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Formal group laws and hypergraph colorings

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Abstract

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This thesis demonstrates a connection between formal group laws and chromatic symmetric functions of hypergraphs, two seemingly unrelated topics in the theory of symmetric functions. A formal group law is a symmetric function of the form

$$f(f^{-1}(x_1) + f^{-1}(x_2) + \cdots) \tag{1}$$

for a formal power series $f(x) \in \mathbb{Q}[[x]]$. A *hypergraph* H is a generalized graph where edges can contain more than two vertices, and a coloring χ of H is *proper* if no edge of H is monochromatic under χ . The *chromatic symmetric function* X_H of hypergraph H on a vertex set V is a sum of monomials corresponding to proper colorings of H .

Our main result gives a combinatorial interpretation for certain formal group laws. In particular, we show that if $f(x)$ is a generating function for a type of combinatorial object with a certain recursive structure then the formal group law (1) is a sum of chromatic symmetric functions. For example, this occurs when $f(x)$ is the generating function for trees, permutations, lattice paths or graphs. We develop a unifying framework by defining *contractible species*, classes of combinatorial objects having generating functions with this property. This will also allow us to give combinatorial interpretations to products of polynomials in some well-known families of polynomials, such as the Bell, Laguerre and Hermite polynomials. Finally, we observe that many of the hypergraphs arising from formal group

laws are special hypergraphs called *hypertrees*, and we show that the chromatic symmetric functions of hypertrees are positive in Gessel's fundamental quasisymmetric functions when they have prime-sized edges.

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Chapter 1

INTRODUCTION

An n -dimensional *formal group law* over a ring R is a tuple of formal power series $F_1, F_2, \dots, F_n \in R[[x_1, \dots, x_n, y_1, \dots, y_n]]$ so that

- $F(x, 0) = x, F(0, y) = y$
- $F(F(x, y), z) = F(x, F(y, z))$.

Here we write x for (x_1, \dots, x_n) , so that

$$F(F(x, y), z) = F(F_1(x, y), F_2(x, y), \dots, F_n(x, y), z_1, \dots, z_n),$$

etc.

Formal group laws were first defined by S. Bochner in 1946 [9], calling them *formal Lie groups*. Formal group laws have a number of applications to class field theory and algebraic topology. In particular, the Quillen [37] determined that the *Lazard ring*, the ring of coefficients of formal group laws, is isomorphic to the coefficient ring of complex cobordism. We refer the reader to Hazewinkel's book [29] for more information on formal group laws, but no background in this area is necessary for what follows.

Given a Lie group G and a chart $f : U \rightarrow \mathbb{R}^n$ defined on a neighborhood U of the identity e of G so that $f(e) = 0$, the multiplication \cdot on G induces a map $F : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ given by

$$F(x, y) = f(f^{-1}(x) \cdot f^{-1}(y)). \tag{1.1}$$

If $f(x)$ is analytic then $F(x, y)$ becomes a formal group law when expanded in a power series about the origin. For example, if $G = \mathbb{R}_{>0}$ the usual multiplication of real numbers, then we

can set f to be the chart $f(u) = u - 1$. Then

$$f(f^{-1}(x) \cdot f^{-1}(y)) = (x + 1)(y + 1) - 1 = x + y + xy, \quad (1.2)$$

the *multiplicative* formal group law.

One motivation for this construction is to give a natural intermediary between the theories of Lie groups and Lie algebras. In fact, the Lie algebra \mathfrak{g} of G is completely determined from the formal group law F of G . The Lie bracket on \mathbb{R}^n is given by $[x, y] = B(x, y) - B(y, x)$ where $B(x, y)$ is the degree-2 homogeneous component of F .

If $G = (\mathbb{R}, +)$ is the real line with addition as the group operation, the formal group law (1.1) becomes

$$F(x, y) = f(f^{-1}(x) + f^{-1}(y)). \quad (1.3)$$

In fact, it is known that if $F(x, y)$ is any one-dimensional formal group law over a field of characteristic 0 that is commutative, meaning that $F(x, y) = F(y, x)$, it is known [40, IV.5] that $F(x, y)$ may be written in the form (1.3) for some power series $f(x) = x + ax^2 + \dots$. For example, if $F(x, y) = x + y + xy$ is the multiplicative formal group law (1.2) we have

$$F(x, y) = f(f^{-1}(x) + f^{-1}(y))$$

where $f(x) = e^x - 1$, $f^{-1}(x) = \log(1 + x)$. For this reason $f^{-1}(x)$ in (1.3) is sometimes called the *logarithm* of F .

Motivated by the expression (1.3) for a formal group law, we consider the power series in infinitely many variables x_1, x_2, \dots of the form

$$F(x_1, x_2, \dots) = f(f^{-1}(x_1) + f^{-1}(x_2) + \dots) \quad (1.4)$$

and from this point on we define formal group laws to be formal power series of the form (1.4) in infinitely many variables x_1, x_2, \dots . The purpose of this thesis is to address the following question.

Main question. Suppose that $f(x)$ is an exponential generating function, so that

$$f(x) = \sum_{n=1}^{\infty} a_n \frac{x^n}{n!}$$

where $a_n = |A_n|$ is the cardinality of a set of labeled combinatorial objects $\sigma \in A_n$. Under what circumstances is there a combinatorial interpretation to the formal group law

$$f(f^{-1}(x_1) + f^{-1}(x_2) + \cdots)?$$

Note that the power series inversion $f^{-1}(x)$ will always have some negative coefficients unless $f(x) = x$. So in general, we would not expect the coefficients of the formal group law $F(x_1, x_2, \dots)$ to be positive. In many cases, however, it turns out that the formal group law does have nonnegative coefficients, and then we may ask if the coefficients have combinatorial meaning.

A *symmetric function* is a power series in x_1, x_2, \dots that is invariant under any finite permutation of the variables. Thus any formal group law $f(f^{-1}(x_1) + f^{-1}(x_2) + \cdots)$ is a symmetric function. Let **Sym** be the set of all symmetric functions of bounded degree over the field \mathbb{Q} . Then **Sym** is a ring with the usual addition and multiplication of multivariate power series. A *partition* is a tuple $\lambda_1, \lambda_2, \dots, \lambda_l$ with $\lambda_1 \geq \dots \geq \lambda_l$. For each partition λ we associate the *monomial symmetric function* m_λ to be the sum of all distinct monomials $x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_l}^{\lambda_l}$. The m_λ 's form a basis of the ring **Sym**, so that the n -th graded component of **Sym** has dimension $p(n)$, the number of partitions of n . The *power sum* symmetric functions

$$p_n = x_1^n + x_2^n + \cdots \tag{1.5}$$

form an algebraically independent set of generators for the ring of symmetric functions, so that **Sym** is in fact a polynomial ring. Formal group laws $f(f^{-1}(x_1) + f^{-1}(x_2) + \cdots)$ are symmetric functions, but they do not have bounded degree in general so they are not necessarily elements of **Sym**. However, the degree- n homogeneous component of a formal group law $F(x_1, \dots, x_n)$ is an element of **Sym** for each n .

Symmetric functions have been an object of much interest since at least the works of Girard [26] and Newton [35] in the 17th century. In the 20th century, symmetric functions were shown to be invaluable in the nascent field of *algebraic combinatorics*. In particular, the *Schur functions* $s_\lambda \in \mathbf{Sym}$, which can be defined as sums of monomials corresponding to certain combinatorial objects called semi-standard Young tableaux, were found to be intimately connected with the representation theory of the symmetric and general linear groups. Much more information on symmetric functions and related algebra and combinatorics can be found in the books by Macdonald [33] or Stanley [46, Chapter 7]. A background in this field is not, however, necessary to understand this thesis.

We will find that the correct interpretation of the coefficients in a formal group law often involves the language of *graph coloring*. A *graph* G is a pair (V, E) where E is a set of pairs $\{u, v\}$ of distinct $u, v \in V$ called the edges of G . A *hypergraph* (sometimes called a *set system*) is a pair $H = (V, E)$ where E is any family of nonempty subsets of V also called the *edges* of H . Thus unlike a graph, the edges of a hypergraph may have any number of vertices. We say that H is *connected* if V is not a disjoint union $V = V_1 \cup V_2$ of nonempty sets V_1, V_2 so that each edge $e \in E$ is contained in either V_1 or V_2 . Every hypergraph H is the disjoint union of connected hypergraphs called the *connected components* of H .

A *coloring* of a hypergraph H is a map $\chi : V \rightarrow \mathbb{P} = \{1, 2, 3, \dots\}$. If e is an edge of H and χ is a coloring of H so that every element of e is given the same color, we say that e is *monochromatic*, and we say that χ is *proper* if no edge $e \in E$ with $|e| > 1$ is monochromatic. Note that in our definition, hypergraphs are allowed to have singleton edges, but they have no effect on the proper colorings of H . There is an enormous literature on hypergraph coloring problems. We direct the reader to the books [6, 10, 52] for much more information.

In the case where each edge $e \in E$ has $|e| = 2$, so H is an ordinary graph, we recover the usual notion of proper graph colorings: no two adjacent vertices may be colored the same. One might expect that we would define proper colorings so that each vertex in an edge must be colored differently. However, these colorings are the same as the colorings of the ordinary graph G given by replacing each edge of H with a clique in G , so nothing new would be

gained by considering hypergraphs under that definition.

The *chromatic symmetric function* X_H of a hypergraph H with vertex set V is defined to be

$$X_H(x_1, x_2, \dots) = \sum_{\chi} \prod_{v \in V} x_{\chi(v)}$$

with the sum taken over all proper colorings χ of H . Thus X_H is indeed a symmetric function, since permuting the colors of a proper coloring gives another proper coloring. The chromatic symmetric function X_G of an ordinary graph G was introduced by Stanley [43], who later generalized the notion to hypergraphs [45]. Chromatic symmetric functions of graphs have been an object of much recent study [3, 19, 34, 39]. In particular, Gasharov [18] showed that chromatic symmetric functions of certain graphs are positive in the basis of Schur functions s_{λ} . Chromatic symmetric functions of hypergraphs are studied in [28, 32].

Note that some edges of H may be redundant. If $e_1, e_2 \in E$ with $e_1 \subsetneq e_2$, and $|e_1| > 1$, then we may delete the edge e_2 from H without changing the set of proper colorings since any coloring of V that is not monochromatic on e_1 will also not be monochromatic on e_2 . Thus we may replace the set E by the set M of *minimal, non-singleton* elements of E to form the *reduced* hypergraph denoted by $\text{red } H = (V, M)$, and then $X_H = X_{\text{red } H}$. Nevertheless, it will be convenient in what follows to retain the flexibility of allowing non-minimal edges.

Chromatic symmetric functions are difficult to compute in general. It is known that determining whether a proper coloring of a hypergraph with a given number of colors exists is NP-hard [13]. However, there is a simple formula to compute the chromatic symmetric function X_H in terms of the power sum symmetric functions (1.5) that is effective if the hypergraph is small. A proof for ordinary graphs was given by Stanley [43, Theorem 2.5], who later extended it to hypergraphs [45, Theorem 3.2]. The proof is an application of the inclusion-exclusion principle.

Theorem 1.1. *Let $H = (V, E)$ be a hypergraph. Then*

$$X_H = \sum_{S \subseteq E} (-1)^{|S|} p_{k_1} p_{k_2} \cdots p_{k_n}$$

where k_1, \dots, k_n are the sizes of the connected components of the hypergraph $H_S = (V, S)$ that has only the edges $e \in S$.

The subject of this thesis is the exploration of a surprising connection between formal group laws and chromatic symmetric functions. If $f(x) = \sum_{n=1}^{\infty} a_n \frac{x^n}{n!}$ is an exponential generating function, then formal group law

$$f(f^{-1}(x_1) + f^{-1}(x_2) + \dots)$$

can frequently be interpreted in terms of chromatic symmetric functions of hypergraphs. Supposing that $a_n = |A_n|$ where A_n is some set of combinatorial structures on the n vertices $\{1, 2, \dots, n\}$, we will see (Theorem 1.3) that in many cases

$$f(f^{-1}(x_1) + f^{-1}(x_2) + \dots) = \sum_{n=1}^{\infty} \sum_{\sigma \in A_n} \frac{X_{H_\sigma}}{n!}$$

where each H_σ is a hypergraph associated with a particular combinatorial object $\sigma \in A_n$.

For example, let A_n be the set of trees of rooted trees with leaves labeled $1, 2, \dots, n$ and all other nodes unlabeled, where no node has exactly one child. (If we allow a node to have exactly one child then A_n becomes infinite.) Fig 1.1 depicts a tree $T \in A_7$. Left-and-right children are not distinguished, so we draw them so that they are increasing left-to-right. These trees are also known as *total partitions*. The sequence $|A_n|$ counting the number of

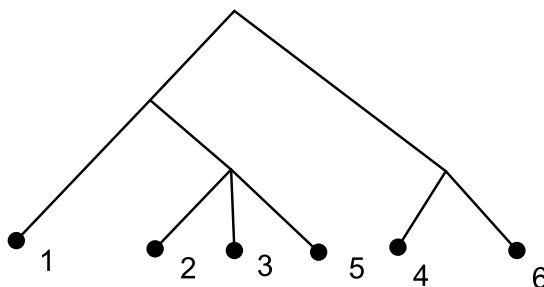


Figure 1.1: A tree with labeled leaves.

total partitions is

$$1, 1, 4, 26, 236, 2752, 39208, 660032, 12818912, \dots \text{ (A000311 in Sloane's [41])}.$$

Given any tree $T \in A_n$, form a hypergraph $H_T = ([n], E(T))$ where the edges $U \in E(T)$ are the set of vertices that form *subtrees* within T . That is, $U \subseteq [n]$ is in $E(T)$ if U consists of the set of all leaves that descended from some fixed node. For example, if T is the tree depicted in Fig. 1.1 then we have (omitting brackets and commas for the edges)

$$E(T) = \{1, 2, 3, 4, 5, 6, 235, 46, 1235, 123456\}.$$

while the minimal, non-singleton edges form the set

$$M(T) = \{235, 46\}.$$

In general, the edges $U \in M(T)$ are the sets of leaves that form a complete set of siblings, all of which are children of the same parent. Thus, X_{H_T} is the sum of monomials corresponding to colorings of the leaves of T so that no subtree is monochromatic, or equivalently that no complete set of siblings is monochromatic.

We now present a surprising fact, which will serve as a prototypical application of our main result. The next theorem is essentially due to Lenart and Ray [31], although it is written there in a different form.

Theorem 1.2. *Let A_n be the set of trees with leaves labeled $\{1, 2, \dots, n\}$ -and let $f(x) = \sum_{n=1}^{\infty} a_n \frac{x^n}{n!}$ where $a_n = |A_n|$. Then*

$$f(f^{-1}(x_1) + f^{-1}(x_2) + \dots) = \sum_{n=1}^{\infty} \sum_{T \in A_n} \frac{X_{H_T}}{n!}. \quad (1.6)$$

The key to our proof of Theorem 1.2 is the recursive structure of trees. Given trees T with the leaf-set V and S with the leaf-set U disjoint from V , there is an obvious way to *insert* S into any leaf $v \in V$ of T to form a new tree $R = T(v \leftarrow S)$. This is illustrated in Figure 1.2. Furthermore, the tree T can be recovered from $R = T(v \leftarrow S)$ if the set of leaves

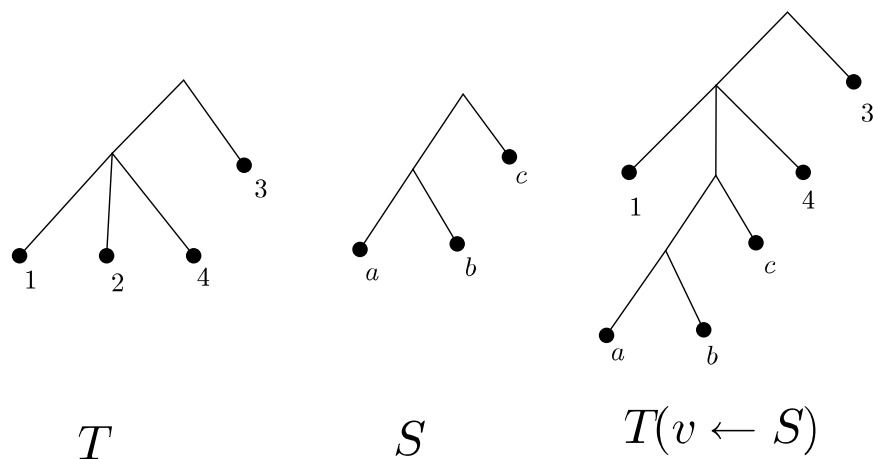


Figure 1.2: Trees T , S , and $R = T(v \leftarrow S)$ where $v = 2$.

U of S is known: simply replace all of S with the single leaf v . We write $T = R(U \rightarrow v)$ and say that \rightarrow is the operation of *contraction*. Similarly, S can be recovered from R by taking S to be the subtree with leaves in U . We write $S = R|_U$ and say that S is the *restriction* of T to U .

In fact, this idea is applicable to several different kinds of combinatorial objects which have similar structure — trees of different kinds, permutations, graphs, lattice paths, and more. In each case, we can define operations of insertion, contraction and restriction that have similar properties to the case of trees - and in each case we will see that a theorem similar to Theorem 1.2 holds. The formal group law $f(f^{-1}(x_1) + f^{-1}(x_2) + \dots)$ corresponding to the generating function $f(x)$ can be written in terms of chromatic symmetric functions of hypergraphs. We will describe a broad generalization of Theorem 1.2 that encompasses all of these examples simultaneously.

The generalization is most easily given using Joyal's theory of *species* [30]. Roughly speaking, a *species* \mathcal{F} is a class of combinatorial objects. A species \mathcal{F} assigns to every finite set V a set $\mathcal{F}[V]$ of combinatorial objects on the set V . For example, we could take $\mathcal{F}[V]$ to be the set of trees with the leaves labeled by V .

In this thesis we define a new class of species which we call *contractible species*. A contractible species is a species \mathcal{F} equipped with an operation of *insertion* that behaves similarly to the insertion for trees that we have described. That is, any \mathcal{F} -structure $\tau \in \mathcal{F}[U]$ may be inserted into any vertex v of another object $\sigma \in \mathcal{F}[V]$ to form a new \mathcal{F} -structure $\sigma(v \leftarrow \tau)$ in such a way that both σ and τ can be recovered from $\sigma(v \leftarrow \tau)$. Figures 3.2, 3.3 and 3.4 in Sections 3.3, 3.5, and 3.6, respectively, are depictions of insertion for various contractible species.

When \mathcal{F} is a contractible species, there will be a hypergraph $H_\sigma = (V, E(\sigma))$ canonically associated with each combinatorial object $\sigma \in \mathcal{F}[V]$. The edges $U \in E(\sigma)$ are, roughly speaking, the vertex sets of *sub-objects* that may be contracted to a point, much like the subtrees in the example of trees with labeled leaves. Our main theorem, answering our main question, says that there is an identity of the form (1.6) for each contractible species.

Theorem 1.3. *Suppose that \mathcal{F} is a contractible species. Let $a_n = |\mathcal{F}[n]|$, the number of \mathcal{F} -structures with labels in $[n]$, and let $f(x) = \sum_{n=1}^{\infty} a_n \frac{x^n}{n!}$ be the exponential generating function for \mathcal{F} . Then*

$$f(f^{-1}(x_1) + f^{-1}(x_2) + \dots) = \sum_{n=1}^{\infty} \sum_{\sigma \in \mathcal{F}[n]} \frac{X_{H_\sigma}}{n!}$$

where each H_σ is a hypergraph associated with $\sigma \in \mathcal{F}[n]$. Furthermore, there is an explicit combinatorial interpretation for $f^{-1}(x)$.

In addition to the trees $T \in A_n$, Theorem 1.3 applies to a number of well-known combinatorial sequences $\{a_n\}$. For example

- $a_n = n^{n-1}$ (rooted trees with all nodes labeled, Section 3.3)
- $a_n = n!$ (permutations, Section 3.4)
- $a_n = 2^{\binom{n}{2}}$ (labeled graphs, Section 3.6).

We also consider *ordinary* generating functions

$$f(x) = \sum_{n=1}^{\infty} a_n x^n$$

and give a version of Theorem 1.3 in this context. Here we can take

- $a_n = C_{n+1}$, where C_n is the n -th *Catalan* number (binary trees, Section 6.2)
- $a_n = M_{n+1}$, where M_n is the n -th *Motzkin* number (lattice paths, Section 6.4)
- $a_n = n!$ (the ordinary generating function for permutations, Section 6.5)

and others. We give a list of all the contractible species mentioned in this thesis in Appendix A.

In Chapter 2 we will give the precise definition of contractible species. Before giving the proof of Theorem 1.3, we will give a number of examples of contractible species in Section 3 as motivation. In Chapter 4, we give the proof of Theorem 1.3.

In Chapter 5 we consider the *Cartesian product* of contractible species, corresponding to the *Hadamard product*

$$\sum_{n=1}^{\infty} a_n b_n \frac{x^n}{n!}$$

of generating functions $f(x) = \sum_{n=1}^{\infty} a_n \frac{x^n}{n!}$, $f_2(x) = \sum_{n=1}^{\infty} b_n \frac{x^n}{n!}$. This allows us to consider formal group laws corresponding to sequences like $2^{\binom{n}{2}}$, $n!^k$, etc. In particular, we describe a contractible species corresponding to *labeled tanglegrams*, pairs of binary trees on the same set of leaves.

We then can give some applications of Theorem 1.3. The examples in Chapter 3 are exponential generating functions $\sum a_n \frac{x^n}{n!}$, but in Chapter 6 we consider *ordinary* generating functions $f(x) = \sum a_n x^n$. To do so, we define *contractible \mathbb{L} -species* and give a number of examples.

In Chapter 7, we examine the closely-related subject of *associated polynomial sequences*, the polynomials $p_n(t)$ defined by the generating function

$$\sum_{n=0}^{\infty} p_n(t)x^n = e^{tf^{-1}(x)}. \quad (1.7)$$

Using umbral methods it is possible to extract the coefficients of monomials in a formal group law

$$f(f^{-1}(x_1) + f^{-1}(x_2) + \dots)$$

from the polynomials $p_n(t)$ defined by (1.7), and conversely the *linearization coefficients* c_{ij}^k defined by

$$p_i(t)p_j(t) = \sum_k c_{ij}^k p_k(t)$$

are determined by the formal group law. We will see that Theorem 1.3 then gives combinatorial interpretations to products of polynomials in a number of well-known polynomial sequences of binomial type, including the Bell, Hermite and Laguerre polynomials.

In Chapter 8, we use associated polynomial sequences to extract asymptotics for the coefficients of monomials in formal group laws. This allows us to determine, for example, the probability that a random coloring of a random \mathcal{F} -structure is proper when \mathcal{F} is a contractible species.

In Chapter 9 we offer some directions for future research. In particular, we give, conjecturally, a necessary and sufficient condition for a formal group law to have nonnegative coefficients. We also conjecture that a certain class of hypergraphs is Schur-positive. In Chapter 10 we give some theoretical evidence toward this conjecture by showing that a larger class of hypergraphs, the *hypertrees* with prime-sized edges, are positive in Gessel's fundamental quasisymmetric functions.

Note that some of the material on formal group laws for ordinary generating function appeared in [50]. Chapter 10 was taken from [49].

Chapter 2

DEFINITION OF CONTRACTIBLE SPECIES

2.1 Definition of species

Before turning to our main object of interest, *contractible species*, we must give the necessary background by briefly discussing the theory of *species*. Species were first defined by André Joyal [30], giving a rigorous modern treatment of the manipulations of generating functions that are often performed by combinatorialists with a bit of hand-waving.

A *species* \mathcal{F} is an operation endowing a finite set with extra structure. That is, \mathcal{F} assigns to each finite set V a finite set $\mathcal{F}[V]$ of “ \mathcal{F} -structures” on the set V . For example, we may let \mathcal{F} denote the species of *graphs*, taking $\mathcal{F}[V]$ to be the set of all ordinary graphs $G = (V, E)$ with vertex set V . We could also let $\mathcal{F}[V]$ be the set of all permutations $\pi : V \rightarrow V$, or the set of all posets P on V , etc.

We require that a species \mathcal{F} be *functorial*. That is, for every bijection $\phi : U \rightarrow V$ between finite sets there is an associated *transport of structure* $\mathcal{F}[\phi] : \mathcal{F}[U] \rightarrow \mathcal{F}[V]$ that maps every \mathcal{F} -structure $\sigma \in \mathcal{F}[U]$ on the vertex set U to an equivalent \mathcal{F} -structure $\mathcal{F}[\phi](\sigma) \in \mathcal{F}[V]$ with the vertex set V . For example, with \mathcal{F} the species of graphs as before, we define $\mathcal{F}[\phi] : \mathcal{F}[U] \rightarrow \mathcal{F}[V]$ by setting $\mathcal{F}[\phi](g)$ to be the graph $g' \in \mathcal{F}[V]$ with the edges $\{\phi(x), \phi(y)\}$ for every edge $\{x, y\}$ of $g \in \mathcal{G}[U]$. In the language of category theory, a species \mathcal{F} is a *covariant functor* from the category of finite sets and bijections to itself. Thus we require that

$$\mathcal{F}[\phi_2] \circ \mathcal{F}[\phi_1] = \mathcal{F}[\phi_2 \circ \phi_1] \tag{2.1}$$

when $\phi_1 : U \rightarrow V$, $\phi_2 : V \rightarrow W$ are bijections between finite sets, and

$$\mathcal{F}[\text{id}_V] = \text{id}_{\mathcal{F}[V]} \tag{2.2}$$

where id_S is the identity map $S \rightarrow S$ for any set S . In the words of Gian-Carlo Rota in [7], the theory of species “expresses in precise terms the commonsense idea of ‘being able to label the vertices of a graph either by integers or by colors, it does not matter,’ and it is the only way of making this commonsense idea precise.”

Functoriality means that if \mathcal{F} is a species then the definition of $\mathcal{F}[V]$ can depend only on V as a set, not on any extra structure that V might possess. In particular, in the common situation where V is a finite set of natural numbers, the set $\mathcal{F}[V]$ cannot depend on the relative ordering of the elements of V . (A variant form of species, \mathbb{L} -species, allows $\mathcal{F}[V]$ to depend on a total ordering of V . We will see this in Chapter 6.)

For convenience we abuse notation slightly by writing $\mathcal{F}[n]$ to mean $\mathcal{F}[[n]]$ where $[n]$ is the set $[n] = \{1, 2, \dots, n\}$. Let $a_n = |\mathcal{F}[n]|$, the number of \mathcal{F} -structures on a set with n elements. We define the (*exponential*) *generating function* associated to a species \mathcal{F} to be

$$f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$$

where $a_n = |\mathcal{F}[n]|$.

2.2 Definition of contractible species

Now we may turn to *contractible species*. We will discuss the general idea of a contractible species before turning to the formal definition. Essentially, a contractible species is a species \mathcal{F} so that each $\sigma \in \mathcal{F}[V]$ is equipped with some extra structure. We have an operation of *insertion*, denoted by \leftarrow , that allows us to substitute an \mathcal{F} -structure $\tau \in \mathcal{F}[U]$ into a vertex $v \in V$ of another \mathcal{F} -structure $\sigma \in \mathcal{F}[V]$ structure to get a third structure which we denote by $\sigma(v \leftarrow \tau) \in \mathcal{F}[V \setminus \{v\} \cup U]$. The insertion in each contractible species is analogous to the insertion defined on trees in Chapter 1 and illustrated in Fig. 1.2.

To each $\sigma \in \mathcal{F}[V]$ we associate a set $E(\sigma) \subseteq \mathcal{P}(V)$ of subsets of V called the *edges* of σ . The edges of σ should be thought of as *substructures* of σ that can be contracted to a point. For example, recall that in Chapter 1 the edges $E(T)$ of a tree T were the vertex sets of subtrees of T . We require that \mathcal{F} possess an operation of *contraction*, denoted by \rightarrow , that is

inverse to insertion in a sense that will be described explicitly in the definition below. That is, for any $U \in E(\sigma)$ and any vertex $x \notin V$, we have an \mathcal{F} -structure $\sigma(U \rightarrow x) \in \mathcal{F}[V \setminus U \cup \{x\}]$. We think of $\sigma(U \rightarrow x)$ as the \mathcal{F} -structure given by contracting U to a single point which we label x .

