The main point of comparison here is that our spectral sequences can be used to compute the cup product, while those of Stefan, Guichardet, and Guccione-Guccione can not (at least after restricting to the case of smash products). However, there are also differences in the methods used in the three sources. As a consequence, the usefulness of the results vary in practice. For example, Stefan produces his spectral sequence as a Grothendieck spectral sequence, whereas those of Guccione and Guccione are derived from filtrations on a certain (rather large) complex. Guichardet shows that the Hochschild cohomology of a crossed product can be computed by the double complex $C^\bullet(G, C^\bullet(A, M))$, where $C^\bullet$ denotes the standard Hochschild cochain complex. Indeed, Guichardet provides quasi-isomorphisms of chain complexes $C^\bullet(G, C(A, M)) \leftrightarrows C^\bullet(A *_\alpha G, M)$. It does not appear that either of the given maps are dg algebra maps, and so the cup product remains obscured.

The spectral sequences produced in this chapter are those associated to the first quadrant double complex (0.2.1), which may in some cases be chosen to be relatively small. Our methods are most closely related those of Guichardet. In fact, by standard techniques, one may move from Guichardet's double Hochschild cochain complex to our double complex(es). To summarize the situation, we have the following chart

	Class of algebras for which the spectral sequences apply	Type of spectral sequences	Accounts for the cup/Yoneda product
Stefan	Hopf Galois extensions	Grothendieck	No
Guichardet	Crossed products with groups	Double complex	No
Guccione-Guccione	Crossed products	Filtration	No
Present paper	Smash products	Double complex	Yes

Of the works discussed, Theorems 2.19, 2.27, and 2.36 below provide the most computationally accessible approach to the cohomology of a smash product, irrespective of the cup product. Grothendieck spectral sequences, for example, require the use of injective resolutions, which are very difficult to come by in general. The methods used here are also more natural than those given in [27] in the sense that many of the constructions we employ are functorial.

0.2.3 Chapter 3: braided structures and Hochschild cohomology

In this chapter we wish to explore, again, the Hochschild cohomology of the smash product $A * \mathcal{G}$. However, in this chapter $\mathcal{G}$ will be finite dimensional with finite exponent, and we assume that $\exp(\mathcal{G})$ is invertible in k. In the case of a group algebra this is equivalent to requiring that $\mathcal{G}$ is semisimple. So, at least in the semisimple case, Stefan's spectral sequence collapses immediately to give an isomorphism

$$HH^\bullet(A * \mathcal{G}) \overset{\cong}{\to} \big(HH^\bullet(A, A * \mathcal{G})\big)^{\mathcal{G}}. \tag{0.2.2}$$

What we do, in particular, is show that the cohomology $HH^\bullet(A, A * \mathcal{G})$ (which appears on the right hand side of the above isomorphism) admits its own bracket and product which recover the Gerstanhaber bracket and cup product on the invariants. The bracket and product on $HH^\bullet(A, A * \mathcal{G})$ are "quantum" versions of the Gerstenhaber bracket and cup product in the sense that they are no longer anti-symmetric and graded commutative, but braided anti-symmetric and braided commutative (with respect to a braiding inherited from some ambient braided tensor category). We further this "quantum" Hochschild cohomology perspective by showing that, in low degree, the cohomology $HH^\bullet(A, A*\mathcal{G})$ classifies braided, or quantum, versions of those structures classified by the usual cohomology $HH^\bullet(A * \mathcal{G})$.

Our motivating example is the case of a finite group G acting on an (affine) scheme X. In this case the skew group ring $k[X]*G$ will represent the category of equivariant sheaves on X in the sense that we have an equivalence of categories between $k[X]*G$-mod and the category of G-equivariant sheaves on X. Since quasi-coherent sheaves on the stack quotient $[X/G]$ are exactly equivariant sheaves, we also get an equivalence $k[X] * G\text{-mod} \overset{\sim}{\to} \mathrm{Qcoh}([X/G])$ (see for example [11]). In the expression

$$HH^\bullet(k[X] * G) = \big(\oplus_{g\in G} \textstyle\bigwedge_{k[X^g]}^{\bullet-\mathrm{codim}(X^g)} T_{X^g}\big)^G, \tag{0.2.3}$$

from (0.1.1), we have $HH^\bullet(k[X], k[X] * G) = \oplus_g \bigwedge_{k[X^g]}^{\bullet-\mathrm{codim}(X^g)} T_{X^g}$, and the above equality is a particular occurrence of Stefan's isomorphism (0.2.2).

Ideally, we would like to gain some HKR-like understanding of the Hochschild cohomology of the quotient $[X/G]$ (or rather of $k[X]*G$). Namely we would like to understand the cohomology, along with the product and bracket, in strictly geometric terms. This project is, however, immediately obstructed by the fact that we do not know what happens to the structure $\oplus_g \bigwedge_{k[X^g]}^{\bullet-\mathrm{codim}(X^g)} T_{X^g}$ after taking invariants. Indeed, it seems improbable that any particular structure is preserved on the invariants. For this reason we seek to move our analysis from the Hochschild cohomology of a smash product to an analysis of the intermediate cohomology $HH^\bullet(k[X],k[X]*G)$, or more generally $HH^\bullet(A,A*\mathcal{G})$. In this chapter we show that many of the algebraic structures on $HH^\bullet(A*\mathcal{G})$ lift to algebraic structures on $HH^\bullet(A,A*\mathcal{G})$.

We already know that a natural graded ring structure on $HH^\bullet(A,A*\mathcal{G})$ exists so that the isomorphism (0.2.2) is one of graded rings. This follows by Theorem III, or can be deduced by a number of ad hoc arguments. Furthermore, a thorough analysis of the cohomology ring $HH^\bullet(k[X],k[X]*\mathcal{G})$ has already been given by Ginzburg-Kaledin and Shepler-Witherspoon [22, 76], at least in some specific cases. So we are much more interested in the Gerstenhaber bracket which, as usual, poses a unique and arguably more challenging problem. First of all, no precedent has been set which would suggest that any kind of natural Lie structure even exists on the cohomology $HH^\bullet(A,A*\mathcal{G})$. We are, however, able to produce such a structure here.

In the statement of the following theorem we let $YD_{\mathcal{G}}^{\mathcal{G}}$ denote the category of (right) Yetter-Drinfeld modules over a Hopf algebra $\mathcal{G}$. One needn't know anything about this category at the moment, except for the fact that it is a braided tensor category.

Theorem VII (Theorem 3.38)**.** *When $\mathcal{G}$ has finite exponent, and $\exp(\mathcal{G})$ is invertible in k, there is a canonical complex $C^\bullet_{\mathrm{Int}}(A*\mathcal{G})$ which*

- *(a) admits a canonical graded Yetter-Drinfeld structure under which it becomes a dg algebra in the tensor category $YD_{\mathcal{G}}^{\mathcal{G}}$,*
- *(b) has cohomology $H^\bullet(C^\bullet_{\mathrm{Int}}(A*\mathcal{G}))$ equal to $HH^\bullet(A,A*\mathcal{G})$ as a graded ring,*

(c) *admits a braided anti-symmetric operation* $[,]_{YD}: C^\bullet_{\text{Int}}(A*\mathcal{G}) \otimes C^\bullet_{\text{Int}}(A*\mathcal{G}) \to C^\bullet_{\text{Int}}(A*\mathcal{G})$, *which then induces a bracket* $[,]_{YD}$ *on the cohomology* $HH^\bullet(A, A*\mathcal{G})$.

Furthermore, when $\mathcal{G}$ *is semisimple*

(d) *the bracket* $[,]_{YD}$ *restricts to the standard Gerstenhaber bracket on the invariants* $HH^\bullet(A, A*\mathcal{G})^{\mathcal{G}} = HH^\bullet(A*\mathcal{G})$.

The complex $C^\bullet_{\text{Int}}(A*\mathcal{G})$ can be identified with the standard Hochschild cochain complex $C^\bullet(A, A*\mathcal{G})$, although this is not necessarily the most appropriate or useful interpretation. Also, the cohomology $HH^\bullet(A, A*\mathcal{G})$ is usually denoted $H^\bullet_{\text{Int}}(A*\mathcal{G})$ in the body of this text, and is referred to as the *intermediate cohomology* therein. The complex $C^\bullet_{\text{Int}}(A*\mathcal{G})$ is called the *intermediate complex* and we call the bracket $[,]_{\mathsf{YD}}$ the *Yetter-Drinfeld bracket.*

To give the reader some idea of what ingredients are involved in the production of the bracket $[,]_{\mathsf{YD}}$ we present the formula here, without an in depth explanation. We have

$$[f,g]_{\mathsf{YD}} = \int_{\mathcal{L}} [f,g]_\circ := \frac{1}{\exp(\mathcal{G})} \sum_{i=0}^{\exp(\mathcal{G})-1} \mathcal{L}^{i*}[f,g]_\circ,$$

where $\mathcal{L}$ is the double braiding on $YD^{\mathcal{G}}_{\mathcal{G}}$ and $[,]_\circ$ is the naive bracket of Definition 3.10. In the process of establishing Theorem VII we also show

Theorem VIII (Corollary 3.21). *The cohomology* $HH^\bullet(A, A*\mathcal{G})$, *along with the cup product and Yetter-Drinfeld structure from Theorem VII, is a braided commutative ring.*

I should mention here that Theorem VIII was proved independently in unpublished work of Shedler-Witherspoon.

Theorems VII and VIII suggest that we may be able to think of the cohomology $HH^\bullet(A, A*\mathcal{G})$ as a quantum Hochschild cohomology, in the sense that all symmetric structures on Hochschild cohomology should exist as braided structures on the intermediate cohomology. In the final section of Chapter 3 we establish one last result which is in line with our quantum cohomology philosophy.

Theorem IX (Theorem 3.39). *In low degree we have*

$$\begin{aligned} H^0_{\mathrm{Int}}(A*\mathcal{G}) = \ & Z^0\big(C^\bullet_{\mathrm{Int}}(A*\mathcal{G})\big) = \textit{the braided center of } A*\mathcal{G} \\ & Z^1\big(C^\bullet_{\mathrm{Int}}(A*\mathcal{G})\big) = \textit{braided (algebra) derivations of } A*\mathcal{G} \\ & Z^2\big(\bar{C}^\bullet_{\mathrm{Int}}(A*\mathcal{G})\big) = \textit{infinitesimal quantum deformations of } A*\mathcal{G} \end{aligned}$$

and

$$\begin{aligned} H^1_{\mathrm{Int}}(A*\mathcal{G}) &= \textit{braided outer derivations of } A*\mathcal{G} \\ H^2_{\mathrm{Int}}(A*\mathcal{G}) &= \textit{isoclasses of quantum deformations.} \end{aligned}$$

An infinitesimal quantum deformation of $A*\mathcal{G}$ is a certain kind of non-associative algebra deformation of $A*\mathcal{G}$. The degree of non-associativity is controlled by the Yetter-Drinfeld structure on $A*\mathcal{G}$. Such objects form a groupoid $\mathscr{Q}\mathscr{D}ef^{\mathcal{G}}_{A*\mathcal{G}}$ and the identification in degree 2 exists as a certain isomorphism of groupoids

$$Z^2\big(\bar{C}^\bullet_{\mathrm{Int}}(A*\mathcal{G})\big)//\bar{C}^1_{\mathrm{Int}}(A*\mathcal{G}) \overset{\cong}{\to} \mathscr{Q}\mathscr{D}ef^{\mathcal{G}}_{A*\mathcal{G}}.$$

(See Theorem 3.54.)

Finally, let me mention that a coarse, but in some sense complete, analysis of the bracket on $HH^\bullet(k[X]*G)$ was given by Shepler-Witherspoon in [77]. They considered the specific case of a group acting by graded automorphisms on the polynomial ring $k[X] = k[\mathbf{A}^n]$. However, none of the structures they produced on the intermediate cohomology $HH^\bullet(k[\mathbf{A}^n], k[\mathbf{A}^n]*G)$ were natural nor did they exist at the cochain level. So their project was somewhat different than what we are trying to do here. We also hope that some of the methods produced here can be used to clarify, and strengthen, some of the results from [77] in this particular case.

0.3 The Hochschild cochain complex, dg Lie algebras, and formal deformations

Let us take a moment to review the relationship between deformation theory and Hochschild cohomology, since this seems to be a point of confusion and interest for the casual observer. None of the material covered here is needed in order to understand the material in the following chapters. We only include the work here for the convenience of the reader, and to synthesis some material which appears not to be expressed clearly in a single resource. The relation between deformation theory and Hochschild cohomology is stated explicitly in Theorem 0.8.

There is an issue about preservation of the unit under deformation, which we ignore for the moment. In this section, and this section only, by an algebra we will mean a not–necessarily–unital algebra. (By coalgebra we will still mean counital coalgebra.) One can see Remark 0.9 for some more information on this apparent problem, an explanation of how to easily fix it, and an explanation of why, in a very explicit sense, it was actually not even a problem.

We assume $\mathrm{char}(k) = 0$ here, and basically follow the work of Manetti [58, 59, 57] on deformation theory via dglas. Most of what is stated below also appears as exercises in [46, Chapter 4]. It would probably aid the reader to have some familiarity with the bar functor $\mathscr{B} : \texttt{Alg} \to \texttt{dgCoalg}$, although such knowledge is not strictly necessary.

For a k-algebra A the Hochschild cohomology can be computed via the Hochschild cochain complex, which we present here. First, let R be a unital commutative base ring and T be a R-algebra. Let $\mathscr{B}T = (\bigoplus_{n\geq 0} T^{\otimes_R n})$ denote the bar coalgebra of T (see Section 1.3).

Lemma 0.1. *There is a bijection*

$$\{R\text{-algebra structures } \eta \text{ on } T\} \longleftrightarrow \left\{ \begin{array}{c} \text{deg 1 coderivations } \Xi \text{ on } \mathscr{B}T \\ \text{satisfying } [\Xi, \Xi] = 0 \end{array} \right\}$$

$$\eta \mapsto f_\eta, \quad \Xi|T^{\otimes_r 2} \mapsfrom \Xi$$

In $\mathscr{B}T$ the degree of T is now -1, so that the degree of $T^{\otimes_R n}$ is $-n$. The bracket $[,]$ is the graded commutator of coderivations $[g, h] = gh - (-1)^{|g||h|}hg$.

Proof. Take $f_\eta|T^{\otimes_R n} := \sum_{i\leq n-2}(-1)^i id_T^{\otimes i} \otimes \eta \otimes id_T^{\otimes n-i-2}$. ■

The Hochschild cochain complex of our R-algebra T is given as the cohomology of the Lie algebra of graded coderivations on $\mathscr{B}T$,

$$\text{(shifted) Hochschild cochain complex} = \Sigma C^\bullet(T) = \mathrm{Coder}^\bullet_R(\mathscr{B}T),$$

with differential given by the graded commutator $[f_\mu, -]$, where μ is the given algebra structure on T. The Lie structure and differential are compatible so that $\Sigma C^\bullet(T)$ becomes a dg Lie algebra. The Hochschild cohomology is the cohomology of this complex $HH^{\bullet+1}(T) = H^\bullet(\Sigma C^\bullet(T))$.[3] One can check that we have a more terrestrial expression of this complex

$$\Sigma C^\bullet(T) = 0 \to T \to \mathrm{Hom}_R(T, T) \to \mathrm{Hom}_R(T^{\otimes_R 2}, T) \to \mathrm{Hom}_R(T^{\otimes_R 3}, T) \to \cdots$$

with the usual differential

$$d(F)(a_1 \otimes \ldots \otimes a_n) = \pm a_1 F(a_2 \otimes \ldots \otimes a_n) \pm F(a_1 \otimes \ldots \otimes a_{n-1}) a_n + \sum_{1 \le i \le n-1} \pm F(a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n)\big).$$

Take $\texttt{Art}_k$ to be the category of unital Artinian local k-algebras with residue field k. For an arbitrary (non-commutative) k-algebra A we define the groupoid valued functor

$$\mathscr{D}ef_A : \texttt{Art}_k \to \texttt{Groupoid}, \quad R \mapsto \left\{ \begin{array}{c} \text{Flat } R\text{-algebras } \mathcal{A} \text{ with} \\ \text{an alg isom } \mathcal{A} \otimes_R k \to A \end{array} \right\}.$$

For the groupoid structure on the set $\{\mathcal{A}\}$ of R-deformations we take all R-algebra isomorphisms $\phi : \mathcal{A} \to \mathcal{A}'$ with $\phi \otimes_R k = id_A$. In the case that $R = k[t]/(t^{n+1})$ we call an element of $\mathscr{D}ef_A(R)$ a nth order deformation of A. Let Def_A denote the set valued functor given by composing $\mathscr{D}ef_A$ with the map $\texttt{Groupoid} \to \texttt{Set}$ identifying all isomorphic objects.

We have yet another functor

$$\mathscr{MC}_{\Sigma C^\bullet(A)} = \mathscr{MC}_A : \texttt{Art}_k \to \texttt{Groupoid}$$

sending each Artinian k-algebra R to the action groupoid of solutions to the Maurer-Cartan equation $\mathscr{MC}_A(\Sigma C^\bullet(A) \otimes m_R)//\mathrm{Gauge}_R$, where Gauge_R is the gauge group. A solution to the Maurer-Cartan equation is a degree 1 element $\xi \in \Sigma C^1(A) \otimes m_R$ satisfying

$$d(\xi) + \frac{1}{2}[\xi, \xi] = [f_\mu, \xi] + \frac{1}{2}[\xi, \xi] = 0.$$

---

[3] This particular interpretation of the Hochschild cochain complex will not be employed in the body of this thesis.

The gauge group $\mathrm{Gauge}_R = \exp(\Sigma C^0(A) \otimes m_R) = \{\exp(f) : f \in \Sigma C^0(A) \otimes m_R\}$, acts on the set of Maurer-Cartan elements by

$$\exp(f) \cdot (\xi) = \xi + \sum_{i \geq 0} \frac{[f, -]^i}{(i+1)!}([f, \xi] - d(f))$$

[59, Section 1]. (Here we've used μ to denote the non-deformed multiplication on the base change $A \otimes R$ for arbitrary R.) Note that since m_R is nilpotent the Lie algebra $\Sigma C^\bullet(A) \otimes m_R$ is nilpotent, and the gauge action therefore makes sense. As with $\mathscr{D}ef_A$, we let MC_A denote the corresponding set valued functor.

The main point is

Theorem 0.2 ([46, Section 4.1.1])**.** *There is an equivalence of functors $\mathscr{MC}_A \overset{\sim}{\to} \mathscr{D}ef_A$ which sends a solution $\xi \in \mathscr{MC}_A(R)$ to the R-module $A \otimes R$ with multiplication $(f_\mu + \xi)|(A \otimes R)^{\otimes_R 2}$.*

By an "equivalence" we mean that the transformation $\mathscr{MC}_A \to \mathscr{D}ef_A$ produces an equivalence $\mathscr{MC}_A(R) \to \mathscr{D}ef_A(R)$ at each R. This relationship is reflected, to a certain degree, in Lemma 0.1, although we elaborate a bit here. The remainder of the section is dedicated to the proof of Theorem 0.2.

We first clarify that our map $\mathscr{MC}_A \to \mathscr{D}ef_A$ is well defined, at least on objects. There is a cochain complex splitting

$$\Sigma C^\bullet(A \otimes R) = \Sigma C^\bullet(A) \oplus \big(\Sigma C^\bullet(A) \otimes m_R\big)$$

under which we find that degree 1 solutions to the equation $[\Xi, \Xi] = 0$ on the left hand side with $\Xi \otimes_R k = f_\mu$, i.e. R-algebra structures on $A \otimes R$ which reduce to (A, μ), correspond to elements of the form $f_\mu + \xi$ on the right hand side. One then finds also

$$[\Xi, \Xi] = 0 \;\Leftrightarrow\; \xi \text{ satisfies the MC equation } [f_\mu, \xi] + \frac{1}{2}[\xi, \xi] = 0.$$

So in this way algebra deformations $(A \otimes R, \Xi|(A \otimes R)^{\otimes_R 2})$ of A correspond to solutions to the Maurer-Cartan equation, and we have that $A \otimes R$ with multiplication $(f_\mu + \xi)|(A \otimes R)^{\otimes_R 2}$ is in fact a R-deformation of A. Also, the fact that each $\mathscr{MC}_A(R) \to \mathscr{D}ef_A(R)$ is essentially surjective on objects follows from the fact that any flat module over an Artinian local ring is free.

We now cover the details of the gauge action. In what follows we abuse language a bit and call a degree 1 solution Ξ to the equation $[\Xi,\Xi]=0$ in $\Sigma C^\bullet(A\otimes R)$ an R-deformation of A, while still using this terminology to refer to flat R-algebras as well.

Lemma 0.3. *1. Let R be any unital base ring and $R[[y]]$ be the complete R-bialgebra specified by $\Delta(y)=y\otimes 1+1\otimes y$. Then the formal exponential $\exp(y)=\sum_{n\geq 0} y^n/n!$ satisfies*

$$\Delta(\exp(y))=\exp(y)\otimes\exp(y).$$

2. Let $R[[x]]$ have the complete R-bialgebra structure specified by taking

$$\Delta(x)=(x\otimes 1+1\otimes x+x\otimes x) \quad\Leftrightarrow\quad \Delta(1+x)=(1+x)\otimes(1+x).$$

Then the formal logarithm $\log(1+x)=\sum_{i\geq 1}(-1)^{i-1}x^i/i$ satisfies

$$\Delta(\log(1+x))=\log(1+x)\otimes 1+1\otimes\log(1+x). \tag{0.3.1}$$

Proof. (1) We have

$$\begin{aligned}\Delta(\exp(y)) &= \textstyle\sum_{n\geq 0}(y\otimes 1+1\otimes y)^n/n!\\ &= \textstyle\sum_n(\sum_l \frac{n!}{l!(n-l)!}y^l\otimes y^{n-l})/n!\\ &= \textstyle\sum_n(\sum_l \frac{1}{l!(n-l)!}y^l\otimes y^{n-l})\\ &= \textstyle\sum_{l,m\geq 0}\frac{1}{l!m!}y^l\otimes y^m=\exp(y)\otimes\exp(y).\end{aligned}$$

(2) It follows from (1) that we have the bialgebra isomorphism

$$\exp: R[[x]]\to R[[y]],\quad x\mapsto \exp(y)-1.$$

Note that, in terms of $\exp(y)$, we have $y=\log(\exp(y))$. So $R[[y]]=R[[\exp(y)-1]]$ and the inverse to exp is the function $y\mapsto\log(1+x)$. It follows that $\log(1+x)$ is primitive in the bialgebra $R[[x]]$, i.e. satisfies (0.3.1). ■

Corollary 0.4. *If C is an R-coalgebra and φ is a nilpotent R-endomorphism of C such that $1+\varphi$ is a coalgebra automorphism, then* $\log(1+\varphi) = \sum_{i\geq 1}(-1)^{i-1}\varphi^i/i$ *is a nilpotent coderivation on C. Conversely, if f is a nilpotent coderivation on C then* $\exp(f) = \sum_{n\geq 0} f^n/n!$ *is a coalgebra automorphism on C.*

Proof. If $1+\varphi$ is an automorphism then we have $\Delta(\varphi) = (\varphi\otimes 1 + 1\otimes\varphi + \varphi\otimes\varphi)\Delta$ and consequently $\Delta(\varphi^n) = (\varphi\otimes 1 + 1\otimes\varphi + \varphi\otimes\varphi)^n\Delta$. Therefore $\Delta(\log(1+\varphi)) = \Phi\Delta$ where Φ is the image of $\log(1+x)$ under the algebra map

$$\begin{tikzcd}
& R[[x]]\widehat{\otimes}R[[x]] = R[[x_1,x_2]] \arrow[dr, dashed] & \\
R[[x]] \arrow[ur, dashed, "\Delta"] \arrow[rr] & & \mathrm{End}_R(C\otimes_R C)
\end{tikzcd}$$

sending x to $\varphi\otimes 1 + 1\otimes\varphi + \varphi\otimes\varphi$ and the x_i to φ. Since $\log(1+x)$ maps to $\log(1+x)\otimes 1 + 1\otimes\log(1+x)$ in $R[[x]]\widehat{\otimes}R[[x]]$ we conclude $\Phi = \log(1+\varphi)\otimes 1 + 1\otimes\log(1+\varphi)$. That is to say, $\log(1+\varphi)$ is a coderivation. The proof that $\exp(f)$ is an automorphism is similar. ∎

Proposition 0.5 ([57, Proposition V.44])**.** *Let R be in Art_k and $\mathcal{A}$ be an R-deformation of A. Then there is a bijection*

$$\left\{\begin{array}{c}\text{degree 0 coderivations } f \text{ on}\\ \mathscr{B}\mathcal{A} \text{ satisfying } f\otimes_R k = 0\end{array}\right\} \underset{\log}{\overset{\exp}{\leftrightarrows}} \left\{\begin{array}{c}\text{coalgebra automorphisms } \phi \text{ on}\\ \mathscr{B}\mathcal{A} \text{ satisfying } \phi\otimes_R k = id_{\mathscr{B}A}\end{array}\right\}$$

By “coderivation” we mean R-linear coderivation, and by “coalgebra automorphism” we mean degree 0 graded coalgebra automorphism.

Proof. Since any such f will be nilpotent we know $\exp(f)$ will be an automorphism with $\exp(f)\otimes_R k = 1$, by the previous lemma. The previous lemma also assures us that the logarithm of any such ϕ is a well defined coderivation. The fact that the two maps are bijections follows from the fact that the exp and log functions are mutually inverse. ∎

As a consequence of this proposition we see that each gauge group Gauge_R is exactly the kernel of the reduction $-\otimes_R k : \mathrm{Aut}_{R\text{-coalg}}(\mathscr{B}\mathcal{A}) \to \mathrm{Aut}_{k\text{-coalg}}(\mathscr{B}A)$, or rather Gauge_R is the collection of automorphisms which reduce to the identity on $\mathscr{B}A$. So we have the exact sequence

$$1 \to \mathrm{Gauge}_R \to \mathrm{Aut}_{R\text{-coalg}}(\mathscr{B}\mathcal{A}) \to \mathrm{Aut}_{k\text{-coalg}}(\mathscr{B}A) \to 1$$

for any R-deformation $\mathcal{A}$ [57, SectionV.5-C]. Finally, we can understand the gauge group action.

Proposition 0.6. *For any R-linear function $\Xi : \mathscr{B}\mathcal{A} \to \mathscr{B}\mathcal{A}$, and derivation f with $f \otimes_R k = 0$, we have $\exp(f)\Xi\exp(-f) = \exp([f,-])(\Xi)$. In the case $\Xi = f_\mu + \xi$ we have*

$$\exp(f)\Xi\exp(-f) - f_\mu = \exp(f)\cdot\xi = \xi + \sum_{i\geq 0}\frac{[f,-]^i}{(i+1)!}([f,\xi] - d(f)).$$

Proof. These are straightforward calculations. See [59, Example 1.3], [57, Exercise V.1]. ■

The previous proposition says that the gauge action simply conjugates a coderivation by a R-linear automorphism to produce another coderivation. By using the fact that the coalgebra automorphisms on $\mathscr{B}\mathcal{A}$ are exactly those maps of the form $\phi = \sum_{n\geq 0}\phi_0^{\otimes_R n}$ for some R-linear automorphism $\phi_0 = \phi|\mathcal{A} : \mathcal{A} \to \mathcal{A}$, one can verify that on the algebra side the action of ϕ corresponding to the map sending a deformed multiplication $m : \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$ to $\phi_0 m(\phi_0^{-1}\otimes\phi_0^{-1})$, as one would assume. If we let $\mathcal{A}'$ denote $\mathcal{A}$ with this new multiplication, the two algebras will be related by the *algebra* automorphism $\phi_0 : \mathcal{A} \to \mathcal{A}'$. Visually

$$\phi\cdot\xi \leftrightsquigarrow \phi\Xi\phi^{-1} \leftrightsquigarrow \phi_0 m(\phi_0^{-1}\otimes\phi_0^{-1}).$$

So the groupoid map $\mathscr{MC}_A(R) \to \mathscr{D}ef(R)$ is defined on morphisms by taking $\exp(f) \in \mathrm{Gauge}_R$ to $\exp(f)|(A\otimes R)$. We can now give the

Proof of Theorem 0.2. We simply reiterate what has already been said. The fact that each map of groupoids $\mathscr{MC}_A(R) \to \mathscr{D}ef_A(R)$ is essentially surjective follows from the fact that each deformation $\mathcal{A}$ will be isomorphic, as an R-module, to $A\otimes R$, since R is Artinian local. We must verify also that each map of homs

$$\mathrm{Hom}_{\mathscr{MC}_A(R)}(\xi,\xi') = \{\exp(f) : \exp(f)\cdot\xi = \xi'\} \to \mathrm{Hom}_{\mathscr{D}ef_A(R)}(\mathcal{A},\mathcal{A}')$$
$$\exp(f) \mapsto \exp(f)|\mathcal{A},$$

is a bijection, where $\mathcal{A} = (A \otimes R, (f_\mu + \xi)|(A \otimes R)^{\otimes 2})$ and $\mathcal{A} = (A \otimes R, (f_\mu + \xi')|(A \otimes R)^{\otimes 2})$. However this is clear from Proposition 0.5, the fact that the map

$$\mathrm{Aut}_{R\text{-coalg}}(\mathscr{B}(A \otimes R)) \to \mathrm{Aut}_{R\text{-mod}}(A \otimes R), \quad \phi \mapsto \phi|(A \otimes R)$$

is an bijection, and the paragraph preceding this proof. ■

We now wish to elaborate on the manner in which the equivalence $\mathscr{MC}_A \overset{\sim}{\to} \mathscr{D}ef_A$ manifests in cohomology. Note that for any $m \leq M$ we have chain inclusions

$$\Sigma C^\bullet(A \otimes k[t]/(t^m)) = \oplus_{i<m} \Sigma C^\bullet(A) \otimes kt^i \to \oplus_{j<M} \Sigma C^\bullet(A) \otimes kt^j = \Sigma C^\bullet(A \otimes k[t]/(t^M)),$$

so any element Ξ in $\Sigma C^\bullet(A \otimes k[t]/(t^m))$ can be seen canonically as an element in $\Sigma C^\bullet(A \otimes k[t]/(t^M))$. This inclusions is *not* a Lie algebra embedding.

Now, given an nth order deformation $\Xi = f_\mu + \xi \in \Sigma C^1(A \otimes k[t]/(t^{n+1}))$, with $\xi = \sum_{0<i\leq n} \xi_i t^i$ we may want to lift Ξ to an $(n+1)$st order deformation $\tilde{\Xi} = f_\mu + \sum_{0<i\leq n+1} \xi_i t^i$. Since Ξ is a deformation we have $[\Xi, \Xi] = 0 \mod t^{n+1}$ in $\Sigma C^\bullet(A \otimes k[t]/(t^{n+2}))$. So for any choice of ξ_{n+1} we have

$$[\tilde{\Xi}, \tilde{\Xi}] = 2([f_\mu, \xi_{n+1}] + \sum_{i>0} [\xi_i, \xi_{n+1-i}]) t^{n+1} = (d(\xi_{n+1}) + \sum_{i>0} [\xi_i, \xi_{n+1-i}]) 2t^{n+1} \tag{0.3.2}$$

So we see that the deformation Ξ is unobstructed, i.e. can be lifted to an order $n+1$ deformation, if and only if the class $[\Xi, \Xi]/2t^{n+1} = \sum_{i>0} [\xi_i, \xi_{n+1-i}]$ is a boundary in $\Sigma C^2(A)$. We would like to know that this element is a cocycle in general.

Lemma 0.7. *If Ξ is a nth order deformation then, in $\Sigma C^\bullet(A \otimes k[t]/(t^{n+2}))$, the element $[\Xi, \Xi]$ is a cocycle.*

Proof. Write $\Xi = f_\mu + \xi$. By the graded Jacobi identity we know $[\Xi, [\Xi, \Xi]] = 0$ and since $[\Xi, \Xi]$ is concentrated in t-degree $n+1$ we know also $[\xi, [\Xi, \Xi]] = 0$. So

$$0 = [\Xi, [\Xi, \Xi]] = [f_\mu, [\Xi, \Xi]] + [\xi, [\Xi, \Xi]] = [f_\mu, [\Xi, \Xi]] = d([\Xi, \Xi]).$$

■

Let $R' \to R$ be a map of algebras in $\mathtt{Art}_k$ and Ξ be an R-deformation of A. In what follows we let $\mathscr{MC}_A(R')_\Xi$ denote the subgroupoid of $\mathscr{MC}_A(R')$ whose objects are those objects mapping to Ξ under the reduction $\mathscr{MC}_A(R') \to \mathscr{MC}_A(R)$ and whose morphisms are those morphisms mapping to id_Ξ under the reduction. We let $\mathrm{MC}_A(R')_\Xi$ denote the set of isoclasses in $\mathscr{MC}_A(R')_\Xi$. We adopt a similar notation for $\mathscr{D}ef_A$ and Def_A.

Theorem 0.8. *Suppose $\Xi = f_\mu + \xi_1 t + \cdots + \xi_n t^n$ is an nth order deformation. Then*

1. *the element $\sum_{0<i\leq n}[\xi_i, \xi_{n+1-i}]$ is a cocycle in $\Sigma C^2(A)$.*

2. *The deformation Ξ is unobstructed if and only if the class of $\sum_i [\xi_i, \xi_{n+1-i}]$ vanishes in $H^2(\Sigma C^\bullet(A)) = HH^3(A)$.*

3. *When Ξ is unobstructed then the set of lifts $\mathrm{obj}\big(\mathscr{MC}_A(k[t]/(t^{n+2}))_\Xi\big)$ is a free $Z^1(\Sigma C^\bullet(A))$-set.*

4. *When Ξ is unobstructed the set of isoclasses $\mathrm{MC}_A(k[t]/(t^{n+2}))_\Xi = \mathrm{Def}_A(k[t]/(t^{n+2}))_\Xi$ is a free $H^1(\Sigma C^\bullet(A)) = HH^2(A)$-set.*

With some more advanced notions of actions of groupoids one should be able to combine statements (3) and (4). By a free G-set, for a group G, we mean a set X with a G-action so that for any $x, y \in X$ there exists a unique $g \in G$ with $gx = y$. Whence, after fixing some element $e \in X$, we get an isomorphism $G \overset{\cong}{\to} X$, $g \mapsto ge$.

Proof. (1) This statement follows from Lemma 0.7, the fact that $[\Xi, \Xi] = (\sum_i [\xi_i, \xi_{n+1-i}])t^{n+1}$, and the fact that the subcomplex $\Sigma C^\bullet(A) \otimes kt^{n+1} \subset \Sigma C^\bullet(A \otimes k[t]/(t^{n+2}))$ is isomorphic to the shifted Hochschild cochain complex. (2) This follows from the fact that equation (0.3.2) holds for any $\tilde{\Xi}$ in $\Sigma C^\bullet(A \otimes k[t]/(t^{n+2}))$ which reduces to Ξ mod t^{n+1} and the fact that $\tilde{\Xi}$ is a deformation if and only if $[\tilde{\Xi}, \tilde{\Xi}] = 0$. (3) This is clear since for any lifts

$$\tilde{\Xi} = f_\mu + \xi_1 t + \cdots + \xi_n t^n + \xi_{n+1} t^{n+1} \quad \text{and} \quad \tilde{\Xi}' = f_\mu + \xi_1 t + \cdots + \xi_n t^n + \xi'_{n+1} t^{n+1}$$

we have $d(\xi_{n+1}) = d(\xi'_{n+1}) = \sum_i [\xi_i, \xi_{n+1-i}]$ so that $\xi_{n+1} - \xi'_{n+1}$ is a cocycle. Whence any two lifts differ by a unique cocycle in $\Sigma C^\bullet(A)$.

(4) A map $\exp(f) \in \mathrm{Gauge}_{k[t]/(t^{n+2})}$ is sent to the identity in $\mathrm{Gauge}_{k[t]/(t^{n+1})}$ if and only if the reduction $\bar{f}$ vanishes in $\Sigma C^0(A) \otimes k[t]/(t^{n+1})$, since the exponential map is a bijection by Lemma 0.5. So $f = qt^{n+1}$ for some coderivation q on $\mathscr{B}A$, $\exp([f,-]) = 1 + [f,-]$, and

$$\exp(f)\tilde{\Xi}\exp(-f) = \exp([f,-])(\tilde{\Xi}) = \tilde{\Xi} + [f, \tilde{\Xi}] = \tilde{\Xi} + [f, f_\mu] = \tilde{\Xi} - d(q)t^{n+1}.$$

So the orbit of any lift $\tilde{\Xi}$ under such automorphisms is exactly $\tilde{\Xi} + B^1 t^{n+1}$, and the free Z^1 action of (3) induces a free $H^1(\Sigma C^\bullet(A)) = HH^2(A)$-action on the quotient space $\mathrm{MC}_A(k[t]/(t^{n+2}))_\Xi$.

■

Remark 0.9. We have ignored the issue of preserving the unit under deformation. In order to incorporate the unit one replaces the Hochschild cochain complex $\Sigma C^\bullet(A)$ with the reduced cochain complex $\Sigma \bar{C}^\bullet(A)$, which is the graded Lie subalgebra of coderivations $\mathscr{B}A \to \mathscr{B}A$ factoring through the reduced bar complex $\bar{\mathscr{B}}A = \oplus_i \bar{A}^{\otimes i}$. Here $\bar{A} = A/k$. It is well known that the inclusion

$$\Sigma \bar{C}^\bullet(A) \to \Sigma C^\bullet(A)$$

is a quasi-isomorphism so that the corresponding map of deformation functors

$$\mathscr{MC}_{\Sigma \bar{C}^\bullet(A)} \to \mathscr{MC}_{\Sigma C^\bullet(A)}$$

is an equivalence [23, Theorem 2.4], [57, Theorem V.51].

0.4 A list of publications

Published:

- C. Negron. *Spectral sequences for the cohomology rings of a smash product.* J. Algebra, 433:73106, 2015

Submitted/preprints:

- C. Negron and S. Witherspoon. *An alternate approach to the Lie bracket on Hochschild cohomology*, arXiv:1406.0036.

- C. Negron. *The cup product on Hochschild cohomology for localizations of filtered Koszul algebras*, arXiv:1304.0527.

None of the material of *An alternate approach. . .* is covered in this thesis, and can be found at the given link.

Chapter 1

TWISTING COCHAINS, THE CUP PRODUCT ON HOCHSCHILD COHOMOLOGY, AND KOSZUL ALGEBRAS

As usual, we let k be a field. In this chapter we give some new relations between twisting cochains and Hochschild cohomology. In particular, we show that for an acyclic twisting cochain $\pi : C \to A$ from a curved dg coalgebra to a dg algebra the cohomology of the resulting twisted hom complex $\mathrm{Hom}_k^\pi(C, A)$ is the Hochschild cohomology ring of A. As explained in the introduction, this result yields some very nice corollaries when applied to the context of Koszul duality (Corollary 1.33).

The general structure of the chapter is as follows: we first give the appropriate framing of Koszul duality as a relation between an algebra and a (curved) dg algebra in Section 1.2. We then introduce twisting cochains and explain how they appear in, and produce, Koszul duality in Sections 1.3-1.4. Section 1.5-1.7 are then dedicated to the Hochschild cohomology in general and a presentation of our main result, Theorems 1.37. Hochschild cohomology of dg algebras is covered Section 1.8, and finally the example of the Heisenberg Lie algebra is presented in Section 1.9.

1.1 Notations and conventions

Let R be an arbitrary ring. By a "R-module" we mean a *left* R-module unless stated otherwise. We will always use the cohomological indexing convention

$$X = \cdots \xrightarrow{d} X^{n-1} \xrightarrow{d} X^n \xrightarrow{d} X^{n+1} \xrightarrow{d} \cdots$$

for chain complexes. Given R-complexes X and Y we write $\mathrm{Hom}_R(X,Y)$ for the standard Hom complex

$$\mathrm{Hom}_R(X,Y)=\bigoplus_{n\in\mathbb{Z}}(\prod_i \mathrm{Hom}_R(X^i,Y^{i+n}))$$

For any homogenous function $\theta\in\mathrm{Hom}_R(X,Y)$ of degree n, the differential d is given by the formula $d(\theta)=d_Y\theta-(-1)^n\theta d_X$.

The grading on a graded ring R will be seen as *internal* and will be denoted by a lower index $R=\oplus_i R_i$. A graded R-module will be referred to as *homologically* graded if it is to be viewed as a chain complex with vanishing differential. When doing computations with graded modules we assume that each element x is homogenous of a particular degree, and we let $|x|$ denote the degree of such an element.

Sweedler's notation will be used to denote the comultiplication on a coalgebra C. So the element $\Delta(c)$ will be written $\Delta(c)=c_1\otimes c_2$, with the sum implicit. To say this more clearly, "$c_1\otimes c_2$" is simply shorthand for some expression of the element

$$\Delta(c)=\sum_i c_{i_1}\otimes c_{i_2}$$

in the tensor product $C\otimes C$. Higher iterations of the comultiplication will be denoted using similar notation. For example, the element

$$(\Delta\otimes id)\Delta(c)=(id\otimes\Delta)\Delta(c)$$

will be denoted $c_1\otimes c_2\otimes c_3$. Again, there is an implicit sum. If C is graded, and $c\in C$ is homogeneous, then the c_1,c_2, etc. will always be taken to be homogeneous.

1.1.1 Our cup product versus Gerstenhabers cup product and the Gerstenhaber bracket

This subsection can be skipped of a first reading, and is meant to preemptively address some subtle points of confusion which may arise in a readers attempts to use the main results of this work. Let A be a k-algebra. On the Hochschild cochain complex

$$0\to A\to\mathrm{Hom}_k(A,A)\to\mathrm{Hom}_k(A^{\otimes 2},A)\to\mathrm{Hom}_k(A^{\otimes 3},A)\to\cdots$$

there are two products one can give. One is Gerstenhabers cup product, which takes functions f and g of respective degrees i and j to the degree $n = i + j$ function

$$f \cdot_{Gerst} g := \big(a_1 \otimes \ldots \otimes a_n \mapsto f(a_1 \otimes \ldots \otimes a_i) g(a_{i+1} \otimes \ldots \otimes a_n)\big),$$

the other of which employs the standard Koszul sign

$$fg := \big(a_1 \otimes \ldots \otimes a_n \mapsto (-1)^{ij} f(a_1 \otimes \ldots \otimes a_i) g(a_{i+1} \otimes \ldots \otimes a_n)\big). \tag{1.1.1}$$

This point is, in some sense, immaterial.

Lemma 1.1. *The graded automorphism sending a degree i map f to $(-1)^{i(i+1)/2} f$ is an algebra isomorphism identifying Gerstenhabers cup product with the signed product (1.1.1). Consequently, the two products on the Hochschild cochain complex, and on the Hochschild cohomology $HH^\bullet(A)$, are canonically isomorphic.*

Proof. Let's denote this map by φ. One simply calculates! For f and g of respective degrees i and j,

$$\varphi(f \cdot_{Gerst} g) = \varphi((-1)^{ij} fg) = (-1)^{ij + \frac{(i+j)(i+j+1)}{2}} fg.$$

Now

$$\begin{aligned} ij + (i+j)(i+j+1)/2 &= ij + (i^2 + j^2 + 2ij + i + j)/2 \\ &\equiv (i^2 + i + j^2 + j)/2 \mod 2 \\ &= i(i+1)/2 + j(j+1)/2 \end{aligned}$$

So $\varphi(f \cdot_{Gerst} g) = \varphi(f)\varphi(g)$. ■

It is this second product which is most naturally identified with the Yoneda product on cohomology. Indeed, if we take C to be the bar dg coalgebra $\mathscr{B}A$ (see section 1.3) our main result Theorem 1.33 implies that the signed product (1.1.1) agrees with the Yoneda product on cohomology. A yet subtler point is the following: the signed product does not produce Gerstenhabers orignal compatibility with the Gerstenhaber bracket given in [18]. Instead, for cocycles f, g, h of respective degrees i, j, k we get the (arguably better looking) relation

$$[f, gh] = [f, g]h + (-1)^{(i-1)j} g[f, h],$$

so that each $[f, -]$ becomes a graded derivation of degree $i-1$. This is the correct relation for the Schouten–Nijenhuis bracket on the Hochschild cohomology $\bigwedge^{\bullet}_{\mathscr{O}_X} \mathscr{T}_X$ of the ring of functions on a smooth affine scheme X.

1.2 Reminders on dg algebras, dg coalgebras, and filtered Koszul algebras

1.2.1 Dg algebras and coalgebras

Recall that a dg algebra is a chain complex (A, d) equipped with a unit $k \to A$ and associative multiplication $\mu : A \otimes A \to A$ which are both chain maps. On elements, this means the unit 1 is a cycle and that d satisfies

$$d(fg) = d(f)g + (-1)^{|f|} f d(g),$$

i.e. that d is a graded derivation. A dg coalgebra is defined dually to be a complex (C, d) with a coalgebra structure such that each structure map $C \to k$, $\Delta : C \to C \otimes C$, is a chain map. We will call a dg (co)algebra locally finite if it is finite dimensional in each homological degree. A dg algebra A (resp. dg coalgebra C) is said to be augmented (resp. coaugmented) if it comes equipped with a dg map $A \xrightarrow{\epsilon} k$ (resp. $k \xrightarrow{u} C$).

Given an arbitrary dg algebra A and dg coalgebra C the hom complex $\mathrm{Hom}_k(C, A)$ becomes a dg algebra under the convolution product

$$f * g := \mu_A(f \otimes g)\Delta_C : c \mapsto (-1)^{|c_1||g|} f(c_1) g(c_2).$$

In particular the dual $C^* = \mathrm{Hom}_k(C, k)$ is a dg algebra. One can check that the dual $A^* = \mathrm{Hom}_k(A, k)$ of any locally finite dg algebra is a dg coalgebra under the coproduct $\Delta(\gamma) = \gamma\mu$. The double dual of a locally finite dg (co)algebra A is naturally isomorphic to A via the standard map

$$\begin{gathered} ev : A \to (A^*)^* \\ a \mapsto (\phi \mapsto (-1)^{|a||\phi|}\phi(a)). \end{gathered} \tag{1.2.1}$$

The tensor product of dg (co)algebras is again a dg (co)algebra under the differential $d_{A \otimes A'} = d_A \otimes id_{A'} + id_A \otimes d_{A'}$, and we can define the opposite dg algebra A^{op} to be the complex A with the opposite multiplication $a \cdot^{op} b := (-1)^{|a||b|} ba$.

Given a dg algebra A, a k-complex M is called a left (resp. right) dg module over A if it is a graded A-module, after we forget the differential, and the action map

$$A \otimes M \to M \quad (\text{resp. } M \otimes A \to M)$$

is a map of chain complexes. Similarly, M is a dg bimodule if it is a graded bimodule over A and the action map $A \otimes M \otimes A \to M$ is one of chain complexes. A bimodule over a dg algebra A can, as in the non-dg case, also be seen as a module over the enveloping algebra $A^e = A \otimes A^{op}$.

Given dg A-modules M and N, we define the hom complex $\mathrm{Hom}_{A\text{-mod}}(M,N)$ in the usual way. That is

$$\mathrm{Hom}_{A\text{-mod}}(M,N) = (\oplus_i \mathrm{Hom}^i_{A\text{-mod}}(M,N), d_{\mathrm{Hom}})$$

with the usual differential $d_{\mathrm{Hom}} : f \mapsto d_N f - (-1)^{|f|} f d_M$. Here the notation $\mathrm{Hom}^i_{A\text{-mod}}(M,N)$ denotes the set of homogenous degree i maps $f : M \to N$ satisfying

$$f(am) = (-1)^{|f||a|} a f(m)$$

for any homogeneous $a, b \in A$. Taking $A = B^e$ for a dg algebra B gives the appropriate definition of the hom complex for bimodules $\mathrm{Hom}_{B\text{-bimod}}(M,N) = \mathrm{Hom}_{B^e\text{-mod}}(M,N)$.

1.2.2 Graded Koszul duality with signs

A Koszul algebra is a finitely generated connected graded algebra B, i.e. a graded algebra of the form

$$B = k \oplus B_1 \oplus B_2 \oplus \cdots ,$$

such that $\mathrm{Ext}_A(k,k)$ is generated by $\mathrm{Ext}^1_B(k,k)$ as an algebra. Here $k = {}_Bk$ denotes the graded simple module $B/(B_{\geq 1})$. The *Koszul dual* of a Koszul algebra B is the algebra $\mathrm{Ext}_B(k,k)$. To avoid confusion with the filtered case, we denote the Koszul dual by E for the moment.

Any Koszul algebra will have a quadratic presentation $B = k\langle V\rangle/(R)$. Let us fix a Koszul algebra with such a presentation. Here $R \subset V \otimes V$ is the subspace of quadratic relations for

B. It is well known that we have a presentation $E \cong k\langle V^*\rangle/(R^\perp)$ given by the identification $E^1 = V^*$. We give here a description of the Koszul dual which takes into account the homological grading on the implicit Koszul resolution of k, which gives rise to the Koszul dual. In particular, we identify E with the dual algebra of some particular homologically graded coalgebra, i.e. a dg coalgebra with vanishing differential.

We let $T\langle V\rangle = \oplus_{n\geq 0} V^{\otimes n}$ denote the tensor coalgebra on V. Recall that the comultiplication on $T\langle V\rangle$ is defined by "separation of tensors"

$$\mathbf{v} = (v_1 \otimes \ldots \otimes v_n) \mapsto (1) \otimes (\mathbf{v}) + (\mathbf{v}) \otimes (1) + \sum_{1\leq j\leq n-1} (v_1 \otimes \ldots \otimes v_j) \otimes (v_{j+1} \otimes \ldots \otimes v_n).$$

We consider $T\langle V\rangle$ to be homologically graded by taking V to be in degree -1. (So the more cumbersome notation $T\langle \Sigma V\rangle$ may be more appropriate here.) The following lemma is well known. See for example [42, Section 4.7], [51, Sections 3.1.3-3.2.2]. In any case, we sketch the proof for the reader's convenience.

Lemma 1.2. *The graded subspace W of $T\langle V\rangle$ defined by $W^0 = k$, $W^{-1} = V$, and*

$$W^{-i} = \bigcap_{i_1+i_2=i-2} V^{\otimes i_1} \otimes R \otimes V^{\otimes i_2} \tag{1.2.2}$$

for all $i \geq 2$, is a graded subcoalgebra of $T\langle V\rangle$.

Proof. We will show that W is closed under the coproduct on $T\langle V\rangle$. We grade $T\langle V\rangle$ by negated tensor degree so that the inclusion $W \to T\langle V\rangle$ is a graded map. Let c be a homogenous element in W. Since W^{-1} and W^0 are equal to $T\langle V\rangle^{-1} = V$ and $T\langle V\rangle^0 = k$ respectively, it is trivial to show that $\Delta(c) \subset W \otimes W$ whenever $|c|$ is $0, 1$, or 2.

Let us assume $|c| = -n \leq -3$. For $i, j \geq 0$, Let $\Delta_{ij}(c)$ denote the component of $\Delta(c)$ in

$$T\langle V\rangle^{-i} \otimes T\langle V\rangle^{-j} = (V^{\otimes i}) \otimes (V^{\otimes j}).$$

So we have $\Delta(c) = \sum_{ij} \Delta_{ij}(c)$, and $\Delta(c) \in W\otimes W$ if and only if each $\Delta(c)_{ij}$ is in $W^{-i}\otimes W^{-j}$.

Since W^{-n} is the intersection (1.2.2), we can write c as a sum

$$c = \sum_l v_{l_1} \otimes \ldots \otimes v_{l_{k-1}} \otimes r_{l_k} \otimes v_{l_{k+2}} \ldots \otimes v_{l_n}$$

for any k between 1 and $n-1$, where the $r_{l_k} \in R$. Now by letting k vary we see that

$$\Delta_{ij}(c) \in (W^{-i} \otimes T\langle V\rangle^{-j}) \cap (T\langle V\rangle^{-i} \otimes W^{-j}).$$

One can verify that this final intersection is equal to $W^{-i} \otimes W^{-j}$. ■

Let us outline our identification $E = W^*$. Consider $k\langle V^*\rangle$, the free algebra on the degree 1 space V^*. We have the canonical algebra isomorphism

$$\begin{aligned} &k\langle V^*\rangle \to (T\langle V\rangle)^* \\ &f_1 \otimes \ldots \otimes f_n \mapsto (v_1 \otimes \ldots \otimes v_n \mapsto (-1)^{n(n-1)/2} f_1(v_1) \ldots f_n(v_n)). \end{aligned} \tag{1.2.3}$$

Here the f_i are in V^*, the v_i are in V, the function $f_1 \otimes \ldots \otimes f_n$ will vanish off $V^{\otimes n}$, and the exponent $n(n-1)/2 = \sum_{l=0}^{n-1} l$ comes from commuting the degree -1 variables v_i past the degree 1 maps f_i.

