The Grothendieck Groups of Module Categories over Coherent Algebras

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Abstract

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Let k be a field and B either a finitely generated free k-algebra, or a regular k-algebra of global dimension two with at least three generators, generated in arbitrary positive degrees. Let $\mathsf{qgr}\, B$ be the quotient category of finitely presented graded right B-modules modulo those that are finite dimensional. We compute the Grothendieck group $K_0(\mathsf{qgr}\, B)$. In particular, if the inverse of the Hilbert series of B (which is a polynomial) is irreducible, then $K_0(\mathsf{qgr}\, B) \cong \mathbb{Z}[\xi] \subset \mathbb{R}$ as ordered abelian groups where ξ is the smallest positive real pole of the Hilbert series of B and where $\mathbb{Z}[\xi]$ inherits its order structure from $\mathbb{R}$. We also obtain general conditions on an algebra B under which our computation of $K_0(\mathsf{qgr}\, B)$ applies.

TABLE OF CONTENTS

Page

Chapter 1: Introduction . . . 1
- 1.1 The main question . . . 1
- 1.2 Background . . . 1
- 1.3 The main results . . . 2
- 1.4 A different question . . . 7
- 1.5 Summary of the dissertation . . . 8

Chapter 2: Preliminaries . . . 9
- 2.1 Graded algebras and modules . . . 9
- 2.2 Coherent algebras . . . 9
- 2.3 Quotient categories . . . 10
- 2.4 The Grothendieck group of an abelian category . . . 11

Chapter 3: The order structure of $K_0(\mathsf{qgr}\, B)$ . . . 14
- 3.1 First conditions on B . . . 14
- 3.2 Second conditions on B . . . 15
- 3.3 Third conditions on B and the main theorem . . . 18
- 3.4 The Hilbert series of finitely presented modules . . . 19

Chapter 4: The case B is a finitely generated free algebra . . . 21
- 4.1 Summary . . . 21
- 4.2 F satisfies C3 . . . 22
- 4.3 The Grothendieck group of $\mathsf{qgr}(F)$ . . . 24
- 4.4 The proof of Proposition 4.2.1 . . . 24
- 4.5 A second way to compute $K_0(\mathsf{qgr}\, F)$ . . . 28
- 4.6 Examples . . . 36

Chapter 5: The case B is a regular algebra of dimension 2 . . . 38
- 5.1 Regular algebras of global dimension 2 . . . 38

5.2 A satisfies C3 39
5.3 The Grothendieck group of $\mathsf{qgr}(A)$ 43
5.4 The proof of Proposition 5.2.1 44
5.5 Examples 53

Bibliography 54

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DEDICATION

to my family and to my friends, Yao and Monica.

Chapter 1

INTRODUCTION

1.1 The main question

Fix a field k. Let B be a non-noetherian regular k-algebra of global dimension two. Let $\mathsf{qgr}\, B$ be the quotient category of finitely presented graded right B-modules modulo those that are finite dimensional. What is the Grothendieck group of $\mathsf{qgr}\, B$ as an ordered abelian group?

1.2 Background

The Grothendieck group $K_0(\mathcal{C})$ of an abelian category $\mathcal{C}$ is an important invariant in many different contexts. For example, Elliott [4] uses Grothendieck groups to classify ultramatricial algebras. In particular, Elliott shows that two ultramatricial algebras are Morita equivalent if and only if the Grothendieck groups of the corresponding module categories are isomorphic as ordered abelian groups. By a result of Smith [14], for B a path algebra of a quiver, $\mathsf{qgr}\, B$ is equivalent to the category of finitely presented right modules over an ultramatricial algebra. Thus $K_0(\mathsf{qgr}\, B)$ will prove a useful tool in the classification of the categories $\mathsf{qgr}\, B$ for B a path algebra.

Classical algebraic geometry provides another example. The Grothendieck group of the category of coherent sheaves plays a central role in the intersection theory of a noetherian scheme. The Artin, Tate and Van den Bergh [1] school of noncommutative projective algebraic geometry substitutes for the category of coherent sheaves over a scheme the category $\mathsf{qgr}\,A$ over a possibly noncommutative algebra A. Mori and Smith [9], [10] and Jørgensen [13] use $K_0(\mathsf{qgr}\,A)$ to construct a noncommutative intersection theory.

Although noetherian algebras generated in degree one have been the primary objects of study in noncommutative projective algebraic geometry, the construction of the category $\mathsf{qgr}\,B$ requires only that the algebra B be coherent. We concern ourselves primarily with describing $K_0(\mathsf{qgr}\,B)$ as an ordered abelian group for coherent non-noetherian algebras B, namely finitely generated free algebras with at least two generators and regular algebras of global dimension two with at least three generators, generated in arbitrary positive degrees. We also provide a set of general conditions for B under which our description of $K_0(\mathsf{qgr}\,B)$ applies.

