

NON-POLAR POINTS FOR REFLECTED BROWNIAN MOTION

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1. Introduction and main results. Our main results are

- (i) a new construction of reflected Brownian motion X in a half-plane with non-smooth angle of oblique reflection and
- (ii) a theorem on existence of some “exceptional” points on the paths of the standard 2-dimensional Brownian motion.

The link between these two seemingly disparate results will be formed by some theorems related to the following question

- (iii) Which points are hit by the reflected Brownian motion X with positive probability?

In order to state our results we introduce some notation. Let $D_* = \{x \in \mathbb{C} : \text{Im } x > 0\}$, identify \mathbb{R}^2 with \mathbb{C} and ∂D_* with \mathbb{R} and suppose that $\theta : \mathbb{R} \rightarrow (-\pi/2, \pi/2)$ is C^2 . Then we may construct a reflected Brownian motion in D_* with the oblique angle of reflection θ in the following way (see, e.g., Rogers (1991)).

Let $Y = Y_1 + iY_2$ be a standard 2-dimensional Brownian motion, $Y(0) \in \overline{D}_*$, and let

$$L_t \stackrel{\text{df}}{=} \max\left(-\inf_{s \leq t} Y_2(s), 0\right),$$

$$(1.1) \quad X_2(t) \stackrel{\text{df}}{=} Y_2(t) + L_t.$$

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Then the equation

$$(1.2) \quad X_1(t) = Y_1(t) + \int_0^t \tan \theta(X_1(s)) dL_s$$

has a solution, possibly exploding in a finite time R . The process $X(t) = X_1(t) + iX_2(t)$, $t \in [0, R)$, is the reflected Brownian motion in D_* with the angle of reflection $\theta(x)$ measured in the clockwise direction from the inward pointing normal at x .

We will consider the space of RCLL (right continuous with left limits) functions equipped with the M_1 topology defined by Skorohod (1960). The original definition deals with the functions defined on $[0, 1]$ and can be extended in an obvious way to an arbitrary non-random interval $[0, t]$. We have to further extend the definition to functions defined on random time intervals. We will say that processes X_k converge a.s. to X in M_1 topology if

- (i) X and $X_k, k \geq 1$, are RCLL;
- (ii) X is defined on a random time interval $[0, R)$, each process X_k is defined on an interval $[0, R_k)$ and $R_k \rightarrow R$ a.s.; and
- (iii) for every fixed $\varepsilon > 0$ and $a < \infty$, X_k converges to X in the M_1 -topology of Skorohod (1960) on the interval $[0, 0 \vee ((R - \varepsilon) \wedge a)]$ a.s.

Note that for every Borel measurable function $\theta : \mathbb{R} \rightarrow [-\pi/2, \pi/2]$ there is a sequence of C^2 functions $\theta_k : \mathbb{R} \rightarrow (-\pi/2, \pi/2)$ which converges to θ almost everywhere.

Theorem 1.1. *Suppose that $\theta : \mathbb{R} \rightarrow [-\pi/2, \pi/2]$ is Borel measurable and the functions $\theta_k : \mathbb{R} \rightarrow (-\pi/2, \pi/2)$ are C^2 and converge almost everywhere to θ as $k \rightarrow \infty$. Then, on some probability space, there exists a sequence of processes $\{X_k\}_{k \geq 1}$ such that*

- (i) for each k , X_k is a reflected Brownian motion with the angle of reflection θ_k and
- (ii) X_k converge a.s. to a process X in M_1 topology.

Note that if X and \tilde{X} are the limit processes corresponding to two different approximating sequences $\{\theta_k\}$ and $\{\tilde{\theta}_k\}$ then the distributions of X and \tilde{X} are identical since the theorem may be applied to the sequence $\{\theta_1, \tilde{\theta}_1, \theta_2, \tilde{\theta}_2, \dots\}$.

We will consider Theorem 1.1 as the definition of the reflected Brownian motion with an arbitrary measurable angle of reflection $\theta : \mathbb{R} \rightarrow [-\pi/2, \pi/2]$. We are not going to present in this paper any other characterization of X as a reflected Brownian motion, for example, as a solution to a submartingale problem or as a solution to equations analogous to (1.1)-(1.2). We plan to discuss these questions in a future article. However, we will briefly indicate in Remark 2.2 how it may be proved that the process X is strong Markov.

The process X is continuous if and only if there are no non-degenerate intervals on which we have almost everywhere $\theta = \pi/2$ or $\theta = -\pi/2$; this will be easily seen from our proof of Theorem 1.1.

If the limiting process X is continuous then X_k converge to X in the topology stronger than M_1 , namely uniformly on $[0, 0 \vee ((R - \varepsilon) \wedge a)]$. If X is discontinuous then the convergence does not hold in \mathbf{J}_1 or \mathbf{J}_2 topologies of Skorohod (1960) (which are stronger than M_1) because continuous functions cannot converge to a discontinuous one in either one of these topologies.

Now we turn to exceptional points on Brownian paths. We will need some more definitions and notation. A domain (i.e. an open and connected set) $D \subset \mathbb{C}$ will be called *monotone* if $x + ib \in D$ whenever $x \in D$ and $b > 0$.

For a Greenian domain D , its Green function will be denoted $G_D(\cdot, \cdot)$. For a monotone domain D , let $\tilde{D} = \{z \in D : |\operatorname{Re} z| < 1\}$ and $I = \{z \in D : \operatorname{Re} z = 0\}$. Let $h = h_D$ be defined by

$$(1.3) \quad h(x) = |\operatorname{Re} x| + \int_{iy \in I} G_{\tilde{D}}(x, iy) \frac{dy}{\pi} \quad \text{for } x \in \tilde{D}.$$

The function h may be characterized as the (unique) bounded harmonic function in \tilde{D} which is equal to $|\operatorname{Re} x|$ on $\partial\tilde{D}$ (see Lemma 3.1).

For $\varepsilon > 0$ let $d_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$ be the smallest non-increasing non-negative function satisfying

$$\{x \in \partial D : |x| < \varepsilon\} \subset \{x \in \mathbb{C} : |x| < \varepsilon, \operatorname{Re} x \leq d_\varepsilon(\operatorname{Im} x)\}.$$

Suppose that D is a monotone domain and $x \in D$. Then $x + ai$ converges as $a \rightarrow \infty$ to a prime end (independent of x) which we will call ∞ . Let $I_* \stackrel{\text{df}}{=} \{z \in \mathbb{C} : \operatorname{Re} z = 0, \operatorname{Im} z > 0\}$.

If $0 \in \partial D$ and $I_* \subset D$ then ai converges as $a \downarrow 0$ to a prime end which will be denoted 0 (see Pommerenke (1975) for the definition of prime ends).

Suppose Y is a standard 2-dimensional Brownian motion and D is a monotone domain such that $0 \in \partial D$ and $I_* \subset D$. We will say that $Y(t_0)$ is a D -point if $t_0 > 0$ and there exists $t_1 \in (0, t_0)$ such that $\operatorname{Re} Y(t_0) = 0$ and $Y(t_1, t_0) \subset D + Y(t_0)$. Here $D + x = \{z \in \mathbb{C} : \exists y \in D \text{ such that } y + x = z\}$. We will call $Y(t_0)$ a *right* D -point if $Y(t_0)$ is a D -point and for some $t_1 \in (0, t_0)$ we have $\operatorname{Re} Y(s) > 0$ for all $s \in (t_1, t_0)$.

Theorem 1.2. *Suppose D is a monotone domain, $0 \in \partial D$ and $I_* \subset D$.*

- (i) *The standard 2-dimensional Brownian motion has D -points with positive probability (and, therefore, with probability 1) if and only if*

$$(1.4) \quad \limsup_{\varepsilon \rightarrow 0^+} h(\varepsilon i)/G_D(i, \varepsilon i) < \infty.$$

- (ii) *The standard 2-dimensional Brownian motion has right D -points with positive probability (and, therefore, with probability 1) if and only if for some $\varepsilon > 0$*

$$(1.5) \quad \int_{-1}^0 d_\varepsilon(r) r^{-2} dr < \infty.$$

We will show in Remark 3.2 below that (1.4) fails if and only if

$$(1.6) \quad \lim_{\varepsilon \rightarrow 0^+} h(\varepsilon i)/G_D(i, \varepsilon i) = \infty.$$

It is not easy to verify (1.4) for an arbitrary domain so let us point out some cases when this can be done. Suppose that $D = \{z \in \mathbb{C} : \operatorname{Im} z \geq a \operatorname{Re} z\}$ for some $a \in \mathbb{R}$. Then D -points do not exist by Theorem 2.1 (iii) of Burdzy (1989); one can also relatively easily check that (1.4) is not satisfied. The property of having D -points is monotone in D so we may want to consider domains of the form

$$(1.7) \quad D = \{z \in \mathbb{C} : \operatorname{Im} z \geq a \operatorname{Re} z - g(\operatorname{Re} z)\}$$

where $a \in \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ is a non-negative Lipschitz function such that $g(0) = 0$. If

$$(1.8) \quad \int_{-1}^1 g(r)r^{-2}dr < \infty$$

then D has an angular derivative at 0 (see Burdzy (1987, Definition 9.1)) and we may show as in the proof of Theorem 1.4 (i), using Theorems 8.2 and 9.2 of Burdzy (1987), that (1.4) does not hold; the converse is not true, i.e., (1.4) need not be satisfied for domains of the form (1.7) even if (1.8) fails. It would take too much space to present an example to this effect. If $g(r) = a_1|r|$ for some $a_1 > 0$ then D -points exist by Theorem 2.1 of Burdzy (1989).

Suppose for a moment that $\tilde{D} = D$. The ratio $G_D(\varepsilon i, y)/G_D(i, \varepsilon i)$ converges to the Martin kernel $K(0, y)$ in D as $\varepsilon \rightarrow 0$ (see Doob (1984)). Hence, by the Fatou lemma, (1.4) implies

$$\int_I K(0, y)dy < \infty.$$

It seems that the converse is false. Let $M = f^{-1}(\{z \in D_* : \operatorname{Re} z = 0\})$ where f is a function defined as in Lemma 2.1. Then

$$\liminf_{\substack{x \rightarrow 0 \\ x \in M}} \frac{|x|}{G_D(i, x)} + \int_I K(0, y)dy < \infty$$

seems to be equivalent to (1.4). Our proof is long and complicated and therefore we omit it.

