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Nonlinear PDEs: regularity, rigidity, and an inverse problem

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Abstract

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Based on joint work with Arunima Bhattacharya, we obtain a sharp regularity result for Lagrangian mean curvature type equations with possibly Hölder continuous Lagrangian phases. Along the way, the constant rank theorem of Bian and Guan is generalized, and a different, lower regularity way to prove strict convexity is developed. Next, based on joint work with Yu Yuan, we show that smooth semiconvex solutions of the sigma-2 equation are quadratic polynomials if they are entire. Finally, a Calderón type inverse problem for quasilinear elliptic equations is discussed, where the author improves a recent result of Muñoz and Uhlmann using boundary jet linearization.

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DEDICATION

to Minnie, Ruby, and Kiki

Chapter 1

INTRODUCTION

This thesis compiles the author's joint works on Lagrangian mean curvature type equations in Bhattacharya-Shankar [7], the sigma-2 equation in Shankar-Yuan [63], and an inverse problem for a quasilinear PDE in Shankar [60]. The introductions of these papers are represented in this section.

1.1 Lagrangian mean curvature type equations

We classify regularity for convex viscosity solutions of Lagrangian mean curvature type equations

$$\sum_{i=1}^{n} \arctan \lambda_i = \psi(x, u, Du) \in [0, n\pi/2] \tag{1.1.1}$$

where λ_i's are the eigenvalues of the Hessian of u. The left hand side $\theta(x) := tr \arctan D^2u(x)$ is called the Lagrangian phase or angle of the Lagrangian submanifold $\{(x, Du(x))\} \subset \mathbb{C}^n = \mathbb{R}^n \times \mathbb{R}^n$. The geometric motivation for studying Lagrangian mean curvature type equations is the presence of several interesting special cases of this equation, which we discuss below; see [79] by Yuan for a recent survey.

Lagrangian mean curvature equation. Suppose $\psi = \psi(x, y)$ is defined on the ambient space $(x, y) \in \mathbb{C}^n$, and u solves

$$\sum_{i=1}^{n} \arctan \lambda_i = \psi(x, Du). \tag{1.1.2}$$

By Harvey-Lawson [32, Proposition 2.17], the Lagrangian angle determines the mean curvature vector of $\{(x, Du(x))\}$:

$$H = J\nabla_g \theta, \tag{1.1.3}$$

where $g = I + (D^2u)^T(D^2u)$ is the induced metric on $\{(x, Du(x))\}$, and J is the almost complex structure on $\mathbb{C}^n$. Since $\theta(x) = \psi(x, Du(x))$, it follows that $J\nabla_g \psi(x, Du(x)) =$

$(J\bar{\nabla}\psi(x,y))^{\perp}$, where $\perp$ is the normal projection, and $\bar{\nabla} = 2(\Re(\partial_{\bar{z}}), \Im(\partial_{\bar{z}})) = (\partial_x, \partial_y)$. Thus (1.1.2) is the potential equation for prescribed Lagrangian mean curvature:

$$H = \Delta_g(x, Du(x)) \overset{(1.1.2)}{=} (J\bar{\nabla}\psi(x, Du(x)))^{\perp}, \tag{1.1.4}$$

The constant case

$$\sum_{i=1}^{n} \arctan \lambda_i = c \tag{1.1.5}$$

is the *special Lagrangian equation* of Harvey-Lawson [32]. In this case, $H = 0$, and $\{(x, Du(x))\}$ is a volume minimizing Lagrangian submanifold.

Lagrangian mean curvature flow. A family of Lagrangian submanifolds $X(x,t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C}^n$ evolves by Lagrangian mean curvature flow if it solves

$$(X_t)^{\perp} = \Delta_g X. \tag{1.1.6}$$

After a change of coordinates, we can locally write $X(x,t) = (x, Du(x,t))$, such that $\Delta_g X = (J\bar{\nabla}\theta(x,t))^{\perp}$ by (1.1.3). This means a local potential $u(x,t)$ evolves by the parabolic equation

$$\begin{aligned} u_t &= \sum_{i=1}^{n} \arctan \lambda_i, \\ u(x,0) &:= u(x). \end{aligned} \tag{1.1.7}$$

Several special cases of (1.1.1) correspond to symmetry reductions of (1.1.6), which reduce (1.1.7) to an elliptic equation for $u(x)$, as demonstrated, for example, by Chau-Chen-He [17]. Such solutions model singularities of the mean curvature flow.

1. *Self-similar solutions.* If $u(x)$ solves

$$\sum_{i=1}^{n} \arctan \lambda_i = c + b(x \cdot Du - 2u) \tag{1.1.8}$$

then $X(x,t) = \sqrt{1-2bt}\,(x, Du(x))$ is a shrinker or expander solution of (1.1.6), if $b > 0$ or $b < 0$, respectively. The initial submanifold $\{(x, Du(x))\}$ has mean curvature $H = -bX^{\perp}$. Entire smooth solutions to (1.1.8) for $b > 0$ are quadratic polynomials, by Chau-Chen-Yuan [18]; see also Huang-Wang [35] for the smooth convex case. The circle $x^2 + u'(x)^2 = 1$ is a closed example of a shrinker $b = 1, c = 0$ in dimension $n = 1$. See e.g. Joyce-Lee-Tsui [40] for other non-graphical examples.

2. *Translating solitons.* If $u(x)$ solves

$$\sum_{i=1}^{n} \arctan \lambda_i = c + k \cdot x + \ell \cdot Du(x), \tag{1.1.9}$$

then $X(x,t) = (x, Du(x)) + t(-\ell, k)$ is a *translator* solution of (1.1.6), with "constant" mean curvature $H = (-\ell, k)^{\perp}$. The "grim reaper" curve $(x, u'(x)) = (x, -\ln\cos(x))$, for $n = 1$ and $k = 1, \ell = c = 0$, is the model example. Entire solutions to (1.1.9) with Hessian bounds are quadratic polynomials, by Chau-Chen-He [17]; see also Ngyuen-Yuan [54] for entire ancient solutions to (1.1.7) with Hessian conditions.

3. *Rotating solitons.* If $A \in su(n)$ is skew-adjoint, then the Hamiltonian vector field $A \cdot z = J\bar{\nabla}\psi$ has a real potential $\psi(x,y) = \frac{1}{2i}\langle z, A \cdot z\rangle_{\mathbb{C}^n}$. Since $\exp(tA) \in U(n)$ preserves the symplectic form $dz \wedge d\bar{z}$, the Hamiltonian flow $X(x,t) = \exp(tA)(x, Du(x))$ is a Lagrangian immersion with $X_t = AX = J\bar{\nabla}\psi$. Provided $\psi(x, Du(x)) = \theta(x)$, i.e. $u(x)$ solves (1.1.2), it follows from this and the isometric property of $\exp(tA)$ that $X(x,t)$ evolves by mean curvature flow (1.1.6). For the example $A = aJ$ and $\psi(x,y) = c + \frac{a}{2}|z|^2$, if $u(x)$ solves

$$\sum_{i=1}^{n} \arctan \lambda_i = c + \frac{a}{2}(|x|^2 + |Du|^2), \tag{1.1.10}$$

then $X(x,t) = \exp(atJ)(x, Du(x))$ is a *rotator* solution of (1.1.6), with $H = a(JX)^{\perp}$. The "Yin-Yang" curve of Altschuler [2] is the solution in dimension $n = 1$. Other examples of such equations include $\psi(x,y) = -\langle x, B \cdot y\rangle$ for $B \in so(n)$ skew-symmetric and $\psi(x,y) = Re(\bar{z}_1 z_2) = x_1 x_2 + y_1 y_2$ in $\mathbb{C}^2$, with $A = diag(B, B)$ and $A \cdot (x,y) = (-y_2, -y_1, x_2, x_1)$, respectively.

Optimal regularity was previously obtained for the dual form of (1.1.5) under the strict convexity condition. The Monge-Ampère equation

$$\ln\det D^2 u = \sum_{i=1}^{n} \ln \lambda_i = c, \tag{1.1.11}$$

is the potential equation for special Lagrangian submanifolds in $(\mathbb{C}^n, dx\, dy)$, as introduced by Hitchin [34]; see also Mealy [49] for an equivalent form. In dimension $n = 2$, Alexandrov

[1] and Heinz [33] established strict convexity of solutions to Monge-Ampère type equations, then regularity for generalized solutions. However, in higher dimensions, the $C^{1,1-2/n}$ and $W^{2,p}$ for $p < \frac{n(n-1)}{2}$ singular solution of Pogorelov [56] shows that strict convexity is necessary for regularity, since its graph contains a line. Pogorelov's interior estimate [56] indicated that strict convexity is sufficient for regularity, and Urbas showed that solutions in $C^{1,\alpha}$ or $W^{2,p}$ for $\alpha > 1 - \frac{2}{n}$ or $p > \frac{n(n-1)}{2}$ are strictly convex, then interior regular. The $W^{2,n(n-1)/2}$ case is due to Collins-Mooney [24]. By Caffarelli [12], regularity holds even for positive Hölder right hand side $\det D^2 u \in C^\alpha$, provided $u = 0$ on the (convex) boundary.

We classify regularity for convex solutions of mean curvature potential equations, for both smooth phases and Hölder phases.

Theorem 1.1.1. *Let u be a convex viscosity solution of (1.1.1) on $B_1 \subset \mathbb{R}^n$. If $\psi \in C^2$ satisfies partial convexity*

$$p \mapsto \psi(x, s, p) \text{ is convex, for each fixed } (x, s), \tag{1.1.12}$$

then $u \in C^{3,\alpha}(B_1)$ for all $\alpha \in (0, 1)$.

Theorem 1.1.2. *Let u be a convex viscosity solution of (1.1.1) on $B_1 \subset \mathbb{R}^n$. If $\psi \in C^\alpha$ for some $\alpha \in (0, 1) \cup (1, 2)$, and $u \in C^{1,\beta}(B_1)$ for some $\beta \in (\frac{1}{1+\alpha}, 1)$, then $u \in C^{2+\alpha}(B_1)$.*

Notation.

1. If k is an integer and α is not an integer, we denote by $C^{k+\alpha}$ the Hölder space $C^{k',\alpha'}$, where $k' = \lfloor k + \alpha \rfloor$ is the integer part, and $\alpha' = k + \alpha - k'$ is the fractional part.

2. The ball B_1 refers to the unit ball centered at the origin. Throughout, we assume all balls to be centered at the origin unless specified otherwise.

Remark 1.1.1. In particular, Theorem 1.1.1 shows convex solutions of mean curvature flow potential equations (1.1.8), (1.1.9), and (1.1.10) for $a \geq 0$ are smooth. Theorem 1.1.2 yields regularity for the $a < 0$ case in rotator equation (1.1.10) provided the graph $\{(x, Du(x))\}$ is $C^{1/3+}$.

Remark 1.1.2. Using singular solutions, we see most of the conditions are optimal:

1.2.1. Partial convexity (1.1.12) is necessary, and the exponent range $\beta > 1/(1+\alpha)$ or $\alpha > \beta^{-1} - 1$ in Theorem 1.1.2 is sharp. Observe that for $\beta > 0$, $u(x) = |x|^{1+\beta}/(1+\beta)$ is a viscosity solution of a mean curvature equation (1.1.2) with non-convex phase

$$\sum_{i=1}^{n} \arctan \lambda_i = \frac{n\pi}{2} - (n-1)\arctan(|Du|^{\beta^{-1}-1}) - \arctan(\beta^{-1}|Du|^{\beta^{-1}-1}), \quad (1.1.13)$$

even when the phase is smooth $\beta = \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$. Similarly, in Chapter 2.6, we prove that for $a < 0$ and $c = n\pi/2$, the smooth, non-convex rotator equation (1.1.10) has a "Yin-Yang" curve-type solution which is convex and $C^{1,1/3}$ near the origin.

1.2.2. The convexity of u in Theorem 1.1.1 is necessary by semiconvex $C^{1,1/3}$ solutions of special Lagrangian equation (1.1.5) in dimension $n = 3$ as shown by Nadirashvili-Vlăduţ in [53]; see also Wang-Yuan [71] for $C^{1,\beta}$ solutions, where $\beta = \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$. In Chapter 2.5, we show a similar solution exists to the shrinker/expander potential equation (1.1.8). In addition, these special Lagrangian singular solutions $v(x_1, x_2, x_3)$ yield solutions of translator equation (1.1.9) for $n \geq 5$ if $k \cdot \ell = 0$, and solutions for $n \geq 4$ if $k \cdot \ell \neq 0$, by adding to v a quadratic polynomial in the $k, \ell \perp \mathbb{R}^3$ directions.

Each condition is also relevant to mean curvature flow. A Lagrangian submanifold $M \subset \mathbb{C}^n$ has compatible normal and almost complex structures, $TM \perp J(TM)$, and it was shown by Smoczyk [64] that this condition is preserved by mean curvature flow. Smoczyk-Wang [65] showed that the convexity condition of the initial potential $D^2u(x) \geq 0$ is preserved by Lagrangian mean curvature flow in the compact periodic setting. In the general entire convex case, the preservation of convexity was reached in Chau-Chen-Yuan [19]. Meanwhile, viscosity solutions are closed under mere uniform limits of the Lagrangian potential, see for example [13, Proposition 2.9]. Self-similar solutions (1.1.8) arise as Type 1 blowup limits (minimal blowup rate) in mean curvature flows, while eternal solutions, such as translators (1.1.9) and rotators (1.1.10), arise from Type 2 blowup limits (larger blowup rate).

Closely related to convexity $D^2u \geq 0$ is the supercriticality condition $|\theta(x)| \geq (n-2)\pi/2$. If either convexity $D^2u \geq 0$ or supercriticality is imposed, then the level set $\{M : tr \arctan(M) = \theta\}$ is convex, and fails convexity if otherwise, as shown by Yuan in [78].

The stronger condition $|\theta(x)| \geq (n-1)\pi/2$ even implies $D^2u \geq 0$. Hessian estimates for constant critical and supercritical phase (1.1.5) were established by Warren-Yuan [74, 75] and Wang-Yuan [71]; see also Li [47]. By the Dirichlet problem solvability due to Caffarelli-Nirenberg-Spruck [11] for constant critical or supercritical phases, these estimates yield interior regularity for (1.1.5). For $C^{1,1}$ supercritical phase $\psi = \psi(x)$, Hessian estimates were proved in [6] for solutions of (1.1.1), and the Dirichlet problem was solved in [5, Theorem 1.1]. The semiconvex $C^{1,1/3}$ solutions of Nadirashvili-Vlăduţ [53] and Wang-Yuan [71] have subcritical phases, so critical or supercritical phase is also an optimal condition.

Unlike supercriticality, the convexity condition $D^2u \geq 0$ is unstable under smooth approximations of boundary data, which prevents using a priori estimates to establish regularity results. It has been an open problem for how to derive interior regularity for convex viscosity solutions to (1.1.5), in light of the a priori estimate in Chen-Warren-Yuan [21]. This was only recently established in [20] using the following low regularity approach. If $u(x)$ were C^2, then the downward $U(n)$ rotation $\bar{z} = e^{-i\pi/4}z$ yields another Lagrangian submanifold $(\bar{x}, D\bar{u}(\bar{x}))$ whose potential $\bar{u}(\bar{x})$ satisfies another special Lagrangian equation (1.1.5) with smaller phase $c - n\pi/4$. For convex Lipschitz $u(x)$, this is still true, but it requires generalizing the canonical transformation $e^{-i\pi/4}(x + iDu(x)) = \bar{x} + iD\bar{u}(\bar{x})$. Using the auxiliary functions $U(x) := su(x) + c|x|^2/2$ and $\bar{U}(\bar{x}) := -s\bar{u}(\bar{x}) + c|\bar{x}|^2/2$, where $s = \sin\pi/4, c = \cos\pi/4$, the C^2 case shows that $\bar{U}(\bar{x}) = U^*(\bar{x})$, where $f^*(\bar{x}) = \sup_x(x \cdot \bar{x} - f(x))$ is the Legendre transform of strictly convex $f(x)$. In the Lipschitz case, defining $\bar{U}(\bar{x}) := U^*(\bar{x})$ supplies the desired rotation operation. The concavity of the equation (1.1.5) for convex u then ensures the existence of subsolution approximations, guaranteeing that rotation preserves the solution property. Because the gradient graph slope $0 \leq D^2u \leq \infty$ drops to the Lipschitz bounds $-I \leq D^2\bar{u} \leq I$ under rotation, methods from geometric measure theory give VMO regularity for $D^2\bar{u}$. Applying perturbation theory for the now uniformly elliptic PDE (1.1.5) then yields C^2 and higher regularity for $\bar{u}(\bar{x})$. At this point, $\{(x, Du(x))\} \subset \mathbb{C}^n$ is regular as a submanifold, but need not be a graph. A geometric calculation shows that the rotated eigenvalue $\sqrt{1 + \bar{\lambda}^2_{max}}$ is a subsolution of $\Delta_{\bar{g}}$, so the strong maximum principle guarantees $\bar{\lambda}_{max} < 1$ everywhere, then $\lambda_{max} < \infty$. In the paper [8], this approach was extended to

$\psi = \psi(x) \in C^{2,\alpha}$ at the cost of a more complicated geometric calculation. In passing, we also mention the Hessian estimates for convex smooth solutions with $C^{1,1}$ phase $\psi = \psi(x)$ in Warren [72, Theorem 8].

Our proof of Theorem 1.1.1 extends the method of [20] and [8] to $\psi = \psi(x, u, Du)$, simplifies the final step of the proof by connecting it to the constant rank theorem of Caffarelli-Guan-Ma [14], and along the way, generalizes the constant rank theorem of Bian-Guan [9]. Our insight is that the final step of showing $\bar{\lambda}_{max} < 1$ is equivalent to showing that $\bar{U}(\bar{x})$ is strongly convex, $D^2\bar{U} > 0$. Once shown, then $D^2U < \infty$ by Legendre duality $DU^* \circ DU(x) = x$. To establish strong convexity, we observe that by convexity condition (1.1.12), the original equation $F(D^2U, DU, U, x) = 0$ (1.1.1) for $U(x)$ is concave if $u(x)$ is convex, so that for $U^*(\bar{x})$ is inverse convex. The dependence of ψ on u corresponds to dependence of the equation for $\bar{U}$ on Legendre transform $\bar{x} \cdot D\bar{U} - \bar{U}$. It turns out the constant rank theorem of Bian-Guan works just as well in this generalized setting, as proved in Section 2.2, even though their result required additional convexity. Overall, this can be thought of as a dual strategy to that for Monge-Ampère (1.1.11), where instead strict convexity of $u(x)$ is key to regularity.

The proof of Theorem 1.1.2 requires a new low regularity approach to showing strong convexity of $U^*(\bar{x})$. In this Hölder ψ setting, we only know $U^*(\bar{x}) \in C^{2,\alpha}$, and the constant rank theorem appears inapplicable. The Monge-Ampère approach to strict convexity in Urbas [70] uses the assumed $C^{1,\alpha}$ or $W^{2,p}$ regularity and the specific determinant structure of the Monge-Ampère equation to construct barriers, which lift the solution along any cylinder. Our approach uses the arctangent structure in (1.1.1) only to deduce that $\bar{u}(\bar{x}) \in C^{2,\alpha}$, as above. Then more generally, if $f(x) \in C^{1,\beta}$ and $f^*(\bar{x}) \in C^{2,\alpha}$, we prove in Theorem 2.3.1 that $f^*(\bar{x})$ is strongly convex, hence $f(x) \in C^{1,1}$. The heuristic idea uses the smoothness reversal of the Legendre transform, interpreted as gradient graph reflection $(x, Df(x)) = (Df^*(\bar{x}), \bar{x})$. If $\{(x, Df(x))\}$ is flat near $x = 0$, then $\{(\bar{x}, Df^*(\bar{x}))\}$ is steep near $\bar{x} = 0$. Quantitatively, if a singular point of $f(x)$ is *no steeper* than $\frac{1}{p}|x|^p$ for $p = 1 + \beta$, then $f^*(\bar{x})$ is *no flatter* than Legendre transform $\frac{1}{q}|\bar{x}|^q$, where $\frac{1}{p} + \frac{1}{q} = 1$ and $q = 1 + \beta^{-1}$. This contradicts $f^*(\bar{x}) \in C^{2,\alpha}$, which implies that $f^*(\bar{x})$ is *at least as flat* as $|\bar{x}|^{2+\alpha}$ at a point

failing strong convexity, if $2+\alpha > 1+\beta^{-1}$.

The structure of Chapter 2 in proving Theorems 1.1.1 and 1.1.2 is as follows. In Section 2.1, we establish regularity for $\bar{u}(\bar{x})$ in both situations of Theorems 1.1.1 and 1.1.2, adapting the above approach of [20] and [8]. Modifications are needed in the $\psi = \psi(x, u, Du)$ case in showing subsolution preservation (Proposition 2.1.1), and in the regularity theory for uniformly elliptic PDEs $F(D^2u) = f(x)$ with Hölder $f(x)$ (Theorem 2.1.2). In Section 2.2, we prove Theorem 2.2.1, which generalizes the constant rank theorem of Bian-Guan [9]. In Section 2.3, we prove Theorem 2.3.1, which establishes "dual" strong convexity in lower regularity situations. We conclude the proofs of Theorems 1.1.1 and 1.1.2 in Section 2.4. In Sections 2.5 and 2.6, we construct singular solutions for self-similar Lagrangian mean curvature flows (1.1.8) and Lagrangian rotator equation (1.1.10), respectively.

1.2 Sigma-2 equation

We next show that every general semiconvex entire solution in $\mathbb{R}^n$ to the Hessian equation

$$\sigma_k\left(D^2u\right) = \sigma_k\left(\lambda\right) = \sum_{1\le i_1<\cdots<i_k\le n} \lambda_{i_1}\cdots\lambda_{i_k} = 1$$

with $k = 2$ must be quadratic. Here $\lambda_i s$ are the eigenvalues of the Hessian D^2u.

Theorem 1.2.1. *Let u be a smooth semiconvex solution to $\sigma_2\left(D^2u\right) = 1$ on $\mathbb{R}^n$ with $D^2u \ge -K\ I$ for some $K > 0$. Then u is quadratic.*

Recall the classical Liouville theorem for the Laplace equation $\sigma_1\left(D^2u\right) = \triangle u = 1$ or Jörgens-Calabi-Pogorelov theorem for the Monge-Ampère equation $\sigma_n\left(D^2u\right) = \det D^2u = 1$: all convex entire solutions to those equations must be quadratic. Theorem 1.2.1 has been settled under an almost convexity condition $D^2u \ge \left(\delta - \sqrt{2/\left[n\left(n-1\right)\right]}\right) I$ for general dimension in Chang-Yuan [16]; and under the general semiconvexity condition $D^2u \ge -KI$ in three dimensions in Yuan [77] by taking advantage of the special Lagrangian form of the equation in this case. Assuming a super quadratic growth condition, Bao-Chen-Ji-Guan [4] demonstrated that all convex entire solutions to $\sigma_k\left(D^2u\right) = 1$ with $k = 1, 2, \cdots, n$ are quadratic polynomials; and Chen-Xiang [22] showed that all "super quadratic" entire solutions to $\sigma_2\left(D^2u\right) = 1$ with $\sigma_1\left(D^2u\right) > 0$ and $\sigma_3\left(D^2u\right) \ge -K$ are also quadratic

polynomials. Warren's rare non-convex entire solutions for the $\sigma_2\left(D^2u\right) = 1$ case [73] confirm the necessity of the semiconvexity assumption. It was "guessed" in the 2009 Chang-Yuan paper [16] that Theorem 1.2.1 should hold true.

The equation $\sigma_2(\kappa) = 1$ prescribes the intrinsic scalar curvature of a Euclidean hypersurface $\{(x, u(x))\}$ in $\mathbb{R}^n \times \mathbb{R}^1$ with extrinsic principal curvatures $\kappa = (\kappa_1, \cdots, \kappa_n)$. The σ_2 function of the Schouten tensor arises in conformal geometry, and complex σ_2-type equations arise from the Strominger system in string theory.