If we then compose the isomorphism (1.2.3) with the dual of the restriction $W \to T\langle V\rangle$ we get an algebra map $k\langle V^*\rangle \to W^*$. One can verify that the kernel of this map is the ideal $(R^\perp)$ and so we get an isomorphism $E \to W^*$. This isomorphism simply sends a monomial $f_1 \ldots f_n$ in E to the function

$$f_1 \ldots f_n : W \to k,\ v_1 \otimes \ldots \otimes v_n \mapsto (-1)^{n(n-1)/2} f_1(v_1) \ldots f_n(v_n).$$

Algebraically, one can see the isomorphism $E \to W^*$ as the unique algebra map defined as the identity on the generators $E^1 = V^* = (W^{-1})^*$. It is via this isomorphism that we identify E with W^* as a graded algebra.

Remark 1.3. The sign conventions we employ here make no difference in the presentation of the Koszul dual, since we will simply replace each relation r produced via the unsigned identification $V^* \otimes V^* \cong (V \otimes V)^*$ with the negated relation under the signed identification $V^* \otimes V^* \cong (V \otimes V)^*$. The identity map on the generators V^* will then provide an isomorphism

$$k\langle V^*\rangle/(R^\perp) \to k\langle V^*\rangle/(-R^\perp).$$

The conventions do make a difference once we start considering differentials and curvature.

1.2.3 (Augmented) filtered Koszul algebras

The class of algebras we will be interested are the following.

Definition 1.4 (Filtered Koszul algebras)**.** A $\mathbb{Z}_{\geq 0}$-filtered algebra $B = \cup_{i\geq 0} F_i B$ such that $\mathrm{gr}B$ is Koszul is called a filtered Koszul algebra.

To distinguish between filtered Koszul and standard Koszul algebras we may refer to standard Koszul algebras as *graded* Koszul. The class of filtered Koszul algebras includes the class of graded Koszul algebras, since we can give any Koszul algebra the filtration $F_n B = \sum_{i=0}^{n} B_i$. In this case $\mathrm{gr}B = B$.

Let B be a filtered Koszul algebra. Let E denote the Koszul dual algebra $\mathrm{Ext}_{\mathrm{gr}B}(k,k)$ of $\mathrm{gr}B$. So we have $\mathrm{gr}B = k\langle V\rangle/(R)$ and $E \cong k\langle V^*\rangle/(R^\perp)$. Recall our identification of E with the dual W^* of the intersection coalgebra of Lemma 1.2. In particular, we have identified E^2 with $W^{*2} = R^*$ by sending a monomial $f_1 f_2$ in E^2 (where the $f_i \in V^*$) to the function $\sum_i r_i \otimes r_i' \mapsto -\sum_i f_1(r_i) f_2(r_i')$.

Suppose for the moment that B is augmented, i.e. has some fixed algebra map $\epsilon : B \to k$. For example, we could consider B to be a universal enveloping algebra of a Lie algebra $\mathfrak{g}$ with the standard augmentation $U(\mathfrak{g}) \to k$ sending $\mathfrak{g}$ to 0. This will provide splittings $A = k \oplus \ker(\epsilon)$ and $F_1 B = V \oplus k$, and hence also provide an embedding $V \to B$. This embedding then produces a second embedding $V \otimes V \to B \otimes B$, and finally a third embedding

$$R \subset V \otimes V \to B \otimes B$$

of the relations for the associated graded ring $\mathrm{gr}B$ into $B \otimes B$. We omit the proof of the following lemma.

Lemma 1.5. *Suppose B is filtered Koszul and augmented, as above. The restriction of the multiplication $\mu : B \otimes B \to B$ on B to the subspace $R \subset B \otimes B$ has image in the set of generators V. Whence we get a canonically defined map*

$$\mu|R : R \to V. \tag{1.2.4}$$

If we take $\alpha_1 := -\mu|R$*, then we get a presentation*

$$B = k\langle V\rangle/(r + \alpha_1(r))_{r\in R}.$$

It is well known that the function $V^* = E^1 \to R^* = E^2$ given by precomposing with α_1, $f \mapsto f\alpha_1$, extends to a dg algebra structure d^B on E, and that the homology of the resulting dg algebra (E, d^B) is the Ext algebra $\mathrm{Ext}_A(k,k)$. This result appears in Priddy's original work on Koszul resolutions [71, Theorem 4.3]. (See also [70, Proposition 2.2], [68, Section 5.4], and Proposition 1.8 below.) In the case that B is the universal enveloping algebra of a Lie algebra $\mathfrak{g}$, for example, the restriction (1.2.4) is given by the Lie bracket and the dg algebra (E, d^B) is the Chevalley-Eilenberg dg algebra of $\mathfrak{g}$, $(E, d^B) = (\bigwedge^\bullet \mathfrak{g}^*, d_{[,]})$.

1.2.4 Nonaugmented filtered Koszul algebras and curved dg structures

The following is just a reiteration of the work of [70]. We take A filtered Koszul with $\mathrm{gr}B = k\langle V\rangle/(R)$.

In the case of a nonaugmented filtered Koszul algebra A, we will have to choose some section $V \to F_1B$ of the sequence $0 \to k \to F_1B \to V \to 0$ in order to identify V with a subspace in B. This will also provide a splitting $F_1B = V \oplus k$. *We consider the choice of section* $V \to F_1B$ *to be implicit in the notion of a filtered Koszul ring.* We have the following weak analog of Lemma 1.5.

Lemma 1.6. *Let* B *be filtered Koszul, but not necessarily augmented, and fix some section* $V \to F_1B$*. Then restricting the multiplication on* B *to the subspace* $R \subset V \otimes V \subset B \otimes B$ *produces a map*

$$\mu|R : R \to F_1B = V \oplus k. \tag{1.2.5}$$

Take $\alpha_1 : R \to V$ *and* $\alpha_0 : V \to k$ *to be the unique functions so that* $\mu|R = [-\alpha_1 \;\; -\alpha_0]^T$*. Then we get a presentation*

$$B = k\langle V\rangle/(r + \alpha_1(r) + \alpha_0(r))_{r\in R}. \tag{1.2.6}$$

One can see from the Lemmas 1.5 and 1.6 that B is augmented if and only if the section $V \to F_1 B$ can be chosen so that $\alpha_0 = 0$.

As in the previous subsection, we take E to be the algebra of extensions $\mathrm{Ext}_{\mathrm{gr}B}(k,k)$ for the associated graded algebra $\mathrm{gr}B$, which we have assumed to be Koszul. The function $V^* = E^1 \to R^* = E^2$ given by precomposition with α_1, $f \mapsto f\alpha_1$, will again extend to a well defined derivation d^B on E [70, Proposition 2.2]. However, we will not be fortunate enough to have that d^B is square 0. Instead, $d^B d^B$ will simply be inner. Whence we introduce the notion of a curved dg (co)algebra, following [70].

Definition 1.7 (Curved dg (co)algebras)**.** A curved dg algebra is a graded algebra $A = \oplus_{i\in\mathbb{Z}} A^i$ along with a degree 1 graded derivation d_A, and a degree 2 element $c_A \in A^2$, so that

$$d_A^2 = [c_A, -] \text{ and } d_A(c_A) = 0.$$

(Here d_A^2 is the square $d_A^2 = d_A d_A$.) Dually, a curved dg coalgebra is a graded coalgebra $C = \oplus_{i\in\mathbb{Z}} C^i$ along with a degree 1 coderivation d_C, and degree 2 function $f_C : C \to k$ satisfying

$$d_C^2 = (f_C \otimes id - id \otimes f_C)\Delta \text{ and } f_C d_C = 0.$$

We may denote a curved dg algebra (resp. coalgebra) as a triple (A, d_A, c_A) (resp. (C, d_C, f_C)).

The etymology here is actually coming from geometric curvature, as one can see from [70, Example 4.1].

Note that the curvature element c_A in a curved dg algebra A is not uniquely determined by d_A in general. Indeed, when A is graded commutative, any degree 2 element $c_A \in A^2$ will have $[c_A, -] = 0$ and whence produce a curved dg algebra $(A, 0, c_A)$.

As with dg algebras and coalgebras, we have some standard constructions. Given a curved dg algebra A and a curved dg coalgebra C the set of graded maps $\mathrm{Hom}_k(C, A) = \bigoplus_n \left(\prod_i \mathrm{Hom}_k(C^i, A^{i+n})\right)$ becomes a curved dg algebra under the convolution product, standard derivation $d(\xi) = d_A\xi - (-1)^{|\xi|}\xi d_C$, and curvature

$$c_{\mathrm{Hom}} = c_A \epsilon_C - 1_A f_C$$

[69, Section 6.2]. In particular, the graded dual C^* of any curved dg coalgebra becomes a curved dg algebra with curvature element $c_{C^*} = -f_C$. The graded dual of any locally finite curved dg algebra A becomes a curved dg coalgebra with the obvious coproduct, derivation $d(\eta) = -(-1)^{|\eta|}\eta d_A$, and curvature function $f_{A^*} = -ev_{c_A}$. When C is locally finite, the evaluation map $ev : C \to C^{**}$ provides an isomorphism of curved dg coalgebras between C and its double dual. Finally, the tensor product $A \otimes A'$ of curved dg algebras will again be a curved dg algebra with $d_{A\otimes A'} = d_A \otimes id_{A'} + id_A \otimes d_{A'}$ and $c_{A\otimes A'} = c_A \otimes 1 + 1 \otimes c_{A'}$.

Theoretically, curved dg structures arise as deformations of dg algebras. For example, a cocycle in the second Hochschild cohomology of a dg algebra A will correspond to a curved dg $k[t]/(t^2)$-algebra, or more generally curved A_∞ $k[t]/(t^2)$-algebra, which reduces to A at $t = 0$.

Returning to our filtered Koszul algebra $B = k\langle V\rangle/(r + \alpha_1(r) + \alpha_0(r))$ of (1.2.6), we note that α_0 defines a function $R \to k$, and hence an element in $E^2 = R^*$. Take $c^B = -\alpha_0$. Recall that the derivation d^B was defined on the generators of E as the function $E^1 = V^* \to E^2 = R^*$, $f \mapsto f\alpha_1$. In [70], Positselski proves the following

Proposition 1.8 ([70, Proposition 2.2])**.** *Suppose B is filtered Koszul. The triple* $(\mathrm{Ext}_{\mathrm{gr}B}(k,k), d^B, c^B)$ *defines a curved dg algebra structure on the algebra of extensions* $\mathrm{Ext}_{\mathrm{gr}B}(k,k)$ *of the Koszul algebra* $\mathrm{gr}B$*.*

Proof. The proof is the same as in [70], which essentially shows that the intersection coalgebra W of Lemma 1.2 is a curved dg coalgebra with $d_W|W^{-1} = \alpha_1$ and $f_W = \alpha_0$. We only note here that the sign on the curvature has changed due to the our signed identification with W^* (see Remark 1.3). ■

It is this structure which we view as the Koszul dual of B. Here we could take B to be a Weyl algebra or Clifford algebra. It is well known that Weyl algebras are simple, and hence admit no augmentation.

Definition 1.9 (The Koszul dual)**.** Let B be a filtered Koszul algebra. The curved dg algebra

$(\mathrm{Ext}_{\mathrm{gr}B}(k,k), d^B, c^B)$ described above will be called the Koszul dual curved dg algebra to B. It will generally be denoted $B^! = (B^!, d_{B^!}, c_{B^!})$.

1.2.5 Example: The nth Weyl algebra

In the case of the nth Weyl algebra

$$A_n(k) = k\langle x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\rangle/([\frac{\partial}{\partial x_j}, x_i] - \delta_{ji}),$$

we have $\mathrm{gr}A = k[x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}]$ and $\mathrm{Ext}_{\mathrm{gr}A}(k,k) = k[\lambda_1, \ldots, \lambda_n, \theta_1, \ldots, \theta_n]$. In the second algebra, the variables λ_i and θ_j are the duals of the x_i and $\frac{\partial}{\partial x_j}$ respectively. We consider these functions to have homological degree 1, and the algebra $k[\lambda_1, \ldots, \lambda_n, \theta_1, \ldots, \theta_n]$ is the free *graded* commutative algebra with these generators. (So the variables anti-commute.)

Recall our identification of $\mathrm{Ext}^2_{\mathrm{gr}A}(k,k) = (V^* \otimes V^*)/R^{\perp}$ with R^* is given by sending a monomial $f_1 f_2$ to the function $\sum_i r_i \otimes r_i' \mapsto -\sum f_1(r_i) f_2(r_i')$. So $\lambda_i\theta_j$ gets identified with the function $R \to k$ defined on basis elements by

$$\lambda_i\theta_j : \begin{array}{c} x_k \otimes x_l - x_l \otimes x_k \mapsto 0 \\ \frac{\partial}{\partial x_k} \otimes x_l - x_l \otimes \frac{\partial}{\partial x_k} \mapsto \delta_{il}\delta_{jk} \\ \frac{\partial}{\partial x_k} \otimes \frac{\partial}{\partial x_l} - \frac{\partial}{\partial x_l} \otimes \frac{\partial}{\partial x_k} \mapsto 0 \end{array}$$

Whence, in this case, the curvature element $c^{A_n(k)} = -\alpha_0$ in $k[\lambda_1, \ldots, \lambda_n, \theta_1, \ldots, \theta_n]^2$ will be the sum

$$c^{A_n(k)} = \sum_{i=1}^{n} \lambda_i\theta_i.$$

The corresponding Koszul dual of $A_n(k)$ will be the curved dg algebra

$$(k[\lambda_1, \ldots, \lambda_n, \theta_1, \ldots, \theta_n], 0, c^{A_n(k)}).$$

1.2.6 Example: PBW deformations of skew polynomial rings

Take $V = \langle x_1, \ldots, x_n\rangle$ and let $k_Q[V]$ denote the skew polynomial ring

$$k_Q[V] = k\langle x_1, \ldots, x_n\rangle/(x_j x_k - q_{jk} x_k x_j)$$

for $Q = [q_{jk}]$ a multiplicatively skew symmetric matrix ($q_{jk} = q_{kj}^{-1}$) with $q_{jj} = 1$. The Koszul dual of the skew polynomial ring in the skew exterior algebra

$$\operatorname{Ext}_{k_Q[V]}(k,k) = \bigwedge\nolimits_Q V^* := k\langle \lambda_1, \ldots, \lambda_n\rangle / (\lambda_k\lambda_j + q_{jk}\lambda_j\lambda_k, \lambda_j^2).$$

An augmented PBW deformation B of $k_Q[V]$ will be given by some constants c_i^{jk} so that the relations on our PBW deformation B will be given by

$$B = k\langle x_1, \ldots, x_n\rangle / (x_jx_k - q_{jk}x_kx_j - \sum_i c_i^{jk} x_i).$$

Thus see that $\alpha_1 : R \to V$ will be the function $x_j \otimes x_k - q_{jk}x_k \otimes x_j \mapsto -\sum_i c_i^{jk} x_i$.

Now, on the Koszul dual, the product $\lambda_i\lambda_j$ in $(\bigwedge_Q V^*)^2$ is identified with the function

$$x_l \otimes x_m - q_{lm}x_m \otimes x_l \mapsto -(\lambda_i(x_l)\lambda_j(x_m) - q_{lm}\lambda_i(x_m)\lambda_j(x_l)) = -\delta_{il}\delta_{jm} + q_{lm}\delta_{im}\delta_{jl},$$

i.e. the negated dual of the relations $x_ix_j - q_{ij}x_jx_i$. So

$$\begin{aligned} d_{B^!}(\lambda_i) &= \lambda_i\alpha_1 \\ &= (x_j \otimes x_k - q_{jk}x_k \otimes x_j \mapsto -\textstyle\sum_l c_l^{jk}\lambda_i(x_l)) \\ &= (x_j \otimes x_k - q_{jk}x_k \otimes x_j \mapsto -c_i^{jk}) \\ &= \textstyle\sum_{j<k} c_i^{jk}\lambda_j\lambda_k. \end{aligned}$$

So, in the final analysis, the differential $d_{B^!}$ is given by the same constants as those defining the map α_1, and hence the relations on B. We will come back to this example in Section 1.5.

1.3 Twisting cochains

We first give the definition in the non-curved setting, then address the curved situation independently. The following definition is standard and can be found, for example, in Loday and Vallette's text [51].

Definition 1.10 (Twisting cochain)**.** Let A be an augmented dg algebra, with augmentation ϵ, and C be a coaugmented coalgebra, with coaugmentation u. A degree 1 linear map $\pi : C \to A$ is called a twisting cochain if

i) There are containments $u(k) \subset \ker \pi$ and $\mathrm{im}(\pi) \subset \ker \epsilon$.

ii) The map π satisfies the equation $-(d_A\pi + \pi d_C) + \mu(\pi \otimes \pi)\Delta = 0$.

Condition i) is equivalent to the requirement that π factors $C \to C/u(k) \to ker(\epsilon) \to A$. In other sources, the formula in ii) may appears as

$$d_A\pi + \pi d_C + \mu(\pi \otimes \pi)\Delta = 0.$$

One can mediate between the two perspectives by replacing π with $-\pi$. Assuming k is of characteristic $\neq 2$, this alternate form of condition ii) is exactly the statement that π is a solution to the Maurer-Cartan equation

$$d(\pi) + \frac{1}{2}[\pi, \pi] = 0,$$

where $[,]$ denotes the graded commutator on the dg algebra $\mathrm{Hom}_k(C, A)$. We are using this alternate formulation simply because it is the formula that arises naturally in our setting.

Remark 1.11. Despite the fact that the use of twisting cochains in noncommutative algebra is something of a novelty, the idea is not at all new. Twisting cochains appear in works of topologists dating back at least to the 1950's (see [10, 7]).

Definition 1.12 (Twisted homs)**.** Given a twisting cochain $\pi : C \to A$, we define the dg algebra of twisted homs $\mathrm{Hom}_k^{\pi}(C, A)$ as the space of graded homs $\mathrm{Hom}_k^{\pi}(C, A) = \oplus_i \mathrm{Hom}_k^i(C, A)$ along with convolution product $*$ and differential

$$\begin{aligned} d_{\mathrm{Hom}_k^{\pi}(C,A)}(f) &:= d_A f - (-1)^{|f|} f d_C - (\pi * f - (-1)^{|f|} f * \pi) \\ &= d_{\mathrm{Hom}_k(C,A)}(f) - [\pi, f]. \end{aligned}$$

One can easily verify that $\mathrm{Hom}_k^{\pi}(C, A)$ is a in fact a dg algebra, or simply see [51, Proposition 2.1.6]. We also have the analogous definition of the twisted homs $\mathrm{Hom}_k^{\pi}(C, M)$, where M is a dg A-bimodule.

Lemma/Definition 1.13. *Suppose $\pi : C \to A$ is a twisting cochain. There is a functor*

$$(-)^\pi : \text{dg Hom}_\text{k}(\text{C, A})\text{-bimodules} \to \text{dg Hom}^\pi_\text{k}(\text{C, A})\text{-bimodules}.$$

This functor takes a dg bimodule (M, d_M) to the bimodule (M^π, d^π_M) which is M as a graded space, has the $\mathrm{Hom}^\pi_k(C, A)$-action given by the algebra identification $\mathrm{Hom}_k(C, A) = \mathrm{Hom}^\pi_k(C, A)$, and differential $d^\pi_M := d_M - [\pi, -]$. For any $\varphi : M \to N$ we simply take $\varphi^\pi := \varphi$.

Proof. Take $d_H = d_{\mathrm{Hom}_k(C,A)}$ and $d^\pi = d_{\mathrm{Hom}^\pi_k(C,A)}$. For any $m \in M$ we have

$$\begin{aligned} (d^\pi_M)^2(m) &= d^2_M(m) - d_M([\pi, m]) - [\pi, d_M(m)] + [\pi, [\pi, m]] \\ &= -[d_H(\pi), m] + [\pi, d_M(m)] - [\pi, d_M(m)] + [\pi, [\pi, m]] \\ &= -[d_H(\pi), m] + [\pi, [\pi, m]]. \end{aligned} \tag{1.3.1}$$

But

$$[\pi, [\pi, m]] = \pi^2 m - m\pi^2 - (-1)^{|m|}\pi m \pi - (-1)^{|m|+1}\pi m \pi = [\pi^2, m]$$

and $\pi^2 = d_H(\pi)$ by the twisting cochain condition. So the final equation of (1.3.1) vanishes and we get that d^π_M is in fact a differential on M.

We need to check now that the equation $d^\pi_M(fm) = d^\pi(f)m + (-1)^{|f|} f d^\pi_M(m)$ for $f \in \mathrm{Hom}^\pi_k(C, A)$ and $m \in M$. But this is clear since $d^\pi_M = d_M + [\pi, -]$ and d_M and $[\pi, -]$ satisfy the equations

$$d_M(fm) = d_H(f)m + (-1)^{|f|} f d_M(m) \quad \text{and} \quad [\pi, fm] = [\pi, f]m + (-1)^{|f|} f[\pi, m].$$

One similarly verifies the equation for $d^\pi_M(mf)$. Finally, one sees that for any $\varphi : M \to N$, $\varphi^\pi = \varphi$ is still a dg map since for any $m \in M$

$$\varphi(d^\pi_M(m)) = \varphi(d_M(m)) + \varphi([\pi, m]) = d_M(\varphi(m)) + [\pi, \varphi(m)] = d^\pi_M(\varphi(m)).$$

■

Lemma 1.14. *Let $\pi : C \to A$ be a twisting cochain. The tensor complex $A \otimes C \otimes A$ is a dg* $\mathrm{Hom}(C, A)$*-bimodule under the left and right actions*

$$f \cdot (a \otimes c \otimes b) := (-1)^{|f|(|a|+|c_1|)} a \otimes c_1 \otimes f(c_2)b$$

and

$$(a \otimes c \otimes b) \cdot f := (-1)^{|f|(|c|+|b|)} af(c_1) \otimes c_2 \otimes b,$$

for $a, b \in A$, $c \in C$.

Proof. The verification of this fact is a sequence of tedious but straightforward calculations. We only check compatibility with the differential under the left action. Take f and $a \otimes c \otimes b$ as above. Then we have

$$\begin{aligned} & d(f \cdot (a \otimes c \otimes b)) \\ & = \pm d(a) \otimes c_1 \otimes f(c_2)b \pm a \otimes d(c_1) \otimes f(c_2)b \pm a \otimes c_1 \otimes d(f(c_2))b \pm a \otimes c_1 \otimes f(c_2)d(b) \\ & = \pm d(a) \otimes c_1 \otimes f(c_2)b \pm a \otimes d(c_1) \otimes f(c_2)b \pm a \otimes c_1 \otimes f(c_2)d(b) \\ & \qquad \pm a \otimes c_1 \otimes d_{\mathrm{Hom}}(f)(c_2)b + (-1)^{(|f|+1)(|a|+|c_1|)+|f|} a \otimes c_1 \otimes f(d(c_2))b \end{aligned}$$

where all the signs $\pm$ are the appropriate Koszul signs, and all the d must be given the appropriate subscript. Using the fact that d_C is a coderivation, this final equation can then be rewritten

$$(-1)^{|f|} f \cdot d_{A\otimes C\otimes A}(a \otimes c \otimes b) + d_{\mathrm{Hom}}(f) \cdot (a \otimes c \otimes b).$$

The verification of the formula

$$\begin{gathered} d_{A\otimes C\otimes A}((a \otimes c \otimes b) \cdot f) \\ = d_{A\otimes C\otimes A}(a \otimes c \otimes b) \cdot f + (-1)^{|f|(|a|+|c|+|b|)} (a \otimes c \otimes b) \cdot d_{\mathrm{Hom}}(f) \end{gathered}$$

is similar. ■

Definition 1.15 (Twisted tensor products)**.** Given a twisting cochain $\pi : C \to A$, we define the twisted tensor product $A \otimes_\pi C \otimes_\pi A$ as the value of the functor $(-)^\pi$ on the bimodule $A \otimes C \otimes A$ (with right and left actions as in described in Lemma 1.14). In other words,

$$A \otimes_\pi C \otimes_\pi A = (A \otimes C \otimes A)^\pi = (A \otimes C \otimes A, d_{A\otimes C\otimes A} - [\pi, -]).$$

On elements, the differential on the twisted tensor product will be given by the formula

$$\begin{aligned}
&d^{\pi}_{A\otimes C\otimes A}(a\otimes c\otimes b)\\
&= d_{A\otimes C\otimes A}(a\otimes c\otimes b)+(-1)^{|a|+|c_1|}a\otimes c_1\otimes \pi(c_2)b\\
&\qquad -(-1)^{|a|+|c|+|b|+|b|+|c|}a\pi(c_1)\otimes c_2\otimes b\\
&= d_{A\otimes C\otimes A}(a\otimes c\otimes b)+(-1)^{|a|+|c_1|}a\otimes c_1\otimes \pi(c_2)b-(-1)^{|a|}a\pi(c_1)\otimes c_2\otimes b.
\end{aligned}$$

So we can write the differential on the twisted product in the more conventional form

$$d_{A\otimes C\otimes A}+\left(\mu(id_A\otimes\pi)\otimes id_C\otimes id_A-id_A\otimes id_C\otimes\mu(\pi\otimes id)\right)\left(id_A\otimes\Delta\otimes id_A\right).$$

The twisted tensor product $A\otimes_\pi C\otimes_\pi A$ will always have a canonical dg A-bimodule structure under the outer A-actions, and we will view the twisted tensor product as an object in dg A-bimod in what follows.

As a basic example we can consider the bar dg coalgebra which, in the case that A is concentrated in degree 0, appears as

$$\mathscr{B}A=\cdots\to A\otimes A\otimes A\to A\otimes A\to A\to k\to 0$$

with differential given by $d_{\mathscr{B}}|A=0$ and

$$d_{\mathscr{B}}|A^{\otimes n}: a_1\otimes\ldots\otimes a_n\mapsto\sum_i(-1)^i a_1\otimes\ldots\otimes a_ia_{i+1}\otimes\ldots\otimes a_n.$$

In the case that A has nonvanishing differential we get $\mathscr{B}A=\oplus_{i\geq 0}(\Sigma A)^{\otimes i}$ with differential

$$d_{\mathscr{B}}|A^{\otimes n}: a_1\otimes\ldots\otimes a_n\mapsto \begin{array}{l}\sum_i(-1)^{\sum_{l\leq i}(|a_l|-1)}a_1\otimes\ldots\otimes a_ia_{i+1}\otimes\ldots\otimes a_n\\ -d_{A^{\otimes n}}(a_1\otimes\ldots\otimes a_n).\end{array}$$

(Here $|a_i|$ denotes the degree of a_i in A.) We have the canonical twisting cochain univ : $\mathscr{B}A\to A$ which is the identity on $A\subset\mathscr{B}A$ and 0 on all other powers $A^{\otimes i}$. The twisted tensor products $A\otimes_{\text{univ}}\mathscr{B}A\otimes_{\text{univ}}A$ will then recover the standard bar resolutions of [21, 17, 56] and the twisted homs $\operatorname{Hom}_k^{\text{univ}}(\mathscr{B}A,A)$ will recover the standard Hochschild cochain complex, for example. More information on twisting cochains and twisted tensor products can be found in [42, Section 4.3] and [51, Chapter 2].

1.3.1 Twisting cochains in the curved/nonaugmented case

As we saw in Section 1.2.4, if we would like to understand the Koszulity of a nonaugmented algebra we will have to deal with curved dg (co)algebras. Here we follow [69, Section 6.2]. The reader may also refer to [65].

Definition 1.16 (Twisting cochains with curvature)**.** A degree 1 linear map $\pi : C \to A$ from a curved dg coalgebra to a curved dg algebra is called a twisting cochain if the equation

$$(c_A\epsilon_C - 1_A f_C) - (d_A\pi + \pi d_C) + \mu(\pi \otimes \pi)\Delta = 0$$

holds.

Here, again, we differ from some other references by a sign. This discrepancy can be alleviated if we replace π with $-\pi$. We will only be interested in the case in which A is a dg algebra. In this setting we still get a twisted tensor products $A \otimes_\pi C \otimes_\pi A$. The differential on this complex is, oddly enough, given by the same formula as in the non-curved setting:

$$d_{A\otimes C\otimes A} + \left(\mu(id_A \otimes \pi) \otimes id_C \otimes id_A - id_A \otimes id_C \otimes \mu(\pi \otimes id)\right)\left(id_A \otimes \Delta \otimes id_A\right). \tag{1.3.2}$$

As one can see from the above formula, the curvature disappears at this level, and the awkward distinction between the augmented and non-augmented algebras becomes irrelevant. Indeed, for the remainder of the paper we will be able to provide a uniform analysis of the augmented and non-augmented situations. We outline below the manner at which one arrives at the above formula for the twisted tensor product in the curved situation.

Definition 1.17 (Curved bimodules)**.** Given a curved dg algebra $A = (A, d_A, c_A)$, a A-bimodule M is called a curved bimodule if M comes equipped with a grading $M = \oplus_i M^i$ and a degree 1 operation d_M satisfying

$$d_M(am) = d_A(a)m + (-1)^{|a|}ad_M(m) \quad \text{and} \quad d_M(ma) = d_M(m)a + (-1)^{|m|}md_A(a)$$

and $d_M^2 = [c_A, -]$. A morphism of curved bimodules is a graded A-bimodule map $\varphi : M \to N$ which satisfies $d_N\varphi = \varphi d_M$.

Lemma/Definition 1.18. *For any twisting cochain $\pi : C \to A$ from a curved dg coalgebra to a curved dg algebra the complex*

$$\mathrm{Hom}_k^\pi(C, A) = (\mathrm{Hom}_k(C, A), d_{\mathrm{Hom}} - [\pi, -])$$

is a dg algebra.

Proof. Take $d^\pi = d_{\mathrm{Hom}} - [\pi, -]$. Since the sum of algebra derivations is again an algebra derivation the operation d^π is an algebra derivation. We need only check that it is square 0. For homogeneous $g \in \mathrm{Hom}_k(C, A)$ we get

$$\begin{aligned}
(d^\pi)^2(g) &= d_{\mathrm{Hom}}^2(g) - d_{\mathrm{Hom}}([\pi, g]) - [\pi, d_{\mathrm{Hom}}(g)] + [\pi[\pi, g]] \\
&= [c_A\epsilon, g] - [1_A f_C, g] - [d_{\mathrm{Hom}}(\pi), g] + [\pi, d_{\mathrm{Hom}}(g)] - [\pi, d_{\mathrm{Hom}}(g)] + [\pi^2, g] \\
&= [c_A\epsilon, g] - [1_A f_C, g] - [d_{\mathrm{Hom}}(\pi), g] + [\pi^2, g] \\
&= [c_A\epsilon - 1_A f_C - d_{\mathrm{Hom}}(\pi) + \pi^2, g] \\
&= 0,
\end{aligned}$$

since π is a twisting cochain and therefore $c_A\epsilon - 1_A f_C - d_{\mathrm{Hom}}(\pi) + \pi^2 = 0$. ■

Lemma/Definition 1.19. *Given a twisting cochain $\pi : C \to A$ from a curved dg coalgebra to a curved dg algebra, we get a functor*

$$(-)^\pi : \text{curved } \mathrm{Hom_k(C, A)}\text{-bimodules} \to \text{dg } \mathrm{Hom_k^\pi(C, A)}\text{-bimodules}.$$

On objects, for a curved bimodule M we take $M^\pi = M$ as a graded bimodule over $\mathrm{Hom}_k^\pi(C, A) = \mathrm{Hom}_k(C, A)$*, and give M^π the differential $d_M^\pi := d_M - [\pi, -]$. Given $\varphi : M \to N$ we take $\varphi^\pi = \varphi$.*

Recall that the natural curvature on the convolution algebra $\mathrm{Hom}_k(C, A)$ is the function $c_A\epsilon - 1_A f_C$.

Proof. Checking that $d_M^2 = 0$ is formally similar to the computation given for Lemma 1.18. The remainder of the proof is exactly the same as that of Lemma 1.13. ■

Lemma 1.20. *For a (non-curved) dg algebra A and curved dg coalgebra C, the tensor complex $A \otimes C \otimes A$ is a curved bimodule over $\mathrm{Hom}_k(C, A)$ under the same actions as in Lemma 1.14.*

Proof. Save for compatibility with the curvature, this is the same as Lemma 1.14. For compatibility with the curvature we have

$$\begin{aligned} d^2(a \otimes c \otimes b) &= a \otimes d_C^2(c) \otimes b \\ &= a \otimes f_C(c_1)c_2 \otimes b - a \otimes c_1 f_C(c_2) \otimes b \\ &= a \otimes c_1 \otimes (-1_A f_C(c_2))b - a(-1_A f_C(c_1)) \otimes c_2 \otimes b \\ &= [c_{\mathrm{Hom}}, a \otimes c \otimes b]. \end{aligned}$$

■

We then get the twisted tensor product, again, as the value of $(-)^\pi$ on this bimodule

$$A \otimes_\pi C \otimes_\pi A = (A \otimes C \otimes A)^\pi = (A \otimes C \otimes A, d_{A \otimes C \otimes A} - [\pi, -]).$$

The twisted tensor product will be taken to be a dg A-bimodule under the outer A-actions.

1.4 Koszul resolutions via twisting cochains

In this section we give a presentation of Koszul resolutions based on the work of Keller and Lefévre-Hasegawa. The original presentation, in the case that A is graded Koszul, appears in [42, Section 4.7] and [40].

For the section we fix B to be a filtered Koszul algebra and $B^! = (B^!, d_{B^!}, c_{B^!})$ its Koszul dual (curved) dg algebra of Proposition 1.8. Let $\mathrm{gr}B = k\langle V\rangle/(R)$ and $B^! = k\langle V^*\rangle/(R^\perp)$ be dual minimal presentations, and let $\{x_i\}$ and $\{\lambda_i\}$ be dual bases for V and V^* respectively. Recall the intersection coalgebra $W = \cdots \oplus (R \otimes V \cap V \otimes R) \oplus R \oplus V \oplus k$ of Section 1.2.2. The following Lemma was essentially proved in [70].

Lemma 1.21. *There is a natural curved dg structure on W given by $f_W = \alpha_0$ and $d_W|W^{-2} = \alpha_1$ so that the identification $B^! = W^*$ is one of curved dg algebras.*

In the statement of the above lemma we are using the fact that $W^{-2} = R$. The definitions of the α_i are recalled in the first lines of the proof.

Proof. Recall from (1.2.6) that we have presentations

$$\mathrm{gr}B = k\langle V\rangle/(R) \text{ and } B = k\langle V\rangle/(r + \alpha_1(r) + \alpha_0(r))_{r\in R},$$

after choosing some embedding $V \to F_1B$.

Let us suppose that such a curved dg coalgebra structure on W exists, i.e. one satisfying $f_W = \alpha_0$ and $d_W|W^{-2} = \alpha_1$. Then the induced curved dg structure on the chain dual W^* will be given by

$$d_{W^*}|W^{*1} : f \mapsto -(-1)^{|f|}fd_W = f\alpha_1$$

and $c_{W^*} = -f_W = -\alpha_0$. Since d_{W^*} and $d^!_B$ agree on the generators, they agree on all of $B^! = W^*$. So $B^! = W^*$ as a curved dg algebra. To see that a curved dg structure on W satisfying the proposed conditions actually exists, one can verify that the graded coalgebra isomorphism $ev : W \to W^{**} = (B^!)^*$ induces the proposed curved dg structure on W. ■

Definition 1.22 (Koszul dual coalgebra)**.** Given a filtered Koszul algebra B, with $\mathrm{gr}B = k\langle V\rangle/(R)$, the Koszul dual (curved) dg coalgebra to A will be the, possibly curved, dg coalgebra (W, d_W, f_W) of Lemma 1.21. We will often write simply W for (W, d_W, f_W).

Lemma/Definition 1.23 (The twisting cochain π)**.** *Let B be filtered Koszul and W be its Koszul dual (curved) dg coalgebra. Let $\pi : W \to B$ be the composition of the projection $W \to W^{-1} = V$ with the inclusion $V \to B$, $v \mapsto v$. The map $\pi : C \to B$ is a twisting cochain.*

Proof. We need to verify the formula

$$-f_W - \pi d_W + \mu(\pi \otimes \pi)\Delta = 0.$$

It suffices to check that the above equation holds when evaluated at a homogeneous degree -2 element in W, since the left hand side vanishes on elements of all other degrees. Recalling that $W^{-2} = R$, we evaluate on a relation $r = \sum_i r_i \otimes r_i'$ to get

$$\begin{aligned}
&(-f_W - \pi d_W + \mu(\pi \otimes \pi)\Delta)(r) \\
&= -f_W(r) - \pi d_W(r) + \mu(\pi(r) \otimes \pi(1) + \pi(1) \otimes \pi(r) - \textstyle\sum_i \pi(r_i) \otimes \pi(r_i')) \\
&= -\alpha_0(r) - \pi(\alpha_1(r)) - \textstyle\sum_i r_i r_i' \\
&= -\alpha_0(r) - \alpha_1(r) + \alpha_1(r) + \alpha_0(r) \\
&= 0.
\end{aligned}$$

■

One can use [49, Proposition 2.2.4.1] to show that the twisted tensor product

$$B \otimes_\pi W \otimes_\pi B$$

provides a resolution for the regular bimodule ${}_B B_B$. Alternatively, in the case where B is graded Koszul the above resolution is easily seen to recover the standard Koszul resolution [85, proof of Proposition 3.3], [40, Section 4.7]. In general, one can employ the filtration

$$F_i(B \otimes_\pi W \otimes_\pi B) = \sum_{i_1+i_2+i_3=i} F_{i_1}B \otimes W^{-i_2} \otimes F_{i_3}B$$

and an easy spectral sequence argument to see that $H^{<0}(B \otimes_\pi W \otimes_\pi B) = 0$. The fact that $H^0(B \otimes_\pi W \otimes_\pi B) = B$ is apparent.

Notation 1.24 (The Koszul resolution K)**.** If B is filtered Koszul we write $K = K(B)$ for the bimodule resolution $B \otimes_\pi W \otimes_\pi B$ of B.

1.5 A complex calculating $HH^\bullet(A, M)$

For the remainder of the paper by a “twisting cochain” $\pi : C \to A$ we will mean a twisting cochain from a, possibly curved, dg coalgebra to a dg algebra. Recall that a morphism $\varphi : M \to N$ of dg bimodules, over a dg algebra A, is a chain map which is also an A-bimodule map (after forgetting the differential). Recall also our definition of the hom complex $\mathrm{Hom}_{A\text{-bimod}}(M, N)$ from Section 1.2 and our definition

$$\mathrm{Hom}_k^\pi(C, M) = (\mathrm{Hom}_k(C, M), d_{\mathrm{Hom}} - [\pi, -]).$$

We often write simple Hom_{A^e} for $\mathrm{Hom}_{A\text{-bimod}}$.

Proposition 1.25. *Suppose $\pi : C \to A$ is a twisting cochain, and that M is any dg A-bimodule. Then the restriction map*

$$\mathrm{rest}_M : \mathrm{Hom}_{A^e}(A \otimes_\pi C \otimes_\pi A, M) \to \mathrm{Hom}_k^\pi(C, M)$$

is an isomorphism of chain complexes.

Proof. Take d to be the differential on $\mathrm{Hom}_{A^e}(A \otimes_\pi C \otimes_\pi A, M)$ and d' to be the differential on $\mathrm{Hom}_k^\pi(C, M)$. We need to check the formula

$$d(f)|C = d_{\mathrm{Hom}}(f|C) - (\pi * (f|C) - (-1)^{|f|}(f|C) * \pi)$$

for any homogenous A^e-linear map $f : A \otimes_\pi C \otimes_\pi A \to M$. We proceed directly. Take $c \in C$. Then

$$\begin{aligned}
& d(f)(c) \\
& = d_M f(c) - (-1)^{|f|}(f d_{A\otimes C\otimes A}(c) + f(\pi(c_1) \otimes c_2 \otimes 1 - (-1)^{|c_1|} 1 \otimes c_1 \otimes \pi(c_2)) \\
& = d_M f(c) - (-1)^{|f|} f d_C(c) - (-1)^{|f|+|f|(|c_1|+1)} \pi(c_1) f(c_2) + (-1)^{|c_1|+|f|} f(c_1)\pi(c_2) \\
& = d_{\mathrm{Hom}_k(C,M)}(f|C)(c) - ((\pi * f)(c) - (-1)^{|f|}(f * \pi)(c)) \\
& = d_{\mathrm{Hom}_k(C,M)}(f|C)(c) - [\pi, f](c) \\
& = d'(f|C)(c).
\end{aligned}$$

■

Note that restriction is natural in M. So we get a natural isomorphism $\mathrm{rest} : \mathrm{Hom}_k^\pi(C, -) \to \mathrm{Hom}_{A^e}(A \otimes_\pi C \otimes_\pi A, -)$ of functors from dg bimodules to chain complexes.

Corollary 1.26. *Suppose B is filtered Koszul and that W is its associated intersection coalgebra, as in Section 1.2.2. Then we have $H^\bullet(\mathrm{Hom}_k^\pi(W, M)) = HH^\bullet(B, M)$.*

Proof. This follows from the previous Proposition, and the fact that $K = B \otimes_\pi W \otimes_\pi B$ is the Koszul bimodule resolution of B in this case. ■

Let B be filtered Koszul with Koszul dual (curved) dg algebra $B^!$. Recall our dual presentations $\mathrm{gr}B = k\langle V\rangle/(R)$ and $B^! = k\langle V^*\rangle/(R^\perp)$ and dual bases $\{x_i\}$ and $\{\lambda_i\}$ for V and V^* respectively.

Definition 1.27 (The identity element)**.** For B and $B^!$ as above, we take $e := \sum_i \lambda_i \otimes x_i$. We call it the identity element in $B^! \otimes B$

To see that e is invariant under change of basis we can note that e is the preimage of the identity map under the canonical isomorphism $V^* \otimes V \to \mathrm{Hom}_k(V, V)$, $f \otimes x \mapsto (y \mapsto f(y)x)$. (Whence the name "identity element".) Alternatively, we can find e as the image of $1 \in k$ under the standard coevaluation map $k \to V^* \otimes V$ that appears in studies of tensor categories.

Proposition 1.28. *Let B be filtered Koszul, and M be any B-bimodule. Let W be the Koszul dual coalgebra to B. Consider the graded k-linear isomorphism*

$$\tau_M : B^! \otimes M \to \mathrm{Hom}_k^\pi(W, M) \quad f \otimes m \mapsto (c \mapsto f(c)m). \tag{1.5.1}$$

1. *When $M = B$, τ is an isomorphism of dg algebras.*

2. *For arbitrary M, τ induces the differential*

   $$d_{B^! \otimes M} + [e, -] : f \otimes m \mapsto d_{B^!}(f) \otimes m + [e, f \otimes m]$$

   on the graded space $B^! \otimes M$.

Proof. We omit the subscript on τ unless some distinction is needed. (1) Let $f \otimes a$ and $g \otimes b$ be elements in $B^! \otimes B$. For any $c \in W$ we have

$$\begin{aligned} \tau(f \otimes a) * \tau(g \otimes b)(c) &= (-1)^{|c_1||g|} f(c_1) a g(c_2) b \\ &= (-1)^{|c_1||g|} f(c_1) g(c_2) ab \\ &= (fg)(c)ab \\ &= \tau(fg \otimes ab)(\phi). \end{aligned} \tag{1.5.2}$$

(2) For $f \in B^!$, $m \in M$ we have

$$\begin{aligned} d_{\mathrm{Hom}^\pi}(\tau(f \otimes m))(c) &= -(-1)^{|f|} f(d_C(c))m - [\pi, \tau(f)](c) \\ &= d_{B^!}(f)(c)m - [\pi, \tau(f \otimes m)](c). \end{aligned}$$

So

$$d_{\mathrm{Hom}^\pi}(\tau(f \otimes m)) - \tau(d_{B^! \otimes M}(f \otimes m)) = -[\pi, \tau(f \otimes m)]. \tag{1.5.3}$$

Now, a computation completely analogous to (1.5.2) shows that for $f \otimes a \in B^! \otimes A$ and $g \otimes m \in B^! \otimes M$ we have

$$\tau_A(f \otimes a) * \tau_M(g \otimes m) = \tau_M(fg \otimes am) \quad \text{and} \quad \tau_M(g \otimes m) * \tau_A(f \otimes a) = \tau(gf \otimes ma).$$

Since $\tau_A(e) = \pi$, we have $-[\pi, \tau(f \otimes m)] = \tau(-[e, f \otimes m])$ and deduce from (1.5.3) that

$$d_{\mathrm{Hom}^\pi}(\tau(f \otimes m)) = \tau(d_{B^! \otimes M}(f \otimes m) - [e, f \otimes m]).$$

■

Definition 1.29 (The functor $B^!\widetilde{\otimes}-$)**.** Let B be filtered Koszul. We define the functor

$$B^!\widetilde{\otimes}- : B\text{-bimod} \to k\text{-complexes}$$

to be the one sending a bimodule M to $(B^! \otimes M, d_{B^! \otimes M} - [e, -])$, and a bimodule map $M \xrightarrow{\phi} N$ to $id_{B^!} \otimes \phi : B^!\widetilde{\otimes}M \to B^!\widetilde{\otimes}N$.

The maps τ_M of the Proposition 1.28 provide a natural isomorphism $\tau : B^!\widetilde{\otimes}- \xrightarrow{\cong} \mathrm{Hom}_k^\pi(W, -)$. Recalling also the result of Proposition 1.25 gives then a triangle of natural isomorphisms

$$\begin{tikzcd} & \mathrm{Hom}_k^\pi(W, -) \arrow[dr, "\mathrm{rest}"] & \\ B^!\widetilde{\otimes}- \arrow[ur, "\tau"] \arrow[rr, dotted, "\cong", "\exists!"'] & & \mathrm{Hom}_{B^e}(K(B), -), \end{tikzcd}$$

The second portion of the following result provides a slight generalization of [85, Theorem 9.1] to allow for filtered, not just graded, Koszul algebras.

Corollary 1.30. *Given a filtered Koszul algebra B, and B bimodule M we have $H^\bullet(B^!\widetilde{\otimes}M) = HH^\bullet(B,M)$. When $M = {}_BB \otimes B_B$ then the $H^\bullet(B^!\widetilde{\otimes}(B\otimes B)) = H^\bullet(B, B\otimes B)$ as a bimodule.*

Proof. The first statement is immediate. The second statement follows from the fact that the natural isomorphisms $\tau_{B\otimes B}$ and $\mathrm{rest}_{B\otimes B}$ will respect the bimodule structures induced by the inner actions on $B \otimes B$. ■

1.5.1 Example continued: PBW deformations of skew polynomials

Take $k = \mathbb{C}$. Consider $k_Q[x_1, x_2, x_3]$ (from Subsection 1.2.6) with skewing constants $q_{12} = q_{23} = q_{31} = q$ for some $q \neq 0$. This algebra has a particular deformation

$$\bar{B}_q = k\langle x_1, x_2, x_3\rangle/(x_ix_j - qx_jx_i - x_k),$$

where the indices $\{i, j, k\}$ are taken to be cyclically ordered in the relations. In the case that $q = 1$ this algebra is isomorphic to the universal enveloping algebra $\bar{B}_1 = U(sl_2)$ of the Lie algebra sl_2, and when $q = -1$ the algebra is isomorphic to the universal enveloping algebra $U(sl_2^c)$ of the color Lie algebra sl_2^c [12, Section 5]. It was shown in [12] that the category of finite dimensional representation of $U(sl_2^c)$ is semisimple, as is the case with $U(sl_2)$. And so we know a bit about its cohomology of $\bar{B}_q$ at these points.

Let us suppose now that $q \neq 0, 1$ (so $\bar{B}_q \neq U(sl_2)$) and employ the isomorphic presentation

$$B_q = k\langle x_1, x_2, x_3\rangle/(x_ix_j - qx_jx_i - (1-q)x_k),$$

where the $\{i, j, k\}$ are cyclically ordered. The algebra B_q has two specific 1-dimensional representation p_0 and p_1, which are both k as a vector space with left actions $x_i \cdot 1 = 0$ and $x_i \cdot 1 = 1$ respectively. We also have the one dimensional bimodule $p_{10} = \mathrm{Hom}_k(p_0, p_1)$ which is k as a vector space with left and right actions $x_i \cdot 1 = 1$ and $1 \cdot x_i = 0$ respectively.

Recall that we have an isomorphism $\mathrm{Ext}_{B_q}(p_0, p_1) = HH^\bullet(B_q, p_{10})$ [88, Lemma 9.1.9]. Recall also that $B^!$ is the dg algebra $\bigwedge_Q\langle\lambda_1, \lambda_1, \lambda_3\rangle$ with differential

$$d_{B^!} : \lambda_i \mapsto (1-q)\lambda_j\lambda_k, \text{ where } \{i, j, k\} \text{ is cyclically ordered}$$

(see Section 1.2.6). This algebra is concentrated in degrees 0 to 3, and $\dim(B^!)^0 = \dim(B^!)^3 = 1$ while $\dim(B^!)^1 = \dim(B^!)^2 = 3$.

One can check $d^2_{B^!} = d^0_{B^!} = 0$. So, according to Corollary 1.30, we can calculate $\mathrm{Ext}_{B_q}(p_0, p_1)$ as the homology of the complex $B^! \otimes p_{10}$ with differential

$$\begin{aligned}
d^0(1\otimes 1) &= -(\textstyle\sum_i \lambda_i \otimes x_i \cdot 1 - \lambda_i \otimes 1 \cdot x_i)\\
&= -(\lambda_1 \otimes 1 + \lambda_2 \otimes 1 + \lambda_3 \otimes 1)\\
d^1(\lambda_i \otimes 1) &= d_{B^!}(\lambda_i) \otimes 1 - \lambda_j\lambda_i \otimes 1 - \lambda_k\lambda_i \otimes 1\\
&= (1-q)\lambda_j\lambda_k \otimes 1 + q\lambda_i\lambda_j \otimes 1 - \lambda_k\lambda_i \otimes 1\\
d^2(\lambda_i\lambda_j \otimes 1) &= \lambda_k\lambda_i\lambda_j \otimes 1,
\end{aligned}$$

where again $\{i, j, k\}$ is taken to be cyclically ordered. The map d^0 is injective while the map d^2 is surjective. So, $\mathrm{Ext}^1_{B_q}(k,k) = \mathrm{Ext}^2_{B_q}(k,k) = 0$ if and only if the map d^1 is rank 2.

In the ordered bases $\{\lambda_1, \lambda_2, \lambda_3\}$ and $\{\lambda_2\lambda_3, \lambda_3\lambda_1, \lambda_1\lambda_2\}$ the differential d^1 is given by the matrix

$$d^1 = \begin{bmatrix} (1-q) & q & -1 \\ -1 & (1-q) & q \\ q & -1 & (1-q) \end{bmatrix}.$$

This matrix is of rank 1 or 2, and is rank 2 if and only if q solves the equation $q^2 - q + 1 = 0$, i.e. if and only if $q = e^{\pm\pi/3}$. So

$$\mathrm{Ext}^1_{B_q}(p_0, p_1) = k \text{ when } q = e^{\pm\pi/3} \text{ and } \mathrm{Ext}^1_{B_q}(p_0, p_1) = 0 \text{ otherwise.}$$

Whence we conclude that there exists some non-split extension $0 \to p_1 \to M \to p_0 \to 0$ exactly when $q = e^{\pm\pi/3}$.

