Polytopes of Cographic Unimodular Systems and Valuated Delta Matroids

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Abstract

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Here we will cover topics arising from matroid theory. To any graphic matroid, we can associate a certain class of polytopes via its representation forming the well-studied generalized permutahedra. When studying these permutehdra, it has been useful understanding both their normal fans and the support functions. We aim to address this in the more unknown case of the cographic matroids, duals of the graphic matroid. To the associated polytopes, will introduce a characterization for both the normal fans and the support functions. Additionally, with valuated delta matroids, we introduce an object that simultaneously extends valuated matroids and Delta matroids. Both these valuated and delta matroids are themselves a generalization of matroids. We will show that our valuated delta matroids satisfy properties we would expect a generalization should. Furthermore, we discuss the representability of these structures and how this relates to what is already known.

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Chapter 1
INTRODUCTION

Matroids are combinatorial objects whose structure generalizes concepts from both linear algebra and graph theory. A common technique to study matroids is with polytopes that we can associate to them. Here we will present two objects that further extend these connections: polytopes of cographic unimodular systems and valuated delta matroids. Doing so both completes preexisiting generalizations as well as opens the possibility of more extensions.

In Chapter 2, we will review key concepts from both matroid theory and polytopes as well as highlighting some of the notation used here. In Chapter 3 we describe the correspondence between cographic unimodular systems and cographic matroids. We also characterize the polytopes we can associate to this system and discuss questions that we could potentially answer using this characterization. Chapter 4 will introduce our motivation and definition for valuated delta matroids. The rest of the chapter will focus on identifying what we can generalize to these structures from valuated matroids.

1.1 Polytopes of Cographic Unimodular Systems

Matroids are combinatorial objects whose structure generalize concepts from both linear algebra and graph theory. A useful technique for studying matroids is to study different classes of polytopes we can associate to them. One such class of polytopes is the *matroid base polytopes*. We define $M = (E, \mathcal{I})$ to be a *matroid* where E is a finite set and $\mathcal{I} \subseteq 2^E$ satisfies the following properties:

(I1) $\emptyset \in \mathcal{I}$

(I2) If $A \in \mathcal{I}$ and $B \subseteq A$ then $B \in \mathcal{I}$.

(I3) If $A, B \in \mathcal{I}$ and $|B| < |A|$ then there exists $x \in A \setminus B$ such that $B \cup \{x\} \in \mathcal{I}$.

We call the inclusionwise maximal independent sets a *basis* of M, and denote the collection of bases $\mathcal{B}$. To each $B \in \mathcal{B}$ we can associate the indicator vector $e_B \in \mathbb{R}^E$ where $e_B = \sum_{i \in B} e_i$. Here, e_i is the standard basis vector with a 1 in the i^{th} entry and 0 everywhere else. Given a matroid $M = (E, \mathcal{I})$, we define the *matroid base polytope* of M as $P_M = \operatorname{conv}\{e_B \mid B \in \mathcal{B}\}$.

Another well-studied class of polyhedra is the *generalized permutahedra.* These are convex polytopes in $\mathbb{R}^n$ all of whose edges are parallel to $e_i - e_j$ for $i, j \in [n]$ and $i \neq j$. These generalized permutahedra relate to matroid polytopes in the following way.

Theorem 1.1.1. *([9]) Let P be a $0/1-$polytope in $\mathbb{R}^n$(a polytope whose vertices have coordinates 0 or 1). Then P is a generalized permutahedron if and only if $P = P_M$ for some matroid $M = (E, \mathcal{B})$.*

Because all edges of a generalized permutahedron are in the direction $e_i - e_j$ for $i, j \in [n]$ and $i \neq j$, each face of a generalized permutahedron is parallel to some subspace spanned by a subset of $\{e_j - e_i \mid i, j \in [n] : i \neq j\}$. In general, given some collection of subspaces $\mathcal{U}$, we can define an associated class of polyhedra as follows. A polyhedron where each face lies in a translation of a subspace in $\mathcal{U}$ is called a *$\mathcal{U}$-polyhedron.* The collection of $\mathcal{U}$-polyhedra with integer vertices is written $\mathscr{P}h(\mathcal{U}, \mathbb{Z})$. If we denote $\{e_j - e_i \mid i, j \in [n] : i \neq j\}$ as $\mathcal{A}_n$, and $\mathcal{U}(\mathcal{R}) = \{\mathbb{R}V \mid V \subseteq \mathcal{R}\}$, then generalized permutahedra are $\mathcal{U}(\mathcal{A}_n)$-polyhedra.

Note that for most arbitrary collections $\mathcal{U}$, $\mathscr{P}h(\mathcal{U}, \mathbb{Z})$ will not contain polytopes. The fact that the $\mathcal{U}(\mathcal{A}_n)$-polyhedra do follows from $\mathcal{A}_n$ being a unimodular system. We call a collection of vectors $\mathcal{R} \subseteq \mathbb{Z}^n$ *unimodular* if for any linearly independent subset $B \subseteq \mathcal{R}$, B forms a basis over $\mathbb{Z}$ of $\mathbb{R}B \cap \mathbb{Z}^n$. By choosing $\mathcal{R}$ to be a unimodular system, we get that $\mathscr{P}h(\mathcal{U}(\mathcal{R}), \mathbb{Z})$ contains polytopes, and the polytopes in $\mathscr{P}h(\mathcal{U}(\mathcal{R}), \mathbb{Z})$ satisfy some nice properties such as being closed under integer translation, reflection, and taking faces. One can refer to [5] for more on the theory of discrete convexity.

Seeing how there is a nice description for the class of polytopes associated to the unimodular system $\mathcal{U}(\mathcal{A}_n)$, one might wonder what we can say for polytopes associated to other unimodular systems. We answer this question for *cographic unimodular system*. Similar to how the nonzero vectors in $\mathcal{A}_n$ form a representation of graphic matroids, up to a sign difference, the nonzero vectors of cographic unimodular systems form a representation of cographic matroids. We denote this system as $\mathcal{R}_G^*$ where G is the graph our cographic matroid comes from. For both $\mathcal{A}_n$ and $\mathcal{R}_G^*$ the nonzero vectors correspond to edges in the graph G. We will give a construction for both of these systems in Chapter 3.

An important way to characterize polytopes is by understanding their normal fan and their support function. For example, it has been shown that the regions of the normal fans for generalized permutahedra correspond to ordered partitions of $[n]$ and their support functions are the submodular functions. We discuss the ordered partitions in more detail in Chapter 2 and the support functions in Chapter 3.

1.1.1 Main Results

Our main result in Chapter 3 is proving an analogous description for the cographic unimodular system case. Both the normal fan and the support functions are related to partially cyclic orientations of G, i.e. orientations on a subset of edges such that any oriented edge forms part of a cycle. To better understand these normal fans, we look at the hyperplane arrangement they coarsen, $\mathcal{H}^*(G) = \{u_\varepsilon^\perp \mid u_\varepsilon \in \mathcal{U}(\mathcal{R}_G^*)\}$.

Theorem 1.1.2. *[12, Lemma 8.1] There exists a bijective map between the cones of $\mathcal{H}^*(G)$ and the partially cyclic orientations on G.*

This was proven by Greene and Zaslavsky in [12]. Because we use a different representation of $\mathcal{U}(\mathcal{R}_G^*)$, in Chapter 3, we will prove what the explicit bijective map is under our representation.

Theorem 1.1.3. *The support functions of $\mathcal{U}(\mathcal{R}_G^*)$-polyhedra are functions f satisfying*

(1) $\sum f(r_{\mathcal{C}_i}) = \sum f(r_{\mathcal{C}'_j})$ *if* $\sum r_{\mathcal{C}_i} = \sum r_{\mathcal{C}'_j}$ *and all cycles* $\{\mathcal{C}_1, \ldots, \mathcal{C}_k, \mathcal{C}'_1, \ldots, \mathcal{C}'_l\}$ *can be completed to the same totally connected orientation on* G*.*

(2) $f(r_{\mathcal{C}_1}) + f(r_{\mathcal{C}_2}) \geq \sum f(r_{\mathcal{C}'_i})$ *where* $\{\mathcal{C}'_i\}$ *is a circulation decomposition of* $r_{\mathcal{C}_1} + r_{\mathcal{C}_2}$*.*

This description of the polytopes of cographic unimodular systems helps us further study these objects.

1.2 Valuated Delta Matroids

While matroids are already generalizations of concepts from linear algebra, there exist generalizations of matroids too. Valuated matroids are one such well-studied generalization. Matroids can be thought of as functions $\binom{[n]}{r} \to \{0, \infty\}$ satisfying a set of "combinatorial Plücker relations", which essentially reduce to the basis exchange axiom. *Valuated matroids* were defined by Dress and Wenzel [8] as a combinatorial abstraction of matroids represented over a valued field. They are functions $\binom{[n]}{r} \to \mathbb{R} \cup \{\infty\}$ satisfying a set of relations called the *tropical Plücker relations*. For example, if K is a field equipped with a nonarchimedean valuation $\nu : K \to \mathbb{R} \cup \{\infty\}$ and $v_1, v_2, \ldots, v_n$ are vectors which span K^r, then the map

$$\{i_1 < i_2 < \cdots < i_r\} \mapsto \nu(\det(v_{i_1}, \ldots, v_{i_r}))$$

is a valuated matroid. Such valuated matroids are called *representable*. Valuated matroids specialize to ordinary matroids by restricting the codomain to $\{0, \infty\}$.

Delta matroids, or Δ-matroids, are a more combinatorial generalization of matroids. A Δ-*matroid* is a family of subsets of $[n]$, called bases, such that if A, B are bases and $a \in A\Delta B$, then there exists $b \in A\Delta B$ such that $A\Delta\{a, b\}$ is a basis. If in the definition we further require that $a \neq b$, then the family is an *even* Δ-*matroid*.

A common and important way to study matroids is through their polytopes. The *matroid polytope* of a matroid is the convex hull of the indicator vectors of the bases of the matroid. A 0-1 polytope is the matroid polytope of a matroid if and only if all of its edges are in the direction $e_i - e_j$ for some i, j. Analogously, the Δ-*matroid polytope* of a Δ-matroid is the convex hull of the indicator vectors of its bases. A 0-1 polytope is the Δ-matroid polytope of a Δ-matroid if and only if all its edges are in the direction e_i, $e_i - e_j$, or $e_i + e_j$ for some i, j. A 0-1 polytope is the Δ-matroid polytope of an *even* Δ-matroid if and only if all its edges are in the direction $e_i - e_j$ or $e_i + e_j$ for some i, j.

Given a finite set of points $X \subset \mathbb{R}^n$ and a function $p : X \to \mathbb{R} \cup \{\infty\}$, we obtain a polytopal complex in $\mathbb{R}^n$ by projecting the lower faces of the polytope $\operatorname{conv}\{(x, p(x)) : x \in X, p(x) \neq \infty\}$. This complex is called the *subdivision induced by* p, and is a polyhedral subdivision of some polytope with vertices in X. A function $\binom{[n]}{r} \to \mathbb{R} \cup \{\infty\}$ is equivalent to a function $\Delta_{n,r} \to \mathbb{R} \cup \{\infty\}$ where $\Delta_{n,r}$ is the set of all 0-1 vectors in $\mathbb{R}^n$ with r 1's, and thus induces a subdivision in $\mathbb{R}^n$. Such a function is a *valuated matroid* if and only if every polytope in the induced subdivision is a matroid polytope.

It is natural to try to give a common generalization of these notions. Wenzel [19] defines a *strongly valuated Δ-matroid* to be a function $\{0,1\}^n \to \mathbb{R} \cup \{\infty\}$ which satisfies a set of relations called the *tropical Wick relations.* For example, given a skew symmetric matrix over a valued field, the valuations of the principal minors form a strongly valuated Δ-matroid. Rincón [15] shows that a function $\{0,1\}^n \to \mathbb{R} \cup \{\infty\}$ is a strongly valuated Δ-matroid if and only if every polytope in the induced subdivision is an *even* Δ-matroid polytope. Note that Rincón, as well as other authors, refer to Wenzel's strongly valuated Δ-matroids as just valuated Δ-matroids.

Example 1.2.1. Consider the 3×3 symmetric matrix $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ over a field of characteristic $\neq 2$. The principal minors of size 1 are zero and all others are nonzero. The corresponding Δ-matroid has bases $\emptyset, \{1,2\}, \{1,3\}, \{2,3\}$, and $\{1,2,3\}$, which is not even. Indeed, taking $A = \{1,2,3\}$, $B = \emptyset$, and any $a \in A$, we see that this fails to satisfy the *strong* exchange axiom, in which there must exist b so that both $A\Delta\{a,b\}$ and $B\Delta\{a,b\}$ are bases. In this example, the only choice of b for which $A\Delta\{a,b\}$ is a basis is $b = a$, in which case $B\Delta\{a,b\} = \{a\}$ is not. To study such representable Δ-matroids and their valuated analogues, we therefore give up the strong exchange axiom.

In this paper, we call a function $p : \{0,1\}^n \to \mathbb{R} \cup \{\infty\}$ a *valuated* Δ*-matroid* if every polytope in the induced subdivision is a Δ-matroid polytope (not necessarily even). In other words, every edge of the induced regular subdivision has direction $\pm e_i$ or $e_i \pm e_j$ for some $i, j \in [n]$. We note that this conflicts with other recent terminology as mentioned above. However, we believe our definition is the natural definition of a valuated Δ-matroid which is consistent with the history and other terminology of Δ-matroids. As we demonstrate in this paper, these objects are interesting in their own right and generalize previous concepts in the study of Δ-matroids, including non-even Δ-matroids.

1.2.1 Main Results

In Chapter 4 we will prove and discuss the properties we have found to extend from ordinary valuated matroids to our valuated Δ-matroids. We first show how the geometric interpretation of a valuated matroids extends to valuated Δ-matroids. Recall that valuated matroids are functions $\binom{[n]}{r} \to \mathbb{R} \cup \{\infty\}$ satisfying the tropical Plücker relations. This can be restated geometrically a function is a valuated matroid if and only if all edges in its induced subdivision are of length 2, where the length of a vector $(x_1, \ldots, x_n)$ is the L_1-length $\sum_{i=1}^n |x_i|$. The valuated Δ-matroid case is the following.

Theorem 1.2.2. *A function* $p : \{0,1\}^n \to \mathbb{R} \cup \{\infty\}$ *is a valuated* Δ*-matroid if and only if* $\operatorname{dom} p$ *is a* Δ*-matroid and the restriction of* p *to every 4-dimensional face of* $[0,1]^n$ *is a valuated* Δ*-matroid.*

Restated in terms of edges in the subdivision, a function is a valuated Δ-matroid if and only if its induced subdivision has no edges of length 3 or 4.

We define the Δ-*Dressian*, $\Delta\text{-Dr}(n)$, to be the set of all function $\{0,1\}^n \to \mathbb{R} \cup \{\infty\}$ that are valuated Δ-matroids. This is analoguous to the polyhedral fan called the *Dressian* which is formed by the set of all functions $\binom{[n]}{r} \to \mathbb{R} \cup \{\infty\}$ that are valuated matroids. We give bounds on this object and show that it is exponential in n.

Theorem 1.2.3. *The dimension of the Δ-Dressian on $[n]$ is between $\frac{1}{2}2^n$ and $\frac{15}{16}2^n$.*

In [18, Proposition 4.4], the rank function for matroids is shown to be a valuated matroid. We show this also holds for the Δ-matroid analogue of a rank funciton.

Theorem 1.2.4. *Let $M = ([n], \mathcal{F})$ be a Δ-matroid and let $r_M : 2^{[n]} \to \mathbb{Z}_{\geq 0}$ be its rank function. Then $-r_M$ is a valuated Δ-matroid.*

Finally, we discuss the realizibility of these valuated Δ-matroids. Not only do the nonzero principal minors of a Hermitian or Skew-Hermitian matrix realize a Δ-matroid, but valuations on these principal minors give valuated Δ-matroids.

