# Aspects of Markov Chains and Particle Systems

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A dissertation submitted in partial fulfillment of the
requirements for the degree of

Doctor of Philosophy

University of Washington

2016

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Program Authorized to Offer Degree:
Mathematics

University of Washington

**Abstract**

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The thesis concerns asymptotic behavior of particle systems and the underlying Markov chains used to model various natural phenomena. The objective is to describe and analyze stochastic models involving spatial structure and evolution over time. Fundamental objects of interest in such systems include the equilibrium measure which the system converges to, the phenomenon of phase transition in the long term behavior and the time taken to converge to stationarity. In this thesis, we present three examples highlighting the above aspects.

In Chapter 2, we will discuss *Competitive Erosion:* a multi-particle system introduced by James Propp in 2003, as a generalization of a fundamental growth model known as *Internal Diffusion Limited Aggregation*. In this model, each vertex of the graph is occupied by a particle, which can be either red or blue. New red and blue particles are emitted alternately from their respective bases and perform random walk. On encountering a particle of the opposite color they remove it and occupy its position. We consider competitive erosion on discretizations of smooth planar simply connected domains. Chapter 2 establishes positively, a conjecture of Propp regarding conformal invariance of the the model at stationarity, by showing that, with high probability the blue and the red regions are separated by an orthogonal circular arc on the disc and by a suitable hyperbolic geodesic

on a general ‘smooth’ simply connected domain.

In Chapter 3, we discuss a family of conservative stochastic processes known as *Activated Random Walk* (ARW) which interpolates between ordinary random walk and the *Stochastic Sandpile*; the latter being a canonical example of *Self Organized Criticality*. These processes are conjectured to exhibit a sharp change in long time behavior, depending on the value of certain parameters. Informally, ARW is a particle system on $\mathbb{Z}$ with mass conservation. One starts with a mass density $\mu > 0$ of initially active particles, each of which performs a symmetric random walk at rate one and falls asleep at rate $\lambda$. Sleepy particles become active on coming in contact with other active particles. We investigate the question of fixation/non-fixation of the process and show for small enough $\lambda$ the critical mass density for fixation is strictly less than one. Moreover, the critical density goes to zero as $\lambda$ tends to zero. This positively answers two open questions from Dickman, Rolla, Sidoravicius (J. Stat. Phys., 2010) and Rolla, Sidoravicius (Invent. Math., 2012).

In Chapter 4, we discuss a model of constrained Glauber dynamics, known as the *East Process*, exhibiting sharp convergence to equilibrium. The East process is a 1D kinetically constrained interacting particle system, introduced in the physics literature in the early 90’s to model liquid-glass transitions. Informally, it is a two spin ($\{0,1\}$) system on $\mathbb{Z}$ where every site at rate one tries to randomize its spin using a fresh $Bernoulli(p)$. However the move is suppressed unless the site to the left is in the $0$ state. Thus the Glauber dynamics move is carried out only in the presence of a certain ‘kinetic’ constraint. Spectral gap estimates of Aldous and Diaconis in 2002 imply that its mixing time on $L$ sites has order $L$. Since the relaxation time is of a smaller order than the mixing time it is natural to expect a sharp convergence to equilibrium. Proving this, is the goal of this chapter, where we establish *Cutoff* for mixing, with an optimal $O(\sqrt{L})$-window.

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## ACKNOWLEDGMENTS

I am grateful to many people who have played a vital role in my development as a mathematician and as an individual. My foremost thanks to my mother who was as much a part of this journey as was I, who is as much interested in academics in general and Mathematics in particular, as am I.

I thank my adviser Ioana Dumitriu for all the help, for always being interested to hear whatever I had to say and for supporting me as an RA so that I could focus on research.

I am deeply grateful to my co-adviser Christopher Hoffman, whom I have had wonderful collaborations with. More importantly, I got an opportunity to talk about various areas of math and beyond. His sharp insight and ideas will continue to shape my development as a researcher.

I also take this opportunity to express my gratitude towards Yuval Peres who acted as my mentor throughout this journey. He was a constant source of inspiration, and I thoroughly cherish all the conversations I had with him about various things. The first chapter in this thesis is based on joint work with him, over the course of which I learnt a host of new things.

I had the good fortune of doing various internships at Microsoft Research, over the summers, which really led to the bulk of the work in this thesis. I thank Christian Borgs, Sébastien Bubeck, Eyal Lubetzky and Yuval Peres for being my mentors during the internships. I was also lucky to be able to work with Fabio Martinelli, joint work with whom, constitutes one chapter of this thesis.

I thank all of my collaborators for giving me the opportunity to be able to work with them, specially Lionel Levine with whom I have worked on several projects.

This acknowledgement would not be complete if I do not thank Krzysztof Burdzy, who has been always willing to lend his advice regarding any of the issues I faced as a student. I also thank Steffen Rohde, for the wonderful courses on SLE and Quasiconformal mappings; those are some of the courses I enjoyed the most during my stay at UW. I thank James R. Lee for being on my thesis committee, and for teaching wonderful courses on spectral graph theory, entropy optimality and so on.

I must also thank my fellow students and postdocs. I am indebted to Wai Tong Fan who was always happy to explain many of the results he obtained as a part of his PhD, which are crucially used in the first chapter. I cherish the discussions with Gerandy Brito and Brent Werness who were always enthusiastic to talk about various aspects of probability and we learnt a lot of things together. I also thank Riddhipratim Basu, Matthew Junge, Bharathwaj Palvannan, and Miklos Racz for various helps throughout graduate school.

Lastly I mention Rabindranath Tagore, whose genius continues to influence me.

*To all those who pursue excellence ...*

# Chapter 1

# INTRODUCTION

## *1.1 Background*

A central approach to understanding statistical mechanics is through interacting particle systems with infinite volume of particles or large but finite number of particles. The field of interacting particle systems began as a branch of probability theory in the late 1960's. The objective was to describe and analyze stochastic models involving spatial structure and evolution over time and develop an understanding of the phenomenon of phase transition in various context and classifying the classical Gibbs states. These are used extensively to model many naturally occurring processes in various contexts: neural networks, tumor growth, spread of infection, and behavioral systems, for example.

From a more mathematical point of view, interacting particle systems represents a natural departure from the established theory of Markov processes.

Quoting from Tom Liggett's *'Interacting Particle systems'*: *"A typical interacting particle system consists of finitely or infinitely many particles which, in the absence of the interaction, would evolve according to independent finite or countable state Markov chains. Superimposed on this underlying motion is some type of interaction. As a result of the interaction, the evolution of an individual particle is no longer Markovian. The system as a whole is of course Markovian. However, it is a large and complex Markov process which differs in many respects from the processes such as Brownian motion on Euclidean spaces which motivated much of the development of standard Markov process theory. Thus, while some connections with the Markovian universe are maintained, substantial departures from it occur as well."*

For the reader unfamiliar with this area of probability, I include some basic examples of such systems, variants of which will recur throughout this thesis.

i. Stochastic Ising model: The (ferromagnetic) Ising Model on a finite graph $G = (V, E)$ with parameter $\beta \geq 0$ and no external magnetic field is defined as follows. Its set of possible configurations is $\Omega = \{1, -1\}^V$, where each configuration $\sigma \in \Omega$ assigns positive or negatives spins to the vertices of the graph. The probability that the system is at a given configuration $\sigma$ is given by the Gibbs distribution

$$\frac{\exp(\beta \sum_{y \sim x} \mathbf{1}(\sigma(x) = \sigma(y)))}{\mathbb{Z}}$$

where $\mathbb{Z}$ is the normalizing constant, ($y \sim x$ denotes that the edge $(x, y) \in E$.) The Gibbs measure is associated with the standard 'Glauber' or 'Heat Bath' dynamics where spin at a uniformly chosen vertex is randomized according to the conditional distribution given the spin of the remaining vertices.

ii. Internal diffusion limited aggregation (IDLA): This is one of the most basic growth models where $n$ particles start at a point and each particle does a continuous time random walk independent of others till it finds an empty site at which point it occupies it and stops, is a fundamental growth model on graphs. This was first proposed by Meakin and Deutch in 1986 as a model of industrial chemical processes such as electropolishing, corrosion and etching and has been studied extensively since.

iii. Abelian Sandpile model: A primary example of self organized criticality (a concept elaborated later in subsection 1.3.2) where one starts with particles/chips at every vertex of an underlying graph. The state of the system tries to reach stability by 'toppling' a site which has at least as many particles as neighbors, it sends one particle to every neighboring site. The system is said to be stable if no site can topple. The final distribution of particles turns out to be independent of the order in which unstable vertices are toppled. In the stochastic version, particles take one step of the random walk on the graph.

Other examples include the Contact process and the Exclusion Process. The interested reader is referred to [67] for a rich introduction to the subject.

Based on non rigorous heuristics, remarkable conjectures about such models exist in the natural sciences literature. This thesis is based on understanding some of these models and making progress towards formal verification of these conjectures. This involves applications of ideas and tools from several other areas of mathematics such as complex analysis, partial differential equations and potential theory.

## *1.2 Main objects of interest*

Particle systems exhibit rich behavior in various situations. Some of the main objects of study about such models are the following:

- **Equilibrium measure:** The long range behavior of the underlying Markov chain and characterization of the invariant or equilibrium measure. In many infinite situations the equilibrium measure is not necessarily unique. In Markov chains exhibiting spatial correlations, the invariant measure typically exhibits rich geometric properties. In Chapter 2 we establish *conformal invariance* (invariance under composition by smooth angle preserving complex valued maps) for a two dimensional competing particle system.

- **Phase transition and critical behavior:** Many particle systems have an underlying parameter in the definition, for e.g. the temperature in Ising model, initial mass density in Sandpile models. Often in such situations the model exhibits strong dependence on the value of these parameters and one observes a sharp phase transition as the parameter crosses a particular critical value. It is well known that such phase transitions occur in the mixing time and uniqueness of Gibbs measures for the stochastic Ising model, on many graphs, as the temperature of the system crosses a critical value. Various models of statistical mechanics are known to exhibit phase transitions out of equilibrium.

  A family of mass conservative stochastic dynamics used to understand the phenomenon of *Self Organized Criticality* including the *Stochastic Sandpile* are believed to exhibit certain absorbing state phase transitions: i.e. the system fixates or stays active depending on the initial mass density. More details can be found in Section 1.3.2.

- **Mixing time:** A quantity of paramount importance in finite systems, is the mixing time: amount of time needed for a Markov chain to reach equilibrium. Precise bounds on such quantities help us bound the running time of algorithms used to sample from the invariant measure. In Chapters 2 and 4 we investigate such questions for various natural systems. In many settings, it is expected that distance to equilibrium describes a sharp transition and it is possible to locate the mixing time precisely, (cf. Chapter 4). This phenomenon was introduced as *Cutoff* in the literature, by Aldous and Diaconis. See [33] for examples of chains showing cutoff and the intuition behind occurrence of this phenomenon.

### *1.3 Summary of the results.*

In this section we present a broad outline of this thesis. All the results in this article will be based on models defined on the Euclidean Lattice resulting in strong dependence. Analogous models on trees are typically easier to understand because of the underlying recursive structure. In chapter 2 we study a competing particle system which models spontaneous interface formation. We prove a conjecture about the invariance of the long term behavior under “composition” by smooth angle preserving (conformal) maps. This is the phenomenon known as *Conformal invariance*, occurring widely in many statistical physics models like percolation, Ising model etc. It is a classical fact that planar Brownian motion is conformally invariant.

In Chapter 3 we discuss results regarding phase transition for certain mass conservative dynamics and in Chapter 4 we discuss precise mixing time results for a class of spin systems and constrained ‘heat bath’ dynamics.

### 1.3.1 Chapter 2: Competing growth model and conformal invariance

A competing version of IDLA (cf. example $ii.$ in Section 1.1) was proposed by James Propp in 2003 with two types of particles doing IDLA on a graph alternately where particles of either color treats sites with particle of the opposite color as an empty site. Many similar models, including competing first passage percolation [50] have been widely used to model behavior of cohabiting species under various natural conditions. One of the primary objects of interest in such models is the interface between various territories. Understanding the behavior of interfaces in various growth and related models has been a major success story with recent proofs of several deep results with connections to the Kardar-Parisi-Zhang universality class.

Based on behavior of Reflecting Brownian Motion, Propp predicted the following:

**Conjecture:** As the mesh size goes to zero at stationarity the territories of the two colors would be separated by the geodesic with respect to the hyperbolic metric which by definition is conformally invariant. On the disc, these geodesics are the circular arcs orthogonal to the boundary. See Fig 2.1.

The last decade and a half has seen tremendous progress in the study of *conformally invariant* two dimensional processes related to percolation, Ising model, self-avoiding polymers. This resulted in the discovery of the Stochastic Loewner evolutions as scaling limits of the above models, and the Fields medal winning works of Lawler, Werner, Schramm, and Smirnov, (c.f. [85]).

(Ganguly, Peres [47]) : We confirm Propp's prediction on 'nice' enough simply connected domains. See Chapter 2 for precise statements and technical assumptions. The technical content of this work involves tools from electrical network theory to understand the behavior of random walk in bounded domains and convergence of Green function in the discrete Neumann setting to the continuous counterpart.

### *1.3.2 Chapter 3: Absorbing state phase transition in conservative dynamics*

Dhar [27, 28] proposed certain collections of communicating finite automata as models of self organized criticality (SOC), which describes a system that has a critical point as an attractor. In most of the examples of SOC simple local moves give rise to complex global properties. These represent systems which under their natural evolution are driven to a state at the boundary between stable and unstable states without any fine-tuning of parameters. Modern statistical mechanics offers a large and important class of driven-dissipative lattice systems that naturally evolve to a critical state, which is characterized by power-law distributions of the sizes of relaxation events (a paradigm example is the emergence of avalanches caused by small perturbations). In many mathematically interesting and physically relevant cases such systems are attracted to a stationary critical state without being specifically tuned to a critical point. In particular, it is believed that this phenomenon lies behind random fluctuations at the macroscopic scale, and creation of self-similar shapes in a variety of growth systems. Due to strong non-locality of correlations and dynamic long-range effects, classical analytic and probabilistic techniques fail in most cases of interest, making the rigorous analysis of such systems a major mathematical challenge. Studies of the above phenomenon are confined to very few models, and its conceptual understanding is extremely fragmented.

One prime example in this context is the so called Abelian Sandpile on a graph where one starts with particles/chips at every vertex. The state of the system tries to reach stability by 'toppling' a site which has at least as many particles as neighbors, it sends one particle to every neighboring site. The system is said to be stable if no site can topple. The final distribution of particles turns out to be independent of the order in which unstable vertices are toppled, a property known as the Abelian property (also shared by IDLA). For a wonderful exposition and list of examples see the survey [17].

*Activated Random Walk:* Notice that the sandpile model described above is deterministic given the initial configuration of particles. Consider the following random version on $\mathbb{Z}$ known as the Stochastic Sandpile (SSM): start with particles at every site on the line by sampling i.i.d. from any distribution with mean $\mu$. Any unstable site topples by emitting two particles which take independently one step of the simple random walk. The most important phenomenon of interest in the study of SSM is the phase transition related to the long term fixation/non-fixation of the system i.e. whether one observes activity in any finite window about a point (say the origin) infinitely often or not. It is known that there exists a critical initial mass density of particles below which the system fixates almost surely. One of the major open problems in this area is to show that the critical density is strictly less than one. However this has turned out to be extremely challenging. The following family of processes were proposed as approximations to the SSM: Start with particles at every site on the line by sampling i.i.d. from a distribution with mean $\mu$ like in SSM. Each particle can be in one of the two following states $A$ (active) and $S$ (sleepy). Initially all the particles are in active state. Each active particle does a continuous time nearest neighbor symmetric random walk on $\mathbb{Z}$ at rate $1$. Sleepy particles do not move. Also each active particle undergoes the transition $A \to S$ at rate $\lambda > 0$ independent of everything else. Note that the case when $\lambda$ is infinity is a variant of SSM. For a fixed sleep rate $\lambda$, as the particle density $\mu$ increases, it is expected that the system shows a transition from almost sure local fixation to staying active forever almost surely.

**Predictions in the physics literature:** The critical density satisfies $\mu_c \to 0$ as $\lambda \to 0$ and $\mu_c \to 1$ as $\lambda \to \infty$. Moreover the value of $\mu_c$ should be universal, i.e. $\mu_c$ should not depend on the particular $\mu$ parametrized distribution of the initial configuration (Geometric, Poisson, etc.). At criticality the system is predicted not to stabilize, even though the density of active particles vanishes as time goes to infinity. Moreover the asymptotic decay of density of activity should obey a power law. For more details see [76].

One of the first mathematically rigorous results about ARW was established in [78] where it is shown that for every $\lambda > 0$ there is a critical particle density $\mu_c \in [\frac{\lambda}{\lambda+1}, 1]$ such that ARW$(\mu, \lambda)$ locally fixates almost surely when $\mu < \mu_c$ and stays active almost surely when $\mu > \mu_c$. Quoting from [78, Section 7],

*"A proof that $\mu_c < 1$ for the SSM and ARW remains as an open problem in any*

*dimension. For the ARW, it should hold for all $\lambda$, and moreover $\mu_c \to 0$ as $\lambda \to 0$. Yet, even a proof that $\mu_c < 1$ for some $\lambda > 0$ is missing."*

Chapter 3 is based on the the following result:

(Basu, Ganguly, Hoffman [13]): We positively resolve the above conjecture for small $\lambda$ and in fact show that indeed $\mu_c$ goes to zero as $\lambda$ tends to zero.

### *1.3.3 Chapter 4: Constrained spin systems: pinpointing mixing time*

Another fundamental quantity of interest about a Markov Chain is its mixing time i.e. the time taken to converge to equilibrium. There are various notions of convergence and in this chapter I will focus on convergence in the total variation metric. Formally for two measures $\mu, \nu$ on a common space $\Omega$, the total variation distance is,

$$||\mu - \nu||_{TV} = \sup_{A \subset \Omega} \mu(A) - \nu(A).$$

For more details, see Chapter 4. Given the above for any $\varepsilon \in (0,1)$, the $\varepsilon-$ mixing time of a Markov Chain is defined to be

$$T_{\text{mix}}(\varepsilon) = \inf\{t : ||P_t(x,\cdot) - \pi||_{TV} \leqslant \varepsilon \text{ for all } x \in \Omega\},$$

where $\pi$ is the stationary measure of the Markov chain and $P_t(x,\cdot)$ is the distribution at time $t$ starting from state $x$.

**Cutoff:** A sequence of Markov chains is said to have cutoff if it shows a sharp transition in its convergence, i.e. there exists $T_n$ (location) and $w_n(\varepsilon)$ (window size for any $\varepsilon \in (0,1)$) with $w_n(\varepsilon) = o(T_n)$ such that for the $n$-th Markov chain in the sequence, the $\varepsilon$-mixing time, $T_{\text{mix}}(\varepsilon)$ satisfies,

$$T_{\text{mix}}(\varepsilon) = T_n + w_n(\varepsilon).$$

Simple random walk on an interval or a cycle is well known to not exhibit cutoff whereas the simple random walk on the hypercube does. See for e.g [66] for excellent exposition, intuition and examples of chains with and without cutoff. There exists natural necessary conditions for cutoff to occur in terms of the spectral gap (smallest non zero eigenvalue) of the associated generator of the Markov chain, i.e. the relaxation time ($1/$spectral gap) should be asymptotically of a smaller order than the mixing time. However it turns out to be not sufficient (for counter examples see [66]). Nevertheless it is still believed to suffice for many 'naturally' occurring chains. Showing cutoff usually turns out to be extremely hard and in the few cases where it has been established, the analysis has been very case specific. Some recent remarkable results about mixing and cutoff for spin systems can be found in [68] and other articles by the same authors.

The East process is a one-dimensional spin system that was introduced in the physics literature by Jäckle and Eisinger [52] in 1991 to model the behavior of cooled liquids near the glass transition point. Each site in $\mathbb{Z}$ has a $\{0,1\}$-value (vacant/occupied), and, denoting this configuration by $\omega = \{\omega_x\}_{x\in\mathbb{Z}}$, the process attempts to update $\omega_x$ to $1$ at rate $0 < p < 1$ (a parameter) and to $0$ at rate $q = 1 - p$, only accepting the proposed update if $\omega_{x-1} = 0$ (a "kinetic constraint"). That the spectral gap of the above chain on finite intervals is uniformly bounded away from 0 for any $p \in (0,1)$ was first proved in a beautiful work of Aldous and Diaconis [2] in 2002. Standard bounds then imply that $T_{\mathrm{mix}}$ for the East process on an interval is up to constants $L$ (the length of the interval). Thus the necessary condition for cutoff is indeed satisfied by the East process. Chapter 4 is based on the following result:

(Ganguly, Lubetzky and Martinelli [46]) : We establish the phenomenon of cutoff for the East process on an interval of length $L$ at location $L/v$, $(v = v(p))$ with a $\sqrt{L}$ window. The window size turns out to be optimal because of a central limit theorem.

Chapter 2

# COMPETING GROWTH MODEL AND CONFORMAL INVARIANCE

## 2.1 *Competitive Erosion*

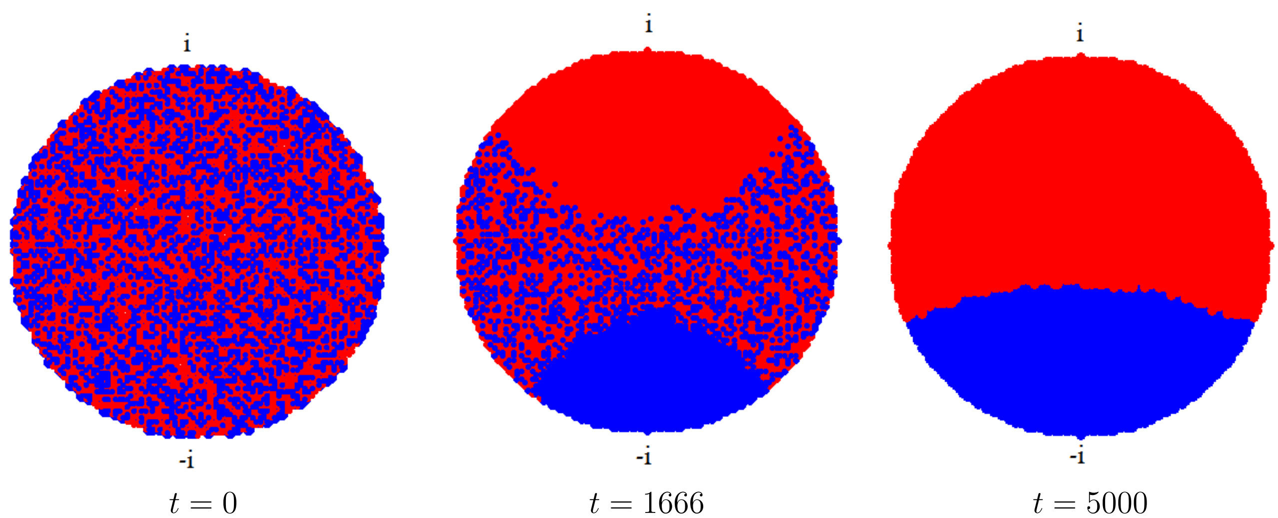


Figure 2.1: Competitive erosion on the unit disc, spontaneously forms an interface. $i$ and $-i$ form the red source and blue source respectively. In the initial state each vertex of a mesh of size $\frac{1}{50}$ is independently colored blue with probability $1/3$ and red otherwise. After $5000$ time steps of competitive erosion, the red and blue regions have separated with the interface being an orthogonal circular arc.

## 2.2 *Interface dynamics*

In 2003, Propp introduced *competitive erosion*: a graph-theoretic model of interface dynamics maintained in equilibrium by equal and opposing forces. The model has the following underlying data:

- A finite connected graph $G = (V, E)$ with vertex set $V$ and edge set $E$.

- Probability measures $\mu_1$ and $\mu_2$ on $V$.

- An integer $0 \leqslant k \leqslant \#V - 1$.

*Competitive erosion* is a discrete-time Markov chain $(S(t))_{t \geqslant 0}$ on the space

$$\{S \subset V : |S| = k\}.$$

One time step is defined by

$$S(t+1) = (S(t) \cup \{X_t\}) - \{Y_t\} \tag{2.2.1}$$

where $X_t$ is the first site in $S(t)^c$ visited by a simple random walk whose starting point has distribution $\mu_1$; and $Y_t$ is the first site in $S(t) \cup \{X_t\}$ visited by an independent simple random walk whose starting point has distribution $\mu_2$. As in Fig. 2.1 we will think of every vertex colored either 'blue' or 'red' and $S(t)$ will denote the set of blue vertices. If the distributions $\mu_1$ and $\mu_2$ have well-separated supports, one expects that these dynamics separate the graph into coherent red and blue territories.

Thus *competitive erosion* is a competing version of Internal DLA which is a fundamental growth model; first proposed by Meakin and Deutch in 1986 as a model of industrial chemical processes such as electropolishing, corrosion and etching. Competing particle systems modeling co-exitence of various species etc, have been the subject of intense study in physical sciences as well as mathematics: competing versions of the Richardson model were considered in [8, 50], processes modeling annihilation between competing species were studied in a series of works, see for e.g. [18, 19]. The study of *competitive erosion* was initiated in [45] where the reader can find a detailed introduction of the process. The underlying graph considered in [45] was the cylinder (the product of a path and a cycle). However the original question asked by Propp ([75]) was in the setting where the underlying graph is a discrete approximation to a general smooth simply connected domain. The process was predicted to exhibit *conformal invariance*. Confirming this is the goal of the chapter. More on comparison with Internal DLA is presented in Section 2.2.4.

Many 2D lattice models of physical phenomena are believed to have conformally invariant scaling limits: percolation, Ising model, self-avoiding polymers, dimer models etc. The last decade and a half has witnessed tremendous progress in understanding such models. Below we mention some of the seminal works; conformal invariance of various observables related to dimer models, was established by Kenyon (see [56–58]), Lawler, Schramm, Werner [64] proved convergence of loop erased random walk to a conformally invariant scaling limit while conformal invariance in percolation and spin systems was shown by Smirnov, see [85] and the references therein.

### *2.2.1 Conformal Invariance and informal setup*

We consider the case when the underlying graph is a discretization of a smooth bounded planar simply connected domain $\mathbb{U}$ i.e. $V = \mathbb{U}_n := \frac{1}{n}\mathbb{Z}^2 \cap \mathbb{U}$ and $k = \lfloor \alpha |\mathbb{U}_n| \rfloor$ for some fixed $\alpha \in (0, 1/2]$. Consider the case when the underlying domain is the unit disc $\mathbb{D}$ thought of as a subset of the complex plane $\mathbb{C}$. Simulations (Fig. 2.1) show if the measures $\mu_1$ and $\mu_2$ are Dirac measures on $-i, i$, then after running the process for some time the blue and red regions are separated with the boundary being an orthogonal circular arc such that the blue region has $\alpha$ fraction of the total area. Let us call this region $\mathbb{D}_{(\alpha)}$. For a general domain $\mathbb{U}$ with two points $(x_1, x_2)$ on the boundary, one can obtain the corresponding $\mathbb{U}_{(\alpha)}$ by conformally mapping $\mathbb{D}$ to $\mathbb{U}$ with $-i, i$ mapped to $x_1, x_2$ respectively and using the area measure on $\mathbb{U}$ to ensure that $\mathbb{U}_{(\alpha)}$ contains $\alpha$ fraction of $\mathrm{area}(\mathbb{U})$, (see Fig 2.3). Precise definitions are provided in subsection 2.2.3.
It was predicted by Propp in 2003 based on the conformally invariant nature of harmonic measure that competitive erosion should exhibit *conformal invariance*. That is, for any reasonably regular domain $\mathbb{U}$ and the erosion chain on $\mathbb{U}_n$ with $k = \lfloor \alpha |\mathbb{U}_n| \rfloor$ and the blue and red sources being $x_1, x_2$ respectively the blue region should "converge" to $\mathbb{U}_{(\alpha)}$ as $n$ grows

large. Our main result confirms this.

However for technical reasons we need $\mu_1, \mu_2$ to be “smooth” and supported in the interior on $\mathbb{U}$ instead of being point masses. We will work with uniform measures on all lattice points inside small ‘blobs’ of radii $\delta/4$ lying inside $\mathbb{U}$, and also at distance $\delta$ from the points $x_1, x_2$ respectively. See Fig 2.2 i.
The main result then involves sending $\delta$ to $0$ to “approximate” the Dirac measures at $x_1, x_2$.
We now present the statement of our main result.
To improve readability at this point we postpone the precise description of the technical assumptions including the definitions of the blobs. These are discussed right after the statement of the theorem (subsection 2.2.3).

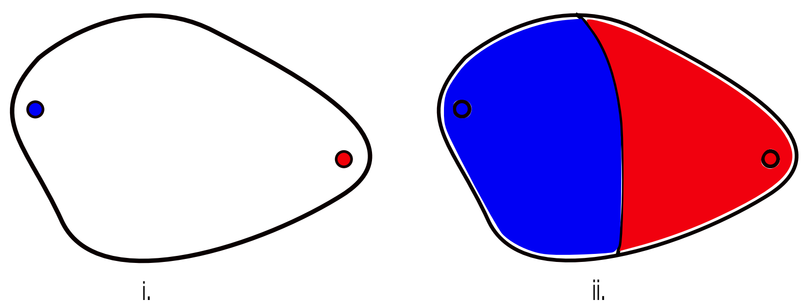


Figure 2.2: On the left: uniform measure on the two blobs close to the points $x_1, x_2$ act as “smooth” approximations of the corresponding Dirac measures. On the right: The red and blue territories have separated out after running the erosion chain with the blue region being “close” to $\mathbb{U}_\alpha$ if the blobs are small enough.

### *2.2.2 Main result*

Recall that $\alpha \in (0, \frac{1}{2}]$ is the fraction of blue vertices. Let a bounded simply connected planar domain “$\mathbb{U} \subset \mathbb{C}$ is smooth” mean that the boundary of $\mathbb{U}$ is an analytic curve (equivalently the conformal map from $\mathbb{U}$ to $\mathbb{D}$ has a conformal extension across the boundary, see [74, Prop 3.1]).

Let

$$\mathcal{G}_{\epsilon,n} := \{S \subset \mathbb{U}_n \, : |S| = \lfloor \alpha |\mathbb{U}_n| \rfloor, \; \mathbb{U}_{(\alpha-\varepsilon)} \cap \mathbb{U}_n \subset S \subset \mathbb{U}_{(\alpha+\varepsilon)} \cap \mathbb{U}_n \}, \tag{2.2.2}$$

Note that because of the smoothness of the domain $\mathbb{U}$, the boundaries of $\mathbb{U}_{(\alpha)}$ and $\mathbb{U}_{(\alpha\pm\varepsilon)}$ are $\Theta(\varepsilon)$ away from each other. $\alpha$ will be fixed throughout the rest of the chapter.

**Theorem 2.2.1.** *(Main Result) Let $\mathbb{U} \subset \mathbb{C}$ be a smooth bounded simply connected domain. Consider the competitive erosion chain on $\mathbb{U}_n = \mathbb{U} \cap \frac{1}{n}\mathbb{Z}^2$ with blob radii $\delta/4$ and $\lfloor \alpha |\mathbb{U}_n| \rfloor$ blue vertices. Then, given $\varepsilon > 0$ for $\delta < \delta_0(\varepsilon)$ there exists a positive constant $d = d(\varepsilon, \delta)$ such that for all large enough $n = 2^m$,*

$$\pi_{\delta,n}(\mathcal{G}_{\varepsilon,n}) \geq 1 - e^{-dn^{1/3}} \tag{2.2.3}$$

*where $S$ is the set of blue vertices sampled from $\pi_{\delta,n}$, the stationary measure of the chain.*

See Fig 2.1 and 2.2 ii. Note that for brevity we suppress the centers of the blobs in the notation $\pi_{\delta,n}$. The centers are points chosen at distance at least $\frac{\delta}{2}$ from the boundary of $\mathbb{U}$ and also at distance $\delta$, from $x_1, x_2$ respectively (see Fig 2.4), formally described in Section 2.2.3.

Informally the above theorem says the following: consider the erosion chain on the discretization of any smooth domain $\mathbb{U}$ with points $x_1, x_2$ on the boundary. Suppose that $\alpha$ fraction of the vertices are blue. If the blobs from which the blue and red random walks start are small enough and close enough to $x_1$ and $x_2$ then as the mesh size goes to zero, at stationarity, the blue region looks like the set $\mathbb{U}_{(\alpha)}$, in the sense that on the '$\mathbb{U}_{(\alpha)}$ side' of a small band around the boundary of $\mathbb{U}_{(\alpha)}$, the vertices are all blue and similarly on the 'other side', all vertices are red. As discussed above, the sets $\mathbb{U}_{(\alpha)}$ for various domains $\mathbb{U}$ can be obtained from each other via conformal maps. Thus Theorem 2.2.1 establishes that competitive erosion is *conformally invariant*.

**Note** that the mesh size goes to $0$ at dyadic scales. This is for technical convenience. Reasons are elaborated in Section 2.2.5 iii.

### 2.2.3 Formal definitions and setup

In this section we make precise all the definitions and the entire setup leading to the statement of Theorem 2.2.1. We will also present a more quantitative version of the result.
For any two points $x, y \in \mathbb{C}$, $d(x,y)$ will be used to denote the euclidean distance between them. For any set $A \subset \mathbb{C}$ and any $x \in \mathbb{C}$ denote by $d(x, A)$, the distance between the point and the set. $\mathbb{D}$ will be used to denote the unit disc centered at the origin in the complex plane. $B(x, \varepsilon)$ denotes the open euclidean ball of radius $\varepsilon$ with center $x$.

*Throughout the rest of the chapter, all our domains will be bounded, simply connected, smooth (as described before the statement of Theorem 2.2.1) and hence we will drop the adjectives.*

**Definition 2.2.1.** For any domain $B \subset \mathbb{C}$ denote by $\partial B$ the boundary of $B$.

For a domain $\mathbb{U}$ and points $x_1, x_2 \in \partial\mathbb{U}$ let

$$\begin{aligned} \phi &: \mathbb{D} \to \mathbb{U} \\ \psi &: \mathbb{U} \to \mathbb{D} \end{aligned} \tag{2.2.4}$$

be conformal maps such that $\phi \circ \psi$, $\psi \circ \phi$ are the identity maps on the respective domains and $\psi(x_1) = -i$, $\psi(x_2) = i$. The existence of such maps is guaranteed by the Riemann Mapping Theorem. See for e.g.: [1, Chapter 6]. In fact there exists more than one pair of $(\phi, \psi)$ since a conformal map between domains has three degrees of freedom and here we have fixed the value at only two points. However we choose a particular pair $(\phi, \psi)$ assumed to be fixed throughout the rest of the chapter. For any $\beta \in \mathbb{R}$ define

$$\mathbb{U}_\beta := \Big\{ z \in \mathbb{U} : \frac{64}{\pi} \log \Big| \frac{\psi(z) - i}{\psi(z) + i} \Big| \geq \beta \Big\}. \tag{2.2.5}$$

The constant $\frac{64}{\pi}$ is not important. It falls out of some natural integrals involving the Brownian motion heat kernel appearing later in the chapter. Thus for the disc, $\mathbb{D}_\beta$ is a region containing $-i$ and enclosed by an orthogonal circular arc (geodesic with respect to the hyperbolic metric) symmetric with respect to $i$ and $-i$. For a general domain $\mathbb{U}$, we transfer the regions via conformal maps and the boundaries still remain geodesics as they are conformally invariant. See Fig. 2.3. However for our purposes we need an area parametriza-

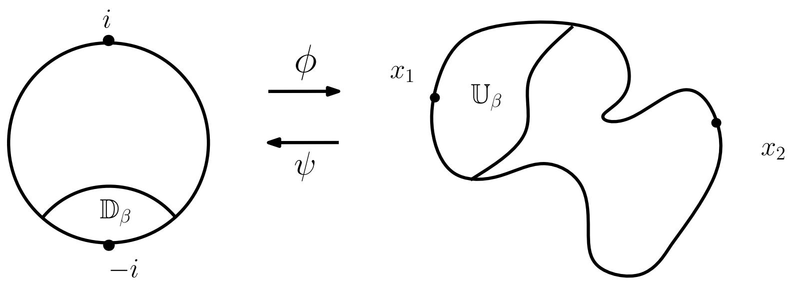


Figure 2.3: Hyperbolic geodesics are circular arcs on the disc. They are invariant under conformal maps. However since conformal maps are not area preserving $\mathbb{D}_\alpha$ can get mapped to $\mathbb{U}_{\alpha'}$ for some $\alpha' \neq \alpha$ .

tion of the regions $\mathbb{U}_\beta$. Given $\alpha \in (0,1)$, let $\beta = \beta(\alpha)$ be such that

$$\text{area}(\mathbb{U}_\beta) = \alpha \ \text{area}(\mathbb{U}). \tag{2.2.6}$$

Let

$$\mathbb{U}_{(\alpha)} := \mathbb{U}_{\beta(\alpha)}. \tag{2.2.7}$$

We now describe the set up in the statement of Theorem 2.2.1 formally.

**Setup 1.** Given $\mathbb{U}$ we take $\mathbb{U}_n = \mathbb{U} \cap (\frac{1}{n}\mathbb{Z}^2)$, as our vertex set. As the edges of our graph we take the usual nearest-neighbor edges of $\mathbb{U}_n$ though of as a subset of $\frac{1}{n}\mathbb{Z}^2$. However we delete every such edge which intersects $\mathbb{U}^c$. Since $\mathbb{U}$ is assumed to be smooth, $\mathbb{U}_n$ will be a connected for large enough $n$. See Remark 2.2.1 below.

Fix $\alpha \in (0, \frac{1}{2}]$, and take $k = \lfloor \alpha |\mathbb{U}_n| \rfloor$. Let $x_1, x_2 \in \partial\mathbb{U}$. The following defines the blobs mentioned in the statement of Theorem 2.2.1. For small enough $\delta > 0$ let $y_1, y_2 \in \mathbb{U}$ be such that

$$\begin{aligned} d(x_i, y_i) &= \delta \\ d(y_i, \partial\mathbb{U}) &> \delta/2. \end{aligned}$$

For $i = 1, 2$, let $\mathbb{U}_i = B(y_i, \frac{\delta}{4})$. As discrete approximations of $\mathbb{U}_i$ we take

$$\mathbb{U}_{i,n} = B(z_{i,n}, \frac{\delta}{4}) \cap \mathbb{U}_n$$

where $z_{i,n} \in \frac{1}{n}\mathbb{Z}^2$ is the closest lattice point to $y_i$. Define $\mu_i$ to be the uniform measure on $\mathbb{U}_{i,n}$.

Thus the blobs are the sets $\mathbb{U}_1$ and $\mathbb{U}_2$ and $\mu_i$'s are uniform over $\mathbb{U}_{i,n}$. Note that in the above, $y_i$'s were just required to satisfy certain properties and other than that were completely arbitrary. Also we abuse notation a little in the definition of the blobs, $n$ should be thought of as large and hence $\mathbb{U}_n$ (the underlying graph) should not be confused with the blobs $\mathbb{U}_1$ and $\mathbb{U}_2$.

**Remark 2.2.1.** The smoothness assumption on $\mathbb{U}$ allows us to choose the $y_i$'s. This is formally proved in Corollary 2.10.1. See Fig. 2.4. The connectedness of $\mathbb{U}_n$ for large enough $n$ follows since near the boundary $\mathbb{U}$ looks like a half plane locally, see (2.10.1).

It will be instructive to prove first the following weaker version of Theorem 2.2.1.

*Statistical version of Theorem 2.2.1*

As mentioned already, to highlight the symmetrical roles played by $S(t)$ and its complement (see (2.2.1)), we will think of the states as 2-colorings (blue and red) rather than sets. In this language $S(t)$ will denote the set of all blue vertices.

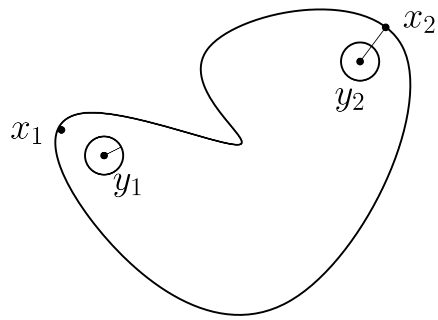


Figure 2.4: $y_i$'s are points at distance $\delta$ from $x_i$'s. They are at distance at least $\frac{\delta}{2}$ from $\partial\mathbb{U}$. The blobs are discs of radius $\frac{\delta}{4}$ centered at the $y_i$'s.

Formally define,

$$\Omega := \Omega_n := \{\sigma \in \{1,2\}^{\mathbb{U}_n}, \#\{x \in \mathbb{U}_n : \sigma(x) = 1\} = \lfloor \alpha|\mathbb{U}_n| \rfloor\} \tag{2.2.8}$$
$$\Omega' := \Omega'_n := \{\sigma \in \{1,2\}^{\mathbb{U}_n}, \#\{x \in \mathbb{U}_n : \sigma(x) = 1\} = \lfloor \alpha|\mathbb{U}_n| \rfloor + 1\}.$$

We let,

$$\sigma_t(x) = \begin{cases} 1 & x \in S(t) \\ 2 & x \notin S(t). \end{cases} \tag{2.2.9}$$

Thus the competitive erosion chain $S(t)$, can be thought of as a Markov chain $\sigma_t, \ \ t = 0, 1, 2, \ldots$, on $\Omega$. However for later purposes we need the following notation which keeps track of the full process,

$$\sigma_t, \quad t = 0, 1/2, 1, 3/2, 2, \ldots \tag{2.2.10}$$

where $\sigma_t \in \Omega$ or $\sigma_t \in \Omega'$ depending on whether $t$ is an integer or a half integer. Clearly a single time step of the chain consists of a step from $\Omega$ to $\Omega'$ followed by a step from $\Omega'$ back to $\Omega$.

Given $\varepsilon > 0$, define $\mathcal{A}_\varepsilon := \mathcal{A}_{n,\varepsilon}$ to be the set of all configurations $\sigma \in \Omega_n$ such that,

$$\#\{x \in \mathbb{U}_{(\alpha)} : \sigma(x) = 2\} \quad \leqslant \quad \varepsilon n^2, \tag{2.2.11}$$
$$\#\{x \in \mathbb{U}^c_{(\alpha)} : \sigma(x) = 1\} \quad \leqslant \quad \varepsilon n^2,$$

where $\mathbb{U}_{(\alpha)}$ was defined in (2.2.7). Thus informally if $\varepsilon$ is small, $\Omega_{n,\varepsilon}$ is the set of all configurations where the amount of "dust" particles of the wrong color has small density. See Fig. 2.5. The next result shows that the stationary measure of $\mathcal{A}_{n,\varepsilon}$ is large.

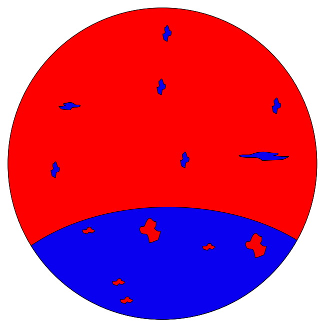

Figure 2.5: A typical configuration in $\Omega_{n,\varepsilon}$ for small $\varepsilon$ with small number of particles of the "wrong" color. By Theorem 2.2.2 the stationary measure of the erosion chain concentrates on such configurations.

**Theorem 2.2.2.** . *Let* $\mathbb{U}$ *be as in Setup 1 and* $\varepsilon > 0$. *Then there exists* $\delta_0 = \delta_0(\varepsilon)$ *such that for all* $\delta \leqslant \delta_0$ *and* $n = 2^m > N(\delta)$,

$$\pi_{\delta,n}(\mathcal{A}_{n,\varepsilon}) \geq 1 - e^{-Dn^2}$$

*where* $D = D(\varepsilon, \delta, \mathbb{U}) > 0$, *where* $\pi_{\delta,n}$, *the stationary measure of the erosion chain on* $\mathbb{U}_n$ *with* $\delta$ *as in Setup 1.*

### *2.2.4 Comparison with IDLA*

Internal diffusion limited aggregation (IDLA) is a fundamental model of a random interface moving in a *monotone* (outward) fashion. IDLA involves only one species with an ever-growing territory

$$I(t+1) = I(t) \cup \{X_t\}$$

where $X_t$ is the first site in $I(t)^c$ visited by a simple random walk whose starting point has distribution $\mu_1$. Competitive erosion can be viewed as a symmetrized version of IDLA: whereas $I(t)$ and $I(t)^c$ play asymmetric roles, $S(t)$ and $S(t)^c$ play symmetric roles in (2.2.1). IDLA on a finite graph is only defined up to the finite time $t$ when $I(t)$ is the entire vertex set. For this reason, the IDLA is usually studied on an infinite graph and the theorems about IDLA are limit theorems: asymptotic shape [63], order of the fluctuations [9, 10, 54], and distributional limit of the fluctuations [55]. In contrast, competitive erosion on a finite graph is defined for all times, so it is natural to ask about its stationary distribution. To appreciate the difference in character between IDLA and competitive erosion, note that the stationary distribution of the latter assigns tiny but positive probability to configurations that look very different to the final figure in Fig. 2.1. Thus competitive erosion will occasionally form these exceptional configurations.

### *2.2.5 Remarks about Theorems 2.2.1, 2.2.2 and Setup 1*

i. Since $\mathbb{U}$ is smooth, using Schwarz reflection $\phi$ and hence $\psi$ can be extended conformally across the boundary onto some neighborhoods of $\overline{\mathbb{D}}$ and $\overline{\mathbb{U}}$. In particular this implies that $|\phi'|$ and $|\psi'|$ are bounded away from $0$ and $\infty$ on $\mathbb{U}$ and $\mathbb{D}$ respectively. See [74, Prop 3.1]. This bi-Lipschitz nature of the maps will be used in several distortion estimates throughout the rest of the chapter.

ii. Note that the blob sources $\mathbb{U}_i$ lie entirely in the interior of $\mathbb{U}$ and the measures $\mu_i$ should be thought of as "smooth" approximations to point masses at points $x_1, x_2$ as $\delta \to 0$. Also by choice $|\mathbb{U}_{1,n}| = |\mathbb{U}_{2,n}|$. This will be technically convenient.

iii. Lastly we discuss the choice of the dyadic mesh sizes in the statements of Theorems 2.2.1 and 2.2.2. The technical core of the proof of the theorems involve convergence of the random walk on $\mathbb{U}_n$ under proper scaling of time to Reflected Brownian motion on $\overline{\mathbb{U}}$. The proof of this appears in [20] and subsequent local CLT estimates have been obtained in [25]. However both the above papers assume dyadic discretization in their proofs. Since the proof of our results rely heavily on the aforementioned convergence results we stick to the dyadic discretization in our statements as well.

## 2.3 Sketch of the proofs and organization of the chapter

To prove Theorem 2.2.2 i.e. the stationary distribution concentrates on the small set $\mathcal{A}_\varepsilon := \mathcal{A}_{n,\varepsilon}$, we identify a *Lyapunov function* $w(\cdot)$ on the state space. That is a function which attains its global maximum in $\mathcal{A}_\varepsilon$ and increases in expectation in one step of the process when starting outside $\mathcal{A}_\varepsilon$, i.e. if $\sigma_0 \notin \mathcal{A}_\varepsilon$,

$$\mathbb{E}(w(\sigma_1) - w(\sigma_0) \mid \sigma_0) \geqslant a > 0. \tag{2.3.1}$$

For more on Lyapunov functions see [43]. To construct such a function $w(\cdot)$ we proceed by defining the following discrete Green function: for any $x \in \mathbb{U}_n$,

$$G_n(x) = \frac{2n^2}{|\mathbb{U}_{1,n}|} \int_0^\infty \mathbb{P}_x(X(t) \in \mathbb{U}_{1,n}) - \mathbb{P}_x(X(t) \in \mathbb{U}_{2,n})\, dt, \tag{2.3.2}$$

where $\mathbb{P}_x(\cdot)$ is the measure induced by the continuous time random walk on the graph $\mathbb{U}_n$ started from $x$ with mean holding time $\frac{1}{2n^2}$. This choice of holding time is made to ensure that the random walk converges to Reflecting Brownian motion on $\overline{\mathbb{U}}$. Thus the above function is the expected difference in the amount of time random walk spends in $\mathbb{U}_{1,n}$ and in $\mathbb{U}_{2,n}$ respectively. Now for any $\sigma \in \Omega \cup \Omega'$ we define the weight function,

$$w(\sigma) := \sum_{x \in B_1(\sigma)} G_n(x),$$

where for $\sigma \in \Omega \cup \Omega'$ and $i = 1, 2$

$$B_i(\sigma) := \{x \in \mathbb{U}_n : \sigma(x) = i\}. \tag{2.3.3}$$

Recalling $S(t)$ from the definition of competitive erosion we note that $S(t) = B_1(\sigma_t)$. Now recall the map $\psi$ from (2.2.4).

Theorem 2.6.1 states that: up to translation by a constant $G_n(x)$ *is close to the function* $\frac{64}{\pi}\log\left|\frac{\psi(x)-i}{\psi(x)+i}\right|$ *if the blob sizes are small and* $n$ *is large.*
This justifies why $w(\sigma)$ is maximized if all the blue vertices are in $\mathbb{U}_{(\alpha)}$ (see (2.2.7)). The proof of the above is technical and involves developing a convergence theory for discrete Green function with Neumann boundary conditions (this is done in [48]). The proofs rely on local central limit theorems for convergence of random walk on $\mathbb{U}_n$ to Reflected Brownian motion in $\mathbb{U}$. Some of these estimates were obtained very recently in [20], [25] and [42].

The technical core in the chapter consists of the following:

- Proof of Theorem 2.5.1, which uses an electrical resistance argument involving Rayleigh's monotonicity principle to establish the drift (2.3.1).

Once (2.3.1) is established we use Azuma's inequality to argue that the process $w(\sigma_t)$ spends nearly all its time in a neighborhood of its maximum. These ingredients together with a general estimate relating hitting times to stationary distributions (Lemma 2.5.7) establish Theorem 2.2.2.
To prove Theorem 2.2.1 from Theorem 2.2.2 we show that starting erosion from a configuration $\sigma \in \mathcal{A}_\varepsilon$, the remaining dust particles (red particles in $\mathbb{U}_{(\alpha-\varepsilon)}$) get wiped off quickly. This is shown using robust IDLA estimates on $\mathbb{U}_n$ starting from the random environment $\sigma$. Proof of such estimates constitutes the remaining technical challenge.

### *2.3.1 The level set heuristic*

We now mention a heuristic that justifies the construction of the function $w(\cdot)$ outlined in the previous subsection and predicts the location of the competitive erosion interface. Consider the case of "well-separated" measures $\mu_1, \mu_2$. on general finite connected graph, which we assume for simplicity to be $r$-regular. Let $g$ be a function on the vertices satisfying

$$\Delta g = \mu_1 - \mu_2 \tag{2.3.4}$$

where $\Delta$ denotes the Laplacian $\Delta g(x) := \frac{1}{r}\sum_{y\sim x}(g(x) - g(y))$ and the sum is over vertices $y$ neighboring $x$. Since the graph is assumed connected, the kernel of $\Delta$ is one-dimensional consisting of the constant functions, so that equation (2.3.4) determines $g$ up to an additive constant.

The *boundary* of a set of vertices $S \subset V$ is the set

$$\partial S = \{v \in V \backslash S \, : \, v \sim w \text{ for some } w \in S\}, \tag{2.3.5}$$

where $v \sim w$ denotes that they are neighboring vertices. Consider a partition of the vertex set $V = S_1 \sqcup B \sqcup S_2$ where $\partial S_1 = B = \partial S_2$. (Think of $S_1$ and $S_2$ as the blue and red territories respectively, of the sort we might expect to see in equilibrium, and the sites in their common boundary $B$ have indeterminate color.)

Let $g_i$ for $i = 1, 2$ be the Green function for random walk started according to $\mu_i$ and stopped on exiting $S_i$. These functions satisfy

$$\begin{aligned} \Delta g_i &= \mu_i \quad \text{on } S_i, \\ g_i &= 0 \quad \text{on } S_i^c. \end{aligned}$$

The probability that simple random walk started according to $\mu_i$ first exits $S_i$ at $x \in B$ is $-\Delta g_i(x)$. To maintain equilibrium in competitive erosion, we seek a partition such that $\Delta g_1 \approx \Delta g_2$ i.e. they are roughly equal, on $B$, that is

$$\Delta(g_1 - g_2) \approx \mu_1 - \mu_2,$$

(exact equality holds except on $B$). Thus by (2.3.4), the function $g-(g_1-g_2)$ is approximately constant. Since $g_i$ vanishes on $B$, the equilibrium interface $B$ should have the property that

$$g \text{ is approximately constant on } B.$$

The partition that comes closest to achieving this goal takes $S_1$ to be the level set

$$S_1 = \{x \,:\, g(x) < K\} \tag{2.3.6}$$

for a cutoff $K$ chosen to make $\#S_1 = k$. An application of the maximum principle shows that for this choice of $S_1$, the maximum and minimum values of $g-(g_1-g_2)$ differ by at most

$$\max_{x\in S_1, y\notin S_1, x\sim y} |g(x)-g(y)|,$$

suggesting that the right notion of "well-separated" measures $\mu_1$ and $\mu_2$ is that the resulting function $g$ has small gradient.

In our case as in Setup 1, $\mu_i$ is the uniform measure on $\mathbb{U}_{i,n}$. It is easy to check and shown in (2.6.4) that $\Delta G_n = \mu_1 - \mu_2$ where $G_n$ was defined in (2.3.2). This shows why the choice of the weight function is "natural".

### *2.3.2 Organization of the chapter*

In Section 2.4 we show that the competitive erosion chain has a unique communicating class of recurrent states. The proof of the main result appears in Section 2.8. Theorem 2.2.2 is proven in Section 2.5. The key idea behind the proof of Theorem 2.2.2 is the construction of a suitable Lyapunov function as discussed in the previous subsection. This construction involves defining a suitable potential function. This is done is Section 2.6. Theorem 2.5.1 states that the constructed function indeed can be used as a Lyapunov function by establishing the drift condition (2.3.1). The proof of this theorem poses the main technical challenge which we address in Section 2.7. Theorem 2.6.1 is one of the main convergence results that we use in this chapter which states that the Lyapunov function is asymptotically conformally invariant. We discuss a sketch of the proof in Section 2.9. The complete proof appears in [48].

Theorem 2.2.2 follows from Theorem 2.5.1 and hitting time estimates for sub-martingales and a general result in markov chains relation hitting times to stationary measure. Section 2.5 is devoted to this. The proof of the main result relies on additional robust IDLA estimates on $\mathbb{U}_n$ which are provided in the Section 2.11. Some basic geometric facts are proved in Section 2.10.

### *2.3.3 Assumptions, notations and conventions*

i. Through out this chapter, by random walk on $\mathbb{U}_n$, we will mean the continuous time random walk with $\exp(2n^2)$ waiting times unless specifically mentioned otherwise. This is done to ensure that the random walk density converges to that of RBM. A fact which would be used heavily through out the chapter.

We now recall some of the notations already used and introduced some new notations to be used throughout the rest of the chapter. We will denote the complex plane by $\mathbb{C}$. As already used earlier for any two points $x, y \in \mathbb{C}$, $d(x, y)$ will denote the euclidean distance between them. Also for any set $A \subset \mathbb{C}$ and any $x \in \mathbb{C}$ denote by $d(x, A)$, the distance between the point and the set and similarly $d(A, B)$ denotes the distance between two sets $A, B \subset \mathbb{C}$. $B(x, \varepsilon)$ denotes the open euclidean ball of radius $\varepsilon$ with center $x$.

For any process and a subset $A$ of the corresponding state space $\tau(A)$ will denote the hitting time of that set (we drop the dependence on the process in the notation since it will be clear from context). $\mathbf{1}(\cdot)$ will be used to denote the indicator function. For any process and a subset $A$ of the corresponding state space $\tau(A)$ will denote the hitting time of that set. When $\omega$ is a state in the space and $A$ is a set of states, we will write $\{\tau(A) \leqslant K \mid \omega\}$ to denote the event that, starting from $\omega$, the process hits $A$ in at most $K$ steps.

We will often use the same letter (generally $C$, $D$, $c$ or $d$) for a constant whose value may change from line to line. We use $\asymp$ to denote that two quantities that are equal up to universal constants. Also $O(\cdot), \Omega(\cdot), \Theta(\cdot)$ are used to denote their usual meaning.

## 2.4 Connectivity properties

Implicit in the statement of Theorem 2.2.1 is the claim that competitive erosion has a unique stationary distribution. In this section we formally define the competitive erosion chain and prove this claim.

### 2.4.1 Formal definition of competitive erosion

Recall $\mathbb{U}_n$ from Setup 1. We will use $\mathbb{U}_n$ to denote both this graph and its set of vertices. For $t = 0, 1, 2, \ldots$ let $(X_s^{(t)})_{s \geqslant 0}$ and $(Y_s^{(t+\frac{1}{2})})_{s \geqslant 0}$ be independent simple random walks in $\mathbb{U}_n$ with

$$P(X_0^{(t)} = z_1) = \frac{1}{|\mathbb{U}_{1,n}|} = \frac{1}{|\mathbb{U}_{2,n}|} = P(Y_0^{(t+\frac{1}{2})} = z_2)$$

for all $z_1, z_2 \in \mathbb{U}_{1,n}, \mathbb{U}_{2,n}$ respectively. That is, each walk $X^{(t)}$ starts uniformly on $\mathbb{U}_{1,n}$ and each walk $Y^{(t+\frac{1}{2})}$ starts uniformly on $\mathbb{U}_{2,n}$. Given the state $S(t)$ of the competitive erosion chain at time $t$, we build the next state $S(t+1)$ in two steps as follows.

$$S(t + \frac{1}{2}) = S(t) \cup \{X_{\tau(t)}^{(t)}\}$$

where $\tau(t) = \inf\{s \geqslant 0 \,:\, X_s^{(t)} \notin S(t)\}$. Let

$$S(t+1) = S(t + \frac{1}{2}) - \{Y_{\tau(t+\frac{1}{2})}^{(t+\frac{1}{2})}\}$$

where $\tau(t + \frac{1}{2}) = \inf\{s \geqslant 0 \,:\, Y_s^{(t+\frac{1}{2})} \in S(t + \frac{1}{2})\}$. As already stated in (2.2.9) this formally defines the evolution of the process $\sigma_t$ on $\Omega_n \cup \Omega_n'$ (see (2.2.8)).