1.6 *The cup product on* $HH^\bullet(A)$

For any twisting cochain $\pi : C \to A$ from a curved dg coalgebra to a dg algebra, we have a canonical map

$$\varepsilon : A \otimes_\pi C \otimes_\pi A \to A, \quad a \otimes c \otimes b \mapsto ab\epsilon_C(c)$$

of dg bimodules. Indeed, we have

$$\begin{aligned}
&\varepsilon(d(a\otimes c\otimes b))\\
&= d_A(a)b\epsilon(c) + (-1)^{|a|+|c|}ad_A(b)\epsilon(c) + (-1)^{|a|}ab\epsilon(d_C(c))\\
&\quad +(-1)^{|a|}a\pi(c_1)\epsilon(c_2)b - (-1)^{|a|+|c_1|}a\epsilon(c_1)\pi(c_2)b\\
&= d_A(a)b\epsilon(c) + (-1)^{|a|+|c|}ad_A(b)\epsilon(c)\\
&\quad +(-1)^{|a|}a\pi(c_1\epsilon(c_2))b - (-1)^{|a|+|c_1|}a\pi(\epsilon(c_1)c_2)b.
\end{aligned}$$

Recalling that $\epsilon(c) = 0$ whenever $|c| \neq 0$ and $c = \epsilon(c_1)c_2 = c_1\epsilon(c_2)$, the final equation reduces to

$$\begin{aligned}
&d_A(a)b\epsilon(c) + (-1)^{|a|}ad_A(b)\epsilon(c) + (-1)^{|a|}a\pi(c)b - (-1)^{|a|}a\pi(c)b\\
&= d_A(ab)\epsilon(c)\\
&= d_A(ab\epsilon(c)) = d_A(\varepsilon(a\otimes c\otimes b)).
\end{aligned}$$

Recall that a coalgebra C is called connected if it has a unique one dimensional simple subcoalgebra $C_0 = k$ [62, Definition 5.1.5]. We say a curved dg algebra $C = (C, d_C, f_C)$ is connected, or "cocomplete", if it is connected as a coalgebra and $d_C|C_0 \subset C_0$. Compatibility of d_C with the counit implies $d_C|C_0 = 0$. In this case the standard coradical filtration

$$\{F_nC := \ker(C \overset{\Delta^{(n)}}{\to} C^{\otimes n} \to (C/C_0)^{\otimes n})\}_n$$

will also satisfy $d_C(F_nC) \subset F_nC$. The following definition is standard [49, Definitions 2.2.1.1], [69, Section 6.5], [42, Section 4.6].

Definition 1.31 (Acyclic twisting cochains)**.** A twisting cochain $\pi : C \to A$ is called admissible if C is a connected curved dg coalgebra and $\pi|C_0 = 0$. A twisting cochain $\pi : C \to A$ is called acyclic if π is admissible and the dg A-bimodule map

$$\varepsilon : A \otimes_\pi C \otimes_\pi A \to A$$

is a quasi-isomorphism.

Our main example of an acyclic twisting cochain will be the canonical twisting cochain $\pi : W \to B$ of Lemma 1.23, which is associated to any filtered Koszul algebra B. In this case the map $\varepsilon : K = B \otimes_\pi W \otimes_\pi B \to B$ simply projects on to the degree 0 component $K^0 = B \otimes B$ then multiplies $B \otimes B \to B$. That is to say, ε is the usual quasi-isomorphism $K \to B$ giving K the structure of a free bimodule resolution of B.

Theorem 1.32. *Let $\pi : C \to A$ be any twisting cochain and take $K = A \otimes_\pi C \otimes_\pi A$. The map $l : \mathrm{Hom}_k^\pi(C, A) \to \mathrm{Hom}_{A^e}(K, K)$ defined by*

$$f \mapsto \big(a \otimes c \otimes b \mapsto (-1)^{|f|(|a|+|c_1|)} a \otimes c_1 \otimes f(c_2)b\big) \tag{1.6.1}$$

is a map of dg algebras. Furthermore, if π is acyclic then the map l is a quasi-isomorphism and we have an identification of graded rings $HH^\bullet(A) = H^\bullet(\mathrm{Hom}_k^\pi(C, A))$.

We prove the second portion of the theorem in Section 1.8 (where the Hochschild cohomology is formally defined for general dg algebras). Here we only give the complete proof in the case that A is concentrated in degree 0 and C is bounded above, which may be sufficient for ring theorists. After establishing the necessary background, the proof for dg algebras will be exactly the same.

(Partial) proof. Take $K = A \otimes_\pi C \otimes_\pi A$. Note that for any dg algebra Π and any left dg module M the formula $d_M(\sigma m) = d_\Pi(\sigma)m + (-1)^{|\sigma|}\sigma d_M(m)$, for any $\sigma \in \Pi$ and $m \in M$, is exactly the statement that the left multiplication map $l_\Pi : \Pi \to \mathrm{Hom}_k(M, M)$ is a chain map. Associativity of the action tells us that the left multiplication map is also an algebra, and hence dg algebra, map. Therefore we get a dg algebra map

$$l_{\mathrm{Hom}_k^\pi} : \mathrm{Hom}_k^\pi(C, A) \to \mathrm{Hom}_k(K, K)$$

given by the formula (1.6.1), since $K = A \otimes_\pi C \otimes_\pi A = (A \otimes C \otimes A)^\pi$ is a left $\mathrm{Hom}_k^\pi(C, A)$-module under the action given in Lemmas 1.19 and 1.20. We simply note that each map $l_{\mathrm{Hom}_k^\pi}(f)$ is left and right A-linear to see that the image of $l_{\mathrm{Hom}_k^\pi}$ lay in the dg subalgebra $\mathrm{Hom}_{A^e}(K, K) \subset \mathrm{Hom}_k(K, K)$. This produces l as the dg algebra map given by restricting the codomain of $l_{\mathrm{Hom}_k^\pi}$.

In the case that π is acyclic, C is bounded above, and A is concentrated in degree 0, the complex K provides a free bimodule resolution of A. Whence the functor $\mathrm{Hom}_{A^e}(K,-)$ preserves quasi-isomorphisms. In particular, the map

$$\varepsilon_*:\mathrm{Hom}_{A^e}(K,K)\to\mathrm{Hom}_{A^e}(K,A)\underset{\mathrm{rest}}{\cong}\mathrm{Hom}^\pi_k(C,A)$$

will be a quasi-isomorphism. (Recall that the restriction map is a chain isomorphism, by Proposition 1.25.) Since $\epsilon(c_1)f(c_2)=f(\epsilon(c_1)c_2)=f(c)$ for each $c\in C$ we see that $\varepsilon_*l=id_{\mathrm{Hom}^\pi_k(C,A)}$. Since ϵ_* is a quasi-isomorphism this then implies that l is also a quasi-isomorphism. ■

Corollary 1.33. *Suppose B is filtered Koszul, W and $B^!$ are its Koszul dual (curved) dg coalgebra and algebra respectively, and $\pi:W\to B$ is the canonical twisting cochain of Lemma 1.23. Then we have identifications of graded algebras*

$$HH^\bullet(B)=H^\bullet(\mathrm{Hom}^\pi_k(W,B))$$

and

$$HH^\bullet(B)=H^\bullet(B^!\widetilde{\otimes}B)=H^\bullet((B^!\otimes B,d_{B^!\otimes B}-[e,-])),$$

where e is the identity element of Definition 1.27.

Proof. As noted above, in this case π will be acyclic and $K=B\otimes_\pi W\otimes_\pi B$ will be bounded above and free. So Theorem 1.32 tells us that the map

$$H^\bullet(l):H^\bullet(\mathrm{Hom}^\pi_k(W,B))\overset{\cong}{\to}H^\bullet(\mathrm{End}_{B^e}(K))=HH^\bullet(B)$$

is an isomorphism of graded algebras. The second statement follows from the fact that $\mathrm{Hom}^\pi_k(W,B)$ is isomorphic to $B^!\widetilde{\otimes}B$ as a dg algebra by Lemma 1.28. ■

We remark that the above corollary enjoys some flexibility. For example, if we consider any (Ore) localization BS^{-1} of B, according to some Ore set S. In this case the map

$W \to BS^{-1}$ given by the composite $W \xrightarrow{\pi} B \to BS^{-1}$ is still an acyclic twisting cochain. So we get

$$HH^\bullet(BS^{-1}) = H^\bullet((B^! \otimes BS^{-1}, d_{B^! \otimes BS^{-1}} - [e, -])).$$

We leave the details to the curious reader.

Remark 1.34. Let $C^\bullet(A) = \mathrm{Hom}_k^{\mathrm{univ}}(\mathscr{B}A, A)$ denote the Hochschild cochain complex for A. Theorem 1.32 can alternately be proved by showing that the quasi-isomorphism $C^\bullet(A) \to \mathrm{Hom}_{A^e}(K, A)$ dual to the canonical embedding of K into the bar resolution for A (see [71, Proposition 3.9]) maps the cup product of elements in $C^\bullet(A)$ to the convolution product of their images. This will be a more appropriate proof if one wishes to address the cup product on Hochschild cohomology with coefficients in some ring extension A' of A.

Remark 1.35. It seems as though the most readily generalizable result is the identification

$$HH^\bullet(A) = H^\bullet(\mathrm{Hom}_k^\pi(C, A)).$$

For example, if one is interested in moving away from the (graded) Koszul case to the general case of connected graded algebras, we should replace the Koszul dual algebra $B^!$ with the Koszul dual A_∞-algebra (see [53]). Taking the dual $B^!$ will give an A_∞-coalgebra W which will be connected to B via an A_∞-twisting cochain $\pi : W \to B$ (similar to the [42, Section 4.4]). It may then be the case that we still have that the cohomology of the twisted homs $\mathrm{Hom}_k^\pi(W, B)$ is the Hochschild cohomology (A_∞-)algebra. This is also the easiest statement to interpret and generalize if one wishes to work over a noncommutative separable base as opposed to a field. This would allow us to address, for example, Koszul path algebras.

1.7 A Remark on A_∞-Structures

Definition 1.36 (A_∞-algebra)**.** An A_∞-algebra is a graded space $\Pi = \oplus_i \Pi^i$ equipped with operations

$$m_n : \Pi^{\otimes n} \to \Pi,$$

for all $n \geq 1$, of respective degrees $2-n$, satisfying the equations

$$0 = \sum_{r+s+t=n} m_{n-s+1}(id^{\otimes r} \otimes m_s \otimes id^{\otimes t})$$

for all $n > 0$.

Some introductory notes on A_∞-algebras can be found in [39, 42, 53]. We refer the reader to these articles for basic results and detailed references. Some applications and calculations of A_∞-structures appear in [45, 54, 55].

A standard fact, due to Kadeishvili, states that the homology of any dg algebra has a unique A_∞-structure (up to non-unique isomorphism). Therefore any Ext algebra $\mathrm{Ext}_A(M, M)$, for any algebra A and A-module M, has an A_∞-structure. This follows from the fact that, if we let P be a projective resolution of M, we have $\mathrm{Ext}_A(M, M) = H^\bullet(\mathrm{Hom}_A(P, P))$. The A_∞-structure on $\mathrm{Ext}_A(M, M)$ is independent of our choice of resolution P. Furthermore, it lifts the algebra structure on $\mathrm{Ext}_A(M, M)$ in the sense that m_2 is simply the Yoneda product. Another standard fact is that a quasi-isomorphism of dg algebras induces an isomorphism of A_∞-algebras on their homologies.

In the Section 1.8 we will extend to above sequence of claims–which apriori only hold for non-dg algebras and modules–to the dg setting. We also prove the following stronger version of Theorem 1.32. Or more precisely, we explain that the completed proof of Theorem 1.32 already verifies the following

Theorem 1.37. *If $\pi : C \to A$ is an acyclic twisting cochain then we have an identification of A_∞-algebras $HH^\bullet(A) = H^\bullet(\mathrm{Hom}_k^\pi(C, A))$. In particular, if $\pi : W \to B$ is the canonical acyclic twisting cochain associated to a filtered Koszul algebra then the identification*

$$HH^\bullet(B) = H^\bullet(\mathrm{Hom}_k^\pi(W, B)) = H^\bullet(B^! \otimes B, d_{B^! \otimes B} - [e, -])$$

is one of A_∞-algebras.

In the non-dg instance, the above theorem immediately follows from the fact that we have the quasi-isomorphism of dg algebras $\mathrm{Hom}_k^\pi(C, A) \to \mathrm{Hom}_{A^e}(K, K)$ and $HH^\bullet(A) = H^\bullet(\mathrm{Hom}_{A^e}(K, K))$ as an A_∞-algebra.

1.8 Hochschild cohomology of dg algebras and the complete proof of Theorem 1.32

Here we give an an overview of some of the definitions and results from Barthel, May, and Riehl's paper [5], and complete the proof of Theorem 1.32. The paper [5] is concerned with analyzing a number of model structures on categories of dg modules. We will, however, avoid discussing model categories at length. For the moment, let us only say that since we are over a field, the (unbounded) derived category of k is equal to the (unbounded) homotopy category of k. This implies that the q-model structure and r-model structure from [5] are actually the same. So we can use the authors results for the r-model structure to address the standard derived category of a dg algebra Π,

$$D(\Pi) = \mathrm{Ho}(\text{dg } \Pi\text{-mod}) = \text{dg } \Pi\text{-mod}[\mathit{Quasi\text{-}isom}^{-1}].$$

We would also like to advertise this recent paper [5] as complementary to Keller's standard reference on derived categories in the dg setting [38]. We fix some dg algebra Π for the moment.

Definition 1.38 (Semi-projective dg modules)**.** A dg Π-module M is called semi-projective (q-semi-projective in [5]) if M is projective as a Π-module, after forgetting the differential, and the hom complex $\mathrm{Hom}_\Pi(M, N)$ is acyclic whenever N is acyclic.

Since the construction of the mapping cone commutes with the hom complex functor, we see that M is semi-projective if and only if it is projective as a (non-dg) Π-module and $\mathrm{Hom}_B(M, -)$ preserves quasi-isomorphism. So, in this case, $\mathrm{Hom}_\Pi(M, -)$ induces a functor on the localizations $D(\Pi) \to D(k)$.

As one might guess, if we would like to derive the functor $\mathrm{Hom}_\Pi(M, -)$, for some possibly non-semi-projective dg module M, we should take a semi-projective approximation $\tilde{M} \overset{\sim}{\to} M$ (defined explicitly below) and take $\mathrm{RHom}_\Pi(M, -) := \mathrm{Hom}_\Pi(\tilde{M}, -)$. Indeed, in the model structure on Π-mod given in [5] the semi-projective modules are the fibrant/cofibrant objects [5, Theorem 9.10]. So this construction of the derived functor agrees with the usual model theoretic construction. We will return to this point below.

Definition 1.39 (Split filtrations)**.** A split filtration of a dg Π-module M is a filtration $M = \cup_i F_i M$ with each $F_i M$ a dg submodule, $F_{-1}M = 0$, and each quotient $F_i M / F_{i-1} M$ isomorphic to a dg bimodule of the form $\Pi \otimes E$ for some k-complex E.

We are principally interested in the following

Proposition 1.40 ([5, Propositions 9.19])**.** *Let M be a dg Π-module. If M admits a split-filtration then M is semi-projective.*

Proof. Since any acyclic k-complex is contractible, and any complex over k is projective over k, we see that the notions of r-semi-projectivity and q-semi-projectivity, in the sense of [5], agree. The statement now follows directly from [5, Propositions 9.19], which states that if M has a split filtration then M is r-semi-projective. ■

We now turn our attention back to bimodules over a dg algebra A, in which case $\Pi = A^e$.

Proposition 1.41. *If $\pi : C \to A$ is an admissible twisting cochain then the dg bimodule $A \otimes_\pi C \otimes_\pi A$ is semi-projective over A^e.*

This result was inspired by [5, Proposition 10.18], which initiated my interest in the paper [5]. The proof is also rather similar.

Proof. We filter by the coradical filtration on C. More specifically, we take

$$F_i = F_i(A \otimes_\pi C \otimes_\pi A) := A \otimes F_i C \otimes A.$$

Since the differential on C is filtered, as is the comultiplication, the differential on the twisted tensor complex does respect this filtration. In fact, from the formula

$$\begin{aligned} d_{A\otimes_\pi C \otimes_\pi A} &= d_{A\otimes C\otimes A} - [\pi, -] \\ &= d_{A\otimes C\otimes A} + \big(\mu(id_A \otimes \pi) \otimes id_C \otimes id_A - id_A \otimes id_C \otimes \mu(\pi \otimes id)\big)\,(id_A \otimes \Delta \otimes id_A) \end{aligned}$$

for the differential on the twisted tensor product, and the fact that $\pi | C_0 = 0$, we see that the portion $[\pi, -]$ of the differential vanishes in the associated graded complex. So we have

$$F_i / F_{i-1} \cong A \otimes (F_i C / F_{i-1} C) \otimes A$$

as an A-bimodule, where the differential is the product differential. Whence, by Proposition 1.40 the complex is semi-projective. ■

We will call a map $p : \tilde{M} \to M$ of dg Π-modules a semi-projective approximation of M if $\tilde{M}$ is semi-projective and p is a surjective quasi-isomorphism. This notions correspond p being an acyclic fibration.

Lemma 1.42. *Let $\pi : C \to A$ be a twisting cochain. If π is acyclic then $\varepsilon : A \otimes_\pi C \otimes_\pi A \to A$ is a semi-projective approximation.*

Proof. In light of the previous Proposition, we need only show that the map ε is surjective. However, this is clear since we have the k-section

$$A \to A \otimes A \cong A \otimes C_0 \otimes A \to A \otimes C \otimes A, \quad a \mapsto a \otimes 1 \otimes 1.$$

■

There is now an obvious definition of the Hochschild cohomology, at least from the perspective of homological algebra and derived functors.

Definition 1.43 (Hochschild cohomology of a dg algebra)**.** Let A be a dg algebra. We define the Hochschild cohomology $HH^\bullet(A)$ as the cohomology of the hom complex $\mathrm{Hom}_{A^e}(\tilde{A}, \tilde{A})$, where $\tilde{A} \to A$ is a semi-projective approximation of A over A^e.

Obviously each of these hom complexes $\mathrm{End}_{A^e}(\tilde{A})$ will be a dg algebra. So the Hochschild cohomology still admits a natural product. This graded ring structure is well defined and independent of choice of resolution. Further, we also find that the A_∞-structure is well defined (up to isomorphism). In fact, the A_∞-structures on extension rings are well defined in general.

Proposition 1.44. *Let Π be a dg algebra and suppose $X \to M$ and $Y \to M$ are semi-projective approximations of a dg module M. Then there is a dg algebra $\mathscr{D}$ which admits a dg algebra quasi-isomorphisms $\mathrm{End}_\Pi(X) \to \mathscr{D}$ and $\mathrm{End}_\Pi(Y) \to \mathscr{D}$ so that $H^\bullet(\mathrm{End}_\Pi(X)) \cong H^\bullet(\mathrm{End}_\Pi(Y))$ as A_∞-algebras.*

Proof. We construct $\mathscr{D}$ and a quasi-isomorphism $\mathrm{End}_\Pi(X) \to \mathscr{D}$. Take $Q = X \oplus Y$. Take any lift $l : Y \to X$ of the map $Y \to M$ and let $f : Q \to X$ be the coproduct map $f = [l\ id_X]$. Let K be a cofibrant (semi-projective) approximation of the kernel of the composite map $Q \to X \to M$, and let i denote the map $K \to Q$.

Now, since $fi : K \to X$ is a chain map which is in the kernel of the induced map $\mathrm{Hom}_\Pi(K, X) \to \mathrm{Hom}_\Pi(K, M)$, and the induced map on homs is a quasi-isomorphisms, we conclude that fi is a coboundary in $\mathrm{Hom}_\Pi(K, X)$. That is to say, there is some degree zero $\sigma : \Sigma K \to X$ with $d_X\sigma - \sigma d_{\Sigma K} = fi$. Now if we take the map $F : \mathrm{cone}(i) \to X$ to be the coproduct map $F = [\sigma\ f]$ we find that F is a chain map with $F|Q = f$ and $F|X = id_X$. Whence X is a summand on $\mathrm{cone}(i)$ with the split inclusion $\mathrm{incl} : X \to \mathrm{cone}(i)$ given by the composite $X \to Q \to \mathrm{cone}(i)$. Furthermore, we have a diagram

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & \mathrm{cone}(i) \arrow[dl] \\ & M & \end{tikzcd} \tag{1.8.1}$$

By basic homological algebra the map $\mathrm{cone}(i) \to M$ is a (surjective) quasi-isomorphism so that $\mathrm{cone}(i) \to M$ is a cofibrant approximation. It follows from the diagram (1.8.1) then that the inclusion $X \to \mathrm{cone}(i)$ is a quasi-isomorphism, as is $F : \mathrm{cone}(i) \to X$. Since quasi-isomorphisms of fibrant/cofibrant objects are homotopy equivalences, we also understand that incl and F are homotopy equivalences. Whence the map

$$\mathrm{End}_\Pi(X) \to \mathrm{End}_\Pi(\mathrm{cone}(i)), \quad \phi \mapsto \mathrm{incl}\phi F$$

is a quasi-isomorphism which also happens to be a dg algebra map. So we can take $\mathscr{D} = \mathrm{End}_\Pi(\mathrm{cone}(i))$ and we are done.

The same argument will also produce a map $\mathrm{End}_\Pi(Y) \to \mathscr{D}$. One should note however that there is some strangeness to consider since the original map $Y \to M$ and the composite $Y \to Q \to X \to M$ will be homotopy equivalent but not necessarily equal. This will cause the diagram (1.8.1) to only be homotopy commutative, rather than commutative–a point which is irrelevant to the argument. ∎

We now complete the proof of Theorem 1.32. This theorem, along with Lemma 1.42 and Proposition 1.44 will verify that when $\pi : C \to A$ is acyclic we have $H^\bullet(\mathrm{Hom}_k^\pi(C, A)) = HH^\bullet(A)$ as A_∞-algebras.

Completed proof of Theorem 1.32. By Lemma 1.42 $\varepsilon : K \to A$ will be a semi-projective approximation of A. Thus the induced map

$$\varepsilon_* : \mathrm{Hom}_{A^e}(K, K) \to \mathrm{Hom}_{A^e}(K, A) \cong \mathrm{Hom}_k^\pi(C, A)$$

will be a quasi-isomorphism, and we can simply repeat the latter half of the proof of Theorem 1.32 to get the desired result. ■

In closing, let us say a few words about the Hochschild cohomology of a dg algebra. For any dg algebra A we will always have the bar dg coalgebra $\mathscr{B}A$ and universal twisting cochain $\mathrm{univ} : \mathscr{B}A \to A$. We then get the standard map

$$\varepsilon : \mathscr{B}ar A = A \otimes_{\mathrm{univ}} \mathscr{B}A \otimes_{\mathrm{univ}} A \to A,$$

which is a quasi-isomorphism since the mapping cone has a canonical contracting homotopy. Then we get, by Theorem 1.32, that

$$HH^\bullet(A) = H^\bullet(\mathrm{Hom}_k^{\mathrm{univ}}(\mathscr{B}A, A)).$$

But $\mathrm{Hom}_k^{\mathrm{univ}}(\mathscr{B}A, A)$ is the standard Hochschild cochain complex. So our derived functor version of the Hochschild cohomology is the same as the deformation theoretic Hochschild cohomology. In particular, $\Sigma HH^\bullet(A)$ admits a graded Lie structure under which the solutions to the Maurer-Cartan equation correspond to infinitesimal deformations of A (see, for example, [46, Section 6.3]).

1.9 Example: the Heisenberg Lie algebra

Suppose k is characteristic 0. Let $U = U(h)$ be the universal enveloping algebra of the Heisenberg Lie algebra h. Explicitly, h is the 3 dimensional Lie algebra $h = \langle x_1, x_2, x_3 \rangle$ with

$x_3 = [x_1, x_2]$ and $[x_1, x_3] = [x_2, x_3] = 0$. The dg algebra dual to U is the exterior algebra $B^! = \bigwedge\langle\lambda_1, \lambda_2, \lambda_3\rangle$, where the element λ_i is dual to x_i. The differential on $B^!$ sends λ_3 to $\lambda_1\lambda_2$ and all other monomials to 0.

It is appropriate to mention, before beginning, that the Hochschild *homology* of $U(h)$ is calculated in Nuss' thesis [66, Theorem 3.2, pg. 48], as a vector space. One can then use Van den Bergh's duality [87] to deduce the vector space structure on Hochschild cohomology. The presentation by Nuss does not look especially similar to the one given here, particularly in degrees 1 and 2. However, the two presentations of $HH^\bullet(U)$ are abstractly isomorphic, simply because they are both of countable dimension in each degree.

In the five subsections below we demonstrate, in detail, the six points of the following theorem.

Theorem 1.45. *Let $h = \langle x_1, x_2, x_3\rangle$ denote the Heisenberg Lie algebra and $U = U(h)$ denote its universal enveloping algebra. Take $\mathcal{Z} = k[x_3]$. We have the following description of $HH^\bullet(U)$:*

1. *$HH^0(U) = \mathcal{Z}$.*
2. *$HH^1(U)$ is the direct sum $FH^1 \oplus TH^1$ of a free, infinitely generated, $\mathcal{Z}$-module and free, infinitely generated, $\mathcal{Z}/(x_3)$-module.*
3. *$HH^2(U)$ is a free, infinitely generated $\mathcal{Z}/(x_3^2)$-module.*
4. *$HH^3(U) = \lambda_1\lambda_2\lambda_3 \otimes U/(x_3)$, i.e. is an infinitely generated $\mathcal{Z}/(x_3)$-module.*
5. *The cup product $HH^1(U) \otimes HH^1(U) \to HH^2(U)$ has image $x_3 \cdot HH^2(U)$.*
6. *The cup product $HH^1(U) \otimes HH^2(U) \to HH^3(U)$ is surjective.*

Note that points (2), (3) and (5) together imply that $HH^\bullet(U)$ is infinitely generated as a $\mathcal{Z}$-algebra with an infinite number of generators in degrees 1 and 2.

Of course, our statement that a $\mathcal{Z}$-module is a free $\mathcal{Z}/(x_3^n)$-module it to say that it is annihilated by x_3^n and free after we mod out by the annihilator. The details are given below.

1.9.1 The dg algebra $U^! \widetilde{\otimes} U$ and $HH^0(U)$

Since it will simplify our presentation, we simply give the center $HH^0(U)$ here, before we begin our computations. The proof is omitted, although it can easily be verified, especially from the presentation of d^0 given below.

Lemma/Definition 1.46. *The center of U, which we denote by $\mathcal{Z}$, is the polynomial ring $k[x_3]$. We have $HH^0(U) = \mathcal{Z}$.*

Note that $[x_1, -]$ is a derivation on U with $[x_1, x_i] = \delta_{i2} x_3$. Similarly, $[x_2, x_j] = -\delta_{j2} x_3$ and $[x_3, -] = 0$, since x_3 is central. Indeed, for a monomial $a = x_1^{n_1} x_2^{n_2} x_3^{n_3}$ we have

$$[x_1, a] = n_2 x_1^{n_1} x_2^{n_2 - 1} x_3^{n_3 + 1} \text{ whenever } n_2 > 0$$

and

$$[x_2, a] = -n_1 x^{n_1 - 1} x_2^{n_2} x_3^{n_3 + 1} \text{ whenever } n_1 > 0.$$

So, under the vector space isomorphism $U = k[x_1, x_2, x_3]$ the operations $[x_1, -]$ and $[x_2, -]$ can be written as the differential operators $x_3 \partial_2$ and $-x_3 \partial_1$ respectively. As far as computing the homology as a $\mathcal{Z}$-module, we identify U with the simultaneous h-representation and $\mathcal{Z}$-module (or simply $\mathcal{Z} \otimes h$-representation) $k[x_1, x_2, x_3]$ where h acts by the prescribed operators.

Observe that $U = k[x_1, x_2, x_3]$ is free over $\mathcal{Z}$ with basis given by monomials in x_1 and x_2. Whence $U^! \widetilde{\otimes} U$ is seen to have a $\mathcal{Z}$-basis of monomials $\{\lambda^M \otimes x^N\}$, for $M = (m_1, m_2, m_3)$ and $N = (n_1, n_2)$ and $\lambda^M \otimes x^N = \lambda_1^{m_1} \lambda_2^{m_2} \lambda_3^{m_3} \otimes x_1^{n_1} x_2^{n_2}$.

The dg algebra $U^! \widetilde{\otimes} U$ calculating $HH^\bullet(U)$ is the tensor algebra

$$\bigwedge \langle \lambda_1, \lambda_2, \lambda_3 \rangle \otimes U = U \bigoplus \langle \lambda_1, \lambda_2, \lambda_3 \rangle \otimes U \bigoplus \langle \lambda_i \lambda_j \rangle_{ij} \otimes U \bigoplus \langle \lambda_1 \lambda_2 \lambda_3 \rangle \otimes U.$$

The differential is specified on the $\mathcal{Z}$-basis $\lambda^M \otimes x^N$ by

$$\begin{aligned} d^0(x^N) &= -(\textstyle\sum_i \lambda_i \otimes x_i x^N - \lambda_i \otimes x^N x_i) \\ &= -\lambda_1 \otimes [x_1, x^N] - \lambda_2 \otimes [x_2, x^N] - \lambda_3 \otimes [x_3, x^N] \\ &= -\lambda_1 \otimes [x_1, x^N] - \lambda_2 \otimes [x_2, x^N] \\ &= -\lambda_1 \otimes x_3 \partial_2(x^N) + \lambda_2 \otimes x_3 \partial_1(x^N). \end{aligned} \tag{1.9.1}$$

in degree 0,

$$\begin{aligned} d^1(\lambda_1 \otimes x^N) &= -\lambda_2\lambda_1 \otimes [x_2, x^N] - \lambda_3\lambda_1 \otimes [x_3, x^N] \\ &= -\lambda_1\lambda_2 \otimes x_3\partial_1(x^N) \\ d^1(\lambda_2 \otimes x^N) &= -\lambda_1\lambda_2 \otimes [x_1, x^N] \\ &= -\lambda_1\lambda_2 \otimes x_3\partial_2(x^N) \\ d^1(\lambda_3 \otimes x^N) &= \lambda_1\lambda_2 \otimes x^N - \lambda_1\lambda_3 \otimes [x_1, x^N] - \lambda_2\lambda_3 \otimes [x_2, x^N] \\ &= \lambda_1\lambda_2 \otimes x^N + \lambda_2\lambda_3 \otimes x_3\partial_1(x^N) + \lambda_3\lambda_1 \otimes x_3\partial_2(x^N) \end{aligned} \tag{1.9.2}$$

in degree 1, and

$$\begin{aligned} d^2(\lambda_1\lambda_2 \otimes x^N) &= 0 \\ d^2(\lambda_2\lambda_3 \otimes x^N) &= \lambda_1\lambda_2\lambda_3 \otimes x_3\partial_2(x^N) \\ d^2(\lambda_3\lambda_1 \otimes x^N) &= -\lambda_1\lambda_2\lambda_3 \otimes x_3\partial_1(x^N) \end{aligned} \tag{1.9.3}$$

in degree 2.

1.9.2 Calculating $HH^1(U)$

In calculating the degree 1 boundaries it will be helpful to have some notation.

Notation 1.47. For a polynomial a in U, which we are identifying with $k[x_1, x_2, x_3]$, we let $\int_i a$ denote the antiderivitive of a with respect to x_i with 0 constant term. For example $\int_2 x_1^3 x_2^5 = \frac{1}{6} x_1^3 x_2^6$.

Lemma 1.48. *An element $\xi = \lambda_1 \otimes a_1 + \lambda_2 \otimes a_2 + \lambda_3 \otimes a_3$ is in B^1 if and only if*

1. $a_3 = 0$

2. x_3 *divides* a_2 *and* a_3

3. $\int_2 a_1 - \int_1 a_2 = 0$.

Proof. From (1.9.1) it is clear that these conditions are necessary in order for ξ to be a boundary, and when they are satisfied we will have $\xi = d^0(\int_1 a_2/x_3)$. ■

Lemma 1.49. *Any degree* 1 *cycle is as sum of the following types of elements:*

Type 1) $\lambda_1 \otimes a$*, where* $a \in \mathcal{Z}k[x_2]$.

Type 2) $\lambda_2 \otimes a$*, where* $a \in \mathcal{Z}k[x_1]$.

Type 3) $\lambda_1 \otimes \int_1 a - \lambda_2 \otimes \int_2 a$*, where* $a \in k[x_1, x_2]$.

Type 3.5) $\lambda_1 \otimes x_3 \int_1 a - \lambda_2 \otimes x_3 \int_2 a$*, where* $a \in \mathcal{Z}k[x_1, x_2]$.

Type 4) $\lambda_1 \otimes f x_1 + \lambda_3 \otimes f x_3$*, where* $f \in \mathcal{Z}$.

Proof. First, note that each of these elements is a cycle. Now, let $\xi = \lambda_1 \otimes b_1 + \lambda_2 \otimes b_2 + \lambda_3 \otimes b_3$ be a cycle. Since applying the differential then composing with the projection onto $\lambda_3\lambda_1 \otimes U \oplus \lambda_2\lambda_3 \otimes U$ yields the element

$$\lambda_3\lambda_1 \otimes x_3\partial_2(b_3) + \lambda_2\lambda_3 \otimes x_3\partial_1(b_3)$$

we conclude that $\partial_2(b_3) = \partial_1(b_3) = 0$. That is to say, b_3 is in $\mathcal{Z}$. So, by subtracting an element of type 4 we may assume $\xi = \lambda_1 \otimes b_1 + \lambda_2 \otimes b_2 + \lambda_3 \otimes \delta$, where δ is a unit or 0. Since the image of $\lambda_1 \otimes b_1 + \lambda_2 \otimes b_2$ will be divisible by x_3, we conclude that δ must also be divisible by x_3 in order for this element to be a cycle, and hence $\delta = 0$. So we may take $\xi = \lambda_1 \otimes b_1 + \lambda_2 \otimes b_2$.

By further subtracting elements of types 1 and 2 we may assume b_1 is divisible by x_1 and b_2 is divisible by x_2. So

$$\xi = \lambda_1 \otimes (\sum_{i,j \geq 0} \frac{a_{ij}}{(i+1)}) x_1 - \lambda_2 \otimes (\sum_{i,j \geq 0} \frac{a'_{ij}}{(j+1)}) x_2,$$

where the a_{ij} and a'_{ij} are of homogenous degree i with respect to x_1 and j with respect to x_2, and

$$d(\xi) = -\lambda_1\lambda_2 \otimes x_3(\sum a_{ij} - \sum a'_{ij}).$$

Since U is torsion free over $\mathcal{Z}$, we must have $\sum a_{ij} = \sum a'_{ij} = a$ for some polynomial a. Hence

$$\xi = \lambda_1 \otimes (\sum_{ij} \frac{a_{ij}}{(i+1)})x_1 - \lambda_2 \otimes (\sum_{ij} \frac{a'_{ij}}{(j+1)})x_2 = \lambda_1 \otimes \int_1 a - \lambda_2 \otimes \int_2 a.$$

So ξ is the sum of elements of types 3 and 3.5. ■

Proposition 1.50. *We have* $HH^1(U) = FH^1 \oplus TH^1$ *where* FH^1 *is the free* $\mathcal{Z}$*-module with generators*

$$\{\lambda_1 \otimes x_2^{n_2}, \lambda_2 \otimes x_1^{n_1}, \lambda_1 \otimes x_1 - \lambda_3 \otimes x_3 : n_i \geq 0\}, \tag{1.9.4}$$

and TH^1 *is the torsion* $\mathcal{Z}$*-module with annihilator* (x_3) *and generators*

$$\{\lambda_1 \otimes \frac{x^{N+(1,0)}}{n_1+1} - \lambda_2 \otimes \frac{x^{N+(0,1)}}{n_2+1} : N = (n_1, n_2),\ n_i \geq 0\}. \tag{1.9.5}$$

Let us remark here that we can substitute the generator $\lambda_2 \otimes x_2 - \lambda_3 \otimes x_3$ for our generator $\lambda_1 \otimes x_1 - \lambda_3 \otimes x_3$ to get the same result.

Proof. First note that, from the description of Z^1 given in Lemma 1.49 and the fact that $(U^!\widetilde{\otimes}U)^1$ is a free $\mathcal{Z}$-module on our given basis, the submodule of cycles Z^1 is free on the basis

$$\{\lambda_1 \otimes x_2^{n_2}, \lambda_2 \otimes x_1^{n_1}, \lambda_1 \otimes x_1 - \lambda_3 \otimes x_3 : n_i \geq 0\}$$

$$\amalg$$

$$\{\lambda_1 \otimes \tfrac{x^{N+(1,0)}}{n_1+1} - \lambda_2 \otimes \tfrac{x^{N+(0,1)}}{n_2+1} : N = (n_1, n_2),\ n_i \geq 0\}.$$

From the description of B^3 above it then becomes clear that we have a surjective map $HH^1 = Z^1/B^1 \to FH^1 \oplus TH^1$, and one similarly constructs a map backwards. ■

1.9.3 Calculating $HH^2(U)$ and $HH^3(U)$

Lemma 1.51. *Any degree 2 boundary is a $\mathcal{Z}$-linear combination of the following types of elements:*

Type 1) $\lambda_1\lambda_2 \otimes x_3 a$, *where* $a \in k[x_1, x_2]$.

Type 2) $\lambda_1\lambda_2 \otimes 1$.

Type 2.5) $\lambda_1\lambda_2 \otimes x^N + \lambda_2\lambda_3 \otimes n_1 x_3 x^{N-(1,0)} + \lambda_3\lambda_1 \otimes n_2 x_3 x^{N-(0,1)}$, *where* x^M *is taken to be zero when an entry in* $M = (m_1, m_2)$ *is negative, and we only consider* $N = (n_1, n_2)$ *with* $n_1 + n_2 > 0$.

Proof. For $i = 1, 2$ and $a \in k[x_1, x_2]$, $d(\lambda_i \otimes a)$ is an element of type 1, and $d(\lambda_3 \otimes a)$ is a sum of elements of types 2 and 2.5. Furthermore, each of these elements are boundaries as we have

$$\begin{aligned}
&d(-\lambda_1 \otimes \textstyle\int_1 a) = \lambda_1\lambda_2 \otimes x_3 a \\
&d(\lambda_3 \otimes 1) = \lambda_1\lambda_2 \otimes 1 \\
&d(\lambda_3 \otimes x^N) = \lambda_1\lambda_2 \otimes x^N + \lambda_2\lambda_3 \otimes n_1 x_3 x^{N-(1,0)} + \lambda_3\lambda_1 \otimes n_2 x_3 x^{N-(0,1)}.
\end{aligned}$$

■

Lemma 1.52. *A degree 2 cycle is a $\mathcal{Z}$-linear combination of the following types of elements:*

Type 1) $\lambda_1\lambda_2 \otimes a$, *where* $a \in k[x_1, x_2]$.

Type 2) $\lambda_2\lambda_3 \otimes x_1^n$, $n \geq 0$.

Type 3) $\lambda_3\lambda_1 \otimes x_2^n$, $n \geq 0$.

Type 4) $\lambda_2\lambda_3 \otimes \int_2 a + \lambda_3\lambda_1 \otimes \int_1 a$, *where* $a \in k[x_1, x_2]$.

Proof. For any cycle ξ, by subtracting a $\mathcal{Z}$-linear combination of elements of types 1, 2, and 3 we may assume

$$\xi = \lambda_2\lambda_3 \otimes a_2 + \lambda_3\lambda_1 \otimes a_1$$

with a_2 divisible by x_2 and a_1 divisible by x_1. Whence

$$\begin{aligned} d(\xi) &= d(\lambda_2\lambda_3 \otimes a_2) + d(\lambda_3\lambda_1 \otimes a_1) \\ &= \lambda_1\lambda_2\lambda_3 \otimes x_3\partial_2(a_2) - \lambda_1\lambda_2\lambda_3 \otimes x_3\partial_1(a_1) \\ &= x_3(\lambda_1\lambda_2\lambda_3 \otimes \partial_2(a_2) - \partial_1(a_1)). \end{aligned}$$

Since U is torsion free over $\mathcal{Z}$ this implies $\partial_2(a_2) = \partial_1(a_1) = a$ and $a_1 = \int_2 a$ and $a_2 = \int_1 a$, since we already know a_1 and a_2 have vanishing constant terms. ■

It will be helpful to have the following Lemma in deducing the degree 2 cohomology.

Lemma 1.53. *The second homology $HH^2(U)$ is annihilated by x_3^2.*

Proof. We simply check that each of the generators, given in Lemma 1.52, is annihilated by x_3^2. A cycle of type 1 becomes a boundary of type 1 after multiplying by x_3, and hence is annihilated on cohomology. A cycle of type 2 multiplied by x_3^2 is of the form

$$\begin{aligned} &\lambda_2\lambda_3 \otimes x_3^2x_1^n \\ &= \lambda_2\lambda_3 \otimes x_3\partial_1(\tfrac{1}{(n+1)}x_3x_1^{n+1}) \\ &= \lambda_2\lambda_3 \otimes x_3\partial_1(\tfrac{1}{(n+1)}x_3x_1^{n+1}) + \lambda_3\lambda_1 \otimes n_2x_3\partial_2(x_3x_1^{n+1}) \\ &= \lambda_2\lambda_3 \otimes x_3\partial_1(\tfrac{1}{(n+1)}x_3x_1^{n+1}) + \lambda_3\lambda_1 \otimes n_2x_3\partial_2(x_3x_1^{n+1}) \\ &= (\lambda_1\lambda_2 \otimes \tfrac{1}{n+1}x_3x_1^{n+1} + \lambda_2\lambda_3 \otimes x_3\partial_1(\tfrac{1}{(n+1)}x_3x_1^{n+1}) + \lambda_3\lambda_1 \otimes n_2x_3\partial_2(\tfrac{1}{(n+1)}x_3x_1^{n+1})) \\ &\quad -\lambda_1\lambda_2 \otimes \tfrac{1}{n+1}x_3x_1^{n+1}, \end{aligned}$$

which is a sum of boundaries, and hence is zero in cohomology. Similarly, we see that elements of type 3 and type 4 are also annihilated by x_3^2. ■

Note that from our $\mathcal{Z}$-basis for $(U^!\widetilde{\otimes}U)^2$, the elements of types 1-4 above are all $\mathcal{Z}$-linearly independent, and so we can produce an easy basis for Z^2 from Lemma 1.52.

Proposition 1.54. *The second Hochschild cohomology $HH^2(U)$ is the free $\mathcal{Z}/(x_3^2)$-module TH^2 with basis*

$$\{\lambda_2\lambda_3 \otimes x_1^l,\ \lambda_3\lambda_1 \otimes x_2^m,\ \lambda_2\lambda_3 \otimes \frac{x^{N+(0,1)}}{(n_2+1)} + \lambda_3\lambda_1 \otimes \frac{x^{N+(1,0)}}{(n_1+1)}\}_{l,m,N}.$$

Proof. By the previous lemma, we have the obvious $\mathcal{Z}$-module map $f : TH^2 \to HH^2(U)$, sending the basis elements to their corresponding generators of types 2, 3, and 4. The generators of type 1 are in the image of this map since we have the boundaries

$$\lambda_1\lambda_2 \otimes a + \lambda_2\lambda_3 \otimes x_3\partial_1(a) + \lambda_3\lambda_1 \otimes x_3\partial_2(a),$$

and $\lambda_1\lambda_2 \otimes 1$. So the map f is surjective. Note that Z^2 is the free $\mathcal{Z}$-module on the basis elements given by Lemma 1.52. So we have the $\mathcal{Z}$-module map $g : Z^2 \to TH^2$ defined on generators of type 1 by

$$g(\lambda_1\lambda_2 \otimes a) = -(\lambda_2\lambda_3 \otimes x_3\partial_1(a) + \lambda_3\lambda_1 \otimes x_3\partial_2(a))$$

and on the generators of types 2, 3, and 4 in the obvious way. This map is clearly surjective and is seen to annihilate all boundaries (since TH^2 is annihilated by x_3^2). Whence we get an induced $\mathcal{Z}$-linear map $\bar{g} : HH^2(U) \to TH^2$. These maps are mutually inverse since they are mutually inverse on the $\mathcal{Z}$-generators. And so we have $HH^2(U) = TH^2$. ■

The calculation of $HH^3(U)$ is trivial, and so we state it here explicitly.

Proposition 1.55. *The third Hochschild cohomology $HH^3(U)$ is the free $\mathcal{Z}/(x_3)$-module with basis $\{\lambda_1\lambda_2 \otimes x^N : N \in \mathbb{Z}^2_{\geq 0}\}$.*

Proof. This is clear from the definition of d^2 given at (1.9.3), and the fact that the partial differentials $\partial_i : k[x_1, x_2] \to k[x_1, x_2]$ are surjective. ■

1.9.4 A multiplication table for $HH^\bullet(U)$

Lemma 1.56. *For any homogeneous ξ in $HH^\bullet(U)$ of degree ≥ 1 we have $\xi^2 = 0$.*

Proof. One can calculate directly that $\xi^2 = 0$ for degree 1 cycles, or simply note that this follows by graded commutativity. For cocycles of degree > 1 this follows simply by the fact that the cohomology vanishes in degree > 3. ■

Below we give a list of products of degree 1 basis elements for $HH^\bullet(U)$. Let us note that the following operations are happening in cohomology, not in $U^! \widetilde{\otimes} U$. The elements below represent classes in $HH^\bullet(U)$.

- $\lambda_1 \otimes x_2^{n_2} \cdot \lambda_2 \otimes x_1^{n_1} = -(\lambda_2\lambda_3 \otimes n_1 x_3 x^{N-(1,0)} + \lambda_3\lambda_2 \otimes n_2 x_3 x^{N-(0,1)})$
- $\lambda_1 \otimes x_2^{n_2} \cdot (\lambda_1 \otimes x_1 - \lambda_3 \otimes x_3) = \lambda_3\lambda_1 \otimes x_3 x_2^{n_2}$
- $\lambda_1 \otimes x_2^{n_2} \cdot (\lambda_1 \otimes \frac{x^{M+(1,0)}}{(m_1+1)} - \lambda_2 \otimes \frac{x^{M+(0,1)}}{(m_2+1)})$
  $= \lambda_2\lambda_3 \otimes \frac{m_1 x_3 x^{M+(-1,n_2+1)}}{(m_2+1)} + \lambda_3\lambda_2 \otimes \frac{(m_2+n_2+1) x_3 x^{M+(0,n_2)}}{(m_2+1)}$
- $\lambda_2 \otimes x_1^{n_1} \cdot (\lambda_1 \otimes x_1 - \lambda_3 \otimes x_3) = -\lambda_2\lambda_3 \otimes x_3 x_1^{n_1}$
- $\lambda_2 \otimes x_1^{n_1} \cdot (\lambda_1 \otimes \frac{x^{M+(1,0)}}{(m_1+1)} - \lambda_2 \otimes \frac{x^{M+(0,1)}}{(m_2+1)})$
  $= \lambda_2\lambda_3 \otimes \frac{(m_1+n_1+1) x_3 x^{M+(n_1,0)}}{(m_1+1)} + \lambda_3\lambda_2 \otimes \frac{m_2 x_3 x^{M+(n_1+1,-1)}}{(m_1+1)}$
- $(\lambda_1 \otimes x_1 - \lambda_3 \otimes x_3) \cdot (\lambda_1 \otimes \frac{x^{M+(1,0)}}{(m_1+1)} - \lambda_2 \otimes \frac{x^{M+(0,1)}}{(m_2+1)})$
  $= \lambda_2\lambda_3 \otimes \frac{m_1 x_3 x^{M+(0,1)}}{(m_2+1)} + \lambda_3\lambda_1 \otimes \frac{m_1 x_3 x^{M+(1,0)}}{(m_1+1)}.$

This list, along with graded commutativity of Hochschild cohomology and the previous lemma, gives a complete multiplication table. We leave it to the reader to verify most of these products, but give a computation of the final product here, as it is the most difficult.

Proof of the final product. In cohomology we have

$$\begin{aligned}
&(\lambda_1 \otimes x_1 - \lambda_3 \otimes x_3) \cdot (\lambda_1 \otimes \tfrac{x^{M+(1,0)}}{(m_1+1)} - \lambda_2 \otimes \tfrac{x^{M+(0,1)}}{(m_2+1)}) \\
&= -\lambda_1\lambda_2 \otimes \tfrac{x^{M+(1,1)}}{(m_2+1)} - \lambda_2\lambda_3 \otimes \tfrac{x_3 x^{M+(0,1)}}{(m_2+1)} - \lambda_3\lambda_1 \otimes \tfrac{x_3 x^{M+(1,0)}}{(m_1+1)} \\
&= \lambda_2\lambda_3 \otimes \tfrac{(m_1+1) x_3 x^{M+(0,1)}}{(m_2+1)} + \lambda_3\lambda_1 \otimes \tfrac{(m_2+1) x_3 x^{M+(1,0)}}{(m_2+1)} \\
&\quad -\lambda_2\lambda_3 \otimes \tfrac{x_3 x^{M+(0,1)}}{(m_2+1)} - \lambda_3\lambda_1 \otimes \tfrac{x_3 x^{M+(1,0)}}{(m_1+1)} \\
&= \lambda_2\lambda_3 \otimes \tfrac{(m_1+1) x_3 x^{M+(0,1)}}{(m_2+1)} + \lambda_3\lambda_1 \otimes x_3 x^{M+(1,0)} \\
&\quad -\lambda_2\lambda_3 \otimes \tfrac{x_3 x^{M+(0,1)}}{(m_2+1)} - \lambda_3\lambda_1 \otimes \tfrac{x_3 x^{M+(1,0)}}{(m_1+1)} \\
&= \lambda_2\lambda_3 \otimes \tfrac{m_1 x_3 x^{M+(0,1)}}{(m_2+1)} + \lambda_3\lambda_1 \otimes \tfrac{m_1 x_3 x^{M+(1,0)}}{(m_1+1)}
\end{aligned}$$

■

From the above list of products we see that for any basic element ξ in $HH^2(U)$ we have $x_3\xi = \eta_1 \cdot \eta_2$ for some basic η_i in degree 1. So we have the following proposition.

Proposition 1.57. *The multiplication map* $HH^1(U) \otimes HH^1(U) \to HH^2(U)$ *has image exactly* $x_3 \cdot HH^2(U)$.

What is left is to present the multiplication table for $HH^1(U) \cdot HH^2(U)$. We have

- $\lambda_1 \otimes x_2^{n_2} \cdot \lambda_2\lambda_3 \otimes x_1^{n_1} = \lambda_1\lambda_2\lambda_3 \otimes x^N$
- $\lambda_1 \otimes x_2^{n_2} \cdot \lambda_3\lambda_1 \otimes x_2^{n_2} = 0$
- $\lambda_1 \otimes x_2^{n_2} \cdot (\lambda_2\lambda_3 \otimes \frac{x^{M+(0,1)}}{(m_2+1)} + \lambda_3\lambda_1 \otimes \frac{x^{M+(1,0)}}{(m_1+1)}) = \lambda_1\lambda_2\lambda_3 \otimes \frac{x^{M+(0,n_2+1)}}{(m_2+1)}$
- $\lambda_2 \otimes x_1^{n_1} \cdot \lambda_2\lambda_3 \otimes x_1^{n_1} = 0$
- $\lambda_2 \otimes x_1^{n_1} \cdot \lambda_3\lambda_1 \otimes x_2^{n_2} = \lambda_1\lambda_2\lambda_3 \otimes x^N$
- $\lambda_2 \otimes x_1^{n_1} \cdot (\lambda_2\lambda_3 \otimes \frac{x^{M+(0,1)}}{(m_2+1)} + \lambda_3\lambda_1 \otimes \frac{x^{M+(1,0)}}{(m_1+1)}) = \lambda_1\lambda_2\lambda_3 \otimes \frac{x^{M+(n_1+1,0)}}{(m_1+1)}$
- $(\lambda_1 \otimes x_1 - \lambda_3 \otimes x_3) \cdot \lambda_2\lambda_3 \otimes x_1^{n_1} = \lambda_1\lambda_2\lambda_3 \otimes x_1^{n_1+1}$
- $(\lambda_1 \otimes x_1 - \lambda_3 \otimes x_3) \cdot \lambda_3\lambda_1 \otimes x_2^{n_2} = 0$
- $(\lambda_1 \otimes x_1 - \lambda_3 \otimes x_3) \cdot (\lambda_2\lambda_3 \otimes \frac{x^{M+(0,1)}}{(m_2+1)} + \lambda_3\lambda_1 \otimes \frac{x^{M+(1,0)}}{(m_1+1)}) = \lambda_1\lambda_2\lambda_3 \otimes \frac{x^{M+(1,1)}}{(m_2+1)}$
- $(\lambda_1 \otimes \frac{x^{L+(1,0)}}{(l_1+1)} - \lambda_2 \otimes \frac{x^{L+(0,1)}}{(l_2+1)}) \cdot \lambda_2\lambda_3 \otimes x_1^{n_1} = \lambda_1\lambda_2\lambda_3 \otimes \frac{x^{L+(n_1+1,0)}}{(l_1+1)}$
- $(\lambda_1 \otimes \frac{x^{L+(1,0)}}{(l_1+1)} - \lambda_2 \otimes \frac{x^{L+(0,1)}}{(l_2+1)}) \cdot \lambda_3\lambda_1 \otimes x_2^{n_2} = -\lambda_1\lambda_2\lambda_3 \otimes \frac{x^{L+(0,n_2+1)}}{(l_2+1)}$
- $(\lambda_1 \otimes \frac{x^{L+(1,0)}}{(l_1+1)} - \lambda_2 \otimes \frac{x^{L+(0,1)}}{(l_2+1)}) \cdot (\lambda_2\lambda_3 \otimes \frac{x^{M+(0,1)}}{(m_2+1)} + \lambda_3\lambda_1 \otimes \frac{x^{M+(1,0)}}{(m_1+1)})$
  $= (\frac{1}{(l_1+1)(m_2+1)} - \frac{1}{(m_1+1)(l_2+1)})\lambda_1\lambda_2\lambda_3 \otimes x^{L+M+(1,1)}$

The computations are more straightforward than those for degree 1 elements, and are left to the interested reader. Our final proposition is clear from the above table.