The current work, as well the previous ones Chang-Yuan [16] and Yuan [77], has been inspired by Nitsche's classical paper [55], where the Legendre-Lewy transform was employed to produced an elementary proof of Jörgens' rigidity for the two dimensional Monge-Ampère equation, and in turn, Bernstein's rigidity for the two dimensional minimal surface equation.

The Legendre-Lewy transform of a general semiconvex solution satisfies a uniformly elliptic, saddle equation. In the almost convex case in Chang-Yuan [16], the new equation becomes concave, thus Evans-Krylov-Safonov theory yields the constancy of the bounded new Hessian, and in turn, the old one. To beat the saddle case, one has to be "lucky". Recall that, in general Evans-Krylov-Safonov fails as shown by the saddle counterexamples of Nadirashvili-Vlăduţ [52]. The earlier obtained trace Jacobi inequality, as an alternative log-convex vehicle, other than the maximum eigenvalue Jacobi inequality, in deriving the Hessian estimates for general semiconvex solutions in Shankar-Yuan [61], could rescue the saddleness. But the trace Jacobi only holds for large enough trace of the Hessian. It turns out that the trace added by a large enough constant satisfies the elusive Jacobi inequality (Proposition 3.1.1)

Equivalently, the reciprocal of the shifted trace Jacobi quantity is superharmonic, and it remains so in the new vertical coordinates under the Legendre-Lewy transformation by a transformation rule (Proposition 3.1.2). Then the iteration arguments developed in Caffarelli-Yuan [15] show the "vertical" solution is close to a "harmonic" quadratic at one small scale (Proposition 3.2.1, two steps in the execution: the superharmonic quantity concentrates to a constant in measure by applying Krylov-Safonov's weak Harnack; a variant of the superharmonic quantity, as a quotient of symmetric Hessian functions of the new potential, is very pleasantly concave and uniformly elliptic, consequently, closeness to a

"harmonic" quadratic is possible by Evans-Krylov-Safonov theory), and the closeness improves increasingly as we rescale (this is a self-improving feature of elliptic equations, no concavity/convexity needed). Thus a Hölder estimate for the bounded Hessian is realized, and consequently so is the constancy of the new and then the old Hessian.

In three dimensions, the proof provides a "pure" PDE approach to establish the rigidity, distinct from the geometric measure theory way used in the earlier work Yuan [77, Theorem 1.3] on the rigidity for special Lagrangian equations.

It is an interesting question if the general semiconvexity condition has meaningful applications to regularity. Let $u(x)$ be a viscosity solution to $\sigma_2 = 1$ on $B_1(0) \subset \mathbb{R}^n$ satisfying the almost-convexity condition $D^2u \geq \left(\delta - \sqrt{2/[n(n-1)]}\right) I$ of Chang-Yuan [16]. The author with Yuan in [62] showed that $u(x)$ is real analytic, using similar Legendre-Lewy transform techniques as Chen-Shankar-Yuan [20]. An effective interior estimate for such solutions is obtained in Shankar-Yuan [61]. For convex viscosity solutions, this result is obtained by Mooney [50] using a quick method based on barriers.

1.3 Quasilinear inverse problem

1.3.1 Discussion

The classical Calderón problem considers an isotropic linear elliptic equation and its Dirichlet-to-Neumann map

$$\begin{aligned} \nabla \cdot (a(x)\nabla u) &= 0, \\ \Gamma_a : u|_{\partial\Omega} &\mapsto a(x)\frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}. \end{aligned} \tag{1.3.1}$$

The inverse problem is to determine $a(x)$ from Γ_a. See [70] for a survey of this problem and its extensions. The matrix extension $a(x) = a_{ij}(x)$, or the anisotropic Calderón problem, is still unsolved in general for dimension three or higher.

Recent work has featured the quasilinear generalization $a = a(x, u, \nabla u)$. The case $a = a(x, u)$ was considered in [66], using a linearization technique similar to that originally introduced in [37]. This was extended in [68] to the anisotropic case $a = a_{ij}(x, u)$ (matrix). See [36] for analysis in the case $a = a_{ij}(x, \nabla u)$ in dimension two, and [41] for the recovery of a Taylor polynomial of $a = a_{ij}(x, \nabla u)$. The linearization technique was also applied

to the semilinear equations $\Delta u + a(x,u) = 0$ and $\Delta u + a(x,\nabla u) = 0$ by [39] and [67], respectively. More recently, a multilinearization technique was developed for $\Delta_g u + a(x,u) = 0$ in [27, 43, 44], where a is analytic in u. The equation $\Delta u + f(x)|\nabla u|^2 + V(x,u) = 0$ with V analytic was determined in the partial data case in [42] using multilinearization. In [38], the equation $\Delta u + c(u,\nabla u) = 0$ was considered using singular solutions of the linearized equation which concentrate at the boundary. The gradient structure was used for the quasilinear equation $\nabla \cdot (a(u)\nabla u) + c(x)u = 0$ with lower order term in [26].

In the work [51] by Muñoz and Uhlmann, attention was brought to the quasilinear case $a = a(u,\nabla u)$, and results were obtained assuming conditions on the holomorphic extension of a. These conditions ensure that linearization is possible around complex-valued affine solutions. The question is raised whether we can allow for *arbitrary* $a(s,p)$ real analytic, $a(s,p)$ smooth but *non-analytic*, and *non-homogeneous* $\partial a(x,s,p)/\partial x \neq 0$.

The purpose of this work is to extend the results of [51] to several new classes of quasilinear conductivities:

1. $a(u,\nabla u)$ real analytic, without additional hypotheses, Theorems 1.3.1 and 1.3.2.

2. $a(u,\nabla u)$ smooth, assuming additional conditions (1.3.7) and (1.3.8), Theorem 1.3.3.

3. Non-homogeneous cases $a = a(x,u,\nabla u)$ with known x part, Corollary 1.3.1.

An apparent *a priori* difficulty of the inverse problem for $a = a(u,\nabla u)$ (and truly one for general $a = a(x,u,\nabla u)$) is that the linearized inverse problem is anisotropic. Not only is the anisotropic problem not understood in dimension three or higher, but the matrix $a_{ij}(x)$ can only be recovered up to pushforward by a boundary-preserving diffeomorphism; see [68] and [67] for similar difficulties. In the p-Laplace case $a = A(x)|\nabla u|^{p-2}\nabla u$, the "nonlinear part" of the quasilinearity is known, unlike for equation (1.3.2), and boundary determination was possible in [59] using explicit Wolff-type oscillatory solutions, with full recovery using a monotonicity identity [31]. If instead the anisotropic and nonlinear part is small, then one may linearize near the zero solution as in [41] and recover the x-dependent Taylor coefficients using linear theory. This is similar to the idea of the multilinearization approach [27, 43, 42, 44] for which dependence on u or ∇u is analytic. Constant solutions were used

for linearizing the conductivity equation in [68], and complex-valued affine solutions were used for linearization in [51]. However, it is not clear what non-constant, explicit solutions quasilinear equation (1.3.2) has in full generality. If $a(s,p)$ is smooth but not real analytic, then complex-valued affine solutions no longer make sense.

Our approach is to use linear boundary determination theory, which is well understood even in the anisotropic case. We first linearize the equation around any solution with a prescribed boundary jet, then recover the tangential part of the linearized conductivity matrix (1.3.3) using boundary determination theory. Next, the isotropic structure of the linearized conductivity matrix yields a solvable algebraic system for the conductivity evaluated at the jet. According to the comparison principle, we can use logarithmic barriers to prescribe boundary jets on a small open subset, which will complete the proof of Theorem 1.3.1. For Theorem 1.3.3, decay condition (1.3.8) ensures that exponential barriers exist, hence solutions with any prescribed boundary jet in $\mathbb{R} \times \mathbb{R}^n$.

The presentation is organized as follows. In Section 4.1.1, we show that the linearized BVP is wellposed and has a well-defined DN map. In Section 4.1.2, we use the comparison principle to construct solutions with prescribed boundary jets. In Section 4.1.3, we geometrically reformulate the linearized problem, which allows us to invoke known boundary determination theory in Section 4.1.4. In Section 4.2, we complete the proofs of Theorems 1.3.1 and 1.3.3.

1.3.2 Statement of results

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a smooth bounded domain, and $u(x) \in C^\infty(\overline{\Omega})$ be the smooth solution of the quasilinear boundary value problem

$$\begin{aligned} \nabla \cdot (a(u, \nabla u)\nabla u) &= 0, \qquad x \in \Omega, \\ u|_{\partial\Omega} &= f, \end{aligned} \tag{1.3.2}$$

where $a(s,p) \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$ satisfies standard conditions to make this BVP wellposed, see [51] and [29]. We assume summation over repeated indices.

Ellipticity:

$$\begin{gathered} 0 < \lambda(s,p)|\xi|^2 \leq a_{ij}(s,p)\xi_i\xi_j, \quad (s,p,\xi) \in \mathbb{R} \times \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}), \\ a_{ij} := a\,\delta_{ij} + \frac{1}{2}(a_{p_i}p_j + a_{p_j}p_i). \end{gathered} \tag{1.3.3}$$

Coercivity:

$$\begin{gathered} \lambda(s,p) \geq \lambda_0(|s|) > 0, \\ a(s,p) \geq 1. \end{gathered} \tag{1.3.4}$$

Growth:

$$\begin{gathered} |p||\nabla_p a| + |a| \leq \mu_0(|s|), \\ (1+|p|)|a_s| \leq \mu_0(|s|)|p|. \end{gathered} \tag{1.3.5}$$

The bounds $\lambda, \mu_0, \lambda_0$ and domain Ω are known quantities. Positive functions λ_0 and μ_0 are non-increasing and non-decreasing in $|s|$, respectively, and we may assume they are continuous. We also assume known bounds on the $C^k(K)$ seminorms of a for all $k \geq 0$ and all compact subsets K of the domain of a.

Now, the Dirichlet-to-Neumann (DN) map is defined by

$$f \mapsto \Gamma_a(f) = a(u, \nabla u)\frac{\partial u}{\partial \nu} dS, \tag{1.3.6}$$

where dS is the Euclidean area form on $\partial\Omega$, and $\frac{\partial u}{\partial \nu} = \nu \cdot \nabla u$ is the Euclidean normal derivative.

Under no additional assumptions, we can recover a from Γ_a along sufficiently small gradient values.

Theorem 1.3.1. *If $\Gamma_a = \Gamma_{\tilde{a}}$, then $a(s,p) = \tilde{a}(s,p)$ on an open set:*

$$\{(s,p) : |p| < \Pi(s)\},$$

where $\Pi : \mathbb{R} \to (0,\infty)$ is a positive continuous function which depends on the known quantities.

We next recover two classes of $a(s,p)$ along large gradients. The first follows immediately from Theorem 1.3.1.

Theorem 1.3.2. *Let $a(s,p)$ and $\tilde{a}(s,p)$ be real-analytic. If $\Gamma_a = \Gamma_{\tilde{a}}$, then $a(s,p) \equiv \tilde{a}(s,p)$.*

Theorem 1.3.2 extends the result of [51], which assumed conditions on the holomorphic extension of real analytic $a(s,p)$ in order to solve a complex-valued BVP.

Theorem 1.3.3. *Let a and $\tilde{a}$ satisfy two additional conditions:*

1. Uniform elliptic lower bound:

$$a_{ij}(s,p), \quad \tilde{a}_{ij}(s,p) \geq \lambda_0 > 0, \qquad (s,p) \in \mathbb{R} \times \mathbb{R}^n, \tag{1.3.7}$$

where the inequality is in the quadratic form sense, and λ_0 is constant.

2. Decay along large gradients:

$$|a_s(s,p)|, \quad |\tilde{a}_s(s,p)| \leq \frac{\varepsilon_0}{|p|}, \qquad (s,p) \in \mathbb{R} \times \mathbb{R}^n, \tag{1.3.8}$$

for some sufficiently small $\varepsilon_0 > 0$ depending on Ω and λ_0.

If $\Gamma_a = \Gamma_{\tilde{a}}$, then $a(s,p) \equiv \tilde{a}(s,p)$.

This seems to be the first example of global uniqueness of a non-analytic conductivity depending on $(u, \nabla u)$. Decay condition (1.3.8) is known from minimal surface type equations $a(s,p) = f(s)/\sqrt{1+|p|^2}$, see [51, page 5].

Finally, as shown in Remarks 4.2.1, 4.2.2, and 4.2.3, the above results hold for non-homogeneous conductivities $a = a(x,s,p)$ as well, provided the "x-dependent part" is either known (i.e. is the same for both a and $\tilde{a}$) or symmetric along a direction, say $(0,\ldots,0,1) \in \mathbb{R}^n$. We assume the known quantities in (1.3.3), (1.3.4), and (1.3.5) are unchanged and do not depend on x, for simplicity.

Corollary 1.3.1. *If $n \geq 3$, suppose also Ω is convex and a and $\tilde{a}$ are of the form*

$$\begin{aligned} a(x,s,p) &= F(x,s,p,b(x^1,\ldots,x^{n-1},s,p)), \\ \tilde{a}(x,s,p) &= F(x,s,p,\tilde{b}(x^1,\ldots,x^{n-1},s,p)), \end{aligned} \tag{1.3.9}$$

where $t \mapsto F(x,s,p,t)$ is injective for each fixed (x,s,p). (If $n=2$, remove convexity of Ω and assume b and $\tilde{b}$ are functions of (s,p) only).

Then the statements of Theorems 1.3.1, 1.3.2, and 1.3.3 are true with replacements of (s,p) by (x,s,p), with $x \in \Omega$.

Chapter 2

LAGRANGIAN MEAN CURVATURE TYPE EQUATIONS

2.1 Regularity of the rotated Lagrangian potential

In this section, we first outline the proof of some preliminary results, adapting methods from [20, 8], followed by regularity theory that we develop for $C^{1,1}$ viscosity solutions with VMO Hessian, modifying the proof in [76]. See Section 2.1.1 below for definitions of rotated potential $\bar{u}(\bar{x})$ and rotated domain $\partial\tilde{u}(B_1)$.

2.1.1 $C^{2+\alpha}$ regularity on the rotated domain.

Theorem 2.1.1. *Let $u(x)$ be a convex viscosity solution to* (1.1.1) *on B_1 for $\psi \in C^\alpha$ for some $\alpha \in (0,1) \cup (1,2)$. If either $u \in C^1$ or ψ satisfies partial convexity* (1.1.12)*, then $\bar{u}(\bar{x}) \in C^{2+\alpha}$ on $\partial\tilde{u}(B_1)$.*

Proof. The proof is explained in the following two subsections.

Lewy-Yuan rotation $\bar{u}(\bar{x})$ solves a better equation

We first recall the Lewy-Yuan rotation, using the formulation developed in [20] for Lipschitz potentials. If $u(x)$ is convex but not C^1, then although its gradient graph $\{(x, Du(x))\}$ no longer makes sense, it still has a subdifferential $(x, \partial u(x))$, the slopes $y \in \partial u(x)$ of planes touching u from below at x. This need not be a graph, but it is isometric to one after a downward rotation of $z = (x, y)$ by $\pi/4$. Namely, there exists a $C^{1,1}$ potential $\bar{u}(\bar{x})$ such that

$$\begin{aligned} \bar{x} &= cx + sy, \\ D\bar{u}(\bar{x}) &= -sx + cy, \end{aligned} \tag{2.1.1}$$

whenever $y \in \partial u(x)$, where $c = \cos \pi/4, s = \sin \pi/4$. Whenever $D^2u(x)$ exists, it transforms according to

$$D^2\bar{u}(\bar{x}) = \left(\frac{-1+t}{1+t}\right) \circ D^2u(x). \tag{2.1.2}$$

Equivalently, the angles of the Hessian decrease by $\pi/4$:

$$\arctan \lambda_i(D^2\bar{u}(\bar{x})) = \arctan \lambda_i(D^2u(x)) - \pi/4. \tag{2.1.3}$$

Accordingly, the new potential satisfies the Hessian bounds

$$-I \leq D^2\bar{u}(\bar{x}) \leq I. \tag{2.1.4}$$

The actual potential is constructed using the Legendre transform:

$$f^*(\bar{x}) = \sup_x \left[x \cdot \bar{x} - f(x)\right].$$

Letting $\tilde{u}(x) = su(x) + \frac{c}{2}|x|^2$, the new potential is given by

$$\bar{u}(\bar{x}) = \frac{c}{2s}|\bar{x}|^2 - \frac{1}{s}(\tilde{u})^*(\bar{x}). \tag{2.1.5}$$

The maximizer x in the Legendre transform satisfies $\bar{x} \in \partial\tilde{u}(x)$. Because $\tilde{u}(x)$ is strictly convex, the domain $\bar{\Omega} = \partial\tilde{u}(\Omega)$ of the rotated potential $\bar{u}(\bar{x})$ is open and connected.

Now by (2.1.1) and (2.1.3), it follows that $\bar{u}(\bar{x})$ satisfies another mean curvature equation for almost every x in Ω:

$$\sum_{i=1}^{n} \arctan \lambda_i(D^2\bar{u}(\bar{x})) = \psi(x(\bar{x}), u(x(\bar{x})), s\bar{x} + cD\bar{u}(\bar{x})) - n\pi/4. \tag{2.1.6}$$

Here, $x(\bar{x}) = cx - sD\bar{u}(\bar{x})$. However, it does not follow from this that (2.1.6) holds for almost every $\bar{x}$ in $\bar{\Omega}$. This difficulty was circumvented in [20] for $\psi = const$ and in [8] for continuous $\psi(x)$, using the additional concavity of $\arctan \lambda$ granted by $\lambda \geq 0$. We now extend this to the present setting.

Proposition 2.1.1. *Let u be a convex viscosity solution of* (1.1.1) *on B_1 for ψ continuous. If either $u \in C^1$ or ψ satsifies partial convexity* (1.1.12)*, then $\bar{u}(\bar{x})$ is a viscosity solution of* (2.1.6) *on $\partial\tilde{u}(B_1)$.*

Proof. First, if convex $u \in C^1$, then $x \mapsto \psi(x, u(x), Du(x)) =: \phi(x)$ is continuous, and u is a viscosity solution of $\theta(D^2u) = \phi(x)$. Since $\phi(x)$ is trivially partial convex in the Du variable, we need only consider the second case.

We next consider the case where u is convex and ψ satisfies (1.1.12). It suffices to show that $\bar{u}(\bar{x})$ is a viscosity subsolution of (2.1.6) on $\partial\tilde{u}(B_{0.8})$, say, since the argument can be localized, and the supersolution case is identical to that in [20, Proposition 2.2]. First, we define continuous $\phi(x, p) = \psi(x, u(x), p)$, and note that $u(x)$ sub-solves $\theta(D^2u) = \phi(x, Du(x))$. Following the argument in [13, Theorems 5.8, 5.5] for the $F(D^2u) = 0$ case, to show that averages are also subsolutions, we define a convex combination $v(x) = \sum_{i=1}^{k} t_i u(x + \varepsilon y_i)$ on $B_{0.9}$ for $\varepsilon > 0$ small enough and some $y_i \in B_1$ and nonnegative $t_i : \sum_1^k t_i = 1$. Then, suppose quadratic $Q(x)$ touches $v(x)$ at the origin $x = 0$ from above near $x = 0$, say. We form an auxiliary function

$$w(x) := Q(x) - v(x) + \frac{\delta}{2}|x|^2 \tag{2.1.7}$$

for small fixed $\delta > 0$. Since $Q - v \geq 0$, it follows that w achieves strictly positive values on small spheres ∂B_r and vanishes at $x = 0$. Moreover, since v is convex, w is $C^{1,1}$ from above in B_r. Hence, letting $\Gamma_w(x)$ be the convex envelope below w, it follows from [13, Lemma 3.5] that $\Gamma_w \in C^{1,1}$ in B_r and $\det D^2\Gamma_w(x) = 0$ outside the contact set $\{w = \Gamma_w\}$, such that the ABP estimate holds:

$$0 < \int_{\{w=\Gamma_w\}\cap B_r} \det D^2\Gamma_w(x)dx. \tag{2.1.8}$$

Choosing z in the integration set such that $D^2u(z + \varepsilon y_i)$ exists for each i, the convexity of Γ_w shows that $D^2Q + \delta I \geq D^2v(z)$. We thus start with

$$\theta(D^2Q + \delta I) - \phi(0, DQ(0)) \geq \theta(D^2v(z)) - \phi(0, DQ(0)). \tag{2.1.9}$$

The concavity of arctangent operator $\theta(D^2v)$ for convex v yields

$$\begin{aligned} \theta(D^2Q + \delta I) - \phi(0, DQ(0)) &\geq \sum_1^k t_i\theta(D^2u(z + \varepsilon y_i)) - \phi(0, DQ(0)) \\ &= \sum_1^k t_i\phi(z + \varepsilon y_i, Du(z + \varepsilon y_i)) - \phi(0, DQ(0)). \end{aligned} \tag{2.1.10}$$

For the second term, we start with

$$DQ(0) = DQ(z) + O(r) = Dv(z) + Dw(z) + O(r). \tag{2.1.11}$$

Since $w(x)$ is semi-concave, it has continuous superdifferential by [58, Corollary 24.5.1], which means $Dw(z) - Dw(0)$ vanishes as $r \to 0$; note that $\partial w(0) = \{Dw(0)\} = \{0\}$. Since $Dw(0) = 0$, it follows $Dw(z) = o_r(1)$ as $r \to 0$. We apply partial convexity (1.1.12) to obtain

$$\begin{aligned}\theta(D^2Q + \delta I) - \phi(0, DQ(0)) &\geq \sum_1^k t_i \phi(z + \varepsilon y_i, Du(z + \varepsilon y_i)) - \phi(0, Dv(z)) - o_r(1) \\ &\geq \sum_1^k t_i \left(\phi(z + \varepsilon y_i, Du(z + \varepsilon y_i)) - \phi(0, Du(z + \varepsilon y_i)\right) - o_r(1) \\ &\geq -o_\varepsilon(1) - o_r(1),\end{aligned} \tag{2.1.12}$$

where, by the local Lipschitz bound for convex u, $o_\varepsilon(1)$ vanishes uniformly in $B_{0.9}$ as $\varepsilon \to 0$. Sending $r \to 0$ and $\delta \to 0$ shows that average v is a subsolution of the equation

$$\theta(D^2 v) \geq \phi(x, Dv(x)) - o_\varepsilon(1), \qquad x \in B_{0.9}(0). \tag{2.1.13}$$

More to the point, letting $\eta_\varepsilon = \varepsilon^{-n}\eta(x/\varepsilon)$ be a standard nonnegative mollifier with $\int \eta = 1$, $supp\, \eta \subset B_1$, we can uniformly approximate the solid average $u_\varepsilon(x) := \int_{B_1} u(x + \varepsilon y)\eta(y)dy$ by some such $v(x)$ as above. It follows from the closedness [13, Proposition 2.9] of subsolutions that $u_\varepsilon(x)$ is a smooth subsolution of (2.1.13) on $B_{0.9}(0)$.

Next, it was shown in [20, Proposition 2.3] the subdifferential $\partial \tilde{u}_\varepsilon$ converges to $\partial \tilde{u}$, and in particular its rotation $\bar{u}_\varepsilon$ is well defined on the open set $\bar{\Omega} := \partial \tilde{u}(B_{0.8}) \subset \partial \tilde{u}_\varepsilon(B_{0.9})$ for small ε. By the gradient and Hessian transformation rules (2.1.1) and (2.1.3), it follows that $\bar{u}_\varepsilon$ is a subsolution of the approximate rotated equation,

$$\theta(D^2 \bar{u}_\varepsilon) \geq \phi(c\bar{x} - sD\bar{u}_\varepsilon(\bar{x}), s\bar{x} + cD\bar{u}_\varepsilon(\bar{x})) - n\pi/4 - o_\varepsilon(1), \qquad \bar{x} \in \bar{\Omega}. \tag{2.1.14}$$

Note that $o_\varepsilon(1)$ is still uniformly small. We now send $\varepsilon \to 0$. As in [20], since the Legendre transform is order reversing, the rotation (2.1.5) is order preserving and respects uniform convergence, $\bar{u}_\varepsilon \to \bar{u}$ on $\bar{\Omega}$. The closedness [13, Proposition 2.9] of viscosity subsolutions under uniformly convergent solutions and locally uniformly convergent PDE operators shows that $\bar{u}$ is a subsolution of rotated equation (2.1.6). $\square$

Rotated regularity $\bar{u}(\bar{x}) \in C^{2+\alpha}$

We recall the notion of VMO functions.

