

©Copyright 2017
Ivana Thng

A Survey of Tverberg Type Problems

Ivana Thng

A thesis
submitted in partial fulfillment of the
requirements for the degree of

Master of Science

University of Washington

2017

Committee:

Isabella Novik

Sara Billey

Program Authorized to Offer Degree:

Mathematics

University of Washington

Abstract

A Survey of Tverberg Type Problems

Ivana Thng

Chair of the Supervisory Committee:
Professor Isabella Novik
Mathematics

Tverberg's theorem, which celebrates its fiftieth anniversary this year, is a central result in the fields of discrete geometry and topological combinatorics. Proved in 1966, it was a major step in solving questions whether, given a complex, all affine (or more generally, continuous) maps into some Euclidean space have some specified intersection property. Many other extensions and variations stem from this first result, such as the topological Tverberg conjecture and the colorful Tverberg theorem. Much work is still being done to generalize and extend Tverberg's theorem, which has resulted in several recent and major breakthroughs. Most notably, Frick's surprising counterexample to the topological Tverberg conjecture was only discovered in 2015, and the "constraint method" used to construct the counterexample lends itself to many other applications in proving different extensions of the topological Tverberg conjecture. In this thesis, we will look at some classical theorems that inspired Tverberg's theorem, the current state of affairs vis-à-vis the Tverberg conjecture, and other closely related problems.

Contents

1	Introduction	1
2	Classical Radon, Helly, and Caratheodory theorems	1
3	Deleted joins and the continuous Radon theorem	5
4	The Borsuk-Ulam theorem and generalizations	7
4.1	The classical version	7
4.2	The Borsuk-Ulam property for finite groups	10
5	Tverberg's theorem	15
6	The topological Tverberg conjecture	18
6.1	$r = 2$	19
6.2	r a prime number	19
6.3	$r = p^k$ for p prime	20
7	The number of Tverberg partitions	22
7.1	The Sierksma configuration	22
7.2	Smaller cases	23
8	Miscellaneous problems	24

List of Symbols

$[n]$ The set of natural numbers from 1 to n

σ^n The n -dimensional simplex

$\mathbf{conv}(S)$ The convex hull of all the points in S

\mathbb{S}^n The n -dimensional sphere

\mathbb{B}^n The n -dimensional ball

Q^{n+1} The $(n + 1)$ -dimensional cross polytope

Γ^n The n -dimensional geometric simplicial complex associated with the boundary of the $(n + 1)$ -dimensional cross polytope

$|\Delta|$ The geometric realization of a simplicial complex Δ

ACKNOWLEDGMENTS

I would like to thank Isabella Novik for her invaluable support and guidance throughout the crafting of this thesis. She made the learning and understanding of mathematics every bit as enjoyable as it should be.

1 Introduction

In this paper, we will delve into the history of Tverberg's theorem, starting with the classical theorems of Radon, Helly, and Caratheodory that Tverberg sought to generalize (see Section 2). Since the $r = 2$ case of the topological Tverberg conjecture is essentially the continuous Radon theorem, we prove the continuous Radon theorem in Section 3 using tools from equivariant homology, concentrating on the \mathbb{Z}_2 case. In Section 4, we discuss Borsuk-Ulam's theorem and Tucker's lemma, as well as a generalization of the Borsuk-Ulam theorem for finite groups. This generalization will later come in useful in the proof of the prime power case of the topological Tverberg conjecture. In Section 5, we discuss Tverberg's theorem and Sarkaria's elegant proof, which uses only linear algebra and the Colorful Caratheodory theorem. In Section 6, we state the topological Tverberg conjecture, highlighting Frick's counterexample, as well as give an overview of the the different cases for which the conjecture has been proved. In Section 7, we consider Sierksma's related problem on the number of Tverberg partitions, if they exist. Finally, we conclude with other open problems and extensions stemming from Tverberg's theorem.

The proofs of many tools used to prove harder theorems are not in the scope of this paper, so where proofs are omitted, references will be provided in their place. Note that throughout the paper, $|\Delta|$ will be used to refer to the geometric realization of a simplicial complex Δ , and where we use τ to represent a face in Δ , τ may refer to the geometric face, the face as a set of vertices, or the equivalence class of all tuples of vertices of the face in a given orientation. Each incarnation τ assumes should hopefully be clear from the context.

2 Classical Radon, Helly, and Caratheodory theorems

Tverberg's theorem is typically seen as a far-reaching generalization of Radon's theorem. To motivate our discussion of Tverberg's theorem, we present Radon's theorem and some related results in this section.

Theorem 2.1 (Radon) *Let x_1, \dots, x_n be points in \mathbb{R}^d , $n \geq d + 2$. Then there is a partition S, T of $\{x_1, \dots, x_n\}$ such that $\text{conv}(S) \cap \text{conv}(T) \neq \emptyset$.*

Proof. Since there are $n > d + 1$ points in \mathbb{R}^d , the points must be affinely dependent, and so there exists $\lambda_1, \dots, \lambda_n$, not all zero, such that

$$\sum_{i=1}^n \lambda_i x_i = 0 \text{ and } \sum_{i=1}^n \lambda_i = 0. \quad (1)$$

We can then partition the points $\{x_1, \dots, x_n\}$ into subsets S and T where

$$S = \{x_i | \lambda_i > 0\},$$

$$T = \{x_j | \lambda_j \leq 0\}.$$

Then by (1), we have

$$\sum_{x_i \in S} \lambda_i x_i = - \sum_{x_j \in T} \lambda_j x_j \text{ and } \sum_{x_i \in S} \lambda_i = - \sum_{x_j \in T} \lambda_j.$$

Define $t = \sum_{x_i \in S} \lambda_i$. Note that $t > 0$ since the λ_i 's are not all zero. Then consider a point $y \in \text{conv}(S)$ given by

$$y := \sum_{x_i \in S} \frac{\lambda_i}{t} x_i = - \sum_{x_j \in T} \frac{\lambda_j}{t} x_j.$$

Since $-\sum_{x_j \in T} \lambda_j = \sum_{x_i \in S} \lambda_i = t$, we obtain that $-\sum_{x_j \in T} \frac{\lambda_j}{t} = 1$ and that $y \in \text{conv}(T)$ as well. Thus, $\text{conv}(S) \cap \text{conv}(T) \neq \emptyset$. \square

The technique of partitioning the points $\{x_1, \dots, x_n\}$ according to the signs of their coefficients λ_i in the affine dependency (1) will later motivate Sarkaria's proof of Tverberg's theorem.

We can use Radon's theorem to prove Helly's Theorem:

Theorem 2.2 (Helly) *Given a family $\mathcal{K} = \{K_1, K_2, \dots, K_n\}$, $n \geq d + 1$ of convex sets in \mathbb{R}^d , if every $d + 1$ of the sets have a point in common, then all of the sets have a point in common.*

Proof. If $n = d + 1$, then the theorem is trivially true. This is our inductive base case. Now suppose $n > d + 1$. Given any $(n - 1)$ -subfamily of \mathcal{K} of convex sets in \mathbb{R}^d , we know that if every $d + 1$ of the sets have a point in common, then they all have a point in common by the inductive hypothesis. So it suffices to prove the following: If every $n - 1$ of the convex sets in \mathcal{K} have a point in common, then all of the sets have a point in common.

For each $i \in [n]$, pick $x_i \in \bigcap \{K_j : 1 \leq j \leq n, j \neq i\}$ to be a point in the intersection of all $K_j \in \mathcal{K} \setminus \{K_i\}$. Now we have a set of n points in \mathbb{R}^d and $n \geq d + 2$ (by assumption) - so we can apply Radon's theorem. Consider a Radon partition S, T of these points such that $\text{conv}(S) \cap \text{conv}(T) \neq \emptyset$. Suppose $z \in \text{conv}(S) \cap \text{conv}(T)$. Then since $z \in \text{conv}(x_i : x_i \in S)$ and $x_i \in K_j$ for all $i \neq j$, it follows that z is also in K_j for each $x_j \notin S$. Similarly, $z \in K_i$ for each $x_i \notin T$. But since $S \cap T = \emptyset$, $z \in K_i$ for all $K_i \in \mathcal{K}$ and all the sets in \mathcal{K} have a point in common. \square

Now recall that the convex hull of $S = \{x_1, \dots, x_n\}$ is defined to be

$$\text{conv}(S) = \left\{ \sum_i \lambda_i x_i : \lambda_i > 0, \sum_i \lambda_i = 1, x_i \in S \right\}.$$

This leads to the natural question: Is there a bound on the number of summands we need to consider for this equality of sets to hold? Caratheodory's theorem provides an upper bound:

Theorem 2.3 (Caratheodory) *For $A \subset \mathbb{R}^d$, if $x \in \text{conv}(A)$, then $x \in \text{conv}(B)$ for some $B \subset A$, with $|B| \leq d + 1$.*

In other words, considering up to $d + 1$ summands is always enough.

Proof. As in Helly's Theorem, this can be proved via Radon's theorem. Given $x \in \text{conv}(A)$, there exist a finite subset $\{a_1, \dots, a_n\} \subseteq A$ and $\lambda_i > 0$ for $i = 1, \dots, n$ with $\sum_{i=1}^n \lambda_i = 1$ such that $x = \sum_i \lambda_i a_i$. Suppose that this representation of x was chosen with n as small as possible. We claim that $n \leq d + 1$.

