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Stable Processes with Opposing Drifts

by

James M. Wright

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Abstract

Stable Processes with Opposing Drifts

by James M. Wright

Chairperson of the Supervisory Committee: Professor Robert M. Blumenthal
Department of Mathematics

A strong Markov process, W^0, is constructed by a natural linking together of two independent stable processes of type (α, β_1) and (α, β_2). The drift for a stable process X of type (α, β) can be measured by β since

$$P^0(X_t > 0) = \frac{1}{2} + \frac{1}{\pi\alpha} \tan^{-1}(-\beta \tan(\frac{\pi\alpha}{2})).$$

Conditions for when W^0 will hit 0 are determined and asymptotics for σ, the time it reaches 0, are obtained.

We then consider the extension problem for W^0 which is to describe all, or at least important classes, of processes W, defined for all time, that agree with W^0 until time σ. It is custumary to require that the extension have no sojourn at 0. Our interest is in scale-invariant extensions since W^0 is scale-invariant. Extensions of the stable processes (α, β) killed at σ are also considered.

Contents

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Dedication

I would like to dedicate this work to my beautiful. kind. and patient wife. She encouraged me often and offered moral and emotional support throughout the years.

Chapter 1

Preliminaries

1.1 Introduction

The problem of finding extensions of a Markov process has been studied by many people. Watanabe [21], Motoo [15], Itô-McKean [10], Itô [9], Lamperti [12], Blumenthal [2], Rogers [17], Salisbury [19], Erickson [6], and Vuolle-Apiala [20] all consider some sort of extension problem.

One well studied problem, initially considered by Feller and then Itô-McKean [10], among others, is as follows: find all strong Markov processes on $[0, \infty)$, with sample paths that are right continuous with left limits, which behave like Brownian motion in $(0, \infty)$. These are the so-called Feller Brownian motions.

Lamperti [12] and Vuolle-Apiala [20] consider self-similar processes (Lamperti calls them semi-stable) in $(0, \infty)$ and extensions to $[0, \infty)$ which are also self-similar. A Markov process (X_t) is said to be self-similar of index α if the transition function $P_t(x, A)$ satisfies

$$P_t(x, A) = P_{at}(a^\alpha x, a^\alpha A),$$

for some $\alpha > 0$. Note that the α stable processes are self-similar of index $1/\alpha$. We will sometimes use the phrase scale-invariant for self-similar.

As another example. let X be a real-valued strictly stable process. In other words. X is a strong Markov process with stationary independent increments and has sample paths that are right continuous with left limits. The characteristic function is given by

$$E^0(e^{i\theta X_t}) = \exp\left(-t|\theta|^\alpha(1 + i\beta sgn(\theta)\tan(\frac{\pi\alpha}{2}))\right)$$

for $\alpha \in (0,1) \cup (1,2]$ and $\beta \in [-1,1]$ or

$$E^0(e^{i\theta X_t}) = \exp\left(-t|\theta|(1 + i\frac{2\beta}{\pi}sgn(\theta)\ln|\theta|)\right)$$

for $\alpha = 1$ and $\beta \in [-1,1]$ We will say X is stable of type (α,β). Stable processes hit points provided $\alpha > 1$, and in this case let X^0 denote X killed at $\sigma = \inf\{t > 0 : X_t = 0\}$. X^0 is called the minimal process. We show there is a unique extension of this minimal process which has no sojourn at zero and exits zero continuously. In addition. we show that there are infinitely many scale-invariant extensions with the property that the excursion start with a jump from zero.

The general theory of excursions from a set can be found in Motoo [15] which deals with a very general situation: find all Markov processes which agree. until hitting the boundary. with the given minimal process X^0 where the boundary is not necessarily a point. Extensions are characterized in terms of the minimal process. the boundary process. and auxiliary factors which relate to the time spent on the boundary (by the extension) and how the extension exits the boundary. These sort of extension results are known when the minimal process involves Brownian motion.

We are interested in real-valued processes and a boundary which is a single point. namely zero. Also. we are interested in extensions which have no sojourn at zero. In other words. $E^0(\int_0^\infty 1_{\{0\}}(X_t)dt) = 0$, where X is the extension of the minimal process. Thus. we are not interested in extensions which remain at the boundary for an exponential amount of time and then exit via a jump kernel or extensions with "sticky" boundaries.

As another example, we construct a Strong Markov process using a technique of Meyer [13] by a natural linking together of two independent processes each with stationary independent increments. We are primarily interested with linking stable processes whose Lévy measure is not concentrated on $(-\infty, 0)$ or $(0, \infty)$. Using a result of Kesten [11] we obtain asymptotics for the hitting time of zero. The scaling properties of stable processes is integral to obtaining these asymptotics. We are able to determine when this new process will hit zero.

Extension results when the minimal process involves Brownian motion or a Cauchy process with linear drift are known, so throughout we take $\alpha \in (0, 2)$ with $\alpha \neq 1$.

1.2 Notation and Terminology

Notation and terminology follows for the most part Blumenthal and Getoor [4]. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ be a Markov Process of function space type with state space $(E, \mathcal{E})$. So we have:

1. Ω is the space of all maps from $[0, \infty)$ to E.
2. X_t is the coordinate map on Ω, i.e., for $\omega \in \Omega$, $X_t(\omega) = \omega(t)$.
3. $\mathcal{F}$ is a σ-field containing $\sigma(X_t : t \geq 0)$.
4. $\{\mathcal{F}_t : t \geq 0\}$ is an increasing family of sub σ-fields of $\mathcal{F}$ and for each t, $\mathcal{F}_t$ is a σ-field containing $\sigma(X_s : s \leq t)$.
5. for each t, $\theta_t : \Omega \to \Omega$ is defined by $\theta_t(\omega(s)) = \omega(t+s)$ or, in other words, $X_s \circ \theta_t(\omega) = \omega(t+s)$.
6. for each $x \in E$, P^x is a probability measure on $(\Omega, \mathcal{F})$.
7. for all $x \in E$ and t and s and $f \in b\mathcal{E}$, $E^x(f \circ X_{t+s} | \mathcal{F}_t) = E^{X_t}(f \circ X_s)$.

We are only interested when E is a subset of the real numbers.

An additive functional of X is a family $A = \{A_t : t \geq 0\}$ of functions from $\Omega \rightarrow [0, \infty)$ satisfying:

1. $t \rightarrow A_t(\omega)$ is non-decreasing, right continuous with $A_0(\omega) = 0$ except for $\omega \in \Lambda$ where $P^{\mu}(\Lambda) = 0$ for all μ,
2. for each t, $A_t \in \mathcal{F}_t$,
3. for each t and s, $A_{t+s} = A_t + A_s \circ \theta_t$ almost surely P^{μ} for each μ.

The canonical example of an additive functional is given by

$$A_t(\omega) = \int_0^t f(X_s)ds$$

where f is a bounded measurable nonnegative function.

For any $b \in E$, set $Z = \{t : X_t = b\}$ and let $\sigma = \inf\{t > 0 : X_t = b\}$. By local time at b we mean a continuous additive functional (CAF), L, which increases only on Z. In other words,

$$\int_0^t 1_{\{b\}}(X_s)dL_s = L_t.$$

L will be normalized to be the unique CAF satisfying

$$E^x(e^{-\sigma}) = E^x(\int_0^\infty e^{-t}dL_t)$$

for all x in E.

A kernel H on $(\Omega, \mathcal{F})$ is a positive function $H(\omega, A)$ defined for $\omega \in \Omega$ and $A \in \mathcal{F}$ such that

$$\omega \mapsto H(\omega, A)$$

is measurable for each $A \in \mathcal{F}$ and

$$A \mapsto H(\omega, A)$$

is a measure on $\mathcal{F}$ for each $\omega \in \Omega$.

The following notation will be used to indicate that the distribution of X under P is the same as the distribution of Y under Q

$$(X, P) \sim (Y, Q).$$

The indicator function of set A will be denoted $\mathbf{1}_A(\cdot)$ and $\mathbb{R}^0$ will denote $\mathbb{R} \setminus \{0\}$. For convenience, we will sometimes write $X(T)$ for X_T.

1.3 Some Excursion Theory

In this section some excursion theory is presented. See Blumenthal [3] or Getoor [8] for more details.

$X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ is a standard Markov process with state space $(E, \mathcal{E})$, where E is a subset of the real numbers. Fix a point $b \in E$ that is recurrent, i.e., $P^x(\sigma < \infty) = 1$ for all $x \in E$, where $\sigma = \inf\{t > 0 : X_t = b\}$. This process can be described in terms of the set $Z = \{t : X_t = b\}$ and the excursions away from b, which are pieces of path defined on the complement of Z. To be precise, let (r, s) be a maximal open interval in the complement of Z, so $s = r + \sigma \circ \theta_r$. An excursion is a piece of the path defined by

$$e(t) = X_{r+t} \quad 0 \le t < s - r.$$

One has $e(t) \neq b$ for all $t \in (r, s)$ and either of the possibilities $e(0) = b$ or $e(0) \neq b$ may occur. It is sometimes useful to define the path for $t \geq s - r$ by setting $e(t) = b$ for $t \geq s - r$.

An excursion measure is a measure $\hat{P}$ on Ω with the following properties:

1. $\hat{P}$ is supported on $\{\omega : \sigma(\omega) > 0\}$and on $\{\omega : \omega(t) = b \ \text{ for } \ t \geq \sigma(\omega)\}$;
2. $\hat{E}(1 - e^{-\sigma}) < \infty$;

3. for each bounded, measurable f and each t and $\Gamma \in \mathcal{F}_t$

$$\hat{E}(f \circ \theta_t; \Gamma \cap \{\sigma > t\}) = \hat{E}(E^{X_t}(f); \Gamma \cap \{\sigma > t\}).$$

Furthermore, if G denotes the set of strictly positive left endpoints of the maximal intervals in the complement of Z, then $\hat{P}$ is the unique measure on Ω satisfying

$$E^x(\sum_{r \in G} Y_r K(r, \theta_r)) = E^x(\int_0^\infty Y_s \hat{P}(K_s) dL_s)$$

for all previsible Y, positive $\mathcal{B} \times \mathcal{F}^*$ measurable K, and $x \in R$, where $\mathcal{B}$ denotes the Borel sets of R^+ and $\mathcal{F}^*$ is the σ-algebra of universally measurable sets over $(\Omega, \mathcal{F})$.