Measurable functions $\theta : \mathbb{R} \rightarrow [-\pi/2, \pi/2]$ correspond to monotone domains $D \subsetneq \mathbb{C}$. Lemmas 2.1 and 2.2 describe precisely the nature of this correspondence and we briefly discuss it here. For a monotone domain $D \subsetneq \mathbb{C}$ there exists a univalent function f mapping D_* onto D and such that for almost all $x \in \partial D_*$, $f'(x)$ exists in the sense of the angular limit and belongs to $[-\pi/2, \pi/2]$. We let $\theta(x) = \arg f'(x)$. Conversely, if $\theta : \mathbb{R} \rightarrow [-\pi/2, \pi/2]$ is given then there exists a univalent function f which maps D_* onto a monotone domain and such that $\arg f'(x) = \theta(x)$ for almost all x . It should be emphasized that the correspondence between θ 's and D 's is not one-to-one. Our next result exploits this relationship

between θ 's and D 's. Let

$$T_A = T(A) = \inf\{t > 0 : X(t) \in A\}$$

with the convention $\inf \emptyset = \infty$.

Theorem 1.3. *Suppose that*

(a) *a measurable function $\theta : \mathbb{R} \rightarrow [-\pi/2, \pi/2]$ is given and a monotone domain D and a function $f : D_* \rightarrow D$ are defined as in Lemma 2.2; if it is possible, D and f are chosen so that $0 \in \partial D$, $I_* \subset D$ and $f(0) = 0$;*

or

(b) *a monotone domain D is given; $0 \in \partial D$, $I_* \subset D$ and $f : D_* \rightarrow D$ and $\theta(x) = \arg f'(x)$ are chosen as in Lemma 2.1 so that $f(0) = 0$.*

Under any of these assumptions we have the following.

- (i) *The reflected Brownian motion in D_* with the oblique angle of reflection θ hits 0 with positive probability if and only if $0 \in \partial D$, $I_* \subset D$ and the standard 2-dimensional Brownian motion has D -points a.s.*
- (ii) *The reflected Brownian motion X in D_* with the oblique angle of reflection θ hits 0 with positive probability and approaches it from the right, i.e.,*

$$(1.9) \quad P(T_{\{0\}} < \infty \text{ and } \exists \varepsilon > 0 : \operatorname{Re} X(t) > 0 \text{ for all } t \in (T_{\{0\}} - \varepsilon, T_{\{0\}})) > 0$$

if and only if $0 \in \partial D$, $I_ \subset D$ and the standard 2-dimensional Brownian motion has right D -points a.s.*

Smooth θ 's correspond to D 's with smooth boundaries and vice versa. Hence, it may be interesting to have a look at D 's with highly irregular boundaries, for example, at a domain D which lies above a "typical" graph of a 1-dimensional Brownian motion. What can we say about the corresponding θ and reflected Brownian motion X in this case? Theorem 1.4 (iii) and its proof provide some information about this process.

Suppose that reflected Brownian motion X in D_* with the oblique angle of reflection θ satisfies $X(0) \in D_*$ a.s. Let \mathcal{A} be the set of all non-polar points for X , i.e., the set of all points $x \in \partial D_*$ such that

$$P(T_{\{x\}} < \infty) > 0.$$

How large is the set \mathcal{A} ? The answer depends on θ and we start our discussion with a few examples.

Example 1.1. (i) If $\theta(x) = 0$ for all x then X has the normal reflection at ∂D_* and it is well known that $\mathcal{A} = \emptyset$.

(ii) Fix some $a > 0$ and let $\theta(0) = 0$, $\theta(x) = a$ for $x < 0$ and $\theta(x) = -a$ for $x > 0$. Then $\mathcal{A} = \{0\}$, by the results of Varadhan and Williams (1985).

(iii) An easy modification of the previous example shows that for some θ , \mathcal{A} may be equal to the set of all integers.

Can \mathcal{A} be uncountably infinite? Can \mathcal{A} be equal to ∂D_* ?

Let λ denote the Lebesgue measure on ∂D_* .

Theorem 1.4. (i) $\lambda(\{x \in \mathcal{A} : \theta(x) \in (-\pi/2, \pi/2)\}) = 0$ for every θ .

(ii) For every θ , the set $\partial D_* \setminus \mathcal{A}$ is dense in ∂D_* .

(iii) There exists θ such that $\lambda(\partial D_* \setminus \mathcal{A}) = 0$. One can choose θ so that all of the following conditions are satisfied.

(a) For every $a < b$

$$\lambda(\{x \in (a, b) : \theta(x) = -\pi/2\}) > 0,$$

$$\lambda(\{x \in (a, b) : \theta(x) = \pi/2\}) > 0,$$

and

$$\lambda(\{x \in (a, b) : |\theta(x)| \neq \pi/2\}) = 0.$$

(b) There is a set $\mathcal{H} \subset \mathcal{A}$ such that $\lambda(\partial D_* \setminus \mathcal{H}) = 0$ and for every $x \in \mathcal{H}$, X can approach x from one side, i.e., for each $x \in \mathcal{H}$ we have either

$$P(T_{\{x\}} < \infty \text{ and } \exists \varepsilon > 0 : \operatorname{Re} X(t) < x \text{ for all } t \in (T_{\{x\}} - \varepsilon, T_{\{x\}})) > 0$$

or

$$P(T_{\{x\}} < \infty \text{ and } \exists \varepsilon > 0 : \operatorname{Re} X(t) > x \text{ for all } t \in (T_{\{x\}} - \varepsilon, T_{\{x\}})) > 0.$$

(c) The process X is recurrent and so

$$(c1) P(T_{\{x\}} < \infty) = 1 \text{ for every } x \in \mathcal{A} \text{ and}$$

$$(c2) P(\lambda(\{x \in \partial D_* : T_{\{x\}} = \infty\}) = 0) = 1.$$

The reflected Brownian motion X discussed in Theorem 1.4 (iii) is continuous despite the fact that $|\theta(x)| = \pi/2$ a.e.; the reason is that θ is not constant on any interval.

In view of Theorem 1.4 (iii) (c2) we propose the following

Problem 1.1. Does there exist θ such that for some set $\mathcal{B} \subset \partial D_*$ we have $\lambda(\partial D_* \setminus \mathcal{B}) = 0$ and

$$P(\{x \in \mathcal{B} : T_{\{x\}} = \infty\} = \emptyset) = 1 ?$$

A construction of reflected Brownian motion with non-smooth angle of reflection was presented by Tsuchiya (1976, 1980) under the assumption that $\theta \in (-\pi/2 + \varepsilon, \pi/2 - \varepsilon)$ for some $\varepsilon > 0$. In recent articles, Motoo (1989, 1990) presented a more general construction. See also Takanobu and Watanabe (1988).

Theorem 1.2 is a more accurate version of Theorem 2.1 of Burdzy (1989).

The question of whether the reflected Brownian motion in a wedge hits the vertex was treated by Varadhan and Williams (1985) under the assumption of the constant angle of reflection on each side of the wedge. Rogers (1991) attacked the same problem for reflected Brownian motion in a half-plane with a variable angle of reflection of class C^1 and obtained some partial results. The complete solution of the last problem was obtained independently by Rogers (1990) and Burdzy and Marshall (1992). A perfectly explicit integral test in terms of θ determines whether the process hits a fixed boundary point with positive probability. The same test applies in our present context, i.e., when $\theta : \mathbb{R} \rightarrow [-\pi/2, \pi/2]$ is allowed to be any measurable function. We omit the proof as it does not require any new ideas.

Let us briefly discuss the sources of our techniques. We use conformal mappings in the manner of Rogers (1989, 1991). A connection between reflected Brownian motion and standard Brownian motion goes back to Lévy (1948) in the 1-dimensional case. El Bachir (1983) and Le Gall (1987) adapted the idea to the 2-dimensional processes. The proof of Theorem 1.2 (i) uses an idea of Davis (1983).

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2. A construction of reflected Brownian motion. The set

$$\mathcal{S}_\varphi = \{z \in D_* : |\arg z - \pi/2| < \varphi\}$$

is called a Stolz angle. Suppose f is a function defined on D_* . If the limit

$$\lim_{\substack{z-x \rightarrow 0 \\ z-x \in \mathcal{S}_\varphi}} f(z)$$

exists for every $\varphi < \pi/2$ then it is called the angular (or non-tangential) limit of f at x and denoted $f(x)$. In particular, the angular derivative $f'(x)$ is defined as the angular limit of f' at x , if it exists. If $f : D_* \rightarrow \mathbb{C}$ is analytic and $f(x)$ exists for some $x \in \partial D_*$ then $f'(x)$ exists if and only if the angular limit of the function $z \rightarrow \frac{f(z)-f(x)}{z-x}$ exists; if both limits exist then they are identical (Pommerenke (1975, Theorem 10.5)).

Lemma 2.1. *Suppose that D is a monotone domain with non-empty complement. Then there exists a univalent analytic mapping f of D_* onto D such that $f(x)$ and $f'(x)$ exist, $f'(x) \neq 0$ and $\theta(x) \stackrel{\text{df}}{=} \arg f'(x) \in [-\pi/2, \pi/2]$, for almost all $x \in \partial D_*$. If 0 is a prime end of D , we may choose f so that $f(0) = 0$ and $\lim_{|z| \rightarrow \infty} |f(z)| = \infty$.*

Proof. Choose regions $D_k \subset D$ such that $D_k \subset D_{k+1}$, $\bigcup D_k = D$, and such that ∂D_k is the graph of a smooth function, $k = 1, 2, \dots$. Fix $w_0 \in D_1$ and choose $z_0 \in D_*$ and conformal maps f_k of D_* onto D_k with $f_k(z_0) = w_0$ and $f'_k(z_0) > 0$. Likewise, let f map D_* conformally onto D with $f(z_0) = w_0$ and $f'(z_0) > 0$. Then, by Theorem IX.13

of Tsuji (1959), the sequence $\{f_k\}$ converges to f normally on D_* (i.e., uniformly on compact subsets of D_*). Since $\operatorname{Re} f'_k(x) > 0$ for $x \in \partial D_*$, we conclude $\operatorname{Re} f'_k(z) > 0$ for $z \in D_*$. Taking normal limits, clearly $\operatorname{Re} f'(z) > 0$ for $z \in D_*$. Thus we can choose a branch of \arg so that $\arg f'(z) \in (-\pi/2, \pi/2)$ for all $z \in D_*$. By Corollary 2.6, p. 114 of Garnett (1981), $f' \in H^p$ for all $p < 1$. Thus, by Theorem 2.2, p. 17 of Duren (1970), the angular limits $f'(x)$ exist a.e. and by Corollary 4.2, p. 65 of Garnett (1981) are non-zero a.e. By Theorem 5.12, p. 88 of Duren (1970), $f \in H^q$ for all $q < \infty$ and thus the angular limits $f(x)$ exist a.e.