Assume towards a contradiction that $n > d + 1$. Then by Radon's theorem applied to $\{a_1, \dots, a_n\}$, we have a partition S, T of the a_i 's such that $\text{conv}(S) \cap \text{conv}(T) \neq \emptyset$, i.e. there exists some $z \in \text{conv}(S) \cap \text{conv}(T)$,

$$z = \sum_{a_i \in S} \gamma_i a_i = \sum_{a_j \in T} \gamma_j a_j \text{ such that}$$

$$\sum_{a_i \in S} \gamma_i = \sum_{a_j \in T} \gamma_j = 1,$$

where $\gamma_i \geq 0$ for $i \in [n]$. Now define $\eta_i = \gamma_i$ for $a_i \in S$ and $\eta_j = -\gamma_j$ for $a_j \in T$. Then we have

$$\sum_{i=1}^n \eta_i a_i = 0 \text{ and } \sum_{i=1}^n \eta_i = 0.$$

Find an index $i_0 \in [n]$ such that

$$\epsilon := \frac{\lambda_{i_0}}{\eta_{i_0}} = \min \left\{ \frac{\lambda_i}{\eta_i} \text{ for } i \in [n] \text{ such that } \frac{\lambda_i}{\eta_i} > 0 \right\}.$$

Then define coefficients $c_i = \lambda_i - \epsilon \eta_i$ for $i \in [n]$. Clearly $c_i \geq 0$ for all $i \in [n]$, $c_{i_0} = 0$, and

$$\sum_{i=1}^n c_i = \sum_{i=1}^n (\lambda_i - \epsilon \eta_i) = \sum_{i=1}^n \lambda_i - \epsilon \sum_{i=1}^n \eta_i = 1 - 0 = 1.$$

Note further that

$$x = \sum_{i=1}^n \lambda_i a_i = \sum_{i=1}^n c_i a_i + \epsilon \sum_{i=1}^n \eta_i a_i = \sum_{i=1}^n c_i a_i$$

with $c_{i_0} = 0$ contradicting the minimality of n . So $n \leq d + 1$. □

We now present a more general version of Caratheodory's theorem, known as the Colorful Caratheodory theorem. This will be used in Sarkaria's proof of Tverberg's theorem.

Theorem 2.4 (Colorful Caratheodory) *Let S_1, S_2, \dots, S_{d+1} be $d + 1$ sets in \mathbb{R}^d . Suppose $x \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \dots \cap \text{conv}(S_{d+1})$. Then there are $d + 1$ points $x_1 \in S_1, x_2 \in S_2, \dots, x_{d+1} \in S_{d+1}$ such that $x \in \text{conv}(x_1, x_2, \dots, x_{d+1})$.*

When $S_1 = S_2 = \dots = S_{d+1}$, this statement reduces to Caratheodory's theorem. If we treat each S_i as a color, then the theorem concludes that x will be contained in the convex hull of a set of points, one of each color.

Proof. For each choice of $x_1 \in S_1, \dots, x_{d+1} \in S_{d+1}$, consider the distance between x and $\text{conv}(x_1, \dots, x_{d+1})$. Let ρ be the minimum distance obtained out of all possible choices of $\{x_i\}_{i \in [d+1]}$, and let p_1, \dots, p_{d+1} be the points in S_1, \dots, S_{d+1} such that $\rho = \text{dist}(x, \text{conv}(p_1, \dots, p_{d+1}))$. Let $z \in \text{conv}(p_1, \dots, p_{d+1})$ be the point that attains this minimum distance, i.e. $\text{dist}(x, z) = \rho$. To complete the proof, it suffices to show that $\rho = 0$, and hence $z = x$.

Assume for contradiction that $z \neq x$. Consider the hyperplane H through z that is perpendicular to the line between z and x . Let H^- and H^+ be the closed half-spaces defined by H . We claim that if $x \in H^-$, then $\text{conv}(p_1, \dots, p_{d+1}) \subset H^+$.

Suppose there exists $y \in \text{conv}(p_1, \dots, p_{d+1})$ in H^- . Then $[y, z] \subseteq \text{conv}(p_1, \dots, p_{d+1})$. Also, the line segments $[y, z]$ and $[x, z]$ span an angle less than $\frac{\pi}{2}$ and there exists a point q in the interval $[y, z) \subset \text{conv}(p_1, \dots, p_{d+1})$ that is closer to x than z , which contradicts our assumption that $z \in \text{conv}(p_1, \dots, p_{d+1})$ gives the minimum distance to x .

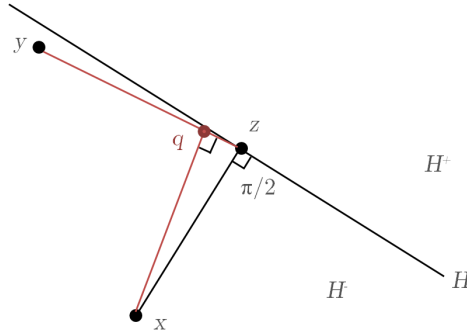


Figure 1: q is nearer to x than z is

Thus H is a supporting hyperplane of $\text{conv}(x_1, \dots, x_{d+1})$. Since $z \in H$, z is contained in the convex set $H \cap \text{conv}(x_1, \dots, x_{d+1})$ that is at most $(d - 1)$ -dimensional. It then follows from Caratheodory's theorem that z is contained in the convex hull of d of the p_i 's, i.e.

$$z \in \text{conv}(p_1, \dots, \widehat{p}_j, \dots, p_{d+1}) \text{ for some } j \in [d + 1].$$

Consider the set S_j . Clearly $S_j \not\subset H^+$ since $x \in \text{conv}(S_j)$, $x \in H^-$. So there exists some $y_j \in S_j$ such that $y_j \in H^-$.

Now replace $\text{conv}(p_1, \dots, p_{d+1})$ with $\text{conv}(p_1, \dots, p_{j-1}, y_j, p_{j+1}, \dots, p_{d+1})$. Then since $y_j \in H^-$, some point on the half open interval $[y_j, z)$ will be closer to x than z , which leads to a contradiction. \square

3 Deleted joins and the continuous Radon theorem

In order to state and prove the continuous version of Radon's theorem, we need to look at deleted joins. For K a simplicial complex, we first define the join $K * K$.

Definition 3.1 Let K be a simplicial complex on the vertex set V . Let K_1 and K_2 be isomorphic copies of K on disjoint vertex sets V_1 and V_2 , so $K_1 = \varphi_1(K)$ and $K_2 = \varphi_2(K)$ where φ_1 and φ_2 are isomorphisms. Define the **twofold join** of K to be

$$K * K := \{\varphi_1(\tau) \sqcup \varphi_2(\sigma) : \tau, \sigma \in K.\}$$

Further, the **twofold deleted join** of K is given by

$$K_{\Delta}^{*2} = K *_{\Delta} K := \{\varphi_1(\tau) \sqcup \varphi_2(\sigma) : \tau, \sigma \in K, \tau \cap \sigma = \emptyset\} \subseteq K * K.$$

Since the vertex sets V_1 and V_2 are disjoint, for simplicity we will always refer to elements of $K *_{\Delta} K$ as $\tau \cup \sigma$ where $\tau \in \varphi_1(K)$ and $\sigma \in \varphi_2(K)$. Note that \mathbb{Z}_2 acts freely on the deleted join K_{Δ}^{*2} by sending $\tau_1 \cup \tau_2$ to $\tau_2 \cup \tau_1$. To show that there exists a \mathbb{Z}_2 -equivariant map from a topological space X to \mathbb{S}^k , we define the \mathbb{Z}_2 index of X :

Definition 3.2 If X is a topological space with a \mathbb{Z}_2 -action, then the **\mathbb{Z}_2 -index** of X is defined to be

$$\text{ind}(X) = \min\{k \geq 0 : \text{there exists a } \mathbb{Z}_2\text{-map } X \rightarrow \mathbb{S}^k\}.$$

For the next lemma, and in subsequent proofs throughout the paper, the definition of the cross polytope will come in very useful:

Definition 3.3 The $(n+1)$ -dimensional **cross polytope** is defined as

$$Q^{n+1} = \text{conv}(\pm e_1, \dots, \pm e_{n+1})$$

where the e_i 's are the standard basis vectors in \mathbb{R}^{n+1} .

Lemma 3.4 Let K be the n -dimensional simplex, i.e., the abstract simplicial complex given by $K = \{\tau : \tau \subseteq [n+1]\}$. Then the deleted join $K *_{\Delta} K$ is \mathbb{Z}_2 -isomorphic to the boundary complex Γ^n of the $(n+1)$ -dimensional cross polytope Q^n and $\text{ind}(|K *_{\Delta} K|) = n$.

Proof. Consider the following map

$$g : K *_{\Delta} K \rightarrow \Gamma^n$$

$$\tau_1 \cup \tau_2 \rightarrow \{e_i : i \in \tau_1\} \cup \{-e_i : i \in \tau_2\}.$$

The map g is clearly a \mathbb{Z}_2 -equivariant isomorphism of complexes and since Γ^n is homeomorphic to \mathbb{S}^n , $\text{ind}(|K *_{\Delta} K|) \leq n$. But note that $\text{ind}(\mathbb{S}^n) = n$ by Theorem 4.1(a), which we will prove later. Suppose there is a \mathbb{Z}_2 -equivariant map $h : K *_{\Delta} K \rightarrow \mathbb{S}^m$ where $m < n$. Then since g is a \mathbb{Z}_2 isomorphism, $h \circ g^{-1}$ would then be a \mathbb{Z}_2 -map from $\mathbb{S}^n \rightarrow \mathbb{S}^m$, which is impossible since $\text{ind}(\mathbb{S}^n) = n > m = \text{ind}(\mathbb{S}^m)$. \square

Recall that Theorem 2.1 states that for any n points x_1, \dots, x_n in \mathbb{R}^d , $n \geq d + 2$, there is a partition S, T of $\{x_1, \dots, x_n\}$ such that $\text{conv}(S) \cap \text{conv}(T) \neq \emptyset$. In particular, we may treat $\{x_1, \dots, x_n\}$ as the images of the vertices of an $(n - 1)$ -simplex σ^{n-1} under an affine map f , so any subset of $\text{vert}(\sigma^{n-1})$ under f is the vertex set of a face of $f(\sigma^{n-1})$. Instead of considering affine maps like f , it turns out that the statement of the theorem holds for continuous maps in general:

Theorem 3.5 (Continuous Radon) *For any continuous map $f : \sigma^{d+1} \rightarrow \mathbb{R}^d$, there exist two disjoint faces τ_1, τ_2 of σ^{d+1} such that $f(\tau_1) \cap f(\tau_2) \neq \emptyset$.*

Theorem 2.1 is then the special case where f is the unique affine map taking the vertices of σ^{d+1} to $\{x_1, \dots, x_{d+2}\}$, so the continuous Radon theorem implies Radon's theorem. To prove the continuous Radon theorem, we need the following proposition:

Proposition 3.6 *Let K be a simplicial complex. If $\text{ind}(|K *_{\Delta} K|) > d$, then for every continuous map $f : |K| \rightarrow \mathbb{R}^d$, there exist disjoint faces $\tau_1, \tau_2 \in K$ such that $f(|\tau_1|) \cap f(|\tau_2|) \neq \emptyset$.*

Proof. See page 101 in [6]. \square

Now we can prove Theorem 3.5.