The excursion measure can be described in terms of an entrance law. If Q_t is the semi-group operator for X killed at σ,

$$Q_t f(x) = E^x(f(X_t); t < \sigma),$$

then an entrance law $\{\eta_s; s > 0\}$ for Q_t is a family of finite measures on the Borel sets of $E \backslash \{b\}$ satisfying

$$\eta_s Q_t = \eta_{s+t}$$

for all s and t and such that $E^{\eta_s}(1 - e^{-\sigma})$ remains bounded as s approaches 0. If $\hat{P}$ is an excursion measure then $\{\eta_s(\cdot)\}$, where $\eta_s(\cdot) = \hat{P}(X_s \in \cdot, s < \sigma)$, is an entrance law.

The process X can be thought of as being comprised of the minimal process, which is the original process killed at b, and the excursion measure or the entrance law.

Chapter 2

Construction

In this chapter a strong Markov process is constructed using a technique of Meyer [13] where independent processes, X and Y, each with stationary independent increments are linked together. In our construction, the Lévy measure for X and for Y is not concentrated on either $(-\infty, 0)$ or $(0, \infty)$. Meyer's procedure involves linking the paths of X and Y together via a stochastic kernel. As we shall see, the notation quickly becomes quite cumbersome when dealing with this simple intuitive idea.

2.1 Meyer's Procedure

Let X and Y be processes with stationary independent increments. We will assume the sample space is $D[0, \infty)$, the space of real-valued paths which are right continuous with left limits. We will let P^x denote probabilities for X starting at x and Q^x denote probabilities for Y starting at x. Define $T = \inf\{t > 0 : X_t < 0\}$ and $S = \inf\{t > 0 : Y_t > 0\}$. We will assume that for $x > 0$, $P^x(X(T) = 0) = 0$ and that for $x < 0$, $Q^x(Y(S) = 0) = 0$. We will also assume that if X and Y do hit points they satisfy the hypotheses of the following theorem due to Millar [14].

Theorem 2.1.1 (Millar) *Let X be a real process with stationary independent increments. For $x \in \mathbb{R}$, define $H_x = \inf\{t > 0 : X_t = x\}$ and for $x > 0$, $T_x = \inf\{t > 0 : X_t > x\}$ and $T_{-x} = \inf\{t > 0 : X_t < -x\}$. If X satisfies*

1. $P^0(H_x < \infty) > 0$ *for all* x,
2. $P^0(T_x < \infty) = P^0(T_{-x} < \infty) = 1$ *for all* $x > 0$,
3. $P^0(X(T_x) = x) = P^0(X(T_{-x}) = -x) = 0$ *for all* $x > 0$,

then, for any $x \neq 0$ and starting from 0, X jumps across level x infinitely often before hitting x.

The stable processes, with $\alpha \in (1, 2)$ and Levy measure not concentrated on $(0, \infty)$ or $(-\infty, 0)$ satisfy the hypotheses of this theorem.

We will define a process which behaves like X when the path is in $(0, \infty)$ and like Y when the path is in $(-\infty, 0)$. For notational convenience we consider a function space realization of the two processes and use Meyer's notation. Thus, we can assume that both X and Y are the coordinate process.

To be specific, let X_t be the coordinate process on $D[0, \infty)$ with shift operator θ_t. For $x \in \mathbb{R}$, (P^x) and (Q^x) are measures such that (X, P^x) and (X, Q^x) are independent processes, starting from x, each with stationary, independent increments. Define $\mathcal{F}_t^0 = \sigma\{X_s; s \leq t\}$ and let $\mathcal{F}_t$ denote the universal completion of $\mathcal{F}_t^0$, and $\mathcal{F}$ that of $\mathcal{F}_\infty^0$. Define $T = \inf\{t > 0 : X_t < 0\}$ and $S = \inf\{t > 0 : X_t > 0\}$. We will define W which evolves as follows. For $x > 0$, W behaves like X under P^x until jumping into $(-\infty, 0)$ at $X(T)$; now W behaves like X under $Q^{X(T)}$ until jumping into $(0, \infty)$; and so on.

More precisely, set $\xi = T \vee S$. Adjoin δ to $\mathbb{R}^0$ as an isolated point. Define X^0 as follows

$$X_t^0 = \begin{cases} X_t & t < \xi \\ \delta & \xi \leq t. \end{cases}$$

Set $\Omega = \{\omega \in D[0,\infty) : \{t : \omega(t) = \delta\} = [\xi(\omega), \infty)\}$. Observe that ξ is the lifetime of ω. $[\delta]$ will denote the "point" in Ω defined by $[\delta](t) = \delta$ for all t. Define the following kernel on $(\Omega, \mathcal{F})$

$$H(\omega, \cdot) = \mathbf{1}_{(-\infty,0)}(X_{\xi}(\omega))Q^{X_{\xi(\omega)}}(\cdot) + \mathbf{1}_{(0,\infty)}(X_{\xi}(\omega))P^{X_{\xi(\omega)}}(\cdot). \tag{2.1}$$

H is called a rebirth kernel. Set

$$\boldsymbol{W} = \{\boldsymbol{w} \in \Omega^{\mathbb{N}} : \text{for } \ \boldsymbol{w} = (\omega_0, \omega_1, \dots) \text{ if } \xi(\omega_i) = 0 \text{ then } \omega_i = \omega_{i+1} = \dots = [\delta]\}$$

and let $\mathcal{G} = \mathcal{F}^{\mathbb{N}}|_{\boldsymbol{W}}$ and set $|\boldsymbol{\delta}| = ([\delta], [\delta], \dots)$. For $\boldsymbol{w} = (\omega_0, \omega_1, \dots)$, let p_n be the projection map, i.e., $p_n(\boldsymbol{w}) = \omega_n$, and let $\eta_n(\boldsymbol{w}) = (\omega_0, ..., \omega_n)$.

Lemma 2.1.2 (Meyer) *For each $x \in \mathbb{R}^0$, there exists a measure Π^x on $\boldsymbol{W}$ for which (p_n) is a Markov chain with transition kernel H and initial law R^x given by*

$$R^x = \mathbf{1}_{(-\infty,0)}(x)Q^x + \mathbf{1}_{(0,\infty)}(x)P^x.$$

For $\boldsymbol{w} \in \boldsymbol{W}$, $\boldsymbol{w} = (\omega_0, \omega_1, ...)$, set $s_{-1} = 0$ and define

$$s_k(\boldsymbol{w}) = \xi(\omega_0) + ... + \xi(\omega_k) \quad \text{for } \ k \geq 0 \tag{2.2}$$

and

$$s_\infty(\boldsymbol{w}) = \lim_{n\to\infty} s_n(\boldsymbol{w}).$$

Now we are in position to define a process with the prescribed behavior. For $\boldsymbol{w} \in \boldsymbol{W}$ define $W_t^0(\boldsymbol{w})$ as follows

$$W_t^0(\boldsymbol{w}) = \begin{cases} X_t^0(\omega_0) & \text{if } 0 \leq t < s_0(\boldsymbol{w}) \\ X_{t-s_0(\boldsymbol{w})}^0(\omega_1) & \text{if } s_0(\boldsymbol{w}) \leq t < s_1(\boldsymbol{w}) \\ \cdot & \\ \cdot & \\ \cdot & \\ \delta & \text{if } s_\infty(\boldsymbol{w}) \leq t \end{cases}$$

To define shift operators $\boldsymbol{\theta}_t$ on $\boldsymbol{W}$ we first observe that for fixed $\boldsymbol{w}$ and any t there is a k with $s_{k-1}(\boldsymbol{w}) \leq t < s_k(\boldsymbol{w})$. Define $\boldsymbol{\theta}_t$ by

$$\boldsymbol{\theta}_t(\boldsymbol{w}) = (\theta_{t-s_{k-1}(\boldsymbol{w})}(\omega_k), \omega_{k+1}, ...).$$

where θ is the shift for X. If $s_\infty(\boldsymbol{w}) \leq t$, then define $\Theta_t(\boldsymbol{w}) = |\boldsymbol{\delta}|$. It is easy to verify that the following hold:

1. $\boldsymbol{\theta}_t \circ \boldsymbol{\theta}_s = \boldsymbol{\theta}_{t+s}$,
2. $W_s^0 \circ \boldsymbol{\theta}_t = W_{s+t}$,
3. $s_0 \circ \boldsymbol{\theta}_t = s_0 - t$ on $\{t < s_0\}$,
4. for $\boldsymbol{w} = (\omega_0, \omega_1, \dots)$, $\boldsymbol{\theta}_{s_0}(\boldsymbol{w}) = (\omega_1, \omega_2, \dots)$ and $\boldsymbol{\theta}_{s_{k-1}}(\boldsymbol{w}) = (\omega_k, \omega_{k+1}, \dots)$.

We now proceed to define fields $\mathcal{G}_t$. For a $\mathcal{G}$ measurable function f, we say it is $\mathcal{G}_t$ measurable if and only if for any k, $f1_{(s_{k-1} \leq t < s_k)}$ is of the form $h \circ \eta_k$ where $h(\omega_0, \omega_1, ..., \omega_k)$ is such that for all fixed $(\omega_0, \omega_1, ..., \omega_{k-1})$ with

$$\xi(\omega_0) + ... + \xi(\omega_{k-1}) \leq t < \xi(\omega_0) + ... + \xi(\omega_k)$$

the function $\omega \to h(\omega_0, ..., \omega_{k-1}, \omega)$ is measurable with respect to $\mathcal{F}_{t-(\xi(\omega_0)+...+\xi(\omega_{k-1}))}$. In other words, $\mathcal{G}_t$ contains information up to $\xi(\omega_0) + \dots + \xi(\omega_k) \leq t$. Observe that $W_t^0 \in \mathcal{G}_t$ and $\{s_i \leq t\} \in \mathcal{G}_t$. The following is Meyer's [13] result.