As for the conditions $f(0) = 0$ and $f(\infty) = \infty$, it is well known that the value of the function f at two boundary points (even three boundary points) may be chosen in an arbitrary way. \square

In order to avoid trivialities we will assume from now on that

$$(2.1) \quad \theta(x) \text{ is neither identically equal to } -\pi/2 \text{ nor identically equal to } \pi/2.$$

Lemma 2.2. *Suppose that $\theta : \mathbb{R} \rightarrow [-\pi/2, \pi/2]$ is a Borel measurable function. Then there exists a univalent analytic mapping f of D_* onto a monotone domain D such that $f(x)$ and $f'(x)$ exist, $f'(x) \neq 0$ and $\arg f'(x) = \theta(x)$ for almost all $x \in \mathbb{R}$. We may choose f so that $\lim_{|z| \rightarrow \infty} |f(z)| = \infty$.*

Proof. Let $\theta : D_* \rightarrow \mathbb{R}$ be the bounded continuous harmonic extension of our original function $\theta : \mathbb{R} \rightarrow [-\pi/2, \pi/2]$ and let $\tilde{\theta}$ be the harmonic conjugate of θ such that $\tilde{\theta}(i) = 0$. Define $f : D_* \rightarrow \mathbb{C}$ by setting $f(i) = i$ and

$$f'(z) = \exp(i(\theta(z) + i\tilde{\theta}(z))).$$

Since we assume (2.1), $\theta(z) \in (-\pi/2, \pi/2)$ for all $z \in D_*$. Hence,

$$\operatorname{Re} f'(z) = e^{-\tilde{\theta}(z)} \cos \theta(z) > 0.$$

As in the proof of Lemma 2.1, the angular limits $f(x)$ and $f'(x)$ exist and $f'(x) \neq 0$ for almost all $x \in \partial D_*$. Let $\gamma(t) = tz + (1-t)w$ where $z, w \in D_1$. Then $\gamma'(t) = z - w$ and

$$(2.2) \quad f(z) - f(w) = \int_0^1 f'(\gamma(t))(z-w)dt = \left[\int_0^1 f'(\gamma(t))dt \right] (z-w).$$

Since the real part of the last integral is strictly positive, $f(z) = f(w)$ if and only if $z = w$. In other words, the function f is univalent.

To see that D is monotone, let $f_\varepsilon = f(z + i\varepsilon)$, $\varepsilon > 0$. By (2.2),

$$\operatorname{Re} f(z) - \operatorname{Re} f(w) = \left(\int_0^1 \operatorname{Re} f'(\gamma(t))dt \right) (\operatorname{Re} z - \operatorname{Re} w)$$

when $\operatorname{Im} z = \operatorname{Im} w$. Thus $\operatorname{Re} f_\varepsilon(x)$ is increasing on ∂D_* . The curve $\gamma_\varepsilon \stackrel{\text{df}}{=} \{\operatorname{Re} f_\varepsilon(x) : -\infty < x < \infty\}$ is an analytic Jordan curve. It must divide the plane into two regions, since f_ε extends to be univalent in a neighborhood of ∂D_* . The boundary of $f_\varepsilon(D_*)$ must be contained in $\gamma_\varepsilon \cup \{\infty\}$, and thus $f_\varepsilon(D_*)$ is monotone. Since ε is arbitrary, $D = f(D_*)$ is also monotone. \square

Lemma 2.3. *Suppose that $\theta : \mathbb{R} \rightarrow [-\pi/2, \pi/2]$ is measurable and $\theta_k : \mathbb{R} \rightarrow (-\pi/2, \pi/2)$ are C^2 -functions which converge almost everywhere to θ as $k \rightarrow \infty$. Let D, f correspond to θ and D_k, f_k correspond to θ_k in the same way as in the proof of Lemma 2.2. The monotone domains D_k converge to D in the following sense.*

- (i) *If B is open and such that $B \cap \partial D \neq \emptyset$, there is a $k_0 = k_0(B)$ such that $B \cap \partial D_k \neq \emptyset$ for all $k \geq k_0$.*
- (ii) *If B is connected and open, with $B \cap D \neq \emptyset$ and $B \subset D_k$ for infinitely many k , then $B \subset D$.*
- (iii) *If K is compact and $K \subset D$ then $K \subset D_k$ for all $k \geq k_0 = k_0(K)$.*

Proof. By the Koebe-1/4 theorem, Theorem 2.3, p. 31 of Duren (1983) and Schwarz's lemma, if g is univalent

$$\operatorname{Im} z |g'(z)|/2 \leq \operatorname{dist}(g(z), \partial g(D_*)) \leq 2 \operatorname{Im} z |g'(z)|$$

for all $z \in D_*$, where $\text{dist}(a, K)$ is the Euclidean distance from a to the set K .

Since $\lim_{k \rightarrow \infty} f_k(z) = f(z)$ and $\lim_{k \rightarrow \infty} f'_k(z) = f'(z)$,

$$\text{dist}(f(z), \partial D)/4 \leq \lim_{k \rightarrow \infty} \text{dist}(f_k(z), \partial D_k) \leq 4\text{dist}(f(z), \partial D).$$

Since $\lim_{k \rightarrow \infty} f_k(z) = f(z)$,

$$\text{dist}(w, \partial D)/4 \leq \lim_{k \rightarrow \infty} \text{dist}(w, \partial D_k) \leq 4\text{dist}(w, \partial D)$$

for all $w \in D$. Actually the 4's can be removed in the above inequalities, for if $\zeta \in D$, choose $w \in D$ with $|\zeta - w| = (1 - \epsilon)\text{dist}(\zeta, \partial D)$. By the above inequalities applied to w ,

$$(1 - 3\epsilon/4)\text{dist}(\zeta, \partial D) \leq \lim_{k \rightarrow \infty} \text{dist}(\zeta, \partial D_k) \leq (1 + 3\epsilon)\text{dist}(\zeta, \partial D).$$

Thus $\lim_{k \rightarrow \infty} \text{dist}(\zeta, \partial D_k) = \text{dist}(\zeta, \partial D)$ for all $\zeta \in D$.

To prove (i), take $w_1 \in D \cap B$ with $\text{dist}(w_1, \partial D) < \text{dist}(w_1, \partial B)/2$, and let $f(z_1) = w_1$. For $k \geq k_0$, $\text{dist}(f_k(z_1), \partial D_k) < \text{dist}(w_1, \partial B)/2$, and $|f_k(z_1) - f(z_1)| < \text{dist}(w_1, \partial B)/4$. Thus $\text{dist}(w_1, \partial D_k) < 3\text{dist}(w_1, \partial B)/4$, proving $\partial D_k \cap B \neq \emptyset$ for $k \geq k_0$.

Statement (ii) follows from statement (i) for if $B \not\subset D$, then $B \cap \partial D \neq \emptyset$.

To prove (iii), note that $K_1 \stackrel{\text{df}}{=} f^{-1}(K)$ is a compact subset of D_* . Choose a large open disc Δ with $\partial\Delta \subset D_*$ and $K_1 \subset \Delta$. The sequence $\{f_k\}$ converges uniformly on $\bar{\Delta}$ to f . For some $\epsilon > 0$ and all $k \geq k_0$, $\text{dist}(f_k(\partial\Delta), K) \geq \epsilon$. The winding number $n(f_k(\partial\Delta), w)$ converges to $n(f(\partial\Delta), w) = 1$ uniformly for $w \in K$. Since winding numbers are integers, $K \subset f_k(D_*)$ for all $k \geq k_0$. \square

Remark 2.1. (i) It follows from Lemma 2.3 (i) that if $x \in \partial D$ then there exists a sequence $\{x_k\}$ such that $x_k \in \partial D_k$ and $x_k \rightarrow x$.

(ii) If $x_k \in \partial D_k$ then x_k 's cannot converge to any point $x \in D$, by Lemma 2.3 (iii).

(iii) Lemma 2.3 (ii) is false if the assumption $B \cap D \neq \emptyset$ is removed.

Proof of Theorem 1.1. Step 1. First we will establish a relationship between reflected Brownian motion in D_* , standard Brownian motion and monotone domains. In this step we will assume that $\theta : \mathbb{R} \rightarrow (-\pi/2, \pi/2)$ is of class C^2 . Let $f : D_* \rightarrow D$ be a mapping corresponding to θ defined in Lemma 2.2. Since θ is of class C^2 , the mapping f is C^2 on \overline{D}_* . This may be proved as in Step 2 of the proof of Theorem 1.1 of Burdzy and Marshall (1992).

Let $V(x) = \tan \theta(x) + i$ for $x \in \partial D_*$, that is let $V(x)$ be the vector of reflection. We may write (1.1)-(1.2) as

$$X(t) = Y(t) + \int_0^t V(X_s) dL_s$$

and the Itô formula yields

$$(2.3) \quad f(X(t)) = f(X(0)) + \int_0^t f'(X(s)) dY_s + \int_0^t f'(X(s)) V(X(s)) dL_s$$

where f' is interpreted as the Jacobian of f . The process X spends zero time on ∂D_* and f is analytic outside ∂D_* so $f(X(0)) + \int_0^t f'(X(s)) dY_s$ is a time-change of a Brownian motion Z , i.e., it is equal to $Z(c(t))$ where

$$c(t) = \int_0^t (f'(X(s)))^2 ds.$$

Note that $c(0) = 0$ and c is continuous and strictly increasing because $f'(x) \in \mathbb{C} \setminus \{0\}$ for $x \in D_*$. The local time L does not increase unless X is at the boundary of D_* and $\operatorname{Re}(f'(x)V(x)) = 0$ for $x \in \partial D_*$ so

$$\int_0^t f'(X(s)) V(X(s)) dL$$

has null real component. Let us time change the process in (2.3); in other words let us define

$$W(c(t)) = f(X(t)),$$

$$A(c(t)) = -i \int_0^t f'(X(s)) V(X(s)) dL_s.$$

Then we have

$$W(t) = Z(t) + iA(t).$$

Hence, W is a reflected Brownian motion in D with the vertical vector of reflection.

We will provide a different representation of $A(t)$, following Skorohod (1961) (see Lemma 3.6.14 of Karatzas and Shreve (1988)). Recall that $D = f(D_*)$. We will show that

$$(2.4) \quad A(t) = -\sup\{a \leq 0 : Z[0, t] \subset D + ai\}.$$

First note that

$$(2.5) \quad \int_0^\infty \mathbf{1}_D(W(t)) dA_t = 0.$$

Let $\tilde{A}(t)$ be the right hand side of (2.4). It is immediate that both $A(t)$ and $\tilde{A}(t)$ are non-decreasing and continuous, they both satisfy (2.5), $A(0) = \tilde{A}(0)$, and $Z(t) + iA(t) \in \bar{D}$ and $Z(t) + i\tilde{A}(t) \in \bar{D}$ for all t .

Suppose that $A(t_1) > \tilde{A}(t_1)$ for some $t_1 > 0$. Let $t_0 = \max\{t < t_1 : A(t) = \tilde{A}(t)\}$.