Theorem 1.2.5. *If A is an $n \times n$ Hermitian or skew-Hermitian matrix over K, then the function $p : \{0,1\}^n \to \mathbb{R} \cup \{\infty\}$ given by $p_S = \nu(A_S)$ is a valuated Δ-matroid.*

Chapter 2

BACKGROUND

We will start with giving background on concepts important for the subsequent chapters. The focus in this chapter will be on introductory ideas about Matroids and Polytopes. Throughout the book, we assume a familiarity with graph theory as given in [7].

Notation. For any vector space V over a field $\mathbb{F}$, we use $V^* = \hom(V, \mathbb{F})$ to denote it's dual space. Unless otherwise specified, we will assume $V = \mathbb{R}^d$. We will use e_S to denote the vector $\sum_{i \in S} e_i$ where $S \subseteq [n]$ and e_i is the i^{th} standard basis vector of V. We call e_S the *indicator* vector for S. Similarly $e^S = \sum_{i \in S} e^i$ where $e^i \in V^*$ such that e^i takes a vector to it's i^{th} component. For any collection of vectors B, we will denote the span of B as $\mathbb{R}B$.

2.1 Matroids

Here we will go over basic definitions and theorems on matroid theory needed throughout the book. For more on matroids, refer to [14].

2.1.1 Basic Definitions

Definition 2.1.1. A *matroid* is an ordered pair $(E, \mathcal{I})$, where E is a finite set and $\mathcal{I} \subseteq 2^E$, satisfying the following properties

(I1) $\emptyset \in \mathcal{I}$.

(I2) If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.

(I3) If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$ then there exists an $e \in I_2 \setminus I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

We call the collection $\mathcal{I}$ the *independent sets* of M and E the *ground set*. These are also often denoted in the literature as $\mathcal{I}(M)$ and $E(M)$. A subset of E not in $\mathcal{I}$ is called *dependent*.

Note that (I3) implies that if $I_1, I_2 \in \mathcal{I}$ are two inclusionwise maximal independent sets, then they must have the same cardinality. Such sets are called the *bases* of M. We will give a few properties about them at the end of this section.

Definition 2.1.2. The *rank* of a matroid M is the cardinality of a basis of M. For $X \subseteq E$, we can also define a *rank of* X in M as the cardinality of the largest independent set contained in X. This gives a well-defined function $r_M : 2^M \to \mathbb{Z}$ where $r_M(X)$ is the rank of X in M.

Definition 2.1.3. For a matroid M, we call $X \subseteq E$ a *flat* if for all $e \in E \setminus X$, we have $r(X) < r(X \cup \{e\})$. A *hyperplane* is a flat $F \subseteq E$ such that $r(F) = r(M) - 1$.

It is no coincidence that the definitions we have given about matroids have analogous ones in linear algebra. In fact, the column vectors of a matrix themselves form a matroid.

Example 2.1.4. Let $E = [n]$ and consider an $m \times n$ matrix over a field $\mathbb{F}$ with the columns labeled 1 through n. If we let $\mathcal{I}$ be all the subsets of E such that the corresponding columns are linearly independent in $\mathbb{F}^m$, then $M = (E, \mathcal{I})$ is a matroid. We call a matroid arising from a matrix A in this way the *vector matroid* of A and denote it by $M[A]$.

For example, if A is the following matrix over $\mathbb{R}$

$$\begin{bmatrix} 1 & 0 & 1 & -1 \\ -1 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

then $M[A]$ would have independent sets $\mathcal{I} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{2,3,4\}\}$.

Example 2.1.5. We can also obtain a matroid from a graph $G = (V, E)$. The edge set E of G will be the ground set of our matroid, and the independent sets will be the subsets of

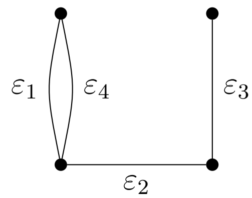


Figure 2.1

E that do not form a cycle. We call this matroid the *cycle matroid* of G, and denote it by $M(G)$. Note that the bases of cycle matroids would be the disjoint unions of spanning trees $T_1 \cup \ldots \cup T_k$ where each T_i is a spanning tree of a different connected component of G.

For example, if G is the graph in Fig. 2.1, then $M(G)$ is the matroid with ground set $E = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ and independent sets $\mathcal{I} = \{\emptyset, \{\varepsilon_1\}, \{\varepsilon_2\}, \{\varepsilon_3\}, \{\varepsilon_4\}, \{\varepsilon_1, \varepsilon_2\}, \{\varepsilon_1, \varepsilon_3\}, \{\varepsilon_2, \varepsilon_3\}, \{\varepsilon_2, \varepsilon_4\}, \{\varepsilon_3, \varepsilon_4\}, \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}, \{\varepsilon_2, \varepsilon_3, \varepsilon_4\}\}$.

Example 2.1.6. Applying Definition 2.1.3 to the cycle matroid of G, we get that a flat of $M(G)$ is a collection X of all the edges within a partition of the vertices, i.e. for all $\varepsilon \in E \backslash X$, there is no path in X connecting the vertices incident to ε. Therefore $r_M(X \cup \{\varepsilon\}) > r_M(X)$ since for all $I \in \mathcal{I}$ such that $I \subseteq X$, $I \cup \{\varepsilon\} \in \mathcal{I}$ as well.

Notice that while the matroids in Examples 2.1.4 and 2.1.5 arise in different ways, they have the same structure, i.e. for any $\{i_1, \ldots, i_k\} \subseteq [4]$, $\{\varepsilon_{i_1}, \ldots, \varepsilon_{i_k}\} \in \mathcal{I}(M(G))$ if and only if $\{i_1, \ldots, i_k\} \in \mathcal{I}(M[A])$.

Definition 2.1.7. Let $M_1 = (E_1, \mathcal{I}_1)$, $M_2 = (E_2, \mathcal{I}_2)$ be two matroids and $\phi : E_1 \to E_2$ be a bijection such that $X \in \mathcal{I}_1$ if and only if $\phi(X) \in \mathcal{I}_2$. Then we call M_1 and M_2 *isomorphic* matroids, written $M_1 \cong M_2$, and the bijection ϕ an *isomorphism*.

Using the G and A from our examples, we can see that $M(G) \cong M[A]$ with the isomorphism $\phi(e_i) = i$. In general, matroids isomorphic to the vector matroid of a matrix over the field $\mathbb{F}$ or the cycle matroid of a graph are called *representable* over $\mathbb{F}$ or *graphic*, respectively.

While not all matroids may be representable or graphic, in our examples, $M[A]$ is both. In fact, if G is a graph, then $M(G)$ is representable over all fields $\mathbb{F}$.

Recall that the incidence matrix A of a directed graph, D, is defined as follows. Its rows are indexed by the vertices of D, columns by the edges of D, and its entries are

$$a_{ij} = \begin{cases} -1 & \text{if the arc } j \text{ is not a loop and points towards the vertex } i \\ 1 & \text{if the arc } j \text{ is not a loop and points away from the vertex } i \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.1.8 ([14])**.** *Let $D(G)$ be any orientation of G, and $\mathbb{F}$ a field. Then $A_{D(G)}$ is a representation of $M(G)$ over $\mathbb{F}$.*

As an immediate corollary we have

Corollary 2.1.9. *For any graph G, $M(G)$ is representable over any field $\mathbb{F}$.*

2.1.2 Basis and Duality

Let us now return to the concept of a basis for a matroid M. We denote the collection of bases by $\mathcal{B}$. They satisfy conditions similar to bases in linear algebra.

Lemma 2.1.10. *[14] Let $\mathcal{B}$ be the set of basis of a matroid $M = (E, \mathcal{I})$. Then*

(B1) $\mathcal{B}$ is nonempty

(B2) For $B_1, B_2 \in \mathcal{B}$ and all $x \in B_1$ there exists a $y \in B_2 \setminus B_1$ such that $B_1 \setminus \{x\} \cup \{y\} \in \mathcal{B}$.

Since all independent sets are contained in a maximally independent one, or a basis, in Example 2.1.4 we could just list out all the bases of $M[A]$, and to find the independent sets we would just look at subsets of these bases. In fact, we only need properties (B1) and (B2) to define a matroid M, i.e. (I1),(I2), and (I3) are exchangeable with (B1) and (B2).

Example 2.1.11. The matroid $M = (E, \mathcal{B})$ where $|E| = n$ and $\mathcal{B} = \{B \subseteq E \mid |B| = r\}$ is called the *uniform matroid*, U_n^r. The independent sets for the uniform matroid on n with rank r is any subset of E with cardinality at most r.

Theorem 2.1.12. *[14] Let E be a set, $\mathcal{B}$ be a subset of 2^E satisfying (B1) and (B2), and $\mathcal{I}$ be the collection of $I \subseteq E$ such that there exists a $B \in \mathcal{B}$ containing I. Then $M = (E, \mathcal{I})$ is a matroid.*

We use this to define the dual of a matroid.

Definition 2.1.13. Let $M = (E, \mathcal{I})$ be a matroid, with $\mathcal{B}$ the set of bases. The *dual matroid* of M is the matroid M^* over ground set E with set of bases

$$\mathcal{B}^* = \{E \setminus B | B \in \mathcal{B}\}.$$

For a graph G, we denote the dual of its graphic matroid $M(G)$ as $M^*(G)$. Matroids isomorphic to $M^*(G)$ for some G are called *cographic matroids*.

Example 2.1.14. Let $G = (V, E)$ be a graph and $M(G)$ its associated cycle matroid. Recall that a basis for $M(G)$ is a union of spanning trees $T_1 \cup \ldots \cup T_\kappa$, where G has κ connected components $\{K_1, \ldots, K_\kappa\}$, and T_i is a spanning tree of K_i. We will refer to this union of spanning trees as a *spanning forest*. By definition of the dual matroid, a basis B^* is a set of edges whose complement is a spanning forest of G. Therefore, the independent sets $\mathcal{I}^*$ of $M^*(G)$ are the set of edges whose complement contains a spanning forest of G.

While the flats of graphic matroids are related to connected components of a subgraph, the flats of cographic matroids are related to edges whose deletion from G increase the number of connected components. We call such an edge a bridge.

Proposition 2.1.15. *Let $M^* = M^*(G) = (E, \mathcal{I}^*)$ for some connected graph $G = (V, E)$. Then $F \subseteq E$ is a flat of M^* if and only if the subgraph $G' = (V, E \setminus F)$ has no bridge.*

Proof. Let $F \subseteq E$ and $G' = (V, E \setminus F)$. Assume that there exists a bridge $\varepsilon \in G'$ connecting connected components K_1 and K_2. Let $P_1, P_2, \dots, P_l, \{\varepsilon\}$ be all the paths from K_1 to K_2 in G and $P = P_1 \cup P_2 \cup \cdots \cup P_l \cup \{\varepsilon\}$. Since ε is a bridge in G', $P \setminus \{\varepsilon\} \subseteq F$. For each path P_i, pick an edge ε_i and let K be the set of all these chosen edges $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_l\}$. Since the subgraph with edges $E \setminus K$ is connected, $K \in \mathcal{I}^*$. Let $I \in \mathcal{I}^*$ be maximal such that $K \subseteq I \subseteq F$. Note that $K \cup \{\varepsilon\}$ disconnected G, and since $K \cup \{\varepsilon\} \subseteq I \cup \{\varepsilon\}$, deleting $I \cup \{\varepsilon\}$ also disconnects our graph G. This implies I is also maximally independent in $F \cup \{\varepsilon\}$, so by definition $r_{M^*}(F) = |I| = r_{M^*}(F \cup \{\varepsilon\})$, and F is not a flat.

Conversely, let $F \subseteq E$ such that the subgraph $G' = (V, E \setminus F)$ does not have any bridges. Pick an arbitrary $\varepsilon \in G'$; the assumptions tell us that there is a cycle $\mathcal{C} \subseteq G'$ containing ε. Since G is connected, there is a spanning tree T of G containing $\mathcal{C} \setminus \{\varepsilon\}$. Let $I \in \mathcal{I}^*$ be maximal so that $(E \setminus T) \cap F \subseteq I \subseteq F$. Since I is independent, its complement contains a spanning tree. In particular one of these spanning trees must contain $\mathcal{C} \setminus \{\varepsilon\}$, implying $I \cup \{\varepsilon\} \in \mathcal{I}$. This means $r(F) = |I| < |I \cup \{\varepsilon\}| \leq r(F \cup \varepsilon)$, and since ε was arbitrary this holds for all $\varepsilon \in E - F$. This is the definition of F being a flat. $\square$

We can also show this description of flats with contractions of a matroid. Refer to [14] for more on matroid contractions. We get the following characterization of maximal proper flats of $M^*(G)$ as a corollary.

Corollary 2.1.16. *The maximal proper flats of a cographic matroid corresponding to the connected graph $G = (V, E)$ are complements of the cycles of G.*

Proof. From Proposition 2.1.15 we know that the flats are subsets of E such that the graph spanned by $E \setminus F$ has no bridge. In other words, if $L \subseteq E$ then by adding all the bridges of $E \setminus L$ we get the smallest flat containing L. Therefore F is a maximally proper flat of M^* if and only if for all $\varepsilon \in E \setminus F$ every edge of $E \setminus (F \cup \{\varepsilon\})$ is a bridge. This is equivalent to $E \setminus (F \cup \{\varepsilon\})$ having no cycles. Since $E \setminus F$ does not contain any bridges, it must also be true that $E \setminus F$ is a cycle. Thus F is a maximally proper flat of M^* if and only if $E \setminus F$ is a cycle. $\square$

There is one more theorem about matroid duals that will be useful.

Theorem 2.1.17. *([14]) Let M be the vector matroid for the matrix $[I_r \mid D]$. Label the columns $c_1, c_2, \ldots, c_n$. Then M^* is the vector matroid of $[-D^T \mid I_{n-r}]$ where the columns are also labeled $c_1, \ldots, c_n$.*

Using the same labeling for both matrices gives us $\{c_1, \ldots, c_n\} \setminus B$ is a basis for M^* if B is a basis for M. For example, if $A = [I_r \mid D]$, then $\{c_1, \ldots, c_r\}$ is a basis for $M[A]$. By definition of the dual matroid, we would want $\{c_{r+1}, \ldots, c_n\}$ to be a basis for $M^*[A]$ which may not be true if we do not fix the column labels.

Any matrix A can be reduced to one of the form $[I_r \mid D]$ by a sequence of the following row operations.

- Interchange two rows.
- Multiply a row by a nonzero element of $\mathbb{F}$.
- Sum a row onto a different one.
- Add or remove a zero row.
- Interchange two columns.

Since applying these row operations onto A do not change the matrix $M[A]$, we can use Theorem 2.1.8 and Theorem 2.1.17 to get the following.

Theorem 2.1.18. *If G is a graph then $M^*(G)$ is representable over any field $\mathbb{F}$.*

We will elaborate more on this representation in later sections.

2.2 Polytopes

Here we cover the necessary definitions about polytopes, and their normal fans. This will include information about the braid arrangement and the generalized permutahedra.

We begin by defining a *polyhedron* P as an intersection of halfspaces. In other words, $P = \{x \in V \mid y_i(x) \leq c_i \ \forall i\}$ where $y_i \in V^*$ and $c_i \in \mathbb{R}$. If the polyhedron is bounded, then we call it a *polytope*. A polytope can equivalently be defined as the convex hull of a finite set of points [20].

Definition 2.2.1. For a polyhedron P, we call a *face of* P a nonempty set $F \subseteq P$ such that there exists a $y \in V^*$, $c \in \mathbb{R}$ such that

- $y(x) = c$ for all $x \in F$
- $y(x) < c$ for all $x \in P \setminus F$.

Its dimension is the dimension of the smallest affine subspace containing it.