Theorem 2.1.3 (Meyer) *$W^0 = (\boldsymbol{W}, \mathcal{G}, \mathcal{G}_t, W_t^0, \boldsymbol{\theta}_t, \Pi^x)$ is a Strong Markov process.*

2.2 Hitting Zero

Our primary interest is when (X, P) and (Y, Q) are independent stable processes of type (α, β_1) and (α, β_2), respectively. The stable processes will be reviewed in the next chapter, but we make a few observations. If $\beta_1 = \beta_2$, then W^0 is equivalent to either of the original processes killed at zero. This follows from the strong Markov property and our choice of rebirth kernel. If $\beta_1 \neq \beta_2$ then W^0 behaves like different stable processes on either side of zero. Of course, the two processes X and Y could have different α's, but our analysis relies on the scaling properties of stable processes, so we do not consider any case where different values of α are used.

A natural question arises: will W^0 hit zero? If so, the existence of extensions beyond the first hit of 0 are what we consider. This will be addressed, for the processes of interest to us, in Chapter 4.

The following result is a step in that direction. Let S_n be the successive times that W^0 jumps across 0.

Theorem 2.2.1 *For each $x \in \mathbb{R}^0$,*

$$\Pi^x(\lim_{n\to\infty} S_n < \infty, \{W^0(S_n) \to 0\}^c) = 0. \tag{2.3}$$

Proof. First observe that for all $\epsilon > 0$, $\eta > 0$, there is $\delta > 0$ such that for $|y| > \epsilon$,

$$\Pi^y(\xi < \delta) < \eta. \tag{2.4}$$

To see this, note that

$$\Pi^y(\xi < \delta) \leq \max(P^0(|X_t| > \epsilon, \text{ for some } t < \delta), Q^0(|X_t| > \epsilon, \text{ for some } t < \delta)).$$

Now for any $\epsilon > 0$ define

$$\Delta_\epsilon = \{\boldsymbol{w} \in \boldsymbol{W} : \lim_{n\to\infty} S_n < \infty, |W^0(S_n)| > \epsilon \ i.o.\}.$$

So for all $\delta > 0$ we have

$$\lim_{N\to\infty}(\Pi^x(\Delta_\epsilon \cap \{S_{n+1} - S_n < \delta \ \forall n \geq N\})) = \Pi^x(\Delta_\epsilon). \tag{2.5}$$

Now suppose that the left side of (2.3) is strictly positive. Pick $\epsilon_1 > 0$ and $\epsilon_2 > 0$ where

$$\Pi^x(\Delta_{\epsilon_1}) = \epsilon_2. \tag{2.6}$$

Pick $\delta > 0$ so that for $|y| > \epsilon_1$

$$\Pi^y(\xi < \delta) < \frac{\epsilon_2}{2}.$$

Using this δ, choose N so that for

$$\Gamma = \{S_{n+1} - S_n < \delta \ \forall n \geq N; |W^0(S_k)| > \epsilon_1 \text{ for some } k \geq N\},$$

we have by (2.6)

$$\Pi^x(\Gamma) \geq \frac{3}{4}\epsilon_2.$$

Now set $M_N = \inf\{k \geq N : |W^0(S_k)| > \epsilon_1\}$. The strong Markov property for $\{W^0(S_n) : n \geq 1\}$ yields

$$\begin{aligned}\frac{3}{4}\epsilon_2 \leq \Pi^x(\Gamma) &\leq \Pi^x(S_{M_N+1} - S_{M_N} < \delta; M_N < \infty)\\ &= \Pi^x(\xi \circ \theta_{S_{M_N}} < \delta; M_N < \infty)\\ &\leq \Pi^x(\Pi^{W^0(S_{M_N})}(\xi < \delta); |W^0(S_{M_N})| > \epsilon_1) < \frac{\epsilon_2}{2},\end{aligned}$$

a contradiction. Thus,

$$\Pi^x(\lim_{n\to\infty} S_n < \infty, \{W^0(S_n) \to 0\}^c) = 0.$$

□

Chapter 3

Stable Processes and the Minimal Process

This chapter contains some facts about stable processes. We also consider when W^0 is constructed using independent stable processes. A result of Kesten [11] is used to obtain asymptotics for the hitting time of zero. This allows us to determine if W^0 will hit zero. We note that the stable processes with $\alpha \leq 1$ do not hit points. Yet, if W^0 is constructed using stable processes with $\alpha \in (\frac{1}{2}, 1)$ and with certain β_i, then W^0 will hit zero. In other words, there is enough "push" towards zero. However, when $\alpha \leq \frac{1}{2}$, W^0 will not hit zero, regardless of the choice of β_i.

3.1 Stable Processes

Let $X = (X_t)_{t \geq 0}$ be a real-valued, strictly stable processes of index α with P^x and E^x denoting probability and expectation starting from x. Recall that we exclude $\alpha = 1$ or 2. In particular, the characteristic function is of the form:

$$E^0(e^{i\theta X_t}) = \exp(-t|\theta|^\alpha(1 + i\beta sgn(\theta)\tan(\frac{\pi\alpha}{2}))).$$

with $|\beta| < 1$. We will say X is stable of type (α, β). If X is symmetric stable it is of type $(\alpha, 0)$. This is justified by the following well known result:

$$q = P^0(X_t > 0) = P^0(X_1 > 0) = \frac{1}{2} + \frac{1}{\pi\alpha}\arctan(-\beta\tan(\frac{\pi\alpha}{2}))$$

where $\arctan(x) \in (-\pi/2, \pi/2)$. In some sense the tendency of the process to be "positive" or "negative" is measured by q or $p = 1 - q$, respectively.

A simple calculation shows $\alpha q \leq 1$ with equality if and only if $\alpha > 1$ and $\beta = 1$. Similarly, $\alpha p \leq 1$ with equality if and only if $\alpha > 1$ and $\beta = -1$. Furthermore, the following bounds are obtained

$$\max(0, 1 - \frac{1}{\alpha}) \leq p, q \leq \min(1, \frac{1}{\alpha}). \tag{3.1}$$

For $\alpha < 1$ and $p = 0$ X is called a stable subordinator or if $q = 0$ then $-X$ is a stable subordinator. If $\beta = -1$ the Lévy measure for X is concentrated on $(0, \infty)$. i.e., X has no negative jumps. Similarly, when $\beta = 1$ X has no positive jumps since the Lévy measure for X is concentrated on $(-\infty, 0)$. We will not be interested when $|\beta| = 1$ since the jumping over hypotheses of Theorem 2.1.1 are not satisfied.

The stable processes have nice scaling properties. Define the following random times

$$\hat{T}_x = \inf\{t : X_t > x\},$$

$$\hat{T} = \inf\{t : X_t < 0\},$$

and

$$\hat{S} = \inf\{t : X_t > 0\}.$$

The following scaling properties hold:

$$(X_t, P^x) \sim (c^{-\frac{1}{\alpha}} X_{ct}, P^{c^{\frac{1}{\alpha}} x}). \tag{3.2}$$

$$(\hat{T}_{|x|}, P^0) \sim (|x|^\alpha \hat{T}_1, P^0). \tag{3.3}$$

$$(\hat{S}, P^{-|x|}) \sim (|x|^{\alpha}\hat{S}, P^{-1}), \tag{3.4}$$

$$\text{if } \beta = 0 \quad (\hat{S}, P^{-|x|}) \sim (\hat{T}, P^{|x|}), \tag{3.5}$$

$$(X_{\hat{T}}, P^{x}) \sim (c^{-1}X_{\hat{T}}, P^{cx}). \tag{3.6}$$

The following result of Bingham [1] will be used in the next section.

Theorem 3.1.1 (Bingham) *Let* (X_t) *be a stable process of type* (α, β) *and* $\hat{T}_x = \inf\{t : X_t > x\}$. *Then there is a positive* k *with*

$$P^0(\hat{T}_{|x|} > t) \sim k(|x|^{\alpha}/t)^{q} \quad \text{as } t \to \infty. \tag{3.7}$$

3.2 Conditions for W^0 to Hit Zero

In this section we consider when independent stable processes of type (α, β_i), $i = 1, 2$ are used to construct W^0. In particular, W^0 behaves like a stable of type (α, β_1) when the path is in $(0, \infty)$ and like a stable of type (α, β_2) when the path is in $(-\infty, 0)$. Set

$$q_2 = Q^0(X_t > 0)$$

and

$$p_1 = P^0(X_t < 0).$$

It seems plausible that W^0 has scaling properties similar to the stable processes since on $[s_j, s_{j+1})$, W^0 behaves as a stable of type (α, β_i) started from $\omega_{j+1}(0) = \omega_j(\xi)$, where $i = 1$ or 2. We have the following.

Lemma 3.2.1 *For $i \geq 0$,*

$$\Pi^x(W^0_{s_i} \in A) = \Pi^{cx}(c^{-1}W^0_{s_i} \in A). \tag{3.8}$$

Proof. This is easily done using induction. For $i = 0$ using (3.6)

$$\Pi^{cx}(c^{-1}W^0_{s_0} \in A) = R^{cx}(c^{-1}X_\xi \in A)$$

$$= R^x(X_\xi \in A)$$

$$= \Pi^x(W^0_{s_0} \in A).$$

Now suppose $\Pi^{cx}(c^{-1}W^0_{s_i} \in A) = \Pi^x(W^0_{s_i} \in A)$ and consider the following.

$$\Pi^{cx}(c^{-1}W^0_{s_{i+1}} \in A) = \Pi^{cx}(W^0_\xi \circ \boldsymbol{\theta}_{s_i} \in cA)$$

$$= \Pi^{cx}(\Pi^{W^0_{s_i}}(W^0_\xi \in A))$$

$$= \Pi^x(\Pi^{c^{-1}W^0_{s_i}}(W^0_\xi \in cA))$$

$$= \Pi^x(\Pi^{W^0_{s_i}}(cW^0_\xi \in cA))$$

$$= \Pi^x(W^0_\xi \circ \boldsymbol{\theta}_{s_i} \in A)$$

$$= \Pi^x(W^0_{s_{i+1}} \in A).$$

□

The next result, due to Kesten [11], will be used to obtain asymptotics for $\sigma = \inf\{t > 0 : W^0_t = 0\}$.