Definition 1 (Vanishing mean oscillation)**.** Let $\Omega \subset \mathbb{R}^n$. A locally integrable function v is in $VMO(\Omega)$ with modulus $\omega(r, \Omega)$ if

$$\omega(r, \Omega) = \sup_{x_0 \in \Omega, 0 < r \leq R} \frac{1}{|B_r(x_0) \cap \Omega|} \int_{B_r(x_0) \cap \Omega} |v(x) - v_{x_0,r}| \to 0, \text{ as } r \to 0$$

where $v_{x_0,r}$ is the average of v over $B_r(x_0) \cap \Omega$.

Now $\bar{u}(\bar{x})$ is $C^{1,1}$, the mean curvature equation (2.1.6) is uniformly elliptic with a continuous right hand side. The improved estimates of such equations allow us to apply a rescaling procedure detailed in [8, Proposition 3.1], in order to locally reduce it to the constant-phase special Lagrangian equation (1.1.5), for which the monotonicity formula of minimal surface theory is available. The rescaling leads to a graphical minimal tangent cone of $(\bar{x}, D\bar{u}(\bar{x}))$; since such cones with the Hessian bounds (2.1.4) are flat by [77, Proposition 2.1] and [76, Proposition 2.2], the conclusion is that the graph $(\bar{x}, D\bar{u}(\bar{x}))$ is "almost" C^1, or that $D^2\bar{u}(\bar{x})$ has vanishing mean oscillation; see [76, Proposition 2.3].

Now that $D^2\bar{u}$ is in VMO, we prepare to invoke the regularity theory in Section 2.1.2. Let $G(D^2\bar{u})$ be a C^2 uniformly elliptic operator which extends the truncation of $F(D^2\bar{u})$ in (2.1.6) to $|D^2\bar{u}| \leq 1$. By the Hessian bounds (2.1.4), it follows that $\bar{u}$ is also a viscosity solution of

$$G(D^2\bar{u}) = g(\bar{x}),$$

where $g(\bar{x}) = \psi(x(\bar{x}), u(\bar{x}), s\bar{x} - cD\bar{u}(\bar{x})) - n\pi/4$ is a $C^{0,\min(\alpha,1)}$ right hand side, since $D\bar{u}(\bar{x})$ is Lipschitz. If $\alpha \in (0,1)$, we apply Theorem 2.1.2 and deduce that $\bar{u} \in C^{2,\alpha}$. If $\alpha \in (1,2)$, then instead $\bar{u} \in C^{2,\beta}$ for all $\beta < 1$, i.e. $\bar{u} \in C^{2,1-}$. Now that $D\bar{u}(\bar{x}) \in C^{1,1-}$, we infer that $g \in C^{\alpha}$, so the linear equation satisfied by the difference quotient $[\bar{u}(\bar{x}+h) - \bar{u}(\bar{x})]/|h|$ has a $C^{\alpha-1}$ right hand side, and the associated second order linear operator has $C^{0,1-}$ coefficients. It follows from Schauder estimates that $D\bar{u}(\bar{x}) \in C^{2,\alpha-1}$, hence $\bar{u} \in C^{3,\alpha-1} = C^{2+\alpha}$. This concludes the proof of Theorem 2.1.1.

□

2.1.2 $C^{2,\alpha}$ estimates for VMO solutions of fully non-linear elliptic equations

In this section, we develop regularity theory for $C^{1,1}$ viscosity solutions of fully non-linear uniformly elliptic equations of the form

$$F(D^2u) = f(x) \text{ in } B_1 \tag{2.1.15}$$

where $F \in C^2$, f is Hölder continuous, and the Hessian of u is in VMO. We assume $|D^2u|_{L^\infty(B_1)} \leq K$. Since the ellipticity constants of the uniformly elliptic Lagrangian mean curvature type equations depend on K, we denote the ellipticity constants of the above equation by K for the sake of simplifying notation.

Theorem 2.1.2. *Suppose that u is a $C^{1,1}$ viscosity solution of (2.1.15) on B_1. If $f \in C^\alpha(B_1)$ and $D^2u \in VMO(B_{3/4})$, then $u \in C^{2,\alpha}(B_{1/2})$ and satisfies the following estimate*

$$||u||_{C^{2,\alpha}(B_{1/2})} \leq C(n, K, \alpha, \|f\|_{C^\alpha(B_1)}). \tag{2.1.16}$$

The following Lemma is an extension of [10, Lemma 3.8] to a variable right hand side. Note that this Lemma holds good under weaker assumptions on equation (2.1.15) as compared to the above Theorem.

Lemma 2.1.1. *Suppose that $u \in C(B_r)$ is a viscosity solution of $G(D^2u) = g(x)$ in B_r, for some $r > 0$, where G is a uniformly elliptic operator and $g \in C^\alpha(B_r)$. If Q is a quadratic polynomial, then there exists a quadratic polynomial P such that*

$$G(D^2P) = g(0),$$
$$||u - P||_{L^\infty(B_r)} \leq C||u - Q||_{L^\infty(B_r)} + C||g||_{C^\alpha(B_r)}r^{2+\alpha}$$

where C is a universal constant.

Proof. We denote the ellipticity constants of G by $0 < \lambda \leq \Lambda$. Let $a = G(D^2Q)$, $b = ||u - Q||_{L^\infty(B_r)}$, and $R(x) = 2r^{-2}b|x|^2$. From [13, Proposition 2.13], we get

$$u - Q \in S(\lambda/n, \Lambda, G(D^2u) - G(D^2Q)) = S(g(x) - a).$$

We observe the following

$$[R - (u - Q)]|_{\partial B_r} = 2b - (u - Q)|_{\partial B_r} \geq 2b - (u - Q)(0) - 2||u - Q||_{L^\infty(B_r)} = [R - (u - Q)](0).$$

Since $u - Q$ is a viscosity subsolution and the minimum of $R - (u - Q)$ is achieved at an interior point of B_r, say x_0, we get $\mathcal{M}^+(D^2R, \lambda/n, \Lambda) \geq g(x_0) - a$. Similarly, using the fact that $u - Q$ is also a viscosity supersolution, we get $|g(x_0) - a| \leq Cr^{-2}b$. This shows

$$|g(0) - G(D^2Q)| \leq Cr^{-2}||u - Q||_{L^\infty(B_r)} + C||g||_{C^\alpha(B_r)}r^\alpha$$

By the ellipticity of G, there exists $s \in \mathbb{R}$ such that $G(D^2Q + sI_n) = g(0)$ and

$$|s| \leq Cr^{-2}||u - Q||_{L^\infty(B_r)} + ||g||_{C^\alpha(B_r)}r^\alpha.$$

Fixing this s we define $P(x) = Q(x) + \frac{1}{2}s|x|^2$. Therefore, we get

$$||u - P||_{L^\infty(B_r)} \leq ||u - Q||_{L^\infty(B_r)} + ||Q - P||_{L^\infty(B_r)}$$
$$= ||u - Q||_{L^\infty(B_r)} + \frac{1}{2}|s|r^2 \leq C||u - Q||_{L^\infty(B_r)} + C||g||_{C^\alpha(B_r)}r^{2+\alpha}.$$

□

Proposition 2.1.2. *Let u be a $C^{1,1}$ viscosity solution of (2.1.15) on $B_1 \subset \mathbb{R}^n$ where $D^2u \in VMO(B_{3/4})$ with VMO modulus $\omega(r)$. Then for any $\varepsilon_0 > 0$ there exists $\eta = \eta(n, \omega, K, ||f||_{C^\alpha(B_1)}, \varepsilon_0)$ and a quadratic polynomial P such that*

$$\sup_{B_1} |\frac{u(\eta x)}{\eta^2} - P(x)| \leq \varepsilon_0$$
$$F(D^2P) = f(0).$$

Proof. For $r > 0$ to be chosen later, let $w_r(x) = \frac{1}{r^2}u(rx)$ and

$$s_r = Tr\frac{1}{|B_1|}\int_{B_1} D^2w_r = Tr\frac{1}{|B_r|}\int_{B_r} D^2u.$$

Note that $|s_r| \leq n|D^2u|_{L^\infty(B_1)} = nK$. We solve the boundary value problem

$$\Delta h(x) = s_r \text{ in } B_1$$
$$h(x) = w_r \text{ on } \partial B_1.$$

Using the Alexandrov-Bakelman-Pucci maximum principle, we observe the following

$$||w_r - h||_{L^\infty}(B_1) \leq C(n)||\Delta w_r - \Delta h||_{L^\infty}(B_1)$$
$$\leq C(n, K)[\int_{B_1} |D^2u(rx) - (D^2u)_{0,r}|]^{1/n}$$
$$= C(n, K)[\frac{1}{|B_r|}\int_{B_r} |D^2u(x) - (D^2u)_{0,r}|]^{1/n} \leq C(n, K)\omega^{1/n}(r).$$

Note that h satisfies the following estimate

$$||D^3h||_{L^\infty}(B_{1/2}) \leq C(n) \sup_{\partial B_1} |w_r(x) - w_r(0) - \langle \nabla w_r(0), x\rangle - \frac{1}{2}s_r|x|^2| \leq C(n,K).$$

We define the quadratic part of h at the origin to be $\bar{P}$

$$|w_r(x) - \bar{P}(x)| \leq C(n,K)\omega^{1/n}(r) + C(n,K)|x|^3.$$

Since w_r satisfies $F(D^2u) = f(x)$, on using Lemma 2.1.1, we see that there exists a quadratic polynomial $\bar{\bar{P}}(x)$ such that $F(D^2\bar{\bar{P}}) = f(0)$ and

$$|w_r(x) - \bar{\bar{P}}(x)| \leq C(n,K)\omega^{1/n}(r) + C(n,K)|x|^3 + ||f||_{C^\alpha(B_1)}r^{2+\alpha}.$$

For $\rho > 0$ to be chosen later, let $x = \rho y$ and $P(y) = \frac{1}{\rho^2}\bar{\bar{P}}(\rho y)$. Given any $\varepsilon_0 > 0$, we choose r, ρ depending on $n, K, \omega, ||f||_{C^\alpha(B_1)}, \varepsilon_0$ such that for $\eta = \eta(n, K, \omega, ||f||_{C^\alpha(B_1)}, \varepsilon_0) = \rho r$ we get

$$|\frac{1}{\eta^2}u(\eta y) - P(y)| \leq \varepsilon_0.$$

□

Lemma 2.1.2. *There exists $\delta > 0$ depending on n, α, K such that if u is a $C^{1,1}$ viscosity solution of (2.1.15) in B_1 with $|D^2u| \leq K$ and $[f]_{C^\alpha(B_1)} \leq \delta$, then there exists a quadratic polynomial P such that*

$$\begin{aligned} ||u - P||_{L^\infty(B_r)} &\leq C_0 r^{2+\alpha} \ \ \forall r \leq 1, \\ |DP(0)| + ||D^2P|| &\leq C_0 \end{aligned} \tag{2.1.17}$$

for some constant $C_0 > 0$ depending on n, α, K.

Proof. The proof follows from the following claim.

Claim 1. *Given the assumptions of the above Lemma there exists $0 < \mu < 1, m$ depending on n, K, α such that if*

$$\begin{aligned} ||u - P||_{L^\infty(B_1)} &\leq \mu^{2+\alpha+m} \\ F(D^2P) &= f(0) \end{aligned} \tag{2.1.18}$$

almost everywhere in B_1, then we have a sequence $P_k(x) = a_k + b_k \cdot x + \frac{1}{2}x^t c_k \cdot x$ satisfying

$$F(D^2 P_k) = f(0) \tag{2.1.19}$$

$$||u - P_k||_{L^\infty(B_{\mu^k})} \leq \mu^{k(2+\alpha)+m} \tag{2.1.20}$$

$$|a_k - a_{k-1}| + \mu^{k-1}|b_k - b_{k-1}| + \mu^{2(k-1)}|c_k - c_{k-1}| \leq C(n,K)\mu^{(k-1)(2+\alpha)+m}. \tag{2.1.21}$$

Proof. Let $P_1 = P$. Then for $k = 1$, the claim holds trivially. For μ determined by (2.1.25), we show that if (2.1.20) holds for $k = i$, then it holds good for $k = i + 1$. We define

$$\begin{aligned} v_i(x) &= \frac{(u - P_i)(\mu^i x)}{\mu^{i(2+\alpha)+m}} \\ F_i(N) &= \frac{F(\mu^{i\alpha+m}N + c_i)}{\mu^{i\alpha+m}} \\ f_i(x) &= \frac{f(\mu^i x)}{\mu^{i\alpha+m}} \end{aligned} \tag{2.1.22}$$

where $P_i(x) = a_i + b_i \cdot x + \frac{1}{2}x^t \cdot c_i x$. So far we have

$$F_i(D^2 v_i(x)) = f_i(x) \text{ and } \|v_i\|_{L^\infty(B_1)} \leq C(n,K). \tag{2.1.23}$$

Denoting ω_n as the volume of a unit ball in n dimensions we observe that

$$||f_i - f_i(0)||_{L^n(B_1)} = \mu^{-i\alpha-m}\mu^{-i}||f - f(0)||_{L^n(B_{\mu^i})} \leq \mu^{-i\alpha-m}\mu^{-i}\left|B_{\mu^i}\right|^{1/n}\delta\mu^{i\alpha} = \mu^{-m}\omega_n^{1/n}\delta.$$

Let h be the solution of the boundary value problem

$$\begin{aligned} DF_i(0) \cdot D^2 h &= 0 \text{ in } B_{1/2} \\ h &= v_i \quad \text{ on } \partial B_{1/2}. \end{aligned}$$

Next, we observe that

$$\begin{aligned} F_i(D^2 h) &= F_i(0) + \sum_{a,b=1}^{n} (F_i)_{ab}(0)h_{ab} + O(|D^2 F||D^2 h|^2 \mu^{i\alpha}) \\ &= \frac{f(0)}{\mu^{i\alpha}} + O(|D^2 F||D^2 h|^2 \mu^{i\alpha}). \end{aligned} \tag{2.1.24}$$

Using Hölder estimates on v_i, with $\beta = \beta(n,K)$, from [13, Proposition 4.10] we see that

$$||v_i||_{C^\beta(\overline{B_{1/4}})} \leq C(n,K)[1 + \mu^{-m}\omega_n^{1/n}\delta].$$

By the global Hölder estimate on h, we have

$$||h||_{C^{2,\beta/2}(\overline{B_{1/2}})} \leq C(n,K)||v_i||_{C^\beta(\partial B_{1/2})} \leq C(n,K)(1+\mu^{-m}\delta).$$

By the Alexandrov-Bakelman-Pucci maximum principle we have for $\varepsilon > 0$ to be chosen later

$$\begin{aligned}
||v_i - h||_{L^\infty(B_{1/2-\varepsilon})} &\leq \sup_{\partial B_{1/2-\varepsilon}} |v_i - h| + C(n,K)||F_i(D^2 v_i) - F_i(D^2 h)||_{L^\infty(B_{1/2-\varepsilon})} \\
&\leq C(n,K)[(\varepsilon^\beta + \varepsilon^{\beta/2})(1+\mu^{-m}\delta) + |D^2F|\varepsilon^{-4}\mu^{i\alpha+m} + \mu^{-m}\delta(1/2-\varepsilon)^\alpha] \\
&\leq C(n,K)(\varepsilon^{\beta/2} + \mu^{i\alpha+m}\varepsilon^{-4} + \mu^{-m}\delta)
\end{aligned}$$

where the second inequality follows from (2.1.24). Since $h \in C^{2,\alpha}$, there exists a quadratic polynomial $\bar{P}$ given by

$$\bar{P}(x) = h(0) + Dh(0)\cdot x + \frac{1}{2}x^t D^2 h(0)\cdot x$$

such that

$$||h - \bar{P}||_{L^\infty(B_\mu)} \leq C(n,K)\mu^3.$$

So far we have

$$||v_i - \bar{P}||_{L^\infty(B_\mu)} \leq C(n,K)(\varepsilon^{\beta/2} + \mu^{i\alpha+m}\varepsilon^{-4} + \mu^{-m}\delta) + C(n,K)\mu^3.$$

From Lemma 2.1.1 we see that there exists a quadratic polynomial $\bar{\bar{P}}$ such that $F_i(D^2\bar{\bar{P}}) = f(0)$ and

$$\begin{aligned}
||v_i - \bar{\bar{P}}||_{L^\infty(B_\mu)} &\leq C(n,K)(\varepsilon^{\beta/2} + \mu^{i\alpha+m}\varepsilon^{-4} + \mu^{-m}\delta + \mu^3) + C(n,K)\|f_i\|_{C^\alpha(B_\mu)}\mu^{2+\alpha} \\
&\leq C(n,K)(\varepsilon^{\beta/2} + \mu^{i\alpha+m}\varepsilon^{-4} + \mu^{-m}\delta + \mu^3 + \mu^{-m}\delta\mu^{2+\alpha}).
\end{aligned}$$

Choosing $\varepsilon^{\beta/2}$ and μ^3, then m, then δ suitably, we get

$$||v_i - \bar{\bar{P}}||_{L^\infty(B_\mu)} \leq \mu^{2+\alpha}. \tag{2.1.25}$$

Rescaling the above back via (2.1.22) we see that

$$|u(x) - P_i(x) - \mu^{i(2+\alpha)+m}\bar{\bar{P}}(\mu^{-i}x)| \leq \mu^{(2+\alpha)(i+1)+m} \text{ for all } x \in B_{\mu^{i+1}}. \tag{2.1.26}$$

We define

$$P_{i+1}(x) = P_i(x) + \mu^{i(2+\alpha)+m}\bar{\bar{P}}(\mu^{-i}x) \tag{2.1.27}$$

and we have

$$c_{i+1} = c_i + \mu^{i\alpha+m}D^2\bar{\bar{P}}. \tag{2.1.28}$$

From (2.1.26) we see that $||u - P_{i+1}||_{L^\infty(B_{\mu^{i+1}})} \leq \mu^{(i+1)(2+\alpha)+m}$ which proves (2.1.20) for $k = i+1$, and from(2.1.28) we get $F(c_{i+1}) = 0$, thereby proving (2.1.19). Evaluating (2.1.27) and its first and second derivates at $x = 0$ we get $a_{i+1} - a_i = \mu^{i(2+\alpha)+m}\bar{\bar{P}}(0), b_{i+1} - b_i = \mu^{i(1+\alpha)+m}D\bar{\bar{P}}(0), c_{i+1} - c_i = \mu^{i\alpha+m}D^2\bar{\bar{P}}(0)$. Thus

$$\begin{aligned} &|a_{i+1} - a_i| + \mu^i|b_{i+1} - b_i| + \mu^{2i}|c_{i+1} - c_i| \\ &= \mu^{i(2+\alpha)+m}(|\bar{\bar{P}}(0)| + |D\bar{\bar{P}}(0)| + |D^2\bar{\bar{P}}(0)|) \\ &\leq \mu^{i(2+\alpha)+m}C(n,K) \end{aligned}$$

proving (2.1.21). This proves claim 1. □

We observe that assumption (2.1.18) in the above claim follows from a rescaling of the result in Proposition (2.1.2) applied to the rescaled function $u(\eta x)/\eta^2$. The remaining proof of the Lemma follows from a standard iterative argument.

□

Proof of Theorem 2.1.2. We may assume $u(0) = Du(0) = 0$. We will first prove that the estimate (2.1.16) holds at the origin. We show that there exists a quadratic polynomial P such that

$$\begin{aligned} ||u - P||_{L^\infty(B_r)} &\leq Cr^{2+\alpha} \quad ,\forall r \leq 1 \\ |DP(0)| + ||D^2P|| &\leq C \end{aligned} \tag{2.1.29}$$

where $C = C(n, K, \alpha, \|f\|_{C^\alpha(B_1)})$. The proof now follows directly from Lemma 2.1.2, if for all $x \in B_1$ we rescale $\tilde{u}(x) = \tau^{-2}u(\tau x)$, so that $\tilde{u}$ solves

$$F(D^2\tilde{u}(x)) = f(x/\tau) =: f_\tau(x).$$

The Hölder seminorm becomes

$$[f_\tau]_{C^\alpha(B_1)} \leq \tau^{-\alpha}[f]_{C^\alpha(B_1)} \leq \delta$$

if $\tau^\alpha = [f]_{C^\alpha(B_1)}/\delta$ is large enough. Therefore the equation $F(D^2\tilde{u}(x)) = f_\tau(x)$ satisfies all the conditions of Lemma 2.1.2 and hence the function $\tilde{u}$ satisfies the estimates in (2.1.17). In particular, there exists $\tilde{P}$ such that

$$||\tilde{u} - \tilde{P}||_{L^\infty(B_r)} \leq C(n, K, \alpha, \|f\|_{C^\alpha(B_1)})r^{2+\alpha}, \quad \forall r \leq 1$$
$$|D\tilde{P}(0)| + ||D^2\tilde{P}|| \leq C(n, K, \alpha, \|f\|_{C^\alpha(B_1)})$$

which proves (2.1.29) on substituting back to the original function. Similarly, we prove this holds at every point $x_0 \in B_{1/2}$, which in turn proves (2.1.16). □

2.2 Generalizing the constant rank theorem

In [9, 14, 69], there are versions of the following constant rank theorem. If $\Omega \subset \mathbb{R}^n$ and equation $G(D^2u, Du, u, x) = 0$ is an elliptic C^2 operator on $Sym^+(n) \times \mathbb{R}^n \times \mathbb{R} \times \Omega$ satisfying "inverse convexity"

$$(A, s, x) \mapsto G(A^{-1}, p, s, x) \text{ is locally convex for each fixed } p \in \mathbb{R}^n, \tag{2.2.1}$$

and u is a C^3 solution, then the rank of the Hessian $D^2u(x)$ is constant in Ω. We generalize this to $G(\,.\,, v) = 0$, where $v := x \cdot Du - u$ can be interpreted as the pullback of the Legendre transform $u^*(\bar{x}) = x \cdot \bar{x} - u(x)$ under the gradient mapping $x \mapsto Du$, or $v = u^* \circ Du$. This new dependence is nontrivial, since unlike for the x or u variables, G need not be convex in v.

Theorem 2.2.1. *Let $G(M, p, s, x, v)$ be C^2, elliptic*

$$G^{ab} := \frac{\partial G}{\partial M_{ab}}(M, p, s, x, v) > 0, \tag{2.2.2}$$

and inverse convex

$$(A, s, x) \mapsto G(A^{-1}, p, s, x, v) \text{ is locally convex for each fixed } (p, v) \in \mathbb{R}^n \times \mathbb{R}. \tag{2.2.3}$$

If u is a C^3 convex solution of $G(D^2u, Du, u, x, x \cdot Du - u) = 0$ in B_1, then $D^2u(x)$ has constant rank in B_1.

Remark 2.2.1. Defining an equation without extra u dependence,

$$H(D^2u, Du, x, x \cdot Du - u) = G(D^2u, Du, u, x, x \cdot Du - u) \tag{2.2.4}$$

we see that H is elliptic and inverse convex in the A, x arguments, with no restrictions on the p, v arguments. In fact, Theorem 2.2.1 holds if and only if the functionally dependent convex u argument in G is eliminated.