A polyhedra is *integer* if every (non-empty) face of P contains an integer point and all it's defining inequalities $y(x) \leq c$ are of the form $y \in (\mathbb{Z}^d)^*$ and $c \in \mathbb{Z}$. This corresponds to polytopes being *integer* if all their vertices are integer.

Example 2.2.2. One particularly useful applications for polytopes is in matroid theory. We can assign to any matroid $M = (E, \mathcal{B})$ a 0/1 polytope $P_M = \text{conv}\{e_B \mid B \in \mathcal{B}\}$ where $\mathcal{B}$ are the bases of M and e_B is the indicator vector. For an example see Fig. 2.2.

Example 2.2.3. We call any nonempty subset $C \subseteq V$ a *cone* if for any $x \in C$, $\lambda x \in C$ for any $\lambda \geq 0$. A *polyhedral cone* is any polyhedra that is also a cone.

For any polyhedra P in V and point $u \in P$, we can associate to it the *normal cone of P at u* which is defined as

$$N_u(P) := \{y \in V^* \mid y(u) = \max_P y\}.$$

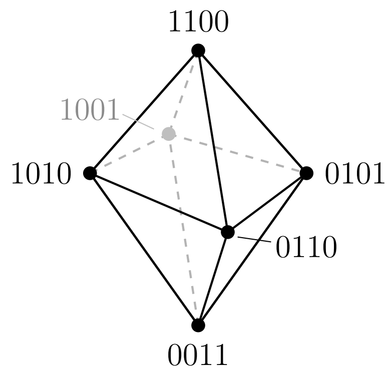


Figure 2.2: The matroid polytope for U_4^2.

For any face of P, we define the normal cone $N_F(P) = N_u(P)$ where $u \in \operatorname{relint} F$. Because along any face of P the normal cone does not change, this is a well-defined definition.

Proposition 2.2.4. *Let $u, v \in \operatorname{relint} F$ where F is a face of P. Then $N_u(P) = N_v(P)$.*

Proof. Let $y \in N_u(P)$. For any point $w \in P$, $y(w) \leq y(u) = \max_P(y)$, so y defines a face F' of P. Since $u \in \operatorname{relint}(F)$, $F \subseteq F'$ and thus $v \in F'$ as well. This implies $y(v) = \max_P(y)$ and $y \in N_v(P)$. Repeat this argument to get the other containment. $\square$

We can define the normal fan of a polyhedra P to be the collection of normal cones along all faces of P and denote this $N(P)$. In Fig. 2.3 there's an example of a polytope, the normal cone of each vertex and the normal fan. In this way, knowing the normal fan is useful when characterizing a class of polyhedra.

More generally, we define *fans* as a family

$$\mathcal{F} = \{A_1, \ldots, A_d\}$$

of nonempty polyhedral cones satisfying

- Every nonempty face of a cone in $\mathcal{F}$ is also a cone in $\mathcal{F}$.

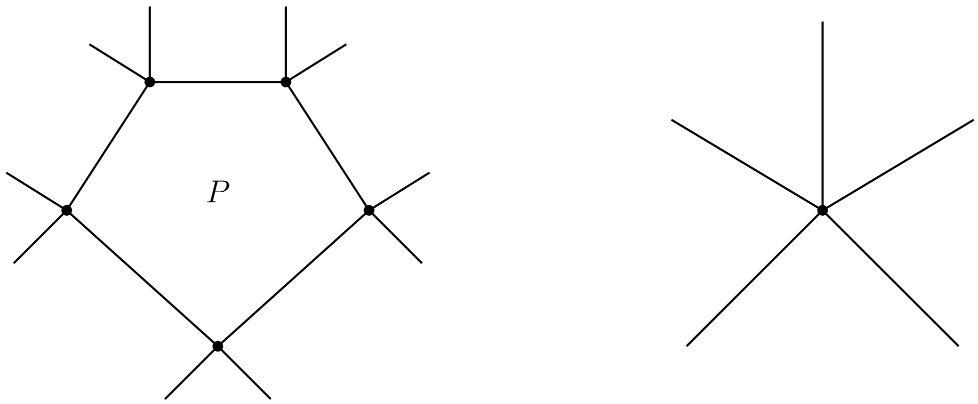


Figure 2.3: On the left is a polytope with the normal fan on the right.

- The intersection of any two cones in $\mathcal{F}$ is a face of both cones.

The *support of* $\mathcal{F}$ is the union of all A_i.

Example 2.2.5. The *braid arrangement* is a d-dimensional hyperplane arrangement in V^* and it is denoted by Σ_n. The hyperplanes in this arrangement are

$$H_{ij} = \{y \in V^* \mid y_i = y_j\}$$

where $y \in V^*$ is written in the dual basis, i.e. $y(e_i) = y_i$. Note that each of these H_{ij} cuts V into 3 spaces

$$H_{ij}^{+} = \{y \in V^* \mid y_i > y_j\}$$
$$H_{ij}^{0} = \{y \in V^* \mid y_i = y_j\}$$
$$H_{ij}^{-} = \{y \in V^* \mid y_i < y_j\}.$$

If we let λ_{ij} be $+, -$, or $=$, the cones of Σ_n have the form $\bigcap_{i<j} H_{ij}^{\lambda_{ij}}$. We can further describe these cones with ordered partition of $[n]$. Let $\pi = \{\pi_1, \ldots, \pi_k\}$ be an ordered partition of $[n]$, the we denote the corresponding cone as $A(\pi)$ and $y \in A(\pi)$ if and only if

- $y_i > y_j$ if $i \in \pi_p$, $j \in \pi_q$ and $p < q$

- $y_i = y_j$ if $i, j \in \pi_p$.

An important class of polyhedra we get from the braid arrangement is the *generalized permutahedra*. These are the polytopes whose normal fans coarsen the braid arrangement. In other words, for all ordered partitions π, there exists some cone A in the normal fan such that $A(\pi) \subseteq A$. These examples will return in Chapter 3 as they motivate the work there.

One final tool we will need for characterizing polytopes is support functions.

Definition 2.2.6. For any polytope P, the *support function of* P is the function $\phi_P : V^* \to \mathbb{R}$ that maps

$$\phi_P(y) = \max_P(y).$$

These support functions are particularly useful because they can be used to uniquely determine our polytope. The following two theorems will be when doing so.

Theorem 2.2.7. *([16, Theorem 13.2.1]) Let P be a polytope in V and ϕ_P its support function. Then*

$$P = \{x \in V : y(x) \leq \phi_P(y) \text{ for all } y \in V^*\}.$$

Theorem 2.2.8. *([16, Theorem 13.2.2] Let M be a polyhedral cone in V^* and $f : M \to \mathbb{R}$ a function. Then f is the support function of a polytope in V if and only if f is nonnegative homoegeneous, convex, and piecewise linear on M.*

Example 2.2.9. We can equivalently characterized the generalized permutahedra in Example 2.2.5 as polyhedra with *submodular* support functions [5, Example 13]. Submodular functions are defined as functions $\phi : V^* \to \mathbb{R}$ such that

$$\phi(e^{S\cap T}) + \phi(e^{S\cup T}) \leq \phi(e^S) + \phi(e^T)$$

for $S, T \subseteq [n]$. We will discuss this example further in Chapter 3.

Chapter 3

COGRAPHIC UNIMODULAR SYSTEMS

In this chapter, we return to Example 2.2.5 and describe its relationship with graphic matroid polytopes. Our main result here is a description for the duals of such polytopes. Specifically, we give a description for their normal fans and support functions. For the rest of the chapter, let M be the free abelian group $\mathbb{Z}^n$. To any collection of vector spaces we can associate a certain class of polyhedra as follows.

3.1 Graphic Unimodular Systems

Definition 3.1.1. Let $\mathcal{U}$ be a collection of vector subspaces of $\mathbb{R}^n$. We call a polyhedron P a *$\mathcal{U}$-polyhedron* if for all faces F of P, the tangent space of F is in $\mathcal{U}$, i.e. $\mathrm{Tan}(F) := \mathbb{R}(F-F) \in \mathcal{U}$.

We will denote the class of $\mathcal{U}$-polyhedra and $\mathcal{U}$-polytopes as $\mathscr{P}h(\mathcal{U})$ and $\mathscr{P}t(\mathcal{U})$ respectively. For integer polyhedra, polyhedra that are equal to the convex hull of their integer points, we denote as $\mathscr{P}h(\mathcal{U},\mathbb{Z})$, and analogously for polytopes as $\mathscr{P}t(\mathcal{U},\mathbb{Z})$.

Example 3.1.2. If $\mathcal{U} = \{\mathbb{R}e_1, \mathbb{R}e_2\}$ then the only $\mathcal{U}$-polytopes are rectangles with sides parallel and perpendicular to either the x or y axis.

In [5], Danilov and Koshevoy present the properties that the collection $\mathcal{U}$ should satisfy so that $\mathscr{P}h(\mathcal{U},\mathbb{Z})$ is closed under summation, as well as the properties that guaranty the class contains polytopes. We are primarily interested in the the latter examples, where there exist polytopes. Although $\mathscr{P}h(\mathcal{U})$ contains polytopes in Example 3.1.2, for most $\mathcal{U}$, $\mathscr{P}h(\mathcal{U})$ will not (excluding the zero-dimensional ones). In order for polytopes to exist, we need enough one dimensional flats. We construct these $\mathcal{U}$ using unimodular systems. In this section, we discuss an example of these systems, called graphic unimodular systems, as well as what is known about their associated polyhedra.

Definition 3.1.3. Let $\mathcal{R} \subseteq M$. We call $\mathcal{R}$ *unimodular* if for any linearly independent $B \subseteq \mathcal{R}$, B is a basis of $\mathbb{R}B \cap M$ over $\mathbb{Z}$. The tuple $(M, \mathcal{R})$ is called a *unimodular system* if $\mathcal{R}$ is a unimodular subset of M. We call *flats* of $(M, \mathcal{R})$ the subspaces of the form $\mathbb{R}B$ where $B \subset \mathcal{R}$, and we denote by $\mathcal{U}(\mathcal{R})$ the collection of all flats of $(M, \mathcal{R})$.

As we explain below, unimodular systems are closely related to *totally unimodular matrices*, matrices whose minors are either 0 or ± 1. For other characterizations of unimodular systems, see [17].

Proposition 3.1.4. *Let A be an $m \times n$ matrix of full rank such that $m \leq n$ and $C = \{c_1, \ldots, c_m\}$ are its columns. Let $\mathcal{R} = \{e_1, \ldots, e_m\} \cup C$. Then A is totally unimodular if and only if $(M, \mathcal{R})$ is a unimodular system.*

Proof. First, let us assume A is totally unimodular. This is true if and only if every $m \times m$ submatrix of $[I \mid A]$ has determinant 0 or ± 1. We want to show that for any linearly independent collection $B = \{e_{i_1}, \ldots e_{i_l}, c_{i_{l+1}}, \ldots, c_{i_m}\}$, B spans $\mathbb{Z}^m$ over $\mathbb{Z}$. Since these columns form an $m \times m$ submatrix of $[I \mid A]$, their determinant is ± 1. Therefore, the n-dimensional volume of the parallelepiped spanned by the vectors of B have determinant 1, and B spans $\mathbb{Z}^m$ over $\mathbb{Z}$.

For the other direction, we can just work the argument in reverse to get that all maximal minors of $[I \mid A]$ have determinant 0 or ± 1. Let B be any square submatrix of A given by deleting rows R and columns C from A. Consider the $m \times m$ matrix M, with columns $\{c_i \mid i \notin C\} \cup \{e_j \mid j \in R\}$. Since this is a maximal submatrix of $[I \mid A]$, $\det(M)$ is 0 or ± 1. Furthermore, due to our choice of e_j, $\det(M) = \det(B)$. Therefore, A is totally unimodular as claimed. $\square$

Corollary 3.1.5. *Let $(M, \mathcal{R})$ be a unimodular system where $\mathcal{R} = \{c_1, \ldots, c_n\}$. Pick a basis $\{c_{i_1}, \ldots, c_{i_m}\}$ of span($\mathcal{R}$). Write each $c_k = \sum_{j=1}^{m} a_{jk} c_{i_j}$. Then the matrix $A = (a_{jk})$ is totally unimodular.*

Using Proposition 3.1.4, the following lemma provides us with many examples of unimodular systems.

Lemma 3.1.6. *[14, Lemma 5.1.4] Let M be a $(0, \pm 1)$ matrix such that each column has at most one 1 and one -1. Then M is a totally unimodular matrix.*

In particular, because the incidence matrix of any directed graph satisfies the conditions of this lemma, we can associate a unimodular system with any graph G. The remainder of this section will be spent on these graphic unimodular systems and how they motivate our work [5, 6]. In the next section we show how we can construct a different unimodular system associated to graphs dual to the graphic unimodular system.

Example 3.1.7. Consider a graph G. Let $D(G)$ is an arbitrary orientation of G. As explained by the previous Lemma and Proposition 3.1.4, $A_{D(G)}$ is a totally unimodular matrix, and its columns are unimodular. If we denote $\{r_1, \ldots, r_k\}$ to be the columns of our incidence matrix, then the unimodular system $(M, \{0, \pm r_1, \ldots, \pm r_k\})$ associated to G is a *graphic unimodular system*, denoted $\mathcal{R}_G$. Note that since flipping the orientation of a edge corresponds to multiplying a column of $A_{D(G)}$ by -1, the orientation on G does not change $\mathcal{R}_G$. We can more precisely define graphic unimodular systems as follows.

Definition 3.1.8. Let $\mathcal{A}_n = \{e_i - e_j | i, j \in [n]\}$. We call *graphic unimodular systems* subsets $\mathcal{R}_G \subseteq \mathcal{A}_n$ such that

- $0 \in \mathcal{R}_G$
- if $r \in \mathcal{R}_G$ then $-r \in \mathcal{R}_G$.

Note that the associated polytopes for graphic unimodular systems, $\mathscr{P}t(\mathcal{U}(\mathcal{R}_G))$, are also known as *generalized permutahedra*. Furthermore, in [10] it was shown that $\mathscr{P}t(\mathcal{U}(\mathcal{R}_G))$ coincides with a different, well-studied class of polytopes called base polytopes. Below we give a proof of for this equivalence in the case of $\mathcal{U}(\mathcal{R}_G)$-polytopes, which will be slightly modified from the one found in [5, Section 7]. Before doing so, we define a base polyhedron. Recall that a *submodular function* is a $\phi : M^* \to \mathbb{R}$ such that for any $S, T \subseteq [n]$, b satisfies

$$\phi\left(e^S\right) + \phi\left(e^T\right) \geq \phi\left(e^{S\cup T}\right) + \phi\left(e^{S\cap T}\right).$$

Definition 3.1.9. A *base polytope* is a polytope whose support functions are submodular, i.e. a polytope of the form

$$P = \left\{x \in M \,|\, e^S(x) \leq \phi\left(e^S\right) \;\forall S \subseteq [n] \text{ and } e^{[n]}(x) = \phi\left(e^{[n]}\right)\right\}$$

where ϕ is a submodular function.

Proposition 3.1.10. *Let $\mathcal{U}$ be the flats generated by a graphic unimodular system, then P is a $\mathcal{U}$-polytope if and only if P is a base polytope.*

Proof. Assume P is a generalized permutahedron and ϕ its support function. To show P is a base polytope, it suffices to show that ϕ is submodular. Recall that since ϕ is the support function it is convex and linear on the cones of Σ_n. Therefore

$$\phi\left(e^S\right) + \phi\left(e^T\right) \geq 2\phi\left(\frac{e^S + e^T}{2}\right).$$

Because $S \cap T \subseteq S \cup T$, $e^{S\cup T}$ is in the same cone as $e^{S\cap T}$ and

$$\phi\left(e^{S\cup T}\right) + \phi\left(e^{S\cup T}\right) = 2\phi\left(\frac{e^S + e^T}{2}\right)$$

which gives us our desired inequality,

$$\phi\left(e^S\right) + \phi\left(e^T\right) \geq \phi\left(e^{S\cup T}\right) + \phi\left(e^{S\cup T}\right).$$

For the other direction, let's assume $P = \left\{x \,|\, e^S(x) \leq \phi\left(e^S\right) \text{ and } e^{[n]}(x) = \phi\left(e^{[n]}\right)\right\}$ where ϕ is a submodular function. We need to show that ϕ is convex and linear on the cones of Σ_n. By definition and extension of ϕ we get linearity, so remains to show that $\phi(v + w) \leq \phi(v) + \phi(w)$. Since we can write any vector in a cone of Σ_n as a linear combination of rays, we can assume v and w are rays e^S and e^T, respectively, for some $S, T \subseteq [n]$.