Theorem 3.2.2 (Kesten) *Let M_n, Q_n , $n > 1$, be real valued random variables such that $(M_n, Q_n)_{n \geq 1}$, are independent and identically distributed and $P(M_n > 0, Q_n \geq 0) = 1$. Assume that*

$$E(\log M_1) < 0,$$

but that for some $\gamma > 0$

$$E(M_1^\gamma) = 1,$$

$$E(M_1^\gamma \log^+ M_1) < \infty,$$

$$0 < E(Q_1^\gamma) < \infty.$$

If in addition $\log M_1$ *is not lattice and* Q_1 *is not a constant times* $(1 - M_1)$, *i.e.*,

$$P(Q_1 = (1 - M_1)r) < 1$$

for each fixed r, *then the series*

$$R = Q_1 + \sum_{i=2}^{\infty} M_1 \dots M_{i-1} Q_i$$

converges with probability one. Moreover, there is a positive constant k *with*

$$\lim_{t \to \infty} t^\gamma P(R > t) = k.$$

Now we are ready to determine when W^0 will hit zero. In other words, determine when $\Pi^x(\sigma < \infty) = 1$. From Lemma 3.2.1, it suffices to consider $W^\cdot$ starting from $x = 1$. Define the following random times for W^0:

$$T = \inf\{t > 0 : W_t^0 < 0\},$$

$$S = \inf\{t > 0 : W_t^0 > 0\},$$

$$R = T + S \circ \theta_T,$$

$$R_0 = 0,$$

and for $n \geq 1$

$$R_n = R_{n-1} + R \circ \theta_{R_{n-1}}.$$

Note that R_n is the nth time that W^0 has jumped into $(0, \infty)$.

We need the distribution of where a stable process, starting from any nonzero x first jumps across zero. The stable processes do jump over levels [14] and the jump distribution is known (see [18], [16]). In particular, when $\alpha q_2 < 1$

$$Q^{-1}(X_S \in dy) = \frac{\sin(\pi\alpha q_2)}{\pi} y^{-\alpha q_2}(1+y)^{-1}dy. \tag{3.9}$$

and when $\alpha p_1 < 1$

$$P^1(X_T \in dy) = \frac{\sin(\pi\alpha p_1)}{\pi} |y|^{-\alpha p_1}(1+|y|)^{-1}dy. \tag{3.10}$$

Define the following random variables for $n \geq 1$

$$M_n = (W^0(R_n)/W^0(R_{n-1}))^\alpha$$

$$\Pi_n = M_1 \cdot \ldots \cdot M_n$$

$$S_n = \log \Pi_n = \sum_{i=1}^{n} \log M_i$$

$$Q_n = \frac{R \circ \theta_{R_{n-1}}}{(W^0(R_{n-1}))^\alpha}.$$

Using (3.8) and the strong Markov property we have the following for any bounded measurable functions f and g

$$\begin{aligned}
&\Pi^1(f(M_2)g(Q_2)) \\
&= \Pi^1\Big(f((W^0(R_2)/W^0(R_1))^\alpha)g\Big(\frac{R \circ \theta_{R_1}}{(W^0(R_1))^\alpha}\Big)\Big) \\
&= \Pi^1\Big(f((W^0(R)/W^0(R_1))^\alpha)g\Big(\frac{R}{(W^0(R_1))^\alpha}\Big) \circ \theta_{R_1}\Big) \\
&= \Pi^1(\Pi^x(f((W^0(R)/x)^\alpha)g(R/x))|_{x=W^0(R_1)}) \\
&= \Pi^1(f(W^0(R))g(R)).
\end{aligned}$$

From this and noting that for $x > 0$ the joint distributions of $W^0(R_1)$ and R_1 under Π^x is the same as that of $xW^0(R_1)$ and $x^\alpha R_1$ under Π^1, it follows that $(M_n, Q_n)_{n \geq 1}$ is an i.i.d. sequence and, hence, so are $(Q_n)_{n \geq 1}$ and $(M_n)_{n \geq 1}$. Thus, $(S_n)_{n \geq 1}$ is a random walk with mean $\Pi^1(\log M_1)$. Also, we note that we have the following decomposition for R_n

$$R_n = \sum_{i=2}^{n} M_1 \dots M_{i-1} Q_i + Q_1.$$

Now consider the random walk $(S_n)_{n\geq 1}$. Use the strong Markov property at time T and use (3.8) to see

$$\Pi^1(\log M_1) = \alpha\Pi^1(\log W^0(R_1))$$

$$= \alpha\Pi^1(\log W^0(T + S \circ \theta_T))$$

$$= \alpha\Pi^{-1}(\log(W^0(S))) + \alpha\Pi^1(\log(|W^0(T)|)).$$

So, using (3.9) and using formula 4.251.1 in *Tables of Integrals, Series, and Products*, Gradshteyn and Ryzhik (4th edition), we have

$$\Pi^{-1}(\log W^0(S)) = \frac{\sin \pi\alpha q_2}{\pi} \int_0^\infty \log(x) x^{-\alpha q_2}(1+x)^{-1} dx$$

$$= \frac{\sin \pi\alpha q_2}{\pi}(\pi) \csc\left((1-\alpha q_2)\pi\right)(-\pi \cot\left((1-\alpha q_2)\pi\right))$$

$$= \pi \cot(\alpha q_2 \pi).$$

Similarly, $\Pi^1(\log|W^0(T)|) = \pi\cot(\alpha p_1 \pi)$. Thus, we have

$$\Pi^1(\log M_1) = \alpha\pi(\cot(\alpha q_2\pi) + \cot(\alpha p_1 \pi))$$

$$= \alpha\pi \frac{\sin(\alpha(q_2 + p_1)\pi)}{\sin(\alpha q_2 \pi)\sin(\alpha p_1 \pi)}.$$

Since $\alpha q_2 < 1$ and $\alpha p_1 < 1$, we will have $\Pi^1(\log M_1) < 0$ provided $\alpha(p_1 + q_2) > 1$. In this case, set $\gamma = p_1 + q_2 - \frac{1}{\alpha}$ and consider $\Pi^1(M_1^\gamma)$. Using (3.8) and the strong Markov property we have

$$\Pi^1(M_1^\gamma) = \Pi^1(W^0(T + S \circ \theta_T)^{\alpha\gamma})$$

$$= \Pi^1((W^0(S)^{\alpha\gamma} \circ \theta_T)$$

$$= \Pi^1(\Pi^x((W^0(S)^{\alpha\gamma}))|_{x=W^0(T)})$$

$$= \Pi^1(\Pi^{-1}(|x|^{\alpha\gamma}W^0(S)^{\alpha\gamma})|_{x=W^0(T)})$$

$$= \Pi^1(|W^0(T)|^{\alpha\gamma})\Pi^{-1}((W^0(S))^{\alpha\gamma}).$$

Using (3.9)

$$\Pi^{-1}(W^0(S)^{\alpha\gamma}) = \frac{\sin \alpha q_2 \pi}{\pi}\int_0^\infty x^{\alpha\gamma}x^{-\alpha q_2}(1+x)^{-1}dx$$

$$= \frac{\sin \alpha q_2 \pi}{\pi}\mathcal{B}(\alpha\gamma - \alpha q_2 + 1, \alpha q_2 - \alpha\gamma)$$

$$= \frac{\sin(\alpha q_2 \pi)}{\sin((\alpha\gamma - \alpha q_2 + 1)\pi)} = \frac{\sin(\pi\alpha q_2)}{\sin(\alpha p_1 \pi)},$$

where $\mathcal{B}(x, y)$ is the Beta function. Similarly,

$$\Pi^1(|W^0(T)|^{\alpha\gamma}) = \frac{\sin(\alpha p_1 \pi)}{\sin(\alpha q_2 \pi)},$$

and so we have

$$\Pi^1(M_1^\gamma) = \frac{\sin(\pi\alpha p_1)}{\sin(\alpha q_2\pi)}\frac{\sin(\pi\alpha q_2)}{\sin(\alpha p_1 \pi)} = 1.$$

Next consider $\Pi^1(R_1^\gamma)$. To show this expectation is finite it will suffice to show that $\Pi^1(T^\gamma)$ and $\Pi^{-1}(S^\gamma)$ are finite since $\gamma < 1$ and

$$\Pi^1(T + S\circ\theta_T)^\gamma < \Pi^1(T^\gamma) + \Pi^1((S\circ\theta_T)^\gamma),$$

where

$$\Pi^1((S\circ\theta_T)^\gamma) = \Pi^1(\Pi^x(S^\gamma)|_{x=W^0(T)})$$

$$= \Pi^1(\Pi^{-1}(|x|^\gamma S^\gamma)|_{x=W^0(T)})$$

$$= \Pi^{-1}(S^\gamma)\Pi^1(|W^0(T)|^{\alpha\gamma}).$$

We show that $\Pi^1(T^\gamma)$ is finite. A similar argument will show that $\Pi^{-1}(S^\gamma)$ is finite. We have

$$\Pi^1(T^\gamma) = \Pi^1(T^\gamma : T \le N) + \Pi^1(T^\gamma : T > N)$$

$$\leq N^\gamma + \int_{N^\gamma}^\infty \Pi^1(T^\gamma > t)dt.$$

Since $\Pi^1(T^\gamma > t) = P^1(T^\gamma > t)$, we have from (3.7) in Theorem 3.1.1, for N large enough

$$\int_{N^\gamma}^\infty \Pi^1(T^\gamma > t)dt = \int_{N^\gamma}^\infty P^1(T^\gamma > t)dt \leq K \int_{N^\gamma}^\infty t^{-p_1/\gamma}dt.$$