Proof. To generalize the proof from [69], it suffices to confirm there are nonnegative supersolutions of $G^{ab}\partial_{ab}$ which vanish according to the rank of $D^2u(x)$, since the Harnack inequality implies such vanishing occurs everywhere. The rest of the proof in [69] involves a $W^{4,n+}$ approximation maneuver that ensures the eigenvalues of $D^2u(x)$ are distinct away from a closed null set and adapting the Harnack inequality to an approximate setting. We refer therein and omit these details, which are the same in our generalized setting.

Let $P(x)$ be a polynomial such that $D^2P(x)$ has distinct positive eigenvalues $0 < \Lambda_1 < \cdots < \Lambda_n$ near $x = x_0$. We assume $D^2P(x_0)$ is diagonal with $\Lambda_i = P_{ii}$. Given integer $k \in [1, n]$, we put

$$Q^k = \Lambda_k + 2\Lambda_{k-1} + \cdots + k\Lambda_1 = \sum_{j=1}^{k}(k + 1 - j)\Lambda_j. \tag{2.2.5}$$

Our objective is to establish the following supersolution recursion estimate, pointwise at x_0:

$$G^{ab}Q^k_{ab} \leq C\sum_{\ell=1}^{k}(Q^\ell + |DQ^\ell| + |\partial_{\ell\ell}G|), \tag{2.2.6}$$

where C depends on the second order Taylor polynomial of G at $P(x_0)$ and the third order Taylor polynomial of P at x_0, and $\partial_\ell G$ is computed at $P(x)$. In the case that P approximates u solving $G = 0$ in $W^{4,n+}$, then the final term is small in L^{n+}. This will verify the approximate supersolution inequality in [69, equation (2.5)] for the case of $\partial_v G = 0$, accounting for their induction hypothesis [69, equation (2.4)] that Q^ℓ, DQ^ℓ are small for $\ell < k$.

A direct calculation is straightforward, as in [69], and the idea is that any v contributions directly absorb into the Q^k and DQ^k terms. Instead, we will show that the v dependence can

be replaced at x_0 by p dependence and affine u, x dependence, then invoke the calculation from [69]. This technique uses the pointwise nature of (2.2.6). The first two jets of $v = x \cdot DP - P$ at x_0 can be expressed in terms of DP and affine P:

$$\begin{aligned} \partial_i v &= x_0 \cdot DP_i = \partial_i (x_0 \cdot DP), \\ \partial_{ij} v &= x_0 \cdot DP_{ij} + P_{ij} = \partial_{ij}(x_0 \cdot DP) + \partial_{ij} P. \end{aligned} \tag{2.2.7}$$

Accordingly, we define a new elliptic operator by replacing v with a p-dependent function, and adding an affine-in-s, x term:

$$\tilde{G}(M, p, s, x) := G(M, p, s, x, x_0 \cdot p - P(x_0)) + \partial_v G|_{x_0}(s - \ell(x)), \tag{2.2.8}$$

where $\ell(x) = P(x_0) + (x - x_0) \cdot DP(x_0)$. For G and $\tilde{G}$ evaluated at $P(x)$, we have $\partial_j \tilde{G} = \partial_j G$ at x_0 since the second term is $O(x - x_0)^2$, while $v = x_0 \cdot DP(x) - P(x_0) + O(x - x_0)^2$, by (2.2.7). Moreover, $\partial_{ij} \tilde{G} = \partial_{ij} G$ at x_0, since we need only check the v_{ij} terms:

$$\partial_{ij}(\tilde{G} - G) \overset{x_0}{=} G_v \, \partial_{ij}(x_0 \cdot DP) + G_v \, \partial_{ij} \, P - G_v \partial_{ij} v, \tag{2.2.9}$$

which vanishes by (2.2.7). Meanwhile, since the second term in (2.2.8) is affine, and the final argument of G only uses p, it follows that $\tilde{G}$ satisfies inverse convexity (2.2.1). We deduce that (2.2.6) holds using $\tilde{G}$. We thus obtain

$$\begin{aligned} G^{ab} Q^k_{ab} \overset{x_0}{=} \tilde{G}^{ab} Q^k_{ab} &\overset{(2.2.6)}{\leq} C \sum_{\ell=1}^{k} (Q^\ell + |DQ^\ell| + |\partial_{\ell\ell} \tilde{G}|) \\ &\overset{x_0}{=} C \sum_{\ell=1}^{k} (Q^\ell + |DQ^\ell| + |\partial_{\ell\ell} G|). \end{aligned} \tag{2.2.10}$$

Since the second order Taylor polynomial of $\tilde{G}$ at $P(x_0)$ is controlled by that of G at $P(x_0)$, the constant C has the required uniformity. This concludes the proof.

□

2.3 Strong convexity without constant rank theorems

The following dual strong convexity criterion is sharp. Considering the example $U(x) = |x|^{1+\beta}/(1+\beta)$ with $\beta \in (0, 1)$, its Legendre transform $U^*(\bar{x}) = |\bar{x}|^{1+\beta^{-1}}/(1+\beta^{-1})$ is $C^{2,\alpha}$ for $\alpha = \beta^{-1} - 1$, or $\beta = 1/(1+\alpha)$, and fails strong convexity at the origin. We assume $U(x)$ is strictly convex so that $\partial U(B_1)$ is open.

Theorem 2.3.1. *Let $U(x)$ be strictly convex on B_1 with Legendre transform $U^*(\bar{x}) \in C^{2+\alpha}$ on $\partial U(B_1)$ for some $\alpha \in (0,1) \cup (1,2)$. If also $U(x) \in C^{1,\beta}$ for some $\beta > 1/(1+\alpha)$, then U^* is strongly convex, $D^2U^* > 0$.*

Proof. Suppose not, or that if λ_i^* are the eigenvalues of D^2U^*, then $\lambda_{min}^*(0) = 0$ for $0 = DU(0) = U(0)$, say. In general, we have the degenerate case

$$0 = \lambda_1^*(0) = \cdots = \lambda_k^*(0) < \lambda_{k+1}^*(0) \leq \cdots \leq \lambda_n^*(0). \tag{2.3.1}$$

We may assume $D^2U^*(0)$ is diagonal, with $U_{ii}^*(0) = \lambda_i^*(0)$. Let us replace $U^*(\bar{x})$ with

$$\hat{U}(\bar{x}) := U^*(\bar{x}) + \frac{1}{2}(2\bar{x}_2^2 + 3\bar{x}_3^2 + \cdots + n\bar{x}_n^2). \tag{2.3.2}$$

Since $D^2\hat{U}(\bar{x})$ is C^α, its spectrum $\hat{\lambda}_i(\bar{x})$ is nondegenerate for small $\bar{x}$:

$$0 \leq \hat{\lambda}_1(\bar{x}) < \hat{\lambda}_2(\bar{x}) < \cdots < \hat{\lambda}_n(\bar{x}). \tag{2.3.3}$$

Since $\hat{\lambda}_1(0) = 0$, it achieves a local minimum, so the fact it is C^α implies the decay

$$\hat{\lambda}_{min}(\bar{x}) \leq C|\bar{x}|^\alpha, \tag{2.3.4}$$

even if $\alpha \in (1,2)$. To find a lower bound to compare with, we use $U \in C^{1,\beta}$ to get

$$U(x) \leq \frac{C_1}{1+\beta}|x|^{1+\beta}. \tag{2.3.5}$$

After a rescaling $C_1^{-1}U(x)$, we can assume $C_1 = 1$. Let us now recall the Legendre transform is order reversing. For $\bar{x} \in \partial U(B_1)$,

$$\begin{aligned} U^*(\bar{x}) &= \sup_{x \in B_1}(x \cdot \bar{x} - U(x)) \geq \sup_{x \in B_1}(x \cdot \bar{x} - \frac{|x|^{1+\beta}}{1+\beta}) \\ &= \frac{|\bar{x}|^{1+\beta^{-1}}}{1+\beta^{-1}}, \end{aligned} \tag{2.3.6}$$

the last equality holding provided $\bar{x} = |x|^{\beta-1}x$ is solvable for $x \in \overline{B}_1$, i.e $|\bar{x}| \leq 1$. Since DU is C^β, it follows that $\partial U(B_r) \subset B_1$ for some r small enough, so for $\bar{x} \in \partial U(B_r)$,

$$\hat{U}(\bar{x}) \geq U^*(\bar{x}) \geq \frac{\beta}{1+\beta}|\bar{x}|^{1+1/\beta}. \tag{2.3.7}$$

Let us convert (2.3.4) into an analogous upper bound on $\hat{U}(x)$. If $f(0) = f'(0) = 0$, then integration by parts yields

$$\begin{aligned} f(t) &= \int_0^t f'(s)ds = \int_0^t \frac{d}{ds}(s-t)f'(s)ds \\ &= \int_0^t (t-s)f''(s)ds. \end{aligned} \tag{2.3.8}$$

Recall that $e_1 = (1, 0 \ldots, 0)$ is a maximum unit eigenvector of $D^2\hat{U}(0)$. Applying (2.3.8) to $f(t) = \hat{U}(te_1)$ gives

$$\hat{U}(te_1) = \int_0^t (t-s)\langle e_1, D^2\hat{U}(se_1)\cdot e_1\rangle \, ds. \tag{2.3.9}$$

From the proof of [3, Theorem 5.1], there is a minimum unit eigenvector $e_{min}(A)$ of symmetric matrix A which depends smoothly on A if $spec(A)$ is nondegenerate. Denoting $e_{min}(\bar{x})$ the one for $A = D^2\hat{U}(\bar{x})$ which satisfies $e_{min}(0) = e_1$, we obtain

$$\begin{aligned} \|e_{min}(\bar{x}) - e_1\| &\le C\|D^2\hat{U}(\bar{x}) - D^2\hat{U}(0)\| \\ &\le C|\bar{x}|^{\min(\alpha,1)}. \end{aligned} \tag{2.3.10}$$

The same will be true for the projection $e(\bar{x})$ of e_1 onto the $e_{min}(\bar{x})$ subspace. That is, denoting $\delta = e(\bar{x}) - e_1$, we have

$$\|\delta\| \le C|\bar{x}|^{\min(\alpha',1)}, \qquad \langle \delta, e\rangle = 0, \qquad |e|^2 = 1 - |\delta|^2, \qquad D^2\hat{U}(\bar{x})\cdot e(\bar{x}) = \hat{\lambda}_{min}(\bar{x})e(\bar{x}).$$

For $\bar{x} = s\, e_1$, we substitute these relations into (2.3.9) and obtain

$$\begin{aligned} \hat{U}(te_1) &= \int_0^t (t-s)(\hat{\lambda}_{min}(se_1)|e|^2 + \langle \delta, D^2\hat{U}(se_1)\cdot\delta\rangle)\, ds \\ &\le \int_0^t (t-s)\hat{\lambda}_{min}(se_1)ds + Ct^{2\min(\alpha,1)}\int_0^t (t-s)ds \\ &= \int_0^t (t-s)\hat{\lambda}_{min}(se_1)ds + Ct^{2+2\min(\alpha,1)}. \end{aligned} \tag{2.3.11}$$

Recalling (2.3.4), we thus obtain the upper bound

$$\hat{U}(te_1) \le Ct^{2+\alpha}. \tag{2.3.12}$$

Combining this with the lower bound (2.3.7), we obtain, for small t,

$$Ct^{1+1/\beta} \le Ct^{2+\alpha}. \tag{2.3.13}$$

Since $\beta > 1/(1+\alpha)$ or $2 + \alpha > 1 + 1/\beta$, it follows that we can choose t small and obtain a contradiction. □

2.4 Proof of Theorems 1.1.1 and 1.1.2

In both cases, it suffices to establish that $\bar{\lambda}_{max}(\bar{x}) < 1$ on $\partial\tilde{u}(B_1)$. By comparing with quadratics as in [20] and [8] and using the order preservation of rotation, we deduce that $\lambda_{max}(x) < \infty$ on B_1, i.e. $u(x) \in C^{1,1}$. We can then either invoke Evans-Krylov-Safonov theory for now-uniformly elliptic (1.1.1) to deduce $u \in C^{2+\alpha}$, or the approach in [8] using the transformation rule

$$D^2u(x) = \left(\frac{1+t}{1-t}\right) \circ D^2\bar{u} \circ \bar{x}(x),$$

where $\bar{x}(x) = cx + sDu(x)$, and the regularity for $D^2\bar{u}(\bar{x})$.

Defining auxiliary convex functions

$$U(x) = su(x) + \frac{c}{2}|x|^2, \qquad \bar{U}(\bar{x}) = -s\bar{u}(\bar{x}) + \frac{c}{2}|\bar{x}|^2, \tag{2.4.1}$$

where again $c = \cos\pi/4$ and $s = \sin\pi/4$, it follows from rotation formula (2.1.5) that $\bar{U}(\bar{x}) = U^*(\bar{x})$. Since $\bar{\lambda}_{max} < 1$ if and only if $D^2\bar{U} > 0$, we will show $\bar{U}(\bar{x})$ is strongly convex.

Proof of Theorem 1.1.2: If $\psi \in C^\alpha$ for some $\alpha \in (0,1) \cup (1,2)$ and $u(x) \in C^{1,\beta}$ for some $\beta > 1/(1+\alpha)$, then rotated regularity [Theorem 2.1.1] shows that $\bar{u}(\bar{x}) \in C^{2+\alpha}$. Moreover, $U(x)$ in (2.4.1) is strictly convex, so the fact that the graph $\{(x, Du(x))\}$ has large enough exponent $\beta > 1/(1+\alpha)$ allows us to apply strong convexity [Theorem 2.3.1] to deduce that $D^2U^* > 0$.

Proof of Theorem 1.1.1: We define a C^2 concave replacement of arctangent (1.1.1) by

$$A(\lambda) = \begin{cases} \arctan\lambda, & \lambda \geq 0, \\ \lambda, & \lambda < 0 \end{cases}. \tag{2.4.2}$$

To rewrite (1.1.1) in terms of auxiliary $U(x)$ in (2.4.1), this changes all u, Du arguments by a rescaling and a function of x. The partial convexity condition (1.1.12) is unchanged by rescaling its arguments or perturbing its arguments by x, so by changing ψ to ψ_1, we rewrite (1.1.1) as

$$\sum_{i=1}^{n} A(s^{-1}\Lambda_i - 1) = \psi_1(x, U, DU), \tag{2.4.3}$$

where Λ_i's are the eigenvalues of $D^2U(x)$. Here, ψ_1 also satisfies partial convexity (1.1.12) in its last argument. Applying the Legendre transform, this equation can be written as

$$\sum_{i=1}^{n} -A(\frac{1}{s\bar{\Lambda}_i} - 1) + \psi_1(D\bar{U}, \bar{x} \cdot D\bar{U} - \bar{U}, \bar{x}) = 0, \tag{2.4.4}$$

where $\bar{\Lambda}_i$'s are the eigenvalues of $D^2\bar{U}(\bar{x})$. More precisely, $\bar{U}(\bar{x})$ is a viscosity solution of this equation, which is the content of subsolution preservation [Proposition 2.1.1], and by rotated regularity [Theorem 2.1.1], we know $\bar{U}(\bar{x}) \in C^3$. We will apply the constant rank theorem 2.2.1 to this equation, where

$$G(M, p, x, v) := \sum_{i=1}^{n} -A(\frac{1}{s\lambda_i(M)} - 1) + \psi_1(p, v, x). \tag{2.4.5}$$

This operator is elliptic and inverse convex since $t \mapsto -A(t)$ is convex, see e.g. [3, Corollary 5.5]. To verify G is C^2 at nonnegative M with zero eigenvalues, we note that as $t \to 0$,

$$A(\frac{1}{st} - 1) = \arctan(\frac{1}{st} - 1) = \pi/2 - \frac{st}{1 - st} + O(t^3), \tag{2.4.6}$$

so A is a smooth function on $[0, \infty)$, and it is known, from e.g. [3, Theorem 5.1], the smoothness of G follows from the permutation symmetry in the λ_i's. Thus, by the constant rank theorem, either $D^2\bar{U}(\bar{x}) > 0$ everywhere, or $\bar{\Lambda}_{min} = 0$ everywhere, in particular. The latter case is impossible: since $U(x)$ is bounded in B_1, we can find a tall quadratic Q touching U from above at some point $p \in B_1$. In fact, the Legendre transform is order reversing: if $\bar{x} \in \partial Q(B_1) \cap \partial U(B_1)$, then

$$Q^*(\bar{x}) = \sup_{x \in B_1} (x \cdot \bar{x} - Q(x)) \leq \sup_{x \in B_1} (x \cdot \bar{x} - U(x)) = U^*(\bar{x}). \tag{2.4.7}$$

Since contact at p shows that $DQ(p) \in \partial Q(B_1) \cap \partial U(B_1)$, the strong convexity of Q and U ensures $\partial Q(B_1) \cap \partial U(B_1)$ contains a ball centered at $\bar{p} := DQ(p)$, so Q^* touches U^* at $\bar{p}$ from below near $\bar{p}$. Since $D^2Q < \infty$, we have $0 < D^2Q^* \leq D^2U^*(\bar{p})$, a contradiction.

2.5 Singular solution to self similar equation (1.1.8)

In dimension $n = 3$, Wang-Yuan [71] constructed a $C^{1,1/3}$ solution $u(x)$ on B_1 to special Lagrangian equation (1.1.5) which is smooth away from the origin $x = 0$, originally found

in Nadirashvili-Vlăduţ [53]. The idea was to find a polynomial approximate solution $P(\bar{x})$ in rotated $(\bar{x}, \bar{y})$ coordinates, invoke the Cauchy-Kovalevskaya Theorem to obtain a full solution, then apply a $U(n)$ transformation to return to (x, y) coordinates.

In this section, we show that this solution, essentially, solves the self-shrinker/expander equation (1.1.8). More precisely, in rotated coordinates, we will see that $P(\bar{x})$ above approximately solves the associated self-similar equation in rotated coordinates. Therefore, the self-similar corrections only occur at higher order.

Proposition 2.5.1. *There exists a $C^{1,1/3}(B_r) \cap C^{\infty}(B_r \setminus \{0\})$ singular viscosity solution to the self-similar Lagrangian mean curvature flow potential equation* (1.1.8) *for some small $r > 0$ and $B_r \subset \mathbb{R}^3$.*

Proof. Given $\Theta \in [0, \pi/2)$, by the construction in [71, pg 1165], there is an analytic solution $\bar{u}(\bar{x})$ to special Lagrangian equation (1.1.5) with phase $c = \pi/2 - \Theta$ near $\bar{x} = 0$. The canonical angles $\bar{\theta}_i = \arctan \lambda_i(D^2\bar{u})$ satisfy near $x = 0$, by [71, Property 3.3, pg 1167],

$$\begin{aligned} \bar{\theta}_1, \bar{\theta}_2 &= (\frac{\pi}{4} - \frac{\Theta}{2})(1 + O(\bar{x})), \\ \bar{\theta}_3 &= -\frac{1}{\tan(\frac{\pi}{4} - \frac{\Theta}{2})} Q(\bar{x})(1 + O(\bar{x})), \end{aligned} \tag{2.5.1}$$

where $Q(\bar{x}) = \frac{1}{2}\langle \bar{x}, D^2 Q \cdot \bar{x} \rangle$ is positive definite. Note that we may assume $\bar{u}(0) = D\bar{u}(0) = 0$, we can write $\bar{u}(\bar{x}) = \bar{u}_2(\bar{x}) + O(|\bar{x}|^3)$, where $\bar{u}_2$ is a homogeneous quadratic polynomial. Substituting this into the self-similar operator (1.1.8) yields

$$\begin{aligned} \theta(D^2\bar{u}) - c - b(\bar{x} \cdot D\bar{u} - 2\bar{u}) &= -b(\bar{x} \cdot D\bar{u} - 2\bar{u}) \\ &= O(|\bar{x}|^3). \end{aligned} \tag{2.5.2}$$

It follows from the Cauchy-Kovalevskaya Theorem that there exists an analytic solution $\bar{v}(\bar{x})$ to the self-similar equation with $c = \pi/2 - \Theta$ which agrees with $\bar{u}(\bar{x})$ up to and including $O(|\bar{x}|^4)$ terms. Since only these terms contribute to the $O(|\bar{x}|^2)$ angle asymptotics (2.5.1), this does not alter the Wang-Yuan construction, and we can continue their procedure unchanged to obtain a singular viscosity solution $v(x)$.

It only needs to be verified that the equation $v(x)$ solves is still a self-similar equation, since in [71], the special Lagrangian equation is considered. This follows from the representation $\bar{X}(\bar{x}, t) = \sqrt{1 - 2bt}\,(\bar{x}, D\bar{v}(\bar{x}))$ of the self-similar solution we obtain in rotated

coordinates. Under a $U(n)$ transformation of space, the solution still has this separated time dependence, $X(x,t) = U \cdot X(\bar{x},t) = \sqrt{1-2bt}\,(x, Dv(x))$, so the spatial potential $v(x)$ still solves a self-similar potential equation, with different constant $c = \Theta$. Note that the solution is only defined on B_r, since the Wang-Yuan solution on B_1 arises from one on B_r after a quadratic rescaling $r^2 u(\frac{x}{r})$, which is not possible in the self-similar setting. □

2.6 Singular solution to rotator equation (1.1.10)

In dimension $n = 1$, Hungerbühler-Smoczyk [36] constructed a global "Yin-Yang" curve rotating soliton solution of mean curvature flow, $H = (JX)^{\perp}$. Locally, its Lagrangian potential will satisfy the rotator potential equation (1.1.10). For all dimensions, we provide a local construction of this solution near the center of the spiral which manifests a singularity in the potential, for a poor choice of coordinates.

Proposition 2.6.1. *There exists a $C^{1,1/3}(B_r) \cap C^{\infty}(B_r \setminus \{0\})$ singular viscosity solution to the rotator Lagrangian mean curvature flow potential equation* (1.1.10) *if $a < 0$ and $c = n\pi/2$, for some small $r > 0$.*

Proof. We first construct an approximately radial analytic solution of the following "rotated" rotator equation with smaller phase constant c:

$$\sum_{i=1}^{n} \arctan \bar{\lambda}_i = n\pi/4 + \frac{a}{2}(|\bar{x}|^2 + |D\bar{u}(\bar{x})|^2). \tag{2.6.1}$$

A radial solution $\bar{u}(\bar{x}) = f(r^2/2)$, $r = |\bar{x}|$, will solve the equation

$$(n-1)\arctan(f'(s)) + \arctan(2sf''(s) + f'(s)) = n\pi/4 + as(1 + f'(s)^2). \tag{2.6.2}$$

We choose $f(0) = 0$, so setting $s = 0$ yields $f'(0) = 1$. Differentiating this equation at $s = 0$ yields

$$(n-1)\frac{f''(0)}{1+f'(0)^2} + \frac{3f''(0)}{1+f'(0)^2} = a(1 + f'(0)^2). \tag{2.6.3}$$

Thus $f''(0) = 4a/(n+2)$. If we denote rotator equation (2.6.1) by $F(\bar{x}, D\bar{u}, D^2\bar{u}) = 0$ and our approximate solution by

$$\bar{v}(\bar{x}) := \frac{1}{2}|\bar{x}|^2 + \frac{a}{2(n+2)}|\bar{x}|^4, \tag{2.6.4}$$

then this calculation shows that $F|_{\bar{v}} = O(s^2)$, hence $\bar{v}(\bar{x})$ solves (2.6.1) to $O(|\bar{x}|^4)$. It follows from the Cauchy-Kovalevskaya Theorem that there exists an analytic solution $\bar{u}(\bar{x})$ to (2.6.1) on a small ball $B_{\bar{r}}$ which agrees with $\bar{v}(\bar{x})$ until $O(|\bar{x}|^6)$ terms.