Proposition 1.58. *The product* $HH^1(U) \otimes HH^2(U) \to HH^3(U)$ *is surjective.*

Chapter 2

SPECTRAL SEQUENCES FOR THE COHOMOLOGY RINGS OF A SMASH PRODUCT

In this chapter we enhance some known spectral sequences with multiplicative structures. Namely, we produce multiplicative spectral sequences

$$E_2 = \mathrm{Ext}_{\mathcal{G}\text{-mod}}(k, HH^\bullet(A,B)) \Rightarrow HH^\bullet(A*\mathcal{G},B)$$

and

$${}'E_1 = \mathrm{Ext}_{\mathcal{G}\text{-mod}}(k, \mathrm{RHom}_{A\text{-bimod}}(A,B)) \Rightarrow HH^\bullet(A*\mathcal{G},B)$$

which may be used to compute the products on the respective cohomologies.

Let us give a synopsis of what is done in the chapter. In Section 2.2 we produce a resolution of the Hopf algebra $\mathcal{G}$ which carries enough structure to admit a smash product construction. In particular, we construct a complex of projective $\mathcal{G}$-bimodules with an additional (compatible) coaction, and quasi-isomorphism to $\mathcal{G}$ which preserves the given structure. We call such a resolution a Hopf bimodule resolution.

In Section 2.3 we propose a smash product construction for complexes of Hopf bimodules and complexes of, so called, equivariant bimodules over A (Definition 2.10). This smash product construction for complexes is used to produce, from the Hopf bimodule resolution of Section 2.2 and an equivariant resolution of A, a bimodule resolution of $A*\mathcal{G}$. In Section 2.4 we use the aforementioned bimodule resolution of $A*\mathcal{G}$ to construct an explicit isomorphism

$$\Xi : \mathrm{RHom}_{A*\mathcal{G}\text{-bimod}}(A*\mathcal{G}, M) \overset{\cong}{\to} \mathrm{RHom}_{\mathcal{G}\text{-mod}}(k, \mathrm{RHom}_{A\text{-bimod}}(A, M))$$

for any complex M of $A*\mathcal{G}$-bimodules.

In Section 2.5 we review the products on both the domain and codomain of the above isomorphism Ξ (when evaluated at an algebra extension of $A*\mathcal{G}$), and in Section 2.6 we show that the map Ξ is an isomorphism of dg algebras when appropriate. Theorem III is also proved in this section. Finally, in Section 2.7 we give versions of our main theorems for the Ext algebras $\mathrm{Ext}_{A*\mathcal{G}\text{-mod}}(M,M)$, for arbitrary M. Some examples are presented in Section 2.8

2.1 Conventions

For any coalgebra $\mathcal{G}$, the coproduct of an element $\gamma \in \mathcal{G}$ will always be expressed using Sweedler's notation

$$\gamma_1 \otimes \gamma_2 = \Delta(\gamma).$$

(So "$\gamma_1 \otimes \gamma_2$" is a symbol representing a sum of elements $\sum_i \gamma_{i_1} \otimes \gamma_{i_2}$ in $\mathcal{G} \otimes \mathcal{G}$.) Given $\mathcal{G}$-modules M and N the tensor product $M \otimes N$ is taken to be a $\mathcal{G}$-module under the standard action

$$\gamma(m \otimes n) := (\gamma_1 m) \otimes (\gamma_2 n).$$

As mentioned previously, *a Hopf algebra will always mean a Hopf algebra with bijective antipode.* Let $\mathcal{G}$ be a Hopf algebra and A be a $\mathcal{G}$-module algebra. Following [76], we denote the action of $\mathcal{G}$ on A by a superscript ${}^{\gamma}a := \gamma \cdot a$. Elements in the smash product $A*\mathcal{G}$ will be denoted by juxtaposition $a\gamma := a \otimes \gamma \in A*\mathcal{G} = A \otimes \mathcal{G}$. Hence, the multiplication on $A*\mathcal{G}$ can be written $(a\gamma)(b\gamma') = a({}^{\gamma_1}b)\gamma_2\gamma'$. For an algebra A we let A^e denote the enveloping algebra $A^e = A^{op} \otimes A$. *All modules are left modules unless stated otherwise.* We do not distinguish between the category of A-bimodules and the category of (right or left) A^e-modules.

In computations, all elements in graded vectors spaces are chosen to be homogenous. For homogeneous x, in a graded space X, we let $|x|$ denote its degree. For any algebra A and A-complexes X and Y we let $\mathrm{Hom}_A(X,Y)$ denote the standard hom complex. Recall that the nth homogenous piece of the hom complex consists of all degree n maps $f : X \to Y$, and for any $f \in \mathrm{Hom}_A(X,Y)$ the differential is given by $f \mapsto d_Y f - (-1)^{|f|} f d_X$.

2.2 A Hopf bimodule resolution of $\mathcal{G}$

Let $\mathcal{G}$ be a Hopf algebra. We have the canonical algebra embedding

$$\begin{gathered}\Delta^{tw}:\mathcal{G}\to\mathcal{G}^e=\mathcal{G}^{op}\otimes\mathcal{G}\\ \gamma\mapsto S(\gamma_1)\otimes\gamma_2.\end{gathered}$$

This map will be referred to as the *twisted diagonal map*. The twisted diagonal map gives $\mathcal{G}^e$ a left $\mathcal{G}$-module structure. On elements, this left action is given by $\delta\cdot(\gamma\otimes\gamma')=\gamma S(\delta_1)\otimes\delta_2\gamma'$, for $\delta\in\mathcal{G}$, $\gamma\otimes\gamma'\in\mathcal{G}^e$.

Note that, since the antipode of $\mathcal{G}$ is bijective, there is an isomorphism of left $\mathcal{G}$-modules $S^{-1}\otimes id:\mathcal{G}^e\to\mathcal{G}\otimes\mathcal{G}$. The module $\mathcal{G}\otimes\mathcal{G}$ is known to be free over $\mathcal{G}$. (One can use the fundamental theorem of Hopf modules [62, Theorem 1.9.4] to show this, for example.) It follows that $\mathcal{G}^e$ is also a free left $\mathcal{G}$-module. This point is made explicit in the following lemma.

Lemma 2.1. *Let V denote the vector space $\mathcal{G}$, after forgetting the $\mathcal{G}$-module structure, and $\mathcal{G}\otimes V$ denote the corresponding free left $\mathcal{G}$-module. There is an isomorphism $\psi:\mathcal{G}^e\to\mathcal{G}\otimes V$ of left $\mathcal{G}$-modules given by*

$$\gamma\otimes\gamma'\mapsto S^{-1}(\gamma_2)\otimes\gamma_1\gamma'.$$

Consequently, the enveloping algebra $\mathcal{G}^e$ is a free, and hence flat, left $\mathcal{G}$-module.

Proof. For any $\delta\in\mathcal{G}$ we have

$$\begin{aligned}\psi(\delta\cdot(\gamma\otimes\gamma')) &=\psi(\gamma S(\delta_1)\otimes\delta_2\gamma'\\ &=S^{-1}(\gamma_2S(\delta_1))\otimes\gamma_1S(\delta_2)\delta_3\gamma'\\ &=\delta_1S^{-1}(\gamma_2)\otimes\gamma_1\epsilon(\delta_2)\gamma'\\ &=\delta S^{-1}(\gamma_2)\otimes\gamma_1\gamma'=\delta\cdot\psi(\gamma\otimes\gamma').\end{aligned}$$

So ψ is left $\mathcal{G}$-linear. Let $\psi':\mathcal{G}\otimes V\to\mathcal{G}^e$ denote the map $\gamma\otimes\gamma'\mapsto S(\gamma_1)\otimes\gamma_2\gamma'$. Then we have

$$\begin{aligned}\psi'\psi(\gamma\otimes\gamma') &= \psi'(S^{-1}(\gamma_2)\otimes\gamma_1\gamma')\\ &= S(S^{-1}(\gamma_3))\otimes S^{-1}(\gamma_2)\gamma_1\gamma'\\ &= \gamma_2\otimes\epsilon(\gamma_1)\gamma'\\ &= \gamma\otimes\gamma'.\end{aligned}$$

So $\psi'\psi = id_{\mathcal{G}^e}$. A similar calculation shown $\psi\psi' = id_{\mathcal{G}\otimes V}$, and hence that ψ is an isomorphism with $\psi^{-1} = \psi'$. ■

Note that the left action of $\mathcal{G}$ on $\mathcal{G}^e$ is compatible with the standard (outer) bimodule structure on $\mathcal{G}^e = {}_{\mathcal{G}}\mathcal{G}\otimes\mathcal{G}_{\mathcal{G}}$. Indeed, the module structure induced by the twisted diagonal map utilizes the inner bimodule structure $\mathcal{G}_{\mathcal{G}}\otimes{}_{\mathcal{G}}\mathcal{G}$ exclusively. So we see that $\mathcal{G}^e$ is a ($\mathcal{G}$-$\mathcal{G}^e$)-bimodule, and that the induced module $M\otimes_{\mathcal{G}}\mathcal{G}^e$ of a right $\mathcal{G}$-module M is a $\mathcal{G}$-bimodule. To be clear, the left and right actions of $\mathcal{G}$ on $M\otimes_{\mathcal{G}}\mathcal{G}^e$ are given by

$$\delta\cdot(m\otimes_{\mathcal{G}}(\gamma\otimes\gamma')) := m\otimes_{\mathcal{G}}(\delta\gamma\otimes\gamma')$$

and

$$(m\otimes_{\mathcal{G}}(\gamma\otimes\gamma'))\cdot\delta := m\otimes_{\mathcal{G}}(\gamma\otimes\gamma'\delta)$$

respectively, where $\delta\in\mathcal{G}$ and $m\otimes_{\mathcal{G}}(\gamma\otimes\gamma')\in M\otimes_{\mathcal{G}}\mathcal{G}^e$. The same analysis holds when we replace M with a complex of right $\mathcal{G}$-modules.

Notation 2.2 (The functor $(-)^{\uparrow}$)**.** Given any right $\mathcal{G}$-module (resp. complex) M, we let $M^{\uparrow}$ denote the induced module (resp. complex) $M\otimes_{\mathcal{G}}\mathcal{G}^e$.

The next lemma gives an alternate expression of the module $M^{\uparrow}$, which may at times be more practical.

Lemma 2.3. *There is an isomorphism of bimodules $M^{\uparrow}\to M\otimes\mathcal{G}$, where the left and right actions of $\mathcal{G}$ on $M\otimes\mathcal{G}$ are given by the formulae*

$$\gamma\cdot(m\otimes x) = mS^{-1}(\gamma_2)\otimes\gamma_1 x \quad \textit{and} \quad (m\otimes x)\cdot\gamma = m\otimes(x\gamma).$$

The proposed isomorphism is given by the formula $m \otimes_{\mathcal{G}} (\gamma \otimes \gamma') \mapsto mS^{-1}(\gamma_2) \otimes \gamma_1\gamma'$ *and it is natural in the sense that, for any* $\mathcal{G}$*-linear map* $f : M \to N$*, the diagram*

$$\begin{tikzcd}
M^{\uparrow} \arrow[r, "f \otimes_{\mathcal{G}} \mathcal{G}^e"] \arrow[d, "\cong"] & N^{\uparrow} \arrow[d, "\cong"] \\
M \otimes \mathcal{G} \arrow[r, "f \otimes \mathcal{G}"] & N \otimes \mathcal{G}
\end{tikzcd} \tag{2.2.1}$$

will commute.

Proof. From the isomorphism ψ of Lemma 2.1, we get a vector space isomorphism $id \otimes \psi : M^{\uparrow} \to M \otimes_{\mathcal{G}} (\mathcal{G} \otimes V) \cong M \otimes V$ given by

$$m \otimes_{\mathcal{G}} (\gamma \otimes \gamma') \mapsto m \otimes_{\mathcal{G}} (S^{-1}(\gamma_2) \otimes \gamma_1\gamma') \mapsto mS^{-1}(\gamma_2) \otimes \gamma_1\gamma'.$$

This isomorphism induces a $\mathcal{G}$-bimodule structure on $M \otimes V = M \otimes \mathcal{G}$ which is given by the proposed formulae. Commutativity of (2.2.1) follows from right $\mathcal{G}$-linearity of f. ■

The following result is proven, in less detail, in [78, Section 3]. However, as we will be needing all the details, a full proof is given here.

Lemma 2.4. *Let* $\xi : L \to k$ *be a resolution of the trivial right* $\mathcal{G}$*-module* $k = \mathcal{G}/ker\epsilon$*.*

1. *The induced complex* $L^{\uparrow}$ *is a complex of projective* $\mathcal{G}$*-bimodules.*
2. *The map* $\xi^{\uparrow} : L^{\uparrow} \to \mathcal{G}$, $\ell \otimes_{\mathcal{G}} (\gamma \otimes \gamma') \mapsto \xi(\ell)\gamma\gamma'$*, is a quasi-isomorphism of complexes of* $\mathcal{G}$*-bimodules.*

Statements (1) and (2) together say that $L^{\uparrow}$ is a projective bimodule resolution of $\mathcal{G}$.

Proof. In each degree i we have the adjunction

$$\mathrm{Hom}_{\mathcal{G}^e}(L^i \otimes_{\mathcal{G}} \mathcal{G}^e, -) = \mathrm{Hom}_{\mathcal{G}}(L^i, \mathrm{Hom}_{\mathcal{G}^e}(\mathcal{G}^e, -)).$$

Whence the functor on the left is seen to be exact. So $L^{\uparrow}$ is a complex of projective bimodules. For (2), exactness of the functor $- \otimes_{\mathcal{G}} \mathcal{G}^e$ ensures that the induced map $\xi \otimes_{\mathcal{G}} id_{\mathcal{G}^e} : L^{\uparrow} \to k^{\uparrow}$

is still a quasi-isomorphism. Now, by the previous lemma we have a bimodule isomorphism $k^{\uparrow} \cong \mathcal{G}$ given by $1 \otimes (\gamma \otimes \gamma') \mapsto \epsilon(S^{-1}(\gamma_2))\gamma_1\gamma' = \gamma\gamma'$. Composing the quasi-isomorphism $\xi \otimes_{\mathcal{G}} id_{\mathcal{G}^e}$ with the bimodule isomorphism $k^{\uparrow} \cong \mathcal{G}$ then gives the proposed quasi-isomorphism $\xi^{\uparrow}$. ■

Definition 2.5 (The coaction on $M^{\uparrow}$)**.** Given any right $\mathcal{G}$-module M, we define the left $\mathcal{G}$-comodule structure ρ_M on $M^{\uparrow}$ by

$$\begin{gathered} \rho_M : M^{\uparrow} \to \mathcal{G} \otimes M^{\uparrow} \\ m \otimes_{\mathcal{G}} (\gamma \otimes \gamma') \mapsto (\gamma_1\gamma'_1) \otimes (m \otimes_{\mathcal{G}} (\gamma_2 \otimes \gamma'_2)). \end{gathered} \tag{2.2.2}$$

Following the standard notation, for any $\mathrm{m} \in M^{\uparrow}$, we denote the element $\rho_M(\mathrm{m})$ by $\mathrm{m}_{-1} \otimes \mathrm{m}_0$.

Under the natural isomorphism $M^{\uparrow} \cong M \otimes \mathcal{G}$ of Lemma 2.3, the induced coaction on $M \otimes \mathcal{G}$ will be given by the simple formula $m \otimes \gamma \mapsto \gamma_1 \otimes (m \otimes \gamma_2)$. Indeed, we have the commutative diagram

$$\begin{tikzcd} M^{\uparrow} \arrow[r] \arrow[d, "\cong"] & \mathcal{G} \otimes M^{\uparrow} \arrow[d, "\cong"] & m \otimes_{\mathcal{G}} (\gamma \otimes \gamma') \arrow[r, mapsto] \arrow[d, mapsto] & \gamma_1\gamma'_1 \otimes (m \otimes_{\mathcal{G}} (\gamma_2 \otimes \gamma'_2)) \arrow[d, mapsto] \\ M \otimes \mathcal{G} \arrow[r] & \mathcal{G} \otimes (M \otimes \mathcal{G}) & mS^{-1}(\gamma_2) \otimes \gamma_1\gamma' \arrow[r, mapsto] & \gamma_1\gamma'_1 \otimes (mS^{-1}(\gamma_3) \otimes \gamma_2\gamma'_2). \end{tikzcd}$$

From this perspective, if we reverse the vertical arrows, it becomes clear that the coaction on $M^{\uparrow}$ is in fact well defined.

Definition 2.6 (Hopf bimodules)**.** By a Hopf bimodule we will mean a $\mathcal{G}$-bimodule N equipped with a left $\mathcal{G}$-coaction $N \to \mathcal{G} \otimes N$ which is a map of $\mathcal{G}$-bimodules (where $\mathcal{G}$ acts diagonally on the tensor product $\mathcal{G} \otimes N$). Maps of Hopf bimodules are maps that are simultaneously $\mathcal{G}$-bimodule maps and $\mathcal{G}$-comodule maps.

The algebra $\mathcal{G}$ itself becomes a Hopf bimodule under the regular bimodule structure and coaction given by the comultiplication. In the notation of [62, Section 1.9], a Hopf bimodule is an object in the category ${}^{\mathcal{G}}_{\mathcal{G}}\mathcal{M}_{\mathcal{G}}$.

Proposition 2.7. *Let M and N be right $\mathcal{G}$-modules and $f : M \to N$ be a morphism of $\mathcal{G}$-modules. For any* $\mathrm{m} \in M^{\uparrow}$ *and* $\gamma \in \mathcal{G}$ *the following equations hold:*

1. $\rho_N(f^{\uparrow}(\mathrm{m})) = \mathrm{m}_{-1} \otimes f^{\uparrow}(\mathrm{m}_0)$
2. $\rho_M(\mathrm{m} \cdot \gamma) = \mathrm{m}_{-1}\gamma_1 \otimes \mathrm{m}_0\gamma_2$
3. $\rho_M(\gamma \cdot \mathrm{m}) = \gamma_1\mathrm{m}_{-1} \otimes \gamma_2\mathrm{m}_0$.

Said another way, $(-)^{\uparrow}$ *is a functor from* mod-$\mathcal{G}$ *to the category of Hopf bimodules.*

The reader should be aware that we will be using the $^{\uparrow}$ notation on maps in a slightly more flexible manner throughout the paper.

Proof. These can all be checked directly from the definitions. For example, for (1), we have

$$\begin{aligned} \rho_M(f^{\uparrow}(m \otimes_{\mathcal{G}} (\gamma \otimes \gamma'))) &= \rho_M(f(m) \otimes_{\mathcal{G}} (\gamma \otimes \gamma')) \\ &= \gamma_1\gamma_1' \otimes f(m) \otimes_{\mathcal{G}} (\gamma_2 \otimes \gamma_2') = \gamma_1\gamma_1' \otimes f^{\uparrow}(m \otimes_{\mathcal{G}} (\gamma_2 \otimes \gamma_2')). \end{aligned}$$

■

Corollary 2.8. *For any complex X of right $\mathcal{G}$-modules the induced complex $X^{\uparrow}$ is a complex of Hopf bimodules.*

Proof. This follows from part (1) of the pervious proposition and the fact that the differentials on X are $\mathcal{G}$-linear. ■

Proposition 2.9. *The quasi-isomorphism $\xi^{\uparrow} : L^{\uparrow} \to \mathcal{G}$ of Lemma 2.4 is a quasi-isomorphism of complexes of Hopf bimodules.*

Proof. This can be checked directly from the definition of $\xi^{\uparrow}$ and the definitions of the coactions on $L^{\uparrow}$ and $\mathcal{G}$. ■

2.3 Bimodule resolutions of $A * \mathcal{G}$ via a smash product construction

Let $\mathcal{G}$ be a Hopf algebra and A be a $\mathcal{G}$-module algebra. We recall here that a k-linear map $M \to N$ of (right or left) $A*\mathcal{G}$-modules is $A*\mathcal{G}$-linear if and only if it is A-linear and $\mathcal{G}$-linear independently. The following definition was given by Kaygun in [36].

Definition 2.10 (Equivariant bimodules)**.** A vector space M is called a $\mathcal{G}$-equivariant A-bimodule if it is both a $\mathcal{G}$-module and A-bimodule, and the structure maps $A \otimes M \to M$ and $M \otimes A \to M$ are maps of $\mathcal{G}$-modules. Morphisms of $\mathcal{G}$-equivariant A-bimodules are maps which are A^e-linear and $\mathcal{G}$-linear independently. The category of such modules will be denoted $\mathrm{EQ}_{\mathcal{G}}A^e$-mod. We define $\mathcal{G}$-equivariant A^e-complexes similarly.

To ease notation we may at times write "equivariant bimodule" instead of the full $\mathcal{G}$-equivariant A-bimodule. One example of an equivariant bimodule is A itself. One can think of an equivariant bimodule as an A-bimodule internal to the monoidal category ($\mathcal{G}$-mod,$\otimes$).

Kaygun has shown that the category $\mathrm{EQ}_{\mathcal{G}}A^e$-mod is actually the module category of a certain smash product $A^e * \mathcal{G}$ [36, Lemma 3.3]. Whence $\mathrm{EQ}_{\mathcal{G}}A^e$-mod is seen to be abelian with enough projectives. Additionally, $\mathrm{EQ}_{\mathcal{G}}A^e$-mod comes equipped with restriction functors (forgetful functors) to A^e-modules and $\mathcal{G}$-modules. Since $A^e * \mathcal{G}$ is free over both A^e and $\mathcal{G}$, one can verify that these restriction functors preserve projectives.

In this section we produce a projective bimodule resolution of $A*\mathcal{G}$ via the smash product construction outlined below.

Definition 2.11 (The smash product of complexes)**.** Let X be any $\mathcal{G}$-equivariant A^e-complex and let Y be any complex of Hopf bimodules. The smash product complex $X * Y$ is defined to be the tensor complex $X \otimes Y$ with the left and right $A * \mathcal{G}$-actions

$$a \cdot (x \otimes y) := (ax) \otimes y,\ \gamma \cdot (x \otimes y) := (\gamma_1 x) \otimes (\gamma_2 y),\ (x \otimes y) \cdot a := x({}^{y_{-1}}a) \otimes y_0$$

and

$$(x \otimes y) \cdot \gamma := x \otimes (y\gamma),$$

for $x \in X$, $y \in Y$, $a \in A$ and $\gamma \in \mathcal{G}$.

Obviously, we can define the smash product of an equivariant bimodule with a Hopf bimodule by considering them to be complexes concentrated in degree 0. The smash product construction is (bi)functorial in the sense of the following

Lemma 2.12. *If $f : X \to X'$ and $g : Y \to Y'$ are maps of complexes of $\mathcal{G}$-equivariant A-bimodules and complexes of Hopf bimodules respectively, then the product map $f \otimes g : X * Y \to X' * Y'$ is a map of complexes of $A * \mathcal{G}$-bimodules.*

Proof. Left A-linearity of $f \otimes g$ follows from left A-linearity of f and right $\mathcal{G}$-linearity follows from right $\mathcal{G}$-linearity of g. Left $\mathcal{G}$-linearity of $f \otimes g$ follows from the fact that both f and g are left $\mathcal{G}$-linear. Finally, right A-linearity of $f \otimes g$ follows from right A-linearity of f and $\mathcal{G}$-colinearity of g. ■

Now, let K be a projective resolution of A as an A-bimodule, with quasi-isomorphism $\tau : K \to A$. We will assume that K has the following additional properties:

(I) there is a $\mathcal{G}$-action on K giving it the structure of a complex of $\mathcal{G}$-equivariant A-bimodules, and the quasi-isomorphism $\tau : K \to A$ is $\mathcal{G}$-equivariant.

(II) K is free over A^e on a graded base space $\bar{K} \subset K$ which is also a $\mathcal{G}$-submodule.

An example of a resolution of A satisfying the above conditions is the bar resolution

$$BA = \cdots \to A \otimes A^{\otimes 2} \otimes A \to A \otimes A \otimes A \to A \otimes A \to 0,$$

with its standard differential

$$b \otimes a_1 \otimes \ldots \otimes a_n \otimes b' \mapsto \begin{array}{l} ba_1 \otimes \ldots \otimes b' + (-1)^n b \otimes \ldots \otimes a_n b' \\ \qquad + \sum_{i=1}^{n-1} (-1)^i b \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes b'. \end{array} \tag{2.3.1}$$

We give BA the natural diagonal $\mathcal{G}$-action

$$\gamma \cdot (b \otimes a_1 \otimes \ldots \otimes a_n \otimes b') = \gamma_1 b \otimes \gamma_2 a_1 \otimes \ldots \otimes \gamma_{n+1} a_n \otimes \gamma_{n+2} b'.$$

In this case, $\overline{BA}$ will be the graded subspace $\overline{BA} = \bigoplus_n k \otimes A^{\otimes n} \otimes k$. One can also use the reduced bar complex or, if A is a Koszul algebra and $\mathcal{G}$ acts by graded endomorphisms, we can take K to be the Koszul resolution.

For any K satisfying (I) and (II), and any resolution L of the trivial right $\mathcal{G}$-module k, we can form the smash product complex $K * L^{\uparrow}$ using the coaction on $L^{\uparrow}$ defined in the previous section. We will see that the smash product complex $K * L^{\uparrow}$ provides a projective resolution of $A * \mathcal{G}$.

Lemma 2.13. *Suppose $\tau : K \to A$ is a bimodule resolution of A satisfying (I) and (II), and let $\xi : L \to k$ be any projective resolution of the trivial right $\mathcal{G}$-module. Let $\xi^{\uparrow} : L^{\uparrow} \to \mathcal{G}$ be the quasi-isomorphism of Lemma 2.4. Then the product map $\tau \otimes \xi^{\uparrow} : K * L^{\uparrow} \to A * \mathcal{G}$ is a quasi-isomorphism of $(A * \mathcal{G})^e$-complexes.*

Proof. The fact that $\tau \otimes \xi^{\uparrow}$ is a quasi-isomorphism follows from the facts that both τ and $\xi^{\uparrow}$ are quasi-isomorphisms, and that the tensor product of any two quasi-isomorphisms (over a field) is yet another quasi-isomorphism. Since τ was chosen to be $\mathcal{G}$-equivariant, and $\xi^{\uparrow}$ is a map of complexes of Hopf bimodules by Proposition 2.9, the product $\tau \otimes \xi^{\uparrow}$ is $(A * \mathcal{G})^e$-linear by Lemma 2.12. ■

Theorem 2.14. *Let K be a bimodule resolution of A satisfying conditions (I) and (II), and let L be any projective resolution of the trivial right $\mathcal{G}$-module k. Then the smash product $K * L^{\uparrow}$ is a projective $A * \mathcal{G}$-bimodule resolution of $A * \mathcal{G}$.*

The proof of the theorem will be clear from the following lemma.

Lemma 2.15. *Let M be a $\mathcal{G}$-equivariant bimodule which is free over A^e on a base space $\bar{M} \subset M$ satisfying $\mathcal{G}\bar{M} = \bar{M}$, and suppose N is a projective right $\mathcal{G}$-module. Then the smash product module $M * N^{\uparrow}$ is projective over $(A * \mathcal{G})^e$.*

Proof. Suppose that N is free on some base $\bar{N} \subset N$. Then we have $N^{\uparrow} \cong (\bar{N} \otimes \mathcal{G}) \otimes_{\mathcal{G}} \mathcal{G}^e \cong \mathcal{G} \otimes \bar{N} \otimes \mathcal{G}$. In particular, we have an embedding $\bar{N} \to N^{\uparrow}$, and $N^{\uparrow}$ is free on the corresponding subspace. Note that $\bar{N}$ is coinvariant in $N^{\uparrow}$. By hypothesis, we also have that M is free on a given subspace $\bar{M}$. Now, the embeddings of $\bar{M}$ and $\bar{N}$ into M and N produce an embedding of vector spaces on the tensor product $\bar{M} \otimes \bar{N} \to M \otimes N^{\uparrow} = M * N^{\uparrow}$. Whence we get an $A * \mathcal{G}$-bimodule map

$$\theta : A * \mathcal{G} \otimes (\bar{M} \otimes \bar{N}) \otimes A * \mathcal{G} \to M * N^{\uparrow} \tag{2.3.2}$$

from the corresponding free (bi)module into $M * N^{\uparrow}$, by freeness of the domain. This map is given on monomials by the formula

$$a\gamma \otimes (m \otimes n) \otimes a'\gamma' \mapsto a(\gamma_1 m)({}^{\gamma_2}a') \otimes \gamma_3 n \gamma',$$

since $n \in \bar{N}$ is coinvariant. We claim that $M * N^{\uparrow}$ is free on the base space $\bar{M} \otimes \bar{N}$. In particular, we claim that the map (2.3.2) is an isomorphism.

To show that θ is an isomorphism, it suffices to show that θ has an k-linear inverse

$$M * N^{\uparrow} \to A * \mathcal{G} \otimes (\bar{M} \otimes \bar{N}) \otimes A * \mathcal{G}.$$

Employing the vector space isomorphism $M * N^{\uparrow} \cong (A \otimes \bar{M} \otimes A) \otimes (\mathcal{G} \otimes \bar{N} \otimes \mathcal{G})$, let us define a proposed inverse map θ' on monomials by

$$\theta' : (a \otimes m \otimes a') \otimes (\gamma \otimes n \otimes \gamma') \mapsto a\gamma_3 \otimes (S^{-1}(\gamma_2)m \otimes n) \otimes {}^{S^{-1}(\gamma_1)}a'\gamma'.$$

Let us check the composition $\theta'\theta$. Recalling that $\bar{M}$ is a $\mathcal{G}$-submodule in M, we have

$$\begin{aligned}
\theta'\theta(a\gamma \otimes (m \otimes n) \otimes a'\gamma') &= \theta'(a \otimes (\gamma_1 m) \otimes ({}^{\gamma_2}a') \otimes \gamma_3 \otimes n \otimes \gamma') \\
&= a\gamma_5 \otimes (S^{-1}(\gamma_4)\gamma_1 m \otimes n) \otimes ({}^{S^{-1}(\gamma_3)\gamma_2}a')\gamma' \\
&= a\gamma_4 \otimes (S^{-1}(\gamma_3)\gamma_1 m \otimes n) \otimes \epsilon(\gamma_2)a'\gamma' \\
&= a\gamma_4 \otimes (S^{-1}(\epsilon(\gamma_2)\gamma_3)\gamma_1 m \otimes n) \otimes a'\gamma' \\
&= a\gamma_3 \otimes (S^{-1}(\gamma_2)\gamma_1 m \otimes n) \otimes a'\gamma' \\
&= a\gamma \otimes m \otimes n \otimes a'\gamma'.
\end{aligned}$$

So $\theta'\theta = id$. As for $\theta\theta'$, we have

$$\begin{aligned}
\theta\theta'((a \otimes m \otimes a') \otimes (\gamma \otimes n \otimes \gamma')) &= \theta(a\gamma_3 \otimes (S^{-1}(\gamma_2)m \otimes n) \otimes {}^{S^{-1}(\gamma_1)}a'\gamma') \\
&= (a \otimes \gamma_3 S^{-1}(\gamma_2)m \otimes {}^{\gamma_4 S^{-1}(\gamma_1)}a') \otimes (\gamma_5 \otimes n \otimes \gamma') \\
&= (a \otimes m \otimes a') \otimes (\gamma \otimes n \otimes \gamma').
\end{aligned}$$

So $\theta\theta' = id$ and we conclude $\theta' = \theta^{-1}$.

In the case that N is not free, we know that N is a summand of some free module $\mathcal{N}$. This will imply that $N^\uparrow$ is a summand of $\mathcal{N}^\uparrow$ as a Hopf bimodule. It follows that $M * N^\uparrow$ is a summand of the free module $M * \mathcal{N}^\uparrow$, and hence projective. ■

Proof of Theorem 2.14. We already know that there is a quasi-isomorphism of $(A * \mathcal{G})^e$-complexes $K * L^\uparrow \to A * \mathcal{G}$, by Lemma 2.13. So we need only show that the smash product complex is projective in each degree. We have chosen K so that each K^i is an equivariant bimodule satisfying the hypotheses of Lemma 2.15, and each L^j is projective by choice. So each $K^i * (L^j)^\uparrow = K^i * (L^\uparrow)^j$ is projective by Lemma 2.15. Now, projectivity of the smash product $K * L^\uparrow$ in each degree follows from the fact that each $(K * L^\uparrow)^n$ is a finite sum of projective modules $K^i * (L^\uparrow)^j$. ■

Remark 2.16. The resolution of $A * \mathcal{G}$ constructed above is one of a number resolutions that have appeared in the literature. In [27], Guccione and Guccione provide a resolution X of the smash $A * \mathcal{G}$ which is the tensor product of the bar resolution of A with the bar resolution of $\mathcal{G}$, along with some explicit differential. In the case that $\mathcal{G}$ is a group algebra, Shepler and Witherspoon have provided a class of resolutions of the smash product [75, Section 4]. Our resolution $K * L^\uparrow$ is a member of their class of resolutions (up to isomorphism). The reader should be aware that the construction given in [75] is somewhat different than the one given here.

2.4 Hochschild cochains as derived invariants

Let $\mathcal{G}$ be a Hopf algebra and A be a $\mathcal{G}$-module algebra.

Definition 2.17 (Our $\mathcal{G}$-action on hom complexes)**.** Let M be a complex of $A * \mathcal{G}$-bimodules and let X be a complex of $\mathcal{G}$-equivariant A-bimodules. We define a right $\mathcal{G}$-module structure on the set of homs $\mathrm{Hom}_{A^e}(X, M)$ by the formula

$$f \cdot \gamma(x) := S(\gamma_1) f(\gamma_2 x)\gamma_3,$$

where $f \in \mathrm{Hom}_{A^e}(X, M)$, $\gamma \in \mathcal{G}$, and $x \in X$.

This action was also considered in [27], and similar actions have appeared throughout the literature (see for example [44, Section 5]). The first portion of the action, $S(\gamma_1)f(\gamma_2 x)$, assures that $f \cdot \gamma$ preserves left A-linearity. The additional right action is necessary to preserve right A-linearity.

Lemma 2.18. *Let M be a complex of $A{*}\mathcal{G}$-bimodules and X be a complex of $\mathcal{G}$-equivariant A-bimodules. The $\mathcal{G}$-module structure on $\mathrm{Hom}_{A^e}(X, M)$ given in Definition 2.17 is compatible with the differential on the hom complex. That is to say, $\mathrm{Hom}_{A^e}(X, -)$ is a functor from $A * \mathcal{G}$-complexes to $\mathcal{G}$-complexes.*

Proof. Recall that the differential on the hom complex is given by $d : f \mapsto d_M f \pm f d_X$. So $\mathcal{G}$-linearity of the differential on the hom complex follows by $\mathcal{G}$-linearity of d_M and d_X. ■

Let L be a projective resolution of the trivial right $\mathcal{G}$-module k, and K be a bimodule resolution of A satisfying conditions (I) and (II) of the previous section. For a complex of $A * \mathcal{G}$-bimodules M, any map $\theta \in \mathrm{Hom}_k(K * L^{\uparrow}, M)$, and any $l \in L^{\uparrow}$, we let $\theta(- \otimes l)$ denote the k-linear map

$$K \to M$$
$$x \mapsto (-1)^{|x||l|}\theta((l_{-1}x) \otimes l_0).$$

Before giving the main theorem of this section let us highlight some points of interest. First, note that there is an embedding of chain complexes $L \to L^{\uparrow}$, $\ell \mapsto \ell \otimes_{\mathcal{G}} 1$. This map becomes $\mathcal{G}$-linear if we take the codomain to be $L^{\uparrow}$ with the adjoint action. It is via this map that we view L as a subcomplex in $L^{\uparrow}$. Second, note that for any $l \in L \subset L^{\uparrow}$ we have $\rho(l) = 1 \otimes l$. Therefore, for all $l \in L \subset L^{\uparrow}$, $\theta(- \otimes l)$ is just the map $x \mapsto (-1)^{|x||l|}\theta(x \otimes l)$.

Theorem 2.19. *Let L be a projective resolution of the trivial right $\mathcal{G}$-module k, and K be a bimodule resolution of A satisfying conditions (I) and (II). Then for any complex M of $A * \mathcal{G}$-bimodules the map*

$$\Xi : \mathrm{Hom}_{(A*\mathcal{G})^e}(K * L^{\uparrow}, M) \to \mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}_{A^e}(K, M))$$
$$\theta \mapsto (l \mapsto \theta(- \otimes l))$$

is a natural isomorphism of chain complexes.

In light of Theorem 2.14, we are claiming that there is an explicit natural isomorphism of derived functors

$$\mathrm{RHom}_{(A*\mathcal{G})^e}(A*\mathcal{G},-)\overset{\cong}{\to}\mathrm{RHom}_{\mathcal{G}}(k,\mathrm{RHom}_{A^e}(A,-)).$$

Proof. To distinguish between the action of $\mathcal{G}$ on L as a subcomplex in $L^{\uparrow}$, and the action of $\mathcal{G}$ on L itself, we will denote the action of $\mathcal{G}$ on $L^{\uparrow}$ by juxtaposition, and the action of $\mathcal{G}$ on L by a dot $\cdot$. So, for $l\in L\subset L^{\uparrow}$ and $\gamma\in\mathcal{G}$, we have

$$l\cdot\gamma=S(\gamma_1)l\gamma_2.$$

It is straightforward to check that Ξ is a map of chain complexes, and we omit the computation. We need to check that, for each θ, the map $\Xi(\theta)$ is right $\mathcal{G}$-linear, that each $\theta(-\otimes l)$ is A^e-linear, and that Ξ is bijective.

Fix a homogeneous $A*\mathcal{G}$-bimodule map $\theta: K*L^{\uparrow}\to M$. Since the coaction on $L^{\uparrow}$ restricts to a trivial coaction on L, the map $\theta(-\otimes l): K\to M$ is seen to be A^e-linear for any $l\in L$. Furthermore, for any $\gamma\in\mathcal{G}$, $l\in L$, and $x\in K$, $\mathcal{G}$-linearity of θ on the left and right gives the sequence of equalities

$$\begin{aligned}\theta(-\otimes l\cdot\gamma)(x)&=(-1)^{|l||x|}\theta(x\otimes l\cdot\gamma)\\&=(-1)^{|l||x|}\theta(x\otimes S(\gamma_1)l\gamma_2)\\&=(-1)^{|l||x|}\theta((S(\gamma_2)\gamma_3x)\otimes S(\gamma_1)l\gamma_4)\\&=(-1)^{|l||x|}S(\gamma_1)\theta((\gamma_2x)\otimes l)\gamma_3\\&=(\theta(-\otimes l)\cdot\gamma)(x).\end{aligned}$$

So we see that $\Xi(\theta)$ is in fact a right $\mathcal{G}$-linear map $L\to\mathrm{Hom}_{A^e}(K,M)$.

To see that Ξ is an isomorphism we provide an explicit inverse. We define, for any $\mathcal{G}$-linear map

$$\chi: L\to\mathrm{Hom}_{A^e}(K,M),$$

a graded vector space map $K \otimes L \otimes \mathcal{G} \to M$ by

$$x \otimes l \otimes \gamma \mapsto (-1)^{|x||l|}\chi(l)(x)\gamma.$$

Precomposing with the isomorphism $K * L^{\uparrow} \cong K \otimes L \otimes \mathcal{G}$ provided by Lemma 2.3 then produces a map $\Phi(\chi) : K * L^{\uparrow} \to M$ defined by

$$\Phi(\chi)(x \otimes \gamma l \gamma') := (-1)^{|x||l|}\chi(l \cdot S^{-1}(\gamma_2))(x)\gamma_1\gamma', \tag{2.4.1}$$

for any $x \in K$, $l \in L \subset L^{\uparrow}$, and $\gamma, \gamma' \in \mathcal{G}$.

The fact that $\Phi(\chi)$ is left A-linear and right $\mathcal{G}$-linear is clear. Right A-linearity follows from right A-linearity of $\chi(l)$ and the fact that coaction on $L^{\uparrow}$ restricts to a trivial coaction on L. For left $\mathcal{G}$-linearity, let $x \in K$, $l \in L \subset L^{\uparrow}$, and $\gamma \in \mathcal{G}$. We have

$$\begin{aligned}
\Phi(\chi)(\gamma(x \otimes l)) &= \Phi(\chi)(\gamma_1 x \otimes \gamma_2 l)\\
&= (-1)^{|x||l|}\chi(l \cdot S^{-1}(\gamma_3))(\gamma_1 x)\gamma_2\\
&= (-1)^{|x||l|}\gamma_5\chi(l)(S^{-1}(\gamma_4)\gamma_1 x)S^{-1}(\gamma_3)\gamma_2 \quad (\mathcal{G}\text{-linearity of } \chi)\\
&= (-1)^{|x||l|}\gamma_3\chi(l)(S^{-1}(\gamma_2)\gamma_1 x)\\
&= (-1)^{|x||l|}\gamma\chi(l)(x)\\
&= \gamma\Phi(\chi)(x \otimes l).
\end{aligned}$$

We can use right $\mathcal{G}$-linearity of $\Phi(\chi)$ to extend the above computation to all of $K * L^{\uparrow} = K \otimes L\mathcal{G}$. Whence we see that $\Phi(\chi)$ is a $A * \mathcal{G}$-bimodule map for arbitrary $\chi : L \to \mathrm{Hom}_{A^e}(K, M)$. The equalities $\Phi(\Xi(\theta)) = \theta$ and $\Xi(\Phi(\chi)) = \chi$ follow by construction. So $\Phi = \Xi^{-1}$ and $\Xi = \Phi^{-1}$. ■

Corollary 2.20. *Let L and K be as in Theorem 2.19, and M be a $A * \mathcal{G}$-bimodule. Then we have a graded isomorphism*

$$HH^{\bullet}(A * \mathcal{G}, M) \cong H^{\bullet}\big(\mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}_{A^e}(K, M))\big).$$

Proof. This follows from Theorem 2.19 and the fact that the Hochschild cohomology is given by the homology of the complex $\mathrm{Hom}_{(A*\mathcal{G})^e}(K * L^{\uparrow}, M)$, since $K * L^{\uparrow}$ is a projective bimodule resolution of the smash product $A * \mathcal{G}$. ■

We can, in fact, replace our resolution K with any equivariant A^e-projective resolution of A. Let $P \to A$ be any $\mathcal{G}$-equivariant A-bimodule resolution of A which is projective over A^e. By a straightforward process, we can produce an equivariant complex Q admitting equivariant quasi-isomorphisms $P \to Q$ and $K \to Q$. First, take d^0 to be the coproduct map $K^0 \oplus P^0 \to A$. Then we construct Q inductively as the complex

$$Q = \cdots \to K^2 \oplus (A \otimes kerd^1 \otimes A) \oplus P^2 \overset{d^2}{\to} K^1 \oplus (A \otimes kerd^0 \otimes A) \oplus P^1 \overset{d^1}{\to} K^0 \oplus P^0 \to 0,$$

where $\mathcal{G}$ acts diagonally on the summands $(A \otimes kerd^i \otimes A)$. Note that Q is a complex of $\mathcal{G}$-equivariant bimodules, and that each Q^i is projective over A^e. The map $d^0 : Q \to A$ is a quasi-isomorphism by construction. Whence we see that the two inclusions $i_K : K \to Q$ and $i_P : P \to Q$ are equivariant quasi-isomorphisms. Taking

$$\mathscr{X} = \mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}_{A^e}(Q, M))$$

then gives the following corollary.

Corollary 2.21. *Let L and K be as in Theorem 2.19, and M be a complex of $A * \mathcal{G}$-bimodules. Let $P \to A$ be an equivariant bimodule resolution of A which is projective over A^e. The complex $\mathscr{X}$ admits quasi-isomorphisms*

$$\mathrm{Hom}_{(A*\mathcal{G})^e}(K * L^{\uparrow}, M) \overset{\sim}{\leftarrow} \mathscr{X} \overset{\sim}{\to} \mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}_{A^e}(P, M)).$$

Proof. Since L is a bounded above complex of projectives, the functor $\mathrm{Hom}_{\mathcal{G}}(L, -)$ preserves quasi-isomorphisms. Whence the proposed quasi-isomorphisms can be given by

$$\mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}_{A^e}(Q, M)) \overset{(i_K^*)_*}{\to} \mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}_{A^e}(K, M)) \cong \mathrm{Hom}_{(A*\mathcal{G})^e}(K * L^{\uparrow}, M)$$

and

$$\mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}_{A^e}(Q, M)) \overset{(i_P^*)_*}{\to} \mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}_{A^e}(P, M)).$$

■

For L and K as above, and any $A*\mathcal{G}$-bimodule M, the complex $\mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}_{A^e}(K, M))$ is the total complex of the first quadrant double complex

$$\begin{tikzcd}
 & \vdots & \vdots & \\
0 \arrow[r] & \mathrm{Hom}(L^0, \mathrm{Hom}(K^{-1}, M)) \arrow[u] \arrow[r, "{d_L}^*"] & \mathrm{Hom}(L^{-1}, \mathrm{Hom}(K^{-1}, M)) \arrow[u] \arrow[r] & \cdots \\
0 \arrow[r] & \mathrm{Hom}(L^0, \mathrm{Hom}(K^0, M)) \arrow[u, "\pm(d_K^*)_*"] \arrow[r, "{d_L}^*"] & \mathrm{Hom}(L^{-1}, \mathrm{Hom}(K^0, M)) \arrow[u, "\pm(d_K^*)_*"] \arrow[r] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & .
\end{tikzcd} \tag{2.4.2}$$

It follows that there are two spectral sequences converging to the Hochschild cohomology of $A*\mathcal{G}$ with coefficients in M. Filtering by the degree on L produces a spectral sequence

$$E_2 = \mathrm{Ext}_{\mathcal{G}}(k, HH^{\bullet}(A*\mathcal{G}, M)).$$

The existence of this spectral sequence is well known. It first appeared in the work of Stefan as a Grothendieck spectral sequence in the setting of a Hopf Galois extension [82], and then in a paper by Guccione and Guccione [27, Corollary 3.2.3]. Since these results are well established, we do not elaborate on the details here. We will show in Section 2.6 that both of these spectral sequences can be used to calculate the cup product on Hochschild cohomology when appropriate. All necessary details will be given there.

Notation 2.22. The filtration induced by the degree of L on the cohomology

$$HH^{\bullet}(A*\mathcal{G}, M) = H^{\bullet}(\mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}_{A^e}(K, M)))$$

will be denoted $F^{\mathcal{G}}$. The filtration induced by the degree on K will be denoted F^A. The associated graded spaces with respect to these filtrations will be denoted

$$\mathrm{gr}_{\mathcal{G}} HH^{\bullet}(A*\mathcal{G}, M) = \bigoplus_i \frac{F_i^{\mathcal{G}}\big(HH^{\bullet}(A*\mathcal{G}, M)\big)}{F_{i-1}^{\mathcal{G}}\big(HH^{\bullet}(A*\mathcal{G}, M)\big)}$$

and

$$\mathrm{gr}_{A} HH^{\bullet}(A*\mathcal{G}, M) = \bigoplus_i \frac{F_i^{A}\big(HH^{\bullet}(A*\mathcal{G}, M)\big)}{F_{i-1}^{A}\big(HH^{\bullet}(A*\mathcal{G}, M)\big)}$$

respectively.

2.5 *Reminder of the cup products on Hochschild cohomology and derived invariant algebras*

The following general approach to the cup product on Hochschild cohomology follows [78]. Let R be any algebra and let B be an algebra extension of R, i.e. an algebra equipped with an algebra map $R \to B$. Let P be a projective R-bimodule resolution of R with quasi-isomorphism $\varphi : P \to R$. Then $P \otimes_R P$ is also a projective resolution of R with quasi-isomorphism $\varphi \otimes_R \varphi : P \otimes_R P \to R$. Whence there exists a quasi-isomorphism $\omega : P \to P \otimes_R P$ which fits into a diagram

$$\begin{tikzcd} P \arrow[rr, "\omega"] \arrow[dr, "\varphi"'] & & P \otimes_R P \arrow[dl, "\varphi\otimes_R\varphi"] \\ & R & \end{tikzcd} \tag{2.5.1}$$

and is unique up to homotopy. From this we get a product map

$$\mathrm{Hom}_{R^e}(P, B) \otimes \mathrm{Hom}_{R^e}(P, B) \to \mathrm{Hom}_{R^e}(P, B)$$
$$f \otimes g \mapsto \mu_B(f \otimes_R g)\omega,$$

and subsequent dg algebra structure on $\mathrm{Hom}_{R^e}(P, B)$. One can check that any choice of ω results in the same product on the cohomology $HH^\bullet(R, B)$. We call this product the *cup product*. Note that the dg algebra $\mathrm{Hom}_{R^e}(P, B)$ need not be associative, but it will be associative up to a homotopy.

Suppose now that $\mathcal{G}$ is a Hopf algebra and L is a projective resolution of $k_{\mathcal{G}} = \mathcal{G}/ker\epsilon$. Let $\mathscr{B}$ be a right $\mathcal{G}$-module dg algebra. (We do not require that $\mathscr{B}$ is strictly associative.) Since $\mathcal{G} \otimes \mathcal{G}$ is free over $\mathcal{G}$, the diagonal action on $L \otimes L$ makes it into a projective resolution of k as well. So, again, we have a quasi-isomorphism $\sigma : L \to L \otimes L$ which is unique up to homotopy and fits into a diagram analogous to (2.5.1). Hence, we get a similarly defined product on the derived invariants

$$\mathrm{Hom}_{\mathcal{G}}(L, \mathscr{B}) \otimes \mathrm{Hom}_{\mathcal{G}}(L, \mathscr{B}) \to \mathrm{Hom}_{\mathcal{G}}(L, \mathscr{B})$$
$$f \otimes g \mapsto \mu_{\mathscr{B}}(f \otimes g)\sigma.$$

This product is unique on cohomology and gives $\mathrm{Hom}_{\mathcal{G}}(L, \mathscr{B})$ the structure of a (not-necessarily-associative) dg algebra.

2.6 Hochschild cohomology as a derived invariant algebra

Let $\mathcal{G}$ be a Hopf algebra and A be a $\mathcal{G}$-module algebra. We also fix a bimodule resolution $\tau : K \to A$ which satisfies conditions (I) and (II) of Section 2.3, and a projective resolution $\xi : L \to k$ of the trivial right $\mathcal{G}$-module. From here on out we assume K also satisfies

(III) there is a quasi-isomorphism $\omega : K \to K \otimes_A K$ of complexes of $\mathcal{G}$-equivariant A-bimodules.