Then

$$\operatorname{Im}(Z(t) + iA(t)) > \operatorname{Im}(Z(t) + i\tilde{A}(t))$$

for $t \in (t_0, t_1)$ and, since D is monotone, $Z(t) + iA(t) \in D$ for $t \in (t_0, t_1)$. By (2.5), $A(t)$ does not increase between t_0 and t_1 , so $A(t_0) = A(t_1)$. Then

$$0 < A(t_1) - \tilde{A}(t_1) = A(t_0) - \tilde{A}(t_1) \leq A(t_0) - \tilde{A}(t_0) = 0.$$

This is a contradiction and it shows that $A(t) \leq \tilde{A}(t)$ for all t . For similar reasons $\tilde{A}(t) \leq A(t)$ and (2.4) is proved.

The mapping of $\{X(t), 0 \leq t < R\}$ onto $\{Z(t), 0 \leq t < \infty\}$ is one-to-one a.s. so the above reasoning may be reversed. Suppose $\theta : \mathbb{R} \rightarrow (-\pi/2, \pi/2)$ is C^2 and let $f : D_* \rightarrow D$

be the corresponding mapping defined as in Lemma 2.2. Suppose that $Z(t)$ is a standard 2-dimensional Brownian motion, $Z(0) \in \overline{D}$ and $A(t)$ is defined by (2.4). Let

$$\begin{aligned} W(t) &= Z(t) + iA(t), \\ \sigma(t) &= \int_0^t ((f^{-1})'(W(s)))^2 ds, \\ X(\sigma(t)) &= f^{-1}(W(t)). \end{aligned}$$

The function $(f^{-1})'$ is bounded away from 0 and ∞ on bounded subsets of D , so $\sigma(t)$ may be infinite only if $\lim_{s \rightarrow t} |W(s)| = \infty$. Hence $X(t)$ is well-defined until possible explosion, i.e., on a random time interval $[0, R)$ such that $\lim_{s \rightarrow R} |X(s)| = \infty$. The process X is a reflected Brownian motion in D_* with the oblique angle of reflection θ .

Step 2. Now suppose that $\theta : \mathbb{R} \rightarrow [-\pi/2, \pi/2]$ is Borel measurable and that $\theta_k : \mathbb{R} \rightarrow (-\pi/2, \pi/2)$ are C^2 functions which converge almost everywhere to θ as $k \rightarrow \infty$. One way of constructing such a sequence is as follows. Extend θ in a bounded, continuous and harmonic way to D_* and let $\theta_k(x) = \theta(x + i/k)$ for $x \in \partial D_*$ (recall our assumption (2.1)).

Let Z be a standard 2-dimensional Brownian motion and let f, A, W, σ and X be defined in the same way as in the last part of Step 1. Define f_k, A_k, W_k, σ_k and X_k in the analogous way relative to Z and θ_k . We do not know any more whether σ takes finite values so at this point $X(t)$ is well-defined only for some values of t .

We will prove in this step that $W_k(t) \rightarrow W(t)$ a.s. for every t . It is obvious that $\operatorname{Re} W_k(t) \rightarrow \operatorname{Re} W(t)$ as $k \rightarrow \infty$. It will suffice to show that $A_k(t) \rightarrow A(t)$ a.s. Fix some $t > 0$.

Suppose that $A(t) = \infty$. Let t_0 be the smallest s such that $A(s) = \infty$. Note that $t_0 < t$ a.s. We must have

$$\{x \in \mathbb{C} : \operatorname{Im} x > a, \operatorname{Re} x \in (\operatorname{Re} Z(t_0) - \varepsilon, \operatorname{Re} Z(t_0) + \varepsilon)\} \cap \partial D \neq \emptyset$$

for all $a \in \mathbb{R}$, $\varepsilon > 0$, and, by Lemma 2.3 (i),

$$\{x \in \mathbb{C} : \operatorname{Im} x > a, \operatorname{Re} x \in (\operatorname{Re} Z(t_0) - \varepsilon, \operatorname{Re} Z(t_0) + \varepsilon)\} \cap \partial D_k \neq \emptyset$$

for all $a \in \mathbb{R}$, $\varepsilon > 0$ and $k > k_0(a, \varepsilon)$. With probability 1, $Z(0, t)$ contains a closed loop around $Z(t_0)$. It follows that $\lim_{k \rightarrow \infty} A_k(t_0 + \varepsilon) = \infty$ a.s. for every $\varepsilon > 0$ and since $t_0 < t$ a.s., we have $\lim_{k \rightarrow \infty} A_k(t) = \infty$ a.s.

Now suppose that $A(t) < \infty$. Then $W(t) \in D$ a.s. because $Z(0, t)$ contains a closed loop around $Z(t)$. Let

$$\begin{aligned}\alpha(t) &= \inf\{s \in [0, t] : W(s, t) \subset D\}, \\ \alpha_k(t) &= \inf\{s \in [0, t] : W_k(s, t) \subset D\}.\end{aligned}$$

Note that $\alpha(t) < t$ a.s.

First we will show that $\tilde{A}(\alpha(t)+) \geq A(\alpha(t))$ where $\tilde{A}(s) \stackrel{\text{df}}{=} \liminf_{k \rightarrow \infty} A_k(s)$. If $\alpha(t) = 0$ then $\tilde{A}(\alpha(t)+) \geq 0 = A(0) = A(\alpha(t))$. Suppose that $\alpha(t) > 0$ and so $x_1 \stackrel{\text{df}}{=} Z(\alpha(t)) + iA(\alpha(t)) \in \partial D$. Assume that $\tilde{A}(\alpha(t)+) < A(\alpha(t))$. Then, for some $\varepsilon_0 > 0$ and all $\varepsilon < \varepsilon_0$ and $\delta = \delta(\varepsilon) > 0$ the interior of the set

$$B(\varepsilon) \stackrel{\text{df}}{=} \{x \in \mathbb{C} : \operatorname{Re} x = \operatorname{Re} Z(s), \operatorname{Im} x \geq \operatorname{Im} Z(s) + \tilde{A}(\alpha(t)+) + \varepsilon \text{ for some } s \in (\alpha(t), \alpha(t) + \delta)\}$$

is contained in

$$\{x \in \mathbb{C} : \operatorname{Re} x = \operatorname{Re} Z(s), \operatorname{Im} x \geq \operatorname{Im} Z(s) + A_k(s) \text{ for some } s \in (\alpha(t), \alpha(t) + \delta)\} \subset D_k$$

for infinitely many k . For each rational $s < t$, $W(s) \in D$ a.s. because $Z(0, t)$ contains a loop around $Z(s)$. Hence $B(\varepsilon) \cap D \neq \emptyset$ and, by Lemma 2.3 (ii), $B(\varepsilon) \subset D$ and

$$Z(\alpha(t) + \delta) + i(\tilde{A}(\alpha(t)+) + \varepsilon) \in \overline{B(\varepsilon)} \subset \overline{D}.$$

It follows that

$$x_2 \stackrel{\text{df}}{=} Z(\alpha(t)) + i\tilde{A}(\alpha(t)+) \in \overline{D}.$$

Therefore the line segment J joining x_1 and x_2 lies in ∂D . By Remark 2.1 (i), there exist vertical line segments $J_k \subset D_k^c$ with endpoints converging to x_1 and x_2 as $k \rightarrow \infty$.

Now we consider two cases. In the first case the range of the process

$$\{\operatorname{Re} Z(s), s \in (\alpha(t) - \varepsilon, \alpha(t) + \varepsilon)\}$$

intersects both intervals

$$(\operatorname{Re} Z(\alpha(t)) - \varepsilon, \operatorname{Re} Z(\alpha(t)))$$

and

$$(\operatorname{Re} Z(\alpha(t)), \operatorname{Re} Z(\alpha(t)) + \varepsilon)$$

for every $\varepsilon > 0$. Then $\liminf_{k \rightarrow \infty} A_k(\alpha(t)) \geq A(\alpha(t))$ because $W_k(t)$ cannot intersect J_k . In the opposite case $\operatorname{Re} Z$ has a local extremum at $\alpha(t)$. This event has probability zero, because $\alpha(t)$ corresponds to a vertical line segment J in ∂D and there is only a countable number of such line segments. This completes the proof that $\tilde{A}(\alpha(t)+) \geq A(\alpha(t))$. We deduce that

$$\liminf_{k \rightarrow \infty} A_k(t) \geq \tilde{A}(\alpha(t)+) \geq A(\alpha(t)) = A(t) \quad \text{a.s.}$$

Now suppose that $\limsup_{k \rightarrow \infty} A_k(t) > A(t)$. We will prove that this assumption leads to a contradiction. By passing to a subsequence if necessary we may assume that $\lim_{k \rightarrow \infty} \alpha_k(t) \stackrel{\text{df}}{=} \tilde{\alpha}(t)$ and $\lim_{k \rightarrow \infty} A_k(\alpha_k(t)) \stackrel{\text{df}}{=} \tilde{A}(\tilde{\alpha}(t))$ exist. We have

$$A(\tilde{\alpha}(t)) \leq A(t) < \limsup_{k \rightarrow \infty} A_k(t) = \limsup_{k \rightarrow \infty} A_k(\alpha_k(t)) = \tilde{A}(\tilde{\alpha}(t))$$

so $\tilde{A}(\tilde{\alpha}(t)) - A(\tilde{\alpha}(t)) > 0$. The fact that $W(\tilde{\alpha}(t)) \in \bar{D}$ and the definition of A imply that $W(\tilde{\alpha}(t)) + ib \in D$ for all $b > 0$. The sequence $\{W_k(\alpha_k(t))\}$ converges to $W(\tilde{\alpha}(t)) + i(\tilde{A}(\tilde{\alpha}(t)) - A(\tilde{\alpha}(t)))$ and its elements belong to ∂D_k so its limit belongs to D^c , by Remark 2.1 (ii), which is a contradiction. This completes the proof that $W_k(t) \rightarrow W(t)$ a.s. for every t .

Step 3. In order to simplify the argument, we will assume that $Z(0) \in \partial D$. There will be no loss of generality. Fix some $a > 0$ and let

$$T = \inf\{t \geq 0 : \operatorname{Im} f^{-1}(W(t)) \geq a\},$$

$$S = \sup\{s < T : \operatorname{Im} f^{-1}(W(t)) = 0\}.$$

It is possible that $T = \infty$ with positive probability. First we will assume that $T < \infty$ a.s. and show that

$$(2.6) \quad \limsup_{k \rightarrow \infty} \sup \{ \operatorname{Im} f^{-1}(W_k(t)) : 0 \leq t \leq S \} < a \quad \text{a.s.}$$

Suppose that (2.6) is not satisfied. By passing to a subsequence if necessary we may assume that there exists a sequence $\{t_k\}$ such that $t_k \rightarrow t_\infty \in [0, S]$ and $\operatorname{Im} f^{-1}W_k(t_k) \rightarrow a_1 \geq a$. By assumption,

$$(2.7) \quad \operatorname{Im} f^{-1}W(t_\infty) < a.$$

Recall that the functions $A_k(u)$ are monotone and they converge a.s. to a monotone function $A(u)$ simultaneously for all rational u . It follows that with probability 1,

$$A(t_\infty -) \leq \liminf_{k \rightarrow \infty} A_k(t_k) \leq \limsup_{k \rightarrow \infty} A_k(t_k) \leq A(t_\infty +).$$

If A is continuous at t_∞ then

$$\begin{aligned} \operatorname{Im} f^{-1}W(t_\infty) &= \operatorname{Im} f^{-1}(Z(t_\infty) + iA(t_\infty)) \\ &= \lim_{k \rightarrow \infty} \operatorname{Im} f^{-1}(Z(t_\infty) + iA_k(t_k)) \\ &= \lim_{k \rightarrow \infty} \operatorname{Im} W_k(t_k) = a_1 > a \end{aligned}$$

and this contradicts (2.7).