Furthermore, each $U \in E(\sigma)$ itself determines an \mathcal{F} -structure, so we have an operation of *restriction*: if $\sigma \in \mathcal{F}[V]$ and $U \in E(\sigma)$ then there is an \mathcal{F} -structure $\tau \in \mathcal{F}[U]$ that we denote by $\tau = \sigma|_U$. The restriction $\sigma|_U$ is generally just given by taking the part of the structure σ that is relevant to the subset U . For the example of trees from Chapter 1, $T|_U$ is simply the subtree on the vertex set U when $U \in E(T)$.

The full definition of contractible species is given below. Note that the operations of contraction and restriction are actually defined in terms of the operation of insertion, so when defining contractible species in the sequel we only need to definition insertion.

Definition. *A contractible species is a species \mathcal{F} , with at least one $\mathcal{F}[V]$ nonempty, with additional structure as follows.*

1. *The species \mathcal{F} is equipped with an insertion operation \leftarrow . For any $\tau \in \mathcal{F}[U]$, $\sigma \in \mathcal{F}[V]$, $v \in V$, where $V \setminus \{v\}$ and $U \setminus \{v\}$ are disjoint, there is an associated \mathcal{F} -structure $\omega \in \mathcal{F}[V \setminus \{v\} \cup U]$ which we denote $\omega = \sigma(v \leftarrow \tau)$.*
2. *For each \mathcal{F} -structure $\sigma \in \mathcal{F}[V]$ define $E(\sigma) \subseteq \mathcal{P}(V)$ to be the set of $U \subseteq V$ so that there is an \mathcal{F} structure $\tau \in \mathcal{F}[U]$ for some v and a $\sigma \in \mathcal{F}[V \setminus U \cup \{v\}]$ with $\omega = \sigma(v \leftarrow \tau)$. We call each $U \in E(\sigma)$ an edge of σ and define $H_\sigma = (V, E(\sigma))$ to be the hypergraph with edge set $E(\sigma)$. We define $\mathcal{F}_U[V]$ to be the set of \mathcal{F} -structures on V with U as an edge, so*

$$\mathcal{F}_U[V] = \{\sigma \in \mathcal{F}[V] : U \in E(\sigma)\}$$

In addition, require that the following axioms are satisfied.

(i) *Bijectivity: The map*

$$I : \mathcal{F}[V] \times \mathcal{F}[U] \rightarrow \mathcal{F}_U[V \setminus \{v\} \cup U]$$

$$(\sigma, \tau) \mapsto \sigma(v \leftarrow \tau)$$

is a bijection. If $I^{-1}(\omega) = (\sigma, \tau)$, then we write $\sigma = \omega(U \rightarrow v)$ and $\tau = \sigma|_U$.

(ii) *Associativity: if $\sigma \in \mathcal{F}[V]$, $\tau \in \mathcal{F}[U]$, $\tau' \in \mathcal{F}[U']$, $u \in U$, and $v \in V$, then*

$$\sigma(v \leftarrow \tau)(u \leftarrow \tau') = \sigma(v \leftarrow (\tau(u \leftarrow \tau'))).$$

(iii) *Contraction preserves edges: If $U', U \in E(\sigma)$, $U' \subseteq U$, then*

$$U \setminus U' \cup \{v\} \in E(\sigma(U' \rightarrow v)).$$

Although the definition of contractible species is somewhat complex, it is not hard to verify that a given species \mathcal{F} becomes a contractible species once the right insertion operation \leftarrow has been defined. In the examples we examine it will generally be fairly evident that Axioms (i) - (iii) are satisfied and we will not give detailed proofs that these properties hold.

The most important axiom is the bijectivity axiom, Axiom (i). It is equivalent to the following identities which will be useful in the proof of our main result, Theorem 1.3. Say $\sigma \in \mathcal{F}[V]$, $\tau \in \mathcal{F}[U]$, $\omega \in \mathcal{F}_U[V \setminus \{v\} \cup U]$ where U, V are finite sets with $v \in V$ and $U \setminus \{v\}$ and $V \setminus \{v\}$ are disjoint. Then we have

$$\sigma(v \leftarrow \tau)|_U = \tau \tag{2.3}$$

$$\sigma(v \leftarrow \tau)(U \rightarrow v) = \sigma \tag{2.4}$$

$$\omega(U \rightarrow v)(v \leftarrow \omega|_U) = \omega. \tag{2.5}$$

The following is our main theorem, justifying the definition of contractible species. It gives a simple combinatorial interpretation to the formal group law associated a contractible

species, as well as a combinatorial interpretation for the compositional inverse of the generating function.

Theorem 1.3. *Let \mathcal{F} be a contractible species and let $f(x) = \sum_{n=1}^{\infty} |\mathcal{F}[n]| \frac{x^n}{n!}$ be the exponential generating function for \mathcal{F} . For each $\sigma \in \mathcal{F}[n]$, let $M(\sigma)$ be the set of minimal, non-singleton edges in $E(\sigma)$ and*

$$\mathcal{C}(\sigma) = \{S \subseteq M(\sigma) : H_S = (V, S) \text{ is a connected hypergraph}\}.$$

Then

$$f^{-1}(x) = \sum_{n=1}^{\infty} \sum_{\sigma \in \mathcal{F}[n]} \sum_{S \in \mathcal{C}(\sigma)} (-1)^{|S|} \frac{x^n}{n!}. \quad (2.6)$$

and

$$f(f^{-1}(x_1) + f^{-1}(x_2) + \dots) = \sum_{n=1}^{\infty} \sum_{\sigma \in \mathcal{F}[n]} \frac{X_{H_\sigma}}{n!}. \quad (2.7)$$

Note that if $\text{red } H_\sigma = ([n], M(\sigma))$ is not connected then $\mathcal{C}(\sigma)$ is the empty set. Thus we may re-write (2.6) as

$$f^{-1}(x) = \sum_{n=1}^{\infty} \sum_{\substack{\sigma \in \mathcal{F}[n] \\ \text{red } H_\sigma \text{ connected}}} \sum_{S \in \mathcal{C}(\sigma)} (-1)^{|S|} \frac{x^n}{n!}.$$

We next give a *weighted version* of Theorem 1.3. Define a *weighted contractible species* to be a contractible species \mathcal{F} so that each \mathcal{F} -structure σ is equipped with a *weight* $w(\sigma) \in R$ for some fixed ring R , where $w(\sigma) = 1$ when $\sigma \in \mathcal{F}[\{v\}]$ for any singleton $\{v\}$, and also satisfying the additional axiom:

(iv) If U, V are finite sets and $v \in V$ with $\sigma \in \mathcal{F}[V], \tau \in \mathcal{F}[U]$ then

$$w(\sigma(v \leftarrow \tau)) = w(\sigma)w(\tau).$$

Then we have the following.

Theorem 2.1. *Let \mathcal{F} be a weighted contractible species with weight w and let*

$$f(x) = \sum_{n=1}^{\infty} \sum_{\sigma \in \mathcal{F}[n]} w(\sigma) \frac{x^n}{n!}.$$

Then

$$f^{-1}(x) = \sum_{n=1}^{\infty} \sum_{\sigma \in \mathcal{F}[n]} \sum_{S \in C(\sigma)} (-1)^{|S|} w(\sigma) \frac{x^n}{n!}. \quad (2.8)$$

Furthermore,

$$f(f^{-1}(x_1) + f^{-1}(x_2) + \cdots) = \sum_{n=1}^{\infty} \sum_{\sigma \in \mathcal{F}[n]} w(\sigma) \frac{X_{H\sigma}}{n!}. \quad (2.9)$$

Clearly Theorem 1.3 is a corollary of the Theorem 2.1, since every contractible species \mathcal{F} can be made into a weighted contractible species with the trivial weight setting $w(\sigma) = 1$ for every \mathcal{F} -structure σ . Thus we will only prove Theorem 2.1. The proof is somewhat technical, so we will delay its proof to Chapter 4. First we will build a collection of example of contractible species, each giving an application of Theorem 2.1.

Chapter 3

EXAMPLES OF CONTRACTIBLE SPECIES

3.1 Trivial examples

Let \mathcal{F} be the species of *singletons* defined by

$$\mathcal{F}[S] = \begin{cases} S & \text{if } |S| = 1 \\ \emptyset & \text{otherwise.} \end{cases}$$

Then \mathcal{F} can be made into a contractible species by setting $E(\sigma) = \{\{v\}\}$ for $\sigma \in \mathcal{F}[\{v\}]$ with the insertion $\sigma(v \leftarrow \{u\}) = \{u\}$ for any u .

It is easy to see that \mathcal{F} trivially satisfies all of the axioms for a contractible species. We can easily verify Theorem 2.1. The only relevant hypergraph is H_σ for $\sigma = \{v\} \in \mathcal{F}[\{v\}]$, given by $H_\sigma = (V, E)$ with singleton vertex set $V = \{v\}$ and edge set $E = \{\{v\}\}$. Then $X_{H_\sigma} = x_1 + x_2 + \dots$ since the single vertex may be colored with any color, and so

$$\sum_{n=1}^{\infty} \sum_{\sigma \in \mathcal{F}[n]} \frac{X_{H_\sigma}}{n!} = \sum_{\sigma \in \mathcal{F}[1]} X_{H_\sigma} = x_1 + x_2 + \dots.$$

The generating function for \mathcal{F} is $f(x) = x$, so

$$f(f^{-1}(x_1) + f^{-1}(x_2) + \dots) = x_1 + x_2 + \dots$$

as well. In particular, $F(x, y) = x + y$ is sometimes called the *additive* formal group law.

A slightly less trivial example is given by taking \mathcal{F} to be the species of *nonempty sets*, where $\mathcal{F}[V] = \{V\}$ for any finite nonempty set V ($\mathcal{F}[\emptyset] = \emptyset$).

We define the insertion operator \leftarrow by $V(v \leftarrow U) = V \setminus \{v\} \cup U$. Again, \mathcal{F} trivially satisfies the axioms (i) - (iii), so \mathcal{F} is a contractible species.

For any subset $U \subseteq V$ and element $v \notin U$, $V = (V \setminus U \cup \{v\})(v \leftarrow U)$. Thus any $U \subseteq V$ satisfies the definition of an edge, so $E(V) = \mathcal{P}(V)$, the power set of V , and $H_\sigma = (V, \mathcal{P}(V))$

is the complete hypergraph. The minimal edges $U \in M(\sigma)$ are all of the pairs $\{v_1, v_2\}$ for $v_1 \neq v_2 \in V$, so the reduced hypergraph $\text{red } H_\sigma = (V, M(\sigma))$ is the complete graph on V . The proper colorings of G are those assigning every vertex a different color, so $X_G = n!e_n$ where $n = |V|$ and e_n is the n -th elementary symmetric function

$$e_n = \sum_{i_1 < i_2 < \dots < i_n} x_{i_1} \cdots x_{i_n}.$$

Thus

$$\sum_{n=1}^{\infty} \sum_{\sigma \in \mathcal{F}[n]} \frac{X_{H_\sigma}}{n!} = e_1 + e_2 + \cdots. \quad (3.1)$$

The generating function for \mathcal{F} is then

$$f(x) = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x - 1,$$

and so

$$\begin{aligned} f(f^{-1}(x_1) + f^{-1}(x_2) + \cdots) &= \exp(\log(1 + x_1) + \log(1 + x_2) + \cdots) - 1 \\ &= \left[\prod_{i=1}^{\infty} (1 + x_i) \right] - 1 \\ &= e_1 + e_2 + \cdots. \end{aligned} \quad (3.2)$$

Comparing (3.1) and (3.2) again confirms Theorem 2.1. This formal group law is the *multiplicative* formal group law from Chapter 1 with $f(f^{-1}(x) + f^{-1}(y)) = x + y + xy$.

3.2 Trees with labeled leaves

We now return to our first example of a contractible species from Chapter 1. Let \mathcal{F} be the species that associates with each V the set of rooted trees with leaves labeled with the elements of V , where the children of a given node are unordered and all other nodes are unlabeled. With the insertion \leftarrow defined in Chapter 1, \mathcal{F} becomes a contractible species. It is easy to see that \leftarrow is injective in the sense of Axiom (i), for we can recover both T and S

from the insertion $T(v \leftarrow S)$ if U and v are known. To find T , we simply replace all of the tree S with the single vertex v , and to find S , we just look at the subtree with vertex set U . Axioms (ii) and (iii) are also clear.

We will say that S is a *subtree* of T if either $S = T$ or, inductively, S is a subtree of one of the trees T_i in the decomposition $T = \{T_1, \dots, T_k\}$. Then we can write $T = R(v \leftarrow S)$ only when S is a subtree of T , and so the edges $U \in E(T)$ are exactly the sets of leaf labels of the subtrees of T .

Let $f(x)$ be the exponential generating function for \mathcal{F} . By Theorem 2.1, we see that

$$f(f^{-1}(x_1) + f^{-1}(x_2) + \dots) = \sum_{T \in \mathcal{F}[n]} \frac{X_{HT}}{n!}. \quad (3.3)$$

We also see by Theorem 2.1 that

$$f^{-1}(x) = \sum_{n=1}^{\infty} \sum_{T \in \mathcal{F}[n]} \sum_{S \in C(T)} (-1)^{|S|} \frac{x^n}{n!}. \quad (3.4)$$

Each minimal edge $U \in M(T)$ is a set of leaves that are all children of a single vertex of T and with every child of that vertex included. In particular, all the edges $U \in M(T)$ of the reduced hypergraph $\text{red } H$ are disjoint. The *only* trees $T \in \mathcal{F}[V]$ on a vertex set V with $|V| > 1$ so that $M(T)$ is connected are the trees where all leaves are siblings. Call such trees *simple*. Recall that $C(T)$ is the set of $S \subseteq M(T)$ so that the hypergraph (V, S) is connected. For a simple tree on a vertex set V , $|V| > 1$, the only choice of $S \subseteq M(T)$ so that (V, S) is a connected hypergraph is $S = M(T) = \{V\}$ and so $C(T) = \{\{V\}\}$. Singleton hypergraphs are always connected, so if $|V| = 1$ then $C(T) = \{\emptyset\}$. Thus we may rewrite (3.4) as

$$\begin{aligned} f^{-1}(x) &= \sum_{n=1}^{\infty} \sum_{T \in \mathcal{F}[n] \text{ simple}} \sum_{S \in C(T)} (-1)^{|S|} \frac{x^n}{n!} \\ &= x + \sum_{n=2}^{\infty} \sum_{T \in \mathcal{F}[n] \text{ simple}} (-1)^1 \frac{x^n}{n!} \\ &= x - \frac{x^2}{2!} - \frac{x^3}{3!} - \dots \\ &= 1 + 2x - e^x. \end{aligned}$$

The identity $f^{-1}(x) = 1 + 2x - e^x$ is well-known. It may be found in [46, Example 5.2.5] and is easy to derive using composition of species.

There is also a *weighted* version of the species \mathcal{F} . Let s_1, s_2, \dots be a set of indeterminates. For each $T \in \mathcal{F}[n]$ define the weight $w(T) = s_2^{k_2} s_3^{k_3} \dots$ where k_i is the number of internal nodes of the tree T with exactly i children. Thus, if T is again the tree from Figure 1.1 then $w(T) = s_2^3 s_3$. We have $w(T) = 1$ when T is the tree on one node, and $w(T(v \leftarrow T')) = w(T)w(T')$, so Axiom (iv) is satisfied. Thus with this weighting \mathcal{F} is a weighted contractible species.

Let $f(x)$ be the weighted generating function for \mathcal{F} , so

$$f(x) = \sum_{n=1}^{\infty} \sum_{T \in \mathcal{F}[n]} w(T) \frac{x^n}{n!}.$$

Each simple tree $T \in \mathcal{F}[n]$ has a weight $w(T) = s_n$, so we have by the Theorem 2.1 that

$$f^{-1}(x) = x - s_2 \frac{x^2}{2!} - s_3 \frac{x^3}{3!} - \dots \quad (3.5)$$

as well as

$$f(f^{-1}(x_1) + f^{-1}(x_2) + \dots) = \sum_{n=1}^{\infty} \sum_{T \in \mathcal{F}[n]} w(T) \frac{X_{HT}}{n!}. \quad (3.6)$$

Equation (3.5) may also be attained directly from the combinatorial definition of $f(x)$ by noting that

$$f(x) = x + s_2 \frac{f(x)^2}{2!} + \frac{f(x)^3}{3!} + \dots$$

Building on work of Parker [36], a broad generalization of (3.5) was given by Drake [14] to find combinatorial interpretations to inverses of generating functions for various kinds of trees. Equation (3.6) is given in a different form by Lenart and Ray [31].

By setting the variables s_i to different values we get more combinatorial interpretations of formal group laws. For example, if $S \subseteq \{2, 3, \dots\}$ then we can set $s_i = 1$ for $i \in S$, $s_i = 0$ otherwise get the generating function $f_S(x)$ for trees whose internal nodes all have outdegrees in S . For example, if $S = \{2\}$, we get the generating function for binary trees,

$f_{\{2\}}(x) = 1 - \sqrt{1 - 2x} = \sum_{n=1}^{\infty} \frac{a_n}{n!} x^n$ where $a_n = \frac{(2n)!}{2^n n!}$, sometimes denoted by the double factorial $a_n = (2n - 3)!! = (2n - 3)(2n - 5) \cdots 1$ for $n > 1$, is the sequence that begins

$$1, 1, 3, 15, 105, 945, 10395, 135135, 2027025, \dots \quad [41, A001147] \quad (3.7)$$

3.3 Trees with all vertices labeled

For another example of a contractible species, let \mathcal{F} be the species of *labeled, rooted trees*. That is, $\mathcal{F}[V]$ is the set of all pairs $T = (G, r)$ where G is a connected graph on V with no cycles and $r \in V$ is a vertex called the *root* of T . Cayley [12] showed that the sequence $a_n = \mathcal{F}[n]$ is given by $a_n = n^{n-1}$. Figure 3.1 depicts a tree $T \in \mathcal{F}[7]$.

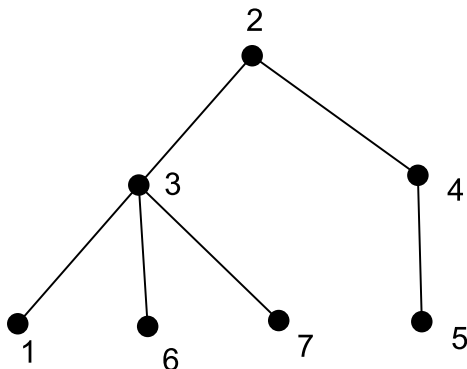


Figure 3.1: A tree $T \in \mathcal{F}[7]$ with root 2.

Given $T \in \mathcal{F}[V]$ and $S \in \mathcal{F}[U]$ with V, U disjoint and $v \in V$, let $T(v \leftarrow S)$ be the tree on $V \setminus \{v\} \cup U$ obtained by replacing v with the root s of S and letting the children of s in $T(v \leftarrow S)$ be the union of the children of r in S and the children of v in T . Figure 3.2 depicts the operation \leftarrow of insertion for trees. It is not hard to check that \mathcal{F} with the insertion operator \leftarrow satisfies all of the axioms of a contractible species. The edges $U \in E(T)$ are again the sets of vertices $U \subseteq V$ that are the vertex set of a subtree S so that we can write $T = R(v \leftarrow S)$ where R is the tree given by contracting S to a single vertex v . Explicitly, $E(T)$ is the set of subsets $U \subseteq V$ attained by choosing a vertex $u \in V$, a subset

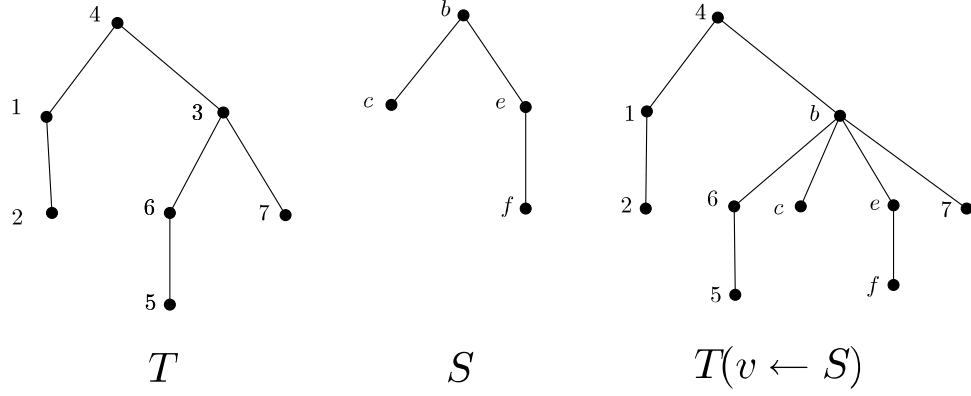


Figure 3.2: Trees T , S and $T(v \leftarrow S)$ where $v = 3$.

u_1, \dots, u_k of the children of u , and letting U consist of u , the children u_1, \dots, u_k , and all of the descendants of u_1, \dots, u_k .

For the example tree T depicted in Figure 3.1, the edge set is

$$E(T) = \{1, 2, 3, 4, 5, 6, 7, 13, 36, 37, 45, 136, 137, 245, 367, 1367, 12367, 1234567\}.$$

while the minimal, non-singleton edges form the set

$$M(T) = \{13, 36, 37, 45\}.$$

For any tree $T \in \mathcal{F}[V]$ the reduced hypergraph $\text{red } H_T = (V, M(T))$ is in fact an ordinary graph. The *minimal* edges $U \in M(T)$ are simply the pairs $\{u, v\}$ where v is a leaf and u is the parent of v . Each connected component of $\text{red } H$ is a “star graph” $K_{1,k}$, $k \geq 0$, where $K_{m,n}$ is the complete bipartite graph.

Letting $f(x)$ be the exponential generating function for \mathcal{F} , so $f(x) = \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} x^n$. It is well-known [46, 5.4] that

$$f^{-1}(x) = xe^{-x}. \tag{3.8}$$

We will demonstrate 3.8 using Theorem 2.1. If T is a tree such that $\text{red } H_T$ is a connected hypergraph, then by the previous discussion, we see that T must be a star graph $K_{1,k}$ consisting only of a root r and leaves v_1, \dots, v_k . Then the only $S \subseteq M(T)$ so that (V, S) is connected is the whole set $M(T)$ of minimal edges, with $(-1)^{|S|} = (-1)^{M(T)} = (-1)^{n-1}$. For each n there are exactly n such trees $T \in \mathcal{F}[n]$, corresponding to the n choices of a root, and so we have by Theorem 2.1

$$\begin{aligned} f^{-1}(x) &= \sum_{n=1}^{\infty} \sum_{T \in \mathcal{F}[n]} \sum_{S \in C(T)} (-1)^{|S|} \frac{x^n}{n!} \\ &= \sum_{n=1}^{\infty} (-1)^{n-1} n \frac{x^n}{n!} = x e^{-x} \end{aligned}$$

as expected.

3.4 Permutations

Let \mathcal{F} be species of *linear orders* on a given set V . Thus if $|V| = n$ we let $\mathcal{F}[V]$ be the set of all possible n -tuples (v_1, v_2, \dots, v_n) listing all elements of V in some order with no repeats. The species of linear orders is not isomorphic to the species of bijections $V \rightarrow V$, referred to as the species \mathcal{S} of permutations in [7]. We will not consider the species \mathcal{S} , so we use the terms *permutation* and *linear order* interchangeably.

There is an obvious definition of insertion in this case. Suppose $\sigma \in \mathcal{F}[V]$, $v \in V$, and $\tau \in \mathcal{F}[U]$ where U is disjoint from V , and let $\sigma = (v_1, v_2, \dots, v_n)$, $\tau = (u_1, u_2, \dots, u_k)$. Then we set

$$\sigma(v \leftarrow \tau) = (v_1, v_2, \dots, v_{i-1}, u_1, \dots, u_k, v_{i+1}, \dots, v_n).$$

With this definition of the insertion operator \leftarrow , we see that $E(\sigma)$ consists exactly of the *intervals* v_i, v_{i+1}, \dots, v_j of σ .

It is easy then to verify the three axioms and determine that \mathcal{F} is indeed a contractible

species. The exponential generating function for \mathcal{F} is

$$f(x) = \sum_{n=1}^{\infty} n! \frac{x^n}{n!} x^n = \frac{x}{1-x}.$$

By Theorem 2.1, we conclude

$$f(f^{-1}(x_1) + f^{-1}(x_2) + \dots) = \sum_{n=1}^{\infty} \sum_{\sigma \in \mathcal{F}[n]} X_{H_\sigma} / n!. \quad (3.9)$$

where H_σ is the hypergraph (V, E) with vertex set $V = [n]$, $E = E(\sigma)$, or — equivalently as far as the chromatic symmetric function X_{H_σ} is concerned — $E = M(\sigma)$, the set of minimal non-singleton edges. Here $M(\sigma)$ just consists of the adjacent pairs of elements of V in their ordering. with σ as before, we have

$$M(\sigma) = \{\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{n-1}, v_n\}\}.$$

Thus for any $\sigma \in \mathcal{F}[n]$ H_σ is an ordinary graph. Specifically, $X_{H_\sigma} = X_{P_n}$ where P_n is the n -element *path* graph with edges $\{1, 2\}, \dots, \{n-1, n\}$. Since there are $n!$ choices of the permutation σ and $f(x) = \frac{1}{1-x} - 1$, so $f^{-1}(x) = \frac{x}{1+x}$, (3.9) becomes

$$\left(1 - \frac{x_1}{1+x_1} - \frac{x_2}{1+x_2} - \dots\right)^{-1} - 1 = \sum_{n=1}^{\infty} X_{P_n}. \quad (3.10)$$

A proper coloring of the path P_n is equivalent to a word $w = w_1 w_2 \dots w_n$ on the alphabet \mathbb{N} so that $w_i \neq w_{i+1}$. Such words are called *Smirnov* or sometimes *Carlitz* words. If we assign a weight $x_{w_1} \dots x_{w_n}$ to the word $w_1 \dots w_n$ we see that (3.10) is the sum of the weights of all nonempty Smirnov words. This result can be found in Goulden and Jackson [27, 2.4.16]. Stanley [43] gives an expression for (3.10) in terms of the elementary symmetric functions e_n showing that it is e -positive.

3.5 Posets

Let $\mathcal{F}[V]$ be the species of *labeled posets* $P = (V, \leq)$ that have a smallest element v_0 and a largest element v_1 so that $P = [v_0, v_1]$. The sequence $a_n = |\mathcal{F}[V]|$ begins

$$1, 2, 6, 36, 380, 6570, \dots$$

Recall that a *covering relation* is a pair $u, v \in V$ so that $u <_P v$ and there is no $w \in V$ satisfying $u <_P w <_P v$. The *Hasse diagram* is the graph with the vertex set V where vertices are connected if they form a covering relation, drawn in the plane so that greater elements are higher. An *interval* of P is a set

$$[u, w] = \{v : u \leq v \leq w\}.$$

Then given $P = (V, \leq_P) \in \mathcal{F}[V], Q = (U, \leq_Q) \in \mathcal{F}[U]$ and $v \in V$, let $R = P(v \leftarrow Q)$ be the poset \leq_R on the vertex set $V \setminus \{v\} \cup U$ given by replacing v with the poset Q as depicted in Figure 3.3. Formally, $R = P(v \leftarrow Q)$ defined so that $x \leq_R y$ if any of the following hold:

- $x, y \in V$ and $x \leq_P y$
- $x, y \in U$ and $x \leq_Q y$
- $x \in V, y \in U$ and $x \leq_P v$
- $x \in U, y \in V$ and $y \geq_P v$.