Here we give $K \otimes_A K$ the diagonal $\mathcal{G}$-action. As was stated in the previous section, there will always be some quasi-isomorphism ω of A^e-complexes. The content of condition (III) is that we may choose ω to be $\mathcal{G}$-linear.

In the case of the bar resolution

$$BA = \cdots \to A \otimes A^{\otimes 2} \otimes A \to A \otimes A \otimes A \to A \otimes A \to 0$$

the map ω is given by

$$\omega : b \otimes a_1 \otimes \ldots \otimes a_n \otimes b' \mapsto \sum_{0 \le i \le n} (b \otimes a_1 \otimes \ldots \otimes a_i \otimes 1) \otimes_A (1 \otimes a_{i+1} \otimes \ldots \otimes a_n \otimes b') \quad (2.6.1)$$

We will denote the image of ω using a Sweedler's type notation, as if ω were a comultiplication. Specifically, on elements we take $\omega_1(x) \otimes_A \omega_2(x) = \omega(x)$, with the sum suppressed. In this notation, $\mathcal{G}$-linearity of ω is equivalent to the equality $\omega_1(\gamma x) \otimes_A \omega_2(\gamma x) = \gamma_1 \omega_1(x) \otimes_A \gamma_2 \omega_2(x)$, for all $\gamma \in \mathcal{G}$ and $x \in K$.

Let us also fix a quasi-isomorphism $\sigma : L \to L \otimes L$. As with ω and K, we denote the image of $l \in L$ under σ by $\sigma_1(l) \otimes \sigma_2(l)$. In this notation $\mathcal{G}$-linearity appears as $\sigma_1(l \cdot \gamma) \otimes \sigma_2(l \cdot \gamma) = \sigma_1(l) \cdot \gamma_1 \otimes \sigma_2(l) \cdot \gamma_2$.

Proposition 2.23. *For any algebra extension B of $A * \mathcal{G}$, the complex $\mathrm{Hom}_{A^e}(K, B)$, with the product of Section 2.5 and $\mathcal{G}$-action of Definition 2.17, is a right $\mathcal{G}$-module dg algebra.*

Proof. Let us denote the multiplication on $\mathrm{Hom}_{A^e}(K, B)$ by juxtaposition. We need to show that for functions $f, g \in \mathrm{Hom}_{A^e}(K, B)$, and $\gamma \in \mathcal{G}$, the formula $(fg) \cdot \gamma = (f \cdot \gamma_1)(g \cdot \gamma_2)$

holds. Let us simply check on elements. We have, for any $x \in K$,

$$\begin{aligned} ((fg)\cdot\gamma)(x) &= S(\gamma_1)(fg)(\gamma_2 x)\gamma_3 \\ &= \pm S(\gamma_1)f(\gamma_2\omega_1(x))g(\gamma_3\omega_2(x))\gamma_4 && \text{(by $\mathcal{G}$-linearity of ω)} \\ &= \pm\big(S(\gamma_1)f(\gamma_2\omega_1(x))\gamma_3\big)\big(S(\gamma_4)g(\gamma_5\omega_2(x))\gamma_6\big) \\ &= (f\cdot\gamma_1)(\omega_1(x))(g\cdot\gamma_2)(\omega_2(x)) \\ &= \big((f\cdot\gamma_1)(g\cdot\gamma_2)\big)(x). \end{aligned}$$

■

According to this proposition, and the material of Section 2.5, the double complex $\mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}_{A^e}(K,B))$ will now carry a natural dg algebra structure.

We now seek to extend the diagonal map σ on L to a diagonal map on the induced complex $L^{\uparrow}$. One can verify that the obvious map $L \otimes L \to L^{\uparrow} \otimes_{\mathcal{G}} L^{\uparrow}$ is an embedding, since the statement holds when L is free. In this way we view $L \otimes L$ as a subcomplex of $L^{\uparrow} \otimes_{\mathcal{G}} L^{\uparrow}$. The complex $L^{\uparrow} \otimes_{\mathcal{G}} L^{\uparrow}$ is taken to be a $\mathcal{G}$-comodule under the standard tensor $\mathcal{G}$-comodule structure $l \otimes_{\mathcal{G}} l' \mapsto (l_{-1}l'_{-1}) \otimes (l_0 \otimes_{\mathcal{G}} l'_0)$. Since $L^{\uparrow}$ is itself a Hopf bimodule over $\mathcal{G}$, this coaction gives $L^{\uparrow} \otimes_{\mathcal{G}} L^{\uparrow}$ the structure of a Hopf bimodule as well. Before giving the next result we also note that, on elements, commutativity of the diagram

$$\begin{tikzcd} L \arrow[rr, "\sigma"] \arrow[dr, "\xi"'] & & L \otimes L \arrow[dl, "\xi\otimes\xi"] \\ & k & \end{tikzcd}$$

produces the equality $\xi(l) = \xi(\sigma_1(l))\xi(\sigma_2(l))$ for each $l \in L$.

Lemma 2.24. *The map $\sigma : L \to L \otimes L \subset L^{\uparrow} \otimes_{\mathcal{G}} L^{\uparrow}$ extends uniquely to a quasi-isomorphism of chain complexes of Hopf-bimodules $\sigma^{\uparrow} : L^{\uparrow} \to L^{\uparrow} \otimes_{\mathcal{G}} L^{\uparrow}$.*

Proof. Let $\cdot$ denote the right action of $\mathcal{G}$ on L, juxtaposition denote the action of $\mathcal{G}$ on the bimodule $L^{\uparrow}$, and l denote an element in $L \subset L^{\uparrow}$ throughout. We employ the expression of $L^{\uparrow}$ as the free right $\mathcal{G}$-module $L^{\uparrow} \cong L \otimes \mathcal{G}$, via Lemma 2.3, and define $\sigma^{\uparrow}$ as

$$\sigma^{\uparrow}(l \otimes \gamma) = \sigma_1(l) \otimes_{\mathcal{G}} \sigma_2(l)\gamma.$$

Colinearity of $\sigma^{\uparrow}$ is clear from the definitions of the coactions given below Definition 2.5 and in the paragraph preceding this lemma.

If we simply view $L^{\uparrow}$ as a complex of bimodules generated by the subspace $L \subset L^{\uparrow}$, the map $\sigma^{\uparrow}$ will be given by

$$\gamma l \gamma' \mapsto \gamma \sigma_1(l) \otimes_{\mathcal{G}} \sigma_2(l)\gamma'. \tag{2.6.2}$$

This follows from a direct calculation using the left and right actions specified at Lemma 2.3,

$$\begin{aligned}
\sigma^{\uparrow}(l \cdot S^{-1}(\gamma_2) \otimes \gamma_1\gamma') &= \sigma_1(l \cdot S^{-1}(\gamma_2)) \otimes_{\mathcal{G}} \sigma_2(l \cdot S^{-1}(\gamma_2))\gamma_1\gamma' \\
&= (\sigma_1(l) \cdot S^{-1}(\gamma_3)) \otimes (\sigma_2(l) \cdot S^{-1}(\gamma_2))\gamma_1\gamma' \\
&= \gamma_5\sigma_1(l)S^{-1}(\gamma_4) \otimes_{\mathcal{G}} \gamma_3\sigma_2(l)S^{-1}(\gamma_2)\gamma_1\gamma' \\
&= \gamma_3\sigma_1(l)S^{-1}(\gamma_2)\gamma_1 \otimes_{\mathcal{G}} \sigma_2(l)\gamma' \\
&= \gamma\sigma_1(l) \otimes_{\mathcal{G}} \sigma_2(l)\gamma'.
\end{aligned}$$

So, from (2.6.2), we see that this map is $\mathcal{G}$-linear on the left and right.

Recall that $\sigma : L \to L \otimes L$ was chosen so that $\xi(l) = \xi(\sigma_1(l))\xi(\sigma_2(l))$, and that $\xi^{\uparrow} : L^{\uparrow} \cong L \otimes \mathcal{G} \to \mathcal{G}$ is defined by $l \otimes \gamma \mapsto \xi(l)\gamma$. So we will have the commutative diagram

$$\begin{tikzcd}
L^{\uparrow} \arrow[rr, "\sigma^{\uparrow}"] \arrow[dr, "\xi^{\uparrow}"'] & & L^{\uparrow} \otimes_{\mathcal{G}} L^{\uparrow} \arrow[dl, "\xi^{\uparrow} \otimes_{\mathcal{G}} \xi^{\uparrow}"] \\
& \mathcal{G} & .
\end{tikzcd}$$

The fact that $\sigma^{\uparrow}$ is a quasi-isomorphism follows from commutativity of the above diagram and the fact that $\xi^{\uparrow}$ and $\xi^{\uparrow} \otimes_{\mathcal{G}} \xi^{\uparrow}$ are quasi-isomorphisms. ■

Now we have a quasi-isomorphism $\omega : K \to K \otimes_A K$ and have produced a quasi-isomorphism $\sigma^{\uparrow} : L^{\uparrow} \to L^{\uparrow} \otimes_{\mathcal{G}} L^{\uparrow}$ from the given map $\sigma : L \to L \otimes L$. We would like to use this information, along with some twisting, to produce an explicit quasi-isomorphism

$$K * L^{\uparrow} \to (K * L^{\uparrow}) \otimes_{A*\mathcal{G}} (K * L^{\uparrow}).$$

The next lemma offers the "twisting" portion of the proposed construction.

Lemma 2.25. *The isomorphism of k-complexes*

$$\begin{gathered}(K \otimes_A K) \otimes (L \otimes L) \to (K \otimes L) \otimes_A (K \otimes L)\\ (x \otimes_A y) \otimes (l \otimes l') \mapsto (-1)^{|l||y|}(x \otimes l) \otimes_A (y \otimes l')\end{gathered}$$

extends uniquely to an isomorphism $\phi : (K \otimes_A K) * (L^{\uparrow} \otimes_{\mathcal{G}} L^{\uparrow}) \to (K * L^{\uparrow}) \otimes_{A*\mathcal{G}} (K * L^{\uparrow})$ of complexes of $A * \mathcal{G}$-bimodules.

Proof. The map ϕ is given by

$$\phi : (x \otimes_A y) \otimes (l \otimes_{\mathcal{G}} l') \mapsto (-1)^{|l||y|}(x \otimes l_0) \otimes_{A*\mathcal{G}} (S^{-1}(l_{-1})y \otimes l'),$$

for $x, y \in K$, $l, l' \in L^{\uparrow}$. The fact that ϕ is well defined follows by standard manipulations, which we do not reproduce here. The fact that ϕ is a chain map can be verified by using the $\mathcal{G}$-linearity and $\mathcal{G}$-colinearity of the differentials on K and $L^{\uparrow}$ respectively.

In order to show that ϕ is an $A * \mathcal{G}$-bimodule map, the only non-trivial things to check are left $\mathcal{G}$-linearity and right A-linearity. For left $\mathcal{G}$-linearity we have, for any $\gamma \in \mathcal{G}$,

$$\begin{aligned}\phi(\gamma((x \otimes_A y) \otimes (l \otimes_{\mathcal{G}} l'))) &= \phi((\gamma_1 x \otimes_A \gamma_2 y) \otimes (\gamma_3 l \otimes_{\mathcal{G}} l'))\\ &= \pm(\gamma_1 x \otimes \gamma_4 l_0) \otimes_{A*\mathcal{G}} (S^{-1}(\gamma_3 l_{-1})\gamma_2 y \otimes l')\\ &= \pm(\gamma_1 x \otimes \gamma_4 l_0) \otimes_{A*\mathcal{G}} (S^{-1}(l_{-1})S^{-1}(\gamma_3)\gamma_2 y \otimes l')\\ &= \pm(\gamma_1 x \otimes \gamma_2 l_0) \otimes_{A*\mathcal{G}} (S^{-1}(l_{-1})y \otimes l')\\ &= \pm\gamma\left((x \otimes l_0) \otimes_{A*\mathcal{G}} (S^{-1}(l_{-1})y \otimes l')\right)\\ &= \gamma\phi((x \otimes_A y) \otimes (l \otimes_{\mathcal{G}} l')).\end{aligned}$$

For right A-linearity we have, for any $a \in A$,

$$\begin{aligned}\phi(((x \otimes_A y) \otimes (l \otimes_{\mathcal{G}} l'))a) &= \phi((x \otimes_A y({}^{l_{-1}l'_{-1}}a)) \otimes (l_0 \otimes_{\mathcal{G}} l'_0))\\ &= \pm(x \otimes l_0) \otimes_{A*\mathcal{G}} (S^{-1}(l_{-1})(y({}^{l_{-2}l'_{-1}}a)) \otimes l'_0)\\ &= \pm(x \otimes l_0) \otimes_{A*\mathcal{G}} ((S^{-1}(l_{-1})y)({}^{S^{-1}(l_{-2})l_{-3}l'_{-1}}a)) \otimes l'_0)\\ &= \pm(x \otimes l_0) \otimes_{A*\mathcal{G}} ((S^{-1}(l_{-1})y)({}^{l'_{-1}}a)) \otimes l'_0)\\ &= \pm\left((x \otimes l_0) \otimes_{A*\mathcal{G}} (S^{-1}(l_{-1})y \otimes l'_0)\right)a\\ &= \phi((x \otimes_A y) \otimes (l \otimes_{\mathcal{G}} l'))a.\end{aligned}$$

The inverse to ϕ is the map

$$(x \otimes l) \otimes_{A*\mathcal{G}} (y \otimes l') \mapsto (-1)^{|l||y|}(x \otimes_A l_{-1}y) \otimes (l_0 \otimes_{\mathcal{G}} l').$$

■

Proposition 2.26. *Let ϕ be the isomorphism of Lemma 2.25, and $\sigma^\uparrow : L^\uparrow \to L^\uparrow \otimes_{\mathcal{G}} L^\uparrow$ be the quasi-isomorphism of Lemma 2.24. Then the map*

$$\phi(\omega \otimes \sigma^\uparrow) : K * L^\uparrow \to K * L^\uparrow \otimes_{A*\mathcal{G}} K * L^\uparrow$$

is a quasi-isomorphism of $(A * \mathcal{G})^e$-complexes.

Proof. Since $\sigma^\uparrow$ is a map of complexes of Hopf bimodules by Lemma 2.24, and ω is a map of complexes of equivariant A-bimodules by choice, $\omega \otimes \sigma^\uparrow$ is $(A * \mathcal{G})^e$-linear, by Lemma 2.12. Also, the product map $\omega \otimes \sigma^\uparrow$ is a quasi-isomorphism since ω and $\sigma^\uparrow$ are themselves quasi-isomorphisms. Now, since ϕ is an isomorphism of $(A * \mathcal{G})^e$-complexes, the claim follows. ■

For any algebra extension $A*\mathcal{G} \to B$, we define the cup product on $\mathrm{Hom}_{(A*\mathcal{G})^e}(K * L^\uparrow, B)$ by way of the diagonal map $K * L^\uparrow \to K * L^\uparrow \otimes_{A*\mathcal{G}} K * L^\uparrow$ given in Proposition 2.26.

The following are the **hypotheses for Theorem 2.27**: K is a bimodule resolution of A equipped with a diagonal map $\omega : K \to K \otimes_A K$ satisfying conditions (I)-(III), and L is a projective resolution of the trivial right $\mathcal{G}$-module k with a quasi-isomorphism $\sigma : L \to L \otimes L$. We give $K * L^\uparrow$ the diagonal quasi-isomorphism of Proposition 2.26.

Theorem 2.27. *For any algebra extension B of the smash product $A * \mathcal{G}$, the isomorphism*

$$\Xi : \mathrm{Hom}_{(A*\mathcal{G})^e}(K * L^\uparrow, B) \overset{\cong}{\to} \mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}_{A^e}(K, B))$$

of Theorem 2.19 is one of (not-necessarily-associative) dg algebras.

Let us note that, if K and L are chosen appropriately, the dg algebra $\mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}_{A^e}(K, B))$ will be associative. It follows, by the theorem, that $\mathrm{Hom}_{(A*\mathcal{G})^e}(K * L^\uparrow, B)$ will also be associative in this case. For example, one can always take K to be the bar resolution BA of A and L to be the bar resolution $k \otimes_{\mathcal{G}} B\mathcal{G}$ of k to get this property.

Proof. We want to verify commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Hom}(K * L^{\uparrow}, B) \otimes \mathrm{Hom}(K * L^{\uparrow}, B) \arrow[r, "\Xi\otimes\Xi"] \arrow[d, "mult"] & \mathrm{Hom}(L, \mathrm{Hom}(K, B)) \otimes \mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}(K, B)) \arrow[d, "mult"] \\
\mathrm{Hom}(K * L^{\uparrow}, B) \arrow[r, "\Xi"] & \mathrm{Hom}(L, \mathrm{Hom}(K, B)).
\end{tikzcd} \tag{2.6.3}$$

There are three multiplications we need to deal with here. For the purpose of this proof we will denote the products on

$$\mathrm{Hom}_{(A*\mathcal{G})^e}(K * L^{\uparrow}, B),\ \mathrm{Hom}(L, \mathrm{Hom}_{A^e}(K, B)),\ \text{and } \mathrm{Hom}_{A^e}(K, B)$$

by a dot $\cdot$, an asterisk $*$, and juxtaposition respectively. Let θ and θ' be functions in $\mathrm{Hom}_{(A*\mathcal{G})^e}(K * L^{\uparrow}, B)$ and fix arbitrary $x \in K$ and $l \in L \subset L^{\uparrow}$.

Following around the top of (2.6.3) sends $\theta\otimes\theta'$ to the function $\Xi(\theta)*\Xi(\theta') \in \mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}_{A^e}(K, B))$. This function sends $l \in L$ to the map

$$(-1)^{|\theta'||\sigma_1(l)|}\theta(- \otimes \sigma_1(l))\theta'(- \otimes \sigma_2(l))$$

in $\mathrm{Hom}_{A^e}(K, B)$, where $\theta(- \otimes \sigma_1(l))$ and $\theta'(- \otimes \sigma_2(l))$ are as defined in the paragraphs preceding Theorem 2.19. Since the coaction on $L^{\uparrow}$ restricts to the trivial coaction on L, the above function evaluated at x is the element

$$(-1)^{\epsilon}\theta\left(\omega_1(x) \otimes \sigma_1(l)\right)\theta'\left(\omega_2(x) \otimes \sigma_2(l)\right) \in B,$$

where

$$\begin{aligned}
\epsilon &= |\theta'||\sigma_1(l)| + |\omega_1(x)|(|\theta'| + |\sigma_2(l)| + |\sigma_1(l)|) + |\omega_2(x)||\sigma_2(l)| \\
&= |\theta'||\sigma_1(l)| + |\omega_1(x)|(|\theta'| + |l|) + |\omega_2(x)||\sigma_2(l)|. \\
&= |\theta'|(|\sigma_1(l)| + |\omega_1(x)|) + |\omega_1(x)||l| + |\omega_2(x)||\sigma_2(l)|.
\end{aligned}$$

Following around the bottom row sends $\theta\otimes\theta'$ to the function $\Xi(\theta\cdot\theta') \in \mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}_{A^e}(K, B))$, which takes our element $l \in L$ to $(\theta \cdot \theta')(- \otimes l) \in \mathrm{Hom}_{A^e}(K, B)$. Evaluating at $x \in K$ produces the element $(-1)^{|x||l|}(\theta \cdot \theta')(x \otimes l) \in B$. Recalling the diagonal map on $K * L^{\uparrow}$ given in Proposition 2.26, the formula for $\phi|(K \otimes_A K) \otimes (L \otimes L)$ given in Lemma 2.25, and the fact that $\sigma^{\uparrow}|L = \sigma$, we have the equality

$$(-1)^{|x||l|}(\theta \cdot \theta')(x \otimes l) = (-1)^{\epsilon'}\theta\left(\omega_1(x) \otimes \sigma_1(l)\right)\theta'\left(\omega_2(x) \otimes \sigma_2(l)\right),$$

where

$$\begin{aligned}\epsilon' &= |x||l| + |\sigma_1(l)||\omega_2(x)| + |\theta'|(|\omega_1(x)| + |\sigma_1(l)|)\\ &= (|\omega_1(x)| + |\omega_2(x)|)|l| + |\sigma_1(l)||\omega_2(x)| + |\theta'|(|\omega_1(x)| + |\sigma_1(l)|)\\ &= |\omega_1(x)||l| + |\omega_2(x)||l| + |\omega_2(x)||\sigma_1(l)| + |\theta'|(|\omega_1(x)| + |\sigma_1(l)|)\\ &= |\omega_1(x)||l| + |\omega_2(x)|(|\sigma_1(l)| + |\sigma_2(l)|) + |\omega_2(x)||\sigma_1(l)| + |\theta'|(|\omega_1(x)| + |\sigma_1(l)|)\\ &\equiv |\omega_1(x)||l| + |\omega_2(x)||\sigma_2(l)| + |\theta'|(|\omega_1(x)| + |\sigma_1(l)|) \mod 2\\ &= \epsilon.\end{aligned}$$

So following around the top or bottom of (2.6.3) produces the same function. ■

Corollary 2.28. *Let B be an algebra extension of $A * \mathcal{G}$, and take K and L as in Theorem 2.27. Then there is an isomorphism of algebras*

$$HH^\bullet(A * \mathcal{G}, B) \cong H^\bullet\left(\mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}_{A^e}(K, B))\right).$$

Proof. This is an immediate consequence of Theorem 2.27 and the fact that

$$HH^\bullet(A * \mathcal{G}, B) = H^\bullet\big(\mathrm{Hom}_{(A*\mathcal{G})^e}(K * L^\uparrow, B)\big)$$

as an algebra. ■

As was the case with Theorem 2.19, we can drop condition (II) on K.

Corollary 2.29. *Let K, L, and B be as in Theorem 2.27. Let $P \to A$ be a $\mathcal{G}$-equivariant bimodule resolution of A which is projective over A^e in each degree. Suppose additionally that P admits a diagonal quasi-isomorphism $P \to P \otimes_A P$ which is $\mathcal{G}$-linear. Then there is third dg algebra $\mathscr{A}$ which admits quasi-isomorphisms*

$$\mathrm{Hom}_{(A*\mathcal{G})^e}(K * L^\uparrow, B) \overset{\sim}{\to} \mathscr{A} \overset{\sim}{\leftarrow} \mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}_{A^e}(P, B)) \tag{2.6.4}$$

which are all algebra maps up to a homotopy.

Let η denote any of the maps of (2.6.4). The main point is that for any cycles f and g in the domain, the difference $\eta(fg) - \eta(f)\eta(g)$ will be a boundary. So all of the maps of (2.6.4) become algebra isomorphisms on homology. The proof of this result is a bit of distraction, and has been relegated to the appendix.

As was discussed in the introduction, a spectral sequence E is called multiplicative if it comes equipped with a bigraded products $E_r^{pq} \otimes E_r^{p'q'} \to E_r^{(p+p')(q+q')}$, for each r, such that each differential $d_r : E_r \to E_r$ is a graded derivation and each isomorphism $E_{r+1} \cong H(E_r)$ is one of algebras. The spectral sequence associated to any filtered dg algebra will be multiplicative, for example. For any multiplicative spectral sequence E, the limiting term E_∞ has the natural structure of a bigraded algebra [88, Multiplicative Structures 5.4.8]. We say that a multiplicative spectral sequence converges to a graded algebra H if H carries an additional filtration and we have an isomorphism of bigraded algebras $E_\infty = \mathrm{grH}$.

Recall the filtrations F^A and $F^{\mathcal{G}}$ on $HH^\bullet(A * \mathcal{G}, B)$ given in Notation 2.22. Since the multiplication on the double complex $\mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}_{A^e}(K, B))$ is bigraded, both the row and column filtrations (i.e. the filtrations induced by the degrees on K and L) give it the structure of a filtered dg algebra. It follows that both of the associated spectral sequences are multiplicative. It also follows that F^A and $F^{\mathcal{G}}$ are algebra filtrations on the Hochschild cohomology.

Corollary 2.30. *For any algebra extension B of the smash product $A * \mathcal{G}$, there are two multiplicative spectral sequences*

$$E_2 = \mathrm{Ext}_{\mathcal{G}}(k, HH^\bullet(A, B)) \Rightarrow HH^\bullet(A * \mathcal{G}, B)$$

and

$$'E_1 = \mathrm{Ext}_{\mathcal{G}}(k, \mathrm{RHom}_{A\text{-bimod}}(A, B)) \Rightarrow HH^\bullet(A * \mathcal{G}, B)$$

which converge to the Hochschild cohomology as an algebra.

Proof. These spectral sequences are induced by the row and column filtrations on the (first quadrant) double complex $\mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}_{A^e}(K, B))$. Since the product on $\mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}_{A^e}(K, B))$

respects both of these filtrations, and its homology is the Hochschild cohomology ring $HH^\bullet(A * \mathcal{G}, B)$, both of the spectral sequences are multiplicative and converge to the Hochschild cohomology.

Filtering by the degree on K produces the spectral sequence with $'E_1 = \mathrm{Ext}_{\mathcal{G}}(k, \mathrm{RHom}_{A^e}(A, B))$, where we take $\mathrm{RHom}_{A^e}(A, B) = \mathrm{Hom}_{A^e}(K, B)$. Filtering by the degree on L produces a spectral sequence with $E_1 = \mathrm{Hom}_{\mathcal{G}}(L, HH^\bullet(A, B))$, since each $\mathrm{Hom}_{\mathcal{G}}(L^i, -)$ is exact and hence commutes with homology. Since the differentials on E_1 are given by d_L^*, it follows that the E_2 term is as described. ■

In the language of Notation 2.22, the E_∞-terms of these spectral sequences are the bigraded algebras $\mathrm{gr}_{\mathcal{G}} HH^\bullet(A * \mathcal{G}, B)$ and $\mathrm{gr}_A HH^\bullet(A * \mathcal{G}, B)$ respectively.

Corollary 2.31. *If the global dimension of* $\mathcal{G}$ *is* ≤ 1 *then we have an isomorphism of algebras*

$$\mathrm{gr}_{\mathcal{G}} HH^\bullet(A * \mathcal{G}, B) \cong \mathrm{Ext}_{\mathcal{G}}(k, HH^\bullet(A, B)).$$

Proof. In this case $E_2 = E_\infty$. ■

2.7 Algebras of extensions as derived invariant algebras

The main purpose of this section is to give some multiplicative spectral sequences converging to the algebra $\mathrm{Ext}_{A*\mathcal{G}}(M, M)$, for any $A * \mathcal{G}$-module M. As was the case with Hochschild cohomology, we will derive our spectral sequences from some explicit isomorphism at the level of cochains. Some related spectral sequences for groups of extensions, without multiplicative structures, were given in [27, Section 3.2.6].

For any algebra R and R-modules M and N, there is a natural bimodule structure on $\mathrm{Hom}_k(M, N)$ induced by the left R-actions on M and N. In the case that $M = N$, the bimodule structure is induced by the associated representation $R \to \mathrm{End}_k(M)$. Since the representation $R \to \mathrm{End}_k(M)$ is an algebra map, the endomorphism algebra of any R-module has the structure of an algebra extension of R.

The Yoneda product on $\mathrm{Ext}_R(M, M)$ is defined in the following manner: first take a projective resolution $Q \to M$, then define $\mathrm{Ext}_R(M, M)$ as the homology algebra of the endomorphism dg algebra $\mathrm{End}_R(Q)$. As the reader may be more familiar with the definition of $\mathrm{Ext}_R(M, M)$ as the homology of the complex $\mathrm{Hom}_R(Q, M)$, let us take a moment to relate the two definitions.

The given quasi-isomorphism $Q \to M$ induces a map $\mathrm{End}_R(Q) = \mathrm{Hom}_R(Q, Q) \to \mathrm{Hom}_R(Q, M)$, which is yet another quasi-isomorphism. To see this, note that if we filter both complexes by the degree induced by Q (in the first coordinate) we get a filtered map of complete filtered complexes. Since each Q^i is projective, and hence each functor is $\mathrm{Hom}_R(Q^i, -)$ exact, the induced map of spectral sequences will be an isomorphism on the E_1 page. Whence, from the Eilenberg-Moore comparison theorem [88, Theorem 5.5.11], we conclude that the original map $\mathrm{End}_R(Q) \to \mathrm{Hom}_R(Q, M)$ is a quasi-isomorphism. Indeed, this is a direct argument for the well known fact that we can use the functor $\mathrm{Hom}_R(Q, -)$ to derive the functor $\mathrm{Hom}_R(M, -)$ [88, Theorem 10.7.4].

Recall that, for any bimodule resolution $\mu : P \to R$, the tensor product $P \otimes_R M$ provides a projective resolution of $\mu \otimes_A id : R \otimes_R M \to M$. This fact is a consequence of Künneth's spectral sequence.

Proposition 2.32 ([88, Lemma 9.1.9])**.** *Let R be any algebra, P be a projective R-bimodule resolution of R, and M and N be any modules over R. The* $\otimes$-Hom *adjunction gives an isomorphism of complexes*

$$\mathrm{Hom}_{R^e}(P, \mathrm{Hom}_k(M, N)) \overset{\cong}{\to} \mathrm{Hom}_R(P \otimes_R M, N).$$

In the case that $M = N$, and P is the bar resolution $P = BR$, there is a quasi-isomorphisms of dg algebras

$$\mathrm{Hom}_{R^e}(P, \mathrm{End}_k(M)) \overset{\sim}{\to} \mathrm{End}_R(P \otimes_R M).$$

Strictly speaking, only the first portion of this proposition is given in Weibel's text. The compatibility of the cup product with the Yoneda product, in the case that $M =$

N, should certainly also be well known. One simply verifies that the map sending $f \in \mathrm{Hom}_{R^e}(P, \mathrm{End}_k(M))$ to the function in $\mathrm{End}_R(P \otimes_R M)$ which is zero on $P^{>-|f|} \otimes_R M$ and maps lower degree monomials as

$$\begin{aligned} & r \otimes r_1 \otimes \ldots \otimes r_n \otimes m \\ & \quad \mapsto (-1)^{|f|(n-|f|)} r \otimes r_1 \otimes \ldots \otimes r_{n-|f|} \otimes \big(f(1 \otimes r_{n-|f|+1} \otimes \ldots \otimes r_n \otimes 1)(m)\big) \end{aligned} \tag{2.7.1}$$

is a morphism dg algebras. The fact that the proposed map is a quasi-isomorphism follows by commutativity of the diagram

$$\begin{tikzcd} & \mathrm{End}_R(P \otimes_R M) \arrow[dr, "(\mu \otimes id)_*"] & \\ \mathrm{Hom}_{R^e}(P, \mathrm{End}_k(M)) \arrow[ur] \arrow[rr, "\cong"] & & \mathrm{Hom}_R(P \otimes_R M, M) \end{tikzcd}$$

and the fact that $(\mu \otimes id)_*$ is a quasi-isomorphism.

Corollary 2.33. *There is a canonical isomorphism of graded vector spaces* $HH^\bullet(R, \mathrm{Hom}_k(M, N)) \cong \mathrm{Ext}_R(M, N)$ *and isomorphism of graded algebras* $HH^\bullet(R, \mathrm{End}_k(M)) \cong \mathrm{Ext}_R(M, M)$.

Note that, for any projective bimodule resolution $P \to R$ with diagonal quasi-isomorphism $\omega : P \to P \otimes_R P$, the dg algebra structure on $\mathrm{Hom}_{R^e}(P, \mathrm{End}(M))$ induces a dg algebra structure on the complex $\mathrm{Hom}_R(P \otimes_R M, M)$ by way of the adjunction isomorphism of Proposition 2.32. For functions $f, g \in \mathrm{Hom}_R(P \otimes_R M, M)$, the product $fg \in \mathrm{Hom}_R(P \otimes_R M, M)$ will be given by

$$fg : x \otimes_R m \mapsto (-1)^{|g||\omega_1(x)|} f(\omega_1(x) \otimes_R g(\omega_2(x) \otimes_R m)),$$

where the notation $\omega(x) = \omega_1(x) \otimes_R \omega_2(x)$ is as in Section 2.6.

Corollary 2.34. *Let M be any R-module, P be any projective R-bimodule resolution over R, and $\omega : P \to P \otimes_R P$ be any R^e-linear quasi-isomorphism. Give* $\mathrm{Hom}_{R^e}(P \otimes_R M, M)$ *the dg algebra structure outlined above. Then*

$$\mathrm{Ext}_R(M, M) \cong \mathrm{H}(\mathrm{Hom}_R(P \otimes_R M, M))$$

as an algebra.

Proof. The algebra structure on $\mathrm{Hom}_{R^e}(P, \mathrm{End}(M))$ is defined so that the isomorphism $\mathrm{Hom}_{R^e}(P, \mathrm{End}(M)) \overset{\cong}{\to} \mathrm{Hom}_R(P \otimes_R M, M)$ is one of dg algebras. In the case that P is the bar resolution, there is an isomorphism of algebras

$$\mathrm{H}(\mathrm{Hom}_R(P \otimes_R M, M)) \cong \mathrm{H}(\mathrm{Hom}_{R^e}(P, \mathrm{End}(M))) \cong \mathrm{Ext}_R(M, M)$$

by Proposition 2.32. The result now follows from the fact that the cup product on

$$\mathrm{H}(\mathrm{Hom}_{R^e}(P, \mathrm{End}(M))) = HH^\bullet(R, \mathrm{End}_k(M))$$

can be computed using any resolution and any quasi-isomorphism $\omega : P \to P \otimes_R P$. ■

Let us return to our analysis of the cohomology of smash products. We fix a Hopf algebra $\mathcal{G}$ and $\mathcal{G}$-module algebra A. Before giving the main results let us clarify a possible point of confusion.

For any $A * \mathcal{G}$-modules M and N there is a standard way to endow $\mathrm{RHom}_A(M, N)$ with a right $\mathcal{G}$-module structure. One simply takes a $A * \mathcal{G}$-complex Q which is a projective resolution of M over A and defines the action on $\mathrm{RHom}_A(M, N) = \mathrm{Hom}_A(Q, N)$ by

$$f \cdot \gamma(q) := S(\gamma_1) f(\gamma_2 q).$$

We would like to know that this $\mathcal{G}$-module structures agree with the $\mathcal{G}$-module structure on $\mathrm{RHom}_{A^e}(A, \mathrm{End}_k(M, N))$ given in Section 2.4.

Lemma 2.35. *Let M and N be modules over $A*\mathcal{G}$ and K be a projective bimodule resolution of A satisfying conditions (I) and (II). Then the complex $K \otimes_A M$ is a $A * \mathcal{G}$-complex under the diagonal $\mathcal{G}$-action and the quasi-isomorphism $K \otimes_A M \to M$ is $\mathcal{G}$-linear. Furthermore, the isomorphism*

$$\mathrm{Hom}_{A^e}(K, \mathrm{Hom}_k(M, N)) \overset{\cong}{\to} \mathrm{Hom}_A(K \otimes_A M, N).$$

is one of complexes of right $\mathcal{G}$-modules.

Since this lemma is not essential to the remainder of the paper, the proof is deferred to the appendix. We now give the main results of the section.

Theorem 2.36. *Let M and N be $A * \mathcal{G}$-modules, L be a projective resolution of the trivial right $\mathcal{G}$-module k, and K be a projective A-bimodule resolution of A satisfying (I) and (II). Then there is an isomorphism of chain complexes*

$$\mathrm{Hom}_{A*\mathcal{G}}(K * L^{\uparrow} \otimes_{A*\mathcal{G}} M, N) \overset{\cong}{\to} \mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}_A(K \otimes_A M, N)).$$

When the hypotheses of Theorem 2.27 are satisfied, the isomorphism

$$\mathrm{Hom}_{A*\mathcal{G}}(K * L^{\uparrow} \otimes_{A*\mathcal{G}} M, M) \overset{\cong}{\to} \mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}_A(K \otimes_A M, M))$$

is one of (not-necessarily-associative) dg algebras.

In the above statement, $\mathrm{Hom}_{A*\mathcal{G}}(K*L^{\uparrow}\otimes_{A*\mathcal{G}}M, M)$ and $\mathrm{Hom}_A(K\otimes_A M, M)$ are supposed to have the algebra structures of Corollary 2.34.

Proof. From Proposition 2.32, Theorem 2.19, and the previous lemma, we get a sequence of isomorphisms of chain complexes

$$\begin{aligned}\mathrm{Hom}_{A*\mathcal{G}}(K * L^{\uparrow} \otimes_{A*\mathcal{G}} M, N) &\cong \mathrm{Hom}_{(A*\mathcal{G})^e}(K * L^{\uparrow}, \mathrm{Hom}_k(M, N))\\ &\cong \mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}_{A^e}(K, \mathrm{Hom}_k(M, N))\\ &\cong \mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}_A(K \otimes_A M, N)).\end{aligned}$$

Suppose now that $N = M$ and that the hypotheses of Theorem 2.27 are met. Then the second isomorphism

$$\mathrm{Hom}_{(A*\mathcal{G})^e}(K * L^{\uparrow}, \mathrm{Hom}_k(M, N)) \cong \mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}_{A^e}(K, \mathrm{Hom}_k(M, N))$$

is one of dg algebras by Theorem 2.27. The isomorphisms

$$\mathrm{Hom}_{A*\mathcal{G}}(K * L^{\uparrow} \otimes_{A*\mathcal{G}} M, N) \cong \mathrm{Hom}_{(A*\mathcal{G})^e}(K * L^{\uparrow}, \mathrm{Hom}_k(M, N))$$

and

$$\mathrm{Hom}_{A^e}(K, \mathrm{Hom}_k(M, N)) \cong \mathrm{Hom}_A(K \otimes_A M, N)$$

are isomorphisms of dg algebras by the definition of the multiplicative structures considered in Corollary 2.34. It follows that the final isomorphism

$$\mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}_{A^e}(K, \mathrm{Hom}_k(M, N)) \cong \mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}_A(K \otimes_A M, N))$$

is one of dg algebras as well. Taking this all together gives the proposed result. ■

In the complex $\mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}_A(K \otimes_A M, N))$, we may replace $K \otimes_A M$ with any $\mathcal{G}$-linear A-projective resolution $Q \to M$. The proof of this fact is the similar to the one given for Corollary 2.21. In the entire statement of Theorem 2.36 we could have also replaces K with any A^e-projective $\mathcal{G}$-equivariant resolution $P \to A$ with an equivariant diagonal, by Corollary 2.29.

Corollary 2.37. *Let M and N be modules over $A * \mathcal{G}$, and K and L be as in Theorem 2.27. Then there is an isomorphism of graded vector spaces*

$$\mathrm{Ext}_{A*\mathcal{G}}(M, N) \cong \mathrm{H}\big(\mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}_A(K \otimes_A M, N))\big)$$

and an isomorphism of graded algebras

$$\mathrm{Ext}_{A*\mathcal{G}}(M, M) \cong \mathrm{H}\big(\mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}_A(K \otimes_A M, M))\big).$$

Proof. This follows by the isomorphism of dg algebras

$$\mathrm{Hom}_{A*\mathcal{G}}(K * L^{\uparrow} \otimes_{A*\mathcal{G}} M, M) \overset{\cong}{\to} \mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}_A(K \otimes_A M, M))$$

of Theorem 2.36 and the fact that

$$\mathrm{Ext}_{A*\mathcal{G}}(M, M) = \mathrm{H}\big(\mathrm{Hom}_{A*\mathcal{G}}(K * L^{\uparrow} \otimes_{A*\mathcal{G}} M, M)\big)$$

as an algebra, by Corollary 2.34. ■

Of course, from the Theorem we can also derive some multaplicative spectral sequences converging to the Ext algebra $\mathrm{Ext}_{A*\mathcal{G}}(M, M)$. We define the filtrations $F^{\mathcal{G}}$ and F^{A} on each $\mathrm{Ext}_A(M, M)$ in the same manner as was done at Notation 2.22.

Corollary 2.38. *For any $A * \mathcal{G}$-module M, there are two multiplicative spectral sequences*

$$E_2 = \mathrm{Ext}_{\mathcal{G}\text{-mod}}(k, \mathrm{Ext}_{A\text{-mod}}(M, M)) \Rightarrow \mathrm{Ext}_{A*\mathcal{G}\text{-mod}}(M, M)$$

and

$$'E_1 = \mathrm{Ext}_{\mathcal{G}\text{-mod}}(k, \mathrm{RHom}_{A\text{-mod}}(M, M)) \Rightarrow \mathrm{Ext}_{A*\mathcal{G}\text{-mod}}(M, M)$$

which converge to $\mathrm{Ext}_{A*\mathcal{G}\text{-mod}}(M, M)$ *as an algebra. In the case that the global dimension of* $\mathcal{G}$ *is* ≤ 1 *we have*

$$\mathrm{gr}_{\mathcal{G}}\mathrm{Ext}_{A*\mathcal{G}}(M, M) = \mathrm{Ext}_{\mathcal{G}}(k, \mathrm{Ext}_A(M, M))$$

as an algebra.

Proof. The two spectral sequences arise from considering the row and column filtrations on the double complex $\mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}_A(K \otimes_A M, M))$. The details are the same as those given in the proofs of Corollaries 2.30 and 2.31. ∎

This corollary is related to the Lyndon-Hochschild-Serre spectral sequence in the following sense: if N and G are groups, and G acts on N by automorphisms, we get an action of $\mathcal{G} = kG$ on $A = kN$. We then have $A * \mathcal{G} = k(N \rtimes G)$, where $N \rtimes G$ denotes the semi-direct product. When $M = k$ is the trivial $N \rtimes G$ module, the multiplicative spectral sequence

$$\mathrm{Ext}_{\mathcal{G}}(k, \mathrm{Ext}_A(M, M)) = H^\bullet(G, H^\bullet(N, k)) \Rightarrow \mathrm{Ext}_{A*\mathcal{G}}(k, k) = H^\bullet(N \rtimes G, k)$$

of Corollary 2.38 is simply the Lyndon-Hochschild-Serre spectral sequence.

2.8 Two Examples With $\mathbb{Z}$

We begin with a quick specification of when the Koszul bimodule resolution $K \to A$ can be used to produce a resolution $K\#L^{\uparrow} \to A\#\mathcal{G}$.

Proposition 2.39. *Suppose $\mathcal{G}$ is a Hopf algebra acting on Koszul algebra $A = k\langle V\rangle/(R)$ by filtered endomorphisms. Then*

1. *the associated graded action induces a $\mathcal{G}$-module coalgebra structure on the Koszul dual coalgebra $C = \cdots \oplus (R \otimes V) \cap (V \otimes R) \oplus R \oplus V \oplus k$ and*

2. *the Koszul resolution $K = A \otimes_\pi C \otimes_\pi A$, along with the diagonal $\mathcal{G}$-action and comultiplication induced by that of C, satisfies conditions (I)-(III).*

The proposition ensured that, under the given hypotheses, the Koszul resolution can be used in Theorem 2.27.

Sketch Proof. The bar resolution $\mathscr{B}ar A$ has a good $\mathcal{G}$-action, and comultiplication, which then induces a good $\mathcal{G}$-action, and comultiplication, on the reduced bar resolution $\bar{\mathscr{B}}ar A$. We then simply note that the embedding $K \subset \bar{\mathscr{B}}ar A$ identifies the Koszul resolution with a $\mathcal{G}$-subcomplex (with the proposed action) which is also a A^e-subcoalgebra. ■

2.8.1 Example 1: $\mathbb{Z}^n$ acting on $\mathbf{A}_k^n$ by translation

Suppose k is of characteristic 0. Let $\mathbb{Z}^n$ act on $\mathbf{A}^n = \mathbf{A}_k^n$ by translation. Specifically, the generator σ_i of the ith copy of $\mathbb{Z}$ in $\mathbb{Z}^n$ will act as the automorphism

$$\begin{gathered} k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n] \\ x_j \mapsto x_j + \delta_{ij} \end{gathered}$$

on the algebra of global functions $k[\mathbf{A}^n] = \Gamma(\mathcal{O}_{\mathbf{A}^n}) = k[x_1, \ldots, x_n]$. Note that the set quotient $\mathbf{A}^n(k)/\mathbb{Z}^n$ is the n-torus $\mathbf{T}^n(k)$. Indeed, if $k = \mathbb{R}$ and $\mathbf{A}^n(k)$ is given the Euclidean topology then $\mathbf{T}^n(k)$ is also the topological quotient. However, the invariant algebra $k[\mathbf{A}^n]^{\mathbb{Z}^n}$ is simply the constants k.

We will see that, at least as far as the cohomology is concerned, the smash product provides a type of intermediate between the algebra of global functions $k[\mathbf{T}^n]$ on the torus and the invariants $k[\mathbf{A}_k^n]^{\mathbb{Z}^n} = k$. In particular, we will see that there is a canonical embedding of Gerstenhaber algebras $HH^\bullet(k[\mathbf{A}^n] * \mathbb{Z}^n) \to HH^\bullet(k[\mathbf{T}^n])$, and subsequent isomorphism of algebras $k[\mathbf{T}^n] \otimes HH^\bullet(k[\mathbf{A}^n] * \mathbb{Z}^n) \xrightarrow{\cong} HH^\bullet(k[\mathbf{T}^n])$.

The following three results will help to simplify the situation.

Lemma 2.40. *There is an isomorphism of algebras*

$$(k[x_1] * \mathbb{Z}) \otimes \ldots \otimes (k[x_n] * \mathbb{Z}) \to k[x_1, \ldots, x_n] * \mathbb{Z}^n$$
$$a_1 s_1 \otimes \ldots \otimes a_n s_n \mapsto (a_1 \ldots a_n)(\sigma_1^{s_1} \ldots \sigma_n^{s_n}),$$

where the a_i are in $k[x_i]$, the s_i are integers, and the σ_i are the generators of $\mathbb{Z}^n$.

Proof. Easy check. ■

Proposition 2.41 ([50])**.** *If A and A' are algebras then there is an isomorphism of algebras $HH^\bullet(A \otimes A') = HH^\bullet(A) \otimes HH^\bullet(A')$.*

Corollary 2.42. *There is an isomorphism of algebras*

$$HH^\bullet(k[x_1, \ldots, x_n] * \mathbb{Z}^n) \to HH^\bullet(k[x_1] * \mathbb{Z}) \otimes \ldots \otimes HH^\bullet(k[x_n] * \mathbb{Z}).$$

Proof. Combine the previous two results. ■

So, all we need to do is calculate the Hochschild cohomology of the smash product $k[x]*\mathbb{Z}$, where $\mathbb{Z}$ acts by translation. Since the action of $\mathbb{Z}$ is filtered, and $k[x]$ is Koszul, we can use Proposition 2.39 to get

$$\mathrm{RHom}_{A^e}(k[x], k[x] * \mathbb{Z}) = (k[\partial] \otimes k[x] * \mathbb{Z}, [e, -]). \tag{2.8.1}$$

Here ∂ is the dual function to x, taken to be in degree 1, and $k[\partial]$ is the *graded commutative* algebra generated by ∂. So, in terms of generators and relations, $k[\partial]$ is the exterior algebra $k\langle\partial\rangle/(\partial^2)$. For simplicity, we let $\mathscr{A}$ denote the dg algebra (2.8.1).

Proposition 2.43. *The differential $[e, -]$ vanishes on the subalgebra $k[\partial, x]$ of $\mathscr{A}$. Furthermore, the embedding of dg algebras $k[\partial, x] \to \mathscr{A}$ induces an equality $k[\partial, x] = HH^\bullet(k[x], k[x] * \mathbb{Z})$.*

Proof. It is clear that the differential $[e,-]$ vanishes on any graded commutative subalgebra of $\mathscr{A}$. Hence, it vanishes on $k[\partial,x]$. Now, for any $y\otimes a\sigma^s$ in $\mathscr{A}$, where $y\in k[y]$, and $a\in k[x]$, we have

$$\begin{aligned}[e,y\otimes a\sigma^s] &= \partial y\otimes xa\sigma^s-(-1)^{|y|}y\partial\otimes a\sigma^s x\\ &= \partial y\otimes xa\sigma^s-\partial y\otimes a(x+s)\sigma^s\\ &= -s\partial y\otimes a\sigma^s.\end{aligned}$$

Whence we see that $\mathscr{A}$ is the $\mathbb{Z}$-graded complex

$$\mathscr{A}=\oplus_{s\in\mathbb{Z}}(k[\partial_y]\otimes k[x]\sigma^s,[e,-]).$$

and each homogeneous piece can be expressed as the complex

$$0\to k[x]\overset{-s\partial\cdot}{\to}\partial\otimes k[x]\to 0.$$

Since the operation $-s\partial\cdot$ is a vector space isomorphism whenever $s\neq 0$ we see, for each $s\neq 0$, the homology $H(k[\partial_y]\otimes k[x]\sigma^s,[e,-])$ vanishes. ■

Proposition 2.44. *For any $y\in k[\partial]$, $a\in k[x]$, and $g\in\mathbb{Z}$, the action of $\mathbb{Z}$ on $HH^\bullet(k[x],k[x]*\mathbb{Z})=k[\partial,x]$ is given by*

$$(ya)\cdot g=y({}^{g^{-1}}a). \tag{2.8.2}$$

Proof. Take $y\in k[\partial]$, $a\in k[x]$, and $g\in\mathbb{Z}$. Recall that $\mathbb{Z}$ acts on $k[\partial]$ by dualizing the associated graded action on $k[x]$. Since the associated graded action on $k[x]$ is trivial, we conclude that the action of $\mathbb{Z}$ on $k[\partial]$ is trivial.

We consider $k[\partial,x]$ as a subalgebra in $\mathscr{A}$. According to the action given at Proposition 2.39, and the fact that g is group like, we have

$$(ya)g=(yg)(g^{-1}ag)=y({}^{g^{-1}}a).$$

So we see that $k[\partial,x]$ is a $\mathbb{Z}$-module subalgebra in $\mathscr{A}$, under the proposed action (2.8.2), and the embedding $k[\partial,x]\overset{\sim}{\to}\mathscr{A}$ is one of $\mathbb{Z}$-module dg algebras. ■

Recall that we have the Koszul resolution of $k_{\mathbb{Z}}$ over $k\mathbb{Z} = k[\sigma, \sigma^{-1}]$ given by

$$0 \to k\mathbb{Z} \to k\mathbb{Z} \to 0$$
$$1 \mapsto (1-\sigma).$$

Proposition 2.45. *Let L be the resolution of $k_{\mathbb{Z}}$ given above. There is a canonical inclusion $k[\partial] \to \mathrm{Hom}_{\mathbb{Z}}(L, k[\partial, x])$ which induces an equality $k[\partial] = HH^{\bullet}(k[x] * \mathbb{Z})$.*

Proof. To distinguish between the two copies of $k\mathbb{Z}$ in L we will write

$$L = 0 \to k\mathbb{Z}[1] \to k\mathbb{Z} \to 0.$$

The canonical inclusion alluded to in the statement of the proposition is given by

$$k[\partial] = \mathrm{Hom}_{\mathbb{Z}}(k\mathbb{Z}, k[\partial]) \to \mathrm{Hom}_{\mathbb{Z}}(L, k[\partial] \otimes k[x]).$$

Note that we can write the hom complex $\mathrm{Hom}_{\mathbb{Z}}(L, k[\partial] \otimes k[x])$ as a double complex. Indeed, using freeness of L, $\mathrm{Hom}_{\mathbb{Z}}(L, k[\partial] \otimes k[x])$ is given by

$$\begin{tikzcd}
 & 0 & 0 & \\
0 \arrow[r] & k[x] \arrow[r, "0"] \arrow[u] & \partial k[x] \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & k[x] \arrow[r, "0"] \arrow[u, "(1-\sigma)\cdot"] & \partial k[x] \arrow[r] \arrow[u, "(1-\sigma)\cdot"] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & .
\end{tikzcd}$$

The map $(1-\sigma)\cdot$ sends a polynomial $f(x)$ in $k[x]$ (resp. $\partial f(x)$ in $\partial k[x]$) to $f(x+1)$ (resp. $\partial f(x+1)$). In particular, for all positive n we have

$$(1-\sigma)x^n = -nx^{n-1} + \text{lower degree terms}.$$

So it is clear that $\ker(1-\sigma) = k$. By induction on degree, or a simple spectral sequence argument, we see that $\mathrm{coker}(1-\sigma) = 0$. So the homology of the double complex $\mathrm{Hom}_{\mathbb{Z}}(L, k[\partial, x])$ is the subalgebra $k[\partial]$. ■

Corollary 2.46. *There is an equality of algebras* $HH^\bullet(k[x_1,\ldots,x_n]*\mathbb{Z}^n) = k[\partial_1,\ldots,\partial_n]$, *where all of the* ∂_i *are of degree* 1.