1.3 The main results

For V a graded k-vector space, let $H_V(t)$ denote the Hilbert series of V. The following theorem is a special case of the main theorem of this dissertation, Theorem 1.3.3.

Theorem 1.3.1. *If B is a regular k-algebra of global dimension two with at least three generators such that the degrees of the generators are positive and relatively prime and $H_B(t)^{-1}$ (which is a polynomial) is irreducible, then the map*

$$K_0(\mathsf{qgr}\,B) \to \mathbb{Z}[\xi], \quad [\pi^* M] \mapsto q_M(\xi)$$

is an isomorphism of ordered abelian groups where ξ is the smallest positive real pole of $H_B(t)$, π^* is the quotient functor of $\mathsf{qgr}\,B$ and $q_M(t) := H_M(t)H_B(t)^{-1}$.

1.3.1 The general conditions

The general conditions are as follows.

Condition C1. We say an $\mathbb{N}$-graded k-algebra B satisfies C1 if it

- is coherent,
- is finitely generated,
- is connected-graded, and
- has finite global dimension.

If B satisfies C1 then $H_B(t)^{-1}$ is a polynomial $h(t) \in \mathbb{Z}[t]$ and for each finitely presented graded right B-module M, the Hilbert series of M is $H_M(t) = q_M(t)H_B(t)$ for some $q_M(t) \in \mathbb{Z}[t, t^{-1}]$.

Condition C2. We say an $\mathbb{N}$-graded k-algebra B satisfies C2 if

- it satisfies C1,
- $\dim_k B_n \geq 1$ for all $n \gg 0$, and
- $h(t) = H_B(t)^{-1}$ has a real root ξ such that
    - ξ is the only root of $h(t)$ in the interval $[0, 1]$,
    - ξ is a simple root, and
    - $\xi < |\lambda|$ for every other root λ of $h(t)$.

Proposition 1.3.2. *If B is an $\mathbb{N}$-graded k-algebra that satisfies C2 and M is a finitely presented graded B-module, then $q_M(\xi) \geq 0$ with equality if and only if M is finite dimensional.*

Condition C3. We say an $\mathbb{N}$-graded k-algebra B satisfies C3 if

- it satisfies C2 and
- for each $p \in \mathbb{Z}[t, t^{-1}]$ such that $p(\xi) > 0$, there is a finitely presented graded right B-module M such that $q_M(t) - p(t) \in (h)$.

1.3.2 The Grothendieck group

Let B be an $\mathbb{N}$-graded k-algebra that satisfies C2. Make $\mathbb{Z}[t^{\pm 1}]/(h)$ an ordered abelian group by defining

$$\left(\frac{\mathbb{Z}[t^{\pm 1}]}{(h)}\right)_{\geq 0} := \left\{\overline{p(t)} \mid p(\xi) > 0\right\} \cup \{0\}$$

where $\overline{p(t)}$ is the image of $p(t) \in \mathbb{Z}[t^{\pm 1}]$ in $\mathbb{Z}[t^{\pm 1}]/(h)$.

Theorem 1.3.3. *Suppose B is an $\mathbb{N}$-graded k-algebra that satisfies C3. The Grothendieck group $K_0(\mathsf{qgr}\, B)$ is isomorphic as an ordered abelian group to*

$$\frac{\mathbb{Z}[t^{\pm 1}]}{(h)}$$

via the map $[\pi^* M] \mapsto \overline{q_M(t)}$ where π^* is the quotient functor of $\mathsf{qgr}\, B$. If $h(t)$ is irreducible, $K_0(\mathsf{qgr}\, B)$ is isomorphic as an ordered abelian group to $\mathbb{Z}[\xi]$ via the map $[\pi^* M] \mapsto q_M(\xi)$.

Furthermore, under the isomorphism(s), the shift functor $\mathcal{M} \mapsto \mathcal{M}(1)$ on $\mathsf{qgr}\, B$ corresponds to multiplication by t^{-1} and multiplication by ξ^{-1}.

1.3.3 Algebras that satisfy C3

Theorem 1.3.4. *If A is either*

1. *a finitely generated free k-algebra such that the degrees of the generators are positive and relatively prime, or*

2. *a regular k-algebra of global dimension two with at least three generators such that the degrees of the generators are positive and relatively prime,*

then A satisfies C3. Consequently, Theorem 1.3.3 describes $K_0(\mathsf{qgr}\, A)$.