If A has a jump at t_∞ then $W(t_\infty) \in \partial D$ and $\operatorname{Im} f^{-1}(W(t_\infty)) = 0$. Since $\limsup_{k \rightarrow \infty} A_k(t_k) \leq A(t_\infty +)$, any cluster point of $W_k(t_k)$ must be contained in D^c . It follows that $\operatorname{Im} f^{-1}(W_k(t_k)) \rightarrow 0$ which again is a contradiction. This completes the proof of (2.6).

We proceed to prove that $\lim_{k \rightarrow \infty} \sigma_k(t) = \sigma(t)$ a.s. for every t . Fix some $\varepsilon > 0$ and

define stopping times

$$\begin{aligned} U_1 &= 0, \\ T_1 &= \inf\{t \geq 0 : \operatorname{Im} f^{-1}(W(t)) \geq \varepsilon\}, \\ U_k &= \inf\{t \geq T_{k-1} : \operatorname{Im} f^{-1}(W(t)) = 0\}, \quad k \geq 2, \\ T_k &= \inf\{t \geq U_k : \operatorname{Im} f^{-1}(W(t)) \geq \varepsilon\}, \quad k \geq 2. \end{aligned}$$

Let

$$\sigma(s, t) = \int_s^t ((f^{-1})'(W(u)))^2 du,$$

and define $\Gamma_k(s)$ by

$$\Gamma_k(\sigma(T_k, t)) = f^{-1}(W(t)), \quad t \in (T_k, U_{k+1}).$$

Note that $\Gamma_k(s)$ is well defined for $s \in (0, \sigma(T_k, U_{k+1}))$ since $\sigma(T_k, \cdot)$ is a strictly increasing function on $(0, \sigma(T_k, U_{k+1}))$. By the strong Markov property of W applied at T_k 's and the conformal invariance of Brownian motion, the processes

$$\{\operatorname{Im} \Gamma_k(s), 0 \leq s \leq \sigma(T_k, U_{k+1})\}, \quad k \geq 1,$$

are independent 1-dimensional Brownian motions starting at $\varepsilon > 0$ and killed upon hitting 0. Let

$$\tau_\varepsilon = \sum_{\substack{k \\ U_{k+1} \leq S}} \sigma(T_k, U_{k+1}).$$

It is evident that when $\varepsilon \rightarrow 0$ then the distribution of τ_ε converges to the distribution (say, \mathcal{L}) of the starting time of the first excursion of height greater or equal to a of reflected 1-dimensional Brownian motion. Moreover, $\tau_\varepsilon \rightarrow \sigma(S)$ a.s. as $\varepsilon \rightarrow 0$. Hence $\sigma(S) < \infty$ a.s. and by considering all $a > 0$ we see that σ is finite for all t .

Since $W_k(t) \rightarrow W(t)$ a.s. for every t , the Fatou lemma implies that $\limsup_{k \rightarrow \infty} \sigma_k(t) \geq \sigma(t)$ a.s. for all t . It follows from (2.6) that the starting points V_k of the first excursion of

height greater or equal to a of $\text{Im } X_k$ converge a.s. to $\sigma(S)$ and by the same argument as in the case of $\sigma(S)$, each V_k has the distribution \mathcal{L} . This and $\limsup_{k \rightarrow \infty} \sigma_k(S) \geq \sigma(S)$ imply that

$$(2.8) \quad \lim_{k \rightarrow \infty} \sigma_k(S) = \sigma(S).$$

If $\limsup_{k \rightarrow \infty} \sigma_k(t) \geq \sigma(t) + a$ then $\limsup_{k \rightarrow \infty} \sigma_k(s) \geq \sigma(s) + a$ for all $s \geq t$. This and (2.8) applied for all $a > 0$ show that $\lim_{k \rightarrow \infty} \sigma_k(t) = \sigma(t)$ for all t a.s.

The case when $T = \infty$ with positive probability may be treated in a similar way.

Step 4. Now we will prove that $X_k \rightarrow X$ in M_1 topology.

Since A is non-decreasing, it has at most a countable number of jumps. We have proved that $W_k(t) \rightarrow W(t)$ a.s. for every t . It follows that $W_k(t) \rightarrow W(t)$ for almost all continuity points of W , with probability 1, by the Fubini theorem. Since $\sigma_k(t) \rightarrow \sigma(t)$ for all t a.s., we see that $X_k(t) \rightarrow X(t)$ a.s. for every continuity point of X . This shows that the condition 2.3.3 (a) of Skorohod (1960) is satisfied. In view of his Theorem 2.4.1 all that remains to be shown is condition 2.4.1 (b).

We will assume that the condition fails, i.e., there exist sequences $\{t_k^1\}_{k \geq 1}$, $\{t_k^2\}_{k \geq 1}$ and $\{t_k^3\}_{k \geq 1}$ such that $t_k^1 \leq t_k^2 \leq t_k^3$, $|t_k^1 - t_k^3| \rightarrow 0$ and the distance d_k from $X_k(t_k^2)$ to the line segment $[X_k(t_k^1), X_k(t_k^3)]$ does not converge to 0 as $k \rightarrow \infty$. By passing to subsequences, we may assume that the following limits exist:

$$\lim_{k \rightarrow \infty} t_k^1 = \lim_{k \rightarrow \infty} t_k^2 = \lim_{k \rightarrow \infty} t_k^3 \stackrel{\text{df}}{=} t_\infty < \infty,$$

$$\lim_{k \rightarrow \infty} X_k(t_k^j) \stackrel{\text{df}}{=} x_j, \quad j = 1, 2, 3,$$

$$\lim_{k \rightarrow \infty} d_k \stackrel{\text{df}}{=} d.$$

If $\sigma^{-1}(t_\infty)$ is a point of continuity of W then it is easy to see that $x_1 = x_2 = x_3$ and $d = 0$.

Suppose that W has a jump at $\sigma^{-1}(t_\infty)$ and let J be the line segment

$$[Z(\sigma^{-1}(t_\infty)) + iA(\sigma^{-1}(t_\infty)-), Z(\sigma^{-1}(t_\infty)) + iA(\sigma^{-1}(t_\infty))].$$

It can be shown using arguments similar to those in the previous steps that any cluster points z_j of $W_k(\sigma^{-1}(t_k^j))$, $j = 1, 2, 3$, can lie only on $J \subset \partial D$ and, moreover, by monotonicity of A_k 's, z_2 lies between z_1 and z_3 . The mapping f^{-1} takes J into a line segment on ∂D_* and it follows that $d = 0$. \square

Remark 2.2. Let us indicate how one may prove that X is a strong Markov process.

Let Z and A be defined as in Step 2 of the proof of Theorem 1.1. The Brownian motion Z is strong Markov and so is the vector (Z, A) since $A(t)$ is defined in terms of $Z(s)$, $s \leq t$. The process W is a function of (Z, A) and it is clear that the distribution of $\{W(s), s \geq t_0\}$ depends only on $W(t_0) = Z(t_0) + iA(t_0)$ but otherwise it does not depend on the values of $Z(t_0)$ and $A(t_0)$. By Theorem 13.5 of Sharpe (1988), W is strong Markov. The process X is obtained from W by first applying a one-to-one mapping and then a time-change. According to Theorem 65.9 of Sharpe (1988) the second operation preserves the strong Markov property. In order to make this argument rigorous one would have to verify carefully the assumptions of Sharpe's theorems and analyze the extension of $f : D_* \rightarrow D$ to a univalent mapping of \bar{D}_* onto the Martin compactification of D . We postpone this discussion to a future article.

3. Exceptional points on Brownian paths. Evidently, the existence of D -points depends only on the local properties of D near the imaginary axis so we will assume without loss of generality that $|\operatorname{Re} z| < 1$ for all $z \in D$, i.e., $\tilde{D} = D$.

In this section, P^z and E^z will denote the distribution and expectation corresponding to the standard 2-dimensional Brownian motion starting from $z \in \mathbb{C}$ and Y will denote its trajectories.

Lemma 3.1. *The function h defined in (1.3) satisfies*

$$(3.1) \quad h(z) = E^z(|\operatorname{Re} Y(T_{D^c})|) \quad \text{for all } z \in D.$$

Proof. Let $g(z)$ be the function on the right hand side of (3.1). Then $g(z) = h(z) = |\operatorname{Re} z|$ for ∂D . It is easy to see that both g and h are bounded so it will suffice to show that

they are harmonic in D . It is a standard fact that g is harmonic in D and h is obviously harmonic in $D \setminus I$. All that remains to be shown is that h satisfies the mean-value property at each point $ib_0 \in I$. (In this proof z will stand for a complex number and b_0, b, r and φ for real numbers.)