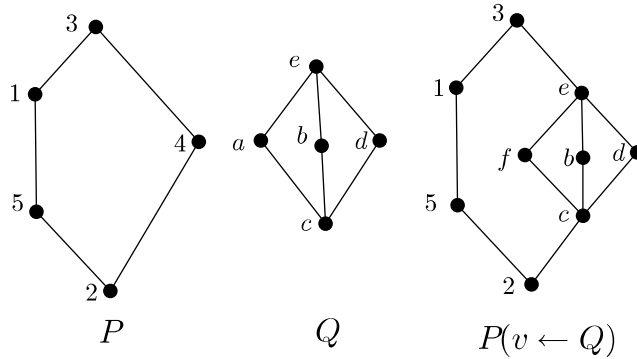


Figure 3.3: Posets P , Q , and $R = P(v \leftarrow Q)$ with $v = 4$.

With the insertion \leftarrow , \mathcal{F} forms a contractible species. The edges $E(P)$ are the intervals $U = [u_0, u_1] \subseteq V$ with the property that if $v \in V \setminus U, u \in U$ then $v \leq_P u$ if and only if $v \leq_P u_0$, and similarly $v \geq_P u$ if and only if $v \geq_P u_1$. This means that in the Hasse diagram of P

there are no edges coming out of U except those coming *down* from u_0 and *up* from u_1 . For example, if R is the poset depicted in Figure 3.3 with vertex set $W = \{1, 2, 3, a, b, c, d, e, 5\}$ the edges $U \in E(R)$ are the singletons, the whole vertex $\{1, 2, 3, a, b, c, d, e, 5\}$, together with the intervals $\{1, 5\}, \{a, b, c, d, e\}$. We can make \mathcal{F} into a weighted contractible species by setting $w(P) = y^{c(P)}$ where $c(P)$ is the number of *covering relations* of P , or equivalently the number of edges in the Hasse diagram of P .

The same insertion for the species \mathcal{F} of *lattices*, posets P so that any two elements x, y have a unique least upper bound $x \vee y$ and greatest lower bound $x \wedge y$. The number of labeled lattices on n vertices begins

$$1, 2, 6, 36, 380, 6390, 157962, 5396888, \dots \text{ (OEISA055512)}.$$

3.6 Graphs

Let \mathcal{F} be the species of *labeled graphs*. Letting $\binom{V}{k}$ be the set of all subsets of V of size k , we set $\mathcal{F}[V]$ to be the set of all subsets of $\binom{V}{2}$, so that $|\mathcal{F}[n]| = 2^{\binom{n}{2}}$.

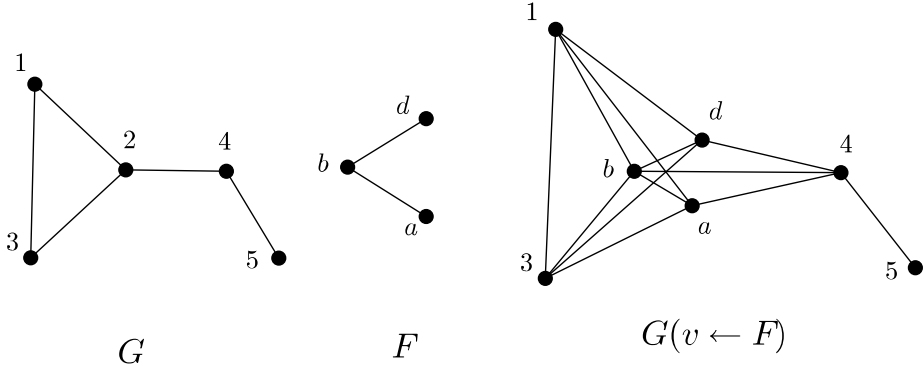


Figure 3.4: Graphs G, F , and $G(v \leftarrow F)$ where $v = 4$.

Here the proper definition of the insertion \leftarrow is not as obvious as in our previous examples. If $G \in \mathcal{F}[V], v \in V$, and $F \in \mathcal{F}[U]$ where U, V are disjoint, define the graph $G(v \leftarrow F)$ on

$V \setminus U \cup \{v\}$ as follows, depicted in Fig 3.4. First, we preserve all the edges of F , so that for $v_1, v_2 \in U$ we have $\{v_1, v_2\} \in G(v \leftarrow F)$ if and only if $\{v_1, v_2\} \in F$. Similarly, if $v_1, v_2 \in V$ with $v_1, v_2 \neq v$ then $\{v_1, v_2\} \in G(v \leftarrow F)$ if and only if $\{v_1, v_2\} \in G$. Finally, if $v_1 \in U$, $v_2 \in V$ then $\{v_1, v_2\} \in G(v \leftarrow F)$ if and only if $\{v, v_2\} \in G$. That is, we connect each $v_1 \in U$ to all of the neighbors v_2 of v in G and no other vertices of V .

This definition of insertion does in fact make \mathcal{F} into a contractible species. The hyperedges $U \in E(G)$ in the hypergraph H_U are the $U \subseteq V$ so that each $v \in V \setminus U$ is either connected to every vertex in U or none of the vertices in U .

Chapter 4

PROOF OF MAIN THEOREM

We will need some background material before we proceed to the proof of Theorem 2.1. Theorem 4.1 below gives the well-known combinatorial interpretation for the composition of exponential generating functions. To see the proof and many applications see [46, Ch. 5].

Define a *set partition* of a set S to be a set $\pi = \{\pi_1, \dots, \pi_k\}$ of disjoint nonempty subsets of S whose union is S , and let Π_n be the set of *set partitions* π of $[n]$.

Theorem 4.1. *Let $f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$ and $g(x) = \sum_{n=1}^{\infty} b_n \frac{x^n}{n!}$. Then*

$$f(g(x)) = \sum_{n=0}^{\infty} \sum_{\pi \in \Pi_n} a_k b_{n_1} b_{n_2} \cdots b_{n_k} \frac{x^n}{n!}$$

where k is the number of parts of the partition π and n_1, n_2, \dots, n_k are the sizes of the parts of π .

We will also need a technical lemma giving some basic properties of contractible species.

Lemma 4.2. *Let \mathcal{F} be a contractible species and let $\sigma \in \mathcal{F}[V]$ for some finite set V .*

- (a) *The whole set V is an element of $E(\sigma)$.*
- (b) *There is a unique $\sigma \in \mathcal{F}[\{v\}]$ for any singleton set $\{v\}$.*
- (c) *For every $v \in V$, the singleton $\{v\}$ is an element of $E(\sigma)$.*
- (d) *For any subsets U and U' of V where $U \in E(\sigma)$, we have $U' \in E(\sigma|_U)$ if and only if $U' \subseteq U$ and $U' \in E(\sigma)$.*

(e) The edge set is closed under non-disjoint unions. That is, if $U, U' \in E(\sigma)$ are disjoint then $U \cup U' \in E(\sigma)$.

(f) Suppose that $S \subseteq M(\sigma)$. If U is the set of vertices of a connected component of the hypergraph $H_S = (V, S)$ then $U \in E(\sigma)$.

(g) If $U, U' \in E(\sigma)$ are disjoint then $U' \in E(\sigma(U \rightarrow v))$.

Proof. (a), (b): Let $\{v\}$ be any singleton set. Recalling that $\mathcal{F}_V[V]$ is the set of $\sigma \in \mathcal{F}[V]$ with $V \in E(\sigma)$, we would like to show that $\mathcal{F}_V[V] = \mathcal{F}[V]$ as well as $|\mathcal{F}[\{v\}]| = 1$. Note that $\{v\}(v \leftarrow V) = V$, and so by Axiom (i), we have

$$|\mathcal{F}[\{v\}] \times \mathcal{F}[V]| = |\mathcal{F}_V[V]|.$$

By definition we have $\mathcal{F}_V[V] \subseteq \mathcal{F}[V]$, so

$$|\mathcal{F}_V[V]| \leq |\mathcal{F}[V]| \leq |\mathcal{F}[\{v\}]| |\mathcal{F}[V]| \leq |\mathcal{F}[\{v\}] \times \mathcal{F}[V]| = |\mathcal{F}_V[V]|.$$

Thus each inequality must in fact be an equality, so we conclude $\mathcal{F}_V[V] = \mathcal{F}[V]$ and $|\mathcal{F}[\{v\}]| = 1$.

((b)): By Axiom (i) we have

$$|\mathcal{F}_{\{v\}}(V)| = |\mathcal{F}[V]| \cdot |\mathcal{F}[\{v\}]|$$

and $|\mathcal{F}[\{v\}]| = 1$ by (b). Thus $\mathcal{F}_{\{v\}}(V) = \mathcal{F}(V)$.

((d)): Say $\tau = \sigma|_U$. If $U' \in E(\tau)$, then by definition $U' \subseteq U$. By surjectivity in the bijectivity axiom, Axiom (i), we may write $\sigma = \omega(v \leftarrow \tau)$ for some v , and similarly $\tau' = \omega'(v' \leftarrow \tau')$ for some $\tau' \in \mathcal{F}[U']$. Then we have

$$\begin{aligned} \sigma &= \omega(v \leftarrow \tau) \\ &= \omega(v \leftarrow \omega'(v' \leftarrow \tau')) \\ &= \omega(v \leftarrow \omega')(v' \leftarrow \tau') \end{aligned}$$

by associativity (Axiom (ii)). Letting $\omega(v \leftarrow \omega') = \omega''$, we have written $\sigma = \omega''(v' \leftarrow \tau')$ for some $\tau' \in \mathcal{F}[U']$, the defining condition for edges of $E(\sigma)$, so $U' \in E(\sigma)$.

Similarly, suppose that $U, U' \in E(\sigma)$, $U' \subseteq U$. Letting $\tau' = \sigma|_{U'}$, we can write $\sigma = \omega''(v' \leftarrow \tau')$ for some v' and $\omega \in \mathcal{F}[V \setminus U' \cup \{v'\}]$. We have $\pi(U) = U \setminus U' \cup \{v'\} \in E(\omega'')$ by Axiom (iii), so we may write $\omega'' = \omega(v \leftarrow \omega')$ for some v and $\omega' \in \mathcal{F}[U \setminus U' \cup \{v'\}]$. Then we have

$$\begin{aligned} \sigma &= \omega''(v' \leftarrow \tau') \\ &= \omega(v \leftarrow \omega')(v' \leftarrow \tau') \\ &= \omega(v \leftarrow \omega'(v' \leftarrow \tau')). \end{aligned}$$

Since $\omega'(v' \leftarrow \tau') \in \mathcal{F}[U]$, we have $\sigma|_U = \omega'(v' \leftarrow \tau')$. Then since $\tau' \in \mathcal{F}[U']$ we conclude $U' \in E(\omega'(v' \leftarrow \tau')) = E(\sigma|_U)$.

(e): Let $U, U' \in E(\sigma)$, $U \cap U' \neq \emptyset$. Letting $U'' = U \setminus U' \cup \{v\}$ for some $v \notin V$, we have $U'' \in \sigma(U \rightarrow v)$, so we may define $\omega = (\sigma(U \rightarrow v))(U'' \rightarrow v')$ for some $v' \notin V, U$. Let $\tau = \sigma|_U$, $\tau' = \sigma(U \rightarrow v)|_{U''}$. Since $U \cap U' \neq \emptyset$, $v \in U''$, so we have

$$\sigma = \omega(v' \leftarrow \tau')(v \leftarrow \tau) = \omega(v' \leftarrow \tau'(v \leftarrow \tau)) \quad (4.1)$$

by (2.5) and Axiom (ii) (associativity). By definition, τ' is an element of $\mathcal{F}[U''] = \mathcal{F}[U \setminus U' \cup \{v\}]$ and so $\tau'(v \leftarrow \tau)$ is an element of $\mathcal{F}[U \cup U']$. Finally, (4.1) together with Axiom (i) tells us that $\sigma \in \mathcal{F}_{U \cup U'}[V]$ which is only defined when $U \cup U' \in E(\sigma)$.

(f): Let U be a component of the hypergraph $H_S = (V, S)$ where $S \subseteq E(\sigma)$. If U is a singleton, $U \in E(\sigma)$ by (b). Otherwise, U must contain at least one edge since it is a non-singleton component. Let $U' \in E(\sigma)$ be a maximal edge with $U' \subseteq U$. We claim that $U' = U$. For suppose not. Then there is a nonempty edge $W \in S$ so that $W \subseteq U$ and $W \cap U' \neq \emptyset$, $W \cap (U \setminus U') \neq \emptyset$, or else $U \setminus U'$ would be a component of H_S . Then have $U' \cup W \in E(\sigma)$ by (e), but this violates the maximality of U . Thus we must have $U' = U$, and $U \in E(\sigma)$ as desired. \square

We are now ready to prove our main result on contractible species, Theorem 2.1.

Proof of Theorem 2.1. Let \mathcal{F} be a contractible species. Recall that

$$\mathcal{C}(\sigma) = \{S \subseteq M(\sigma) : H_S = (V, S) \text{ is a connected hypergraph}\}.$$

Let

$$g(x) = \sum_{n=1}^{\infty} \sum_{\sigma \in \mathcal{F}[n]} w(\sigma) \sum_{S \in \mathcal{C}(\sigma)} (-1)^{|S|} \frac{x^n}{n!}.$$

We will show that

$$f(g(x_1) + g(x_2) + \cdots) = \sum_{n=1}^{\infty} \sum_{\sigma \in \mathcal{F}[n]} \frac{X_{H_\sigma}}{n!}. \quad (4.2)$$

Both (2.6) and (2.7) follow from (4.2) as follows. Set $x_1 = x$, $x_2 = x_3 = \cdots = 0$ in (4.2). On the left-hand side of (4.2), we get $f(g(x))$. On the right-hand side, we note that for any $\sigma \in \mathcal{F}[n]$, $n > 1$, we have $X_{H_\sigma}(x, 0, 0, \dots) = 0$: there is no proper coloring of H_σ that uses only one color since we have $[n] \in E(\sigma)$ by (b) in Lemma 4.2. At the same time, there is a unique $\sigma \in \mathcal{F}[1]$, by Lemma 4.2, and in that case $X_{H_\sigma} = x_1 + x_2 + \cdots$ so that $X_{H_\sigma}(x, 0, 0, \dots) = x$. Thus the right-hand side of (4.2) becomes x , so that $f(g(x)) = x$ and $g(x) = f^{-1}(x)$, proving (2.6). Substituting $f^{-1}(x)$ for $g(x)$ in (4.2) then establishes (2.7).

To establish (4.2), it is convenient to use an additional variable x so that we can apply Theorem 4.1. We will establish the identity

$$f(g(x_1x) + g(x_2x) + \cdots) = \sum_{n=1}^{\infty} \sum_{\sigma \in \mathcal{F}[n]} X_{H_\sigma} \frac{x^n}{n!} \quad (4.3)$$

from which (4.2) follows. To prove (4.3), we will re-write both sides of (4.3) as weighted sums over sets of combinatorial objects and then find a weight-preserving bijection between the sets.

First, we re-write the left-hand side of (4.3). Let $b_n = \sum_{\sigma \in \mathcal{F}[n]} \sum_{S \in \mathcal{C}(\sigma)} (-1)^{|S|}$, so that $g(x) = \sum_{n=1}^{\infty} b_n \frac{x^n}{n!}$. We have

$$\begin{aligned} g(x_1x) + g(x_2x) + \cdots &= \sum_{n=1}^{\infty} b_n \frac{(x_1x)^n}{n!} + \sum_{n=1}^{\infty} b_n \frac{(x_2x)^n}{n!} + \cdots \\ &= \sum_{n=1}^{\infty} b_n p_n \frac{x^n}{n!}. \end{aligned}$$

Then by Theorem 4.1, we have

$$f(g(x_1x) + g(x_2x) + \cdots) = \sum_{n=1}^{\infty} \frac{x^n}{n!} \sum_{\pi \in \Pi_n} a_{|\pi|} b_{n_1} \cdots b_{n_k} p_{n_1} \cdots p_{n_k} \quad (4.4)$$

where n_1, \dots, n_k are the sizes of the parts of the set partition π .

Let A_n be the set of tuples $z = (\pi, \omega, \tau_1, \dots, \tau_k, S_1, \dots, S_k)$ where $\pi = \{\pi_1, \dots, \pi_k\}$ is a partition of $[n]$ with the parts ordered according to their least elements using the natural ordering on $[n]$, $\omega \in \mathcal{F}[k]$, $\tau_i \in \mathcal{F}[\pi_i]$, and $S_i \in \mathcal{C}(\tau_i)$. Define a weight on A_n by setting $w(z) = w(\omega)w(\tau_1) \cdots w(\tau_k) \cdot (-1)^{|S_1| + \cdots + |S_k|} p_{n_1} \cdots p_{n_k}$ where $n_i = |\pi_i|$. Then (4.4) becomes

$$f(g(x_1x) + g(x_2x) + \cdots) = \sum_{n=1}^{\infty} \frac{x^n}{n!} \sum_{z \in A_n} w(z). \quad (4.5)$$

Now we re-write the right-hand side of (4.3). Let $B_n = \{(\sigma, S) : \sigma \in \mathcal{F}[n], S \subseteq M(\sigma)\}$, and define the weight of $(\sigma, S) \in B_n$ to be $w(\sigma, S) = w(\sigma) \cdot (-1)^{|S|} p_{n_1} p_{n_2} \cdots p_{n_k}$ where n_1, n_2, \dots, n_k are the sizes of the connected components of the hypergraph $H_S = ([n], S)$. Recall from Chapter 1 that we may replace $E(\sigma)$ with $M(\sigma)$ without changing the set of proper colorings. Then applying Theorem 1.1 to the hypergraph $([n], M(\sigma))$ gives

$$\sum_{n=1}^{\infty} \sum_{\sigma \in \mathcal{F}[n]} w(\sigma) X_{H_\sigma} \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{x^n}{n!} \sum_{(\sigma, S) \in B_n} w(\sigma, S). \quad (4.6)$$

Comparing (4.5) and (4.6), we see that (4.3) will be proven once we know that

$$\sum_{z \in A_n} w(z) = \sum_{(\sigma, S) \in B_n} w(\sigma, S). \quad (4.7)$$

Thus the proof is finished after we have established the following claim.

Claim. *There is a weight preserving bijection $\Gamma : A_n \rightarrow B_n$ for each n .*

Proof of claim. Given $z = (\pi, \omega, \tau_1, \dots, \tau_k, S_1, \dots, S_k) \in A_n$, set $\sigma = \omega(1 \leftarrow \tau_1) \cdots (k \leftarrow \tau_k)$ and $S = S_1 \cup \cdots \cup S_k$, and define $\Gamma : A_n \rightarrow B_n$ by $\Gamma(z) = (\sigma, S)$. (Technically, we should replace $1, \dots, k$ with a disjoint set $\hat{1}, \dots, \hat{k}$ to avoid vertices being shared by σ and the τ_i 's, but we will suppress this.) Note that the Lemma 4.2(d) guarantees that each edge $U \in S_i$

belongs to $E(\sigma)$, so $(\sigma, S) \in A_n$. The bijection Γ is illustrated in Figs. 4.1 and 4.2 below for two different examples of contractible species we have seen in Chapter 3.

We will show that Γ is weight-preserving. Say $\Gamma(z) = (\sigma, S)$, and let n_1, \dots, n_k be the sizes of the connected components of the hypergraph $H_S = ([n], S)$. By repeated application of the weight axiom, Axiom (iv), we have

$$w(\sigma) = w(\omega(1 \leftarrow \tau_1) \cdots (k \leftarrow \tau_k)) = w(\omega)w(\tau_1) \cdots w(\tau_k).$$

Then we have

$$\begin{aligned} w(\Gamma(z)) &= w(\sigma, S) = w(\sigma) \cdot (-1)^{|S_1| + \cdots + |S_k|} p_{n_1} \cdots p_{n_k} \\ &= w(\omega)w(\tau_1) \cdots w(\tau_k) \cdot (-1)^{|S_1| + \cdots + |S_k|} p_{n_1} \cdots p_{n_k} \\ &= w(z) \end{aligned}$$

as desired.

Next, we will show Γ is a bijection by explicitly constructing its inverse $\Lambda : B_n \rightarrow A_n$. Given $(\sigma, S) \in B_n$, form the hypergraph $H_S = (V, E)$ with vertex set $V = [n]$ and edge set $E = S$, and define a partition $\pi = \{\pi_1, \dots, \pi_k\}$ where the parts π_i are the connected components of H_S . It is convenient to have an ordering on these parts, so we assume they are ordered according to their least elements, so that $i < j$ implies $\min \pi_i < \min \pi_j$. Define ω_i for $0 \leq i \leq k$ by $\omega_0 = \sigma$, and $\omega_i = \omega(U_1 \rightarrow 1) \cdots (U_i \rightarrow i)$. Lemma 4.2(g) guarantees that $U_i \in E(\omega_{i-1})$, so this contraction is allowed. Set $\tau_i = \omega_{i-1}|_{U_i}$ and $S_i = \{U \in S : U \subseteq \pi_i\}$. Set $z = (\pi, \omega, \tau_1, \dots, \tau_k, S_1, \dots, S_k)$. Lemma 4.2 (d) guarantees that $S_i \subseteq E(\tau_i)$. Since the π_i are the connected components, we have $S_i \in \mathcal{C}(\pi_i)$, and so $z \in A_n$ as desired. We set $\Lambda(\sigma, S) = z$.

Next, we show that $\Gamma \circ \Lambda(\sigma, S) = (\sigma, S)$. Let $z = (\pi, \omega, \tau_1, \dots, \tau_k, S_1, \dots, S_k) = \Gamma(\sigma, S)$, $\Lambda(z) = (\hat{\sigma}, \hat{S})$, and $\omega_i = \sigma(U_1 \rightarrow 1) \cdots (U_i \rightarrow i)$ as in the definition of Λ . Recall that S_1, \dots, S_k are by definition the edges of S contained in the connected components of the hypergraph $([n], S)$, so we have $\hat{S} = S_1 \cup \dots \cup S_k = S$. It remains to show that $\hat{\sigma} = \sigma$. By

definition, we have $\tau_i = \omega_{i-1}|_{U_i}$, so the identity (2.5) gives

$$\omega_{i-1}(U_i \rightarrow i)(k \leftarrow \tau_i) = \omega_{i-1}(U_i \rightarrow i)(k \leftarrow \omega_{i-1}|_{U_i}) = \omega_{i-1}$$

for any i . Repeatedly using this identity gives

$$\begin{aligned} \hat{\sigma} &= \omega(k \leftarrow \tau_k) \cdots (1 \leftarrow \tau_1) \\ &= \sigma(U_1 \rightarrow 1) \cdots (U_k \rightarrow k)(k \leftarrow \tau_k) \cdots (1 \leftarrow \tau_1) \\ &= \omega_{k-1}(U_k \rightarrow k)(k \leftarrow \tau_k) \cdots (1 \leftarrow \tau_1) \\ &= \omega_{k-1}(k-1 \leftarrow \tau_{k-1}) \cdots (1 \leftarrow \tau_1) \\ &\quad \vdots \\ &= \omega_0 = \sigma. \end{aligned}$$

Similarly, we show $\Delta \circ \Gamma(z) = z$. Let $z = (\pi, \omega, \tau_1, \dots, \tau_k, S_1, \dots, S_k)$, $\Gamma(z) = (\sigma, S)$ and $\Delta(\sigma, S) = \hat{z} = (\hat{\pi}, \hat{\omega}, \hat{\tau}_1, \dots, \hat{\tau}_k, \hat{S}_1, \dots, \hat{S}_k)$. By definition, $S = S_1 \cup \cdots \cup S_k$, and each (π_i, S_i) is a connected hypergraph. The $\hat{\pi}_i$ are by definition the connected components of the hypergraph $([n], S)$, so $\hat{\pi}_i = \pi_i$ and $\hat{S}_i = S_i$ for all i .

Again letting $\omega_i = \sigma(1 \leftarrow \tau_1) \cdots (i \leftarrow \tau_i)$, we have

$$\begin{aligned} \hat{\tau}_i &= \omega_i|_{\pi_i} && \text{(definition of } \Delta) \\ &= \omega_{i-1}(i \leftarrow \tau_i)|_{\pi_i} \\ &= \tau_i && \text{(by the identity (2.3).)} \end{aligned}$$

Finally, we have

$$\begin{aligned} \hat{\omega} &= \sigma(U_1 \rightarrow 1) \cdots (U_k \rightarrow k) \\ &= \omega(k \leftarrow \tau_k) \cdots (1 \leftarrow \tau_1)(U_1 \rightarrow 1) \cdots (U_k \rightarrow k) \\ &= \omega(k \leftarrow \tau_k) \cdots (1 \leftarrow \tau_2)(U_2 \rightarrow 2) \cdots (U_k \rightarrow k) && \text{(by the identity (2.4).)} \\ &\quad \vdots \\ &= \omega. \end{aligned}$$

□

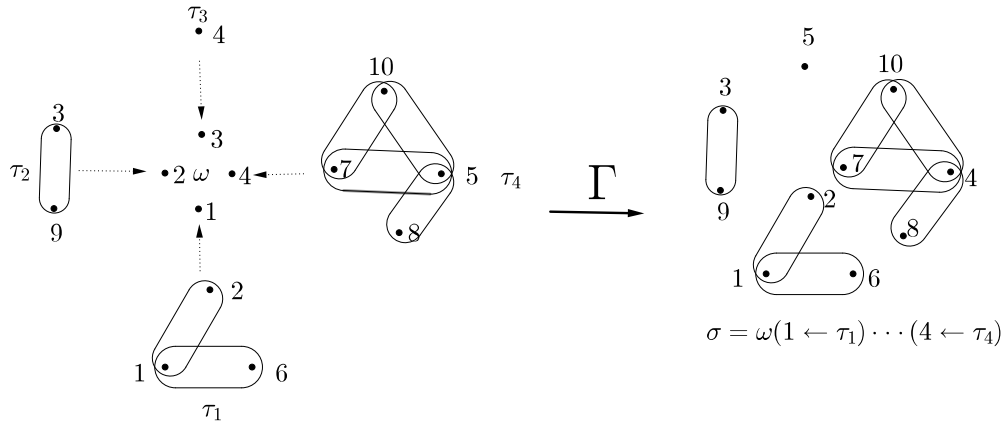


Figure 4.1: The bijection Γ illustrated for the species of sets, $\mathcal{F}[V] = \{V\}$, from Section 3.1. In this case the minimal edges of V are all of the pairs of elements of V . On the left, a $z \in A_n$ for $n = 10$: $z = (\pi, \omega, \tau_1, \dots, \tau_k, S_1, \dots, S_k)$, where $\pi = \{126, 39, 4, 5789\}$, $S_1 = \{12, 16\}$, $S_2 = \{39\}$, $S_3 = \emptyset$, etc., while $\omega, \tau_1, \dots, \tau_4$ are just sets. On the right, the pair $\Gamma(z) = (\sigma, S) \in B_n$ with $\sigma = [10]$.

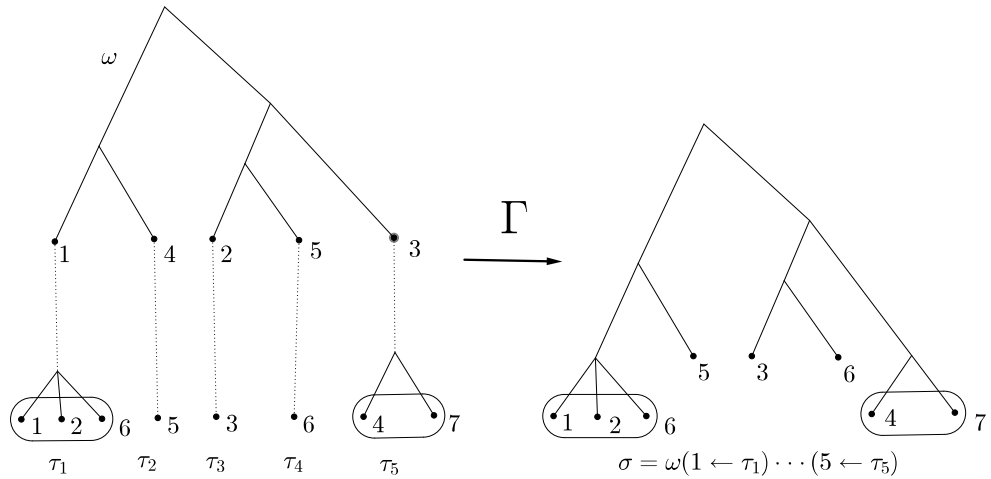


Figure 4.2: The bijection Γ illustrated for the species of trees with labeled leaves from Section 3.2. On the left, a $z \in A_n$ for $n = 7$: $z = (\pi, \omega, \tau_1, \dots, \tau_k, S_1, \dots, S_k)$, where $\pi = \{126, 3, 5, 47, 5\}$, $S_1 = \{126\}$, $S_4 = \{47\}$, and $S_2 = S_3 = S_5 = \emptyset$. On the right, the pair $\Gamma(z) = (\sigma, S) \in B_n$ with $S = \{126, 47\}$.