The last integral is finite since $\gamma < p_1$. So we have

$$\Pi^1(R_1^\gamma) < \infty.$$

Next, we will verify $\Pi^1(M_1^\gamma \log^+ M_1) < \infty$. Making use of (3.8) and the strong Markov property we have the following

$$\Pi^1(M_1^\gamma \log^+ M_1) \leq \Pi^1(M_1^\gamma |\log(M_1)|)$$

$$= \alpha\Pi^1(W^0(R_1)^{\alpha\gamma}|\log(W^0(R_1))|)$$

$$= \alpha\Pi^1(W^0(T + S \circ \boldsymbol{\theta}_T)^{\alpha\gamma}|\log(W^0(T + S \circ \boldsymbol{\theta}_T))|)$$

$$= \alpha\Pi^1(W^0(S)^{\alpha\gamma}|\log(W^0(S))| \circ \boldsymbol{\theta}_T)$$

$$= \alpha\Pi^1(\Pi^x(W^0(S)^{\alpha\gamma}|\log(W^0(S))|)|_{x=W^0(T)})$$

$$= \alpha\Pi^1(\Pi^{-1}(|x|^{\alpha\gamma}W^0(S)^{\alpha\gamma}|\log(|x|W^0(S))|)|_{x=W^0(T)})$$

$$\leq \alpha\Pi^1(\Pi^{-1}(|x|^{\alpha\gamma}W^0(S)^{\alpha\gamma}(|\log|x|| + |\log(W^0(S))|))|_{x=W^0(T)})$$

$$= \alpha\Pi^1(|W^0(T)|^{\alpha\gamma}|\log(|W^0(T)|)|)\Pi^{-1}(W^0(S)^{\alpha\gamma})$$

$$+\alpha\Pi^1(|W^0(T)|^{\alpha\gamma})\Pi^{-1}(W^0(S)^{\alpha\gamma}|\log(W^0(S))|).$$

But we have

$$\Pi^{-1}(W^0(S)^{\alpha\gamma}|\log(W^0(S))|) = c\int_0^\infty x^{\alpha\gamma}|\log(x)|x^{-\alpha q_2}(1+x)^{-1}dx$$

$$= c\int_0^\infty |\log(x)|x^{\alpha p_1 - 1}(1+x)^{-1}dx.$$

The last integral is finite since $-1 < \alpha p_1 - 1 < 0$.

Observe that $\log(M_1)$ has a continuous distribution so is not lattice. Also, $Q_1 \geq 0$ whereas $1 - M_1$ takes on negative and positive values, so Q_1 is not a constant multiple of $(1 - M_1)$.

Using Kesten's result (Theorem 3.2.2), we have proved

Theorem 3.2.3 *Set* $R = \sum_{i=2}^{\infty} M_1 \dots M_{i-1} Q_i + Q_1$. *Then* $\Pi^1(R < \infty) = 1$. *Moreover, for some* $k > 0$

$$\Pi^1(R > t) \sim kt^{-\gamma} \quad \text{as } t \to \infty. \tag{3.11}$$

Corollary 3.2.4 *The following holds, with k as above,*

$$\Pi^1(1 - e^{-|x|^\alpha R}) \sim k\Gamma(1-\gamma)|x|^{\alpha\gamma} \quad \text{as } x \to 0. \tag{3.12}$$

Proof. Let μ be the distribution of R under Π^1. Then

$$\Pi^1(1 - e^{\theta R}) = \int_0^\infty \int_0^s \{\theta e^{-\theta t} dt\} \mu(ds) = \int_0^\infty \mu(t, \infty) \theta e^{-\theta t} dt.$$

By (3.11) we can pick k_1, k_2 with $k_1 < k < k_2$ and choose N large enough so that for $t > N$, $k_1 t^{-\gamma} < \mu(t, \infty) < k_2 t^{-\gamma}$. Then we have for $\theta > 0$,

$$\int_0^N \mu(y, \infty)\theta e^{-\theta y} dy + k_1 \int_N^\infty y^{-\gamma} \theta e^{-\theta y} dy \tag{3.13}$$

$$\leq \Pi^1(1 - e^{-\theta R})$$

$$\leq \int_0^N \mu(y, \infty)\theta e^{-\theta y} dy + k_2 \int_N^\infty y^{-\gamma} \theta e^{-\theta y} dy. \tag{3.14}$$

where the first terms in (3.13) and (3.14) are $o(\theta^\gamma)$ and the second terms are

$$k_i \theta^\gamma \int_{\theta N}^\infty y^{-\gamma} e^{-y} dy.$$

□

Now we will determine when $\Pi^1(\sigma < \infty) = 1$. Recall μ is the mean for the random walk (S_n) where $S_n = \log \Pi_n = \alpha \log W(R_n)$ and so

$$\begin{aligned}\mu = \Pi^1(\log(M_1)) &= \pi\alpha(\cot(\alpha q_2 \pi) + \cot(\alpha p_1 \pi)) \\ &= \frac{\sin(\alpha(q_2 + p_1)\pi)}{\sin(\alpha q_2 \pi)\sin(\alpha p_1 \pi)}.\end{aligned}$$

Thus, we see that

$$\mu > 0 \text{ if and only if } \alpha(p_1 + q_2) < 1,$$

$$\mu = 0 \text{ if and only if } \alpha(p_1 + q_2) = 1,$$

and

$$\mu < 0 \text{ if and only if } \alpha(p_1 + q_2) > 1.$$

If $\mu > 0$ then the random walk $S_n = \alpha \log(W^0(R_n))$ drifts to ∞, so then does $W(R_n)$. If $\mu = 0$, then consider

$$\begin{aligned}\Pi^1((\log M_1)^2) &= \Pi^1(\log(W(T + S \circ \Theta_T)^2)) \\ &\leq 2\Pi^1(\log(|W(T)|)^2) + 2\Pi^{-1}(\log(W(S))^2)\end{aligned}$$

but from (3.9)

$$\Pi^{-1}(\log(W(S)^2)) = c\int_0^\infty \log(x)^2 x^{\alpha p_1 - 1}(1+x)^{-1}dx.$$

This integral if finite since $-1 < \alpha p_1 - 1 < 0$. Similarly, $\Pi^1(\log(|W(T)|)^2) < \infty$ and so by the central limit theorem,

$$\limsup \Pi^1(S_n > 0) > 0.$$

Thus, when $\mu \geq 0$, W will not hit the origin. If $\mu < 0$, then Theorem 3.2.3 applies. In this case, from our construction and the fact that stable processes satisfy the hypotheses of Theorem 2.1.1 we can identify R with $\sigma = \inf\{t > 0 : W_t = 0\}$ and we have the following.

Theorem 3.2.5 $\Pi^1(\sigma < \infty) = 1$ *if and only if* $\mu < 0$.

Chapter 4

Extensions

This chapter has some extension results. Let X^0, with transition operators P_t^0, be a Markov process started at a point in $\mathbb{R}^0 = \mathbb{R} \setminus \{0\}$ and killed at the time, σ, when it reaches $\{0\}$. The process X^0 is called the minimal process. The extension problem is to describe all, or at least important classes of, processes X defined for all time that agree with X^0 up until time σ. To get uniqueness theorems it is customary to require that the extension has no sojourn at $\{0\}$, that is

$$P^x(\int_0^\infty \mathbf{1}_{\{0\}}(X_t)dt = 0) = 1$$

for all starting points x. Another restriction often placed on the extension, X, is that it reenters $\mathbb{R}^0$ from $\{0\}$ continuously, that is

$$P^x(\text{for some } s, X_{s-} = 0, X_s \neq 0) = 0$$

for each x. Processes with this property are said to have continuous entrance (from the distinguished state 0). At the other end of the spectrum are extensions that begin all their excursions away from $\{0\}$ by means of jumps. We will give some uniqueness of extension theorems that apply to extensions with no sojourn and continuous entrance, and a theorem concerning jumping in extensions of a particular type.

We will consider extensions of a stable process killed at zero as well as the case where the minimal process is a mixture of stable (α, β_1) and (α, β_2) processes, as constructed in Chapter 3. Naturally, it was necessary to determine conditions on α, β_1, β_2 so that σ is finite. In addition, the University of Washington thesis of Andrew Booker [5], concerning the case $\alpha = 1$, indicates that the asymptotics of Corollary 3.2.4 will play an important role in this investigation (see Theorem 4.2.3). That explains our interest in the problems of Chapter 3.

4.1 Motoo's Theory

The description of the possible extensions of a given minimal process was given, in a very general setting, by Motoo [15]. Let

$$V_\lambda^0 f(x) = E^x(\int_0^\sigma e^{-\lambda t} f(X_t)dt)$$

denote the resolvent operator for the minimal process. In Motoo's theory a key role is played by the ratio

$$\hat{H}^\lambda f(x) = \frac{V_\lambda^0 f(x)}{E^x(1 - e^{-\sigma})} \tag{4.1}$$

defined for $x \in \mathbb{R}^0, \lambda > 0$, and f bounded and Borel measurable. Note that $\hat{H}^\lambda$ depends only on the minimal process. Motoo defines $\hat{H}^\lambda f(0)$ to be the limit

$$\hat{H}^\lambda f(0) = \lim_{x \to 0} \hat{H}^\lambda f(x) \tag{4.2}$$

for those f for which the limit exists. We will call "Motoo's hypothesis" the assertion that the limit in (4.2) exists for every bounded continuous function f on $\mathbb{R}$ which vanishes at 0. In [15] Motoo proves that if $\hat{H}^\lambda f(0)$ exists for a given f and if X is an extension of X^0 with continuous entrance and no sojourn then

$$E^0(\int_0^\infty e^{-\lambda t} f(X_t)dt) = \hat{H}^\lambda f(0).$$

See V-(4.2) of [3] for a proof. In particular. if Motoo's hypothesis holds then there can be only one such extension since

$$E^x\left(\int_0^\infty e^{-\lambda t} f(X_t)dt\right) = V_\lambda^0 f(x) + E^x e^{-\lambda\sigma} \hat{H}^\lambda f(0),$$

and the right side of this equality is determined by the minimal process. and the potential operators on the left side determine the law of X.

We will show that Motoo's hypothesis holds when the minimal process is stable of type (α, β) killed at σ. Note that it does not hold for Brownian motion killed at σ, since one obtains different limits in (4.2) according to whether 0 is approached from below or above.

We also will consider a minimal process where the state space is $(0, \infty)$ and we seek extensions to $[0, \infty)$. Then in (4.2) we are considering limits as x decreases to 0.

Finally. one can see without difficulty that for applying Motoo's theorem it is necessary to verify the limit in (4.2) only for a single value of λ. say $\lambda = 1$.