By submodularity of ϕ we then get

$$\begin{aligned}\phi(v+w) = \phi\left(e^{S\cup T} + e^{S\cap T}\right) &= \phi\left(e^{S\cup T}\right) + \phi\left(e^{S\cap T}\right)\\ &\leq \phi\left(e^{S}\right) + \phi\left(e^{T}\right).\end{aligned}$$

Therefore, P is a generalized permutahedron as claimed. □

These base polytopes are also closely related to matroids via matroid polytopes.

Definition 3.1.11. Let $M = (E, \mathcal{B})$ be a matroid with bases $\mathcal{B}$. We define its *matroid polytope*, P_M, to be the 0-1 polytope $\operatorname{conv}\{e_B \mid B \in \mathcal{B}\}$.

This allows us to identify matroids with a unique polytope (up to isomorphism), and the following theorem shows that these polytopes are exactly $\mathscr{P}t(\mathcal{U}(\mathcal{R}_G), \mathbb{Z})$ when restricted to $\{0,1\}^n$.

Theorem 3.1.12. *([9],[11]) Let P be a 0-1 polytope in $\mathbb{R}^n$, then P is the matroid polytope for some matroid if and only if P is a $\mathcal{U}(\mathcal{R}_G)$-polytope for some graph G.*

This connection between matroids and the polytopes of our graphic unimodular system motivate the work on the dual unimodular system in the next section. We present our findings on cographic unimodular systems, giving a description of the support functions as the cographic analogue to submodular functions and base polyhedra.

3.2 Cographic Unimodular Systems

There are various ways of understanding the cographic unimodular system for a connected graph $G = (V, E)$ where $V = \{v_1, \ldots, v_n\}$ and $E = \{\varepsilon_1, \ldots, \varepsilon_m\}$. We will proceed by taking the dual of the graphical representation of G. For another method, see [1, Section 3].

First assign some orientation on G, and let T be a spanning tree for G. We call edges not in a given spanning tree the *co-tree links* of that tree. Order the co-tree links of T $\{\lambda_1, \ldots, \lambda_{m-n+1}\}$, and pick some ordering for the edges of T. Reorder $E = \{\varepsilon_1, \ldots, \varepsilon_m\}$ so that the first $n-1$ edges belong to the tree and $\lambda_i = \varepsilon_{n-1+i}$. If $\varepsilon = (v_i, v_j) \in E$ we let w_ε

be the indicator vector of edge ε, i.e. $(w_\varepsilon)_i = 1$ and $(w_\varepsilon)_j = -1$. By letting $-\varepsilon = (v_j, v_i)$ we have $w_{-\varepsilon} = -w_\varepsilon$. Note that for each co-tree link λ_i there is a unique directed cycle $\{\lambda_i, \varepsilon_{i_1}, \ldots \varepsilon_{i_{k_i}}\} \subseteq T \cup \lambda_i$. Let B_i be a directed cycle containing λ_i such that for all $j \in [k]$ we orient edges ε_{i_j} or $-\varepsilon_{i_j}$ so that they all lie in the same direction as λ_i in B_i. Notice that the edges in the directed cycle B_i may have a different orientation than the one given in G and are chosen so that B_i has a well-defined direction. This gives us a collection of directed cycles $\{B_1, \ldots, B_{m-n+1}\}$ that we will call the *fundamental cycles of* G. In this chapter, we will reserve the use of B_i for the recently defined fundamental cycles of our graph, while we use $\mathcal{C}$ or $\mathcal{C}_i$ to denote arbitrary directed cycles. Since $w_{\lambda_i} + \sum_{j=1}^{k_i} c_j w_{\varepsilon_{i_j}} = 0$ where c_j is 1 if B_i contains ε_{i_j} and -1 if it contains $-\varepsilon_{i_j}$ rewriting our representation of G in terms of basis $\{w_{e_1}, \ldots, w_{e_{n-1}}\}$ we get.

$$\left[\begin{array}{c|c|c|c} w_{\varepsilon_1} & w_{\varepsilon_2} & \cdots & w_{\lambda_{m-n}} & w_{\lambda_{m-n+1}} \end{array}\right] \longrightarrow \left[\begin{array}{c|c} I_{n-1} & D \end{array}\right]$$

where the column of D corresponding to λ_i is $-\sum_{j=1}^{k_i} c_j e_{i_j}$. Therefore in D, the i^{th} entry of row j of D is -1 if B_i traverses ε_j, 1 if it traverses $-\varepsilon_j$ and 0 if neither. Now using Theorem 2.1.17, the columns of

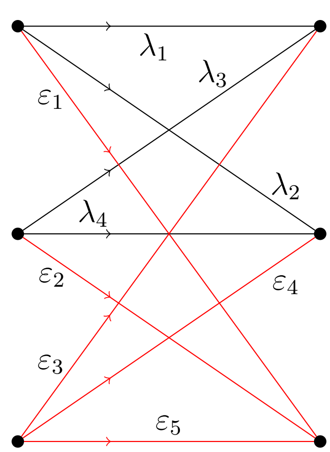
$$\left[\begin{array}{c|c} -D^T & I_{m-n+1} \end{array}\right]$$

form a representation of our cographic matroid $M^*(G)$. The collection of vectors are the columns of the matrix representing each edge $\varepsilon \in E$ as $u_\varepsilon = \sum_i a_i e_i \in \mathbb{R}^{m-n+1}$ where

$$a_i = \begin{cases} 1 & \text{if } B_i \text{ traverses } \varepsilon \text{ in the same direction} \\ -1 & \text{if } B_i \text{ traverses } \varepsilon \text{ in the opposite direction} \\ 0 & \text{if } B_i \text{ does not traverse } \varepsilon. \end{cases}$$

Here each co-tree link λ_i is realized as $e_i \in \mathbb{R}^{m-n+1}$. Using Corollary 3.2 and Lemma 3.3, we have that the collection of our column vectors u_ε form a unimodular system. In particular $\{\pm u_\varepsilon\}_{\varepsilon \in E} \cup \{0\}$ is also a unimodular system which we call $\mathcal{R}_G^*$, the cographic unimodular system associated to G. By this construction we see that the nonzero vectors of $\mathcal{R}_G^*$ form a representation of $M^*(G)$ up to a sign, and because these vectors realize our matroid, more than one ε can correspond to the same u_ε. Note that for the remainder of the section u_ε will refer to these vectors.

Example 3.2.1. Consider the directed graph on the following page with spanning tree $T = \{\varepsilon_1, \ldots, \varepsilon_5\}$ and co-tree links $\{\lambda_1, \ldots, \lambda_4\}$. The unique B_i corresponding to each λ_i are given in Fig. 3.1.



Note that only B_1 and B_2 traverse ε_1. Furthermore, they both traverse ε_1 in the opposite

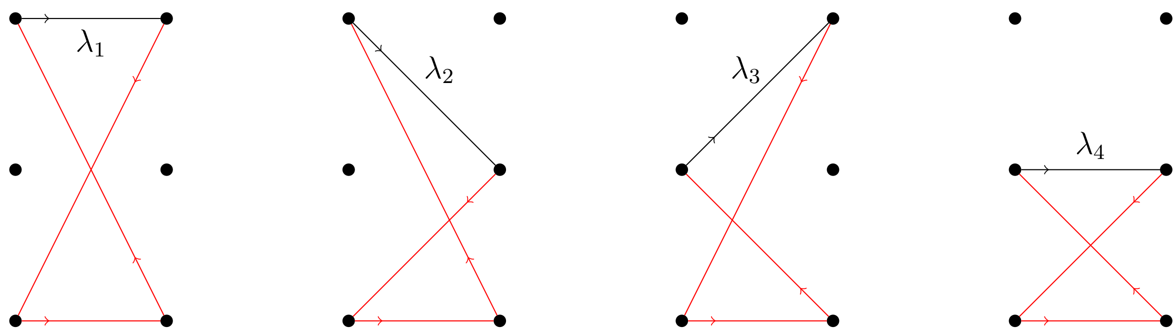


Figure 3.1: Curves B_i for each λ_i co-tree link

direction. This gives us $u_{\varepsilon_1} = -e_1 - e_2$. Repeating this for all other eges of $K_{3,3}$ gives us

$$\begin{aligned}
u_{\varepsilon_2} &= -e_3 - e_4 \\
u_{\varepsilon_3} &= -e_1 - e_3 \\
u_{\varepsilon_4} &= -e_2 - e_4 \\
u_{\varepsilon_5} &= e_1 + e_2 + e_3 + e_4 \\
u_{\lambda_i} &= e_i.
\end{aligned}$$

Our cographic unimodular system is then $\mathcal{R}^*_{K_{3,3}} = \{0, \pm e_1, \ldots, \pm e_4, \pm u_{\varepsilon_1}, \ldots, \pm u_{\varepsilon_5}\}$.

While we define our cographic unimodular system with this represenation for edges, there is a different representation for directed cycles that will be important for understanding the normal fan and support function. We will represent a directed cycle with the vector $\gamma_{\mathcal{C}} \in \mathbb{R}^m$ where

$$(\gamma_{\mathcal{C}})_i = \begin{cases} 1 & \text{if } \varepsilon_i \in \mathcal{C} \\ -1 & \text{if } -\varepsilon_i \in \mathcal{C} \\ 0 & \text{if } \varepsilon_i \notin \mathcal{C} \end{cases}.$$

Definition 3.2.2. Let $v \in V$. We let $\delta_{in}(v) = \{\varepsilon \in E \mid \varepsilon = (v_j, v) \text{ for some } j \in [n]\}$ and $\delta_{out}(v) = \{\varepsilon \in E \mid \varepsilon = (v, v_j) \text{ for some } j \in [n]\}$. We call $\gamma \in \mathbb{R}^E$ a *circulation* on G if for all

$v \in V$

$$\sum_{\varepsilon \in \delta_{in}(v)} (\gamma)_\varepsilon = \sum_{\varepsilon \in \delta_{out}(v)} (\gamma)_\varepsilon.$$

In particular, if $\mathcal{C}$ is a directed cycle then $\gamma_\mathcal{C}$ is a circulation on G. A nice feature of this representation, is that for any directed cycle $\mathcal{C}$, $\gamma_\mathcal{C}$ is in the span of $\{\gamma_{B_i}\}_i$ where our B_i's are the fundamental cycles described above.

Lemma 3.2.3. *For any cycle $\mathcal{C}$ in G,*

$$\gamma_\mathcal{C} = \sum_{i:\lambda_i \in \mathcal{C}} \gamma_{B_i} - \sum_{j:-\lambda_j \in \mathcal{C}} \gamma_{B_j}$$

Proof. Let $\mathbf{C} = \{B_1, \ldots, B_{m-n+1}\}$. We want to show there exists constant b_i's such that

$$\gamma_\mathcal{C} - \sum_{B_i \in \mathbf{C}} b_i \gamma_{B_i} = \vec{0}.$$

By construction of our B_i, it is the only directed cycle in $\mathbf{C}$ containing λ_i. Because of this, by picking the correct b_i's we can cancel out the λ_i entries of $\gamma_\mathcal{C}$. In addition, because we are only adding and subtracting cycles, this difference must be a circulation on T. Since T is a tree this is only possible if the difference is $\vec{0}$ as desired.

Finally, notice that in order to cancel out the λ_i entries,

$$b_i = \begin{cases} 1 & \text{if } \mathcal{C} \text{ traverses } \lambda_i \text{ in the same direction} \\ -1 & \text{if } \mathcal{C} \text{ traverses } \lambda_i \text{ in the opposite direction} \\ 0 & \text{if } \mathcal{C} \text{ does not traverse } \lambda_i \end{cases}$$

giving us our desired linear combination for $\mathcal{C}$. □

3.3 Normal fans and support functions

In this section, we give the main result of the chapter, a description of the support functions of $\mathcal{U}(\mathcal{R}_G^*)$-polytopes. First, we give a bijection between the regions of the hyperplane arrangement $\mathcal{H}^*(G) = \{u_\varepsilon^\perp \mid u_\varepsilon \in \mathcal{U}(\mathcal{R}_G^*)\}$ and a particular set of orientations on G. This bijection was presented by Greene and Zaslavsky in [12, Lemma 8.1]. Note that we can equivalently define $\mathscr{P}t(\mathcal{U}(\mathcal{R}_G^*))$ as the polyhedra whose normal fans coarsen $\mathcal{H}^*(G)$. Therefore, describing the hyperplane arrangement $\mathcal{H}^*(G)$ characterizes the normal fans of $\mathscr{P}t(\mathcal{U}(\mathcal{R}_G^*))$. We will use $\mathcal{O}$ to denote an arbitrary orientation on the edges of a graph.

Definition 3.3.1. Let $G = (V, E)$ be a graph. A *partial cyclic orientation* on G is an orientation on a subset $S \subseteq E$ of edges, such that if $\varepsilon \in S$, then ε forms part of a directed cycle. If $S = E$, then we call the orientation *totally cyclic*.

Given some graph G and region R of $\mathcal{H}^*(G)$, we can fix a partially cyclic orientation $\mathcal{O}_{\mathcal{R}} = \tau(R)$ on G. Pick $x \in \operatorname{relint}(R)$, let $S = \{\varepsilon \in E \,|\, x \cdot u_\varepsilon \neq 0\}$ and orient the edges in S so that $x \cdot u_\varepsilon > 0$. For edges such that $x \cdot u_\varepsilon = 0$, assign no orientation. Below, we give a proof that this $\mathcal{O}_{\mathcal{R}}$ is a partially cyclic orientation and that τ is indeed a bijection. Since our representation for cographic matroids differs from the one found in [12], the description for our hyperplane arrangement is different. The proof found here is theirs retrofitted for our representation of cographic matroids.