Proof. This follows by Proposition 2.45 and Corollary 2.42. ■

In order to address the Gerstenhaber structure on the Hochschild cohomology of $k[x_1,\ldots,x_n]*\mathbb{Z}^n$ we produce a quasi-isomorphism from $k[\partial_1,\ldots,\partial_n]$ to the Hochschild cochain complex

$$C^\bullet(k[\mathbf{A}^n]*\mathbb{Z}^n) = \mathrm{Hom}(B(k[\mathbf{A}^n]*\mathbb{Z}^n), k[\mathbf{A}^n]*\mathbb{Z}^n)$$

$$= 0 \to k[\mathbf{A}^n]*\mathbb{Z}^n \to \mathrm{Hom}_k(k[\mathbf{A}^n]*\mathbb{Z}^n, k[\mathbf{A}^n]*\mathbb{Z}^n) \to \mathrm{Hom}_k(k[\mathbf{A}^n]*\mathbb{Z}^n\otimes k[\mathbf{A}^n]*\mathbb{Z}^n, k[\mathbf{A}^n]*\mathbb{Z}^n) \to \cdots.$$

First, we identify each ∂_i with the derivation

$$\begin{gathered}\partial_i : k[x_1,\ldots,x_n]*\mathbb{Z}^n \to k[x_1,\ldots,x_n]*\mathbb{Z}^n\\ ag \mapsto \tfrac{\partial}{\partial x_i}(a)g\end{gathered}$$

in $C^1(k[\mathbf{A}^n]*\mathbb{Z}^n) = \mathrm{End}_k(k[\mathbf{A}^n]*\mathbb{Z}^n)$. Now define the map

$$\Sigma : k[\partial_i] \to C^\bullet(k[\mathbf{A}^n]*\mathbb{Z}^n)$$

by

$$\partial_{i_1}\ldots\partial_{i_r} \mapsto \frac{1}{n!}\sum_{\tau\in S_r}\partial_{i_{\tau(1)}}\cup\cdots\cup\partial_{i_{\tau(r)}},$$

where $\cup$ denote the product on $C^\bullet(k[\mathbf{A}^n]*\mathbb{Z}^n)$. Recall that the product on the Hochschild cochain complex $C^\bullet(k[\mathbf{A}^n]*\mathbb{Z}^n)$ is graded commutative up to homotopy [18].

Proposition 2.47. *The map* Σ *is a quasi-isomorphism.*

Proof. Since the ∂_i are derivations on $k[\mathbf{A}^n]*\mathbb{Z}^n$ they are cycles in $C^1(k[\mathbf{A}^n]*\mathbb{Z}^n)$. It is easy to check that any nonzero $\sum_i c_i\partial_i$ is a non-inner derivation. So $H^1(\Sigma)$ is an injective map from $k[\partial_1,\ldots,\partial_n]^1 = \langle\partial_1,\ldots,\partial_n\rangle$ to

$$HH^1(k[\mathbf{A}^n]*\mathbb{Z}^n) = \mathrm{Der}(k[\mathbf{A}^n]*\mathbb{Z}^n)/\mathrm{InnDer}(k[\mathbf{A}^n]*\mathbb{Z}^n).$$

Now, we already know $HH^\bullet(k[\mathbf{A}^n]*\mathbb{Z}^n) = k[\partial_1,\ldots,\partial_n]$ by Corollary 2.46. So $H^1(\Sigma)$ is in fact an isomorphism.

Since $C^\bullet(k[\mathbf{A}^n] * \mathbb{Z}^n)$ is homotopy commutative, the map Σ is in fact an algebra map up to homotopy. This implies that $H(\Sigma)$ is an algebra map $k[\partial_1, \ldots, \partial_n] \to HH^\bullet(k[\mathbf{A}^n] * \mathbb{Z}^n) = k[\partial_1, \ldots, \partial_n]$. Now, since $H(\Sigma)$ is an isomorphism on the generators, we can conclude that $H(\Sigma)$ is an isomorphism. ■

Our final result refers to the graded Lie bracket on the (shifted) Hochschild cohomology. Let us list here only the details we will need to know in order to understand the proof below. Firstly, we recall that the Lie bracket on

$$C^1(k[x_1, \ldots, x_n] * \mathbb{Z}^n) = \mathrm{End}_k(k[x_1, \ldots, x_n] * \mathbb{Z}^n)$$

is simply the graded commutator $[f, g] = fg - (-1)^{|f||g|} gf$. Secondly, we note that, on homology, each operation $[f, -]$ is a degree $|f| - 1$ graded derivation, i.e.

$$[f, gh] = [f, g]h + (-1)^{(|f|-1)|g|} g[f, h]$$

for all $f, g, h \in HH^\bullet(k[x_1, \ldots, x_n] * \mathbb{Z}^n)$.

Theorem 2.48. *The Lie bracket on* $HH^\bullet(k[x_1, \ldots, x_n] * \mathbb{Z}^n) = k[\partial_1, \ldots, \partial_n]$ *vanishes.*

Proof. One can check directly that, for all i, j, $[\partial_i, \partial_j] = 0$ in $\mathrm{Der}(k[\mathbf{A}^n] * \mathbb{Z}^n)$. Hence, the Lie subalgebra $k[\partial_1, \ldots, \partial_n]^1$ is abelian. Now, one can use the fact that, for any f in the Hochschild cohomology, the map $[f, -]$ is a graded degree $|f| - 1$ derivation to get $[\partial_j, k[\partial_1, \ldots, \partial_n]] = 0$. Use this fact one more time to get $[k[\partial_1, \ldots, \partial_n], k[\partial_1, \ldots, \partial_n]] = 0$. ■

Let us, in conclusion, explain some of the motivation behind the previous two results and address the claims concerning the Hochschild cohomology of the n-torus. First, the action of $\mathbb{Z}^n$ on $k[x_1, \ldots, x_n]$ induces a natural action of $\mathbb{Z}^n$ on the polyvector fields

$$T_{poly}(\mathbf{A}^n) = \bigwedge\nolimits_{k[\mathbf{A}^n]} \mathrm{Der}(k[\mathbf{A}^n]).$$

Recall that [32] tells us we have a quasi-isomorphism

$$HKR : T_{poly}(\mathbf{A}^n) \xrightarrow{\sim} C^\bullet(k[\mathbf{A}^n])$$
$$Y_1 \wedge \cdots \wedge Y_r \mapsto \tfrac{1}{n!} \textstyle\sum_{\tau \in S_r} Y_{\tau(1)} \cup \cdots \cup Y_{\tau(r)},$$

and subsequent isomorphism $HH^\bullet(k[\mathbf{A}^n]) \cong T_{poly}(\mathbf{A}^n)$.

It is rather easy to check that there is an equality $k[\frac{\partial}{\partial x_i}] = T_{poly}(\mathbf{A}^n)^{\mathbb{Z}^n}$. So we have an algebra isomorphism

$$\begin{gathered} HH^\bullet(k[x_1, \ldots, x_n] * \mathbb{Z}^n) \xrightarrow{\cong} T_{poly}(\mathbf{A}^n)^{\mathbb{Z}^n} \\ \partial_i \mapsto \tfrac{\partial}{\partial x_i}. \end{gathered}$$

Since all the $\frac{\partial}{\partial x_i}$ are orthogonal, this map is an isomorphism of Gerstenhaber algebras. Also, as was shown above, any $\mathbb{Z}^n$-invariant derivation on $k[\mathbf{A}^n]$ extends to a derivation on the smash product. So we can define the map Σ as the invariant map $\Sigma = HKR^{\mathbb{Z}^n}$, up to some abuse of notation.

Finally, there is a natural embedding of Gerstenhaber algebras

$$HH^\bullet(k[\mathbf{A}^n] * \mathbb{Z}^n) \cong T_{poly}(\mathbf{A}^n)^{\mathbb{Z}^n} \to T_{poly}(\mathbf{T}^n) = HH^\bullet(k[\mathbf{T}^n]).$$

If X_i denotes the standard nonvanishing vector field on the ith copy of S^1 in $\mathbf{T}^n = (S^1)^n$, the above embedding is given by $\partial_i \mapsto X_i$. This map identifies $HH^\bullet(k[\mathbf{A}^n] * \mathbb{Z}^n)$ with a $k[\mathbf{T}^n]$-basis for $T_{poly}(\mathbf{T}^n)$. From this we get the proposed isomorphism of algebras

$$\begin{gathered} k[\mathbf{T}^n] \otimes HH^\bullet(k[\mathbf{A}^n] * \mathbb{Z}^n) \xrightarrow{\cong} T_{poly}(\mathbf{T}^n) = HH^\bullet(k[\mathbf{T}^n]) \\ f \otimes Y \mapsto fY. \end{gathered}$$

It is somewhat tempting to conjecture that the analogous result holds if we replace $k[\mathbf{A}^n]$ with the algebra $k[\mathbf{A}^n]^\infty$ of C^∞-functions. In this case we would expect an equality of Gerstenhaber algebras

$$HH^\bullet(k[\mathbf{A}^n]^\infty * \mathbb{Z}^n) = T^\infty_{poly}(\mathbf{A}^n)^{\mathbb{Z}^n} = T^\infty_{poly}(\mathbf{T}^n).$$

Indeed, it seems likely that there will be an equivalence of categories $k[\mathbf{A}^n]^\infty * \mathbb{Z}\text{-mod} \xrightarrow{\sim} k[\mathbf{T}^n]^\infty\text{-mod}$ given by restriction.

2.8.2 Example 2: The Untwisted Skew Polynomial Ring

Let k be of characteristic 0 and q be a nonzero scalar which is not a root of unity. Let $k_q[x, y]$ denote the skew polynomial ring in 2-variables,

$$k_q[x, y] = \frac{k\langle x, y\rangle}{(yx - qxy)}.$$

This algebra is Koszul and twisted Calabi-Yau with Koszul dual given by the skew *graded* commutative ring

$$k_{q^{-1}}[s, t] = \frac{k\langle s, t\rangle}{(s^2, t^2, st + qts)},$$

where the generators s and t are taken to be degree 1. One can check directly that $k_{q^{-1}}[s, t]$ is Frobenius with Frobenius automorphism $s \mapsto -qs$, $t \mapsto -q^{-1}t$. Then, according to [86, Theorem 9.2], $k_q[x, y]$ will be twisted CY with twisting automorphism $\phi : x \mapsto q^{-1}x$, $y \mapsto qy$. (Note that one needs to invert the automorphism on the rigid dualizing complex to get the twisting automorphism.) We let the generator of $\mathbb{Z}$ act on $k_q[x, y]$ by ϕ. Now, according to [24] and [72, Corollary 0.6], the smash product $k_q[x, y] * \mathbb{Z}$ will be Calabi-Yau.

Let λ be a variable of homological degree 0, ε, ξ_1, and ξ_2 be variables of homological degree 1, and ζ and all η_i be of homological degree 2. We give all elements internal degree 0 except for ε, which we give internal degree 1. In this subsection we show that

$$HH^\bullet(k_q[x, y] * \mathbb{Z}) = \frac{k[\varepsilon, \lambda, \xi_1, \xi_2, \zeta]}{(\lambda\zeta - \xi_1\xi_2, \xi_i\zeta, \zeta^2)} \times_{k[\varepsilon]} \frac{k[\varepsilon, \eta_i : i \in \mathbb{Z} - \{-1\}]}{(\eta_i^2, \eta_i\eta_j)}$$

as bigraded algebras. Here the algebra $k[A, B, C, D, E]$ denotes the free graded commutative algebra with respect to the homological grading.

We consider the same resolution L as in the previous example,

$$0 \to k\mathbb{Z} \overset{(1-\phi)\cdot}{\to} k\mathbb{Z} \to 0$$

of k, where ϕ denotes the generator of $\mathbb{Z}$. We will give a diagonal map $L \to L \otimes L$ below. Let us for now simply focus on computing the cohomology $HH^\bullet(k_q[x, y] * \mathbb{Z})$ as a vector space.

Let us note before beginning that $k_q[x, y] * \mathbb{Z}$ has a basis given by ordered monomials $\{x^i y^j \phi^k : i, j \geq 0, k \in \mathbb{Z}\}$. According to Proposition 2.39, the dg algebra computing

$HH^\bullet(k_q[x,y], k_q[x,y]*\mathbb{Z})$ is

$$0 \to k_q[x,y]*\mathbb{Z} \overset{[e,-]}{\to} k_q[x,y]*\mathbb{Z}s \oplus k_q[x,y]*\mathbb{Z}t \overset{[e,-]}{\to} k_q[x,y]*\mathbb{Z}st \to 0,$$

where the differentials in degrees 0 and 1 are given by

$$x^iy^j\phi^k \mapsto (1-q^{j-k})x^{i+1}y^j\phi^k s + (q^i - q^k)x^iy^{i+1}\phi^k t \tag{2.8.3}$$

and

$$ax^iy^j\phi^k s + bx^ly^m\phi^n t \mapsto \big(a(q^k - q^{i-1})x^iy^{j+1}\phi^k + b(1-q^{m-1-n})x^{l+1}y^m\phi^n\big)st \tag{2.8.4}$$

respectively. We let $\mathscr{A}$ denote this dg algebra.

The right action of $\mathbb{Z}$ is given by

$$\begin{aligned} (x^iy^j\phi^k)\cdot\phi &= q^{i-j}x^iy^j\phi^k, & (x^iy^j\phi^k s)\cdot\phi &= q^{i-j-1}x^iy^j\phi^k s \\ (x^iy^j\phi^k t)\cdot\phi &= q^{i-j+1}x^iy^j\phi^k t, & (x^iy^j\phi^k st)\cdot\phi &= q^{i-j}x^iy^j\phi^k st. \end{aligned}$$

Finally, the double complex computing $HH^\bullet(k_q[x,y]*\mathbb{Z})$ is

$$\begin{tikzcd}
0 \arrow[r] & k_q[x,y]*\mathbb{Z} \arrow[r, "-{[e,-]}"] & k_q[x,y]*\mathbb{Z}s \oplus k_q[x,y]*\mathbb{Z}t \arrow[r, "{[e,-]}"] & k_q[x,y]*\mathbb{Z}st \arrow[r] & 0 \\
0 \arrow[r] & k_q[x,y]*\mathbb{Z} \arrow[r, "{[e,-]}"] \arrow[u, "\cdot(1-\phi)"] & k_q[x,y]*\mathbb{Z}s \oplus k_q[x,y]*\mathbb{Z}t \arrow[r, "{[e,-]}"] \arrow[u, "\cdot(1-\phi)"] & k_q[x,y]*\mathbb{Z}st \arrow[r] \arrow[u, "\cdot(1-\phi)"] & 0
\end{tikzcd}$$

We will compute the Hochschild cohomology of the smash product using the column filtration on the above double complex. Note that the E_1 page in this case has columns $H^\bullet(\mathbb{Z}, \mathscr{A}^i)$, where $H^\bullet(\mathbb{Z}, -)$ denotes the Hopf cohomology $\mathrm{Ext}_{\mathbb{Z}}(k,-)$. Recall that in degree 0 the Hopf cohomology is simply the invariants $(-)^{\mathbb{Z}}$.

Proposition 2.49. *We have*

$$\begin{aligned} H^0(\mathbb{Z}, \mathscr{A}^0) &= k[xy]*\mathbb{Z}, \\ H^0(\mathbb{Z}, \mathscr{A}^1) &= k[xy]x*\mathbb{Z}s \oplus k[xy]y*\mathbb{Z}t, \\ H^0(\mathbb{Z}, \mathscr{A}^2) &= k[xy]*\mathbb{Z}st. \end{aligned}$$

There are also canonical isomorphisms

$$\begin{aligned} H^1(\mathbb{Z}, \mathscr{A}^0) &\cong k[xy] * \mathbb{Z}, \\ H^1(\mathbb{Z}, \mathscr{A}^1) &\cong k[xy]x * \mathbb{Z}s \oplus k[xy]y * \mathbb{Z}t, \\ H^1(\mathbb{Z}, \mathscr{A}^2) &\cong k[xy] * \mathbb{Z}st. \end{aligned}$$

given by the natural embeddings $k[xy] * \mathbb{Z} \to k_q[x,y] * \mathbb{Z}$, $k[xy]x * \mathbb{Z}s \to k_q[x,y] * \mathbb{Z}s$, *etc.*

Proof. This follows from the fact that $q^n = 1$ if and only if $n = 0$, and so $(1 - q^n) = 0$ if and only if $n = 0$. ■

As a consequence of the above Proposition we see that the E_1 page of the spectral sequence associated to the column filtration is given by

$$E_1 = \begin{tikzcd} 0 \arrow[r] & k[xy] * \mathbb{Z} \arrow[r, "-{[e,-]}"] & k[xy]x * \mathbb{Z}s \oplus k[xy]y * \mathbb{Z}t \arrow[r, "{[e,-]}"] & k[xy] * \mathbb{Z}st \arrow[r] & 0 \\ 0 \arrow[r] & k[xy] * \mathbb{Z} \arrow[r, "{[e,-]}"] & k[xy]x * \mathbb{Z}s \oplus k[xy]y * \mathbb{Z}t \arrow[r, "{[e,-]}"] & k[xy] * \mathbb{Z}st \arrow[r] & 0 \end{tikzcd}$$

From the formulas (2.8.3) and (2.8.4), the restrictions of the differentials in degree 0 and 1 are now given by

$$x^i y^i \phi^k \mapsto (1 - q^{i-k}) x^{i+1} y^i \phi^k s + (q^i - q^k) x^i y^{i+1} \phi^k t \tag{2.8.5}$$

and

$$a x^{i+1} y^i \phi^k s + b x^j y^{j+1} \phi^n t \mapsto \big(a(q^k - q^i) x^{i+1} y^{i+1} \phi^k + b(1 - q^{j-n}) x^{j+1} y^{j+1} \phi^n\big) st \tag{2.8.6}$$

Lemma 2.50. *The spectral sequence E stabilizes at the 2nd page, i.e. $E_2 = E_\infty$.*

Proof. We can see from the differentials given above, the obvious inclusion

$$E_1 \to \mathrm{Hom}_{\mathbb{Z}}(L, \mathrm{Hom}_{A^e}(K, k_q[x,y] * \mathbb{Z}))$$

is a embedding of (double) complexes. Furthermore, this map is a quasi-isomorphism since the spectral sequences associated to the column filtrations agree at the E_1 page. So we have

$$H^\bullet\big(\mathrm{Hom}_{\mathbb{Z}}(L, \mathrm{Hom}_{A^e}(K, k_q[x,y] * \mathbb{Z}))\big) = H^\bullet(E_1).$$

■

Recall that the bottom row is of E_1 is the invariant dg algebra $\mathscr{A}^{\mathbb{Z}}$ and that the top row is canonically isomorphic to $\mathscr{A}^{\mathbb{Z}}$ as a complex. This will simply help ease notation.

Proposition 2.51. *Let λ be a variable in degree 0, ξ_i be variables in degree 1, and ζ be variables in degree 2. Let η_i also be variables in degree 2. Then the algebra embedding*

$$\frac{k[\lambda,\xi_1,\xi_2,\zeta]}{(\lambda\zeta-\xi_1\xi_2,\xi_i\zeta,\zeta^2)}\to\mathscr{A}^{\mathbb{Z}},\quad \begin{aligned}\lambda&\mapsto xy\phi\\ \xi_1&\mapsto xs\\ \xi_2&\mapsto yt\\ \zeta&\mapsto\phi^{-1}st\end{aligned}\quad, \tag{2.8.7}$$

is a quasi-isomorphism in degrees 0 and 1 (where the domain is taken to have vanishing differential). The chain embedding

$$\Omega:\frac{k[\lambda,\xi_1,\xi_2,\zeta]}{(\lambda\zeta-\xi_1\xi_2,\xi_i\zeta,\zeta^2)}\times_k\frac{k[\eta_i:i\in\mathbb{Z}-\{-1\}]}{(\eta_i^2,\eta_i\eta_j)}\to\mathscr{A}^{\mathbb{Z}},$$

sending η_i to $\phi^i st$ and mapping all the other generators as in (2.8.7), is a quasi-isomorphism.

Proof. Let ι denote the corresponding map on the free graded commutative algebra $k[\lambda,\xi_1,\xi_2,\zeta]$ to $\mathscr{A}^{\mathbb{Z}}$. From (2.8.6), and the fact that q is not a root of unity, we see that a monomial $x^iy^i\phi^k$ is in the kernel of d^0 if and only if $k=i$. So we see that ι is an isomorphism in degree 0. Similarly, we see that

$$ax^{i+1}y^i\phi^ks+bx^jy^{j+1}\phi^nt$$

is in the kernel if and only if $k=i$ and $n=j$ or, assuming a is nonzero, $k=n,i=j$, $k\neq i$, and $b=a(q^k-q^i)/(1-q^{i-k})$. In this second case, the cycle appears as

$$(1-q^{i-k})x^{i+1}y^i\phi^ks+(q^i-q^k)x^iy^{i+1}\phi^kt,$$

up to multiplication by a constant. As is clear from the expression (2.8.5), all such elements are boundaries. So we see

$$H^1(\mathscr{A}^{\mathbb{Z}})=\text{image of}\big(k[\lambda]\xi_1\oplus k[\lambda]\xi_2\big).$$

Whence ι is a quasi-isomorphism in degree 1 as well. By the same analysis,

$$d^1(\mathscr{A}^{\mathbb{Z}}) = \langle x^{i+1}y^{i+1}\phi^k st : i, j \geq 0, k \neq i\rangle$$

and so

$$H^2(\mathscr{A}^{\mathbb{Z}}) = \langle x^{i+1}y^{i+1}\phi^i st : i \geq 0\rangle \oplus \langle \phi^k st : k \in \mathbb{Z}\rangle.$$

It follows that the image of the coproduct map

$$k[\lambda, \xi_1, \xi_2, \zeta] \oplus \langle \phi^k st : k \in \mathbb{Z}\rangle \to \mathscr{A}^{\mathbb{Z}}$$

is a quasi-isomorphic subcomplex in $\mathscr{A}^{\mathbb{Z}}$ on which the differential vanishes. Obviously, we can omit $\phi^{-1}st$ in the second summand, since $\phi^{-1}st$ is the image of ζ. In fact, $\mathrm{im}(\iota) \cap \langle \phi^k st : k \in \mathbb{Z}\rangle = k\phi^{-1}st$. Now one simply checks that the kernel of ι is the ideal $(\lambda\zeta - \xi_1\xi_2, \xi_i\zeta, \zeta^2)$ and that

$$\frac{k[\lambda, \xi_1, \xi_2, \zeta]}{(\lambda\zeta - \xi_1\xi_2, \xi_i\zeta, \zeta^2)} \times_k \frac{k[\eta_i : i \in \mathbb{Z} - \{-1\}]}{(\eta_i^2, \eta_i\eta_j)} = \frac{k[\lambda, \xi_1, \xi_2, \zeta]}{(\lambda\zeta - \xi_1\xi_2, \xi_i\zeta, \zeta^2)} \oplus \langle \phi^k st : k \in \mathbb{Z} - \{0\}\rangle$$

to get the desired result. ■

Note that the image of Ω is contained in the subalgebra of cycles $Z(\mathscr{A}^{\mathbb{Z}})$. So we get induced maps onto the homology via the projection $Z(\mathscr{A}^{\mathbb{Z}}) \to H(\mathscr{A}^{\mathbb{Z}})$.

Corollary 2.52. *The isomorphism*

$$H^\bullet(\Omega) : \frac{k[\lambda, \xi_1, \xi_2, \zeta]}{(\lambda\zeta - \xi_1\xi_2, \xi_i\zeta, \zeta^2)} \times_k \frac{k[\eta_i : i \in \mathbb{Z} - \{-1\}]}{(\eta_i^2, \eta_i\eta_j)} \to H^\bullet(\mathscr{A}^{\mathbb{Z}})$$

derived from Proposition 2.51 is an isomorphism of algebras

Proof. Since $\lambda\eta_i$ is the boundary $xy\phi^{i+1}st$, we see that the implicit relation $\lambda\eta_i = 0$ is respected on in $H(\mathscr{A})$. All the other relations are respected at the cochain level. ■

For the sake of notation, we will take

$$T = \frac{k[\lambda, \xi_1, \xi_2, \zeta]}{(\lambda\zeta - \xi_1\xi_2, \xi_i\zeta, \zeta^2)} \times_k \frac{k[\eta_i : i \in \mathbb{Z} - \{-1\}]}{(\eta_i^2, \eta_i\eta_j)}$$

Note that Ω can also be considered a map into $\mathscr{A}$ since we have $\mathscr{A}^{\mathbb{Z}} \subset \mathscr{A}$.

Corollary 2.53. *Let $\mathbb{Z}$ act trivially on the algebra T. The induced map*

$$\Omega_* : \mathrm{Hom}_{\mathbb{Z}}(L, T) \to \mathrm{Hom}_{\mathbb{Z}}(L, \mathscr{A})$$

is a quasi-isomorphism and induces an algebra isomorphism on homology.

Proof. Note that the differential on the domain vanishes. By construction, the map Ω induces an algebra isomorphism on page 2 of the corresponding spectral sequences. By Lemma 2.50 the E_2 page of the codomain is equal to the homology as an algebra. ■

All that is left to do is give a diagonal map $L \xrightarrow{\sim} L \otimes L$. We can then compute the algebra structures on the hom complexes $\mathrm{Hom}_{\mathbb{Z}}(L, T)$ and $\mathrm{Hom}_{\mathbb{Z}}(L, \mathscr{A})$.

Lemma 2.54. *There is $\mathbb{Z}$-linear quasi-isomorphism $L \to L \otimes L$ given in degree 0 by the comultiplication and in degree -1 by*

$$\begin{gathered} k\mathbb{Z}[1] \to L \otimes L \\ 1[1] \mapsto \tfrac{1}{2}(1[1] \otimes (1+\phi) + (1+\phi) \otimes 1[1]). \end{gathered}$$

Proof. The complex $L \otimes L$ is given by

$$\cdots \to k\mathbb{Z}[1] \otimes k\mathbb{Z} \oplus k\mathbb{Z} \otimes k\mathbb{Z}[1] \to k\mathbb{Z} \otimes k\mathbb{Z} \to 0$$

with differential $1[1] \otimes 1 \mapsto (1-\phi) \otimes 1$ and $1 \otimes 1[1] \mapsto 1 \otimes (1-\phi)$. We need to check that the proposed diagonal map is a chain map. Let σ denote the proposed diagonal map. We have

$$\begin{aligned} d_{L\otimes L}(\sigma(1[1])) &= d_{L\otimes L}(\tfrac{1}{2}(1[1] \otimes (1+\phi) + (1+\phi) \otimes 1[1]) \\ &= \tfrac{1}{2}((1-\phi) \otimes (1+\phi) + (1+\phi) \otimes (1-\phi)) \\ &= \tfrac{1}{2}(2(1 \otimes 1) - \phi \otimes 1 - 1 \otimes \phi + 2(\phi \otimes \phi)) \\ &= 1 \otimes 1 + \phi \otimes \phi \\ &= \sigma(1-\phi) \\ &= \sigma(d_L(1[1])). \end{aligned}$$

So φ is a chain map. The fact that it is a quasi-isomorphism follows from the fact that it sends 1 to $1 \otimes 1$ in degree 0. ■

Theorem 2.55. *Let* ε *be a variable of degree* 1*, and let all other variables be of the degrees specified in Proposition 2.51. There is an isomorphism of graded algebras*

$$\frac{k[\varepsilon,\lambda,\xi_1,\xi_2,\zeta]}{(\lambda\zeta-\xi_1\xi_2,\xi_i\zeta,\zeta^2)}\times_{k[\varepsilon]}\frac{k[\varepsilon,\eta_i : i\in\mathbb{Z}-\{-1\}]}{(\eta_i^2,\eta_i\eta_j)}\cong HH^\bullet(k_q[x,y]*\mathbb{Z}).$$

Proof. Recall our notation

$$T=\frac{k[\lambda,\xi_1,\xi_2,\zeta]}{(\lambda\zeta-\xi_1\xi_2,\xi_i\zeta,\zeta^2)}\times_k\frac{k[\eta_i : i\in\mathbb{Z}-\{-1\}]}{(\eta_i^2,\eta_i\eta_j)}$$

Let ε denote the degree 1 map

$$L\to L^{-1}=k\mathbb{Z}[1]\to T$$
$$1[1]\mapsto 1$$

in the hom complex $\mathrm{Hom}_{\mathbb{Z}}(L,T)$ and let the product on the hom complex be denoted by juxtaposition. For any $f\in T$ let f denote the map

$$L\to L^0=k\mathbb{Z}\to T$$
$$1\mapsto f$$

by abuse of notation. We now have that $f\varepsilon$ is the degree 1 map

$$f\varepsilon : L\to L\otimes L\to T$$
$$1\mapsto 1\otimes 1\mapsto 0$$
$$1[1]\mapsto \tfrac{1}{2}(1[1]\otimes(1+\phi)+(1+\phi)\otimes 1[1])\mapsto \tfrac{1}{2}2f=f$$

One can check easily that εf is the map $1\mapsto 0$, $1[1]\mapsto(-1)^{|f|}f$. So $\varepsilon f=(-1)^{|f|}f\varepsilon$. Obviously $\varepsilon^2=0$ since there are no degree 2 maps from L into T. Whence we get the bigraded algebra isomorphism

$$\frac{k[\varepsilon,\lambda,\xi_1,\xi_2,\zeta]}{(\lambda\zeta-\xi_1\xi_2,\xi_i\zeta,\zeta^2)}\times_{k[\varepsilon]}\frac{k[\varepsilon,\eta_i : i\in\mathbb{Z}-\{-1\}]}{(\eta_i^2,\eta_i\eta_j)}\to\mathrm{Hom}_{\mathbb{Z}}(L,T),\qquad f\varepsilon^i\mapsto f\varepsilon^i.$$

Finally, one can deduce from Corollary 2.53 that the isomorphism

$$\mathrm{Hom}_{\mathbb{Z}}(L,T)\cong HH^\bullet(k_q[x,y]*\mathbb{Z})$$

is one of algebras. ∎

As was discussed in the beginning of the section, an easy exercise shows that

$$HH^{\bullet}(k_q[x,y]) = k[xs, yt, st] = k[\xi_1, \xi_2, \eta_0].$$

Whence we, at least, have a canonical algebra map $HH^{\bullet}(k_q[x,y]) \to HH^{\bullet}(k_q[x,y] * \mathbb{Z})$.

Appendix

Proof of Corollary 2.29. Let $\mathscr{P} \to P$ be a projective resolution of P as a complex in $\mathrm{EQ}_{\mathcal{G}}A^e$-mod. Since the restriction functor $\mathrm{EQ}_{\mathcal{G}}A^e$-mod$\to A^e$-mod preserves projectives, $\mathscr{P}$ will be a projective bimodule resolution of A as well. The tensor product of the quasi-isomorphism $\mathscr{P} \to P$ then produces a quasi-isomorphism $\mathscr{P} \otimes_A \mathscr{P} \to P \otimes_A P$ and we get a diagram

$$\begin{tikzcd}
\mathscr{P} \otimes_A \mathscr{P} \arrow[r] & P \otimes_A P \\
\mathscr{P} \arrow[r] \arrow[u] \arrow[ur, phantom, "h \circlearrowleft"] & P, \arrow[u]
\end{tikzcd}$$

which commutes up to a $\mathcal{G}$-equivariant homotopy $h : \mathscr{P} \to P \otimes_A P[1]$. Here the two two vertical maps are any choice of diagonal quasi-isomorphisms on $\mathscr{P}$ and P. The existence of this homotopy follow by projectivity of $\mathscr{P}$ and the fact that the diagram commutes on homology. We will then have a diagram

$$\begin{tikzcd}
\mathrm{Hom}_{A^e}(P,B) \otimes \mathrm{Hom}_{A^e}(P,B) \arrow[r] \arrow[d] \arrow[dd, dotted, bend right=60, "mult"'] & \mathrm{Hom}_{A^e}(\mathscr{P},B) \otimes \mathrm{Hom}_{A^e}(\mathscr{P},B) \arrow[d] \arrow[dd, dotted, bend left=60, "mult"] \\
\mathrm{Hom}_{A^e}(P \otimes_A P, B) \arrow[r] \arrow[d] \arrow[dr, phantom, "h^* \circlearrowleft"] & \mathrm{Hom}_{A^e}(\mathscr{P} \otimes_A \mathscr{P}, B) \arrow[d] \\
\mathrm{Hom}_{A^e}(P,B) \arrow[r] & \mathrm{Hom}_{A^e}(\mathscr{P},B),
\end{tikzcd}$$

where the top square commutes and the bottom square commutes up to the $\mathcal{G}$-linear homotopy

$$h^* : \mathrm{Hom}_{A^e}(P \otimes_A P, B) \to \mathrm{Hom}_{A^e}(\mathscr{P}, B)[1].$$

To be clear, the top most vertical maps take a product of functions $f \otimes g$ to the function sending a monomial $x \otimes_A y$ in $P \otimes_A P$, or $\mathscr{P} \otimes_A \mathscr{P}$, to $(-1)^{|x||g|} f(x) g(y)$. It follows that the map $\mathrm{Hom}_{A^e}(P,B) \to \mathrm{Hom}_{A^e}(\mathscr{P},B)$ is an algebra map, up to a homotopy, as is the induced map

$$\mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}_{A^e}(P,B)) \to \mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}_{A^e}(\mathscr{P},B)).$$

Since P and $\mathscr{P}$ are projective A-bimodule resolutions of A, both of these maps are also seen to be quasi-isomorphisms.

Repeat the process with K to get a resolution $\mathscr{K} \to K$ and quasi-isomorphism

$$\mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}_{A^e}(K,B)) \to \mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}_{A^e}(\mathscr{K},B))$$

which is an algebra map up to a homotopy. Finally, since both $\mathscr{K}$ and $\mathscr{P}$ are projective resolutions of A in $\mathrm{EQ}_{\mathcal{G}}A^e$, there is a $\mathcal{G}$-equivariant quasi-isomorphism $\mathscr{K} \to \mathscr{P}$. We repeat the above argument a third time to deduce a quasi-isomorphism

$$\mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}_{A^e}(\mathscr{P},B)) \to \mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}_{A^e}(\mathscr{K},B))$$

which is an algebra map up to a homotopy. Taking $\mathscr{A} = \mathrm{Hom}_{\mathcal{G}}(L, \mathrm{Hom}_{A^e}(\mathscr{K},B))$ then provides the desired result. ■

Proof of Lemma 2.35. The first claim follows from the fact that K is a left $A*\mathcal{G}$-complex itself. Now, for any f in $\mathrm{Hom}_A(K\otimes_A M, N)$ let f_{Hom} denote its image in $\mathrm{Hom}_{A^e}(K, \mathrm{Hom}_k(M,N))$ under the adjunction isomorphism of Proposition 2.32. Then we have, for any $\gamma \in \mathcal{G}$, $x \in K$, and $m \in M$,

$$\begin{aligned}
\big((f\gamma)_{\mathrm{Hom}}(x)\big)(m) &= f\gamma(x \otimes_A m)\\
&= S(\gamma_1)f(\gamma_2 x \otimes_A \gamma_3 m)\\
&= S(\gamma_1)f_{\mathrm{Hom}}(\gamma_2 x)(\gamma_3 m)\\
&= \big(S(\gamma_1)f_{\mathrm{Hom}}(\gamma_2 x)\gamma_3\big)(m)\\
&= (f_{\mathrm{Hom}}\gamma(x))(m).
\end{aligned}$$

So $f\gamma$ maps to $f_{\mathrm{Hom}}\gamma$ under the adjunction isomorphism of the proof of Theorem 2.32 and, consequently, the isomorphism

$$\mathrm{RHom}_{A^e}(A, \mathrm{Hom}_k(M,N)) \overset{\cong}{\to} \mathrm{RHom}_A(M,N)$$

is $\mathcal{G}$-linear. ■

Chapter 3

BRAIDED STRUCTURES AND HOCHSCHILD COHOMOLOGY

Let k be a field. In this chapter we produce a natural braided anti-symmetric operation $[,]_{\mathsf{YD}}$ on the intermediate cohomology $H^\bullet_{\mathrm{Int}}(A*\mathcal{G}) = HH^\bullet(A, A*\mathcal{G})$. Although we are still interested in the cohomology of smash products, our set-up is a bit different than it was in Chapter 2. We will require here at least that $\mathcal{G}$ is finite dimensional and, at some point, we will require additionally that $\mathcal{G}$ has finite exponent $\exp(\mathcal{G})$ which is invertible in the base k. In the case of a group ring, this condition on the exponent is equivalent to the requirement that $\mathcal{G} = kG$ is semisimple, although in general the two properties are not the same (see Corollary 3.34).

In Section 3.1 we produce a Hochschild cochain complex for algebras over a noncommutative separable base. We show that this complex embeds as a dg Lie subalgebra in the usual Hochschild cochain complex. We then introduce the intermediate complex $C^\bullet_{\mathrm{Int}}(A*\mathcal{G})$, show that it is a dg algebra in the category of Yetter-Drinfeld modules (Proposition 3.9), set the groundwork necessary for producing the Yetter-Drinfeld bracket $[,]_{\mathsf{YD}}$, and show that the cohomology $H^\bullet_{\mathrm{Int}}(A*\mathcal{G})$ is a braided commutative algebra (Corollary 3.21). This material is given in Section 3.2. In Sections 3.3 and 3.4 we give some background on the exponent of Hopf algebras and present the Yetter-Drinfeld bracket $[,]_{\mathsf{YD}}$, which exists as a braided anti-symmetric map at the cochain level

$$[,]_{\mathsf{YD}} : C^\bullet_{\mathrm{Int}}(A*\mathcal{G}) \otimes C^\bullet_{\mathrm{Int}}(A*\mathcal{G}) \to C^\bullet_{\mathrm{Int}}(A*\mathcal{G}).$$

Finally, in Section 3.5 we give some natural interpretations of the cohomology $H^\bullet_{\mathrm{Int}}(A*\mathcal{G})$ in low degree.

3.1 Algebras over a separable base and the intermediate cohomology

Let S be a (noncommutative) separable algebra and R be an S-algebra in the sense that it comes equipped with an algebra map $S \to R$. We do not require that the map $S \to R$ is injective.

Lemma 3.1 ([63, Lemma 5.5])**.** *If S is separable then S^e is semisimple.*

We let $\mathscr{B}ar_S R$ denote the relative bar resolution for R over S,

$$\mathscr{B}ar_S R = (\oplus_i R \otimes_S R^{\otimes_S i} \otimes_S R, \text{with standard differential}).$$

Lemma 3.2. *The relative bar resolution provides a projective resolution of R as a bimodule, and the projection $\mathscr{B}ar R \to \mathscr{B}ar_S R$ is a homotopy equivalence.*

Proof. By the previous lemma, the action map $S \otimes V \otimes S \to V$ of any S-bimodule V is a split epimorphism of S-bimodules. So the surjection $R \otimes V \otimes R = R \otimes_S S \otimes V \otimes S \otimes_S R \to R \otimes_S V \otimes_S R$ is a split epimorphism of R-bimodules. In particular, taking $V = R^{\otimes_S n}$, we see that $\mathscr{B}ar_S R$ is projective in each degree.

The fact that the map $\mathscr{B}ar_S R \to R$ is a quasi-isomorphism follows from the fact that the usual contracting homotopy on the mapping cone still exists in this setting. Now we see that the projection is a homotopy equivalence since it fits into a diagram

$$\begin{tikzcd}
\mathscr{B}ar R \arrow[rr] \arrow[dr] & & \mathscr{B}ar_S R \arrow[dl] \\
& R &
\end{tikzcd},$$

is therefore a quasi-isomorphism, and quasi-isomorphisms of bounded above complexes on projectives are homotopy equivalences. ■

Notation 3.3. Take

$$C^\bullet_S(R) := \mathrm{Hom}_{R^e}(\mathscr{B}ar_S R, R) = \mathrm{Hom}_{S^e}(R^{\otimes_S \bullet}, R)$$

and

$$C^\bullet(R) := \mathrm{Hom}_{R^e}(\mathscr{B}ar R, R) = \mathrm{Hom}_k(R^{\otimes \bullet}, R).$$

We have the canonical embedding $C^\bullet_S(R) \to C^\bullet(R)$ dual to the projection $\mathscr{B}ar R \to \mathscr{B}ar_S R$. This identifies $C^\bullet_S(R)$ with the subcomplex of functions $f : R^{\otimes n+2} \to R$ which vanish on the relations

$$r_0 \otimes \ldots \otimes r_i s \otimes r_{i+1} \otimes \ldots \otimes r_{n+1} - r_0 \otimes \ldots \otimes r_i \otimes s r_{i+1} \otimes \ldots \otimes r_{n+1}, \tag{3.1.1}$$

for each i and $s \in S$. In the following lemma Σ denotes the shift automorphism of the category of chain complexes.

Lemma 3.4. *The subcomplex $C^\bullet_S(R)$ is closed under the circle operation on $C^\bullet(R)$. More importantly, $\Sigma C^\bullet_S(R)$ is a quasi-isomorphic Lie subalgebra of $\Sigma C^\bullet(R)$. Whence $H^\bullet(\Sigma C^\bullet_S(R)) = \Sigma HH^\bullet(R)$ as a graded Lie algebra.*

Proof. It is easy to see that Gerstenhaber's circle product $f \circ g$ of functions vanishing on the relation (3.1.1) already vanishes on (3.1.1). Whence the bracket $[f, g] = f \circ g \mp g \circ f$ of functions in $C^\bullet_S(R)$ is still in $C^\bullet_S(R)$. ■

Another way to understand the bracket of $C^\bullet_S(R)$ is as follows: let $T_S R$ denote the tensor S^e-coalgebra $T_S R = \oplus_i R^{\otimes_S i}$ with the standard comultiplication and grading given by taking R to be degree -1. Then we have an identification

$$\Sigma C^\bullet_S(R) \cong \mathrm{Coder}_{S^e}(T_S R)$$

in the usual way and recover the Lie bracket on the shifted complex $\Sigma C^\bullet_S(R)$ as the graded commutator of coderivations.

3.1.1 When $S = \mathcal{G}$ and $R = A * \mathcal{G}$: the complex $C^\bullet_{\mathrm{Int}}(A * \mathcal{G})$ and its $\mathcal{G}$-coaction

Suppose we have a finite dimensional Hopf algebra $\mathcal{G}$ acting on an algebra A. We denote the action on A by a left superscript $\gamma \cdot a = {}^\gamma a$. In this case we have $\mathscr{B}ar_{\mathcal{G}}(A * \mathcal{G}) = (\mathscr{B}ar A) * \mathcal{G}$, where $\mathcal{G}$ acts on the bar resolution diagonally, and the smash product is as in Chapter 2. The material of Chapter 2 Section 2.4 then gives

Proposition 3.5. *Let $\mathcal{G}$ act on the right of the hom complex $C^\bullet(A, A*\mathcal{G})$ according to the formula*

$$f\cdot\gamma(a_1\dots a_n) := S(\gamma_1)f({}^{\gamma_2}a_1\dots{}^{\gamma_{n+1}}a_n)\gamma_{n+2}.$$

This action commutes with the differential and, in the case that $\mathcal{G}$ is semisimple, we have an identification of complexes $C^\bullet(A, A*\mathcal{G})^{\mathcal{G}} = C^\bullet_{\mathcal{G}}(A*\mathcal{G})$

Proof. This is a particular occurrence of Theorem 2.19. In short, we will have a sequence of equalities

$$\begin{aligned} C^\bullet_{\mathcal{G}}(A*\mathcal{G}) &= \mathrm{Hom}_{A*\mathcal{G}^e}(\mathscr{B}ar_{\mathcal{G}}(A*\mathcal{G}), A*\mathcal{G}) \\ &= \mathrm{Hom}_{(A*\mathcal{G})^e}\big((\mathscr{B}arA)*\mathcal{G}, A*\mathcal{G}\big) \\ &= \mathrm{Hom}_{A\otimes(A*\mathcal{G})^{op}}\big((\mathscr{B}arA)*\mathcal{G}, A*\mathcal{G}\big)^{\mathcal{G}} \\ &= \mathrm{Hom}_{A^e}(\mathscr{B}arA, A*\mathcal{G})^{\mathcal{G}} \qquad\qquad \text{(restrict to } \mathscr{B}arA) \\ &= C^\bullet(A, A*\mathcal{G})^{\mathcal{G}}. \end{aligned} \tag{3.1.2}$$

■

In the case that $\mathcal{G}$ is not semisimple we will still have the right action of $\mathcal{G}$ on $C^\bullet(A, A*\mathcal{G})$, although we will not have the identification of the invariants with the complex $C^\bullet_{\mathcal{G}}(A*\mathcal{G})$.

Notation 3.6. We let $C^\bullet_{\mathrm{Int}} = C^\bullet_{\mathrm{Int}}(A*\mathcal{G})$ denote the complex $C^\bullet(A, A*G)$ and $H^\bullet_{\mathrm{Int}} = H^\bullet_{\mathrm{Int}}(A*\mathcal{G})$ denote the cohomology of this complex. This complex and its cohomology will be referred to as the intermediate cochain complex and intermediate cohomology respectively.

Obviously $H^\bullet_{\mathrm{Int}}(A*\mathcal{G}) = HH^\bullet(A, A*\mathcal{G})$. However, we will also be interested in identifying the intermediate cohomology with other useful cohomologies. For example, if we take

$$C^\bullet_{\mathrm{Int}} = \mathrm{Hom}_{A\otimes(A*\mathcal{G})^{op}}\big((\mathscr{B}arA)*\mathcal{G}, A*\mathcal{G}\big) \text{ and thus } H^\bullet_{\mathrm{Int}} = \mathrm{Ext}_{A\otimes(A*\mathcal{G})^{op}}(A*\mathcal{G}, A*\mathcal{G})$$

when $\mathcal{G}$ is semisimple then we get actual equalities $C^\bullet_{\mathcal{G}}(A*\mathcal{G}) = C^\bullet_{\mathrm{Int}}(A*\mathcal{G})^{\mathcal{G}}$ and $HH^\bullet(A*\mathcal{G}) = H^\bullet_{\mathrm{Int}}(A*\mathcal{G})^{\mathcal{G}}$. This will not occur with other expressions. So we do not resign ourselves to any particular interpretation of the intermediate cohomology at this point.

There is a natural right (cofree) $\mathcal{G}$-comodule structure ρ on this complex induced by the k-linear identification

$$C^\bullet(A) \otimes \mathcal{G} \to C^\bullet_{\mathrm{Int}}(A * \mathcal{G}), \quad \xi \otimes \gamma \mapsto (x \mapsto \xi(x)\gamma). \tag{3.1.3}$$

That is to say, the coaction is induced by the coaction on the codomain $A*\mathcal{G}$. So, if we identify elements of $C^\bullet_{\mathrm{Int}} \otimes \mathcal{G}$ with $A*\mathcal{G}\otimes\mathcal{G}$ valued functions on $\mathscr{B}ar A$ by taking $\xi \otimes \gamma(x) := \xi(x) \otimes \gamma$, we can then give the function $\rho(\xi)(x) := \rho_{A*\mathcal{G}}(\xi(x))$.

Notation 3.7. We write $C^\bullet_\gamma$ for the image of $C^\bullet(A) \otimes k\gamma$ under the map (3.1.3) and f_γ for the image of $f \otimes \gamma$ under (3.1.3).

So, $C^\bullet_\gamma$ denotes the subspace of functions which take values in $A\gamma \subset A * \mathcal{G}$. In this notation we have $\rho(f_\gamma) = f_{\gamma_1} \otimes \gamma_2$.

3.2 The Yetter-Drinfeld structure on $C^\bullet_{\mathrm{Int}}(A * \mathcal{G})$, naive bracket, and braided commutativity

Let $\mathcal{G}$ be a finite dimensional Hopf algebra. We consider the categories of right $\mathcal{G}$-co/modules almost exclusively (as opposed to left co/modules).

Definition 3.8 (Yetter-Drinfeld modules, maps, and complexes)**.**

1. A simultaneous right $\mathcal{G}$-module and comodule M, with coaction $\rho : M \to M \otimes \mathcal{G}$, is called a Yetter-Drinfeld module if the action and coaction satisfy

   $$\rho(m\gamma) = m_0\gamma_2 \otimes S(\gamma_1)m_1\gamma_3, \tag{3.2.1}$$

   for any $m \in M$, $\gamma \in \mathcal{G}$.

2. A morphism of Yetter-Drinfeld modules is a simultaneous module/comodule map. the category of (right) Yetter-Drinfeld modules over $\mathcal{G}$ is denoted $YD^{\mathcal{G}}_{\mathcal{G}}$.

3. A chain complex X in which each X^i is a Yetter-Drinfeld module is said to be a complex in $YD^{\mathcal{G}}_{\mathcal{G}}$ if the differentials are morphism in $YD^{\mathcal{G}}_{\mathcal{G}}$ and the action and coaction are maps of chain complexes, where $X \otimes \mathcal{G}$ is given the standard differential $d(x \otimes \gamma) := d(x) \otimes \gamma$.

The category $YD_{\mathcal{G}}^{\mathcal{G}}$ is well known to be a braided category under the non-trivial braiding

$$\mathsf{braid}_{MN} : M \otimes N \to N \otimes M, \quad m \otimes n \mapsto n_0 \otimes m \cdot n_1.$$

This braiding will play an essential role in the production of a good bracket on the intermediate complex $C^\bullet_{\mathrm{Int}}(A * \mathcal{G})$.

Proposition 3.9. *The complex $C^\bullet_{\mathrm{Int}}(A * \mathcal{G})$, with the right $\mathcal{G}$-action and coaction of Section 3.1.1, becomes a complex in the category of Yetter-Drinfeld modules over $\mathcal{G}$.*

Proof. Let ρ denote the coaction on the intermediate complex. We identify elements of $C^\bullet_{\mathrm{Int}} \otimes \mathcal{G}$ as $A * \mathcal{G} \otimes \mathcal{G}$ valued function on $\mathscr{B}ar A$ by

$$(\xi \otimes \alpha)(x) = \xi(x) \otimes \alpha.$$

This gives $\rho(\xi)(x) = \rho(\xi(x))$ for each $x \in \mathscr{B}ar A$. By Lemma 2.18 of Chapter 2, we already know that the $\mathcal{G}$-action commutes with the differential. As for the coaction, we have for any $f_\gamma \in C^\bullet_\gamma$ that $\rho(f_\gamma) = f_{\gamma_1} \otimes \gamma_2$ and therefore

$$\begin{aligned}
&(-1)^{|f|+1} d_{C^\bullet_{\mathrm{Int}} \otimes \mathcal{G}}(f_{\gamma_1} \otimes \gamma_2)(a_1 \otimes \dots a_n)\\
&= a_1 f_{\gamma_1}(a_2 \otimes \dots \otimes a_n) \otimes \gamma_2 \pm f_{\gamma_1}(a_1 \otimes \dots \otimes a_{n-1}) a_n \otimes \gamma_2\\
&\quad + \textstyle\sum_i \pm f_{\gamma_1}(a_1 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n) \otimes \gamma_2\\
&= a_1 f(a_2 \otimes \dots \otimes a_n)\gamma_1 \otimes \gamma_2 \pm f(a_1 \otimes \dots \otimes a_{n-1})^{\gamma_1} a_n \gamma_2 \otimes \gamma_3\\
&\quad + \textstyle\sum_i \pm f(a_1 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n)\gamma_1 \otimes \gamma_2\\
&= \rho\big(d_{C^\bullet_{\mathrm{Int}}}(f_\gamma)(a_1 \otimes \dots \otimes a_n)\big)\\
&= \rho(d_{C^\bullet_{\mathrm{Int}}}(f))(a_1 \otimes \dots \otimes a_n).
\end{aligned}$$

This shows that $C^\bullet_{\mathrm{Int}}(A*\mathcal{G})$ is a complex of $\mathcal{G}$-modules and comodules. To see that the action and coaction satisfy the appropriate compatibility we simply calculate

$$\begin{aligned}
\rho(f_\gamma \cdot \sigma(x)) &= \rho(S(\sigma_1) f_\gamma({}^{\sigma_2}x)\sigma_3)\\
&= \rho({}^{S(\sigma_2)} f({}^{\sigma_3}x) S(\sigma_1)\gamma\sigma_4)\\
&= {}^{S(\sigma_3)} f({}^{\sigma_4}x) S(\sigma_2)\gamma_1\sigma_5 \otimes S(\sigma_1)\gamma_2\sigma_6\\
&= S(\sigma_2) f({}^{\sigma_3}x)\gamma_1\sigma_4 \otimes S(\sigma_1)\gamma_2\sigma_5\\
&= f_{\gamma_1} \cdot \sigma_2(x) \otimes S(\sigma_1)\gamma_2\sigma_3.
\end{aligned}$$

■

With this new understanding, we define a preliminary "bracket" on $C^\bullet_{\mathrm{Int}}(A*\mathcal{G})$. This bilinear operation turns out to be poorly behaved, but still rather useful. A well behaved bracket on $C^\bullet_{\mathrm{Int}}$ will be defined in Section 3.4.