Fix some $ib_0 \in I$ and choose $r > 0$ so that $ib_0 + re^{i\varphi} \in D$ for every φ . Elementary calculations show that

$$(3.2) \quad \int_0^{2\pi} |\operatorname{Re}(ib_0 + re^{i\varphi})| \frac{d\varphi}{2\pi} = \frac{2r}{\pi}.$$

If $|b - b_0| > r$ then $G_D(\cdot, ib)$ is harmonic in $\{z : |z - ib_0| \leq r\}$ and, by the mean value property,

$$(3.3) \quad \int_0^{2\pi} G_D(ib_0 + re^{i\varphi}, ib) \frac{d\varphi}{2\pi} = G_D(ib_0, ib).$$

If $|b - b_0| < r$, write

$$G_D(z, ib) = -\log|z - ib| + f(z)$$

where f is harmonic in D . Thus for $|b - b_0| < r$

$$(3.4) \quad \begin{aligned} \int_0^{2\pi} G_D(ib_0 + re^{i\varphi}, ib) \frac{d\varphi}{2\pi} &= f(ib_0) - \int_0^{2\pi} \log|ib_0 + re^{i\varphi} - ib| \frac{d\varphi}{2\pi} \\ &= G_D(ib_0, ib) - \int_0^{2\pi} \log \left| \frac{ib_0 + re^{i\varphi} - ib}{i(b_0 - b)} \right| \frac{d\varphi}{2\pi}. \end{aligned}$$

Note that

$$\log \left| \frac{ib_0 + re^{i\varphi} - ib}{i(b_0 - b)} \right| = \log \left| -ie^{i\varphi} + \frac{r}{b_0 - b} \right|$$

and the function $z \rightarrow \log \left| -iz + \frac{r}{b_0 - b} \right|$ is harmonic in $\{z : |z| < 1\}$ since $r > |b_0 - b|$. Hence the last integral in (3.4) is equal to $\log \left| \frac{r}{b_0 - b} \right|$. We have

$$(3.5) \quad \int_{\{|b-b_0|<r\}} \log \left| \frac{r}{b_0 - b} \right| \frac{db}{\pi} = \frac{2r}{\pi}.$$

Now we combine (3.2)-(3.5) to obtain

$$\begin{aligned}
\int_0^{2\pi} h(ib_0 + re^{i\varphi}) \frac{d\varphi}{2\pi} &= \int_0^{2\pi} |\operatorname{Re}(ib_0 + re^{i\varphi})| \frac{d\varphi}{2\pi} + \int_I \int_0^{2\pi} G_D(ib_0 + re^{i\varphi}, ib) \frac{d\varphi}{2\pi} \frac{db}{\pi} \\
&= \frac{2r}{\pi} + \int_I G_D(ib_0, ib) \frac{db}{\pi} - \int_{\{|b-b_0|<r\}} \log \left| \frac{r}{b_0 - b} \right| \frac{db}{\pi} \\
&= |\operatorname{Re}(ib_0)| + \int_I G_D(ib_0, ib) \frac{db}{\pi} = h(ib_0). \quad \square
\end{aligned}$$

Remark 3.1. The following probabilistic argument may elucidate the relationship between formulae (1.3) and (3.1). We leave it to the reader to supply the rigorous justification of the argument.

The local time of Y on I (say, $L(t)$) is also the local time of $|\operatorname{Re} Y|$ at 0. The process $V(t) \stackrel{\text{df}}{=} |\operatorname{Re} Y(t)| - L(t)$ is a Brownian motion and hence a martingale. By the optional sampling theorem and Theorem V.1 of Revuz (1970)

$$\begin{aligned}
E^x |\operatorname{Re} Y(T_{D^c})| &= E^x (|\operatorname{Re} Y(T_{D^c})| - L(T_{D^c}) + L(T_{D^c})) \\
&= E^x V(T_{D^c}) + E^x L(T_{D^c}) \\
&= E^x V(0) + E^x L(T_{D^c}) \\
&= |\operatorname{Re} x| + \int_I G_D(x, y) \frac{dy}{\pi}.
\end{aligned}$$

Let $D_1 = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. Recall that D stands for a monotone domain and d_ε was defined in Section 1. See Doob (1984) or Burdzy (1987) for the definition of minimal thinness and its relationship to Brownian motion.

Lemma 3.2. *The set $D^c \cap D_1$ is minimal thin in D_1 at 0 if and only if there is an $\varepsilon > 0$ such that*

$$(3.6) \quad \int_{-1}^0 d_\varepsilon(r) r^{-2} dr < \infty.$$

Proof. (i) Suppose that (3.6) is satisfied for some $\varepsilon > 0$ and let

$$D_2 = \{x \in \mathbb{C} : \operatorname{Re} x > d_\varepsilon(\operatorname{Im} x)\}.$$

By Theorems 9.1 (b) (ii) and 9.2 (i) and (iii) of Burdzy (1987), $D_1 \setminus D_2$ is minimal thin in D_1 at 0. Since for some neighborhood U of 0 we have

$$D^c \cap D_1 \cap U \subset (D_1 \setminus D_2) \cap U,$$

$D^c \cap D_1$ is also minimal thin in D_1 at 0.

(ii) Now suppose that (3.6) is not satisfied for any $\varepsilon > 0$. Let

$$M_k = \{x \in D^c \cap D_1 : |x| \leq 1/k\},$$

and let $\tilde{d}_k : \mathbb{R} \rightarrow \mathbb{R}$ be the smallest non-increasing non-negative function such that

$$M_k \subset \{x \in \mathbb{C} : \operatorname{Re} x \leq \tilde{d}_k(\operatorname{Im} x)\}.$$

Let

$$D_k = \{x \in \mathbb{C} : \operatorname{Re} x > \tilde{d}_k(\operatorname{Im} x)\}.$$

Note that $\tilde{d}_k = d_{1/k}$ so

$$\int_{-1}^0 \tilde{d}_k(r) r^{-2} dr = \infty$$

for every k . By Theorems 9.1 and 9.2 of Burdzy (1987), for each k , the set $D_1 \setminus D_k$ is not minimal thin in D_1 at 0.

For $x \in \overline{D_1}$, P_f^x will denote the distribution of a Brownian motion Z in D_1 starting from x and conditioned by the harmonic function $f(x) \stackrel{\text{df}}{=} \operatorname{Re} x$. For $t > 0$, let

$$T(t) = \inf\{s > t : Z(s) \in D_k^c\}.$$

Since the set $D_1 \setminus D_k$ is not minimal thin in D_1 at 0 it follows that $\inf\{s > 0 : Z(s) \in D_k^c\} = 0$, P_f^0 -a.s. (Doob (1984, Theorem 3.III.3)) and, therefore, there exists $t = t(k) > 0$ such that $P_f^0(T(t) < \infty) > 1/2$.

For each $x \in D_1 \setminus D_k$, there is an $a \geq \operatorname{Re} x$ such that

$$\{z \in \mathbb{C} : \operatorname{Im} z \leq \operatorname{Im} x, \operatorname{Re} z = a\} \subset D^c \cap M_k.$$

By symmetry, the process Z under P_f^x , will hit the line $\{z \in \mathbb{C} : \operatorname{Re} z = a\}$ with equal probability below and above $a + i\operatorname{Im} x$. Hence by the strong Markov property applied at $T(t)$, the P_f^0 -chance of hitting $D^c \cap M_k$ is at least $1/4$. Let $k \rightarrow \infty$ to see that

$$\inf\{t > 0 : Z(t) \in D^c\} = 0$$

with P_f^0 -probability greater or equal to $1/4$. By the 0–1 law, this probability is equal to 1. According to Theorem 3.III.3 of Doob (1984), the set $D_1 \setminus D$ is minimal thin in D_1 at 0. \square

Proof of Theorem 1.2. If D -points exist with positive probability then they exist with probability 1 because Brownian motion returns to the imaginary axis for arbitrarily large times and it is strong Markov. The same remark applies to right D -points.

(i) *Step 1.* We start with an estimate of the expectation of a stopping time. Recall that we assume that $|\operatorname{Re} x| < 1$ for $x \in D$. Let $T_1 = \min(1, T(J))$, where J is the imaginary axis. We will show that there exist constants $0 < c_1 \leq c_2 < \infty$ such that for all $x \in \overline{D}$

$$(3.7) \quad c_1 |\operatorname{Re} x| \leq E^x T_1 \leq c_2 |\operatorname{Re} x|.$$

The density of $T(J)$ under E^x is given by (Theorem 7.5.3 of Karlin and Taylor (1975))

$$f_{x_1}(t) = x_1 (2\pi t^3)^{-1/2} \exp(-x_1^2/2t)$$

where $x_1 = |\operatorname{Re} x|$. Then

$$\begin{aligned} E^x T_1 &\geq \int_{x_1^2/2}^1 x_1 (2\pi t^3)^{-1/2} \exp(-x_1^2/2t) dt \\ &\geq x_1 \int_{x_1^2/2}^1 (2\pi t^3)^{-1/2} e^{-1} dt \\ &\geq x_1 \int_{1/2}^1 (2\pi t^3)^{-1/2} e^{-1} dt = c_1 x_1. \end{aligned}$$

On the other hand, with $u = t/x_1^2$,

$$\begin{aligned}
E^x T_1 &\leq \left(\int_0^{x_1^2} + \int_{x_1^2}^1 + \int_1^\infty \right) x_1 (2\pi t^3)^{-1/2} \exp(-x_1^2/2t) t dt \\
&\leq \int_0^{x_1^2} x_1 (2\pi t^3)^{-1/2} \exp(-x_1^2/2t) t dt + \int_{x_1^2}^1 x_1 (2\pi t^3)^{-1/2} t dt + \int_1^\infty x_1 (2\pi t^3)^{-1/2} dt \\
&= \int_0^1 (2\pi)^{-1/2} u^{-1/2} \exp(-1/2u) x_1^2 du + x_1 (2\pi)^{-1/2} 2t^{1/2} \Big|_{t=x_1^2}^{t=1} + c_3 x_1 \\
&= c_4 x_1^2 + c_5 x_1 (1 - x_1) + c_3 x_1 \leq c_2 x_1.
\end{aligned}$$

Step 2. In this step we will estimate the expectation of another stopping time. Let

$$\begin{aligned}
T_2 &= \inf\{t > T(D^c) : \operatorname{Re} Y(t) = 0\}, \\
T_3 &= \inf(1, T_2), \\
T_4 &= \inf(T_2, T(D^c) + 1).
\end{aligned}$$

We will prove that for some $0 < c_1, c_2 < \infty$ we have

$$c_1 h(x) \leq E^x T_3 \leq c_2 h(x)$$

for $x \in D \cap J$. First, observe that $T_3 \geq T_4 - T(D^c)$. Use this fact, the strong Markov property applied at $T(D^c)$, (3.7) and Lemma 3.1 to see that

$$\begin{aligned}
E^x T_3 &\geq E^x (T_4 - T(D^c)) \\
&= E^x (E^{Y(T(D^c))} (T(J) \wedge 1)) \\
(3.8) \quad &\geq E^x (c_1 |\operatorname{Re} Y(T(D^c))|) = c_1 h(x).
\end{aligned}$$

We also have

$$(3.9) \quad T_3 \leq T_4 = T(D^c) + (T_4 - T(D^c)).$$

The same reasoning which leads to (3.8) yields

$$(3.10) \quad \begin{aligned} E^x(T_4 - T(D^c)) &= E^x(E^{Y(T(D^c))}(T(J) \wedge 1)) \\ &\leq E^x(c_2|\operatorname{Re} Y(T(D^c))|) = c_2h(x). \end{aligned}$$

The process $(\operatorname{Re} Y(t))^2 - t$ is a martingale so by the optional sampling theorem we have for $a < \infty$

$$E^x(\operatorname{Re} Y(T(D^c) \wedge a))^2 = E^x(T(D^c) \wedge a).$$

By the dominated convergence theorem and the monotone convergence theorem we can let $a \rightarrow \infty$. Since $|\operatorname{Re} z| \leq 1$ for $z \in \partial D$, we have by Lemma 3.1,

$$(3.11) \quad \begin{aligned} E^x T(D^c) &= E^x(\operatorname{Re} Y(T(D^c)))^2 \\ &\leq E^x|\operatorname{Re} Y(T(D^c))| = h(x). \end{aligned}$$

By combining (3.9)-(3.11) we obtain $E^x T_3 \leq c_3 h(x)$.