Theorem 3.3.2. *([12, Lemma 8.1]) The map τ is a bijection between regions of $\mathcal{H}^*(G)$ and partially cyclic orientations on G.*

For our proof of this theorem, the following Lemma will be useful. We will let $\mathcal{C}$ be any cycle directed by $\mathcal{O}_1$, and define $r_{\mathcal{C}} \in \mathbb{R}^{m-n+1}$ to be $r_{\mathcal{C}} = \sum b_i e_i$ where

$$b_i = \begin{cases} 1 & \text{if in } \mathcal{O}_2 \ \lambda_i \in \mathcal{C} \\ -1 & \text{if in } \mathcal{O}_2 \ -\lambda_i \in \mathcal{C} \\ 0 & \text{otherwise} \end{cases} .$$

Lemma 3.3.3. *Let G be a directed graph and the collection $\{u_\varepsilon\}$ represent the edges as described in Section 3.2. We have that $r_{\mathcal{C}} \cdot u_\varepsilon = (\gamma_{\mathcal{C}})_\varepsilon$, Equivalently, $r_{\mathcal{C}} \cdot u_\varepsilon = 1$ if $\varepsilon \in \mathcal{C}$, -1 if $-\varepsilon \in \mathcal{C}$, and 0 otherwise.*

Proof. Since $u_\varepsilon = \sum a_i e_i$ where $a_i = 1$ if $\varepsilon \in B_i$ and $a_i = -1$ if $-\varepsilon \in B_i$, we get

$$r_{\mathcal{C}} \cdot u_\varepsilon = \sum_{\substack{\lambda_i \in \mathcal{C}:\\ \varepsilon \in B_i}} 1 \;-\; \sum_{\substack{\lambda_i \in \mathcal{C}:\\ -\varepsilon \in B_i}} 1 \;-\; \sum_{\substack{-\lambda_i \in \mathcal{C}:\\ \varepsilon \in B_i}} 1 \;+\; \sum_{\substack{\lambda_i \in \mathcal{C}:\\ -\varepsilon \in B_i}} 1$$

Since the right hand side coincides with the equation from Lemma 3.2.3, $r_{\mathcal{C}} \cdot u_\varepsilon = (\gamma_{\mathcal{C}})_\varepsilon$. □

Proof. (of theorem) Let $G = (V, E)$ be our graph with m edges and n vertices, and R a region of our normal fan. Give G the orientation defined above by $\tau(R) = \mathcal{O}_{\mathcal{R}}$. Let $x \in \operatorname{relint}(R)$ and define $v \in \mathbb{R}^E$ such that $v_\varepsilon = x \cdot u_\varepsilon$. Note that by construction of $\mathcal{O}_{\mathcal{R}}$, $v_\varepsilon > 0$ for all $\varepsilon \in S$. Assume there exists an edge $\varepsilon \in S$ such that ε is not part of a directed cycle. Then under the orientation given by $\mathcal{O}_{\mathcal{R}}$, there must be a bond $E(P, P^c)$ from P to P^c. Notice that by our construction for u_ε, for all vertices $p \in V$, we have $\sum_{\varepsilon : p \in \varepsilon} k_\varepsilon u_\varepsilon = 0$ where $k_\varepsilon = 1$ if $\varepsilon = (\,\cdot\,, p)$ and $k_\varepsilon = -1$ if $\varepsilon = (p, \,\cdot\,)$. Therefore, if we sum the net inflow of P, we get

$$\begin{aligned}
\sum_{p \in P} \left(\sum_{\varepsilon : p \in \varepsilon} k_\varepsilon v_\varepsilon \right) &= \sum_{p \in P} \left(\sum_{\varepsilon : p \in \varepsilon} k_\varepsilon x \cdot u_\varepsilon \right) \\
&= \sum_{p \in P} x \cdot \left(\sum_{\varepsilon : p \in \varepsilon} k_\varepsilon u_\varepsilon \right) \\
&= \sum_{p \in P} x \cdot 0 \\
&= 0.
\end{aligned}$$

Because P is a strongly connected region and all edges in $E(P, P^c)$ point towards P^c, the first term of our sum simplifies to

$$\sum_{p \in P} \left(\sum_{\varepsilon : p \in \varepsilon} k_\varepsilon v_\varepsilon \right) = \sum_{p \in E(P, P^c)} \left(\sum_{\varepsilon : p \in \varepsilon} k_\varepsilon v_\varepsilon \right) = \sum_{p \in E(P, P^c)} \left(\sum_{\varepsilon : p \in \varepsilon} -v_\varepsilon \right).$$

This implies that $\sum_{p \in E(P, P^c)} \left(\sum_{\varepsilon : p \in \varepsilon} -v_\varepsilon \right) = 0$ which is a contradiction because $v_\varepsilon > 0$ for all $\varepsilon \in S$. Thus, $\mathcal{O}_{\mathcal{R}}$ is partially cyclic orientation.

Next, give G some partially cyclic orientation $\mathcal{O}_1$. It remains to show that we get a unique region of our normal fan. Recall that we fixed some orientation $\mathcal{O}_2$ on the edges of

G when constructing u_ε. For notation sake, for any edge ε that has a different orientation in $\mathcal{O}_1$ compared to $\mathcal{O}_2$, we will replace u_ε with $-u_\varepsilon$.

Construct a vector $v \in \mathbb{R}^E$ so that v_ε is the number of directed cycles under the orientation fixed by $\mathcal{O}_1$ that traverse ε or $-\varepsilon$. Note that all such cycles will traverse ε or they will all traverse $-\varepsilon$. We can now choose $x \in \mathbb{R}^{m-n+1}$ so that $x \cdot u_{\lambda_i} = v_{\lambda_i}$. Because $u_{\lambda_i} = e_i$ or $-e_i$, depending on the orientation of λ_i, this is equivalent to letting $|x_{\lambda_i}|$ be the number of cycles traversing λ_i. I claim $x \cdot u_\varepsilon = v_\varepsilon$ for all ε. If so, we are done since the region R, containing this x, is the corresponding region to $\mathcal{O}_1$, i.e. $\tau(R)$ is $\mathcal{O}_1$.

If one directed cycle traverses λ_i, then so do all the other directed cycles (similarly with $-\lambda_i$). Therefore, the λ_i component for all $r_\mathcal{C}$ have the same sign, and, by construction of $r_\mathcal{C}$, we get $\sum_\mathcal{C} (r_\mathcal{C})_{\lambda_i} = x_{\lambda_i}$. Since this holds for each component of x, we get $x = \sum_\mathcal{C} r_\mathcal{C}$. We can therefore deduce $x \cdot v_\varepsilon$ by analyzing $r_\mathcal{C} \cdot v_\varepsilon$. By Lemma 3.3.3, $r_\mathcal{C} \cdot u_\varepsilon$ is ± 1, depending on the orientation of ε in $\mathcal{O}_2$, or 0. However, because we replaced u_ε with $-u_\varepsilon$ if the orientation for ε in $\mathcal{O}_1$ was flipped from $\mathcal{O}_2$, we get that

$$r_\mathcal{C} \cdot u_\varepsilon = \begin{cases} 1 & \text{if } |\varepsilon| \in \mathcal{C} \\ 0 & \text{otherwise} \end{cases}.$$

This means that

$$x \cdot u_\varepsilon = \sum r_\mathcal{C} u_\varepsilon = \text{ number of cycles that contain } \varepsilon$$

which is v_ε, as claimed. As explained above, our partially cyclic orientation $\mathcal{O}_1$ determines a region (the one containing x) giving us a bijection as desired. □

Corollary 3.3.4. *For any partially cyclic orientation $\mathcal{O}$ of G, the dimension of the corresponding region is*

$$(\#\ \textit{of directed edges}) - |V(G)| + (\#\ \textit{of connected components})$$

We can see from this that the rays of our normal fan correspond to directed cycles of G, and these rays span the region of our fan as follows.

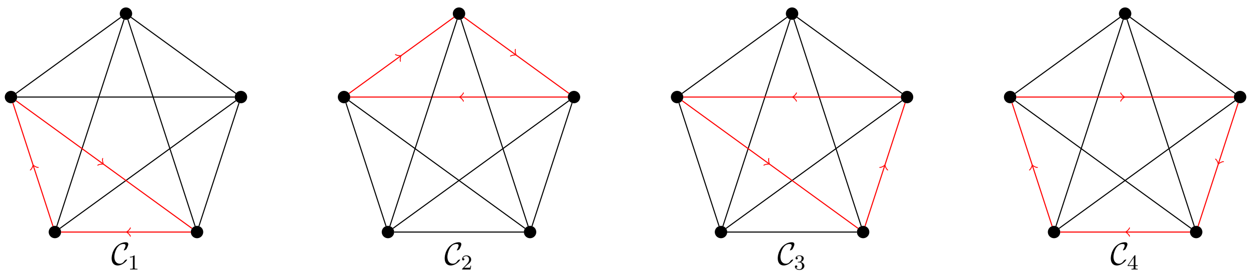


Figure 3.2: $\mathcal{C}_1, \mathcal{C}_2$, and $\mathcal{C}_3$ can all be completed to the same totally cyclic orientation. $\mathcal{C}_4$ can only be completed to the same totally cyclic orienation as $\mathcal{C}_1$.

Proposition 3.3.5. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be directed cycles of G with orientations possibly different from the one originally fixed on the edges of G. The rays $r_{\mathcal{C}_1}$ and $r_{\mathcal{C}_2}$, as defined in the proof of Theorem 3.3.2, are then in the same cone if and only if $\mathcal{C}_1$ and $\mathcal{C}_2$ can be completed to the same totally cyclic orientation on G.*

Proof. Assume $r_{\mathcal{C}_1}$ and $r_{\mathcal{C}_2}$ lie in the same cone and ε is an edge in both $\mathcal{C}_1$ and $\mathcal{C}_2$. From the proof of Theorem 3.3.2, we have shown that $r_{\mathcal{C}_i} \cdot u_\varepsilon = 1$ if $\varepsilon \in \mathcal{C}_i$ and -1 if $-\varepsilon \in \mathcal{C}_i$. Since $r_{\mathcal{C}_1}$ and $r_{\mathcal{C}_2}$ are in the same cone, $r_{\mathcal{C}_1} \cdot u_\varepsilon > 0$ if and only if $r_{\mathcal{C}_2} \cdot u_\varepsilon > 0$. Thus either $\varepsilon \in \mathcal{C}_1 \cap \mathcal{C}_2$ or $-\varepsilon \in \mathcal{C}_1 \cap \mathcal{C}_2$. Since $\mathcal{C}_1$ and $\mathcal{C}_2$ give edges they share a well-defined direction, we can complete both cycles to the same orientation on G.

If $\mathcal{C}_1$ and $\mathcal{C}_2$ can both be completed to the same orientation on G, all edges traversed by both $\mathcal{C}_1$ and $\mathcal{C}_2$ are done so in the same direction for both cycles. This implies that $r_{\mathcal{C}_1} \cdot u_\varepsilon \geq 0$ if and only if $r_{\mathcal{C}_2} \cdot u_\varepsilon \geq 0$, i.e. $r_{\mathcal{C}_1}$ and $r_{\mathcal{C}_2}$ are in the same cone. $\square$

Figure 3.2 shows examples of when cycles may or may not be completed to the same totally cyclic orienation. Furthermore, Corollary 3.3.4 shows that the full-dimensional regions of the hyperplane arrangement correspond to totally cyclic orientations on G, and the boundary separating two of these regions is given by the following.

Lemma 3.3.6. *Let R_1 and R_2 be full dimensional regions of $\mathcal{H}^*(G)$ that share a wall. Then we get from $\tau(R_1)$ to $\tau(R_2)$ by flipping all edges realized by u_ε where $u_\varepsilon^\perp$ is the hyperplane containing that wall.*

Proof. Since the hyperplanes of our arrangement are of the form $u_\varepsilon^\perp$ by construction, we know that $R_1 \cap R_2$ must span some $u_\varepsilon^\perp$, i.e. if $x \in \operatorname{relint} R_1 \cap R_2$ then $x \cdot u_\varepsilon = 0$. The associated partially cyclic orientation to $R_1 \cap R_2$ thus does not give any edge realized by u_ε an orientation. Without loss of generality assume that perturbing x so that $x \cdot u_\varepsilon > 0$ means $x \in \operatorname{relint}(R_1)$, and $x \cdot u_\varepsilon < 0$ means $x \in \operatorname{relint}(R_2)$. As in the proof of Theorem 3.3.2, this implies if we assign one orientation to edges represented by u_ε we get $\mathcal{O}_1$ and if we flip this orientation, we get $\mathcal{O}_2$. $\square$

Similarly to how we identified the vectors $\gamma \in \mathbb{R}^E$ from the previous section with circulations, we also identify $\sum r_{\mathcal{C}}$ with circulations. This is possible because of Lemma 3.3.3 by defining the corresponding γ component wise: $\gamma_\varepsilon = \sum r_{\mathcal{C}} \cdot u_\varepsilon$.

Proposition 3.3.7. *Let γ be a circulation on G. There exists $k_i \geq 0$'s and a collection of directed cycles $\{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$ such that all $r_{\mathcal{C}_i}$'s lie in the same region of our normal fan and*

$$\gamma_\varepsilon = \sum k_i r_{\mathcal{C}_i} \cdot u_\varepsilon.$$

In particular, this means even if two directed cycles, $\mathcal{C}_1$ and $\mathcal{C}_2$, have corresponding rays $r_{\mathcal{C}_1}$ and $r_{\mathcal{C}_2}$ that do not lie in the same cone, we can find directed cycles $\{\mathcal{C}'_1, \ldots, \mathcal{C}'_k\}$ such that our $r_{\mathcal{C}'_i}$'s lie in the same region of the normal fan and $r_{\mathcal{C}'_1} + \ldots + r_{\mathcal{C}'_k}$ gives the same circulation as $r_{\mathcal{C}_1} + r_{\mathcal{C}_2}$, i.e

$$r_{\mathcal{C}_1} + r_{\mathcal{C}_2} = r_{\mathcal{C}'_1} + \cdots + r_{\mathcal{C}'_k}.$$

We call such a set of cycles a *circulation decomposition* for γ, and describe an algorithm to find these decompositions as follows:

1. Select some edge $\varepsilon = \{a, b\}$ with nonzero flow

2. if $\gamma_\varepsilon > 0$:

   (a) add $\{a, b\}$ to our path

   (b) pick a new edge $\{b, c\}$ with positive flow in γ or $\{c, b\}$ with negative flow in γ

   (c) add edge $\{b, c\}$ to our path

   (d) repeat until we revist a vertex v and delete all edges prior to visiting v the first time

   (e) this returns some cycle $\mathcal{C}_1'$ and subtract $r_{\mathcal{C}_1'} \cdot u_\varepsilon$ from γ_ε

   (f) repeat until γ is the zero vector

3. else repeat steps (a)-(f) with edge $\{b, a\}$.

By construction, the orientation of edges that appear in a direced cycle $\mathcal{C}_i'$ is consistent across all directed cycles in $\{\mathcal{C}_1', \ldots, \mathcal{C}_k'\}$. This is equivalent to saying that all $\mathcal{C}_i'$ can be completed to the same totally connected orientation, and so by Lemma 3.3.5 we have that $\{r_{\mathcal{C}_1'}, \ldots r_{\mathcal{C}_k'}\}$ lie in the same cone as claimed. Additionally, this construction allows for directed cycles to appear more than once in the decomposition.

With this description of the hyperplane arrangement $\mathcal{H}^*(G)$ we give characterize the support function of polytopes in $\mathscr{P}h(\mathcal{U}(\mathcal{R}_G^*))$.

Theorem 3.3.8. *The support functions of $\mathcal{U}(\mathcal{R}_G^*)$-polytopes are functions f linear on the rays of $\mathcal{H}^*(G)$ satisfying*

(1) $\sum f(r_{\mathcal{C}_i}) = \sum f(r_{\mathcal{C}_j'})$ if $\sum r_{\mathcal{C}_i} = \sum r_{\mathcal{C}_j'}$ and all directed cycles $\{\mathcal{C}_1, \ldots, \mathcal{C}_k, \mathcal{C}_1', \ldots, \mathcal{C}_l'\}$ can be completed to the same totally connected orientation on G.

(2) $f(r_{\mathcal{C}_1}) + f(r_{\mathcal{C}_2}) \geq \sum f(r_{\mathcal{C}_i'})$ where $\{\mathcal{C}_i'\}$ is a circulation decomposition of $r_{\mathcal{C}_1} + r_{\mathcal{C}_2}$.