4.2 A Unique Extension

Let X^0 denote X killed at zero where X is a real stable process of type (α, β) with $\alpha > 1$ and $\beta \in (-1, 1)$. The hitting time of zero is denoted by σ. We are interested in extensions with continuous entrance and no sojourn at zero.

Let U_λ denote the resolvent operator for X and let V_λ^0 denote the resolvent for X^0, i.e.,

$$U_\lambda f(x) = E^x\left(\int_0^\infty e^{-\lambda t} f(X_t)dt\right)$$

$$V_\lambda^0 f(x) = E^x\left(\int_0^\sigma e^{-\lambda t} f(X_t)dt\right).$$

The matter at hand is to verify that (4.2) exists for every $f \in C_0(\mathbb{R})$.

Two preliminary Lemmas are needed. The first is a consequence of (3.7) in Theorem 3.1.1. and the second is a special case of Corollary 3.2.4.

Lemma 4.2.1 *Let X be a real-valued stable process of type (α, β). Set $S = \inf\{t : X_t > 0\}$ and $q = P^0(X_1 > 0)$. There exists $k_1 > 0$ such that for $x \in \mathbb{R}^0$*

$$E^{-|x|}(1 - e^{-S}) \sim k_1 |x|^{\alpha q} \quad \textit{as } x \to 0. \tag{4.3}$$

Proof. Using scaling (3.4)

$$\begin{aligned} E^{-|x|}(1 - e^{-S}) &= E^{-1}(1 - e^{-|x|^\alpha S}) = \\ &= \int_0^\infty e^{-t} P^{-1}(|x|^\alpha S > t)dt = \\ &= |x|^\alpha \int_0^\infty e^{-|x|^\alpha t} P^{-1}(S > t)dt. \end{aligned}$$

In other words, the Laplace Transform of the measure with density function $P^{-1}(S > t)$ is given by

$$\frac{E^{-1}(1 - e^{-|x|S})}{|x|}.$$

By an Abelian Theorem (Theorem 4 in Feller [7, XIII.5]) the result follows from Theorem 3.1.1 since $P^{-1}(S > t) = P^0(T_1 > t)$. □

Lemma 4.2.2 *Let X be a real-valued stable process of type (α, β) with $\alpha \in (1, 2)$ and set $\sigma = \inf\{t > 0 : X_t = 0\}$. There exists $k_2 > 0$ such that for $x \in \mathbb{R}^0$*

$$E^x(1 - e^{-\sigma}) \sim k_2 |x|^{\alpha - 1} \quad \textit{as } x \to 0. \tag{4.4}$$

We can now establish the following.

Theorem 4.2.3 *Let X_t be a stable process of type (α, β) with $\alpha \in (1, 2)$. Let U_λ and V_λ^0 denote the resolvent operators for X and for X killed at $\sigma = \inf\{t : X_t = 0\}$, respectively. Then for any $f \in C_0(\mathbb{R})$*

$$\lim_{x \to 0} \frac{V_1^0 f(x)}{E^x(1 - e^{-\sigma})} = U_1 f(0). \tag{4.5}$$

Proof. Functions in $C_0(\mathbb{R})$ which are differences of bounded increasing functions generate a dense subset of $C_0(\mathbb{R})$ so it suffices to prove (4.5) for increasing f.

By the strong Markov property

$$U_1f(x) = V_1^0 f(x) + U_1f(0)E^x(e^{-\sigma}) \tag{4.6}$$

so for $x < 0$

$$V_1^0 f(x) \le U_1f(0) - U_1f(0)E^x(e^{-\sigma}).$$

Thus, it follows that

$$\limsup_{x\uparrow 0} \frac{V_1^0 f(x)}{E^x(1-e^{-\sigma})} \le U_1f(0).$$

Recall $S = \inf\{t : X_t > 0\}$ and so

$$\begin{aligned} U_1f(x) &= E^x\Big(\int_0^S e^{-t} f(X_t)dt\Big) + E^x(e^{-S}U_1f(X_S)) \\ &\ge E^x\Big(\int_0^S e^{-t} f(X_t)dt\Big) + U_1f(0)E^x(e^{-S}), \end{aligned}$$

since f is increasing. So we have

$$\begin{aligned} V_1^0 f(x) &= U_1f(x) - U_1f(0)E^x(e^{-\sigma}) \\ &\ge E^x\Big(\int_0^S e^{-t} f(X_t)dt\Big) + U_1f(0)E^x(e^{-S}) - U_1f(0)E^x(e^{-\sigma}) \\ &= E^x\Big(\int_0^S e^{-t} f(X_t)dt\Big) - U_1f(0)E^x(1-e^{-S}) + U_1f(0)E^x(1-e^{-\sigma}). \end{aligned}$$

Thus,

$$\frac{V_1^0 f(x)}{E^x(1-e^{-\sigma})} \ge \frac{E^x(\int_0^S e^{-t} f(X_t)dt)}{E^x(1-e^{-\sigma})} - U_1f(0)\frac{E^x(1-e^{-S})}{E^x(1-e^{-\sigma})} + U_1f(0).$$

Using (4.3) and (4.4) we have

$$\lim_{x\to 0} \frac{E^x(1-e^{-S})}{E^x(1-e^{-\sigma})} = k \lim_{x\to 0} \frac{|x|^{\alpha q}}{|x|^{\alpha-1}} = k \lim_{x\to 0} |x|^{1-\alpha q} = 0,$$

and so

$$\liminf_{x\uparrow 0} \frac{V_1^0 f(x)}{E^x(1-e^{-\sigma})} \ge U_1f(0).$$

Thus,

$$\lim_{x\uparrow 0}\frac{V_1^0 f(x)}{E^x(1-e^{-\sigma})}=U_1 f(0).$$

By symmetry, a similar argument yields

$$\lim_{x\downarrow 0}\frac{V_1^0 f(x)}{E^x(1-e^{-\sigma})}=U_1 f(0).$$

□

We can now establish the following extension result.

Theorem 4.2.4 *Let X be a stable process of type (α,β) with $\alpha>1$ and $\beta\in(-1,1)$. Then X has no sojourn at zero, leaves zero continuously, and is the only extension of the minimal process with these two properties.*

Proof. Since the Fourier transform of X_t is integrable X_t has a density so $P^0(X_t=0)=0$ for all $t>0$. Thus, there is no sojourn at zero. Assuming the continuous entrance has been established, the uniqueness follows from the continuity at 0 of Motoo's ratio (4.2), which we have just established. For the continuous entrance property, let $f(x,y)=1$ if $x=0, y\neq 0$ and $f(x,y)=0$ otherwise. The Lévy system formula (see VII-2(d) in [3]),

$$E^0\sum_s f(X_{s-},X_s)=E^0\int_0^\infty ds\int f(X_s,X_s+y)\mu(dy)$$

where μ is the Lévy measure for X, yields

$$E^0\sum_s f(X_{s-},X_s)=0.$$

Clearly,

$$\sum_{s\in G}\mathbf{1}_{\{0\}^c}(X_s)\le\sum_s f(X_{s-},X_s)$$

and so $\hat{P}(X_0\neq 0)=0$ as required. □

We also have the following.

Theorem 4.2.5 *Let X be a stable process of type $(\alpha, 0)$ with $\alpha > 1$. Then $|X|$ has no sojourn at zero, leaves zero continuously, and is the only extension of the minimal process ($|X|$ killed at σ) with these two properties.*

Proof. The monotoneity argument in the previous proof will not suffice. Let U_λ and V_λ^0 denote the resolvent operators for $|X|$ and $|X|$ killed at σ, respectively. Using the strong Markov property we have for $x > 0$

$$U_1 f(x) = V_1^0 f(x) + E^x(e^{-\sigma}) U_1 f(0)$$

and by rearranging terms

$$\frac{U_1 f(x) - U_1 f(0)}{E^x(1 - e^{-\sigma})} = \frac{V_1^0 f(x)}{E^x(1 - e^{-\sigma})} - U_1 f(0).$$

Now for continuous f with compact support and continuous derivative

$$|U_1 f(x) - U_1 f(0)| \leq \int_0^\infty e^{-t} E^0(|f(|X_t + x|) - f(|X_t|)|dt = O(x)$$

while $E^x(1 - e^{-\sigma}) \sim k|x|^{1-1/\alpha}$ where $1 - 1/\alpha \in (0, \frac{1}{2})$. Thus,

$$\lim_{x \downarrow 0} \frac{V_1^0 f(x)}{E^x(1 - e^{-\sigma})} = U_1 f(0).$$

Motoo's ratio is continuous at zero and, as in the above proof, we see that there is continuous entrance and no sojourn at zero. □

Note that this argument would have sufficed for Theorem 4.2.3. We included the argument given there with a view to other possible applications.

4.3 A Jumping In Extension

In this section we consider extensions of W^0 whose excursions from zero begin with a jump into $\mathbb{R}^0$. Recall that W^0 is constructed from stables of type (α, β_1) and (α, β_2) with $\Pi^x(\sigma < \infty) = 1$, for all $x \in \mathbb{R}^0$. Set $\gamma = p_1 + q_2 - 1/\alpha > 0$. Recall that W^0 has a scaling property (Lemma 3.2.1) at the "jump over" times s_i. In fact, (W^0, Π^x) has scaling, i.e., the processes $(c^{1/\alpha}W^0_{t/c}, \Pi^x)$ and $(W^0_t, \Pi^{c^{1/\alpha}x})$ are equal in law.

To see this, note that having established the existence of (W^0, Π^x) it can be realized in the following way: take an independent family of processes $(X^n, Y^m)_{n\geq 1, m\geq 1}$ where the X's are stable of type (α, β_1) starting at 0 and the Y's are stable of type (α, β_2) starting at 0. Then given an x, say $x > 0$, define as follows

$$W^0_t = \begin{cases} X^1_t + x & \text{if } 0 \leq t < T_1 \\ & T_1 = \inf\{t | X^1_t < -x\} \\ Y^1_{t-T_1} + X^1_{T_1} + x & \text{if } T_1 \leq t < T_1 + S_1 \\ & S_1 = \inf\{r | Y^1_r > -(X^1_{T_1} + x)\} \\ X^2_{t-(T_1+S_1)} & \text{if } T_1 + S_1 \leq t < T_1 + S_1 + T_2 \\ \quad +Y^1_{S_1} + X^1_{T_1} + x & \\ & T_2 = \inf\{t | X^2_t < -(Y^1_{S_1} + X^1_{T_1} + x)\} \\ \cdot & \\ \cdot & \\ \cdot & \end{cases}$$

Call Π^x the law of that process.