Chapter 5

CARTESIAN PRODUCT OF CONTRACTIBLE SPECIES

Define the *Hadamard product* of $f_1(x) = \sum_{n=1}^{\infty} a_n \frac{x^n}{n!}$, $f_2(x) = \sum_{n=1}^{\infty} b_n \frac{x^n}{n!}$ to be the formal power series $f_1 \times f_2(x)$ given by multiplying f_1, f_2 coefficient by coefficient, so that

$$(f_1 \times f_2)(x) = \sum_{n=1}^{\infty} a_n b_n \frac{x^n}{n!}.$$

The Hadamard product lifts naturally to a product of species. Given two species $\mathcal{F}_1, \mathcal{F}_2$, we define the Cartesian product $\mathcal{F}_1 \times \mathcal{F}_2$ to be the species

$$(\mathcal{F}_1 \times \mathcal{F}_2)(V) = \mathcal{F}_1[V] \times \mathcal{F}_2[V]$$

where \times on the right hand side is the usual Cartesian product, so that $\mathcal{F}_1[V] \times \mathcal{F}_2[V]$ is the set of pairs (σ_1, σ_2) where $\sigma_1 \in \mathcal{F}_1[V], \sigma_2 \in \mathcal{F}_2[V]$. Thus if $f_1(x), f_2(x)$ are the exponential generating functions of $\mathcal{F}_1, \mathcal{F}_2$, respectively, then the Hadamard product $(f_1 \times f_2)(x)$ is the generating function for $\mathcal{F}_1 \times \mathcal{F}_2$.

In the case where \mathcal{F}_1 and \mathcal{F}_2 are contractible species, $\mathcal{F}_1 \times \mathcal{F}_2$ can be made into a contractible species as well.

Theorem 5.1. *Let $\mathcal{F}_1, \mathcal{F}_2$ be contractible species. Define the insertion operation \leftarrow on $\mathcal{F}_1 \times \mathcal{F}_2$ by*

$$(\sigma_1, \sigma_2)(v \leftarrow (\tau_1, \tau_2)) = (\sigma_1(v \leftarrow \tau_1), \sigma_2(v \leftarrow \tau_2)).$$

With this operation $\mathcal{F}_1 \times \mathcal{F}_2$ is a contractible species, and

$$E(\sigma_1, \sigma_2) = E(\sigma_1) \cap E(\sigma_2) = \{U \subseteq V : U \in E(\sigma_1), U \in E(\sigma_2).\}$$

Furthermore, if $\mathcal{F}_1, \mathcal{F}_2$ are weighted contractible species with weights w_1, w_2 then $\mathcal{F}_1 \times \mathcal{F}_2$ is a weighted contractible species with the weight $w(\sigma_1, \sigma_2) = w_1(\sigma_1)w_2(\sigma_2)$.

Proof. The axioms of a weighted contractible species are easily verified. We will show that $E(\sigma_1, \sigma_2) = E(\sigma_1) \cap E(\sigma_2)$. By definition, the edges $U \in E(\sigma_1, \sigma_2)$ are those for which we have $(\tau_1, \tau_2) \in (\mathcal{F}_1 \times \mathcal{F}_2)(U)$ and $(\omega_1, \omega_2) \in (\mathcal{F}_1 \times \mathcal{F}_2)[V \setminus U \cup \{v\}]$ so that

$$(\sigma_1, \sigma_2) = (\omega_1, \omega_2)(v \leftarrow (\tau_1, \tau_2)). \quad (5.1)$$

By definition, (5.1) means that $\sigma_i = \omega_i(v \leftarrow \tau_i)$ for $i = 1, 2$. This happens exactly when U is an edge in both $E(\sigma_1)$ and $E(\sigma_2)$. □

Theorem 5.1 allows us to give combinatorial interpretations to the formal group laws corresponding to generating functions $f(x)$ attained by taking Hadamard products of our previous examples. For example, we could take

$$f(x) = \sum_{n=1}^{\infty} (2n-3)!! n^{n-1} 2^{\binom{n}{2}} \frac{x^n}{n!}$$

corresponding to tuples (T, T', G) where T is a rooted, binary tree with labeled leaves, T' is a rooted tree with all vertices labeled, and G is any labeled graph.

Note in particular that if \mathcal{L} is the species of *linear orders* from 3.4 then $\mathcal{F} \times \mathcal{L}$ has the generating function

$$\sum_{n=1}^{\infty} a_n n! \frac{x^n}{n!} = \sum_{n=1}^{\infty} a_n x^n,$$

the *ordinary* generating function for \mathcal{F} . We will turn to the theory of formal group laws for ordinary generating functions in Chapter 6.

If we consider the contractible species $\mathcal{L} \times \mathcal{L}$, we get a combinatorial interpretation to the formal group law corresponding to

$$f(x) = \sum_{n=1}^{\infty} n! x^n. \quad (5.2)$$

We could even consider the contractible species of the form $\mathbb{L} \times \mathbb{L} \cdots \times \mathbb{L}$ corresponding to the generating functions

$$f(x) = \sum_{n=1}^{\infty} n!^k x^n. \quad (5.3)$$

We will return to this example in Section 6.5.

5.1 Example: Labeled tanglegrams and tangled chains

Let \mathcal{F} be the species of binary trees with labeled leaves from Section 3.2. Then $(\mathcal{F} \times \mathcal{F})[V]$ is the set of pairs (S, T) of binary trees with leaves labeled by the same vertex set V . We call such a pair (S, T) a *labeled tanglegram*. More generally, we denote by $\mathcal{F}^{\times k}$ the Cartesian product $\mathcal{F}^{\times k} = F \times F \times \cdots \times F$ of k copies of F . The tuples $(T_1, T_2, \dots, T_k) \in \mathcal{F}^{\times k}[V]$ are called *labeled tangled chains* of length k . Recall that there are

$$|\mathcal{F}[n]| = 1 \cdot 3 \cdot \cdots \cdot (2n - 3) = (2n - 3)!!$$

binary trees with leaves labeled $1, 2, \dots, n$. Thus there are

$$|(\mathcal{F}^{\times k})[V]| = (2n - 3)!!^k$$

labeled tangled chains of length k on n labeled leaves. Figure 5.1 depicts two labeled tanglegrams.

The definition of tanglegrams was motivated by the study of phylogenetic trees in evolutionary biology. More background on the genetic motivation is given in [16]. In recent work, Billey, Konvalinka and Matsen [17] gave a surprisingly simple formula for the number of *unlabeled* tangled chains of length k . Gessel [22] gives a species-theoretic interpretation of this formula. For the rest of this section we will only refer to *labeled* tanglegrams and tangled chains.

Theorem 5.1 shows that the species of tangled chains of length k is a contractible species. The edges $U \in E(T_1, T_2, \dots, T_k)$ are the subsets $U \subseteq V$ so that for each i there is a subtree T' of T_i with U the set of leaves of T' . For example, if S and T are the two tanglegrams in Figure 5.1 then

$$E(T) = \{1, 2, 3, 4, 5, 24, 124, 35, 12345\}$$

while

$$E(S) = \{1, 2, 3, 4, 5, 12345\}.$$

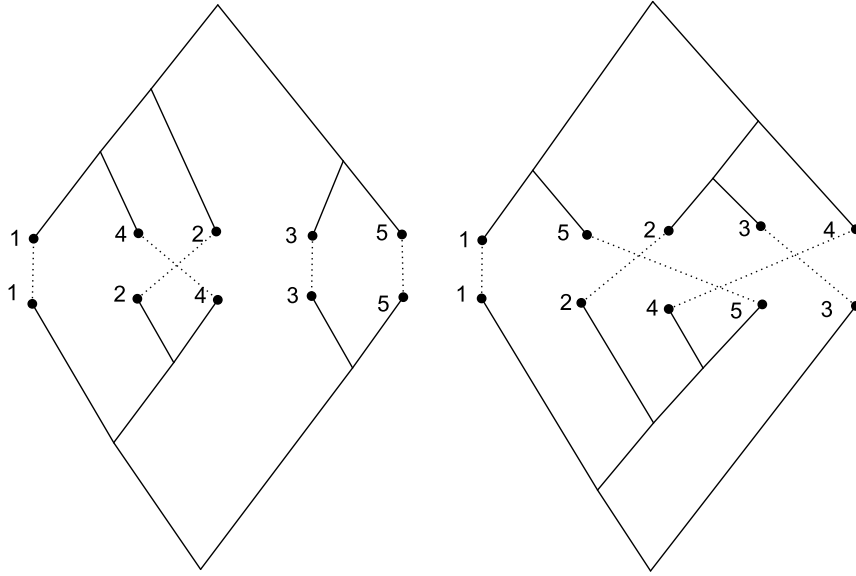


Figure 5.1: Two labeled tanglegrams in $(\mathcal{F} \times \mathcal{F})[5]$.

Given any two edges of a tangled chain, either one is a subset of the other or they are disjoint. It follows that the only tangled chains T so that the reduced hypergraph $\text{red } H_T = (V, M(T))$ is connected are those where V is a minimal edge, so that $M(T) = \{V\}$. Call these tangled chains *simple*. Simple tangled chains $(T_1, \dots, T_k) \in \mathcal{F}^{\times k}[V]$ are those for which there is no non-singleton $U \subsetneq V$ so that each T_i has a subtree with the leaves U . Of the two tanglegrams in Figure 5.1, the first is not simple but the second is.

For every tangled chain $T \in \mathcal{F}^{\times k}[V]$ that is not simple, $C(T) = \emptyset$, while for every simple T , $C(T)$ is the singleton $\{V\}$. From Theorem 2.1, it follows that if

$$f_k(x) = \sum_{n=1}^{\infty} (2n-3)!!^k \frac{x^n}{n!}$$

is the generating function for $\mathcal{F}^{\times k}$ then

$$f_k^{-1}(x) = x - \sum_{n=1}^{\infty} b_{n,k} \frac{x^n}{n!}$$

where $b_{n,k}$ is the number of simple tangled chains of length k on n leaves. The sequence $b_{n,2}$

counting simple tanglegrams on n leaves when $n \geq 2$ begins

1, 6, 150, 7680, 650160, 81572400, 14177252880, . . .

Chapter 6

FORMAL GROUP LAWS FOR ORDINARY GENERATING FUNCTIONS

6.1 Contractible \mathbb{L} -species

Ordinary generating functions are sometimes more useful than exponential generating functions for combinatorial objects that are defined on an *ordered* set. Thus an alternative theory of species based on totally ordered sets, \mathbb{L} -species, is more appropriate here.

A *totally ordered set* is an n -tuple $V = (v_1, v_2, \dots, v_n)$ for some n , where we write $|V| = n$ and say $v_i < v_j$ if $i < j$. An \mathbb{L} species is a functor \mathcal{F} from the category of totally ordered finite sets and increasing bijections to the category of finite sets and bijections. Thus if \mathcal{F} is an \mathbb{L} -species then for every ordered set V there is an associated set $\mathcal{F}[V]$. The transport of structure here is trivial: for every two ordered sets U, V with $|U| = |V|$ there is a *unique* increasing bijection $\pi : U \rightarrow V$ and then there is an associated map $\mathcal{F}[\pi] : \mathcal{F}[U] \rightarrow \mathcal{F}[V]$. In particular, there is only one increasing bijection $\pi : V \rightarrow V$, namely the identity. Thus there are no distinct \mathcal{F} -structures on V that are isomorphic.

If \mathcal{F} is an \mathbb{L} -species then we let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be its associated (ordinary) generating function, where $a_n = |\mathcal{F}[n]|$, writing write $\mathcal{F}[n]$ for $\mathcal{F}[[n]]$ where $[n] = \{1, 2, \dots, n\}$ with its usual order.

To define *contractible* \mathbb{L} -species, we need only to define an insertion operation for ordered sets. There is an obvious way to do this. If $U = (u_1, \dots, u_m)$ and $V = (v_1, \dots, v_n)$ are totally ordered sets and $v = v_i$ is an element of V , we define $V(v \leftarrow U)$ to be the set $V \setminus \{v\} \cup U$ with the total order attained by replacing v by U , so

$$V(v \leftarrow U) = (v_1, v_2, \dots, v_{i-1}, u_1, \dots, u_m, v_{i+1}, \dots, v_n).$$

Definition. A contractible \mathbb{L} -species is an \mathbb{L} -species \mathcal{F} equipped with an operation \leftarrow of

insertion on \mathcal{F} -structures, so that if $\sigma \in \mathcal{F}[V]$ and $\tau \in \mathcal{F}[U]$ for two disjoint totally ordered sets U, V , then for any $v \in V$ we have an \mathcal{F} -structure $\sigma(v \leftarrow \tau) \in \mathcal{F}[V(v \leftarrow U)]$, and all the axioms (i) - (iii) for a contractible species be satisfied. A weighted contractible \mathbb{L} -species is an \mathbb{L} -species \mathcal{F} equipped with a weight w satisfying (iv).

The edges $E(\sigma)$ of an \mathcal{F} -structure $\sigma \in \mathcal{F}[V]$ for a contractible \mathbb{L} -species \mathcal{F} are defined as they are for an ordinary species: $U \subseteq V$ is an element of $E(\sigma)$ if there is an ordered set W , an element $w \in W$, and \mathcal{F} -structures $\omega \in \mathcal{F}[W], \tau \in \mathcal{F}[U]$ so that $\omega(w \leftarrow \tau) = \sigma$. Note that this implies that $W(v \leftarrow U) = V$ as ordered sets. An *interval* of an ordered set V is a subset $[a, b] = \{x \in V : a \leq x \leq b\}$. By definition of the insertion operator \leftarrow for totally ordered sets, *each edge* $U \in \mathcal{F}[\sigma]$ *is an interval of* V . As before, we let H_σ be the hypergraph $(V, E(\sigma))$. A hypergraph H so that there is an ordering on its vertex set so that all of its edges are intervals is sometimes called an *interval hypergraph*. We see that when \mathcal{F} is a contractible \mathbb{L} -species then every $\sigma \in \mathcal{F}[n]$ determines an interval hypergraph H_σ .

Note that any \mathbb{L} -species \mathcal{F} may be turned into an ordinary species $\hat{\mathcal{F}}$ in a natural way. For any *unordered* finite set V , we let $\hat{\mathcal{F}}(V)$ be the set of pairs (ρ, σ) where $\rho : V \rightarrow [n]$ is a bijection and $n = |V|$, and σ is an \mathcal{F} -structure $\sigma \in \mathcal{F}[V_\rho]$ where V_ρ is the ordered set with the elements of V and the order induced by ρ , so $V_\rho = (\rho^{-1}(1), \dots, \rho^{-1}(n))$. Then $|\hat{\mathcal{F}}[n]| = |\mathcal{F}[n]| \cdot n!$, so the *exponential* generating function $f(x)$ for $\hat{\mathcal{F}}$ coincides with the *ordinary* generating function \mathcal{F} .

If \mathcal{F} is a contractible \mathbb{L} -species then it is not hard to show that $\hat{\mathcal{F}}$ is a contractible species. This gives the following version of Theorem 2.1 for ordinary generating functions.

Theorem 6.1. *Let \mathcal{F} be a contractible \mathbb{L} -species and let $f(x)$ be its ordinary generating function*

$$f(x) = \sum_{n=1}^{\infty} a_n x^n$$

where $a_n = |\mathcal{F}[n]|$. Then

$$f(f^{-1}(x_1) + f^{-1}(x_2) + \dots) = \sum_{\sigma \in \mathcal{F}[n]} X_{H_\sigma}. \quad (6.1)$$

Proof. By applying Theorem 2.1 to $\hat{\mathcal{F}}$ we get

$$f(f^{-1}(x_1) + f^{-1}(x_2) + \cdots) = \sum_{(\rho, \sigma) \in \hat{\mathcal{F}}[n]} \frac{X_{H_{\rho, \sigma}}}{n!}.$$

The hypergraphs $H_{\rho, \sigma}$ associated to the \hat{F} structures (ρ, σ) are all isomorphic to the hypergraph H_σ associated with the \mathcal{F} -structure σ for any given ρ , so we get (6.1). \square

We also have a combinatorial interpretation of the inverse function $f^{-1}(x)$. Here it is somewhat simpler than in the exponential generating function case from Theorem 2.1.

Theorem 6.2. *Let \mathcal{F} be a contractible \mathbb{L} -species and let $f(x)$ be its associated ordinary generating function. Then*

$$f^{-1}(x) = \sum_{n=1}^{\infty} \sum_{\sigma \in \mathcal{F}_c[n]} (-1)^{|M(\sigma)|} \tag{6.2}$$

where $\mathcal{F}_c[n]$ is the set of $\sigma \in \mathcal{F}[n]$ so that H_σ is a connected hypergraph, and $M(\sigma)$ is the set of minimal non-singleton edges in $E(\sigma)$.

Proof. Applying (6.2) in the statement of 2.1 to the contractible species \hat{F} gives

$$f^{-1}(x) = \sum_{n=1}^{\infty} \sum_{(\rho, \sigma) \in \hat{\mathcal{F}}[n]} \sum_{S \in C(\rho, \sigma)} (-1)^{|M(\rho, \sigma)|} \frac{x^n}{n!}$$

recalling that $C(\rho, \sigma)$ is the set of $S \subseteq M(\rho, \sigma)$ so that the hypergraph $H_S = ([n], S)$ is connected. In particular, if $H_{\rho, \sigma}$ is not connected then $C(\rho, \sigma) = \emptyset$. But by our previous discussion, the hypergraphs $H_{\rho, \sigma}$ are interval hypergraphs. Minimal edges cannot contain each other, so we may write $M(\rho, \sigma) = \{U_1, U_2, \dots, U_k\}$ where each U_i , $1 < i < k$, intersects U_{i-1} and U_{i+1} but is disjoint from all other edges. Hence deleting any edge from $M(\rho, \sigma)$ would disconnect the hypergraph, so we see

$$C(\sigma, \rho) = \{M(\sigma, \rho)\}.$$

Then applying Theorem 2.1 gives

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{(\rho, \sigma) \in \hat{F}[n]} \sum_{S \in C(\rho, \sigma)} (-1)^{|M(\rho, \sigma)|} \frac{x^n}{n!} &= \sum_{n=1}^{\infty} \sum_{(\rho, \sigma) \in \hat{F}_c[n]} (-1)^{|M(\rho, \sigma)|} \frac{x^n}{n!} \\ &= \sum_{n=1}^{\infty} \sum_{\sigma \in F_c[n]} (-1)^{|M(\sigma)|} x^n. \end{aligned}$$

□

We now turn to some specific examples of contractible \mathbb{L} -species.

6.2 Plane trees with labeled leaves

We will describe a variant of the trees with labeled leaves from 3.2 for which ordinary generating functions are appropriate. A *plane tree* (also called *embedded tree*) is a rooted tree so that each node is equipped with an ordering of its children. A *leaf* of a tree T is a node that has no children. Let $\mathcal{F}[n]$ be the set of plane trees with n leaves labeled $1, 2, \dots, n$ left-to-right, with all other nodes unlabeled and with no node having exactly one child.

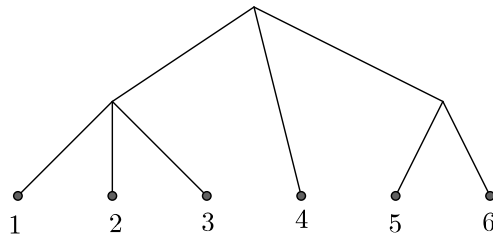


Figure 6.1: A plane tree $T \in \mathcal{F}[6]$.

If $T \in A_n$, i is a leaf of T , and $T' \in A_m$, we form the tree $T(i \leftarrow T')$ by replacing i with the root of T' . With this insertion, \mathcal{F} is a contractible \mathbb{L} -species. As in Section 3.2, the edges of T are the leaf sets of subtrees, and the minimal edges are the complete sets of siblings. For example, if T is the tree depicted in Figure 6.1 then $M(T) = \{123, 56\}$.

Let w be the same weighting described in 3.2, so that $w(T) = s_2 s_3^2$ if T is the tree in Figure 6.1. Then let $f(x)$ be the generating function

$$f(x) = \sum_{n=1}^{\infty} x^n \sum_{T \in A_n} w(T).$$

As in Section 3.2, the only trees T so that $\text{red } H_T$ is connected are the simple trees, those with all leaves having the same parent. Then from 6.2 we see that

$$f^{-1}(x) = x - s_2 x^2 - s_3 x^3 - \dots .$$

If we set $s_2 = 1$ and $s_i = 0$ for $i > 2$ then the coefficient of x^n in $f(x)$ is the number of *binary* trees with n leaves, the Catalan number C_{n-1} , with

$$f(x) = \frac{1 - \sqrt{1 - 4x}}{2}.$$

6.3 Plane trees with all vertices labeled

We give a second interpretation to the formal group law corresponding to Catalan numbers from Section 6.2, which there counted binary trees with labeled leaves. This alternate interpretation naturally leads to a different weighting.

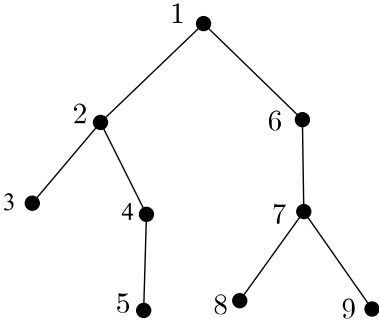


Figure 6.2: A plane tree $T \in \mathcal{F}[9]$.

Given an ordered set V , let $\mathcal{F}[V]$ be the set of all plane trees T with vertex set V so that

1. If v is the child of u then $v < u$.
2. If u_1, \dots, u_k are the children of v then $u_1 < \dots < u_k$, and furthermore if w_i, w_j are descended from u_i, u_j respectively and $i < j$ then $w_i < w_j$.

Given an *unlabeled* plane tree T , there is only one labeling of the vertices of T with the elements of V satisfying the conditions (1) and (2). This labeling is sometimes called the *depth-first search* or *preorder* labeling. Since there is only one possible labeling of each unlabeled tree, $|\mathcal{F}[V]|$ is the number of unlabeled plane trees with $n = |V|$ vertices, which is the Catalan number C_{n+1} . Figure 6.2 depicts a plane tree $T \in \mathcal{F}[n]$.

Given trees $T \in \mathcal{F}[V]$, $S \in \mathcal{F}[U]$, and $v \in V$, let $R = T(v \leftarrow S) \in \mathcal{F}[V(v \leftarrow U)]$ be the tree attained by removing v and its descendants, replacing v with the root of the tree S , and re-attaching the descendants of v to the right-most leaf of S , necessarily $\max U$. An example of insertion of plane trees is depicted in Figure 6.3.

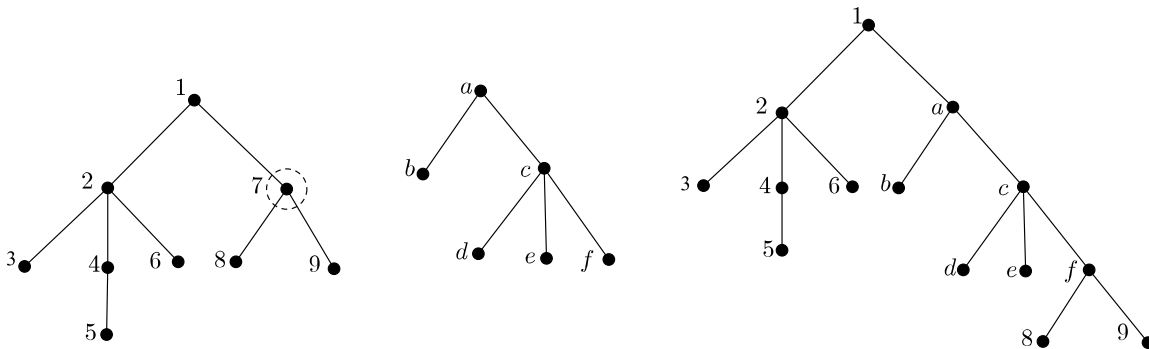


Figure 6.3: Plane trees T, S and $R = T(v \leftarrow S)$ where $v = 7$.

The edges $U \in E(T)$ are those intervals of V that have the property that whenever $v \in V \setminus U$, if v is a descendant of any element of V then v is a descendant of the right-most leaf of U . For example, if T is the tree depicted in Figure 6.2, then we have

$$E(T) = \{1, 2, 3, 4, 5, 7, 8, 9, 45, 789, 67, 234, 2345, 6789, 123456789\}.$$

As in previous examples, we can weigh trees according to the degrees of the vertices. We set $w(T) = s_1^{n_1} s_2^{n_2} \cdots$ where n_i is the number of degrees of vertex i . If we set

$$G(x) = 1 + s_1 x + s_2 x^2 + s_3 x^3 + \cdots,$$

then is not hard to see that the weighted generating function

$$f(x) = \sum_{n=1}^{\infty} \sum_{T \in \mathcal{F}[n]} w(T)$$

satisfies

$$f(x) = xG(f(x)) \tag{6.3}$$

or equivalently

$$f^{-1}(x) = x/G(x).$$

The functional equation (6.3) can be found in [46, Section 5.4], where it is attributed to Etherington [15].

6.4 Lattice paths

Let L be a finite subset of \mathbb{Z} . Define a L -admissible path to be a map $P : [n] \rightarrow \mathbb{N}$ so that $P(1) = P(n) = 0$ and $P(i+1) - P(i) \in L$ for $i = 1, \dots, n-1$, and let $\mathcal{F}_L[n]$ be the set of L -admissible paths P on $[n]$. Let $a_{n,L} = |\mathcal{F}_L[n]|$ and define the generating function $f_L(x) = \sum_{n=1}^{\infty} a_{n,L} x^n$. If $L = \{-1, 1\}$ then an L -admissible path is called a *Dyck path*. Then $a_{n,L} = 0$ if n is even while $a_{2n+1} = C_n$, the n -th Catalan number, and

$$f_L(x) = \frac{1 - \sqrt{1 - 4x^2}}{2x}.$$

If $L = \{-1, 0, 1\}$ then L -admissible paths are called *Motzkin paths* and $a_{n,L} = M_{n+1}$ where M_n is the n -th Motzkin number, with

$$f_L(x) = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x}.$$

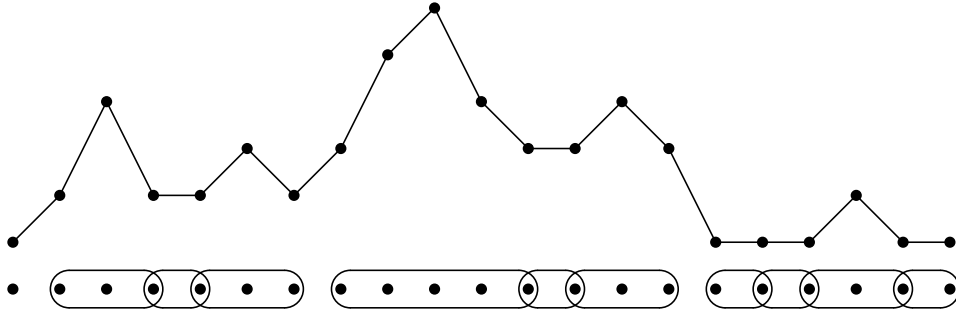


Figure 6.4: A path $P \in \mathcal{F}_L[n]$ where $L = \{-2, -1, 0, 1, 2\}$ and $n = 21$. The minimal edges $U \in M(P)$, circled below, correspond to minimal excursions of P .

Figure 6.4 depicts an L -admissible path for $L = \{-2, -1, 0, 1, 2\}$.