Proof. Let f be the support function for a $\mathcal{U}(\mathcal{R}_G^*)$-polyhedra. Let $\{\mathcal{C}_{i_1}, \ldots, \mathcal{C}_{i_k}, \mathcal{C}_{j_1}, \ldots, \mathcal{C}_{j_l}\}$ be a collection of cycles satisfying the constraints of (1). Lemma 3.3.5 shows all the corresponding $r_{\mathcal{C}_i}$ lie in the same region of the normal fan, so by linearity of support functions on the regions of the normal fan, we get $\sum f(r_{\mathcal{C}_{i_k}}) = \sum f(r_{\mathcal{C}_{j_l}})$. For (2), We can use linearity on the fan and convexity of f to get

$$\begin{aligned}\sum f(r_{\mathcal{C}'_i}) &= 2f\left(\sum \frac{r_{\mathcal{C}'_i}}{2}\right) \\ &= 2f\left(\frac{r_{\mathcal{C}_1} + r_{\mathcal{C}_2}}{2}\right) \\ &\leq f(r_{\mathcal{C}_1}) + f(r_{\mathcal{C}_2}).\end{aligned}$$

Now assume f is a function satisfying (1) and (2). To show linearity on the regions of the fan, we want to show that the equalities of (1) are complete. Let

$$r_{\mathcal{C}_1} + \cdots + r_{\mathcal{C}_k} = r_{\mathcal{C}'_1} + \cdots + r_{\mathcal{C}'_l}$$

be an arbitrary equality on the rays of a cone of the normal fan. According to Lemma 3.3.5, since $r_{\mathcal{C}_i}$ and $r_{\mathcal{C}'_j}$ are in the same cone, all cycles $\mathcal{C}_i$ and $\mathcal{C}'_j$ can be completed to the same totally cyclic orientation. Property (1) then implies that $\sum f(r_{\mathcal{C}_i}) = \sum f(r_{\mathcal{C}'_j})$, so our list of equalities is comprehensive.

To show the convexity of f, consider $v, w \in \mathbb{R}^{m-n+1}$. Because it is piecewise linear on each cone, f must be convex or concave. Furthermore, since f is piecewise linear, local convexity implies global convexity. Therefore, it suffices to show that it is convex on the walls, so we can assume $v = r_{\mathcal{C}_1}$ and $w = r_{\mathcal{C}_2}$ for some cycles $\mathcal{C}_1$ and $\mathcal{C}_2$ Let $\{\mathcal{C}'_1, \ldots, \mathcal{C}'_k\}$ be the cycle decomposition of the circulation associated with $r_{\mathcal{C}_1} + r_{\mathcal{C}_2}$. By property (2), we get

$$\begin{aligned}f(r_{\mathcal{C}_1} + r_{\mathcal{C}_2}) &= f\left(\sum r_{\mathcal{C}'_i}\right) \\ &= \sum f(r_{\mathcal{C}'_i}) \\ &\leq f(r_{\mathcal{C}_1}) + f(r_{\mathcal{C}_2}).\end{aligned}$$

Therefore, properties (1) and (2) completely characterize the support functions of $\mathcal{U}(\mathcal{R}^*_G)$-polyhedra as desired. $\square$

3.4 Remaining questions

Notice that the defining equalities for our support function in Theorem 3.3.8 is an infinite list of equalities. Additionally, the condition in (1) for Theorem 3.3.8 is equivalent to saying the equalities $\sum f(r_{\mathcal{C}_i}) = \sum f(r_{\mathcal{C}'_j})$ are generated by circulation γ with more than one circulation decomposition $\{\mathcal{C}_i\}$ and $\{\mathcal{C}'_j\}$. The next natural step would be to search for a finite list of generating equalities. We do so by eliminating equalities coming from circulations where any of its decompositions contain a cycle more than once.

Conjecture 3.4.1. *Let $\{\gamma_i\}$ be a collection of circulations with only positive components on a totally cyclic orientation satisfying*

1. *For each γ_i there exists more than one decomposition $\{\mathcal{C}_j\}$.*

2. *If $\{\mathcal{C}_j\}$ is a decomposition for any of our γ_i, then $\mathcal{C}_j \neq \mathcal{C}_k$ for all $j \neq k$.*

Then, $\{\gamma_i\}$ generates all the equalities of Theorem 3.3.8.

Example 3.4.2. We give an example of Conjecture 3.4.1 for the region of $\mathcal{H}^*(K_5)$ corresponding to Figure 3.3. One can use a computer to show that the circulations

- $r_{\mathcal{C}_7} + r_{\mathcal{C}_6} = r_{\mathcal{C}_2} + r_{\mathcal{C}_5}$
- $r_{\mathcal{C}_1} + r_{\mathcal{C}_2} = r_{\mathcal{C}_8} + r_{\mathcal{C}_9}$
- $r_{\mathcal{C}_1} + r_{\mathcal{C}_6} = r_{\mathcal{C}_3} + r_{\mathcal{C}_9}$
- $r_{\mathcal{C}_5} + r_{\mathcal{C}_8} = r_{\mathcal{C}_3} + r_{\mathcal{C}_7}$

generate the equalities of our support function. Notice that each of these circulations can be decomposed in exactly two ways, each of which is comprised of unique cycles. Because these circulations satisfy Conjecture 3.4.1, the potentially larger list given by this conjecture must also generate the equalities, i.e. Conjecture 3.4.1 must hold for this example.

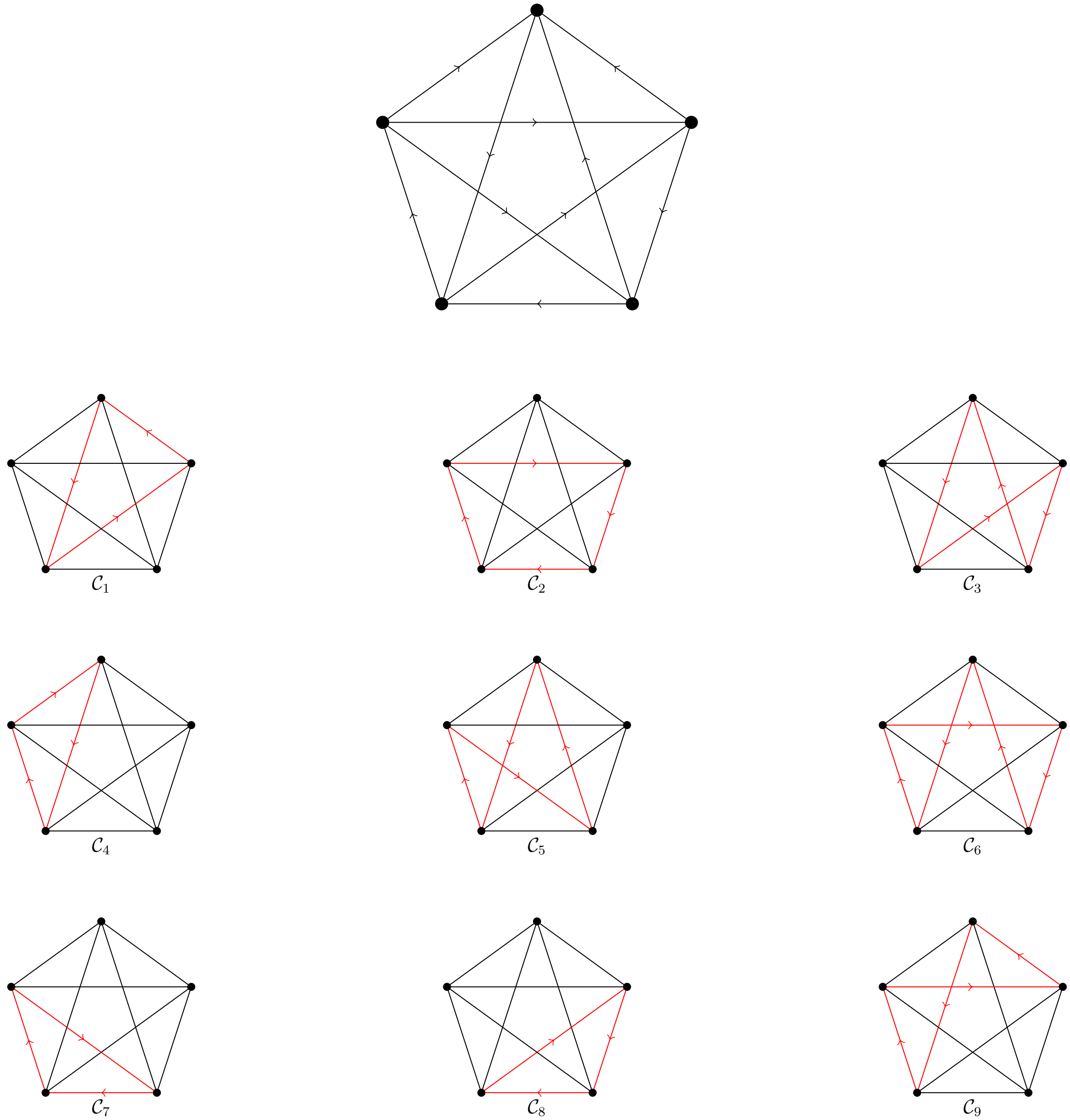


Figure 3.3: The top graph gives a totally cyclic orientation corresponding to a full dimensional region of our fan. The cycles below it correspond to the spanning rays of this particular region.

As there are only finitely many cycles in a finite graph, this conjecture gives us a finite list of generating equations. If proven, this would have implications beyond these cographic polyhedra.

For example, first note that all the linear dependencies on the representation of cycles in G have the form $\sum \alpha_i r_{\mathcal{C}_i} = \sum \beta_i r_{\mathcal{C}'_i}$. Consider then a field $\mathbb{K}$ and the polynomial ring $\mathbb{K}[\ t_{\mathcal{C}} \mid \mathcal{C}$ is a cycle of G]. We call the ideal generated by polynomials of the form

$$\prod t_{\mathcal{C}_i}^{\alpha_i} - \prod t_{\mathcal{C}'_i}^{\beta_i}$$

the *toric ideal* associated with circulation with nonunique decompositions. Our conjecture would therefore imply that this toric ideal is generated by square-free polynomials. Next, one could ask whether a square-free quadratic Gröbner basis also exists for this ideal. I anticipate that these questions could lead to research on Gröbner bases for other toric ideals.

Another potential avenue for research would be to use this description to discover the properties of these polytopes. In particular, in [2], Liu and Backman have shown that all integral generalized permutahedra have regular unimodular triangulations, and I would be interested in determining whether this holds in our dual case as well.

Chapter 4
VALUATED DELTA MATROIDS

The work in this chapter was done with Tracy Chin, Gaku Liu, and Cynthia Vinzant. The content here is nearly identical to what is found in [4]. While in the previous chapter we presented an analogous structure to generalized permutahedra and matroid polytopes, this chapter focuses on matroids and their polytopes. Here we will introduce an object that unifies two common extensions of matroids.

4.1 Valuated and Delta Matroids

In this section we will introduce both valuated and delta matroids, the objects motivating our definition of a valuated delta matroid. Both objects are generalizations of matroids. There are multiple ways to understand these objects, and we are primarily interested in the geometric interpretations.

Delta matroids, or Δ-matroids, can be thought of as the type B analogue of matroids (which are "type A").

Definition 4.1.1. A Δ-*matroid* is a collection, $\mathcal{F}$, of subsets of $[n]$ such that if $A, B \in \mathcal{F}$ then for all $a \in A\Delta B$ there exists $b \in B\Delta A$ such that $A\Delta\{a, b\} \in \mathcal{F}$. We call elements of $\mathcal{F}$ *bases*. If we add the restriction that $a \neq b$ to our condition, then $\mathcal{F}$ is an *even* Δ-*matroid*.

Notice that this is a less restrictive property compared to the basis exchange axiom for matroids. In particular, this definition does not require $a \in A$ and $b \in B$. It is possible for both to belong to A, both to belong to B, or $a = b$. By defining the Δ-matroid polytopes analogously to the matroid polytopes in Example 2.2.2, we get a more geometric interpretation of this generalization.

Definition 4.1.2. The Δ-*matroid polytope* of a Δ-matroid is the convex hull of the indicator vectors of its bases. The exchange property for Δ-matroids means all edges are in the direction e_i, $e_i - e_j$, or $e_i + e_j$ for some i, j. With even Δ-matroid, the edge directions exclude e_i.

These new edge directions arise from the ability to add or delete up to two elements from a basis. From now on we will identify Δ-matroids and Δ-matroid polytopes, referring to both as Δ-matroids.

In this chapter, length of edges will mean the hamming distance between it's endpoints. In other words, given a line segment with endpoints in $\{0, 1\}^n$, it's length is the number of differing components between the endpoints. Using this measure of length, we can restate the condition in Definition 4.1.2 to no edge of our polytope having length greater than 2.

When defining valuated matroids, we use $\mathbb{R} \cup \{\infty\}$, the real line extended by positive ∞. Here we define $+$, max, min and multiplication by a scalar naturally with ∞, $a + \infty = \infty$ and $c \cdot \infty = \infty$.

Definition 4.1.3. A *polytopal complex* is a collection $\mathcal{S}$ of polytopes in $\mathbb{R}^n$ satisfying the following properties.

1. If $P \in \mathcal{S}$ and F is a face of P, then $F \in \mathcal{S}$.

2. If P, $Q \in \mathcal{S}$ and $P \neq Q$, then the relative interiors of P and Q are disjoint.

The elements of $\mathcal{S}$ are called *cells*, and the *support* of $\mathcal{S}$ is $\bigcup_{P \in \mathcal{S}} P$. A polytopal complex whose support is a polytope Q is called a *polytopal subdivision*, or *subdivision*, of Q.

Let $X \subset \mathbb{R}^n$ be a finite set of points and $p : X \to \mathbb{R} \cup \{\infty\}$ any function. The *effective domain* of p is $\operatorname{dom} p := \{x \in X : p(x) \neq \infty\}$. Define a polytope P_p in $\mathbb{R}^{n+1}$ by

$$P_p := \operatorname{conv}\{(x, p(x)) : x \in \operatorname{dom} p\}.$$

We use P_p to construct a subdivision of $\operatorname{conv} \operatorname{dom} p$ in the following manner. Note that projecting $\mathbb{R}^{n+1} \to \mathbb{R}^n$ by deleting the last coordinate maps P_p onto $\operatorname{conv} \operatorname{dom} p$. Moreover, it maps each of the lower faces of P_p (that is, faces of P_p supported by a hyperplane whose inner normal has positive last coordinate) to a polytope in $\operatorname{conv}\operatorname{dom} p$. The collection of the projected lower faces form a polytopal subdivision of $\operatorname{conv}\operatorname{dom} p$, which is called the subdivision *induced* by p. We denote it by $\mathcal{S}_p$.

In [8], *valuated matroids* are defined as functions $\binom{[n]}{r} \to \mathbb{R} \cup \{\infty\}$ satisfying a set of relations called the tropical Plücker relations. By identifying $\binom{[n]}{r}$ with the set of all 0-1 vectors in $\mathbb{R}^n$ with r 1's, $\Delta_{n,r}$, we define a *valuated matroid* as the functions $\Delta_{n,r} \to \mathbb{R} \cup \{\infty\}$ such that every polytope in the induced subdivision is a matroid polytope.

These polytopal subdivisions are important to our definition of valuated delta matroids, so it will be useful to characterize when $\mathcal{S}_p$ contains a simplex. Recall that if x is in the relative interior of the convx hull of an affinely independent set of points I, then x can be expressed uniquely as $x = \sum_{i \in I} \lambda_i i$ where $0 < \lambda_i < 1$ and $\sum_{i \in I} \lambda_i = 1$. We call this the *convex circuit representation* of x with support I. If the elements of I have weights given by a function p, we define

$$p(x, I) := \sum_{i \in I} \lambda_i p(i).$$

Proposition 4.1.4. *Let $X \subset \mathbb{R}^n$ be a finite set of points in convex position, and let $p : X \to \mathbb{R} \cup \{\infty\}$ be any function. Let I be an affinely independent subset of X. The following are equivalent.*

1. *I is the set of vertices of a simplex in $\mathcal{S}_p$.*

2. *For any x in the relative interior of $\operatorname{conv} I$, and for any convex circuit representation of x with different support $J \subseteq X$, we have $p(x, I) < p(x, J)$.*

3. *There exists x in the relative interior of $\operatorname{conv} I$, such that for any convex circuit representation of x with different support $J \subseteq X$, we have $p(x, I) < p(x, J)$.*

Proof. (1) $\implies$ (2). Suppose I is the set of vertices of a simplex in $\mathcal{S}_p$. Then $I_p := \{(x, p(x)) : x \in I\}$ is the set of vertices of a lower face of P_p. Hence, there is a linear functional $(\phi, 1) \in$

$(\mathbb{R}^n)^* \times \mathbb{R}$ and real number b such that $(\phi, 1)(x, y) = b$ for all $(x, y) \in \operatorname{conv}(I_p) \subset \mathbb{R}^n \times \mathbb{R}$, and $(\phi, 1)(x, y) > b$ for all $(x, y) \in P_p \setminus \operatorname{conv}(I_p)$.