The condition that the the degrees of the generators be relatively prime is mild. If B is an $\mathbb{N}$-graded k-algebra, there exists an $\mathbb{N}$-graded k-algebra B' with degrees of generators relatively prime such that $\mathsf{qgr}\, B$ is equivalent to a finite direct sum of copies of $\mathsf{qgr}\, B'$.

1.3.4 Algebras that do not satisfy C3

Suppose B and B' are $\mathbb{N}$-graded k-algebras such that $\mathsf{qgr}\, B$ and $\mathsf{qgr}\, B'$ are equivalent and B satisfies C3. It is not necessarily the case that B' satisfies C3.

Theorem 1.3.5 ([7, Theorem 1.1])**.** *If $\mathbf{C}$ and $\mathbf{C}'$ are two of the five classes below and B belongs to $\mathbf{C}$, then there is an algebra B' in $\mathbf{C}'$ and an equivalence $F : \mathsf{qgr}\, B \to \mathsf{qgr}\, B'$.*

- *Path algebras of finite quivers with grading induced by declaring that all arrows have degree 1; this implies that the degree of a path is equal to its length.*

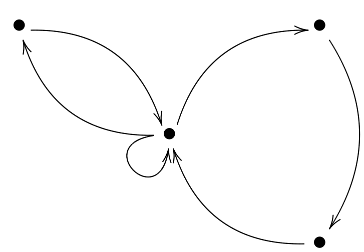
- *Weighted path algebras of finite quivers—this is a path algebra with grading given by assigning each arrow a degree ≥ 1.*

- *Monomial algebras: these are algebras of the form kQ/I where kQ is a weighted path algebra of a finite quiver and I is an ideal generated by a finite set of paths.*

- *Connected monomial algebras: these are monomial algebras kQ/I in which Q has only one vertex.*

- *Connected monomial algebras that are generated by elements of degree 1.*

A **wedge of cycles** is a union of cycles that share a vertex. For example, the quiver



is a wedge of three cycles of lengths one, two and three.

Theorem 1.3.6. *If Q is a wedge of n cycles of lengths $c_1, \dots, c_n$ and* $\gcd\{c_1, \dots, c_n\} = 1$ *then*

$$K_0(\mathsf{qgr}\, kQ) \cong \mathbb{Z}[\xi]$$

as ordered abelian groups where ξ is the smallest positive real root of $1 - \sum_{i=1}^{n} t^{c_i}$.

Theorem 1.3.6 follows from the proof of Theorem 1.3.5 (which shows

$$\mathsf{qgr}\, kQ \equiv \mathsf{qgr}\, k\langle x_1, \dots, x_n\rangle$$

where $\deg x_i = c_i$) and Theorem 1.3.3 applied to $k\langle x_1, \dots, x_n\rangle$. Since $\dim_k (kQ)_0 \geq 2$ if $c_i \geq 2$ for some i, kQ does not satisfy C3 in general.

1.4 A different question

Let B be a graded k-algebra and let $\mathsf{gr}\, B$ denote the category of finitely presented graded right B-modules. What are the possible Hilbert series of objects in $\mathsf{gr}\, B$?

Theorem 1.4.1. *Let B be an $\mathbb{N}$-graded k-algebra that satisfies condition C3. Let $h(t) = (H_B(t))^{-1} \in \mathbb{Z}[t]$ and let ξ be the smallest positive real root of $h(t)$. If $M \in \mathsf{gr}\, B$, then $H_M(t) = q(t)/h(t)$ for some $q(t) \in \mathbb{Z}[t^{\pm 1}]$ such that $q(\xi) \geq 0$, with equality if and only if M is finite dimensional.*

The converse is not true. In particular, suppose $q(t) \in \mathbb{Z}[t^{\pm 1}]$ such that $q(\xi) > 0$ and let $g(t)$ be the formal Laurent series $g(t) = q(t)/h(t)$. It is not necessarily the case that there exists $M \in \mathsf{gr}\, B$ such that $g(t) = H_M(t)$. In fact, $g(t)$ may have negative integer coefficients. However, we prove the following partial converse.

If $g(t)$ is a Laurent series and $n \in \mathbb{Z}$, define $g(t)_{\geq n}$ to be the sub-series of $g(t)$ containing the terms of degree at least n.

Theorem 1.4.2. *Let B be as in Theorem 1.4.1. If $q(t) \in \mathbb{Z}[t^{\pm 1}]$ such that $q(\xi) > 0$, then*

$$\left(\frac{q(t)}{h(t)}\right)_{\geq N} = H_M(t)$$

for some $M \in \mathsf{gr}\, B$ and some $N \in \mathbb{Z}$.