Proof. Let $K = \{\tau : \tau \subseteq [d+2]\}$ be the abstract simplicial complex with $|K| = \sigma^{d+1}$. Then by Lemma 3.4, we know that $\text{ind}(|K *_{\Delta} K|) = d + 1$. But Proposition 3.6 tells us that since $\text{ind}(|K *_{\Delta} K|) = d + 1 > d$, for every continuous map $f : \sigma^{d+1} \rightarrow \mathbb{R}^d$, there exist disjoint faces $\tau_1, \tau_2 \in \sigma^{d+1}$ such that $f(\tau_1) \cap f(\tau_2) \neq \emptyset$. \square

There is a far-reaching generalization of Radon's theorem known as Tverberg's theorem, which we will discuss in Section 5, and in Section 6 we will discuss the topological Tverberg conjecture, which is a generalization of Tverberg's theorem to continuous maps analogous to how the continuous Radon theorem generalizes Radon's theorem.

4 The Borsuk-Ulam theorem and generalizations

The Borsuk-Ulam theorem is an example of a topological statement that can be proved both by homological and combinatorial methods. In this section, we present a combinatorial proof via Tucker's lemma, as well as a generalization of the Borsuk-Ulam property for groups other than \mathbb{Z}_2 .

4.1 The classical version

For the rest of the paper, let \mathbb{S}^n denote the n -dimensional sphere and \mathbb{B}^n the n -dimensional ball. The Borsuk-Ulam theorem can be stated in any of the following ways:

Theorem 4.1 (Borsuk-Ulam) *The following statements hold, and are equivalent:*

- (a) *If $f : \mathbb{S}^n \rightarrow \mathbb{S}^m$ is a continuous antipodal map, then $n \leq m$.*
- (b) *If $f : \mathbb{S}^n \rightarrow \mathbb{R}^n$ is a continuous antipodal map, then there exists an $x \in \mathbb{S}^n$ such that $f(x) = 0$.*
- (c) *If $f : \mathbb{S}^n \rightarrow \mathbb{R}^n$ is a continuous map, then there exists $x \in \mathbb{S}^n$ such that $f(x) = f(-x)$.*
- (d) *If \mathbb{S}^n is covered by $n + 1$ subsets U_1, \dots, U_{n+1} such that each of the U_i 's is open or closed, then one of the sets contains an antipodal pair of points, i.e. there exist an $i \in [n + 1]$ and $x \in \mathbb{S}^n$ such that $x, -x \in U_i$.*

To show that the statements hold, we require Tucker's lemma:

Lemma 4.2 (Tucker) *The following statements hold, and are equivalent:*

- (i) *Let K be an antipodally symmetric subdivision of Γ^n and let $\lambda : \text{vert}(K) \rightarrow \{\pm 1, \dots, \pm n\}$ be any antipodally symmetric labeling of K , i.e. $\lambda(v) = -\lambda(v)$ for all v . Then there exists a complementary edge, i.e. an edge $uv \in K$ such that $\lambda(u) + \lambda(v) = 0$.*
- (ii) *Let T be a triangulation of \mathbb{B}^n that is antipodally symmetric on the boundary. Then there is no continuous map $g : \text{vert}(T) \rightarrow \text{vert}(\sigma^{n-1})$ that is both a simplicial map of T into σ^{n-1} and antipodal on the boundary.*

For a proof of Tucker's lemma, see page 36 in [9]. In addition to Tucker's lemma, we will need the notion of barycentric subdivision:

Definition 4.3 Given a simplicial complex Δ , a complex Δ' is a **subdivision** of Δ if $\text{vert}(\Delta) \subseteq \text{vert}(\Delta')$ and $|\Delta| = |\Delta'|$.

Definition 4.4 Given an abstract simplicial complex Δ , the **first barycentric subdivision** of Δ is defined to be the complex

$$\text{sd } \Delta = \{ \{ \sigma_0, \dots, \sigma_k \} : \sigma_0, \dots, \sigma_k \in \Delta \setminus \{ \emptyset \}, \sigma_0 \subset \sigma_1 \subset \dots \subset \sigma_k \},$$

i.e. the simplices of Δ become the vertices of $\text{sd } \Delta$. The k -th barycentric subdivision of Δ is written $\text{sd}^k \Delta$ and is the result of iterating the procedure above k times.

Now, we have everything we need to prove Theorem 4.1.

Proof. First, we prove the statement of Theorem 4.1(b), and then show that the four statements are in fact equivalent.

Denote by Γ^n the boundary complex of the $(n+1)$ -dimensional cross polytope. Since $|\Gamma^n|$ and \mathbb{S}^n are homeomorphic, we can assume towards a contradiction that there exists an antipodal map $f : |\Gamma^n| \rightarrow \mathbb{R}^n$ that does not have a zero. So there exists some $\epsilon > 0$ such that $\|f\|_\infty \geq \epsilon$. Since we assumed f to be continuous, there exists a k -subdivision $K = \text{sd}^k \Gamma^n$ such that for all edges uv of K , we have

$$\|f(u) - f(v)\|_\infty < \epsilon.$$

For any vertex $v \in \text{vert}(K)$, define

$$i(v) = \min\{i : |f_i(v)| \geq \epsilon\}$$

where $f_i(v)$ denotes the i -th coordinate of $f(v)$. This is well-defined since $\|f\|_\infty \geq \epsilon$ implies that for every $x \in |\Gamma^n|$, there is some coordinate i such that $|f_i(x)| \geq \epsilon$. Then we can construct an antipodally symmetric labeling $\lambda : \text{vert}(K) \rightarrow \{\pm 1, \dots, \pm n\}$:

$$\lambda(v) = \begin{cases} +i(v) & \text{if } f_{i(v)}(v) \geq \epsilon \\ -i(v) & \text{if } f_{i(v)}(v) \leq -\epsilon. \end{cases}$$

Since f is antipodal, $|f_i(-v)| = |-f_i(v)| = |f_i(v)|$ which implies $i(-v) = i(v)$. Antipodality of f again implies $\lambda(-v) = -\lambda(v)$. So λ is in fact an antipodally symmetric labeling of the vertices of K .

By Tucker's lemma, there exists some edge uv in K such that $\lambda(u) + \lambda(v) = 0$, i.e. there exists an edge uv in K such that for some $i \in [n]$, $\lambda(u) = +i$ and $\lambda(v) = -i$. Then $f_i(u) \geq \epsilon$, $f_i(v) \leq -\epsilon$, which contradicts $\|f(u) - f(v)\|_\infty < \epsilon$.

Hence, Theorem 4.1(b) holds. Now we show the equivalence between the four statements.

(a) \implies (b): Suppose we have a continuous antipodal map $f : \mathbb{S}^n \rightarrow \mathbb{R}^n$ with no zero. Then we can define a map

$$g : \mathbb{S}^n \rightarrow \mathbb{S}^{n-1} \\ x \mapsto \frac{f(x)}{\|f(x)\|}$$

which is also continuous and antipodal. But $n-1 < n$, contradicting (a).

(b) \implies (c): Let $f : \mathbb{S}^n \rightarrow \mathbb{R}^n$ be a continuous map. Define another continuous map

$$g : \mathbb{S}^n \rightarrow \mathbb{R}^n \\ x \mapsto f(x) - f(-x).$$

The map g is clearly antipodal. Part (b) implies that g has a zero at some $x \in \mathbb{S}^n$, which in turn implies that $f(x) = f(-x)$ by our definition of g .

(c) \implies (d): Cover \mathbb{S}^n with $n + 1$ subsets U_1, \dots, U_{n+1} such that each of the U_i 's is either open or closed. Reindex them such that none of the first n of the U_i 's contain an antipodal pair of points. We want to show that U_{n+1} contains an antipodal pair of points.

We claim that $x \in U_i$ implies that $\text{dist}(-x, U_i) > 0$ for each $i \in [n]$ and each $x \in \mathbb{S}^n$.

- Case 1: U_i is closed
Then $x \in U_i$ and $-x \notin U_i$, so $\text{dist}(-x, U_i) > 0$.
- Case 2: U_i is open
Suppose $x \in U_i$, then since $\mathbb{S}^n \setminus U_i$ is closed and $x \notin \mathbb{S}^n \setminus U_i$, $\text{dist}(x, \mathbb{S}^n \setminus U_i) > 0$. Since $-U_i \subset \mathbb{S}^n \setminus U_i$,

$$\text{dist}(-x, U_i) = \text{dist}(x, -U_i) \geq \text{dist}(x, \mathbb{S}^n \setminus U_i) > 0.$$

Now we will exhibit an antipodal pair of points in U_{n+1} . Consider the continuous map

$$f : \mathbb{S}^n \rightarrow \mathbb{R}^n \\ x \mapsto \begin{pmatrix} \text{dist}(x, U_1) \\ \vdots \\ \text{dist}(x, U_n) \end{pmatrix}.$$

By (c) we know that there exists $x \in \mathbb{S}^n$ such that $f(x) = f(-x)$. We want to show that for this x , both $x, -x \notin U_i$ for $i \in [n]$. Since $f(x) = f(-x)$, we know that $\text{dist}(x, U_i) = \text{dist}(-x, U_i)$.

- Case 1: $\text{dist}(x, U_i) = \text{dist}(-x, U_i) > 0$
Clearly $x, -x \notin U_i$.
- Case 2: $\text{dist}(x, U_i) = \text{dist}(-x, U_i) = 0$
This is not possible since $x \in U_i \implies \text{dist}(-x, U_i) > 0$. So $x, -x \notin U_i$.

Since $x, -x \notin U_1 \cup \dots \cup U_n$ and the U_i 's cover \mathbb{S}^n , $x, -x \in U_{n+1}$.

(d) \implies (a): Suppose there is an antipodal map $f : \mathbb{S}^n \rightarrow \mathbb{S}^{n-1}$. We claim that \mathbb{S}^{n-1} can be covered with $n + 1$ closed sets, none of which contain an antipodal

pair. Consider the simplex σ^n centered at 0. Define X_1, \dots, X_{n+1} to be the radial projections of the $n + 1$ facets of σ^n to \mathbb{S}^{n-1} . Define $U_i = f^{-1}(X_i)$. Since f is continuous, the U_i 's are closed and f antipodal implies that none of the sets contain an antipodal pair of points.