We now perform the inverse rotation operation $z = e^{i\pi/4}\bar{z}$. Let us first verify that a local potential exists. The auxiliary function $\bar{U}(\bar{x})$ in (2.4.1) has the expansion

$$\bar{U}(\bar{x}) = -\frac{as}{2(n+2)}|\bar{x}|^4 + O(|\bar{x}|^6). \tag{2.6.5}$$

Since $a < 0$, we see $\bar{U}$ is strictly convex and has an inverse Legendre transform $U(x)$ on some small ball B_r, with a fractional power leading term:

$$U(x) = C|x|^{4/3} + O(|x|^2). \tag{2.6.6}$$

Moreover, since $\bar{U}(\bar{x})$ is strongly convex away from the origin, it follows $U(x)$ is smooth away from the origin. Defining $u(x)$ in terms of $U(x)$, as in (2.4.1), we recall the (x, Du) and Hessian transformation rules (2.1.1) and (2.1.3). The Euclidean length $|x|^2 + |y|^2$ is preserved under rotations, and the angles $\theta_i = \arctan \lambda_i$ increase under rotations, so $u(x)$ solves the rotator equation

$$\sum_{i=1}^{n} \arctan \lambda_i = n\pi/2 + \frac{a}{2}(|x|^2 + |Du(x)|^2), \tag{2.6.7}$$

in the classical sense away from the origin. Moreover, by the expansion (2.6.6), it follows that it is also a viscosity solution at the origin. □

Let us note another way to see the "rotator" property of the potential is preserved. The rotator potential $\bar{u}(\bar{x})$ leads to a mean curvature flow by J, $\bar{X}(\bar{x}, t) = e^{atJ}(\bar{x}, D\bar{u}(\bar{x}))$, which commutes with J-rotations: $X(x,t) := e^{\frac{\pi}{4}J}\bar{X}(\bar{x}, t) = e^{atJ}(x, Du(x))$, where $X(x, 0) = e^{\frac{\pi}{4}J}\bar{X}(\bar{x}, 0)$ is the rotation of the initial submanifold. This means $X(x,t)$ is also a rotator solution of Lagrangian mean curvature flow, so the initial potential $u(x)$ of $X(x,t)$ solves the rotator potential equation, with a phase larger by $n\pi/4$.

Chapter 3

RIGIDITY FOR SIGMA-2 EQUATION

3.1 Shifted trace Jacobi inequality and superharmonicity under Legendre-Lewy transform

Taking the gradient of both sides of the quadratic Hessian equation

$$F\left(D^2u\right)=\sigma_2\left(\lambda\right)=\frac{1}{2}\left[\left(\triangle u\right)^2-\left|D^2u\right|^2\right]=1, \tag{3.1.1}$$

we have

$$\triangle_F Du=0, \tag{3.1.2}$$

where the linearized operator is given by

$$\triangle_F=\sum_{i,j=1}^{n}F_{ij}\partial_{ij}=\sum_{i,j=1}^{n}\partial_i\left(F_{ij}\partial_j\right), \tag{3.1.3}$$

with

$$\left(F_{ij}\right)=\triangle u\ I-D^2u=\sqrt{2+\left|D^2u\right|^2}\ I-D^2u>0. \tag{3.1.4}$$

Here without loss of generality, we assume $\triangle u>0$ in the remaining. Otherwise the smooth Hessian D^2u would be in the $\triangle u<0$ branch of the equation (3.1.1). Given the semiconvexity condition, the conclusion in Theorem 1.2.1 would be straightforward by Evans-Krylov-Safonov.

The gradient square $\left|\nabla_F v\right|^2$ for any smooth function v with respect to the inverse "metric" $\left(F_{ij}\right)$ is defined as

$$\left|\nabla_F v\right|^2=\sum_{i,j=1}^{n}F_{ij}\partial_i v\partial_j v.$$

3.1.1 Shifted trace Jacobi inequality

Proposition 3.1.1. *Let u be a smooth solution to $\sigma_2\left(\lambda\right)=1$ with $D^2u\geq -KI$. Set $b=\ln\left(\triangle u+J\right)$. Then for $J=8nK/3$ and $\varepsilon=1/3$, we have*

$$\triangle_F b\geq\varepsilon\left|\nabla_F b\right|^2. \tag{3.1.5}$$

Proof:

Step 1. Differentiation of the trace

We derive the following formulas for function $b = \ln(\sigma_1 + J) = \ln(\triangle u + J)$:

$$|\nabla_F b|^2 = \sum_i f_i \frac{(\triangle u_i)^2}{(\sigma_1 + J)^2} \tag{3.1.6}$$

and

$$\triangle_F b = \tag{3.1.7}$$
$$\frac{1}{(\sigma_1 + J)} \left\{ 6 \sum_{i>j>k} u_{ijk}^2 + \left[3 \sum_{i \neq j} u_{jji}^2 + \sum_i u_{iii}^2 - \sum_i \left(1 + \frac{f_i}{\sigma_1 + J} \right) (\triangle u_i)^2 \right] \right\}$$

at $x = p$, where, without loss of generality, $D^2 u(p)$ is assumed to be diagonalized and $f(\lambda) = \sigma_2(\lambda)$.

Noticing (3.1.4), it is straightforward to have the identity (3.1.6) and at p

$$\triangle_F b = \sum_{i=1}^n f_i \left[\frac{\partial_{ii} \triangle u}{(\sigma_1 + J)} - \frac{(\partial_i \triangle u)^2}{(\sigma_1 + J)^2} \right]. \tag{3.1.8}$$

Next we substitute the fourth order derivative terms $\partial_{ii} \triangle u = \sum_{k=1}^n \partial_{ii} u_{kk}$ in the above by lower order derivative terms. Differentiating equation (3.1.2) $\sum_{i,j=1}^n F_{ij} \partial_{ij} u_k = 0$ and using (3.1.4), we obtain at p

$$\begin{aligned} \sum_{i=1}^n f_i \partial_{ii} \triangle u = \triangle_F u_{kk} &= \sum_{i,j=1}^n F_{ij} \partial_{ij} u_{kk} = \sum_{i,j=1}^n -\partial_k F_{ij} \partial_{ij} u_k \\ &= \sum_{i,j=1}^n -(\triangle u_k \ \delta_{ij} - u_{kij}) u_{kij} = \sum_{i,j=1}^n \left[u_{ijk}^2 - (\triangle u_k)^2 \right]. \end{aligned}$$

Plugging the above identity in (3.1.8), we have at p

$$\triangle_F b = \frac{1}{(\sigma_1 + J)} \left[\sum_{i,j,k=1}^n u_{ijk}^2 - \sum_{k=1}^n (\triangle u_k)^2 - \sum_{i=1}^n \frac{f_i \ (\triangle u_i)^2}{\sigma_1 + J} \right]$$

Regrouping those terms $u_{\heartsuit \spadesuit \clubsuit}$, $u_{\spadesuit \spadesuit \heartsuit}$, $u_{\heartsuit \heartsuit \heartsuit}$, and $\triangle u_\heartsuit$ in the last two expressions, we obtain (3.1.7).

Subtracting (3.1.6)$*\varepsilon$ from (3.1.7), we have

$$\left(\triangle_F b - \varepsilon |\nabla_F b|^2 \right) (\sigma_1 + J) \geq 3 \sum_{i \neq j} u_{jji}^2 + \sum_i u_{iii}^2 - \sum_i \left(1 + \delta \frac{f_i}{\sigma_1 + J} \right) (\triangle u_i)^2$$

with $\delta = 1 + \varepsilon$.

Fix i and denote $t = (u_{11i}, \cdots, u_{nni})$ and e_i the i'th basis vector in $\mathbb{R}^n$, then the i'th term above can be written as

$$Q = 3\left|t\right|^2 - 2\left\langle e_i, t\right\rangle^2 - \left(1 + \delta\frac{f_i}{\sigma_1 + J}\right)\left\langle (1, \cdots, 1), t\right\rangle^2 \tag{3.1.9}$$

Step 2. Tangential projection

Equation (3.1.2) at p yields that t is tangential to the level set of the equation $\sigma_2(\lambda) = 1$, $\langle Df, t\rangle = 0$. Then by projecting e_i and $(1, \cdots, 1)$ to the tangential space,

$$E = (e_i)_T = e_i - \frac{f_i}{\left|Df\right|^2}Df \quad \text{and} \quad L = (1, \cdots, 1)_T = (1, \cdots, 1) - \frac{(n-1)\sigma_1}{\left|Df\right|^2}Df.$$

The coefficients of the two negative terms in the quadratic form (3.1.9)

$$Q = 3\left|t\right|^2 - 2\left\langle E, t\right\rangle^2 - \left(1 + \delta\frac{f_i}{\sigma_1 + J}\right)\left\langle L, t\right\rangle^2$$

decrease, as simple symmetric computation shows

$$\begin{gathered}\left|E\right|^2 = 1 - \frac{f_i^2}{\left|Df\right|^2} < 1, \ \left|L\right|^2 = 1 - \frac{2(n-1)}{\left|Df\right|^2} < 1, \\ \text{and } E \cdot L = 1 - \frac{(n-1)\sigma_1 f_i}{\left|Df\right|^2}.\end{gathered} \tag{3.1.10}$$

Step 3. Two anisotropic and non-orthogonal directions

We proceed to show that the quadratic form Q is positive definite. When t is perpendicular to both E and L, $Q = 3\left|t\right|^2 \geq 0$. So we only need to deal with the anisotropic case, when t is along $\{E, L\}$-space. The corresponding matrix of the quadratic form Q is

$$Q = 3I - 2E \otimes E - \eta L \otimes L$$

with $\eta = 1 + \delta\frac{f_i}{\sigma_1 + J} = 1 + (1 + \varepsilon)\frac{f_i}{\sigma_1 + J}$. The *real* ξ-eigenvector equation for (symmetric) Q under *non-orthogonal* basis $\{E, L\}$ is

$$\begin{pmatrix} 3 - 2\left|E\right|^2 & -2E \cdot L \\ -\eta L \cdot E & 3 - \eta\left|L\right|^2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \xi\begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

where corresponding real eigenvalues

$$\xi = \frac{1}{2}\left(tr \pm \sqrt{tr^2 - 4\det}\right) \quad \text{with}$$

$$tr = 6 - 2\left|E\right|^2 - \eta\left|L\right|^2 \quad \text{and} \quad \det = 9 - 6\left|E\right|^2 - 3\eta\left|L\right|^2 + 2\eta\left[\left|E\right|^2\left|L\right|^2 - (E\cdot L)^2\right].$$

Now by (3.1.10)

$$\begin{aligned} tr &= 6 - 2\left(1 - \frac{f_i^2}{|Df|^2}\right) - \left(1 + \delta\frac{f_i}{\sigma_1 + J}\right)\left(1 - \frac{2(n-1)}{|Df|^2}\right) \\ &> 3 - \delta\frac{f_i}{\sigma_1 + J} = \frac{(3-\delta)\sigma_1 + \delta\lambda_i + 3J}{\sigma_1 + J} > 0 \end{aligned} \tag{3.1.11}$$

for any $\delta \le 1.5$ and $J \ge 0$, given $\sigma_1 = \sqrt{|\lambda|^2 + 2} > |\lambda_i|$ in the nontrivial remaining case.

Next again by (3.1.10)

$$\begin{aligned} \det &= 6\frac{f_i^2}{|Df|^2} - 3\delta\frac{f_i}{\sigma_1 + J} + 3\left(1 + \delta\frac{f_i}{\sigma_1 + J}\right)\underbrace{\frac{2(n-1)}{|Df|^2}} \\ &+2\left(1 + \delta\frac{f_i}{\sigma_1 + J}\right)\left[\frac{2(n-1)\sigma_1 f_i}{|Df|^2} - \frac{nf_i^2}{|Df|^2} - \underbrace{\frac{2(n-1)}{|Df|^2}}\right] \\ &> -3\delta\frac{f_i}{\sigma_1 + J} + 4\left(1 + \delta\frac{f_i}{\sigma_1 + J}\right)\frac{(n-1)\sigma_1 f_i}{|Df|^2} + \left[6 - 2n\left(1 + \delta\frac{f_i}{\sigma_1 + J}\right)\right]\frac{f_i^2}{|Df|^2}. \end{aligned}$$

Then for $\delta = 1 + \varepsilon = 4/3$, we have

$$\begin{aligned} &\det \cdot \frac{(\sigma_1 + J)|Df|^2}{f_i} \ge \\ &-3\delta\underbrace{\left[(n-1)\sigma_1^2 - 2\right]}_{|Df|^2} + \left\{\begin{array}{c} 4(\sigma_1 + J + \delta f_i)(n-1)\sigma_1 + \\ \left[(6-2n)(\sigma_1 + J) - 2n\delta f_i\right] f_i \end{array}\right\} \\ &= \left\{\begin{array}{c} 6\delta + 4(n-1)J\sigma_1 + 2(3-n)J\underbrace{f_i}_{\sigma_1 - \lambda_i} \\ + (n-1)(4-3\delta)\sigma_1^2 + \left[2n(2\delta - 1) + 6 - 4\delta\right]\sigma_1 f_i - 2n\delta f_i\underbrace{f_i}_{\sigma_1 - \lambda_i} \end{array}\right\} \\ &\overset{\delta = 4/3}{=} 8 + 2(n+1)J\sigma_1 + 2(n-3)J\lambda_i + \frac{2(n+1)}{3}\sigma_1 f_i + \frac{8}{3}n\lambda_i f_i \qquad (3.1.12) \\ &> 2(n+1)J\sigma_1 + 2(n-3)J\lambda_i + \frac{8}{3}n\lambda_i f_i. \qquad (3.1.13) \end{aligned}$$

Case $\lambda_i \ge 0$: (3.1.13) is positive by the ellipticity $f_i > 0$ from (3.1.4).

Case $0 > \lambda_i \ge -K$:

$$(3.1.13) = 2nJ\underbrace{(\sigma_1 + \lambda_i)}_{\sqrt{2+|\lambda|^2} + \lambda_i > 0} - 6J\lambda_i + 2J\sigma_1 + \frac{8}{3}n\lambda_i\underbrace{f_i}_{\sigma_1 - \lambda_i < 2\sigma_1} > 0$$

if $J = 8nK/3$.

Therefore, the quadratic form Q is positive definite, and we have derived the shifted Jacobi inequality (3.1.5) in the semiconvex case.

Remark. *In three dimensions, the Jacobi inequality (3.1.5) still holds for any* $J \geq 0$ *and* $\varepsilon = 1/3$ *without the semiconvexity assumption* $D^2u \geq -KI$. Actually, we only need to show that, in Step 3, (3.1.12) with $\delta = 1 + \varepsilon = 4/3 < 1.5$ is also positive for negative λ_i. We would have the desired lower bound for (3.1.12)

$$\det \cdot \frac{(\sigma_1 + J)\left|Df\right|^2}{f_i} > 8f_i\left(\frac{\sigma_1}{3} + \lambda_i\right) > 0,$$

if we know $\lambda_i > -\sigma_1/3$. Without loss of generality, we assume $\lambda_1 \geq \lambda_2 \geq \lambda_3$. Because $\lambda_2 + \lambda_3 = f_1 > 0$, only the smallest eigenvalue λ_3 could be negative. In such a negative case $\lambda_3 = \frac{1-\lambda_1\lambda_2}{\lambda_1+\lambda_2}$ with $\lambda_1\lambda_2 > 1$, we do have $\lambda_i > -\sigma_1/3$ or $\frac{\sigma_1}{-\lambda_3} > 3$. Because

$$\begin{aligned}\frac{\sigma_1}{-\lambda_3} &= -1 + \frac{(\lambda_1 + \lambda_2)^2}{\lambda_1\lambda_2 - 1} \\ &\geq -1 + \frac{4\lambda_1\lambda_2}{\lambda_1\lambda_2 - 1} > 3.\end{aligned}$$

Note that, in three dimensions, the Jacobi inequality for the log-convex $b = \ln \triangle u = \ln\sqrt{2 + |\lambda|^2}$ (with $\varepsilon = 1/100$) was derived by Qiu [57, Lemma 3] for solutions to (3.1.1) along with variable right hand side; and the Jacobi inequality with $\varepsilon = 1/3$ for the log-max $b = \ln \lambda_{\max}$ (with $\varepsilon = 1/3$) was derived for solutions to (3.1.1) in Warren-Yuan [74, Lemma 2.2].

In general dimensions, a Jacobi inequality for sufficiently large $b = \ln u_{11}$, at points where $u_{11} = \lambda_{\max}$, was obtained for solutions having $\sigma_3\left(D^2u\right)$ lower bound to (3.1.1) along with variable right hand side by Guan-Qiu [30, p.1650]; and another Jacobi inequality for sufficiently large $b = \ln \lambda_{\max}$ was derived for semiconvex solutions to (3.1.1) in Shankar-Yuan [61, Proposition 2.1], as mentioned in section 1.2.

3.1.2 Superharmonicity under Legendre-Lewy transform

Set $\tilde{u}(x) = u(x) + \bar{K}\left|x\right|^2/2$ for our K-semiconvex entire solution u and say, $\bar{K} = J/n > K + 1$, where $J = 8nK/3$ is from Proposition 3.1.1. The $\bar{K}$-convexity of $\tilde{u}$ ensures that the

smallest canonical angle of the "Lewy-sheared" "gradient" graph is larger than $\pi/4$. This means we can make a well defined Legendre reflection about the origin,

$$(x, D\tilde{u}(x)) = (Dw(y), y) \in \mathbb{R}^n \times \mathbb{R}^n \tag{3.1.14}$$

where $w(y)$ is the Legendre transform of $u + \frac{\bar{K}}{2}|x|^2$; see [46]. Note that $y(x) = Du(x) + Kx$ is a diffeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$ and

$$0 < D^2 w = \left(D^2 u + \bar{K} I\right)^{-1} < I.$$

More precisely, by Chang-Yuan [16, p.663] or Shankar-Yuan [61, (2.11)], the eigenvalues $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ of $D^2 w$ satisfy

$$0 < \mu_1 \leq c\,(n) < 1 \ \text{ and } \ 0 < c\,(n, K) \leq \mu_i < 1 \ \text{ for } i \geq 2. \tag{3.1.15}$$

As shown in Chang-Yuan [16, p.663] or Shankar-Yuan [61, proof of Proposition 2.4], the equation solved by the vertical coordinate Lagrangian potential $w(y)$,

$$G(D^2 w) = -F(D^2 u) = -\sigma_2(-\bar{K} I + (D^2 w)^{-1}) = -1,$$

is conformally, uniformly elliptic for K-convex solutions u, in the sense that for $H_{ij} := \sigma_n(\mu(D^2 w))G_{ij}$, the linearized operator $H_{ij}\partial_{ij}$ of equation

$$\begin{aligned} 0 = H\left(D^2 w\right) &= \sigma_n\left(D^2 w\right)\left[G\left(D^2 w\right) + 1\right] \\ &= -\sigma_{n-2}\left(\mu\right) + \underbrace{(n-1)\,\bar{K}}_{A_1}\,\sigma_{n-1}\left(\mu\right) - \underbrace{\left[\frac{n\,(n-1)}{2}\bar{K}^2 - 1\right]}_{A_2}\sigma_n\left(\mu\right) \end{aligned} \tag{3.1.16}$$

is uniformly elliptic:

$$c(n, K)I \leq (H_{ij}) = \sigma_n(\mu)(G_{ij}) \leq C(n, K)I.$$

Proposition 3.1.2. *Let $u\,(x)$ be a smooth solution to $\sigma_2\,(\lambda) = 1$ with $D^2 u \geq -KI$. Set*

$$a\,(y) = \left(\frac{1}{\mu_1} + \cdots + \frac{1}{\mu_n}\right)^{-1/3} = \left[\frac{\sigma_n}{\sigma_{n-1}}\,(\mu)\right]^{1/3}$$

with $\mu_i s$ being the eigenvalues of the Hessian $D^2 w\,(y)$ of the Legendre-Lewy transform of $u\,(x) + \bar{K}\,|x|^2\,/2$. Then we have

$$\triangle_H a \leq 0.$$

Proof:

The trace Jacobi inequality (3.1.5) in Proposition 3.1.1 with $J = n\bar{K}$ is equivalent to

$$\triangle_F \left(\triangle u + J\right)^{-1/3} = \triangle_F e^{-b/3} \leq 0.$$

Noticing that $\triangle u + n\bar{K} = \frac{1}{\mu_1} + \cdots + \frac{1}{\mu_n}$, and applying the transformation rule in Shankar-Yuan [61, Proposition 2.3], we immediately obtain the desired superharmonicity

$$\triangle_H a = \sigma_n \left(\mu\right) \triangle_G a \leq 0.$$

3.2 Hölder Hessian estimate for saddle equation and rigidity

The Hessian bound $0 < D^2 w(y) \leq I$ ensures that establishing a local $C^{2,\alpha}$ estimate for such solutions to (3.1.16) will prove, by scaling, that $w(y)$ is a quadratic polynomial. By the iteration arguments developed in Caffarelli-Yuan [15] for such smooth PDEs $H(D^2 w) = 0$ with solutions satisfying Hessian bounds, proving $C^{2,\alpha}$ regularity at a point, say the origin, reduces to showing that $w(y)$ is close to a uniform quadratic polynomial, namely:

Proposition 3.2.1. *Let $u\left(x\right)$ be a smooth solution to $\sigma_2\left(\lambda\right) = 1$ with $D^2 u \geq -KI$ in $\mathbb{R}^n$. Let $w(y)$ be its Legendre-Lewy transform defined in (3.1.14) solving (3.1.16) in $\mathbb{R}^n$ with $0 < D^2 w \leq I$. Given any $\theta > 0$, there exists small $\eta = \eta(n, K, \theta) > 0$ and a quadratic polynomial $P(y)$ whose coefficients only depend on n, K, θ such that*

$$\left| \frac{1}{\eta^2} w(\eta z) - P(z) \right| \leq \theta$$

is valid for $|z| \leq 1$.

In the case that the level set $\{H\left(D^2 w\right) = 0\}$ were convex (in fact saddle from Chang-Yuan [16, p.661]), the alternative way in Caffarelli-Yuan [15] other than Evans-Krylov-Safonov is the following. The Laplacian $\Delta w(y)$ is a sub or supersolution of the linearized operator $\Delta_H = H_{ij}\, \partial^2 / \partial y_i \partial y_j$ of $H(D^2 w)$. The weak Harnack inequality shows that $\Delta w(y)$ concentrates in measure at a level c on a small ball $B = B_r(0)$. Solving the equation $\Delta v = c$ on B with $v = w$ on ∂B furnishes the desired smooth approximation, which is uniform by the ABP estimate. The Laplacian can be replaced with any elliptic slice of the Hessian,

such that the elliptic slice is a supersolution of Δ_H, and the corresponding elliptic equation both has $C^{2,\alpha}$ interior regular solutions and allows for the ABP estimate.

However, it is not clear if the saddle level set $\{H\left(D^2w\right) = 0\}$ of (3.1.16) is any of trace-convex [15], max-min [10], or twisted [25, 23], so it is not clear if there are good PDEs which super-solve Δ_H. Now that the remarkable superharmonic quantity $\sigma_n(\mu)/\sigma_{n-1}(\mu)$ in Proposition 3.1.2 is available, the core method in [15, pg 687-690] becomes more realistic.