### *2.4.2 Blocking sets and transient states*

We show that the erosion chain has an unique irreducible class and hence has a well defined stationary measure.

**Definition 2.4.1.** Call a subset $A \subset \mathbb{U}_n$ **blocking** if $\mathbb{U}_n \backslash A$ is disconnected and the subsets $\mathbb{U}_{1,n} \backslash A$ and $\mathbb{U}_{2,n} \backslash A$ lie in different components.

**Definition 2.4.2.** For two disjoint blocking subsets $A, B \subset \mathbb{U}_n$ we say that $A$ is **over** $B$ if

1. $A$ and $\mathbb{U}_{1,n} \backslash B$ lie in different components of $\mathbb{U}_n \backslash B$; and

2. $B$ and $\mathbb{U}_{2,n} \backslash A$ lie in different components of $\mathbb{U}_n \backslash A$.

**Lemma 2.4.1.** *The competitive erosion chain has exactly one irreducible class. Moreover any $\sigma \in \Omega$ that has a blue blocking set over a red blocking set is transient.*

*Proof.* We first claim the following: there exists $\sigma_* \in \Omega_n$ such that for $i = 1, 2$, $\sigma_*^{-1}(i)$ is a connected set containing $\mathbb{U}_{i,n}$. This is true in the continuous setting clearly by taking the sets to be $\mathbb{U}_\alpha$ and $\mathbb{U}_\alpha^c$ (see (2.2.7)). To find such subsets in $\mathbb{U}_n$ we look at $\mathbb{U}_\alpha \cap \mathbb{U}_n$ and its complement. These sets might not be connected and might not have the exact number of points. However using the smoothness of $\mathbb{U}$, and considering the sets bounded by a lattice path approximating the boundary of $\mathbb{U}_\alpha$ and making minor perturbations adding and removing only $O(n)$ many vertices gives us such sets. The details are omitted.

Now to prove the first part notice that from any $\sigma$ one can reach $\sigma_*$. Since in the target configuration there is exactly one blue component $B$ and red component $R$, we look at the closest vertex with $\sigma$ value $2$ from $\mathbb{U}_{1,n}$ in $B$. Since there is a blue path from $\mathbb{U}_{1,n}$ to that point the Markov chain allows us to change it to $1$ and similarly at the other end in $R$. Thus we are done by repeating this.

To prove the second part we prove that starting from $\sigma_*$ one cannot reach any configuration like the one on the right which has one blue blocking set over another red blocking set. We formally prove this by contradiction. Let $\sigma$ be a configuration with a blue blocking set

$B$ over a red blocking set $A$. Now the erosion chain evolves as $\sigma_0, \sigma_{1/2}, \sigma_1, \sigma_{3/2}\dots$ where for every non-negative integer $k$ $\sigma_k \in \Omega$ and $\sigma_{k+1/2} \in \Omega'$.

Assume now that there is a path $\sigma_0, \sigma_{1/2}, \sigma_1, \sigma_{3/2}\dots, \sigma_t = \sigma$. Let $\tau'$ and $\tau''$ be the last times along the path such that at least one vertex of $A$ is blue and similarly $B$ contains at least one red vertex respectively. If such a time does not exist let us call it $-\infty$. Since $\sigma_0$ does not have a blue blocking set over a red blocking set, $\max(\tau', \tau'') > -\infty$.

$\tau'$ must be a half integer (since at half integers there is one more blue particle) and similarly $\tau''$ must be an integer. Thus $\tau' \neq \tau''$. Next we see that we cannot have $\tau' < \tau''$. This is because then for all times greater than $\tau'$, $A$ is entirely red. Thus no blue walker crosses $A$ at any time greater than $\tau'$. So $B$ must already be blue at $\tau'$ and stays blue through till $t$ implying that $\tau'' < \tau'$. By similar argument we cannot have $\tau' > \tau''$. Hence we arrive at a contradiction. □

### 2.5 *Proof of Theorem 2.2.2*

The goal of this section is to prove Theorem 2.2.2 modulo some results whose proofs will be deferred to later sections. We first restate formally the crucial statistics mentioned in Section 3.2.1 to be used throughout the chapter. Recall from (2.3.2),

- The discrete Green function : for any $x \in \mathbb{U}_n$,

$$G_n(x) = \frac{2n^2}{|\mathbb{U}_{1,n}|}\int_0^\infty \mathbb{P}_x(X(t) \in \mathbb{U}_{1,n}) - \mathbb{P}_x(X(t) \in \mathbb{U}_{2,n})\,dt - c. \tag{2.5.1}$$

  where for $i = 1, 2$, $\mathbb{P}_x(X(t) \in \mathbb{U}_{i,n})$ is the probability that random walk on $\mathbb{U}_n$ started from $x$, is in $\mathbb{U}_{i,n}$ at time $t$. $c = c(\delta)$ is a centering constant we introduce to ensure that if $\delta$ is small and $n$ is large then the Green function is close to the function

$$\frac{64}{\pi}\log\left|\frac{\psi(x) - i}{\psi(x) + i}\right|. \tag{2.5.2}$$

  This follows from Theorem 2.6.1 and Lemma 2.6.3. $c(\delta)$ can be written in terms of the RBM heat kernel on $\mathbb{U}$. The explicit form of $c$ does not affect the arguments and appears later in (2.9.2).

- The weight function: for $\sigma \in \Omega \cup \Omega'$,

$$w(\sigma) = \sum_{x \in B_1} G_n(x) \tag{2.5.3}$$

where $B_1(\sigma)$ is defined in (2.3.3).

### *2.5.1 Technical results*

We start by defining the following 'good set'. Consider the following way to divide our domain $\mathbb{U}$ according to the geodesics defined in (2.6.8). Let

$$\ldots a_{-1}, a_0, a_1, a_2, \ldots$$

be such that $d(\gamma_{a_i}, \gamma_{a_{i+1}}) = \frac{100}{n}$, for all $i = \ldots -1, 0, 1, 2, \ldots$ and $a_0 = \beta$ (recall the definition $\mathbb{U}_\beta$ from (2.2.5)). Thus number of $a_i$'s is linear in $n$. Also away from the points $x_1, x_2$ on the boundary the function $\log\left|\frac{\psi(z)-i}{\psi(z)+i}\right|$ (see (2.6.8)) is a bounded function with bounded non zero derivative and hence as long as $d(\gamma_{a_i}, \{x_1, x_2\}) \geq a$ for some $a > 0$ (independent of $n$),

$$|a_i - a_{i-1}| = \Theta(\frac{1}{n}) \tag{2.5.4}$$

where the constants in the $\Theta(\cdot)$ notation depends on $a$ and $\mathbb{U}$. Let us define

$$\mathbb{U}^{(i)} := \mathbb{U}_{a_i} \backslash \mathbb{U}_{a_{i+1}}. \tag{2.5.5}$$

Given $\varepsilon > 0$, define,

$$\Omega_{(\varepsilon)} := \Omega_{n,(\varepsilon)} \text{ to be the set of all configurations } \sigma \in \Omega_n \tag{2.5.6}$$

such that there exists at most $\varepsilon n$ many negative $i's$ such that $\mathbb{U}^{(i)} \cap B_2 \neq \varnothing$, (see (2.3.3) for definition of $B_2$) and at most $\varepsilon n$ many positive $i's$ such that $\mathbb{U}^{(i)} \cap B_1 \neq \varnothing$. Thus in words $\Omega_{(\varepsilon)}$ consists of configurations where at most $2\varepsilon n$ many shells $\mathbb{U}^{(i)}$ contain vertices of the 'wrong' color.

Recall the erosion chain $\sigma_t$. As mentioned in Section 3.2.1 the next result is one of the key technical ingredients of the chapter. It shows that the weight function has a positive drift if the configuration is outside the set $\Omega_{(\varepsilon)}$.

**Theorem 2.5.1.** *Given $\varepsilon > 0$, there exists $a = a(\varepsilon) > 0$ such that for all small enough $\delta$, and all $n = 2^m$ large enough, (depending on $\delta$) if $\sigma \in \Omega^c_{(\varepsilon)}$ then*

$$\mathbb{E}(w(\sigma_1) - w(\sigma) \mid \sigma_0 = \sigma) > a.$$

The proof is quite involved and is deferred to Section 2.7. Next we state a few properties of the Green function (see (2.5.1)) and the weight function $w(\cdot)$ whose proofs are deferred to subsection 2.6.3. The following result proves a uniform upper bound of the weight function independent of $\delta$.

**Lemma 2.5.1.** *There exists a constant $D$ such that for all $\delta$ small enough and $n = 2^m > N(\delta)$, for all $\sigma \in \Omega$, $|w(\sigma)| \leqslant Dn^2$.*

The above statement is a consequence of the following fact: recall from (2.5.2) that as $\delta$ goes to $0$ the Green function approaches the function $\frac{64}{\pi} \log \left| \frac{\psi(z)-i}{\psi(z)+i} \right|$. Now if $n$ is chosen large enough the weight function divided by $n^2$ should approximate the integral of the above function over the blue region. The lemma now follows from the fact that the function $\log \left| \frac{\psi(z)-i}{\psi(z)+i} \right|$ is integrable.

Let us denote by

$$w_{\max} := \sup_{\sigma \in \Omega} w(\sigma). \tag{2.5.7}$$

The next lemma shows that the weight function is uniformly close to its maximum value over the set $\mathcal{A}_\varepsilon$ (Table 2.1)

**Lemma 2.5.2.** *For $\varepsilon > 0$. There exists constants $\delta_0(\varepsilon), \zeta(\varepsilon)$ such that for $\delta \leqslant \delta_0$ and large enough $n = 2^m > N(\delta)$,*

$$\inf_{\sigma \in \mathcal{A}_\varepsilon} w_\sigma \geq w_{\max} - \zeta n^2 \tag{2.5.8}$$

*where $\zeta$ goes to $0$ with $\varepsilon$.*

As mentioned above the Green function approaches the function $\frac{64}{\pi} \log \left| \frac{\psi(z)-i}{\psi(z)+i} \right|$, as $n \to \infty$ followed by $\delta$ going to $0$. Thus in particular it blows up in the limit. The next lemma established the "logarithmic singularity" of the limit by determining the rate of blow up of the function as $\delta$ goes to $0$.

**Lemma 2.5.3.** *For all small enough $\delta$,*

$$\limsup_{n=2^m} \sup_{z \in \mathbb{U}_n} |G_n(z)| \asymp |\log(\delta)|. \tag{2.5.9}$$

We define another 'good' set. Given a number $\varepsilon_1$ let,

$$\Gamma_{\varepsilon_1} = \{\sigma \in \Omega : w(\sigma) \geq w_{\max} - 2\varepsilon_1 n^2\}. \tag{2.5.10}$$

**Remark 2.5.1.** Given $\varepsilon > 0$ there exists $\varepsilon_2 < \varepsilon < \varepsilon_1$ such that for all $\delta < \delta_0(\varepsilon)$ and $n = 2^m \geq N(\delta)$,

$$\Gamma_{\varepsilon_2} \subset \mathcal{A}_\varepsilon \subset \Gamma_{\varepsilon_1}.$$

Moreover both $\varepsilon_1, \varepsilon_2$ go to $0$ as $\varepsilon$ goes to $0$. Also there exists $\varepsilon_3$ such that $\Omega_{(\varepsilon_3)} \subset \mathcal{A}_\varepsilon$.

Thus the above remark strengthens Lemma 2.5.2. It says that not only does the weight function stay close to its maximum on $\mathcal{A}_\varepsilon$, but $\mathcal{A}_\varepsilon$ is roughly a level set of the weight

| Set | Description | Defined in |
|---|---|---|
| $\mathcal{G}_\varepsilon$ | all the sites out side an $\varepsilon$ band are the right color | (2.2.2) |
| $\mathcal{A}_\varepsilon$ | at most $2\varepsilon n^2$ sites have the wrong color | (2.2.11) |
| $\Gamma_\varepsilon$ | weight function within $\varepsilon n^2$ of its maximum | (2.5.10) |
| $\Omega_{(\varepsilon)}$ | at most $\varepsilon n$ different 'levels' have sites of the wrong color | (2.5.6) |

Table 2.1: Different kinds of good sets.

function i.e. the actual level sets $\Gamma_\varepsilon$'s can be approximated by the sets $\mathcal{A}_\varepsilon$'s and vice versa. For proofs of Lemmas 2.5.1, 2.5.2, 2.5.3 and Remark 2.5.1 see subsection 2.6.3.

### *2.5.2 Hitting time estimates*

The next result proves a uniform bound on the hitting time of the set $\mathcal{A}_\varepsilon$ .

**Lemma 2.5.4.** *Given $\varepsilon_1, \varepsilon_2 > 0$ for all $\delta \leqslant \delta_0(\varepsilon_1, \varepsilon_2)$, there exists $C, d$ such that for all $n = 2^m > N(\delta)$ for all $\sigma \in \Omega$,*

$$\mathbb{P}_\sigma(\tau(\mathcal{A}_{\varepsilon_1} \cap \Omega_{(\varepsilon_2)}) \geq Cn^2) \leqslant e^{-dn^2} \tag{2.5.11}$$
$$\mathbb{E}_\sigma(\tau(\mathcal{A}_{\varepsilon_1} \cap \Omega_{(\varepsilon_2)})) \leqslant Cn^2$$

*where $\delta$ is the parameter appearing in Setup 1.*

Now we show that once the process has hit $\mathcal{A}_\varepsilon$ it tends to stay "close" to the set for a long time. By Remark 2.5.1 the next result suffices.

**Lemma 2.5.5.** *Given $\varepsilon_1 > 0$ there exists $\delta_0(\varepsilon_1)$ such that for all $\delta \leqslant \delta_0$ there exists a positive constant $d = d(\varepsilon_1, \delta)$ such that for all large enough $n = 2^m > N(\delta)$ and any $\sigma \in \Omega$*

$$\inf_{\sigma \in \Gamma_{\varepsilon_1}} \mathbb{P}_\sigma(\tau(\Omega \backslash \Gamma_{2\varepsilon_1}) > e^{dn^2}) \geq 1 - e^{-dn^2}.$$

The above results follow from a general lemma about hitting times for submartingales.

The statement has a few parameters and could be difficult to read. However for subsequent applications it would allow us to plug in various values for the parameters.

Let $\omega(t)$ be a stochastic process taking values in an abstract set $\mathcal{D}$. Also let $g : \mathcal{D} \to \mathbb{R}$ be a real valued function. Let $\mathcal{F}_t$ be the filtration generated by the process up to time $t$ and $X_t := g(\omega(t))$.

**Lemma 2.5.6.** [45, Lemma 9] *Let $A_1, A_2, a_1 > 0$. Suppose*

$$|g(\omega)| \leqslant A_1 \text{ for all } \omega \in \mathcal{D} \tag{2.5.12}$$

$$|X_t - X_{t-1}| \leqslant A_2 \text{ for all } t. \tag{2.5.13}$$

*Also suppose that $B \subset \mathcal{D}$ is such that for any time $t$*

$$\mathbb{E}(X_t - X_{t-1}|\mathcal{F}_{t-1}) \geq a_1 \mathbf{1}(\omega(t-1) \notin B) \tag{2.5.14}$$

*then*

*i.* $\mathbb{E}_w(\tau(b)) \leqslant \frac{a_2}{a_1}$. *Moreover* $P_\omega(\tau(B) \geq T) \leqslant \exp\left(-\frac{(a_2 - a_1 T)^2}{4A_2^2 T}\right)$ *for all $\omega \in \mathcal{D}$ such that $g(w) \geq A_1 - a_2$ for some $a_2 > 0$ and any $T$ such that $a_2 - a_1 T < 0$.*

*ii. Now consider the special case when $B$ is a level set i.e suppose for some $a_4 > 2A_2$, $B = \{\omega : g(w) \geq A_1 - a_4\}$ and $B' = \{\omega : g(w) \leqslant A_1 - 2a_4\}$. Then for all $\omega \in B$ and all $T > \frac{2A_2}{a_1}$*

$$\mathbb{P}_\omega(\tau(B') \geq T') \geq 1 - \left[\exp\left(-\frac{a_4^2}{32A_2^2 T}\right) + \exp\left(-\frac{a_1^2 T^2}{32 A_2^2 T}\right)\right]$$

*where* $T' = \exp\left(\frac{\min(a_4^2, a_1^2 T^2)}{32 A_2^2 T}\right)$.

Using the above result the proofs of Lemmas 2.5.4 and 2.5.5 follows easily.

**Proof of Lemma 2.5.4.** The proof follows from Lemma 2.5.6 $i$. The stochastic process we consider is the erosion chain $\sigma_t$. $X_t = w(\sigma_t)$ is the weight function defined in (2.5.3). We make the following choice of parameters.

$$A_1 = Dn^2 \qquad \text{appearing in Lemma 2.5.1}$$
$$A_2 = C\log(\delta) \qquad \text{for a large enough universal constant } C$$
$$a_1 = a(\varepsilon) \qquad \text{appearing in Theorem } 2.5.1$$
$$a_2 = 2Dn^2, \quad T = \frac{(3D+1)n^2}{a_1}.$$

(2.5.12) is satisfied satisfied by Lemma 2.5.1. That (2.5.13) is satisfied by our choice of $A_2$ follows from Lemma 2.5.3. Thus by Lemma 2.5.6 $i.$ the result follows. □

**Proof of Lemma 2.5.5.** The proof follows from Lemma 2.5.6 $ii$. Let

$$X(t) = w(\sigma_t)$$

where $w(\cdot)$ is the weight function defined in (2.5.3). We make the following choice of parameters:

$$B = \Gamma_{\varepsilon_1}, B' = \Gamma_{2\varepsilon_1}, a_4 = \varepsilon_1 n^2, T = n^2.$$
$$A_1 = w_{\max} \quad \text{(see (2.5.7))},$$
$$A_2 = C\log(\delta) \qquad \text{for a large enough universal constant } C,$$
$$a_1 = a(\varepsilon_3) \text{ appearing in Theorem } 2.5.1,$$

The drift condition (2.5.14) is satisfied by $a_1 = a(\varepsilon_3)$ where $a(\cdot)$ appears in Theorem 2.5.1 and $\varepsilon_3$ in Remark 2.5.1. Now by the above choice of parameters $T' = \frac{1}{2}e^{\frac{\min(\varepsilon_1^2, a_1^2)n^2}{16}}$. Thus by

Lemma 2.5.6 $ii.$ for all $\sigma \in \Gamma_{\varepsilon_1}$,

$$\mathbb{P}_\sigma(\tau(\Gamma_{2\varepsilon_1}) \geq T') \geq 1 - [e^{-\frac{\varepsilon_1^2 n^2}{16}} + e^{-\frac{a(\varepsilon)^2 n^2}{16}}].$$

Hence the proof is complete. □

### *2.5.3 Proof of Theorem 2.2.2.*

We will use the following general lemma relating hitting times and stationary measure. The result roughly says the following: consider any Markov chain and subsets $A \subset B$ of the state space . If the hitting time of the set $A$ is uniformly "small" compared to the exit time of $B$ after hitting $A$, then $B$ has large stationary measure.

**Lemma 2.5.7.** [49, Prop 1.4] *Let $\omega(\cdot)$ be an irreducible Markov chain on a finite state space $\mathcal{M}$. Suppose $A, B \subset \mathcal{M}$. Let*

$$t_1 = \max_{x \in B} \mathbb{E}_x(\tau(A)) \tag{2.5.15}$$

$$t_2 = \min_{x \in A} \mathbb{E}_x(\tau(B)) \tag{2.5.16}$$

*Then,*

$$\pi(B) \leqslant \frac{t_1}{t_1 + t_2}.$$

*where $\pi$ is the stationary distribution of the Markov chain $\omega(\cdot)$ on $\mathcal{M}$.*

We now finish off the proof of Theorem 2.2.2. Notice that by the lower containment in Remark 2.5.1 it suffices to prove that for given small enough $\varepsilon$ for all large enough $n = 2^m$

$$\pi(\Gamma_{2\varepsilon}) \geq 1 - e^{-cn^2} \tag{2.5.17}$$

for some $c = c(\varepsilon, \delta, \mathbb{U}) > 0$. The proof now follows immediately from Lemma 2.5.7 with the following choices of the parameters:

$$A = \Gamma_\varepsilon, B = \Omega \backslash \Gamma_{2\varepsilon}, t_1 = d_1 n^2, t_2 = e^{d_2 n^2},$$

where $c_1, c_2, d_1, d_2$ are chosen such that the hypotheses (2.5.15) and (2.5.16) of Lemma 2.5.7 are satisfied. Lemmas 2.5.4 and 2.5.5 allow us to do that. □

## *2.6 Technical Preliminaries*

In this section we will develop some preliminary tools needed for the proof of Theorem 2.5.1.

### *2.6.1 Discrete Green Function*

Recall that we consider continuous time random walk on $\mathbb{U}_n$ (Section 2.3.3 ii). Call it $X(t)$. For $x, y \in \mathbb{U}_n$ let

$$\mathbb{P}_x(X(t) = y) \tag{2.6.1}$$

denote the chance that the random walk on $\mathbb{U}_n$ starting from $x$ is at $y$ at time $t$. For notational simplicity we suppress the $n$ dependence in $\mathbb{P}$ since the graph will be clear from context. Similarly for any set $A \subset \mathbb{U}_n$, let $\mathbb{P}_x(X(t) \in A)$ denote the chance that the random walk is in $A$ at time $t$ starting from $x$. Recall the definition of the function $G_n$ on $\mathbb{U}_n$ from (2.5.1): for any $x \in \mathbb{U}_n$

$$G_n(x) = \frac{2n^2}{|\mathbb{U}_{1,n}|} \int_0^\infty [\mathbb{P}_x(X(t) \in \mathbb{U}_{1,n}) - \mathbb{P}_x(X(t) \in \mathbb{U}_{2,n})] dt - c. \tag{2.6.2}$$

where $c = c(\delta)$ is some constant explicitly mentioned in (2.9.2). The fact that the integral in $G_n$ is absolutely integrable follows immediately from the following lemma.

**Lemma 2.6.1.** [48, Lemma 3.1] *Given $\mathbb{U}$ as in Setup 1 there exists a constant $D = D(\mathbb{U})$*

*and a time $T = T(\mathbb{U})$ such that, for all large enough $n$,*

$$\sup_{x\in\mathbb{U}_n} |\mathbb{P}_x(X(t) \in \mathbb{U}_{1,n}) - \mathbb{P}_x(X(t) \in \mathbb{U}_{2,n})| \leqslant 2e^{-Dt}$$

*for all $t \geq T$.*

The above is a straightforward consequence of the mixing rate of random walk and the fact that since by choice $|\mathbb{U}_{1,n}| = |\mathbb{U}_{2,n}|$, (see subsection 2.2.5 ii.),

$$\pi_{RW}(\mathbb{U}_{1,n}) = \pi_{RW}(\mathbb{U}_{2,n}),$$

where $\pi_{RW}$ is the stationary measure for the random walk on $\mathbb{U}_n$.

For any function $f : \mathbb{U}_n \to \mathbb{R}$ define the laplacian $\Delta f : \mathbb{U}_n \to \mathbb{R}$: for any $x \in \mathbb{U}_n$

$$\Delta f(x) = f(x) - \frac{1}{d_x}\sum_{y\sim x} f(y) \tag{2.6.3}$$

where $d_x$ is the degree of the vertex $x$ and $y \sim x$ denotes that $y$ is a neighbor of $x$.

**Lemma 2.6.2.** *Consider the function $G_n(\cdot)$ on $\mathbb{U}_n$. Then,*

$$\Delta(G_n) = \frac{1}{|\mathbb{U}_{1,n}|}(\mathbf{1}(\mathbb{U}_{1,n}) - \mathbf{1}(\mathbb{U}_{2,n})), \tag{2.6.4}$$

*where for any subset $A \subset \mathbb{U}_n$, $\mathbf{1}(A)$ denotes the indicator of the set $A$.*

*Proof.* Proof follows from definition of $G_n$ and looking at the first step of random walk which by definition is of expected duration $\frac{1}{2n^2}$. Thus we have

$$G_n(x) = \frac{1}{|\mathbb{U}_{1,n}|}(\mathbf{1}(\mathbb{U}_{1,n}) - \mathbf{1}(\mathbb{U}_{2,n}))(x) + \frac{1}{d_x}\sum_{y\sim x} G_n(y)$$

and hence the lemma. □

### 2.6.2 Conformal invariance of Green function

In this section we state a key technical result establishing *conformal invariance* of the Green function. We define the following function on the smooth domain $\mathbb{U}$ :

$$\tilde{f} := \frac{16}{\text{area}(\mathbb{U}_1)}\big(\mathbf{1}(\mathbb{U}_2) - \mathbf{1}(\mathbb{U}_1)\big), \tag{2.6.5}$$

where $\mathbb{U}_1$ and $\mathbb{U}_2$ are defined in Setup 1. Again $16$ is a constant that appears for the same reason that $64$ appears in (2.2.5) and is not important.

Recall the functions $\phi$ and $\psi$ from (2.2.4). Now let us call $\psi(\mathbb{U}_i) = A_i$. Then

$$\tilde{f} \circ \phi = \frac{16}{\text{area}(\mathbb{U}_1)}\big(\mathbf{1}(A_2) - \mathbf{1}(A_1)\big). \tag{2.6.6}$$

For any $\mathbb{U}$ as in Setup 1, define the function $G_* : \overline{\mathbb{U}} \to \mathbb{R}$ such that for all $z \in \overline{\mathbb{U}}$ if $y \in \overline{\mathbb{D}}$ is such that $\phi(y) = z$, then

$$G_*(z) = \frac{1}{\pi}\int_{|\zeta|<1} \tilde{f} \circ \phi(\zeta)|\phi'(\zeta)|^2 \log(|(\zeta - y)(1 - \bar{\zeta}y)|^2)d\xi d\eta, \tag{2.6.7}$$

where $\zeta = \xi + i\eta$. The above expression is well defined on $\overline{\mathbb{U}}$ since by Section 2.2.5 i., the maps $\phi, \psi$ can be extended to neighborhoods containing $\overline{\mathbb{D}}, \overline{\mathbb{U}}$ respectively. Notice the dependence of $G_*$ on $\delta$ through $\tilde{f}$. However for brevity we choose to suppress the dependence on $\delta$ in the notation. The definition of the function might be a little mysterious at this point (the reason for our choice is explained in Section 2.9.1). Nevertheless Lemma 2.6.3 shows that as $\delta$ goes to $0$, the function approaches the function $\log\left|\frac{\psi(z)-i}{\psi(z)+i}\right|$, up to a multiplicative constant. Since the above function is conformally invariant by definition, this establishes that the function $G_*(\cdot)$ is "asymptotically" conformally invariant.

Even though the function $G_n$ in (2.6.2) is defined on the graph $\mathbb{U}_n$, in this section we use interpolation to think of it as a function on the closure of the whole domain, $\overline{\mathbb{U}}$. We are now ready to state one of the main convergence results of this chapter establishing asymptotic conformal invariance of $G_n(\cdot)$. We start with the interpolation scheme. We follow the scheme mentioned in [25, Pf of Theorem 2.12] ( also appears in [42, Pf of Theorem 2.2.8]).

For all $x \in \frac{1}{n}\mathbb{Z}^2\backslash\mathbb{U}_n$, define $G_n(x) = 0$. Now having defined $G_n(x)$ for all $x, \in \frac{1}{n}\mathbb{Z}^2$ we define it on $\mathbb{C}$ and hence by restricting on $\overline{\mathbb{U}}$. This is done by interpolating $G_n(x)$ by a sequence of harmonic extensions along simplices,

$i.$ First extend it along the edges using the value on the verices in $\frac{1}{n}\mathbb{Z}^2$.

$ii.$ Then extend it to the squares using the value on the edges.

Thus the function is extended to the entire complex plane $\mathbb{C}$. By abuse of notation in the following result we call the extended function $G_n(\cdot)$ as well.

**Theorem 2.6.1.** [48, Theorem 3.1] *For all small enough* $\delta$,

$$\lim_{\substack{m\to\infty \\ n=2^m}} \sup_{z\in\overline{\mathbb{U}}} |G_n(z) - G_*(z)| = 0.$$

The basic outline is sketched in Section 2.9. For the complete proof see [48]. We end this section by making a few observations about the function $G_*(\cdot)$ and hence by the aforementioned theorem, about the asysmptotics of the function $G_n$.

Let us define the following one parameter family of curves for any $\mathbb{U}$ which form the boundary of the sets $\mathbb{U}_\beta$ defined in (2.2.5). For any $\beta \in \mathbb{R}$ define

$$\gamma_\beta(\mathbb{U}) = \{z \in \mathbb{U} : \frac{64}{\pi} \log \left| \frac{\psi(z) - i}{\psi(z) + i} \right| = \beta\}. \tag{2.6.8}$$

We will often suppress the dependence on $\mathbb{U}$ and denote the above just by $\gamma_\beta$ for brevity, if the underlying domain is clear from context. It is well known that for any $\beta \in \mathbb{R}$, $\gamma_\beta$ is a hyperbolic geodesic. In case of the disc, $\gamma_\beta$'s are circular arcs orthogonal to $\partial\mathbb{D}$ symmetric with respect to $-i$ and $i$. For more details regarding the hyperbolic metric see [1].

The following is a uniform convergence result of $G_*(\cdot)$ away from the points $x_1, x_2$.

**Lemma 2.6.3.** *Let $\mathbb{U}$ be as in Setup 1. Given $a < 1$,*

$$\lim_{\delta\to 0} \sup_{\substack{z\in\overline{\mathbb{U}}:\\ d(z,x_1)\geq\delta^a\\ d(z,x_2)\geq\delta^a}} \left|G_*(z) - \frac{64}{\pi}\log\left|\frac{\psi(z)-i}{\psi(z)+i}\right|\right| = 0$$

*Proof.* Recall that the map $\phi$ is bi-Lipschitz by subsection 2.2.5 i. Thus it suffices to show that

$$\lim_{\delta\to 0} \sup_{\substack{z\in\overline{\mathbb{D}}:\\ d(z,-i)\geq\delta^a\\ d(z,i)\geq\delta^a}} \left|G_*\circ\phi(z) - \frac{64}{\pi}\log\left|\frac{z-i}{z+i}\right|\right| = 0.$$

Recall the sets $A_1, A_2 \subset \mathbb{D}$ from (2.6.6). Note that by hypothesis

$$\mathbb{U}_1 \subset B(x_1, 2\delta)\cap\mathbb{U}$$

and hence

$$A_1 \subset B(-i, C\delta)\cap\mathbb{D} \tag{2.6.9}$$

for some constant $C$ ( where $C/2$ is the Lipschitz constant of $\psi$). Similar containment holds for $\mathbb{U}_2$ and $A_2$. Now notice that by the change of variable formula for $i = 1, 2$,

$$\left|\frac{1}{\text{area}(\mathbb{U}_1)}\int_{A_i}\tilde{f}\circ\phi(\zeta)|\phi'(\zeta)|^2 d\xi d\eta\right| = 16. \tag{2.6.10}$$

These facts along with (2.6.7) clearly imply that

$$G_*\circ\phi(z) = \frac{64}{\pi}\log\left[\frac{|z-i|+O(\delta)}{|z+i|+O(\delta)}\right], \tag{2.6.11}$$

where the constant in the $O(\cdot)$ term depends only on $C$ in (2.6.9). Thus

$$\left|G_*\circ\phi(z) - \frac{64}{\pi}\log\left|\frac{z-i}{z+i}\right|\right| = O\left(\left|\log\left(1+\frac{O(\delta)}{|z-i|}\right)\right| + \left|\log\left(1+\frac{O(\delta)}{|z+i|}\right)\right|\right)$$

Since $a < 1$ we have $\lim_{\delta\to 0}\frac{\delta}{\delta^a} = 0$. Thus sending $\delta$ to $0$ we see that the RHS uniformly converges to $0$ on the set $\{z \in \overline{\mathbb{D}} : \min(|z-i|, |z+i|) \geq \delta^a\}$. Hence we are done. $\square$

Since the function $\log\left|\frac{z-i}{z+i}\right|$, has logarithmic singularities at $i$ and $-i$ as $\delta \to 0$ by Lemma 2.6.3 the function $G_*(z)$ blows up near the singularities $x_1, x_2$. The following lemma determines the rate at which the blow up occurs.

**Lemma 2.6.4.** *For $a < 1$ and all small enough $\delta$, for all $z \in \overline{\mathbb{U}}$ such that $d(z, x_1) \leqslant \delta^a$*

$$G_*(z) \asymp |\log(\delta)|$$

*and similarly if $d(z, x_2) \leqslant \delta^a$,*

$$G_*(z) \asymp -|\log(\delta)|.$$

*The constants in the $\asymp$ notation depend on $a$ only.*

*Proof.* We only show the first case since the argument for the other case is similar. Hence we assume $d(x_1, z) \leqslant \delta^a$. Also using the bi-Lipschitz property of $\phi$ it suffices to show that for all $z \in \overline{\mathbb{D}}$ such that $d(z, -i) \leqslant \delta^a$,

$$(G_* \circ \phi)(z) \asymp |\log(\delta)|.$$

Using (2.6.7) we do it in two steps. We first show the following:

(i).
$$\frac{1}{\mathrm{area}(\mathbb{U}_1)} \frac{1}{\pi} \int_{|\zeta|<1} \big(\mathbf{1}(A_2) - \mathbf{1}(A_1)\big)(\zeta)|\phi'(\zeta)|^2 \log|\zeta - z| d\zeta \asymp (|\log(\delta)|).$$

Clearly
$$\frac{1}{\mathrm{area}(\mathbb{U}_1)} \frac{1}{\pi} \int_{|\zeta|<1} \mathbf{1}(A_2)(\zeta)|\phi'(\zeta)|^2 \log|\zeta - z| d\zeta = O(1)$$

since $d(z, A_2) > d(x_1, x_2)/2$ for $\delta$ small enough. Since $\psi$ is bi-Lipschitz, $A_1$ is contained in a ball of radius $C\delta$ and contains a ball of radius $c\delta$ for some constants $C, c$ not depending on $\delta$ which implies that $\mathrm{area}(\mathrm{A}_1) = \Theta(\delta^2)$. Also $|\phi'|$ is bounded away from $0$ and $\infty$.

Thus for any $z \in \overline{\mathbb{D}}$ we have

$$\begin{aligned}\left|\frac{1}{\mathrm{area}(\mathbb{U}_1)}\frac{1}{\pi}\int_{|\zeta|<1}\mathbf{1}(A_1)(\zeta)|\phi'(\zeta)|^2\log|\zeta-z|d\zeta\right| &= O\left(\left|\frac{1}{\delta^2}\int_0^{C\delta}\int_{|\zeta-z|=r}\log|\zeta-z|d\theta dr\right|\right)\\ &= O\left(\left|\frac{1}{\delta^2}\int_0^{C\delta}r\log(r)dr\right|\right)\\ &= O\left(|\log(\delta)|\right).\end{aligned}$$

The lower bound follows immediately since $d(z,A_1)\leqslant\delta^a$.

(ii). We now prove a similar bound for the term

$$\frac{1}{\mathrm{area}(\mathbb{U}_1)}\frac{1}{\pi}\int_{|\zeta|<1}\big(\mathbf{1}(A_1)-\mathbf{1}(A_2)\big)(\zeta)|\phi'(\zeta)|^2\log|1-\bar{\zeta}z|d\zeta.$$

Again it is easy to check that

$$\frac{1}{\mathrm{area}(\mathbb{U}_1)}\frac{1}{\pi}\int_{|\zeta|<1}\mathbf{1}(A_2)(\zeta)|\phi'(\zeta)|^2\log|1-\bar{\zeta}z|d\zeta=O(1).$$

To find the exact order of the term

$$\frac{1}{\mathrm{area}(\mathbb{U}_1)}\frac{1}{\pi}\int_{|\zeta|<1}\mathbf{1}(A_1)(\zeta)|\phi'(\zeta)|^2\log|1-\bar{\zeta}z|d\zeta$$

recall from Setup 1 $d(\mathbb{U}_i,\partial\mathbb{U})=\Theta(\delta)$ and hence $d(A_i,\partial\mathbb{D})=\Theta(\delta)$. Thus for any $\zeta\in A_1$, we have $C\delta\leqslant|1-\bar{\zeta}z|$. Also since $d(z,x_1)\leqslant\delta^a$ by hypothesis $|1-\bar{\zeta}z|<O(\delta^a)$. Therefore,

$$\frac{1}{\mathrm{area}(\mathbb{U}_1)}\frac{1}{\pi}\int_{|\zeta|<1}\big(\mathbf{1}(A_1)\big)(\zeta)|\phi'(\zeta)|^2\log|1-\bar{\zeta}z|d\zeta\asymp(|\log(\delta)|).$$

Thus putting (i) and (ii) together we are done. □

### *2.6.3 Proofs of some earlier statements.*

In this section we provide the proof of some of the statements stated earlier.

*Proof of Lemma 2.5.3.* It follows immediately from Lemmas 2.6.3, 2.6.4 and Theorem 2.6.1. □

*Proof of Lemma 2.5.1.* Clearly by (2.6.7) and Lemmas 2.6.3, 2.6.4, $G_*(z)$ is a bounded continuous function on $\mathbb{U}$ (where the bound is $O(|\log(\delta)|)$. Thus approximating integral by Riemann sums we get

$$\int_{\mathbb{U}} |G_*(z)|\, d\xi d\eta = \lim_{n\to\infty} \sum_{z_n \in \mathbb{U}_n} \frac{1}{n^2} |G_*(z_n)|$$

where $z = \xi + i\eta$. The proof will then immediately follow from Theorem 2.6.1 if we can show that

$$\limsup_{\delta\to 0} \int_{\mathbb{U}} |G_*(z)|\, d\xi d\eta < \infty.$$

Now notice that

$$\begin{aligned} \lim_{\delta\to 0} \int_{\mathbb{U}} |G_*(z)|\, d\xi d\eta &= \int_{\mathbb{U}} \frac{64}{\pi} \left| \log \left| \frac{\psi(z) - i}{\psi(z) + i} \right| \right| d\xi d\eta \\ &= \frac{64}{\pi} \int_{\mathbb{D}} |\phi'(z)|^2 \left| \log \left| \frac{z - i}{z + i} \right| \right| d\xi d\eta \\ &< \infty. \end{aligned} \tag{2.6.12}$$

The first equality is a consequence of Lemmas 2.6.3 and 2.6.4. The second equality is just the change of measure formula. The last inequality holds since $|\phi'(z)|$ is bounded and the function $\log\left|\frac{z-i}{z+i}\right|$ is an integrable function on the disc. Thus we are done. □

*Proof of Lemma 2.5.2.* Let us denote by $\sigma_*$ a configuration where every vertex in $\mathbb{U}_{(\alpha)}$ except possibly $O(n)$ many, is colored blue. The $O(n)$ correction is needed since every configuration has exactly $\lfloor \alpha |\mathbb{U}_n| \rfloor$ many blue vertices. By Lemma 2.6.3 for all $z \in \overline{\mathbb{U}}$ such that $\min(d(z,x_1), d(z,x_2)) \geq \delta^a$ we have,

$$G_*(z) = \frac{64}{\pi} \log \Big| \frac{\psi(z) - i}{\psi(z) + i} \Big| + o(1). \tag{2.6.13}$$

By Lemma 2.5.3 on the remaining set of measure $O(\delta^a)$, one has $|G_*| = O(|\log(\delta)|)$. This along with Theorem 2.6.1 implies that given any number $c$ for small enough $\delta$ and $n = 2^m > N(\delta)$ we have,

$$\begin{aligned} \left| w_{\max} - n^2 \frac{64}{\pi} \int_{\mathbb{U}_\alpha} \log \Big| \frac{\psi(z) - i}{\psi(z) + i} \Big| \, d\xi d\eta \right| &\leqslant cn^2 \\ |w(\sigma_*) - w_{\max}| &\leqslant cn^2. \end{aligned} \tag{2.6.14}$$

From Lemmas 2.6.3 and 2.6.4 it follows that the function $G_*(z)$ (as $\delta$ goes to $0$) is uniformly integrable i.e. given any $a$ there exists $b = b(a)$ ($b(a) \to 0$ as $a$ goes to $0$) such that,

$$\limsup_{\delta \to 0} \sup_{A} \int_A G_*(z) d\xi d\eta \leqslant b$$

where the supremum is taken over all subsets $A$ of $\mathbb{U}$ of measure at most $a$. Using this and Theorem 2.6.1 we thus have the following : Given any constant $a$ there exists a constant $b = b(a)$ such that for all small enough $\delta$ and $n = 2^m > N(\delta)$:

$$\sup_{\substack{A \subset \mathbb{U}_n \\ |A| \leqslant an^2}} \sum_{z_n \in A} |G_n(z_n)| \leqslant bn^2 \tag{2.6.15}$$

where $b = b(a)$ tends to $0$ as $a$ tends to $0$. Thus for any $\sigma \in \mathcal{A}_\varepsilon$ by (2.6.15),

$$|w(\sigma) - w(\sigma_*)| \leqslant b(2\varepsilon) n^2.$$

Hence we are done by (2.6.14). □

*Proof of Remark 2.5.1.* The upper containment follows from (2.5.8). The lower containment follows from (2.6.13) and Theorem 2.6.1. □

We are now ready to prove the main technical result of this chapter.

## *2.7 Proof of Theorem 2.5.1*

The proof is quite long and involved and constitutes the technical core of the chapter. The argument has to be divided into cases depending on $\sigma$. Thus we split the result into two theorems. We will suppress the $\sigma$ dependence in the notations we define subsequently for brevity whenever there is no scope for confusion. We start with some definitions and notation.

**Definition 2.7.1.** Given $\sigma \in \Omega \cup \Omega'$ (see (2.2.8) for definition) for $i = 1, 2$, let $R_i = R_i(\sigma)$ be the set of all points in $\mathbb{U}_n$ of color $i$ reachable by a monochromatic path of color $i$ from a point in $\mathbb{U}_{i,n}$ ( we do not allow the endpoint to be the opposite color). Note that $R_1$ and $R_2$ are disjoint.

Clearly by definition

$$\mathbb{E}[w(\sigma_1) - w(\sigma_0) \mid \sigma = \sigma_0] = \mathbb{E}[G_n(X_{\tau(R_1(\sigma_0))}) - G_n(Y_{\tau(R_2(\sigma_{1/2}))})] \tag{2.7.1}$$

where $X_{\tau(R_1(\sigma_0))}$ is the point at which the random walk exits $R_1(\sigma_0)$ and similarly the second term. The expectation in the first term is over $\mu_1$ (starting distribution of the blue random walk, see Setup 1.) Note the expectation in the second term is over $\mu_2$ as well as the random intermediate configuration $\sigma_{1/2}$, (see (2.2.10)).

For any $x \in \overline{\mathbb{U}}, a > 0$ define

$$C(x, a) := \partial B(x, a) \cap \mathbb{U},\ \text{i.e. the part of the boundary inside } \mathbb{U}. \tag{2.7.2}$$

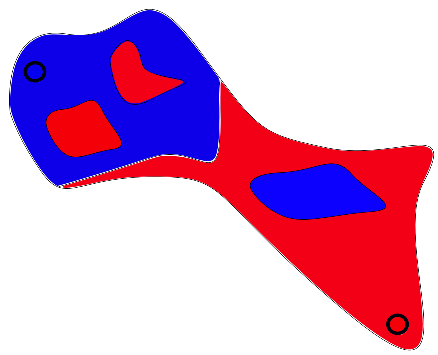

Figure 2.6: The blue connected region containing the blue blob is $R_1$ and similarly $R_2$. Note that the two red islands and the blue island are not included in either set. Also recall (even though not in figure) that if a part of the blue blob is colored red, it is not included in the set $R_1$.

Recall the weight function $w(\cdot)$ from (2.5.3). We now state the first theorem towards the proof of Theorem 2.5.1. The theorem roughly says that if either diameter of $R_1$ is "small" or $R_2$ connects a region "close" to $x_2$ to a region "close" to $x_1$ then the drift in Theorem 2.5.1 is like $|\log(\delta)|$. See Fig 2.7 and 2.8.

**Theorem 2.7.1.** *Let $\mathbb{U}$ be as in Setup 1. Then there exists constants*

$$0 < a_2 < a_1 < 1$$

*and $\delta_0$ such that for $\delta \leqslant \delta_0$ and all $n = 2^m > N(\delta)$ if $\sigma \in \Omega$ is such that either*

*i. $R_1$ does not intersect $C(y_1, \delta_2)$ or,*

*ii. $R_1$ intersects $C(y_1, \delta_2)$ and $R_2$ intersects $C(y_1, \delta_1)$ or*

*switching the roles of $R_1$ and $R_2$ in the above two cases*

*iii. $R_2$ does not intersect $C(y_2, \delta_2)$ or,* *iv.* $R_2$ *intersects* $C(y_2, \delta_2)$ *and* $R_1$ *intersects* $C(y_2, \delta_1)$

*then*

$$\mathbb{E}[w(\sigma_1) - w(\sigma_0) \mid \sigma_0 = \sigma] \asymp |\log(\delta)|. \tag{2.7.3}$$

The reason for the above is roughly the following: (we only discuss the first two cases.)

- In case $i.$ diameter of $R_1$ is small implies that the blue random walk stops before exiting a small ball. Thus the first term in (2.7.1) is roughly $|\log(\delta)|$ by Lemma 2.6.4. Now by standard random walk estimates one can show that the red walk is likely to exit $R_2$ before coming close to $x_1$. Thus the second term in (2.7.1) is much smaller and hence the result follows.

- In case $ii.$ by hypothesis there is a red path which connects $\mathbb{U}_{2,n}$ to a small ball around $y_1$ of radius $\delta^a$ for some $a$. Then it becomes likely that the blue random walk starting from $\mathbb{U}_{1,n}$ will hit this path and hence exit $R_1$ before going too far from $x_1$ again makes the first term in (2.7.1) roughly $|\log(\delta)|$. The rest of the arguments are then the same as in the previous case which shows that the second term in (2.7.1) is much smaller.

Before providing the formal proof we first state a lemma regarding random walk on $\mathbb{U}_n$. Let $\mathbb{P}_x(\cdot)$ denote the random walk probability measure started from $x$.

**Lemma 2.7.1.** [48, Lemma 5.2] *For all* $\mathbb{U}, x_1$ *as in Setup 1,*

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \sup_{\sigma \in \Omega \cup \Omega'} \left[ \sup_{z \in \mathbb{U}_n \setminus B(x_1, \frac{1}{2})} \mathbb{P}_z\big\{\tau(B(x_1, \varepsilon)) \leqslant \tau(B_1)\big\} \right] = 0$$

*where* $B_1 = B_1(\sigma)$ *was defined in* (2.3.3).

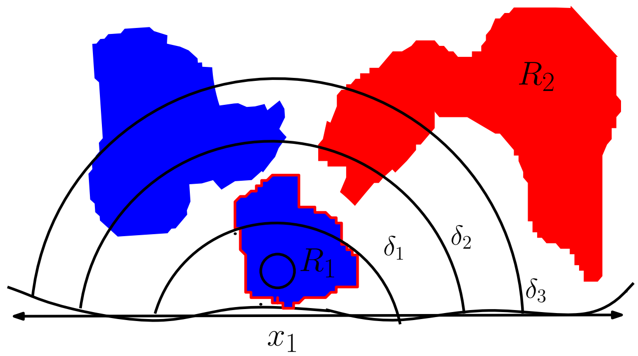


Figure 2.7: Illustrating the case when the diameter of $R_1$ is less than $\delta_2$ and hence the blue walker stops within $B(x_1, \delta_2)$.

The above lemma says that uniformly from any point $z$ at distance $1/2$ (any constant would work) from $x_1$ and all configurations $\sigma \in \Omega \cup \Omega'$ the chance that the random walk does not hit $B_1$ before reaching at distance $\varepsilon$ from $x_1$ goes to $0$ as $\varepsilon$ goes to $0$. Note that since $B_1$ is the set of all blue particles by definition $\pi_{RW}(B_1) = \alpha - o(1)$ where the $o(1)$ goes to $0$ as $n$ goes to infinity. The proof of the above lemma for general sets of measure at least $\alpha$ appears in [48]. We are now ready to prove Theorem 2.7.1.

**Proof of Theorem 2.7.1.**

By symmetry it suffices to prove the theorem for the first two cases.

*Proof of case i.* Let $a_3 < a_2 < a_1 < 1$ be constants to be specified later. Let $\delta_3 = \delta^{a_3}$ and $p(\delta_3)$ be the probability that the random walk started uniformly from $\mathbb{U}_{2,n}$ hits $B(x_1, \delta_3)$ before hitting $B_1(\sigma)$ (the set of blue sites) which has size $\alpha\,\mathrm{area}(\mathbb{U})\mathrm{n}^2 + \mathrm{O}(\mathrm{n})$. Now observe that in this case there exists universal $D_1, D_2 > 0$ (not depending on $\delta, n$) such that for all small enough $\delta$, and large enough $n$

$$\mathbb{E}\big(w(\sigma_1) - w(\sigma_0)|\sigma_0\big) \geq D_1|\log(\delta_2)| - \big[p(\delta_3)D_2|\log(\delta)| + (1 - p(\delta_3))D_2|\log(\delta_3)|\big]. \quad (2.7.4)$$

To see (2.7.4) we first recall (2.7.1).

$$\mathbb{E}[w(\sigma_1) - w(\sigma_0) \mid \sigma = \sigma_0] = \mathbb{E}[G_n(X_{\tau(R_1(\sigma_0))}) - G_n(X_{\tau(R_2(\sigma_{1/2}))})].$$

By hypothesis $R_1$ does not intersect $B^c(x_1, \delta_2)$. See Fig 2.7. Hence the blue random walk stops before exiting $B(x_1, \delta_2)$. Thus from Theorem 2.6.1, Lemma 2.6.4 it follows that the first term is at least $D_1|\log(\delta_2)|$. Now the second term is at most $O(|\log(\delta)|)$ by Theorem 2.6.1 and Lemma 2.5.3 if the red random walk enters the ball $B(x_1, \delta_3)$ before hitting $B_1(\sigma_{1/2})$ which happens with probability at most $p(\delta_3)$. Otherwise it is at most $D_2|\log(\delta_3)|$ by Lemma 2.6.3 and Theorem 2.6.1.
Since $p(\delta_3)$ goes to $0$ as $\delta$ goes to $0$ we can suitably choose $a_3 < a_2$ to be done. Note that we do not need $a_1$ for this argument. It appears in the proof of the next case.

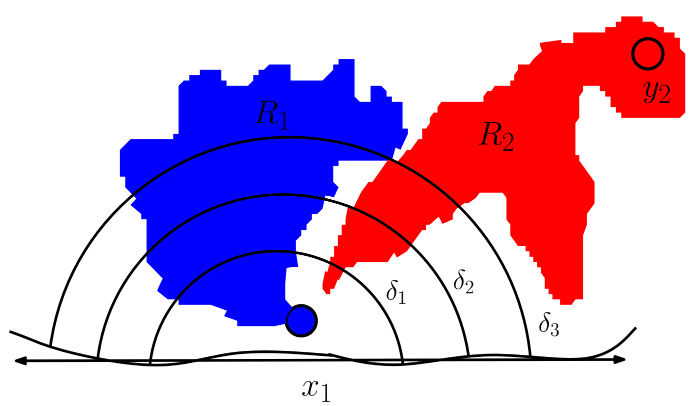


Figure 2.8: Illustrating the case when $R_2$ connects $B(y_2, \delta)$ to a point in the neighborhood of $x_1$.

*Proof of case ii.* By hypothesis there exists a path $\gamma$ of red vertices connecting $B(x_1, \delta_1)$ and $B^c(x_1, \delta_2)$. Thus it should be likely that the blue random walk started from $\mathbb{U}_{1,n}$ will hit $\gamma$ before exiting $B(x_1, \delta_2)$. This is precisely the content of the next lemma which roughly says if a connected set $A$ of large enough diameter is close enough to $\mathbb{U}_{1,n}$ then random walk starting from $\mathbb{U}_{1,n}$ is more likely to hit the set $A$ before exiting a large enough ball.

**Lemma 2.7.2.** [48, Lemma 5.5] *Let* $0 < \varepsilon_1 < \varepsilon_2$. *Assume* $A \subset \mathbb{U}_n$ *is a connected set such that*

$$d(\mathbb{U}_{1,n}, A) \leqslant \varepsilon_1.$$

*Also assume* $A \cap \big(\mathbb{U}_n \backslash B(y_1, \varepsilon_2)\big) \neq \varnothing$. *Then*

$$\sup_{x \in \mathbb{U}_{1,n}} \mathbb{P}_x(\tau(\mathbb{U}_n \backslash B(y_1, \varepsilon_2)) \leqslant \tau(A)) \leqslant C^{\log\left(\frac{\varepsilon_2}{\varepsilon_1}\right)},$$

*for some* $C = C(\mathbb{U}) < 1$ *independent of* $n$.

Thus by the above lemma,

$$\inf_{x \in \mathbb{U}_{1,n}} \mathbb{P}_x\big(\tau(\gamma) \leqslant \tau(B^c(x_1, \delta_2))\big) \geq 1 - C^{\log(\delta_2/\delta_1)} \tag{2.7.5}$$

for some $C = C(U) < 1$. Thus

$$\mathbb{E}\big(w(\sigma_1) - w(\sigma_0) | \sigma_0\big) \geq D_1 |\log(\delta_2)| - C^{\log(\delta_2/\delta_1)} D_2 |\log(\delta)| - \big[p(\delta_3)|\log(\delta)| + (1 - p(\delta_3)) D_2 |\log(\delta_3)|\big]. \tag{2.7.6}$$

This is because $G_n(X_{\tau(R_1(\sigma_0))})$ is at least $|\log(\delta_2)|$ if the blue random walk stops before exiting $B(x_1, \delta_2))$ which happens with chance at least $1 - C^{\log(\delta_2/\delta_1)}$ by (2.7.5). Otherwise it is at least $-O(|\log(\delta)|)$ since that is the minimum value $G_n$ takes by Theorem 2.6.1 and Lemma 2.5.3. Bound on $G_n(X_{\tau(R_2\sigma_{1/2})})$ is exactly as in case $i$.

Now one can suitably choose constants $a_3 < a_2 < a_1$ such that the RHS in both (2.7.4) and (2.7.6) are $\Theta(|\log(\delta)|)$. Thus like case $i$. the proof follows by suitably choosing $a_i$'s for $i = 1, 2, 3$. The proof of Theorem 2.7.1 is hence complete. □

Now to finish the proof of Theorem 2.5.1 we have to consider the cases not considered in Theorem 2.7.1. Thus we first make the assumption,

**Assumption 1.** : $\sigma$ does not fall in any of the four cases in the statement of Theorem 2.7.1.

**Throughout the sequel till the completion of the proof of Theorem 2.5.1 we will work under the above assumption.**

That is both $R_1, R_2$ extend beyond $B(x_1,\delta_2), B(x_2,\delta_2)$ respectively and also $R_2$ does not reach $B(x_1,\delta_1)$ and similarly $R_1$ does not reach $B(x_2,\delta_1)$. Thus under Assumption 1,

$$\begin{aligned} B(x_1,\delta_1)\bigcap R_2 &= \varnothing \\ B(x_2,\delta_1)\bigcap R_1 &= \varnothing. \end{aligned} \tag{2.7.7}$$

Notice that for $i = 1, 2$,

$$\mathbb{U}_{i,n} \subset B(y_i,\delta_1)$$

(see Setup 1 for definition of $\mathbb{U}_{i,n}$) for small enough $\delta$, since $\delta_1 = \delta^a$ for some $a < 1$.

**Remark 2.7.1.** We now claim the following. Given any small enough $\delta$ for all $n \geq N(\delta)$ there exists a set $A_{1,n} \subset \mathbb{U}_n$ such that $A_{1,n}$ is connected and,

$$B(y_1,\frac{\delta_1}{4}) \cap \mathbb{U}_n \subset A_{1,n} \subset B(y_1,\delta_1) \cap \mathbb{U}_n.$$

Similarly one has $A_{2,n}$ corresponding to $y_2$. This is a simple consequence of the fact that the locally near the boundary $\mathbb{U}$ looks like a half plane, see (2.10.1). We omit the details.

Now since

$$\mathbb{U}_{i,n} \subset B(y_i,\frac{\delta_1}{4}) \cap \mathbb{U}_n \subset A_{i,n},$$

it immediately follows from definition of $R_i$ that for $i = 1, 2$

$$R_i \bigcup A_{i,n}$$

are two disjoint connected subsets of $\mathbb{U}_n$. For technical reasons sometimes it will be convenient to work with the following larger set instead of $R_1$. Let $\mathbb{U}_n \backslash \{R_1 \bigcup A_{1,n}\} = \bigcup_{i=1}^{k} \mathcal{C}_{1,i}$ where $\mathcal{C}_{1,i}$ are the connected components of $\mathbb{U}_n \backslash \{R_1 \bigcup A_{1,n})\}$. Let $\{R_2 \bigcup A_{2,n}\} \subseteq \mathcal{C}_{1,k}$. Define

$$\mathring{R}_1 = R_1 \bigcup A_{i,n} \bigcup_{i \neq k} \mathcal{C}_{1,i}. \tag{2.7.8}$$

In words $\mathring{R}_1$ is the set $R_1 \bigcup A_{i,n}$ along with all the red holes filled in. Similarly define $\mathring{R}_2$. Note that for $i = 1, 2$, $\mathring{R}_i$ is connected. Note that $\mathring{R}_i$ depend on $\sigma$ which we choose to suppress for notational cleanliness. See Fig. 2.9.