There is an obvious way to insert one path into the vertex of another path. If P is an L -admissible path on $[n]$, $j \in [n]$, and Q is an L -admissible path on $[m]$, define $R = P(j \leftarrow Q)$ to be the path on $[n + m - 1]$ given by

$$R(i) = \begin{cases} P(i) & \text{if } i < j \\ Q(i - j + 1) + P(j) & \text{if } j \leq i < j + m \\ P(i - m + 1) & \text{if } i \geq j + m. \end{cases}$$

It is clear that $R = P(j \leftarrow Q)$ is an L -admissible path when P and Q are. Figure 6.5 depicts an example of path insertion.

Under this definition of insertion it is clear that the edges $E(P)$ of a path P are exactly the *excursions* of P , defined to be the intervals $I = \{a, a+1, \dots, b\} \subseteq [n]$ so that $P(a) = P(b)$ and $P(i) \geq P(a)$ for $i \in I$. It is easy to see then that \mathcal{F} satisfies the axioms of a contractible \mathbb{L} -species.

Note that in the case $L = \{0\}$, the only allowed path on $[n]$ is the constant path $P = 0$, so

$$f_L(x) = \frac{1}{1-x} - 1, f_L^{-1}(x) = \frac{x}{1+x}.$$

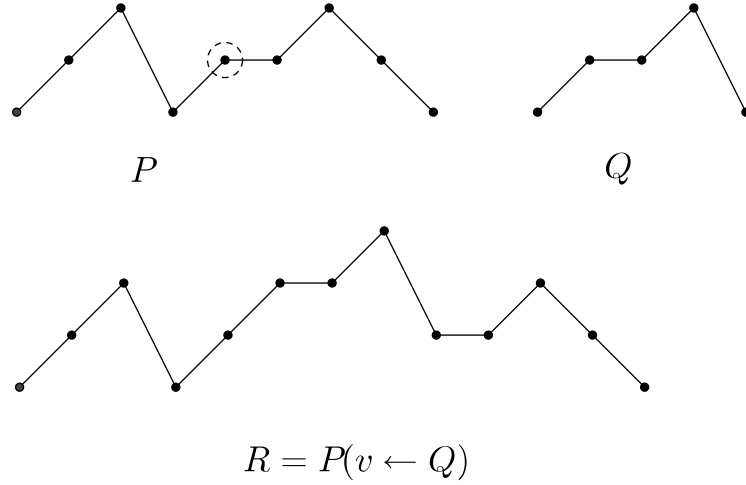


Figure 6.5: L -admissible paths P , Q , and $R = P(j \leftarrow Q)$ where $L = \{-2, -1, 0, 1\}$ and $j = 5$ corresponds to the circled point.

The excursions of the constant path P are exactly the intervals $I \subseteq [n]$, so the formal group law from Section 6.5 counting Smirnov words is recovered. If we take $L = \emptyset$ then only the trivial path $P : [1] \rightarrow \mathbb{N}$ is allowed and get $f_L(x) = x$ corresponding to the trivial formal group law from Section 3.1.

6.5 Permutations

Let \mathcal{F} be the \mathbb{L} -species that assigns to each ordered set $V = (v_1, \dots, v_n)$ the set of permutations $(v_{j_1}, v_{j_2}, \dots, v_{j_n})$. Then the ordinary generating function associated to \mathcal{F} is

$$f(x) = \sum_{n=1}^{\infty} n! x^n.$$

The power series $f(x)$ is nowhere convergent, but as a formal power series it still has a well-defined inverse $f^{-1}(x)$. Then \mathcal{F} can be given the structure of a contractible \mathbb{L} -species. In this case, the insertion \leftarrow is most easily described in terms of permutation matrices. Let M_σ be the permutation matrix of $\sigma \in S_n$, given by the entries $a_{ij} = 1$ if $\sigma(j) = i$, with $a_{ij} = 0$ otherwise. Then if $\sigma \in S_m$, $j \in [m]$, and $\sigma' \in S_k$, we define the permutation $\sigma(j \leftarrow \sigma')$ by

letting $M_{\sigma(j \leftarrow \sigma')}$ be the matrix given by inserting $M_{\sigma'}$ as a $k \times k$ block into the entry $(j, \sigma(j))$ of M_{σ} . For example, if $\sigma = 41523$, $j = 4$, and $\sigma' = 213$ then

$$M_{\sigma} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad M_{\sigma'} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad M_{\sigma(4 \leftarrow \sigma')} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{0} & \boxed{1} & \boxed{0} & 0 \\ 0 & 0 & 0 & \boxed{1} & \boxed{0} & \boxed{0} & 0 \\ 0 & 0 & 0 & \boxed{0} & \boxed{0} & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and so $\sigma(4 \leftarrow \sigma') = 6173245$. From here we see how to define the associated hypergraph H_{σ} for a permutation $\sigma \in S_n$. We let \hat{E}_{σ} be the set of intervals $I \subseteq [n]$ so that σ maps I to another interval $J \subseteq [n]$. Thus $\{(j, \sigma(j)) : j \in I\}$ is the set of entries with 1's in an $|I| \times |I|$ block within M_{σ} which is a permutation matrix in itself. We then define E_{σ} to be the minimal non-singleton elements of \hat{E}_{σ} . For example, if $\sigma = 659421387$ then E_{σ} consists of the intervals $\{1, 2\}$, $\{4, 5, 6, 7\}$, $\{8, 9\}$ since these map to the intervals $\{5, 6\}$, $\{1, 2, 3, 4\}$, and $\{7, 8\}$ respectively and they are minimal with respect to this property.

One application of Theorem 2.1 is to count *simple permutations*. A permutation $\sigma \in \mathfrak{S}_n$ is called *simple* if $E_{\sigma} = \{[n]\}$, so that σ does not map any proper non-singleton subinterval of $[n]$ to another subinterval of $[n]$. For example, the permutations 12 and 24153 are simple, but 253641 is not simple because it maps the interval $\{2, 3, 4, 5\}$ to the interval $\{3, 4, 5, 6\}$. We can also state this in terms of the permutation matrix M_{σ} : if σ is simple then M_{σ} has no $k \times k$ block that is itself a permutation matrix, unless $k = 1$ or n . The following proposition was first found by found by Albert and Atkinson [2].

Proposition 6.3. *Let $f(x) = \sum_{n=1}^{\infty} n! x^n$ and let s_n be the number of simple permutations in \mathfrak{S}_n . Then*

$$f^{-1}(x) = x - 2x^2 + \sum_{n=3}^{\infty} (2(-1)^{n-1} - s_n) x^n \quad (6.4)$$

Proof. From Theorem 2.1 we get a combinatorial interpretation of $f^{-1}(x)$.

Clearly if $\sigma \in S_n$ is simple then H_σ is connected and so $\sigma \in C_n$. In fact, we will show that all but two of the permutations in C_n are simple for $n > 2$. It is not hard to show that if I_1, I_2 are distinct elements of E_σ with $I_1 \cap I_2 \neq \emptyset$ then $|I_1| = |I_2| = 2$. It follows if H_σ is a connected hypergraph then H_σ is either simple, the identity $123 \cdots n$, or the reverse $n \cdots 321$. In the latter two cases we have $E_\sigma = \{\{1, 2\}, \{2, 3\}, \dots, \{n-1, n\}\}$, so $|E_\sigma| = n-1$. Then using

$$f^{-1}(x) = x - 2x^2 + \sum_{n=3}^{\infty} (2(-1)^{n-1} - s_n) x^n \quad (6.5)$$

where s_n is the number of simple permutations in S_n , which is the sequence 1, 2, 0, 2, 6, 46, 338, ... (Sequence A059372 in [41].) □

Chapter 7

EXTRACTING COEFFICIENTS OF FORMAL GROUP LAWS

7.1 Associated polynomials

Formal group laws arise naturally from the study of certain polynomial sequences. If $(p_n(x))_{n=0}^{\infty}$ is a sequence of polynomials with $\deg p_n = n$, then by triangularity these polynomials form a basis for the ring $\mathbb{Q}[x]$ of polynomials, and we may ask for the coefficients c_{ij}^n in the expansion

$$p_i(x)p_j(x) = \sum_{n=0}^{i+j} c_{ij}^n p_n(x).$$

These coefficients c_{ij}^k are called the *linearization* or *structure* coefficients of the polynomial sequence $(p_n(x))$. More generally, if α is a *weak composition*, a tuple $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ of nonnegative integers $\alpha_i \in \mathbb{N}$, we may consider the coefficients $c_{\alpha,n}$ in the expansion

$$p_{\alpha_1}(x)p_{\alpha_2}(x) \cdots p_{\alpha_k}(x) = \sum_n c_{\alpha,n}^k p_n(x). \quad (7.1)$$

It is often useful to categorize polynomial sequences by the form of their generating functions. In what follows we will be interested in polynomial sequences $\{p_n(t)\}$ defined by a generating function

$$\sum_{n=0}^{\infty} p_n(t)x^n = e^{tf^{-1}(x)} \quad (7.2)$$

where $f(x)$ is a formal power series. We say that $\{p_n(t)\}$ is the *associated sequence* of $f(x)$. These sequences are important in the *umbral calculus* developed by Rota and Roman [38]. (We normalize the polynomials differently from Roman, whose $p_n(t)$ is our $n!p_n(t)$. The polynomials $n!p_n(t)$ are also known as *polynomials of binomial type*.)

There is a simple relationship between the associated polynomials and the formal group law. The linearization coefficients of associated sequence are the exactly the monomial

coefficients of the formal group law and its powers. This was shown by Lenart [?]. We use the notation $[x^\alpha]X$ to mean the coefficient of $x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_l^{\alpha_l}$ in X when α is a tuple $(\alpha_1, \dots, \alpha_l)$ of nonnegative integers.

Proposition 7.1. *Let c_α^k be the linearization coefficients of the polynomial sequence $(p_n(t))$ associated to the power series $f(x)$ as defined by (7.1). Then*

$$c_\alpha^k = [x^\alpha] \left(f(f^{-1}(x_1) + f^{-1}(x_2) + \cdots) \right)^k.$$

Proof. We have

$$\begin{aligned} \sum_{\alpha} p_{\alpha_1}(t) \cdots p_{\alpha_l}(t) x^\alpha &= \prod_{i=1}^{\infty} \sum_{n=0}^{\infty} p_n(t) x_i^n \\ &= \exp \left(t(f^{-1}(x_1) + f^{-1}(x_2) + \cdots) \right) \\ &= \exp \left(t f^{-1} \left(f(f^{-1}(x_1) + f^{-1}(x_2) + \cdots) \right) \right) \\ &= \sum_{k=0}^{\infty} p_k(t) \left(f(f^{-1}(x_1) + f^{-1}(x_2) + \cdots) \right)^k \end{aligned}$$

and the statement follows by extracting the coefficient of x^α of both sides. \square

Thus knowing a formal group law is equivalent to knowing the linearization coefficients of the associated polynomial sequence. In particular, for each contractible species or \mathbb{L} -species we have discussed so far we have a combinatorial interpretation for these linearization coefficients. In Section 7.2 we examine these individual examples further.

The associated polynomial sequence also gives an efficient way of extracting coefficients of the formal group law in the monomial basis. Given a power series $f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$ define a linear functional $\Phi_f : \mathbb{Q}[t] \rightarrow \mathbb{Q}$ on the space of polynomials $\mathbb{Q}[t]$ by the rule $\Phi_f(t^n) = a_n$. The lemma below tells us that $\Phi_f(p(t))$ is the coefficient of $p_1(t)$ in the expansion of $p(t)$ in the basis $(p_n(t))$.

Lemma 7.2. *Let $\{p_n(t)\}$ be the associated sequence of a power series $f(x)$. Then for any $k \in \mathbb{N}$,*

$$\Phi_{f^k}(p_n(t)) = 1$$

for $n = k$ and otherwise $\Phi_{f^k}(p_n(t)) = 0$. (Here $f^k(x) = f(x)^k$.)

Proof. Extend Φ_{f^k} to be a linear map $\mathbb{Q}[[x]][t] \rightarrow \mathbb{Q}[[x]]$ by setting $\Phi(x^m t^n) = x^m a_{n,k}$ where $a_{n,k} = \left[\frac{x^n}{n!}\right]f(x)^k$. Then we have

$$\begin{aligned} \Phi\left(\sum_{n=0}^{\infty} p_n(t)x^n\right) &= \Phi\left(e^{tf^{-1}(x)}\right) \\ &= \Phi\left(\sum_{n=0}^{\infty} \frac{(tf^{-1}(x))^n}{n!}\right) \\ &= \sum_{n=0}^{\infty} \frac{a_{n,k} (f^{-1}(x))^n}{n!} \\ &= f(f^{-1}(x))^k = x^k \end{aligned}$$

and extracting the coefficient of x^k from both sides finishes the proof. □

Using Lemma 7.2 together with Proposition 7.1 immediately gives a formula for the coefficient of x^α in a formal group law.

Theorem 7.3. *Let $\{p_n(t)\}$ be the associated sequence of a power series $f(x)$. Then for any weak composition $\alpha = (\alpha_1, \dots, \alpha_l)$ we have*

$$\Phi_{f^k}(p_{\alpha_1}(t) \cdots p_{\alpha_l}(t)) = [x^\alpha]f(f^{-1}(x_1) + f^{-1}(x_2) + \cdots)^k.$$

In particular,

$$\Phi_f(p_{\alpha_1}(t) \cdots p_{\alpha_l}(t)) = [x^\alpha]f(f^{-1}(x_1) + f^{-1}(x_2) + \cdots).$$

Thus if formulas are known for the associated polynomials $p_n(t)$ and the coefficients a_n of a power series $f(x)$ then the coefficients of the corresponding formal group law can be quickly evaluated.

7.2 Examples

We now turn to the associated sequences for some of the examples we have seen so far. A complete list is given in the Appendix.

7.2.1 Sets - binomials

Let \mathcal{F} be the contractible \mathbb{L} -species of *sets* from Section 3.1: $\mathcal{F}[V] = \{V\}$. The (exponential) generating function for \mathcal{F} is then $f(x) = e^x - 1$, so the associated polynomial sequence $p_n(t)$ defined by

$$\begin{aligned} \sum_{n=1}^{\infty} p_n(t)x^n &= e^{t(\log(1+x))} \\ &= (1+x)^t. \end{aligned}$$

We then see from Newton's binomial theorem that $p_n(t) = \binom{t}{n} = \frac{t(t-1)\cdots(t-n+1)}{n!}$.

Recall that

$$f(f^{-1}(x_1) + f^{-1}(x_2) + \cdots) = (1+x_1)(1+x_2)\cdots - 1 = e_1 + e_2 + \cdots, \quad (7.3)$$

so that (7.3) is the sum of all positive-degree squarefree monomials in x_1, x_2, \dots . Theorem 7.3 then becomes the obvious statement that

$$\binom{1}{n_1} \binom{1}{n_2} \cdots \binom{1}{n_k}$$

is 1 if $n_i \leq 1$ for each i , and is 0 otherwise.

7.2.2 Trees with labeled leaves: Bell polynomials

Let \mathcal{F} be the contractible species of *trees with labeled leaves* from 3.2 with the associated weight $w(T) = s_2^{k_2} s_3^{k_3} \cdots$ where k_i is the number of internal nodes of the tree T with exactly i children. Recall from equation (3.5) that if $f(x)$ is the weighted generating function for \mathcal{F} then

$$f^{-1}(x) = x - s_2 \frac{x^2}{2!} - s_3 \frac{x^3}{3!} - \cdots.$$

The associated polynomials $p_n(t)$, actually functions of t and s_2, s_3, \dots , are then defined by

$$\sum_{n=1}^{\infty} p_n(t)x^n = e^{t(x - s_2 \frac{x^2}{2!} - \cdots)}$$

The (complete) *Bell polynomials* B_n are the sequence of polynomials defined by

$$\sum_{n=1}^{\infty} B_n(s_1, s_2, \dots) \frac{x^n}{n!} = e^{x+s_2 \frac{x^2}{2!} + \dots}. \quad (7.4)$$

and therefore we see

$$p_n(t) = \frac{1}{n!} B_n(t, -s_2 t, -s_3 t, \dots).$$

The Bell polynomials represent the universal associated polynomial sequence, in the sense that every polynomial sequence $p_n(t)$ associated to some $f(x)$ is a specialization of $B_n(t)$.

7.2.3 Binary trees - Hermite polynomials

A specialization of the above polynomials $p_n(t)$ is of some interest. If we set $s_2 = 1$ and $s_i = 0$ for $i > 2$ in the generating function for trees above we get the generating function

$$f(x) = \frac{1 - \sqrt{1 - 4x}}{2} = \sum_{n=1}^{\infty} C_{n-1} x^n$$

for the Catalan numbers C_n (with a shift). Then $f^{-1}(x) = x - x^2/2$ so the associated polynomials $p_n(t)$ are defined by

$$\sum_{n=0}^{\infty} p_n(t) x^n = e^{t(x-x^2/2)}. \quad (7.5)$$

The polynomials $p_n(t)$ are essentially the same as the classical sequence of polynomials known as the Hermite polynomials, defined by

$$\sum_{n=0}^{\infty} \frac{H_n(t)}{n!} x^n = e^{tx-x^2/2}. \quad (7.6)$$

Note that Hermite polynomials are sometimes normalized differently. The polynomials $H_n(t)$ are sometimes called the *probabilists'* Hermite polynomials.

Comparing (7.5) and (7.6), we see that

$$p_n(t) = \frac{1}{n!} (\sqrt{t})^n H_n(\sqrt{t}).$$

7.2.4 Permutations - Laguerre polynomials

Let \mathcal{F} be the contractible species of *permutations* from 3.4. Then the associated generating function is $f(x) = \frac{x}{1-x}$. (Equivalently, as far as the generating function $f(x)$ goes, we could let \mathcal{F} be the trivial contractible \mathbb{L} -species with $\mathcal{F}[V] = \{V\}$ for any V .)

Then the associated polynomials $p_n(t)$ are a form of Laguerre polynomial. Specifically, $p_n(t) = (-1)^n L_n^{(-1)}(t)$ where

$$L_n^{(\alpha)}(t) = \sum_{i=0}^n (-1)^i \binom{n+\alpha}{n-i} \frac{t^i}{i!}$$

is the *generalized Laguerre polynomial*. From Theorem 7.3 shows that

$$\Phi_f(p_{n_1}(t) \cdots p_{n_k}(t)) \tag{7.7}$$

is the coefficient of $x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$ in the formal group law $f(f^{-1}(x_1) + f^{-1}(x_2) + \cdots)$. From 3.4 we know that (7.7) is the number of Smirnov words using n_1 ones, n_2 twos, etc. This fact was first shown by Gessel [21] using a variant of rook theory. Some further applications of Gessel's formula to counting words with various restrictions are given in [48].

Chapter 8

ASYMPTOTICS OF FORMAL GROUP LAW COEFFICIENTS

Using the tools from Chapter 7, we can determine the asymptotics of the coefficients in a formal group law. Fix a power series $f(x)$. For a weak composition $\alpha = (\alpha_1, \dots, \alpha_l)$ let c_α be the coefficient of $x_1^{\alpha_1} \cdots x_l^{\alpha_l}$ in $f(f^{-1}(x_1) + f^{-1}(x_2) + \cdots)$. For fixed k we will demonstrate how to determine the asymptotics of the sequence

$$c_{(k)^n} = [x_1^k x_2^k \cdots x_n^k] f(f^{-1}(x_1) + f^{-1}(x_2) + \cdots). \quad (8.1)$$

In the where $f(x) = \sum_{n=1}^{\infty} a_n x^n$ is the ordinary generating function for a contractible \mathbb{L} -species \mathcal{F} , so that $a_n = |\mathcal{F}[n]|$, (8.1) is the number of pairs (σ, χ) where χ is a proper coloring of $\sigma \in \mathcal{F}[kn]$ using exactly k of each color. The total number of colorings of \mathcal{F} -structures on a set with $[kn]$ elements using k each of n colors is $a_{kn} \frac{(kn)!}{(k!)^n}$. Thus the probability that a random permutation of the multiset with k copies each of $1, \dots, n$ is a proper coloring of a random \mathcal{F} -structure $\sigma \in \mathcal{F}[kn]$ is

$$\frac{c_{(k)^n}}{a_{kn} \frac{(kn)!}{(k!)^n}}.$$

We consider what happens to this probability as n tends to infinity.

Let $\{p_k(t)\}$ be the associated sequence of polynomials to $f(x)$. In the case of $\alpha = (k)^n$, we have $c_{(k)^n} = \Phi_f(p_k(t)^n)$ by Theorem 7.3. Thus it will suffice to compute the asymptotic behavior of sequences of the form $\Phi_f(p(t)^n)$ where $p(t)$ is a polynomial. The strategy is to first handle the case when $p(t)$ has only the two terms of highest power and then show that, under certain conditions, any other terms of $p(t)$ have negligible effect.

Theorem 8.1. *Let $f(x) = a_0 + a_1 x + a_2 x^2 + \dots$ be a power series, where $\{a_n\}$ is a sequence of real numbers and the limit*

$$r = \lim_{n \rightarrow \infty} \frac{a_{n-1}}{a_n}$$

exists, so that $f(x)$ is analytic for $|x| < r$. Let $p(t) = b_k t^k + b_{k-1} t^{k-1}$ where $b_k \neq 0$ and $k > 0$.

Then

$$\Phi_f(p(t)^n) \sim a_{kn}(kn)! b_k^n e^{\frac{r b_{k-1}}{k b_k}}$$

as $n \rightarrow \infty$.

Proof. By scaling we can assume that $b_k = 1$. In this case, we have

$$\begin{aligned} \Phi(p(t)^n) &= \Phi\left(\sum_i \binom{n}{i} t^{k(n-i)} (b_{k-1} t^{k-1})^i\right) \\ &= \Phi\left(\sum_i \binom{n}{i} t^{kn-i} b_{k-1}^i\right) \\ &= \sum_i \binom{n}{i} a_{kn-i} (kn-i)! b_{k-1}^i \end{aligned}$$

and so

$$\frac{\Phi(p(t)^n)}{a_{kn}(kn)!} = \sum_i \frac{a_{kn-i}}{a_{kn}} \frac{n(n-1)\cdots(n-i+1)}{kn(kn-1)\cdots(kn-i+1)} \frac{b_{k-1}^i}{i!}. \quad (8.2)$$

The term corresponding to index i in (8.2) is eventually bounded by $\frac{|(r+1)b_{k-1}|^i}{i!}$ and tends to $\frac{1}{i!} \left(\frac{r b_{k-1}}{k}\right)^i$ as n tends to ∞ . Applying the dominated convergence theorem then proves the claim. \square

Corollary 8.2. *Let \mathcal{F} be a contractible \mathbb{L} -species with an ordinary generating function $f(x) = \sum_{n=1}^{\infty} a_n x^n$ having radius of convergence r . Suppose that $\sigma \in \mathcal{F}[2n]$ is chosen uniformly at random, as is a coloring of $[2n]$ with the colors $1, 1, 2, 2, \dots, n, n$. Then the probability that χ is a proper coloring of H_σ tends to $e^{-a_2 r}$ as $n \rightarrow \infty$.*

Proof. We calculate $p_2(t) = \frac{t^2}{2!} - a_2 t$. Then by 8.1 the limiting probability that the coloring is proper is

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{c_{(2)^n}}{a_{2n} \frac{(2n)!}{(2!)^n}} &= \lim_{n \rightarrow \infty} \frac{\Phi_f\left(\frac{t^2}{2!} - a_2 t\right)^n}{a_{2n} \frac{(2n)!}{(2!)^n}} \\ &= e^{-a_2 r}. \square \end{aligned}$$

For example, let $f(x) = x/(1-x)$. Then $r = a_2 = 1$. Corollary 8.2 then provides the limit

$$\lim_{n \rightarrow \infty} \frac{c_{(2)^n}}{(2n)!/2^n} = e^{-1}.$$

Recall from 3.4 that the coefficient c_α in the formal group law counts the number of Smirnov words with α_1 copies of 1, α_2 copies of 2, etc. Thus if you are given a large number of pairs of distinct symbols and scramble them, the chance you will form a word with no repeated letters is about $\frac{1}{e}$.

For another example, let $f(x) = \frac{1-\sqrt{1-4x}}{2}$, the generating function for Catalan numbers. Then we see that $r = \frac{1}{4}$ and $a_2 = 1$. Thus if we have n pairs of colors and form a random binary tree with these $2n$ colored nodes as leaves, the chance that no pair are “siblings” is about $e^{-\frac{1}{4}} = 0.77880\dots$. Indeed, for $n = 4000$ we calculate

$$\frac{c_{(2)^n}}{a_{2n} \frac{(2n)!}{(2)^n}} = 0.77876\dots$$

Now we handle the general case. The asymptotics of $c_{(k)^n}$ can be determined from Theorem 8.1 by showing that it suffices to approximate $p(t)$ by the two highest power terms. To do so, we write $p(t) = q(t) + s(t)$ where $q(t) = b_k t^k + b_{k-1} t^{k-1}$ and $s(t) = b_{k-2} t^{k-2} + \dots$ and show that the terms

$$\binom{n}{i} \Phi(q(t)^{n-i} r(t)^i)$$

in the binomial expansion of $\Phi((q(t) + r(t))^n)$ are negligible for $i > 1$. To get an estimate of these terms, we use a version of Hölder’s inequality. This requires that we put some restriction on the coefficients a_n of $f(x)$. We will assume that there is a nonnegative measure μ on $[0, \infty)$ so that

$$n! a_n = \int_0^\infty t^n d\mu(t).$$

Then

$$\Phi_f(p(t)) = \int_0^\infty p(t) d\mu(t)$$

The problem of finding which sequences a_n have this property is called the *Stieltjes moment problem*. More information can be found in [1, 5].

For example, we have the well-known integral

$$\int_0^\infty e^{-t} t^n dt = n!.$$

Thus if $f(x) = x/(1-x)$ then so that if $p(0) = 0$ we have

$$\Phi_f(p(t)) = \int_0^\infty e^{-t} p(t) d\mu(t)$$

where $d\mu(t) = e^{-t} dt$. Similarly, using well-known evaluations of the Gamma function gives

$$n!C_{n-1} = \frac{(2n-2)!}{(n-1)!} = \int_0^\infty t^n \left(e^{t/4} t^{3/2} 2\sqrt{2\pi} \right)^{-1} dt.$$

For a polynomial $p(t) = b_k t^k + b_{k-1} t^{k-1} + \dots$, we use the notation $|p|(t) = |b_k| t^k + |b_{k-1}| t^{k-1} + \dots$. Then we have the following version of Hölder's inequality.

Lemma 8.3. *Hölder's inequality for polynomials*

Let μ be a nonnegative measure on $[0, \infty)$ and define a linear functional Φ on polynomials by

$$\Phi(p(t)) = \int_0^\infty p(t) d\mu(t).$$

Then if $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\Phi(r(t)s(t)) \leq (\Phi(|r|(t)^p))^{1/p} (\Phi(|s|(t)^q))^{1/q}.$$

for any polynomials $r(t), s(t)$.

Theorem 8.4. Let $f(x) = a_0 + a_1 x + a_2 x^2 + \dots$ be a power series with real coefficients so that $n! a_n = \int_0^\infty t^n d\mu(x)$ for some nonnegative measure μ on $[0, \infty)$ and that the limit

$$r = \lim_{n \rightarrow \infty} \frac{a_{n-1}}{a_n}$$

exists, so that $f(x)$ is analytic for $|x| < r$. Let $p(t) = b_k t^k + b_{k-1} t^{k-1} + \dots$ be a polynomial of degree $k > 0$. Then

$$\Phi_f(p(t)^n) \sim a_{kn} (kn)! b_k^n \exp(kr b_{k-1}/b_k)$$

as $n \rightarrow \infty$.

Proof. We induct on the degree k . The case $k = 1$ is trivial and $k = 2$ is handled by Theorem 8.1. Write $p(t) = q(t) + s(t)$, where $q(t) = b_k t^k + b_{k-1} t^{k-1}$ and $s(t) = b_{k-2} t^{k-2} + \dots$. After rescaling, we may take $b_k = 1$.