Step 3. Choose $\delta > 0$ so small that the set

$$M \stackrel{\text{df}}{=} \{x \in D : \operatorname{Im} x \geq 1 - \delta, |\operatorname{Re} x| \leq \delta\}$$

is contained in D . Let Θ denote the usual shift operator for the Markov process Y and let

$$A_0 = \{\exists t \in [0, 1] : \operatorname{Re} Y(t) = 0, T(M + Y(t)) \circ \Theta_t + 1 < T(D^c + Y(t)) \circ \Theta_t\}.$$

Fix some $\varepsilon > 0$. Let $T_0 = 0$ and

$$\begin{aligned} U_k &= \inf\{t > T_k : Y(t) \in D^c + Y(T_k) - \varepsilon i\}, \quad k \geq 0, \\ T_k &= \inf\{t > U_{k-1} : \operatorname{Re} Y(t) = \operatorname{Re} Y(T_{k-1})\} \wedge (T_{k-1} + 1), \quad k \geq 1, \\ B_k &= \{T(M + Y(T_k) - \varepsilon i) \circ \Theta_{T_k} + 1 < T(D^c + Y(T_k) - \varepsilon i) \circ \Theta_{T_k}\}, \quad k \geq 0, \\ A_\varepsilon &= \bigcup_{k \geq 0} (\{T_k \leq 1\} \cap B_k). \end{aligned}$$

A moment's thought confirms that $A_0 \subset A_\varepsilon$ for every $\varepsilon > 0$ so $A_0 \subset \bigcap_{\varepsilon > 0} A_\varepsilon$. Now suppose that $\bigcap_{\varepsilon > 0} A_\varepsilon$ holds, in particular, for every positive integer m , the event $A_{1/m}$ holds. Let $k = k(m)$ be the smallest k such that $\{T_k \leq 1\} \cap B_k$ holds with $\varepsilon = 1/m$. By compactness, a subsequence of $\{T_{k(m)}\}_{m \geq 1}$ converges to a point $t_0 \in [0, 1]$. It is easy to verify that t_0 satisfies the definition of A_0 . We conclude that $\bigcap_{\varepsilon > 0} A_\varepsilon \subset A_0$ and, therefore, $\bigcap_{\varepsilon > 0} A_\varepsilon = A_0$.

Let $B = \{T(M) + 1 < T(D^c)\}$. The functions $x \rightarrow P^x(B)$ and $x \rightarrow G_D(i, x)$ are positive and harmonic in $D \setminus M$ and vanish on ∂D . Thus, for $x \in D$ in a neighborhood of 0,

$$c_1 G_D(i, x) \leq P^x(B) \leq c_2 G_D(i, x).$$

To see this, one may map conformally D onto the half-plane D_* . The conformal invariance of harmonic functions and the boundary Harnack principle applied in D_* (see, e.g., Theorem 4 of Dahlberg (1977) or Theorem 2.1 of Burdzy (1987)) yield the above inequality.

By the strong Markov property applied at T_k and the translation invariance of Brownian motion we have $P^0(B_k) = P^{\varepsilon i}(B)$ and, consequently,

$$(3.12) \quad c_1 G_D(i, \varepsilon i) \leq P^0(B_k) \leq c_2 G_D(i, \varepsilon i).$$

Let N be the largest k such that $T_k \leq 1$.

$$\begin{aligned} P^0(A_\varepsilon) &= P^0 \left(\bigcup_{k \geq 1} (\{T_k \leq 1\} \cap B_k) \right) \\ &= \sum_{k \geq 1} P^0(\{T_k \leq 1\} \cap B_k) \quad (\text{disjoint events}) \\ &= \sum_{k \geq 1} P^0(T_k \leq 1) P^0(B_k) \quad (\text{independence}) \\ &= E^0(N + 1) P^0(B_k). \end{aligned}$$

In view of (3.12),

$$(3.13) \quad c_1 G_D(i, \varepsilon i) E^0(N + 1) \leq P^0(A_\varepsilon) \leq c_2 G_D(i, \varepsilon i) E^0(N + 1).$$

The random variables $T_k - T_{k-1}$ are i.i.d., positive and $T_{N+1} = \sum_{k=1}^{N+1} (T_k - T_{k-1})$. By Wald's identity (Feller (1971) XVIII 2)

$$E^0 T_{N+1} = E^0(N+1)E^0(T_1 - T_0).$$

This and the fact that $1 \leq T_{N+1} \leq 2$ a.s. imply that

$$1/E^0(T_1 - T_0) \leq E^0(N+1) \leq 2/E^0(T_1 - T_0).$$

The P^0 -distribution of $T_1 - T_0$ is the same as the $P^{\varepsilon i}$ -distribution of the stopping time T_3 of Step 2 so

$$c_3/h(\varepsilon i) \leq E^0(N+1) \leq c_4/h(\varepsilon i).$$

This and (3.13) imply that

$$(3.14) \quad c_5 G_D(i, \varepsilon i)/h(\varepsilon i) \leq P^0(A_\varepsilon) \leq c_6 G_D(i, \varepsilon i)/h(\varepsilon i).$$

Now we see that (1.4) is equivalent to $P^0(A_0) > 0$.

Step 4. It is now easy to finish the proof. First of all, it is evident that $P^0(A_0) = 0$ if and only if with probability 1, there are no points $Y(t)$ such that $\operatorname{Re} Y(t) = 0$ and $Y(t, t_1) \subset D + Y(t)$ for some $t_1 > t$. Finally, the last condition is equivalent to almost sure non-existence of D -points, by the time reversal.

(ii) We will assume without loss of generality that the Brownian motion Y starts from i . Assume that (1.5) is satisfied. Let

$$K = \{z \in \mathbb{C} : \operatorname{Re} z = 0, \operatorname{Im} z \leq -1\},$$

$$T = \sup\{t < T_K : \operatorname{Re} Y(t) = 0\}.$$

Note that $T < T_K < \infty$ a.s. Hence $\{Y(t), T < t < T_K\}$ is an excursion of Y from the imaginary axis. Since (1.5) holds, the set $D^c + Y(T_K)$ is minimal thin in $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$

at $Y(T_K)$, by Lemma 3.2. By the time reversal, the local path behavior of Brownian excursions is identical at both ends of an excursion, so Lemma 8.1 (ii) of Burdzy (1987) shows that $\{Y(t), T < t < T_K\}$ does not intersect $D^c + Y(T_K)$ immediately prior to T_K a.s. Now standard arguments show that, with positive probability, the whole piece $Y[0, T_K)$ of the Brownian path is disjoint with $D^c + Y(T_K)$ and this means that $Y(T_K)$ is a right D -point with positive probability.

Now suppose that (1.5) does not hold. Assume that $Y(t_0)$ is a right D -point. Then $Y(t_0)$ is the endpoint of an excursion of Y from the imaginary axis. By Lemma 8.1 (ii) of Burdzy (1987) and Lemma 3.2, every excursion $\{Y(s), s_1 < s < s_2\}$ of Y from the imaginary axis must hit $D^c + Y(s_2)$ immediately prior to s_2 and this holds for all excursions simultaneously with probability 1 since there is only a countable number of excursions. In particular, Y must hit $D^c + Y(t_0)$ immediately prior to t_0 and, therefore, $Y(t_0)$ is not a right D -point. This proves our assertion. \square

Remark 3.2. It is not hard to see that $A_{\varepsilon_1} \subset A_{\varepsilon_2}$ for $\varepsilon_1 \leq \varepsilon_2$. Hence, $P^0(A_\varepsilon)$ is a monotone function of ε and (3.14) implies that (1.6) is equivalent to the negation of (1.4).

Proof of Theorem 1.3. (i) First suppose that the prime end 0 is well defined in D and, therefore, we may find f with $f(0) = 0$. Then Theorem 1.3 (i) follows immediately from the construction of reflected Brownian motion given in the proof of Theorem 1.1.

Now suppose that we cannot find D and f as in Lemma 2.2 and such that the prime end 0 is well defined in D . Let f be any function defined as in Lemma 2.2 and let γ be the prime end in D which is the image of 0 under f . If γ corresponded to an $x_0 \in \partial D$ such that $x_0 + ai \in D$ for all $a > 0$ then we could define $f_1(y) \stackrel{\text{df}}{=} f(y - x_0)$ and we would have $f_1(0) = 0$, contrary to our assumption. Hence γ must correspond either to a point $x_0 \in \partial D$ which is in the middle of a vertical line segment $J \subset \partial D$ or γ is the limit of $x_0 + ai$ as $a \rightarrow -\infty$ for some $x_0 \in D$. In either case the process W defined in the proof of Theorem 1.1 does not approach γ a.s. and therefore X does not approach 0 a.s.

(ii) We can prove as in part (i) that the a.s. existence of right D -points is equivalent to

$P(T_{\{0\}} < \infty \text{ and } \exists \varepsilon > 0 : \text{Re } X(t) > 0 \text{ for all } t \in (T_{\{0\}} - \varepsilon, T_{\{0\}}) \text{ such that } \text{Im } X(t) = 0) > 0.$

It has been shown in Step 1 of the proof of Corollary 1.3 of Burdzy and Marshall (1992) that the last condition holds if and only if (1.9) is satisfied. \square

4. Sets of non-polar points for reflected Brownian motion.

Proof of Theorem 1.4. (i) Let f be the function defined in Lemma 2.2 and $D = f(D_*)$. For λ -almost all $x \in \partial D_*$ such that $\theta(x) \in (-\pi/2, \pi/2)$ both $f(x)$ and $f'(x)$ exist and $\arg f'(x) = \theta(x)$. Consider a point x with all these properties and assume without loss of generality that $f(x) = 0$. In view of Theorem 1.3, in order to show that $x \notin \mathcal{A}$ it will suffice to prove that D -points do not exist a.s.