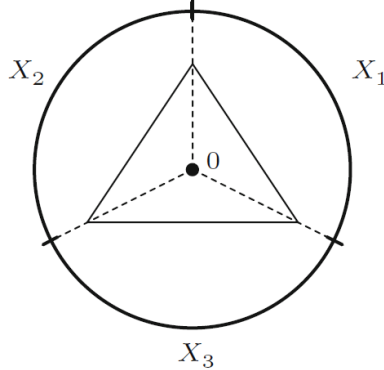


Figure 2: Radial projection of the facets of the 3-simplex to the 2-sphere (page 13 in [6])

□

4.2 The Borsuk-Ulam property for finite groups

Recall that the $(n + 1)$ -dimensional **cross polytope** is defined as

$$Q^{n+1} = \text{conv}(\pm e_1, \dots, \pm e_{n+1})$$

where the e_i 's are the standard basis vectors in \mathbb{R}^{n+1} . In particular, its boundary is the $(n + 1)$ -fold join of two-point sets, i.e.

$$\Gamma^n = \{\pm e_1\} * \{\pm e_2\} * \dots * \{\pm e_{n+1}\}.$$

So we see that $|\Gamma^n| \cong |(\mathbb{Z}_2)^{*(n+1)}| \cong \mathbb{S}^n$ and Theorem 4.1(b) is equivalent to saying that every continuous antipodal map $f : |(\mathbb{Z}_2)^{*(n+1)}| \rightarrow \mathbb{R}^n$ has a zero. In this section, we want to generalize the Borsuk-Ulam theorem by replacing the group \mathbb{Z}_2 by an arbitrary finite group G .

Definition 4.5 Let G be a finite group considered as a 0-dim geometric simplicial complex and let $N \geq 1$ be an integer. Then define $E_N G$ to be the geometric simplicial complex given by the $(N + 1)$ -fold join of G , i.e.

$$E_N G = G * \dots * G,$$

and let its geometric realization be given by

$$|E_N G| = \left\{ (g_0 t_0, \dots, g_N t_N) : g_i \in G, t_i \geq 0, \sum_{i=0}^N t_i = 1 \right\}.$$

Note that $|E_N G|$ is a compact G -space with the following action of G on $|E_N G|$:

$$\begin{aligned} G \times |E_N G| &\rightarrow |E_N G| \\ (g, (h_0 t_0, \dots, h_N t_N)) &\mapsto ((gh_0)t_0, \dots, (gh_N)t_N). \end{aligned}$$

Since $E_N G$ was constructed to emulate \mathbb{S}^n in the Borsuk-Ulam theorem, we highlight some key properties of $E_N G$ that allow a generalized statement of the Borsuk-Ulam theorem to go through.

Proposition 4.6 $|E_N G|$ is a free G -space, that is, G acts freely on $|E_N G|$.

Proof. Suppose

$$g \cdot (h_0 t_0, \dots, h_N t_N) = (h_0 t_0, \dots, h_N t_N).$$

Then since there exists at least one j such that $t_j \neq 0$, we get $gh_j = h_j$ which implies that g must be the identity element in G . \square

Theorem 4.7 $E_N G$ is a pure N -dimensional shellable simplicial complex, and hence it has the homotopy type of a wedge of N -dimensional spheres. In particular, it is $(N - 1)$ -connected.

Proof. See page 25 in [6]. \square

We will use the fact the $E_N G$ is the wedge of N -spheres and the fact that $E_N G$ is $(N - 1)$ -connected in the proof of a generalized Borsuk-Ulam theorem.

Now we want to generalize the Borsuk-Ulam theorem to $|E_N G|$. Let \mathbb{E} be an N -dimensional real vector space with norm-preserving G -action, i.e. $\|gx\| = \|x\|$ for all $g \in G$, $x \in \mathbb{E}$. Assume further that $\mathbb{E}^G := \{x \in \mathbb{E} : gx = x \text{ for all } g \in G\} = \{0\}$, the origin in \mathbb{E} .

Definition 4.8 The group G has the **Borsuk-Ulam property** if for any $N \geq 1$ and N -dim space \mathbb{E} with a norm-preserving G -action and $\mathbb{E}^G = \{0\}$, every continuous G -equivariant map $g : |E_N G| \rightarrow \mathbb{E}$ has a zero.

This is clearly analogous to Theorem 4.1(b), which states that every continuous antipodal map $f : |E_N \mathbb{Z}_2| \rightarrow \mathbb{R}^N$ has a zero.

Theorem 4.9 Let $G = \mathbb{Z}_p$ be the cyclic group of prime order $p \geq 2$. Then G has the Borsuk-Ulam property.

For the rest of the paper, we will use multiplicative notation for the elements of \mathbb{Z}_p . This theorem was proved by Bárány, Shlosman, and Szűcs in 1981. The proof will require the following propositions, which we state without proof:

Proposition 4.10 Let G be a finite group and X, Y be G -spaces. Assume that the action on X is free and that X has a G -invariant triangulation. If $\dim(X) \leq n$ and Y is at least $(n - 1)$ -connected, then there exists a G -equivariant map $g : X \rightarrow Y$.

Proposition 4.11 (Equivariant simplicial approximation theorem) *Let K be a G -complex¹, L a regular G -complex², and $f : |K| \rightarrow |L|$ be a G -equivariant map. Then there exists $r \geq 0$ and a G -equivariant simplicial approximation $g : sd^r K \rightarrow L$ such that $|g| : |K| \rightarrow |L|$ is G -equivariantly homotopic to f .*

Proposition 4.12 (Hopf trace formula) *Let K be a simplicial complex, $f : K \rightarrow K$ be a simplicial map, and F be a field. Then f induces the following F -linear maps:*

$$\begin{aligned} f_\Delta : C_*(K; F) &\rightarrow C_*(K; F), \\ f_* : H_*(K; F) &\rightarrow H_*(K; F). \end{aligned}$$

Suppose K has dimension n . Then

$$\sum_{i=0}^n (-1)^i \operatorname{tr}_i(f_\Delta) = \sum_{i=0}^n (-1)^i \operatorname{tr}_i(f_*).$$

Now, we're ready to prove Theorem 4.9.

Proof. Suppose there is an N -dimensional space \mathbb{E} with norm-preserving G -action such that $\mathbb{E}^G = \{0\}$ and assume towards a contradiction that there is a continuous G -equivariant map $f : |E_N G| \rightarrow \mathbb{E}$ without a zero.

Define $S(\mathbb{E})$ to be the $(N - 1)$ -dimensional unit sphere in \mathbb{E} . Since f does not have a zero, we can define

$$\begin{aligned} \tilde{f} : |E_N G| &\rightarrow S(\mathbb{E}) \\ x &\mapsto \frac{f(x)}{\|f(x)\|}. \end{aligned}$$

Then for $g \in G$ we have

$$\begin{aligned} \tilde{f}(gx) &= \frac{f(gx)}{\|f(gx)\|} \\ &= \frac{g \cdot f(x)}{\|g \cdot f(x)\|} && \text{since } f \text{ is } G\text{-equivariant} \\ &= \frac{g \cdot f(x)}{\|f(x)\|} && \text{since the } G\text{-action is norm-preserving} \\ &= g \cdot \tilde{f}(x). \end{aligned}$$

So \tilde{f} is also G -equivariant.

We claim that G acts freely on $S(\mathbb{E})$. Suppose there exists $g \in G$, $g \neq e$, and $x \in S(\mathbb{E})$ such that $gx = x$. Then $g^k x = x$ for all $k \in \mathbb{Z}$. Since \mathbb{Z}_p is generated by g

¹I.e., a simplicial complex with a simplicial G -action

²A G -complex has a regular G -action if for each subgroup H of G , for each $g_0, \dots, g_n \in H$ and $\{v_0, \dots, v_n\} \in L$ such that $\{g_0 v_0, \dots, g_n v_n\} \in L$, there exists an element $g \in H$ such that $g v_i = g_i v_i$ for all $i = 0, \dots, n$.

(because p is prime), $g^k x = x$ for all $k \in \mathbb{Z}$ implies that $x \in \mathbb{E}^G$, which contradicts $\mathbb{E}^G = \{0\}$.

Since $|E_N G|$ is $(N - 1)$ -connected by Theorem 4.7, it follows from Proposition 4.10 that there exists a G -equivariant map $\rho : S(\mathbb{E}) \rightarrow |E_N G|$. Consider the following composition of G -equivariant maps:

$$|E_N G| \xrightarrow{\tilde{f}} S(\mathbb{E}) \xrightarrow{\rho} |E_N G|$$

By Proposition 4.11, there exists a subdivision K of $E_N G$ and a simplicial map $\psi : K \rightarrow E_N G$ such that $|\psi|$ is G -equivariantly homotopic to $\rho \circ \tilde{f}$.

We know that $\rho \circ \tilde{f}$ is continuous so it induces the following map in homology (see Proposition 4.12)

$$H_*(E_N G; \mathbb{Q}) \xrightarrow{\tilde{f}_*} H_*(S(\mathbb{E}); \mathbb{Q}) \xrightarrow{\rho_*} H_*(E_N G; \mathbb{Q}).$$

Further we know that since ψ is a simplicial map, it induces the following map of simplicial chain complexes (see Proposition 4.12)

$$C_*(K; \mathbb{Q}) \xrightarrow{\psi_\Delta} C_*(E_N G; \mathbb{Q}).$$

Define a map $\varphi : C_*(E_N G; \mathbb{Q}) \rightarrow C_*(K; \mathbb{Q})$ that maps an i -simplex σ of $E_N G$ to the oriented sum of i -simplices contained in σ , e.g. see Figure 3.