Our goal is to show that the estimate $p(t) \approx q(t)$ suffices. If $k > 2$, then by Theorem 8.1 and our induction hypothesis we may choose $M > 0$ so that $\Phi(|q|(t)^n) \leq 2a_{kn}(kn)!$ and $\Phi(|s|(t)^n) \leq a_{kn-2n}(kn-2n)!M^n$ for all sufficiently large n . Then for $i > 0$, applying Hölder's inequality with $p = \frac{n}{n-i}$, $q = \frac{n}{i}$ gives, for large n ,

$$\begin{aligned}
\frac{|\Phi(q(t)^{n-i} s(t)^i)|}{a_{kn}(kn)!} &\leq \frac{\Phi(|q|(t)^n)^{(n-i)/n} \Phi(|s|(t)^n)^{i/n}}{a_{kn}(kn)!} \\
&\leq \frac{(2a_{kn}(kn)!)^{(n-i)/n} (a_{kn-2n}(kn-2n)!M^n)^{i/n}}{a_{kn}(kn)!} \\
&= 2 \left(\frac{M^n a_{kn-2n}(kn-2n)!}{2a_{kn}(kn)!} \right)^{i/n} \\
&\leq 2 \left(M^n \frac{a_{kn-2n}}{a_{kn}(kn)(kn-1)\dots(kn-2n-1)} \right)^{i/n} \\
&\leq 2 \left(\frac{a_{kn-2n}}{a_{kn}(kn-2n-1)^{2n}} \right)^{i/n} \\
&= 2 \left(\left(\frac{a_{kn-2n}}{a_{kn}} \right)^{1/n} \frac{M}{a_{kn}(k-2)^2 n^2} \right)^i.
\end{aligned}$$

We have

$$\lim_{n \rightarrow \infty} \left(\frac{a_{kn-2n}}{a_{kn}} \right)^{1/n} = r^2$$

so we may assume

$$\left(\frac{a_{kn-2n}}{a_{kn}} \right)^{1/n} \leq 2r^2.$$

Finally, we have

$$\begin{aligned}
\left| \frac{\Phi(p(t)^n - q(t)^n)}{a_{kn}(kn)!} \right| &\leq \sum_{i=1}^n \binom{n}{i} \frac{\Phi(|q|(t)^{n-i}|s|(t)^i)}{a_{kn}(kn)!} \\
&\leq 2 \sum_{i=1}^n \binom{n}{i} \left(\frac{2r^2 M}{c(k-2)^2 n^2} \right)^i \\
&= 2 \left(1 + \frac{2r^2 M}{c(k-2)^2 n^2} \right)^n - 2 \\
&\leq 2 \exp \left(\frac{2r^2 M}{c(k-2)^2 n} \right) - 2
\end{aligned} \tag{8.3}$$

where we have used the inequality $(1 + x/n)^n \leq e^x$ in the last step. The expression (8.3) tends to 0 as $n \rightarrow \infty$, so $\Phi(p(t)^n)$ is asymptotically equal to $\Phi(r(t)^n)$ and are finished after applying Theorem 8.1. \square

Now to compute the asymptotic behavior of coefficients $c_{(k)^n}$ of a formal group law it is enough to compute two highest order terms in the associated polynomial $p_k(t)$.

Corollary 8.5. *Let $f(x) = x + a_2 x^2 + \dots$ so that the sequence $m_n = n! a_n$ for $n > 0$, $m_0 = 1$, is a solution to the Stieltjes moment problem. Define c_α by*

$$\sum_{\alpha} c_{\alpha} x^{\alpha} = f(f^{-1}(x_1) + f^{-1}(x_2) + \dots).$$

Then

$$c_{(k)^n} \sim a_{kn}(kn)! e^{-a_2 r(k-1)}$$

as $n \rightarrow \infty$, where $(k)^n$ denotes the multi-index $\alpha = (k, k, \dots, k)$ with n parts.

Proof. Letting $p_k(t)$ be the polynomial sequence associated to $f(x)$, we have

$$\sum_k p_k(t) x^k = e^{t f^{-1}(x)} = \sum_k \frac{t^k}{k!} f^{-1}(x)^k$$

and we compute $[t^i] p_k(t) = [x^i] \frac{f^{-1}(x)^k}{k!}$ for any i . Since $f^{-1}(x) = x - a_2 x^2 + \dots$, we compute $p_k(t) = \frac{t^k}{k!} - \frac{a_2 t^{k-1}}{(k-2)!} + \dots$. Then applying Theorem 8.4 finishes the proof. (Since $p_k(0) = 0$ for $k > 0$ we can ignore the fact that $\Phi_f(1) = 0$.) \square

Chapter 9

CONJECTURES AND SUGGESTIONS FOR FURTHER WORK

9.1 A characterization of positivity for formal group laws?

We have considered a number of examples of formal group laws that can be given combinatorial interpretations showing positivity. If a symmetric function has nonnegative coefficients in a basis $\{b_\lambda\}$ of Λ we will say that it is *b-positive*. Thus all the examples of formal group laws we have seen so far have been *m-positive*, where m_λ is the monomial symmetric function associated to a partition λ .

A question presents itself. Given a power series $f(x)$, is there a way to determine if the formal group law $f(f^{-1}(x_1) + f^{-1}(x_2) + \dots)$ is *m-positive*? We give, conjecturally, an answer.

Conjecture A. *For any power series $f(x) \in \mathbb{R}[[x]]$ with $f(0) = 0$, $f'(0) = 1$, the formal group law $f(f^{-1}(x_1) + f^{-1}(x_2) + \dots)$ is *m-positive* if and only if*

$$\frac{1}{(f^{-1})'(x)}$$

has nonnegative coefficients.

There is strong theoretical and empirical support for Conjecture A. In particular, necessity of the condition is easily shown.

Theorem 9.1. *Let $f(x) \in \mathbb{R}[[x]]$ with $f(0) = 0$, $f'(0) = 1$, be a power series so that the formal group law $f(f^{-1}(x_1) + f^{-1}(x_2) + \dots)$ is *m-positive*. Then*

$$\frac{1}{(f^{-1})'(x)}$$

has nonnegative coefficients.

Proof. Note that

$$\frac{\partial}{\partial y} f(f^{-1}(x) + f^{-1}(y)) = f'(f^{-1}(x) + f^{-1}(y)) \cdot (f^{-1})'(y). \quad (9.1)$$

The properties $f(0) = 0$, $f'(0) = 1$ imply $f^{-1}(0) = 0$, $(f^{-1})'(0) = 1$ as well. Then setting $y = 0$ in (9.1) gives

$$\left. \frac{\partial}{\partial y} f(f^{-1}(x) + f^{-1}(y)) \right|_{y=0} = f'(f^{-1}(x)) = \frac{1}{(f^{-1})'(x)}. \quad (9.2)$$

Since the operation of differentiation preserves nonnegativity of coefficients, we see that $\frac{1}{(f^{-1})'(0)}$ has nonnegative coefficients when $f(f^{-1}(x) + f^{-1}(y))$ does. \square

We also have evidence of the converse to Theorem 9.1. Let

$$\phi(x) = 1 + s_1 x + s_2 \frac{x^2}{2!} + \cdots$$

where s_1, s_2, \dots are indeterminates. Define $f(x)$ by $f(0) = 0$ and

$$\frac{1}{(f^{-1})'(x)} = \phi(x)$$

or, equivalently, the differential equation

$$f'(x) = \phi(f(x)).$$

Then we compute

$$f(x) = x + s_1 \frac{x^2}{2!} + (s_1^2 + s_2) \frac{x^3}{3!} + (s_1^3 + 4s_1 s_2 + s_3) \frac{x^4}{4!} + \cdots. \quad (9.3)$$

We then calculate the associated formal group law $F(x_1, x_2, \dots) = f(f^{-1}(x_1) + f^{-1}(x_2) + \cdots)$ to be

$$F(x_1, x_2, \dots) = m_1 + \frac{1}{2!} (2s_1 m_{11}) + \frac{1}{3!} (6s_1^2 + 6s_2 m_{111} + 3s_2 m_{21}) + \cdots. \quad (9.4)$$

This computation leads us to the following conjecture.

Conjecture B. Define $f(x)$ by $f'(x) = \phi(f(x))$, $f(0) = 0$, where

$$\phi(x) = 1 + s_1x + s_2\frac{x^2}{2!} + s_3x\frac{x^3}{3!} + \cdots.$$

Then the coefficient of $\frac{m\lambda}{n!}$ in the formal group law $f(f^{-1}(x_1) + f^{-1}(x_2) + \cdots)$ is a polynomial with nonnegative integral coefficients for each partition λ of n .

Using the mathematics software Sage [47], we have confirmed Conjecture B up to degree 25.

Conjecture B implies Conjecture A. For suppose Conjecture B holds and $f(x) \in \mathbb{R}[x]$ has

$$\frac{1}{(f^{-1})'(x)} = 1 + a_1x^2 + a_2x^3 + \cdots$$

with each $a_i \geq 0$. Then $f(f^{-1}(x_1) + f^{-1}(x_2) + \cdots)$ is given by setting $s_i = a_i$ for all i in (9.4) which must result in nonnegative coefficients.

There is a simple combinatorial interpretation for (9.3) in terms of *increasing trees*. An increasing tree is a rooted tree whose vertices are integers with the property that if v is the child of u then $u < v$. Let A_n be the set of increasing trees with vertices labeled $[n]$. To each $T \in A_n$ define the weight $w(T)$ of T as in Example 3.3, so that $w(T) = s_1^{n_1}s_2^{n_2}\cdots s_l^{n_l}$ where n_i is the number of nodes with exactly i children. An increasing tree $T \in A_{11}$ is depicted in Figure 9.1. It was shown by Bergeron, Flajolet and Salvy [8] that if $f(x)$ satisfies the ODE $f(0) = 0$, $f'(x) = \phi(f(x))$, where $\phi(x) = 1 + s_1x + s_2x^2/2! + \cdots$, then

$$f(x) = \sum_{n=1}^{\infty} \sum_{T \in A_n} w(T) \frac{x^n}{n!}$$

is the generating function for increasing trees.

It seems very likely that there is a simple combinatorial interpretation for the formal group law (9.4) that would give a proof of Conjectures A and B. We give such a combinatorial interpretation in two special cases.

First, we consider the unweighted case, where we set $s_1 = s_2 = \cdots = 1$. We compute

$$f(x) = -\log(1-x) = \sum_{n=1}^{\infty} (n-1)! \frac{x^n}{n!}.$$

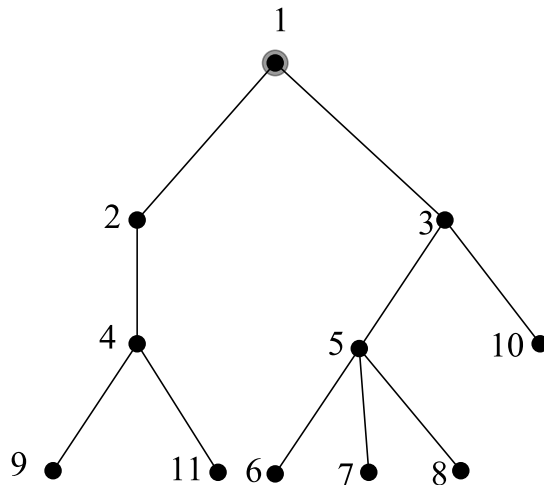


Figure 9.1: An increasing plane tree $T \in A_{11}$ with weight $w(T) = s_1 s_2^3 s_3$.

Let $\mathcal{F}[n] = A_n$ be the set of all increasing trees using each of the labels $1, 2, \dots, n$ exactly once, so that $|\mathcal{F}[n]| = (n-1)!$. Technically, \mathcal{F} is an \mathbb{L} -species, not a species, since its definition depends on the ordering of the vertices, but we consider its exponential generating function $f(x)$ rather than its ordinary generating function. The insertion defined in Section 3.3 for labeled, rooted trees preserves the set of increasing trees, and the proof of Theorem 2.1 works in this case. Using the same definition of H_T as in Section 3.3, we find that

$$f(f^{-1}(x_1) + f^{-1}(x_2) + \dots) = \sum_{T \in \mathcal{F}[n]} \frac{X_{H_T}}{n!}. \quad (9.5)$$

However, the weight axiom (Axiom (iv)) fails under this insertion, so there is no weighted version of (9.5).

We can also give a combinatorial interpretation in the case of *unary-binary* increasing trees, where each node has two or fewer children. It is convenient to work with *plane* trees, meaning that the children of a given node are ordered. Thus we take $\phi(x) = 1 + s_1 x + s_2 x^2$. The proof is somewhat ad-hoc, as the theorem does not seem to be amenable to the methods we have developed so far.

Define a *semistandard increasing tree* to be a rooted, unary-binary plane tree T whose vertices are positive integers so that

- T is an increasing tree, so $j \geq i$ whenever j is a child of i .
- $j > i$ if j is the only child of i , or else j is the rightmost of two children of i .

Let B_n be the set of all semistandard trees with n nodes. Figure 9.2 depicts a semistandard increasing tree $T \in B_9$.

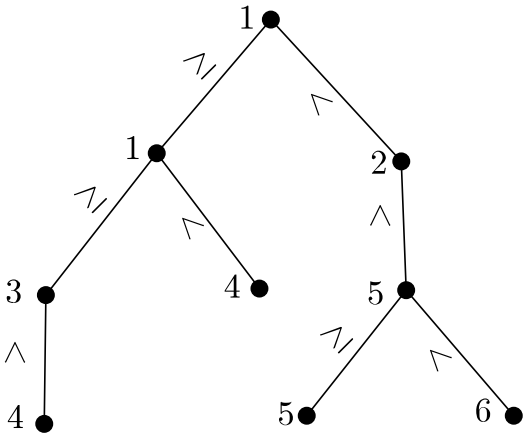


Figure 9.2: A semistandard increasing plane tree $T \in B_9$ with weight $w(T) = s_1^2 s_2^3$ and the necessary inequalities depicted. Labels weakly increase downward, and strictly increase to the right and from a parent to an only child.

We have the following fact. It appears to be new, although it is a variant of a result described by Ardila and Serrano [4, Proposition 3.4] and Goulden and Jackson [27, Corollary 4.2.20].

Theorem 9.2. Define $f(x)$ by $f(0) = 0$, $f'(x) = \phi(f(x))$ where $\phi(x) = 1 + s_1 x + s_2 x^2$. Then

$$f(f^{-1}(x_1) + f^{-1}(x_2) + \dots) = \sum_{n=1}^{\infty} \sum_{T \in B_n} w(T) \prod_{i \in T} x_i \tag{9.6}$$

Proof. Note that although $f(x)$ is an exponential generating function, the right-hand side has integer coefficients in this special case. Explicitly solving the differential equation $f'(x) = \phi(f(x))$, $f(0) = 0$, gives

$$f = \frac{2 \tan(\omega x)}{2\omega - s_1 \tan(\omega x)} \quad (9.7)$$

where $\omega = \sqrt{s_2 - s_1^2/4}$. We then calculate

$$f(f^{-1}(x) + f^{-1}(y)) = \frac{x + y + s_1 xy}{1 - s_2 xy}. \quad (9.8)$$

Let

$$F = F(x_1, x_2, \dots) = \sum_{n=1}^{\infty} \sum_{T \in B_n} w(T) \prod_{i \in T} x_i$$

and set $F_2 = F_2(x_2, x_3, \dots) = F(0, x_2, x_3, \dots)$. Then $F - F_2$ is the sum over trees T so that with $\chi(r) = 1$ where r is the root of T . Such a tree T may be

- the single vertex r ,
- a root r with a single child v , where v and its descendants are all labeled 2 or higher,
or
- a root r labeled 1 with two children, where the rightmost child and its descendants must have all labels ≥ 2 .

On the level of generating functions, this gives

$$F - F_2 = x_1 + s_1 x_1 F_2 + s_2 x_1 F_2 F. \quad (9.9)$$

Rearranging (9.9) and applying (9.8) gives

$$\begin{aligned} F &= \frac{x_1 + F_2 + s_1 x_1 F_2}{1 - s_2 x_1 F_2} \\ &= f(f^{-1}(x_1) + f^{-1}(F_2)). \end{aligned} \quad (9.10)$$

By the same argument, if $F_3 = F_3(x_3, x_4, \dots) = F(0, 0, x_3, x_4, \dots)$ we get

$$F_2 = f(f^{-1}(x_2) + f^{-1}(F_3)). \quad (9.11)$$

Substituting (9.11) into (9.10) then gives

$$F = f(f^{-1}(x_1) + f^{-1}(x_2) + f^{-1}(F_3)).$$

Continuing in this way completes the proof. \square

Note that setting $s_1 = 0, s_2 = 1$ in (9.7) gives $f(x) = \tan(x)$. Thus

$$\tan(\tan^{-1}(x_1) + \tan^{-1}(x_1) + \dots) \quad (9.12)$$

is a sum of monomials corresponding to semistandard binary trees. Equation (9.12) can also be described in terms of the *Schur functions*.

Recall that a *partition* λ is a tuple $(\lambda_1, \lambda_2, \dots, \lambda_l)$ of positive integers with $\lambda_1 \geq \dots \geq \lambda_l$. The *Young diagram* of shape λ is given by l rows of left-justified boxes, where the i th row has λ_i boxes, which we also denote by λ . Formally, the Young diagram is $\lambda = \{(i, j) : i \leq l, j \leq \lambda_i\}$. Given two partitions λ, μ with $\mu \subseteq \lambda$ as Young diagrams, the *skew shape* λ/μ is the set of pairs $\{(i, j) : i \leq l, \mu_i < j \leq \lambda_i\}$. A *semistandard young tableau* is a mapping $T : \lambda/\mu \rightarrow \mathbb{P}$ so that T weakly increases left-to-right and strictly increases top-to-bottom, so $T(i+1, j) \leq T(i, j)$ and $T(i, j) < T(i, j-1)$. The *skew Schur function* of shape λ/μ is then

$$s_{\lambda/\mu} = \sum_T \prod_{ij \in T} x_{T(i,j)}.$$

In the case where μ is empty we write $s_{\lambda/\mu} = s_\lambda$. The function s_λ form an important basis of the ring **Sym** of symmetric functions that arise from representation theory and have many applications in algebraic combinatorics. See the texts [19, 33, 46] for much more information. Figure 9.3 depicts a semistandard Young tableau.

Letting δ_n be the staircase partition $(n, n-1, \dots, 2, 1)$, we have the following alternate form of (9.12).

$$\tan(\tan^{-1}(x_1) + \tan^{-1}(x_2) + \dots) = \sum_{n=1}^{\infty} s_{\delta_n/\delta_{n-2}}. \quad (9.13)$$

The proof of (9.13), which is similar to the proof of Theorem 9.2, is given in [4]. It is also not hard to prove directly from Theorem 9.2 using an appropriate bijection.

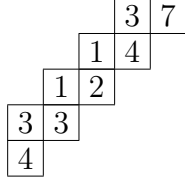


Figure 9.3: A semistandard young tableau of shape δ_n/δ_{n-2} for $n = 5$.

9.2 A Schur positivity conjecture

There is one property that is true in all our examples of contractible species, although it may not be provable from our definition. In each example we have discussed so far of a contractible (\mathbb{L} -) species \mathcal{F} , the edges $U \in \mathcal{F}[\sigma]$ are *closed under non-disjoint intersection*, so that if $U_1, U_2 \in E(\sigma)$, $U_1 \cap U_2 \neq \emptyset$, then $U_1 \cap U_2 \in E(\sigma)$.

We say that a hypergraph $H = (V, E)$ is *linear* if edges meet in at most a singleton: $|e_1 \cap e_2| \leq 1$ for any distinct $e_1, e_2 \in E$. It follows that if the edges $U \in E(\sigma)$ are closed under non-disjoint intersection then the reduced hypergraph $([n], M(\sigma))$ is linear. To see this, recall that $M(\sigma)$ is the set of minimal non-singleton edges. If $U_1, U_2 \in M(\sigma)$ with $|U_1 \cap U_2| > 1$ then $U_1 \cap U_2$ is a proper non-singleton subset of U_1 that is an edge in $E(\sigma)$, contradicting the definition of $M(\sigma)$. Thus in all our examples $\text{red } H(\sigma)$ is a linear hypergraph for any \mathcal{F} -structure σ .

Recall from 6 that if \mathcal{F} is a contractible \mathbb{L} -species then every $\sigma \in \mathcal{F}[n]$ is an *interval hypergraph*, meaning that every edge $U \in E(\sigma)$ is an interval $\{i, i+1, \dots, i+k\} \subseteq [n]$. Based on numerical evidence, we make the following conjecture.

Conjecture C. *Linear interval hypergraphs are Schur-positive.*

In particular, this conjecture would imply that all of the formal group laws we have seen thus far corresponding to contractible \mathbb{L} -species are Schur-positive.

In some cases we can prove Schur-positivity directly. For example, let $f(x) = x/(1-x)$. Stanley has shown [46, Exercise 7.47(k)] that (3.10) can be rewritten

$$f(f^{-1}(x_1) + f^{-1}(x_2) + \cdots) = \sum_{n=1}^{\infty} X_{P_n} = \frac{\sum_{i=1}^{\infty} e_i}{1 - \sum_{i=1}^{\infty} (i-1)e_i} - 1 \quad (9.14)$$

where e_i is the i th elementary symmetric function. It follows that the formal group law F corresponding to $f(x) = x/(1-x)$ is e -positive and hence Schur-positive, and all the paths P_n are e -positive as well. A linear interval hypergraph that is actually an ordinary graph is a disjoint union of paths, and so must also be e -positive. The Schur positivity of a disjoint union of paths also follows from results of Gessel [25] and Gasharov [18], where a combinatorial interpretation of the coefficients of X_G in the Schur basis is given when G is the incomparability graph of a $(3+1)$ -free poset.

If H is the hypergraph with vertex set $[n]$ whose only edge is the whole set $[n]$ then $X_H = p_1^n - p_n$ since the only colorings of H that are not proper are the ones that assign all of H to a single color. It is not hard to see that X_H is Schur-positive in this case, and it follows that any hypergraph with all edges disjoint is Schur-positive. Recall from Section 6.2 that if

$$f^{-1}(x) = x - s_2x^2 - s_3x^3 - \cdots$$

then

$$f(f^{-1}(x_1) + f^{-1}(x_2) + \cdots) = \sum_T w(T)X_{H_T}$$

with the sum taken over plane trees T where $w(T)$ is a monomial in s_2, s_3, \dots and H_T is a hypergraph with all of its edges disjoint. It follows we may set s_2, s_3, \dots to be any sequence of nonnegative real numbers and the resulting formal group law will be Schur-positive, giving the following.

Theorem 9.3. *If $f(x) \in \mathbb{R}[[x]]$ so that $f^{-1}(x) = x - s_2x^2 - s_3x^3 - \cdots$, with each $s_i \geq 0$, then the corresponding formal group law is Schur-positive.*

It is not the case that the hypergraphs arising from contractible species in general are Schur-positive. For example, in the case of labeled, rooted trees from 3.3, we saw that the star graphs $K_{1,n}$ arise, and these are not Schur-positive. The graph $C = K_{1,3}$ is sometimes called the *claw graph*. Stanley has conjectured in [43] that if G is *clawfree* then X_G is Schur-positive, where a graph G is clawfree when it has no induced subgraphs isomorphic to the claw C .

Chapter 10

CHROMATIC SYMMETRIC FUNCTIONS OF HYPERTREES

10.1 Definitions

In this chapter, we prove a weaker form of Conjecture C which stated that linear interval hypergraphs are Schur-positive. In particular, we will show that linear interval hypergraphs are F -positive, meaning that they are positive in the fundamental quasisymmetric functions F_S of Gessel [20], when their edges have prime cardinality. In fact, we show a larger class of hypergraphs are F -positive: the *hypertrees* with prime-sized edges. Furthermore, we give an explicit combinatorial interpretation for the F -coefficients of X_H .

As their name suggests, Gessel's fundamental quasisymmetric functions are not symmetric functions, but form a basis for the larger ring $\mathbf{QSym} \supset \mathbf{Sym}$ of *quasisymmetric functions*. A formal power series of bounded degree X in the variables x_1, x_2, \dots is called *quasisymmetric* if the coefficient of $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$ in X is the same as the coefficient of $x_{j_1}^{\alpha_1} x_{j_2}^{\alpha_2} \cdots x_{j_k}^{\alpha_k}$ in X whenever $i_1 < \dots < i_k$ and $j_1 < \dots < j_k$. The vector space \mathbf{QSym}_n of quasisymmetric functions of degree n has dimension 2^{n-1} and has the basis of *fundamental quasisymmetric functions* F_S^n indexed by subsets $S \subseteq [n-1]$, defined by

$$F_S^n = \sum_{i_1, i_2, \dots, i_n} x_{i_1} x_{i_2} \cdots x_{i_n}$$

with the sum over all weakly increasing sequences $i_1 \leq i_2 \leq \dots \leq i_n$ of positive integers with the restriction that if $j \in S$ then $i_j < i_{j+1}$. In what follows all the symmetric and quasisymmetric functions are homogeneous, so we will write $F_S = F_S^n$ without ambiguity. If X is a symmetric function then it is also quasisymmetric, so we may consider the coefficients a_S in the expansion $X = \sum_{S \subseteq [n-1]} a_S F_S$. If each a_S is nonnegative, we will say that X is F -positive.

In [43], Stanley used the theory of P -partitions to show that X_G is always F -positive for any ordinary graph G . In that case the F -coefficients count linear extensions of posets defined by acyclic orientations of G . However, X_H is not always Schur-positive when H is a hypergraph. For example, if $H = (V, E)$ where $V = \{1, 2, 3, 4\}$ and $E = \{\{1, 2, 3\}, \{2, 3, 4\}\}$ then

$$X_H = 2F_{\{1\}} + 6F_{\{2\}} + 2F_{\{3\}} + 4F_{\{1,2\}} + 8F_{\{1,3\}} + 4F_{\{2,3\}} - 2F_{\{1,2,3\}} \quad (10.1)$$

is not F -positive. The reader might observe that the coefficients in (10.1) sum to $24 = 4!$, and this is not a coincidence. The sum of the F -coefficients in a chromatic symmetric function X_H will always be $n!$ where $n = |V|$, and this can be seen by considering the coefficient of $x_1 x_2 \dots x_n$. Thus when X_H is F -positive, we might expect then to be able to write X_H as a sum of fundamental quasisymmetric functions indexed by permutations, and this is what we proceed to do for a certain class of hypergraphs: the *hypertrees* with prime-sized edges. There are a number of closely-related definitions of *hypertree* occurring in the literature; we adopt the definition given in [23].

Let $H = (V, E)$ be a hypergraph. A *path* in H is a nonempty sequence

$$v_1, e_1, v_2, e_2, \dots, e_m, v_{m+1}$$

where each $e_i \in E$ with $v_i, v_{i+1} \in e_i$ and the edges e_i and vertices v_i of the path are distinct, except that we allow $v_1 = v_{m+1}$. If $v_1 = v_{m+1}$ and $m > 1$ we say the path is a *cycle*. We say that a hypergraph $H = (V, E)$ is *connected* if there is a path from v to v' for any given $v, v' \in V$. A *hypertree* is a hypergraph that is connected and has no cycles. Thus in a hypertree there is a *unique* path between any two distinct vertices. A hypergraph H is called *linear* if $|e \cap e'| \leq 1$ for any distinct edges $e, e' \in E$. Hypertrees are linear, for if there are distinct $v_1, v_2 \in e_1 \cap e_2$ for $e_1 \neq e_2$ then there is a cycle v_1, e_1, v_2, e_2, v_1 . Figure 1 depicts a hypertree.

Our main result is the following fact, appearing as Theorem 10.8 in Section 10.4.

Theorem. *Let $H = (V, E)$ be a hypertree so that $|e|$ is a prime number for each edge $e \in E$.*

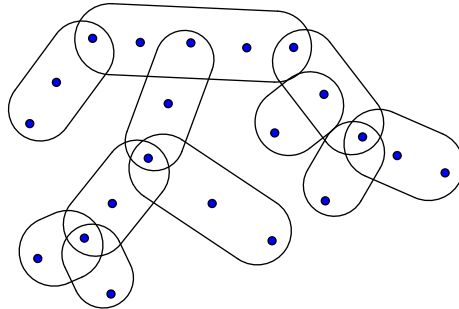


Figure 10.1: A hypertree with edges $e \in E$ circled.