Let x be in the relative interior of $\operatorname{conv} I$, and let $\sum_{i \in I} \lambda_i i$ be its convex circuit representation with support I. We have

$$\sum_{i \in I} \lambda_i (\phi, 1)(i, p(i)) = b \implies \phi(x) + p(x, I) = b \implies p(x, I) = -\phi(x) + b.$$

Suppose that x is also in the relative interior of $\operatorname{conv} J$, where $J \subseteq X$ and $J \neq I$. Since X is in convex position, at least one element of J is not in $\operatorname{conv} I$. Let $\sum_{j \in J} \lambda'_j j$ be the convex support representation of x with support J. Then

$$\sum_{j \in J} \lambda'_j (\phi, 1)(j, p(j)) > b \implies \phi(x) + p(x, J) > b \implies p(x, J) > -\phi(x) + b.$$

It follows that $p(x, I) < p(x, J)$, as desired. (If any element of J is not in $\operatorname{dom} p$, then this inequality is obvious as the right side is finite and the left side is ∞.)

(2) $\implies$ (3). Immediate.

(3) $\implies$ (1). We prove the contrapositive. Suppose I is not the set of vertices of a simplex in $\mathcal{S}_p$. Choose any x in the relative interior of $\operatorname{conv} I$. Let $F \subset X$ be the set of vertices of the cell of $\mathcal{S}_p$ whose relative interior contains x. Since x is in the relative interior of $\operatorname{conv} I$, F is not a proper subset of I. In addition, $F \neq I$ by assumption. Hence F contains a point not in I. It follows that there exists $J \subset F$, $J \neq I$, such that J is affinely independent and x is in the relative interior of $\operatorname{conv} J$. (For example, one can consider a pulling triangulation of $\operatorname{conv} F$ starting with a vertex not in I.) Using the same argument as above, we have $p(x, J) \leq p(x, I)$, and thus (3) is not true. $\square$

Definition 4.1.5. We call a function $p : \{0, 1\}^n \to \mathbb{R} \cup \{\infty\}$ a *valuated* Δ*-matroid* if every cell in the induced subdivision $\mathcal{S}_p$ is a Δ-matroid. Equivalently, a function $p : \{0, 1\}^n \to \mathbb{R} \cup \{\infty\}$ is a valuated Δ-matroid if and only if there does not exist edges (1-dimensional cells) of its induced subdivision with length greater than 2.

Slightly more generally, given a subset M of $\{0, 1\}^n$ such that M is a Δ-matroid, a function $p : M \to \mathbb{R} \cup \{\infty\}$ is a *valuated* Δ*-matroid* if every cell in the induced subdivision

$\mathcal{S}_p$ is a Δ-matroid. Crucially, if $p : \{0,1\}^n \to \mathbb{R} \cup \{\infty\}$ is a valuated Δ-matroid, then the restriction of p to any face of $[0,1]^n$ is also a valuated Δ-matroid.

We set some final notation. Given a function $p : \{0,1\}^n \to \mathbb{R} \cup \{\infty\}$, we will often abbreviate the value $p(e_S)$ as p_S. In addition, we will often abbreviate a set $\{a_1, \ldots, a_k\}$ as $a_1 \ldots a_k$; for example $p(e_{\{1,2,3\}})$ would be written p_{123}. For a set $S \subseteq [n]$, we define $S^C := [n] \setminus S$ to be its complement in $[n]$.

4.2 Valuated Delta matroids in n=3 and n=4

Before we describe a characterization for valuated Δ-matroids on $\{0,1\}^n$ for all n, we first work with $n = 3$ and 4. In Section 4.3, we will show that our constraints for $n = 3$ and 4 are enough to characterize valuated Δ-matroids for all n.

Lemma 4.2.1. *A function $p : \{0,1\}^3 \to \mathbb{R} \cup \{\infty\}$ is a valuated Δ-matroid on $[3]$ if and only if the minimum*

$$\min(2p_\emptyset + 2p_{123}, 2p_1 + 2p_{23}, 2p_2 + 2p_{13}, 2p_3 + 2p_{12},$$
$$p_\emptyset + p_{12} + p_{13} + p_{23}, p_1 + p_2 + p_3 + p_{123})$$

is achieved on at least two of its arguments, or on one of the last two arguments.

Proof. We will show the contrapositive in both directions.

First, suppose that $p : \{0,1\}^3 \to \mathbb{R} \cup \{\infty\}$ is not a valuated Δ-matroid. Then the induced subdivision $\mathcal{S}_p$ has an edge of length 3, so $[e_S, e_{S^C}]$ is an edge for some $S \subseteq [3]$. Without loss of generality, assume $[0, e_{123}]$ is an edge. Let $x = (1/2, 1/2, 1/2)$ be the midpoint of this edge. We list all of the convex circuit representations of x using subsets of $\{0,1\}^3$:

$$\frac{1}{2}(e_\emptyset + e_{123}), \frac{1}{2}(e_1 + e_{23}), \frac{1}{2}(e_2 + e_{13}), \frac{1}{2}(e_3 + e_{12}),$$
$$\frac{1}{4}(e_\emptyset + e_{12} + e_{13} + e_{23}), \frac{1}{4}(e_1 + e_2 + e_3 + e_{123})$$

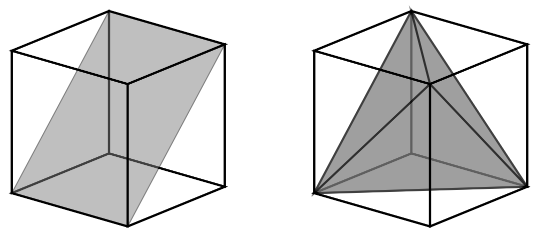


Figure 4.1: Illustration of two cases in Lemma 4.2.1. The left cube is the case where the minimum is achieved on two of the first four arguments, and the right cube is the case where the minimum is achieved uniquely on one of the last two arguments.

By Proposition 4.1.4, $[0, e_{123}]$ being an edge means that in the list of numbers

$$\frac{1}{2}(p_\emptyset + p_{123}), \frac{1}{2}(p_1 + p_{23}), \frac{1}{2}(p_2 + p_{13}), \frac{1}{2}(p_3 + p_{12}),$$
$$\frac{1}{4}(p_\emptyset + p_{12} + p_{13} + p_{23}), \frac{1}{4}(p_1 + p_2 + p_3 + p_{123})$$

the minimum is achieved uniquely at $\frac{1}{2}(p_\emptyset + p_{123})$. In particular, the minimum is not achieved twice nor at one of the last two arguments, as desired.

Conversely, suppose the above minimum is not achieved on at least two arguments nor on one of the last two arguments. Then it must be achieved uniquely on one of the first four arguments, that is, $\frac{1}{2}(p_S + p_{S^C})$ for some $S \subseteq [3]$. By Proposition 4.1.4 applied to the point $x = (1/2, 1/2, 1/2)$, it follows that $[e_S, e_{S^C}]$ is an edge in $\mathcal{S}_p$. Thus $\mathcal{S}_p$ is not a Δ-matroid. $\square$

Before stating the characterization for $n = 4$, we first need to describe the convex circuit representations of $(1/2, 1/2, 1/2, 1/2)$ with support in $\{0,1\}^4$. Let B_n denote the symmetry group of the n-cube. It is generated by permutations of the coordinates and flipping the bits of coordinates. The following can be verified computationally:

Proposition 4.2.2. *The convex circuit representations of* $(1/2, 1/2, 1/2, 1/2)$ *are exactly the* *images of the following formal linear combinations under B_4:*

$$\frac{1}{2}(e_\emptyset + e_{1234}),\ \frac{1}{4}(e_\emptyset + e_{14} + e_{123} + e_{234}),\ \frac{1}{3}e_\emptyset + \frac{1}{6}(e_{123} + e_{124} + e_{134} + e_{234}).$$

Let $\mathcal{J}$ be the set of all supports of convex circuit representations from Proposition 4.2.2. We now state the result for $n = 4$.

Theorem 4.2.3. *A function $p : \{0,1\}^4 \to \mathbb{R} \cup \{\infty\}$ is a valuated Δ-matroid on $[4]$ if and only if both of the following hold:*

1. *p restricts to a valuated Δ-matroid on each facet of $[0,1]^4$.*

2. *For $x = (1/2, 1/2, 1/2, 1/2)$, the minimum*

$$\min_{J \in \mathcal{J}} p(x, J)$$

is achieved on at least two of its arguments.

Proof. Let $\mathcal{J} = \mathcal{J}_1 \sqcup \mathcal{J}_2 \sqcup \mathcal{J}_2$, where $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}_3$ are the orbits of $\{\emptyset, 1234\}$, $\{\emptyset, 14, 123, 234\}$, and $\{\emptyset, 123, 124, 134, 234\}$, respectively, under the action of B_4. We again prove the contrapositive in both directions.

Suppose $p : \{0,1\}^4 \to \mathbb{R} \cup \{\infty\}$ is not a valuated Δ-matroid. Then its induced subdivision $\mathcal{S}_p$ either it has an edge of length 3 or an edge of length 4. If it has an edge of length 3, then this edge is contained in some facet of $[0,1]^4$. Therefore (1) is not true, as desired. Suppose $\mathcal{S}_p$ has an edge of length 4. The set of endpoints of this edge is an element $J \in \mathcal{J}_1$. Therefore, by Proposition 4.1.4, the minimum $\min_{J \in \mathcal{J}} p(x, J)$ is achieved uniquely on J. Hence (2) is not true, proving one direction of Theorem 4.2.3.

For the other direction, suppose that one of (1) or (2) is not true. If (1) is not true, then some facet of $[0,1]^4$ contains an edge of length 3, so p is not a valuated Δ-matroid. Suppose (2) is not true. Let $J \in \mathcal{J}$ be the argument at which the minimum is achieved. If $J \in \mathcal{J}_1$, then by Proposition 4.1.4 $\mathcal{S}_p$ contains an edge of length 4, so p is not a valuated Δ-matroid. If $J \in \mathcal{J}_2$ or $\mathcal{J}_3$, then $\mathcal{S}_p$ contains an image of $[e_\emptyset, e_{123}]$ under B_n, and hence an edge of length 3. So p is not a valuated Δ-matroid, completing the proof. $\square$

4.3 A finite characterization for all n

Here we show that in order to check if $(p_S)_{S\subseteq[n]}$ is a valuated Δ-matroid, it is sufficient to only consider the 4-dimensional faces of $[0,1]^n$. This is analogous to ordinary valuated matroids, where only the 3-dimensional faces of the hypersimplex need to be considered.

Proposition 4.3.1. *Let $\mathcal{S}$ be a subdivision of $[0,1]^n$ with vertices in $\{0,1\}^n$. Let k be a positive integer. If $\mathcal{S}$ has no edges of length ℓ for all $k<\ell\leq 2k$, then it has no edges of length $>2k$.*

Proof. Suppose for the sake of contradiction that $\mathcal{S}$ has no edges of length $k<\ell\leq 2k$ and an edge of length $m>2k$. Choose the smallest such edge e. By applying symmetry, we may assume e has endpoints $e_\emptyset$ and $e_{[m]}$. Now let $\mathcal{S}'$ be the restriction of $\mathcal{S}$ to the face $[0,1]^m$ of $[0,1]^n$. Consider a 2-dimensional cell F of $\mathcal{S}'$ which contains e. The polygon F contains an edge between $e_\emptyset$ and some point of $\{0,1\}^m$ other than $e_{[m]}$, say e_S. By assumption e is the shortest edge of length $>2k$ and $\mathcal{S}$ has no edges of length $k<\ell\leq 2k$, so $|S|\leq k$. Since $\dim F=2$, we have that F lies in the linear span of e_S and $e_{[m]}$. Hence, the only other possible vertex of F is $e_{[m]\setminus S}$. Since $[0,e_{[m]}]$ is an edge, F cannot contain $e_{[m]\setminus S}$, since $[e_\emptyset,e_{[m]}]$ and $[e_S,e_{[m]\setminus S}]$ have the same midpoint. Thus F only has three vertices, so it contains the edge $[e_S,e_{[m]}]$. This edge has length $m-|S|>2k-|S|\geq k$, and also length strictly less than m, a contradiction. $\square$

Applying Proposition 4.3.1 with $k=2$ gives the following immediate corollary.

Corollary 4.3.2. *If a subdivision of $[0,1]^n$ with vertices in $\{0,1\}^n$ has no edges of length 3 or 4, then it has no edges of length >4.*

This gives us a characterization of valuated Δ-matroids whose effective domain is the full cube.

Corollary 4.3.3. *A function $p:\{0,1\}^n\to\mathbb{R}$ is a valuated Δ-matroid if and only if its restriction to every 4-dimensional face of $[0,1]^n$ is a valuated Δ-matroid.*

For general valued Δ-matroids, we have the following characterization.

Theorem 4.3.4. *A function $p : \{0,1\}^n \to \mathbb{R} \cup \{\infty\}$ is a valuated Δ-matroid if and only if* $\operatorname{dom} p$ *is a Δ-matroid and the restriction of p to every 4-dimensional face of* $[0,1]^n$ *is a valuated Δ-matroid.*

Proof. One direction is immediate: if p is a valuated Δ-matroid, then $\operatorname{conv} \operatorname{dom} p$ has no edges of length > 2 and hence is a Δ-matroid polytope. Moreover, the restriction of p to any 4-face of the cube is also a valuated Δ-matroid.

To show the reverse direction, let $p : \{0,1\}^n \to \mathbb{R} \cup \{\infty\}$ be a function such that $\operatorname{dom} p$ is a Δ-matroid and the restriction of p to every 4-dimensional face of $[0,1]^n$ is a valuated Δ-matroid. Suppose $\mathcal{S}_p$ has an edge of length greater than or equal to 5, and let e be such an edge of minimum length m. We may assume e has endpoints $e_\emptyset$ and $e_{[m]}$. Let $\mathcal{S}'$ be the restriction of $\mathcal{S}$ to the face $[0,1]^m$ of $[0,1]^n$. Since $\operatorname{dom} p$ is a Δ-matroid and $\varnothing$, $[m] \in \operatorname{dom} p$, by the symmetric exchange property there exists some $j \in [m]$ such that $\{1, j\} \in \operatorname{dom} p$. In particular, $\mathcal{S}'$ is not the single edge e, and therefore e is contained in a two-dimensional cell of $\mathcal{S}'$. The same argument as in Propostion 4.3.1 leads to a contradiction. $\square$

4.4 Dimension

Recall that the set of all functions $\binom{[n]}{r} \to \mathbb{R} \cup \{\infty\}$ that are valuated matroids form a polyhedral fan called the *Dressian*. In this section we define analogously the set of all functions $\{0,1\}^n \to \mathbb{R} \cup \{\infty\}$ that are valuated Δ-matroids to be the Δ-*Dressian*, Δ-$\mathrm{Dr}(n)$. (This conflicts slightly with the usage in [15].) Δ-$\mathrm{Dr}(n)$ is a polyhedral conical complex in the vector space of all functions $p : \{0,1\}^n \to \mathbb{R}$.