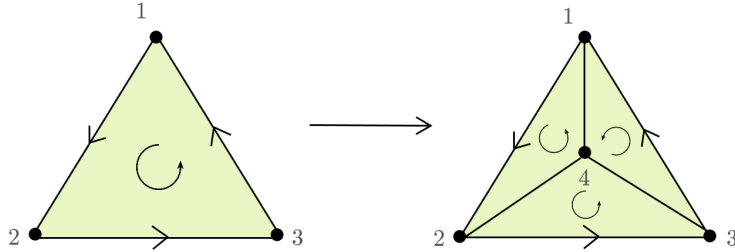
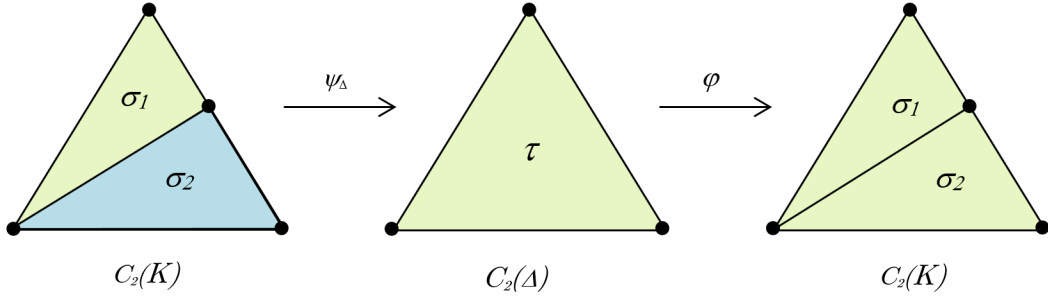


Figure 3: φ_2 sends (123) to $(124) + (234) + (143)$

Define $\bar{\psi} : C_*(K; \mathbb{Q}) \rightarrow C_*(K; \mathbb{Q})$ to be the composition $\varphi \circ \psi_\Delta$.

For any i , consider the square matrix representation of $\bar{\psi}_i : C_i(K; \mathbb{Q}) \rightarrow C_i(K; \mathbb{Q})$ with respect to the basis given by the oriented i -dimensional simplices of K . For example, suppose Δ is the filled in triangle and K is a triangulation of Δ . If $C_2(K; \mathbb{Q}) = \mathbb{Q}[\sigma_1] + \mathbb{Q}[\sigma_2]$ as shown in Figure 4, and if

$$\begin{aligned} \psi_\Delta([\sigma_1]) &= [\tau], \quad \varphi([\tau]) = [\sigma_1] + [\sigma_2], \\ \psi_\Delta([\sigma_2]) &= 0, \quad \varphi(0) = 0, \end{aligned}$$

Figure 4: The $\bar{\psi}_2 = \varphi \circ \psi_\Delta$ map

then the square matrix representation of $\varphi \circ C_2(\psi)$ is given by $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. The square matrix representation always has integer entries by our definition of φ and the fact that ψ is simplicial.

Since G acts freely on $|E_N G|$ by Proposition 4.6, G also acts freely on $|K|$. Let K_i denote the i -dimensional faces of K . If ψ_i fixes some $\sigma \in K_i$, then the fact that ψ is G -equivariant implies that ψ_i also fixes $g\sigma, g^2\sigma, \dots, g^p\sigma$ since

$$\psi_i(g^k\sigma) = g^k\psi_i(\sigma) = g^k\sigma \text{ for all } k \in [p].$$

Since G acts freely, the $g^k\sigma$'s are distinct, so there are p of them, i.e. each diagonal entry of the square matrix representation of $\bar{\psi}_i$ appears p times. Thus p divides $\text{tr}(\bar{\psi}_i)$.

But

$$\sum_{i=0}^N (-1)^i \text{tr}(\bar{\psi}_i) = \sum_{i=0}^N (-1)^i \text{tr}(H_i(\psi)) \quad (2)$$

$$= \sum_{i=0}^N (-1)^i \text{tr}(H_i(\rho \circ \tilde{f})) \quad (3)$$

$$= \text{tr}(H_0(\rho \circ \tilde{f})) + (-1)^N \text{tr}(H_N(\rho \circ \tilde{f})) \quad (4)$$

$$= \text{tr}(H_0(\rho \circ \tilde{f})) \quad (5)$$

$$= 1, \quad (6)$$

which is not divisible by p . Indeed,

(2) follows from Proposition 4.12;

(3) holds because $|\psi|$ is homotopic to $\rho \circ \tilde{f}$;

(4) is a consequence of $E_N G$ being a wedge of N -spheres by Theorem 4.7, so the intermediate homologies $H_i(E_N G; \mathbb{Q})$ vanish;

(5) holds because $H_N(g \circ \tilde{f})$ factors through $H_N(S(\mathbb{E}); \mathbb{Q}) = 0$, so its trace is 0;

(6) follows from $g \circ \tilde{f}$ being a non-trivial map between two connected, non-empty spaces.

Here we have a contradiction, and so $f : |E_N G| \rightarrow \mathbb{E}$ has a zero. Thus \mathbb{Z}_p has the Borsuk-Ulam property. \square

Theorem 4.13 *Let $p \geq 2$ be a prime and $k \geq 1$. The group $G = (\mathbb{Z}_p)^k$ has the Borsuk-Ulam property.*

This theorem was proved independently by Özaydin, Volovikov, and Sarkaria. The proof goes beyond the scope of this thesis. Sarkaria's proof can be found in [12].

5 Tverberg's theorem

We are now ready to discuss Tverberg's theorem.

Theorem 5.1 (Tverberg) *Let $d \geq 1$ and $r \geq 2$ be fixed numbers, and let x_1, \dots, x_m be points in \mathbb{R}^d where $m \geq (r-1)(d+1) + 1$. Then there is a partition S_1, S_2, \dots, S_r of $\{x_i : i \in [m]\}$ such that*

$$\bigcap_{i=1}^r \text{conv}(S_i) \neq \emptyset.$$

Note that Tverberg's theorem is very similar to Radon's theorem, except that we require r parts (for an arbitrary fixed r) instead of 2 parts. In the case where $r = 2$, we simply get Radon's theorem. Observe also that the bound given in Tverberg's theorem is sharp: Suppose we have $(r-1)(d+1)$ points X in \mathbb{R}^d in sufficiently general position³. Then for every partition of these points into r parts, even the affine spans of the parts will have empty intersection.

For example, suppose $r = 3$ and $d = 2$, and let X be a set of $(r-1)(d+1) = 6$ points in \mathbb{R}^2 in general position. We claim that the affine hulls of any three pairwise disjoint subsets of X have empty intersection. Indeed, let a 3-partition of the points be $\{\mu_1, \mu_2, \mu_3\}$, where $|\mu_1| \geq |\mu_2| \geq |\mu_3|$.

- Case 1: $|\mu_1| = 4, |\mu_2| = |\mu_3| = 1$. Then clearly $\text{aff}(\mu_2) \cap \text{aff}(\mu_3) = \emptyset$.
- Case 2: $|\mu_1| = 3, |\mu_2| = 2, |\mu_3| = 1$. Then $\text{aff}(\mu_2) \cap \text{aff}(\mu_3) = \emptyset$ or there would be three collinear points.
- Case 3: $|\mu_1| = |\mu_2| = |\mu_3| = 2$. Then $\text{aff}(\mu_1), \text{aff}(\mu_2),$ and $\text{aff}(\mu_3)$ are three lines in general position in \mathbb{R}^2 , so their intersection is empty.

³A set of points X in \mathbb{R}^d are said to be in **sufficiently general position** if they satisfy the following conditions:

1. Given any subset $A \subset X$, the dimension of its affine hull is maximal, i.e. $\dim \text{aff}(A) = \min\{|A| - 1, d\}$.
2. Given any k disjoint subsets $\mu_1, \dots, \mu_k \subset X$, the dimension of the intersection of their affine hulls is minimal, i.e.

$$\dim(\text{aff}(\mu_1) \cap \dots \cap \text{aff}(\mu_k)) = \min\{|\mu_1| + \dots + |\mu_k| - (k-1)d - k, d\}.$$

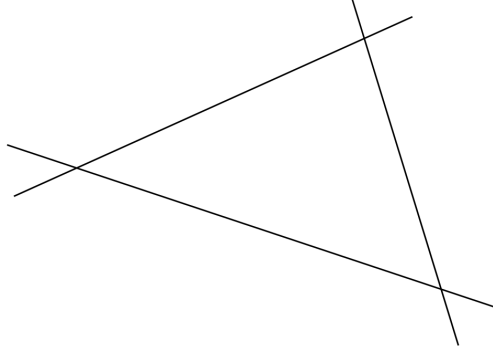


Figure 5: Three lines in general position in \mathbb{R}^2 .

The first proof [14] of Tverberg's theorem published in 1966 involved proving the theorem for specially arranged points, continuously perturbing those points, then showing that the theorem still held true. Several other simplified proofs [15, 10] were published after the initial paper, but one of the most elementary proofs was formulated by Sarkaria [11]. His proof uses only linear algebra and the Colorful Caratheodory theorem. Sarkaria's proof most closely emulates the simplicity of the partitioning argument given in the proof for Radon's theorem, and we present it here below.

Proof. Assume without loss of generality that $m = (r - 1)(d + 1) + 1$. If m is larger, we can just distribute the additional points into the existing parts, and adding these points to any of the S_j 's would not take away any points from the intersection. Without loss of generality, suppose v_1, \dots, v_m are points belonging to the d -dimensional affine space H in $V = \mathbb{R}^{d+1}$ given by $\{(x, \dots, x_{d+1}) : x_1 + \dots + x_{d+1} = 1\}$. In other words, we assume that

$$v_i = (v_i[1], v_i[2], \dots, v_i[d + 1]) \text{ satisfy } \sum_{k=1}^{d+1} v_i[k] = 1$$

where $v[s]$ denotes the s th coordinate of v . This may be done by embedding \mathbb{R}^d into \mathbb{R}^{d+1} via the map

$$(x[1], \dots, x[d]) \mapsto (x[1], \dots, x[d], 1 - (x[1] + \dots + x[d])).$$

Let W be an $(r - 1)$ -dimensional vector space, and let w_1, \dots, w_r be r vectors in W such that

$$w_1 + \dots + w_r = 0$$

is their only non-trivial dependence. For example, we may define w_1, \dots, w_{r-1} to be the standard basis vectors in \mathbb{R}^{r-1} and then define $w_r = -w_1 - w_2 - \dots - w_{r-1}$.

Then $V \otimes W$ is a $(d + 1)(r - 1)$ -dimensional vector space with elements defined in the following way: given $x \in V$, $y \in W$, we define $x \otimes y = (x_i \cdot y_j)$ to be a

$(d+1) \times (r-1)$ matrix whose (i, j) -th entry is the number $x_i \cdot y_j$. Note that this matrix has rank 1. (Indeed, by definition each row is a scalar multiple of the first row and each column is a scalar multiple of the first column). For each $i \in [m]$, define $S_i = \{v_i \otimes w_j : j = 1, \dots, r\}$ to be a set of r points in $V \otimes W$. Since $\sum_{j=1}^r w_j = 0$,

$$\frac{1}{r} \sum_{j=1}^r v_i \otimes w_j = \frac{1}{r} v_i \otimes \sum_{j=1}^r w_j = 0,$$

so $0 \in \text{conv}(S_i)$ for each i .