The function $c^{1/\alpha}W^0_{t/c}$ is defined exactly as was the function W^0_t except that x is replaced by $c^{1/\alpha}x$ and X^i_t is replaced by $c^{1/\alpha}X^i_{t/c}$ and Y^i_t is replaced by $c^{1/\alpha}Y^i_{t/c}$. Since that does not change the laws of the building blocks, we see that W^0 has scaling.

For η a measure on $\mathbb{R}^0$, define Π^η by

$$\Pi^\eta(A) = \int_{\mathbb{R}^0} \Pi^x(A)\eta(dx). \tag{4.7}$$

We are interested in the existence of scale invariant extensions W (with laws P^x) of W^0 with excursion measure of the form (4.7). A few observations are in order.

First, note that $\hat{P}$ is an excursion measure for the excursions away from zero for a process W if (recall the notation from Section 1.3)

$$E^0(\sum_{s\in G} Z_s F\circ\theta_s) = \hat{P}(F)E^0(\int_0^\infty Z_s dL_s) \tag{4.8}$$

where L is local time at zero normalized so that $E^0\int_0^\infty e^{-s}dL_s = 1$. If we take $Z_s = e^{-\lambda s}$ and $F = \int_0^\sigma e^{-\lambda t}g(X_t)dt$ the left side of (4.8) is

$$\begin{aligned} &E^0\sum_{s\in G} e^{-\lambda s}\int_0^{\sigma\circ\theta_s} e^{-\lambda t}g(W_{t+s})dt \\ &= E^0\sum_{s\in G}\int_s^{s+\sigma\circ\theta_s} e^{-\lambda r}g(W_r)dr \\ &= U_\lambda\bar{g}(0) \end{aligned}$$

where $\bar{g} = g\mathbf{1}_{\{0\}^c}$. Thus, with this choice of Z_s and F, (4.8) becomes

$$U_\lambda\bar{g}(0) = \hat{P}(\int_0^\sigma e^{-\lambda t}g(W_t)dt)E^0(\int_0^\infty e^{-\lambda s}dL_s). \tag{4.9}$$

Taking $\lambda = 1$ we get

$$U_1\bar{g}(0) = \hat{P}(\int_0^\sigma e^{-t}g(W_t)dt).$$

Thus, $\hat{P}$ determines $U_1 g(0)$, (since $U_1\mathbf{1}(0) = 1$), and hence $\hat{P}$ determines the law of W.

Taking $g \equiv 1$, we see that $\hat{P}(1-e^{-\sigma}) \leq 1$, and $\hat{P}(1-e^{-\sigma}) = 1$ if and only if W has no sojourn at 0.

Second, a characteristic measure compatible with a minimal process (W^0, Π^x) is a measure n on path space such that relative to n the coordinate process is Markovian with Π^x as laws, killed at σ, and $n(1-e^{-\sigma}) < \infty$, i.e.,

$$n(uv\circ\theta_t; t < \sigma) = n(\Pi^{W_t}(v); u\cap\{t<\sigma\}),$$

where u is $\sigma\{W_r^0 : r \leq t\}$ measurable.

According to a theorem of Salisbury [19], if n is a characteristic measure satisfying certain other conditions then there is an extension W of the minimal process (necessarily unique in law) having n as the excursion measure for the excursions away from zero.

Let η be a measure such that $||\eta|| = \infty$ and $\Pi^\eta(1 - e^{-\sigma})$ is finite. Clearly Π^η is a characteristic measure. Note that $\Pi^\eta(\sigma = 0) = \int \Pi^x(\sigma = 0)\eta(dx)$ and so $\Pi^\eta(\sigma = 0) = 0$. The characteristic measure Π^η satisfies the following additional conditions:

1. Π^η is σ-finite,
2. $||\Pi^\eta|| = \infty$,
3. If n' is a characteristic measure and $n' \leq \Pi^\eta$ then n' attributes no mass to the set of paths which start at 0,
4. If V is any neighborhood of 0 then

$$\Pi^\eta(W_t^0 \notin V \text{ for some } t < \sigma) < \infty.$$

Condition (1) follows from $\infty > \Pi^\eta(1 - e^{-\sigma}) \geq (1 - e^{-\epsilon})\Pi^\eta(\sigma \geq \epsilon)$. Condition (2) is clear since $||\Pi^x|| = 1$ for all x and $||\eta|| = \infty$. Condition (3) is clear since $\Pi^\eta(W^0(0) = 0) = 0$. As to condition (4), let $\tau = \inf\{t|W_t^0 \notin V\}$. Note that there exists $\delta > 0$ such that $\Pi^x(1 - e^{-\sigma}) \geq \delta$ for all $x \notin V$ since for $x > 0$, $\Pi^x(1-e^{-\sigma}) \geq \Pi^x(1-e^{-T})$ and for $x < 0$, $\Pi^x(1-e^{-\sigma}) \geq \Pi^x(1-e^{-S})$ and by scaling these increase with $|x|$. So we have

$$\begin{aligned}\Pi^\eta(1 - e^{-\sigma}) &\geq \Pi^\eta(1 - e^{-\sigma}; \tau < \sigma)\\ &= \Pi^\eta(\Pi^{W^0(\tau)}(1 - e^{-\sigma}); \tau < \sigma)\\ &\geq \delta \cdot \Pi^\eta(\tau < \sigma),\end{aligned}$$

and so

$$\Pi^\eta(\tau < \sigma) < \infty. \tag{4.10}$$

These conditions are sufficient conditions for Salisbury's theorem. According to his theorem if we normalize η so that $\Pi^{\eta}(1-e^{-\sigma}) \leq 1$, there is a recurrent extension of W^0 and the excursion measure of the extension is Π^{η}.

Now let us take η of the form

$$\eta(dx) = \frac{c_1 \mathbf{1}_{(-\infty,0)}(dx)}{|x|^{\epsilon}} + \frac{c_2 \mathbf{1}_{(0,\infty)}(dx)}{|x|^{\epsilon}}.$$

$\Pi^{\eta}(1-e^{-\sigma})$ will be finite provided $\epsilon > 1$ and since $\Pi^{x}(1-e^{-\sigma}) \sim k|x|^{\alpha\gamma}$ for x near zero, provided $\alpha\gamma - \epsilon > -1$ or $\epsilon < \alpha(p_1+q_2)$. We have the following.

Theorem 4.3.1 *Let W^0 be a mixture of stable processes of type (α, β_1) and (α, β_2) such that $\Pi^{x}(\sigma < \infty) = 1$ for all $x \in \mathbb{R}^0$. For $\epsilon \in (1, \alpha(p_1+q_2))$ define $\eta(dx)$ by*

$$\eta(dx) = \frac{c_1 \mathbf{1}_{(-\infty,0)}(dx)}{|x|^{\epsilon}} + \frac{c_2 \mathbf{1}_{(0,\infty)}(dx)}{|x|^{\epsilon}} \tag{4.11}$$

with c_1 and c_2 such that $\Pi^{\eta}(1-e^{-\sigma}) = 1$. Then Π^{η} is the excursion measure of a recurrent scale invariant extension of W^0 which has no sojourn at zero. Conversely, a recurrent scale invariant extension of W^0 with no sojourn at zero and with excursion measure given by Π^{η} must have η of the form (4.11) with $\epsilon \in (1, \alpha(p_1+q_2))$.

Proof. For the first half of the proof, it follows from the discussion preceeding the statement of the theorem that for $\eta(dx)$ of the form (4.11) there is an extension with excursion measure Π^{η} so it suffices to verify that the extension is scale invariant.

Scaling can be characterized in terms of the resolvent operators for the extension, i.e., for $a > 0$

$$U_\lambda f(x) = \frac{1}{a} U_{\lambda/a} f \circ a^{-\frac{1}{\alpha}}(a^{\frac{1}{\alpha}} x),$$

and since $U_\lambda f(x)$ is determined by the minimal process and $U_\lambda f(0)$, i.e.,

$$U_\lambda f(x) = V_\lambda^{0} f(x) + \Pi^{x}(e^{-\lambda\sigma}) U_\lambda f(0), \tag{4.12}$$

it suffices to verify

$$U_\lambda f(0) = \frac{1}{a} U_{\lambda/a} f \circ a^{-\frac{1}{\alpha}}(0) \tag{4.13}$$

to see that the extension is scale invariant.

It will be convenient to introduce Rogers's [17] operators n_λ, the Laplace transform of the entrance law, or equivalently

$$n_\lambda(f) = \Pi^\eta(\int_0^\sigma e^{-\lambda t} f(W_t)dt).$$

Note that n_λ satisfies the "resolvent equation", i.e., for $\lambda > 0$, $\mu > 0$ and $\lambda \neq \mu$

$$n_\lambda V_\mu^0 = \frac{n_\lambda - n_\mu}{\mu - \lambda}. \tag{4.14}$$

Taking $g \equiv 1$ in (4.9) and multiplying by λ yields

$$1 = \lambda n_\lambda(\mathbf{1})\Pi^0 \int_0^\infty e^{-\lambda s} dLs.$$

In other words,

$$U_\lambda f(0) = \frac{n_\lambda(f)}{\lambda n_\lambda(\mathbf{1})}.$$

Thus, we have the following

$$\begin{aligned}
n_\lambda(f) &= \Pi^\eta(\int_0^\sigma e^{-\lambda t} f(W_t)dt) \\
&= \int_{\mathbb{R}^0} \Pi^x(\int_0^\sigma e^{-\lambda t} f(W_t)dt)\eta(dx) \\
&= \int_{\mathbb{R}^0} \Pi^{a^{1/\alpha}x}(\int_0^{\frac{1}{a}\sigma} e^{-\lambda t} f(a^{-\frac{1}{\alpha}} W_{at})dt)\eta(dx) \\
&= \int_{\mathbb{R}^0} \Pi^x(\int_0^\sigma e^{-\frac{\lambda}{a}t} f(a^{-\frac{1}{\alpha}} W_t)dt)\eta(dx)a^{-1+(\epsilon-1)/\alpha} \\
&= a^{-1+(\epsilon-1)/\alpha}\Pi^\eta(\int_0^\sigma e^{-\frac{\lambda}{a}} f \circ a^{-\frac{1}{\alpha}}(W_t)dt) \\
&= a^{-1+(\epsilon-1)/\alpha} n_{\lambda/a}(f \circ a^{-\frac{1}{\alpha}}).
\end{aligned}$$

And so

$$\begin{aligned}
\frac{1}{a} U_{\lambda/a} f \circ a^{-\frac{1}{\alpha}}(0) &= \frac{n_{\lambda/a}(f \circ a^{-\frac{1}{\alpha}})}{\lambda n_{\lambda/a}(\mathbf{1})} \\
&= \frac{n_\lambda(f)}{\lambda n_\lambda(\mathbf{1})}
\end{aligned}$$

$$= U_\lambda f(0).$$

It follows that the extension is scale invariant.