Then X_H is F -positive. In particular,

$$X_H = \sum_{\pi \in \mathfrak{S}_V} F_{\text{Des}_H(\pi)}$$

where $n = |V|$ and $\text{Des}_H(\pi)$ is the set of H -descents of the permutation π , to be defined in Section 10.4.

It is not true that X_H is F -positive whenever H is linear. For example, X_H is not F -positive when H consists of the edges $\{1, 2, 3\}$, $\{1, 4\}$, $\{2, 4\}$, $\{3, 4, 5\}$. On the other hand, we guess (Conjecture D) that the primality condition is not necessary for F -positivity, although our proof relies on primality in a crucial way. We also note that it is easy to extend Theorem 10.8 to disjoint unions of hypertrees, or *hyperforests*, but for simplicity we only consider the connected case.

In Section 10.2, we use a standardization procedure due to Gessel and Reutenauer [24] to show F -positivity of X_H when H consists of a single prime-sized edge. In Section 10.3, we combine the result of Section 10.2 with the theory of P -partitions due to Stanley [42] and Gessel [20] to show F -positivity of X_H for a hypertree H with prime-size edges. In Section 10.4 we describe a combinatorial interpretation of the results in Section 10.3 by giving the definition of H -descents and proving Theorem 10.8. We conclude in Section 10.5 by giving some conjectures and suggestions for further work.

10.2 The single edge case

If $H = (V, E)$ only has a single edge e consisting of all of V , so that $E = \{V\}$, then a proper coloring of H is any coloring that is not constant. Thus if $|V| = n$ then $X_H = p_1^n - p_n$, where p_i is the i th power sum symmetric function

$$p_i = x_1^i + x_2^i + \cdots .$$

In this case it is not difficult to show that X_H is F -positive using standard results on symmetric functions. In fact, X_H is *Schur-positive*, which implies F -positivity. This in itself is not enough to show F -positivity for other chromatic symmetric functions of hypergraphs. However, in this section we will prove F -positivity of $X_H = p_1^n - p_n$ when n is prime by exhibiting an explicit partition of the set of all nonconstant colorings where each set in the partition has a generating function that is a fundamental quasisymmetric function F_S . We will see in Section 10.3 that if such a partition can be found for each edge e in a hypertree then there is a similar partition of the set of proper colorings of that hypertree.

If V is a finite set with $|V| = n$, let \mathfrak{S}_V be the set of permutations of V realized as bijections $\pi : V \rightarrow [n]$; if $V = [n]$ we write \mathfrak{S}_n for $\mathfrak{S}_{[n]}$. Given a permutation $\pi \in \mathfrak{S}_V$ and a subset $S \subseteq [n - 1]$, let $A(\pi, S)$ be the set of colorings $\chi : V \rightarrow \mathbb{P}$ satisfying the conditions

$$\chi(\pi^{-1}(1)) \leq \chi(\pi^{-1}(2)) \leq \dots \leq \chi(\pi^{-1}(n))$$

and $\chi(\pi^{-1}(i)) < \chi(\pi^{-1}(i + 1))$ when $i \in S$. Thus for any $\pi \in \mathfrak{S}_V$ and $S \subseteq [n - 1]$, $A(\pi, S)$ has the quasisymmetric generating function

$$\sum_{\chi \in A(\pi, S)} x_{\chi(\pi^{-1}(1))} x_{\chi(\pi^{-1}(2))} \cdots x_{\chi(\pi^{-1}(n))} = F_S.$$

Recall that the *descent set* $\text{Des}(\pi)$ of a permutation $\pi \in \mathfrak{S}_n$ is the set of $i < n$ so that $\pi(i) > \pi(i + 1)$. The goal of this section is to prove the following fact.

Theorem 10.1. *Let V be a set with $|V| = n$ prime and let $c : V \rightarrow V$ be a cyclic permutation of V . Then the set of nonconstant colorings $\chi : V \rightarrow \mathbb{P}$ is the disjoint union*

$$\bigsqcup_{\pi \in \mathfrak{S}_V} A(\pi, \text{Des}(\pi c \pi^{-1})).$$

Theorem 10.1 immediately gives the F -expansion of the symmetric function $p_1^n - p_n$ when n is prime; we have

$$p_1^n - p_n = \sum_{\pi \in \mathfrak{S}_n} F_{\text{Des}(\pi c \pi^{-1})}. \quad (10.2)$$

If we think of each $\pi \in \mathfrak{S}_V$ as a *labeling* of V with the labels $1, \dots, n$, identifying v with $\pi(v)$, then $\pi c \pi^{-1}$ is the same as c when viewed as a permutation of the labels. Each cyclic permutation of the labels $[n]$ will appear n times in this sum, so we have

$$\frac{p_1^n - p_n}{n} = \sum_c F_{\text{Des}(c)} \quad (10.3)$$

where the sum is taken over all cyclic permutations $c : [n] \rightarrow [n]$. The identity (10.3) was shown by Gessel and Reutenauer [24], and in fact if n is prime then $(p_1^n - p_n)/n$ is both the generating function for *primitive necklaces* of length n and the Frobenius characteristic of the \mathfrak{S}_n -representation given by the degree- n multilinear part of the free Lie algebra on a set of size n .

In the proof that follows it will be convenient to assume without loss of generality that $V = [n]$ and c is the particular cyclic permutation $c(i) = i + 1$ for $1 \leq i < n$ with $c(n) = 1$. We think of a coloring $\chi : V \rightarrow \mathbb{P}$ as a *word* $w = \chi$, where we write $w = w(1) \cdots w(n) \in \mathbb{P}^n$. To prove Theorem 10.1, we will use a method of obtaining a permutation $\pi \in \mathfrak{S}_n$ from a nonconstant word $w \in \mathbb{P}^n$ when n is prime due to Gessel and Reutenauer [24]. Let $w = w(1)w(2) \cdots w(n) \in \mathbb{P}^n$ be a word that uses at least two distinct letters from \mathbb{P} , so that we do not have $w(1) = w(2) = \cdots = w(n)$. Let $r_i(w)$ be the rotation

$$r_i(w) = w(i)w(i+1) \cdots w(n)w(1)w(2) \cdots w(i-1)$$

of w . The rotations $r_1(w), r_2(w), \dots, r_n(w)$ need not be distinct in general. For example, if $w = 1212$ then $r_1(w) = r_3(w) = 1212$. If n is prime, however, this cannot occur and the rotations of w are all distinct as long as w is a nonconstant word.

Assuming n is prime, define the *cyclic standardization* of w , which we'll denote $\text{cstd}(w)$, to be the permutation obtained by ordering these rotated words lexicographically: we say

$\pi = \text{cstd}(w)$ if π is the unique permutation in \mathfrak{S}_n so that $r_x(w) <_{\text{lex}} r_y(w)$ whenever $\pi(x) < \pi(y)$. That is, we find $\pi = \text{cstd}(w)$ by setting $\pi(i) = j$ when $r_i(w)$ is the j th smallest rotation of w . For example, if $w = 2114132$ then $\pi = \text{cstd}(w) = 4137265$; we have $\pi(2) = 1$ since $r_2(w) = 1141321$ is the least rotation of w lexicographically, $\pi(5) = 2$ since $r_5(w) = 1322114$ is the next smallest, etc.

By the primality of n , the set of words $w \in \mathbb{P}^n$ that are not constant is the disjoint union

$$\bigsqcup_{\pi \in \mathfrak{S}_n} \{w \in \mathbb{P}^n : \text{cstd}(w) = \pi\}.$$

Thus the proof of Theorem 10.1 is immediate from the following lemma.

Lemma 10.2. *Let n be prime and let $w \in \mathbb{P}^n$ be a nonconstant word. Then $\text{cstd}(w) = \pi$ if and only if $w \in A(\pi, S)$ where $S = \text{Des}(\pi c \pi^{-1})$.*

Proof. First, suppose that $\text{cstd}(w) = \pi$. For any $1 \leq i < n$, suppose $\pi(x) = i$ and $\pi(y) = i+1$; then $r_y(w) = w(y)w(y+1)\cdots$ is the next largest rotation of w in lexicographic order after $r_x(w) = w(x)w(x+1)\cdots$, where we take $n+1 = 1$, $n+2 = 2$, etc. In particular, we must have $w(x) \leq w(y)$, or $w(\pi^{-1}(i)) \leq w(\pi^{-1}(i+1))$ as desired. Now suppose that i is a descent of $\pi c \pi^{-1}$; we must show that $w(x) < w(y)$. We have $\pi c \pi^{-1}(i) > \pi c \pi^{-1}(i+1)$; that is, $\pi(x+1) > \pi(y+1)$. Then we have

$$r_{x+1}(w) = w(x+1)w(x+2)\cdots >_{\text{lex}} r_{y+1}(w) = w(y+1)w(y+2)\cdots.$$

But we also know that $w(x)w(x+1)\cdots <_{\text{lex}} w(y)w(y+1)\cdots$, and the only way both of these lexicographic inequalities can occur is if $w(x) < w(y)$.

Conversely, suppose that $w \in A(\pi, S)$. To show that $\text{cstd}(w) = \pi$, we will show that $\pi(x) < \pi(y)$ implies $r_x(w) <_{\text{lex}} r_y(w)$. Since n is prime, all rotations of w are distinct, and so it is enough to show that $\pi(x) < \pi(y)$ implies $r_x(w) \leq_{\text{lex}} r_y(w)$; we may also assume that $\pi(x) = i$ and $\pi(y) = i+1$. We will proceed inductively. For any word $v = v(1)v(2)\cdots v(n) \in \mathbb{P}^n$, let $v|_m$ be the truncation $v(1)\cdots v(m)$. We will show that for each m , if $\pi(x) < \pi(y)$ we must have $r_x(w)|_m \leq_{\text{lex}} r_y(w)|_m$. If $m = 1$, the truncations $r_x(w)|_m, r_y(w)|_m$ are the

single-character words $w(x), w(y)$; and $w(x) = w(\pi^{-1}(i)) \leq w(\pi^{-1}(i+1)) = w(y)$ since $w \in A(\pi, S)$.

Now suppose that the statement holds for m . We will show that $r_x(w)|_{m+1} \leq_{\text{lex}} r_y(w)|_{m+1}$. By the argument for the base case, we know $w(x) \leq w(y)$; if $w(x) < w(y)$ we are done, so assume $w(x) = w(y)$. Since $w \in A(\pi, S)$, i must be an ascent of $\pi c \pi^{-1}$; that is, $\pi c \pi^{-1}(i) < \pi c \pi^{-1}(i+1)$, or $\pi(x+1) < \pi(y+1)$. By our inductive hypothesis, we must have $r_{x+1}(w)|_m \leq_{\text{lex}} r_{y+1}(w)|_m$. Then $r_x(w)|_{m+1} = w(x)r_{x+1}(w)|_m \leq_{\text{lex}} w(x)r_{y+1}(w)|_m = r_y(w)|_{m+1}$. \square

10.3 Proof of F -positivity

Now we are in a position to show the F -positivity of X_H when $H = (V, E)$ is a hypertree with prime-sized edges. The primality gives us the decomposition described in Theorem 10.1 for each edge $e \in E$, and the hypertree structure will enable us to glue these decompositions together to get a similar decomposition of the set of all proper colorings of H . The glue, in this case, is the theory of P -partitions.

Given a poset P on a vertex set V , a mapping $f : V \rightarrow \mathbb{P} = \{1, 2, \dots\}$ is a P -partition if $x \leq_P y$ implies $f(x) \leq f(y)$. If P is the poset $[n]$ with the usual order, then a P -partition is a sequence of increasing integers $f(1) \leq f(2) \leq f(3) \leq \dots \leq f(n)$. This is equivalent to the usual definition of a *partition* of the integer $f(1) + f(2) + \dots + f(n)$. Traditionally a partition of an integer is written in descending order, so what we call P -partitions were called *reverse* P -partitions by Stanley [42, 46].

Suppose that $\omega \in \mathfrak{S}_V$ is a bijection $V \rightarrow [n]$. In what follows it will be convenient to identify ω with the total order $<_\omega$ put on the vertices of P where $x <_\omega y$ means that $\omega(x) < \omega(y)$. A (P, ω) -partition is a P -partition f that has strict inequalities where the orders P and ω disagree. That is, if $x <_P y$ and $x <_\omega y$ then $f(x) \leq f(y)$, but if $x <_P y$ and $x >_\omega y$ then $f(x) < f(y)$.

A *linear extension* of P is an order-preserving bijection $f : V \rightarrow [n]$. The main result on (P, ω) -partitions we need is the following fact, sometimes called the Fundamental Theorem of (P, ω) -Partitions. See [44, Lemma 3.15.3] for a proof when (P, ω) -partitions are taken to

be order-reversing; it is given without proof in [46, 7.19.4] for (P, ω) -partitions taken to be order-preserving as we do.

Theorem 10.3. *Let $P = (V, \leq_P)$ be a finite poset with $|V| = n$ and let $\omega : V \rightarrow [n]$ be any bijection. Then the set of (P, ω) -partitions is exactly the disjoint union*

$$\bigsqcup_{\pi} A(\pi, \text{Des}(\omega\pi^{-1}))$$

where $A(\pi, S)$ is as defined in Section 10.2, and the union is taken over all linear extensions $\pi : V \rightarrow [n]$ of P .

The key fact we will use about hypertrees is that posets on different edges are compatible with each other.

Lemma 10.4. *Let $H = (V, E)$ be a hypertree with $E = \{e_1, \dots, e_k\}$. Suppose that each edge $e_i \in E$ has an associated poset P_i with vertex set e_i and relation $<_i$. Define the relation $<$ on V by taking the transitive closure of all the relations $<_e$, so that $x < y$ in V if there is a chain $x = v_1 <_{i_1} v_2 <_{i_2} \dots <_{i_l} v_l = y$. Then $P = (V, <)$ is a poset.*

Proof. Form a directed graph G on V by setting $x \rightarrow y$ when there is an edge $e \in E$ with $x, y \in e$ and $x <_e y$. Then G is easily seen to be acyclic since H is a hypertree, and any directed acyclic graph determines a poset after extending transitively. \square

Theorem 10.5. *Let $H = (V, E)$ be a hypertree so that $|e|$ is prime for each $e \in E$. Then X_H is F -positive.*

Proof. Say $E = \{e_1, \dots, e_k\}$. For each edge $e_i \in E$ fix a particular bijection $c_i : e_i \rightarrow e_i$ that is cyclic. Since each edge $e_i \in |E|$ has $|e_i|$ prime, by Theorem 10.1 the set of nonconstant colorings $\chi : e_i \rightarrow \mathbb{P}$ is the disjoint union

$$\bigsqcup_{\pi \in \mathfrak{S}_{e_i}} A(\pi, \text{Des}(\pi c_i \pi^{-1})). \tag{10.4}$$

Let $PC(H)$ be the set of proper colorings χ of H . A coloring χ of H is proper if and only if each restriction $\chi|_{e_i} : e_i \rightarrow \mathbb{P}$ is not constant, so we have

$$PC(H) = \bigsqcup_{\pi_1, \dots, \pi_k} A(\pi_1, \pi_2, \dots, \pi_k) \quad (10.5)$$

where the union is taken over all k -tuples (π_1, \dots, π_k) with $\pi_i \in \mathfrak{S}_{e_i}$ and

$$A(\pi_1, \pi_2, \dots, \pi_k) = \{\chi : \mathbb{P} \rightarrow V : \chi|_{e_i} \in A(\pi_i, \text{Des}(\pi_i c_i \pi_i^{-1})) \text{ for all } i\}.$$

The union (10.5) is disjoint since each union (10.4) is: a coloring $\chi \in A(\pi_1, \pi_2, \dots, \pi_k)$ uniquely determines each π_i since the restriction $\chi|_{e_i}$ uniquely determines π_i .

Given an edge e_i and a bijection $\pi_i : e_i \rightarrow [m] \in \mathfrak{S}_{e_i}$ where $m = |e_i|$, define a poset P_{π_i} (actually a total order) on the vertex set e_i by $x <_{P_{\pi_i}} y$ when $\pi_i(x) < \pi_i(y)$, and let $\omega_{\pi_i} : e_i \rightarrow [m]$ be the labeling $\pi_i c_i$. Then π_i is the unique linear extension of P_{π_i} , so by Theorem 10.3 the set of $(P_{\pi_i}, \omega_{\pi_i})$ -partitions is exactly the set $A(\pi_i, \text{Des}(\omega_{\pi_i} \pi_i^{-1})) = A(\pi_i, \text{Des}(\pi_i c_i \pi_i^{-1}))$. Thus $A(\pi_1, \pi_2, \dots, \pi_k)$ is the set of colorings χ so that $\chi|_{e_i}$ is a $(P_{\pi_i}, \omega_{\pi_i})$ -partition for each i .

Fix a choice of $\pi_i \in \mathfrak{S}_{e_i}$ for each edge e_i . By Lemma 10.4 there are well-defined posets $P_{\pi_1, \dots, \pi_k}, Q_{\pi_1, \dots, \pi_k}$ given by taking the transitive closure of the relations of the posets $P_{\pi_i}, \omega_{\pi_i}$ respectively. Let $\omega_{\pi_1, \dots, \pi_k}$ be any linear extension of Q_{π_1, \dots, π_k} . We claim that $A(\pi_1, \pi_2, \dots, \pi_k)$ is exactly the set of $(P_{\pi_1, \dots, \pi_k}, \omega_{\pi_1, \dots, \pi_k})$ -partitions. It is clear from the definitions that if χ is a $(P_{\pi_1, \dots, \pi_k}, \omega_{\pi_1, \dots, \pi_k})$ -partition then $\chi|_{e_i}$ is a $(P_{\pi_i}, \omega_{\pi_i})$ -partition for each i , so $\chi \in A(\pi_1, \pi_2, \dots, \pi_k)$. Conversely, if $\chi \in A(\pi_1, \pi_2, \dots, \pi_k)$ it is not hard to see that χ is a $(P_{\pi_1, \dots, \pi_k}, \omega_{\pi_1, \dots, \pi_k})$ -partition by repeatedly applying the ‘‘local conditions’’ that each $\chi|_{e_i}$ is a $(P_{\pi_i}, \omega_{\pi_i})$ -partition.

Combining Theorem 10.3 with (10.5) then gives

$$PC(H) = \bigsqcup_{\pi_1, \dots, \pi_k} \bigsqcup_{\sigma} A(\sigma, \text{Des} \omega_{\pi_1, \dots, \pi_k} \sigma^{-1}) \quad (10.6)$$

where the union is taken over all tuples π_1, \dots, π_k with $\pi_i \in \mathfrak{S}_{e_i}$ and linear extensions $\sigma : V \rightarrow [n]$ of P_{π_1, \dots, π_k} . Since the quasisymmetric generating function of $A(\sigma, S)$ is F_S we are done. \square

By rewriting (10.6) in a simpler form we can give an expression for X_H as a sum of fundamental quasisymmetric functions indexed by permutations $\pi : V \rightarrow [n]$.

Corollary 10.6 (of the proof of Theorem 10.5). *Let $H = (V, E)$ be a hypertree. Given a permutation $\pi : V \rightarrow [n]$, define posets $P(\pi), Q(\pi)$ on V so that when x, y both belong to the same edge e then $x <_{P(\pi)} y$ iff $\pi(x) < \pi(y)$ and $x <_{Q(\pi)} y$ iff $\pi c(x) < \pi c(y)$. Fix a linear extension ω_π of $Q(\pi)$ for each $\pi \in \mathfrak{S}_V$. Then*

$$X_H = \sum_{\pi \in \mathfrak{S}_V} F_{\text{Des}(\omega_\pi \pi^{-1})}. \quad (10.7)$$

Proof. Say $E = \{e_1, \dots, e_k\}$. Then for each $\pi : V \rightarrow [n] \in \mathfrak{S}_V$ there is a unique choice of π_1, \dots, π_k so that π is a linear extension of P_{π_1, \dots, π_k} : let $\pi_i = g \circ \pi|_{e_i}$ where g is the unique increasing function from $\pi(e_i)$ to $[[e_i]]$. Applying this fact to (10.6) and taking the quasisymmetric generating function gives (10.7). \square

In Corollary 10.6 the choice of the linear extension ω_π is arbitrary. Every poset has a linear extension, and this fact is enough to prove F -positivity. From a combinatorial standpoint, however, it would be desirable to find a specific choice of ω_π that is natural in some sense. In the next section we do so, giving a simple combinatorial interpretation to the F -coefficients of X_H .

10.4 Combinatorial interpretation

Before we can define the H -descents alluded to in the introduction, we need to show the existence of a particularly nice ordering of the edges of a hypertree. The following lemma says that any hypertree may be constructed by adding one edge at a time, each new edge intersecting the others in a single vertex.

Lemma 10.7. *Let $H = (V, E)$ be a hypertree. Then there is an ordering of its edges so that $E = \{e_1, e_2, \dots, e_k\}$ with*

$$|(e_1 \cup e_2 \cup \dots \cup e_i) \cap e_{i+1}| = 1 \text{ for } i = 1, \dots, k-1. \quad (10.8)$$

Proof. It is enough to find an $e \in E$ and $v \in e$ so that $e' \cap e \subseteq \{v\}$ for any $e' \in E$ with $e' \neq e$. Once such an e is found, let $H' = (V', E')$ with $V' = V \setminus e \cup \{v\}$, $E' = E \setminus \{e\}$. It is easy to check that H' is a hypertree with $k - 1$ edges, so we may assume inductively that H' has an ordering e_1, e_2, \dots, e_{k-1} of E' satisfying (10.8). Setting $e_k = e$, we see that e_1, \dots, e_k is the desired order of E .

To find such an e and $v \in e$, let $v_1, f_1, v_2, f_2, \dots, v_l, f_l, v_{l+1}$ be a path of maximal length l in H . We claim that $e = f_l$, $v = v_l$ satisfies the desired property. Suppose that there is $e' \in E$ with $e' \cap f_l \not\subseteq \{v_l\}$; then there is $u \in e' \cap f_l$ with $u \neq v$. We must have $e' = f_j$ for some $j < l$, or else we would have a longer path $v_1, f_2, \dots, v_l, f_l, u, e', u'$ where $u \in e'$ with $u \neq u'$. Say that j is as large as possible. Then we have a cycle $u, f_j, v_{j+1}, f_{j+1}, \dots, f_l, u$ which violates the definition of a hypertree. \square

In fact, the converse of Lemma 10.7 is easily seen to hold as well, so that the existence of edge-orderings satisfying (10.8) characterizes hypertrees. From now on we will generally assume that H is equipped with some choice of such an edge-ordering and our subsequent definitions are all based on this edge-ordering.

Suppose that $H = (V, E)$ is a hypertree with $E = \{e_1, \dots, e_k\}$ satisfying (10.8), and fix a choice of cyclic permutation $c_i : e_i \rightarrow e_i$ for each edge $e \in E$. Suppose also that $V = [n]$, so that V is equipped with the order $<$. Since H is a hypertree, for each i there is a *unique* path $i = v_1, e_{j_1}, v_2, e_{j_2}, \dots, e_{j_l}, v_{l+1} = i + 1$ from i to $i + 1$, where the vertices and edges in the path are all distinct. Let $j_r = \min(j_1, j_2, \dots, j_l)$. Then we say that i is an H -descent if $c_{j_r}(v_r) > c_{j_r}(v_{r+1})$.

For example, let H be the hypergraph in Fig. 2. The cyclic permutations c_i are given by reading along the indicated direction in cycle notation, so that c_3 is $(12, 1, 2)$ with $12 \mapsto 1 \mapsto 2 \mapsto 12$. The unique path from 1 to 2 is just $1, e_3, 2$ since 1 and 2 are both contained in e_3 . Then $c_3(1) = 2 < c_3(2) = 12$, so 1 is not an H -descent. The unique path from 2 to 3 is given by $2, e_2, 4, e_1, 10, e_4, 13, e_7, 3$ and the edge with the smallest index occurring in this path is e_1 . Then $c_1(4) = 10 > c_1(10) = 7$ and so 2 is an H -descent. Continuing, we find the

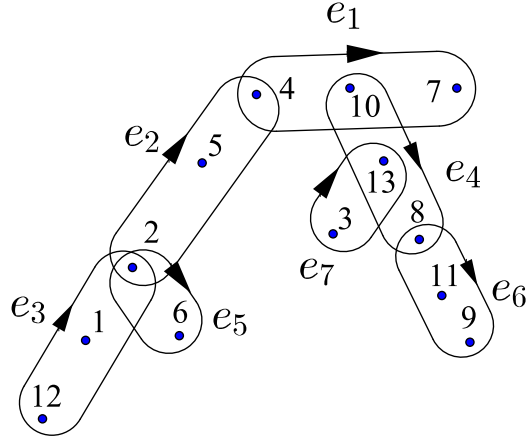


Figure 10.2: A hypertree with labeled vertices, a suitable ordering of its edges, and a cyclic permutation of each edge.

H -descents are $\{2, 6, 8, 10, 12\}$.

Now let $H = (V, E)$ be a hypertree as before with an edge-ordering satisfying (10.8) and a cyclic permutation of each edge, but now allow V to be an arbitrary finite set which we will think of as being unordered. Given a permutation $\pi : V \rightarrow [n] \in \mathfrak{S}_V$, we will consider π a *labeling* of V , identifying v with $\pi(v)$. We then denote the corresponding set of H -descents by $\text{Des}_H(\pi)$ and call them the H -descents of π . With these definitions in hand, we state our main theorem.

Theorem 10.8. *Let $H = (V, E)$ be a hypertree so that $|e|$ is prime for each edge $e \in E$. Fix an ordering of the edges so $E = \{e_1, \dots, e_k\}$ with the property that $|(e_1 \cup \dots \cup e_i) \cap e_{i+1}| = 1$ for all $1 \leq i < k$, and also fix a choice of cyclic permutation $c_i : e_i \rightarrow e_i$ of each edge $e_i \in E$. Then*

$$X_H = \sum_{\pi \in \mathfrak{S}_V} F_{\text{Des}_H(\pi)}$$

where $\text{Des}_H(\pi)$ is the set of H -descents of π with respect to the chosen edge-ordering and cyclic permutations.

Note that in the case where H consists of a single edge e with a cyclic permutation $c : e \rightarrow e$, $\text{Des}_H(\pi)$ is exactly $\text{Des}(\pi^{-1}c\pi)$, so Theorem 10.8 reduces to Corollary 10.2.

To prove Theorem 10.8, we will need a systematic way of combining total orders together. Given totally ordered sets (U, ω_U) , (V, ω_V) where U, V share a single element, say $U \cap V = \{x\}$, we define $\omega_U \leftarrow \omega_V$ to be the total order of the union $U \cup V$ given by “inserting” V with its total order ω_V into the place of x in U . That is, ω is the unique total order agreeing with ω_U, ω_V on U, V so that when $u \in U, v \in V$ we have $u <_\omega v$ if and only if $u <_{\omega_U} x$. Thus if U consists of elements $u_1 <_{\omega_U} u_2 <_{\omega_U} \cdots <_{\omega_U} u_m$, with $u_i = x$, and V has elements $v_1 <_{\omega_V} v_2 <_{\omega_V} \cdots <_{\omega_V} v_n$, then $\omega = \omega_U \leftarrow \omega_V$ is the total order of $U \cup V$ with

$$u_1 <_\omega u_2 <_\omega \cdots <_\omega u_{i-1} <_\omega v_1 <_\omega \cdots <_\omega v_n <_\omega u_{i+1} <_\omega \cdots <_\omega u_m.$$

For example, if $U = \{x <_{\omega_U} b <_{\omega_U} y\}$, $V = \{a <_{\omega_V} b <_{\omega_V} c\}$ are totally ordered sets then $\omega = \omega_U \leftarrow \omega_V$ is the total order $x <_\omega a <_\omega b <_\omega c <_\omega y$.

We now consider the total orders that arise from repeated insertion.

Lemma 10.9. *Let $H = (V, E)$ be a hypertree with $E = \{e_1, \dots, e_k\}$ so that (10.8) holds. Suppose there is a total order ω_i on each e_i , and define a total order ω on V by*

$$\omega = (\cdots (\omega_1 \leftarrow \omega_2) \leftarrow \cdots) \leftarrow \omega_k.$$

Then for any distinct $x, y \in V$, $x <_\omega y$ if and only if $v_r <_{\omega_{j_r}} v_{r+1}$ where

$$x = v_1, e_{j_1}, v_2, e_{j_2}, \dots, v_l, e_{j_l}, v_{l+1} = y \tag{10.9}$$

is the unique path from x to y in H and $j_r = \min(j_1, j_2, \dots, j_l)$.