Theorem 2.4 says that there exists some $s_i \in S_i$ for each $i \in [m]$ such that $0 \in \text{conv}(s_1, \dots, s_m)$, i.e. there exist $\lambda_1, \dots, \lambda_m$ such that

$$0 = \sum_{i=1}^m \lambda_i s_i, \quad 0 \leq \lambda_i \leq 1, \quad \text{and} \quad \sum_{i=1}^m \lambda_i = 1.$$

Suppose $s_i = v_i \otimes w_{j_i}$. Define $\Omega_k = \{i : j_i = k\}$ for $k = 1, 2, \dots, r$. We claim that $\Omega_1, \dots, \Omega_r$ is a partition of $[m]$ such that

$$\bigcap_{k=1}^r \text{conv}(v_i : i \in \Omega_k) \neq \emptyset.$$

Note that

$$\begin{aligned} 0 &= \sum_{i=1}^m \lambda_i s_i \\ &= \sum_{i=1}^m \lambda_i v_i \otimes w_{j_i} \\ &= \sum_{i \in \Omega_1} \lambda_i v_i \otimes w_1 + \dots + \sum_{i \in \Omega_r} \lambda_i v_i \otimes w_r. \end{aligned} \tag{7}$$

Since each $v_i \otimes w_j$ is a $(d+1) \times (r-1)$ matrix, we can look at the k th row of these matrices – if we define $v_i[k]$ to be the k th coordinate of v_i , then the k th row of (7) is given by

$$\sum_{j=1}^r \left(\sum_{i \in \Omega_j} \lambda_i v_i[k] \right) w_j = 0.$$

But we defined the w_j 's such that the only non-trivial linear dependence (up to scalar multiples) was $\sum_{j=1}^r w_j = 0$, so we may define a point $y \in \mathbb{R}^{d+1}$ with the following coordinates:

$$y[k] := \sum_{i \in \Omega_1} \lambda_i v_i[k] = \dots = \sum_{i \in \Omega_r} \lambda_i v_i[k].$$

Since we assumed $\sum_{k=1}^{d+1} v_i[k] = 1$ for all $i \in [m]$, for any j ,

$$\begin{aligned} \sum_{k=1}^{d+1} y[k] &= \sum_{k=1}^{d+1} \sum_{i \in \Omega_j} \lambda_i v_i[k] \\ &= \sum_{i \in \Omega_j} \lambda_i \left(\sum_{k=1}^{d+1} v_i[k] \right) \\ &= \sum_{i \in \Omega_j} \lambda_i \text{ for all } j \in [r]. \end{aligned}$$

Since $\sum_{i=1}^m \lambda_i = 1$, we have that $\sum_{i \in \Omega_j} \lambda_i = \frac{1}{r}$ for each $j \in [r]$. Then $ry = r \sum_{i \in \Omega_j} \lambda_i v_i \in \text{conv}(v_i : i \in \Omega_j)$ for each $j \in [r]$ and

$$\bigcap_{j=1}^r \text{conv}(v_i : i \in \Omega_j) \neq \emptyset.$$

□

Now the natural question arises: Can Tverberg's theorem be extended to a continuous Tverberg theorem, in the same way Radon's theorem was generalized to the continuous Radon theorem? It turns out that such a generalization holds for some values of parameters (e.g. for r a prime power); it is false in general, as we will see in the next section.

6 The topological Tverberg conjecture

Recall that the proof of Tverberg's Theorem shows the existence of a partition of $m = (r-1)(d+1) + 1$ points into r parts whose convex hulls intersect. We may view these m points as the images of the vertices of an $(m-1)$ -simplex σ^{m-1} under some affine map f , so each part is simply the image of the vertex set of a face of σ^{m-1} . What if f were not required to be affine?

Conjecture 6.1 Let $d \geq 1, r \geq 2$ and let $N \geq (r-1)(d+1)$. For any continuous map $f : \sigma^N \rightarrow \mathbb{R}^d$, there exist r pairwise disjoint faces of σ^N whose images have a point in common.

This conjecture was posited by Bárány, Shlosman, and Szűcs in 1978 [3]. If we impose that f is affine, then we just get Tverberg's theorem. Tverberg's theorem is thus sometimes referred to as the "affine Tverberg theorem" whereas the conjecture above is referred to as the "continuous Tverberg conjecture" or the "topological Tverberg conjecture". In general, the conjecture is not true – in the thirty years after Conjecture 6.1 was published, a lot of effort was put into proving various cases of the conjecture without attaining full generality. In 2015, Florian Frick provided a

spectacular counterexample: Conjecture 6.1 is false for all values of r such that r is not a prime power!

Theorem 6.2 (Frick) *Let $r \geq 6$ be an integer that is not a prime power, and let $k \geq 3$ be an integer. Let $N = (r - 1)(rk + 2)$. Then there exists a continuous map $f : \sigma^N \rightarrow \mathbb{R}^{rk+1}$ such that for any r pairwise disjoint faces τ_1, \dots, τ_r of σ^N , we have*

$$f(\tau_1) \cap \dots \cap f(\tau_r) = \emptyset.$$

Proof. See [7]. □

The proof is beyond the scope of this thesis. In the rest of this section, we will concentrate on discussing the cases where the topological Tverberg conjecture holds.

6.1 $r = 2$

The case where $r = 2$ is essentially the continuous Radon theorem and was proved by Bajmoczy and Bárány in [1] using the Borsuk-Ulam theorem. In the same paper, they also extended Conjecture 6.1 by replacing the simplex σ^N with any convex polytope of the same dimension with non-empty interior.

Theorem 6.3 *Let $d \geq 1$. Let $P \subset \mathbb{R}^{d+1}$ be a convex polytope with non-empty interior. Then given any continuous map $f : \partial P \rightarrow \mathbb{R}^d$, there exist two disjoint faces B and C of P such that $f(B) \cap f(C) \neq \emptyset$.*

Note that when P is the $(d + 1)$ -dimensional simplex σ^{d+1} , we get Conjecture 6.1 for $r = 2$.

Instead of proving Theorem 6.3 directly, they instead proved the following:

Theorem 6.4 *Given a convex polytope $P \subset \mathbb{R}^{d+1}$ with non-empty interior and a continuous map $f : \partial P \rightarrow \mathbb{R}^d$, there exist two opposite points⁴ x and y in P such that $f(x) = f(y)$.*

Note that two opposite points have to lie in two opposite faces⁵ B and C in P , so $f(x) = f(y)$ implies that $f(B) \cap f(C) \neq \emptyset$, which in turn implies Theorem 6.3 since opposite faces of a simplex are clearly disjoint. Theorem 6.4 was proved by observing that the Minkowski sum of P and $-P$ has boundary homeomorphic to a sphere, and then applying Theorem 4.1(c). The proof can be found in [1].

6.2 r a prime number

The case where r is a prime number was proved by Bárány, Schlosman and Szűcs in [3].

⁴Given a polytope $P \subset \mathbb{R}^{d+1}$ and $a \in \mathbb{R}^{d+1}$, we can define $P(a) = \{x \in P : \langle a, x \rangle = \max_{p \in P} \langle a, p \rangle\}$. Two points $x, y \in P$ are **opposite** if there exists some $a \in \mathbb{R}^{d+1}$ such that $x \in P(a)$ and $y \in P(-a)$.

⁵Opposite faces of a polytope P are faces of P with parallel supporting hyperplanes.

6.3 $r = p^k$ for p prime

The case where r is a prime power was proved independently by Özaydin, Volovikov, and Sarkaria and we will present one of the proofs in this section. First, we need a few preliminaries.

Recall the definition of a twofold deleted join of abstract simplicial complexes. We can define the r -fold deleted join analogously:

Definition 6.5 Given K an abstract simplicial complex, the r -fold deleted join K_{Δ}^{*r} is given by

$$K_{\Delta}^{*r} = \{\tau_1 \cup \cdots \cup \tau_r : \tau_1, \dots, \tau_r \in K, \text{ for all } i, j \in [r] : \tau_i \cap \tau_j = \emptyset\}.$$

Lemma 6.6 Let Θ_r be the simplicial complex with r vertices and no higher dimensional simplices, i.e. $\Theta_r = \{\emptyset, \{x_1\}, \dots, \{x_r\}\}$. Let $r \geq 1$ and $N \geq 0$. Then there is an isomorphism of simplicial complexes $(2^{[N+1]})_{\Delta}^{*r} \cong (\Theta_r)^{*N+1}$.

Proof. The isomorphism is given by the following map:

$$\begin{aligned} (2^{[N+1]})_{\Delta}^{*r} &\rightarrow (\Theta_r)^{*N+1}, \\ \tau_1 \cup \cdots \cup \tau_r &\mapsto \sigma_1 \cup \cdots \cup \sigma_{N+1} \end{aligned}$$

where for $i \in [N+1]$,

$$\sigma_i = \begin{cases} \{x_j\} & \text{if } i \in \tau_j \\ \emptyset & \text{otherwise.} \end{cases}$$

In other words, the simplices τ_j and σ_i can be read off the adjacency matrix whose rows correspond to faces τ_1, \dots, τ_r in $(2^{[N+1]})_{\Delta}^{*r}$ and whose columns correspond to vertices $\{1, \dots, N+1\}$ in $2^{[N+1]}$. \square

Corollary 6.7 If G is a group with r elements considered as a 0-dimensional simplicial complex, then $E_N G$ is equivariantly simplicially homeomorphic to $|(2^{[N+1]})_{\Delta}^{*r}|$.

Now we are ready to prove Conjecture 6.1 for the case where r is a prime power p^k .

Theorem 6.8 Let $d \geq 1, r = p^k$ a prime power greater than 2, and let $N \geq (r-1)(d+1)$. For any continuous map $f : \sigma^N \rightarrow \mathbb{R}^d$, there exist r pairwise disjoint faces of σ^N whose images have a point in common.