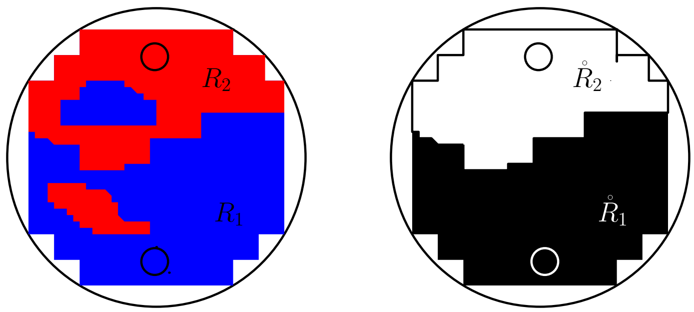


Figure 2.9: Illustrating the difference between the sets $R_i$ and $\mathring{R}_i$ for $i = 1, 2$. Note that in the latter the holes are filled in.

Given $A \subset \mathbb{U}_n$, we define two types of boundary of $A$,

$$\begin{aligned} \partial_{out} A &:= \{y \in A^c : \exists\, x \in A \text{ such that } x \sim y\} \qquad (2.7.9) \\ \partial_{in} A &:= \{x \in A : \exists\, y \in A^c \text{ such that } x \sim y\}. \end{aligned}$$

**Definition 2.7.2.** Let $\mathbb{U}_n^*$ denote the graph $\mathbb{U}_n$ along with all the diagonals of the squares that are entirely in $\mathbb{U}_n$. We will call connected subsets of $\mathbb{U}_n^*$ as $*-$ connected subsets of $\mathbb{U}_n$.

**Remark 2.7.2.** Since both $\mathring{R}_1$ and $\mathring{R}_1^c$ are connected by Remark 2.7.1 it follows by [88, Lemma 2] that both $\partial_{out}\mathring{R}_1$ and $\partial_{in}\mathring{R}_1$ are $*-$connected.

We now proceed towards completing the proof of Theorem 2.5.1. The proof will be based on an electrical resistance argument. We first review some basic facts about energy of flows on graphs.

### *2.7.1 Energy and flows on finite graphs.*

For a graph $G = (V, E)$, let $w \sim v$ signify that $w$ is a neighbor of $v$. Also let $\underline{E}$ be the set of directed edges where each edge in $E$ corresponds to two directed edges in $\underline{E}$, one in each direction (except for self-loops, which correspond to just one directed edge in $\underline{E}$). A flow is an antisymmetric function $f : \underline{E} \to \mathbb{R}$ (i.e. a function satisfying $f(w, v) = -f(v, w)$). Note that by definition the value of a flow on a self loop is $0$. Define the energy of a flow $f$ by,

$$\mathcal{E}(f) = \frac{1}{2} \sum_{(v,w)\in\underline{E}} f(v, w)^2. \tag{2.7.10}$$

For any flow $f : \underline{E} \to \mathbb{R}$ define the divergence, $\operatorname{div} f : V \to \mathbb{R}$ by,

$$\operatorname{div} f(v) = \sum_{w\sim v} f(w, v). \tag{2.7.11}$$

Note that for any flow,

$$\sum_{x\in V} \operatorname{div} f = \sum_{\substack{x,y\in V \\ y\sim x}} f(x, y) + f(y, x) = 0, \tag{2.7.12}$$

since $f$ is antisymmetric. For disjoint subsets $A, B \subset V$ and a flow $f$ we say that the flow is from $A$ to $B$ if $\operatorname{div} f(z) = 0$ for all vertices except vertices of $A$ and $B$ and the divergences across vertices of $A$ and $B$ are non negative and non positive respectively. For more about flows see [66, Chapter 9]. For any function $F : V \to \mathbb{R}$ define the gradient $\nabla F : \underline{E} \to \mathbb{R}$ by

$$\nabla F(v, w) = F(w) - F(v).$$

Recall the definition of laplacian from (2.6.3). Thus clearly for any $F : V \to \mathbb{R}$,

$$[\operatorname{div} (\nabla F)](v) = d_v.(\Delta F)(v), \tag{2.7.13}$$

where $d_v$ is the degree of the vertex $v$. The next result is a standard summation by parts formula.

**Lemma 2.7.3.** *For any function* $F : V \to \mathbb{R}$,

$$\mathcal{E}(\nabla F) = \sum_{v \in V} F(v) d_v . \Delta F(v).$$

The proof follows by definition and expanding the terms. We omit the details.

We now discuss some energy interpretations of the Green function $G_n(\cdot)$ defined on $\mathbb{U}_n$ in (2.6.2). The next lemma is a standard result adapted to our setting.

**Lemma 2.7.4** (Thomson's principle)**.** $\mathcal{E}(\nabla G_n) \leqslant \mathcal{E}(\theta)$ *for all flows* $\theta$ *on* $\mathbb{U}_n$ *such that*

$$\operatorname{div} \theta = \frac{4}{|\mathbb{U}_{1,n}|} \big[ \mathbf{1}(\mathbb{U}_{1,n}) - \mathbf{1}(\mathbb{U}_{2,n}) \big].$$

*Proof.* First observe that by (2.6.4) and (2.7.13), the flow $\nabla G_n$ has the same divergence condition as above. Note the $4$ appears since every vertex in $\mathbb{U}_{1,n}$ and $\mathbb{U}_{2,n}$ has degree $4$. The proof now follows by standard arguments. See [66, Theorem 9.10]. We sketch the main steps. One begins by observing that the flow $\nabla G_n$ satisfies the cycle law i.e. sum of the flow along any cycle is $0$. To see this notice that for any cycle

$$x_1, x_2, \ldots x_k = x_1$$

where $x_i$'s $\in \mathbb{U}_n$,

$$\sum_{i=1}^{k-1} \nabla G_n(x_i, x_{i+1}) = \sum_{i=1}^{k-1} (G_n(x_{i+1}) - G_n(x_i)) = 0.$$

The proof is then completed by first showing that the flow with the minimum energy must satisfy the cycle law, followed by showing that there is a unique flow satisfying the given divergence conditions and the cycle law which implies that $\nabla G_n$ is the unique minimizer.

□

For notational cleanliness we introduce the following notation.

$$\frac{1}{|\mathbb{U}_{1,n}|}\mathbf{1}(\mathbb{U}_{1,n}) = f_1 \quad (2.7.14)$$

$$\frac{1}{|\mathbb{U}_{1,n}|}\mathbf{1}(\mathbb{U}_{2,n}) = f_2 \quad (2.7.15)$$

$$f = f_1 - f_2. \quad (2.7.16)$$

Thus rewriting (2.6.4) in this notation we get

$$\Delta G_n = f. \quad (2.7.17)$$

*Stopped Green functions.*

For $i = 1, 2,$ and $\sigma \in \Omega \cup \Omega'$ define

$$G_{i,n}(x) := G_{i,n}(\sigma)(x) := 2n^2 \mathbb{E}_x \int_0^{\tau_i} f(X(t))dt, \quad (2.7.18)$$

where $X(t)$ is the continuous time random walk on $\mathbb{U}_n$, as defined in (2.6.1) and

$$\tau_i := \tau_i(\sigma) := \tau(R_i^c(\sigma)), \quad (2.7.19)$$

i.e. hitting time of $R_i^c(\sigma)$ for the random walk $X_t$, where $R = R_i(\sigma)$ was defined in Definition 2.7.1. Similar to (2.5.1) we again suppress the dependence on $\delta$ in the notation $G_{i,n}$. Now notice,

$$\Delta G_{i,n}(\sigma)(x) = f(x) \text{ for } x \in R_i(\sigma) \quad (2.7.20)$$

$$G_{i,n}(\sigma)|_{\mathbb{U}_n \setminus R_i(\sigma)} = 0. \quad (2.7.21)$$

Recalling notation for the energy of a flow from (2.7.10), we state the following key lemma, which translates Theorem 2.5.1 to a statement about difference in energies of flows.

**Lemma 2.7.5.** *Under Assumption 1,*

$$\mathbb{E}(w(\sigma_1) - w(\sigma_0) \mid \sigma_0 = \sigma) \;\; \geq \;\; \frac{1}{4}[\mathcal{E}(\nabla G_n(\sigma_0) - \mathcal{E}(\nabla G_{1,n}(\sigma_0)) - \mathcal{E}(\nabla G_{2,n}(\sigma_0))] \tag{2.7.22}$$

Thus using the above result our approach to proving Theorem 2.5.1 involves lower bounding the RHS in the above equation.

Before proving Lemma 2.7.5 we make a few more observations about the functions $G_{i,n}$ to be used later. First notice that if $X_k$ was a discrete time random walk then (2.7.18) is exactly the same as

$$\mathbb{E}_x \sum_{k=0}^{\tau_i - 1} f(X_k), \tag{2.7.23}$$

where $\tau_i$ is now the exit time of the discrete random walk.

Thus for the rest of the section for notational convenience we will switch to the sum notation and hence think of the random walk as discrete time.

The next remark gives us a crude upper bound for the stopped Green functions $G_{i,n}$.

**Remark 2.7.3.** For $i = 1, 2$ and any $\sigma \in \Omega \cup \Omega'$,

$$\limsup_{\substack{n=2^m \\ m\to\infty}} \sup_{x \in \mathbb{U}_{i,n}} G_{i,n}(x) = O\left(\frac{1}{\delta^2}\right).$$

A sharper upper bound of $O(\log(\frac{1}{\delta}))$ can be obtained with a little more work. However for our purposes this will suffice.

*Proof.* Recall $B_i(\sigma)$ from (2.3.3). Since $R_i \subseteq B_i$ it follows that $\tau(R_i^c) \leqslant \tau(B_i^c)$. As a simple consequence of the fact that the $\mathbb{L}_\infty$ mixing time of the discrete time random walk on $\mathbb{U}_n$ is $O(n^2)$ (for a proof see [48, Lemma 5.4]) and the fact that both $B_1, B_2$ have $\Theta(n^2)$ vertices, one gets that for $i = 1, 2$,

$$\sup_{x \in \mathbb{U}_n} \mathbb{E}(\tau(B_i)) = O(n^2),$$

Since $B_1 = B_2^c$ from the above discussion we get that,

$$\sup_{x\in\mathbb{U}_n} \mathbb{E}(\tau_i) = O(n^2),$$

where $\tau_i$ appears in (2.7.19). Now the remark follows once we observe that by (2.7.23),

$$G_{i,n}(x) \leqslant \frac{1}{|\mathbb{U}_{i,n}|}\mathbb{E}_x(\tau_i).$$

□

**Remark 2.7.4.** By (2.7.7) under Assumption 1, $R_1$ does not intersect $\mathbb{U}_{2,n}$ which is the support of $f_2$ and similarly $R_2$ does not intersect $\mathbb{U}_{1,n}$ which is the support of $f_1$.

Thus (2.7.23) becomes

$$G_{1,n}(\sigma)(x) = \mathbb{E}_x\Big[\sum_{k=0}^{\tau_1(\sigma)-1} f_1(X_k)\Big] \tag{2.7.24}$$

$$G_{2,n}(\sigma)(x) = \mathbb{E}_x\Big[\sum_{k=0}^{\tau_2(\sigma)-1} f_2(X_k)\Big],$$

and (2.7.20) and (2.7.21) become

$$\Delta G_{i,n}(\sigma)(x) = f_i(x) \text{ for } x \in R_i(\sigma) \tag{2.7.25}$$

$$G_{i,n}(\sigma)|_{\mathbb{U}_n\setminus R_i(\sigma)} = 0. \tag{2.7.26}$$

An easy but useful observation is the following:

**Lemma 2.7.6.** *Under Assumption 1 for any $\sigma \in \Omega$,*

$$\mathbb{E}[(G_{2,n}(\sigma_{1/2})(x) \mid \sigma_0 = \sigma)] \leqslant G_{2,n}(\sigma)(x) \tag{2.7.27}$$

*for all $x \in \mathbb{U}_n$ where the expectation in the first term is over the random intermediate state $\sigma_{1/2}$, (see* (2.2.10)*).*

*Proof.* The proof immediately follows from (2.7.24) since $\tau_2(\sigma_{1/2}) \leqslant \tau_2(\sigma_0)$ as $\sigma_{1/2}$ has one more blue particle than $\sigma_0$. □

*Proof of Lemma 2.7.5*

Let us denote the random walk starting uniformly from $\mathbb{U}_{1,n}$ by $X_k$ and the random walk starting uniformly from $\mathbb{U}_{2,n}$ by $Y_k$. Recall (2.7.1),

$$\mathbb{E}(w(\sigma_1) - w(\sigma_0) \mid \sigma_0 = \sigma) = \mathbb{E}(G_n(X_{\tau_1(\sigma_0)}) - G_n(Y_{\tau_2(\sigma_{1/2})}),$$

where the expectation on the right is over the random walks $X_k, Y_k$. We start with the following telescopic sum,

$$G_n(X_{\tau_1(\sigma_0)}) = G_n(X_0)+[G_n(X_1)-G_n(X_0)]+[G_n(X_2)-G_n(X_1)]+\ldots+[G_n(X_{\tau_1(\sigma_0)})-G_n(X_{\tau_1(\sigma_0)-1})].$$

Now clearly

$$\mathbb{E}(G_n(X_{i+1}) - G_n(X_i) \mid \mathcal{F}_i) = -\Delta G_n(X_i)\mathbf{1}(\tau_1(\sigma_0) > i)$$

where $\mathcal{F}_i$ is the filtration generated by $\sigma_0$ and the random walk $X_k$ up to time $i$. Taking expectation on both sides we get,

$$\mathbb{E}(G_n(X_{\tau_1(\sigma_0)})) = \mathbb{E}\big[G_n(X_0) - \sum_{k\,\geqslant\,0} \Delta G_n(X_{(k\wedge\tau_1(\sigma_0))-1})\big].$$

A similar equation holds for $\mathbb{E}(G_n(X_{\tau_2(\sigma_{1/2})}))$. Plugging in both of them we get,

$$\mathbb{E}(G_n(X_{\tau_1(\sigma_0)}) - G_n(Y_{\tau_2(\sigma_{1/2})})) = \mathbb{E}\big[G_n(X_0) - \sum_{k\,\geqslant\,0} \Delta G_n(X_{(k\wedge\tau_1(\sigma_0))-1}) - G_n(Y_0) + \sum_{k\,\geqslant\,0} \Delta G_n(Y_{(k\wedge\tau_2(\sigma_{1/2}))-1})\big] \tag{2.7.28}$$

Now

$$\mathbb{E}(G_n(X_0) - G_n(Y_0)) = \sum_{x \in \mathbb{U}_n} G_n(x)(f_1(x) - f_2(x)) = (G_n, \Delta G_n) = \frac{1}{4}\mathcal{E}(\nabla G_n). \qquad (2.7.29)$$

where all inequalities but the last are by definition. The last inequality is by Lemma 2.7.3 and the fact that all vertices in $\mathbb{U}_{1,n} \cup \mathbb{U}_{2,n}$ have degree $4$. Also,

$$\mathbb{E}_x \sum_{k \geqslant 0} \Delta G_n(X_{(k \wedge \tau_1(\sigma_0))-1}) = \mathbb{E}_x \sum_{k \geqslant 0} f_1(X_{(k \wedge \tau_1(\sigma_0))-1}) = G_{1,n}(\sigma_0)(x),$$

where the first equality is by (2.7.25) and the second equality is by definition (2.7.24). Similarly

$$\mathbb{E}_y \sum_{k \geqslant 0} \Delta G_n(Y_{(k \wedge \tau_2(\sigma_{1/2}))-1}) = -\mathbb{E}(G_{2,n}(\sigma_{1/2})(y)).$$

Hence by the above discussion, notice in (2.7.28),

$$\begin{aligned}\mathbb{E}[\sum_{k \geqslant 0} \Delta G_n(X_{(k \wedge \tau_1(\sigma_0))-1})] &= \mathbb{E}\left[\mathbb{E}[\sum_{k \geqslant 0} \Delta G_n(X_{(k \wedge \tau_1(\sigma_0))-1}) \mid X_0]\right] \\ &= \mathbb{E}[G_{1,n}(\sigma_0)(X_0)].\end{aligned}$$

Similarly we have $\mathbb{E}[\sum_{k \geqslant 0} \Delta G_n(Y_{(k \wedge \tau_2(\sigma_0))-1})] = -\mathbb{E}[G_{2,n}(\sigma_{1/2})(Y_0)]$. Thus

$$\begin{aligned}\mathbb{E}(w(\sigma_1) - w(\sigma_0) \mid \sigma_0) = &\left[\sum_{\mathbb{U}_{1,n}} G_n(x) f_1(x) - \sum_{\mathbb{U}_{2,n}} G_n(x) f_2(x) - \sum_{\mathbb{U}_{1,n}} G_{1,n}(\sigma_0)(x) f_1(x)\right] \\ &- \mathbb{E}[\sum_{\mathbb{U}_{2,n}} G_{2,n}(\sigma_{1/2})(x) f_2(x)], \\ \geq &\left[\sum_{\mathbb{U}_{1,n}} G_n(x) f_1(x) - \sum_{\mathbb{U}_{2,n}} G_n(x) f_2(x) - \sum_{\mathbb{U}_{1,n}} G_{1,n}(\sigma_0)(x) f_1(x)\right] \\ &- \sum_{\mathbb{U}_{2,n}} G_{2,n}(\sigma_0)(x) f_2(x),\end{aligned}$$

where for the inequality we replace the last term using Lemma 2.7.6. Now by Lemma 2.7.3 and (2.7.25)

$$\sum_{x\in\mathbb{U}_{1,n}} G_{1,n}(x)f_1(x) = \frac{1}{4}\mathcal{E}(\nabla G_{1,n})$$
$$\sum_{x\in\mathbb{U}_{2,n}} G_{2,n}(x)f_2(x) = \frac{1}{4}\mathcal{E}(\nabla G_{2,n}).$$

Thus we are done using these and (2.7.29) in the above inequality. □

We need a few more definitions and results to use the inequality in Lemma 2.7.5 to prove Theorem 2.5.1. Recall the standard gluing operation on any multigraph $G = (V, E)$ where certain subsets of vertices are identified ("glued") and they act as a single vertex. Any edge between two identified vertices now act as a self loop in the glued graph. For more on glued graphs see [69]. Also in the sequel we will often specify a graph by just referring to the set of vertices which will be a subset of the vertex set of $\mathbb{U}_n$. The graph then would be the sub graph induced on the set of vertices from $\mathbb{U}_n$.
For the following definitions, recall (2.7.8) and (2.7.9).

**Definition 2.7.3.** i. For $i = 1, 2$, let $\mathring{R}_i^{out}$ denote the graph with vertex set $\mathring{R}_i \cup \partial_{out}\mathring{R}_i$ with the following subset of vertices,

$$\left\{\mathring{R}_i \cup \partial_{out}\mathring{R}_i\right\} \backslash R_i$$

glued.

ii. For $i = 1, 2$, let $\mathring{R}_i^{in}$ denote the graph with vertex set $\mathring{R}_i$ with the following subset of vertices,

$$\{\mathring{R}_i \backslash R_i\} \bigcup \partial_{in}\mathring{R}_i$$

glued.

The set of glued vertices now acting as a single vertex will be denoted by $\mathfrak{v}$ (the underlying graph will be clear from context).

With the above definition, $\nabla(G_{i,n})$ is a flow on $\mathring{R}_i^{out}$. (2.7.25) and (2.7.26) are the same as

$$\begin{aligned} \Delta(G_{i,n})(x) &= f_i \text{ for all } x \in R_i && (2.7.30)\\ G_{i,n}(\mathfrak{v}) &= 0. && (2.7.31)\end{aligned}$$

Analogous to $G_{i,n}$ (see (2.7.24)) for $i = 1, 2,$ we define $G^*_{i,n}$ to be the function on $\mathring{R}_i^{in}$ such that

$$G^*_{i,n}(x) = \mathbb{E}_x \sum_{k=0}^{\infty} f_i(X_k)\mathbf{1}((k < \tau(R_i^c \cup \partial_{in}\mathring{R}_i))), \tag{2.7.32}$$

i.e. the random walk is stopped on hitting $R_i^c$ or the boundary $\partial_{in}\mathring{R}_i$. It is easy to verify that

$$\begin{aligned} \Delta(G^*_{i,n})(x) &= f_i \text{ for all } x \in R_i \backslash \partial_{in}\mathring{R}_i, && (2.7.33)\\ G^*_{i,n}(\mathfrak{v}) &= 0. && (2.7.34)\end{aligned}$$

Notice that by (2.7.12), the above information determines the laplacian (equivalently divergence) at $\mathfrak{v}$ in the glued graphs $\mathring{R}_i^{out}$, $\mathring{R}_i^{in}$ respectively.
Let us form the following abbreviations

$$\mathcal{E}(\nabla G_n) = \mathcal{E}. \tag{2.7.35}$$

Also for $i = 1, 2,$

$$\begin{aligned} \mathcal{E}(\nabla G_{i,n}) &= \mathcal{E}_i && (2.7.36)\\ \mathcal{E}(\nabla G^*_{i,n}) &= \mathcal{E}^*_i. && (2.7.37)\end{aligned}$$

For brevity we write (2.7.22) in terms of the above abbreviations,

$$\mathbb{E}(w(\sigma_1) - w(\sigma_0) \mid \sigma_0) \geq \frac{1}{4}[\mathcal{E} - \mathcal{E}_1 - \mathcal{E}_2]. \tag{2.7.38}$$

**Remark 2.7.5.** Notice that the arguments in the proof of Lemma 2.7.4 are not special to the potential function $G_n$, i.e. [66, Theorem 9.10] also gives us the following two facts:

i. $\mathcal{E}_i = \min_\theta \mathcal{E}(\theta)$ where the minimum is taken over all flows $\theta$ on $\mathring{R}_i^{out}$ such that

$$\operatorname{div} \theta(x) = 4f_i$$

for all $x \in R_i$.

ii. $\mathcal{E}_i^* = \min_\theta \mathcal{E}(\theta)$ where the minimum is taken over all flows $\theta$ on $\mathring{R}_i^{in}$ such that

$$\operatorname{div} \theta(x) = 4f_i(x)$$

for all $x \in R_i \backslash \partial_{in} \mathring{R}_i$.

With the preceeding preparation, the proof of Theorem 2.5.1 follows from a series of lemmas which we prove next. We first prove a monotonicity result. It is standard but we include it for completeness.

**Lemma 2.7.7.** *If $\sigma \in \Omega$ satisfies Assumption 1 then for $i = 1, 2$,*

$$\mathcal{E}_i^* \leqslant \mathcal{E}_i,$$

*where $\mathcal{E}_i, \mathcal{E}_i^*$ are defined in (2.7.36) and (2.7.37) respectively.*

*Proof.* Wlog we take $i = 1$. By Lemma 2.7.3 and (2.7.30) we have $\mathcal{E}_1 = \sum_{x \in \mathring{R}_1^{out}} G_{1,n}(x) 4 f_1(x)$, and similarly by (2.7.33) we have $\mathcal{E}_1^* = \sum_{x \in \mathring{R}_1^{in}} G_{1,n}^*(x) 4 f_1(x)$. The lemma now follows since

$$G^*_{i,n} \leqslant G_{i,n} \tag{2.7.39}$$

which immediately follows from definitions of $G_{1,n}, G^*_{1,n}$((2.7.24) and (2.7.32)) and the simple fact that,

$$\tau(R_1^c \cup \partial_{in}\mathring{R}_1) \leqslant \tau(R_1^c).$$

□

The previous lemma shows that $\mathcal{E}_i \geq \mathcal{E}_i^*$. The next lemma says that nevertheless they are close to each other.

**Lemma 2.7.8.** *There exists a $\beta > 0$ such that if $\sigma \in \Omega$ satisfies Assumption 1 then for $i = 1, 2$,*

$$|\mathcal{E}_i - \mathcal{E}_i^*| = O(\frac{1}{n^\beta}),$$

*where the constant in the $O$ depends on $\delta$ appearing in Setup 1.*

To prove the above we need a few preliminary lemmas first. Let,

$$\theta_* \quad = \quad \nabla(G^*_{1,n}). \tag{2.7.40}$$

Thus (2.7.37) is the same as $\mathcal{E}_1^* := \mathcal{E}(\theta_*)$. Hence by Remark 2.7.5, $ii.$ $\theta_*$ has minimum energy among all the flows on $\mathring{R}_1^{in}$ with the same divergence condition. Even though $\mathring{R}_1^{in}$ is obtained from a subgraph of $\mathbb{U}_n$ by identifying some vertices we will denote the edges of this graph by the corresponding edges of $\mathbb{U}_n$. The next lemma is a conservation of flow result. It says that the total flow across $\partial_{in}\mathring{R}_1$ is at most a constant.

**Lemma 2.7.9.**

$$\sum_{x \in \partial_{in}\mathring{R}_1} \sum_{\substack{y \sim x \\ y \in R_1 \setminus \partial_{in}\mathring{R}_1}} \theta_*(x,y) \leqslant 4.$$

*Proof.* We start with the observation that for all such $x, y$ as in the sum, $G^*_{1,n}(x) = 0$ and $G^*_{1,n}(y) \geq 0$ which follows by definition. Hence $\theta_*(x,y) \geq 0$ for all such $(x,y)$. Now by

(2.7.12)

$$\sum_{z\in \mathring{R}_1^{in}} \mathrm{div}\,(\theta_*)(z) = 0.$$

By (2.7.33) for any $z \in \mathring{R}_1^{in}$ such that $z \in R_1 \backslash \partial_{in} \mathring{R}_1$

$$\mathrm{div}\,(\theta_*)(z) = 4f_1.$$

Note above that even though the graph is not regular the $4$ appears since every vertex in the support of $f_1$ has degree $4$. Also

$$\sum_{z\in \mathring{R}_1^{in}} \mathrm{div}\,(\theta_*)(z) = \sum_{z\in R_1\backslash \partial_{in}\mathring{R}_1} 4f_1 + \sum_{x\in \partial_{in}\mathring{R}_1 \bigcup R_1^c} \sum_{\substack{y\sim x \\ y\in R_1\backslash \partial_{in}\mathring{R}_1}} \theta_*(y,x).$$

The LHS is $0$ and the first sum on the RHS is at most $4$. Thus we are done. □

We state two more lemmas. The first result is a Beurling type estimate which says that for any connected subset $A$ of $\mathbb{U}_n$ with large enough diameter which is at a certain distance away from $\mathbb{U}_{1,n}$ the probability that random walk started from a neighboring site of $A$, hits $\mathbb{U}_{1,n}$ before hitting $A$ decays as a power law in $n$. The result is standard on the whole lattice. The proof in our case with the necessary adaptions is provided in [48].

**Lemma 2.7.10.** [48, Lemma 5.1] *Fix $c > 0$. Consider $A \subset \mathbb{U}_n$ be $*-$connected (Definition 2.7.2). Also suppose that $\min(diam(A), d(\mathbb{U}_1, A)) \geq c$ (here the diameter is in terms of euclidean distance not graph distance). Then for large $n$, for all such $A$,*

$$\sup_{x\sim A} \mathbb{P}_x(\tau(\mathbb{U}_{1,n}) \leqslant \tau(A)) \leqslant \frac{C}{n^{\beta}}$$

*for some positive $\beta, C$ depending only on $c$ and $\mathbb{U}$. Here $x \sim A$ means that $x \notin A$ and there exists $y \in A$ such that $x$ is a neighbor of $y$.*

The next result is a basic isoperimetry inequality saying that the boundary of the set $\mathring{R}_1$ is not small.

**Lemma 2.7.11.** *There exists a* $c = c(\delta, \mathbb{U})$ *such that*

$$\min(diam(\partial_{in}\mathring{R}_1), diam(\partial_{out}\mathring{R}_1)) > c$$

*where* $\delta$ *appears in Setup 1.*

*Proof.* Clearly it suffices to just show that

$$diam(\partial_{in}\mathring{R}_1) > c$$

since every vertex in $\partial_{in}\mathring{R}_1$ has a neighbor in $\partial_{out}\mathring{R}_1$. Recall that by definition $\mathbb{U}_{1,n} \subset \mathring{R}_1$ and $\mathbb{U}_{2,n} \subset \mathring{R}_1^c$. We first prove the lemma for the disc $\mathbb{D}$. It is easy to notice that for the disc there exists a $c = c(\delta)$ such that for any ball $B$ of radius $c$, $\mathbb{D}_1$ and $\mathbb{D}_2$ are connected by a lattice path in $\mathbb{D}_n \backslash B$. Now since $\partial_{in}\mathring{R}_1$ separates $\mathbb{D}_{1,n}, \mathbb{D}_{2,n}$ this shows that $diam(\partial_{in}\mathring{R}_1) > c$. The general proof now follows by using the bi-Lipschitz nature of the conformal map $\phi$ in (2.2.4). □

We are now ready to prove Lemma 2.7.8. Roughly the following will be our strategy: wlog we take $i = 1$. Recall $\theta_*$ (2.7.40) is a flow on $\mathring{R}_1^{in}$. We will construct a flow $\theta_1$ on $\mathring{R}_1^{out}$ from $\theta_*$ by defining the value of the flow on the edges from $\partial_{in}\mathring{R}_1$ to $\partial_{out}\mathring{R}_1$ (the extra edges in $\mathring{R}_1^{out}$ not in $\mathring{R}_1^{in}$) and keeping the value of the flow on all other edges same as $\theta_*$. $\theta_1$ will have the same divergence as $\nabla G_{i,n}$. Thus

$$\mathcal{E}_1^* \leqslant \mathcal{E}_1 \leqslant \mathcal{E}(\theta_1), \tag{2.7.41}$$

where the first inequality is by Lemma 2.7.7 and the second inequality by Remark 2.7.5 $i$. Then we would show that

$$\mathcal{E}(\theta_1) - \mathcal{E}_1^* = O(\frac{1}{n^\beta})$$

for some $\beta$ and thus complete the proof. Formally we do the following.

**Proof of Lemma 2.7.8.** We construct a flow $\theta_1$ on $\mathring{R}_1^{out}$: For each $y$ in $R_1 \cap \partial_{in}\mathring{R}_1$ choose

$x \in \partial_{out}\mathring{R}_1$ such that $y \sim x$ (such an $x$ exists by definition of $\partial_{in}\mathring{R}_1$). Let

$$\begin{aligned}\theta_1(y,x) &= \sum_{\substack{z\in R_1\\ z\sim y}} \theta_*(z,y), \text{ for each } y \in R_1 \cap \partial_{in}\mathring{R}_1 \\ \theta_1(y',x') &= 0 \text{ for all other edges with } y' \text{ in } \partial_{in}\mathring{R}_1, x' \in \partial_{out}\mathring{R}_1 \\ &= \theta_* \text{ everywhere else.}\end{aligned}$$

We first claim that

$$\mathcal{E}_1 \leqslant \mathcal{E}(\theta_1). \tag{2.7.42}$$

By Remark 2.7.5 $i$. this will follow if we can show for all $y \in R_1$

$$\operatorname{div}(\theta_1)(y) = d_y(\Delta G_{1,n})(y) = 4f_1(y), \tag{2.7.43}$$

where $d_y$ is the degree of $y$ in $\mathring{R}_1^{out}$. We know by definition that

$$\operatorname{div}(\theta_*)(y) = d_y(\Delta G^*_{1,n})(y) = 4f_1(y) \text{ for all } y \in R_1\backslash\partial_{in}\mathring{R}_1.$$

Also by construction $\theta_1 = \theta_*$ on all the edges except the boundary edges of $\partial_{in}\mathring{R}_1$. Now by (2.7.8)

$$d(\mathbb{U}_{1,n}, \partial_{in}\mathring{R}_1) \geq \delta_1/4. \tag{2.7.44}$$

Thus $f_1 \mid_{\partial_{in}\mathring{R}_1} = 0$. Hence to verify (2.7.43) it suffices to show that

$$\operatorname{div}(\theta_1) \mid_{\partial_{in}\mathring{R}\cap R_1} = 0.$$

Let $y \in \partial_{in}\mathring{R}_1 \cap R_1$. By construction we know that there exists exactly one $x \in \partial_{out}\mathring{R}_1$ such that

$$\theta_1(y,x) \neq 0.$$

Also

$$\theta_1(y,x) = \sum_{\substack{z\in R_1\\ z\sim y}} \theta_*(z,y) = \sum_{\substack{z\in R_1\\ z\sim y}} \theta_1(z,y),$$

where the first equality is by definition and for the second inequality we use the fact that $\theta_1 = \theta_*$ for all the edges in the sum by construction. Thus

$$\mathrm{div}\,(\theta_1)(y) = \sum_{\substack{w\in \mathring{R}_1^{out}\\ w\sim y}} \theta_1(w,y) = 0,$$

and (2.7.42) is verified. By (2.7.41) the proof of the lemma will be complete once we show that

$$\mathcal{E}(\theta_1) - \mathcal{E}_1^* \leqslant O(\frac{1}{n^\beta}).$$

Now we claim that

$$\sup_{y\in\partial_{in}\mathring{R}_1} \sum_{\substack{z\in R_1\\ z\sim y}} \theta_*(y,z) = O(\frac{1}{n^\beta}). \tag{2.7.45}$$

First notice that $\partial_{in}\mathring{R}_1$ satisfy the hypotheses of Lemma 2.7.10:

- By (2.7.44), $d(\partial_{in}\mathring{R}_1, \mathbb{U}_1) \geq \delta_1/4$.
- The connectedness hypothesis is satisfied by Remark 2.7.2.
- The diameter lower bound follows from Lemma 2.7.11.

Thus by Lemma 2.7.10, for any $y \in \partial_{in}\mathring{R}_1$ and $z \sim y$,

$$\mathbb{P}_z(\tau(\mathbb{U}_{1,n}) < \tau(\partial_{in}\mathring{R}_1)) = O\left(\frac{1}{n^\beta}\right),$$

where the constant in the $O$ term depends on $\delta_1$ and $\mathbb{U}$. Since $f_1$ is positive only on $\mathbb{U}_{1,n}$

and $0$ everywhere else by (2.7.32) it follows that

$$\begin{aligned} G^*_{1,n}(z) &\leqslant \mathbb{P}_z\left(\tau(\mathbb{U}_{1,n}) < \tau(\partial_{in}\mathring{R}_1)\right) \sup_{x\in\mathbb{U}_{1,n}} G^*_{1,n}(x) \\ &\leqslant \mathbb{P}_z\left(\tau(\mathbb{U}_{1,n}) < \tau(\partial_{in}\mathring{R}_1)\right) \sup_{x\in\mathbb{U}_{1,n}} G_{1,n}(x). \end{aligned}$$

The last inequality follows since for all $x \in \mathbb{U}_n$, by (2.7.39), $G^*_{1,n}(x) \leqslant G_{1,n}(x)$. Thus by Remark 2.7.3 we get,

$$\theta_*(y,z) = G^*_{1,n}(z) \leqslant O(\frac{1}{n^\beta}),$$

where the first equality follows from the fact that $\theta_* = \nabla G^*_{1,n}$ and $G^*_{1,n}(y) = 0$. Since $\theta_1 = \theta_*$ on all but the boundary edges,

$$\begin{aligned} \mathcal{E}(\theta_1) - \mathcal{E}(\theta_*) &= O(\sum_{y\in\partial_{in}\mathring{R}_1} \big(\sum_{z\sim y} |\theta_*(z,y)|\big)^2) \\ &\leqslant \big(\sup_{y\in\partial_{in}\mathring{R}_1} \sum_{z\sim y} |\theta_*(z,y)|\big)\big(\sum_{y\in\partial_{in}\mathring{R}_1} \sum_{z\sim y} |\theta_*(z,y)|\big) \\ &\leqslant O(\frac{1}{n^\beta}) \end{aligned}$$

where the last inequality is due to Lemma 2.7.9 and (2.7.45). Hence

$$\mathcal{E}_1 - \mathcal{E}_1^* \leqslant \mathcal{E}(\theta_1) - \mathcal{E}(\theta_*) = O(\frac{1}{n^\beta})$$

where the first inequality is by (2.7.42). □

**Remark 2.7.6.** From Lemma 2.7.8 it follows that

$$\mathcal{E} - \mathcal{E}_1 - \mathcal{E}_2 \geq \mathcal{E} - \mathcal{E}_1^* - \mathcal{E}_2 - O(\frac{1}{n^\beta}). \tag{2.7.46}$$

By (2.7.38) and the above remark, to prove Theorem 2.5.1, it suffices to prove a lower bound for the quantity

$$\mathcal{E} - \mathcal{E}_1^* - \mathcal{E}_2.$$

We start with the following weaker version which shows it is always non negative.

*Weak version of Theorem 2.5.1.*

**Lemma 2.7.12.** *If $\sigma \in \Omega$ satisfies Assumption 1 then*

$$\mathcal{E} - \mathcal{E}_1^* - \mathcal{E}_2 \geq 0$$

*Proof.* Let us take the optimal flow $\theta$ for $\mathcal{E}$ on $\mathbb{U}_n$ i.e.

$$\theta = \nabla G_n. \tag{2.7.47}$$

For $i = 1, 2$, let,

$$\begin{aligned} \tilde{\theta}_i &:= \theta \mid_{\mathring{R}_i^{out}} \text{ and} \\ \tilde{\theta}_i^* &:= \theta \mid_{\mathring{R}_i^{in}} \end{aligned}$$

be the restrictions on $\mathring{R}_i^{out}$ and $\mathring{R}_i^{in}$ respectively. Now by (2.7.17) and Remark 2.7.4 it follows that,

$$div(\tilde{\theta}_i)(x) = 4f_i \text{ for all } x \in R_i \tag{2.7.48}$$

$$div(\tilde{\theta}_i^*)(x) = 4f_i \text{ for all } x \in R_i \backslash \partial_{in} \mathring{R}_i \tag{2.7.49}$$

for $i = 1, 2$ where $f_i$ was defined in (2.7.14) and (2.7.15). Thus the flows $\tilde{\theta}_i$ and $\tilde{\theta}_i^*$ on $\mathring{R}_i^{out}$ and $\mathring{R}_i^{in}$ have the same divergence as $\nabla G_{i,n}$ and $\nabla G_{i,n}^*$ respectively. Hence by Remark 2.7.5,

$$\mathcal{E}_i \leqslant \mathcal{E}(\tilde{\theta}_i), \mathcal{E}_i^* \leqslant \mathcal{E}(\tilde{\theta}_i^*).$$

Recall the definitions of $C_{1,k}$ from (2.7.8). Since $\mathring{R}_1 = \mathbb{U}_n \backslash C_{1,k}$ is connected and $\{R_2 \bigcup A_{2,n}\} \subseteq \mathcal{C}_{1,k}$, by definition $\mathring{R}_2 \subset C_{1,k}$. Thus the edge set of the graphs $\mathring{R}_1^{in}, \mathring{R}_2^{out}$ are disjoint. Therefore

$$\mathcal{E} = \mathcal{E}(\theta) \geq \mathcal{E}(\theta\mid_{\mathring{R}_1^{in}}) + \mathcal{E}(\theta\mid_{\mathring{R}_2^{out}}) = \mathcal{E}(\tilde{\theta}_1^*) + \mathcal{E}(\tilde{\theta}_2) \geq \mathcal{E}_1^* + \mathcal{E}_2. \quad (2.7.50)$$

Hence we are done. □

However as already mentioned before, proof of Theorem 2.5.1 demands a stronger quantitative version of Lemma 2.7.12. We now proceed to obtain that. Let $\overset{wired}{\mathbb{U}_n}$ be the same graph as $\mathbb{U}_n$ but with the following set of vertices

$$\partial_{in}\mathring{R}_1 \bigcup \{x \in \mathring{R}_1 : \sigma(x) = 2\} \bigcup \{x \in \mathring{R}_1^c : \sigma(x) = 1\} \quad (2.7.51)$$

glued. The points that are glued are all the red vertices in $\mathring{R}_1$ (which cause the blue random walk to stop) and similarly all the blue vertices in $\mathbb{U}_n \backslash \mathring{R}_1$. Now in $\overset{wired}{\mathbb{U}_n}$ the glued vertices act as a single vertex. Let us denote it by $\mathfrak{w}$.

In the proof of Lemma 2.7.12 we took the flow $\nabla G_n$ on $\mathbb{U}_n$ and restricted it to $\mathring{R}_1^{in}$ and $\mathring{R}_2^{out}$. To prove Theorem 2.5.1 instead of restricting $\nabla G_n$. we will construct another flow on $\overset{wired}{\mathbb{U}_n}$ and then restrict it. We first construct the flow on $\overset{wired}{\mathbb{U}_n}$. Recall $f$ from (2.7.16).

**Lemma 2.7.13.** *There exists a unique flow* $\overset{wired}{\theta}$ *on* $\overset{wired}{\mathbb{U}_n}$ *with the following properties:*

*i.*

$$\begin{aligned} \operatorname{div}(\overset{wired}{\theta})\mid_{\{R_1 \cup R_2\}\backslash \mathfrak{w}} &= 4f \mid_{\{R_1 \cup R_2\}\backslash \mathfrak{w}} \\ \operatorname{div}(\overset{wired}{\theta})(\cdot) &= 0 \textit{ for all other vertices except } \mathfrak{w}. \end{aligned}$$

*ii.*

$$\overset{wired}{\theta} = \min_g \mathcal{E}(g)$$

*where the infimum is taken over all flows* $g$ *on* $\overset{wired}{\mathbb{U}_n}$ *which has the same divergence as* $\overset{wired}{\theta}$.

Note that using the above data and (2.7.12), $div(\overset{wired}{\theta})(\mathfrak{w})$ is determined exactly.

*Proof.* For brevity let us call the set of flows $g$ on $\overset{wired}{\mathbb{U}_n}$ satisfying the above divergence conditions as $\mathcal{F}$. That the set is non empty follows since clearly by (2.7.17) the flow $\nabla(G_n)\mid_{\overset{wired}{\mathbb{U}_n}}$ is in this set.

Now let

$$\overset{wired}{\mathcal{E}} = \inf_{g\in\mathcal{F}} \mathcal{E}(g). \tag{2.7.52}$$

By standard arguments using compactness, see proof of [66, Theorem 9.10], there exists a unique minimizer $\overset{wired}{\theta}$ of the energy minimization problem. □

**Remark 2.7.7.** Let $\overset{wired}{\theta}$ be as in Lemma 2.7.13. The restrictions,

$$\begin{aligned} \overset{wired}{\theta}_1 &= \overset{wired}{\theta}\mid_{\mathring{R}_1^{in}} \\ \overset{wired}{\theta}_2 &= \overset{wired}{\theta}\mid_{\mathring{R}_2^{out}} \end{aligned}$$

under Assumption 1 satisfy the following divergence conditions

$$\begin{aligned} div(\overset{wired}{\theta}_1)(x) &= 4f_1(x) \text{ for all } x \in R_1\backslash\partial_{in}\mathring{R}_1 \\ div(\overset{wired}{\theta}_2)(y) &= 4f_2(y) \text{ for all } y \in R_2. \end{aligned}$$

*Proof.* Note that no vertex in either $R_1\backslash\partial_{in}\mathring{R}_1$ or $R_2$ was in the set of glued vertices in (2.7.51). Now by Remark 2.7.4 for $i = 1, 2$, $f\mid_{R_i} = f_i$. The remark thus follows since $\overset{wired}{\theta}$ by definition satisfies the divergence conditions. □

Hence by Remark 2.7.5 we have $\mathcal{E}_1^* \leqslant \mathcal{E}(\overset{wired}{\theta}_1)$, and $\mathcal{E}_2 \leqslant \mathcal{E}(\overset{wired}{\theta}_2)$.

**Remark 2.7.8.** Similar to (2.7.50) we have

$$\overset{wired}{\mathcal{E}} = \mathcal{E}(\overset{wired}{\theta}) \geq \mathcal{E}(\overset{wired}{\theta}\mid_{\mathring{R}_1^{in}}) + \mathcal{E}(\overset{wired}{\theta}\mid_{\mathring{R}_2^{out}}) = \mathcal{E}(\overset{wired}{\theta}_1) + \mathcal{E}(\overset{wired}{\theta}_2) \geq \mathcal{E}_1^* + \mathcal{E}_2.$$

Therefore we have

$$\begin{aligned}\mathcal{E}-\mathcal{E}_1-\mathcal{E}_2 \geq \mathcal{E}-\mathcal{E}_1^*-\mathcal{E}_2-O(\frac{1}{n^\beta}) &= \mathcal{E}-\overset{wired}{\mathcal{E}}+\left[\overset{wired}{\mathcal{E}}-\mathcal{E}_1^*-\mathcal{E}_2\right]-O(\frac{1}{n^\beta}) \\ &\geq \mathcal{E}-\overset{wired}{\mathcal{E}}-O(\frac{1}{n^\beta})\end{aligned}$$

where the first inequality follows from (2.7.46). Hence to prove Theorem 2.5.1 by (2.7.38) it suffices to show that

$$\mathcal{E}-\overset{wired}{\mathcal{E}} \geq c \tag{2.7.53}$$

for some constant $c = c(\varepsilon)$.

Notice that $\mathcal{E}-\overset{wired}{\mathcal{E}}$ is nothing but the drop in voltage when points at various voltages are glued together to get $\overset{wired}{\mathbb{U}_n}$ from $\mathbb{U}_n$. This is the well known *Rayleigh's monotonicity principle*. For more details see [69, Chap 2]. However for our purposes we need a quantitative version. To estimate such voltage drops we state and prove two technical lemmas. Let $\tilde{\mathbb{U}}_n$ be a graph obtained from $\mathbb{U}_n$ by gluing certain pairs of points. Let $A$ be a set of $k$ pairs of points $(z_1, z_1'), \ldots (z_k, z_k')$ which are among the pairs glued to obtain $\tilde{\mathbb{U}}_n$. Recall from (2.7.47), $\theta = \nabla G_n$. Consider the restricted flow $\nabla G_n \mid_{\tilde{\mathbb{U}}_n}$, (the flow that $\nabla G_n$ induces on $\tilde{\mathbb{U}}_n$). Let $\tilde{\theta}$ denote the flow on $\tilde{\mathbb{U}}_n$ such that

$$\mathcal{E}(\tilde{\theta}) = \inf_g \{\mathcal{E}(g)\}, \tag{2.7.54}$$

where the infimum is taken over all flows $g$ on $\tilde{\mathbb{U}}_n$ which has the same divergence as $\theta \mid_{\tilde{\mathbb{U}}_n}$. Existence of $\tilde{\theta}$ follows by a standard compactness argument as in the proof of [66, Theorem 9.10], (this has already been used in the proof of Lemma 2.7.13).

**Lemma 2.7.14.** *Let $A \subset \mathbb{U}_n,\ \theta$ and $\tilde{\theta}$ be as defined above. Suppose for each $1 \leqslant i \leqslant k$, there exists simple paths in $\mathbb{U}_n$ joining $(z_i, z_i')$ of length $d_i$ then,*

$$\mathcal{E}(\tilde{\theta}) \leqslant \mathcal{E}(\theta) - \frac{[\sum_{i=1}^k G_n(z_i) - G_n(z_i')]^2}{D^2 \sum_{i=1}^k d_i}$$

*where $D$ is the maximum number of paths that intersect some edge.*

Thus the lemma measures the amount of voltage drop when points at different potentials are glued together.

*Proof.* W.l.o.g. assume for all $i$, that $G_n(z_i) \geq G_n(z_i')$. Take a path joining $z_i'$ to $z_i$ of length $d_i$. Think of it as oriented towards $z_i$. Since on $\tilde{\mathbb{U}}_n$, $z_i$ and $z_i'$ are glued, the path becomes a directed cycle. Let us denote it by $\overrightarrow{\gamma_i}$. Also let $\overleftarrow{\gamma_i}$ be the same cycle in the reverse orientation. We first create a new flow $\theta_A$ on $\mathbb{U}_n$ by sending an additional amount of flow $\beta$ along $\overrightarrow{\gamma_i}$ for all $i = 1, \ldots, k$. Thus

$$\theta_A(\underline{e}) = \theta(\underline{e}) + \sum_{i=1}^{k} \beta \mathbf{1}(\underline{e} \in \overrightarrow{\gamma_i}) + \sum_{j=1}^{k} -\beta \mathbf{1}(\underline{e} \in \overleftarrow{\gamma_j})$$

for all directed edges $\underline{e}$, where every undirected edge in $\mathbb{U}_n$ appears with both orientations. Now the additional flow along an edge is at most $D\beta$ since by hypothesis every edge is in at most $D$ many cycles. An easy computation by expanding the squares now shows that

$$\mathcal{E}(\theta_A) \leqslant \mathcal{E}(\theta) + 2\beta \sum_{i=1}^{k} (G_n(z_i) - G_n(z_i')) + D^2\beta^2 \sum_{i}^{k} d_i.$$

Optimizing the RHS over $\beta$ we see that,

$$\inf_{\beta} \mathcal{E}(\theta_A) \leqslant \mathcal{E}(\theta) - \frac{[\sum_{i=1}^{k} (G_n(z_i) - G_n(z_i'))]^2}{D^2 \sum_{i}^{k} d_i}.$$

Notice that since all the paths $\overrightarrow{\gamma_i}$ become cycles in $\tilde{\mathbb{U}}_n$ and sending additional mass along cycles does not change the divergence of the flow, $\operatorname{div}(\theta_A |_{\tilde{\mathbb{U}}_n}) = \operatorname{div}(\theta |_{\tilde{\mathbb{U}}_n})$, (note however that $\theta_A$ and $\theta$ do not have the same divergence on $\mathbb{U}_n$). Hence by (2.7.54), $\mathcal{E}(\tilde{\theta}) \leqslant \mathcal{E}(\theta_A |_{\tilde{\mathbb{U}}_n}) \leqslant \mathcal{E}(\theta_A)$ and the proof is complete. □

By the above discussion and (2.7.53), to use Lemma 2.7.14 in the proof of Theorem 2.5.1, one has to estimate the $d_i$'s and $D$ appearing in the statement of the lemma. That is the purpose of the next lemma. The statement of the lemma is rather crude but will suffice for our purposes.

Recall the definition of $\mathbb{U}^{(i)}$ from (2.5.5).

**Lemma 2.7.15.** *Given a smooth domain $\mathbb{U}$ for any small enough $d > 0$ (depending on $\mathbb{U}$), there exists $D, a > 0$ such that for all $\delta < \delta_0(d)$ and all $n = 2^m > N(\delta)$ the following is true: For any set $A \subset \mathbb{U}_n$ containing $dn$ points in distinct $\mathbb{U}^{(i)}$'s , there exists at least $dn/10$ disjoint pairs $(z_1, z'_1), \dots (z_{dn/10}, z'_{dn/10})$ of elements of $A$ such that:*

$i.$ $|G_n(z_j) - G_n(z'_j)| > a$ *for* $j = 1, \dots, dn/10,$

$ii.$ *There exist paths in $\mathbb{U}_n$ between $z_j$ and $z'_j$ of length at most $Dn$, such that no edge in $\mathbb{U}_n$ intersects more than 2 paths.*

**Remark 2.7.9.** It will be useful later, to note that any set $A \subset \mathbb{U}_n$ with $|A| \geq cn^2$ for some constant $c$ independent of $n$ satisfies the hypothesis of the above lemma for a certain value of $d = d(c, U)$.

*Proof of Lemma 2.7.15 $ii$.* Recall from (2.5.5) that all the $\mathbb{U}^{(i)}$'s have width at least $100/n$. Now let $z_i \in A \cap \mathbb{U}^{(i)}$. We will pair $z_i$ from different $\mathbb{U}^{(i)}$'s Now the only difficulty can come if the points lie close to the boundary where the graph does not look like $\mathbb{Z}^2$. However in that case for each such $z_i$ one can take a path in $\mathbb{U}_n$ away from the boundary of the set $\mathbb{U}$, ending at say $y_i$, which lies entirely in $\mathbb{U}^{(i-1)} \cup \mathbb{U}^{(i)} \cup \mathbb{U}^{(i+1)}$, (smoothness of $\mathbb{U}$ (see (2.10.1)) allows us to do that. Now to avoid intersection of these paths we will only consider points $z_i$ in $\mathbb{U}^{(i)}$ that are not consecutive.

The proof is now completed by pairing points $y_i$ into pairs $(y_j, y'_j)$ such that corresponding $\mathbb{U}^{(i)}$ for the elements of a pair are at least $d/10$ apart. That the elements of each such pair $(y_j, y'_j)$ can be joined using disjoint lattice paths of length at most $Dn$ for some $D$ depending only on $\mathbb{U}$ is easy to see. We omit the details. See Fig 2.10.

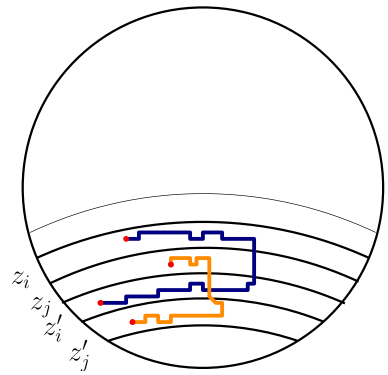


Figure 2.10: Figure illustrating the proof of Lemma 2.7.15 for the case of the circle where points in different $\mathbb{U}^{(i)}$ are connected by almost edge disjoint paths.

*Proof of i.* Recall that by the above construction for any pair $z_j$ and $z_j'$ the corresponding $\mathbb{U}^{(i)}$'s are at least $d/10$ apart. Thus by Theorem 2.6.1 and Lemma 2.6.3 there exists $\delta_0$ such that given $\delta < \delta_0$

$$\liminf_{\substack{n=2^m \\ m\to\infty}} [\inf_j (G_n(z_j) - G_n(z_j')] > c,$$

for some constant $c$ depending only on $d$ and $\mathbb{U}$. □

### *2.7.2 Proof of Theorem 2.5.1*

In combination with Theorem 2.7.1 it suffices to prove Theorem 2.5.1 under Assumption 1. To this end we will show: for all $\sigma \in \Omega^c_{(\varepsilon)}$ there is a set $A$ which is a subset of the set in (2.7.51) (hence is glued to get $\overset{wired}{\mathbb{U}_n}$ from $\mathbb{U}_n$) such that: $A$ satisfies the hypothesis of Lemma 2.7.15.

Before showing the above we first discuss why it suffices. Immediately using Lemma 2.7.15 we have pairs $(z_1, z_1'), \ldots (z_k, z_k')$ with $k = \Omega(n)$ which are glued and have paths connecting them of length $O(n)$ such that no edge appears in more than $2$ paths. Also, $G_n(z_j) - G_n(z_j') \geq c'$, for $j = 1 \ldots k$. Recall $\overset{wired}{\mathcal{E}}$ from (2.7.52). Thus using Lemma 2.7.14 we get, $\overset{wired}{\mathcal{E}} \leqslant \mathcal{E} - d$, for some constant $d = d(\varepsilon)$. Hence we are done by (2.7.53).

We now proceed to showing existence of such sets $A$ (the arguments will be rather crude and not optimal in any sense. However since we are interested only in statements up to constants this suffices):

Recall $\beta$ from (2.2.6) Let

$$\beta_{-4} > \beta_{-3} > \beta_{-2} > \beta_{-1} > \beta_0 = \beta > \beta_1 > \beta_2 > \beta_3 > \beta_4 \tag{2.7.55}$$

such that for all $i = -4, -3, \ldots, 3$,

$$d(\gamma_{\beta_i}, \gamma_{\beta_{i+1}}) = \varepsilon_1 \; (\varepsilon_1 \text{ is specified soon}).$$

See Fig. 2.11. Recall (2.7.8) and (2.7.9). Now consider the following cases:

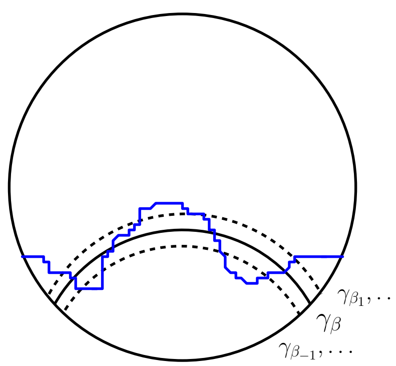


Figure 2.11: $\gamma_{\beta_i}$'s on either side of $\gamma_\beta$ to measure how far the boundary of $\mathring{R}_1$, the blue curve , deviates from $\gamma_\beta$.

$i.$ $\partial_{in}\mathring{R}_1$ is a subset of $\mathbb{U}_{\beta_{-2}}$,
$ii.$ $\partial_{in}\mathring{R}_1$ is a subset of $\mathbb{U}^c_{\beta_2}$,
$iii.$ $\partial_{in}\mathring{R}_1$ intersects $\mathbb{U}_{\beta_i}$ and $\mathbb{U}^c_{\beta_{i+1}}$ for some $i \in \{-4, \ldots, 3\}$,
$iv.$ $\partial_{in}\mathring{R}_1$ is a subset of $\mathbb{U}^c_{\beta_{-3}} \cap \mathbb{U}_{\beta_3}$,

We choose $\varepsilon_1 \leqslant \frac{\varepsilon}{100}$ such that,

$$\text{area}(\mathbb{U}^{\text{c}}_{\beta_{\text{j}-1}} \cap \mathbb{U}_{\beta_{\text{j}}}) \; 6 \; \varepsilon/100 \tag{2.7.56}$$

for all $j = -4, \ldots 3$. ($100$ is just a large enough number and is nothing special) Also let $\varepsilon' > 0$ be such that the left hand side in (2.7.56) is at least $\varepsilon'$.

The arguments for $cases\ i.$, $ii.$ are symmetric and hence to avoid repetition we provide arguments only for case $i.$

By hypothesis $\partial_{in}\mathring{R}_1$ lies entirely in $\mathbb{U}_{\beta_{-2}}$. Now by arguments similar to the proof of Lemma 2.7.15 $ii.$ for large enough $n$, any two points in $\mathbb{U}_n \cap \mathbb{U}^c_{\beta_{-1}}$ are connected by a path in $\mathbb{U}_n$ which lies entirely in $\mathbb{U}_n \cap \mathbb{U}^c_{\beta_{-1}+\varepsilon_1/2}$ (hence does not intersect $\partial_{in}\mathring{R}_1$). Thus $\mathbb{U}_n \cap \mathbb{U}^c_{\beta_{-1}}$ is in $\mathring{R}_1^{\,c}$. Recall by definition $\text{area}(\mathbb{U}^{\text{c}}_{\beta_0}) = (1-\alpha)\text{area}(\mathbb{U})$. Thus, $\text{area}(\mathbb{U}^{\text{c}}_{\beta_{-1}}) \geq (1-\alpha)\text{area}(\mathbb{U})+\varepsilon'$. Hence number of blue vertices in $\mathbb{U}_n \cap \mathbb{U}^c_{\beta_{-1}}$ is atleast $e'n^2$. Recall from (2.7.51) that both $\sigma^{-1}(2) \cap \mathring{R}_1$ and $\sigma^{-1}(1) \cap \mathring{R}_1^c$ are glued to obtain $\overset{wired}{\mathbb{U}_n}$ from $\mathbb{U}_n$. Thus at least $e'n^2$ vertices are glued and hence we are done by Remark 2.7.9.