The proofs of the converse in the case that B is free and the case that B is regular of global dimension two are not strictly constructive (though they provide a loose algorithm). Thus no formula for N or M is given.

1.5 Summary of the dissertation

The dissertation is organized as follows. In Chapter 2, we give preliminary definitions and results.

In Chapter 3, we define general conditions on an algebra B under which our method for computing $K_0(\mathsf{qgr}\, B)$ works. We then describe $K_0(\mathsf{qgr}\, B)$ for B an algebra satisfying those conditions.

In Chapter 4, we show that if F is a finitely generated free algebra such that the degrees of the generators are positive and relatively prime, then F satisfies condition C3, and as a corollary to the general result, we compute $K_0(\mathsf{qgr}\, F)$. A second method for computing $K_0(\mathsf{qgr}\, F)$ involves presenting $\mathsf{qgr}\, F$ as the category of modules over an ultramatricial algebra. We show that these two methods produce isomorphic ordered abelian groups. Finally, we apply our computation to specific examples.

In Chapter 5, we show that if A is a regular algebra of global dimension two with at least three generators such that the degrees of the generators are positive and relatively prime, then A satisfies condition C3, and thus we compute $K_0(\mathsf{qgr}\, A)$. We also apply our computation to specific examples.

Chapter 2

PRELIMINARIES

We fix a field k for the rest of the dissertation.

2.1 Graded algebras and modules

Definition 2.1.1. Let $V = \oplus_{i\in\mathbb{Z}} V_i$ be a $\mathbb{Z}$-graded k-vector space.

The **Hilbert series** of V is the formal Laurent series $H_V(t) := \sum_{i\in\mathbb{Z}}(\dim_k V_i)t^i$.

For $n \in \mathbb{Z}$, define $V(n)$ to be the $\mathbb{Z}$-graded k-vector space that is equal to V as a k-vector space but is graded by $V(n)_i = V_{n+i}$. We call $V(n)$ the **shift** of V by n.

Definition 2.1.2. If $g(t) = \sum_{i\in\mathbb{Z}} a_i t^i$ and $n \in \mathbb{Z}$, define

$$g(t)_{\geq n} := \sum_{i\geq n} a_i t^i.$$

Definition 2.1.3. Let B be a graded k-algebra. We denote by $\mathsf{gr}\, B$ the category of finitely presented graded right B-modules. We denote by $\mathsf{fdim}\, B$ the subcategory of $\mathsf{gr}\, B$ consisting of finite dimensional graded right B-modules.

2.2 Coherent algebras

Definition 2.2.1. A graded k-algebra B is **graded right coherent** if every homogeneous finitely generated right-sided ideal of B is finitely presented.

Remark 2.2.2. From now on, by coherent we mean graded right coherent.

Example 2.2.3. Every graded noetherian algebra is coherent. Free algebras and path algebras are coherent. By [11, Theorem 4.1], every algebra defined by a single homogeneous quadratic relation is coherent.

Theorem 2.2.4 ([11, Theorem 2.1])**.** *Let B be a graded k-algebra. The following are equivalent:*

- *B is coherent;*
- *every finitely generated graded submodule of a finitely presented graded right B-module is finitely presented;*
- *$\mathsf{gr}\, B$ is abelian.*

2.3 Quotient categories

Let $\mathcal{C}$ be an abelian category.

Definition 2.3.1. A full abelian subcategory $\mathcal{B} \subset \mathcal{C}$ is called a **dense** or **Serre subcategory** if it is closed under subobjects, quotients and extensions: that is, if $0 \to M \to N \to P \to 0$ is exact in $\mathcal{C}$ then $N \in \mathcal{B}$ if and only if $M, P \in \mathcal{B}$.

Example 2.3.2. If B is a graded coherent k-algebra, then $\mathsf{fdim}\, B \subset \mathsf{gr}\, B$ is a Serre subcategory.

Definition 2.3.3. Let $\mathcal{B} \subset \mathcal{C}$ be a Serre subcategory. The **quotient category** $\mathcal{C}/\mathcal{B}$ is an abelian category with an exact functor $\pi^* : \mathcal{C} \to \mathcal{C}/\mathcal{B}$ (called the **quotient functor**) such that

- $\pi^*(M) \cong 0$ for all $M \in \mathcal{B}$, and
- if $F : \mathcal{C} \to \mathcal{A}$ is an exact functor such that $F(M) \cong 0$ for all $M \in \mathcal{B}$, then there is a unique exact functor $F' : \mathcal{C}/\mathcal{B} \to \mathcal{A}$ such that $F = F' \circ \pi^*$.