There is still one more hurdle to overcome. The superharmonic, "one-step" Hessian quotient $a^3 = \sigma_n(\mu)/\sigma_{n-1}(\mu)$ is well known to be concave, but not uniformly elliptic, because $\sigma_n(\mu)$ could be arbitrarily close to zero. This prevents applying Evans-Krylov-Safonov theory. We resolve this by substituting the concentration of a into the "conformal" equation (3.1.16). This implies concentration of a better quantity. Observe that equation (3.1.16) can be written as

$$q(\mu) := \frac{\sigma_{n-1}(\mu)}{\sigma_{n-2}(\mu)} = \left[A_1 - A_2\frac{\sigma_n(\mu)}{\sigma_{n-1}(\mu)}\right]^{-1}. \tag{3.2.1}$$

Thus, the concentration of the higher quotient $a^3 = \sigma_n/\sigma_{n-1}$ implies concentration of the lower quotient $\sigma_{n-1}/\sigma_{n-2}$, which is also a concave operator [48, Theorem 15.18]. The almost-convex case, $D^2u \geq (-K+\delta)I$ for $K^{-2} = n(n-1)/2$ and any $\delta > 0$ considered in Chang-Yuan [16], corresponds to $A_2 = 0$. There, it was shown that (3.1.16) is uniformly elliptic for arbitrarily large K, in particular, the lower quotient $\sigma_{n-1}/\sigma_{n-2} = A_1^{-1}$ for $K^{-2} = n(n-1)/2$. For arbitrary K, using the bound for μ in (3.1.15) and the result in [48, Theorem 15.18], we deduce the uniform ellipticity of $q(\mu)$

$$\partial_{\mu_i} q \in \frac{\sigma_{n-1}(\mu)}{\sigma_{n-2}^3(\mu)}\sigma_{n-2,i}^2(\mu)\left[c(n), 1\right] \subset \left[c(n,K), C(n,K)\right]. \tag{3.2.2}$$

Proof of Proposition 3.2.1

Given any small $\xi, \delta > 0$, we denote $a_k = \min_{B_{1/2^k}} a$ and define a "bad set" $E_k = \{y \in B_{1/2^k} : a > a_k + \xi\}$. Using Krylov-Safonov's weak Harnack inequality [29, Theorem 9.22] for supersolution $a(y)$ from Proposition 3.2.1, the iteration argument in Caffarelli-Yuan [15, pg 687-690] shows that there exists $k_0(n, K, \xi, \delta)$ large enough such that $|E_\ell| < \delta|B_{1/2^\ell}|$ for some $\ell \in [1, k_0]$. By applying the quadratic scaling $2^{2\ell}w(2^{-\ell}y)$ and modifying the small

constant η in Proposition 3.2.1, we may assume $\ell = 0$. This is without loss of generality and is possible, since we already assumed $w(0) = 0$ and $Dw(0) = 0$.

Using uniform ellipticity (3.2.2), let us extend $\sigma_{n-1}/\sigma_{n-2}$ to a uniformly elliptic concave operator $q(\mu)$ outside the eigenvalue rectangle (3.1.15), $\mu \in [0, c(n)] \times [c(n,K), C(n,K)]^{n-1}$. The notation q in (3.2.1) was abused for the sake of notation simplicity. Let $v(y) \in C^\infty(B_1)$ solve the concave equation $q[v] = q\left(\mu(D^2 v)\right) = a_\ell$ in B_1 with $v = w$ on ∂B_1. Then from the quotient representation (3.2.1) for the equation that w solves, the ABP estimate [29, Theorem 9.1] yields on B_1,

$$\begin{aligned} |w - v| &\le C(n,K)\|q[w] - q[v]\|_{L^n(B_1)} \\ &\le C(n,K)\delta + C(n,K)\left\| \frac{a^3 - a_\ell^3}{(A_1 - A_2\, a^3)(A_1 - A_2\, a_\ell^3)} \right\|_{L^n(E_\ell^c)} \\ &\le C(n,K)(\delta + \xi), \end{aligned}$$

where, in the last inequality, we used the boundedness of $(A_1 - A_2\, a)^{-1}$ via (3.2.1) and (3.1.15). By Evans-Krylov-Safonov theory applied to the smooth equation $q[v] = a_\ell$, $v(y)$ has uniform interior estimates, so v can be replaced by its quadratic part at the origin up to a uniform $O(|x|^3)$ term, which is $O(r^3)$ on $B_r(0)$. The conclusion of this proposition follows for $\eta = r$ in a standard way by successively choosing the small constants ξ, δ, r depending on the small parameter θ and n, K.

Proof of Theorem 1.2.1

As indicated in the beginning of Section 3.1, we only need to handle the positive branch $\triangle u > 0$ of the quadratic equation $\sigma_2\left(D^2 u\right) = 1$. Because the only other possibility is that $D^2 u$ is on the negative branch $\triangle u < 0$ of the still elliptic and concave equation $\sigma_2\left(D^2 u\right) = 1$. Then the semiconvex solutions must have bounded Hessian, and consequently, the conclusion in Theorem 1.2.1 is straightforward by Evans-Krylov-Safonov.

Now armed with Proposition 3.2.1, the initial closeness of w to a "harmonic" quadratic on the unit ball, and repeating the proof of Proposition 2 in Caffarelli-Yuan [15] with the equation there replaced by our smooth uniformly elliptic equation (3.2.1), we see that the closeness to "harmonic" quadratics accelerates. As in [15, p.692], we obtain that $D^2 w$ is

Hölder at the origin. Similarly, one proves that D^2w is Hölder in the half ball

$$\left[D^2w\right]_{C^\alpha\left(B_{1/2}\right)} \leq C\left(n,K\right),$$

where $\alpha = \alpha\left(n,K\right) > 0$.

By quadratic scaling $R^2w\left(y/R\right)$, we get

$$\left[D^2w\right]_{C^\alpha(B_R)} \leq \frac{C\left(n,K\right)}{R^\alpha} \longrightarrow 0, \quad \text{as } R \to \infty.$$

We conclude that D^2w is a constant matrix, and in turn, so is D^2u.

Chapter 4

QUASILINEAR INVERSE PROBLEM

4.1 Preliminaries

4.1.1 The linearized problem

For arbitrary solutions to the quasilinear equation (1.3.2), the linearized Dirichlet problem centered at that solution may not be solvable. Under a small gradient assumption or the decay assumption (1.3.8), we will show the uniqueness of weak solutions to the linearized equation, and standard techniques will again apply after invoking higher regularity. Note that the non-homogeneous case $a = a(x, u, \nabla u)$ applies here without change as well.

We denote by $u = u[f]$ the solution of the BVP (1.3.2). If $f, h \in C^\infty(\partial\Omega)$, then formally, we let

$$v(x) := \frac{d}{dt}\Big|_{t=0} u[f + th](x) = \lim_{t\to 0} \frac{u[f + th](x) - u[f](x)}{t} \tag{4.1.1}$$

solve the linearized Dirichlet problem

$$\begin{aligned} \partial_i(a_s v u_i + a_{p_j} v_j u_i + a v_i) = 0, \\ v|_{\partial\Omega} = h. \end{aligned} \tag{4.1.2}$$

Here, $u_i = \partial u/\partial x^i$ and $v_j = \partial v/\partial x^j$. The linearized DN map:

$$\Gamma_a[u](h) := \frac{d}{dt}\Big|_{t=0} \Gamma_a(f + th) = (a_s v \frac{\partial u}{\partial \nu} + a_{p_j} v_j \frac{\partial u}{\partial \nu} + a\frac{\partial v}{\partial \nu}) dS. \tag{4.1.3}$$

For brevity, we say that *the linearization of* (1.3.2) *exists at* u if the limit (4.1.1) is well defined in $C^1(\overline{\Omega})$ and the Dirichlet problem (4.1.2) for v is uniquely solvable in $W^{1,2}(\Omega)$. If $\Gamma_a = \Gamma_{\tilde{a}}$ and the linearization exists at u and $\tilde{u}$, it then follows from $C^1(\overline{\Omega})$ convergence that $\Gamma_a[u] = \Gamma_{\tilde{a}}[\tilde{u}]$.

To show that the linearization of (1.3.2) exists, we first specialize to solutions u with small gradients, under no additional assumptions on $a(s, p)$, proved below.

Proposition 4.1.1. *Given $s \in \mathbb{R}$, there exists $\Pi_1 > 0$ sufficiently small depending on known quantities and s such that if $f \in C^\infty(\partial\Omega)$ satisfies*

$$\|f - s\|_{C^{2,\alpha}(\partial\Omega)} \leq \Pi_1,$$

then the linearization of (1.3.2) *exists at $u = u[f]$. Here, $\alpha \in (0,1)$ is as in [51, Theorem 2.10] and depends on the known quantities and s.*

To linearize *near* constant solutions, we need to verify the solution operator is Lipschitz continuous *at* constants.

Lemma 4.1.1. *Given $k \in \mathbb{R}$ and $h \in C^{2,\alpha}(\partial\Omega)$, there exists $\alpha \in (0,1)$ sufficiently small such that the following estimate holds:*

$$\|u[k+h] - k\|_{C^{2,\alpha}(\overline{\Omega})} \leq C(k, \|h\|_{C^2})\|h\|_{C^{2,\alpha}(\partial\Omega)}. \tag{4.1.4}$$

Proof. Letting $f = k + h$ and $u = k + w$ in (1.3.2), we obtain the BVP

$$\begin{aligned} \nabla \cdot (a(k+w, \nabla w)\nabla w) &= 0, \\ w|_{\partial\Omega} &= h. \end{aligned} \tag{4.1.5}$$

The structural conditions (1.3.3), (1.3.4), and (1.3.5) are still satisfied, with lower and upper bounds now also depending on a parameter k. It was shown in [51] that (1.3.2) has an *a priori* $C^{1,\alpha}(\overline{\Omega})$ estimate in terms of $\|f\|_{C^2(\partial\Omega)}$ for some $\alpha \in (0,1)$ depending on a and $\|f\|_{C^2(\partial\Omega)}$, which means each coefficient in the nondivergence form of (4.1.5),

$$a_{ij}(k+w, \nabla w)w_{ij} + (a_s(k+w, \nabla w)w_i)w_i = 0, \tag{4.1.6}$$

has a $C^\alpha(\overline{\Omega})$ estimate also in terms of $\|f\|_{C^2(\partial\Omega)}$. Therefore, the linear Schauder $C^{2,\alpha}$ estimate [29, Theorem 6.6] yields

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leq C(a, \Omega, \|f\|_{C^2(\partial\Omega)})\|f\|_{C^{2,\alpha}(\partial\Omega)}. \tag{4.1.7}$$

In the case of (4.1.5), the same estimate is therefore true, with $C(a, \Omega, \|f\|)$ replaced by $C(k, a, \Omega, \|h\|) =: C(k, \|h\|)$, and α depending also on k and $\|h\|_{C^2}$. Substitution of $w = u - k$ into the above completes the proof. □

We also need the continuity of the solution operator near *any* small solution, which we record for completeness. Given $M > 0$ and the sufficiently small $\alpha = \alpha(M)$ of (4.1.7), we give the subset $B_M = \{f \in C^{2,\alpha}(\partial\Omega) : \|f\|_{C^2(\partial\Omega)} < M\}$ the $C^{2,\alpha}(\partial\Omega)$ norm.

Lemma 4.1.2. *For each $M > 0$ and $\beta \in (0, \alpha(M))$, the operator $B_M \to C^{2,\beta}(\overline{\Omega})$ given by $f \mapsto u[f]$ is continuous.*

Proof. If not, then there exists a sequence $f_k \in C^{2,\alpha}(\partial\Omega)$ with $\|f_k\|_{C^2} \leq M$ converging to $f \in C^{2,\alpha}(\partial\Omega)$ but with $\|u[f_k] - u[f]\|_{C^{2,\beta}(\Omega)} \geq 1$, say. By (4.1.7), the solutions $u[f_k]$ are uniformly bounded in $C^{2,\alpha}(\overline{\Omega})$, so the Arzela-Ascoli theorem yields a subsequence, also named $u[f_k]$, which converges in $C^{2,\beta}(\overline{\Omega})$ to some v. It follows that $\|v - u[f]\|_{C^{2,\beta}(\Omega)} \geq 1$. However, since each $u[f_k]$ solves (1.3.2) with boundary data f_k, sending $k \to \infty$ in (1.3.2) shows that v solves (1.3.2) as well with boundary data f. By the comparison principle [29, Theorem 10.7] for quasilinear divergence form operators, it follows that $C^{2,\beta}$ solutions are unique, hence $v = u[f]$, a contradiction. □

Proof of Proposition 4.1.1. Defining t-dependent quantities,

$$\begin{aligned} u(x;t) := u[f+th](x), \quad v(x;t) := \frac{u(x;t) - u(x;0)}{t}, \\ a(x;t) := a[u(x;t)], \quad u(;\sigma;t) = u(;0) + \sigma(u(;t) - u(;0)), \end{aligned} \tag{4.1.8}$$

with $a[u] := a(u, \nabla u)$, we take a "discrete derivative":

$$\begin{aligned} a(;t)u_i(;t) - a(;0)u_i(;0) &= ta(;t)v_i(;t) + (a(;t) - a(;0))u_i(;0) \\ &= tav_i + ta_s(;t)v(;t)u_i(;0) + ta_{p_j}(;t)v_j(;t)u_i(;0), \end{aligned} \tag{4.1.9}$$

where, by the fundamental theorem of calculus,

$$a_s(;t) := \int_0^1 a_s[u(;\sigma;t)]d\sigma, \quad a_{p_j}(;t) := \int_0^1 a_{p_j}[u(;\sigma;t)]d\sigma. \tag{4.1.10}$$

We see that $v(;t)$ then solves the Dirichlet problem

$$\begin{aligned} \partial_i \left(a_s(;t)v(;t)u_i(;0) + a_{p_j}(;t)v_j(;t)u_i(;0) + a(;t)v_i(;t)\right) = 0, \\ v(;t)|_{\partial\Omega} = h, \end{aligned} \tag{4.1.11}$$

with associated Cauchy data (DN map)

$$\left(a_s(;t)v(;t)\frac{\partial u}{\partial \nu}(;0) + a_{p_j}(;t)v_j(;t)\frac{\partial u}{\partial \nu}(;0) + a(;t)\frac{\partial v}{\partial \nu}(;t)\right) dS. \tag{4.1.12}$$

Step 1:

We verify the Dirichlet problem (4.1.11) for $v(;t)$ has a unique weak solution in $W^{1,2}(\Omega)$ which is uniformly bounded in t. Let $w \in W_0^{1,2}(\Omega)$ solve

$$\partial_i \left(a_s(;t) w u_i(;0) + a_{p_j}(;t) w_j u_i(;0) + a(;t) w_i \right) = 0. \tag{4.1.13}$$

Multiplying (4.1.13) by w and integrating by parts, we obtain

$$\int_\Omega a_{ij}(;t) w_i w_j \, dx = - \int_\Omega a_s(;t) u_i(;0) w w_i \, dx, \tag{4.1.14}$$

where we defined

$$a_{ij}(;t) := a(;t)\delta_{ij} + \frac{1}{2} \int_0^1 u_i(;0) a_{p_j}[u(;\sigma;t)] + u_j(;0) a_{p_i}[u(;\sigma;t)] \, d\sigma. \tag{4.1.15}$$

For $|t| \leq 1$, estimate (4.1.7) shows $u(;t)$ is uniformly bounded in $C^{2,\alpha}(\overline{\Omega})$ with respect to t, for α sufficiently small. Moreover, Lemma 4.1.2 shows that $t \mapsto u(;t)$ is continuous in $C^{2,\beta}(\overline{\Omega})$ if $\beta \in (0, \alpha)$, so by the smoothness of a, we can write $a_{ij}(;t) = a_{ij}(;0) + r_1(t)$, where remainder $r_1(t)$ vanishes in $C^{1,\beta}(\overline{\Omega})$ as $t \to 0$. By (1.3.4), we can therefore suppose t is sufficiently small so that

$$a_{ij}(;t)\xi_i\xi_j \geq \frac{1}{2}\lambda_0(|u(;0)|)|\xi|^2, \qquad \xi \in \mathbb{R}^n. \tag{4.1.16}$$

Similarly, we write $a_s(;t) = a_s(;0) + r_2(t)$, and recalling (1.3.5), we suppose t is sufficiently small so that

$$|a_s(;t)| \leq 2\mu_0(|u(;0)|). \tag{4.1.17}$$

We then apply the Cauchy-Schwarz inequality to (4.1.14) using (4.1.16) and (4.1.17):

$$\int_\Omega \lambda_0(|u(;0)|)|Dw|^2 \, dx \leq 4 \int_\Omega \frac{\mu_0^2}{\lambda_0}(|u(;0)|)|Du(;0)|^2 w^2 \, dx \tag{4.1.18}$$

Since $Du = D(u - s)$, the estimate (4.1.4) for $k = s$ and $h = f - s$ implies the gradient is small:

$$\|Du(;0)\|_{L^\infty} \leq C(s)\|f - s\|_{C^{2,\alpha}} \leq C(s)\Pi_1. \tag{4.1.19}$$

Here Π_1 is to be fixed. If $\Pi_1 \leq 1$, then $\|f\|_{C^{2,\alpha}} \leq |s| + C(s)$, and $C^{2,\alpha}$ estimate (4.1.7) yields

$$\|u(;0)\|_{L^\infty} \leq C(s), \quad \lambda_0(|u(;0)|) \geq \lambda_0(C(s)), \quad \mu_0(|u(;0)|) \leq \mu_0(C(s)), \tag{4.1.20}$$

for another $C(s)$; we recall the monotonicity of λ_0, μ_0. Substituting these and the Poincaré inequality for $w \in W_0^{1,2}(\Omega)$ into (4.1.18) yields

$$\int_\Omega |Dw|^2 \, dx \leq \frac{\mu_0^2}{\lambda_0^2}(C(s)) \cdot C(s, \Omega) \cdot \Pi_1^2 \int_\Omega |Dw|^2 \, dx. \tag{4.1.21}$$

Choosing Π_1 small depending on s, Ω confirms that $w = 0$. The Fredholm alternative [29, Theorem 5.3] shows that Dirichlet problem (4.1.11) has a unique solution $v(;t) \in W^{1,2}(\Omega)$, and [29, Corollary 8.7] yields the $W^{1,2}$ estimate

$$\|v(;t)\|_{W^{1,2}(\Omega)} \leq C((4.1.11), \Omega)\|h\|_{W^{1,2}(\Omega)}, \tag{4.1.22}$$

where $C((4.1.11), \Omega)$ indicates (uniform) dependence on the coefficients of the differential operator in (4.1.11). For small t, the coefficients and their derivatives have controlled t dependence in the $C^\beta(\overline{\Omega})$ norm, so we conclude that $v(;t)$ has a uniform-in-t $W^{1,2}$ bound. Moreover, since $f, h \in C^\infty$, standard Schauder bootstrapping yields higher regularity $C^{k,\alpha}(\partial\Omega) \to C^{k,\alpha}(\overline{\Omega})$ estimates for $u(;t)$ as in (4.1.7), so the t dependence of the coefficients in (4.1.11) is uniform in any C^k norm. Therefore, the higher regularity $W^{k+2,2}(\Omega)$ estimate [29, Theorem 8.13] combined with Sobolev embeddings yield uniform-in-t estimates for $v(;t)$ in any C^ℓ norm.

Step 2:

We now show $v(;t)$ converges to v in $C^1(\overline{\Omega})$, where v is the unique solution of Dirichlet problem (4.1.2). Note that the unique solvability for v is similar to that for $v(;t)$ in Step 1, so we omit the proof. In (4.1.11), as before, we split

$$a_s(;t) = a_s(;0) + r_2(t), \quad a_{p_j}(;t) = a_{p_j}(;0) + r_3(t), \quad a(;t) = a(;0) + r_4(t), \tag{4.1.23}$$

where each $r_i(t)$ vanishes in $C^{1,\beta}(\overline{\Omega})$ as $t \to 0$. Thus, omitting $t = 0$ arguments, $v(;t)$ solves the inhomogeneous equation

$$\partial_i(a_s v(;t) u_i + a_{p_j} v_j(;t) u_i + a v_i(;t)) = o(1)_{C^\beta(\overline{\Omega})}, \tag{4.1.24}$$

where $o(1)_{C^\beta(\overline{\Omega})}$ indicates a quantity vanishing as $t \to 0$ in $C^\beta(\overline{\Omega})$, and its uniformity in t with respect to terms including v or v_i follows from Step 1. We then put $w(;t) = v(;t) - v$

and see that $w(;t)$ uniquely solves the Dirichlet problem

$$\begin{aligned} \partial_i(a_s w(;t)u_i + a_{p_j} w(;t)_j u_i + a w(;t)_i) &= o(1)_{C^\beta(\overline{\Omega})}, \\ w(;t)|_{\partial\Omega} &= 0. \end{aligned} \tag{4.1.25}$$

The right hand side clearly vanishes in L^p for any $p > 1$, so it follows from the strong $W^{2,p}$ estimate [29, Theorem 9.13] that $\|w(;t)\|_{W^{2,p}(\Omega)} \to 0$ as $t \to 0$. By Sobolev embedding, we deduce that $v(;t) \to v$ in $C^1(\overline{\Omega})$, as desired. □

We next linearize around solutions u with large gradients if we assume that $a(s,p)$ satisfies additional conditions.

Proposition 4.1.2. *Let a satisfy uniform lower bound* (1.3.7) *and decay condition* (1.3.8). *Then for any solution u to* (1.3.2) *with smooth boundary data f, the linearization of* (1.3.2) *exists at u.*

Proof. The proof of Step 2 in Proposition 4.1.1 is unchanged, and we repeat Step 1 until (4.1.16). To proceed further, we invoke additional conditions (1.3.7) and (1.3.8). We suppose t is sufficiently small so that

$$|a_s(;t)Du(;0)| \leq \varepsilon_0 + o(1)_{C^{1,\beta}} \|Du(;0)\|_{L^\infty} \leq 2\varepsilon_0. \tag{4.1.26}$$

Applying the Cauchy-Schwartz inequality and the Poincaré inequality, we obtain

$$\int_\Omega |Dw|^2 \, dx \leq \frac{4\varepsilon_0^2}{\lambda_0} \cdot C(\Omega) \int_\Omega |Dw|^2 \, dx. \tag{4.1.27}$$

Choosing ε_0 small depending on λ_0, Ω yields the desired conclusion. The rest of the proof proceeds identically. □

4.1.2 Solutions with prescribed boundary jets

We first introduce a coordinate normalization, with $x^i, i = 1, \ldots, n$ the usual Euclidean coordinates in this section.

Definition 2. We say that Ω *sits above the origin*, if $\{x^n = 0\}$ is a supporting hyperplane for Ω with $0 \in \partial\Omega$, and $\Omega \subset \{x^n > 0\}$.

The class of PDEs (1.3.2) with conditions (1.3.3) and (1.3.5) considered is invariant under Euclidean isometries. Let $I = R \circ T$ be a Euclidean isometry (rotation R plus translation T) such that $I^{-1}(\Omega)$ sits above the origin. The resulting PDE for $I^*u(x) := u(Ix)$ is also of class (1.3.2):

$$\nabla \cdot [R_* a(I^*u, \nabla I^*u) \nabla I^*u] = 0, \qquad x \in I^{-1}(\Omega), \tag{4.1.28}$$

where $R_* a(s,p) = a(s, R^{-1}p)$. It is clear that $R_* a$ satisfies any of conditions (1.3.3), (1.3.5), or (1.3.8) if and only if a does.

We now construct solutions with prescribed boundary jet, provided the gradient at the boundary point is sufficiently small compared to the solution. No additional assumptions on a are needed. Recall $\Pi_1(s)$ as in Proposition 4.1.1.

Proposition 4.1.3. *Let $\Pi : \mathbb{R} \to (0,\infty)$ be the positive continuous function defined in (4.1.46). Then for any $x_0 \in \partial\Omega$ and $(s,p) \in \mathbb{R} \times \mathbb{R}^n$ such that $|p| < \Pi(s)$, there exists boundary data $f \in C^\infty(\partial\Omega)$ for (1.3.2) such that $u(x_0) = s$ and $\nabla u(x_0) = p$, with $\|f - s\|_{C^{2,\alpha}} \leq \Pi_1(s)$.*

Proof. **Step 1.** We construct sub/super-solutions.