In this section we give bounds on the dimension of Δ-$\mathrm{Dr}(n)$, i.e. the dimension of the largest cone in Δ-$\mathrm{DR}(n)$. Note that even when $n = 3$, we can see that not all maximal cones are the same dimension. In Fig. 4.2, we give an example of two maximal subdivisions. Since the subdivision on the right is a triangulation it's corresponding cone is full-dimensional. The subdivision on the left, on the other hand, is not a triangulation and thus it's maximal

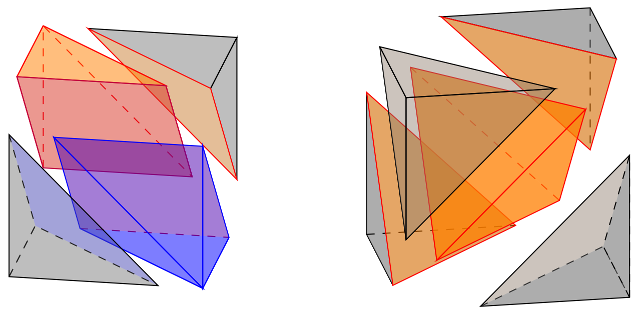


Figure 4.2: Two Δ-matroidal subdivisions of the cube, each corresponding to a maximal cone in Δ-Dr(3).

cone is not full-dimensional.

Since the largest cone is eight dimensional, Δ-Dr(3) is full-dimensional, i.e. the dimension of Δ-Dr(3) is eight. For $n = 4$ we can show that the codimension of Δ-Dr(4) is 1, and in general Δ-Dr(4) is exponential in n.

Proposition 4.4.1. Δ-Dr(4) *has codimension* 1 *in* $\mathbb{R}^{2^4} \cong \mathbb{R}^{16}$.

Proof. Consider the subdivision induced by $p_\emptyset = p_{12} = p_{34} = p_{1234} = 0$, $p_i = p_{ijk} = 100$ for all distinct $i, j, k \in [4]$, and $p_{ij} = 1$ for all $ij \neq 12, 34$. We claim that this induces a Δ-matroid subdivision, and that the associated cone in the Δ-Dressian has codimension 1.

First, we claim that this induces a Δ-matroid subdivision. Indeed, since $\min_{S \subseteq [4]} p_S + p_{S^C}$ is obtained by $[e_\emptyset, e_{1234}]$ and $[e_{12}, e_{34}]$, there are no edges of length 4. Moreover, on any 3-dimensional face of the form $x_l = 0$, setting $\{i, j, k\} = [4] \setminus l$, we have

$$\begin{aligned} 2(p_\emptyset + p_{ijk}) &= 200 \\ 2(p_m + p_{\{i,j,k\}\setminus m}) &\geq 200 \text{ for all } m \in \{i, j, k\} \\ p_\emptyset + p_{ij} + p_{ik} + p_{jk} &\leq 3 \\ p_i + p_j + p_k + p_{ijk} &= 400, \end{aligned}$$

so the minimum is attained uniquely by $p_\emptyset + p_{ij} + p_{ik} + p_{jk}$. Similarly, on the 3-dimensional face $x_l = 1$, setting $\{i, j, k\} = [4] \setminus l$, we have

$$\begin{aligned} 2(p_l + p_{1234}) &= 200 \\ 2(p_{lm} + p_{[4]\setminus m}) &\geq 200 \text{ for all } m \in \{i, j, k\} \\ p_l + p_{ijl} + p_{jkl} + p_{ikl} &= 400 \\ p_{il} + p_{jl} + p_{kl} + p_{1234} &\leq 3, \end{aligned}$$

so the minimum is attained uniquely by $p_{il} + p_{jl} + p_{kl} + p_{1234}$. Thus, by Lemma 4.2.1, there are no edges of length 3, so p is a valuated Δ-matroid.

Moreover, we claim that the functions which induce $\mathcal{S}_p$ form a cone of codimension 1 in Δ-Dr(4). Let $x = (1/2, 1/2, 1/2, 1/2)$. First, we claim that $\min_{J\in\mathcal{J}} p(x, J)$ is attained exactly twice. Indeed, since $p_S > 0$ for all $S \neq \varnothing, 12, 34, 1234$, then $p(x, J) > 0$ for all $J \in \mathcal{J}$ except for $[e_\emptyset, e_{1234}]$ and $[e_{12}, e_{34}]$, and so the minimum is attained exactly twice, as desired.

Moreover, we showed above that the minimum from Lemma 4.2.1 on each three dimensional face is attained uniquely. Hence, the only linear relation satisfied by functions which induce $\mathcal{S}_p$ is $p_\emptyset + p_{1234} = p_{12} + p_{34}$. Thus, the cone of such functions has codimension 1.

Finally, Δ-Dr(4) is not full-dimensional, because any Δ-matroid must satisfy (2) from Theorem 4.2.3. □

In fact, the dimension of the Δ-Dressian is exponential in n.

Theorem 4.4.2. *(Theorem 1.2.3 revisited) The dimension of the Δ-Dressian on $[n]$ is between $\frac{1}{2}2^n$ and $\frac{15}{16}2^n$.*

Proof. First, we claim that the Δ-Dressian has codimension at least 2^{n-4}, which gives the upper bound of $\frac{15}{16}2^n$ on the dimension. Indeed, we note that we can find 2^{n-4} disjoint 4-faces of $[0, 1]^n$. For each $b_5, \ldots, b_n \in \{0, 1\}$, we consider the 4-dimensional face $\{x \in [0, 1]^n : x_5 = b_5, \ldots, x_n = b_n\}$. Each of these faces contributes a codimension-1 constraint on the Δ-Dressian, and these constraints are independent since the faces are pairwise disjoint. Thus, the dimension of the Δ-Dressian is at most $2^n - 2^{n-4} = \frac{15}{16}2^n$.

Next, we claim the dimension is at least $\frac{1}{2}2^n$. Consider the subdivision induced by taking $p_S = 0$ whenever $|S|$ is even, and $p_S > 0$ otherwise. We claim that this is a Δ-matroid subdivision. Indeed, on any 3-face or 4-face of the cube, the minimum of p is attained uniquely on the convex hull of all vertices with an even sum of coordinates, which is a polytope of dimension 3 or 4 respectively. Hence the induced subdivision has no edges of length 3 or 4, so it Δ-matroid subdivision as claimed. Since we allow any positive value on the odd vertices, the cone corresponding to this subdivision has dimension at least 2^{n-1}, as desired. $\square$

4.5 Example: The rank function

One way to characterize matroids is via rank functions. In fact, it has been shown in [18, Proposition 4.4] that the rank function of any matroid gives a valuated matroid. Explicitly, this means any matroid polytope appears as a cell in some matroidal subdivision. Here we will extend these ideas to the rank function for a Δ-matroid.

Definition 4.5.1. Let $M = ([n], \mathcal{F})$ be a Δ-matroid, where $\mathcal{F}$ is the set of bases. Following [13, Section 2.2], we define the *rank function* for M as $r_M : 2^{[n]} \to \mathbb{Z}_{\geq 0}$ where

$$r_M(S) = \max_{B \in \mathcal{F}}(|B \cap S| + |B^C \cap S^C|).$$

By construction, $r_M(S) \leq n$ for all $S \subseteq [n]$, with equality if and only if $S \in \mathcal{F}$.

In this section, we show that if M is a Δ-matroid, then $-r_M$ is a valuated Δ-matroid. Thus, just as in the matroid case, any Δ-matroid polytope appears as a cell in some Δ-matroidal subdivision.

First, we give a geometric interpretation of r_M.

Proposition 4.5.2. *The rank function $r_M(S)$ is equal to*

$$r_M(S) = n - \min_{B \in \mathcal{F}} |e_B - e_S|_1.$$

That is, $r_M(S)$ also has a geometric interpretation in terms of how far e_S is from the closest vertex of P_M.

Proof. We note that $|e_B - e_S|_1 = |B \setminus S| + |S \setminus B|$. Since $|B \cap S| + |B^C \cap S^C| + |B \setminus S| + |S \setminus B| = n$, then

$$\begin{aligned} r_M(S) &= \max_{B \in \mathcal{F}}(|B \cap S| + |B^C \cap S^C|) \\ &= \max_{B \in \mathcal{F}}(n - (|B \setminus S| + |S \setminus B|)) \\ &= n - \min_{B \in \mathcal{F}}(|B \setminus S| + |S \setminus B|) \\ &= n - \min_{B \in \mathcal{F}} |e_B - e_S|_1. \end{aligned}$$

□

In particular, this means that $|r_M(S) - r_M(S\Delta i)| \leq 1$ by the triangle inequality, so adding or removing a single element from S changes the rank by at most one.

Lemma 4.5.3. *Suppose $n \geq 3$, and let $M = ([n], \mathcal{F})$ be a Δ-matroid. If $\max_{S \subseteq [n]}(r_M(S) + r_M(S^C))$ is attained uniquely by some $S \subseteq [n]$, then $S, S^C \in \mathcal{F}$.*

Proof. Assume $r_M(S) + r_M(S^C)$ attains its maximum at S and suppose $S \notin \mathcal{F}$. We wish to show that $\max_{S \subseteq [n]}(r_M(S) + r_M(S^C))$ is not attained uniquely.

Let $B \in \mathcal{F}$ be a set in $\mathcal{F}$ closest to S (i.e. B attains $\min_{B \in \mathcal{F}} |e_B - e_S|_1$), and let $i \in S\Delta B$. Then $|e_B - e_{S\Delta i}| = |e_B - e_S| - 1$, so $r(S\Delta i) = r(S) + 1$. Moreover, adding or removing an element can drop the rank by at most one, so $r(S^C \Delta i) \geq r(S^C) - 1$. Since $S^C \Delta i = (S\Delta i)^C$,

$$r(S\Delta i) + r((S\Delta i)^C) \geq r(S) + r(S^C).$$

We know that $S\Delta i \neq S$, and since $n \geq 3$, we have $S\Delta i \neq S^C$, so $r_M(S) + r_M(S^C)$ cannot attain $\max_{S \subseteq [n]}(r_M(S) + r_M(S^C))$ uniquely. □

Now, we are ready to show that $-r_M$ is a valuated Δ-matroid.

Theorem 4.5.4. *Let $M = ([n], \mathcal{F})$ be a Δ-matroid and let $r_M : 2^{[n]} \to \mathbb{Z}_{\geq 0}$ be its rank function. Then $-r_M$ is a valuated Δ-matroid.*

Proof. If $n = 2$, all functions on $2^{[2]}$ are valuated Δ-matroids, so it suffices to consider the case where $n \geq 3$.

Let $M = ([n], \mathcal{F})$ be a Δ-matroid and suppose that the subdivision induced by $-r_M$ has an edge of length greater than 2. By replacing M with its restriction to the face of $[0,1]^n$ containing this edge, we may assume that this subdivision contains an edge of length n. It follows that $\max_{S \subseteq [n]}(r_M(S) + r_M(S^C))$ is attained uniquely. By Lemma 4.5.3, this implies that it is attained uniquely by some $T, T^C \in \mathcal{F}$.

Since $r_M(S) \leq n$ for all S with equality if and only if $S \in \mathcal{F}$, the Δ-matroid polytope $P_M = \text{conv}\{e_S : S \in \mathcal{F}\}$ must appear as a cell in the induced subdivision, and this cell contains both e_T and e_{T^C}. Since M is a Δ-matroid, all edges of P_M have length at most two, so $[e_T, e_{T^C}]$ cannot be an edge of the subdivision, a contradiction. Thus, $-r_M$ cannot induce an edge of length n, completing the proof. $\square$

4.6 Further Questions

In the previous section we give an example of how properties from valuated matroids translate to valuated Δ-matroids as well. I would be interested in continuing in this direction, and seeing what else translates.

Another example of this would be representability. In [3, Corollary 4.3], it is shown that the subsets of $[n]$ corresponding to nonzero principal minors of Hermitian and skew-Hermitian over $\mathbb{C}$ form Δ-matroids. We give the following analogous theorem. Let K be a field with a *nonarchimedian valuation* $\nu : K \to \mathbb{R} \cup \{\infty\}$, i.e. ν is a map satisfying

1. $\nu(x) = \infty$ if and only if $x = 0$.

2. $\nu(xy) = \nu(x) + \nu(y)$ for all $x, y \in K$.

3. $\nu(x + y) \geq \inf\{\nu(x), \nu(y)\}$.

We will also assume that K has an automorphic involution $a \mapsto \bar{a}$ such that $\nu(a) = \nu(\bar{a})$. We will include the possibility of the involution being the identity map.

Theorem 4.6.1. *(Theorem 1.2.5 revisited) If A is an $n \times n$ Hermitian or skew-Hermitian matrix over K, then the function $p : \{0,1\}^n \to \mathbb{R} \cup \{\infty\}$ given by $p_S = \nu(A_S)$ is a valuated Δ-matroid.*

Proof. Omitted, see [4]. □

There is potential to find other ways to represent valuated delta matroids. We give the following conjecture as an example, and refer the reader to [4] for partial progress on this.

Conjecture 4.6.2. *Let K be a valued field with an automorphic involution preserving the valuation. Let $A = B + C$ where $B \in K^{n\times n}$ is skew-Hermitian, and $C \in K^{n\times n}$ is Hermitian of rank one. Then the valuations of the principal minors of A form a valuated Δ-matroid.*

BIBLIOGRAPHY

[1] I. V. Artamkin. Geometry of unimodular systems, 2023.

[2] Spencer Backman and Gaku Liu. A regular unimodular triangulation of the matroid base polytope. *Duke Mathematical Journal*, pages 1 – 15, 2025.

[3] Petter Brändén. Polynomials with the half-plane property and matroid theory. *Adv. Math.*, 216(1):302–320, 2007.

[4] Nathan Cheung, Tracy Chin, Gaku Liu, and Cynthia Vinzant. Valuated delta matroids and principal minors of hermitian matrices, 2025.

[5] V. I. Danilov and G. A. Koshevoy. Discrete convexity and unimodularity-i. *Advances in Mathematics*, 189(2):301–324, 10 2004.

[6] Vladimir Danilov and Viatcheslav Grishukhin. Maximal unimodular system of vectors. *European Journal of Combinatorics*, 20(6):507–526, 8 1999.

[7] Reinhard Diestel. *Graph Theory.* Graduate texts in Mathematics. Springer Berlin, Heidelberg, 2025.

[8] Andreas W.M. Dress and Walter Wenzel. Valuated matroids. *Adv. Math.*, 93:214–250, 1992.

[9] Jack Edmonds. *Submodular functions, matroids, and certain polyhedra*, pages 69–87. Gordon and Breach, New York, 1970.

[10] András Frank and Éva Tardos. Generalized polymatroids and submodular flows. *Mathematical programming*, 42(1-3):489–563, 1988.

[11] I.M Gelfand, R.M Goresky, R.D MacPherson, and V.V Serganova. Combinatorial geometries, convex polyhedra, and schubert cells. *Advances in Mathematics*, 63(3):301–316, 1987.

[12] Curtis Greene and Thomas Zaslavsky. On the interpretation of whitney numbers through arrangements of hyperplanes, zonotopes, non-radon partitions, and orientations of graphs. *Transactions of the American Mathematical Society*, 280(1):97–126, 1983.

[13] Matt Larson. Rank functions and invariants of delta-matroids, 2023.

[14] James Oxley. *Matroid Theory*. Oxford University Press, 02 2011.

[15] Felipe Rincón. Isotropical linear spaces and valuated Delta-matroids. *J. Combin. Theory Ser. A*, 119(1):14–32, 2012.

[16] R. T. Rockafellar. *Convex Analysis*. Princeton Landmarks in Mathematics and Physics. Princeton University Press, 1970.

[17] Alexander Schrijver. *Theory of linear and integer programming*. John Wiley & Sons, Inc., USA, 1986.

[18] David E. Speyer. Tropical linear spaces. *SIAM J. Discrete Math.*, 22(4):1527–1558, 2008.

[19] Walter Wenzel. Pfaffian forms and δ-matroids. *Discrete Math.*, 115(1–3):253–266, 1993.

[20] Günter M. Ziegler. *Lectures on Polytopes*. Graduate texts in Mathematics. Springer New York, NY, 1994.