$Case\ iii.$ By hypothesis in this case $\partial_{in}\mathring{R}_1$ (which is a $*-$ connected set by Remark 2.7.2) intersects different level sets of the function $G_n$. Thus the intersection with the different level sets could be taken to be $A$ in the usage of Lemma 2.7.15.

Formally we do the following: by hypothesis $\partial_{in}\mathring{R}_1$ intersects both $\mathbb{U}_{\beta_i}$ and $\mathbb{U}^c_{\beta_{i+1}}$ Now by compactness there is some constant $c = c(\beta, \varepsilon_1, \mathbb{U})$ such that for all $i = -4, \ldots, 3$,

$$d(\gamma_{\beta_i}, \gamma_{\beta_{i+1}}) \geq c$$

Since $\partial_{in}\mathring{R}_1$ is $*-$connected (Remark 2.7.2), it intersects at least $dn$ many $\mathbb{U}^{(i)}$'s for some $d = d(c)$. Thus $A = \partial_{in}\mathring{R}_1$ satisfies the hypothesis of Lemma 2.7.15 and is also glued by (2.7.51). Therefore we are done.

$Case\ iv.$ Recall that by hypothesis $\sigma \in \Omega^c_{(\varepsilon)}$ and $\partial_{in}\mathring{R}_1$ is a subset of $\mathbb{U}^c_{\beta_{-3}} \cap \mathbb{U}_{\beta_3}$. Since

$\sigma \in \Omega^c_{(\varepsilon)}$ and $\varepsilon_1 \leqslant \frac{\varepsilon}{100}$, there exists at least $dn$ many $\mathbb{U}^{(i)}$'s containing at least one red vertex all of which are in $\mathbb{U}_n \cap \mathbb{U}_{\beta_{-3}}$ or similarly there exists at least $dn$ many $\mathbb{U}^{(i)}$'s containing at least one blue vertex all of which are in $\mathbb{U}_n \cap \mathbb{U}^c_{\beta_3}$. We take the set of such points to be $A$. The proof of Theorem 2.5.1 is thus complete. □

## *2.8 Proof of Theorem 2.2.1*

Given the preceding results the proof of the main theorem follows by similar arguments as in the proof of the main result in [45]. We first state the following two hitting time results. Recall the sets from Table 2.1.

**Lemma 2.8.1.** *Given any $\varepsilon > 0$ there exist a positive constant $\varepsilon_1$ such that for all small enough $\delta$ (blob size) there exist positive constants $c = c(\varepsilon,\delta), d = d(\varepsilon,\delta),$ such that for all large enough $n = 2^m$ and $\sigma \in \Omega_{(\varepsilon)} \cap \mathcal{A}_{\varepsilon_1}$*

$$\mathbb{P}_\sigma(\tau(\mathcal{G}_{\varepsilon^{1/4}}) > dn^2) \quad \leqslant \quad e^{-cn^{1/3}}. \tag{2.8.1}$$

**Lemma 2.8.2.** *Given $\varepsilon > 0$ there exist a positive constant $\varepsilon'$ such that for all small enough $\delta$ (blob size), there exists $b = b(\varepsilon,\delta), d = d(\varepsilon,\delta) > 0$ such that for all large enough $n = 2^m$ and $\sigma \in \mathcal{G}_\varepsilon \cap \Gamma_{\varepsilon'/2}$*

$$\mathbb{P}_\sigma(\tau(\mathcal{G}^c_{\varepsilon^{1/4}}) > e^{dn^{1/3}}) > 1 - e^{-bn^{1/3}}.$$

The proof of the above two lemmas involve arguments involving IDLA on $\mathbb{U}_n$ and is postponed to Section 2.11. We now show how to finish the proof of Theorem 2.2.1 using the above.

**Lemma 2.8.3.** *Let $\tau_\varepsilon = \inf\{t : S(t) \in \mathcal{G}_\epsilon\}$. For any $\varepsilon > 0$, for small enough $\delta$, there exist positive constants $c = c(\varepsilon,\delta), d = d(\varepsilon,\delta), N = N(\delta)$ such that for all $n = 2^m > N$ and all subsets $S \subset \mathbb{U}_n$ of cardinality $\lfloor \alpha n^2 \rfloor$ we have*

$$\mathbb{P}(\tau_\varepsilon > dn^2 \mid S(0) = S) < e^{-cn^{1/3}}.$$

*Proof.* We will prove something stronger which will be used in the proof of Theorem 2.2.1. However for brevity we first need some notation. We start by recalling the sets in Table 2.1. Given any positive $\varepsilon_1$ and $\varepsilon_2$ define the set

$$\mathcal{C}_{\varepsilon_1,\varepsilon_2} = \mathcal{G}_{\varepsilon_1} \cap \Gamma_{\varepsilon_2}.$$

We claim that for small enough $\varepsilon, \varepsilon' > 0$ there exists constants $D, D' > 0$ such that for all large enough $n = 2^m$ and any $\sigma \in \Omega$,

$$\mathbb{P}_\sigma\big(\tau(\mathcal{C}_{\varepsilon^{1/4},\varepsilon'/2}) \geq Dn^2\big) \leqslant e^{-D'n^{1/3}} \tag{2.8.2}$$

$$\mathbb{E}_\sigma\big(\tau(\mathcal{C}_{\varepsilon^{1/4},\varepsilon'/2}) \leqslant 2Dn^2\big) \tag{2.8.3}$$

(2.8.3) clearly follows from (2.8.2) as the latter holds for any starting $\sigma$ and hence the expected hitting time is clearly dominated by $Dn^2$ times a geometric variable with mean $2$. Also showing (2.8.2) clearly proves Theorem 2.8.3 since $\mathcal{C}_{\varepsilon^{1/4},\varepsilon'/2} \subset \mathcal{G}_{\varepsilon^{1/4}}$. Let

$$\tau' = \tau(\Gamma_{\varepsilon'/4}), \tau'' = \tau(\Gamma^{\mathsf{c}}_{\varepsilon'/2})$$
$$\tau''' = \tau(\Omega_{(\varepsilon)} \cap \mathcal{A}_{\varepsilon_1}), \tau'''' = \tau(\mathcal{G}_{\varepsilon^{1/4}}).$$

where the $\varepsilon_1$ in the definition of $\tau'''$ is the same as the one in the hypothesis of Lemma 2.8.1. Now to show (2.8.2) we notice the following containment of events for any positive $A$:

$$\big[\{\tau' \leqslant An^2\} \cap \{\tau'' \geq e^{cn} \mid \sigma_{\tau'}\} \cap \{\tau''' \leqslant An^2 \mid \sigma_{\tau'}\} \cap \{\tau'''' \leqslant An^2 \mid \sigma_{\tau'''}\}\big] \subset \{\tau(\mathcal{C}_{\varepsilon^{1/4},\varepsilon'/2}) \leqslant 3An^2\}.$$

To see why this containment holds we first notice that $\Gamma_{\varepsilon'/4} \subset \Gamma_{\varepsilon'/2}$. Thus the first two events imply that the process hits the set $\Gamma_{\varepsilon'/2}$ in $An^2$ steps and stays inside for an exponential (in $n$) amount of time, and in particular stays in $\Gamma_{\varepsilon'/2}$ from time $An^2$ through time $3An^2$. In addition, the third and fourth events together imply that regardless of where the chain is at time $An^2$, the chain enters $\Omega_{(\varepsilon)} \cap \mathcal{A}_{\varepsilon_1}$ by time $2An^2$ and then enters $\mathcal{G}_{\varepsilon^{1/4}}$ by time $3An^2$. Hence in the intersection of the four events, the hitting time of $\mathcal{C}_{\varepsilon^{1/4},\varepsilon'/2} = \mathcal{G}_{\varepsilon^{1/4}} \cap \Gamma_{\varepsilon'/2}$ is at most $3An^2$. Let $D = 3A$. Now for a large enough constant $A$, there exists $D' > 0$ such that the probabilities of of all the four events on the left hand side are at least $1 - e^{-D'n^{1/3}}$. This follows by Lemmas 2.5.4, 2.5.5, and 2.8.1 respectively. Hence by the union bound, (2.8.2) follows for a slightly smaller value of $D'$. □

### *2.8.1 Proof of Theorem 2.2.1*

Note that (2.8.2) is true for all small enough choices of $\varepsilon$ and $\varepsilon'$. However given $\varepsilon > 0$, by Lemma 2.8.2 there exist $\varepsilon', b, d > 0$ such that for all large enough $n = 2^m$, and $\sigma \in \mathcal{C}_{\varepsilon,\varepsilon'/2}$

$$\mathbb{P}_\sigma(\tau(\mathcal{G}^c_{\varepsilon^{1/4}}) > e^{dn^{1/3}}) > 1 - e^{-bn^{1/3}}. \tag{2.8.4}$$

For such an $\varepsilon'$ the proof of Theorem 2.2.1 follows by using Lemma 2.5.7 with the following choices of parameters:

$$A = \mathcal{C}_{\varepsilon,\varepsilon'/2}, B = \mathcal{G}^c_{\varepsilon^{1/4}}, t_1 = 2Dn^2, t_2 = e^{dn^{1/3}}$$

The above choice of parameters satisfy the hypotheses of Lemma 2.5.7 by (2.8.2) and (2.8.4). □

The remainder of the chapter provides a sketch of the proof of Theorem 2.6.1 and finally in Section 2.11 we finish off with the proofs of Lemmas 2.8.1 and 2.8.2.

## ***2.9 Outline of the Proof of Theorem 2.6.1***

We provide the sketch of the main steps in the proof of Theorem 2.6.1. The full proof appears in [48]. The proof relies on results based on convergence of random walk on $\mathbb{U}_n$ to Reflected Brownian motion on $\mathbb{U}$. For a formal definition of Reflected Brownian motion on $\overline{\mathbb{U}}$ see [25, Definition 2.7], [11, 20, 24]. Throughout the rest of the section we will denote it by $B_t$. The basic idea behind proving Theorem 2.6.1 is to show that the function

$G_*(z)$ is the same as,

$$G(z) = \frac{2}{\text{area}(\mathbb{U}_1)} \int_0^\infty [P_z(B_t \in \mathbb{U}_1) - P_z(B_t \in \mathbb{U}_2)]\, dt - c. \tag{2.9.1}$$

The constant $c$ is chosen such that the integral of $G$ along $\partial\mathbb{U}$ is $0$. Formally, we parametrize the boundary $\partial U$ by $\theta \in [0, 2\pi)$ via the conformal map $\phi$ (2.2.4). We then fix $c$ such that

$$\int_{|\zeta|=1} G \circ \phi(\zeta) \frac{d\zeta}{\zeta} = 0. \tag{2.9.2}$$

Compare the above expression with $G_n(\cdot)$ defined in (2.5.1) (recall that the constant $c$ is the same in both expressions). The proof of Theorem 2.6.1 has two parts. One part shows that $G_n$ converges to $G,$ (see Lemma 2.9.2). This will follow by convergence of the random walk measure on $\mathbb{U}_n$ to $B_t.$ For more on this see [20], [25] and the references therein. The remaining step then is to show that indeed $G_* = G.$ This will be proved using the fact that the density for Reflected Brownian motion is a fundamental solution to the Neumann problem and hence the function $G$ is roughly satisfies,

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) G = \frac{4}{\text{area}(\mathbb{U}_1)} (\mathbf{1}(\mathbb{U}_2) - \mathbf{1}(\mathbb{U}_1)) \tag{2.9.3}$$

with Neumann boundary conditions. One then checks that $G_*$ is a solution to the above problem. The proof is then complete by uniqueness of the solution of such a problem. For brevity we will adopt the following notation

$$\Delta :\equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Recall (2.6.3). Thus we use $\Delta$ to denote the laplacian in both the continuous and discrete setting since there will be no scope of confusion.

*2.9.1 Closed form of the limit.*

**Theorem 2.9.1.** [48, Theorem 4.2] *Let $\mathbb{U}$ be as in Setup 1. For all $z \in \overline{\mathbb{U}}$,*

$$G(z) = G_*(z). \tag{2.9.4}$$

As discussed above, to show this informally one identifies $G$ as a solution to a second order partial differential equation also satisfied by $G_*$. The result then follows by uniqueness of such a solution. To give a general idea we quote a result in the theory of boundary value problems with Neumann boundary condition which is used in the proof.

**Lemma 2.9.1.** *Let function $w \in C^2(\mathbb{U}) \cap C^1(\overline{\mathbb{U}})$ be a function on $\mathbb{U}$ satisfying the following properties*

$$\begin{aligned} \Delta(w) &= \frac{f}{4} \\ \frac{\partial w}{\partial \nu} &= 0 \text{ on } \partial\mathbb{U} \end{aligned}$$

*where $\frac{\partial}{\partial \nu}$ denotes the normal derivative and $f \in L^1(\mathbb{R}, \mathbb{U})$. Then*

$$w \circ \phi(z) = d + \frac{1}{\pi} \int_{|\zeta|<1} f \circ \phi(\zeta) |\phi'|^2(\zeta) \log(|(\zeta - z)(1 - \bar{\zeta} z)|^2) d\xi d\eta, \tag{2.9.5}$$

*where*

$$\frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{w \circ \phi(\zeta)}{\zeta} d\zeta = d.$$

*Proof.* The result for the case $\mathbb{U} = \mathbb{D}$ (the unit disc) is stated as [14, Theorem 8]. The above lemma now follows by change of variable under composing with the map $\phi$. □

Theorem 2.9.1 now follows roughly by arguing that the function $G(\cdot)$ satisfies Lemma 2.9.1 with $\Delta G = \frac{4}{\text{area}(\mathbb{U}_1)}(\mathbf{1}(\mathbb{U}_2) - \mathbf{1}(\mathbb{U}_2))$ and noticing that $G_*$ is nothing but $f$ in Lemma 2.9.1 replaced by $\frac{16}{\text{area}(\mathbb{U}_1)}(\mathbf{1}(\mathbb{U}_2) - \mathbf{1}(\mathbb{U}_2))$. However formally one works with "smoother" versions of $(\mathbf{1}(\mathbb{U}_2) - \mathbf{1}(\mathbb{U}_2))$. The detailed proof appears in [48].

**Remark 2.9.1.** Recall the constant $c$ in (2.9.1) such that.

$$\int_{|\zeta|=1} G \circ \phi(\zeta) \frac{d\zeta}{\zeta} = 0.$$

Consider the the special case of the unit disc i.e. $\mathbb{U} = \mathbb{D}$ with point $x_1 = -i, x_2 = i$ and $y_1 = -(1-\delta)i$, $y_2 = (1-\delta)i$. Now owing to the invariance of Reflected Brownian motion on $\mathbb{D}$ under the transformation $\zeta \to -\zeta$, one sees that the integral in (2.9.1)

$$\frac{2}{\text{area}(\mathbb{D}_1)} \int_0^\infty \left[P_z(B_t \in \mathbb{D}_1) - P_z(B_t \in \mathbb{D}_2)\right] dt$$

is odd, (because with the choice of $y_1, y_2$, we have $\mathbb{D}_1 = -\mathbb{D}_2$). Thus $c$ is $0$ and

$$G(z) = \frac{2}{\text{area}(\mathbb{D}_1)} \int_0^\infty \left[P_z(B_t \in \mathbb{D}_1) - P_z(B_t \in \mathbb{D}_2)\right] dt.$$

### *2.9.2 Convergence*

The last missing piece is to show that the function $G_n$ converges to $G$. To this end we now state the following lemma which along with Theorem 2.9.1 proves Theorem 2.6.1.

**Lemma 2.9.2.** [48, Lemma 4.8] *For all small enough* $\delta$,

$$\lim_{\substack{m\to\infty \\ n=2^m}} \sup_{x\in\overline{\mathbb{U}}} |G_n(x) - G(x)| = 0$$

*where* $G$ *is defined in* (2.9.1).

Note again the suppressing of $\delta$ in the above statement. The proof of the lemma uses the local CLT estimates for random walk approximation of Reflected Brownian motion obtained in [25, Theorem 2.12]. The complete proof appears in [48].

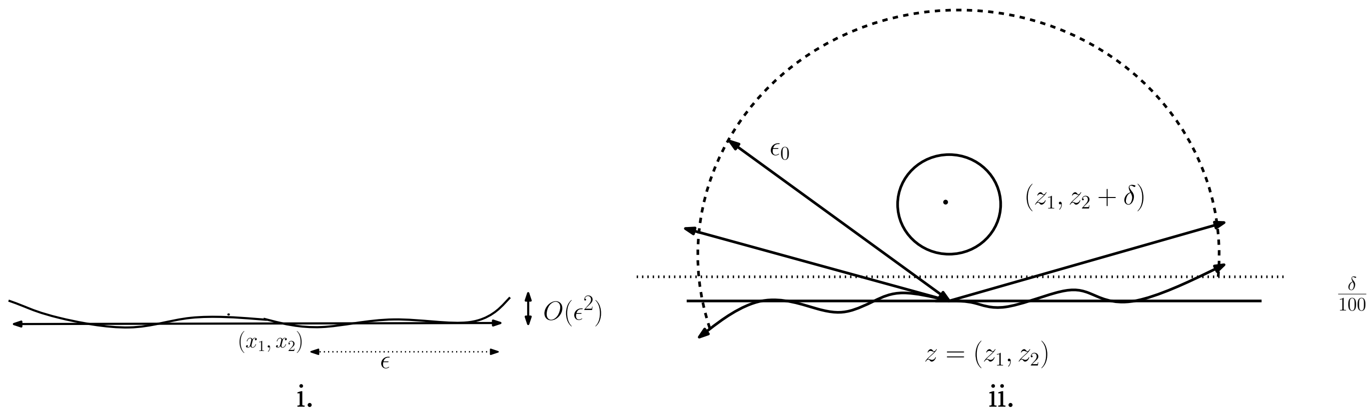


Figure 2.12: i. Locally the region near the boundary looks like a half plane. ii. Illustrating the proof of Corollary 2.10.1.

**Proof of Theorem 2.6.1.** The proof follows immediately from Lemma 2.9.2 and Theorem 2.9.1. □

### *2.10 Basic geometric properties of* $\mathbb{U}$

We include a short discussion of some basic geometric properties of $\mathbb{U}_n$ used before in the chapter. The following is a well known property of the boundary of $\mathbb{U}$ as in Setup 1: since the boundary $\partial\mathbb{U}$ is analytic there exists a $C > 0$ and an $\varepsilon_0$ such that for all $x \in \partial\mathbb{U}$ there exists an orthogonal system of coordinates centered at $x = (x_1, x_2)$ such that for all $\varepsilon \leqslant \varepsilon_0$

$$B(x, \varepsilon) \cap \mathbb{U} = \{(x_1', x_2') \in B(x, \varepsilon) : x_1' \in (x_1 - \varepsilon, x_1 + \varepsilon), x_2' \geq f(x_1')\} \tag{2.10.1}$$

and

$$|f(x_1') - x_2| \leqslant C|x_1' - x_1|^2.$$

The above is a simple consequence of Taylor expansion up to second order of the curve locally near $x$. See Fig 2.12 i. As a simple corollary of the above fact we see that $\mathbb{U}$ satisfies the following property which shows that the $y_i$'s in Setup 1 can indeed be chosen.

**Corollary 2.10.1.** *Let* $\mathbb{U}$ *be as in Setup 1. Then there exists* $\delta_0 = \delta_0(\mathbb{U})$ *such that for all* $x \in \overline{\mathbb{U}}$ *and* $\delta < \delta_0$ *there exists* $y \in \mathbb{U}$ *such that* $d(y, x) \leqslant \delta$ *and* $B(y, \delta/2) \subset \mathbb{U}$.

Recall that $d(y, x)$ is the euclidean distance between $x$ and $y$. $B(y, \delta)$ denotes the euclidean ball of radius $\delta$ with center at $y$.

*Proof.* Choose $\delta_0 \leqslant \varepsilon_0/4$ such that $C\delta_0^2 \leqslant \frac{\delta_0}{100}$ where $\varepsilon_0$ and $C$ appear in (2.10.1). For any $\delta < \delta_0$ the lemma is immediate if $d(x, \partial\mathbb{U}) > \delta/2$. since then we can choose $y = x$. Otherwise let $z = (z_1, z_2) \in \partial U$ be the closest point on the boundary to $x$. Now in the local coordinate system centered at $z$ as in (2.10.1) choose $y = (z_1, z_2 + \delta/2)$. Then

$$d(x, y) \leqslant d(x, z) + d(z, y) \leqslant \delta.$$

Also clearly $B(y, \delta/2) \subset \mathbb{U}$ and hence we are done. See Fig 2.12 ii.

□

***Concluding remarks***

Note that Theorem 2.2.1 only proves concentration of the interface.

**Fluctuations**. A natural next step would be to find the order of magnitude of fluctuations of the interface.

**Higher dimensions**. One can also study the process in higher dimensions and try to characterize the hyper surface separating the two colors at stationarity. It is natural to guess that it should be the level set of the analogous higher dimensional potential function which is harmonic in the interior of the corresponding domain and has positive and negative singularities at the two sources with Neumann boundary conditions.

**Multiple particles**. Proving a theorem analogous to Theorem 2.2.1 for *Competitive Erosion* with more than two particles remains an intriguing challenge.

### 2.11 Generalized IDLA estimates

We finish off with the proofs of Lemmas 2.8.1 and 2.8.2 that were used in Section 2.8. We start with a generalized IDLA estimate. For definitions and other results on standard IDLA on the square lattice see [63]. However our setting will be different in two aspects:

- The graph will be $\mathbb{U}_n$ instead of $\mathbb{Z}^2$.
- At time $0$ some of the sites will be filled unlike the standard setting where every site is vacant initially.

Formally we have the following: recall $\mathbb{U}^{(i)}$ from (2.5.5) and $\mathbb{U}_{(\alpha)}$ from (2.2.7). We start an IDLA process on $\mathbb{U}_n$ with the random walks starting uniformly from $\mathbb{U}_{2,n}$ (see Setup 1) with the following initial condition: All the sites in $\mathbb{U}_n \backslash \{\mathbb{U}_n \cap \mathbb{U}_{(\alpha)}\}$ are filled along with an arbitrary $\varepsilon' n$ many $\mathbb{U}^{(i)}$'s for some small $\varepsilon' > 0$. Let us now fix an $\varepsilon \geq D\varepsilon'$ ($D$ is just a large enough constant to be specified later). The next result will bound the size of the cluster after $\varepsilon n^2$ particles have been released similar to standard bounds (see for eg: [63]). Let the IDLA cluster (set of filled sites) at time $t$ be $A_t$.

**Theorem 2.11.1.** *(IDLA upper bound) There exists $c, C$ depending only on $\mathbb{U}, \alpha$ such that given small enough $\varepsilon > 0$ and starting with any initial condition as described above, for all large enough $n$, with probability at least $1 - e^{-c(\sqrt{\varepsilon} n)^{1/3}}$,*

$$A_{\varepsilon n^2} \backslash A_0 \subset [\{\mathbb{U} \backslash \mathbb{U}_{(\alpha - C\sqrt{\varepsilon})}\} \cap \mathbb{U}_n].$$

**Remark 2.11.1.** Note that the distance between the geodesics which act as boundaries of $\mathbb{U}_{(\alpha)}$ and $\mathbb{U}_{(\alpha-\varepsilon)}$ are up to universal constants, $\varepsilon$ apart for all small $\varepsilon$. Thus the above theorem is rather crude and says that the aggregate of $\varepsilon n^2$ particles cannot grow beyond an 'annulus' of width $\sqrt{\varepsilon}$ (recall that the initial cluster was inside $\mathbb{U} \backslash \mathbb{U}_{(\alpha)}$ except for $\varepsilon' n$ many $\mathbb{U}^{(i)}$'s) even though one should expect it to be $\varepsilon$. However it suffices for our purpose.

The precise definition of IDLA is provided along with the proof of the above.

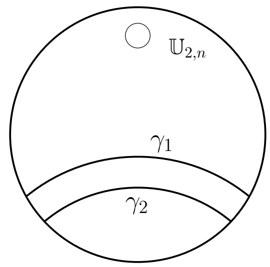


Figure 2.13: $\gamma_1$ and $\gamma_2$ denote the boundaries of $\mathbb{U}_{(\alpha)}$ and $\mathbb{U}_{(\alpha-C\sqrt{\varepsilon})}$ respectively. If the intial aggregate contains all of the region above $\gamma_1$ i.e. $(\mathbb{U}\backslash\mathbb{U}_{(\alpha)})$ and at most $\varepsilon' n$ where $\varepsilon' \ll \varepsilon$ many thin strips $\mathbb{U}^{(i)}$ (see (2.5.5)) then in $\varepsilon n^2$ rounds of IDLA on $\mathbb{U}_n$ where the random walks start from $\mathbb{U}_{2,n}$, the aggregate does not occupy any site below $\gamma_2$ i.e. the new part of the aggregate stays in $\{\mathbb{U}\backslash\mathbb{U}_{(\alpha-C\sqrt{\varepsilon})}\}$.

### *2.11.1 Proofs of Lemmas 2.8.1 and 2.8.2*

With the help of the above theorem we now finish all the remaining proofs. A corresponding lower bound analogous to Theorem 2.11.1 will be the content of the proof of Lemma 2.8.1. We follow the approach in [63]. Similar statements for IDLA on the entire lattice $\mathbb{Z}^2$ have cleaner proofs owing to symmetry of $\mathbb{Z}^2$. However in our setting we need more robust approaches and we will rely on certain heat kernel bounds for the random walk on $\mathbb{U}_n$ from [25], which are summarized in [48]. We first quote the following hitting time estimate.

**Lemma 2.11.1.** [48, Lemma 5.3] *Given small enough $\varepsilon > 0$, for all $z \in \mathbb{U}_n \cap \mathbb{U}_{(\alpha-\sqrt{\varepsilon})}$, and $y \in \mathbb{U}_n$ such that $d(y,z) \leqslant \varepsilon^2$,*

$$\mathbb{P}_y(\tau(z) < \tau(\mathbb{U}_n\backslash\{\mathbb{U}_n \cap \mathbb{U}_{(\alpha-\varepsilon)}\})) = \Theta(\frac{\log(1/d(y,z))}{\log n}),$$

*where the constant in the $\Theta$ notation depend on $\varepsilon, \alpha, \mathbb{U}$.*

*Proof of Lemma 2.8.1.* Since by hypothesis $\sigma \in \Omega_{(\varepsilon)} \cap \mathcal{A}_{\varepsilon_1}$ it has at most $\varepsilon_1 n^2$ red vertices in $\mathbb{U}_{(\alpha)} \cap \mathbb{U}_n$ and similarly at most $\varepsilon_1 n^2$ blue vertices in $\mathbb{U}^c_{(\alpha)} \cap \mathbb{U}_n$. To show that the configuration reaches $\mathcal{G}_{\varepsilon^{1/4}}$ we need to show that the red particles in $\mathbb{U}_{\alpha-\varepsilon^{1/4}}$ gets wiped off and similarly the blue particles on the other side. Because of the obvious similarity in the situations we will provide argument only for the former case.

Now notice that the red cluster in erosion is clearly dominated by the IDLA process as described in Theorem 2.11.1 with the initial aggregate being the set of red vertices in $\sigma$. Thus by Theorem 2.11.1 with probability at least $1 - e^{c(\sqrt{\varepsilon}n)^{1/3}}$ no red walker reaches $\mathbb{U}_{(\alpha - C_2\sqrt{\varepsilon})}$ in the next $C_1\varepsilon n^2$ rounds where $C_2$ depends on $C_1$. Observe that on the above event the blue cluster insider $\mathbb{U}_{(\alpha - C_2\sqrt{\varepsilon})}$ dominates a 'killed' IDLA process where the blue walkers start uniformly from $\mathbb{U}_{1,n}$ and perform IDLA but are killed on hitting the boundary of the set $\mathbb{U}_n \cap \mathbb{U}_{(\alpha - C_2\sqrt{\varepsilon})}$ (since there is no interaction with red random walkers). Thus it suffices to show that in the next $C_1\varepsilon n^2$ rounds of the 'killed' IDLA process the blue walkers will occupy and hence wipe off all the red vertices in $\mathbb{U}_{(\alpha - \varepsilon^{1/4})}$ (at most $\varepsilon_1 n^2$ ), which amounts to proving a lower bound for the 'killed' IDLA aggregate for the blue walkers. We adapt the proof in [63]. For any positive integer $i$ we associate the following stopping times to the $i^{th}$ blue walker:

- $\sigma^i$: the stopping time in the killed IDLA process
- $\tau_z^i$: the hitting time of $z \in \mathbb{U}_n$
- $\tau^i$: the exit time of the set $\mathbb{U}_{(\alpha - C_2\sqrt{\varepsilon})} \cap \mathbb{U}_n$.

For $z \in \mathbb{U}_n \cap \mathbb{U}_{(\alpha - \varepsilon^{1/4})}$ we define the random variables,

$$
\begin{aligned}
N &= \sum_{i=1}^{C_1\epsilon n^2} \mathbf{1}_{(\tau_z^i < \sigma^i)}, \text{ the number of blue walkers that visit } z \text{ before stopping} \\
M &= \sum_{i=1}^{C_1\epsilon n^2} \mathbf{1}_{(\tau_z^i < \tau^i)}, \text{ the number of blue walkers that visit } z \text{ before exiting } \mathbb{U}_{(\alpha - C_2\sqrt{\varepsilon})} \\
L &= \sum_{i=1}^{C_1\epsilon n^2} \mathbf{1}_{(\sigma^i < \tau_z^i < \tau^i)}, \text{ the number of blue walkers that visit } z \text{ before exiting } \mathbb{U}_{(\alpha - C_2\sqrt{\varepsilon})} \\
&\quad \text{but after stopping.}
\end{aligned}
$$

Note that random walkers can stop in two ways: either by occupying an 'empty' (red) vertex or by exiting $\mathbb{U}_{(\alpha - C_2\sqrt{\varepsilon})}$. Thus $N \geq M - L$. Hence,

$$\mathbb{P}\Big(z \notin A_{C_1 \varepsilon n^2}\Big) = \mathbb{P}(N = 0) \leqslant \mathbb{P}(M < a) + \mathbb{P}(L > a). \tag{2.11.1}$$

where the last inequality holds for any $a$. Now by definition,

$$\mathbb{E}(M) = C_1 \varepsilon n^2 \mathbb{P}_{\mathbb{U}_{1,n}}(\tau(z) < \tau(\mathbb{U}_n \cap \{\mathbb{U} \backslash \mathbb{U}_{(\alpha - C_2\sqrt{\varepsilon})}\})), \tag{2.11.2}$$

where $\mathbb{P}_{\mathbb{U}_{1,n}}(\cdot)$ denotes the measure when the random walk starts uniformly from $\mathbb{U}_{1,n}$. We now bound the expectation of $L$. Define the following quantity: let independent random walks start from each 'empty' (red) site in $\mathbb{U}_{(\alpha - C_2\sqrt{\varepsilon})}$ and let

$$\tilde{L} = \sum_{\substack{w \in \mathbb{U}_{(\alpha - C_2\sqrt{\varepsilon})} \cap \mathbb{U}_n \\ w: empty}} \mathbf{1}(\tau(z) < \tau(\mathbb{U}_n \cap \{\mathbb{U} \backslash \mathbb{U}_{(\alpha - C_2\sqrt{\varepsilon})}\}) \text{ for the walker starting at } w).$$

Clearly $L$ is stochastically dominated by $\tilde{L}$. Hence the RHS of (2.11.1) can be upper bounded by $\mathbb{P}(M < a) + \mathbb{P}(\tilde{L} > a)$. Now

$$\mathbb{E}(\tilde{L}) = \sum_{\substack{w \in \mathbb{U}_{(\alpha - C_2\sqrt{\varepsilon})} \cap \mathbb{U}_n \\ w: empty}} \mathbb{P}_w(\tau(z) < \tau(\mathbb{U}_n \cap \{\mathbb{U} \backslash \mathbb{U}_{(\alpha - C_2\sqrt{\varepsilon})}\})).$$

Note that by hypothesis there exists at most $\varepsilon_1 n^2$ empty (red) vertices in $\mathbb{U}_{(\alpha - C_2\sqrt{\varepsilon})} \cap \mathbb{U}_n$. Since the above sum is over a set of size at most $\varepsilon_1 n^2$, by Lemma 2.11.1 we have the following,

$$\mathbb{E}(\tilde{L}) \leqslant C' \sqrt{\varepsilon_1} \frac{n^2}{\log n}$$
$$\mathbb{E}(M) \geq C'' C_1 \varepsilon \frac{n^2}{\log n}$$

for some constants $C', C''$. The first bound follows by summing the bound in Lemma 2.11.1

over a set of points $y$, of size $\varepsilon_1 n^2$ (this is maximized when all the points are in a ball of radius $\sqrt{\varepsilon_1}$ around $z$). The latter bound follows since uniformly for any point $z \in \mathbb{U}_n \cap \mathbb{U}_{(\alpha-\varepsilon^{1/4})}$, the blue random walk starting uniformly from $\mathbb{U}_{1,n}$ has a chance $C_1 = C(\varepsilon, \mathbb{U})$ of reaching a point $\varepsilon^2$ close to $z$ before exiting $\mathbb{U}_{(\alpha - C_2\sqrt{\varepsilon})}$.[1] Now once the random walker reaches such a point, using Lemma 2.11.1 we get the bound of $C''\frac{n^2}{\log n}$. We now choose $\varepsilon_1$ small enough so that $\mathbb{E}(\tilde{L}) \leqslant \mathbb{E}(M)/4$. Choose $a = \mathbb{E}(M)/2$. Now using Azuma's inequality for indicator variables we get

$$\mathbb{P}(\tilde{L} > a) \leqslant \exp(-dn) \text{ and } \mathbb{P}(M < a) \leqslant \exp(-dn)$$

for some constant $d > 0$. Thus in (2.11.1) we get

$$\mathbb{P}(M < a) + \mathbb{P}(L > a) \leqslant 2\exp(-dn).$$

The proof of the lower bound now follows by taking the union bound of the at most $\varepsilon_1 n^2$, empty $z \in \mathbb{U}_n \cap \mathbb{U}_{(\alpha-\varepsilon^{1/4})}$ i.e. such that $\sigma(z)$ is red. □

### *2.11.2 Proof of Lemma 2.8.2*

For any set $A \subset \Omega$ define the positive return time $\tau^+(A)$ to be $\inf\{t \geq 1 : \sigma_t \in A\}$. We break the proof into a couple of lemmas. Recall Table 2.1.

**Lemma 2.11.2.** *Given positive numbers $\varepsilon, c$ there exist $\varepsilon'$ such that for all small enough $\delta$ (blob size) there exists $D = D(\varepsilon, c, \delta)$ such that for any $\sigma \in \Omega_{(\varepsilon)} \cap \Gamma_{\varepsilon'}$*

$$\mathbb{P}_\sigma(\tau^+(\Omega_{(\varepsilon)}) \geq cn^2) < e^{-Dn^2}, \tag{2.11.3}$$

[1] Formally one can take a tube of a small enough width joining a point in the 'interior of' $\mathbb{U}_n$ (say $10\varepsilon$ away from the boundary) to any point $z$ and see that the random walk stays in that tube and hits the end with the point $z$ with constant probability $C_1 = C_1(\varepsilon, \mathbb{U})$, independent of $n$. That the random walk starting from $\mathbb{U}_{1,n}$ reaches such an interior point before exiting $\mathbb{U}_{(\alpha - C_2\sqrt{\varepsilon})}$ is a straightforward consequence of the heat kernel estimates [48, Theorem 5.2].

*for all large enough* $n = 2^m$.

*Proof.* The proof follows from Lemma 2.5.6 $i$. First let $\sigma' = \sigma_1$ for the competitive erosion chain starting from $\sigma_0 = \sigma$. If $\sigma' \in \Omega_{(\varepsilon)}$ we are done. Otherwise since $\sigma \in \Gamma_{\varepsilon'}$,

$$x := w(\sigma') \geq w_{max} - \varepsilon' n^2.$$

Now let

$$X(t) = w(\sigma_{t+1})$$

where $w(\cdot)$ is the weight function defined in (2.5.3). Thus $X(0) = x$. We make the following choice of parameters:

$$B = \Omega_{(\varepsilon)}, A_1 = w_{max}, A_2 = \log(1/\delta)$$
$$a_1 = a(\varepsilon) \text{ appearing in Theorem } 2.5.1$$
$$a_2 = \varepsilon' n^2 \text{ with } \varepsilon' < a_1 c/2, T = cn^2.$$

The choice of $a_2$ works since $\sigma \in \Gamma_{\varepsilon'}$ by hypothesis. Also the choice of $\varepsilon'$ ensures that $a_2 - a_1 T < 0$ and hence the hypothesis of Lemma 2.5.6 $i$. is satisfied. Thus for $\varepsilon' = \frac{a_1 c}{2}$ by Lemma 2.5.6 $i$. for all $\sigma \in \Omega_{(\varepsilon)} \cap \Gamma_{\varepsilon'}$,

$$\mathbb{P}_{\sigma'}(\tau(\Omega_{(\varepsilon)}) \geq T) \leqslant \exp\left(-Dn^2\right),$$

for some $D = D(\varepsilon, c, \delta)$. Since $\tau^+(\Omega_{(\varepsilon)})$ starting from $\sigma$ is one more than $\tau(\Omega_{(\varepsilon)})$ starting from $\sigma'$ we are done. □

**Lemma 2.11.3.** *For all small* $\varepsilon > 0$ *there exists* $\varepsilon' > 0$ *such that for all small enough* $\delta$ *(blob size) there exists a constant* $b(\delta, \varepsilon)$ *such that for large enough* $n = 2^m$ *and any* $\sigma \in \Omega_{(\varepsilon)} \cap \Gamma_{\varepsilon'} \cap \mathcal{G}_{\varepsilon^{1/4}}$,

$$\mathbb{P}_\sigma(\tau^+(\Omega_{(\varepsilon)}) \leqslant \tau(\Omega \backslash \mathcal{G}_{\varepsilon^{1/4}})) > 1 - e^{-bn^{1/3}}.$$

*Proof.* Since $\sigma \in \Omega_{(\varepsilon)} \cap \mathcal{G}_{\varepsilon^{1/4}}$, by Theorem 2.11.1 (since clearly IDLA dominates erosion),

$$\mathbb{P}_\sigma(\tau(\Omega \backslash \mathcal{G}_{\varepsilon^{1/4}}) \geq dn^2) \geq 1 - e^{-hn^{1/3}}$$

for some $d, h$ depending on $\varepsilon, \delta$. Now using $c = d/2$ in Lemma 2.11.2 we get that we can choose $\varepsilon'$ such that for any $\sigma \in \Omega_{(\varepsilon)} \cap \Gamma_{\varepsilon'} \cap \mathcal{G}_{\varepsilon^{1/4}}$

$$\mathbb{P}_\sigma(\tau^+(\Omega_{(\varepsilon)}) \geq dn^2/2) < e^{-Dn^2}. \tag{2.11.4}$$

for some positive constant $D$. Thus for such an $\varepsilon'$,

$$\begin{aligned} \mathbb{P}_\sigma(\tau^+(\Omega_{(\varepsilon)}) \leqslant \tau(\Omega \backslash \mathcal{G}_{\varepsilon^{1/4}})) &\geq \mathbb{P}_\sigma(\tau^+(\Omega_{(\varepsilon)}) \leqslant dn^2/2) - \mathbb{P}_\sigma(\tau(\Omega \backslash \mathcal{G}_{\varepsilon^{1/4}}) \leqslant dn^2) \\ &\geq 1 - e^{-hn^{1/3}} - e^{-Dn^2}. \end{aligned}$$

Hence we are done by choosing $b = \frac{h}{2}$. □

### *2.11.3 Proof of Lemma 2.8.2*

We will specify $\varepsilon'$ later. However notice that for any small enough $\varepsilon'$ if $\sigma \in \Gamma_{\varepsilon'/2}$ then by Lemma 2.5.5 there exist positive constants $c, d$ depending on $\varepsilon'$ and the blob size such that for large enough $n = 2^m$,

$$\mathbb{P}_\sigma(\tau' > e^{cn^2}) \geq 1 - e^{-dn^2} \tag{2.11.5}$$

where $\tau' = \tau(\Omega \backslash \Gamma_{\varepsilon'})$. Now notice that by hypothesis $\sigma \in \mathcal{G}_\varepsilon$ and hence $\sigma \in \Omega_{(c\varepsilon)}$, for some constant $c = c(\mathbb{U})$. Let $\tau_{(1)}, \tau_{(2)} \ldots$ be successive return times to $\Omega_{(c\varepsilon)}$, i.e. $\tau_{(1)} = 0$ and for all $i \geq 0$,

$$\tau_{(i+1)} = \inf\{t : t > \tau_{(i)}, \sigma_t \in \Omega_{(c\varepsilon)}\}.$$

The following containment is true for any positive $b'$: Let $s := e^{b'n^{1/3}}$. Then

$$\{\tau(\mathcal{G}^c_{\varepsilon^{1/4}}) \leqslant s\} \subset \{\exists i \leqslant s \text{ such that } \sigma_i \notin \Gamma_{\varepsilon'}\} \cup \left( \bigcup_{j=0}^{s} \left\{ \{\tau_{(j)} < \tau(\mathcal{G}^c_{\varepsilon^{1/4}}) \leqslant \tau_{(j+1)}\} \cap \{\sigma_{\tau_{(j)}} \in \Gamma_{\varepsilon'}\} \right\} \right).$$

The above follows by first observing whether there exists any $i < e^{b'n^{1/3}}$ such that $\sigma_i \notin \Gamma_{\varepsilon'}$. Also let $j$ be the first index such that

$$\tau_{(j)} < \tau(\mathcal{G}^c_{\varepsilon^{1/4}}) \leqslant \tau_{(j+1)}.$$

Notice that on the event $\{\tau(\mathcal{G}^c_{\varepsilon^{1/4}}) \leqslant s\}$, we have $j \leqslant s$. Also by definition on the event $\{\sigma_i \in \Gamma_{\varepsilon'} \text{ for all } i \leqslant s\}$, we have $\sigma_{\tau_{(j)}} \in \Gamma_{\varepsilon'} \cap \Omega_{(c\varepsilon)} \cap \mathcal{G}_{\varepsilon^{1/4}}$. Thus by the union bound,

$$\begin{aligned}\mathbb{P}_\sigma\big(\tau(\mathcal{G}^c_{\varepsilon^{1/4}}) \leqslant e^{b'n^{1/3}}\big) \quad &\leqslant \quad \mathbb{P}(\exists\, i \leqslant s \text{ such that } \sigma_i \notin \Gamma_{\varepsilon'}) \\ &+ \quad \sum_{j=1}^{e^{b'n^{1/3}}} \mathbb{P}_\sigma\left(\left\{\sigma_{\tau_{(j)}} \in \Gamma_{\varepsilon'} \cap \Omega_{(c\varepsilon)} \cap \mathcal{G}_{\varepsilon^{1/4}}\right\} \cap \left\{\tau(\mathcal{G}^c_{\varepsilon^{1/4}}) \leqslant \tau_{(j+1)}\right\}\right).\end{aligned}$$

Recall that by hypothesis $\sigma \in \Gamma_{\varepsilon'/2}$. Thus by (2.11.5) for any small enough $\varepsilon'$ there exists $d = d(\varepsilon', \delta) > 0$ (where $\delta$ is the blob size) such that for any $b' > 0$ the first term is less than $e^{-dn^2}$ for large enough $n$. Also notice that by Lemma 2.11.3 we can choose $\varepsilon'$ such that every term in the sum is at most $e^{-bn^{1/3}}$ for some constant $b > 0$. Thus for such an $\varepsilon'$, putting everything together we get that for any $b' > 0$ and large enough $n$,

$$\mathbb{P}_\sigma(\tau(\mathcal{G}^c_{\varepsilon^{1/4}}) \leqslant e^{b'n^{1/3}}) \leqslant e^{-dn^2} + \sum_{i=1}^{e^{b'n^{1/3}}} e^{-bn^{1/3}}.$$

Hence by choosing $b' < b$ we get that for large enough $n$,

$$\mathbb{P}(\tau(\mathcal{G}^c_{\varepsilon^{1/4}}) \leqslant e^{b'n^{1/3}}) \leqslant 2e^{(b'-b)n^{1/3}}.$$

The proof is thus complete. □

All that is left to be done is to provide the proof of Theorem 2.11.1. The argument will be a modification of the argument in [39] adapted to our setting (for the sake of reference we will follow the same notation as in [39] throughout this section). Let $S \subset \mathbb{U}_n$. We now precisely define the IDLA process in our setting. In order to define IDLA, we first define adding one particle started at $x$ to an existing aggregate $S$. For $x \in \mathbb{U}_n$, denote by $A(S;x)$ the IDLA aggregate obtained as follows: Let $\xi = (\xi(0), \xi(1), ...)$ be a random walk on $\mathbb{U}_n$ started at $\xi(0) = x$ and let $t_S$ be the first time this walk is not in $S$. Define $A(S;x) := S \cup \{\xi(t_S)\}$. Also consider a slightly more general process, where the growth of the aggregate is stopped at certain stopping times, e.g., upon exiting a set $T$. Denote by $A(S; x \to T)$, the aggregate obtained by letting a particle randomly walk from $x$, but pausing it if it exits $T$, which is defined as follows. Let $\xi$ be a random walk on $\mathbb{U}_n$ started at $x$. Let $t_S$ be the first time this walk is not in $S$ as above, and let $t_T$ be the first time $\xi$ exits $T$. Define $A(S; x \to T) := S \cup \{\xi(t_S \wedge (t_T - 1))\}$. To keep track of the position of a paused particle, define

$$P(S; x \to T) = \begin{cases} \xi(t_T) & \text{if } t_T < t_S, \\ \perp & \text{otherwise,} \end{cases}$$

so $\perp$ means that the particle is already absorbed, in the aggregate. Given vertices $x_1, \ldots, x_k$ in $\mathbb{U}_n$ and a set $T$, define $A(S; x_1, \ldots, x_k \to T)$ to be the IDLA aggregate formed from an existing aggregate $S$ by $k$ particles, started at $x_1, x_2, \ldots x_k$ and paused upon exiting $T$. Formally we use induction: $S_0 = S$, $S_j = A(S_{j-1}; x_j \to T)$ for $j \in \{1, 2, \ldots k\}$ and $A(S; x_1, \ldots, x_k \to T) = S_k$. Again to keep track of paused particles, define $P(S; x_1, \ldots, x_k \to T)$ to be the sequence of particles paused in this process, i.e. if $p_j = P(S_{j-1}; x_j \to T)$ for $j \in \{1, \ldots, k\}$, then $P(S; x_1, \ldots, x_k \to T)$ is the sequence $(p_j : p_j \neq \perp)$. We recall the *Abelian property* of IDLA: $A(S; x_1, \ldots, x_k)$ has the same distribution as $A(A(S; x_1, \ldots, x_k \to T); P(S; x_1, \ldots, x_k \to T))$. For more details about the Abelian property see [63] and the references therein.

**Proof of Theorem 2.11.1**. We begin by stating two lemmas from [39] adapted to our setting. Recall $\mathbb{U}_\beta$ from (2.2.5), (should not be confused with $\mathbb{U}_n$ which denotes the graph) and the 'shells' $\mathbb{U}^{(i)}$ from (2.5.5).

**Lemma 2.11.4.** [39, Lemma 5] *Fix an interval $I \in \mathbb{R}$. There exists small constants $\varepsilon, \varepsilon_1$ such that for all large enough $n$ and positive integer $r$ with $n^{1/6} < r < \varepsilon n$ and $\beta \in I$, the following holds: Let $x \in \mathbb{U}_n \cap \{\mathbb{U} \backslash \mathbb{U}_\beta\}$ and let $S \subset \{\mathbb{U} \backslash \mathbb{U}_{\beta+\frac{r}{n}}\} \cap \mathbb{U}_n$ be so that $|S \cap [\mathbb{U}_n \cap \mathbb{U}_\beta]| \leqslant \varepsilon_1 r^2$. Moreover let $A$ be a set which intersects at most $\varepsilon_1$ fraction of the $\mathbb{U}^{(i)}$'s between $\mathbb{U}_\beta$ and $\mathbb{U}_{\beta+\frac{r}{n}}$, Let $\xi$ be a random walk started at $x$ and stopped upon hitting $\mathbb{U}_{\beta+\frac{r}{n}}$. Then,*

$$\mathbb{P}(\xi \cap \left(\mathbb{U}_\beta \backslash (S \cup A \cup \mathbb{U}_{\beta+\frac{r}{n}})\right) \neq \varnothing) \geq \eta,$$

*for $\eta = \eta(I, \mathbb{U})$.*

Recall $\mathbb{U}_{\beta'} \subset \mathbb{U}_\beta$ for any $\beta' \geq \beta$. The above lemma says that if the size of the intersection of the set $S$ with the region between $\mathbb{U}_\beta$ and $\mathbb{U}_{\beta+\frac{r}{n}}$ is not large then there is a constant chance that the random walk starting from $\mathbb{U}_n \cap \{\mathbb{U} \backslash \mathbb{U}_\beta\}$ will visit sites in $\mathbb{U}_\beta \backslash \mathbb{U}_{\beta+\frac{r}{n}}$ which are neither in $S$ nor in $A$ before hitting $\mathbb{U}_{\beta+\frac{r}{n}}$. After analyzing the behavior of a single particle, we can analyze the behavior of the whole aggregate. The following lemma says that, with high probability, a constant fraction of the aggregate is absorbed in a wide enough (yet still very fine) annulus. Recall the hypothesis about the initial cluster in Theorem 2.11.1.

**Lemma 2.11.5.** [39, Lemma 6] *Let $I$ be as above. There exist $\varepsilon, \varepsilon_1, \varepsilon_2 > 0$ and $p < 1$ such that for all $n$ large enough, for all $n^{1/3} < k < \varepsilon n^2$ and $x_1, ..., x_k \in \mathbb{U}_n \cap \{\mathbb{U} \backslash \mathbb{U}_\beta\}$, and for all $S \subset [\mathbb{U}_n \cap \{\mathbb{U} \backslash \mathbb{U}_\beta\}]$ where $\beta \in I$. Moreover as in the previous lemma, let $B$ be a set which intersects at most $\varepsilon_1$ fraction of the $\mathbb{U}^{(i)}$'s between $\mathbb{U}_\beta$ and $\mathbb{U}_{\beta+\frac{k^{1/2}}{n}}$, then,*

$$\mathbb{P}(|A\left(S \cup B; x_1, ..., x_k \to \mathbb{U}_n \cap \{\mathbb{U} \backslash \mathbb{U}_{\beta+\frac{k^{1/2}}{n}}\}\right) \backslash S \cup B| \leqslant \varepsilon_2 k) \leqslant p^k$$

The above lemma says that if at any time the aggregate is a subset of $\mathbb{U} \backslash \mathbb{U}_\beta$ along with some $\varepsilon_1$ fraction of the $\mathbb{U}^{(i)}$'s between $\mathbb{U}_\beta$ and $\mathbb{U}_{\beta+\frac{k^{1/2}}{n}}$, then in the next $k$ rounds a constant fraction of particles fill up sites in $\mathbb{U} \backslash \mathbb{U}_{\beta+\frac{k^{1/2}}{n}}$ with high probability.

Before proving the above lemmas we prove Theorem 2.11.1.

*Proof of Theorem 2.11.1.* Recall from (2.2.6) that $\mathbb{U}_\beta = \mathbb{U}_{(\alpha)}$. And clearly $\mathbb{U}_{(\alpha - C\sqrt{\varepsilon})} \subset \mathbb{U}_{\beta + C_1\sqrt{\varepsilon}}$ for some $C_1$ depending on $C$, (due to the same reasoning as in (2.5.4)). Thus it suffices to prove that the growth cluster will not touch any new points in $\subset \mathbb{U}_{\beta + C_1\sqrt{\varepsilon}}$ for some $C_1$.

The proof is along the lines of [39] and consists of inductively constructing a sequence of aggregates $A_{(j)}$ by pausing the particles at different distances $m_j$ from the starting set $\mathbb{U}_{2,n}$. If $k_j$ is the number of paused particles, we choose the next distance $m_{j+1}$, at which we pause the particles again, in terms of $m_j$ and $k_j$. We iterate this procedure until there are less than $n^{1/3}$ paused particles. Since there are too few particles now, we naively bound the growth of the cluster from this point on.

Define $A_{(j)}, m_j, P_j, k_j$ as follows: $A_{(0)}$ be the initial set of filled sites as in the hypothesis of Theorem 2.11.1. $m_1 = \sqrt{\varepsilon} n$ and $A_{(1)}$ be the cluster after $\varepsilon n^2$ random walks were released uniformly from $\mathbb{U}_{2,n}$ (see Setup 1) and stopped on hitting $\mathbb{U}_{\beta_1}$ where $\beta_1 = \beta + \frac{m_1}{n}$. Correspondingly let $P_1$ be the the sequence of stopped particles and $k_1 = |P_1|$. We now define $A_{(j)}, P_j$ for all $j > 1$: For $j > 1$ define $m_{j+1}$ in the following way: (this is where we make the adaptations from [39] since in our setting the initial cluster has some particles and hence not all the sites in $\mathbb{U}_n \cap \mathbb{U}_{\beta_j}$ are empty. Consider the sequence

$$m_j + k_j^{1/2}, m_j + 2k_j^{1/2}, m_j + 3k_j^{1/2}, \ldots.$$

Recall $\mathbb{U}^{(i)}$ from (2.5.5). Let $\ell$ be the smallest integer such that at most $\varepsilon_1$'th fraction of the 'shells' $\mathbb{U}^{(i)}$ between $\mathbb{U}_{\beta+[m_j+(\ell-1)k_j^{1/2}]/n}$ and $\mathbb{U}_{\beta+[m_j+\ell k_j^{1/2}]/n}$ contain particles from the initial cluster $A_{(0)}$ ($\varepsilon_1$ is the same as in the hypothesis of Lemma 2.11.5). We define $m'_j$ and $m_{j+1}$ to be $m_j + (\ell-1)k_j^{1/2}$ and $m_j + \ell k_j^{1/2}$ respectively. That is we define $m_{j+1}$ to be the smallest number of the right scale such that there are enough empty sites between $\mathbb{U}_{\beta+m_{j+1}/n}$ and $\mathbb{U}_{\beta+m'_j/n}$.

Let $A_{(j+1)} = A(A_{(j)}; P_j \to \mathbb{U}_n \cap \{\mathbb{U}\backslash\mathbb{U}_{\beta+m_{j+1}/n}\})$; and $P_{j+1} = P(A_{(j)}; P_j \to \mathbb{U}_n \cap \{\mathbb{U}\backslash\mathbb{U}_{\beta+m_{j+1}/n}\})$ and let $k_{j+1} = |P_{j+1}|$. Also let $J$ be the (random) first time at which

$k_j \leqslant n^{1/3}$. We do not stop the particles after this point and hence by construction, $A_{(j)} = A_{(J+1)}$ for any $j \geq J+1$. As mentioned earlier the Abelian property guarantees that $A_{(J+1)}$ has the same law as the IDLA cluster with $m_0 = \varepsilon n^2$ particles. Naively $A_{(J+1)} \subset \mathbb{U}_n \cap \{\mathbb{U} \backslash \mathbb{U}_{\beta+m_J/n+C(\varepsilon' n+n^{1/3})/n}\}$ for some constant $C = C(\mathbb{U})$,. This follows by the following crude bound: since there are at most $n^{1/3}$ particles left and initially at most $\varepsilon' n$ many shells $\mathbb{U}^{(i)}$ filled up, the growth from $J$ onwards cannot be more than by $\varepsilon' n + n^{1/3}$ shells $\mathbb{U}^{(i)}$. By Lemma 2.11.5 for some $\varepsilon_2 \leqslant 1$,

$$\mathbb{P}[\exists\, 2 \leqslant j \leqslant J \wedge n \,:\, k_j \geq (1-\varepsilon_2)^j k_1] \leqslant \mathbb{P}[\exists\, 2 \leqslant j \leqslant J \wedge n \,:\, k_j \geq (1-\varepsilon_2) k_{j-1}] \leqslant np^{n^{1/3}}.$$

Since $k_1 \leqslant \varepsilon n^2$ by hypothesis this implies that with probability at least $1 - np^{n^{1/3}}$, we have $J \leqslant n$ and $k_j$ decreases geometrically at each step, hence,

$$\begin{aligned} m_J &= m_1 + (m_1' - m_1) + k_1^{1/2} + (m_2' - m_2) + \ldots + (m_{J-1}' - m_{J-1}) + k_{J-1}^{1/2} \\ &\leqslant m_1 + k_1^{1/2} \frac{1}{1-(1-\varepsilon_2)^{1/2}} + \sum_{\ell=1}^{J-1} (m_\ell' - m_\ell). \end{aligned} \tag{2.11.6}$$

Now we finish by bounding $\sum_{\ell=1}^{J-1}(m_\ell' - m_\ell)$. Notice that by construction at least $\varepsilon_1$'th fraction of all the shells between $m_\ell' - m_\ell$ intersect the initial cluster. Also by hypothesis on the initial cluster number of such shells is at most $\varepsilon' n \leqslant \varepsilon n/D$ (recall the hypothesis of Theorem 2.11.1, we specify the value of $D$ here). Thus $\varepsilon_1 \sum_{\ell=1}^{J-1}(m_\ell' - m_\ell) \leqslant \frac{\varepsilon}{D} n$ and hence choosing $D = \frac{1}{\varepsilon_1}$ we get that $\sum_{\ell=1}^{J-1}(m_\ell' - m_\ell) \leqslant \varepsilon n$. Also naively $k_1 \leqslant \varepsilon n^2$ and hence putting everything together with probability at least $1 - np^{n^{1/3}}$, $m_J \leqslant C m_1 = C\sqrt{\varepsilon} n$, for some absolute constant $C$. Thus $A_{(J+1)}$ does not intersect $\mathbb{U}_{\beta + C_1\sqrt{\epsilon}}$ and hence we are done. □

We now provide the proofs of Lemmas 2.11.4 and 2.11.5.