The quotient category exists. See for example [12, §4.3] for an explicit construction.

In this dissertation, we study the quotient category

$$\mathsf{qgr}\, B := \frac{\mathsf{gr}\, B}{\mathsf{fdim}\, B}$$

for B a graded coherent k-algebra.

2.4 The Grothendieck group of an abelian category

Let $\mathcal{C}$ be an abelian category.

Definition 2.4.1 ([16, Definition 6.1.1])**.** The **Grothendieck group** of $\mathcal{C}$, denoted $K_0(\mathcal{C})$, is the free abelian group on generators $\{[M] \mid M \in \mathcal{C}\}$ modulo the relations $[M] = [N] + [P]$ for all short exact sequences $0 \to N \to M \to P \to 0$ in $\mathcal{C}$.

Definition 2.4.2. Let G be an abelian group. An **additive function** from $\mathcal{C}$ to G is a function $f : \mathcal{C} \to G$ such that $f(M) = f(N) + f(P)$ for all short exact sequences $0 \to N \to M \to P \to 0$ in $\mathcal{C}$.

Example 2.4.3. The dimension function $\dim_k(V)$ from the category of finite dimensional k-vector spaces to $\mathbb{N}$ is an additive function.

The map $\mathcal{C} \to K_0(\mathcal{C})$, $M \mapsto [M]$, is additive by definition, and satisfies the following univeral property.

Theorem 2.4.4. *[16, Universal Property 6.1.2] If $f : \mathcal{C} \to K_0(\mathcal{C})$ is an additive function, then there is a unique group homomorphism $g : K_0(\mathcal{C}) \to G$ such that $f(M) = g([M])$ for every $M \in \mathcal{C}$.*

2.4.1 Order structure

Definition 2.4.5. An **ordered abelian group** $(G, G_{\geq 0})$ is an abelian group G and a semigroup $G_{\geq 0} \subset G$ (called the **positive cone** of G) such that $G_{\geq 0} - G_{\geq 0} = G$ and $G_{\geq 0} \cap -G_{\geq 0} = \{0\}$.

The Grothendieck group of $\mathcal{C}$ is an ordered abelian group with positive cone

$$K_0(\mathcal{C})_{\geq 0} := \{[M] \mid M \in \mathcal{C}\}.$$

2.4.2 Dévissage

Theorem 2.4.6 ([16, Theorem 6.3])**.** *Let $\mathcal{B} \subset \mathcal{A}$ be (skeletally) small abelian categories. Suppose that*

- *$\mathcal{B}$ is closed in $\mathcal{A}$ under subobjects and quotient objects, and*
- *every object $M \in \mathcal{A}$ has a finite filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ with all quotients M_{i+1}/M_i in $\mathcal{B}$.*

Then the inclusion functor $\mathcal{B} \subset \mathcal{A}$ is exact and induces an isomorphism $K_0(\mathcal{B}) \cong K_0(\mathcal{A})$.

2.4.3 Localization

Theorem 2.4.7 ([16, Theorem 6.4]). *Let $\mathcal{A}$ be a (skeletally) small abelian category, and $\mathcal{B}$ a Serre subcategory of $\mathcal{A}$. Then the sequence*

$$K_0(\mathcal{B}) \to K_0(\mathcal{A}) \to K_0(\mathcal{A}/\mathcal{B}) \to 0$$

is exact, where $K_0(\mathcal{B}) \to K_0(\mathcal{A})$ is the homomorphism induced by the exact inclusion functor $\mathcal{B} \hookrightarrow \mathcal{A}$ and $K_0(\mathcal{A}) \to K_0(\mathcal{A}/\mathcal{B})$ is the homomorphism induced by the exact quotient functor $\pi^* : \mathcal{A} \to \mathcal{A}/\mathcal{B}$.

Chapter 3

THE ORDER STRUCTURE OF $K_0(\mathsf{qgr}\, B)$

In this chapter, we prescribe conditions on an algebra B under which our method for computing $K_0(\mathsf{qgr}\, B)$ works. We then describe $K_0(\mathsf{qgr}\, B)$ for B an algebra satisfying those conditions.

3.1 First conditions on B

Condition C1. We say an $\mathbb{N}$-graded k-algebra B satisfies C1 if it

- is coherent (i.e. $\mathsf{qgr}(B)$ is an abelian category),
- is finitely generated,
- is connected-graded, and
- has finite global dimension.

3.1.1

Suppose B satisfies C1. Then each $M \in \mathsf{gr}(B)$ has a finite graded resolution by free B-modules of finite rank, hence $H_B(t)^{-1}$ is a polynomial in t and for all $M \in \mathsf{gr}(B)$,

$$H_M(t) = q_M(t)H_B(t)$$

for some $q_M(t) \in \mathbb{Z}[t^{\pm 1}]$.