Proof. Let $G = (\mathbb{Z}_p)^k$. Recall that Theorem 4.13 states that G has the Borsuk-Ulam property, that is, for any $M \geq 1$ and M -dimensional vector space \mathbb{E} with a norm-preserving G -action and $\mathbb{E}^G = \{0\}$, every continuous G -equivariant map $|E_M G| \rightarrow \mathbb{E}$ has a zero. So taking $M = N$, our aim is to construct a G -equivariant map from $E_N G \rightarrow \mathbb{E}$ such that any zero of this map corresponds to a set of pairwise disjoint faces $\tau_1, \dots, \tau_r \leq \sigma^N$ whose images under the given continuous map f intersect.

Define $\mathbb{A}^d \subset \mathbb{R}^{d+1}$ to be the d -dimensional affine subspace of vectors with coordinate sum equal to 1. Without loss of generality, we can assume that $f : \sigma^N \rightarrow \mathbb{A}^d$ via the embedding $(x_1, \dots, x_d) \mapsto (x_1, \dots, x_d, 1 - (x_1 + \cdots + x_d))$.

Since $|(2^{[N+1]})_{\Delta}^{*r}|$ is a subcomplex of $|(2^{[N+1]})^{*r}|$, which is simplicially homeomorphic to $(\sigma^N)^{*r}$, we have the following sequence of G -equivariant maps

$$\iota : |(2^{[N+1]})_{\Delta}^{*r}| \hookrightarrow |(2^{[N+1]})^{*r}| \xrightarrow{\cong} (\sigma^N)^{*r}.$$

We can define the following map \tilde{f} induced by f :

$$\begin{aligned} \tilde{f} : |(2^{[N+1]})_{\Delta}^{*r}| &\rightarrow \mathbb{R}^{(d+1) \times r}, \\ (t_1 x_1, \dots, t_r x_r) &\mapsto [t_1 f(x_1), \dots, t_r f(x_r)] \end{aligned}$$

where we write $f(x_1), \dots, f(x_r)$ as column vectors. Note that \tilde{f} is equivariant with respect to the symmetric group S_r on r elements which acts on $|(2^{[N+1]})_{\Delta}^{*r}|$ by permuting the coordinates of its elements and acts on $\mathbb{R}^{(d+1) \times r}$ by permuting the columns of matrices in it. Note that G can be identified with a subgroup of the symmetric group on G , $\text{Sym}(G)$, which is isomorphic to S_r . So we can identify G with a subgroup of S_r by bijecting elements of G arbitrarily into $[r]$. Thus \tilde{f} is also G -equivariant.

Suppose there exist pairwise disjoint simplices $\tau_1, \dots, \tau_r \leq \sigma^N$ and $x_1 \in \tau_1, \dots, x_r \in \tau_r$ such that $f(x_1) = \dots = f(x_r)$, i.e. the $f(\tau_i)$'s have a non-empty intersection. Then

$$\tilde{f} \left(\frac{1}{r} x_1, \dots, \frac{1}{r} x_r \right) = \left[\frac{1}{r} f(x_1), \dots, \frac{1}{r} f(x_r) \right]$$

is a constant-row matrix. The converse is also true: Suppose $x_1 \in \tau_1, \dots, x_r \in \tau_r$ for pairwise disjoint $\tau_1, \dots, \tau_r \leq \sigma^N$. Suppose also we have a constant-row matrix $[t_1 f(x_1), \dots, t_r f(x_r)]$, i.e. $t_1 f(x_1) = \dots = t_r f(x_r)$ where each $f(x_i)$ is a point in \mathbb{A}^d . Since we assumed $f : \sigma^N \rightarrow \mathbb{A}^d$, we know that $\sum_{j=1}^{d+1} f(x_i)_j = 1$ where $f(x_i)_j$ denotes the j th coordinate of $f(x_i)$. So we have $t_1 = \dots = t_r$ and hence $f(x_1) = \dots = f(x_r)$ and the $f(\tau_i)$'s have a non-empty intersection.

Now let \mathbb{E} be the subspace in $\mathbb{R}^{(d+1) \times r}$ of matrices whose row sums are equal to zero, i.e.

$$\mathbb{E} = \left\{ (a_{ij}) \in \mathbb{R}^{(d+1) \times r} : \sum_{j=1}^r a_{ij} = 0 \text{ for all } i = 1, \dots, d+1 \right\}.$$

Observe that \mathbb{E} is in fact the orthogonal complement of all matrices with constant rows under the standard inner product on matrices: $\langle (a_{ij}), (b_{ij}) \rangle = \sum_i \sum_j a_{ij} b_{ij}$. The group S_r , and hence G , act on \mathbb{E} by permuting columns of matrices in \mathbb{E} . Thus G acts norm-preservingly. Define $\pi : \mathbb{R}^{(d+1) \times r} \rightarrow \mathbb{E}$ be the orthogonal projection of all $(d+1) \times r$ matrices onto \mathbb{E} .

We claim that this projection is equivariant with respect to S_r (and hence also G). If we denote by \mathbb{K} the set of all constant-row $(d+1) \times r$ matrices, since \mathbb{E} is the orthogonal complement to \mathbb{K} , any matrix $A \in \mathbb{R}^{(d+1) \times r}$ can be written as $A = Z + C$ for $Z \in \mathbb{E}$ and $C \in \mathbb{K}$. Given any $g \in S_r$, g permutes the columns of matrices in $\mathbb{R}^{(d+1) \times r}$, and $gA = gZ + gC$. But permuting the columns of any matrix $Z \in \mathbb{E}$ does not change the zero row sums, similarly, permuting the columns of a constant-row

matrix yields another constant-row matrix. So $\pi(gA) = gZ = g(\pi(A))$, and π is hence S_r -equivariant.

Note that $\dim \mathbb{E} = (d+1)r - (d+1) = (d+1)(r-1) = N$, so \mathbb{E} is an N -dimensional vector space with \mathbb{E}^G = the set of matrices in \mathbb{E} that are invariant under all column permutations, i.e. constant-row matrices, so $\mathbb{E}^G = \{0\}$. Composing π with \tilde{f} , we get a G -equivariant map

$$\pi \circ \tilde{f} : |(2^{[N+1]})_{\Delta}^{*r}| \rightarrow \mathbb{E}.$$

Note that zeros of π correspond precisely to constant row matrices in $\mathbb{R}^{(d+1) \times r}$ and we know that the pre-images of constant-row matrices under \tilde{f} yield an r -tuple of points in r pairwise disjoint faces of σ^N . Since $|(2^{[N+1]})_{\Delta}^{*r}|$ is equivariantly simplicially homeomorphic to $E_N G$ by Corollary 6.7, $\pi \circ \tilde{f}$ has a zero by Theorem 4.13 and there exist r pairwise disjoint faces of σ^N whose faces under f have a point in common. \square

7 The number of Tverberg partitions

According to Tverberg's theorem, any given set of $(r-1)(d+1) + 1$ points in \mathbb{R}^d has an r -partition that intersects non-trivially. We call such a partition a **Tverberg partition**. The natural question arises: If a Tverberg partition exists, how many such Tverberg partitions are there? Given a set of points X in \mathbb{R}^d , let $T_r(X)$ represent the number of Tverberg r -partitions of X . The following conjecture by Gerard Sierksma can be found in [13]:

Conjecture 7.1 If X is a set of $(r-1)(d+1) + 1$ points in \mathbb{R}^d , then $T_r(X) \geq ((r-1)!)^d$.

This conjecture is yet unproved in general, but modest advances have been made for some small cases, which we will highlight in Section 7.2. This conjecture is also sometimes referred to as Sierksma's 'Dutch cheese problem' because a Dutch cheese was offered as a reward for its solution!

7.1 The Sierksma configuration

The Sierksma configuration is a way of arranging $(r-1)(d+1) + 1$ points in \mathbb{R}^d in a way that attains the conjectured lower bound on the number of Tverberg partitions. The construction is simple: consider the $d+1$ vertices of the simplex σ^d in \mathbb{R}^d . Choose some small $\epsilon > 0$, where ϵ is much smaller than half of the length of an edge in σ^d . For each vertex $v \in \text{vert}(\sigma^d)$, we place $r-1$ points in an ϵ -neighborhood of v . Finally, place a point in the barycenter of σ^d . See Figure 6.

It is not hard to see that each Tverberg r -partition $\{\mu_1, \dots, \mu_r\}$ of points in the Sierksma configuration has the following form: for $i \in [r-1]$, μ_i consists of one point from each of the $d+1$ ϵ -balls, while μ_r consists solely of the point at the barycenter. There are $((r-1)!)^{d+1}$ ways to distribute the $r-1$ points in $d+1$ ϵ -balls

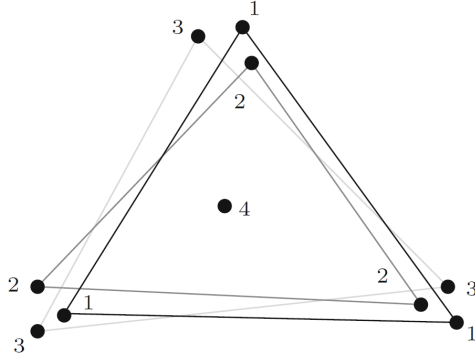


Figure 6: The Sierksma configuration for $d = 2$, $r = 4$ (page 107 in [6])

into the μ_i 's such that $|\mu_1| = \dots = |\mu_{r-1}| = d + 1$. Clearly $\bigcap_{j=1}^r \text{conv}(\mu_j) = \mu_r$ is non-empty, so $\{\mu_1, \dots, \mu_r\}$ is indeed a Tverberg partition. Furthermore, the total number of Tverberg partitions obtained in this manner is given by

$$\frac{(\# \text{ ways to distribute } r - 1 \text{ points into } r - 1 \text{ sets})^{|\text{vert}(\sigma^d)|}}{\# \text{ ways to permute the first } r - 1 \text{ parts}} = \frac{((r - 1)!)^{d+1}}{(r - 1)!} = ((r - 1)!)^d,$$

which is precisely the desired lower bound.