*Proof of Lemma 2.11.4* We stop the random walk $\xi$ when it hits $\mathbb{U}_{\beta+r/2n}$ (call the point at which it stops as $y$) and look at the ball of radius $cr/n$ (for some small constant $c$) around it. Because of the smoothness assumption on $\mathbb{U}$ even if $y$ lies close to the boundary of $\mathbb{U}$, $|\{B(y, cr/n) \cap \mathbb{U}_n\}| \geq 2dr^2$, for some $d = d(c, \mathbb{U})$. Now if $\varepsilon_1 \leqslant d$ then $|\{B(y, cr/n) \cap \mathbb{U}_n\} \backslash S| \geq dr^2$. Also by hypothesis there are at most $\varepsilon_1 r$ shells $\mathbb{U}^{(i)}$ initially with particles at time $0$ and hence $|\{B(y, cr/n) \cap \mathbb{U}_n\} \backslash \{S \cup \text{initial particles}\}| \geq dr^2/2$ by choosing suitably small $\varepsilon_1 = \varepsilon_1(c)$.The result now follows from the heat kernel estimate in [48, Theorem 5.2] which shows that the probability that the random walk started at $y$ hits the set $\{B(y, cr/n) \cap \mathbb{U}_n\} \backslash \{S \cup \text{initial particles}\}$ before exiting $B(y, r/2n)$ is a constant independent of $r, n$. □

*Proof of Lemma 2.11.5* The proof of [39, Lemma 6] uses [39, Lemma 5]. The same arguments as in [39, Lemma 6] using Lemma 2.11.4 above instead of [39, Lemma 5] completes the proof. □

# Chapter 3

## ABSORBING STATE PHASE TRANSITION IN CONSERVATIVE DYNAMICS

### *3.1 The model description and main result*

Consider the following continuous time interacting particle system on $\mathbb{Z}$. Start with particles at every site on the line by sampling i.i.d. from a certain distribution with mean $\mu$. Each particle can be in one of the two following states $A$ (active) and $S$ (sleepy). Initially all the particles are in active state. Each active particle does a continuous time nearest neighbour symmetric random walk on $\mathbb{Z}$ at rate $1$. Sleepy particles do not move. Also each active particle undergoes the transition $A \to S$ at rate $\lambda > 0$ independent of everything else. Sleepy particles undergo the transition $S + A \to 2A$ instantaneously, i.e., a sleepy particle at $x \in \mathbb{Z}$ becomes active instantaneously when an active particle visits the site $x$. Also the transition $A \to S$ is observed only if at that time the active particle was the only particle at its site, i.e. the instantaneous transitions $2A \to A + S \to 2A$ is not observed.

**Definition 3.1.1.** The above process is known as Activated Random Walk, which we denote by ARW$(\mu, \lambda)$ or simply ARW throughout the rest of the chapter.

Formal definition appears in Section 3.2. It is possible to view ARW as a particular case of the general class of reaction-diffusion system that were introduced in the late 1970's by F. Spitzer and has since been studied extensively. In the non-equilibrium statistical mechanics literature, ARW and its closed variants (especially with $\lambda \to \infty$) have been studied in connection with the the phenomenon of self-organised criticality. Before providing proper definitions, we review the pertinent questions and state the main results of this chapter. The most important phenomenon of interest in the study of ARW is the long term fixation/non-fixation of the system i.e. whether one observes sustained activity in any finite window about a point (say the origin) infinitely often or not.

Clearly the answer should depend on the initial particle density $\mu$ and the sleep rate $\lambda$. The primary focus of interest in studying ARW centres around the phase transition phenomenon. For a fixed sleep rate $\lambda$, as the particle density $\mu$ increases, it is expected that the system shows a transition from almost sure local fixation to staying active forever almost surely. One of the first mathematically rigorous results about ARW was established in [78] where it is shown that for every $\lambda > 0$ there is a critical particle density $\mu_\lambda \in [\frac{\lambda}{\lambda+1}, 1]$ such that ARW$(\mu, \lambda)$ locally fixates almost surely when $\mu < \mu_\lambda$ and stays active almost surely when $\mu > \mu_\lambda$. That the above probabilities are $0$ or $1$ is a simple consequence of the ergodicity of the starting distribution and translation invariant nature of the events. However, even though it is believed that for all $\lambda < \infty$, $\mu_\lambda$ is strictly less than $1$, a proof has so far been missing.

These problems have been re-iterated in [38], [22] and [87]. In this chapter we will only consider ARW and for easy reference purpose we now record below the conjectures in our setting.

**Conjecture 1.** For $\mathrm{ARW}(\mu, \lambda)$ on $\mathbb{Z}$, we have that $\mu_\lambda < 1$ for all $\lambda > 0$.

**Conjecture 2.** In the same set-up as **Conjecture 1**, $\mu_\lambda \to 0$ as $\lambda \to 0$.

The main result of this chapter immediately provides a positive resolution of **Conjecture 1** for small $\lambda$.

**Theorem 3.1.1.** *Given $\mu > 0$ there exists $\lambda_\mu > 0$ such that $\mathrm{ARW}(\mu, \lambda)$ on $\mathbb{Z}$ with particle density $\mu$ and sleep rate $\lambda$ stays active almost surely for all $\lambda < \lambda_\mu$.*

It was also established in [78] that $\mu_\lambda$ in a non-decreasing function of $\lambda$. As a consequence we obtain the following corollary resolving **Conjecture 2**:

**Corollary 3.1.2.** For $\mathrm{ARW}(\mu, \lambda)$ on $\mathbb{Z}$, the critical particle density $\mu_\lambda \to 0$ as $\lambda \to 0$.

**Remark 3.1.3.** For any $\lambda > 0$, it is easy to heuristically explain why $\mu_\lambda$ should be at most one. If $\mu > 1$, then on an average "there are more particles than sites", and since at most one particle can fall asleep at a site, the system should not fixate. This argument is formalized in [6, 78, 83] in different settings, see Section 3.1.1 for details. However, establishing $\mu_\lambda < 1$ requires understanding how activity is being sustained forever due to particle interaction, i.e. sleepy particles being woken up by active particles over and over. This makes the analysis substantially more difficult.

### *3.1.1 Background*

In the recent years, studies in non-equilibrium statistical mechanics have offered up a number of mathematically challenging interacting particle processes which exhibit phase transitions far away from equilibrium. A particular class of conservative models that has drawn significant attention is one where though the particles are conserved, they can exist in one of the two states: active and inactive. Typically an inactive particle becoming active requires interaction with one or more active ones. Paradigm examples in this class includes conserved lattice gases [81] and Sandpile models with stochastic update rules [36, 71, 72]. In infinite volume, these models are believed to exhibit so called *absorbing-state phase transition* from an active phase as some model parameter (typically particle density) is varied. In finite volume, when run with a carefully controlled driven-dissipative mechanism, these systems are believed to exhibit the phenomenon of *self-organised criticality* [35, 37], where the system is attracted to a critical state, even though it is not explicitly tuned to this critical value.

The transitions in these models are believed to belong to a universality class, the so-called *Manna class*. Whether the Manna class exists as an autonomous universality class separate from the universality class of directed percolation (DP) seems not to have a broadly accepted answer among physicists at this point [12, 65], it appears beyond the state-of-the-art techniques to obtain mathematically rigorous results on the critical or near-critical behaviour of these models (see [22] for some progress). Even the more basic questions of existence of phase transitions seem challenging and has only been settled in a few particular cases. One of the main challenges in studying these systems is the intricate long-range interaction caused by the conservation of particles, which makes it harder to apply some of the standard techniques in rigorous statistical mechanics.

A major motivation to study ARW in the above general class of models is the following Stochastic Sandpile Model (SSM), a variant of Manna's model [71, 72]. In SSM, started with an initial particle configuration of product measure with density $\mu$, a site with an isolated particle instantaneously becomes inactive, whereas at any site containing $d > 2$ remain active, and at rate $1$, emits two particles using independent symmetric random walk steps, leaving $(d-2)$ particles at the site (observe the contrast to the deterministic sandpile model). Compared to the deterministic sandpile model (see e.g. [28]), much less is rigorously known about SSM. It was only recently proved in [78] that there exists $\mu_c \in [\frac{1}{4}, 1]$ such that the system fixates for $\mu < \mu_c$ and remains active for $\mu > \mu_c$. Numerical simulations suggest that $\mu_c \approx 0.9489$, it remains a major mathematical challenge to prove $\mu_c < 1$. It is reasonable to expect that ARW is a sufficiently good (and possibly more mathematically tractable) approximation to SSM and thus captures some of its crucial aspects. In particular as $\lambda \to \infty$, ARW corresponds exactly to the model studied in [53].

ARW can also be viewed as a special case of driven-diffusive epidemic processes. In this process a healthy particle does a simple random walk at rate $D_A \geqslant 0$ and each infected particle does a simple random walk with rate $D_B > 0$, infecting all particles that it steps on, and recuperating at rate $\lambda$. ARW corresponds to the special case $D_A = 0$. This model, in the case $D_A = D_B$ was introduced by Spitzer in late 1970s, but was rigorously studied in detail much later in [59–62]. Although numerical studies have predicted different regimes of critical behaviour for $D_A < D_B$ and $D_A > D_B$, not much is rigorously known for the case $D_A \neq D_B$ (see [21] for the study of a related model), in partcular, it is not understood whether the behaviour for $D_A = 0$ (i.e. ARW) should be the same as the behaviour for $0 < D_A < D_B$.

For more background and a fuller history of ARW and related processes, the interested reader is referred to [38, 78] and the references therein; see [73] for a detailed account of non-equilibrium phase transitions in lattice models.

There have been a flurry of rigorous results on ARW in the last few years, mostly in the wake of the breakthrough paper [78], where, as mentioned above, the existence of an absorbing-state phase transition was established on $\mathbb{Z}$ for both ARW and SSM. This paper crucially uses a construction of the ARW process using the Diaconis-Fulton representation and the so-called Abelian property (see the next section). An upper bound of one on the critical density for ARW was also established in [78] for any $\lambda > 0$. The same upper bound was also established in [6] for ARW on unimodular graphs using mass transport principle and in [83] for general bounded degree graphs using a comparison with certain internal aggregation models. The results of [83] also implies that critical density is positive for ARW for any $\mathbb{Z}^d$ for $\lambda = \infty$. This result has recently been established for any $\lambda > 0$ and any $d$ in [84] using a multi-scale argument. As far as the critical value is concerned the only result is by [22] for the case $\lambda = \infty$, where it is shown that the system on $\mathbb{Z}^d$ fixates for all $\mu < 1$, thus in conjunction with the results of [6, 83] establishing $\mu_c = 1$. The behaviour of ARW at criticality is also largely open, the essentially only result is for $\lambda = \infty$, in which case the system does not fixate at criticality [22]. For more detailed predictions on the behaviour of ARW, see [38].

So far we have concerned ourselves with the model where the active particles perform a simple symmetric random walk, and that scenario will be the focus of the present work. However one could consider a situation where the random walk steps are biased, and in certain cases, such models are better understood. The study of the biased ARW started with an unpublished argument of Hoffman and Sidoravicius (see [22, Theorem 1]) that considers the case of totally asymmetric walks and establishes that the critical density is $\mu_c = \frac{\lambda}{\lambda+1}$ and further that the system does not fixate at criticality. More recently ARW on $\mathbb{Z}^d$ with asymmetric (but not necessarily totally asymmetric) jump distribution has been studied in [80, 87]. It is shown in [87] that in $d = 1$, when the jump distribution is biased, Conjecture 2 holds and Conjecture 1 holds for small sleep rate. In a very recent paper

[80] the same has been established for $d \geqslant 2$ sharpening a previous result of [87] which stated that the system does not fixate for small $\lambda$ and $\mu$ sufficiently close to one. We refer the reader to the lecture notes of Leonardo Rolla [79] for more details on the previously known results on ARW.

Despite this impressive progress in the study of biased ARW, Conjecture 1 and Conjecture 2 has so far remained open for symmetric ARW (we shall always mean the symmetric case by ARW unless otherwise mentioned), and it appears that the methods employed cannot be (easily) adapted to understand Conjecture 1 and Conjecture 2 for symmetric ARW on $\mathbb{Z}$.

In this chapter we partly solve Conjecture 1 and as a consequence prove Conjecture 2. The proof considers truncated versions (with finite universe) of $\mathrm{ARW}$ as has been used in several other arguments regarding this model in the literature . The new ingredients include a novel use of the Abelian property (see Section 3.3) which allows us to study a slightly different labeled variant of $\mathrm{ARW}$ which is dominated by the actual $\mathrm{ARW}$ (i.e. under a coupling, fixation of the latter implies the same for the former). More details appear in Section 3.2.1.

## *3.2 Formal definitions and setup*

We follow [78] in formally describing the set-up of ARW. Let $\mathbb{N}_{0\rho} = \mathbb{N}_0 \cup \rho$ where $\rho$ is a formal symbol (whose meaning will be clear soon). The state space $\Omega$ will be the space of all bi-infinite sequence with elements of $\mathbb{N}_{0\rho,}$ i.e., $\Omega = \mathbb{N}_{0\rho}^{\mathbb{Z}}$. For any time $t \geq 0$, $\eta_t \in \Omega$ will denote the state of the system i.e. $\eta_t(x)$ denotes the number of particles at $x \in \mathbb{Z}$ at time $t$. $\eta_t(x) = \rho$ denotes that the only particle at $x$ at time $t$ is in state $S$ (is asleep). Following notation from [78] we formally let $|\rho| = 1$ so that irrespective of the state of the particles, $|\eta_t(x)|$ denotes the number of particles at site $x$ at time $t$. For notational convenience we define the following addition and multiplication operations on $\mathbb{N}_{0\rho}$, to describe the $A+S \to 2A$ and $A \to S$ transitions.

$$\begin{aligned} \rho + 0 &= 0 + \rho = \rho \\ \rho + n &= n + \rho = n+1. \\ \rho.1 &= \rho \\ \rho.n &= n \text{ for } n > 1. \end{aligned}$$

Let for $x \in \mathbb{Z}$ and $\eta \in \Omega$, $A(\eta(x))$ denote the number of active particles at site $x$ in configuration $\eta$. With the above notation formally the process evolves as follows: for each site $x$, we have the transitions $\eta \to \tau_{x,y}\eta$ at rate $A(\eta_t(x))\frac{1}{2}\mathbf{1}_{|y-x|=1}$, and $\eta \to \tau_{x,\rho}\eta$ at rate $\lambda A(\eta_t(x))$, where

$$\tau_{x,y}(\eta)(z) = \begin{cases} \eta(z)+1 & z = y \\ \eta(z)-1 & z = x \\ \eta(z) & \text{otherwise.} \end{cases} \tag{3.2.1}$$

Similarly

$$\tau_{x,\rho}(\eta)(z) = \begin{cases} \rho.\eta(x) & z = x \\ \eta(z) & \text{otherwise.} \end{cases} \tag{3.2.2}$$

We shall refer to this models as **Activated Random Walk** and call it ARW for brevity throughout the rest of the chapter.

Let $\mathbb{P}^\nu$ denote the law of the process started from an initial configuration distributed according to $\nu$. Throughout the rest of the chapter we will focus on $\nu$ being a product measure with identical coordinate projections which in particular implies ergodicity.

One needs to argue that such a process is well-defined even starting with infinitely many particles. To this end we could use the general theory of interacting particle systems developed in [67], or appeal to a construction due to Andjel [7] which shows the process is well-defined under some mild finiteness condition on $\nu$, and can be approximated in a suitable sense by its finite truncations. See [78] for more details on this.

We say that the system has particle density $\mu$ if $\mathbb{E}_\nu(\eta_0(0)) = \mu$. Thus $\mathrm{ARW}(\mu, \lambda)$ will denote the activated random walk process with initial density $\mu$ and sleep rate $\lambda$ (note that we choose to suppress the dependence of the distribution $\nu$ and just keep track of the particle density).

**Definition 3.2.1.** ARW started from any configuration is said to **locally fixate** if for every $x \in \mathbb{Z}$, $\eta_t(x)$ becomes constant eventually. Otherwise we say that the system stays **active**.

### *3.2.1 Key ideas and outline of the proofs*

To prove Theorem 3.1.1 we first adopt the usual approach of approximating the infinite system by considering 'truncated' $\mathrm{ARW}$ on large but finite boxes (see Lemma 3.3.1). The goal is to then show that by choosing the box to be large enough, the number of times a particle gets emitted from the origin can be made larger than any finite constant with probability bounded away from $0$. More specifically we run several rounds of the truncated process with growing intervals, with the origin as one of the end points. The intervals are chosen to be growing exponentially in size and it is argued that the total number of times particles gets emitted from the origin in the $\ell^{th}$ round dominates a $\mathrm{Ber}(\frac{1}{3})$ random variable. Also by construction we ensure that the rounds are independent. Thus the total number of particles emitted at the origin up to the $\ell^{th}$ round dominates a $\mathrm{Bin}(\ell, \frac{1}{3})$ random variable. Using Lemma 3.3.1 one then argues that since the activity at the origin in the finite volume is arbitrarily large the origin does not fixate in the infinite volume system almost surely. Thus Theorem 3.1.1 follows. This argument is made precise in Section 3.3.3.

Proving the claim that the total number of times particles gets emitted from the origin in the $\ell^{th}$ round dominates a $\mathrm{Ber}(\frac{1}{3})$ random variable is done by showing that for a large enough interval with not too few particles it is exponentially unlikely (in the size of the interval) for the finite volume system to stabilize inside the interval without any particle touching the boundary (recall that the origin was chosen to be one of the end points of the interval). Thus with probability almost $1/2$ particles hit the origin (probability of hitting either of the boundary points is close to $1$, and both are equally likely to happen by the left right symmetry of the starting configuration and the dynamics).

The technical core of this chapter consists of proving the exponential upper bound on the probability of the event mentioned above. The proof includes a novel use of Abelian Property which allows the reduction to a study of a labeled variant of the $\mathrm{ARW}$ which makes things technically feasible (see Section 3.4). Roughly we show the following which is made precise as the content of Lemma 3.4.1. We start by renormalizing space: i.e. consider the lattice points $K\mathbb{Z} = \{\ldots, -2K - K, 0, K, 2K, \ldots\}$. Also consider a large interval $[-r, r]$, with at least $\frac{\mu r}{2}$ particles (recall that $\mu$ is the particle density). For convenience of explanation, for the moment we assume that the initial particles are supported only on $K\mathbb{Z}$. We now show that for $\lambda$ is small enough, the probability that this finite system stabilizes without hitting either $-r$ or $r$ is $e^{-cr}$ for some $c > 0$ depending only on $\mu$. We choose $K \gg \frac{1}{\mu}$ and then show that at the end of the stabilization process for $\lambda$ small enough it is unlikely to have many particles asleep in any interval of length $K$. Since we start with not too few particles this implies that particles must escape through the boundary and in the process hit them with high probability.

A statistic of fundamental importance in our analysis of the ARW and the study of other abelian systems is the so called 'Odometer function' : the number of times a particle was emitted from a site till stabilization. (see (3.3.5) for a formal definition). For other uses of the odometer see [70]. The proof proceeds as the following: fix a sequence $z = \{0 = z_{-r}, \ldots, z_{-2K}, z_{-K}, z_0, z_K, z_{2K}, \ldots z_r = 0\}$. We bound the probability that the odometer function at the points $K\mathbb{Z} \cap [-r, r]$ is the sequence $z$ after the truncated process on $[-r, r]$ has run till stabilization (this happens in finite time almost surely due to the finiteness of the universe). Since by choice $z_r = z_{-r} = 0$, on the the above event, $-r, r$ do not emit particles. The probability bound as a function of $z$ turns out to be small enough to allow us to take an union bound over all such sequences $z$ and end up with $e^{-cr}$.

The proof of the above relies heavily on a new and novel renormalized variant of the standard Diaconis Fulton representation of abelian systems (see [29]). Roughly one way to run the $\mathrm{ARW}$ dynamics, as done in [78], is to start with a probability space where at each site on $\mathbb{Z}$ one has 'stacks' of left, right or sleep instructions used by particles when they are emitted, (see (3.3.2)). The odometer function at a site is then the number of elements from the corresponding stack used till stabilization. For our purposes we introduce 'renormalized' variants of the above where the stacks are only at the renormalized lattice points $K\mathbb{Z}$ and instead of being steps of length one, are lazy random walk paths stopped on hitting the nearest point in the set $K\mathbb{Z}$. The laziness of the walk corresponds to the sleep instructions. See (3.4.3) for precise definitions.

### *3.2.2 Organisation of the chapter*

In Section 3.3 we collect all the basic preliminaries about the truncated (finite universe) ARW and Abelian property. In Section 3.3.3, we show how to complete the proof of Theorem 3.1.1 assuming the statement of the key Lemma 3.3.7. In Section 3.4 we develop a labeled variant of the ARW dynamics and reduce Lemma 3.3.7 to Lemma 3.4.1 about the labeled ARW dynamics. Section 3.5 develops a few technical preliminaries needed for the proof of Lemma 3.4.1, which is completed in Section 3.6 using a careful union bound over the possible odometer values. We finish with some open questions in Section 3.7.

## ***3.3 Finite volume dynamics and Abelian property***

For our purposes we now define a truncated version of the $\mathrm{ARW}(\mu, \lambda)$ dynamics. Recall that the initial configuration $\eta_0$ is distributed according to some product measure $\nu$ with particle density $\mu$. Let $\nu_M$ be the restriction of $\nu$ on the finite box $[-M, M]$ and $\mathbb{P}^M := \mathbb{P}^{\nu_M}$ be the corresponding law of the process. The next result shows that to prove Theorem 3.1.1 it suffices to consider the measures $\mathbb{P}^M$.

To see this consider the following modification of the ARW dynamics. Given a configuration $\eta$ on $\mathbb{Z}$, and $M \in \mathbb{N}$, let $\widehat{\eta}_M$ denote the restriction of $\eta$ on $[-M, M]$. With the initial configuration $\widehat{\eta}_M$, consider ARW process with the restriction that particles that move out of $[-M, M]$ are deleted from the system (alternatively they fall asleep immediately irrespective of everything else). Since initially there are (almost surely) finitely many particles in the system, and since any particle not falling asleep will eventually exit $[-M, M]$ it is easy to argue that this process will stabilize almost surely, i.e., all particles will fall asleep in finite time.

The object of interest at this point is the number of transitions happening at the origin, we call this $u_M(0)$ (formal definitions appear in the next subsection). One now considers a coupling all the truncated ARWs on one common probability space. We denote the underlying measure by $\mathscr{P}$ (formal definition appears in § 3.3.1). As an important consequence of the so called 'Abelian Property' it follows that under the above coupling the sequence $u_M(0)$ is nondecreasing in $M$. Let

$$u(0) := \lim_{M\to\infty} u_M(0). \tag{3.3.1}$$

We then have the following lemma.

**Lemma 3.3.1.** [78, Lemma 4] *Let $\nu$ be a translation invariant ergodic measure with finite particle density $\mathbb{E}_\nu(\eta(0)) < \infty$. Then*

$$\mathbb{P}^\nu(\textit{the system locally fixates}) = \mathscr{P}(u(0) < \infty) \in \{0, 1\}.$$

We refer to [77], which is an older arXiv version of [78], for a proof of the above Lemma. As already mentioned before the $0-1$ law is a direct consequence of ergodicity.

In the next subsection we formally define the coupling of the truncated processes mentioned above and precisely state the monotonicity property used to define $u(0)$.

### *3.3.1 Coupling of truncated ARW's*

To construct the coupling of truncated ARWs we shall take resort to a Diaconis-Fulton representation [29, 40] of this process where the process is implemented through a sequence of instructions attached to the sites. The advantage of this representation is the so called Abelian Property, which allows one to disregard the order in which different steps were performed in certain settings. The relevant elements of the Diaconis-Fulton representation in our context was developed in [78, Section 3], and we closely follow their treatment for the rest of this and the next subsection. We start by introducing a series of notations. Recall the transitions $\tau_{x,y}$ and $\tau_{x,\rho}$ from (3.2.1) and (3.2.2). Now consider the following array of random variables:

$$\mathscr{I} = \begin{matrix} \dots & \xi_{(-2,1)} & \xi_{(-1,1)} & \xi_{(0,1)} & \xi_{(1,1)} & \xi_{(2,1)} & \dots \\ \dots & \xi_{(-2,2)} & \xi_{(-1,2)} & \xi_{(0,2)} & \xi_{(1,2)} & \xi_{(2,2)} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{matrix} \tag{3.3.2}$$

where $\xi_{(x,j)}$ are independent for any $x \in \mathbb{Z}$ and $j \in \mathbb{N}$ and moreover,

$$\xi_{(x,j)} = \begin{cases} \tau_{x,x-1} & \text{with probability } \frac{1}{2(\lambda+1)} \\ \tau_{x,x+1} & \text{with probability } \frac{1}{2(\lambda+1)} \\ \tau_{x,\rho} & \text{with probability } \frac{\lambda}{\lambda+1}. \end{cases} \tag{3.3.3}$$

We would call the $\xi_{(x,j)}$'s instructions at the site $x$ and the underlying product measure $\mathscr{P}$. Using these instructions one can define a discrete time version of the ARW process in the following way: We start by defining the notion of stability of a site $x$ with respect to a configuration $\eta \in \Omega$.

**Definition 3.3.1.** For any $\eta \in \Omega$ we say a site $x \in \mathbb{Z}$ is **stable** if $\eta(x) = 0$ or $\rho$, and otherwise we call it **unstable**.

Thus given a configuration $\eta \in \Omega$ at each discrete time step $t$, one can choose an unstable site $x$ and use the first unused element from the stack $\xi_{(x,\cdot)}$ and use it to perform the transition to a configuration $\eta'$ at time step $(t+1)$. We call such an operation "toppling" at site $x$. Formally we keep track of the number of topplings at every site as a function of time $t$. Let

$$h := (h(x) : x \in \mathbb{Z}) \tag{3.3.4}$$

where $h(x)$ denote the number of topplings at $x$, which we will call the odometer function at $x$. $h_t(\cdot)$ will be used to denote the odometer function at time $t$. For later purposes it will be convenient to keep track of both the odometer function $h_t$ and the configuration $\eta_t$. To this end define the toppling operation at $x$ acting on the pair $(\eta, h)$ by

$$\Phi_x(\eta, h) = (\xi_{(x,h(x)+1)}(\eta), h + \delta_x),$$

i.e. we use $\xi_{(x,h_x+1)}$, the first unused element of the stack at $x$ to topple, and $h$ increases by $1$ at $x$.

**Definition 3.3.2.** We say that $\Phi_x$ is **legal** if $x$ is unstable i.e. $\eta(x) \geq 1$. For any sequence $\alpha = (x_1, x_2, \ldots, x_k)$ we define the sequence of topplings at $x_1$, followed by $x_2$ and so on through till $x_k$ by $\Phi_\alpha$, i.e. $\Phi_\alpha = \Phi_{x_k} \ldots \Phi_{x_1}$. We now say that $\alpha$ is a **legal sequence** if $\Phi_{x_i}$ is legal for $\Phi_{i-1} \ldots \Phi_1$ for all $i = 1, \ldots k$.

We abuse notation a little to denote by $h_\alpha$ the odometer function after performing the sequence of toppling given by $\alpha$, i.e. for any $x \in \mathbb{Z}$,

$$h_\alpha(x) = \sum_{i=1}^{k} \mathbf{1}(\alpha_i = x). \tag{3.3.5}$$

We also define the natural ordering on $\Omega$ and the space of odometer functions: for $\eta, \tilde{\eta} \in \Omega$ we say that $\eta \geq \tilde{\eta}$ if $\eta(x) \geq \tilde{\eta}(x)$ for all $x \in \mathbb{Z}$. Similarly for odometer functions $h, \tilde{h}$, we say $h \geq \tilde{h}$ if $h(x) \geq \tilde{h}(x)$ for all $x \in \mathbb{Z}$. We also write $(\eta, h) \geq (\tilde{\eta}, \tilde{h})$ if $\eta \geq \tilde{\eta}$ and $h = \tilde{h}$.

The most basic property of the above process is the so called Abelian Property which says that given two sequence of legal topplings which result in the same odometer function (see (3.3.5)), the final configuration is the same in both the cases i.e. the order in which topplings are performed does not matter.

**Lemma 3.3.2.** *(Abelian Property) Given any two legal sequence of topplings $\alpha$ and $\alpha'$ such that $h_\alpha = h_{\alpha'}$, then*

$$\Phi_\alpha(\eta) = \Phi_{\alpha'}(\eta).$$

To see why the above is true notice that for each site $x, y \in \mathbb{Z}$ with $x \neq y$ any pair of legal topplings, one at $x$ and the other at $y$ commutes. The sequence of topplings at a single site $x$ do not commute however since the compositions of $\tau_{x,\rho}$ and $\tau_{x,x+1}$ clearly depends on the order of composition. However given the stacks $\{\xi_{(x,j)}\}$ (see (3.3.2)) and the function $h_\alpha(\cdot)$ one knows in which order the topplings at a single site $x$ occurs.

Before providing a formal proof of the above we now list some basic facts about the toppling operation in the following observation.

**Observation 3.3.3.** The toppling operation as described above has the following properties:

i. If $\alpha$ is a legal sequence for $\eta$, then $\Phi_\alpha(\eta)$ depends on $\alpha$ only through $h_\alpha(\cdot)$. This is immediate from Lemma 3.3.2.

ii. $\Phi_\alpha(\eta)$ is non-increasing in $h_\alpha(x)$ and non-decreasing in $h_\alpha(z)$, $z \neq x$. This is easy by observing that a toppling at $x$ can only decrease the number of particles at $x$, whereas a toppling at another site can only increase it.

iii. If $x$ is unstable in $\eta$ and $\eta_0(x) > \eta(x)$, then $x$ is unstable in $\eta_0$.

iv. Moreover if $\eta_0 > \eta$, then $\Phi_\alpha(\eta_0) > \Phi_\alpha(\eta)$.

Items iii. and iv. above are obvious.

*Proof of Lemma 3.3.2* Let $|\alpha|$ denote the length of the sequence $\alpha$. Since $h_\alpha = h_{\alpha'}$ clearly $|\alpha| = |\alpha'|$. The proof now follows by induction on $|\alpha|$. Clearly if $|\alpha| = 1$, there is nothing to prove. Otherwise let $\alpha = (x_1, x_2, \ldots, x_k)$ and $\alpha' = (y_1, y_2, \ldots y_k)$. Now if $x_1 = y_1$ we are done by induction, considering the sequences $(x_2, \ldots x_k)$ and $(y_2, \ldots y_k)$ with initial configuration $\Phi_{x_1}(\eta) = \Phi_{y_1}(\eta)$. If $x_1 \neq y_1$ notice that since $h_{\alpha'}(y_1) \geq 1$ and $h_\alpha = h_{\alpha'}$, there exists a $j \leqslant k$ such that $x_j = y_1$. Let $j_0$ be the minimum such $j$. Now consider the sequence $\alpha_{j_0} = (x_{j_0}, x_1, x_2 \dots x_{j_0-1}, x_{j_0+1}, \dots, x_k)$. We now notice that $\Phi_\alpha(\eta) = \Phi_{\alpha_{j_0}}(\eta)$. To see this first observe that $\alpha_{j_0}$ is a legal sequence. This is because initially the toppling $\Phi_{x_{j_0}}$ is legal since $\alpha'$ is legal. Also since $x_{j_0}$ does not occur in $(x_1, \dots x_{j_0-1})$ by the discussion before the proof, $\Phi_{x_{j_0}}$ commutes with $\Phi_{x_i}$ $1 \leqslant i \leqslant j_0 - 1$. Thus $\Phi_\alpha(\eta) = \Phi_{\alpha_{j_0}}(\eta)$. Now considering $\alpha_{j_0}$ and $\alpha'$ we are done by the previous case since both the sequences have the same first element. Thus

$$\Phi_\alpha(\eta) = \Phi_{\alpha_{j_0}}(\eta) = \Phi_{\alpha'}(\eta).$$

□

### *3.3.2 Consequences of the Abelian Property*

As mentioned before, for our purposes we will be often interested in finite ARW dynamics restricted to an interval. We start with the following definition.

**Definition 3.3.4.** Let $V$ be a finite subset of $\mathbb{Z}$. A configuration $\eta$ is said to be stable in $V$ if all the sites $x \in V$ are stable. We say that a sequence of topplings $\alpha$ is contained in $V$ if all its elements are in $V$, and we say that $\alpha$ stabilizes $\eta$ in $V$ if every $x \in V$ is stable in $\Phi_\alpha(\eta)$.

**Lemma 3.3.3.** *(Least Action Principle,* [78, Lemma 1]*) Given a set $V$, if $\alpha, \beta$ are two legal sequences of topplings such that $\beta$ is contained in $V$ and $\alpha$ stabilizes $\eta$ in $V$ then $h_\beta \leqslant h_\alpha$, i.e. all the topplings in $\beta$ are also needed in $\alpha$.*

*Proof.* The proof is similar to the proof of Lemma 3.3.2 and uses induction on $|\beta|$. Let $\alpha = (x_1, x_2, \dots x_k)$ and $\beta = (y_1, y_2, \dots y_j)$. Since $\beta$ is legal, $y_1$ is unstable and hence must occur in $\alpha$. Let $j_0$ be the first index such that $x_{j_0} = y_1$. Thus $\alpha_{j_0} = (x_{j_0}, x_1, x_2 \dots x_{j_0-1}, x_{j_0+1}, \dots, x_k)$ is also legal and has the same end configuration as $\alpha$. The proof now follows by induction by looking at the sequences $(x_1, \dots x_{j_0-1}, x_{j_0+1}, \dots, x_k)$ and $(y_2, \dots y_j)$ and the starting configuration $\Phi_{y_1}(\eta)$. □

As a simple corollary we obtain the following.

**Lemma 3.3.4.** *(*[78, Lemma 2]*) If $\alpha$ and $\beta$ are both legal toppling sequences for $\eta$ that are contained in $V$ and stabilize $\eta$ in $V$, then $h_\alpha = h_\beta$ . In particular, $\Phi_\alpha(\eta) = \Phi_\beta(\eta)$.*

*Proof.* By Lemma 3.3.3, $h_\beta \leqslant h_\alpha$ as well as $h_\alpha \leqslant h_\beta$ and hence we are done. □

We now state a simple but crucial monotonicity result that allows us define $u(0)$ in (3.3.1). First we define formally the odometer function appearing in (3.3.1).

**Definition 3.3.5.** (Truncated odometer) Given $\eta \in \Omega$, a set of vertices $V$, and any legal stabilizing sequence $\alpha$ as in Lemma 3.3.4 define the truncated odometer,

$$u_V(\cdot) := u_{\eta,V}(\cdot) := h_\alpha(\cdot). \tag{3.3.6}$$

Note that by Lemma 3.3.4, $u_V(\cdot)$ is a well defined quantity. We also define the final configuration

$$\eta_{\infty,V} := \Phi_\alpha(\eta) \tag{3.3.7}$$

which is also well defined by the above lemma. For our purposes we will only consider finite boxes i.e. $V = [-M, M]$, for some positive integer $M$. For brevity we define

$$u_M := u_{[-M,M]}. \tag{3.3.8}$$

**Lemma 3.3.5.** *(Monotonicity,* [78, Lemma 3]*) For vertex sets $V \subset V'$ and configurations $\eta \leqslant \eta'$,*

$$u_{\eta,V} \leqslant u_{\eta',V'}.$$

*Proof.* Let $\alpha$ and $\beta$ be legal stabilizing sequences for the two systems. By iii. and iv. in Observation 3.3.3 we see that $\alpha$ is also a legal sequence for $\eta'$. Thus we are done by Lemma 3.3.3. □

Using Lemma 3.3.5 one can define $u_{\eta,\infty} = \lim_{M\to\infty} u_M$. Note also that Lemma 3.3.5 implies that $u_{\eta,\infty}$ does not depend on the increasing sets $[-M, M]$, and any sequence of sets increasing to $\mathbb{Z}$ would have the same limit. When the underlying configuration is clear from context we will denote $u_{\eta,\infty}(\cdot)$ just by $u(\cdot)$ as stated in (3.3.1).

A configuration $\eta$ is said to be stabilizable if $u(x) < \infty$ for all $x \in \mathbb{Z}$. (one can easily see that almost surely $u(\cdot)$ is finite everywhere or infinite everywhere).

We now finish off the section with the statement of some technical lemmas and deduce the proof of Theorem 3.1.1 from them. The rest of this chapter will be devoted to the proof of the lemmas.

### *3.3.3 Proof of Theorem 3.1.1*

Before proceeding further with technical arguments, we now discuss the key steps in the proof of Theorem 3.1.1. By Lemma 3.3.1 it suffices to show the following: for any $\mu > 0$ for small enough sleep rate, almost surely, the sequence $\{u_M(0)\}_{M\in\mathbb{N}}$ (see (3.3.8)) takes arbitrarily large values. To show this we look at the following sequence of intervals of exponentially growing size.

$$\begin{aligned} V_1 &= [0, 2^5] \\ V_2 &= [0, 2^8] \\ \vdots &= \vdots \\ V_\ell &= [0, 2^{3\ell+2}] \end{aligned}$$

Recall $\eta$ is the initial configuration on $\mathbb{Z}$ distributed as the product measure $\nu$. Let $\eta^{(\ell)}$ denote the restriction of $\eta$ to the interval $I_\ell = [2^{3\ell}, 3 \cdot 2^{3\ell}]$. Observe that $I_\ell$'s are just the middle halves of the interval $V_\ell$ and are mutually disjoint for $\ell \in \mathbb{Z}$. Since $\nu$ is a product measure, $\{\eta^{(\ell)}\}_{\ell\in\mathbb{N}}$ are independent samples from measures $\{\nu_{I_\ell}\}_{\ell\in\mathbb{N}}$ respectively where $\nu_{I_\ell}$ denotes the coordinate projection of the product measure $\nu$ onto the interval $I_\ell$.

We then consider running rounds of truncated ARWs with initial data $\eta^{(\ell)}$ for and vertex set $V_\ell$ for $\ell = 1, 2, \ldots$. That is, in the $\ell$-th round, we ignore all the particles that are not initially in $I_\ell$ and stabilize the initial configuration $\eta^{(\ell)}$ on the interval $V_\ell$ using the elements of the stacks (see (3.3.2)) that have not been used upto $(\ell-1)$-th round. Notice that in this sequential process, we are ignoring certain active particles from previous rounds, however using Lemma 3.3.5 this will suffice for our purpose of obtaining a lower bound on the odometer counts. Let $w_\ell$ denote the increase in the odometer count at $0$ during the $\ell$-th round of the process described above. We shall show that $\sum_\ell w_\ell$ becomes arbitrarily large. To this end we have the following lemma.

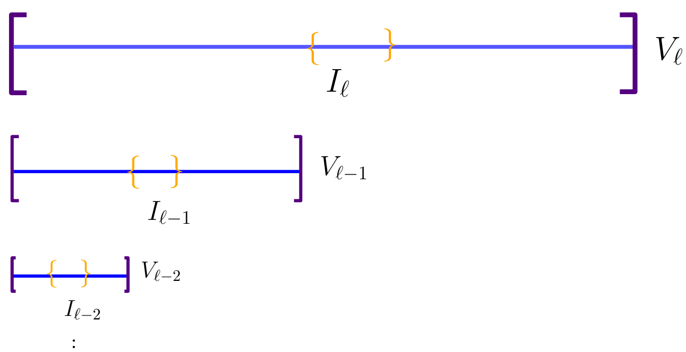


Figure 3.1: The interval of activity $V_\ell$ in the $\ell^{th}$ round.

**Lemma 3.3.6.** $\{w_\ell\}_{\ell\in\mathbb{N}}$ *are mutually independent. Further, for* $\mu > 0$ *and* $\lambda$ *sufficiently small depending on* $\mu$ *we have that for all large* $\ell$*,*

$$\nu \otimes \mathscr{P}(w_\ell > 0) \geq \frac{1}{4}.$$

We postpone the proof of Lemma 3.3.6 for the moment and instead show first how this implies Theorem 3.1.1.

*Proof of Theorem 3.1.1.* It follows from Lemma 3.3.6 that $\sum_{\ell=1}^{\infty} w_\ell = \infty$, $\nu \otimes \mathscr{P}$-almost surely. Observe that at the end of the $\ell$-th round, the odometer function at the origin takes value $\sum_{i=1}^{\ell} w_i$. Now recall $u_{V_\ell}(0)$ from (3.3.6). Thus by Lemma 3.3.1 we would be done once we show for $\ell \geqslant 1$,

$$u_{V_\ell}(0) \geq \sum_{i=1}^{\ell} w_i. \tag{3.3.9}$$

To see the above notice the following: all the steps up to the $\ell^{th}$ round are legal even though they need not be stabilizing in $V_\ell$. Indeed while running the rounds we ignore particles from previous rounds which can affect the process in two ways. In our sequential operation described above, a particle that is asleep at the end of a round is static throughout all the next rounds, whereas while stabilizing $\eta$ on $V_\ell$ these particles can be woken up by movements of particles in later rounds. Also some of the sleep instructions implemented during the $\ell^{th}$ round might not be implemented in the actual stabilization process on $V_\ell$. This is because other particles from previous rounds might be present at the site (recall that these particles are ignored while running the $\ell$-th round, so in the $\ell$-th round these sleep instructions might actually have been implemented). However crucially, even though the sequence of topplings up to round $\ell$ need not be stabilizing, all the steps are legal with respect to the stabilization process on the set $V_\ell$ with initial particle configuration being $\eta$ restricted to $V_\ell$. Thus (3.3.9) follows by Lemma 3.3.3. □

It remains to prove Lemma 3.3.6.The main step is to show that if the number of particles in the configuration $\eta^{(\ell)}$ is not too small, it is exponentially unlikely in the length of the interval $V_\ell$ that during the $\ell$-th round of the operation described above (i.e., stabilizing the initial configuration $\eta^{(\ell)}$ of particles in the interval $V_\ell$) none of the particles actually move out $V_\ell$. Formally we prove the following key lemma.

**Lemma 3.3.7.** *Fix $\mu > 0$. Let $\eta_{(r)}$ be a particle distribution supported on $[-r, r]$ such that the total number of particles in $\eta_r$ is at least $\mu r$. Let $\mathscr{P}_{\eta_{(r)}}$ denote the law of the ARW dynamics with initial configuration $\eta_{(r)}$ and the stacks with law $\mathscr{P}$ (as in* (3.3.2)*). Then for $\lambda$ sufficiently small depending on $\mu$ then there exists $c = c(\mu, \lambda) > 0$, such that for all large $r$*

$$\mathscr{P}_{\eta_{(r)}}\left(\max\{u_{(r)}(-2r), u_{(r)}(2r)\} = 0\right) \leqslant e^{-cr},$$

*where* $u_{(r)}(\cdot) := u_{[-2r,2r]}(\cdot)$ *denotes the odometer function (see* (3.3.8)*).*

We finish off this section by proving Lemma 3.3.6 using Lemma 3.3.7.

*Proof of Lemma 3.3.6.* Recall that since the intervals $I_\ell$ are disjoint, the initial configurations $\eta^{(\ell)}$ are independent samples from the distributions $\nu_{I_\ell}$ for $\ell = 1, 2, \ldots$. Also observe that, because of the independence of the elements of the stack, the joint distribution of the unused elements of the stack at the end of the $(\ell-1)$-th round (i.e. the stack that is being used for the $\ell$-th round) is the same as the law $\mathscr{P}$ of the original stacks, and further this is independent of the elements of the stack that have been used in the first $(\ell-1)$ rounds. This implies that $w_\ell$ are mutually independent and further that for each $\ell \geqslant 1$, the distribution of $w_\ell$ (under $\nu \otimes \mathscr{P}$) is the same as the law of $u_{V_\ell}(0)$ under $\nu_{I_\ell} \otimes \mathscr{P}$.

Let $|\eta^{(\ell)}|$ denote the number of particles in the configuration distributed as $\nu_{I_\ell}$. Due to an obvious translation invariance in the system, Lemma 3.3.7 implies that

$$\nu_{I_\ell} \otimes \mathscr{P}\left(\max\{u_{V_\ell}(0), u_{V_\ell}(2^{3\ell+2})\} = 0, |\eta^{(\ell)}| \geqslant \mu 2^{3\ell}\right) \leqslant e^{-c2^{3\ell+2}}.$$

Observe that by strong law of large numbers for $\ell$ sufficiently large we have $\nu_{I_\ell}(|\eta^{(\ell)}| \geqslant \mu 2^{3\ell}) \geqslant 0.9$ and hence for $\ell$ sufficiently large

$$\nu_{I_\ell} \otimes \mathscr{P}\left(\max\{u_{V_\ell}(0), u_{V_\ell}(2^{3\ell+2})\} > 0\right) \geqslant 0.9 - e^{-c2^{3\ell+2}} \geqslant \frac{1}{2}.$$

Notice that the interval $I_\ell$ and $V_\ell$ are both symmetric about $2^{2\ell+1}$. Because $\nu_{I_\ell}$ is a product measure with identical marginals, and the dynamics of ARW possesses an inherent left-right symmetry, it follows by a reflection about $2^{3\ell+1}$ that $u_{V_\ell}(0)$ and $u_{V_\ell}(2^{3\ell+2})$ has identical distributions under the measure $\nu_{I_\ell} \otimes \mathscr{P}$. The lemma follows. $\square$

The rest of this chapter is devoted to the proof of Lemma 3.3.7. Recall the stacks and the underlying measure $\mathscr{P}$ from (3.3.2). In case of finite vertex sets (which will be the focus of our analysis ) by Lemma 3.3.2, the final odometer function and the particle distribution is just a function of the stacks and the initial particle configuration. However for our purposes we need a different set of stacks, labeled particles, and a certain toppling rule. This is described elaborately in the next section and termed as **Labeled ARW dynamics**. Thus to distinguish the two, we denote the description of ARW so far (equivalently the measure space corresponding to $\mathscr{I}$ (see (3.3.2) and the initial particle configuration $\eta$) as **Unlabeled ARW dynamics**[1] The next section also provides a coupling between the two 'processes'.

## *3.4 Labeled ARW Dynamics*

We begin by trying to understand why Lemma 3.3.7 should be true, i.e., why is it unlikely that upon stabilizing the configuration in $[-2r, 2r]$ with initial particle distribution in $[-r, r]$ the odometer at both $-2r$ and $2r$ is $0$. Typically we shall have about $2\mu r$ many particles in our system. Notice that on the event $\max\{u_{(r)}(-2r), u_{(r)}(2r)\} = 0$, all the $2\mu r$ particles eventually fall asleep in the interval $[-2r, 2r]$. Now consider $K$ a very large integer $\gg \mu^{-1}$. Divide the interval $[-2r, 2r]$ into subintervals $[iK, (i+1)K]$ of length $K$ each. In the final configuration, we typically expect only a small number of particles asleep on each interval $[iK, (i+1)K]$ if $\lambda$ is sufficiently small depending on $K$. Indeed, whenever a particle is released from the site $iK$ it is extremely likely (by taking $\lambda$ small) that this particle will reach either $(i-1)K$ or $(i+1)K$ without falling asleep in between. Further, in such a case, it wakes up all the sleepy particles in either $[iK, (i+1)K]$ or $[(i-1)K, iK]$. Hence it is unlikely that $2\mu r$ many particles will fall asleep in $4r/K$ many intervals of length $K$ each.

To make this intuition into a proof, we shall introduce a different set of stacks containing instructions, where, instead of a single step, the instructions now will consist of a random walk path that tells the particle (started at $iK$) its steps until it reaches $iK \pm K$.

---

[1]Note the abuse of notation since there is no canonical process in the discrete time setting and any order of toppling eventually lead to the same statistics by Lemma 3.3.2.

### *3.4.1 Random Walk Stacks*

Let $\lambda > 0$ and consider the lazy symmetric random walk $S_\lambda(\cdot)$ with laziness $\frac{\lambda}{\lambda+1}$ on $\mathbb{Z}$ started from the origin, i.e. $S_\lambda(0) = 0$ and in each discrete time step, $t = 0, 1, 2, \ldots$

$$S_\lambda(t+1) - S_\lambda(t) = \begin{cases} 0 & \text{with probability } \frac{\lambda}{\lambda+1} \\ 1 & \text{with probability } \frac{1}{2(\lambda+1)} \\ -1 & \text{with probability } \frac{1}{2(\lambda+1)} \end{cases} \tag{3.4.1}$$

independent of everything. Now for any positive integer $K$ let $\tau(\{-K, K\}$ denote the hitting time of the set $\{-K, K\}$ for the walk $S_\lambda(\cdot)$ and let

$$S_{K,\lambda}(\cdot) = S_\lambda(\cdot \wedge \tau(\{-K, K\}))$$

be the killed random walk, i.e. $S_\lambda(\cdot)$ stopped on hitting $\{-K, K\}$.

We now define a probability space where instructions will be the killed random walk paths instead of single steps as in (3.3.2). We consider the renormalized lattice,

$$K\mathbb{Z} = \ldots - 2K, -K, 0, K, -2K, \ldots, \tag{3.4.2}$$

and the following set of instructions at the points $K\mathbb{Z}$ (which we shall sometimes refer to as lattice points):

$$\mathscr{I}_* = \begin{matrix} \cdots & \zeta_{(-2,1)} & \zeta_{(-1,1)} & \zeta_{(0,1)} & \zeta_{(1,1)} & \zeta_{(2,1)} & \cdots \\ \cdots & \zeta_{(-2,2)} & \zeta_{(-1,2)} & \zeta_{(0,2)} & \zeta_{(1,2)} & \zeta_{(2,2)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{matrix} \tag{3.4.3}$$

where $\zeta_{(x,j)}$ are independent for any $x \in \mathbb{Z}$ and $j \in \mathbb{N}$ and moreover,

$$\zeta_{(x,j)} = xK + S_{K,\lambda}(\cdot), \tag{3.4.4}$$

i.e. the elements $\zeta_{(x,j)}$ are i.i.d copies of random walk paths $S_{K,\lambda}(\cdot)$ started at $xK$ and stopped on hitting $xK \pm K$. We will still continue to think of the random walk steps to be distributed as the elements $\xi_{(\cdot,\cdot)}$ in (3.3.2) namely:

$$\begin{array}{cl} \tau_{z,z-1} & \text{with probability } \frac{1}{2(\lambda+1)} \\ \tau_{z,z+1} & \text{with probability } \frac{1}{2(\lambda+1)} \\ \tau_{z,\rho} & \text{with probability } \frac{\lambda}{\lambda+1}. \end{array} \tag{3.4.5}$$

where $z$ is the current location of the random walk. That is left and right steps in the random walks would be interpreted as instruction to go left or right, whereas lazy steps would be interpreted as sleep instructions. However, how these instructions act upon the configurations will be slightly different from the standard ARW dynamics of § 3.1 as explained below. We call the above product measure $\mathscr{P}_*$ analogous to $\mathscr{P}$.

### *3.4.2 Labeled Particles and Labeled ARW Dynamics*

With the above stacks we now define a variant of the ARW dynamics from the discussion following Definition 3.3.1. To define the modified dynamics, we shall restrict ourselves to initial particle configurations supported on $K\mathbb{Z}$. In this setting the particles will be labeled. The labels will keep changing over time and will be from the set $\mathbb{Z} \times \mathbb{N}$.

**Labeling Scheme:** Suppose we start with a configuration $\eta_*$ supported on $K\mathbb{Z}$. The first co-ordinate of the label of all the particles starting at $iK$ will be $i$. The second label is arbitrarily chosen to be: $1$ to the number of points on that lattice site. That is, if there are $n_i$ particles at $iK$ to start with they are labeled $(i,1),(i,2),\ldots,(i,n_i)$ in an arbitrary way. See Figure 3.2.

As mentioned above the labels are not static and evolve with time. Any particle of label $(i,j)$ will change its label when it hits the lattice points $(i-1)K$ or $(i+1)K$ in which case the label would change to $(i',j)$ where $i' = i \pm 1$ according to the lattice point hit and $j$ is the smallest positive integer such that the label $(i',j)$ has never been used before in the

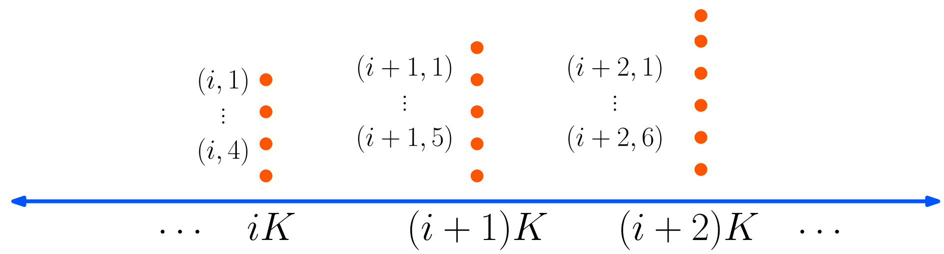


Figure 3.2: Initial labeling

history of the process. A particle with label $(i, \cdot)$ will sometimes be referred to as a particle **emitted** from $iK$.

We now introduce a toppling scheme for the labeled process analogous to $\S$ 3.3.1. To do that we first associate with each particle a random walk from the stacks (3.4.3).

**Associating Random Walks with Particles:** The first time a particle acquires label $(a, b)$ (recall that our labeling rule implies that at this time the particle must be at $aK$), it gets associated with the random walk $\zeta_{(a,b)}$ which stays associated with the particle till it changes its label next.

Next we define how to use the random walks to topple particles. Informally we do the following. Whenever we decide to topple a particle with label $(a, b)$, we use the first unused step of the random walk $\zeta_{(a,b)}$. Let $\tau$ be the total number of steps in $\zeta_{(a,b)}$ (i.e., until the random walk hits $aK \pm K$). Whenever the particle is toppled later before $\tau$, say for the $i$-th time, the toppling step is the $i$-th step of $\zeta_{(a,b)}$. We now provide a formal description.

**Toppling Rule and Labeled particle dynamics:** Recall the transitions $A \to S$ and $A + S \to 2A$ between particles in active and sleepy states from $\S$ 3.3.1 and that only $A$ particles could be toppled. We introduce a particle interaction which is a slight variant of that dynamics, where the particles with different first labels do not interact. We shall call this variant, the labeled ARW dynamics and it will always be defined on a finite universe $V$. That is, we shall fix a finite subset $V$ of $\mathbb{Z}$ and and particles outside $V$ become instantaneously inactive and stay inactive for ever. To distinguish from the standard ARW dynamics from $\S$ 3.3.1, we denote the state of active and sleepy particles in the labeled

ARW dynamics by $\tilde{A}$ and $\tilde{S}$ respectively.

Consider stacks distributed according to $\mathscr{P}_*$ and a particle configuration $\eta_*$ distributed supported on $K\mathbb{Z}$. Fix $V$ and let all particles in $V$ initially be in $\tilde{A}$ state. Consider the labeling scheme described above. For our purposes we shall work with particular order of topplings. To this end we start by defining the lexicographic order on the labels: i.e. $(a, b) < (c, d)$ if $a < b$ or $a = b$ and $c < d$.

At each time $t$, we topple the $\tilde{A}$ particle with the smallest label, say $(a, b)$, which is well defined as we are restricted to the finite set $V$.[2] To topple, use the first unused instruction of random walk $\zeta_{(a,b)}$. The left and right step instructions from the random walks are interpreted as before and used to move the particle one step to left or right respectively. The lazy steps in the random walks are interpreted as sleep instructions. However, in the labeled dynamics:

$\mathbf{\tilde{A}} + \mathbf{\tilde{S}} \to \mathbf{2\tilde{A}}$ transition occurs only when the label of the two particles have same first co-ordinate. Let us illustrate this with an example. Consider an $\tilde{S}$ particle at $y \in (aK, (a+1)K)$ with label $(x, j)$. Notice that by definition either $x = a$ or $x = a + 1$. Now if an $\tilde{A}$ particle is toppled from $y \pm 1$ and moves to $y$, the transition $\tilde{A} + \tilde{S} \to 2\tilde{A}$ occurs only if the $\tilde{A}$ particle has label of the form $(x, \cdot)$. Thus in words the above says that a particle which was emitted from a lattice site $aK \in K\mathbb{Z}$ can be woken up only by another particle emitted from $aK$.

$\mathbf{\tilde{A}} \to \mathbf{\tilde{S}}$ transition is allowed to happen only when the label of no other particle at that site has the same first co-ordinate. Notice the difference with $A \to S$ transition. Thus as observed before the particles with different first labels do not interact. [3]

The system is said to stabilize if all the particles in $V$ are in $\tilde{S}$ state (recall that a particle that moves out of $V$ is inactive forever). As before it is easy to argue that as $V$ is finite

[2]Note that in Unlabeled ARW dynamics, particles were unlabeled and hence the definition of the process involved toppling sites rather than particles unlike here.

[3]As a consequence that now we are allowed to have two $\tilde{S}$ particle per site provided they have different first label. Moreover, if there are two $\tilde{S}$ particles at $y \in (iK, (i+1)K)$ (one with first label $i$ and another with $(i+1)$) and an $\tilde{A}$-particle with first label $i$ moves to $y$. It then wakes up the $\tilde{S}$-particle at $y$ with label $i$, but not the one with label $i+1$, i.e., in such a situation we have a $\tilde{A} + 2\tilde{S} \to 2\tilde{A} + \tilde{S}$ transition.

and the number of particles is finite the system stabilizes almost surely in finite time. We define the odometer function and the final configuration analogous to (3.3.6) and (3.3.7) respectively.

Note in Unlabeled ARW dynamics there was no toppling rule unlike here and hence to ensure that (3.3.6) and (3.3.7) are well defined we had to invoke the abelian property (Lemma 3.3.2). However in the labeled dynamics the toppling rule is fixed and hence there is no ambiguity in the definitions of the final odometer count and the final configuration. Let us denote the odometer function and the final configuration by

$$u_{*,V}(\cdot) \text{ and } \eta_{*,\infty,V} \tag{3.4.6}$$

respectively, (note the dependence on the intitial configuration is suppressed).