3.2 Second conditions on B

Condition C2. We say an $\mathbb{N}$-graded k-algebra B satisfies C2 if

- it satisfies C1,
- $\dim_k B_n \geq 1$ for all $n \gg 0$, and
- $h(t) = H_B(t)^{-1}$ has a real root ξ such that
  - ξ is the only root of $h(t)$ in the interval $[0, 1]$,
  - ξ is a simple root, and
  - $\xi < |\lambda|$ for every other root λ of $h(t)$.

Lemma 3.2.1. *Let $\sum_{n=0}^{\infty} c_n z^n$ be a power series in which $c_n > 0$ for all $n \gg 0$. Suppose*

1. *$\sum_{n=0}^{\infty} c_n z^n$ has radius of convergence $R > 0$ and on the disk $|z| < R$ it converges to a rational function $s(z)$ that has a simple pole at $z = R$;*
2. *all other poles of $\sum_{n=0}^{\infty} c_n z^n$ have modulus $> R$.*

Then

$$\lim_{n\to\infty} \frac{c_n}{c_{n+1}} = R.$$

Proof. There are polynomials $p(z)$ and $q(z)$, neither divisible by $R - z$, such that

$$s(z) = \frac{p(z)}{(R-z)q(z)} = \frac{\alpha}{R-z} + \frac{r(z)}{q(z)}$$

where $\alpha \in \mathbb{C}^{\times}$, $r(z)$ is a polynomial, and $r(z)/q(z)$ has a Taylor series expansion $\sum_{n=0}^{\infty} b_n z^n$ with radius of convergence $> R$ by (2). Since $\sum_{n=0}^{\infty} b_n R^n$ converges $\lim_{n\to\infty} b_n R^n = 0$.

Since

$$s(z) = \frac{\alpha}{R}\sum_{n=0}^{\infty}\frac{z^n}{R^n} + \sum_{n=0}^{\infty} b_n z^n$$

for $|z| < R$,

$$c_n = \frac{\alpha}{R^{n+1}} + b_n.$$

Therefore

$$\lim_{n\to\infty}\left(\frac{c_n}{c_{n+1}}\right) = \lim_{n\to\infty}\left(\frac{\alpha R + b_n R^{n+2}}{\alpha + b_{n+1}R^{n+2}}\right) = \frac{\alpha R}{\alpha} = R,$$

as claimed. □

For B an $\mathbb{N}$-graded k-algebra, we write $b_n := \dim_k(B_n)$ for all $n \in \mathbb{N}$.

Lemma 3.2.2. *Suppose B is an $\mathbb{N}$-graded k-algebra that satisfies C2. For all $m \geq 1$,*

$$\lim_{n\to\infty}\frac{b_n}{b_{n+m}} = \xi^m.$$

Proof. Since B is connected-graded and finitely generated, $b_n < \infty$ for all $n \in \mathbb{N}$. Since B satisfies C2, $b_n \geq 1$ for all $n \gg 0$. Since

$$\frac{b_n}{b_{n+m}} = \frac{b_n}{b_{n+1}}\frac{b_{n+1}}{b_{n+2}}\cdots\frac{b_{n+m-1}}{b_{n+m}}$$

for $n \gg 0$, it suffices to prove the result for $m = 1$. By C2, $H_B(t) = \sum_{i=0}^{\infty} b_i t^i$ satisfies the conditions of Lemma 3.2.1 for $R = \xi$ so the result follows from the conclusion of Lemma 3.2.1. □

Proposition 3.2.3. *Suppose B is an $\mathbb{N}$-graded k-algebra that satisfies C2. If $M \in \mathsf{gr}(B)$, then $q_M(\xi) \geq 0$.*

Proof. Write $q_M(t) = \sum_{i=-s}^{s} p_i t^i$ and define $e_i := \sum_{j=-s}^{s} p_j b_{i-j}$. Then

$$H_M(t) \;=\; q_M(t) H_B(t) = \left(\sum_{i=-s}^{s} p_i t^i \right) \left(\sum_{i=0}^{\infty} b_i t^i \right) \;=\; \sum_{i=-s}^{\infty} e_i t^i.$$

By C2, $b_m \neq 0$ for all $m \gg 0$. Thus, as $m \to \infty$,

$$\frac{e_m}{b_m} \;=\; \sum_{j=-s}^{s} \left(\frac{b_{m-j}}{b_m} \right) p_j \;\longrightarrow\; \sum_{j=-s}^{s} p_j \xi^j \;=\; q_M(\xi).$$