7.2 Smaller cases

In [8], Hell proves a much weaker version of Sierksma's conjecture in general, as well as Sierksma's conjecture for $d = 2$ and $r = 3$. He does this by considering **constraint graphs**, or subgraphs C of $\sigma^{(d+1)(r-1)}$ such that every continuous map $f : \|\sigma^{(d+1)(r-1)}\| \rightarrow \mathbb{R}^d$ has a Tverberg partition of disjoint faces not using any edge of C . Some examples of constraint graphs are as follows:

Theorem 7.2 *Let $r > 2$ be a prime power. Then the following subgraphs of $\sigma^{(d+1)(r-1)}$ are constraint graphs:*

- (i) Complete graphs K_n on n vertices for $2n < r + 2$,
- (ii) complete bipartite graphs $K_{1,n}$ for $n < r - 1$,
- (iii) paths P_n on $n + 1$ vertices for $n \leq (d + 1)(r - 1)$ and $r > 3$,
- (iv) cycles C_n on n vertices for $n \leq (d + 1)(r - 1) + 1$ and $r > 4$,
- (v) and arbitrary disjoint unions of graphs from (i) - (iv).

Let X be a set of $(d + 1)(r - 1) - 1$ points in general position in \mathbb{R}^d . Hell sorts Tverberg partitions of X into two types: Type I is precisely the partitions gotten

from points arranged in a similar manner to the Sierksma configuration, i.e. one part consists of a central vertex v , and the remaining $r - 1$ parts consist of the vertex sets of d -simplices containing v ; Type II consists of k intersecting simplices of dimension less than d , and $(r - k)$ d -simplices containing the intersection point for some $1 < k \leq \min(d, r)$. In both types, the point corresponding to the intersection point is called a **Tverberg point**. Then exhaustively considering all configurations of Tverberg points and finding Tverberg partitions based on constraint graphs, he proves the following three theorems:

Theorem 7.3 *Let X be a set of $(d + 1)(r - 1) + 1$ points in general position in \mathbb{R}^d where $d \geq 1$. Then*

(i) $T_r(X)$ is even for $r > d + 1$.

(ii) $T_r(X) \geq (r - d)!$

Theorem 7.4 *Let X be a set of $(d + 1)(r - 1) + 1$ points in general position in \mathbb{R}^d where $d \geq 1$, and let $r = p^k$ be a prime power. Then*

$$T_r(X) \geq \frac{1}{(r - 1)!} \cdot \left(\frac{r}{k + 1} \right)^{\lceil \frac{(d+1)(r-1)}{2} \rceil}.$$

Theorem 7.5 *If $d = 2$ and $r = 3$, then $T_3(X) \geq 4$. In particular, Sierksma's conjecture holds in this case.*

Note that the lower bound $T_r(X) \geq (r - d)!$ in Theorem 7.3 is much lower than that in Sierksma's conjecture, where $T_r(X)$ was conjectured to be greater than or equal to $((r - 1)!)^d$. Also observe that for large d and r , the bound given in Theorem 7.4 is roughly the square root of the bound in Sierksma's conjecture.

More recently, Bukh, Loh, and Nivasch found a large family of $((d + 1)(r - 1) + 1)$ -point sets which have exactly $((r - 1)!)^d$ Tverberg partitions by characterizing unavoidable Tverberg partitions [5].

8 Miscellaneous problems

One interesting variation of Tverberg's theorem involves assigning a color to each point and then requiring that each part in a Tverberg partition attains all possible colors, as in the colorful Caratheodory theorem.

Let C_1, \dots, C_{d+1} be disjoint subsets of \mathbb{R}^d , each of cardinality $\geq t$. We will treat each C_i as a color.

Definition 8.1 A $(d + 1)$ -subset S of $\bigcup_{i=1}^{d+1} C_i$ is **multicolored** if $S \cap C_i \neq \emptyset$ for $i = 1, \dots, d + 1$.

Let $r \in \mathbb{Z}_{>0}$ and define $N(r, d)$ to be the smallest value of n such that for n points P_n in \mathbb{R}^d that are r -properly colored⁶, there exists r pairwise disjoint multicolored

⁶A set of points is called **r -properly colored** if the cardinality of each color is at least r

subsets $\mu_1, \dots, \mu_r \subset P_n$ such that $\bigcap_{j=1}^r \text{conv}(\mu_j) \neq \emptyset$. Define $T(r, d)$ to be the smallest value of t such that for every collection of colors C_1, \dots, C_{d+1} of size at least t , there are r pairwise disjoint multicolored sets S_1, \dots, S_r such that $\bigcap_{i=1}^r \text{conv}(S_i) \neq \emptyset$. The colored Tverberg problem sought to determine $T(r, d)$.

In the planar case where $d = 2$, this problem (also referred to as the ‘colored Radon problem’) was solved by Bárány and Larman in [2]:

Theorem 8.2 (Bárány, Larman) *Given positive integers r and d ,*

$$\begin{aligned} N(r, 1) &= 2r, \\ N(r, 2) &= 3r, \\ N(2, d) &= 2(d + 1) \end{aligned}$$

Clearly when $d = 2$, $N(r, 2) = 3r$ implies that r is the smallest possible cardinality of each color such that there are r pairwise disjoint multicolored sets $S_1, \dots, S_r \subset P_{N(r,2)}$ with $\bigcap_{i=1}^r \text{conv}(S_i) \neq \emptyset$, so $T(r, 2) = r$. In the same paper, Bárány and Larman made the following conjecture:

Conjecture 8.3 $T(r, d) = r$ for all $r \geq 2$ and $d \geq 1$.

The case of arbitrary d was studied by Zivaljevic and Vrećica in [16]:

Theorem 8.4 (Zivaljevic, Vrećica) $T(r, d) \leq 4r - 1$ for all r and $T(r, d) \leq 2r - 1$ for r a prime.

In 2014, Blagojević, Matschke and Ziegler proved Conjecture 8.3 that $T(r, d) = r$ for the case where $r + 1$ is prime and also provided the following asymptotic estimate [4]:

Theorem 8.5 *For $d \geq 1$ and an arbitrary $r \geq 2$,*

$$r \leq T(r, d) \leq 2r - 2.$$

Further,

$$r \leq T(r, d) \leq (1 + o(1))r \text{ for } d \geq 1 \text{ and } r \rightarrow \infty.$$

It is yet unknown if Conjecture 8.3 is true in full generality.

Apart from looking at colorful Tverberg problems, one could also explore the counterexamples to the topological Tverberg conjecture. After the first counterexample was found in 2015, Frick combined Mabillard’s and Wagner’s r -fold Whitney trick with the constraint method described in [7] to further obtain counterexamples to the conjecture for all $r \geq 6$ that are not prime powers. All the counterexamples that have been found so far exist in sufficiently higher dimensions, but we do not yet know if the conjecture also fails for $d = 2$. As already highlighted in Section 7, even in the case where the topological Tverberg conjecture holds, we do not know how many

Tverberg partitions exist. One could also look into the computational complexity of finding a Tverberg partition where the conjecture holds – is there a polynomial time algorithm to find such a partition?

Another interesting consideration is that of the dimension of Tverberg points.

Definition 8.6 Given a set A , we call $T_k(A)$ the set of points in \mathbb{R}^d which belong to the intersection of the convex hulls of k pairwise disjoint subsets of A .

Kalai conjectured the following in 1974:

Conjecture 8.7 (Kalai) For every $A \subset \mathbb{R}^d$,

$$\sum_{k=1}^{|A|} \dim T_k(A) \geq 0$$

where $\dim \emptyset = -1$. Note that when $|A| = (r-1)(d+1) + 1$ and $\dim A = d$, if $T_r(A)$ were empty, then clearly $T_k(A)$ would be empty for $k \geq r$ and so we would have

$$\sum_{k=1}^{|A|} \dim T_k(A) \leq \sum_{k=1}^{r-1} d + \sum_{k=r}^{|A|} (-1) = (r-1)d + (|A| - r + 1)(-1) = -1,$$

which contradicts the conjecture. Thus the conjecture, if true, implies $T_r(A) \neq \emptyset$, which is Tverberg's theorem. This conjecture was proved by Kadari for $d = 2$, but no further progress has been made.

References

- [1] Ervin Bajmóczy and Imre Bárány. On a common generalization of Borsuk's and Radon's theorem. *Acta Mathematica Hungarica*, 34(3-4):347–350, 1979.
- [2] Imre Bárány and David Larman. A colored version of Tverberg's theorem. *Journal of the London Mathematical Society*, 2(2):314–320, 1992.
- [3] Imre Bárány, Senya Shlosman, and András Szücs. On a topological generalization of a theorem of Tverberg. *Journal of the London Mathematical Society*, 2(1):158–164, 1981.
- [4] Pavle V. M. Blagojević, Benjamin Matschke, and Günter M. Ziegler. Optimal bounds for the colored Tverberg problem. *J. Eur. Math. Soc. (JEMS)*, 17(4):739–754, 2015.
- [5] Boris Bukh, Po-Shen Loh, and Gabriel Nivasch. Classifying unavoidable Tverberg partitions. *arXiv preprint arXiv:1611.01078*, 2016.
- [6] Mark De Longueville. *A course in topological combinatorics*. Springer Science & Business Media, 2012.

- [7] Florian Frick. Counterexamples to the topological Tverberg conjecture. *arXiv preprint arXiv:1502.00947*, 2015.
- [8] Stephan Hell. Tverberg's theorem with constraints. *Journal of Combinatorial Theory, Series A*, 115(8):1402–1416, 2008.
- [9] Jiri Matousek. *Using the Borsuk-Ulam theorem: lectures on topological methods in combinatorics and geometry*. Springer Science & Business Media, 2008.
- [10] Jean-Pierre Roudneff. Partitions of points into simplices with k -dimensional intersection. Part I: The conic Tverberg's theorem. *European Journal of Combinatorics*, 22(5):733–743, 2001.
- [11] Karanbir Sarkaria. Tverberg's theorem via number fields. *Israel journal of mathematics*, 79(2):317–320, 1992.
- [12] Karanbir Sarkaria. Tverberg partitions and Borsuk-Ulam theorems. *Pacific Journal of Mathematics*, 196(1):231–241, 2000.
- [13] Gerard Sierksma. Convexity without linearity; the Dutch cheese problem. *Mimeographed notes*, 1979.
- [14] Helge Tverberg. A generalization of Radon's theorem. *J. London Math. Soc.*, 41(1):123–128, 1966.
- [15] Helge Tverberg and Siniša Vrećica. On generalizations of Radon's theorem and the Ham Sandwich theorem. *European journal of combinatorics*, 14(3):259–264, 1993.
- [16] Rade Živaljević and Siniša Vrećica. The colored Tverberg's problem and complexes of injective functions. *Journal of Combinatorial Theory, Series A*, 61(2):309–318, 1992.