First suppose Ω sits above the origin and $x_0 = 0$. Given $(s,p) \in \mathbb{R}\times\mathbb{R}^n$ with $p = p' + p_n e_n$ and $p' \perp e_n$, we define

$$u^{s,p}(x) := s + p' \cdot x - A p_n \log(1 - x^n/A), \tag{4.1.29}$$

where A is the sufficiently large constant

$$A := 2 \text{ diam } \Omega \geq 2 \max_{x\in\Omega} x^n. \tag{4.1.30}$$

We have $u^{s,p}(0) = s$ and $Du^{s,p}(0) = p$. The derivatives:

$$Du^{s,p}(x) = p' + \frac{p_n}{A - x^n} e_n, \quad D^2 u^{s,p}(x) = \frac{p_n}{(A-x^n)^2} e_n \otimes e_n. \tag{4.1.31}$$

The expanded form of equation (1.3.2) is

$$a_{ij} u_{ij} + a_s |\nabla u|^2 = 0. \tag{4.1.32}$$

For $\pm 1 = \text{sign } p_n$, we thus obtain

$$\begin{aligned}\pm(a_{ij}u^{s,p}_{ij} + a_s|Du^{s,p}|^2) &\geq \lambda_0(|u^{s,p}|)\frac{|p_n|}{(A-x^n)^2} - \mu_0(|u^{s,p}|)(|p'|^2 + \frac{p_n^2}{(A-x^n)^2}) \\ &\geq \frac{\mu_0(U')}{(A-x^n)^2}\left[\frac{\lambda_0(U')}{\mu_0(U')}|p_n| - (\frac{A^2}{4}|p'|^2 + p_n^2)\right],\end{aligned} \tag{4.1.33}$$

where

$$U' := s + \frac{A}{2}|p'| + A|p_n|\log 2$$

satisfies $|u^{s,p}(x)| \leq U'$ on Ω.

Let us now restrict p to satisfy

$$\max(|p'|, |p_n|) \leq 1, \qquad U' \leq U := s + 2A. \tag{4.1.34}$$

Then we obtain

$$\pm(a_{ij}u^{s,p}_{ij} + a_s|Du^{s,p}|^2) \geq \frac{\mu_0(U)}{(A-x^n)^2}\left[\frac{\lambda_0(U)}{\mu_0(U)}|p_n| - (\frac{A^2}{4}|p'|^2 + p_n^2)\right]. \tag{4.1.35}$$

Choosing now p to be any vector defined in the paraboloid

$$\left(\frac{2\lambda_0(U)}{A^2\mu_0(U)}\right)^{-1}|p'|^2 \leq |p_n| \leq \frac{\lambda_0(U)}{2\mu_0(U)}, \qquad \max(|p'|, |p_n|) \leq 1, \tag{4.1.36}$$

we find that $u^{s,p}$ is a one-sided solution:

$$\pm(a_{ij}u^{s,p}_{ij} + a_s|Du^{s,p}|^2) \geq 0. \tag{4.1.37}$$

To remove the "corners" of the paraboloid (4.1.36), we denote by $U = U^s$ the constant in (4.1.34), and denote the various constants by

$$C^1_s := \min(\frac{\lambda_0(U^s)}{2\mu_0(U^s)}, 1), \qquad C^2_s := \min(\frac{2\lambda_0(U^s)}{A^2\mu_0(U^s)}, 1),$$

so that after shrinking the bounds, the paraboloid becomes

$$\frac{1}{C^2_s}|p'|^2 \leq |p_n| \leq C^1_s. \tag{4.1.38}$$

To satisfy the smallness condition of Proposition 4.1.1, we shrink the bounds again. Defining

$$B^1_s := \min(C^1_s, \frac{\Pi_1(s)}{N \text{ diam } \Omega}), \qquad B^2_s := C^2_s,$$

for some sufficiently large N, it is easy to verify that $u^{s,p}$ satisfies the desired estimate

$$\|u^{s,p} - s\|_{C^{2,\alpha}(\partial\Omega)} \leq \Pi_1(s)$$

provided $p = (p', p_n)$ lies within the paraboloid

$$\frac{1}{B_s^2}|p'|^2 \leq |p_n| \leq B_s^1. \tag{4.1.39}$$

We next construct solutions with boundary jet inside this paraboloid, and more generally in the smallest rectangle containing it.

Step 2. A comparison principle argument.

Given s, we let $p \in \mathbb{R}^n$ be any vector in the vertical boundary of the paraboloid (4.1.39):

$$|p'|^2 < B_s^1 B_s^2, \qquad p_n = B_s^1. \tag{4.1.40}$$

Since $p_n > 0$, it follows that $u^{s,p}$ is a subsolution of (1.3.2). Choosing boundary data $f \in C^\infty(\partial\Omega)$ of the form

$$f^{s,p}(x) := u^{s,p}(x), \qquad x \in \partial\Omega, \tag{4.1.41}$$

the comparison principle [29, Theorem 10.7] shows that

$$u[f^{s,p}](x) \geq u^{s,p}(x), \qquad x \in \Omega, \tag{4.1.42}$$

with equality at $x = 0$, so the inner normal derivatives satisfy the same inequality:

$$\frac{\partial u[f^{s,p}]}{\partial x^n}(0) \geq \frac{\partial u^{s,p}}{\partial x^n}(0) = p_n. \tag{4.1.43}$$

If we instead choose

$$p_n = -B_s^1,$$

then the inequalities reverse because $u^{s,p}$ becomes a supersolution. This means that the set of normal derivatives with tangential jet (s, p') and *small* boundary data,

$$\begin{aligned}\mathcal{N}^{s,p'} := \{\frac{\partial u[g]}{\partial x^n}(0)\Big| g \in C^\infty(\partial\Omega), g(0) = s, \nabla g(0) = p', \text{ and} \\ \|g - s\|_{C^{2,\alpha}(\partial\Omega)} \leq \Pi_1(s)\},\end{aligned} \tag{4.1.44}$$

contains numbers at least as large as B_s^1 and at least as small as $-B_s^1$. Since the set of g with boundary jet $(g, \nabla g)(0) = (s, p')$ is convex, it is connected. So, of course, is the set of such g also satisfying the smallness estimate

$$\|g - s\|_{C^{2,\alpha}(\partial\Omega)} \leq \Pi_1(s). \tag{4.1.45}$$

By the smoothness of solutions to (1.3.2), we deduce that $\mathcal{N}^{s,p'}$ is connected, and it therefore contains the interval $[-B_s^1, B_s^1]$. Varying p' according to (4.1.40), we conclude that for each s and any vector p in the rectangle

$$|p'|^2 \leq B_s^1 B_s^2, \qquad |p_n| \leq B_s^1,$$

there exists a solution with boundary jet $(u(0), \nabla u(0)) = (s, p)$ with boundary data g satisfying (4.1.45). Defining Π by the radius of the largest sphere inside the rectangle,

$$\Pi(s) = \min(\sqrt{B_s^1 B_s^2}, B_s^1). \tag{4.1.46}$$

This completes the proof of the Proposition, in the case that Ω sits above the origin.

Step 3. Rotational invariance.

Now suppose $\Omega \subset \mathbb{R}^n$ is any smooth domain, and choose s, p such that $|p| < \Pi(s)$, and $x_0 \in \partial\Omega$. Letting $T(x) = x - x_0$ translate x_0 to the origin, we choose a rotation $R \in O(n)$ such that the inner normal vector is mapped to e_n, i.e. $R(-\nu|_{x_0}) = e_n$. Then for Euclidean isometry $I = R \circ T$, we see that $I^{-1}(\Omega)$ sits above the origin. Moreover, inequalities (1.3.3), (1.3.4), and (1.3.5) and the constant A in (4.1.30) are preserved under the isometry. Since $|Rp| < \Pi(s)$, we can find a solution I^*u of the rotated equation (4.1.28) such that $I^*u(0) = s$ and $\nabla I^*u(0) = Rp$. In original variables, we have $u(x_0) = s$ and $\nabla u(x_0) = p$, which completes the proof. □

If we also assume the decay condition (1.3.8), then we can choose $\Pi = +\infty$. That is, we can find a solution with any prescribed boundary jet. We can also change conditions (1.3.7) and (1.3.8), in this proof, to

$$|a_s(s, p)| \leq C \frac{\lambda(s, p)}{|p|}, \tag{4.1.47}$$

where $\lambda(s, p)$ is in (1.3.3), and $C > 0$ is arbitrary. This condition makes (1.3.2) behave similar to a linear equation which admits exponential solutions.

Proposition 4.1.4. *Let a satisfy* (4.1.47)*, or alternatively* (1.3.7) *and* (1.3.8)*. For any $(s,p) \in \mathbb{R} \times \mathbb{R}^n$ and $x_0 \in \partial\Omega$, there exists boundary data $f \in C^\infty(\partial\Omega)$ for* (1.3.2) *such that $u(x_0) = s$ and $\nabla u(x_0) = p$.*

Proof. As in Step 3 in the proof of Proposition 4.1.3, by the invariance of the problem under Euclidean isometries, it suffices to assume that Ω sits above the origin.

Fix $(s,p) \in \mathbb{R} \times \mathbb{R}^n$ with $p_n \neq 0$ and $p' := p - p_n e_n$, and for $h > 0$ small depending on the constant C in (4.1.47) and p, we introduce the function

$$u^{s,p}(x) := s + p' \cdot x + h\, p_n\, (e^{x^n/h} - 1). \tag{4.1.48}$$

If $p_n > 0$ (resp. $p_n < 0$) then we claim that $u^{s,p}$ is a subsolution (supersolution) of (1.3.2), with jet

$$u^{s,p}(0) = s, \qquad \nabla u^{s,p}(0) = p. \tag{4.1.49}$$

We have

$$|\nabla u^{s,p}|^2 = |p'|^2 + p_n^2 e^{2x^n/h}, \qquad u^{s,p}_{ij} = \frac{p_n}{h} e^{x^n/h} \delta_{in}\delta_{jn}, \tag{4.1.50}$$

so ellipticity (1.3.3) yields, for $\pm 1 = \operatorname{sign} p_n$,

$$\pm(a_s|\nabla u^{s,p}|^2 + a_{ij}u^{s,p}_{ij}) \geq \pm a_s|\nabla u^{s,p}|^2 + \lambda \frac{|p_n|}{h} e^{x^n/h}. \tag{4.1.51}$$

Applying decay condition (4.1.47), we obtain

$$\pm(a_s|\nabla u^{s,p}|^2 + a_{ij}u^{s,p}_{ij}) \geq \lambda e^{x^n/h} \left(\frac{|p_n|}{h} - C\sqrt{|p'|^2 e^{-2x^n/h} + p_n^2} \right). \tag{4.1.52}$$

Choosing h such that $|p_n|/h \geq C\sqrt{|p'|^2 + p_n^2}$ verifies the claim.

We next observe that by continuity, the set of possible normal derivatives with tangential jet (s, p')

$$\{ \frac{\partial u[g]}{\partial x^n}(0) | \quad g \in C^\infty(\partial\Omega) \text{ and } g(0) = s, \nabla' g(0) = p' \} \tag{4.1.53}$$

is connected and nonempty, so to show it is $\mathbb{R}$, we need only show it is unbounded from above and below. If $p_n \gg 1$ is large, we let $f = u^{s,p}|_{\partial\Omega}$ and u solve (1.3.2) with boundary value f. Then $u^{s,p}$ is a subsolution, and the comparison principle [29, Theorem 10.7] says

$$u(x) \geq u^{s,p}(x), \qquad x \in \Omega,$$

with equality at $x = 0$, so

$$\frac{\partial u}{\partial x^n}(0) \geq \frac{\partial u^{s,p}}{\partial x^n}(0) = p_n.$$

Repeating the argument with $p_n \ll -1$ and using supersolution $u^{s,p}$ yield the desired conclusion. □

Remark 4.1.1. The results of this section hold true unchanged for non-homogeneous conductivities $a = a(x, s, p)$, provided the inequalities (1.3.3) and (1.3.5) remain independent of x and we include in (1.3.5) the structural condition (1.7) from [51], namely

$$|\nabla_x a(x, s, p)| \leq \mu_0(|s|)|p|.$$

Under this assumption, Proposition 4.1.3 goes through, since estimate (4.1.33) is only changed by a factor of 2. To capture the global jet prescription Proposition 4.1.4, we could instead use the condition

$$|\nabla_x a(x, s, p)| \leq C\lambda(s, p),$$

for some $C > 0$, or just $\leq C$ if $\lambda(s, p) \geq \lambda_0$ is uniformly elliptic. In these cases, estimate (4.1.52) goes through. At the final step of rotational invariance, we instead define the transformed conductivity in (4.1.28) by $I_* a(x, s, p) = a(Ix, s, R^{-1}p)$.

4.1.3 Geometric reformulation of the linearized problem

We next geometrically reformulate the linearized equation (4.1.2) and its DN map (4.1.3), see e.g. [45] for the same with the conductivity equation via the Laplace operator. We assume throughout this section that the linearization of (1.3.2) exists at a given solution u, which we addressed in Section 4.1.1.

Let us first rewrite the equation. We can decompose the principal part of (4.1.2), namely

$$\partial_i(a_{p_j} v_j u_i + a v_i),$$

into its symmetric and anti-symmetric parts and rewrite the linearized differential operator

as:

$$\begin{aligned}\mathcal{L}_a[u]v &:= \partial_i(a_s v u_i + a_{p_j} v_j u_i + a v_i) \\ &= \partial_i(a_{ij} v_j + (a_s u_i v + A_{ij} v_j)) \\ &= \partial_i(a_{ij} v_j + b^i v),\end{aligned} \tag{4.1.54}$$

where linearized conductivity a_{ij}, given in (1.3.3), is the symmetric part, and

$$b^i := a_s u_i - \partial_j A_{ij}, \qquad A_{ij} := \frac{1}{2}(a_{p_j} u_i - a_{p_i} u_j) \tag{4.1.55}$$

accounts for the anti-symmetric part; the last equality in (4.1.54) follows from anti-symmetry. We can also rewrite the linearized DN map (4.1.3):

$$\Gamma_a[u](h) = (a_{ij}\nu_i v_j + A_{ij}\nu_i v_j + a_s v\, u_i \nu_i) dS. \tag{4.1.56}$$

Henceforth in this section only, we distinguish Euclidean coordinates from ordinary subscripts (which will denote local coordinates) unless otherwise stated.

We next rewrite this equation in terms of geometric quantities. Let x^i denote local coordinates on Ω, and denote x^i_e to be global Euclidean coordinates. We define a $(2,0)$ tensor field $G \in \mathcal{T}^2_0(\Omega)$ by its values in Euclidean coordinates, using the symmetric part a_{ij} in (1.3.3) of the linearized operator. If $n \geq 3$,

$$G^{ij}_e := G(dx^i_e, dx^j_e) = (\det a_{k\ell})^{\frac{1}{2-n}} a_{ij}, \qquad G = G^{ij}\partial_i\partial_j = G^{ij}_e \partial_{i,e}\partial_{j,e}. \tag{4.1.57}$$

If $n = 2$, we first define a conductivity function $\sigma \in C^\infty(\Omega)$ by

$$\sigma := \det a_{k\ell}. \tag{4.1.58}$$

We can then define G as follows:

$$G^{ij}_e := G(dx^i_e, dx^j_e) = (\det a_{k\ell})^{-1/2} a_{ij}, \qquad G = G^{ij}\partial_i\partial_j. \tag{4.1.59}$$

In each case, the relationship can be written as

$$\begin{aligned} n \geq 3: &\quad \frac{1}{\sqrt{\det G^{k\ell}_e}} G^{ij}_e = a_{ij}, \\ n = 2: &\quad \sigma\, G^{ij}_e = a_{ij}, \qquad \det G^{k\ell}_e = 1. \end{aligned} \tag{4.1.60}$$

We next define a Riemannian metric $g \in \mathcal{T}^0_2(\Omega)$ by the inverse of G in Euclidean coordinates, which is well defined by ellipticity (1.3.3)

$$g_{e,ij} := g(\frac{\partial}{\partial x^i_e}, \frac{\partial}{\partial x^j_e}) = (G_e^{-1})_{ij}, \qquad g = g_{ij} dx^i dx^j = g_{ij,e} dx^i_e dx^j_e. \tag{4.1.61}$$

We will instead denote $\partial_{e,i} = \partial/\partial x^i_e$, along with $g^{ij} = (g_{k\ell})^{-1}_{ij}$ and $\sqrt{g} = \sqrt{\det g_{ij}}$. The divergence of the linearized conductivity takes two forms in various dimensions:

$$\begin{aligned} n \geq 3: &\quad \partial_{e,i}(a_{ij}\partial_{e,j} v) = \sqrt{g_e}\, \Delta_g v, \\ n = 2: &\quad \partial_{e,i}(a_{ij}\partial_{e,j} v) = \mathrm{div}_g(\sigma \nabla_g v) = \sigma \Delta_g v + \langle \nabla_g \sigma, \nabla_g v \rangle_g, \end{aligned} \tag{4.1.62}$$

where, in local coordinates, the Laplace operator, gradient, and divergence are given by

$$\Delta_g v = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g}\, g^{ij} \partial_j v), \quad \nabla_g v = g^{ij} \partial_j v\, \partial_i, \quad \mathrm{div}_g B = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g}\, B^i). \tag{4.1.63}$$

To account for the lower order terms in (4.1.54), we define the *magnetic Schrödinger operator with potential*:

$$\begin{aligned} \Delta_{g,A,q} v &:= \frac{1}{\sqrt{g}} (\partial_i + A_i) \sqrt{g}\, g^{ij} (\partial_j + A_j) v + qv \\ &= \Delta_g v + 2\langle A^\#, \nabla_g v \rangle + \Big(\mathrm{div}_g A^\# + |A|^2_g + q \Big) v, \end{aligned} \tag{4.1.64}$$

where $A = A_i dx^i, q$ are to be chosen, $A^\# = g^{ij} A_j \partial_i$, and $|A|^2_g = g^{ij} A_i A_j$. Recalling (4.1.62), we obtain the following relations:

$$\begin{aligned} n \geq 3: &\quad \mathcal{L}_a[u] v = \sqrt{g_e}\, \Delta_{g,A,q} v, \\ n = 2: &\quad \mathcal{L}_a[u] v = \sigma\, \Delta_{g,A,q} v, \end{aligned} \tag{4.1.65}$$

provided the following relations are true:

$$\begin{aligned} n \geq 3: &\quad b^i = 2\sqrt{g_e}\, g^{ij}_e A_{e,j}, \qquad \partial_{e,i} b^i = \sqrt{g_e} (\mathrm{div}_g A^\# + |A|^2_g + q), \\ n = 2: &\quad b^i + g^{ij}_e \partial_{e,j} \sigma = 2\sigma g^{ij}_e A_{e,j}, \qquad \partial_{e,i} b^i = \sigma (\mathrm{div}_g A^\# + |A|^2_g + q). \end{aligned} \tag{4.1.66}$$

This system can be solved for A and q in terms of b, σ, and g. This completes the reformulation of the linearized PDE (4.1.2) via (4.1.65).

We next reformulate the linearized DN map (4.1.56), and start with the anti-symmetric part of the linearized operator (4.1.55). We define a $(1,1)$ tensor $\alpha \in \mathcal{T}^1_1(\Omega)$ in terms of g:

$$\alpha^i_{e,j} := \alpha(dx^i_e, \partial_{e,j}) = \frac{1}{\sqrt{g_e}}\, g_{e,jk} A_{ik}, \qquad \alpha = \alpha^i_j \partial_i dx^j. \tag{4.1.67}$$

Here, $\sqrt{g_e} = \sqrt{\det g_{e,ij}}$, and A_{ik} is as in (4.1.55). The tensor α is anti-symmetric in the inner product induced by g:

$$\langle \alpha \cdot V, W\rangle_g = -\langle V, \alpha \cdot W\rangle_g, \quad V, W \in T\Omega, \tag{4.1.68}$$

since in Euclidean coordinates, with $V = V_e^i \partial_{e,i}, W = W_e^i \partial_{e,i}$, we have

$$\begin{aligned} \langle \alpha \cdot V, W\rangle_g &= g_{e,i\ell} \left(\frac{1}{2\sqrt{g_e}} g_{e,jk} (a_{p_k} u_i - a_{p_i} u_k) V_e^j \right) W_e^\ell \\ &=: (g_{e,i\ell} T^{ik} g_{e,jk}) V_e^j W_e^\ell, \end{aligned} \tag{4.1.69}$$

where T^{ik} is anti-symmetric in its indices, so the claim follows.

The Riemannian volume form dV_g has the Euclidean representation

$$dV_g = \sqrt{g_e}\, dx_e^1 \wedge \cdots \wedge dx_e^n = \sqrt{g_e}\, dx_e. \tag{4.1.70}$$

The induced area form dS_g on $\partial\Omega$ is

$$dS_g = \nu_g \lrcorner\, dV_g|_{\partial\Omega}, \tag{4.1.71}$$

where ν_g is the unit normal to $\partial\Omega$. Letting f be a defining function for $\partial\Omega$, or a smooth function $\mathbb{R}^n \to \mathbb{R}$ with $\partial\Omega = f^{-1}(0)$ and $df|_{\partial\Omega} \neq 0$, the unit normals for Euclidean and g metrics can be written in Euclidean coordinates as

$$\nu = (|df|_e)^{-1} \partial_{e,i} f \partial_{e,i}, \qquad \nu_g = (G(df, df))^{-1/2} G_e^{ij} \partial_{e,j} f \partial_{e,i}. \tag{4.1.72}$$

As sections of the product bundle $\partial(T\Omega) \otimes \Lambda^0_{n-1}(\partial\Omega)$, the following relationship between the normal vectors holds:

$$\begin{aligned} n \geq 3: &\quad (a_{ij} \nu_j \partial_{e,i}) \otimes dS = \nu_g \otimes dS_g, \\ n = 2: &\quad (a_{ij} \nu_j \partial_{e,i}) \otimes dS = \sigma\, \nu_g \otimes dS_g. \end{aligned} \tag{4.1.73}$$

In particular, the main part of the linearized DN map (4.1.56) has the form

$$\begin{aligned} n \geq 3: &\quad (a_{ij} \nu_i v_i + A_{ij} \nu_i v_j) dS = \langle \nabla_g v + \alpha \cdot \nabla_g v, \nu_g \rangle_g dS_g, \\ n = 2: &\quad (a_{ij} \nu_i v_i + A_{ij} \nu_i v_j) dS = \langle \sigma\, \nabla_g v + \alpha \cdot \nabla_g v, \nu_g \rangle_g dS_g, \end{aligned} \tag{4.1.74}$$

where $v \in C^\infty(\overline{\Omega})$, a_{ij}, σ are in (4.1.60), A_{ij} is in (4.1.55), and α is in (4.1.67).

For the final term in (4.1.56), we define a vector field $\beta = \beta^i \partial_i$ so that the following equation holds on $\partial\Omega$:

$$\begin{aligned} n \geq 3: \quad & (a_s u_i \nu_i) dS = \langle A^{\#} + \beta, \nu_g \rangle dS_g, \\ n = 2: \quad & (a_s u_i \nu_i) dS = \langle \sigma A^{\#} + \beta, \nu_g \rangle dS_g. \end{aligned} \tag{4.1.75}$$

We now recall the *magnetic Dirichlet-to-Neumann map*, $C^\infty(\partial\Omega) \to C^\infty(\partial\Omega)$:

$$\begin{aligned} \Lambda_{g,A,q}(h): \quad & h \mapsto \langle \nabla_g v[h] + A^{\#} v[h], \nu_g \rangle, \\ \text{where:} \quad & \Delta_{g,A,q} v = 0, \\ & v|_{\partial\Omega} = h. \end{aligned} \tag{4.1.76}$$

Then we have the following geometric reformulation. The linearized PDE (4.1.54) corresponds to a magnetic Schrödinger operator with potential, and the linearized DN map (4.1.56) corresponds to an oblique magnetic normal derivative with lower order term.

Proposition 4.1.5. *Let u be a solution of* (1.3.2) *and $f \in C^\infty(\partial\Omega)$ such that the linearization of* (1.3.2) *exists at u. Let also $h \in C^\infty(\partial\Omega)$ and v solve the linearized problem* (4.1.2).