Since $e_i = \dim(M_i)$, $\{e_m/b_m\}_{m \gg 0}$ is a sequence of non-negative numbers its limit, $q_M(\xi)$, is ≥ 0. $\square$

Lemma 3.2.4. *Suppose B is an $\mathbb{N}$-graded k-algebra that satisfies C2. Let $M \in \mathsf{gr}(B)$. The following are equivalent:*

1. *$M \in \mathsf{fdim}(B)$;*

2. *$h(t)$ divides $q_M(t)$;*

3. *$q_M(\xi) = 0$.*

Proof. (1) $\Rightarrow$ (2) If $\dim_k(M) < \infty$, then $H_M(t) \in \mathbb{N}[t, t^{-1}]$ so $q_M(t)$ is a multiple of $h(t)$.

(2) $\Rightarrow$ (3) If $h(t)$ divides $q_M(t)$ then $q_M(\xi) = 0$ since ξ is a root of $h(t)$.

(3) $\Rightarrow$ (1) Suppose $q_M(\xi) = 0$ but $\dim_k(M) = \infty$. The Laurent series $H_M(t)$ has non-negative coefficients and a finite radius of convergence $R \leq 1$. Since $H_M(t) = q_M(t) H_B(t)$, $q_M(\xi) = 0$ and ξ is a simple pole of $H_B(t)$ and the only pole of $H_B(t)$ in the interval $[0, 1]$, $H_M(t)$ has no poles in the interval $[0, 1]$. This contradicts Pringsheim's Theorem [5, Theorem IV.6] which says that $H_M(t)$ has a pole at $t = R$. We therefore conclude that $\dim_k(M) < \infty$. $\square$

3.3 Third conditions on B and the main theorem

Condition C3. We say an $\mathbb{N}$-graded k-algebra B satisfies C3 if

- B satisfies C2 and
- for each $p \in \mathbb{Z}[t, t^{-1}]$ such that $p(\xi) > 0$, there is an $M \in \mathsf{gr}(B)$ such that $q_M(t) - p(t) \in (h)$.

Let B be an $\mathbb{N}$-graded k-algebra that satisfies C2. We make $\mathbb{Z}[t^{\pm 1}]/(h)$ an ordered abelian group by defining

$$\left(\frac{\mathbb{Z}[t,t^{-1}]}{(h)}\right)_{\geq 0} := \{\overline{p} \mid p(\xi) > 0\} \cup \{0\}\,.$$

Let $\pi^* : \mathsf{gr}(B) \to \mathsf{qgr}(B)$ be the quotient functor.

Theorem 3.3.1. *Suppose B is an $\mathbb{N}$-graded k-algebra that satisfies C3. The Grothendieck group $K_0(\mathsf{qgr}(B))$ is isomorphic as an ordered abelian group to*

$$\frac{\mathbb{Z}[t,t^{-1}]}{(h)}$$

via the map $[\pi^* M] \mapsto \overline{q_M(t)}$. If h is irreducible, $K_0(\mathsf{qgr}(B))$ is isomorphic as an ordered abelian group to $\mathbb{Z}[\xi]$ via the map $[\pi^* M] \mapsto q_M(\xi)$.

Furthermore, under the isomorphism(s), the functor $\mathcal{M} \mapsto \mathcal{M}(1)$ on $\mathsf{qgr}(B)$ corresponds to multiplication by t^{-1} and multiplication by ξ^{-1}.

Proof. By localization and dévissage, the map

$$K_0(\mathsf{qgr}(B)) \to \frac{\mathbb{Z}[t,t^{-1}]}{(h)}, \qquad [\pi^* M] \mapsto \overline{q_M(t)}, \tag{3.3-1}$$

is an isomorphism of abelian groups and $[\mathcal{M}(1)] = t^{-1}[\mathcal{M}]$ under this isomorphism.

Under the isomorphism (3.3-1), the positive cone in $K_0(\mathsf{qgr}(B))$ is mapped to $\{\overline{q_M(t)} \mid M \in \mathsf{gr}(B)\}$. To show that (3.3-1) is an isomorphism of ordered abelian groups we must show that

$$\{\overline{p} \mid p(\xi) > 0\} \cup \{0\} = \{\overline{q_M(t)} \mid M \in \mathsf{gr}(B)\}. \tag{3.3-2}$$

Let $M \in \mathsf{gr}(B)$. By Proposition 3.2.3, $q_M(\xi) \geq 0$. If $q_M(\xi) > 0$, then $\overline{q_M(t)}$ is in the left-hand side of (3.3-2). If $q_M(\xi) = 0$, then $h(t)$ divides $q_M(t)$ by Lemma 3.2.4, whence $\overline{q_M(t)} = 0$. Thus, the right-hand side of (3.3-2) is contained in the left-hand side of (3.3-2).