Definition 3.10 (The naive bracket)**.** The circle operaton on $C^\bullet_{\mathrm{Int}}(A*\mathcal{G})$ is defined as follows: for $f_\alpha \in C^\bullet_\alpha$ and $g_\beta \in C^\bullet_\beta$ we take

$$\begin{aligned} f_\alpha \circ g_\beta(x) &:= (-1)^{|x_1|(|g|-1)} f_\alpha(x_1 \otimes g(x_2) \otimes {}^{\beta_1}x_3)\beta_2 \\ &= (-1)^{|x_1|(|g|-1)} f(x_1 \otimes g(x_2) \otimes {}^{\beta_1}x_3)\alpha\beta_2. \end{aligned} \tag{3.2.2}$$

we define the naive bracket by

$$[f_\alpha, g_\beta]_\circ = f_\alpha \circ g_\beta - (-1)^{(|f|-1)(|g|-1)} g_{\beta_1} \circ (f_\alpha \cdot \beta_2).$$

On generic functions $\xi, \eta \in C^\bullet_{\mathrm{Int}}$ we can write also

$$[\xi, \eta]_\circ = \xi \circ \eta - (-1)^{|\xi||\eta|} \eta_0 \circ (\xi \cdot \eta_1).$$

Here we employ the identification $C^\bullet_{\mathrm{Int}}(A*\mathcal{G}) = \mathrm{Hom}_k(A^{\otimes \bullet}, A*\mathcal{G}) = \mathrm{Hom}_k(\mathscr{B}A, A*\mathcal{G})$ so that $x_1 \otimes x_2 \otimes x_3$ denotes the second iterated comultiplication on an element x in the tensor coalgebra $\mathscr{B}A$. We also employ the identification $A^{\otimes i} \otimes A \otimes A^{\otimes j} \cong A^{\otimes(i+j+1)}$ to view the $x_1 \otimes g(x_2) \otimes {}^{\beta}x_3$ as elements in the bar complex. One has an alternate expression of the circle operation (3.2.2) which mirrors Gerstenhabers original formula. We see below that this bracket respects the Yetter-Drinfeld structure on the intermediate complex.

Lemma 3.11. *For any $\gamma \in \mathcal{G}$, and $\xi, \eta \in C^\bullet_{\mathrm{Int}}(A*\mathcal{G})$, we have $[\xi,\eta]_\circ \cdot \gamma = [\xi \cdot \gamma_1, \eta \cdot \gamma_2]_\circ$ and $\rho([\xi,\eta]_\circ) = [\xi_0, \eta_0]_\circ \otimes \xi_1\eta_1$.*

Proof. Take $f_\alpha \in C^\bullet_\alpha$ and $g_\beta \in C^\bullet_\beta$. Then we have

$$\begin{aligned}
&f_\alpha \cdot \gamma_1 \circ g_\beta \cdot \gamma_2(x) \\
&= \sum \pm S(\gamma_1) f\big({}^{\gamma_2}(x_1 \otimes {}^{S(\gamma_6)}g({}^{\gamma_7}x_2) \otimes {}^{S(\gamma_5)\beta_1\gamma_8}x_3)\big)\alpha\gamma_3 S(\gamma_4)\beta_2\gamma_9 \\
&= \sum \pm S(\gamma_1) f\big({}^{\gamma_2}(x_1 \otimes {}^{S(\gamma_4)}g({}^{\gamma_5}x_2) \otimes {}^{S(\gamma_3)\beta_1\gamma_6}x_3)\big)\alpha\beta_2\gamma_7 && \text{erase } \gamma_3 S(\gamma_4) \\
&= \sum \pm S(\gamma_1) f\big({}^{\gamma_2}x_1 \otimes {}^{\gamma_3 S(\gamma_6)}g({}^{\gamma_7}x_2) \otimes {}^{\gamma_4 S(\gamma_5)\beta_1\gamma_8}x_3)\big)\alpha\beta_2\gamma_9 && \text{distribute } \gamma_2 \\
&= \sum \pm S(\gamma_1) f\big({}^{\gamma_2}x_1 \otimes {}^{\gamma_3 S(\gamma_4)}g({}^{\gamma_5}x_2) \otimes {}^{\beta_1\gamma_6}x_3)\big)\alpha\beta_2\gamma_7 && \text{erase } \gamma_4 S(\gamma_5) \\
&= \sum \pm S(\gamma_1) f\big({}^{\gamma_2}x_1 \otimes g({}^{\gamma_3}x_2) \otimes {}^{\beta_1\gamma_4}x_3)\big)\alpha\beta_2\gamma_5 && \text{erase } \gamma_3 S(\gamma_4) \\
&= S(\gamma_1)(f_\alpha \circ g_\beta)({}^{\gamma_2}x)\gamma_3 \\
&= \left((f_\alpha \circ g_\beta) \cdot \gamma\right)(x).
\end{aligned}$$

So $(f_\alpha \circ g_\beta) \cdot \gamma = (f_\alpha \cdot \gamma) \circ (g_\beta \cdot \gamma)$ for all f_α and g_β. Since each function in $C^\bullet_{\mathrm{Int}}$ is a finite sum of such f_α, g_β, the result holds for arbitrary maps. Now, using the Yetter-Drinfeld identity, we get

$$\begin{aligned}
&[\xi, \eta]_\circ \cdot \gamma \\
&= (\xi \circ \eta) \cdot \gamma - (-1)^{(|\xi|-1)(|\eta|-1)}(\eta_0 \circ (\xi \cdot \eta_1)) \cdot \gamma \\
&= \xi \cdot \gamma_1 \circ \eta \cdot \gamma_2 - (-1)^{(|\xi|-1)(|\eta|-1)}\eta_0 \cdot \gamma_1 \circ (\xi \cdot \eta_1\gamma_2) \\
&= \xi \cdot \gamma_1 \circ \eta \cdot \gamma_2 - (-1)^{(|\xi|-1)(|\eta|-1)}\eta_0 \cdot \gamma_3 \circ (\xi \cdot \gamma_1 S(\gamma_2)\eta_1\gamma_4) \\
&= \xi \cdot \gamma_1 \circ \eta \cdot \gamma_2 - (-1)^{(|\xi|-1)(|\eta|-1)}\eta_0 \cdot \gamma_3 \circ \big((\xi \cdot \gamma_1) \cdot S(\gamma_2)\eta_1\gamma_4\big) \\
&= \xi \cdot \gamma_1 \circ \eta \cdot \gamma_2 - (-1)^{(|\xi|-1)(|\eta|-1)}(\eta \cdot \gamma_2)_0 \circ \big((\xi \cdot \gamma_1) \cdot (\eta \cdot \gamma_2)_1\big) \\
&= [\xi \cdot \gamma_1, \eta \cdot \gamma_2]_\circ.
\end{aligned}$$

As for compatibility with the coaction, on elements we get

$$f_\alpha \circ g_\beta(x) = \sum \pm f(x_1 \otimes g(x_2) \otimes {}^{\beta_1}x_3)\alpha\beta_2.$$

Applying the coaction to the final element makes it clear that

$$\rho(f_\alpha \circ g_\beta) = f_{\alpha_1} \circ g_{\beta_1} \otimes \alpha_2\beta_2,$$

and on general functions in $C^\bullet_{\mathrm{Int}}$ we get

$$\rho(\xi \circ \eta) = \xi_0 \circ \eta_0 \otimes \xi_1\eta_1.$$

and so

$$\begin{aligned}
&\rho([\xi,\eta]_\circ) = \rho(\xi\circ\eta) \mp \rho(\eta_0\circ\xi\cdot\eta_1)\\
&= \xi_0\circ\eta_0\otimes\xi_1\eta_1 \mp \eta_0\circ(\xi\cdot\eta_2)_0\otimes\eta_1(\xi\cdot\eta_2)_1\\
&= \xi_0\circ\eta_0\otimes\xi_1\eta_1 \mp \eta_0\circ(\xi_0\cdot\eta_3)\otimes\eta_1 S(\eta_2)\xi_1\eta_4 \quad \text{by YD identity}\\
&= \xi_0\circ\eta_0\otimes\xi_1\eta_1 \mp \eta_0\circ(\xi_0\cdot\eta_1)\otimes\xi_1\eta_2 \qquad\qquad \text{erase } \eta_1 S(\eta_2)\\
&= [\xi_0,\eta_0]_\circ\otimes\xi_1\eta_1,
\end{aligned}$$

as proposed. ■

Lemma 3.12. *For invariant functions* $\xi,\eta\in C^\bullet_{\mathrm{Int}}(A*\mathcal{G})$ *the bracket* $[\xi,\eta]_\circ$ *agrees with the Gerstenhaber bracket, after making the identification* $C^\bullet_{\mathrm{Int}}(A*\mathcal{G})^{\mathcal{G}}\cong C^\bullet_G(A*G)$.

Proof. For invariant functions we have

$$\begin{aligned}
[\xi,\eta]_\circ &= \xi\circ\eta-(-1)^{(|\xi|-1)(|\eta|-1)}\eta_0\circ(\xi\cdot\eta_1)\\
&= \xi\circ\eta-(-1)^{(|\xi|-1)(|\eta|-1)}\eta_0\circ\xi\epsilon(\eta_1)\\
&= \xi\circ\eta-(-1)^{(|\xi|-1)(|\eta|-1)}\eta\circ\xi.
\end{aligned}$$

Now, under the sequence of equalities (3.1.2) the circle operation on invariant functions in $C^\bullet_{\mathrm{Int}}$ agrees with the standard circle operation on $C^\bullet_{\mathcal{G}}(A*\mathcal{G})$. So the final expression is equal to the Gerstenhaber bracket on $C^\bullet_{\mathcal{G}}(A*\mathcal{G})$. ■

We now, essentially, have all the machinery needed to continue with our analysis of the intermediate cohomology, and prove our main results. However, our current presentation of the intermediate complex is rather unnatural. The lack of naturality will sometimes emerge as an impediment in giving clear proofs/constructions. In the next section we give an alternate, and preferred, expression of the intermediate complex and naive bracket (see Lemmas 3.13, 3.16, 3.19, 3.18).

3.2.1 A reconsideration of the complex $C^\bullet_{\mathrm{Int}}(A*\mathcal{G})$

For each n have an isomorphisms

$$\mathrm{Hom}_{\mathcal{G}^{op}}\big((A*\mathcal{G})^{\otimes_{\mathcal{G}} n},A*\mathcal{G}\big)\overset{\cong}{\to}\mathrm{Hom}_k(A^{\otimes n},A*\mathcal{G})=C^n(A,A*\mathcal{G})$$

given by the restriction map. These isomorphisms induce a differential d_{Int} on the graded space $\bigoplus_n \mathrm{Hom}_{\mathcal{G}^{op}}(A * \mathcal{G}^{\otimes_{\mathcal{G}} n}, A * \mathcal{G})$. Whence we have the complex

$$(\bigoplus_n \mathrm{Hom}_{\mathcal{G}^{op}}(A * \mathcal{G}^{\otimes_{\mathcal{G}} n}, A * \mathcal{G}), d_{\mathrm{Int}})$$

and the following trivial result.

Lemma 3.13. *Restriction gives a canonical identification*

$$\mathrm{restrict} : (\bigoplus_n \mathrm{Hom}_{\mathcal{G}^{op}}(A * \mathcal{G}^{\otimes_{\mathcal{G}} n}, A * \mathcal{G}), d_{\mathrm{Int}}) \to C^{\bullet}_{\mathrm{Int}}(A * \mathcal{G}).$$

A formula for the differential d_{Int} is given in Lemma 3.19. This alternate expression of the intermediate complex will clarify, among other things, the right $\mathcal{G}$-action and coaction on $C^{\bullet}_{\mathrm{Int}}$. Note that the right adjoint action on $A * \mathcal{G}$,

$$(a\gamma) \cdot \sigma = S(\sigma_1)a\gamma\sigma_2 = {}^{S(\sigma_2)}aS(\sigma_1)\gamma\sigma_3$$

and obvious coaction, $a\gamma \mapsto a\gamma_1 \otimes \gamma_2$, give it the structure of an algebra in $YD^{\mathcal{G}}_{\mathcal{G}}$. The induced diagonal co/action on the higher tensor products $A * \mathcal{G}^{\otimes n}$ give them Yetter-Drinfeld structures as well.

Lemma 3.14. *The kernel of the projection $A*\mathcal{G}^{\otimes n} \to A*\mathcal{G}^{\otimes_{\mathcal{G}} n}$ is a Yetter-Drinfeld submodule in the n-fold tensor product, and hence induces a canonical Yetter-Drinfeld structure on $A * \mathcal{G}^{\otimes_{\mathcal{G}} n}$.*

Proof. This follows from the fact that $\mathcal{G}$ is a Yetter-Drinfeld subalgebra in $A * \mathcal{G}$. So, for example, for any $\alpha, \beta \in A * \mathcal{G}$ and $\gamma, \sigma \in \mathcal{G}$ the relation $\alpha\gamma \otimes \beta - \alpha \otimes \gamma\beta$ we will have

$$\rho(\alpha\gamma \otimes \beta - \alpha \otimes \gamma\beta) = (\alpha_0\gamma_1 \otimes \beta_0 - \alpha_0 \otimes \gamma_1\beta_0) \otimes \alpha_1\gamma_2\beta_1$$

and

$$(\alpha\gamma \otimes \beta - \alpha \otimes \gamma\beta) \cdot \sigma = (\alpha \cdot \sigma_1)(\gamma \cdot \sigma_2) \otimes (\beta \cdot \sigma_3) - (\alpha \cdot \sigma_1) \otimes (\gamma \cdot \sigma_2)(\beta \cdot \sigma_3).$$

■

We can express the action on $A * \mathcal{G}^{\otimes_{\mathcal{G}} n}$ as a large conjugation

$$\begin{aligned}(\alpha_1 \otimes \ldots \otimes \alpha_n) \cdot \sigma &= (S(\sigma_1)\alpha_1) \otimes \alpha_2 \otimes \ldots \otimes \alpha_{n-1} \otimes (\alpha_n \sigma_2) \\ &= (S(\sigma_1)\alpha_1\sigma_2) \otimes (S(\sigma_3)\alpha_2\sigma_4) \otimes \ldots \otimes (S(\sigma_{2n-1})\alpha_n\sigma_{2n}),\end{aligned} \tag{3.2.3}$$

where the α_i are in $A * \mathcal{G}$ and the tensor products are over $\mathcal{G}$. The coaction is given by

$$\alpha_1 \otimes \ldots \otimes \alpha_n \mapsto ((\alpha_1)_1 \otimes \ldots \otimes (\alpha_n)_1) \otimes (\alpha_1)_2 \ldots (\alpha_n)_2.$$

The following lemma is well known. See for example [60, Remark 3.5].

Lemma 3.15. *Let M and N be Yetter-Drinfeld modules over $\mathcal{G}$. Then there is a canonical Yetter-Drinfeld structure on the set of k-linear maps* $\mathrm{Hom}_k(M, N)$ *so that the standard* $\otimes$-Hom *adjunction induces a natural isomorphism*

$$\mathrm{Hom}_{YD^{\mathcal{G}}_{\mathcal{G}}}(L \otimes M, N) \cong \mathrm{Hom}_{YD^{\mathcal{G}}_{\mathcal{G}}}(L, \mathrm{Hom}_k(M, N)).$$

The right action of $\mathcal{G}$ on $\mathrm{Hom}_k(M, N)$ *is given explicitly by the formula $(f \cdot \sigma)(m) = f(mS^{-1}(\sigma_2))\sigma_1$, while the right $\mathcal{G}$-coaction is specified by the left $\mathcal{G}^*$-action $(\xi \cdot f)(m) = \xi_1 f(S(\xi_2)m)$.*

Said another way, the category of Yetter-Drinfeld modules has inner homs. In the proof we use the fact that the category of Yetter-Drinfeld modules is equal to the category of modules over the Drinfeld double $D(\mathcal{G})$. The unfamiliar reader is invited to skip the proof momentarily, or see Section 3.3.1 for the appropriate background.

Proof. Yetter-Drinfeld modules over $\mathcal{G}$ are the same thing as left modules over the Drinfeld double $D(\mathcal{G})$. So it suffices to show that the category of left H-modules has inner homs for any finite dimensional Hopf algebra.

Let M and N be left H-modules for any such H and give the set of k-linear homs the proposed action $hf(m) := h_1 f(S(h_2)m)$. One can check that this is a well defined action.

Suppose we have any right H-linear map $\phi : L \otimes M \to N$ and and let $f_\phi : L \to \mathrm{Hom}_k(M, N)$ be the adjoint map $l \mapsto (m \mapsto \phi(l \otimes m))$. Then we have for any $h \in H$

$$\begin{aligned} f_\phi(hl)(m) &= \phi(hl \otimes m) \\ &= \phi(h_1(l \otimes S(h_2)m)) \\ &= h_1\phi(l \otimes S(h_2)m) \\ &= h_1 f_\phi(l)(S(h_2)m) = \big(h \cdot (f_\phi(l))\big)(m). \end{aligned}$$

A similar calculation shows that the map $\phi^f : L \otimes M \to N$ adjoint to a H-linear map $f : L \to \mathrm{Hom}_k(M, N)$ is H-linear. So the map

$$\mathrm{Hom}_{YD}(L \otimes M, N) \to \mathrm{Hom}_{YD}(L, \mathrm{Hom}_k(M, N)), \quad \phi \mapsto f_\phi.$$

is a bijection. The formula for the the right $\mathcal{G}$-action in the case $H = D(\mathcal{G})$ follows from the $\mathcal{G}$-action is given by restricting along the Hopf algebra embedding $\mathcal{G}^{op} \to D(\mathcal{G})$ and the fact that $S_{D(\mathcal{G})}|\mathcal{G}^{op} = S_{\mathcal{G}}^{-1}$. We deduce the formula for the coaction similarly. ■

For Yetter-Drinfeld modules M and N we let $\underline{\mathrm{Hom}}_k(M, N)$ denote the set of k-linear homs with its natural Yetter-Drinfeld structure. We will not need to understand the right coaction on $\underline{\mathrm{Hom}}_k(A * \mathcal{G}^{\otimes_{\mathcal{G}} n}, A * \mathcal{G})$ directly. For our purposes, it will suffice to understand the $\mathcal{G}^*$-action on $\underline{\mathrm{Hom}}_k(V \otimes M, W \otimes M)$ for vector spaces V and W and a single $\mathcal{G}$-comodules M. Here we have the natural isomorphism

$$\mathrm{Hom}_k(V, W) \otimes M \otimes M^* \overset{\cong}{\to} \underline{\mathrm{Hom}}_k(V \otimes M, W \otimes M), \quad f \otimes m \otimes \lambda \mapsto (v \otimes l \mapsto f(v)m\lambda(l)).$$

Under this identification the induced $\mathcal{G}^*$-action on a monomial $f \otimes m \otimes \lambda$ is given by

$$\xi \cdot (f \otimes m \otimes \lambda) = f \otimes (\xi_1 m) \otimes (\lambda S(\xi_2)).$$

Lemma 3.16. *Each subspace*

$$\mathrm{Hom}_{\mathcal{G}^{op}}(A * \mathcal{G}^{\otimes_{\mathcal{G}} n}, A * \mathcal{G}) \subset \underline{\mathrm{Hom}}_k(A * \mathcal{G}^{\otimes_{\mathcal{G}} n}, A * \mathcal{G})$$

is a Yetter-Drinfeld submodule, and hence inherits a canonical Yetter-Drinfeld structure.

Proof. Suppose $f : A * \mathcal{G}^{\otimes_{\mathcal{G}} n} \to A * \mathcal{G}$ is right $\mathcal{G}$-linear. Then we have, for any $\alpha_i \in A * \mathcal{G}$ and $\sigma, \gamma \in \mathcal{G}$,

$$\begin{aligned}(f \cdot \sigma)(\alpha_1 \otimes \ldots \otimes \alpha_n \gamma) &= S(\sigma_1) f(\sigma_4 \alpha_1 \otimes \ldots \otimes \alpha_n \gamma S^{-1}(\sigma_3))\sigma_2 \\ &= S(\sigma_1) f(\sigma_4 \alpha_1 \otimes \ldots \otimes \alpha_n)\gamma S^{-1}(\sigma_3)\sigma_2 \\ &= S(\sigma_1) f(\sigma_2 \alpha_1 \otimes \ldots \otimes \alpha_n)\gamma \\ &= S(\sigma_1) f(\sigma_4 \alpha_1 \otimes \ldots \otimes \alpha_n S^{-1}(\sigma_3))\sigma_2 \gamma \\ &= (f \cdot \sigma)(\alpha_1 \otimes \ldots \otimes \alpha_n)\gamma.\end{aligned}$$

As for the coaction we have, as comodules, $(A * \mathcal{G})^{\otimes_{\mathcal{G}} n} = A^{\otimes n} \otimes \mathcal{G}$ and $A * \mathcal{G} = A \otimes \mathcal{G}$ so that

$$\underline{\mathrm{Hom}}_k(A * \mathcal{G}^{\otimes_{\mathcal{G}} n}, A * \mathcal{G}) \xrightarrow{\cong} \mathrm{Hom}_k(A^{\otimes n}, A) \otimes \mathcal{G} \otimes \mathcal{G}^*,$$

with the $\mathcal{G}^*$-action as described above. Let $\{e_i\}$ and $\{e^i\}$ be dual bases for $\mathcal{G}$ and $\mathcal{G}^*$ so that $e = \sum e_i \otimes e^i$ is the identity element in $\mathcal{G} \otimes \mathcal{G}^*$. That is, the unique element identified with the identity map under the obvious $\mathcal{G}^*$-module isomorphism $\mathcal{G} \otimes \mathcal{G}^* \cong \mathrm{End}_k(\mathcal{G})$. This element is always $\mathcal{G}^*$-invariant (since the identity map is). We can now locate the right $\mathcal{G}$-linear maps in $\mathrm{Hom}_k(A^{\otimes n}, A) \otimes \mathcal{G} \otimes \mathcal{G}^*$ as sums of maps of the form $f \otimes (\sum_i \gamma e_i \otimes e^i)$ for some $f \in \mathrm{Hom}_k(A^{\otimes n}, A)$ and $\gamma \in \mathcal{G}$. Indeed, we have for any $x \in A^{\otimes n}$, $\sigma \in \mathcal{G}$,

$$(f \otimes (\sum_i \gamma e_i \otimes e^i))(x \otimes \sigma) = f(x)\gamma(\sum_i e_i e^i(\sigma)) = f(x)\gamma\sigma.$$

Now, since $\mathcal{G}$ is a comodule algebra over itself, it is a $\mathcal{G}^*$-module algebra. So, for any $\xi \in \mathcal{G}^*$ we have

$$\begin{aligned}\xi \cdot (f \otimes (\textstyle\sum_i \gamma e_i \otimes e^i)) &= f \otimes (\textstyle\sum_i (\xi_1 \gamma)(\xi_2 e_i \otimes e^i S(\gamma_2))) \\ &= f \otimes (\textstyle\sum_i (\xi_1 \epsilon(\xi_2)\gamma) e_i \otimes e^i) \qquad (3.2.4) \\ &= f \otimes (\textstyle\sum_i (\xi \cdot \gamma) e_i \otimes e^i).\end{aligned}$$

So we see that the subspace of right $\mathcal{G}$-linear maps is indeed closed under the $\mathcal{G}^*$-action, and hence a $\mathcal{G}$-subcomodule in $\underline{\mathrm{Hom}}_k(A * \mathcal{G}^{\otimes_{\mathcal{G}} n}, A * \mathcal{G})$. ■

Equation (3.2.4) actually tells us what the coaction on $\mathrm{Hom}_{\mathcal{G}^{op}}(A * \mathcal{G}^{\otimes_{\mathcal{G}} n}, A * \mathcal{G})$ will be. Namely, if we take $f_\gamma \in C^\bullet_\gamma \subset C^\bullet_{\mathrm{Int}}(A * \mathcal{G}) = \mathrm{Hom}_{\mathcal{G}^{op}}(A * \mathcal{G}^{\otimes_{\mathcal{G}} n}, A * \mathcal{G})$ then $\rho(f_\gamma) = f_{\gamma_1} \otimes \gamma_2$, since

$$f_\gamma = f \otimes (\sum_i \gamma e_i \otimes e^i)$$

under the isomorphism $\mathrm{Hom}_{\mathcal{G}^{op}}(A * \mathcal{G}^{\otimes_{\mathcal{G}} n}, A * \mathcal{G}) = \mathrm{Hom}_k(A^{\otimes n}, A) \otimes \mathcal{G} \otimes \mathcal{G}^*$ employed in the proof of Lemma 3.16.

Proposition 3.17. *Give $\bigoplus_n \mathrm{Hom}_{\mathcal{G}^{op}}(A * \mathcal{G}^{\otimes_{\mathcal{G}} n}, A * \mathcal{G})$ the Yetter-Drinfeld structure induced by the embeddings of Lemma 3.16. Then the restriction map*

$$\mathrm{restrict} : \bigoplus_n \mathrm{Hom}_{\mathcal{G}^{op}}(A * \mathcal{G}^{\otimes_{\mathcal{G}} n}, A * \mathcal{G}) \to C^\bullet_{\mathrm{Int}} \tag{3.2.5}$$

is and isomorphism of Yetter-Drinfeld modules (where $C^\bullet_{\mathrm{Int}}$ has the YD-structure described in Section 3.2).

Proof. It is clear that the restriction map is an isomorphism of comodules, by the discussion preceding the proposition. As for the action, we have for any $a_1 \otimes \ldots \otimes a_n \in A^{\otimes n}$, and $f \in \mathrm{Hom}_{\mathcal{G}^{op}}(A * \mathcal{G}^{\otimes_{\mathcal{G}} n}, A * \mathcal{G})$,

$$\begin{aligned}
& f \cdot \gamma(a_1 \otimes \ldots \otimes a_n) = f((a_1 \otimes \ldots \otimes a_n) \cdot S^{-1}(\gamma_2)) \cdot \gamma_1 \\
& = S(\gamma_1) f({}^{\gamma_4}a_1 \otimes \ldots \otimes {}^{\gamma_{n+3}}a_n \gamma_{n+4} S^{-1}(\gamma_3))\gamma_2 \\
& = S(\gamma_1) f({}^{\gamma_4}a_1 \otimes \ldots \otimes {}^{\gamma_{n+3}}a_n)\gamma_{n+4} S^{-1}(\gamma_3)\gamma_2 && \text{(by right } \mathcal{G}\text{-linearity)} \\
& = S(\gamma_1) f({}^{\gamma_2}a_1 \otimes \ldots \otimes {}^{\gamma_{n+1}}a_n)\gamma_{n+2},
\end{aligned}$$

as desired. ■

As a consequence of Proposition 3.17 we see that the complex

$$(\bigoplus_n \mathrm{Hom}_{\mathcal{G}^{op}}(A * \mathcal{G}^{\otimes_{\mathcal{G}} n}, A * \mathcal{G}), d_{\mathrm{Int}})$$

enjoys its own natural Yetter-Drinfeld structure, under which the differential d_{Int} is $\mathcal{G}$-linear and colinear (by Proposition 3.9). Proposition 3.17 also gives an alternate expression for the intermediate complex $C^\bullet_{\mathrm{Int}}(A * \mathcal{G})$ and intermediate cohomology and a natural embedding

$$C^\bullet_{\mathrm{Int}}(A * \mathcal{G}) \to \underline{\mathrm{Hom}}_k(A * \mathcal{G}^{\otimes_{\mathcal{G}} \bullet}, A)$$

of Yetter-Drinfeld modules, by way of Lemma 3.16.

Under this new interpretation of the intermediate complex the circle operation, and naive bracket, become more natural. Before stating the result we note that we have an obvious identification

$$\mathrm{Hom}_{\mathcal{G}^{op}}(\mathscr{B}ar_{\mathcal{G}}(A*\mathcal{G}), A*\mathcal{G}) = \bigoplus_n \mathrm{Hom}_{\mathcal{G}^{op}}(A*\mathcal{G}^{\otimes_{\mathcal{G}} n}, A*\mathcal{G})$$

and natural comultiplication $\Delta : \mathscr{B}ar_{\mathcal{G}}(A*\mathcal{G}) \to \mathscr{B}ar_{\mathcal{G}}(A*\mathcal{G})\otimes_{\mathcal{G}}\mathscr{B}ar_{\mathcal{G}}(A*\mathcal{G})$ on $\mathscr{B}ar_{\mathcal{G}}(A*\mathcal{G})$ given by separating tensors. This comultiplication induces a multiplication on the associated hom complex so that the restriction isomorphism (3.2.5) becomes an isomorphism of dg algebras $YD^{\mathcal{G}}_{\mathcal{G}}$ and can be used to express the circle operation and bracket.

Lemma 3.18. *The circle operation on* $(\bigoplus_n \mathrm{Hom}_{\mathcal{G}^{op}}(A*\mathcal{G}^{\otimes_{\mathcal{G}} n}, A*\mathcal{G}), d_{\mathrm{Int}})$ *induced by the identification with* $C^{\bullet}_{\mathrm{Int}}(A*\mathcal{G})$*, and original operation of Definition 3.10, are given by the formula*

$$f \circ g(\omega) = (-1)^{|\omega_1|(|g|-1)} f((\omega_1)_0 \otimes (g\cdot(\omega_1)_1)(\omega_2)\otimes \omega_3).$$

The naive bracket is given by the braided commutator under this operation,

$$[f,g] = f\circ g - (-1)^{(|f|-1)(|g|-1)} g_0 \circ (f\cdot g_1).$$

Proof. Straightforward check. ■

We give now a formula for the differential d_{Int}.

Lemma 3.19. *The differential* d_{Int} *is given by the formula*

$$\begin{aligned}&(-1)^{|f|+1} d_{\mathrm{Int}}(f)(\alpha_1\otimes\ldots\otimes\alpha_n)\\ &= (\alpha_1)_0 (f\cdot(\alpha_1)_1)(\alpha_2\otimes\ldots\otimes\alpha_n) + (-1)^n f(\alpha_1\otimes\cdots\otimes\alpha_{n-1})\alpha_n\\ &\quad +(\textstyle\sum_i (-1)^i f(\alpha_1\otimes\ldots\otimes\alpha_i\alpha_{i+1}\otimes\ldots\otimes\alpha_n))\end{aligned}$$

for elements $\alpha_i \in A*\mathcal{G}$*.*

Proof. We replace our lower indices α_i with upper indices α^i to ease notation. It suffices to consider the case in which $\alpha^i = a^i\gamma^i$ for each i, where $a^i \in A$, $\gamma^i \in \mathcal{G}$. Now we have

$$\begin{aligned}
&(-1)^{|f|+1}d_{\mathrm{Int}}(f)(a^1\gamma^1 \otimes \ldots \otimes a^n\gamma^n)\\
&= (-1)^{|f|+1}d_{\mathrm{Int}}(f)(a^1 \otimes \ldots \otimes {}^{\gamma^1_n\cdots\gamma^{n-1}_1}a^n\gamma^1_{n+1}\cdots\gamma^{n-1}_2\gamma^n) \quad \text{(since the $\otimes$ are over $\mathcal{G}$)}\\
&= (-1)^{|f|+1}d_{\mathrm{Int}}(f)(a^1 \otimes \ldots \otimes {}^{\gamma^1_n\cdots\gamma^{n-1}_1}a^n)\gamma^1_{n+1}\cdots\gamma^{n-1}_2\gamma^n \quad \text{(by right $\mathcal{G}$-linearity)}\\
&= (-1)^{|f|+1}d_{C^\bullet(A,A*\mathcal{G})}(f)(a^1 \otimes \ldots \otimes {}^{\gamma^1_n\cdots\gamma^{n-1}_1}a^n)\gamma^1_{n+1}\cdots\gamma^{n-1}_2\gamma^n.\\
&= a^1 f({}^{\gamma^1_1}a^2 \otimes \ldots \otimes {}^{\gamma^1_n\cdots\gamma^{n-1}_1}a^n)\gamma^1_{n+1}\cdots\gamma^n\\
&+(\textstyle\sum_i(-1)^i f(a^1 \otimes \ldots ({}^{\gamma^1_n\cdots\gamma^{i-1}_1}a^i)({}^{\gamma^1_n\cdots\gamma^i_1}a^{i+1}) \ldots \otimes {}^{\gamma^1_n\cdots\gamma^{n-1}_1}a^n)\gamma^1_{n+1}\cdots\gamma^n)\\
&+(-1)^n f(a^1 \otimes \cdots \otimes {}^{\gamma^1_n\cdots\gamma^{n-2}_1}a^{n-1}){}^{\gamma^1_n\cdots\gamma^{n-1}_1}a^n\gamma^1_{n+1}\cdots\gamma^n.
\end{aligned}$$

Using right $\mathcal{G}$-linearity of f again this final expression simplifies to

$$\begin{aligned}
&a^1 f({}^{\gamma^1_1}a^2 \otimes \ldots \otimes {}^{\gamma^1_n\cdots\gamma^{n-1}_1}a^n)\gamma^1_{n+1}\cdots\gamma^n + (-1)^n f(a^1\gamma^1 \otimes \cdots \otimes a^{n-1}\gamma^{n-1})a^n\gamma^n\\
&+\big(\textstyle\sum_i(-1)^i f(a^1\gamma^1 \otimes \ldots a^i\gamma^i a^{i+1}\gamma^{i+1} \ldots \otimes a^n\gamma^n)\big).
\end{aligned}$$

We need only deal with the first term to verify the proposed formula. As for this term we have

$$\begin{aligned}
&a^1 f({}^{\gamma^1_1}a^2 \otimes \ldots \otimes {}^{\gamma^1_n\cdots\gamma^{n-1}_1}a^n)\gamma^1_{n+1}\cdots\gamma^n\\
&= a^1 f(\gamma^1 a^2 \otimes \ldots \otimes {}^{\gamma^2_{n-1}\cdots\gamma^{n-1}_1}a^n)\gamma^2_n\cdots\gamma^n\\
&= a^1 f(\gamma^1 a^2\gamma^2 \otimes \ldots \otimes a^n\gamma^n)\\
&= a^1\gamma^1_1 S(\gamma^1_2) f(\gamma^1_3 a^2\gamma^2 \otimes \ldots \otimes a^n\gamma^n)\\
&= a^1\gamma^1_1 S(\gamma^1_2) f(\gamma^1_5 a^2\gamma^2 \otimes \ldots \otimes a^n\gamma^n S^{-1}(\gamma_4)\gamma^1_3)\\
&= a^1\gamma^1_1 S(\gamma^1_2) f(\gamma^1_5 a^2\gamma^2 \otimes \ldots \otimes a^n\gamma^n S^{-1}(\gamma^1_4))\gamma^1_3\\
&= a^1\gamma^1_1 (f \cdot \gamma_2)(a^2\gamma^2 \otimes \ldots \otimes a^n\gamma^n).
\end{aligned}$$

■

This formula will become useful in the following subsection.

3.2.2 Identities for the naive bracket: braided commutativity of $H^\bullet_{\mathrm{Int}}(A * \mathcal{G})$

In breaking with the tradition established in the previous two chapters, we denote the product on the intermediate cohomology, and intermediate complex, by a cup $f \cup g$. By Proposition

2.23 of Chapter 2, the complex $C^\bullet_{\mathrm{Int}}$ is a $\mathcal{G}$-module dg algebra under the cup product and given action. One can check also that, for $f, g \in C^\bullet_{\mathrm{Int}}$, we have $\rho(f \cup g) = (f_0 \cup g_0) \otimes f_1 g_1$ so that $C^\bullet_{\mathrm{Int}}$ is additionally a $\mathcal{G}$-comodule dg algebra, and hence a dg algebra in $YD^{\mathcal{G}}_{\mathcal{G}}$. It follows then that $H^\bullet_{\mathrm{Int}}$ is a graded algebra in $YD^{\mathcal{G}}_{\mathcal{G}}$.

We employ the expression

$$C^\bullet_{\mathrm{Int}}(A * \mathcal{G}) = (\bigoplus_n \mathrm{Hom}_{\mathcal{G}^{op}}(A * \mathcal{G}^{\otimes_{\mathcal{G}} n}, A * \mathcal{G}), d_{\mathrm{Int}})$$

with the differential as described in Lemma 3.19.

Proposition 3.20. *For arbitrary f and g in $C^\bullet_{\mathrm{Int}}(A * \mathcal{G})$ we have*

$$(-1)^{|f|+1} d(f \circ g) + (-1)^{|f|} d(f) \circ g - f \circ d(g) = f \cup g - (-1)^{|f||g|} g_0 \cup (f \cdot g_1). \tag{3.2.6}$$

In particular, if f and g are cocycles then

$$(-1)^{|f|+1} d(f \circ g) = f \cup g - (-1)^{|f||g|} g_0 \cup (f \cdot g_1). \tag{3.2.7}$$

Sketch proof. The proof is essentially the same as Gerstenhabers original proof in the unbraided setting [18, Theorem 3], and so we gloss over some of the subtle points. The reader should also be aware that [18] generally employs no Koszul signs, which accounts for some of the cosmetic differences.

Note, first of all, that all operations are determined by their value on the generators $A^{\otimes n} \subset A * \mathcal{G}^{\otimes_{\mathcal{G}} n}$, and that under the coaction $\rho(g(y)) = g_0(y) \otimes g_1$ for any $y \in A^{\otimes |g|}$. Take

$$\partial(a_1 \otimes \ldots \otimes a_n) = \sum_{i=1}^{n-1} (-1)^i a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n$$

so that

$$(-1)^{|h|+1} d(h)(a_1 \otimes \ldots \otimes a_n) = a_1 h(a_2 \otimes \ldots \otimes a_n) + (-1)^n h(a_1 \otimes \ldots \otimes a_{n-1}) a_n + h(\partial(a_1 \otimes \ldots \otimes a_n)).$$

For any monomial $x = a_1 \otimes y = y' \otimes a_n \in A^{\otimes n}$ we have

$$\begin{aligned}
&-(-1)^{|f|} d(f \circ g)(x) \\
&= -(-1)^{(|x_1|+1)(|g|-1)+|g|} f(\partial(x_1) \otimes g(x_2) \otimes x_3) - (-1)^{|x_1||g|+|g|} f(x_1 \otimes g(\partial(x_2)) \otimes x_3) \\
&\quad -(-1)^{|x_1||g|+|x_2|+|g|} f(x_1 \otimes g(x_2) \otimes \partial(x_3)) \\
&\quad -(-1)^{|y_1|(|g|-1)+|g|} a_1 f(y_1 \otimes g(y_2) \otimes y_3) - (-1)^{|y'_1|(|g|-1)+n+|g|} f(y'_1 \otimes g(y'_2) \otimes y'_3) a_n
\end{aligned}$$

while, according to the formula for d_{Int} given in Lemma 3.19,

$$\begin{aligned}
&((-1)^{|f|}d(f)\circ g - f\circ d(g))(x)\\
&= -g_0(x_1)(f\cdot g_1)(x_2) - (-1)^{|x_1|(|g|-1)+|x_1|+1}f(x_1)g(x_2)\\
&\quad -(-1)^{|x_1|(|g|-1)}f(\partial(x_1)\otimes g(x_2)\otimes x_3) - (-1)^{|x_1|(|g|-1)+|x_1|+1}f(x_1\otimes g(x_2)\otimes \partial(x_3))\\
&\quad -(-1)^{(|y_1|+1)(|g|-1)}a_1 f(y_1\otimes g(y_2)\otimes y_3) - (-1)^{|y_1'|(|g|-1)+n-|g|+1}f(y_1'\otimes g(y_2')\otimes y_3')a_n\\
&\quad +(-1)^{|x_1||g|+|g|}f(x_1\otimes g(\partial(x_2))\otimes x_3).
\end{aligned} \tag{3.2.8}$$

One then matches signs, using the fact $g(w)\neq 0 \Rightarrow |w| = -|g|$, to get

$$\begin{aligned}
&((-1)^{|f|+1}d(f\circ g) + (-1)^{|f|}d(f)\circ g - f\circ d(g))(x)\\
&= -g_0(x_1)(f\cdot g_1)(x_2) - (-1)^{|x_1|(|g|-1)+|x_1|+1}f(x_1)g(x_2)\\
&= (f\cup g - (-1)^{|f||g|}g_0\cup(f\cdot g_1))(x).
\end{aligned}$$

A point we've glossed over here is that elements of the form

$$\pm f(a_1\otimes\ldots\otimes a_i g(a_{i+1}\otimes\ldots\otimes a_j)\otimes a_{j+1}\otimes\ldots\otimes a_n)$$

and

$$\pm f(a_1\otimes\ldots\otimes g(a_i\otimes\ldots\otimes a_j)a_{j+1}\otimes\ldots\otimes a_n)$$

do not appear in the expression (3.2.8). However, one can check that such contributions from $(-1)^{|f|}d(f)\circ g$ and $f\circ d(g)$ cancel, as was the case in [18]. ■

From (3.2.7) we get immediately that $H^{\bullet}_{\mathrm{Int}}$ is braided commutative.

Corollary 3.21. *The intermediate cohomology $H^{\bullet}_{\mathrm{Int}}(A*\mathcal{G})$ is braided commutative.*

Another curious corollary, which will actually become very useful, is the following result.

Corollary 3.22. *Suppose $f,g\in C^{\bullet}_{\mathrm{Int}}(A*\mathcal{G})$ are cocycles. The bracket $[f,g]_\circ$ satisfies*

$$(-1)^{|f|+1}d([f,g]_\circ) = f\cup g - f_0\cdot g_2\cup\big(g_0\cdot(S(g_1)f_1g_2)\big),$$

and hence is not necessarily a cycle.

Proof. We have

$$\begin{aligned}(-1)^{|f|+1}d([f,g]_\circ) &= f \cup g - (-1)^{|f||g|} g_0 \cup (f \cdot g_1) \\ &\quad -(-1)^{|g||f|+1} g_0 \cup (f \cdot g_1) + (-1)^1 f_0 \cdot g_2 \cup \big(g_0 \cdot (S(g_1) f_1 g_2)\big) \\ &= f \cup g - f_0 \cdot g_2 \cup \big(g_0 \cdot (S(g_1) f_1 g_2)\big).\end{aligned}$$

■

3.3 (In)finiteness of the square braiding on $YD^{\mathcal{G}}_{\mathcal{G}}$

We saw in the last section that, although the circle operation proved useful in addressing the graded ring structure on the intermediate cohomology, the naive bracket failed to preserve cocycles and coboundaries. This section marks the beginnings our investigation into a new bracket $[,]_{\mathsf{YD}}$ on the cochain complex $C^\bullet_{\mathrm{Int}}$ which will be well behaved. We first need to think a bit deeper about the tensor category $YD^{\mathcal{G}}_{\mathcal{G}}$, with the help of [15].

Let M and N be complexes over $YD^{\mathcal{G}}_{\mathcal{G}}$. Then, as we have seen, we have the braiding

$$\mathsf{braid}_{MN} : M \otimes N \to N \otimes M,\ m \otimes n \mapsto (-1)^{|m||n|} n_0 \otimes m \cdot n_1.$$

We can square this operation to get an automorphism on $M \otimes N$. We denote this automorphism by

$$\mathcal{L}_{MN} := \mathsf{braid}_{NM}\mathsf{braid}_{MN} : M \otimes N \to M \otimes N$$

and call it the square braiding on $M \otimes N$. By functoriality of the braiding, all of these automorphism come together to produce a automorphism on the bifunctor $- \otimes -$ which we denote simply

$$\mathcal{L} : (- \otimes -) \to (- \otimes -).$$

So we may, with some authority, speak simply of $\mathcal{L}$ and drop the subscripts. The main question is the following

Question 3.23. Is the transformation $\mathcal{L}$ of finite order? If so, what is the relation between the order $\mathcal{L}$ and the (vector space) dimension of $\mathcal{G}$?

This question is answered, in part, in the work of Etingof and Gelaki [15]. They show that $\mathcal{L}$ is not finite order in general [15, Example 2.4]. However for group algebras (in arbitrary characteristic), as well as Hopf algebras which are simultaneously semisimple and cosemisimple, the order of $\mathcal{L}$ is finite and divides either $\dim \mathcal{G}$ or some low power of $\dim \mathcal{G}$. We review the necessary background below.

3.3.1 The Drinfeld double

Here we recall the definition of the Drinfeld double, and its relation to the category of Yetter-Drinfeld modules. The familiar reader can skip this section. In the following definition by $\mathcal{G}^{op}$ we mean $\mathcal{G}$ with the opposite multiplication, regular (non-opposite) comultiplication, and antipode $S_{\mathcal{G}^{op}} = S_{\mathcal{G}}^{-1}$.

Definition 3.24 (The Drinfeld double)**.** For any finite dimensional Hopf algebra $\mathcal{G}$ we define the Drinfeld double $D(\mathcal{G})$ to be the unique Hopf algebra satisfying the following:

1. $D(\mathcal{G})$ contains $\mathcal{G}^{op}$ and $\mathcal{G}^*$ as Hopf subalgebras.

2. The restriction of the multiplication map on $D(\mathcal{G})$ gives a vector space isomorphism $\mathcal{G}^* \otimes \mathcal{G}^{op} \xrightarrow{\cong} D(\mathcal{G})$.

3. Any $\alpha \in \mathcal{G}^{op}$ and $f \in \mathcal{G}^*$ satisfy the relation

$$\alpha f = (S(\alpha_3) \cdot f \cdot \alpha_1)\alpha_2$$

in $D(\mathcal{G})$, where the actions of $\mathcal{G}$ on $\mathcal{G}^*$ are the standard ones so that

$$\beta \cdot f' \cdot \beta' : x \mapsto f'(\beta' x \beta).$$

As compared with [62], [15], and Drinfeld's original work [14], our Drinfeld double is actually the Drinfeld double of of $\mathcal{G}^{op}$. It can alternatively be seen as the dual of the quantum double presented in [67] and [84]. We recall a result of Takeuchi.

Lemma 3.25 ([84, Proposition 3.2])**.** *There is an equality of monoidal categories between left $D(\mathcal{G})$-modules and right Yetter-Drinfeld modules over $\mathcal{G}$.*

One can see this result, at least in part, by noting that any left $D(\mathcal{G})$-module becomes a left $\mathcal{G}^{op}$-module and left $\mathcal{G}^*$-module, or rather a right $\mathcal{G}$-module and a right $\mathcal{G}$-comodule, by way of restriction. One can check the Yetter-Drinfeld property (3.2.1) by employing the alternate relation

$$\alpha_2(\alpha_3 \cdot f \cdot S(\alpha_1)) = f\alpha$$

to the one given in part (3) of the definition.

It is well known that the Drinfeld double is quasi-triangular [62, Definition 10.1.5] with universal R-matrix

$$R = \sum_i e_i \otimes e^i \quad \text{in} \quad D(\mathcal{G}) \otimes D(\mathcal{G}),$$

where $\{e_i\}$ and $\{e^i\}$ are dual bases for $\mathcal{G}$ and $\mathcal{G}^*$ [62, Theorem 10.4.2]. This element R is a unit with

$$R^{-1} = (S \otimes id)(R) = (id \otimes S)(R).$$

Following [15] we let R_{21} denote the element $\sum_i e^i \otimes e_i$ obtained by applying the trivial symmetry τ to R.

Lemma 3.26. *For arbitrary Yetter-Drinfeld modules M and N, the braiding $M \otimes N \to N \otimes M$ is given by applying the trivial symmetry τ, then multiplying (on the left) by R_{21}. The automorphism $\mathcal{L} : M \otimes N \to M \otimes N$ is given by (left) multiplication by $R_{21}R$. Additionally, $|\mathcal{L}| = |R_{21}R|$.*

Proof. For any monomial $m \otimes n \in M \otimes N$ we have

$$\begin{aligned} R_{21}(n \otimes m) &= \textstyle\sum_i e^i n \otimes m e_i \\ &= \textstyle\sum_i n_0 e^i(n_1) \otimes m e_i \\ &= \textstyle\sum_i n_0 \otimes m e^i(n_1) e_i \\ &= n_0 \otimes m \cdot n_1, \end{aligned}$$

since $\sum_i e^i(n')e_i = n'$ for any $n' \in N$. Since $\mathcal{L}$ is simply given by applying the braiding twice we have

$$\mathcal{L}(m \otimes n) = R_{21}(\sum_i me_i \otimes e^i n) = R_{21}R \cdot (m \otimes n).$$

By applying $\mathcal{L}$ to the module $D(\mathcal{G}) \otimes D(\mathcal{G})$ one verifies that the order of $\mathcal{L}$ is the smallest (positive) N with $(R_{21}R)^N = 1$ in $D(\mathcal{G}) \otimes D(\mathcal{G})$, i.e. the order of $R_{21}R$. ■

3.3.2 The exponent of a Hopf algebra and the order of $\mathcal{L}$: results from Etingof and Gelaki

We let m_n denote the $(n+1)$st iteration of the multiplication $m_n : \mathcal{G} \otimes \ldots \otimes \mathcal{G} \to \mathcal{G}$ and define $\Delta_n : \mathcal{G} \to \mathcal{G} \otimes \ldots \otimes \mathcal{G}$ similarly.

Definition 3.27 ([15])**.** For any finite dimensional Hopf algebra $\mathcal{G}$ we define the exponent of $\mathcal{G}$ as

$$\exp(\mathcal{G}) := \min\{n \in \mathbb{N} : m_n(id_{\mathcal{G}} \otimes S^{-1} \otimes \ldots \otimes S^{-2n+2})\Delta_n = \epsilon\}.$$

If $m_n(id_{\mathcal{G}} \otimes S^{-1} \otimes \ldots \otimes S^{-2n+2})\Delta_n$ is not equal to ϵ for any n, we take $\exp(\mathcal{G}) = \infty$.

In the case in which $S = S^{-1}$ (e.g. in the case that $\mathcal{G}$ is semisimple and cosemisimple, cocommutative, commutative, etc.) the operation

$$m_n(id \otimes S^{-1} \otimes \ldots \otimes S^{-2n+2})\Delta_n$$

reduces to the operation $m_n\Delta_n$. This simplified operation was studied by Kashina in [33, 34]. Indeed, her work served as the catalyst for that of Etingof and Gelaki. The essential result of [15], at least as far as the present work is concerned, is

Theorem 3.28 ([15, Theorem 2.5])**.** *The order of the element $R_{21}R$ in the second tensor power of the Drinfeld double $D(\mathcal{G}) \otimes D(\mathcal{G})$ is equal to* $\exp(\mathcal{G})$*. We also have $|\mathcal{L}| =$* $\exp(\mathcal{G})$*.*

Proof. The fact that $|R_{21}R| = \exp(\mathcal{G})$ is stated directly in [15]. The second portion of the Theorem simply follows from the fact that $|R_{21}R| = |\mathcal{L}|$ from Lemma 3.26 (and is not new). ■

The significance of Theorem 3.28 can be seen in the deceptively simple corollary

Corollary 3.29. *In the case that $\mathcal{G}$ is the group algebra $\mathcal{G} = kG$ of some finite group G (in arbitrary characteristic) the order of the automorphism $\mathcal{L}$ divides $|G|$.*

Proof. In this case, $\exp(kG)$ is the standard exponent of G by [15, Proposition 2.2], i.e. the least common multiplie of the order of the elements of G. From this description it is clear that $\exp(kG)$ divides the order of G. Now one simply notes that $|\mathcal{L}| = \exp(kG)$ by Theorem 3.28. ■

If one simply tries to calculate the order of $R_{21}R$ in $D(kG)^{\otimes 2}$ s/he will find that the relation proposed in the corollary is not at all apparent. As was noted previously, the exponent of a Hopf algebra, and hence the order of the square braiding, is not always finite [15, Example 2.4]. Some more general results on the exponent are given in [15].