Proof. We proceed by induction on the number of edges of H . If H has only one edge, the statement is trivial, so suppose that the statement holds for hypertrees with fewer than k edges and that H has exactly k edges. Let $H' = (V', E')$ be the hypertree with $V' = e_1 \cup \cdots \cup e_{k-1}$ and $E' = E \setminus \{e_k\}$, and let ω' be the total order on V' given by

$$\omega' = (\cdots (\omega_1 \leftarrow \omega_2) \leftarrow \cdots) \leftarrow \omega_{k-1},$$

so that $\omega = \omega' \leftarrow \omega_k$. If both x and y are in V' then we are done by the inductive hypothesis. Similarly if $x, y \in e_k$ then there is nothing to show. So assume that $x \in e_k \setminus V'$ and $y \in V' \setminus e_k$ and let (10.9) be the path from x to y , so that $\{v_2\} = (e_1 \cup \dots \cup e_{k-1}) \cap e_k$. From the definition of the insertion $\omega' \leftarrow \omega_k$ we have $x <_{\omega} y$ if and only if $v_2 <_{\omega'} y$. Then

$$v_2, e_{j_2}, \dots, v_l, e_{j_l}, v_{l+1}$$

is the unique path from v_2 to y in H' and clearly $j_r = \min(j_1, j_2, \dots, j_l) = \min(j_2, \dots, j_l)$ since $j_1 = k$ is the highest index of any edge in H . Thus by our inductive hypothesis we see that $v_2 <_{\omega'} y$ is equivalent to $v_r <_{\omega_{j_r}} v_{r+1}$. \square

Proof of Theorem 10.8. Given a bijection $\pi : V \rightarrow [n]$, let $P(\pi)$ and $Q(\pi)$ be as in the statement of Corollary 10.6. For each i let ω_i be the total order of e_i given by restricting Q to e_i , so that $x <_{\omega_i} y$ in e if $\pi c_i(x) < \pi c_i(y)$, and let ω_π be the total order on V given by $\omega_\pi = (\dots(\omega_1 \leftarrow \omega_2) \leftarrow \dots) \leftarrow \omega_k$. Then ω_π is a linear extension of $Q(\pi)$, and by Lemma 10.9 we see that i is a descent of $\omega_\pi \pi^{-1}$ if and only if $v_r >_{\omega_{j_r}} v_{r+1}$, that is, $\pi c_{j_r}(v_r) > \pi c_{j_r}(v_{r+1})$ where j_r is the least-index edge in the path $\pi^{-1}(i) = v_1, e_{j_1}, \dots, e_{j_l}, v_{l+1} = \pi^{-1}(i+1)$ from $\pi^{-1}(i)$ to $\pi^{-1}(i+1)$. After identifying v with $\pi(v)$, the descents of $\omega_\pi \pi^{-1}$ become the H -descents, so that $\text{Des}_H(\pi) = \text{Des}(\omega_\pi \pi^{-1})$. Applying Corollary 10.6 then finishes the proof. \square

10.5 Are all hypertrees \mathcal{F} -positive?

It is likely that the condition that the edges have prime size could be removed. A closer examination of the proof of F -positivity (Theorem 10.5) reveals that it does not depend on primality *per se*, but only on the existence of partitions of colorings of the form

$$\{\chi : [e_i] \rightarrow \mathbb{P} : \chi \text{ not constant}\} = \bigsqcup_{\pi \in \mathfrak{S}_{e_i}} A(\pi, S(\pi)). \quad (10.10)$$

for each edge e_i of H , where $S(\pi) \subseteq [n-1]$ is a choice of subset for each $\pi \in \mathfrak{S}_{e_i}$, with $n = |e_i|$. We need only to give the role played by the maps $\pi_i c_i$ in the proof of Theorem 10.5 to an appropriate choice of bijections $\omega_{\pi_i} \in \mathfrak{S}_{e_i}$ so that $\text{Des}(\omega_{\pi_i} \pi_i^{-1}) = S(\pi)$ for each $\pi_i \in \mathfrak{S}_{e_i}$. Thus we have:

Theorem 10.10. *Suppose that $H = (V, E)$ is a hypertree so that for each $e \in E$ there is a partition of the form (10.10). Then X_H is F -positive.*

The fact that a partition of the form (10.10) exists when $n = |e_i|$ is prime gives a proof of F -positivity of hypertrees with prime-sized edges. In fact, such a partition of the nonconstant colorings of a set of $n = 4$ elements does exist as well; it was found with a search algorithm using the software package Sage [47]. Finding such a partition for each n would then constitute a proof of the following.

Conjecture D. *Let H be a hypertree. Then X_H is F -positive.*

We can rephrase this idea in terms of *simplicial complexes*. A simplicial complex Δ is a family of subsets of a finite vertex set V so that if $F \in \Delta$ and $F' \subseteq F$ then $F' \in \Delta$. If Δ is a simplicial complex and $S \subseteq \Delta$ is any subset of Δ , then S is a *partial simplicial complex* and we say that S is *partitionable* if S is a disjoint union

$$S = \bigsqcup_i [G_i, F_i]$$

where the F_i are facets (maximal faces) of Δ , $G_i \subseteq F_i$, and $[G_i, F_i] = \{F \in \Delta : G_i \subseteq F \subseteq F_i\}$. Then the existence of a partition of the nonconstant colorings of the form (10.10) when $|e_i| = n$ is equivalent to the statement that $\Delta_n \setminus \{\emptyset\}$ is partitionable where Δ_n is the *Coxeter complex* of type A_{n-1} , a simplicial complex whose facets are in natural bijection with permutations $\pi \in \mathfrak{S}_n$. The problem of partitionability for a partial simplicial complex $S \subseteq \Delta_n$ is discussed by Breuer and Klivans in [11], where S is thought of as a *scheduling problem*.

The optimistic reader might hope that chromatic symmetric functions of hypertrees would in fact be Schur-positive, but this is not the case even for ordinary graphs. For example, if $C = (V, E)$ is the “claw” with $V = \{1, 2, 3, 4\}$, $E = \{\{1, 2\}, \{1, 3\}, \{1, 4\}\}$ then X_C is not Schur-positive. Stanley has conjectured in [43] that if G is *clawfree* then X_G is Schur-positive, where a graph G is clawfree when it has no induced subgraphs isomorphic to the claw C .

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Appendix A

LIST OF FORMAL GROUP LAWS

A.1

For reference, we list the formal group laws we have seen so far with related information. First we consider the case of exponential generating functions. For each contractible species \mathcal{F} , we list the ordinary generating function (OGF)

$$f(x) = \sum_{n=1}^{\infty} a_n \frac{x^n}{n!}$$

as well as the coefficients $a_n = |\mathcal{F}[n]|$, or $a_n = \sum_{\sigma \in \mathcal{F}[n]} w(\sigma)$ if \mathcal{F} is weighted, starting from $n = 1$. (We always have $a_0 = 0$.) We also list the associated polynomials $p_n(t)$ defined by

$$\sum_{n=0}^{\infty} p_n(t) x^n = e^{t f^{-1}(x)}$$

and the expansion of the formal group law $f(f^{-1}(x_1) + f^{-1}(x_2) + \dots)$ in the monomial basis m_λ . When possible, we give a measure μ on $[0, \infty)$ so that $a_n = \int_0^\infty t^n d\mu(t)$ for $n \geq 1$. We refer the reader to Neil Sloane's Online Encyclopedia of Integer Sequences [41] and the references given there for clarification of any undefined terms.

| Singletons (Section 3.1) | |
|---------------------------|---|
| EGF $f(x)$ | x |
| a_n | $a_n = 1$ if $n = 1$, else 0. $1, 0, 0, 0, \dots$ (OEIS A000007) |
| $p_n(t)$ | $p_n(t) = \frac{t^n}{n!}$ $1, t, \frac{t^2}{2!}, \frac{t^3}{3!}, \frac{t^4}{4!}, \frac{t^5}{5!}, \frac{t^5}{5!}$ |
| FGL | $x_1 + x_2 + \dots = m_1$ |

Sets (Section 3.1)

| | |
|--------------------|---|
| Sets (Section 3.1) | |
| EGF $f(x)$ | e^x |
| a_n | $a_n = 1$ $1, 1, 1, 1, \dots$ OEIS A000012 |
| $p_n(t)$ | $\binom{t}{k}$ $1, t, \frac{1}{2!}(t^2 - t), \frac{1}{3!}(t^3 - 3t^2 + 2t), \frac{1}{4!}(t^4 - 6t^3 + 11t^2 - 6t), \dots$ |
| FGL | $(\prod_{i=1}^{\infty} (1 + x_i)) - 1 = e_1 + e_2 + e_3 + \dots$ $= m_1 + \frac{1}{2!}(2m_{11}) + \frac{1}{3!}(6m_{111}) + \frac{1}{4!}(24m_{1111}) + \dots$ |
| $d\mu(t)$ | $\delta_1(t) dt$ (Dirac delta function centered at $t = 1$) |

Trees with labeled leaves (Section 3.2)

| | |
|---|--|
| Trees with labeled leaves (Section 3.2) | |
| EGF $f(x)$ | $f(x)$, where $f^{-1}(x) = 1 + 2x - e^x$ |
| EGF (weighted) | $f(x)$, where $f^{-1}(x) = x - s_2 \frac{x^2}{2!} - s_3 \frac{x^3}{3!} - \dots$ |
| a_n | $1, 1, 4, 26, 236, 2752, 39208, 660032, 12818912, \dots$ (OEIS A000311) |
| a_n (weighted) | $1, s_2, 3s_2^2 + s_3, 15s_2^3 + 10s_2s_3 + s_4, 105s_2^4 + 105s_2^2s_3 + 10s_3^2 + 15s_2s_4, \dots$ |
| $p_n(t)$ | $1, t, \frac{1}{2!}(t^2 - t), \frac{1}{3!}(t^3 - 3t^2 - t), \frac{1}{4!}(t^4 - 6t^3 - t^2 - t), \dots$ (OEIS A135494) |
| $p_n(t)$ (weighted) | $p_n(t) = \frac{1}{n!} B_n(t, -s_2t, -s_3t, \dots)$, B_n the n th complete Bell polynomial $1, t, \frac{1}{2!}(t^2 - ts_2), \frac{1}{3!}(t^3 - 3t^2s_2 - ts_3), \frac{1}{4!}(t^4 - 6t^3s_2 + 3t^2s_2^2 - 4t^2s_3 - ts_4), \dots$ |
| FGL | $m_1 + \frac{1}{2!}(2m_{11}) + \frac{1}{3!}(24m_{111} + 9m_{21})$ $+ \frac{1}{4!}(264m_{211} + 52m_{31} + 624m_{1111} + 114m_{22}) + \dots$ |
| FGL (weighted) | $m_1 + \frac{1}{2!}(2s_2m_{11}) + \frac{1}{3!}((18s_2^2 + 6s_3)m_{111} + (6s_2^2 + 3s_3)m_{21}) + \dots$ |

| Binary trees with labeled leaves (Section 3.2) | |
|--|--|
| EGF $f(x)$ | $1 - \sqrt{1 - 2x}$ |
| a_n | $(1)(3)(5) \cdots (2n - 3) = (2n - 3)!!$ $1, 1, 3, 15, 105, 945, 10395, 135135, \dots$ (OEIS A001147) |
| $p_n(t)$ | $\frac{1}{n!}(\sqrt{t})^n H_n(\sqrt{t})$ where $H_n(t)$ is the n th Hermite polynomial $1, t, \frac{1}{2!}(t^2 - t), \frac{1}{3!}(t^3 - 3t^2), \frac{1}{4!}(t^4 - 6t^3 + 3t^2), \dots$ OEIS A104556 |
| FGL | $m_1 + \frac{1}{2!} \cdot 2m_{11} + \frac{1}{3!}(18m_{111} + 6m_{21}) + \frac{1}{4!}(60m_{22} + 24m_{31} + 360m_{1111} + 144m_{211}) + \dots$ |
| $d\mu(t)$ | $e^{-t/2}/\sqrt{2\pi t^3} dt$ |

| Trees with all vertices labeled (Section 3.3) | |
|---|---|
| EGF $f(x)$ | $f(x)$ where $f^{-1}(x) = xe^{-x}$ |
| a_n | n^{n-1} $1, 2, 9, 64, 625, 7776, 117649, 2097152, 43046721, \dots$ (OEIS A000169) |
| $p_n(t)$ | $\sum_{i=0}^n \binom{n}{i} (-i)^{n-i} t^i$ $1, t, \frac{1}{2!}(t^2 - 2t), \frac{1}{3!}(t^3 - 6t^2 + 3t), \frac{1}{4!}(t^4 - 12t^3 + 24t^2 - 4t), \dots$ (OEIS A059297) |
| FGL | $m_1 + \frac{1}{2!}(4m_{11}) + \frac{1}{3!}(54m_{111} + 15m_{21}) + \frac{1}{4!}(552m_{211} + 64m_{31} + 1536m_{1111} + 216m_{22}) + \dots$ |
| $d\mu(t)$ | See [51]. |

| Permutations (Section 3.4) | |
|----------------------------|---|
| EGF $f(x)$ | $x/(1-x)$ |
| a_n | $n!$ $1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, \dots$ (OEIS A000142) |
| $p_n(t)$ | $(-1)^k L_k^{(-1)}(t)$ where $L_k^{(\alpha)}$ is the generalized Laguerre polynomial $\sum_{i=0}^k (-1)^{k-i} \binom{k-1}{k-i} \frac{t^i}{i!}$ $1, t, \frac{1}{2!}(t^2 - 2t), \frac{1}{3!}(t^3 - 6t^2 + 6t), \frac{1}{4!}(t^4 - 12t^3 + 36t^2 - 24t), \dots$ (OEIS A111596) |
| FGL | $m_1 + \frac{1}{2!}(4m_{11}) + \frac{1}{3!}(36m_{111} + 6m_{21}) + \frac{1}{4!}(144m_{211} + 576m_{1111} + 48m_{22}) + \dots$ |
| $d\mu(t)$ | $e^{-t} dt$ |

| Labeled posets with minimum and maximum (Section 3.5) | |
|---|---|
| EGF $f(x)$ | $x + 2x + 6\frac{x^2}{2!} + 36\frac{x^3}{3!} + 380\frac{x^4}{4!} + 6570\frac{x^5}{5!} + \dots$ |
| a_n | $1, 2, 6, 36, 380, 6570, \dots$ |
| $p_n(t)$ | $1, t, \frac{1}{2!}(t^2 - 2t), \frac{1}{3!}(t^3 - 6t^2 + 6t), \frac{1}{4!}(t^4 - 12t^3 + 36t^2 - 36t), \dots$ |
| FGL | $m_1 + \frac{1}{2!}(4m_{11}) + \frac{1}{3!}(36m_{111} + 6m_{21}) + \frac{1}{4!}(288m_{211} + 48m_{31} + 864m_{1111} + 120m_{22}) + \dots$ |

| Lattices (Section 3.5) | |
|------------------------|---|
| EGF $f(x)$ | $x + 2x + 6\frac{x^2}{2!} + 36\frac{x^3}{3!} + 380\frac{x^4}{4!} + 6390\frac{x^5}{5!} + \dots$ |
| a_n | $1, 2, 6, 36, 380, 6390, 157962, 5396888, 243179064, 13938711210 \dots$ (OEIS A055512) |
| $p_n(t)$ | $1, t, \frac{1}{2!}(t^2 - 2t), \frac{1}{3!}(t^3 - 6t^2 + 6t), \frac{1}{4!}(t^4 - 12t^3 + 36t^2 - 36t), \dots$ |
| FGL | $m_1 + \frac{1}{2!}(4m_{11}) + \frac{1}{3!}(36m_{111} + 6m_{21}) + \frac{1}{4!}(288m_{211} + 48m_{31} + 864m_{1111} + 120m_{22}) + \dots$ |

| Labeled graphs (Section 3.5) | |
|------------------------------|---|
| EGF $f(x)$ | $x + 2\frac{x^2}{2} + 8\frac{x^3}{3} + 64\frac{x^4}{4} + 1024\frac{x^5}{5} + \dots$ |
| a_n | $2^{\binom{n}{2}}$ 1, 2, 8, 64, 1024, 32768, 2097152, 268435456, 68719476736, ... (OEIS A006125) |
| $p_n(t)$ | $1, t, \frac{1}{2!}(t^2 - 2t), \frac{1}{3!}(t^3 - 6t^2 + 4t), \frac{1}{4!}(t^4 - 12t^3 + 28t^2 - 24t), \dots$ |
| FGL | $m_1 + \frac{1}{2!}(4m_{11}) + \frac{1}{3!}(48m_{111} + 12m_{21}) + \frac{1}{4!}(576m_{211} + 96m_{31} + 1536m_{1111} + 240m_{22}) + \dots$ |

| Labeled tanglegrams (Section 5.1) | |
|-----------------------------------|---|
| EGF $f(x)$ | $x + \frac{x^2}{2!} + 9\frac{x^3}{3!} + 225\frac{x^4}{4!} + 11025\frac{x^5}{5!} + \dots$ |
| a_n | $(2n - 3)!!^2 = (1 \cdot 3 \cdot \dots \cdot (2n - 3))^2$ 1, 1, 9, 225, 11025, 893025, 108056025, ... (OEIS A001818) |
| $p_n(t)$ | $1, t, \frac{1}{2!}(t^2 - t), \frac{1}{3!}(t^3 - 3t^2 - 6t), \frac{1}{4!}(t^4 - 6t^3 - 21t^2 - 150t), \dots$ |
| FGL | $m_1 + \frac{1}{2!} \cdot 2m_{11} + \frac{1}{3!}(54m_{111} + 24m_{21}) + \frac{1}{4!}(2592m_{211} + 768m_{31} + 5400m_{1111} + 1248m_{22}) + \dots$ |
| $d\mu(t)$ | $t^{-3/2}K_0(\sqrt{t})/\pi, K_\nu(x)$ the modified Bessel function of the second kind |

| Increasing trees (Section 9.1) | |
|--------------------------------|---|
| EGF $f(x)$ | $-\log(1-x)$ |
| EGF $f(x)$ (weighted) | $f(x)$ so that $f(0) = 0$, $f'(x) = \phi(f(x))$ where $\phi(x) = 1 + s_1x + s_2\frac{x^2}{2!} + \dots$ |
| a_n | $(n-1)!$ $1, 1, 2, 6, 24, 120, 720, 5040, 40320, 362880, \dots$ (OEIS A001818) |
| a_n (weighted) | $1, s_1, s_1^2 + s_2, s_1^3 + 4s_1s_2 + s_3, s_1^4 + 11s_1^2s_2 + 4s_2^2 + 7s_1s_3 + s_4, \dots$ |
| $p_n(t)$ | $(-1)^n T_n(-t)/n!$, $T_n(t)$ the n th <i>Touchard polynomial</i> $\sum_{i=1}^n S(n, i)(-1)^{n-i} t^i$, $S(n, k)$ the Stirling numbers of the second kind $1, t, \frac{1}{2!}(t^2 - t), \frac{1}{3!}(t^3 - 3t^2 + t), \frac{1}{4!}(t^4 - 6t^3 + 7t^2 - t), \dots$ (OEIS A080417 , OEIS A008277) |
| $p_n(t)$ (weighted) | $1, t, \frac{1}{2!}(-s_1t + t^2), \frac{1}{3!}((2s_1^2 - s_2)t - 3s_1t^2 + t^3), \dots$ |
| FGL | $m_1 + \frac{1}{2!}(2m_{11}) + \frac{1}{3!}(12m_{111} + 3m_{21}) + \frac{1}{4!}(48m_{211} + 4m_{31} + 144m_{1111} + 18m_{22}) + \dots$ |
| $d\mu(t)$ | $e^{-t}/t dt$ |
| FGL (weighted) | $m_1 + \frac{1}{2!}(2s_1m_{11}) + \frac{1}{3!}(6s_1^2 + 6s_2)m_{111} + \dots$ |

| Increasing binary trees (Section 9.1) | |
|---------------------------------------|---|
| EGF $f(x)$ | $\tan(x)$ |
| a_n | $(n-1)!$ $1, 1, 2, 6, 24, 120, 720, 5040, 40320, 362880, \dots$ (OEIS A001818) |
| $p_n(t)$ | $1, t, \frac{1}{2!}(t^2), \frac{1}{3!}(t^3 - 2t), \frac{1}{4!}(t^4 - 8t^2), \dots$ (OEIS A049218) |
| FGL | $m_1 + 2m_{111} + m_{21} + 4m_{221} + 16m_{11111} + m_{32} + 2m_{311} + 8m_{2111} + \dots$ |
| $d\mu(t)$ | $e^{-t}/t dt$ |

| Increasing unary-binary trees (Section 9.1) | |
|---|---|
| EGF $f(x)$ | $\frac{2 \tan(\omega x)}{2\omega - \tan(\omega x)}, \omega = \sqrt{3}/2$ |
| EGF $f(x)$ (weighted) | $\frac{2 \tan(\omega x)}{2\omega - s_1 \tan(\omega x)}, \omega = \sqrt{s_2 - s_1^2/4}$ |
| a_n | $1, 1, 3, 9, 39, 189, 1107, 7281, 54351, \dots$ (OEIS A080635) |
| a_n (weighted) | $1, s_1, s_1^2 + 2s_2, s_1^3 + 8s_1s_2, s_1^4 + 22s_1^2s_2 + 16s_2^2, s_1^5 + 52s_1^3s_2 + 136s_1s_2^2, \dots$ |
| $p_n(t)$ | $1, t, \frac{1}{2!}(t^2 - t), \frac{1}{3!}(t^3 - 3t^2), \frac{1}{4!}(t^4 - 6t^3 + 3t^2 + 6t), \dots$ |
| $p_n(t)$ (weighted) | $1, t, \frac{1}{2!}(-s_1t + t^2), \frac{1}{3!}(2s_1^2t - 3s_1t^2 + t^3 - 2s_2t), \dots$ |
| FGL | $m_1 + (1)m_{11} + (3)m_{111} + (1)m_{21} + (3)m_{211} + (9)m_{1111} + (1)m_{22} + \dots$ |
| FGL (weighted) | $m_1 + (s_1)m_{11} + (s_1^2 + 2s_2)m_{111} + (s_2)m_{21} + (3s_1s_2)m_{211} + (s_1^3 + 8s_1s_2)m_{1111} + (s_1s_2)m_{22} + \dots$ |

A.2 Ordinary generating functions

We list information about the contractible L-species we have encountered. The information provided is the same as in Section A.1 for contractible species, except that for each we list

the ordinary generating function

$$f(x) = \sum_{n=1}^{\infty} a_n x^n$$

where $a_n = |\mathcal{F}[n]|$ rather than the exponential generating function, and we find a measure μ so that

$$n!a_n = \int_0^{\infty} t^n d\mu(t).$$

| Plane trees with labeled leaves (Section 6.2) | |
|---|--|
| OGF $f(x)$ | $\frac{1+x-\sqrt{x^2-6x+1}}{4}$ |
| OGF $f(x)$ (weighted) | $f(x)$ where $f^{-1}(x) = x - s_2 \frac{x^2}{2!} - s_3 \frac{x^3}{3!} - \dots$ |
| a_n | Little Schroeder numbers 1, 1, 3, 11, 45, 197, 903, 4279, 20793, ... (OEIS A001003) |
| a_n (weighted) | 1, s_2 , $2s_2^2 + s_3$, $5s_2^3 + 5s_2s_3 + s_4$, $14s_2^4 + 21s_2^2s_3 + 3s_3^2 + 6s_2s_4 + s_5, \dots$ |
| $p_n(t)$ | 1, t , $\frac{1}{2!}(t^2 - 2t)$, $\frac{1}{3!}(t^3 - 6t^2 - 6t)$, $\frac{1}{4!}(t^4 - 12t^3 - 12t^2 - 24t)$, ... |
| $p_n(t)$ (weighted) | 1, t , $\frac{1}{2!}(t^2 - 2ts_2)$, $\frac{1}{3!}(t^3 - 6t^2s_2 - 6ts_3)$, $\frac{1}{4!}(t^4 - 12t^3s_2 + 12t^2s_2^2 - 24t^2s_3 - 24ts_4)$, ... |
| FGL | $m_1 + 2m_{11} + 18m_{111} + 7m_{21} + 114m_{211} + 24m_{31} + 264m_{1111} + 50m_{22} + \dots$ |
| FGL (weighted) | $m_1 + 2s_2m_{11} + (12s_2^2 + 6s_3)m_{111} + (4s_2^2 + 3s_3)m_{21} + \dots$ |

| Plane trees with all vertices labeled, Section 6.3 | |
|--|--|
| OGF $f(x)$ | $\frac{1-\sqrt{1-4x}}{2}$ |
| OGF $f(x)$ (weighted) | $f(x)$ so that $x/f^{-1}(x) = 1 + s_1x + s_2x^3 + \dots$ |
| a_n | Catalan numbers 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, ... (OEIS A000108) |
| a_n (weighted) | $1, s_1, s_1^2 + s_2, s_1^3 + 3s_1s_2 + s_3, s_1^4 + 6s_1^2s_2 + 2s_2^2 + 4s_1s_3 + s_4, \dots$ |
| $p_n(t)$ | $1, t, \frac{1}{2!}(t^2 - 2t), \frac{1}{3!}(t^3 - 6t^2),$ (OEIS 119275) |
| $p_n(t)$ (weighted) | $1, t, \frac{1}{2!}(t^2 - 2ts_1), \frac{1}{3!}(t^3 - 6t^2s_1 + 6ts_1^2 - 6ts_2), \frac{1}{4!}(t^4 - 12t^3s_1 + 36t^2s_1^2 - 24ts_1^3 - 24t^2s_2 + 48ts_1s_2), \dots$ |
| FGL | $m_1 + 2m_{11} + 12m_{111} + 4m_{21} + 20m_{22} + 8m_{31} + 120m_{1111} + 48m_{211} + \dots$ |
| FGL (weighted) | $m_1 + 2s_1m_{11} + (6s_1^2 + 6s_2)m_{111} + (s_1^2 + 3s_2m_{21} + 6s_1^3 + 30s_1s_2 + 12s_3)m_{211} + \dots$ |
| $d\mu(t)$ | $(e^{t/4}t^{3/2}2\sqrt{2\pi})^{-1} dt$ |

| Motzkin paths (Section 6.4) | |
|-----------------------------|---|
| OGF $f(x)$ | $\frac{1-x-\sqrt{1-2x-3x^2}}{2x}$ |
| a_n | Motzkin numbers 1, 1, 2, 4, 9, 21, 51, 127, 323, 835, ... (OEIS A000106) |
| $p_n(t)$ | $1, t, \frac{1}{2!}(t^2 - 2t), \frac{1}{3!}(t^3 - 6t^2), \frac{1}{4!}(t^4 - 12t^3 + 12t^2 + 24t), \dots$ (OEIS A119275) |
| FGL | $m_1 + 2m_{11} + 12m_{111} + 4m_{21} + 14m_{22} + 4m_{31} + 96m_{1111} + 36m_{211} + \dots$ |

Permutations (Section 6.4)

| | |
|------------|---|
| OGF $f(x)$ | $x + 2x^2 + 6x^3 + 120x^4 + \dots$ |
| a_n | n! 1, 2, 6, 24, 120, 720, 5040, 40320, 362880 (OEIS A000106) |
| $p_n(t)$ | $1, t, \frac{1}{2!} (t^2 - 4t), \frac{1}{3!} (t^3 - 12t^2 + 12t), \frac{1}{4!} (t^4 - 24t^3 + 96t^2 - 96t), \dots$ (OEIS A119275) |
| FGL | $m_1 + 4m_{11} + 36m_{111} + 10m_{21} + 216m_{211} + 32m_{31} + 576m_{1111} + 88m_{22} + \dots$ |
| $d\mu(t)$ | $2K_0(2\sqrt{t})/t dt, K_\nu(x)$ the modified Bessel function of the second kind |