Now we want to relate the Labeled ARW dynamics with Unlabeled ARW dynamics. Since by definition in the labeled process, particles with different first labels do not interact, one would expect more active to sleepy transitions and less sleepy to active transitions, than the unlabeled process. So started with the same initial configuration it is expected that the labeled dynamics is quicker to stabilize (in a stochastic sense). To make this formal we need to define a coupling between the measures $\mathscr{P}$ and $\mathscr{P}_*$ (the law of the stacks used in the unlabeled and the labeled dynamics respectively, see (3.3.2), (3.4.3)).

### *3.4.3 Coupling Labeled ARW Dynamics with Unlabeled ARW dynamics*

Fix a finite subset $V \subseteq \mathbb{Z}$. Given an initial particle configuration $\eta$ supported on $V$, we define a natural coupling $\mathcal{P}_\eta$ of the stacks $\mathscr{I}$ and $\mathscr{I}_*$ having laws $\mathscr{P}$ and $\mathscr{P}_*$ respectively. The coupling is run in three rounds. Note the labeled process is only defined when the initial configuration is supported on $K\mathbb{Z}$. This is what the first round achieves.

**First Round:** We run Unlabeled ARW dynamics in $V$ where we topple all active particles in $\mathbb{Z}\backslash K\mathbb{Z}$. After the end of this round the system only has active particles at $V \cap K\mathbb{Z}$. Call the configuration of active particles $\tilde{\eta}$ (supported on $K\mathbb{Z}$).

**Second Round:** In this round we describe a 'natural' coupling of Unlabeled ARW dynamics and the Labeled ARW dynamics starting from $\tilde{\eta}$. The coupling $\mathcal{P}$ will be Markovian and will ensure that the ARW processes are exactly the same for all times. This is done in the following way: recall the labeling of the particles and toppling rule described in Section 3.4.2. Also recall that by the Abelian property (Lemma 3.3.2) the final statistics of Unlabeled ARW dynamics are independent of the order in which particles or sites are toppled. Thus in order to couple with the labeled process we choose the same toppling rule as the latter. Namely, we topple the same particle in both the processes and couple the stacks $\mathscr{I}$ and $\mathscr{I}_*$ such that the two processes stay identical throughout. Note that this is possible due to the following two reasons:

- The steps of the random walk $\zeta_{(\cdot,\cdot)}$ in (3.4.3) have the same distribution as $\xi_{(\cdot,\cdot)}$ in (3.3.2), as already remarked in (3.4.5).

- Arguing inductively, if the particle configuration stays the same in both the versions under $\mathcal{P}$ up to the $t^{th}$ toppling, since an $\tilde{A}$ particle is always an $A$ particle, the $t+1^{th}$ toppling would be legal in both the processes.

As a consequence of the above the stabilizing sequence in the Labeled dynamics would always be a legal sequence in Unlabeled ARW dynamics. Thus this round of the coupling runs till the labeled dynamics has been stabilized in $V$.
Note that potentially many of the particles in $\eta$ that were not in $\tilde{\eta}$ and hence not toppled in the second round could have woken up in the process rendering the configuration running in Unlabeled ARW dynamics unstable. The last round takes care of these remaining particles.

**Third round:** Independently sample the rest of the stacks $\mathscr{I}$ and $\mathscr{I}_*$. Use the former and run Unlabeled ARW dynamics till stabilization toppling arbitrarily (by Lemma 3.3.2 does not change the final distribution).

Thus we have completed the description between the two processes. The following observation is trivial from the definition of $\mathcal{P}_\eta$.

**Observation 3.4.1.** Under the coupling $\mathcal{P}_\eta$ described above, $u_V(\cdot)$ started from $\eta$ is lower bounded by $u_{*,V}(\cdot)$ (see (3.4.6)) starting from $\tilde{\eta}$.

Thus to prove Lemma 3.3.7 it suffices to lower bound $u_{*,[-2r,2r]}(\cdot)$ started from $\tilde{\eta}$ (notice that $\tilde{\eta}$ is a particle configuration supported on $K\mathbb{Z}$ whose distribution depends on $\eta$, but independent of the stack $\mathscr{I}_*$) where $\eta$ is an initial configuration supported on $[-r,r]$ with at least $\mu r$ particles. This is the content of the next two lemmas.

**Lemma 3.4.1.** *Fix $\mu > 0$. There exists $K = K(\mu)$ and $\lambda = \lambda(\mu, K) > 0$ such that the following is true for all $r$ sufficiently large. Consider an initial particle configuration $\widehat{\eta}$ supported on $[-r,r] \cap K\mathbb{Z}$ with at least $\frac{\mu r}{2}$ particles. Then there exists $c > 0$ such that for all sufficiently large $r$,*

$$\mathscr{P}_{*,\widehat{\eta}}\left(\max\{u_{(*,r)}(-2r), u_{(*,r)}(2r)\} = 0\right) \leqslant e^{-cr} \tag{3.4.7}$$

*where $\mathscr{P}_{*,\widehat{\eta}}$ denotes the law of the Labeled ARW Dynamics started with the initial particle configuration $\widehat{\eta}$ and $u_{(*,r)}(\cdot) := u_{*,[-2r,2r]}(\cdot)$.*

**Lemma 3.4.2.** *Recall the coupling $\mathcal{P}_\eta$ from above. For any $\eta$ supported on $[-r,r]$ with at least $\mu r$ many particles, for $\lambda$ sufficiently small there exist $c > 0$ such that*

$$\mathcal{P}_\eta(|\tilde{\eta}| < \frac{\mu r}{2}) \leqslant e^{-cr}$$

*where $|\tilde{\eta}|$ denote the number of particles in $\tilde{\eta}$.*

*Proof.* For any particle in $\eta$, at $y \in ((x-1)K, xK)$, notice that for $\lambda$ sufficiently small depending on $K$, each particle has a chance at least $\frac{3}{4}$ of reaching $\{(x-1)K, xK\}$ before falling asleep independent of each other. Since all such particles are in $\tilde{\eta}$, the conclusion now follows from the fact that there are at least $\mu r$ particles in $\eta$ and a large deviation estimate. □

We finish off this Section with the proof of Lemma 3.3.7. The rest of this chapter is devoted to proving Lemma 3.4.1.

*Proof of Lemma 3.3.7.* The proof follows from a combination of Lemma 3.4.2, Lemma 3.4.1 and Observation 3.4.1. □

In the following section we develop the remaining technical results needed for the proof of Lemma 3.4.1.

### 3.5 Technical Preliminaries

Recall the set-up of Lemma 3.4.1: fix $K = K(\mu)$ sufficiently large and $r$ sufficiently large depending on $K$ and a particle configuration $\widehat{\eta}$ supported on $[-r, r] \cap K\mathbb{Z}$ that contains at least $\frac{\mu r}{2}$ many particles. Consider Labeled ARW dynamics on $[-2r, 2r]$ starting with initial configuration $\widehat{\eta}$. We shall denote the law of this process by $\mathscr{P}_{*,\widehat{\eta}}$. Let $\widehat{\eta}_\infty$ denote the final configuration and for $xK \in (-2r, 2r)$ let $\widehat{\eta}_{x,\infty}$ denote the final configuration of the particles labeled $(x, \cdot)$. It is trivial to observe that $\widehat{\eta}_{x,\infty}$ is supported on $((x-1)K, (x+1)K)$.

Recall also from (3.4.7) that $u_{(*,r)}(x)$ counts the number of times a labeled particle is toppled at a site $x$ up to stabilization. However for our purposes we need to determine how many elements (random walk paths) from the stacks in (3.4.3) have been used. For $xK \in K\mathbb{Z}$, define the **renormalized odometer** as:

$$M(x) := \max\{j : \text{ a particle labeled } (x, j) \text{ was emitted}\}, \tag{3.5.1}$$

i.e. the number of random walk paths $\zeta_{(x,\cdot)}$ in (3.4.3) that were used until stabilization.

Now we define certain statistics of the process which will be used in the proof of Lemma 3.4.1. For each $xK \in [-2r, 2r]$, we keep track of the net number of particles labeled $(x, \cdot)$

(emitted from $xK$) hitting $(x-1)K$, $xK-1$, $xK+1$ and $(x+1)K$ respectively by

$$\begin{aligned} L(x-1 \leftarrow x) &= L(x-1 \leftarrow x)(\widehat{\eta}), \\ L(\leftarrow x) &= L(\leftarrow x)(\widehat{\eta}), \\ R(x \rightarrow) &= R(x \rightarrow)(\widehat{\eta}), \\ R(x \rightarrow x+1) &= R(x \rightarrow x+1)(\widehat{\eta}) \end{aligned} \tag{3.5.2}$$

respectively. Note that we are counting the net number of particles in all the counts, i.e. if a particle moved back and forth across $xK$ a few times to eventually land somewhere in the interval $(xK, (x+1)K)$, it just contributes $1$ to $R(x \rightarrow)$ but not to $R(x \rightarrow x+1)$. On the other hand he particle since and $0$ to others. See Figure 3.3. Next we make a

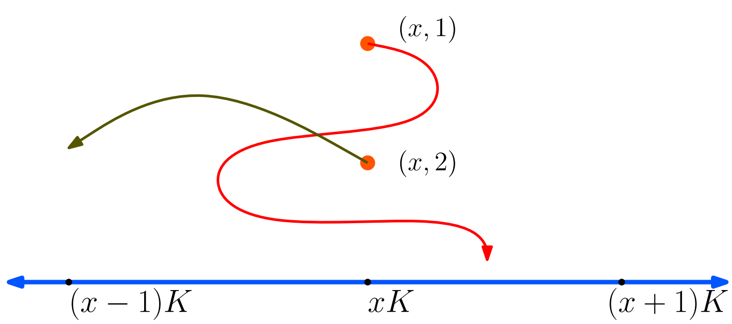


Figure 3.3: The particle $(x, 1)$ eventually lands to the right of $xK$ and hence contributes one to $R(x \rightarrow)$. On the other hand the particle $(x, 2)$ hits $(x-1)K$ and hence contributes one to $L(\leftarrow x)$ and $L(x-1 \leftarrow x)$.

crucial observation that the statistics in (3.5.2) are "local" functions of the Labeled ARW dynamics. This is the reason behind defining the Labeled dynamics in such a way that particles with different first labels do not interact. We make things formal below.

### *3.5.1 Single site dynamics*

For any $x \in \mathbb{Z}$ consider Labeled ARW dynamics in $V_x = ((x-1)K, (x+1)K)$ started with $m$ particles at $x$. Then the quantities analogous to (3.5.2) will be denoted by

$$L^{SS}_{x-1\leftarrow x}(m), L^{SS}_{\leftarrow x}(m), R^{SS}_{x\rightarrow}(m), R^{SS}_{x\rightarrow x+1}(m) \tag{3.5.3}$$

respectively. Thus the above can be thought as a localized version of the labeled ARW dynamics. The final configuration in this process will be denoted by $\eta^{SS}_x(m)$.

Consider labeled ARW dynamics started with initial configuration $\hat{\eta}$ stabilised in $(-2r, 2r)$. The particles with label $(x, \cdot)$ are toppled only at certain times say $\tau_1, \tau_2, \ldots$. Thus the quantities in (3.5.2) are functions of the configurations at these times. By definition

$$\{(x,1), (x,2), \cdots, (x, M(x))\}$$

are the different labels of such particles.

Now compare the above with the restricted system where the vertex set is $((x-1)K, (x+1)K)$ and the initial configuration of $M(x)$ labeled active particles at $xK$. Notice that owing to the labeling scheme and the labeled toppling rule the movement at times $\tau_1, \tau_2, \ldots$ as described above is exactly the same as in the single site dynamics. All this can be formalized by an easy induction on $M(x)$, hence we have the following observation.

**Observation 3.5.1.** For a fixed initial configuration $\hat{\eta}$ recall the definition of $M(\cdot)$ given in (3.5.1). For each $x \in \mathbb{Z}$ such that $xK \in (-2r, 2r)$, the quantities in (3.5.2) are the same as the respective quantities in (3.5.3) with $m = M(x)$, where both dynamics use the same stack $\mathscr{I}_*$ (see (3.4.3)). Moreover $\eta^{SS}_x(M(x)) = \widehat{\eta}_{x,\infty}$, i.e., the eventual distribution of the particles labeled $(x, \cdot)$ is same in both processes.

Notice the restricted process clearly is just a function of $M(x)$ and the stack at $x$( $\{\zeta_{(x,\cdot)}\}$). Hence the above observation says that after the labeled system has fixated the distribution of the particles labeled $(x,\cdot)$ and the other local statistics defined in (3.5.2) are just functions of the stack $\{\zeta_{(x,j)} : j \in \mathbb{N}\}$ at $x$ and $M(x)$ (the renormalized odometer value at $x$). We also note that there might be particles labeled $(x \pm 1, \cdot)$ in the interval $((x-1)K, (x+1)K)$ after fixation. However the stack at $x$ allows us to determine the distribution of particles labeled $(x,\cdot)$. This fact will be crucially used in the proof of Lemma 3.4.1.

## 3.6 *Proof of Lemma 3.4.1*

The proof follows by a careful union bound over the possible values that the function $M(\cdot)$ (see (3.5.1)). We start by describing the rough idea before providing formal arguments. Recall the event from (3.4.7).

$$\mathcal{A} := \left\{\max\{u_{(*,r)}(-2r), u_{(*,r)}(2r)\} = 0\right\} \tag{3.6.1}$$

and that the initial configuration $\hat{\eta}$ is supported on $[-r,r]$ with $2\mu r \geqslant |\hat{\eta}| \geqslant \frac{\mu r}{2}$. Note that in (3.4.7) we only assume the lower bound on $|\hat{\eta}|$. However by monotonicity (Observation 3.3.3 iv.) it suffices to assume the upper bound as well.

**Observation 3.6.1.** On $\mathcal{A}$, no particle hits $\{-2r, 2r\}$ and hence at least $\frac{\mu r}{2}$ particles fall asleep inside $(-2r, 2r)$.

We shall need the following notation to keep track of the net flux of particles across lattice sites. Namely for every $x \in \mathbb{Z}$, let

$$\begin{aligned} F_x^+ &= R(x \rightarrow) - L(x \leftarrow x+1) \\ F_x^- &= L(\leftarrow x) - R(x-1 \rightarrow x) \end{aligned} \tag{3.6.2}$$

where $R(\cdot)$ and $L(\cdot)$ are defined in (3.5.2). Thus $F_x^-$ and $F_x^+$ denote the net number of particles moving across the points $xK - .5$ and $xK + .5$ respectively.

For a positive integer $a_0$ and integers $f_0^+, f_0^-$, let $\mathcal{D} = \mathcal{D}(a_0, f_0^+, f_0^-)$ denote the event that the following equalities hold.

$$M(0) = a_0, F_0^+ = f_0^+, F_0^- = f_0^-. \tag{3.6.3}$$

For a fixed triple $(a_0, f_0^+, f_0^-)$ we shall show that $\mathcal{D} \cap \mathcal{A}$ is unlikely, and Lemma 3.4.1 shall follow from a union bound over different triples. It turns out that we need to consider two cases, we start with the following case, where $M(0) = a_0$ is 'small'. i.e. bounded by a polynomial in $r$.

### *3.6.1* $M(0)$ *is small.*

Recall the probability measure $\mathscr{P}_{*,\hat{\eta}}$ from Lemma 3.4.1.

**Lemma 3.6.1.** *Fix $(a_0, f_0^+, f_0^-)$ as above with $a_0 \leqslant r^6$ and $|f_0^+|, |f_0^-| \leqslant 2\mu r$. There exists $K = K(u)$ sufficiently large such that for $\lambda$ sufficiently small and for some constant $c = c(\mu, \lambda) > 0$, for all large enough $r$,*

$$\mathscr{P}_{*,\hat{\eta}}(\mathcal{A} \cap \mathcal{D}(a_0, f_0^+, f_0^-)) \leqslant e^{-cr}. \tag{3.6.4}$$

**For the rest of this subsection, fix $(a_0, f_0^+, f_0^-)$ as in Lemma 3.6.1. Recall also that $\hat{\eta}$, the initial configuration of active particles, supported on $K\mathbb{Z} \cap [-r, r]$ has already been fixed.**

Given the stack at $0$, (i.e. $\{\zeta_{(0,j)} : j \in \mathbb{N}\}$) using Observation 3.5.1 and the fact that $M(0) = a_0$, one can compute $r_0 := R(0 \to)$. Since $F_0^+ = f_0^+$ by definition (3.6.2),

$$\ell_1 := L(0 \leftarrow 1) = r_0 - f_0^+. \tag{3.6.5}$$

Now $\ell_1$, together with the stack at $K$, i.e., $\{\zeta_{1,j} : j \in \mathbb{N}\}$, can be used to calculate $M(1)$ upto some (typically small) error. In general, given $\ell_x$, one can work out the value of $M(x)$

upto some error. Let $\ell_x(\cdot) := L^{SS}_{x-1\leftarrow x}(\cdot)$ where the latter was defined in (3.5.3). We have the following lemma.

**Lemma 3.6.2.** *For any $x \in \mathbb{Z}$ and every realization of the stack at $xK$ (i.e. $\{\zeta_{(x,j)} : j \in \mathbb{N}\}$, (see* (3.4.3) *) $\ell_x(\cdot)$ is a non decreasing function with jump size at most $2K$.*

*Proof.* Notice that for $j \geqslant 0$, we have

$$\ell_x(j+1) = \ell_x(j) + \ell_*$$

where $\ell_*$ denotes the net number of particles going to $(x-1)K$ in the following process: Start with initial configuration of $\tilde{S}$ particles distributed according to $\eta^{SS}_x(j)$. Add an additional $\tilde{A}$ particle with label $(x, j+1)$ at $xK$ (this will wake up $\tilde{S}$ particle present at $xK$, if any), call this initial configuration $\eta_*$. Now run labeled ARW dynamics with $\eta_*$ stabilizing on $((x-1)K, (x+1)K)$ using the remaining parts of corresponding stack elements. This immediately shows monotonicity. To see the bound on the jump length note that $\ell_* \leqslant |\eta_*|$. Since $\eta_*$ can have at most two particles at $xK$, and at most one particles at any other site in $((x-1)K, (x+1)K)$, the second assertion of the lemma follows. □

Now it might turn out that for some realization of the stack $\{\zeta_{(1,j)} : j \in \mathbb{N}\}$, $\ell_1(\cdot)$ does not take the value $r_0 - f_0^+$ and hence $\mathcal{D}(a_0, f_0^+, f_0^-)$ cannot hold. On the other hand since the function $\ell_1(\cdot)$ is not strictly increasing it might also happen that $\ell_1(\cdot) = r_0 - f_0^+$ on a non empty interval of integers. Let the interval be

$$[M_*(1), M_*(1) + w(1)], \tag{3.6.6}$$

for some positive integer $M_*(1)$ and $w(1) \geqslant 0$, i.e. for all $y \in [M_*(1), M_*(1) + w(1)]$, $\ell_1(y) = r_0 - f_0^+$. In this case, based on the information so far (i.e. $\mathcal{D}(a_0, f_0^+, f_0^-)$ and the stacks at $0$ and $K$) one cannot determine the exact value of $M(1)$ and it could potentially be any element in the interval $[M_*(1), M_*(1) + w(1)]$.

Now, observe that if we knew the value of $M(1) \in [M_*(1), M_*(1)+w(1)]$, it would also be possible to work out the value $f_1^+$ from the stack at $K$ and hence the value of $\ell_2$ as well. So recursively, assuming we know that value $M(x)$ and using stack at $(x+1)K$, using Lemma 3.6.2 and Observation 3.5.1, we can identify a interval $[M_*(x+1), M_*(x+1)+w(x+1)]$ in which $M(x+1)$ must be contained. Such an interval being empty implies our assumption on the values $M(0), M(1), \dots, M(x)$ (and also $F_0^+$) must be incorrect. Formally we use the following parametrization.

**Definition 3.6.2.** Let $\tilde{S}_{i,i+1}(j)$ be the number of labeled sleepy particles in the configuration $\eta_x^{SS}(j)$ (by definition all of them have label $(i,\cdot)$) between $(Ki, K(i+1))$ and similarly $\tilde{S}_{i,i-1}(j)$ denotes the number of sleepy $(i,\cdot)$ labeled particles between $(K(i-1), K(i)]$. Also following Observation 3.5.1 let $\tilde{S}_{i,i+1} = \tilde{S}_{i,i+1}(M(i))$ and $\tilde{S}_{i,i-1} = \tilde{S}_{i,i-1}(M(i))$ be the $(i,\cdot)$ labeled sleepy particles in $\hat{\eta}_{x,\infty}$.

Fix a sequence of non negative integers,

$$\underline{y} = \left\{ y(-\frac{2r}{K}+1), \dots, y(-2), y(-1), y(0)=0, y(1), y(2), \dots, y(\frac{2r}{K}-1) \right\}. \tag{3.6.7}$$

Given the triple $(a_0, f_0^+, f_0^-)$, we now define a filtration $\{\mathscr{F}_i\}_{i\geq 0}$ generated by the variables $\zeta_{(\cdot,\cdot)}$ (also depending on the initial particle configuration $\hat{\eta}$ and $\underline{y}$) as follows. For convenience of reading, we shall denote the filtration by the set of random variables that generate it. We start by defining,

$$\mathscr{F}_0 := \{\zeta_{(0,1)}, \zeta_{(0,2)}, \dots \zeta_{(0,a_0)}\}. \tag{3.6.8}$$

We now recursively define a sequence of functions all of which will be progressively measurable with respect to the filtration $\mathscr{F}_i$ (yet to be fully defined). For $i \geqslant 0$, assume $\ell_1, \dots, \ell_i$; $a_1, \dots, a_i$ and $f_1^+, \dots, f_i^+$ have already been defined. Let

$$\ell_{i+1} := R_{i\to}^{SS}(a_i) - f_i^+ \tag{3.6.9}$$

We now define the filtration $\mathscr{F}_i$ recursively for $\underline{y}$ such that $y(0), y(1), \dots, y(i)$ is **realiz-**

**able** (also recursively defined below) . Assume $\mathscr{F}_0, \ldots \mathscr{F}_i$, have been defined. Analogous to (3.6.6), define $M_*(i+1)$ and $w(i+1)$ by

$$M_*(i+1) = \min\{j \geqslant 0 : \ell_{i+1}(j) = \ell_{i+1}\}; \;\; M_*(i+1) + w(i+1) = \max\{j \geqslant 0 : \ell_{i+1}(j) = \ell_{i+1}\}$$

if such numbers exist. Call the sequence $\{y(0), y(1), \ldots y(i+1)\}$ **realizable** if (a) either $i = 0$ or $\{y(0), y(1), \ldots y(i)\}$ is **realizable** and (b) $M_*(i+1)$ and $w(i+1)$ as above exist and $y(i+1) \leqslant w(i+1)$. In such a situation define

$$a_{i+1} = M_*(i+1) + y(i+1). \tag{3.6.10}$$

For a realizable sequence $\{y(0), y(1), \ldots y(i+1)\}$, define

$$\mathscr{F}_{i+1} := \mathscr{F}_i \bigcup \{\zeta_{(i+1,1)}, \zeta_{(i+1,2)}, \ldots \zeta_{(i+1,a_{i+1})}\}.$$

Now by definition $R_{i+1\to}(a_{i+1}), \tilde{S}_{i+1,i+2}(a_{i+1})$ are $\mathscr{F}_{i+1}$ measurable. Also define

$$f^+_{i+1} := \left|\hat{\eta}_{(iK+1,(i+1)K]}\right| + f^+_i - \tilde{S}_{i,i+1}(a_i) - \tilde{S}_{i+1,i}(a_{i+1}); \tag{3.6.11}$$

where $|\hat{\eta}_B|$ denote the number of particles of $\hat{\eta}$ contained in $B \subseteq \mathbb{Z}$. That $f^+_{i+1}$ is $\mathscr{F}_{i+1}$ measurable is now a simple consequence of (3.6.11). Thus by the above discussion, using (3.6.9),(3.6.11) and (3.6.10) repeatedly we inductively construct the filtration $\mathscr{F}_i$ for realizable sequences.

Now for $i < 0$, we construct in the exactly similar way a filtration $\mathscr{F}_i$, and a sequence of functions $M_*(i)$, $w(i)$, $a_i$, $r_i$ and $f^-_i$, recursively going from right to left and starting with $(a_0, f^-_0)$. The sequence $\{y(i), \ldots, y(0)\}$ is called realizable if $y(i) \leqslant w(i)$. Finally, we call $\underline{y}$ realizable if $\{y(-\frac{2r}{K}+1), \ldots, y(-2), y(-1), y(0)\}$ and $\{y(0), y(1), y(2), \ldots, y(\frac{2r}{K}-1)\}$ are both realizable.

Following is a consequence of Observation 3.5.1: suppose $(y(0), \ldots, y(i))$ (resp. $(y(i), \ldots, y(0))$) is realizable, and $M(i) = a_i$. Then, on $\mathcal{D}(a_0, f^+_0, f^-_0) \cap \mathcal{A}$, there exists $y(i+1) \geqslant 0$ (resp.

$y(i-1) \geqslant 0$) such that $(y(0), \ldots, y(i+1))$ (resp. $(y(i-1), \ldots, y(0))$) is also realizable. This leads to the following observation.

**Observation 3.6.3.** On $\mathcal{D}(a_0, f_0^+, f_0^-) \cap \mathcal{A}$, there exists a (unique) $\underline{y}$ as in (3.6.7) such that $\underline{y}$ is realizable and $M(i) = M_*(i) + y(i)$ for each $i$. Further for such an $\underline{y}$, we have $\ell_j = L(j-1 \leftarrow j)$ and $f_j^+ = F_j^+$ for all $j \geqslant 0$ where $\ell_j$ and $f_j^+$ are defined as in (3.6.9) and (3.6.11) respectively.

It might be instructive to think of the path $y$ as a path in a tree of possibilities of the odometer values at various points, out of which the process picks only one.

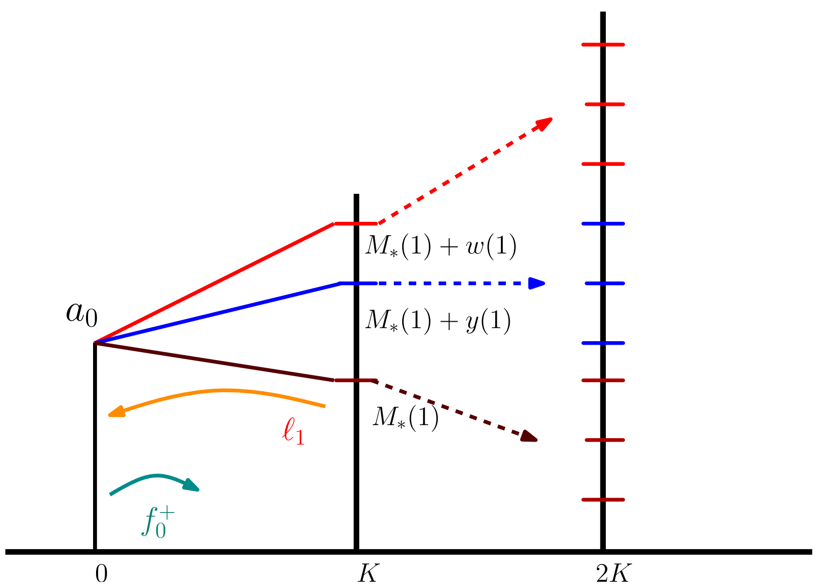


Figure 3.4: The paths of various colors denote the different realizable sequences.

Denote by $\mathcal{R}(\underline{y})$ the event that $\underline{y}$ is realizable and $M(i) = M_*(i) + y_i$ for all $i$. Also for $j > 0$ (resp. $j < 0$), let $\mathcal{R}(\underline{y}; j)$ denote the event that $\{y(0), \ldots y(j)\}$ is realizable (resp. $\{y(j), \ldots, y(0)\}$ is realizable) and $M(i) = M_*(i) + y_i$ for all $i \in \{0, 1, \ldots, j\}$ (resp.for all $i \in \{j, j+1, \ldots, 0\}$). We now state for any $\underline{y}$ and the triple $(a_0, f_0^+, f_0^-)$ as in Lemma 3.6.1, a quantitative upper bound on the probability of the event $\mathcal{R}(\underline{y}) \cap \mathcal{D} \cap \mathcal{A}$.

**Lemma 3.6.3.** *There exists $c > 0$ such that for any $K$ sufficiently large depending on $\mu$, there exists $\lambda_K$ sufficiently small, such that for $\lambda \leqslant \lambda_K$, and $\underline{y}$ as in* (3.6.7)*,*

$$\mathscr{P}_{*,\hat{\eta}}(\mathcal{R}(\underline{y}) \cap \mathcal{D} \cap \mathcal{A}) \leqslant e^{2K} e^{-\mu r/2} e^{-c\sum_i y_i}.$$

To start with recall from Observation 3.6.1, on $\mathcal{A}$,

$$\sum_{i=-\frac{2r}{K}+1}^{\frac{2r}{K}-1} [\tilde{S}_{i,i+1} + \tilde{S}_{i,i-1}] \geqslant \frac{\mu r}{2}. \tag{3.6.12}$$

Fix $\underline{y}$ as in (3.6.7). To prove Lemma 3.6.3 we prove the following,

$$\mathbb{E}\left(e^{\sum_{i=-2r/K+1}^{2r/K-1}[\tilde{S}_{i,i+1}+\tilde{S}_{i,i-1}]}\mathbf{1}[\mathcal{R}(\underline{y}) \cap \mathcal{D} \cap \mathcal{A}]\right) \leqslant e^{2K} e^{-c\sum_i y_i}. \tag{3.6.13}$$

The proof of Lemma 3.6.3 then follows by Markov's inequality and the observation that on the event $\mathcal{D} \cap \mathcal{A}$

$$e^{\sum_{i=-2r/K+1}^{2r/K-1}[\tilde{S}_{i,i+1}+\tilde{S}_{i,i-1}]} \geq e^{\mu r/2}. \tag{3.6.14}$$

We now proceed toward proving (3.6.13). We shall prove the following for all $j \in \{1, \ldots, 2r/K - 1\}$:

$$\mathbb{E}\left[e^{[\tilde{S}_{j,j+1}+\tilde{S}_{j,j-1}]}\mathbf{1}\big[\mathcal{R}(\underline{y}; j) \cap \mathcal{D} \cap \mathcal{A}\big] \mid \mathscr{F}_{j-1}\right] \leqslant e^{-cy_j}. \tag{3.6.15}$$

Similarly for $j \in \{-2r/K + 1, \ldots, -1\}$,

$$\mathbb{E}\left[e^{[\tilde{S}_{j,j+1}+\tilde{S}_{j,j-1}]}\mathbf{1}\big[\mathcal{R}(\underline{y}; j) \cap \mathcal{D} \cap \mathcal{A}\big] \mid \mathscr{F}_{j+1}\right] \leqslant e^{-cy_j}. \tag{3.6.16}$$

Clearly the above bounds for all $j$ implies (3.6.13). Note that in the above equations we do not consider $\tilde{S}_{0,1} + \tilde{S}_{0,-1}$. However we naively bound it by $2K$ (since at most $2K$ particles with first label $0$ can fall asleep in the interval $(-K, K)$ ). This accounts for the extra $e^{2K}$ term in (3.6.13). We shall only show (3.6.15), the other bound will follow similarly. Before proceeding further we need some notation: recall the random walk paths $\zeta_{(\cdot,\cdot)}$ in (3.4.3) are lazy (with parameter $\lambda$) stopped on hitting the nearest renormalized lattice point (multiple of $K$) to the left or to the right. By symmetry each random walk path has probability $1/2$ of hitting either neighbor. Call a random walk path $\zeta_{(x,\cdot)}$ a **left instruction** (resp. **right instruction**) if it is killed at hitting $(x-1)K$ (resp. $(x+1)K$) before taking any lazy step. Clearly for any $\chi > 0$ and $\lambda = \lambda(K,\chi)$ sufficiently small $\mathscr{P}_*(\zeta_{(x,\cdot)}$ is a left instruction$) \geqslant \frac{1}{2} - \chi$ and similarly for a right instruction. Fix any $\varepsilon > 0$. Let $R$ be such that the probability of having a right instruction in $R$ i.i.d. samples of the random walk paths is at least $1-\varepsilon$.

For every $x \in \mathbb{Z}$, given $\ell_x$, we define the following stopping times measurable with respect to the filtration

$$\mathscr{G}_x(t) = \{\zeta_{(x,1)}, \dots \zeta_{(x,t)}\}. \tag{3.6.17}$$

Recall $\ell_x(\cdot)$ from Lemma 3.6.2 and define

$$\tau_1 = \tau_1(x) := \inf\{t : \ell_x(t) \geq \ell_x - 3KR\} \tag{3.6.18}$$

Let $\tau_3 = \tau_1 + R$. Also define $\tau_4$ to be the first time after $\tau_1$ such that there are at least $3KR$ many left instructions among the random walk paths $\zeta_{(x,\tau_1+1)} \dots \zeta_{(x,\tau_4)}$. Clearly $\tau_4 \geq \tau_1 + 3KR$ and hence $\tau_4 > \tau_3$. Also notice that using Lemma 3.6.2, on the event that $M_*(x)$ exists,

$$M_*(x) \in [\tau_3, \tau_4].$$

We now bound the maximum number of sleepy particles with first label $x$ ever to be on the interval $\big((x-1)K, (x+1)K\big)$ between times $[\tau_3, \tau_4]$, i.e.

$$\max_{\tau_3 \leqslant t \leqslant \tau_4} \tilde{S}_{x,x+1}(t) + \tilde{S}_{x,x-1}(t).$$

Condition on $\mathscr{G}_j(\tau_1)$ and consider $\eta_x^{SS}(\tau_1)$. Clearly $|\eta_x^{SS}(\tau_1)| \leqslant 2K$, and all these particles have parts of random walk paths attached to it, which have not been revealed yet, and has the laws of independent lazy random walks (started at their respective current locations), and killed on hitting $\{(x-1)K, (x+1)K\}$. Choose $\lambda$ small enough such that with probability at least $1-\varepsilon$ all these remaining random walk paths corresponding to the particles in $\eta_x^{SS}(\tau_1)$ will not take any lazy steps at all (i.e., before hitting $(x-1)K$ or $(x+1)K$). Further, choose $\lambda$ sufficiently small so that with probability $1-\varepsilon$, $\zeta_{x,j}$ is either a left instruction or a right instruction for all $j \in \{\tau_1+1, \tau_1+2, \ldots, \tau_4\}$. Again this is possible since for $\lambda$ sufficiently small, the chance of observing either a left instruction or a right instruction can be made arbitrarily close to one.

By choice of $R$ with probability at least $1-2\varepsilon$ there has been at least a left instruction and a right instruction in the interval $(\tau_1, \tau_3]$. Let

$$s_1 = \min\{k > \tau_1 : \zeta_{x,k} \text{ is a left instruction}\},\ s_2 = \min\{k > \tau_1 : \zeta_{x,k} \text{ is a right instruction}\}$$

and set $s = s_1 \vee s_2$. Now consider $\eta_x^{SS}(s)$. By definition all the intermediate sleepy particles from $\eta_x^{SS}(\tau_1)$ have been woken up. Now since with probability $1-\varepsilon$ none of their remaining paths contain any sleep instruction. Moreover with failure probability at most $\varepsilon$ no random walk $\zeta_{x,j}$ for $j \in [s_1, \tau_4]$ will have no lazy steps and hence no new particles fall asleep. Hence with probability at least $1-4\varepsilon$, we have $\tilde{S}_{x,x-1}(t) = 0$ for all $t \in [s, \tau_4]$. Taking care of $\tilde{S}_{x,x+1}(t)$ in a similar manner it follows that with conditional probability at least $1-8\varepsilon$

$$\max_{\tau_3 \leqslant t \leqslant \tau_4} \tilde{S}_{x,x-1}(t) + \tilde{S}_{x,x+1}(t) = 0. \tag{3.6.19}$$

With this preparation, let us now return to the proof of (3.6.15).

*Proof of* (3.6.15).

Fix $j > 0$. We want to prove

$$\mathbb{E}\left[e^{[\tilde{S}_{j,j+1}+\tilde{S}_{j,j-1}]}\mathbf{1}\left[\mathcal{R}(\underline{y};j) \cap \mathcal{D} \cap \mathcal{A}\right] \mid \mathscr{F}_{j-1}\right] \leqslant e^{-cy_j}.$$

Recall from previous discussion and using (3.6.9) $\ell_j$ is $\mathscr{F}_{j-1}$ measurable. Recall also the filtration $\mathscr{G}_j(t)$ from (3.6.17) and the stopping time $\tau_1 = \tau_1(j)$ from (3.6.18). Define $\mathscr{F}_{j-1,\tau_1} := \mathscr{F}_{j-1} \cup \mathscr{G}_j(\tau_1)$. We compute

$$\mathbb{E}\left[e^{[\tilde{S}_{j,j+1}+\tilde{S}_{j,j-1}]}\mathbf{1}\left[\mathcal{R}(\underline{y};j)\cap\mathcal{D}\cap\mathcal{A}\right] \mid \mathscr{F}_{j-1,\tau_1}\right].$$

Let $\mathcal{B}_1$ denote the event that $M_*(j)$ exists and $\tilde{S}_{j,j-1}(M_*(j)) + \tilde{S}_{j,j+1}(M_*(j)) > 0$. It follows from (3.6.19) that for $\mathcal{B}_2 := \mathcal{R}(\underline{y};j)\cap\mathcal{D}\cap\mathcal{A}\cap\mathcal{B}_1$ we have

$$\mathbb{P}[\mathcal{B}_2 \mid \mathscr{F}_{j-1,\tau_1}] \leqslant 8\varepsilon. \tag{3.6.20}$$

Now starting from $M_*(j)$, we try to bound the probability that $R(\underline{y};j)$ holds, i.e., in particular,

$$\ell_j(M_*(j)+y_j) = \ell_j(M_*(j)).$$

Clearly for this to happen, $\zeta_{j,k}$ cannot be a left instruction for any $k \in [M_*(j)+1, M_*(j)+y_j]$. Let $\mathcal{B}_3$ denote this event. Clearly $\mathcal{B}_3$, is independent of $\mathcal{B}_1$ and $\mathscr{F}_{j-1,\tau_1}$ and for $\lambda$ sufficiently small,

$$\mathbb{P}[\mathcal{B}_3] \leqslant (0.51)^{y_j}. \tag{3.6.21}$$

Now consider a constant $C$ to be specified later (whose purpose will be clear soon). Let $\mathcal{B}_4$ denote the event that at least one of the random walks $\zeta_{(j,M*(j)+1)},\ldots,\zeta_{(j,M*(j)+C)}$ takes a lazy step (see (3.4.1)). By taking $\lambda = \lambda(C)$ small enough

$$\mathbb{P}[\mathcal{B}_4 \mid \mathscr{F}_{j-1,\tau_1}] \leqslant \varepsilon. \tag{3.6.22}$$

We then conclude,

$$\mathbb{E}\left[e^{[\tilde{S}_{j,j+1}+\tilde{S}_{j,j-1}]}\mathbf{1}\left[\mathcal{R}(\underline{y};j)\cap\mathcal{D}\cap\mathcal{A}\right] \mid \mathscr{F}_{j-1,\tau_1}\right]$$

$$\leqslant \begin{cases} e^{2K}(0.51)^{y_j} & \text{if } y_j \geq C \\ 9\varepsilon e^{2K} + (1-8\varepsilon)(0.51)^{y_j} & \text{if } y_j < C \end{cases} \tag{3.6.23}$$

The first case in the above computation is straightforward: we use the fact $\tilde{S}_{j,j+1} + \tilde{S}_{j,j-1}$ is at most $2K$ and from (3.6.21), $\mathbb{P}[\mathcal{R}(\underline{y};j) \cap \mathcal{D} \cap \mathcal{A} \mid\mid \mathscr{F}_{j-1,\tau_1}] \leqslant (0.51)^{y_j}$. For the second case, notice for $y_j < C$,

$$\mathbb{E}\left[e^{[\tilde{S}_{j,j+1}+\tilde{S}_{j,j-1}]}\mathbf{1}\left[\mathcal{R}(\underline{y};j)\cap\mathcal{D}\cap\mathcal{A}\right] \mid \mathscr{F}_{j-1,\tau_1}\right] \leqslant \mathbb{P}(\mathcal{B}_2\cup\mathcal{B}_4 \mid \mathscr{F}_{j-1,\tau_1})e^{2K}+\mathbb{P}(B_2^c\cap B_3 \mid \mathscr{F}_{j-1,\tau_1}).$$

The second conclusion in (3.6.23) now follows from (3.6.20), (3.6.21) and (3.6.22). Now $\varepsilon$ in (3.6.23) can be chosen to be small enough (depending on $C$) such that, averaging over $\mathscr{G}_j(\tau_1)$ we get that

$$\mathbb{E}\left[e^{[\tilde{S}_{j,j+1}+\tilde{S}_{j,j-1}]}\mathbf{1}\left[\mathcal{R}(\underline{y};j)\cap\mathcal{D}\cap\mathcal{A}\right] \mid \mathscr{F}_{j-1}\right] \leqslant (0.52)^{y_j}.$$

This completes the proof of (3.6.15) and (3.6.16) can be established similarly.

□

Now we complete the proofs of Lemma 3.6.3 and Lemma 3.6.1.

*Proof of Lemma 3.6.3.* Using (3.6.15) and (3.6.16) and the naive bound of $2K$ on $\tilde{S}_{0,1} + \tilde{S}_{0,-1}$ as discussed right after (3.6.16), we have

$$\mathbb{E}\left(e^{\sum_{i=-r/K+1}^{r/K-1}[\tilde{S}_{i,i+1}+\tilde{S}_{i,i-1}]}\mathbf{1}\left[\mathcal{R}(\underline{y};j)\cap\mathcal{D}\cap\mathcal{A}\right]\right) \leqslant e^{2K}(0.52)^{\sum_i y_i}$$

Thus using (3.6.14) and Markov's inequality we get,

$$\mathscr{P}_*(\mathcal{R}(\underline{y})\cap\mathcal{D}\cap\mathcal{A}) \leqslant e^{2K}e^{-\mu r/2}(0.52)^{\sum_i y_i},$$

completing the proof of Lemma 3.6.3. □

*Proof of Lemma 3.6.1.* Using Lemma 3.6.3 and summing over all possible sequences $\underline{y}$ (see

(3.6.7)) and using Observation 3.6.3 we get

$$\mathscr{P}_{*,\hat{\eta}}(\mathcal{D}(a_0, f_0^+, f_0^-) \cap \mathcal{A}) \leqslant e^{2K} e^{-\mu r/2} \left(\frac{1}{0.48}\right)^{r/K} \leqslant e^{-cr}. \tag{3.6.24}$$

for some $c > 0$ choosing $K = K(\mu)$ to be large enough, where the last inequality holds for large enough $r$. This establishes (3.6.4) and completes the proof. □

Next we analyse the remaining case i.e. when $M(0)$ is large.

### *3.6.2 $M(0)$ is large.*

Recall the renormalized odometer $M(\cdot)$ from (3.5.1). To complete the proof of Lemma 3.4.1, we still need to deal with the case when $M(0) \geqslant r^6$. We have the following lemma.

**Lemma 3.6.4.** *In the set-up of Lemma 3.4.1, we have*

$$\mathscr{P}_{*,\hat{\eta}}(M(0) \geqslant r^6, \mathcal{A}) \leqslant e^{-cr} \tag{3.6.25}$$

*for some constant $c > 0$.*

*Proof.* For $i \in \mathbb{Z}$ and $T \geqslant 1$, let $L_i'(T)$ (resp. $R_i'(T)$) denote the number of random walks among $\{\zeta_{i,j}; j \in \{1, \ldots, T\}\}$ that reach $(i-1)K$ (resp. $(i+1)K$). Clearly, for each $i$ and $T$, $L_i'(T) \sim \text{Bin}(T, \frac{1}{2})$. By Azuma-Hoeffding inequality and a union bound, there exists a constant $c > 0$ such that with probability at least $1 - e^{-cr}$, the following event occurs.

$$\mathcal{H} := \left\{ \forall \imath \in \{\frac{-2r}{K}, \ldots, \frac{2r}{K}\},\ \forall T \geqslant r^4 |L_i'(T) - R_i'(T)| \leqslant \frac{T^{2/3}}{2} \right\}. \tag{3.6.26}$$

We show below that on $\mathcal{H} \cap \mathcal{A}$, $M(0) < r^6$, clearly this implies the statement of the lemma.

Observe that on $\mathcal{H} \cap \{M(0) \geqslant r^6\}$, we have

$$R(0 \to 1) \geq \frac{M(0)}{2} - \frac{M(0)^{2/3}}{2} - 2K.$$

Also recall that, by the assumption on $\hat{\eta}$, $|F_i^+| \leqslant 2\mu r$, $|F_i^-| \leqslant 2\mu r$ for each $i$. Using (3.6.2) this implies that for any $i \geqslant 0$,

$$L(i \leftarrow i+1) \geq R(i \to i+1) - 4\mu r.$$

We can thus conclude that for each $i \geqslant 0$,

$$\frac{M(i+1)}{2} + \frac{M(i+1)^{2/3}}{2} \geq M(i)/2 - M(i)^{2/3}/2 - 2K - 4\mu r.$$

The above simplifies to,

$$M(i+1) \geq M(i) - \max(M(i), M(0))^{2/3} - 2K - 4\mu r.$$

Let $i_* \in \{1, 2, \ldots, \frac{2r}{K}\}$ be the index at which $M(\cdot)$ is maximized. It follows that $M(\frac{2r}{K}) \geq M(i_*) - M(i_*)^{2/3}.r - (2K + 4\mu)r \geq 0$ for $r$ sufficiently large, since by hypothesis $M(i_*) \geq M(0) \geq r^6$. Hence $\mathcal{H} \cap \mathcal{A} \cap \{M(0) \geqslant r^6\} = \varnothing$, and we are done. □

Finally we put everything together to establish Lemma 3.4.1.

### *3.6.3 Proof of Lemma 3.4.1*

By the assumption on $\hat{\eta}$, notice that $|F_0^+|, |F_0^-| \leqslant 2\mu r$. Hence taking a union bound over all triples $(a_0, f_0^+, f_0^-)$ satisfying the conditions in the statement of Lemma 3.6.1, it follows from Lemma 3.6.1 that

$$\mathbb{P}(\mathcal{A}, M(0) \leqslant r^6) \leqslant e^{-cr/2}.$$

Lemma 3.4.1 now follows from the above and Lemma 3.6.4. □

## 3.7 Concluding Remarks and Open Questions

Thus we have shown that ARW on $\mathbb{Z}$ remains in active phase even starting with arbitrarily low density of particles provided the sleep rate is sufficiently small. In particular this implies that the critical density $\mu_\lambda < 1$ for small enough sleep rate $\lambda$. However our understanding of the process is still far from complete. Investigating the following seem the natural next step.

1. **What happens when $\lambda$ is large?** It is believed that for any $\lambda$ finite, $\mu_\lambda < 1$. However the only known results in this direction is that $\mu_\lambda \leqslant 1$ for all $\lambda$ ([6, 78, 83]) and $\mu_\lambda = 1$ for $\lambda = \infty$ [22]. Observe that, our arguments, though wasteful at many places, uses crucially that $\lambda$ can be taken to be arbitrarily small, it is not clear whether it is possible to improve upon this.

2. **ARW on $\mathbb{Z}^d$:** Does Theorem 3.1.1 hold for ARW on $\mathbb{Z}^d$ for $d \geqslant 2$? It is believed that the answer to this question is affirmative.

3. **Critical density for SSM:** For the Stochastic Sandpile Model, is the critical density $\mu_c$ strictly less than one? As mentioned before, numecial evidence suggests an affirmative answer to this question, while the best known rigorous bound in [78] gives $\mu_c \leqslant 1$. A strict inequality here would be a substantial progress because of the same reasons as in Remark 3.1.3.

# Chapter 4

## CONSTRAINED SPIN SYSTEMS: PINPOINTING MIXING TIME

### *4.1 Background*

The East process is a one-dimensional spin system that was introduced in the physics literature by Jäckle and Eisinger [51] in 1991 to model the behavior of cooled liquids near the glass transition point, specializing a class of models that goes back to [44]. Each site in $\mathbb{Z}$ has a $\{0,1\}$-value (vacant/occupied), and, denoting this configuration by $\omega$, the process attempts to update $\omega_x$ to $1$ at rate $0 < p < 1$ (a parameter) and to $0$ at rate $q = 1 - p$, only accepting the proposed update if $\omega_{x-1} = 0$ (a "kinetic constraint").

It is the properties of the East process before and towards reaching equilibrium — it is reversible w.r.t. $\pi$, the product of Bernoulli($p$) variables — which are of interest, with the standard gauges for the speed of convergence to stationarity being the inverse spectral-gap and the total-variation mixing time ($\mathrm{gap}^{-1}$ and $T_{\mathrm{mix}}$) on a finite interval $\{0,\ldots,L\}$, where we fix $\omega_0 = 0$ for ergodicity (postponing formal definitions to §4.2). That the spectral-gap is uniformly bounded away from 0 for any $p \in (0,1)$ was first proved in a beautiful work of Aldous and Diaconis [2] in 2002. This implies that $T_{\mathrm{mix}}$ is of order $L$ for any fixed threshold $0 < \epsilon < 1$ for the total-variation distance from $\pi$.

For a configuration $\omega$ with $\sup\{x : \omega_x = 0\} < \infty$, call this rightmost 0 its *front* $X(\omega)$; key questions on the East process $\omega(t)$ revolve the law $\mu^t$ of the sites behind the front at time $t$, basic properties of which remain unknown. One can imagine that the front advances to the right as a biased walk, behind which $\mu^t \approx \pi$ (its trail is mixed). Indeed, if one (incorrectly!) ignores dependencies between sites as well as the randomness in the position of the front, it is tempting to conclude that $\mu^t$ converges to $\pi$, since upon updating a site $x$ its marginal is forever set to Bernoulli($p$). Whence, the positive vs. negative increments to $X(\omega)$ would have rates $q$ (a 0-update at $X(\omega)+1$) vs. $pq$ (a 1-update at $X(\omega)$ with a 0 at its left), giving the front an asymptotic speed $v = q^2 > 0$.

Of course, ignoring the irregularity near the front is problematic, since it is precisely the distribution of those spins that governs the speed of the front (hence mixing). Still, just as a biased random walk, one expects the front to move at a positive speed with normal fluctuations, whence its concentrated passage time through an interval would imply total-variation *cutoff* — a sharp transition in mixing — within an $O(\sqrt{L})$-window.

To discuss the behavior behind the front, let $\Omega_{\mathrm{F}}$ denote the set of configurations $\omega^{\mathrm{F}}$ on the negative half-line $\mathbb{Z}_-$ with a fixed 0 at the origin, and let $\omega^{\mathrm{F}}(t)$ evolve via the East process constantly re-centered (shifted by at most 1) to keep its front at the origin. Blondel [15] showed (see Theorem 4.2.1) that the process $\omega^{\mathrm{F}}(t)$ converges to an invariant measure $\nu$, on which very little is known, and that $\frac{1}{t}X(\omega(t))$ converges in probability to a positive limiting value $v$ as $t \to \infty$ (an asymptotic velocity) given by the formula

$$v = q - pq^* \qquad \text{where} \qquad q^* := \nu(\omega_{-1} = 0).$$

(We note that $q < q^* < q/p$ by the invariance of the measure $\nu$ and the fact that $v > 0$.)

The East process $\omega(t)$ of course entails the joint distribution of $\omega^{\mathrm{F}}(t)$ and $X(\omega(t))$; thus, it is crucial to understand the dependencies between these as well as the rate at which $\omega^{\mathrm{F}}(t)$ converges to $\nu$ as a prerequisite for results on the fluctuations of $X(\omega(t))$.

Our first result confirms the biased random walk intuition for the front of the East process $X(\omega(t))$, establishing a CLT for its fluctuations around $vt$ (illustrated in Fig. 4.1).

**Theorem 4.1.1.** *There exists a non-negative constant $\sigma_* = \sigma_*(p)$ such that for all $\omega \in \Omega_{\mathrm{F}}$,*

$$\lim_{t\to\infty} \tfrac{1}{t}X(\omega(t)) = v \quad \mathbb{P}_\omega\text{-}a.s, \tag{4.1.1}$$

$$\mathbb{E}_\omega\left[X(\omega(t))\right] = vt + O(1), \tag{4.1.2}$$

$$\lim_{t\to\infty} \tfrac{1}{t}\operatorname{Var}_\omega\left(X(\omega(t))\right) = \sigma_*^2. \tag{4.1.3}$$

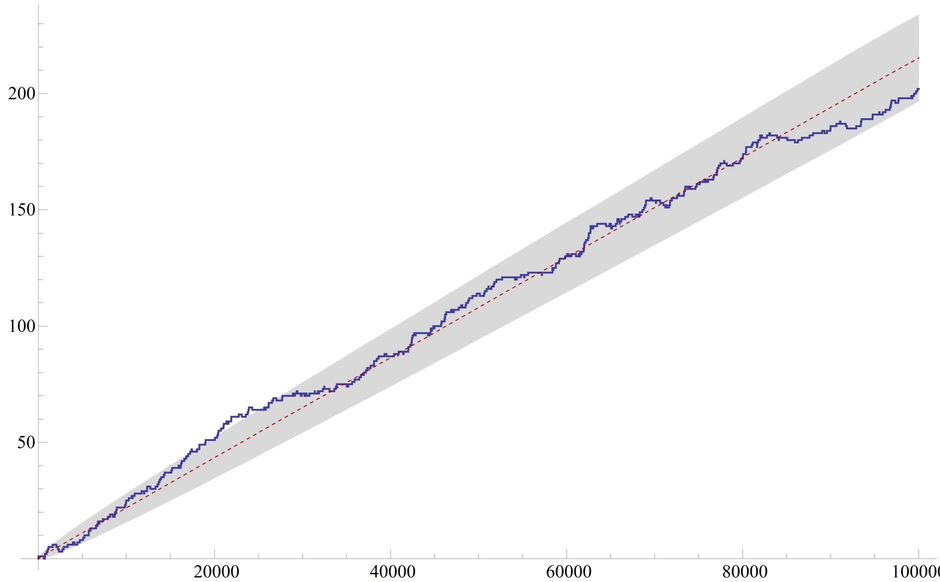


Figure 4.1: Trajectory of the front of an East process for $p = \frac{1}{4}$ along a time interval of $10^5$, vs. its mean and standard deviation window.

*Moreover,* $X(\omega(t))$ *obeys a central limit theorem:*

$$\frac{X(\omega(t)) - vt}{\sqrt{t}} \xrightarrow{d} \mathcal{N}(0, \sigma_*^2) \quad w.r.t. \quad \mathbb{P}_\omega \text{ as } t \to \infty. \tag{4.1.4}$$

A key ingredient for the proof is a quantitative bound on the rate of convergence to $\nu$, showing that it is exponentially fast (Theorem 4.3.1). We then show that the increments

$$\xi_n := X(\omega(n)) - X(\omega(n-1)) \qquad (n \in \mathbb{N}) \tag{4.1.5}$$

behave (after an initial burn-in time) as a stationary sequence of weakly dependent random variables (Corollary 4.3.1), whence one can apply an ingenious Stein's-method based argument of Bolthausen [16] from 1982 to derive the CLT.

Moving our attention to finite volume, recall that the *cutoff phenomenon* (coined by Aldous and Diaconis [5]; see [4, 31] as well as [30] and the references therein) describes a sharp transition in the convergence of a finite Markov chain to stationarity: over a negli-

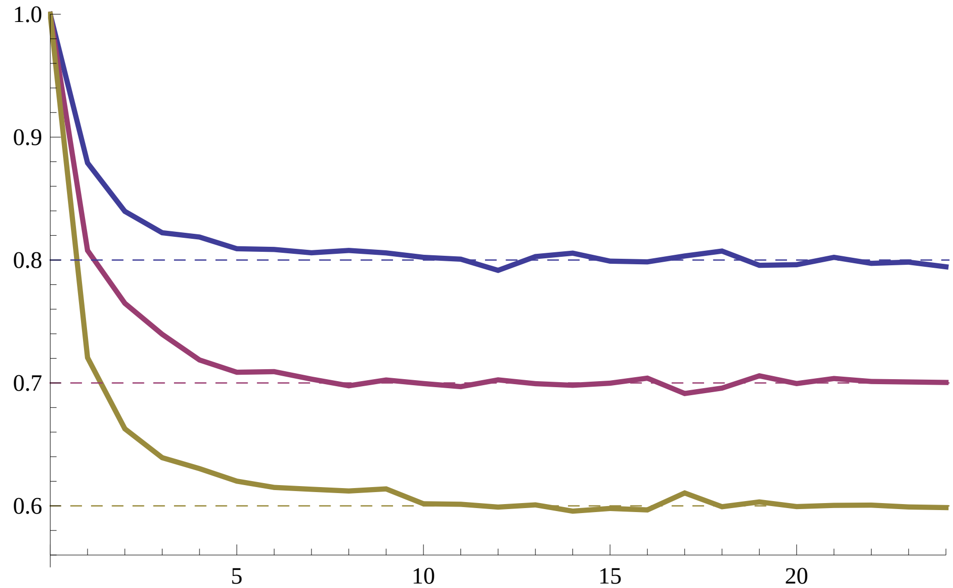


Figure 4.2: The invariant measure $\nu$ behind the front of the East process (showing $\nu(\omega_{-i} = 0)$ simulated via Monte-Carlo for $p \in \{0.2, 0.3, 0.4\}$.)

gible period of time (the cutoff window) the distance from equilibrium drops from near 1 to near $0$. Formally, a sequence of chains indexed by $L$ has cutoff around $t_L$ with window $w_L = o(t_L)$ if $T_{\text{mix}}(L, \epsilon) = t_L + O_\epsilon(w_L)$ for any fixed $0 < \epsilon < 1$.

It is well-known (see, e.g., [32, Example 4.46]) that a biased random walk with speed $v > 0$ on an interval of length $L$ has cutoff at $v^{-1}L$ with an $O(\sqrt{L})$-window due to normal fluctuations. Recalling the heuristics that depicts the front of the East process as a biased walk flushing a law $\mu^t \approx \pi$ in its trail, one expects precisely the same cutoff behavior. Indeed, the CLT in Theorem 4.1.1 supports a result exactly of this form.