Now assume that Y is a recurrent scale invariant extension with excursion measure given by (4.7). From scaling, for any $a > 0$ (4.13) holds, so that

$$\frac{n_\lambda(f)}{\lambda n_\lambda(\mathbf{1})} = \frac{n_{\lambda/a}(f \circ a^{-\frac{1}{\alpha}})}{\lambda n_{\lambda/a}(\mathbf{1})}. \tag{4.15}$$

In particular, with $f = 1_A$

$$n_\lambda(A) = \frac{n_\lambda(\mathbf{1})}{n_{\lambda/a}(\mathbf{1})} n_{\lambda/a}(a^{\frac{1}{\alpha}} A). \tag{4.16}$$

Using (4.16) and the (4.14) we have the following:

$$\frac{n_\lambda(\mathbf{1}) - n_\mu(\mathbf{1})}{\mu - \lambda} = n_\lambda V^{0}_\mu(\mathbf{1}) = \int_{\mathbb{R}^0} V^{0}_\mu(\mathbf{1})(x) n_\lambda(dx)$$

$$= \int_{\mathbb{R}^0} \frac{1}{a} V^{0}_{\mu/a}(\mathbf{1})(a^{\frac{1}{\alpha}} x) \frac{n_\lambda(\mathbf{1})}{n_{\lambda/a}(\mathbf{1})} n_{\lambda/a}(d(a^{\frac{1}{\alpha}} x))$$

$$= \frac{1}{a} \frac{n_\lambda(\mathbf{1})}{n_{\lambda/a}(\mathbf{1})} \int_{\mathbb{R}^0} V^{0}_{\mu/a}(\mathbf{1})(x) n_{\lambda/a}(dx)$$

$$= \frac{1}{a} \frac{n_\lambda(\mathbf{1})}{n_{\lambda/a}(\mathbf{1})} n_{\lambda/a} V^{0}_{\mu/a}(\mathbf{1})$$

$$= \frac{1}{a} \frac{n_\lambda(\mathbf{1})}{n_{\lambda/a}(\mathbf{1})} \frac{n_{\lambda/a}(\mathbf{1}) - n_{\mu/a}(\mathbf{1})}{\mu/a - \lambda/a}.$$

Simplifying the above

$$n_\mu(\mathbf{1}) n_{\lambda/a}(\mathbf{1}) = n_\lambda(\mathbf{1}) n_{\mu/a}(\mathbf{1}).$$

Thus, as a function of λ, we must have $n_\lambda(\mathbf{1}) = k\lambda^\delta$ for some δ, and (4.15) becomes

$$n_\lambda(f) = a^\delta n_{\lambda/a}(f \circ a^{-\frac{1}{\alpha}}).$$

In terms of the excursion measure

$$\Pi^\eta\left(\int_0^\sigma e^{-\lambda t} f(W_t) dt\right) = a^\delta \Pi^\eta\left(\int_0^\sigma e^{-\frac{\lambda}{a} t} f(a^{-\frac{1}{\alpha}} W_t) dt\right).$$

Thus, we have

$$\begin{aligned}
&\int_{\mathbb{R}^0} \Pi^x\Big(\int_0^{\sigma} e^{-\lambda t} f(W_t)dt\Big)\eta(dx) \\
&= a^{\delta} \int_{\mathbb{R}^0} \Pi^x\Big(\int_0^{\sigma} e^{-\frac{\lambda}{a}t} f(a^{-\frac{1}{\alpha}}W_t)dt\Big)\eta(dx) \\
&= a^{\delta+1} \int_{\mathbb{R}^0} \Pi^x\Big(\int_0^{\sigma/a} e^{-\lambda t} f(a^{-\frac{1}{\alpha}}W_{at})dt\Big)\eta(dx) \\
&= a^{\delta+1} \int_{\mathbb{R}^0} \Pi^{a^{1/\alpha}x}\Big(\int_0^{\sigma/a} e^{-\lambda t} f(a^{-\frac{1}{\alpha}}W_{at})dt\Big)\eta(d(a^{\frac{1}{\alpha}}x)) \\
&= a^{\delta+1} \int_{\mathbb{R}^0} \Pi^x\Big(\int_0^{\sigma} e^{-\lambda t} f(W_t)dt\Big)\eta(d(a^{\frac{1}{\alpha}}x)).
\end{aligned}$$

In other words, with $\eta_a(dx) = a^{\delta+1}\eta(d(a^{\frac{1}{\alpha}}x))$,

$$\int \eta(dx)V_\lambda^0 f(x) = \int \eta_a(dx)V_\lambda^0 f(x).$$

Now for continuous f with compact support

$$\begin{aligned}
\lambda V_\lambda^0 f(x) &= \Pi^x\Big(\int_0^{\sigma} \lambda e^{-\lambda t} f(W_t)dt\Big) \\
&= \Pi^x\Big(\int_0^{\lambda\sigma} e^{-t} f(W_{t/\lambda})dt\Big),
\end{aligned}$$

which clearly goes to $f(x)$ as $\lambda \to \infty$, and does so boundedly since $||\lambda V_\lambda^0 f|| \leq ||f||$. Furthermore, for V a neighborhood of 0 such that f vanishes inside V and with $||f|| \leq 1$, and setting $\tau = \sigma_{V^c}$, we have

$$\lambda V_\lambda^0 f(x) \leq \Pi^x\Big(\int_0^{\sigma} e^{-\lambda t}dt; \tau < \sigma\Big) \leq \Pi^x(\tau < \sigma).$$

But, from (4.10)

$$\int \Pi^x(\tau < \sigma)\eta(dx) < \infty.$$

It is straightforward to see that $\Pi^x(\tau < \sigma)$ is also integrable with respect to $\eta_a(dx)$. Thus it must be that $\eta(dx) = a^{\delta+1}\eta(d(a^{\frac{1}{\alpha}}x))$, i.e., η is scale invariant, so it follows that

$$\eta(dx) = \frac{c_1 \mathbf{1}_{(-\infty,0)}(dx)}{|x|^{\epsilon}} + \frac{c_2 \mathbf{1}_{(0,\infty)}(dx)}{|x|^{\epsilon}}$$

where $\epsilon = \alpha(\delta + 1) + 1$. As before, ϵ needs to be in the interval $(1, \alpha(p_1 + q_2))$ for $\Pi^\eta(1 - e^{-\sigma})$ to be finite. $\square$

Bibliography

[1] N.H. Bingham. Maxima of sums of random variables and suprema of stable processes. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, 26(2):273–296, 1973.

[2] Robert M. Blumenthal. On construction of Markov processes. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, 63:433–444, 1983.

[3] Robert M. Blumenthal. *Excursions of Markov Processes.* Birkhäuser, Boston, 1993.

[4] Robert M. Blumenthal and Ronald Getoor. *Markov Processes and Potential Theory.* Academic Press, New York, 1968.

[5] Andrew Booker. *Construction of Some Reflecting Brownian Motions.* Ph.D. Thesis, University of Washington, 1986.

[6] K. Bruce Erickson. Continuous extensions of skew product diffusions. *Prob. Th. Rel. Fields*, 85:73–89, 1990.

[7] William Feller. *An Introduction to Probability Theory and Its Applications, vol. II.* John Wiley & Sons, Inc., New York, 1968.

[8] Ronald Getoor. Excursions of Markov processes. *Annals of Probability*, 7(2):244–266, 1979.

[9] K. Itô. Poisson point processes attached to Markov processes. *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability*, pages 225–239, 1970.

[10] K. Itô and H. P. McKean. Brownian motions on a half-line. *Ill. J. Math.*, 7:181–231, 1963.

[11] Harry Kesten. Random difference equations and renewal theory for products of random matrices. *Acta Mathematica*, 131:207–248, 1973.

[12] John Lamperti. Semi-stable Markov processes. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, 22(2):205–225, 1972.

[13] P.A. Meyer. Renaissance, recollements, mélanges, ralentissement de processus de Markov. *Ann. Inst. Fourier, Grenoble*, 25:465–497, 1975.

[14] P.W. Millar. Exit properties of stochastic processes with stationary independent increments. *Trans. Amer. Math. Soc.*, 178:459–479, 1973.

[15] M. Motoo. Applications of additive functionals to the boundary problem of Markov processes. *Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability*, II:75–110, 1965.

[16] Daniel Ray. Stable processes with an absorbing barrier. *Trans. Amer. Math. Soc.*, 89:16–24, 1958.

[17] L.C.G. Rogers. Itô excursion theory via resolvents. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, 63:237–255, 1983.

[18] B.A. Rogozin. The distribution of the first ladder moment and height and fluctuation of a random walk. *Theory of Probability and its Applications*, 16(4):575–595, 1971.

[19] T. S. Salisbury. On the Itô excursion process. *Prob. Th. Rel. Fields*, 73:319–350, 1986.

[20] J. Vuolle-Apiala. Itô excursion theory for self-similar Markov processes. *Annals of Probability*, 22(2):546–565, 1992.

[21] Shinzo Watanabe. On stable processes with boundary conditions. *J. Math. Soc. Japan*, 14:170–198, 1962.

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