Theorem 3.30 ([15, Theorems 4.2, 4.3 and Corollary 4.10])**.**

1. *If $\mathcal{G}$ is a triangular Hopf algebra [62, definition 10.1.5] in characteristic 0, then* $\exp(\mathcal{G})$ *divides* $\dim \mathcal{G}$.

2. *If $\mathcal{G}$ is both semisimple and cosemisimple (in any characteristic) then* $\exp(\mathcal{G})$ *divides* $(\dim \mathcal{G})^3$.

3. *If $\mathcal{G}$ is any finite dimensional Hopf algebra in finite characteristic then* $\exp(\mathcal{G})$ *is finite.*

In general, we expect a slightly better bound for semisimple and cosemisimple Hopf algebras as proposed by Kashina's conjecture.

Conjecture 3.31 (Kashina)**.** If $\mathcal{G}$ is both semisimple and cosemisimple (in any characteristic) then $\exp(\mathcal{G})$ divides $\dim \mathcal{G}$.

3.4 The Yetter-Drinfeld bracket $[,]_{\mathsf{YD}}$ on $C^{\bullet}_{\mathrm{Int}}(A * \mathcal{G})$

From this point on we suppose $\exp(\mathcal{G}) < \infty$ and that the exponent is invertible in the base k. We fix an algebra A on which $\mathcal{G}$ acts, and take $C^{\bullet}_{\mathrm{Int}} = C^{\bullet}_{\mathrm{Int}}(A * \mathcal{G})$.

3.4.1 A remark on characteristic

We will, at some point, want to divide by the exponent of $\mathcal{G}$. We can divide by the exponent exactly when $\mathrm{char}(k)$ does not divide $\exp(\mathcal{G})$. So we really just need to avoid the primes appearing in the exponent of $\mathcal{G}$. However, in the case of a group ring, Cauchy's theorem implies that the primes appearing in the exponent are exactly those primes appearing in the order of G. So we may divide by the exponent exactly when kG is semisimple. Whence we ask

Question 3.32. Suppose for a finite dimensional Hopf algebra over a field k that $\exp(\mathcal{G})$ is a unit in the base field. Does it then follow that $\mathcal{G}$ is semisimple?

We can say at this moment that the converse implication

$$\mathcal{G} \text{ is semisimple} \Rightarrow \exp(\mathcal{G}) \text{ is a unit in } k$$

does *not* hold, as we will be illustrated in the next proposition and corollary. Semisimplicity appears in our homological inquires in that it implies the identifications

$$C^{\bullet}_{\mathrm{Int}}(A * \mathcal{G})^{\mathcal{G}} = C^{\bullet}_{\mathcal{G}}(A * \mathcal{G}) \quad \text{and} \quad H^{\bullet}_{\mathrm{Int}}(A * \mathcal{G})^{\mathcal{G}} = HH^{\bullet}(A * \mathcal{G}).$$

Proposition 3.33. *Let k be a field of characteristic p. If $u(L)$ is the restricted enveloping algebra of an abelian restricted Lie algebra L then* $\exp(u(L)) = p$.

Proof. Since $S^2 = id$ for $u(L)$, the exponent of $u(L)$ is the minimal integer N so that $m_N\Delta_N = \epsilon$. We note that, since $u(L)$ is is commutative, we have

$$m_n\Delta_n(xy) = (m_n\Delta_n(x))(m_n\Delta_n(y))$$

for all $x, y \in u(L)$. Indeed, $\Delta_n(xy) = \Delta_n(x)\Delta_n(y)$ even in the non-commutative case, and commutativity gives $m_n(\Delta_n(x)\Delta_n(y)) = m_n(\Delta_n(x))m_n(\Delta_n(y))$. So it suffices to prove $m_n\Delta_n(x) = \epsilon(x)$ for each of the generators $x \in L$. Or, more to the point, we must show $m_n\Delta_n(x) = 0$.

Take any $x \in L$. Then $\Delta(x) = x \otimes 1 - 1 \otimes x$ and we see, by induction, that $\Delta_n = \sum_{0 \leq i < n} 1^{\otimes i} \otimes x \otimes 1^{\otimes(n-i-1)}$. So

$$m_n \Delta_n(x) = m_n(\sum 1^{\otimes i} \otimes x \otimes 1^{\otimes(n-i-1)}) = nx$$

and $m_p \Delta_p(x) = px = 0$. ■

Corollary 3.34. *If $u(L)$ is semisimple then $\exp(L)$ is 0 in the base field.*

Proof. A theorem of Hochschild states that any semisimple $u(L)$ has L abelian necessarily [31], [62, Theorem 2.3.3]. ■

To see that such an L exists we can simply consider $u(L) = \mathbb{F}_p[x_1, \dots, x_n]/(x_i^p - x_i)_i$, $L = k\{x_1, \dots, x_n\}$. So we see that the condition $\exp(\mathcal{G}) \in k^\times$ is not equivalent to semisimplicity in general, although invertability of the exponent may imply semisimplicity. We note that the $u(L)$ is not cosemisimple so, although the above calculation was not difficult, it is not covered by the results of [15].

As a related question, we can also ask if the analog of Cauchy's theorem holds for Hopf algebras. This question is already under active investigation, and was answered positively by Kashina, Sommerhäuser, and Zhuin the case that $\mathcal{G}$ is both semisimple and cosemisimple. Namely, in this case any prime which divides the dimension of $\mathcal{G}$ also divides the exponent of $\mathcal{G}$ [35, Corllary 3.4]. Combining this result with Etingof and Gelaki's result Theorem 3.30 (2) we see that the primes which appear in $\dim \mathcal{G}$ are exactly those which appear in the exponent.

3.4.2 The Yetter-Drinfeld bracket

For any function $\xi : M \otimes N \to L$ of Yetter-Drinfeld modules, and integer i, we have the function $(\mathcal{L}^i)^*(\xi) := \xi \mathcal{L}^i : M \otimes N \to L$. To avoid cumbersome notation we simply write $\mathcal{L}^{i*}\xi$ for this function. We can take an average of all of the resulting functions to get

$$\int_{\mathcal{L}} \xi = \frac{1}{|\mathcal{L}|} \sum_{i=0}^{|\mathcal{L}|-1} \mathcal{L}^{i*}\xi = \frac{1}{\exp(\mathcal{G})} \sum_{i=0}^{\exp(\mathcal{G})-1} \mathcal{L}^{i*}\xi.$$

In particular, we can apply this averaging process to the naive bracket $[,]_\circ$ to produce yet another operation on the complex $C^\bullet_{\mathrm{Int}}(A * \mathcal{G})$.

Definition 3.35 (Yetter-Drinfeld bracket)**.** Suppose that $\mathcal{G}$ has finite exponent with $\exp(\mathcal{G})$ invertible in k. Then we define the bracket $[,]_{\mathsf{YD}} : C^\bullet_{\mathrm{Int}} \otimes C^\bullet_{\mathrm{Int}} \to C^\bullet_{\mathrm{Int}}$ as

$$[,]_{\mathsf{YD}} = \int_{\mathcal{L}} [,]_\circ := \frac{1}{\exp(\mathcal{G})} \sum_{i=0}^{\exp(\mathcal{G})-1} \mathcal{L}^{i*}[,]_\circ.$$

If we can have our way, we would like that the bracket $[,]_{\mathsf{YD}}$

(a) is a map of Yetter-Drinfeld modules $[,]_{\mathsf{YD}} : C^\bullet_{\mathrm{Int}} \otimes C^\bullet_{\mathrm{Int}} \to C^\bullet_{\mathrm{Int}}$ (Lemma 3.37),

(b) is braided anti-commutative (Lemma 3.36),

(c) preserves cocycles and coboundaries (Theorem 3.38),

(d) satisfies a braided Jacobi identity,

(e) satisfies some compatibility with the cup product so that the bracket $[f, gh]_{\mathsf{YD}}$ can be determined from the brackets $[f, g]_{\mathsf{YD}}$ and $[f, h]_{\mathsf{YD}}$, and

(f) when $\mathcal{G}$ is semisimple, its invariants should recover the Gerstenhaber bracket on $C^\bullet_{\mathrm{Int}}(A * \mathcal{G})^{\mathcal{G}} = C^\bullet_{\mathcal{G}}(A * \mathcal{G})$ (Lemma 3.37).

In short, we would like that $C^\bullet_{\mathrm{Int}}$ with the Yetter-Drinfeld bracket and cup product is a braided Gerstenhaber algebra.

The points with associated Lemmas/Theorems are verified below. The other points have yet to be verified or contradicted. It is relatively clear that $\mathcal{L}^*[f, g]_{\mathsf{YD}} = [f, g]_{\mathsf{YD}}$. There is something to check, however, for (braided) anti-commutativity of this new bracket.

Lemma 3.36. *There is an equality*

$$[f, g]_{\mathsf{YD}} = -(-1)^{(|f|-1)(|g|-1)}[g_0, f \cdot g_1]_{\mathsf{YD}}$$

for any $f, g \in C^\bullet_{\mathrm{Int}}(A * \mathcal{G})$.

Proof. Recall that $\mathcal{L}$ is the second power of the braiding transformation $\mathcal{L} = \mathsf{braid}^2$. If we consider the bracket $[,]_{\mathsf{YD}}$ as a function on the shifted complex $\Sigma C^{\bullet}_{\mathrm{Int}}(A * \mathcal{G})$, we have

$$\begin{aligned}
&[,]_{\mathsf{YD}}\\
&= \tfrac{1}{\exp(\mathcal{G})} \textstyle\sum_{i=0}^{\exp(\mathcal{G})-1} [-,-]_{\circ}\mathcal{L}^i\\
&= \tfrac{1}{\exp(\mathcal{G})} \textstyle\sum_{i=0}^{\exp(\mathcal{G})-1} \left((- \circ -)\mathsf{braid}^{2i} - (- \circ -)\mathsf{braid}^{2i+1}\right)\\
&= \tfrac{1}{\exp(\mathcal{G})} \textstyle\sum_{i=0}^{\exp(\mathcal{G})-1} (- \circ -)\mathsf{braid}^{2i} - \tfrac{1}{\exp(\mathcal{G})} \textstyle\sum_{i=0}^{\exp(\mathcal{G})-1} (- \circ -)\mathsf{braid}^{2i+1}
\end{aligned}$$

Since $\mathcal{L}$ is order $\exp(\mathcal{G})$, the braid transformation is of order dividing $2\exp(\mathcal{G})$. So $\mathsf{braid}^{2\exp(\mathcal{G})} = id_{C^{\bullet}_{\mathrm{Int}} \otimes C^{\bullet}_{\mathrm{Int}}}$ and the final sum can be reindexed to get

$$\begin{aligned}
&[,]_{\mathsf{YD}}\\
&= \tfrac{1}{\exp(\mathcal{G})} \textstyle\sum_{i=1}^{\exp(\mathcal{G})} (- \circ -)\mathsf{braid}^{2i} - \tfrac{1}{\exp(\mathcal{G})} \textstyle\sum_{i=0}^{\exp(\mathcal{G})-1} (- \circ -)\mathsf{braid}^{2i+1}\\
&= -\tfrac{1}{\exp(\mathcal{G})} \textstyle\sum_{i=0}^{\exp(\mathcal{G})-1} (- \circ -)\mathsf{braid}^{2i}\mathsf{braid} + \tfrac{1}{\exp(\mathcal{G})} \textstyle\sum_{i=0}^{\exp(\mathcal{G})-1} (- \circ -)\mathsf{braid}^{2i+1}\mathsf{braid}\\
&= -\left(\tfrac{1}{\exp(\mathcal{G})} \textstyle\sum_{i=0}^{\exp(\mathcal{G})-1} (- \circ -)\mathsf{braid}^{2i} - \tfrac{1}{\exp(\mathcal{G})} \textstyle\sum_{i=0}^{\exp(\mathcal{G})} (- \circ -)\mathsf{braid}^{2i+1}\right)\mathsf{braid}\\
&= -\left(\tfrac{1}{\exp(\mathcal{G})} \textstyle\sum_{i=0}^{\exp(\mathcal{G})-1} \left((- \circ -)\mathsf{braid}^{2i} - (- \circ -)\mathsf{braid}^{2i+1}\right)\right)\mathsf{braid}\\
&= -[,]_{\mathsf{YD}}\mathsf{braid}.
\end{aligned}$$

Applying this to elements gives the proposed equality. ■

Lemma 3.37. *The bracket $[,]_{\mathsf{YD}}$ is a map of map of Yetter-Drinfeld modules and the restriction of $[,]_{\mathsf{YD}}$ to the invariants recovers the Gerstenhaber bracket, after making the identification $C^{\bullet}_{\mathrm{Int}}(A * \mathcal{G})^{\mathcal{G}} = C^{\bullet}_{\mathcal{G}}(A * \mathcal{G})$.*

Proof. Recall that $[,]_{\circ}$ is a map in $YD^{\mathcal{G}}_{\mathcal{G}}$, by Lemma 3.11, so that each $\mathcal{L}^i[,]_{\circ}$ is therefore a map in $YD^{\mathcal{G}}_{\mathcal{G}}$. It follows that the final sum

$$[,]_{\mathsf{YD}} = \frac{1}{\exp(\mathcal{G})} \sum_{i=0}^{\exp(\mathcal{G})-1} \mathcal{L}^{i*}[,]_{\circ}$$

is a map of Yetter-Drinfeld modules. As for the statement about invariants, we simply note that $\mathcal{L}|(C^\bullet_{\mathrm{Int}})^{\mathcal{G}} \otimes (C^\bullet_{\mathrm{Int}})^{\mathcal{G}} = id$. Whence, by Lemma 3.12,

$$\begin{aligned} [f,g]_{\mathsf{YD}} &= \tfrac{1}{\exp(\mathcal{G})} \textstyle\sum_{i=0}^{\exp(\mathcal{G})-1} \mathcal{L}^i[f,g]_\circ \\ &= \tfrac{1}{\exp(\mathcal{G})} \textstyle\sum_{i=0}^{\exp(\mathcal{G})-1} [f,g]_\circ \\ &= [f,g]_\circ \\ &= [f,g] \end{aligned}$$

whenever f and g are invariant, where the last bracket is the standard Gerstenhaber bracket.

■

Finally, we are able to give our second major declaration about the intermediate cohomology (following Corollary 3.21).

Theorem 3.38. *Suppose $f, g \in C^\bullet_{\mathrm{Int}}(A * \mathcal{G})$ are cocycles. Then*

1. *$[f,g]_{\mathsf{YD}}$ is another cocycle.*
2. *If f or g is a boundary, $[f,g]_{\mathsf{YD}}$ is a boundary.*
3. *If $\mathcal{G}$ is semisimple, the induced operation on cohomology*

   $$[,]_{\mathsf{YD}} : H^\bullet_{\mathrm{Int}}(A*\mathcal{G}) \otimes H^\bullet_{\mathrm{Int}}(A*\mathcal{G}) \to H^\bullet_{\mathrm{Int}}(A*\mathcal{G})$$

   is such that the invariants $[,]^{\mathcal{G}}_{\mathsf{YD}}$ agrees with the standard Gerstenhaber bracket, after making the identification $H^\bullet_{\mathrm{Int}}(A*\mathcal{G})^{\mathcal{G}} = HH^\bullet(A*\mathcal{G})$.

Proof. We let m denote the cup product $(- \cup -) : C^\bullet_{\mathrm{Int}} \otimes C^\bullet_{\mathrm{Int}} \to C^\bullet_{\mathrm{Int}}$, to avoid a cumbersome notation. (1) According to Corollary 3.22 we have

$$d([f,g]_\circ) = (-1)^{|f|-1} m\big(f \otimes g - \mathcal{L}(f \otimes g)\big). \tag{3.4.1}$$

Now, since the action and coaction of $\mathcal{G}$ preserves cocycles by Proposition 3.9, each element $\mathcal{L}^i(f \otimes g)$ is a sum of monomials $f' \otimes g'$ with f' and g' cocycles. So, in general, the same calculation as above gives

$$d(\mathcal{L}^{i*}[f,g]_\circ) = (-1)^{|f|-1} m\big(\mathcal{L}^i(f \otimes g) - \mathcal{L}^{i+1}(f \otimes g)\big)$$

and we have

$$\begin{aligned}
&(-1)^{|f|-1}d([f,g]_{\mathsf{YD}})\\
&= (-1)^{|f|-1}\tfrac{1}{\exp(\mathcal{G})}\textstyle\sum_{i=0}^{\exp(\mathcal{G})-1} d(\mathcal{L}^i[f,g]_\circ)\\
&= \tfrac{1}{\exp(\mathcal{G})}\textstyle\sum_{i=0}^{\exp(\mathcal{G})-1} m\big(\mathcal{L}^i(f\otimes g)-\mathcal{L}^{i+1}(f\otimes g)\big)\\
&= \tfrac{1}{\exp(\mathcal{G})}m\big(\textstyle\sum_{i=0}^{\exp(\mathcal{G})-1}\mathcal{L}^i(f\otimes g)-\sum_{j=1}^{\exp(\mathcal{G})}\mathcal{L}^j(f\otimes g)\big)\\
&= \tfrac{1}{\exp(\mathcal{G})}m\big((f\otimes g)-\mathcal{L}^{\exp(\mathcal{G})}(f\otimes g)\big)
\end{aligned}\tag{3.4.2}$$

Since $|\mathcal{L}| = \exp(\mathcal{G})$, we have $\mathcal{L}^{\exp(\mathcal{G})} = id$ and we get finally

$$(-1)^{|f|-1}d([f,g]_{\mathsf{YD}}) = \frac{1}{\exp(\mathcal{G})}m\big((f\otimes g)-(f\otimes g)\big) = 0.$$

(2) By braided anti-commutativity (Lemma 3.36) we may assume that f is a boundary. Take F with $d(F) = f$. First, from the general expression (3.2.6) of Proposition 3.20 we get

$$\begin{aligned}
&d([F,g]_\circ)\\
&= d(F)\circ g - (-1)^{|g|-1+(|g|-1)(|F|-1)}g_0\circ d(F)g_1 + (-1)^{|F|+1}m\big(F\otimes g-\mathcal{L}(F\otimes g)\big)\\
&= f\circ g + (-1)^{(|g|-1)|F|}g_0\circ f\cdot g_1 + (-1)^{|f|}m\big(F\otimes g-\mathcal{L}(F\otimes g)\big)\\
&= f\circ g + (-1)^{(|g|-1)(|f|-1)}g_0\circ f\cdot g_1 + (-1)^{|f|}m\big(F\otimes g-\mathcal{L}(F\otimes g)\big)\\
&= [f,g]_\circ + (-1)^{|f|}m\big(F\otimes g-\mathcal{L}(F\otimes g)\big).
\end{aligned}$$

Now by augmenting (3.4.2) we can get

$$\begin{aligned}
&d([F,g]_{\mathsf{YD}})\\
&= \tfrac{1}{\exp(\mathcal{G})}\textstyle\sum_{i=0}^{\exp(\mathcal{G})-1} d(\mathcal{L}^i[F,g]_\circ)\\
&= \tfrac{1}{\exp(\mathcal{G})}\textstyle\sum_{i=0}^{\exp(\mathcal{G})-1}\mathcal{L}^i[f,g]_\circ + (-1)^{|f|}\tfrac{1}{\exp(\mathcal{G})}\textstyle\sum_{i=0}^{\exp(\mathcal{G})-1} m\big(\mathcal{L}^i(F\otimes g)-\mathcal{L}^{i+1}(F\otimes g)\big)\\
&= \tfrac{1}{\exp(\mathcal{G})}\textstyle\sum_{i=0}^{\exp(\mathcal{G})-1}\mathcal{L}^i[f,g]_\circ\\
&= [f,g]_{\mathsf{YD}}.
\end{aligned}$$

(3) Now that we understand that the bracket $[,]_{\mathsf{YD}}$ preserves cocycles and coboundaries, it is clear that we do get an induced operation on the cohomology $H^\bullet_{\mathrm{Int}}(A*\mathcal{G})$. The fact that this operation agrees with the Gerstenhaber bracket after taking invariants follows by Lemma 3.37 above. ■

3.5 Interpretations of the intermediate cohomology in low degree

We adopt thought the presentation

$$C^\bullet_{\mathrm{Int}}(A*\mathcal{G}) = (\bigoplus_n \mathrm{Hom}_{\mathcal{G}^{op}}(A*\mathcal{G}^{\otimes_{\mathcal{G}} n}, A*\mathcal{G}), d_{\mathrm{Int}}).$$

In this section we prove

Theorem 3.39. *In low degree we have*

$$\begin{aligned} H^0_{\mathrm{Int}}(A*\mathcal{G}) = \quad & Z^0\big(C^\bullet_{\mathrm{Int}}(A*\mathcal{G})\big) = \textit{the braided center of } A*\mathcal{G} \\ & Z^1\big(C^\bullet_{\mathrm{Int}}(A*\mathcal{G})\big) = \textit{braided (algebra) derivations of } A*\mathcal{G} \\ & Z^2\big(\bar{C}^\bullet_{\mathrm{Int}}(A*\mathcal{G})\big) = \textit{infinitesimal quantum deformations of } A*\mathcal{G} \end{aligned}$$

and

$$\begin{aligned} H^1_{\mathrm{Int}}(A*\mathcal{G}) &= \textit{braided outer derivations of } A*\mathcal{G} \\ H^2_{\mathrm{Int}}(A*\mathcal{G}) &= \textit{isoclasses of quantum deformations.} \end{aligned}$$

The identification for H^0_{Int} follows from the fact that the differential on $C^\bullet_{\mathrm{Int}}$ sends an element $\beta \in A*\mathcal{G} = \mathrm{Hom}_{\mathcal{G}^{op}}(\mathcal{G}, A*\mathcal{G})$ to the function $[\beta,-]_{br} : \alpha \mapsto \beta\alpha - \alpha_0(\beta\cdot\alpha_1)$ (Lemma 3.19), and the definition of the braided center of $A*\mathcal{G}$ as the collection of β for which the operation $[\beta,-]_{br}$ vanishes. The identifications for H^1_{Int} and H^2_{Int} are given in Proposition 3.42/Corollary 3.45 and Theorem 3.54.

3.5.1 Braided derivations and $H^1_{\mathrm{Int}}(A*\mathcal{G})$

We fix some algebra B in $YD^{\mathcal{G}}_{\mathcal{G}}$.

Definition 3.40. An endomorphism $f \in \underline{\mathrm{Hom}}_k(B,B)$ is said to be a braided derivation if it satisfies the equality

$$f(ab) = f(a)b + a_0(f\cdot a_1)(b)$$

for each $a, b \in B$. Given some YD-subalgebra $E \subset B$, we say a braided derivation f is E-relative if $f|E = 0$.

We are most interested in the case $B = A * \mathcal{G}$, $E = \mathcal{G}$. Recall that the action here is the right adjoint action $b \cdot \sigma = S(\sigma_1)b\sigma$ for each $b \in A * \mathcal{G}$ and $\sigma \in \mathcal{G}$. The coaction is given by $\rho(b\gamma) = b\gamma_1 \otimes \gamma_2$ when $b \in A$, $\gamma \in \mathcal{G}$.

Lemma 3.41. *For any $b \in B$ the braided commutator*

$$[b, -]_{br} : a \mapsto ba - a_0(b \cdot a_1)$$

is a braided derivation. When $B = A * \mathcal{G}$ each such $[b, -]_{br}$ satisfies $[b-,]_{br}|\mathcal{G} = 0$, i.e. is a $\mathcal{G}$-relative derivation.

Proof. We have

$$\begin{aligned} [b, aa']_{br} &= baa' - a_0a'_0(b \cdot a_1a'_1) \\ &= baa' - a_0(b \cdot a_1)a' + a_0(b \cdot a_1)a' - a_0a'_0(b \cdot a_1a'_1) \\ &= [b, a]_{br}a' + a_0[b \cdot a_1, a']_{br}. \end{aligned}$$

Now we have for any $\sigma \in \mathcal{G}$

$$\begin{aligned} ([b, -]_{br} \cdot \sigma)(a') &= [b, a' \cdot S^{-1}(\sigma_2)]_{br} \cdot \sigma_1 \\ &= (b(a' \cdot S^{-1}(\sigma_2))) \cdot \sigma_1 - \big((a'_0 \cdot S^{-1}(\sigma_3))(b \cdot \sigma_4 a'_1 S^{-1}(\sigma_2))\big) \cdot \sigma_1 \\ &= (b \cdot \sigma)a' - (a'_0)(b \cdot \sigma a'_1) \\ &= [b \cdot \sigma, a']_{br}. \end{aligned}$$

Whence

$$[b, aa']_{br} = [b, a]_{br}a' + a_0[b \cdot a_1, a']_{br} = [b, a]_{br}a' + a_0([b, -]_{br} \cdot a_1)(a')$$

and $[b, -]_{br}$ is seen to be a braided derivation. In the case that $B = A * \mathcal{G}$, $b \in A * \mathcal{G}$, and $\sigma \in \mathcal{G}$, we have

$$[b, \sigma]_{br} = b\sigma - \sigma_1(b \cdot \sigma_2) = b\sigma - \sigma_1 S(\sigma_2)b\sigma_3 = b\sigma - b\sigma = 0.$$

∎

We will call braided derivations of the form $[b,-]_{\mathrm{br}}$ *inner (braided) derivations.* We let $\mathrm{BrDer}(B)$ (resp. $\mathrm{BrInn}(B)$) denote the set of braided derivations (resp. inner derivations) on B. Note that the set of inner derivations is a k-subspace since $[cb,-]_{br} + [c'b',-]_{br} = [cb + c'b',-]_{br}$. We take

$$\mathrm{BrOut}(B) = \mathrm{BrDer}(B)/\mathrm{BrInn}(B)$$

Let $\mathrm{BrDer}_E(B)$ denote the subspace of E-relative braided derivations. Note that E-relativity is equivalent to right E-linearity of f under the module structure induced by the algebra embedding $E \to B$, since we have for any $b \in B$ and $e \in E$

$$f(be) = f(b)e + b_0 f(eS^{-1}(b_2))b_1 = f(b)e.$$

When $B = A * \mathcal{G}$, and $E = \mathcal{G}$, we define $\mathrm{BrOut}_{\mathcal{G}}(A * \mathcal{G})$ in the obvious manner. (The definition makes sense here since all the inner braided derivations on $A * \mathcal{G}$ will already be $\mathcal{G}$-relative.)

Proposition 3.42. *We have*

1. $Z^1(C^{\bullet}_{\mathrm{Int}}(A * \mathcal{G})) = \mathrm{BrDer}_{\mathcal{G}}(A * \mathcal{G})$
2. $B^1(C^{\bullet}_{\mathrm{Int}}(A * \mathcal{G})) = \mathrm{BrInn}(A * \mathcal{G})$
3. $H^1_{\mathrm{Int}}(A * \mathcal{G}) = H^1(C^{\bullet}_{\mathrm{Int}}(A * \mathcal{G})) = \mathrm{BrOut}_{\mathcal{G}}(A * \mathcal{G})$

Proof. According to the formula for d_{Int} given in Lemma 3.19 we have, for any function $f \in C^1_{\mathrm{Int}}(A * \mathcal{G})$ and $\alpha, \alpha' \in A * \mathcal{G}$,

$$f \text{ is a cocycle} \Leftrightarrow 0 = \alpha_0(f \cdot \alpha_1)(\alpha') - f(\alpha\alpha') + f(\alpha)\alpha' \Leftrightarrow f \quad \text{is a braided derivation.}$$

This gives (1). For any $\beta \in C^0_{\mathrm{Int}}(A * \mathcal{G}) \cong A * \mathcal{G}$ one also checks that we have

$$d_{\mathrm{Int}}(\beta)(\alpha) = -(\alpha_0(\beta \cdot \alpha_0) - \beta\alpha) = [\beta, \alpha]_{br}.$$

This gives (2) and (3). ■

It turns our that, at least some of the time, every braided derivation on $A * \mathcal{G}$ will automatically be $\mathcal{G}$-relative.

Proposition 3.43. *Suppose $\mathcal{G}$ is a cocommutative and let $f : A * \mathcal{G} \to A * \mathcal{G}$ be any braided derivation. Then $f|\mathcal{G} = 0$ and $\mathrm{BrDer}_{\mathcal{G}}(A * \mathcal{G}) = \mathrm{BrDer}(A * \mathcal{G})$.*

Proof. First note that for $\sigma, \tau \in \mathcal{G}$ we have

$$f(\sigma\tau) = f(\sigma)\tau + \sigma_1(S(\sigma_2)f(\sigma_5\tau\sigma^{-1}(\sigma_4))\sigma_3 = f(\sigma)\tau + f(\sigma_3\tau S^{-1}(\sigma_2))\sigma_1.$$

For $\sigma \in \mathcal{G}$ take $\sigma^{[0]} = \epsilon(\sigma)$ and $\sigma^{[n]} = \sigma_1 \ldots \sigma_n$ for all positive n. I claim $f(\sigma^{[n]}) = nf(\sigma_1)\sigma_2^{[n-1]}$. Suppose the claim is true for the moment. Since $\mathcal{G}$ is cocommutative and hence $S^2 = 0$, we have $\sigma^{[\exp(\mathcal{G})]} = \epsilon(\sigma)$ for each $\sigma \in \mathcal{G}$. Then for any $\sigma \in \mathcal{G}$ we will have

$$f(\sigma) = f(\sigma^{[\exp(\mathcal{G})+1]}) = (\exp(\mathcal{G}) + 1)f(\sigma_1)\sigma_2^{[\exp(\mathcal{G})]} = (\exp(\mathcal{G}) + 1)f(\sigma)$$

$$\Rightarrow f(\sigma) = \exp(\mathcal{G})f(\sigma) + f(\sigma) \Rightarrow \exp(\mathcal{G})f(\sigma) = 0 \Rightarrow f(\sigma) = 0,$$

since the exponent is invertible in the base. So we would like to prove the claim, $f(\sigma^{[n]}) = nf(\sigma)\sigma^{[n-1]}$.

We prove the claim by induction, with the base case being $f(\sigma^{[1]}) = f(\sigma_1)\epsilon(\sigma_2)$. Now take some positive $n > 1$ and suppose the claim holds for all $i < n$. Then we have

$$\begin{aligned}
f(\sigma^{[n]}) &= f(\sigma_1\sigma_2^{[n-1]}) \\
&= f(\sigma_1)\sigma_2^{[n-1]} + f(\sigma_3\sigma_4^{[n-1]}S^{-1}(\sigma_2))\sigma_1 \\
&= f(\sigma_1)\sigma_2^{[n-1]} + f(\sigma_1\sigma_2^{[n-1]}S(\sigma_3))\sigma_4 && \text{(by cocommutativity)} \\
&= f(\sigma_1)\sigma_2^{[n-1]} + f(\sigma_1^{[n]}S(\sigma_2))\sigma_3 \\
&= f(\sigma_1)\sigma_2^{[n-1]} + f(\sigma_1^{[n-1]})\sigma_2 \\
&= f(\sigma_1)\sigma_2^{[n-1]} + (n-1)f(\sigma_1)\sigma_2^{[n-2]}\sigma_3 && \text{(by induction hypothesis)} \\
&= f(\sigma_1)\sigma_2^{[n-1]} + (n-1)f(\sigma_1)\sigma_2^{[n-1]} = nf(\sigma_1)\sigma_2^{[n-1]}.
\end{aligned}$$

Thus the claim is proved. ■

Recall that a finite group scheme Θ acting on an (affine) scheme X corresponds to a coaction of the commutative Hopf algebra $k[\Theta]$ on $k[X]$. This then corresponds to an action of the cocommutative Hopf algebra $(k[\Theta])^*$ on $k[X]$. So the above proposition is applicable in the case of a finite group scheme acting on a scheme, provided $\exp(k[\Theta]) \in k^\times$. (The exponent of a Hopf algebra and its dual are equal [15, Proposition 2.2 (4)].)

There is some question about what kinds of Hopf algebras will be cocommutative with $\exp(\mathcal{G})$ a unit in the base. Supposing Question 3.32 is answered in the positive, then we know that in characteristic 0 any such Hopf algebra will be cosemisimple, by Larson and Radford [48]. So after changing base to $\bar{k}$ our Hopf algebra $\mathcal{G}$ will be a group algebra. In finite characteristic we have already seen that restricted Lie algebras of abelian Lie algebras are such that $\exp(u(L)) = 0 \in k$. We ask the following question with no suggestion for a possible answer.

Question 3.44. If $\mathcal{G}$ is cocommutative with $\exp(\mathcal{G})$ a unit in the base k, does it follow that $\mathcal{G}$ is cosemisimple?

As a corollary of Propositions 3.42 and 3.43 we get

Corollary 3.45. *When $\mathcal{G}$ is cocommutative we have*

1. $Z^1(C^\bullet_{\mathrm{Int}}(A * \mathcal{G})) = \mathrm{BrDer}(A * \mathcal{G})$

2. $B^1(C^\bullet_{\mathrm{Int}}(A * \mathcal{G})) = \mathrm{BrInn}(A * \mathcal{G})$

3. $H^1_{\mathrm{Int}}(A * \mathcal{G}) = H^1(C^\bullet_{\mathrm{Int}}(A * \mathcal{G})) = \mathrm{BrOut}(A * \mathcal{G})$

This result improves upon Proposition 3.42 in that we see *all* braided derivations appearing as cocycles, not just the $\mathcal{G}$-relative ones.

Any naive attempt to produce a commutator bracket on the set of braided derivations seems not to yield another braided derivation. However, from Theorem 3.38 we deduce some algebraic structure on $\mathrm{BrDer}(A * \mathcal{G})$.

Corollary 3.46. *Suppose $\mathcal{G}$ has finite exponent which is invertible in k. Then*

1. *for any ($\mathcal{G}$-relative) braided derivations f, g on $A * \mathcal{G}$ we have that the endomorphism*

$$[f, g]_{\mathsf{YD}} = \frac{1}{\exp(\mathcal{G})} \sum_{i=1}^{\exp(\mathcal{G})} (\mathcal{L}^i)^* [f, g]_{br}$$

is another braided derivation and

2. *when either f or g is inner, then so is $[f, g]_{\mathsf{YD}}$.*

Proof. This follows from the fact $f \circ g$ is simply composition of functions for degree 1 elements in $C^\bullet_{\mathrm{Int}}(A * \mathcal{G})$. From this we get $[f, g]_\circ = [f, g]_{br}$. ■

The second point can be interpreted as the statement that $\mathrm{BrInn}(A * \mathcal{G})$ is an "ideal" in $\mathrm{BrDer}(A * \mathcal{G})$.

3.5.2 Quantum deformations and $H^2_{\mathrm{Int}}(A * \mathcal{G})$

In this section it will be convient to replace the the intermediate complex $C^\bullet_{\mathrm{Int}}(A * \mathcal{G})$ with the reduced intermediate complex $\bar{C}^\bullet_{\mathrm{Int}}(A * \mathcal{G})$. This will be the subcomplex of functions $f : A * \mathcal{G}^{\otimes_{\mathcal{G}} n} \to A * \mathcal{G}$ with

$$f|\big((A * \mathcal{G})^{\otimes_{\mathcal{G}} i} \otimes_{\mathcal{G}} \mathcal{G} \otimes_{\mathcal{G}} (A * \mathcal{G})^{\otimes_{\mathcal{G}} (n-i-1)}\big) = 0$$

for each $i \leq n-1$. This complex is identified with the standard reduced Hochschild cochain complex under the identification $C^\bullet_{\mathrm{Int}}(A * \mathcal{G}) = C^\bullet(A, A * \mathcal{G})$ and so we see that the inclusion $\bar{C}^\bullet_{\mathrm{Int}}(A * \mathcal{G}) \to C^\bullet_{\mathrm{Int}}(A * \mathcal{G})$ is a quasi-isomorphism.

Let B be an algebra in $YD^{\mathcal{G}}_{\mathcal{G}}$ and take $R = k[t]/(t^2)$. Let $\mathcal{G}_R$ denote the Hopf R-algebra $R \otimes \mathcal{G}$ and B_R be the Yetter-Drinfeld $\mathcal{G}_R$-module $R \otimes B$. We call an R-bilinear operation

$$* : B_R \otimes_R B_R \to B_R, \quad b \otimes b' \mapsto b * b'$$

an (infinitesimal) quantum deformation of B if

(a) under the identification $B_R \otimes_R k = B$ the map $*$ reduces to the multiplication on B, $(b \bar{*} b') = \bar{b}\bar{b}'$,

(b) $b * 1 = 1 * b = b$ for all $b \in B$,

(c) the operation $*$ satisfies the associativity relation

$$(b * b') * b'' - b_0 * \big((b' S^{-1}(b_3) * b'' S^{-1}(b_2))b_1\big) = 0, \tag{3.5.1}$$

A morphism between two quantum deformations $\mathcal{B} = (B_R, *)$ and $\mathcal{B}' = (B_R, *')$ is an R-linear map $\Phi : \mathcal{B} \to \mathcal{B}'$ which satisfies

$$\Phi(b_0) *' \big((\Phi \cdot b_1)(b')\big) = \Phi(b * b') \ \text{ for each } b, b' \in \mathcal{B} \text{ and } \ \Phi \otimes_R k = id_B.$$

The identity morphism is simply id_{B_R} and composition is simply composition of functions (See Lemma 3.47).

In general the multiplication map $* : \mathcal{B} \otimes_R \mathcal{B} \to \mathcal{B}$ will not be $\mathcal{G}$-linear. However, we should still think of a quantum deformation $\mathcal{B}$ as an object in $YD^{\mathcal{G}_R}_{\mathcal{G}_R}$, if for no other reason than to make sense of the relation (3.5.1). Note that if the map $*$ is a map of right $\mathcal{G}$-modules then we will get

$$b_0 * \big((b' S^{-1}(b_3) * b'' S^{-1}(b_2))b_1\big) = b_0 * \big((b' S^{-1}(b_4)b_1) * (b'' S^{-1}(b_3)b_2)\big) = b * (b' * b'').$$

So, in this case, the quantum associativity (3.5.1) reduces to the standard associativity relation $(b * b') * b'' - b * (b' * b'') = 0$. Whence $\mathcal{G}$-invariant quantum deformations are exactly deformations of B as a $\mathcal{G}$-module algebra.

Given a YD-subalgebra $E \subset B$ we say a quantum deformation $(B_R, *)$ is a *E-relative deformation* if for each $e \in E_R$ and $b \in B_R$ we have $e * b = eb$ and $b * e = be$. In the case that $\mathcal{B}$ and $\mathcal{B}'$ are E-relative deformations, we say a morphism of quantum deformations $\Phi : \mathcal{B} \to \mathcal{B}'$ is a map of E-relative deformations if $\Phi | E_R = id_{E_R}$.

Lemma 3.47. *For maps of quantum deformations $\Phi : \mathcal{B} \to \mathcal{B}'$ and $\Psi : \mathcal{B}' \to \mathcal{B}''$ the composition $\Psi\Phi$ is in fact a map of quantum deformations.*

This can be verified by way of Lemma 3.51 below. We give here an alternate proof.

Sketch proof. We have $B_R = B \oplus Bt$ so that $\Phi = (id + \phi t)$ for some $\phi : B \to B$. Similarly, $\Psi = (id + \psi t)$. Now, in the R-module $\mathrm{End}_R(B_R) \otimes_R \mathrm{End}_R(B_R)$ we have

$$\begin{aligned} \Phi \otimes \Psi &= id \otimes id + (\phi \otimes id)t + (id \otimes \psi)t \\ &= id_0 \otimes (id \cdot id_1) + (\phi_0 \otimes (id \cdot \phi_1))t + (id_0 \otimes (\psi \cdot id_1))t \\ &= \Phi_0 \otimes (\Psi \cdot \Phi_1) \end{aligned}$$

since the identity map is $\mathcal{G}$-invariant and coinvariant. So

$$\begin{aligned} \Psi\Phi(b * b') &= \Psi(\Phi(b_0) *' (\Phi \cdot b_1)(b')) \\ &= ((\Psi\Phi_0)(b_0) *'' \big((\Psi \cdot (\Phi_1 b_1))((\Phi \cdot b_2)(b'))\big) \\ &= ((\Psi\Phi_0)(b_0) *'' \big((\Psi \cdot \Phi_1) \cdot b_1))((\Phi \cdot b_2)(b'))\big) \\ &= ((\Psi\Phi)(b_0) *'' \big((\Psi \cdot b_1))((\Phi \cdot b_2)(b'))\big) \\ &= ((\Psi\Phi)(b_0) *'' \big(((\Psi\Phi) \cdot b_1)(b')\big). \end{aligned}$$

Furthermore, since $- \otimes_R k$ is a functor, $(\Psi\Phi) \otimes_R k = (\Psi \otimes_R k)(\Psi \otimes_R k) = id$. So the composite $\Psi\Phi$ is a map of quantum deformations. ■

The condition $\Phi \otimes_R k = id_B$ implies that each morphism Φ is a R-linear isomorphism, and we will see below (Proposition 3.52) that the R-linear inverse Φ^{-1} supplies an inverse to Φ in the category of quantum deformations.

Definition 3.48 (The groupoid of quantum deformations)**.** We let $\mathscr{QD}ef_B$ denote the groupoid of infinitesimal quantum deformations of B, and QDef_B denote the set of isoclasses from $\mathscr{QD}ef_B$. Given a YD-subalgebra $E \subset B$, we let $\mathscr{QD}ef_B^E$ denote the groupoid of E-relative deformations and QDef_B^E denote its set of isoclasses.

Given a quantum deformation $\mathcal{B} = (B_R, *)$, R-linearity of the operation $*$ implies that it is specified by its value on the generating subset $B \subset B_R$. If we split $B_R = B \oplus Bt$ then we find that $*$ is specified by a map $F : B \otimes B \to B$ so that, for any $b, b' \in B$, $b * b' = bb' + F(b, b')t$.

The identity condition demands $F(1,b) = F(b,1) = 0$ and the quantum associativity (3.5.1) then is equivalent to the relation

$$\begin{aligned} 0 &= F(bb',b'') + F(b,b')b'' - F\big(b_0,(b'S^{-1}(b_3)b''S^{-1}(b_2))b_1\big) - b_0\big(F(b'S^{-1}(b_3),b''S^{-1}(b_2))b_1\big) \\ &= F(bb',b'') + F(b,b')b'' - F\big(b,b'b''\big) - b_0\big(F(b'S^{-1}(b_3),b''S^{-1}(b_2))b_1\big) \\ &= -b_0\big(F\cdot b_1\big)(b',b'') + F(bb',b'') - F\big(b,b'b''\big) + F(b,b')b''. \end{aligned} \tag{3.5.2}$$

When $B = A*\mathcal{G}$ this final equation of (3.5.2) is the statement that $F : (A*\mathcal{G})\otimes(A*\mathcal{G}) \to A*\mathcal{G}$ is a degree 2 cocycle in $\bar{C}^{\bullet}_{\text{Int}}(A*\mathcal{G})$. Or rather, F would be a cocycle in the intermediate complex if it satisfied the additional relations

$$F(\alpha,\gamma) = F(\gamma,\alpha) = 0, \tag{3.5.3}$$

that is, $\mathcal{G}$-relativity, as well as

$$F(\alpha\gamma,\beta) = F(\alpha,\gamma\beta) \text{ and } F(\alpha,\beta\gamma) = F(\alpha,\beta)\gamma \tag{3.5.4}$$

for each $\alpha,\beta \in A*\mathcal{G}$ and $\gamma \in \mathcal{G}$.

Lemma 3.49. *If a map $F : B\otimes B \to B$ produces an E-relative deformation, then for each $a,a' \in B$, and $e \in E$, the relations*

$$F(ae,a') = F(a,ea') \quad \text{and} \quad F(a,a'e) = F(a,a')e$$

are satisfied. In particular, for $B = A*\mathcal{G}$ and $E = \mathcal{G}$, (3.5.3) implies the relations (3.5.4).

Proof. Supposing E-relativity, we can take $b = a, b' = e$, and $b'' = a'$ in the formula (3.5.2) to recover the first relation of (3.5.4). Subsequently taking $b = a, b' = a'$, and $b'' = e$ implies the second relation of (3.5.2). ■

The above discussion, as well as the above lemma, then give

Proposition 3.50. *Let m denote the non-deformed multiplication on the B_R. There is a natural bijection*

$$Z^2(\bar{C}^{\bullet}_{\text{Int}}(A*\mathcal{G})) \to \text{obj}\big(\mathscr{Q}\mathscr{D}ef^{\mathcal{G}}_{A*\mathcal{G}}\big), \quad F \mapsto (B_R, m+Ft)$$

and subsequent inclusion $Z^2(\bar{C}^{\bullet}_{\text{Int}}(A*\mathcal{G})) \to \text{obj}\big(\mathscr{Q}\mathscr{D}ef_{A*\mathcal{G}}\big)$.

Proof. Take $B = A * \mathcal{G}$. The function F associated to a deformation $(B_R, *)$ satisfies (3.5.3) if and only if $*$ satisfies $\alpha * \gamma = \alpha\gamma$ and $\gamma * \alpha = \gamma\alpha$ for each $\alpha \in B_R$ and $\gamma \in \mathcal{G}_R$. ∎

Now we return to the discussion of morphisms of quantum deformations. Any $\Phi : B_R \to B_R$ satisfying $\Phi \otimes_R k = id_B$ will be of the form $\Phi = id_{B_R} + \phi t$ for some $\phi : B \to B$. Indeed this gives a bijection $\mathrm{End}_k(B) \to \mathrm{End}_R(B_R)_{id_B}$, $\phi \mapsto (id + \phi t)$, where $\mathrm{End}_R(B_R)_{id_B}$ denotes those endomorphisms reducing to the identity on B under $- \otimes_R k$.

Suppose we are given a R-linear map $\Phi = (id + \phi t) : \mathcal{B} \to \mathcal{B}'$ between quantum deformations with multiplications specified by maps $F : B \otimes B \to B$ and $F' : B \otimes B \to B$ respectively.

Lemma 3.51. *The map Φ is a quantum deformation if and only if, for each $b, b' \in B \subset B_R$, we have*

$$F(b, b') - F'(b, b') = b_0(\phi \cdot b_1)(b') - \phi(bb') + \phi(b)b'.$$

Proof. First note that id_{B_R} is $\mathcal{G}_R$-invariant, so that

$$\Phi \cdot \gamma = (id_{B_R} + \phi t) \cdot \gamma = \epsilon(\gamma) id_{B_R} + (\phi \cdot \gamma)t$$

for each $\gamma \in \mathcal{G}_R$. It follows that

$$\Phi(b_0) \otimes (\Phi \cdot b_1)(b') = b \otimes b' + \phi(b) \otimes b't + b_0 \otimes (\phi \cdot b_1)(b')t$$

and then

$$\Phi(b_0) *' (\Phi \cdot b_1)(b') = bb' + (F'(b, b') + \phi(b)b' + b_0(\phi \cdot b_1)(b'))t.$$

On the other side we get $\Phi(b * b') = bb' + \phi(bb')t + F(b, b')t$ so that

$$\Phi(b_0) *' (\Phi \cdot b_1)(b') - \Phi(b * b') = (F'(b, b') + \phi(b)b' + b_0(\phi \cdot b_1)(b') - \phi(bb') - F(b, b'))t.$$

Whence $0 = \Phi(b_0) *' (\Phi \cdot b_1)(b') - \Phi(b * b')$ if and only if $F(b, b') - F'(b, b') = b_0(\phi \cdot b_1)(b') - \phi(bb') + \phi(b)b'$. Now by R-linearity of Φ and $*$ we see that, for any $r, r' \in R$

$$\Phi(b_0 r) *' (\Phi \cdot b_1)(b'r') - \Phi(br * b'r') = \big(\Phi(b_0) *' (\Phi \cdot b_1)(b') - \Phi(b * b')\big)rr' = 0$$

so that the proper relation is satisfied on all of $B_R \otimes_R B_R$. ∎

Proposition 3.52. *If $\Phi : \mathcal{B} \to \mathcal{B}'$ is a morphism of quantum deformations then it is an R-linear isomorphism and its inverse $\Phi^{-1} : \mathcal{B}' \to \mathcal{B}$ is also a morphism of quantum deformations.*

Proof. If $\Phi = (id + \phi t)$ then it is invertible with inverse $\Phi^{-1} = (id - \phi t)$. Now since ϕ satisfies the equation of Lemma 3.51 we see that $-\phi$ satisfies

$$F'(b, b') - F(b, b') = b_0(-\phi \cdot b_1)(b') - (-\phi)(bb') + (-\phi)(b)b'$$

so that $\Phi^{-1} : \mathcal{B}' \to \mathcal{B}$ is a map of deformations. ■

Now we wish to investigate the subgroupoid $\mathscr{QD}ef^E_B$.

Lemma 3.53. *If $\Phi : \mathcal{B} \to \mathcal{B}'$ is a morphism of E-relative deformations, then for each $e \in E_R$, $b \in B_R$, we have $\Phi(be) = \Phi(b)e$.*

Proof. We already know $\Phi|E_R = id_{E_R}$ so that $\phi|E = 0$. From Lemma 3.49 we know that both F and F' will satisfy $F(b, e) = F'(b, e) = 0$. So from the formula

$$0 = F(b, e) - F'(b, e) = b_0(\phi \cdot b_1)(e) - \phi(be) + \phi(b)e,$$

and the fact E is a $\mathcal{G}$-submodule so that $b_0(\phi \cdot b_1)(e) = 0$, we find $\phi(be) = \phi(b)e$. ■

For any vector space V, and map $d : W \to V$, we let $V//W$ denote the groupoid with objects $\mathrm{obj}(V//W) = V$ and morphisms $\mathrm{Hom}_{V//W}(v, v') = \{w \in W : v - v' = d(w)\}$.

Theorem 3.54. *There is an isomorphism of groupoids*

$$Z^2(\bar{C}^\bullet_{\mathrm{Int}}(A * \mathcal{G}))//\bar{C}^1_{\mathrm{Int}}(A * \mathcal{G}) \to \mathscr{QD}ef^{\mathcal{G}}_{A*\mathcal{G}}, \quad \begin{array}{l} F \mapsto (m + F) \\ \phi \mapsto (id_{B_R} + \phi t) \end{array}$$

and subsequent identification

$$H^2_{\mathrm{Int}}(A * \mathcal{G}) \xrightarrow{\cong} \mathrm{QDef}^{\mathcal{G}}_{A*\mathcal{G}}.$$

Proof. The fact that this is a map of groupoids follows from Lemma 3.51 and the description of the differential on the intermediate complex from Lemma 3.19. The fact that it's a bijection on objects was covered in Proposition 3.50 and fully faithfulness follows from Lemma 3.51 again. ■

We do not know at this moment when the inclusion $\mathscr{Q}\mathscr{D}ef_B^E \to \mathscr{Q}\mathscr{D}ef_B$ is fully faithful, essentially surjective, etc., nor do we know anything about the induced map $\mathrm{QDef}_B^E \to \mathrm{QDef}_B$. We end with a conjecture and a question.

Conjecture 3.55. When $\mathcal{G}$ is cocommutative the map $\mathrm{QDef}_{A*\mathcal{G}}^{\mathcal{G}} \to \mathrm{QDef}_{A*\mathcal{G}}$ is a bijection, so that $H^2_{\mathrm{Int}}(A*\mathcal{G}) = \mathrm{QDef}_{A*\mathcal{G}}$.

Question 3.56. If $\mathcal{G}$ is cocommutative, is the map of groupoids $\mathscr{Q}\mathscr{D}ef_{A*\mathcal{G}}^{\mathcal{G}} \to \mathscr{Q}\mathscr{D}ef_{A*\mathcal{G}}$ fully faithful?

Our knowledge from homological algebra would suggest that if E is sufficiently semisimple–possibly separable–then we should have that the map $\mathscr{Q}\mathscr{D}ef_B^E \to \mathscr{Q}\mathscr{D}ef_B$ is an equivalence. However, the usefulness of cocommutativity in the previous section confounds this principle.

Fin!

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