**Theorem 4.1.2.** *The East process on $\Lambda = \{1, 2, \ldots, L\}$ with parameter $0 < p < 1$ exhibits cutoff at $v^{-1}L$ with an $O(\sqrt{L})$-window: for any fixed $0 < \epsilon < 1$ and large enough $L$,*

$$T_{\text{mix}}(L, \epsilon) = v^{-1}L + O\left(\Phi^{-1}(1 - \epsilon)\sqrt{L}\right),$$

*where $\Phi$ is the c.d.f. of $\mathcal{N}(0, 1)$ and the implicit constant in the $O(\cdot)$ depends only on $p$.*

While these new results relied on a refined understanding of the convergence of the process behind the front to its invariant law $\nu$ (shown in Fig. 4.2), various basic questions on $\nu$ remain unanswered. For instance, are the single-site marginals of $\nu$ monotone in the distance from the front? What are the correlations between adjacent spins? Can one explicitly obtain $q^* = \nu(\omega_{-1} = 0)$, thus yielding an expression for the velocity $v$? For the latter, we remark that the well-known upper bound on $T_{\rm mix}$ in terms of the spectral-gap (Eq. (4.2.2)), together with Theorem 4.1.2, gives the lower bound (cf. also [26])

$$v \geq \limsup_{L\to\infty} \frac{\operatorname{gap}(\mathcal{L}_{[0,L]})}{\log\left(1/(p \wedge q)\right)} = \frac{\operatorname{gap}(\mathcal{L})}{\log\left(1/(p \wedge q)\right)}.$$

## *4.2 Preliminaries and tools for the East process*

### *4.2.1 Setup and notation*

Let $\Omega = \{0,1\}^{\mathbb{Z}}$ and let $\Omega^* \subset \Omega$ consist of those configurations $\omega \in \Omega$ such that the variable $X(\omega) := \sup\{x : \omega_x = 0\}$ is finite. In the sequel, for any $\omega \in \Omega^*$ we will often refer to $X(\omega)$ as the *front* of $\omega$. Given $\Lambda \subset \mathbb{Z}$ and $\omega \in \Omega$ we will write $\omega_\Lambda$ for the restriction of $\omega$ to $\Lambda$.

(i) *The East process.* For any $\omega \in \Omega$ and $x \in \mathbb{Z}$ let $c_x(\omega)$ denote the indicator of the event $\{\omega_{x-1} = 0\}$. We will consider the Markov process $\{\omega(t)\}_{t\geq 0}$ on $\Omega$ with generator acting on local functions (*i.e.* depending on finitely many coordinates) $f : \Omega \mapsto \mathbb{R}$ given by

$$\mathcal{L}f(\omega) = \sum_{x\in\mathbb{Z}} c_x(\omega)\left[\pi_x(f)(\omega) - f(\omega)\right],$$

where $\pi_x(f)(\omega) := pf(\omega^{(x,1)}) + qf(\omega^{(x,0)})$ and $\omega^{(x,1)}, \omega^{(x,0)}$ are the configurations in $\Omega$ obtained from $\omega$ by fixing equal to $1$ or to $0$ respectively the coordinate at $x$. In the sequel the above process will be referred to as the *East process on* $\mathbb{Z}$ and we will write $\mathbb{P}_\omega(\cdot)$ for its law when the starting configuration is $\omega$. Average and variance w.r.t. to $\mathbb{P}_\omega(\cdot)$ will be denoted by $\mathbb{E}_\omega[\cdot]$ and $\operatorname{Var}_\omega(\cdot)$ respectively. Similarly we will write $\mathbb{P}^t_\omega(\cdot)$ and $\mathbb{E}^t_\omega[\cdot]$ for the law and average at a fixed time $t > 0$. If the starting configuration is distributed according to an initial distribution $\eta$ we will simply write $\mathbb{P}_\eta(\cdot)$ for $\int d\eta(\omega)\mathbb{P}_\omega(\cdot)$ and similarly for $\mathbb{E}_\eta[\cdot]$.

It is easily seen that the East process has the following graphical representation. To each $x \in \mathbb{Z}$ we associate a rate-1 Poisson process and, independently, a family of independent Bernoulli$(p)$ random variables $\{s_{x,k} : k \in \mathbb{N}\}$. The occurrences of the Poisson process associated to $x$ will be denoted by $\{t_{x,k} : k \in \mathbb{N}\}$. We assume independence as $x$ varies in $\mathbb{Z}$. That fixes the probability space. Notice that almost surely all the occurrences $\{t_{x,k}\}_{k\in\mathbb{N},x\in\mathbb{Z}}$ are different. On the above probability we construct a Markov process according to the following rules. At each time $t_{x,n}$ the site $x$ queries the state of its own constraint $c_x$. If and only if the constraint is satisfied ($c_x = 1$) then $t_{x,n}$ is called a *legal ring* and the configuration resets its value at site $x$ to the value of the corresponding Bernoulli variable $s_{x,n}$. Using the graphical construction it is simple to see that if $\omega \in \Omega^*$ then

$$\mathbb{P}_\omega(\omega(t) \in \Omega^* \, \forall t \geq 0) = 1.$$

(ii) *The half-line East process.* Consider now $a \in \mathbb{Z}$ and let $\Omega^a$ consist of those configurations $\omega \in \Omega$ with a *leftmost* zero at $a$. Clearly, for any $\omega \in \Omega^a$, $\mathbb{P}_\omega(\omega(t) \in \Omega^a \, \forall t > 0) = 1$ because $c_x(\omega) = 0$ for any $x \leqslant a$. We will refer to the corresponding process in $\Omega^a$ as the East process on the half-line $(a, \infty)$. Notice that in this case the variable at $a+1$ will always be unconstrained because $c_a(\omega) = 1$ for all $\omega \in \Omega^a$. The corresponding generator will be denoted by $\mathcal{L}_{(a,\infty)}$.

(iii) *The finite volume East process.* Finally, if $\Lambda \subset \mathbb{Z}$ is a discrete interval of the form $\Lambda = [a+1, \dots a+L]$, the projection on $\Omega_\Lambda \equiv \{0,1\}^\Lambda$ of the half-line East process on $(a, \infty)$ is a continuous time Markov chain because each vertex $x \in \Lambda$ only queries the state of the spin to its left. In the sequel the above chain will be referred to as the *East process* in $\Lambda$. Let $\mathcal{L}_\Lambda$ denote the corresponding generator.

The main properties of the above processes can be summarized as follows (cf. [41] for a survey). They are all ergodic and reversible w.r.t. to the product Bernoulli($p$) measure $\pi$ (on the corresponding state space). Their generators $\mathcal{L}, \mathcal{L}_{(a,\infty)}, \mathcal{L}_\Lambda$ are self-adjoint operators on $L^2(\pi)$ satisfying the following natural ordering:

$$\mathrm{gap}(\mathcal{L}) \leqslant \mathrm{gap}(\mathcal{L}_{(a,\infty)}) \leqslant \mathrm{gap}(\mathcal{L}_\Lambda).$$

**Remark.** By translation invariance the value of $\mathrm{gap}(\mathcal{L}_{(a,\infty)})$ does not depend on $a$ and, similarly, $\mathrm{gap}(\mathcal{L}_\Lambda)$ depends only on the cardinality of $\Lambda$.

As mentioned before, the fact that $\mathrm{gap}(\mathcal{L}) > 0$ (but only for $p \sim 1$) was first proved by Aldous and Diaconis [2], where it was further shown that

$$e^{-(\frac{1}{\log 2}+o(1))\log^2(1/q)} \leqslant \mathrm{gap}(\mathcal{L}) \leqslant e^{-(\frac{1}{2\log 2}+o(1))\log^2(1/q)} \quad \text{as } q \downarrow 0, \tag{4.2.1}$$

the order of the exponent in the lower bound matching non-rigorous predictions in the physics literature. The positivity of $\mathrm{gap}(\mathcal{L})$ was rederived and extended to all $p \in (0,1)$ in [23] by different methods, and the correct asymptotics of the exponent as $q \downarrow 0$ — matching the *upper bound* in (4.2.1) — was very recently established in [26]. It is easy to check (e.g., from [23]) that $\lim_{p\to 0} \mathrm{gap}(\mathcal{L}) = 1$, a fact that will be used later on.

For the East process in $\Lambda$ it is natural to consider its mixing times $T_{\mathrm{mix}}(L,\epsilon)$, $\epsilon \in (0,1)$, defined by

$$T_{\mathrm{mix}}(L,\epsilon) = \inf\Big\{t : \max_{\omega\in\Omega_\Lambda} \|P^t_\omega(\cdot) - \pi\| \leqslant \epsilon\Big\},$$

where $\|\cdot\|$ denotes total-variation distance. It is a standard result for reversible Markov chains (see e.g. [3, 66, 82]) that

$$T_{\mathrm{mix}}(L,\epsilon) \leqslant \frac{1}{2}\mathrm{gap}(\mathcal{L}_\Lambda)^{-1}\left(2 + \log\frac{1}{\pi^*_\Lambda}\right)\log\frac{1}{\epsilon}, \tag{4.2.2}$$

where $\pi^*_\Lambda := \min_{\omega\in\Omega_\Lambda} \pi(\omega)$. In particular $T_{\mathrm{mix}}(L,\epsilon) \leqslant c(p)L\log 1/\epsilon$. A lower bound which also grows linearly in the length $L$ of the interval $\Lambda$ follows easily from the *finite speed of* *information propagation*: If we run the East model in $\Lambda$ starting from the configuration of $\omega \equiv 1$ except for a zero at the origin, then, in order to create zeros near the right boundary of $\Lambda$ a sequence of order $L$ of successive rings of the Poisson clocks at consecutive sites must have occurred. That happens with probability $O(1)$ iff we allow a time which is linear in $L$ (see §4.2.4 and in particular Lemma 4.2.2).

### *4.2.2 The process behind the front*

Given two probability measures $\nu, \mu$ on $\Omega$ and $\Lambda \subset \mathbb{Z}$ we will write $\|\mu - \nu\|_\Lambda$ to denote the total variation distance between the marginals of $\mu$ and $\nu$ on $\Omega_\Lambda = \{0,1\}^\Lambda$.

When the process starts from a initial configuration $\omega \in \Omega^*$ with a front, it is convenient to define a new process $\{\omega^{\mathrm{F}}(t)\}_{t\geq 0}$ on $\Omega_{\mathrm{F}} := \{\omega \in \Omega^* : \; X(\omega) = 0\}$ as *the process as seen from the front* [15]. Such a process is obtained from the original one by a random shift $-X(\omega(t))$ which forces the front to be always at the origin. More precisely we define on $\Omega_{\mathrm{F}}$ the Markov process with generator $\mathcal{L}^{\mathrm{F}} = \mathcal{L}^{\mathrm{E}} + \mathcal{L}^{\mathrm{S}}$ given by

$$\mathcal{L}^{\mathrm{E}} f(\omega) = \sum_{x<0} c_x(\omega) \left[\pi_x(f)(\omega) - f(\omega)\right],$$
$$\mathcal{L}^{\mathrm{S}} f(\omega) = (1-p) \left[f(\vartheta^- \omega) - f(\omega)\right] + p\, c_0(\omega) \left[f(\vartheta^+ \omega) - f(\omega)\right],$$

where

$$\left(\vartheta^{\pm}\omega\right)_x = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \\ \omega_{x\mp 1} & \text{otherwise.} \end{cases}$$

That is, the generator $\mathcal{L}^{\mathrm{F}}$ incorporates the moves of the East process behind the front plus $\pm 1$ shifts corresponding to whenever the front itself jumps forward/backward.

**Remark.** The same graphical construction that was given for the East process $\omega(t)$ applies to the process $\omega^{\mathrm{F}}(t)$: this is clear for the East part of the generator $\mathcal{L}^{\mathrm{E}}$; for the shift part $\mathcal{L}^{\mathrm{S}}$, simply apply a positive shift $\vartheta^+$ when there is a ring at the origin and the corresponding

Bernoulli variable is one. If the Bernoulli variable is zero, operate a negative shift $\vartheta^-$.

With this notation, the main result of Blondel [15] can be summarized as follows.

**Theorem 4.2.1** ([15]). *The front of the East process, $X(\omega(t))$, and the process as seen from the front, $\omega^{\mathrm{F}}(t)$, satisfy the following:*

*(i) There exists a unique invariant measure $\nu$ for the process $\{\omega^{\mathrm{F}}(t)\}_{t\geq 0}$. Moreover, $\|\nu - \pi\|_{(-\infty,-x]}$ decreases exponentially fast in $x > 0$.*

*(ii) Let $q^* := \nu(\omega_{-1} = 0)$ and let $v = q - pq^*$. Then $v > 0$ and for any $\omega \in \Omega_{\mathrm{F}}$,*

$$\lim_{t\to\infty} \frac{X(\omega(t))}{t} \xrightarrow{\mathbb{P}_\omega} v.$$

Thus, if the East process has a front at time $t = 0$ then it will have a front at any later time. The latter progresses in time with an asymptotically constant speed $v$.

### *4.2.3 Local relaxation to equilibrium*

In this section we review the main technical results on the local convergence to the stationary measure $\pi$ for the (infinite volume) East process. The key message here is that *each* vacancy in the starting configuration, in a time lag $t$, induces the law $\pi$ in an interval in front of its position of length proportional to $t$. That explains why the distance between the invariant measure $\nu$ and $\pi$ deteriorates when we approach the front from behind.

**Definition 4.2.1.** Given a configuration $\omega \in \Omega$ and an interval $I$ we say that $\omega$ satisfies the **Strong Spacing Condition (SSC)** in $I$ if the largest sub-interval of $I$ where $\omega$ is identically equal to one has length at most $10\log|I|/(|\log p|\wedge 1)$. Similarly, given $\delta, \epsilon \in (0, 1/4)$, we will say that $\omega$ satisfies the $(\delta,\epsilon)$**-Weak Spacing Condition (WSC)** in $I$ if the largest sub-interval of $I$ where $\omega$ is identically equal to one has length at most $\delta|I|^\epsilon$.

For brevity, we will omit the $(\delta,\epsilon)$ dependence in WSC case when these are made clear from the context.

**Proposition 4.2.2.** There exist universal positive constants $c^*, m$ independent of $p$ such that the following holds. Let $\Lambda = [1, 2, \dots, \ell]$ and let $\omega \in \Omega$ be such that $\omega_0 = 0$. Further let $\Delta(\omega)$ be largest between the maximal spacing between two consecutive zeros of $\omega$ in $\Lambda$ and the distance of the last zero of $\omega$ from the vertex $\ell$. Then

$$\|\mathbb{P}^t_\omega - \pi\|_\Lambda \leqslant \ell\, (c^*/q)^{\Delta(\omega)}\, e^{-t\,(\mathrm{gap}(\mathcal{L}) \wedge m)}.$$

To prove this proposition, we need the following lemma.

**Lemma 4.2.1.** *There exist universal positive constants $c^*, m$ independent of $p$ such that the following holds. Fix $\omega \in \Omega$ with $\omega_0 = 0$, let $\ell \in \mathbb{N}$ and let $f : \Omega_{(-\infty,\ell]} \mapsto \mathbb{R}$ with $\|f\|_\infty \leqslant 1$. Let also $\pi_\ell(f)$ denote the new function obtained by averaging $f$ w.r.t. the marginal of $\pi$ over the spin at $x = \ell$. Then,*

$$|\mathbb{E}_\omega\left[f(\omega(t)) - \pi_\ell(f)(\omega(t))\right]| \leqslant (c^*/q)^\ell e^{-t\,(\mathrm{gap}(\mathcal{L}) \wedge m)}. \tag{4.2.3}$$

**Remark.** If we replaced the r.h.s. of (4.2.3) with $\left(2\sqrt{2}/(p \wedge q)\right)^\ell e^{-t\,\mathrm{gap}(\mathcal{L})}$, then the statement would coincide with that in [15, Proposition 4.3]. Notice that as $p \downarrow 0$, the term $c^*/q$ does not blow up— unlike $2\sqrt{2}/(p \wedge q)$—and as remarked below (4.2.1), $\mathrm{gap}(\mathcal{L})$ stays bounded away from $0$. Hence, as $p \downarrow 0$, the time after which the r.h.s. in (4.2.3) becomes small is bounded from above by $C_0 \times \ell$ for some universal $C_0 > 0$ not depending on $p$. This fact will be crucially used in the proofs of some of the theorems to follow.

*Proof of Lemma 4.2.1.* As mentioned in the remark using [15, Proposition 4.3] it suffices to assume that $p < 1/3$. Fix $\omega$ as in the lemma and let $\Omega^\omega_{(-\infty,\ell]}$ be the set of all configurations $\omega' \in \Omega_{(-\infty,\ell]}$ which coincides with $\omega$ on the half line $(-\infty, 0]$. The special configuration in $\Omega^\omega_{(-\infty,\ell]}$ which is identically equal to one in the interval $[1, \ell]$ will be denoted by $\omega^*$. Observe that, using reversibility together with the fact that the updates in $(-\infty, 0]$ do not check the spins to the right of the origin,

$$\sum_{\omega' \in \Omega^{\omega}_{(-\infty,\ell\,]}} \pi_{[1,\ell]}(\omega')\mathbb{E}_{\omega'}\left[f(\omega'(t)\right] = \mathbb{E}_{\omega}\left[\pi_{[1,\ell]}(f)(\omega(t)\right]$$

$$\sum_{\omega' \in \Omega^{\omega}_{(-\infty,\ell\,]}} \pi_{[1,\ell]}(\omega')\mathbb{E}_{\omega'}\left[\pi_{\ell}(f)(\omega'(t)\right] = \mathbb{E}_{\omega}\left[\pi_{[1,\ell]}(f)(\omega(t)\right]. \tag{4.2.4}$$

Using the graphical construction as a grand coupling for the processes with initial condition in $\Omega^{\omega}_{(-\infty,\ell\,]}$, it is easy to verify that, at the hitting time $\tau_\ell$ of the set $\{\omega' \in \Omega_{(-\infty,\ell]} :\ \omega'_\ell = 0\}$ for the process started from $\omega^*$, the processes starting from *all* possible initial conditions in $\Omega^{\omega}_{(-\infty,\ell\,]}$ have coupled. Let $\omega' \in \Omega^{\omega}_{(-\infty,\ell\,]}$ be distributed according to $\pi_{[1,\ell]}$. Then using the grand coupling,

$$\begin{aligned}
|\mathbb{E}_{\omega}\left[f(\omega(t)) - \pi_\ell(f)(\omega(t))\right]| &= |\mathbb{E}_{\omega,\pi_{[1,\ell]}}\left[f(\omega(t)) - f(\omega'(t)) + \pi_\ell(f)(\omega'(t)) - \pi_\ell(f)(\omega(t))\right]| \\
&\leqslant 4 \sup_{\omega' \in \Omega^{\omega}_{(-\infty,\ell\,]}} \mathbb{P}(\exists\, x \in [1,\ell] :\ \omega_x(t) \neq \omega'_x(t)) \\
&\leqslant 4\mathbb{P}_{\omega^*}(\tau_\ell > t) \\
&\leqslant 4\mathbb{P}_{\omega^*}(X(\omega^*(t)) < \ell).
\end{aligned}$$

The first equality follows by adding and subtracting $\mathbb{E}_{\omega}\left[\pi_{[1,\ell]}(f)(\omega(t)\right]$ from the l.h.s. and then using (4.2.4). The rest of the inequalities are immediate from the above discussion. In order to bound the above probability, we observe that the front $X(\omega^*(t))$, initially at $x = 0$, can be coupled to an asymmetric random walk $\xi(t)$, with $q(\text{resp.p})$ as jump rate to the right(resp. left), in such a way that $X(\omega^*(t)) \geq \xi(t)$ for all $t \geq 0$. Since we have assumed that $p < 1/3,$ by standard hitting time estimates for biased random walk there exist universal constants $c, m$ such that, for $t \geq c\ell$, the above probability is smaller than $e^{-mt}$. $\square$

*Proof of Proposition 4.2.2.* Let $\omega \in \Omega$ be such that $\omega_0 = 0$. Then

$$\begin{aligned}
&\max_{\substack{f:\Omega_\Lambda \mapsto \mathbb{R} \\ \|f\|_\infty \leqslant 1}} |\mathbb{E}_\omega \left[ f(\omega(t)) - \pi(f) \right]| \\
&\qquad \leqslant \max_{\substack{f:\Omega_\Lambda \mapsto \mathbb{R} \\ \|f\|_\infty \leqslant 1}} |\mathbb{E}_\omega \left[ f(\omega(t)) - \pi_\ell(f)(\omega(t)) \right]| + \max_{\substack{f:\Omega_\Lambda \mapsto \mathbb{R} \\ \|f\|_\infty \leqslant 1}} |\mathbb{E}_\omega \left[ \pi_\ell(f)(\omega(t)) - \pi(f) \right]| \\
&\qquad \leqslant (c^*/q)^{\Delta(\omega)} e^{-t\,(\mathrm{gap}(\mathcal{L}) \wedge m)} + \max_{\substack{f:\Omega_\Lambda \mapsto \mathbb{R} \\ \|f\|_\infty \leqslant 1}} |\mathbb{E}_\omega \left[ \pi_\ell(f)(\omega(t)) - \pi(f) \right]|,
\end{aligned}$$

where we applied the above lemma to the shifted configuration in which the origin coincides with the rightmost zero in $\Lambda$ of $\omega$.
We now observe that the new function $\pi_\ell(f)$ depends only on the first $\ell - 1$ coordinates of $\omega$ and that $\|\pi_\ell(f)\|_\infty \leqslant 1$. Thus we can iterate the above bound $(\ell - 1)$ times to get that

$$\|\mathbb{P}^t_\omega - \pi\|_\Lambda \leqslant 2 \max_{\substack{f:\Omega_\Lambda \mapsto \mathbb{R} \\ \|f\|_\infty \leqslant 1}} |\mathbb{E}_\omega \left[ f(\omega(t)) - \pi(f) \right]| \leqslant \ell (c^*/q)^{\Delta(\omega)} e^{-t\,(\mathrm{gap}(\mathcal{L}) \wedge m)}. \qquad \square$$

**Corollary 4.2.3.** Fix $\omega \in \Omega^*$, $\ell \in \mathbb{N}$ and let $I^\ell_\omega = [X(\omega), X(\omega) + \ell - 1]$. Then

$$\sup_{\omega \in \Omega^*} \| \mathbb{P}^t_\omega - \pi\|_{I^\ell_\omega} \leqslant (c^*/q)^\ell e^{-t\,(\mathrm{gap}(\mathcal{L}) \wedge m)}. \tag{4.2.5}$$

$$\sup_{\omega \in \Omega^*} \mathbb{P}_\omega \left( \omega(t) \text{ does not satisfy SSC in } I^\ell_\omega \right) \leqslant \ell (c^*/q)^\ell e^{-t\,(\mathrm{gap}(\mathcal{L}) \wedge m)} + \ell^{-9}. \tag{4.2.6}$$

$$\sup_{\omega \in \Omega^*} \mathbb{P}_\omega \left( \omega(t) \text{ does not satisfy WSC in } I^\ell_\omega \right) \leqslant (c^*/q)^\ell e^{-t\,(\mathrm{gap}(\mathcal{L}) \wedge m)} + \ell p^{\delta \ell^{\varepsilon/2}}. \tag{4.2.7}$$

*Proof.* By construction, $\Delta_{I^\ell_\omega}(\omega) = \ell$ for any $\omega \in \Omega^*$. Thus the first statement follows at once from Proposition 4.2.2. The other two statements follow from the fact that

$$\pi \left( \{\omega : \omega \text{ does not satisfy SSC in } [1, \ldots, \ell]\} \right) \leqslant \ell^{-9}$$

and

$$\pi \left( \{\omega : \omega \text{ does not satisfy the WSC in } [1, \ldots, \ell]\} \right) \leqslant \ell p^{\delta \ell^{\varepsilon/2}}. \qquad \square$$

### 4.2.4 Finite speed of information propagation

As the East process is an interacting particle system whose rates are bounded by one, it is well known that in this case information can only travel through the system at finite speed. A quantitative statement of the above general fact goes as follows.

**Lemma 4.2.2.** *For $x < y \in \mathbb{Z}$ and $0 \leqslant s < t$, define the "linking event" $F(x, y; s, t)$ as the event that there exists a ordered sequence $s \leqslant t_x < t_{x+1} < \cdots < t_y < t$ or $s \leqslant t_y < t_{y-1} < \cdots < t_x < t$ of rings of the Poisson clocks associated to the corresponding sites in $[x, y] \cap \mathbb{Z}$. Then there exists a constant $v_{\max}$ such that, for all $|y - x| \geq v_{\max}(t - s)$,*

$$\mathbb{P}(F(x, y; s, t)) \leqslant e^{-|x-y|}.$$

*Proof.* The probability of $F(x, y; s, t)$ is equal to the probability that a Poisson process of intensity $1$ has at least $|x - y|$ instances within time $t - s$. □

**Remark 4.2.1.** An important consequence of the above lemma is the following fact. Let $0 < s < t$ and let $\mathcal{F}_s$ be the $\sigma$-algebra generated by all the rings of the Poisson clocks and all the coin tosses up to time $s$ in the graphical construction of the East process. Fix $x < y < z$ and let $A, B$ be two events depending on $\{\omega_a\}_{a \leqslant x}$ and $\{\omega_a\}_{a \geq z}$ respectively. Then

$$\begin{aligned} &\mathbb{P}_\omega\left(\{\omega(t) \in A \cap B\} \cap F(y, z; s, t)^c \mid \mathcal{F}_s\right) \\ &= \mathbb{P}_\omega\left(\{\omega(t) \in A\} \mid \mathcal{F}_s\right) \mathbb{P}_\omega\left(\{\omega(t) \in B\} \cap F(y, z; s, t)^c \mid \mathcal{F}_s\right). \end{aligned}$$

This is because: (i) on the event $F(y, z; s, t)^c$ the occurrence of the event $B$ does not depend anymore on the Poisson rings and coin tosses to the left of $y$; (ii) the occurrence of the event $A$ depends only on the Poisson rings and coin tosses to the left of $x$ because of the oriented character of the East process.

The finite speed of information propagation, together with the results of [2], implies the following rough bound on the position of the front $X(\omega(t))$ for the East process started from $\omega \in \Omega^*$ (also see, e.g., [15, Lemma 3.2]).

**Lemma 4.2.3.** *There exists constants* $v_{\min} > 0$ *and* $\gamma > 0$ *such that*

$$\sup_{\omega \in \Omega^*} \mathbb{P}_\omega \left( X(\omega(t)) \in [X(\omega) + v_{\min} t, X(\omega) + v_{\max} t] \right) \geq 1 - e^{-\gamma t}.$$

**Remark 4.2.2.** When $p \downarrow 0$ one can obtain the above statement with $v_{\min} \to 1$ and $\gamma$ uniformly bounded away from $0$ by using our Proposition 4.2.2 instead of [15, Proposition 4.3] in the proof of [15, Lemma 3.2].

The second consequence of the finite speed of information propagation is a kind of mixing result behind the front $X(\omega(t))$ for the process started from $\omega \in \Omega^*$. We first need few additional notation.

**Definition 4.2.4.** For any $a \in \mathbb{Z}$, we define the *shifted* configuration $\vartheta_a \omega$ by

$$\vartheta_a \omega_x = \omega_{x+a}, \ \forall x \in \mathbb{Z}.$$

**Proposition 4.2.5.** Let $\Lambda \subset (-\infty, -\ell] \cap \mathbb{Z}$ and let $B \subset \{0,1\}^\Lambda$. Assume $\ell \geq 2v_{\max}(t-s)$. Then for any $\omega \in \Omega^*$ and any $a \in \mathbb{Z}$ the following holds:

$$\left| \mathbb{E}_\omega \left[ \mathbb{1}_{\{\vartheta_{X(\omega(s))}[\omega(t)] \in B\}} \mathbb{1}_{\{X(\omega(t)) = a\}} \mid \mathcal{F}_s \right] - \mathbb{E}_\omega \left[ \mathbb{1}_{\{\vartheta_{X(\omega(s))}[\omega(t)] \in B\}} \mid \mathcal{F}_s \right] \mathbb{E}_\omega \left[ \mathbb{1}_{\{X(\omega(t)) = a\}} \mid \mathcal{F}_s \right] \right| \\ = O(e^{-\ell}).$$

To see what the proposition roughly tells we first assume that the front at time $s$ is at $0$. Then the above result says that at a later time $t$ any event supported on $(-\infty, -\ell]$ is almost independent of the location of the front.

*Proof.* Recall the definition of the event $F(x, y; s, t)$ from Lemma 4.2.2 and let

$$B_1 := F\left( X(\omega(s)) - \ell, X(\omega(s)) - \ell/2 - 1; s, t \right)$$
$$B_2 := F\left( X(\omega(s)) - \ell/2, X(\omega(s)); s, t \right).$$

We now write

$$\mathbb{1}_{\{\vartheta_{X(\omega(s))}[\omega(t)]_\Lambda \in B\}}\mathbb{1}_{\{X(\omega(t))=a\}} = \mathbb{1}_{\{\vartheta_{X(\omega(s))}[\omega(t)]_\Lambda \in B\}}\mathbb{1}_{\{X(\omega(t))=a\}}\mathbb{1}_{\{B_1^c\}}\mathbb{1}_{\{B_2^c\}} + \mathbb{1}_{\{\vartheta_{X(\omega(s))}[\omega(t)]_\Lambda \in B\}}\mathbb{1}_{\{X(\omega(t))=a\}}\left[1 - \mathbb{1}_{\{B_1^c\}}\mathbb{1}_{\{B_2^c\}}\right].$$

We first note that given $\mathcal{F}_s$ for any $a < X(\omega(s)) - \ell/2$,

$$\mathbb{1}_{\{X(\omega(t))=a\}}\mathbb{1}_{\{B_2^c\}} = 0,$$

and hence

$$\mathbb{E}_\omega\left[\mathbb{1}_{\{X(\omega(t))=a\}}\mathbb{1}_{\{B_2^c\}} \mid \mathcal{F}_s\right] = 0.$$

Thus, we may assume that $a \geq X(\omega(s)) - \ell/2$. Now

$$\begin{aligned}&\mathbb{E}_\omega\left[\mathbb{1}_{\{\vartheta_{X(\omega(s))}[\omega(t)]_\Lambda \in B\}}\mathbb{1}_{\{X(\omega(t))=a\}}\mathbb{1}_{\{B_1^c\}}\mathbb{1}_{\{B_2^c\}} \mid \mathcal{F}_s\right]\\ &\quad= \mathbb{E}_\omega\left[\mathbb{1}_{\{\vartheta_{X(\omega(s))}[\omega(t)]_\Lambda \in B\}}\mathbb{1}_{\{B_1^c\}} \mid \mathcal{F}_s\right]\mathbb{E}_\omega\left[\mathbb{1}_{\{X(\omega(t))=a\}}\mathbb{1}_{\{B_2^c\}} \mid \mathcal{F}_s\right]\end{aligned}$$

because under the assumption that $a \geq X(\omega(s)) - \ell/2$, the two events are functions of an independent set of variables in the graphical construction (cf. Remark 4.2.1). By Lemma 4.2.2 we know that $\mathbb{P}(B_i^c \mid \mathcal{F}_s) \leqslant e^{-\ell},\ i = 1,2$ and the proof is complete. □

## *4.3 The law behind the front of the East process*

Our main result in this section is a quantitative estimate on the rate of convergence as $t \to \infty$ of the law $\mu_\omega^t$ of the process seen from the front to its invariant measure $\nu$. Consider the process $\{\omega^{\mathrm{F}}(t)\}_{t\geq 0}$ seen from the front (recalling §4.2.2) and let $\mu_\omega^t$ be its law at time $t$ when the starting configuration is $\omega$.

**Theorem 4.3.1.** *For any $p \in (0,1)$ there exist $\alpha \in (0,1)$ and $v^* > 0$ such that*

$$\sup_{\omega\in\Omega_F} \| \mu_\omega^t - \nu\|_{[-v^*t,\,0]} = O(e^{-t^\alpha}).$$

*Moreover, $\alpha$ and $v^*$ can be chosen uniformly as $p \to 0$.*

A corollary of this result — which will be key in the proof of Theorem 4.1.1 — is to show that, for any $\omega \in \Omega_F$, the increments in the position of the front (the variables $\xi_n$ below) behave asymptotically as a stationary sequence of weakly dependent random variables with exponential moments.

Fix $\Delta$[1] $> 0$ and let $t_n = n\Delta$ for $n \in \mathbb{N}$. Define

$$\xi_n := X(\omega(t_n)) - X(\omega(t_{n-1})),$$

so that

$$X(\omega(t)) = \sum_{n=1}^{N_t} \xi_n + [X(\omega(t)) - X(\omega(t_N))], \quad N = \lfloor t/\Delta \rfloor. \tag{4.3.1}$$

Recall also that $\alpha, v^*$ are the constants appearing in Theorem 4.3.1.

**Corollary 4.3.1.** Let $f : \mathbb{R} \mapsto [0,\infty)$ be such that $e^{-|x|}f^2(x) \in L^1(\mathbb{R})$. Then

$$C_f \equiv \sup_{\omega\in\Omega_F} \mathbb{E}_\omega \left[f(\xi_1)^2\right] < \infty. \tag{4.3.2}$$

Moreover, there exists a constant $\gamma > 0$ such that

$$\sup_{\omega\in\Omega_F} |\mathbb{E}_\omega [f(\xi_n)] - \mathbb{E}_\nu [f(\xi_1)]| = O(e^{-\gamma n^\alpha}) \quad \forall n \geq 1, \tag{4.3.3}$$

$$\sup_{\omega\in\Omega_F} |\mathrm{Cov}_\omega (\xi_j, \xi_n) - \mathrm{Cov}_\nu (\xi_1, \xi_{n-j})| = O(e^{-\gamma j^\alpha}) \wedge O(e^{-\gamma(n-j)^\alpha}) \quad \forall j < n \tag{4.3.4}$$

[1] In the sequel we will always use the letter $\Delta$ to denote a time lag. Its value will depend on the context and will be specified in advance.

and

$$\sup_{\omega\in\Omega_{\rm F}} |\mathrm{Cov}_\omega\left(f(\xi_j), f(\xi_n)\right)| = O(e^{-\gamma(n-j)^\alpha}), \quad \forall j \leqslant n, \tag{4.3.5}$$

where the constants in the r.h.s. of (4.3.3) and (4.3.5) depend on $f$ only through the constant $C_f$. Finally, for any $k, n \in \mathbb{N}$ such that $v^* k > n v_{\max}$ and for any bounded $F : \mathbb{R}^n \mapsto \mathbb{R}$ ,

$$\sup_{\omega\in\Omega_{\rm F}} \left|\mathbb{E}_\omega\left[F\Big(\xi_k, \xi_{k+1}, \ldots, \xi_{k+n-1}\Big)\right] - \mathbb{E}_\nu\left[F\Big(\xi_1, \xi_2, \ldots, \xi_n\Big)\right]\right| = O\left(e^{-\gamma t_k^\alpha}\right). \tag{4.3.6}$$

To prove Theorem 4.3.1 we will require a technical result, Theorem 4.3.2 below, which can informally be summarized as follows:

- Starting from $\omega \in \Omega^*$, at any fixed large time $t$, with high probability the configuration satisfies WSC apart from an interval behind the front $X(\omega(t))$ of length proportional to $t^\epsilon$ .

- If the above property is true at time $t$, then at a later time $t' = t + \text{const} \times t^\epsilon$ the law of the process will be very close to $\pi$ apart from a small interval behind the front where the strong spacing property will occur with high probability.

Formally, fix a constant $\kappa$ to be chosen later on and $t > 0$. Let $\ell \equiv t^\epsilon$, where $\epsilon$ appears in the WSC and let $t_\ell = t - \kappa\ell/v_{\min}$. Let $\mathcal{S}_\ell$ denotes the set of configurations which fail to satisfy SSC in the interval $[-3(v_{\max}/v_{\min})\kappa\,\ell, -\kappa\log\ell) \cap \mathbb{Z}$ and let $\mathcal{W}_{\ell,t}$ be those configurations which fail to satisfy WSC in the interval $[-v_{\min}t, -\ell) \cap \mathbb{Z}$.

**Theorem 4.3.2.** *It is possible to choose $\delta$ small enough and $\kappa$ large enough depending only* *on $p$ in such a way that for all $t$ large enough the following holds:*

$$\sup_{\omega\in\Omega_{\mathrm{F}}} \mu_\omega^t\left(\mathcal{W}_{\ell,t}\right) = O(e^{-t^{\epsilon/2}}), \tag{4.3.7}$$

$$\sup_{\omega\in\Omega_{\mathrm{F}}} \mu_\omega^t\left(\mathcal{S}_\ell \mid \mathcal{F}_{t_\ell}\right) = O(t^{-7\epsilon}) + \mathbb{1}_{\mathcal{W}_{\ell,t}}(\omega(t_\ell)), \tag{4.3.8}$$

$$\sup_{\omega\in\Omega_{\mathrm{F}}} \|\mu_\omega^t(\cdot \mid \mathcal{F}_{t_\ell}) - \pi\|_{[-v_{\min}t, -3(v_{\max}/v_{\min})\kappa\ell]} = O(e^{-t^{\epsilon/2}}) + \mathbb{1}_{\mathcal{W}_{\ell,t}}(\omega(t_\ell)). \tag{4.3.9}$$

*Moreover, $\kappa$ stays bounded as $p \downarrow 0$.*

### *4.3.1 Non-equilibrium properties of the law behind the front: Proof of Theorem 4.3.2*

We begin by proving (4.3.7). Bounding $\sup_{\omega\in\Omega_{\mathrm{F}}} \mu_\omega^t\left(\mathcal{W}_{\ell,t}\right)$ from above is equivalent to bounding $\sup_{\omega\in\Omega_{\mathrm{F}}} \mathbb{P}_\omega(\omega(t) \in \mathcal{W}_{\ell,t}^*)$ from above, where $\mathcal{W}_{\ell,t}^*$ denotes the set of configurations $\omega \in \Omega^*$ which do not satisfy the spacing condition in $[X(\omega) - v_{\min}t, X(\omega) - \ell]$.

Using Lemma 4.2.3, with probability greater than $1-e^{-\gamma t}$ we can assume that $X(\omega(t)) \in [v_{\min}t, v_{\max}t]$. Next we observe that, for any $a \in [v_{\min}t, v_{\max}t]$, the events $\{X(\omega(t)) = a\}$ and $\{\omega(t) \in \mathcal{W}_{\ell,t}^*\}$ imply that there exists $x \in \mathbb{Z}$ with the following properties:

- $0 \leqslant x \leqslant a - \ell$;
- The hitting time $\tau_x := \inf\{s > 0 : X(\omega(s)) = x\}$ is smaller than $t$;
- $\omega(t)$ is identically equal to one in the interval $I_x := [x, x + \delta(v_{\min}\, t)^\epsilon/2]$;
- The linking event $F(x, a; \tau_x, t)$ defined in Lemma 4.2.2 occurred.

In conclusion, using twice a union bound (once for the choice of $a \in [v_{\min}t, v_{\max}t]$ and once for the choice of $x \in [0, a-\ell]$) together with the strong Markov property at time $\tau_x$, we get

$$\begin{aligned}
&\mathbb{P}_\omega(\omega(t) \in \mathcal{W}^*_{\ell,t}) \\
&\leqslant e^{-\gamma t} + \sum_{a=v_{\min}t}^{v_{\max}t} \sum_{x=0}^{a-\ell} \mathbb{P}\left(F(x,a;\tau_x,t)\right) \mathbb{1}_{\{|x-a| \geq v_{\max}(t-\tau_x)\}} \\
&\qquad + \sum_{a=v_{\min}t}^{v_{\max}t} \sum_{x=0}^{a-\ell} \left[ \|\mathbb{P}^t_{\omega(\tau_x)} - \pi\|_{I_x} + p^{\frac{\delta}{2}(v_{\min}t)^\epsilon} \right] \mathbb{1}_{\{|x-a| \leqslant v_{\max}(t-\tau_x)\}} \\
&\leqslant (v_{\max}t)^2 \left[ e^{-\gamma t} + 2e^{-\ell} + \sqrt{2}\delta (v_{\min}t)^\epsilon \left(\frac{c^*}{q}\right)^{\frac{\delta}{2}(v_{\min}t)^\epsilon} e^{-\frac{\ell}{v_{\max}}(\mathrm{gap}(\mathcal{L}) \wedge m)} + p^{\frac{\delta}{2}(v_{\min}t)^\epsilon} \right].
\end{aligned}$$

Above we used Lemma 4.2.2 in the case $|x-a| \geq v_{\max}(t-\tau_x)$ and (4.2.5) of Corollary 4.2.3 otherwise. The statement (4.3.7) now follows by taking $\delta$ small enough.

We now prove (4.3.8). As before we give the result in the East process setting (*i.e.* for the law $\mathbb{P}^t_\omega(\cdot \mid \mathcal{F}_s)$ and $\mathcal{S}_\ell$ replaced by its random shifted version $\mathcal{S}^*_\ell$). We decompose the interval $[X(\omega(t)) - 3(v_{\max}/v_{\min})\kappa\,\ell, X(\omega(t)) - \kappa \log \ell)] \cap \mathbb{Z}$ where we want SSC to hold into $[X(\omega(t_\ell)), X(\omega(t)) - \kappa \log \ell]$ and $[X(\omega(t)) - 3(v_{\max}/v_{\min})\kappa\,\ell, X(\omega(t_\ell))]$.
Note that by Lemma 4.2.3 we can ignore the events $\{X(\omega(t_\ell)) > X(\omega(t)) - \kappa \log \ell\}$ and $\{X(\omega(t)) - 3(v_{\max}/v_{\min})\kappa\,\ell > X(\omega(t_\ell))\}$.

We now proceed in two steps: (1) we show that SSC occurs with high probability in the first interval. Here we do not use the condition that $\omega(t_\ell) \notin \mathcal{W}^*_{\ell,t}$. (2) we prove the same statement for the second interval. Here instead the fact that $\omega(t_\ell) \notin \mathcal{W}^*_{\ell,t}$ will be crucial.

- *Step (1).* Let $\Delta \equiv 5\log \ell / (|\log p| \wedge 1)$. For any intermediate time $s \in [t_\ell, t -$

$(\kappa/v_{\max})\log\ell]$, Corollary 4.2.3 together with the Markov property at time $s$ show that

$$\begin{aligned}
&\mathbb{P}_\omega\left(\omega(t)_x = 1\ \forall x \in [X(\omega(s)), X(\omega(s))+\Delta] \mid \mathcal{F}_s\right) \\
&\quad \leqslant \|\mathbb{P}_\omega(\cdot \mid \mathcal{F}_s) - \pi\|_{[X(\omega(s)),X(\omega(s))+\Delta]} + \pi\left(\omega_x = 1\ \forall x \in [X(\omega(s)), X(\omega(s))+\Delta]\right) \\
&\quad \leqslant \Delta\left(\frac{c^*}{q}\right)^{\Delta} e^{-(t-s)(\operatorname{gap}(\mathcal{L})\wedge m)} + p^{|\Delta|} = O(t^{-10\epsilon}). \qquad (4.3.10)
\end{aligned}$$

Above we used the fact that $t - s \geq \kappa/v_{\max}\log\ell$. Hence, $\kappa$ can be chosen depending only on $p$ such that (4.3.10) holds and $\kappa$ stays bounded as $p \downarrow 0$.

We now take the union of the random intervals $[X(\omega(s)), X(\omega(s))+\Delta]$ over discrete times $s$ of the form $s_j = t_\ell + j/\ell^2$, $j = 0, 1, \ldots, n$ and $n$ such that $s_n = t - (\kappa/v_{\max})\log\ell$. Thus $n = O(\ell^3) = O(t^{3\epsilon})$. The aim here is to show that, with high probability, the above union is actually an interval containing the target one $[X(\omega(t_\ell)), X(\omega(t)) - \kappa\log\ell]$, with the additional property that it does not contain a sub-interval of length $\Delta$ where $\omega(t)$ is constantly equal to one (which will then imply (4.3.8), with room to spare).

We now upper bound the probability that the set $\cup_{j=0}^{n}[X(\omega(s_j)), X(\omega(s_j))+\Delta]$ is not an interval. It is an easy observation that if $X(\omega(s_n)) > X(\omega(s_0))$ then the aforementioned event occurs if $X(\omega(s_{j+1})) - X(\omega(s_j)) \leqslant \Delta$ for $j = 0, 1, \ldots n$. Now by the lower bound in Lemma 4.2.3

$$\mathbb{P}(X(\omega(s_n)) > X(\omega(s_0))) \geq 1 - e^{-ct^{\epsilon}}$$

for some constant $c$. Also

$$\begin{aligned}
&\sum_j \mathbb{P}_\omega(X(\omega(s_{j+1})) - X(\omega(s_j)) \geq \Delta) \\
&\qquad \leqslant \sum_j \mathbb{E}\left[\mathbb{P}_\omega\left(F(X(\omega(s_j)), X(\omega(s_j))+\Delta; s_j, s_{j+1}) \mid \mathcal{F}_{s_j}\right)\right] \\
&\qquad \leqslant ne^{-\Delta} = O(t^{-8\epsilon}).
\end{aligned}$$

Above $F(X(\omega(s_j)), X(\omega(s_j)) + \Delta; s_j, s_{j+1})$ is the linking event and we used Lemma 4.2.2 because $\Delta \gg (s_{j+1} - s_j)$.

Moreover, Lemma 4.2.3 implies that $\kappa$ can be chosen (bounded as $p \downarrow 0$), such that with probability greater than

$$1 - e^{-\gamma(t-s_0)} - e^{-\gamma(t-s_n)} = 1 - O(t^{-10\epsilon}),$$

the front $X(\omega(t))$ satisfies

$$X(\omega(t)) \leqslant X(\omega(s_n)) + v_{\max}(t - s_n) \leqslant X(\omega(s_n)) + \kappa \log \ell.$$

Thus

$$[X(\omega(t_\ell)), X(\omega(t)) - \kappa \log \ell] \subset \cup_{j=0}^{n} [X(\omega(s_j)), X(\omega(s_j)) + \Delta].$$

with probability $1 - O(t^{-10\epsilon})$.

Finally, using (4.3.10) and union bound, the probability that there exists $j \leqslant n$ such that $\omega(t)$ is identically equal to one in $[X(\omega(s_j)), X(\omega(s_j)) + \Delta]$ is $O(t^{-7\epsilon})$ uniformly in the configuration at time $t_\ell$.

In conclusion we proved that SSC holds with probability $1 - O(t^{-8\epsilon})$ in an interval containing $[X(\omega(t_\ell)), X(\omega(t) - \kappa \log \ell]$. The first step is complete.

- *Step (2)*. Let $x^* = \max\{x \leqslant X(\omega(t_\ell)) - 3\kappa(v_{\max}/v_{\min})\ell : \omega(t_\ell)_x = 0\}$. Since $\omega(t_\ell) \notin \mathcal{W}^*_{\ell,t}$ such a zero exists. Moreover, $\omega(t_\ell) \notin \mathcal{W}^*_{\ell,t}$ implies that $\omega(t_\ell)$ has a zero in every sub-interval of $[x^*, X(\omega(t_\ell)) - \ell]$ of length $\delta t^\epsilon = \delta \ell$. Hence we can apply Proposition 4.2.2 to the interval $[x^*, X(\omega(t_\ell))]$ to get that

$$\| \mathbb{P}^t_\omega(\cdot \mid \mathcal{F}_{t_\ell}) - \pi \|_{[x^*, X(\omega(t_\ell))]} = O(e^{-t^{\epsilon/2}}),$$

by choosing $\kappa$ large enough. Since by Remark 4.2.2 $v_{\min} \to 1$ as $p \downarrow 0$, we can choose $\kappa$ to be bounded as $p \downarrow 0$. Also

$$\pi\left(\omega:\ \omega \text{ violates SSC in } [x^*, X(\omega(t_\ell))]\right) = O(t^{-7\epsilon}).$$

Thus we have proved that SSC holds in $[x^*, X(\omega(t_\ell))]$ with probability $1 - O(t^{-7\epsilon})$.

Finite speed of propagation in the form of Lemma 4.2.3 guarantees that, with probability $1 - O(e^{-\gamma(t-t_\ell)})$, $x^* < X(\omega(t)) - 2\kappa(v_{\max}/v_{\min})\ell$. The proof of (4.3.8) is complete.

It remains to prove (4.3.9). Let $\Lambda := [-v_{\min}t, -3(v_{\max}/v_{\min})\kappa\ell] \cap \mathbb{Z}$ and let $A \subset \{0,1\}^\Lambda$. Recall Definition 4.2.4 of the shifted configuration $\vartheta_a\omega$ and that $t_\ell = t - \kappa\ell/v_{\min}$. Then (4.3.9) follows once we show that

$$|\mathbb{P}_\omega(\vartheta_{X(\omega(t)}\omega(t)_\Lambda \in A \mid \mathcal{F}_{t_\ell}) - \pi(A)| \leqslant e^{-t^{\epsilon/2}}$$

whenever $\omega(t_\ell)$ satisfies WSC in the interval $I = [X(\omega(t_\ell)) - v_{\min}t, X(\omega(t_\ell)) - \ell]$. This property is assumed henceforth. Let us decompose $\mathbb{P}_\omega(\vartheta_{X(\omega(t)}\omega(t)_\Lambda \in A \mid \mathcal{F}_{t_\ell})$ according to the value of the front:

$$\mathbb{P}_\omega\left(\vartheta_{X(\omega(t))}\omega(t) \in A \mid \mathcal{F}_{t_\ell}\right) = \sum_{a\in\mathbb{Z}} \mathbb{E}_\omega\left[\mathbb{1}_{\{\vartheta_a\omega(t)_\Lambda \in A\}}\,\mathbb{1}_{\{X(\omega(t))=a\}} \mid \mathcal{F}_{t_\ell}\right].$$

Using Lemma 4.2.3, $0 < X(\omega(t)) - X(\omega(t_\ell)) \leqslant v_{\max}(t - t_\ell)$ occurs with probability greater than $1 - e^{-\gamma(t-t_\ell)}$. Thus

$$\begin{aligned}&\sum_{a\in\mathbb{Z}} \mathbb{E}_\omega\left[\mathbb{1}_{\{\vartheta_a\omega(t)_\Lambda \in A\}}\,\mathbb{1}_{\{X(\omega(t))=a\}} \mid \mathcal{F}_{t_\ell}\right]\\ &= \sum_{\substack{a\in\mathbb{Z}\\ 0<a-X(\omega(t_\ell))\leqslant v_{\max}(t-t_\ell)}} \mathbb{E}_\omega\left[\mathbb{1}_{\{\vartheta_a\omega(t)_\Lambda \in A\}}\,\mathbb{1}_{\{X(\omega(t))=a\}} \mid \mathcal{F}_{t_\ell}\right] + e^{-\gamma(t-t_\ell)}.\end{aligned}$$

By definition, the event $\{\vartheta_a\omega(t)_\Lambda \in A\}$ is the same as the event $\{\omega(t)_{\Lambda+a} \in A\}$. Using the restriction that $|a - X(\omega(t_\ell))| \leqslant v_{\max}(t - t_\ell)$, the choice of $\Lambda$ and the fact that $(v_{\max}/v_{\min})\kappa\ell \geq v_{\max}(t - t_\ell)$, we get $\Lambda + a \subset (-\infty, X(\omega(t_\ell)) - 2(v_{\max}/v_{\min})\kappa\ell]$. Thus, the event $\{\omega(t)_{\Lambda+a} \in A\}$

satisfies the hypothesis of Proposition 4.2.5, which can then be applied to each term in the above sum to get

$$\begin{aligned}&\sum_{\substack{a\in\mathbb{Z}\\0<a-X(\omega(t_\ell))\leqslant v_{\max}(t-t_\ell)}} \mathbb{E}_\omega\left[\mathbb{1}_{\{\vartheta_a\omega(t)_\Lambda\in A\}}\,\mathbb{1}_{\{X(\omega(t))=a\}}\mid\mathcal{F}_{t_\ell}\right]\\ =&\sum_{\substack{a\in\mathbb{Z}\\0<a-X(\omega(t_\ell))\leqslant v_{\max}(t-t_\ell)}} \mathbb{E}_\omega\left[\mathbb{1}_{\{\vartheta_a\omega(t)_\Lambda\in A\}}\mid\mathcal{F}_{t_\ell}\right]\mathbb{E}_\omega\left[\mathbb{1}_{\{X(\omega(t))=a\}}\mid\mathcal{F}_{t_\ell}\right]+O(\ell\, e^{-\ell}).\end{aligned}$$

Finally we claim that, for any $a$ such that $0<a-X(\omega(t_\ell))\leqslant v_{\max}(t-t_\ell)$, if $\delta$ is chosen small enough and $\kappa$ large enough depending on $p$ (bounded as $p\downarrow 0$),

$$\mathbb{E}_\omega\left[\mathbb{1}_{\{\vartheta_a\omega(t)_\Lambda\in A\}}\mid\mathcal{F}_{t_\ell}\right]=\pi(A)+O(e^{-t^{\epsilon/2}}). \tag{4.3.11}$$

To prove it we apply Proposition 4.2.2 to the interval $I=[X(\omega(t_\ell))-v_{\min}t, X(\omega(t_\ell))-\ell])$ to get that

$$\|\mathbb{P}^t_\omega(\cdot\mid\mathcal{F}_{t_\ell})-\pi\|_I\leqslant|I|\left(\frac{c^*}{q}\right)^{\delta|I|^\epsilon}e^{-(t-t_\ell)(\text{gap}(\mathcal{L})\wedge m)}, \tag{4.3.12}$$

where $|I|=O(t)$ is the length of $I$, since by assumption $\omega(t_\ell)$ satisfies WSC in $I$. Because of our choice of the parameters $(\ell,t_\ell)$ the r.h.s. of (4.3.12) is $O(e^{-t^{\epsilon/2}})$ if $\delta,\kappa$ are chosen small enough and large enough respectively depending on $p$. Since by Remark 4.2.2 $v_{\min}\to 1$ as $p\downarrow 0$, $\kappa$ can be chosen to be bounded as $p\downarrow 0$.

The claim now follows because $\{\omega:\vartheta_a\omega\in A\}\subset\{0,1\}^{\Lambda+a}$, with

$$\begin{aligned}\Lambda+a&=[-v_{\min}t+a,-3(v_{\max}/v_{\min})\kappa\ell+a]\\&\subset[X(\omega(t_\ell))-v_{\min}t_\ell,\,X(\omega(t_\ell))-(v_{\max}/v_{\min})\kappa\ell\,]\subset I,\end{aligned}$$

together with the translation invariance of $\pi$ expressed by $\pi\left(\{\omega:\vartheta_a\omega\in A\}\right)=\pi(A)$. This establishes (4.3.9) and concludes the proof of Theorem 4.3.2. Notice that at all points in the proof, $\kappa$ was chosen to be bounded as $p\downarrow 0$. □

### *4.3.2 On the rate of convergence to the invariant measure $\nu$: Proof of Theorem 4.3.1*

The proof is based on a coupling argument. There exists $v^* > 0$ such that, for any $t$ large enough and for any pair of starting configurations $(\omega, \omega') \in \Omega_{\mathrm{F}} \times \Omega_{\mathrm{F}}$,

$$\|\mu^t_\omega - \mu^t_{\omega'}\|_{[-v^*t,\,0]} \leqslant c' e^{-t^\alpha}, \tag{4.3.13}$$

with $(c', \alpha)$ independent of $(\omega, \omega')$. Also $v^*, \alpha$ can be chosen uniformly as $p \downarrow 0$. Once this step is established and using the invariance of the measure $\nu$ under the action of the semigroup $e^{t\mathcal{L}^{\mathrm{F}}}$,

$$\begin{aligned}\| \mu^t_\omega - \nu\|_{[-v^*t,\,0]} &= \| \mu^t_\omega - \int d\nu(\omega')\mu^t_{\omega'} \|_{[-v^*t,\,0]} \\ &\leqslant \int d\nu(\omega')\| \mu^t_\omega - \mu^t_{\omega'} \|_{[-v^*t,\,0]} \leqslant c' e^{-t^\alpha}.\end{aligned}$$

We now prove (4.3.13). We first fix a bit of notation.

Given $\epsilon \in (0,1)$ and a large $t > 0$, let $\Delta_1 = (\kappa/v_{\min})t^\epsilon$ where $\kappa$ is the constant appearing in Theorem 4.3.2, let $\Delta_2 = \kappa\epsilon \log t$ and define $\Delta = \Delta_1 + \Delta_2$. We then set

$$t_0 = (1-\epsilon)t, \quad t_n = t_{n-1} + \Delta, \quad n = 1, \dots N, \quad N = \lfloor \epsilon\, t/\Delta \rfloor.$$

It will be convenient to refer to the time lag $[t_{n-1}, t_n)$ as the $n^{\mathrm{th}}$-round. In turn we split each round into two parts: from $t_{n-1}$ to $s_n := t_{n-1} + \Delta_1$ and from $s_n$ to $t_n$. We will refer to the first part of the round as the *burn-in part* and to the second part as the *mixing part*. We also set $I_n = [-v_{\min} t_n + 2v_{\max}\Delta n, 0]$. Observe that $I_n \neq \varnothing$ for any $n \leqslant N+1$ if $\epsilon$ is chosen smaller than $v_{\min}/v_{\max}$ and $t$ is large enough depending on $\epsilon$.

Next, for any pair $(\mu, \mu')$ of probability measures on a finite set, we denote by $\mathrm{MC}(\mu, \mu')$ their *maximal coupling*, namely the one that achieves the variation distance between $\mu, \mu'$ in the variational formula (see, e.g., [66]),

$$\|\mu - \mu'\| = \inf\{M(\omega \neq \omega') : \ M \text{ a coupling of } \mu, \mu'\}.$$

If $(\mu, \mu')$ are probability measures on $\Omega$ and $\Lambda$ is a finite subset of $\mathbb{Z}$, we define the $\Lambda$-*maximal coupling* $\mathrm{MC}_\Lambda(\mu, \mu')$ as follows:

a) first sample $(\omega_\Lambda, \omega'_\Lambda)$ according to the maximal coupling of the marginals of $\mu, \mu'$ on $\Omega_\Lambda$;

b) then sample *independently* $(\omega_{\mathbb{Z}\setminus\Lambda}, \omega'_{\mathbb{Z}\setminus\Lambda})$ according to their respective conditional distribution $\mu(\cdot \mid \omega_\Lambda), \mu'(\cdot \mid \omega'_\Lambda)$.