Then for $n \geq 3$:

$$\begin{aligned} \mathcal{L}_a[u] v &= \sqrt{g_e}\, \Delta_{g,A,q} v, \\ \Gamma_a[u](h) &= \Lambda_{g,A,q}(h) dS_g + \langle \alpha \cdot \nabla_g h + \beta h, \nu_g \rangle_g dS_g. \end{aligned} \tag{4.1.77}$$

For $n = 2$:

$$\begin{aligned} \mathcal{L}_a[u] v &= \sigma\, \Delta_{g,A,q} v, \\ \Gamma_a[u](h) &= \sigma \Lambda_{g,A,q}(h) dS_g + \langle \alpha \cdot \nabla_g h + \beta h, \nu_g \rangle_g dS_g. \end{aligned} \tag{4.1.78}$$

Here, metric g is given by (4.1.60),(4.1.61), *and σ by* (4.1.58) *for $n = 2$. Magnetic potential A and potential q are determined by* (4.1.66), *anti-symmetric tensor α is given by* (4.1.67), *and vector field β is defined according to* (4.1.75). *The map Λ is defined by* (4.1.76), *and $\Gamma_a[u]$ is defined by* (4.1.56).

In particular, if $\Gamma_a = \Gamma_{\tilde{a}}$, then for $n \geq 3$:

$$\begin{aligned} &\Lambda_{g,A,q}(h) dS_g + \langle \alpha \cdot \nabla_g h + \beta h, \nu_g \rangle_g dS_g = \\ &\Lambda_{\tilde{g},\tilde{A},\tilde{q}}(h) dS_{\tilde{g}} + \langle \tilde{\alpha} \cdot \nabla_{\tilde{g}} h + \tilde{\beta} h, \nu_{\tilde{g}} \rangle_{\tilde{g}} dS_{\tilde{g}}. \end{aligned} \tag{4.1.79}$$

For $n = 2$:

$$\begin{aligned} \sigma\Lambda_{g,A,q}(h)dS_g + \langle \alpha \cdot \nabla_g h + \beta h, \nu_g\rangle_g dS_g = \\ \tilde{\sigma}\Lambda_{\tilde{g},\tilde{A},\tilde{q}}(h)dS_{\tilde{g}} + \langle \tilde{\alpha} \cdot \nabla_{\tilde{g}} h + \tilde{\beta} h, \nu_{\tilde{g}}\rangle_{\tilde{g}} dS_{\tilde{g}}. \end{aligned} \tag{4.1.80}$$

Proof. The only claim remaining to prove is the identity

$$\langle \alpha \cdot \nabla_g v[h], \nu_g\rangle_g dS_g = \langle \alpha \cdot \nabla_g h, \nu_g\rangle_g dS_g, \tag{4.1.81}$$

or that this term depends only on the tangential projection of $\nabla_g v[h]$. This follows from the anti-symmetry of α, since the normal contribution is

$$\langle \nabla_g v[h], \nu_g\rangle \langle \alpha \cdot \nu_g, \nu_g\rangle = 0,$$

which follows from (4.1.68). □

4.1.4 Boundary determination of the linearized conductivity

In this section, we determine the tangential part of the linearized conductivity a_{ij} at the boundary $\partial\Omega$. The main technical tool we need is the calculation of the symbol of the magnetic DN map (4.1.76) in boundary normal coordinates, [28, Lemma 8.6]. See [45] for the construction in the case of the ordinary Laplace operator.

Letting $x^1, \ldots, x^{n-1}$ be local coordinates on a neighborhood $\Sigma \subset \partial\Omega$, we denote boundary normal coordinates in a neighborhood of Σ in Ω by $(x^1, \ldots, x^n)$, with x^n the g-distance function from $\partial\Omega$. The metric and its dual in boundary normal coordinates are

$$g = (dx^n)^2 + \sum_{i,j<n} g_{ij} dx^i dx^j, \quad G = \partial_n \otimes \partial_n + \sum_{i,j<n} g^{ij} \partial_i \otimes \partial_j. \tag{4.1.82}$$

Note that

$$\sum_{i,j<n} g_{ij} dx^i dx^j = g|_{T\partial\Omega}, \qquad \sum_{i,j<n} g^{ij} \partial_i \otimes \partial_j = G|_{T^*\partial\Omega}, \tag{4.1.83}$$

if $x \in \partial\Omega$. For two metrics $g, \tilde{g}$, we use the *same* local coordinates $x^1, \ldots, x^{n-1}$ for Σ when defining boundary normal coordinates.

We also need the principal symbol of pseudo-differential operators on a manifold using left quantization, see e.g. [33, Definition 18.1.20]. The differential operator $C^\infty(\partial\Omega) \to$

$C^\infty(\partial\Omega)$ defined by

$$h \mapsto \langle \alpha \cdot \nabla_g h + \beta h, \nu_g \rangle_g = \sum_{k<n} (-\alpha \cdot \nu_g)^k h_k + \langle \beta, \nu_g \rangle h,$$

clearly has principal symbol

$$i\langle \alpha \cdot \xi^\#, \nu_g \rangle_g = \sum_{k<n} (-\alpha \cdot \nu_g)^k (i\xi)_k, \qquad (x,\xi) \in T^*_x(\partial\Omega). \tag{4.1.84}$$

To compute the final term in the DN maps (4.1.77),(4.1.78), we now apply [28, Lemma 8.6], see also [45], where the symbol of the magnetic DN map (4.1.76) is computed in boundary normal coordinates in $n \geq 3$ dimensions. The proof there goes through unchanged for the case of $n = 2$, although the subsequent boundary determination problem is different due to conformal invariance; see the remark in [45].

Lemma 4.1.3 (From Lemma 8.6 in [28])**.** *Let (Ω, g) be a compact Riemannian manifold with boundary with dimension $n \geq 2$. Then $\Lambda_{g,A,q}$ is a pseudo-differential operator of order 1 on $\partial\Omega$, and its principal symbol is*

$$-|\xi|_g := -\sqrt{\sum_{i,j<n} g^{ij}(x)\xi_i\xi_j}, \qquad (x,\xi) \in T^*_x\partial\Omega. \tag{4.1.85}$$

We thus obtain the principal symbol of $\Gamma_a[u]$, thought of as a form-valued pseudo-differential operator:

$$\begin{aligned} n \geq 3: &\quad (-|\xi|_g + i\langle \alpha \cdot \xi^\#, \nu_g \rangle) dS_g, \\ n = 2: &\quad (-\sigma|\xi|_g + i\langle \alpha \cdot \xi^\#, \nu_g \rangle) dS_g. \end{aligned} \tag{4.1.86}$$

In dimension $n \geq 3$, we can recover the tangential projection of the metric on $\partial\Omega$. In dimension $n = 2$, we can recover the "effective conductivity" σ on the boundary. In both cases, we can recover the normal elements of the anti-symmetric tensor.

Lemma 4.1.4. *If $\Gamma_a = \Gamma_{\tilde{a}}$, then $G|_{T^*\partial\Omega} = \tilde{G}|_{T^*\partial\Omega}$ if $n \geq 3$, where G is given in terms of a in* (4.1.60)*. If $n = 2$, then $\sigma = \tilde{\sigma}$ on $\partial\Omega$, where we defined σ by* (4.1.58)*. In both cases, we have*

$$(\alpha \cdot \nu_g)\, dS_g = (\tilde{\alpha} \cdot \nu_{\tilde{g}})\, dS_{\tilde{g}} \tag{4.1.87}$$

on $\partial\Omega$.

Proof. By Proposition 4.1.5, equalities (4.1.77), (4.1.78) hold, so the principal symbols on $T^*\partial\Omega$ of the associated pseudo-differential operators on $\partial\Omega$ must coincide. By (4.1.86) and (4.1.79)-(4.1.80), we obtain

$$\begin{aligned} &n \geq 3: \quad (-|\xi|_g + i\langle \alpha \cdot \xi^{\#}, \nu_g \rangle) dS_g = (-|\xi|_{\tilde{g}} + i\langle \tilde{\alpha} \cdot \tilde{\xi}^{\#}, \nu_{\tilde{g}} \rangle) dS_{\tilde{g}}, \\ &n = 2: \quad (-\sigma|\xi|_g + i\langle \alpha \cdot \xi^{\#}, \nu_g \rangle) dS_g = (-\tilde{\sigma}|\xi|_{\tilde{g}} + i\langle \tilde{\alpha} \cdot \tilde{\xi}^{\#}, \nu_{\tilde{g}} \rangle) dS_{\tilde{g}}, \end{aligned} \tag{4.1.88}$$

where we denote $\tilde{\xi}^{\#} = \tilde{g}^{ij}\xi_j\partial_i$. In boundary normal coordinates, we have

$$dS_g = \partial_n \lrcorner dV_g|_{x^n=0} = \sqrt{g}\, dx^1 \wedge \cdots \wedge dx^{n-1} = \sqrt{g|_{T\partial\Omega}}\, dx^1 \wedge \cdots \wedge dx^{n-1},$$

so equating the real parts in (4.1.88) yields

$$\begin{aligned} &n \geq 3: \quad (\det g) g^{ij} = (\det \tilde{g}) \tilde{g}^{ij}, \qquad i, j < n, \\ &n = 2: \quad \sigma = \tilde{\sigma}. \end{aligned} \tag{4.1.89}$$

The imaginary parts in (4.1.88) yield (4.1.87) in both cases. Note that the dimension $n = 2$ equality in (4.1.89) follows from $|\xi_g| dS_g = dx^1$.

Taking the determinant of both sides of (4.1.89) shows that $(\det g)^{n-2} = (\det \tilde{g})^{n-2}$, hence $g^{ij} = \tilde{g}^{ij}$ for $i, j < n$. The $n \geq 3$ case thus recovers $G|_{T^*\partial\Omega}$. □

We thus obtain the main result of this section, a boundary determination result for the quasilinear PDE (1.3.2) in Euclidean coordinates.

Proposition 4.1.6. *Let u be a solution of* (1.3.2) *and $f \in C^\infty(\partial\Omega)$ such that the linearization of* (1.3.2) *exists at u. Similarly with $\tilde{u}$, and $\tilde{a}$ replacing a.*

If $n \geq 3$ and $\Gamma_a = \Gamma_{\tilde{a}}$, then the following holds on $\partial\Omega$:

$$\begin{aligned} [aI + \frac{1}{2}(\nabla_p a \otimes \nabla f + \nabla f \otimes \nabla_p a)]\Big|_{T^*\partial\Omega} = \\ [\tilde{a}I + \frac{1}{2}(\nabla_p \tilde{a} \otimes \nabla f + \nabla f \otimes \nabla_p \tilde{a})]\Big|_{T^*\partial\Omega}. \end{aligned} \tag{4.1.90}$$

If $n \geq 2$ and $\Gamma_a = \Gamma_{\tilde{a}}$, then we have on $\partial\Omega$,

$$(\nabla_p a \otimes \nabla u - \nabla u \otimes \nabla_p a) \cdot \nu = (\nabla_p \tilde{a} \otimes \nabla \tilde{u} - \nabla \tilde{u} \otimes \nabla_p \tilde{a}) \cdot \nu. \tag{4.1.91}$$

If $n = 2$ and $\Gamma_a = \Gamma_{\tilde{a}}$, then we also have

$$\det a_{ij} = \det \tilde{a}_{ij} \tag{4.1.92}$$

on $\partial\Omega$, where a_{ij} is in (1.3.3).

Proof. If $\Gamma_a = \Gamma_{\tilde{a}}$, then Lemma 4.1.4 applies. Since for $n \geq 3$ we have $G|_{T^*\partial\Omega} = \tilde{G}|_{T^*\partial\Omega}$, the characterization (4.1.60) of a_{ij} implies (4.1.90); indeed, the restriction of ∇u to $T^*\partial\Omega$ is precisely ∇f by (1.3.2). Recalling the definition (4.1.58) of σ shows that $\det a_{ij} = \det \tilde{a}_{ij}$ on $\partial\Omega$.

For (4.1.87), we recall Lemma 4.1.4 and (4.1.87), in the form

$$-(\alpha \cdot \nu_g)^s dS_g = (A_{js}\nu_j) dS, \tag{4.1.93}$$

and (4.1.87) yields

$$A_{ij}\nu_j = \tilde{A}_{ij}\nu_j, \tag{4.1.94}$$

which is precisely (4.1.91), recalling the definition (4.1.55) of A_{ij}. □

4.2 Proofs of Theorems

Suppose first that Ω sits above the origin, see Definition 2. Evaluating equalities (4.1.90), (4.1.91), (4.1.92) at $x = 0$ and identifying $T_0^*\partial\Omega$ with $\mathbb{R}^{n-1} \times \{0\}$, we obtain the following equalities at $x = 0$:

If $n \geq 3$:

$$a\delta_{ij} + \frac{1}{2}(a_{p_j} f_i + a_{p_i} f_j) = \tilde{a}\delta_{ij} + \frac{1}{2}(\tilde{a}_{p_j} f_i + \tilde{a}_{p_i} f_j), \quad i, j < n. \tag{4.2.1}$$

If $n \geq 2$:

$$a_{p_i} u_n - a_{p_n} f_i = \tilde{a}_{p_i}\tilde{u}_n - \tilde{a}_{p_n} f_i, \quad i < n. \tag{4.2.2}$$

If $n = 2$:

$$\begin{aligned}(a + a_{p_1} f_1)(a + a_{p_2} u_2) - \frac{1}{4}(a_{p_1} u_2 + a_{p_2} f_1)^2 = (\tilde{a} + \tilde{a}_{p_1} f_1)(\tilde{a} + \tilde{a}_{p_2}\tilde{u}_2) \\ - \frac{1}{4}(\tilde{a}_{p_1}\tilde{u}_2 + \tilde{a}_{p_2} f_1)^2.\end{aligned} \tag{4.2.3}$$

Idea of Proving Theorem 1.3.1: Our approach below is to evaluate the above equalities at solutions with prescribed boundary jet. After exploiting the algebraic structure of these matrix equalities, we eventually deduce $a = \tilde{a}$ at this boundary jet. By varying the jet, we obtain equality along a subspace of $\mathbb{R} \times \mathbb{R}^n$. By using Euclidean isometry invariance to rotate this subspace, we conclude the full result. We consider the dimension cases $n = 2$ and $n = 3$ separately below.

Proof of Theorem 1.3.2: This follows from Theorem 1.3.1 since $a - \tilde{a}$ is an analytic function which vanishes on an open set.

Idea of Proving Theorem 1.3.3: This is proved exactly as in Theorem 1.3.1 below. The only difference is we can prescribe an arbitrarily large boundary jet, by Proposition 4.1.4, which ensures global uniqueness.

Idea of Proving Corollary 1.3.1: This is identical to proving Theorem 1.3.1, except that the homogeneity assumption $\partial a(x, s, p)/\partial x = 0$ is used in the last step of Euclidean invariance. In Remarks 4.2.1 and 4.2.2 for dimension $n = 3$ and Remark 4.2.3 for dimension $n = 2$, we show how this can be handled to obtain Corollary 1.3.1.

The case $n \geq 3$

Fixing $s \in \mathbb{R}$, we choose $p \in \mathbb{R}^{n-1} \times \{0\}$ such that $0 < |p| < \Pi(s)$. Using Proposition 4.1.3, we can find $f \in C^\infty(\partial\Omega)$ such that the solution $u = u[f]$ to (1.3.2) has the following properties:

$$f(0) = s, \quad \nabla f(0) = p, \quad \frac{\partial u}{\partial x^n}(0) = 0, \quad \|f - s\|_{C^{2,\alpha}(\partial\Omega)} \leq \Pi_1(s). \tag{4.2.4}$$

By Proposition 4.1.1, the linearization thus exists at $u = u[f]$, and we can invoke Proposition 4.1.6.

We now use this same f as boundary data to create solution $\tilde{u}[f]$ to (1.3.2) with $\tilde{a}$ in place of a. Since $\Gamma_a = \Gamma_{\tilde{a}}$, we infer at $x = 0$,

$$\tilde{a}\frac{\partial \tilde{u}}{\partial x^n}(0) = a\frac{\partial u}{\partial x^n}(0) = 0, \tag{4.2.5}$$

and hence $\tilde{u}_n(0) = 0$ as well. Substituting this into (4.2.1) yields

$$a\delta_{ij} + \frac{1}{2}(a_{p_j}p_i + a_{p_i}p_j)|_{(s,p)} = \tilde{a}\delta_{ij} + \frac{1}{2}(\tilde{a}_{p_j}p_i + \tilde{a}_{p_i}p_j)|_{(s,p)}, \quad i, j < n. \tag{4.2.6}$$

Let us put $\mathcal{A} = a - \tilde{a}$:

$$\mathcal{A}\delta_{ij} + \frac{1}{2}(\mathcal{A}_{p_j}p_i + \mathcal{A}_{p_i}p_j)|_{(s,p)} = 0, \quad i,j < n. \tag{4.2.7}$$

To find the spectrum, we first note that $\mathcal{A}$ is the eigenvalue in directions orthogonal to $D_p\mathcal{A}$ and p. Afterwards, we expand an eigenvector in this basis and diagonalize the resulting 2×2 matrix. The spectrum of the matrix on the left hand side of (4.2.7), listed in increasing order, is as follows

$$\mathcal{A} + \frac{1}{2}p\cdot\nabla'_p\mathcal{A} - \frac{1}{2}|p||\nabla'_p\mathcal{A}|, \quad \mathcal{A},\ldots,\mathcal{A}, \quad \mathcal{A} + \frac{1}{2}p\cdot\nabla'_p\mathcal{A} + \frac{1}{2}|p||\nabla'_p\mathcal{A}|, \tag{4.2.8}$$

where $\nabla'_p\mathcal{A} = (\mathcal{A}_{p_1},\ldots,\mathcal{A}_{p_{n-1}})$, and middle eigenvalue $\mathcal{A}$ is listed $n-3$ times. From (4.2.7), these eigenvalues are all zero. Subtracting the largest and smallest eigenvalues yields

$$|\nabla'_p\mathcal{A}(s,p)| = 0, \qquad s\in\mathbb{R}, \quad p\in\mathbb{R}^{n-1}\times\{0\}, \quad |p| < \Pi(s) \tag{4.2.9}$$

so the smallest eigenvalue yields the result on a hyperplane, even if $n = 3$,

$$\mathcal{A}(s,p) = 0, \qquad s\in\mathbb{R}, \quad p\in\mathbb{R}^{n-1}\times\{0\}, \quad |p| < \Pi(s). \tag{4.2.10}$$

We can also conclude this equality from the vanishing of the middle eigenvalue in (4.2.8). Now choose any Euclidean isometry $I = R\circ T$ such that $I^{-1}(\Omega)$ sits above the origin. Then the above argument applies to the rotated PDE (4.1.28), and we obtain

$$R_*\mathcal{A}(s,p) = 0, \qquad s\in\mathbb{R}, \quad p\in\mathbb{R}^{n-1}\times\{0\}, \quad |p| < \Pi(s). \tag{4.2.11}$$

The set of such rotations R is all of $O(n)$. Recalling that

$$R_*\mathcal{A}(s,p) = \mathcal{A}(s,R^{-1}p)$$

and varying $R\in O(n)$, we conclude in general that

$$\mathcal{A}(s,p) = 0, \qquad s\in\mathbb{R}, \quad p\in\mathbb{R}^{n}, \quad |p| < \Pi(s). \tag{4.2.12}$$

This completes the proof of Theorem 1.3.1 if $n \geq 3$.

Remark 4.2.1. Theorem 1.3.1 still holds if $n \geq 3$ and we replace $a(s,p)$ and $\tilde{a}(s,p)$ by

$$F(x,s,p,b(s,p)) \quad \text{and} \quad F(x,s,p,\tilde{b}(s,p)),$$

where $t \mapsto F(x,s,p,t)$ is injective for each fixed (x,s,p). To see this, we merely observe that (4.2.10) and the injectivity of F imply

$$b(s,p) = \tilde{b}(s,p), \qquad s \in \mathbb{R}, \quad p \in \mathbb{R}^{n-1} \times \{0\}, \quad |p| < \Pi(s).$$

The Euclidean invariance argument completes the proof.

Remark 4.2.2. Another non-homogeneous extension. Let a be non-homogeneous with a symmetry in one direction:

$$a = a(x^1, \ldots, x^{n-1}, s, p).$$

If Ω is convex, then Theorem 1.3.1 still holds. Indeed, instead choosing a general vector $p \in \mathbb{R}^n$ with $|p| < \Pi(s)$, we simply repeat the proof starting at (4.2.7) and deduce

$$\mathcal{A}(0,0,\ldots,0,s,p) = 0$$

directly. Since Ω is convex, we can place Ω above a hyperplane at any point $x \in \partial\Omega$, so a Euclidean invariance argument means there holds

$$\mathcal{A}(\pi_{\mathbb{R}^{n-1}\times\{0\}}(x), s, p) = 0$$

for any $x \in \partial\Omega$. Since $\partial\mathcal{A}/\partial x^n = 0$, this completes the proof. It is not clear whether this works for $n = 2$ or Ω non-convex.

The case $n = 2$

Fixing $s \in \mathbb{R}$, we choose $p \in \mathbb{R}$ such that $0 < |p| < \Pi(s)$. Using Proposition 4.1.3, we can find $f \in C^\infty(\partial\Omega)$ such that the solution $u = u[f]$ to (1.3.2) has the following properties:

$$f(0) = s, \quad \frac{\partial f}{\partial x^1}(0) = p, \quad \frac{\partial u}{\partial x^2}(0) = 0, \quad \|f - s\|_{C^{2,\alpha}(\partial\Omega)} \leq \Pi_1(s). \tag{4.2.13}$$

We now use this same f as boundary data to create solution $\tilde{u}[f]$ to (1.3.2) with $\tilde{a}$ in place of a. Since $\Gamma_a = \Gamma_{\tilde{a}}$, we infer at $x = 0$,

$$\tilde{a}\frac{\partial \tilde{u}}{\partial x^2}(0) = a\frac{\partial u}{\partial x^2}(0) = 0, \tag{4.2.14}$$

and hence $\tilde{u}_2(0) = 0$ as well. Substituting into (4.2.2) yields

$$pa_{p_2}(s,p,0) = p\tilde{a}_{p_2}(s,p,0). \tag{4.2.15}$$

Substituting this and $u_2 = \tilde{u}_2 = 0$ into (4.2.3) yields

$$(a + pa_{p_1})a|_{(s,p,0)} = (\tilde{a} + p\tilde{a}_{p_1})\tilde{a}|_{(s,p,0)}. \tag{4.2.16}$$

Putting

$$\mathcal{A}(s,p) := a(s,p,0)^2 - \tilde{a}(s,p,0)^2, \tag{4.2.17}$$

we deduce

$$\mathcal{A} + \frac{1}{2}p\mathcal{A}_p = 0, \tag{4.2.18}$$

and hence

$$\mathcal{A}(s,p) = c(s)|p|^{-2}. \tag{4.2.19}$$

Since $\mathcal{A}(s,p)$ is smooth, we conclude $c = 0$:

$$\mathcal{A}(s,p) = 0, \qquad s \in \mathbb{R}, \quad |p| < \Pi(s). \tag{4.2.20}$$

As in the proof for $n \geq 3$, since $O(2) \cdot e_1 = S^1$, where $e_1 = (1,0)$ and $O(2)$ is the group of orthogonal transformations on $\mathbb{R}^2$, invoking Euclidean isometries completes the proof.

Remark 4.2.3. As in Remark 4.2.1, Theorem 1.3.1 still holds for $n = 2$ if we replace a and $\tilde{a}$ by

$$F(x,s,p,b(s,p)) \quad \text{and} \quad F(x,s,p,\tilde{b}(s,p)),$$

where $t \mapsto F(x,s,p,t)$ is injective for each fixed (x,s,p). Indeed, from (4.2.20), we obtain

$$F(0,s,p,0,b(s,p,0))^2 = F(0,s,p,0,\tilde{b}(s,p,0))^2, \qquad s,p \in \mathbb{R}, \quad |p| < \Pi(s).$$

By (1.3.4), we have $F \geq 1$, so the map $t \mapsto F(x,s,p,t)^2$ is also injective for fixed (x,s,p), and we obtain

$$b(s,p,0) = \tilde{b}(s,p,0), \quad s,p \in \mathbb{R}, \quad |p| < \Pi(s).$$

As before, the Euclidean invariance argument completes the proof.

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