We have the following explicit formula (Doob (1984) 1.VIII (9.3)).

$$G_{D_*}(x, y) = \log \frac{|x - y - 2i\operatorname{Im} x|}{|x - y|} = \log \left| 1 - 2i \frac{x}{x - y} \right|.$$

For every Stolz angle

$$\mathcal{S}_\varphi = \{z \in D_* : |z| < 1, |\arg z - \pi/2| < \varphi\}$$

with $\varphi < \pi/2$ and $x, y \in \mathcal{S}_\varphi$, $|y| \geq 2|x|$, we have

$$1 - 2 \frac{|x|/\arccos \varphi}{|y|} \leq \left| 1 - 2i \frac{x}{x - y} \right| \leq 1 + 2 \frac{|x|}{|y|/2}.$$

Hence

$$c_1|x/y| \leq G_{D_*}(x, y) \leq c_2|x/y|$$

for $x, y \in \mathcal{S}_\varphi$, $|x/y| \leq \alpha$ where $\alpha = \min(1/8, (\arccos \varphi)/4)$. We will take

$\varphi = \pi/2 - (\pi/2 - |\theta(x)|)/2$. Then it follows from the definition of the angular derivative and conformal invariance of the Green function that

$$c_3|x/y| \leq G_D(x, y) \leq c_4|x/y|$$

for $x, y \in f(\mathcal{S}_\varphi)$, $|y| \geq 2|x|/\alpha$, in particular, for $x, y \in I$, $|y| \geq 2|x|/\alpha$. Hence, for small $\varepsilon > 0$,

$$h(\varepsilon i) = \int_I G_D(\varepsilon i, y) dy \geq \int_{2\varepsilon/\alpha}^1 c_3\varepsilon/y dy = -c_3\varepsilon \log(2\varepsilon/\alpha).$$

We see that $h(\varepsilon i)/G_D(i, \varepsilon i) \geq -c_5 \log(2\varepsilon/\alpha) \rightarrow \infty$ as $\varepsilon \rightarrow 0$ and by Theorem 1.2 D -points do not exist a.s.

(ii) Consider any interval (a, b) , $a < b$. If

$$\lambda(\{x \in (a, b) : |\theta(x)| < \pi/2\}) > 0$$

then $(a, b) \setminus \mathcal{A} \neq \emptyset$ by part (i).

If $\theta(x) = -\pi/2$ for almost all $x \in (a, b)$ then the function f defined in Lemma 2.2 maps (a, b) onto a vertical line segment $K \subset \partial D$. It follows from the construction of the reflected Brownian motion (see especially (2.4)) that with probability 1, W never takes values in K , so X does not take values in (a, b) a.s. The same argument takes care of the case when $\theta(x) = \pi/2$ for $x \in (a, b)$.

Finally, suppose that $|\theta(x)| = \pi/2$ for almost all $x \in (a, b)$ but θ is not almost everywhere constant in (a, b) . Note that D lies above the graph of a function g , possibly taking infinite values. The interval (a, b) is mapped onto a part of ∂D . If g is monotone on any interval between $\operatorname{Re} f(a)$ and $\operatorname{Re} f(b)$ then θ must take values of one sign on the corresponding interval and the previous argument may be applied. If g has a jump then ∂D contains a vertical line segment and again we may apply the previous argument. If g is not monotone on a subinterval of $(\operatorname{Re} f(a), \operatorname{Re} f(b))$ and g has no jumps on this interval then g must have a finite local minimum, say at $\operatorname{Re} f(x_0)$, $x_0 \in (a, b)$. We may suppose without loss of generality that $f(x_0) = 0$. One can easily check using Theorem 1.2 that D_* -points do not exist; this has been already proved in Burdzy (1989, Theorem 2.1 (iii)). Since the property of having D -points is monotone in D and $D \subset D_*$ locally, we conclude that D -points do not exist a.s. Hence, $x_0 \notin \mathcal{A}$.

(iii) *Step 1.* For a domain D_1 , $x \in \mathbb{C}$ and $\varepsilon > 0$ let $h = h_{x, \varepsilon, D_1, \eta}$ be the smallest Lipschitz function with the Lipschitz constant $\eta > 0$ such that

$$\begin{aligned} & \{y \in \partial D_1 : |\operatorname{Re} x - \operatorname{Re} y| \leq \varepsilon, |\operatorname{Im} x - \operatorname{Im} y| \leq \varepsilon\} \\ & \subset \{y \in \mathbb{C} : |\operatorname{Re} x - \operatorname{Re} y| \leq \varepsilon, |\operatorname{Im} x - \operatorname{Im} y| \leq \varepsilon, \operatorname{Re} y \leq h(\operatorname{Im} y)\}. \end{aligned}$$

Next, let

$$K_1(x, \varepsilon, D_1, \eta) = \{y \in \mathbb{C} : |\operatorname{Re} x - \operatorname{Re} y| < \varepsilon, |\operatorname{Im} x - \operatorname{Im} y| < \varepsilon, \operatorname{Re} y > h(\operatorname{Im} y)\},$$

$$K(x, \varepsilon, D_1, \eta, c) = e^{-ci} K_1(e^{ci} x, \varepsilon, e^{ci} D_1, \eta).$$

Suppose that D is a monotone domain and let $f : D_* \rightarrow D$ be defined as in Lemma 2.1.

Let

$$M_{a,b} = \{x \in \partial D_* : \arg f'(x) \in (a, b)\}$$

and suppose that $\lambda(M_{a,b}) > 0$ for some $-\pi/2 + \eta_1 < a < b < \pi/2 - \eta_1$, $\eta_1 > 0$. Then we can argue as in the proof of Theorem 2.6 (i) of Burdzy (1989) to show that for $\eta = \eta_1/2$ there exist $x \in \mathbb{C}$, $\varepsilon > 0$ and $c \in \mathbb{R}$ such that the harmonic measure of

$$N \stackrel{\text{df}}{=} f(M_{a,b}) \cap \partial K(x, \varepsilon, D, \eta, c)$$

is strictly positive in a connected component of $K(x, \varepsilon, D, \eta, c)$. Since $K(x, \varepsilon, D, \eta, c)$ is a union of Lipschitz domains, Theorem 1 of Dahlberg (1977) implies that the linear measure of N is positive and, therefore the Lebesgue measure of the projection of N onto ∂D_* is positive, by our choice of η_1 and η .

Hence, if $\lambda(M_{-\pi/2, \pi/2}) > 0$ then there exists a set $F \subset \mathbb{R}$ of positive measure such that if $x \in \partial D$ and $\operatorname{Re} x \in F$ then there is $\delta = \delta(x) > 0$ such that

$$\{y \in \mathbb{C} : \arg(y - x) \in (\pi/2 - \delta, \pi/2 + \delta), |y - x| < \delta\} \subset D.$$

One can show in a similar way that if for some set $A \subset \partial D_*$ we have $\lambda(A) > 0$ and $|\arg f'(x)| = \pi/2$ for $x \in A$ then the projection of $f(A)$ on the imaginary axis has a strictly positive measure.

Step 2. Let $B^1(t)$ and $B^2(t)$ be independent standard 1-dimensional Brownian motions starting from 0 and let D be a (random) monotone domain defined by

$$D = \{x \in \mathbb{C} : \operatorname{Re} x \geq 0, \operatorname{Im} x > B^1(\operatorname{Re} x)\} \cup \{x \in \mathbb{C} : \operatorname{Re} x < 0, \operatorname{Im} x > B^2(\operatorname{Re} x)\}.$$

Let $f : D_* \rightarrow D$ be the function defined in Lemma 2.1 and let X be the reflected Brownian motion in D with the oblique angle of reflection $\theta(x) \stackrel{\text{df}}{=} \arg f'(x)$. We will show that with probability 1 (with respect to the distribution of B^1 and B^2) the function θ satisfies Theorem 1.4 (iii) (a)-(b).

For a fixed $t > 0$, $j = 1, 2$,

$$\limsup_{\varepsilon \rightarrow 0^+} \frac{B^j(t + \varepsilon) - B^j(t)}{\varepsilon} = \limsup_{\varepsilon \rightarrow 0^-} \frac{B^j(t + \varepsilon) - B^j(t)}{-\varepsilon} = \infty \quad \text{a.s.}$$

By the Fubini theorem, the set of all $x \in \partial D$ such that there is $\delta > 0$ with

$$\{y \in \mathbb{C} : \arg(y - x) \in (\pi/2 - \delta, \pi/2 + \delta), |y - x| < \delta\} \subset D$$

has measure 0, with probability 1. By Step 1, $|\arg f'(x)| = \pi/2$ for almost all $x \in \partial D_*$.

Consider an $x_0 \in \partial D_*$ such that $|\arg f'(x_0)| = \pi/2$ and suppose it is mapped by f onto $t_0 + B^1(t_0)i$ for some $t_0 > 0$ (the argument is analogous for $t_0 < 0$). It follows from the definition of the angular derivative $f'(x_0)$ that $B^1(s) < B^1(t_0)$ for all $s \in (t_0, t_0 + \varepsilon)$ or for all $s \in (t_0 - \varepsilon, t_0)$ for some $\varepsilon > 0$. Hence, t_0 is the endpoint of an excursion of B^1 from $a \stackrel{\text{df}}{=} B^1(t_0)$. For definiteness, suppose that $B^1(s) < B^1(t_0)$ for all $s \in (t_0, t_0 + \varepsilon)$.

It is well known that excursions of 1-dimensional Brownian motion have the same local properties as the 3-dimensional Bessel process (see, e.g., Williams (1979)). By Theorem 3.3 (ii) of Shiga and Watanabe (1973),

$$(4.1) \quad \limsup_{\varepsilon \rightarrow 0^+} \frac{B^1(t_0 + \varepsilon) - B^1(t_0)}{\varepsilon^{3/4}} \leq -1$$

and since there is only a countable number of excursions of B^1 from a , the same property holds for all endpoints of all excursion from a a.s. If we move D so that $t_0 + B^1(t_0)i$ becomes 0, we see that (4.1) implies that (1.5) (or its mirror image) holds and therefore right (or left) $D - (t_0 + B^1(t_0)i)$ -points exist, by Theorem 1.2. Thus the reflected Brownian motion X hits x_0 with positive probability and approaches it from one side, by Theorem 1.3.

By the Fubini theorem, the set of levels a such that (4.1) is violated for at least one excursion from a has measure 0. It follows from Step 1 that the set of $x \in \partial D_*$ which cannot be approached from one side with positive probability has measure 0.

Since B^j is not monotone on any interval with probability 1, it follows that $\theta(x) = \arg f'(x)$ cannot have one sign almost everywhere on any interval (a, b) , $a < b$. This completes the proof of (a) and (b).

Suppose that reflected Brownian motions X_1 and X_2 in D_* have oblique angles of reflection θ_1 and θ_2 which correspond to two monotone domains D_1 and D_2 such that $D_1 \subset D_2$. It follows from our construction of reflected Brownian motion that if X_1 is recurrent then X_2 is recurrent as well. If $D_1 = D_*$ then X_1 is a reflected Brownian motion with the normal reflection on ∂D_* and it is well known that it is recurrent. Now replace in the definition of D Brownian motions $B^j(t)$, $j = 1, 2$, with Brownian motions with drift $\tilde{B}^j(t) \stackrel{\text{df}}{=} B^j(t) - t$. Then $D_* + ai \subset D$ for some $a > 0$ since $\tilde{B}^j(t) \rightarrow -\infty$ as $t \rightarrow \infty$. The reflected Brownian motion X corresponding to this new domain D is recurrent and it also satisfies (a) and (b) since the constant drift does not change the local properties of the 1-dimensional Brownian motion.

The recurrence of X implies (c1). Condition (c2) follows from (c1) and the Fubini theorem. \square

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