Finally the *basic coupling* for the East process will be the one in which two configurations evolve according to the graphical construction using the same Poisson clocks and the same coin tosses.

We are now ready to recursively construct the coupling $M^t_{\omega,\omega'}$ of $\mu^t_\omega, \mu^t_{\omega'}$ satisfying (4.3.13). For lightness of notation, in the sequel the starting configurations $(\omega, \omega')$ will be sometimes omitted.

**Definition 4.3.2** (The coupling $M^t_{\omega,\omega'}$). We first define a family $\{M^{(n)}\}$ of couplings for $\{(\mu^{t_n}_\omega, \mu^{t_n}_{\omega'})\}_{n=0}^N$ as follows. $M^{(0)}$ is the trivial product coupling. Given $M^{(n)}$, the coupling $M^{(n+1)}$ at time $t_{n+1}$ is constructed according to the following algorithm:

(a) Sample $(\omega(t_n), \omega'(t_n))$ from $M^{(n)}$. If they coincide in the interval $I_n$ then let them evolve according to the basic coupling for a time lag $\Delta$;

(b) otherwise, sample $(\omega(s_n), \omega'(s_n))$ at the end of the burn-in part of round$(n+1)$ via the $\Lambda_n$-maximal coupling $MC_{\Lambda_n}$ for the laws $\mu^{s_n}_\omega(\cdot \mid \mathcal{F}_{t_n})$ and $\mu^{s_n}_{\omega'}(\cdot \mid \mathcal{F}_{t_n})$ at the configurations $(\omega(t_n), \omega'(t_n))$ from step (a). Here $\Lambda_n = [-v_{\min} s_n, -3\,(v_{\max}/v_{\min})\,\kappa t^\epsilon]$.

   (i) If $(\omega(s_n), \omega'(s_n))$ are not equal in the interval $\Lambda_n$, then let them evolve for the mixing part of the round (i.e., from time $s_n$ to time $t_{n+1}$) via the basic coupling.

(ii) If instead they agree on $\Lambda_n$, then search for the rightmost common zero of $(\omega(s_n), \omega'(s_n))$ in $\Lambda_n$ and call $x_*$ its position. If there is no such a zero, define $x_*$ to be the right boundary of $\Lambda_n$. Next sample a Bernoulli random variable $\xi$ with $\mathrm{Prob}(\xi = 1) = e^{-2\Delta_2}$. The value $\xi = 1$ has to be interpreted as corresponding to the event that the two Poisson clocks associated to $x_*$ and to the origin in the graphical construction did not ring during the mixing part of the round.

(1) If $\xi = 1$, set $\omega(t_{n+1})_{x_*} = \omega(s_n)_{x_*}$ and similarly for $\omega'$. The remaining part of the configurations at time $t_{n+1}$ is sampled using the basic coupling to the left of $x_*$ and the maximal coupling for the East process in the interval $[x_*+1, -1]$ with boundary condition at $x_*$ equal to $\omega(s_n)_{x_*}$.

(2) If $\xi = 0$ we let evolve $(\omega(s_n), \omega'(s_n))$ with the basic coupling conditioned to have at least one ring either at $x_*$ or at the origin or both.

The final coupling $M^t_{\omega,\omega'}$ will be obtained by first sampling $(\omega(t_N), \omega'(t_N))$ from $M^{(N)}$ and then by applying the basic coupling for the time lag $(t - t_N)$.

It is easy to check that $\{M^{(n)}\}$ is indeed a family of couplings for $\{\big(\mu^{t_n}_{\omega}, \mu^{t_n}_{\omega'}\big)\}_{n=0}^{N}$. Define now

$$p_n := M^{(n)}\left(\omega \neq \omega' \text{ in the interval } I_n\right)$$

and recall that $\epsilon$ is the exponent entering in the definition of the round length $\Delta$.

**Claim 4.3.3.** There exist $\epsilon_0 > 0$ such that, for all $\epsilon < \epsilon_0$ and all $t$ large enough depending on $\epsilon$,

$$p_N = O(e^{-t^{\alpha}}),$$

for some positive $\alpha = \alpha(\epsilon)$.

*Proof.* The claim follows from the recursive inequality:

$$p_{n+1} \leqslant Ce^{-t^{\epsilon/2}} + p_n(1 - e^{-2\Delta_2}/2), \tag{4.3.14}$$

for some constant $C$. In fact, if we assume (4.3.14) and recall that $e^{-2\Delta_2} = t^{-2\kappa\epsilon}$, we get

$$\begin{aligned} p_N &\leqslant Ce^{-t^{\epsilon/2}}[1 + (1 - e^{-2\Delta_2}/2) + (1 - e^{-2\Delta_2}/2)^2 + \ldots] + \left(1 - e^{-2\Delta_2}/2\right)^N \\ &\leqslant 2Ce^{-t^{\epsilon/2}}t^{2\kappa\epsilon} + \left(1 - e^{-2\Delta_2}/2\right)^N = O(e^{-t^{\epsilon/3}}), \end{aligned}$$

provided that $1 - \epsilon(1 + 2\kappa) > \epsilon/3$, *i.e.* $\epsilon < 3/(4 + 6\kappa)$, since $N > ct^{1-\epsilon}$ for some constant $c$. Notice crucially that since $\kappa$ was bounded as $p \downarrow 0$ in the statement of Theorem 4.3.2, $\epsilon_0$ and $\alpha(\epsilon)$ can be chosen uniformly as $p \downarrow 0$.

To prove (4.3.14) we use Lemma 4.2.2 together with Theorem 4.3.2. We begin by examining the possible occurrence of two very unlikely events each of which will contribute to the constant term in (4.3.14).

- The first possibility is that $\omega(t_n) = \omega'(t_n)$ in the interval $I_n$ and $F(a_n, a_{n+1}; t_n, t_{n+1})$ occurred. Here $a_n = -v_{\min}t_n + 2v_{\max}\Delta n$ is the left boundary of $I_n$ and similarly for $a_{n+1}$. The linking event could in fact move possible discrepancies between $\omega(t_n), \omega'(t_n)$ sitting outside $I_n$ to the inside of $I_{n+1}$. Since $|a_n - a_{n+1}| \geq v_{\max}(t_{n+1} - t_n)$, Lemma 4.2.2 shows that this case gives a contribution to $p_{n+1}$ which is $O(e^{-|a_n - a_{n+1}|}) = O(e^{-v_{\max}t^{\varepsilon}})$.

- The second possibility is that either $\omega(t_n)$ or $\omega'(t_n)$ do not satisfy the $(\delta, \epsilon)$-weak spacing condition in $[-v_{\min}t_n, -t_n^{\epsilon}]$. The bound (4.3.7) of Theorem 4.3.2 shows that the contribution of such a case is $O(e^{-t^{\epsilon/2}})$.

Having discarded the occurrence of the above "extremal" situations, we now assume that $(\omega(t_n), \omega'(t_n))$ are such that: (i) they are different in the interval $I_n$; (ii) they satisfy the $(\delta, \epsilon)$-weak spacing condition in $[-v_{\min}t_n, -t_n^{\epsilon}]$. It will be useful to denote by $\mathcal{G}_n$ the set of pairs $(\omega, \tilde{\omega})$ fulfilling (i) and (ii) above.

We will show that, *uniformly* in $(\omega, \tilde{\omega}) \in \mathcal{G}_n$, the probability that at the end of the round $(\omega(\Delta), \tilde{\omega}(\Delta))$ are not coupled inside the interval $I_{n+1}$ is smaller than $(1 - \frac{1}{2}e^{-2\Delta_2})$. That clearly proves the second term in (4.3.14).

To prove that, recall the definition of the $\Lambda_n$-maximal coupling $MC_{\Lambda_n}$, fix $(\omega, \tilde{\omega}) \in \mathcal{G}_n$ and consider the event $\mathcal{B}$ that:

(i) at the end of the *burn-in* part of the round $\omega(\Delta_1) = \tilde{\omega}(\Delta_1)$ in $\Lambda_n$,

(ii) the vertex $x_*$ appearing in (ii) of step (b) of Definition 4.3.1 is within $\epsilon \log t$ from the right boundary of $\Lambda_n$ and $\omega(\Delta_1)_{x_*} = \tilde{\omega}(\Delta_1)_{x_*} = 0$,

(iii) $\omega(\Delta_1)$ and $\tilde{\omega}(\Delta_1)$ satisfy SSC in the interval $[-3(v_{\max}/v_{\min})\kappa t^{\epsilon}, -\kappa\epsilon \log t]$.

Theorem 4.3.2 proves that, uniformly in $\omega, \tilde{\omega} \in \mathcal{G}_n$,

$$MC_{\Lambda_n}(\mathcal{B}) \geq 1 - O(e^{-t^{\epsilon/2}}) - O(t^{-7\epsilon}) - p^{\epsilon \log t} = 1 - O(p^{\epsilon \log t}).$$

The first error term takes into account the variation distance from $\pi$ of the marginals in $\Lambda_n$ of $\mathbb{P}^{\Delta_1}_{\omega}$ and $\mathbb{P}^{\Delta_1}_{\tilde{\omega}}$, the second error term bounds the probability that either $\omega(\Delta_1)$ or $\tilde{\omega}(\Delta_1)$ do not satisfy the SSC condition in the interval $[-3(v_{\max}/v_{\min})\kappa t^{\epsilon}, -\kappa\epsilon \log t]$ and the third term bounds the $\pi$-probability that the event in item (ii) does not occur.

Next we claim that, for any $\kappa$ large enough and any $z \in \Lambda_n$ at distance at most $\epsilon \log t$ from the right boundary of $\Lambda_n$,

$$\begin{aligned} &\sup_{\omega, \tilde{\omega} \in \mathcal{G}_n} \mathbb{P}\left(\omega(\Delta) \neq \tilde{\omega}(\Delta) \text{ in } I_{n+1} \mid \mathcal{B}, \{x_* = z\}, \{\xi = 1\}\right) \\ &\leqslant e^{-|a_n - a_{n+1}|} + 3\kappa t^{\epsilon} \left(\frac{c^*}{q}\right)^{\epsilon \log t} e^{-\Delta_2(\mathrm{gap}(\mathcal{L}) \wedge m)} = O(t^{-2\epsilon}). \end{aligned} \tag{4.3.15}$$

The first term in the r.h.s. is the probability that the linking event $F(a_n, a_{n+1}; \Delta_1, \Delta)$ occurred. The second term comes from Proposition 4.2.2 and it bounds from above the probability that, under the maximal coupling for the East process in the interval $[x_* + 1, -1]$ and in a time lag $\Delta_2$, we see a discrepancy.

In conclusion, the probability that $\omega(\Delta) = \tilde{\omega}(\Delta)$ in $I_{n+1}$ is larger than

$$MC_{\Lambda_n}(\mathcal{B})(1 - o(1))\mathbb{P}(\xi = 1) \geq \frac{1}{2}e^{-2\Delta_2},$$

thus proving the claim. □

We are now in a position to finish the proof of Theorem 4.3.1. Let $v^* \equiv v_{\min} - 3\epsilon v_{\max}$ and let $a_N = -v_{\min} t_N + \epsilon v_{\max} t$ be the left boundary of the interval $I_N = [a_N, 0]$. Since by Remark 4.2.2, $v_{\min}$ converges to $1$ as $p \downarrow 0$, $v^*$ can be chosen uniformly as $p \downarrow 0$.

Pick two configurations $\omega(t_N), \omega'(t_N)$ at time $t_N$ and make them evolve under the basic coupling until time $t$. Clearly the events $\{\omega(t_N)_x = \omega'(t_N)_x \ \forall x \in I_N\}$ and $\{\exists\, x \in [-v^* t, 0] : \omega_x(t) \neq \omega'_x(t)\}$ imply the linking event $F(a_N, -v^*t; t_N, t)$ from Lemma 4.2.2. By construction $|v^*t - a_N| = \epsilon v_{\max} t \geq v_{\max}(t - t_N)$ for large enough $t$. Therefore,

$$\begin{aligned} M^t_{\omega,\omega'}(\exists\, x \in [-v^*t, 0] :\ \omega_x \neq \omega'_x) &\leqslant p_N + \mathbb{P}\left(F(a_N, -v^*t; t_N, t)\right) \\ &\leqslant O(e^{-t^{\alpha}}) + e^{-\epsilon v_{\max} t}, \end{aligned}$$

as required. Moreover, by the proof of Claim 4.3.3, $\alpha$ can be chosen uniformly as $p \downarrow 0$. Thus we are done. □

### *4.3.3 Mixing properties of the front increments: Proof of Corollary 4.3.1*

To prove (4.3.2) we observe that, for any $n \geq v_{\max}\Delta$, the event $|\xi_1| \geq n$ implies the occurrence of the linking event $F(0, n; 0, \Delta)$. Lemma 4.2.2 now gives that

$$\mathbb{E}_\omega\left[f(\xi_1)^2\right] \leqslant \max_{|x| \leqslant v_{\max}\Delta} f(x)^2 + \sum_{n \geq v_{\max}\Delta} f(n+1)^2 e^{-n} < \infty. \tag{4.3.16}$$

In order to prove (4.3.3) we apply the Markov property at time $t_{n-1}$ and write

$$\mathbb{E}_\omega\left[f(\xi_n)\right] = \int d\mu^{t_{n-1}}_\omega(\omega')\, \mathbb{E}_{\omega'}\left[f(\xi_1)\right].$$

At this stage we would like to appeal to Theorem 4.3.1 to get the sought statement. However Theorem 4.3.1 only says that, for any $t$ large enough, $\mu^t_\omega$ is very close to the invariant measure $\nu$ in the interval $[-v^*t, 0]$. In order to overcome this problem, for any $\omega \in \Omega_{\rm F}$ and any $t > 0$ we define $\Phi_t(\omega) \in \Omega_{\rm F}$ as that configuration which is equal to $\omega$ in $[-v^*t, 0]$ and identically equal to $1$ elsewhere. Then, under the basic coupling, the front at time $t$ starting from $\Phi_t(\omega)$ is different from the front starting from $\omega$ iff the linking event $F(-v^*t, 0; 0, \Delta)$ occurred.

In conclusion, if $v^*t_{n-1} \geq v_{\max}\Delta$,

$$\begin{aligned}
\sup_{\omega\in\Omega_{\rm F}} \left| \int d\mu_\omega^{t_{n-1}}(\omega')\, \mathbb{E}_{\omega'}\left[f(\xi_1)\right] - \int d\mu_\omega^{t_{n-1}}(\omega')\, \mathbb{E}_{\Phi_{t_{n-1}}(\omega')}\left[f(\xi_1)\right] \right| \\
\leqslant \mathbb{P}(F(-v^*t_{n-1}, 0; 0, \Delta))^{1/2} \sup_{\omega\in\Omega_{\rm F}} \mathbb{E}_\omega\left[f(\xi_1)^2\right]^{1/2} \\
\leqslant e^{-v^*t_{n-1}/2} \sup_{\omega\in\Omega_{\rm F}} \mathbb{E}_\omega\left[f(\xi_1)^2\right]^{1/2}.
\end{aligned}$$

We can now apply Theorem 4.3.1 to get that

$$\begin{aligned}
\left| \int d\mu_\omega^{t_{n-1}}(\omega')\, \mathbb{E}_{\Phi_{t_{n-1}}(\omega')}\left[f(\xi_1)\right] - \mathbb{E}_\nu\left[f(\xi_1)\right] \right| \\
\leqslant \left[\sup_{\omega\in\Omega_{\rm F}} \|\mu_\omega^{t_{n-1}} - \nu\|^{1/2}_{[-v^*t_{n-1}, 0]} + e^{-v^*t_{n-1}/2}\right] \sup_{\omega\in\Omega_{\rm F}} \mathbb{E}_\omega\left[f(\xi_1)^2\right]^{1/2} = O(e^{-t_{n-1}^\alpha/2}).
\end{aligned}$$

To prove (4.3.4) suppose first that $v^*(j-1) \geq v_{\max}(n-j)$ where $v^*$ is the constant appearing in Theorem 4.3.1. Then we can use the Markov property at time $t_{j-1}$ and repeat the previous steps to get the result. If instead $v^*(j-1) \leqslant v_{\max}(n-j)$ it suffices to write

$$\mathrm{Cov}_\omega\left(\xi_j, \xi_n\right) = \mathrm{Cov}_\omega\left(\xi_j, \mathbb{E}_\omega[\xi_n \mid \mathcal{F}_{t_j}]\right)$$

and apply (4.3.3) to $\mathbb{E}_\omega[\xi_n \mid \mathcal{F}_{t_j}]$ to get that in this case

$$\sup_{\omega\in\Omega_{\rm F}} \left|\mathrm{Cov}_\omega\left(\xi_j, \xi_n\right)\right| = O(e^{-\gamma(n-j)^\alpha}) \tag{4.3.17}$$

for some constant $\gamma$ depending on $v^*, v_{\max}$. Following the exact steps as above after replacing $\xi_j, \xi_n$ by $f(\xi_j), f(\xi_n)$ yields (4.3.5). Finally, (4.3.6) follows from exactly the same steps leading to the proof of (4.3.3). □

## 4.4 Proofs of main results

### 4.4.1 Proof of Theorem 4.1.1

We begin with the proofs of (4.1.1) and (4.1.2).

As far as (4.1.2) is concerned, this follows directly from observing that

$$\frac{d}{dt}\mathbb{E}_\omega\left[X(\omega(t))\right] = q - p\mu_\omega^t(\omega(-1)=0) = v + O(e^{-t^\alpha}).$$

Appealing to (4.3.1) and Corollary 4.3.1 we get immediately that for any $\omega \in \Omega_{\mathrm{F}}$

$$\mathbb{E}_\omega\left[((X(\omega(t)) - vt)/t)^4\right] = O(t^{-2})$$

and (4.1.1) follows at once.

We next prove (4.1.3). Using Corollary 4.3.1 with $f(x) = x^2$, we get that, for any $n$ large enough,

$$\mathrm{Var}_\omega(\xi_n) = \mathrm{Var}_\nu(\xi_1) + O(e^{-n^\alpha}).$$

Hence

$$\lim_{t\to\infty}\frac{1}{t}\left[\sum_{n=1}^{N_t}\mathrm{Var}_\omega(\xi_n) + \mathrm{Var}_\omega(X(\omega(t)) - X(\omega(t_N)))\right] = \Delta^{-1}\,\mathrm{Var}_\nu(\xi_1).$$

Moreover, (4.3.4) implies that

$$\lim_{t\to\infty}\frac{2}{t}\left[\sum_{j<n}^{N_t}\mathrm{Cov}_\omega(\xi_j,\xi_n) + \sum_{n=1}^{N_t}\mathrm{Cov}_\omega(\xi_n, X(\omega(t)) - X(\omega(t_N)))\right]$$
$$= \frac{2}{\Delta}\sum_{n\geq 2}\mathrm{Cov}_\nu\left(\xi_1,\xi_n\right),$$

the series being absolutely convergent because of (4.3.17). In conclusion, for any $\omega \in \Omega_{\mathrm{F}}$

$$\lim_{t\to\infty} \frac{1}{t} \operatorname{Var}_\omega \left(X(\omega(t))\right) = \Delta^{-1} \left[ \operatorname{Var}_\nu(\xi_1) + 2 \sum_{n\geq 2} \operatorname{Cov}_\nu \left(\xi_1, \xi_n\right) \right]. \tag{4.4.1}$$

Next we show that for $p$ small enough the r.h.s. of (4.4.1) is positive. We first observe that there exists $c = c(p)$ such that $\limsup_{p\to 0^+} c(p) < \infty$ and

$$\sup_{\Delta} \sum_{n\geq 2} |\operatorname{Cov}_\nu \left(\xi_1, \xi_n\right)| \leqslant c(p). \tag{4.4.2}$$

To prove (4.4.2) assume without loss of generality that $\Delta \in \mathbb{N}$ and write $\xi_1 = \sum_{i=1}^{\Delta} \xi'_i$ and $\xi_n = \sum_{i=(n-1)\Delta+1}^{n\Delta} \xi'_i$, where the increments $\xi'_i$'s refer to a unit time lag. Thus

$$\sum_{n\geq 2} |\operatorname{Cov}_\nu \left(\xi_1, \xi_n\right)| \leqslant \sum_{n\geq 2} \sum_{i=1}^{\Delta} \sum_{j=(n-1)\Delta+1}^{n\Delta} |\operatorname{Cov}_\nu \left(\xi'_i, \xi'_j\right)|$$

The claim now follows from (4.3.4) together with the fact that the constants $\alpha, v^*$ are uniformly bounded away from zero as $p \to 0$.

Thus, in order to show that the r.h.s. of (4.4.1) is positive, it is enough to show that it is possible to choose $\Delta$ and $p$ such that $\operatorname{Var}_\nu\left(\xi_1\right) > \limsup c(p)$.

Recall that $q^* = \nu(\omega_{-1} = 0)$. Then a little computation shows that

$$\begin{aligned}
\frac{d}{dt} \operatorname{Var}_\nu \left(X(\omega(t))\right) &= q + pq^* - 2p \operatorname{Cov}_\nu \left(X(\omega(t)), \mathbb{1}_{\{\omega(t)\in\Omega^{**}\}}\right) \\
&\geq q + pq^* - 2p \operatorname{Var}_\nu \left(X(\omega(t))\right)^{1/2} \left(q^*(1-q^*)\right)^{1/2} \qquad (4.4.3) \\
&\geq q + pq^* - p \operatorname{Var}_\nu \left(X(\omega(t))\right)^{1/2}, \qquad (4.4.4)
\end{aligned}$$

where $\Omega^{**} = \{\omega \in \Omega^* : \ \omega_{X(\omega)-1} = 0\}$.

If $\left[\operatorname{Var}_\nu\left(\xi_1\right)\right]^{1/2} \leqslant \frac{q+pq^*}{2p}$ for all $\Delta > 0$, then (4.4.4) implies that

$$\lim_{\Delta\to\infty} \operatorname{Var}_\nu\left(\xi_1\right) = \infty.$$

Otherwise there exists $\Delta > 0$ such that $[\mathrm{Var}_\nu(\xi_1)]^{1/2} \geq \frac{q+pq^*}{2p}$; hence, the desired inequality (4.1.3) follows by taking $p$ small enough.

It remains to prove (4.1.4). If $\sigma^* = 0$, then necessarily

$$\sup_{\Delta} \mathrm{Var}_\nu(\xi_1) < \infty.$$

In this case the Chebyshev inequality suffices to prove that, for any $\omega \in \Omega_{\mathrm{F}}$,

$$(X(\omega(t)) - vt)/\sqrt{t} \xrightarrow{\mathbb{P}_\omega} 0, \quad \text{as } t \to \infty.$$

If instead $\sigma^* > 0$, we appeal to an old result on the central limit theorem for mixing stationary random fields [16]. Unfortunately our mixing result, as expressed e.g. in Corollary 4.3.1 (cf. (4.3.6)), is not exactly what is needed there and we have to go through some of the steps of [16] to prove the sought statement.

Consider the sequence $\{\xi_j\}$ defined above (with e.g. $\Delta = 1$) and let $\bar{\xi}_j := \xi_j - v\Delta$. Further let $S_n = \sum_{j=1}^n \bar{\xi}_j$. It suffices to prove that, for all $\omega \in \Omega_{\mathrm{F}}$, the law of $S_n/\sigma_*\sqrt{n}$ converges to the normal law $\mathcal{N}(0,1)$. As in [16] let $f_N(x) = \max[\min(x,N), -N]$ and let $\tilde{f}_N(x) := x - f_N(x)$. Clearly $\mathrm{Var}(\tilde{f}_N(\bar{\xi}_j)) \to 0$ as $N \to \infty$ uniformly in $j$.

Then Corollary 4.3.1 (4.3.5) implies that

$$\mathbb{E}_\omega \left[ \frac{\sum_{j=1}^n \tilde{f}_N(\bar{\xi}_j) - \mathbb{E}_\omega[\tilde{f}_N(\bar{\xi}_j)]}{n^{1/2}} \right]^2 = \frac{1}{n} \sum_{j,k=1}^n \mathrm{Cov}_\omega \left( \tilde{f}_N(\bar{\xi}_j), \tilde{f}_N(\bar{\xi}_k) \right)$$

converges to $0$ as $N \to \infty$ uniformly in $n$. Hence it is enough to prove the result for the truncated variables $f_N(\bar{\xi}_j)$. For lightness of notation we assume henceforth that the $\bar{\xi}_j$'s are bounded.

Let now $\ell_n = n^{1/3}$ and let

$$S_{j,n} = \sum_{k=1}^n \mathbb{1}_{|k-j| \leqslant \ell_n} \bar{\xi}_k, \quad \alpha_n = \sum_{j=1}^n \mathbb{E}_\omega \left[ \bar{\xi}_j S_{j,n} \right], \quad j \in \{1, \ldots, n\}.$$

The decay of covariances (4.3.4) implies that $\alpha_n = \mathrm{Var}_\omega(S_n) + o(1)$. Hence it is enough to show that $S_n/\sqrt{\alpha_n}$ is asymptotically normal. The main observation of [16], in turn inspired by the Stein method [86], is that the latter property of $S_n/\sqrt{\alpha_n}$ follows if

$$\lim_{n\to\infty} \mathbb{E}_\omega \left[ (i\lambda - S_n) e^{i\lambda \frac{S_n}{\sqrt{\alpha_n}}} \right] = 0, \quad \forall \lambda \in \mathbb{R}. \tag{4.4.5}$$

In turn (4.4.5) follows if (see [16, Eqs. (4)–(5)])

$$\lim_{n\to\infty} \mathbb{E}_\omega \Big[ \big(1 - \frac{1}{\sqrt{\alpha_n}} \sum_{j=1}^n \bar{\xi}_j S_{j,n} \big)^2 \Big] = 0 \tag{4.4.6}$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{\alpha_n}} \mathbb{E}_\omega \Big[ \Big| \sum_{j=1}^n \bar{\xi}_j \big( 1 - e^{-i\lambda \frac{S_n}{\sqrt{\alpha_n}}} - i\lambda S_{j,n} \big) \Big| \Big] = 0 \tag{4.4.7}$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{\alpha_n}} \sum_{j=1}^n \mathbb{E}_\omega \Big[ \bar{\xi}_j \, e^{i\lambda \frac{(S_n - S_{j,n})}{\sqrt{\alpha_n}}} \Big] = 0. \tag{4.4.8}$$

As in [16], the mixing properties (4.3.4) and (4.3.6) easily prove that (4.4.6) and (4.4.7) hold. As far as (4.4.8) is concerned the formulation of Theorem 4.3.1 forces us to argue a bit differently than [16]. We first observe that, using the boundedness of the variables $\bar{\xi}_j$'s, (4.4.8) is equivalent to

$$\lim_{n\to\infty} \frac{1}{\sqrt{\alpha_n}} \sum_{j=\ell_n}^n \mathbb{E}_\omega \left[ \bar{\xi}_j \, e^{i\lambda \frac{(S_n - S_{j,n})}{\sqrt{\alpha_n}}} \right] = 0, \quad \forall \lambda \in \mathbb{R}. \tag{4.4.9}$$

Fix two numbers $M$ and $L$ with $L \leqslant M/10$ (eventually they will be chosen logarithmically increasing in $n$) and write

$$\begin{aligned} e^{i\lambda \frac{(S_n - S_{j,n})}{\sqrt{\alpha_n}}} &= \sum_{m=0}^{M} \frac{(i\lambda)^m}{m!} \left( \frac{(S_n - S_{j,n})}{\sqrt{\alpha_n}} \right)^m \\ &+ \sum_{m=M+1}^{\infty} \frac{(i\lambda)^m}{m!} \left( \frac{(S_n - S_{j,n})}{\sqrt{\alpha_n}} \right)^m \mathbb{1}_{\{|\frac{(S_n - S_{j,n})}{\sqrt{\alpha_n}}| \leqslant L\}} \\ &+ \left[ e^{i\lambda \frac{(S_n - S_{j,n})}{\sqrt{\alpha_n}}} - \sum_{m=0}^{M} \frac{(i\lambda)^m}{m!} \left( \frac{(S_n - S_{j,n})}{\sqrt{\alpha_n}} \right)^m \right] \mathbb{1}_{\{|\frac{(S_n - S_{j,n})}{\sqrt{\alpha_n}}| > L\}} \\ &=: \; Y_1^{(j)} + Y_2^{(j)} + Y_3^{(j)}. \end{aligned}$$

Let us first examine the contribution of $Y_2^{(j)}$ and $Y_3^{(j)}$ to the covariance term (4.4.9). Using the boundedness of the variables $\{\bar{\xi}_j\}_{j=1}^n$ there exists a positive constant $c$ such that:

$$\begin{aligned} \frac{1}{\sqrt{\alpha_n}} \sum_{j=\ell_n}^{n} |\mathbb{E}_\omega \left[ \bar{\xi}_j \, Y_2^{(j)} \right] | &\leqslant c\sqrt{n}\, \frac{L^{M+1}}{M!}, \\ \frac{1}{\sqrt{\alpha_n}} \sum_{j=\ell_n}^{n} |\mathbb{E}_\omega \left[ \bar{\xi}_j \, Y_3^{(j)} \right] | &\leqslant c\sqrt{n} \max_j \mathbb{E}_\omega \left[ e^{2|\lambda| \frac{|S_n - S_{j,n}|}{\sqrt{\alpha_n}}} \right]^{1/2} \mathbb{P}_\omega \left( |\frac{(S_n - S_{j,n})}{\sqrt{\alpha_n}}| > L \right). \end{aligned}$$

**Lemma 4.4.1.** *There exists $c > 0$ such that, for all $n$ large enough and any $\beta = O(\log n)$,*

$$\mathbb{E}_\omega \left[ e^{\beta \frac{|S_n - S_{j,n}|}{\sqrt{\alpha_n}}} \right] \leqslant 2e^{c\beta^2}. \tag{4.4.10}$$

*Moreover, there exists $c' > 0$ such that, for all $n$ large enough and all $L \leqslant \log n$,*

$$\mathbb{P}_\omega \left( |\frac{(S_n - S_{j,n})}{\sqrt{\alpha_n}}| > L \right) \leqslant e^{-c'L^2}. \tag{4.4.11}$$

Assume for the moment the lemma and choose $L = M/10$ and $M = \log n$. We can conclude that

$$\frac{1}{\sqrt{\alpha_n}} \sum_{j=\ell_n}^{n} |\mathbb{E}_\omega \left[ \bar{\xi}_j \, (Y_2^{(j)} + Y_3^{(j)}) \right] | \leqslant C\sqrt{n} \left[ e^{-c'L^2} + \frac{L^{M+1}}{M!} \right],$$

so that

$$\lim_{n\to\infty} \frac{1}{\sqrt{\alpha_n}} \sum_{j=\ell_n}^{n} |\mathbb{E}_\omega \left[\bar{\xi}_j \, (Y_2^{(j)} + Y_3^{(j)})\right] | = 0.$$

We now examine the contribution of $Y_1^{(j)}$ to (4.4.9). Recall

$$S_n - S_{j,n} = \sum_{\substack{1 \leqslant i \leqslant n \\ |i-j|>\ell_n}} \bar{\xi}_i.$$

Thus clearly,

$$\frac{1}{\sqrt{\alpha_n}} \sum_{j=\ell_n}^{n} \mathbb{E}_\omega \left[\bar{\xi}_j \, (Y_1^{(j)})\right] = \frac{1}{\sqrt{\alpha_n}} \sum_{j=\ell_n}^{n} \sum_{m=1}^{M} \left(\frac{i\lambda}{\sqrt{n}}\right)^m \sum_{\substack{i_1,\ldots,i_m \\ \min_k |i_k - j| \geq \ell_n}} \mathbb{E}_\omega \left[\bar{\xi}_j \prod_{i=k}^{m} \bar{\xi}_{i_k}\right],$$

where the labels $i_1, \ldots i_m$ run in $\{1, 2, \ldots, n\}$.

**Lemma 4.4.2.** *Let $M = \log n$. Then, for any $m \leqslant M$, any $j \in \{\ell_n, \ldots, n\}$ and any $\{i_1, \ldots i_m\}$ satisfying $\min_k |i_k - j| \geq \ell_n$, it holds that*

$$|\mathbb{E}_\omega \Big[\bar{\xi}_j \prod_{i=k}^{m} \bar{\xi}_{i_k}\Big]| = O(e^{-n^{\alpha/6}}).$$

*Here $\alpha$ is the mixing exponent appearing in Theorem 4.3.1.*

Assuming the lemma we get immediately that also

$$\lim_{n\to\infty} \frac{1}{\sqrt{\alpha_n}} \sum_{j=\ell_n}^{n} \mathbb{E}_\omega \left[\bar{\xi}_j \, (Y_1^{(j)})\right] = 0$$

and (4.4.9) is established. In conclusion, (4.1.4) would follow from Lemmas 4.4.1–4.4.2.

*Proof of Lemma 4.4.1.* Let us begin with (4.4.10). For simplicity we prove that, for any constant $\beta = O(\log n)$, $\mathbb{E}_\omega \left[\exp(\beta S_n/\sqrt{n})\right] \leqslant e^{c\beta^2}$ for some constant $c > 0$. Similarly one could proceed for $\mathbb{E}_\omega\left[\exp(-\beta S_n/\sqrt{n})\right]$ and get that

$$\mathbb{E}_\omega\left[\exp(\beta|S_n|/\sqrt{n})\right] \leqslant \mathbb{E}_\omega\left[\exp(\beta S_n/\sqrt{n})\right] + \mathbb{E}_\omega\left[\exp(-\beta S_n/\sqrt{n})\right] \leqslant 2e^{c\beta^2}.$$

We partition the discrete interval $\{1, 2, \ldots, n\}$ into disjoints blocks of cardinality $n^{1/3}$. Given a integer $\kappa$, by applying the Cauchy-Schwarz inequality a finite number of times depending on $\kappa$, it is sufficient to prove the result for $S_n$ replaced by the sum $S_{\mathcal{B}}^{(\kappa)}$ of the $\bar{\xi}_j$'s restricted to an arbitrary collection $\mathcal{B}$ of blocks with the property that any two blocks in $\mathcal{B}$ are separated by at least $\kappa$ blocks.

Fix one such collection $\mathcal{B}$ and let $B$ be the rightmost block in $\mathcal{B}$. Let $n_B$ be the largest label in $\mathcal{B}$ which is not in the block $B$ and let $t_B = n_B\Delta$ be the corresponding time. Further let $Z_B = \sum_{j\in B}\bar{\xi}_j$. If $c\kappa > v_{\max}$ where $c$ is the constant appearing in Theorem 4.3.1, we can appeal to (4.3.6) to obtain

$$\mathbb{E}_\omega\left[\exp(\beta Z_B/\sqrt{n})\,|\,\mathcal{F}_{t_B}\right] = \mathbb{E}_\nu\left[\exp(\beta Z_B/\sqrt{n})\right] + O(e^{-n^{\alpha/3}}e^{\beta n^{-1/6}}).$$

Using the trivial bound $Z_B/\sqrt{n} = O(n^{-2/3})$ we have

$$\mathbb{E}_\nu\left[\exp(cZ_B/\sqrt{n})\right] = 1 + \frac{\beta^2}{2n}\operatorname{Var}_\nu(Z_B) + O(\beta^3 n^{-7/6})\operatorname{Var}_\nu(Z_B),$$

where $\operatorname{Var}_\nu(Z_B) = O(n^{1/3})$ thanks to (4.3.4). Above we used the trivial bound

$$\mathbb{E}_\nu\left[|Z_B|^3\right] \leqslant c\,n^{1/3}\operatorname{Var}_\nu(Z_B).$$

In conclusion, using the apriori bound $\beta \leqslant \log n$, we get that

$$\mathbb{E}_\omega\left[\exp(\beta Z_B/\sqrt{n})\,|\,\mathcal{F}_{t_B}\right] \leqslant 1 + c\frac{\beta^2}{n^{2/3}}.$$

The Markov property and a simple iteration imply that,

$$\mathbb{E}_\nu\left[\exp\left(\beta S^{(\kappa)}_{\mathcal{B}}/\sqrt{n}\right)\right] \leqslant \left[1+c\frac{\beta^2}{n^{2/3}}\right]^{|\mathcal{B}|} \leqslant \exp(c'\beta^2),$$

uniformly in the cardinality $|\mathcal{B}|$ of the collection. The bound (4.4.10) is proved.

The bound (4.4.11) follows at once from (4.4.10) and the exponential Chebyshev inequality

$$\mathbb{P}_\omega\left(|\frac{(S_n-S_{j,n})}{\sqrt{\alpha_n}}|>L\right) \leqslant e^{-\beta L}\,\mathbb{E}_\omega\left[\exp(|\frac{(S_n-S_{j,n})}{\sqrt{\alpha_n}}|)\right],$$

with $\beta=\varepsilon L$, $\varepsilon$ being a sufficiently small constant. □

*Proof of Lemma 4.4.2.* Fix $j\in[1,\ldots,n]$ and $m\leqslant \log n$, together with a choice of labels $1\leqslant i_1\leqslant i_2\leqslant \ldots \leqslant i_m\leqslant n$ such that $\min_k|i_k-j|\geq \ell_n$. Let $t_{i_k}=i_k\Delta$.

• If $i_m\leqslant j-\ell_n$ then we can apply the Markov property at time $t_{i_m}$ together with Corollary 4.3.1 to get

$$\left|\mathbb{E}_\omega\Big[\bar{\xi}_j\prod_{i=k}^{m}\bar{\xi}_{i_k}\Big]\right| \leqslant e^{-n^{\alpha/3}}\mathbb{E}_\omega\Big[\prod_{i=k}^{m}|\bar{\xi}_{i_k}|\Big] \leqslant c^m e^{-n^{\alpha/3}}.$$

• If instead there exists $b\leqslant m-1$ such that $i_b<j<i_{b+1}$ we need to distinguish between two sub-cases.

(a) For all $k\geq b+2$ it holds that $i_k-i_{k-1}\leqslant n^{1/6}$ and in particular $t_m-t_{b+1}\leqslant n^{1/6}\Delta$. In this case the fact that $t_{b+1}-j\Delta\geq \ell_n$ and $v_{\max}(t_m-t_{b+1})\ll \ell_n$ together with (4.3.6), imply that

$$\mathbb{E}_\omega\Big[\bar{\xi}_j\prod_{i=k}^{m}\bar{\xi}_{i_k}\Big]=\mathbb{E}_\omega\Big[\bar{\xi}_j\prod_{i=k}^{b}\bar{\xi}_{i_k}\Big]\left[\mathbb{E}_\nu\Big[\prod_{i=b+1}^{m}\bar{\xi}_{i_k}\Big]+O\Big(e^{-n^{\alpha/3}}\mathrm{poly}(n)\Big)\right].$$

The conclusion of the lemma then follows from the previous case $i_m\leqslant j-\ell_n$.

(b) We now assume that $k^*:=\max\{k\geq b+1:\ i_{k+1}\geq i_k+n^{1/6}\}<n$. By repeating the previous step with the Markov property applied at time $t_{i_{k^*}}$ we get

$$\mathbb{E}_\omega\Big[\bar{\xi}_j\prod_{i=k}^{m}\bar{\xi}_{i_k}\Big]=\mathbb{E}_\omega\Big[\bar{\xi}_j\prod_{i=k}^{k^*}\bar{\xi}_{i_k}\Big]\left(\mathbb{E}_\nu\Big[\prod_{i=k^*+1}^{m}\bar{\xi}_{i_k}\Big]+O\Big(e^{-n^{\alpha/3}}\mathrm{poly}(n)\Big)\right).$$

By iterating the above procedure we can reduce ourselves to case (a) and get the sought result. □

As Lemmas 4.4.1–4.4.2 imply (4.1.4), this concludes the proof of Theorem 4.1.1. □

**Remark 4.4.1.** The above proof also established that the limiting variance $\sigma_*^2$ is strictly positive for all $p$ small enough.

### *4.4.2 Proof of Theorem 4.1.2*

Given the interval $\Lambda = [1, \dots, L]$ and $\omega \in \Omega_\Lambda$, let $\mathbb{P}_\omega^{\Lambda,t}$ be the law of the process started from $\omega$. Recall that

$$\|\mathbb{P}_\omega^{\Lambda,t} - \mathbb{P}_{\omega'}^{\Lambda,t}\| = \inf\{M(\omega(t) \neq \omega'(t)) : \ M \text{ a coupling of } \mathbb{P}_\omega^{\Lambda,t} \text{ and } \mathbb{P}_{\omega'}^{\Lambda,t}\},$$

and introduce the hitting time

$$\tau(L) = \inf\{t : X(\omega(t)) = L\},$$

where the initial configuration is identically equal to one (in the sequel **1**). It is easy to check (see, e.g., [41]) that at time $\tau(L)$ the basic coupling (cf. §4.2.1) has coupled all initial configurations. Thus

$$d_{\text{TV}}(t) \leqslant \sup_{\omega,\omega'} \|\mathbb{P}_\omega^{\Lambda,t} - \mathbb{P}_{\omega'}^{\Lambda,t}\| \leqslant \mathbb{P}^\Lambda(\tau(L) > t).$$

Using the graphical construction, up to time $\tau(L)$ the East process in $\Lambda$ started from the configuration **1** coincides with the infinite East process started from the configuration $\omega^* \in \Omega_{\text{F}}$ with a single zero at the origin. Therefore

$$\mathbb{P}_{\mathbf{1}}^\Lambda(\tau(L) > t) \leqslant \mathbb{P}_{\omega^*}(X(\omega(t)) < L),$$

thus establishing a bridge with Theorem 4.1.1. Recall now the definition of $\sigma_*$ from Theorem 4.1.1 and distinguish between the two cases $\sigma_* > 0$ and $\sigma_* = 0$.

• The case $\sigma_* > 0$. Here we will show that

$$T_{\text{mix}}(L,\epsilon) = v^{-1}L + (1+o(1))\frac{\sigma_*}{v^{3/2}}\Phi^{-1}(1-\epsilon)\sqrt{L}\,. \tag{4.4.12}$$

For $s \in \mathbb{R}$, let $t_\star = L/v + s\sqrt{L}$. Then (4.1.3) implies that

$$\mathbb{P}_{\omega^*}\left(X(\omega(t_\star)) < L\right) = \mathbb{P}_{\omega^*}\Big(\frac{X(\omega(t_\star)) - vt_\star}{\sqrt{L/v}} < -v^{3/2}s\Big) \to \Phi\Big(-\frac{v^{3/2}s}{\sigma_*}\Big)$$

as $L \to \infty$. Hence,

$$\limsup_{L\to\infty} d_{\text{TV}}(L/v + s\sqrt{L}) \leqslant \Phi\Big(-\frac{v^{3/2}s}{\sigma_*}\Big). \tag{4.4.13}$$

To prove a lower bound on the total variation norm, set $a_L = \log L$ (any diverging sequence which is $o(\sqrt{L})$ would do here) and define the event

$$A_t = \big(\omega_x(t) = 1 \text{ for all } x \in (L - a_L, L]\big).$$

Then

$$\mathbb{P}^{\Lambda}_{\mathbf{1}}(A_t) \geq \mathbb{P}_{\omega^*}\left(X(\omega(t)) \leqslant L - a_L\right) \quad \text{and} \quad \pi(A_t) = p^{a_L} = o(1),$$

and so any lower bound on $\mathbb{P}_{\omega^*}(X(\omega(t_\star)) \leqslant L - a_L)$ would translate to a lower bound on $d_{\text{TV}}(t_\star)$ up to an additive $o(1)$-term. Again by (4.1.3),

$$\mathbb{P}_{\omega^*}\Big(X(\omega(t_\star)) \leqslant L - a_L\Big) = \mathbb{P}_{\omega^*}\left(\frac{X(\omega(t_\star)) - vt_\star}{\sqrt{L/v}} \leqslant -v^{3/2}s - a_L\sqrt{v/L}\right) \to \Phi\Big(-\frac{v^{3/2}s}{\sigma_*}\Big)$$

as $L \to \infty$. Thus we conclude that

$$\liminf_{L\to\infty} d_{\text{TV}}(L/v + s\sqrt{L}) \geq \Phi\Big(-\frac{v^{3/2}s}{\sigma_*}\Big). \tag{4.4.14}$$

Eq. (4.4.12) now follows from (4.4.13) and (4.4.14) by choosing $s = \sigma_* v^{-3/2}\Phi^{-1}(1-\varepsilon)$.

• The case $\sigma_* = 0$. Here a similar argument shows that

$$T_{\mathrm{mix}}(L,\epsilon) = v^{-1}L + O_\epsilon(1),$$

using the fact (following the results in §4.3) that $\sup_\omega \sup_t \mathrm{Var}_\omega(X(\omega(t))) < \infty$ if $\sigma_* = 0$.

This concludes the proof of Theorem 4.1.2. $\square$

# BIBLIOGRAPHY

[1] L. V Ahlfors, *Complex analysis: an introduction to the theory of analytic functions of one complex variable*.

[2] D. Aldous, *The asymmetric one-dimensional constrained ising model: rigorous results*, J. Stat. Phys. **107** (2002), no. 5-6, 945–975.

[3] D. Aldous, *Reversible markov chains and random walks on graphs*. In preparation, `http://www.stat.berkeley.edu/∼aldous/RWG/book.html`.

[4] D. Aldous, *Random walks on finite groups and rapidly mixing Markov chains* **986** (1983), 243–297.

[5] D. Aldous, *Shuffling cards and stopping times*, Amer. Math. Monthly **93** (1986), 333–348.

[6] G. Amir and O. Gurel-Gurevich, *On fixation of activated random walks*, Electron. Commun. Probab. (2010), no. 12, 119–123.

[7] E. D. Andjel, *Invariant measures for the zero range processes,* Ann. Probab. **10** (1982), 525–547.

[8] T. Antunović, Y. Dekel, E. Mossel, and Y. Peres, *Competing first passage percolation on random regular graphs*, arXiv preprint arXiv:1109.2575 (2011).

[9] A. Asselah and A. Gaudilliere, *From logarithmic to subdiffusive polynomial fluctuations for internal DLA and related growth models*, Annals of Probability **41** (2013), no. 3A, 1115–1159.

[10] A. Asselah and A. Gaudillière, *Lower bounds on fluctuations for internal DLA*, Probability Theory and Related Fields **158** (2014), no. 1-2, 39–53.

[11] R. F Bass, *Brownian motion, heat kernels, and harmonic functions*, Proceedings of the international congress of mathematicians, 1995, pp. 980–985.

[12] M. Basu, U. Basu, S. Bandyopadhyay, P. K. Mohanty, and H. Hinrichsen, *Fixed-energy sandpiles belong generically to directed percolation*, Phys. Rev. Lett. **109** (2012), 44–48.

[13] R. Basu, S. Ganguly, and C. Hoffman, *Non-fixation of symmetric activated random walk on the line for small sleep rate*, http: (2015).

[14] H. Begehr, *Boundary value problems in complex analysis ii*, Boletın de la Asociación Matemática Venezolana (2005), 217.

[15] O. Blondel, *Front progression for the east model*, Stochastic Process. Appl. **123**.

[16] E. Bolthausen, *On the central limit theorem for stationary mixing random fields*, Ann. Probab. **10** (1982), no. 4, 1047.

[17] B. Bond and L. Levine, *Abelian networks i. foundations and examples*, arXiv preprint arXiv:1309.3445 (2013).

[18] M. Bramson and J. L Lebowitz, *Asymptotic behavior of densities in diffusion-dominated annihilation reactions*, Physical review letters **61** (1988), no. 21, 2397.

[19] M. Bramson and J. L Lebowitz, *Asymptotic behavior of densities for two-particle annihilating random walks*, Journal of statistical physics **62** (1991), no. 1-2, 297–372.

[20] K. Burdzy and Z.-Q. Chen, *Discrete approximations to reflected Brownian motion*, Annals of Probability **36** (2008), no. 2, 698–727.

[21] M. Cabezas, L. T. Rolla, and V. Sidoravicius, *Recurrence and density decay for diffusion limited annihilating systems* (2013). Preprint, arXiv:1309.4387.

[22] M. Cabezas, L. T. Rolla, and V. Sidoravicius, *Non-equilibrium phase transitions: Activated random walks at criticality*, Journal of Statistical Physics **155** (2014), no. 6, 1112–1125.

[23] N. Cancrini, *Kinetically constrained spin models*, Probab. Theory Related Fields **140** (2008), no. 3-4, 459–504.

[24] Z.-Q. Chen, *On reflecting diffusion processes and skorokhod decompositions*, Probability theory and related fields **94** (1993), no. 3, 281–315.

[25] Z.-Q. Chen and W.-T. Fan, *Hydrodynamic limits and propagation of chaos for interacting random walks in domains*, arXiv preprint arXiv:1311.2325 (2013).

[26] P. Chleboun, *Time scale separation and dynamic heterogeneity in the low temperature East model*.

[27] D. Dhar, *Self-organized critical state of sandpile automaton models*, Physical Review Letters **64** (1990), no. 14, 1613.

[28] D. Dhar, *The abelian sandpile and related models*, Physica A **263** (1999), 4–25.

[29] P. Diaconis and W. Fulton, *A growth model, a game, an algebra, Lagrange inversion, and characteristic classes*, Rend. Sem. Mat. Univ. Politec. Torino **49** (1991), no. 1, 95–119.

[30] P. Diaconis, *The cutoff phenomenon in finite Markov chains*, Proc. Nat. Acad. Sci. U.S.A.

[31] P. Diaconis, *Generating a random permutation with random transpositions*, Z. Wahrsch. Verw. Gebiete.

[32] P. Diaconis, *Strong stationary times via a new form of duality*, Ann. Probab.

[33] P. Diaconis, *The cutoff phenomenon in finite markov chains*, Proceedings of the National Academy of Sciences **93** (1996), no. 4, 1659–1664.

[34] P. Diaconis and W. Fulton, *A growth model, a game, an algebra, lagrange inversion, and characteristic classes*, Rend. Sem. Mat. Univ. Politec. Torino **49** (1991), no. 1, 95–119.

[35] R. Dickman, *Nonequilibrium phase transitions in epidemics and sandpiles*, Physica A **306** (2002), 90–97.

[36] R. Dickman, M. Alava, M. A. Muñoz, J. Peltola, A. Vespignani, and S. Zapperi, *Critical behaviour of a one-dimensional fixed-energy sandpile*, Phys. Rev. E **64** (2001), 56104.

[37] R. Dickman, M. A. Muñoz, A. Vespignani, and S. Zapperi, *Paths to self-organized criticality*, Braz. J. Phys. **30** (2000), 27.

[38] R. Dickman, L. T. Rolla, and V. Sidoravicius, *Activated random walkers: Facts, conjectures and challenges*, Journal of Statistical Physics **138** (2010), no. 1-3, 126–142.

[39] H. Duminil-Copin, C. Lucas, A. Yadin, and A. Yehudayoff, *Containing internal diffusion limited aggregation*, arXiv preprint arXiv:1111.0486 (2011).

[40] K. Eriksson, *Chip-firing games on mutating graphs*, SIAM Journal on Discrete Mathematics **9** (1996), no. 1, 118–128.

[41] A. Faggionato, *The east model: recent results and new progresses*.

[42] W.-T. Fan, *Interacting particle systems with partial annihilation through membranes*, Ph.D. Thesis, 2014.

[43] G. Fayolle, V. A. Malyshev, and M. V. Menshikov, *Topics in the constructive theory of countable markov chains*, Cambridge university press, 1995.

[44] G. H. Fredrickson, *Kinetic ising model of the glass transition*, Phys. Rev. Lett. **53** (1984), no. 13, 1244–1247.

[45] S Ganguly, L Levine, Y Peres, and J Propp, *Formation of an interface by competitive erosion*, arxiv preprint, arxiv:1501.03584 (2015).

[46] S. Ganguly, E. Lubetzky, and F. Martinelli, *Cutoff for the east process*, Communications in Mathematical Physics **335** (2015), no. 3, 1287–1322.

[47] S. Ganguly and Y. Peres, *Competitive erosion is conformally invariant*, arXiv preprint arXiv:1503.06989 (2015).

[48] S. Ganguly and Y. Peres, *Convergence of discrete green functions with neumann boundary conditions*, arXiv preprint arXiv:1503.06948 (2015).

[49] S. Griffiths, R. Kang, R. Oliveira, and V. Patel, *Tight inequalities among set hitting times in markov chains*, Proceedings of the American Mathematical Society **142** (2014), no. 9, 3285–3298.

[50] O. Häggström, R. Pemantle, et al., *First passage percolation and a model for competing spatial growth*, Journal of Applied Probability **35** (1998), no. 3, 683–692.

[51] J. Jäckle, *A hierarchically constrained kinetic ising model*, Zeitschrift fr Physik B Condensed Matter **84** (1991), no. 1.

[52] J. Jäckle, *A hierarchically constrained kinetic ising model*, Zeitschrift fr Physik B Condensed Matter **84** (1991), no. 1.

[53] K. Jain, *Simple sandpile model of active-absorbing phase transitions*, Phys. Rev. E **72** (2005), 017105.

[54] D. Jerison, L. Levine, and S. Sheffield, *Logarithmic fluctuations for internal DLA*, Journal of the American Mathematical Society **25** (2012), no. 1, 271–301.

[55] D. Jerison, L. Levine, and S. Sheffield, *Internal DLA and the gaussian free field*, Duke Mathematical Journal **163** (2014), no. 2, 267–308.

[56] R. Kenyon, *Conformal invariance of domino tiling*, Annals of probability (2000), 759–795.

[57] R. Kenyon, *Dominos and the gaussian free field*, Annals of probability (2001), 1128–1137.

[58] R. Kenyon, *Conformal invariance of loops in the double-dimer model*, Communications in Mathematical Physics **326** (2014), no. 2, 477–497.

[59] H. Kesten and V. Sidoravicius, *Branching random walk with catalysts*, Elec. J. Probab. **8** (2003), 1–51.

[60] H. Kesten and V. Sidoravicius, *The spread of a rumor or infection in a moving population*, Ann. Probab. **33** (2005), 2402–2462.

[61] H. Kesten and V. Sidoravicius, *A phase transition in a model for the spread of an infection*, Illinois J. Math. **50** (2006), 547–634.

[62] H. Kesten and V. Sidoravicius, *A shape theorem for the spread of an infection*, Ann. Math. **167** (2008), 701–766.

[63] G. F Lawler, M. Bramson, and D. Griffeath, *Internal diffusion limited aggregation*, Annals of Probability (1992), 2117–2140.

[64] G. F Lawler, O. Schramm, and W. Werner, *Conformal invariance of planar loop-erased random walks and uniform spanning trees*, Selected works of oded schramm, 2011, pp. 931–987.

[65] S. B. Lee, *Comment on "Fixed-energy sandpiles belong generically to directed percolation"*, Phys. Rev. Lett. **110** (2013).

[66] D. A. Levin, *Markov chains and mixing times*, American Mathematical Society. With a chapter by James G. Propp and David B. Wilson.

[67] T. M. Liggett, *Interacting particle systems*, Springer, New York, USA, 1985.

[68] E. Lubetzky and A. Sly, *Cutoff for the ising model on the lattice*, Inventiones mathematicae **191** (2013), no. 3, 719–755.

[69] R. Lyons and Y. Peres, *Probability on trees and networks*, Citeseer, 2005.

[70] A. Lytova and L. Pastur, *Central limit theorem for linear eigenvalue statistics of random matrices with independent entries*, Ann. Probab. **37** (2009), no. 5, 1778–1840. MR2561434 (2011c:60024)

[71] S. S. Manna, *Large-scale simulation of avalanche cluster distribution in sand pile model*, J. Stat. Phys. **59** (1990), 509–521.

[72] S. S. Manna, *Two-state model of self-organized criticality*, J.Phys. A, Math. Gen. **24** (1991), L363–L369.

[73] J. Marro and R. Dickman, *Nonequilibrium phase transitions in lattice models*, Cambridge University Press, 1999.

[74] C. Pommerenke, *Boundary behaviour of conformal maps*, Springer, 1992.

[75] J Propp, *Private communication* (2003).

[76] L. T Rolla, *Activated random walks*, arXiv preprint arXiv:1507.04341 (2015).

[77] L. T. Rolla and V. Sidoravicius, *Absorbing-state phase transition for driven-dissipative stochastic dynamics on* $\mathbb{Z}$ (2011). arXiv:0908.1152v2.

[78] L. T Rolla and V. Sidoravicius, *Absorbing-state phase transition for driven-dissipative stochastic dynamics on*, Inventiones mathematicae **188** (2012), no. 1, 127–150.

[79] L. T. Rolla, *Activated random walks* (2015). Lecture Notes, arXiv:1507.04341.

[80] L. T. Rolla and L. Tournier, *Sustained activity for biased activated random walks at arbitrarily low density* (2015). Preprint, arXiv:1507.04732.

[81] M. Rossi, R. Pastor-Satorras, and A. Vespignani, *Universality class of absorbing phase transitions with a conserved field*, Phys. Rev. Lett. **85** (2000), 1803–1806.

[82] L. Saloff-Coste, *Lectures on finite markov chains*.

[83] E. Shellef, *Nonfixation for activated random walks*, ALEA **7** (2010), 137149.

[84] V. Sidoravicius and A. Teixeira, *Absorbing-state transition for stochastic sandpiles and activated random walks* (2014). Preprint, arXiv 1412.7098.

[85] S. Smirnov, *Towards conformal invariance of 2d lattice models*, arXiv preprint arXiv:0708.0032 (2007).

[86] C. Stein, *A bound for the error in the normal approximation to the distribution of a sum of dependent random variables* (1972), 583–602.

[87] L. Taggi, *Absorbing-state phase transition in biased activated random walk*, 2014. Preprint, arXiv 1403.1986.

[88] Á. Timár, *Boundary-connectivity via graph theory*, Proceedings of the American Mathematical Society **141** (2013), no. 2, 475–480.