If $p \in \mathbb{Z}[t^{\pm 1}]$ and $p(\xi) > 0$, then $\overline{p} = \overline{q_M}$ for some $M \in \mathsf{gr}(B)$ by C3 so $\overline{p}$ is in the right-hand side of (3.3-2). It is clear that 0 is in the right-hand side of (3.3-2). Thus, the left-hand side of (3.3-2) is contained in the right-hand side of (3.3-2). Hence (3.3-1) is an isomorphism of ordered abelian groups.

Suppose h is irreducible. The composition

$$K_0(\mathsf{qgr}(B)) \to \frac{\mathbb{Z}[t, t^{-1}]}{(h)} \to \mathbb{Z}[\xi], \quad [\pi^* M] \mapsto q_M(\xi), \tag{3.3-3}$$

is certainly an isomorphism of abelian groups. By (3.3-2), the image of the positive cone in $K_0(\mathsf{qgr}(B))$ under this compsition is $\mathbb{R}_{\geq 0} \cap \mathbb{Z}[\xi]$, the positive cone in $\mathbb{Z}[\xi]$. Hence (3.3-2) is an isomorphism of ordered abelian groups and $[\mathcal{M}(1)] = \xi^{-1}[\mathcal{M}]$ under the isomorphism. $\square$

3.4 The Hilbert series of finitely presented modules

Our work to compute $K_0(\mathsf{qgr}\, B)$ for an $\mathbb{N}$-graded k-algebra B that satisfies C3 suggests an answer to the following, simpler question: what are the possible Hilbert series of finitely presented graded B-modules?

The Hilbert series of a finite dimensional module is a Laurent polynomial with nonnegative integer coefficients. Conversely, a Laurent polynomial with nonnegative integer coefficients is the Hilbert series of a sum of shifts of copies of the trivial module $B/B_{\geq 1}$. The next result completes the answer.

Theorem 3.4.1. *Let B be an $\mathbb{N}$-graded k-algebra that satisfies C3. Let $h(t) = (H_B(t))^{-1} \in \mathbb{Z}[t]$ and let ξ be the smallest positive real root of $h(t)$. If $M \in \mathsf{gr}\, B$, then $H_M(t) = p(t)/h(t)$ for some $p(t) \in \mathbb{Z}[t^{\pm 1}]$ such that $p(\xi) \geq 0$, with equality if and only if M is finite dimensional.*

Conversely, if $p(t) \in \mathbb{Z}[t^{\pm 1}]$ and $p(\xi) > 0$, then

$$\left(\frac{p(t)}{h(t)}\right)_{\geq N} = H_M(t)$$

for some $M \in \mathsf{gr}\, B$ and some $N \in \mathbb{Z}$.

Proof. Suppose $M \in \mathsf{gr}\, B$. By the discussion in §3.1.1, $H_M(t) = p(t)/h(t)$ for some $p(t) \in \mathbb{Z}[t^{\pm 1}]$. By Proposition 3.2.3, $p(\xi) \geq 0$, and $p(\xi) = 0$ if and only if M is finite dimensional by Lemma 3.2.4.

Suppose $p(t) \in \mathbb{Z}[t^{\pm 1}]$ such that $p(\xi) > 0$. By condition C3, there exists a module $M \in \mathsf{gr}\, B$ such that $p(t) - q_M(t) \in (h)$. In other words, the formal Laurent series $p(t)/h(t)$ and $H_M(t)$ differ only in a finite number of terms. Hence

$$\left(\frac{p(t)}{h(t)}\right)_{\geq N} = H_M(t)_{\geq N} = H_{M_{\geq N}}(t)$$

for some $N \in \mathbb{Z}$. $\square$

Chapter 4

THE CASE B IS A FINITELY GENERATED FREE ALGEBRA

4.1 Summary

In this chapter, we will show that an $\mathbb{N}$-graded finitely generated free algebra F over k with generators in positive degrees satisfies condition C3 of chapter 3. Consequently we compute $K_0(\mathsf{qgr}(F))$ as an ordered abelian group. We then discuss another method for computing $K_0(\mathsf{qgr}(F))$ and compare it to our method. Finally, we discuss examples.

Let k be a field, g a positive integer, $D = \{d_1, \ldots, d_g\} \subset \mathbb{N}_{\geq 1}$ with $\gcd D = 1$ and $F = k\langle x_1, \ldots, x_g\rangle$ the $\mathbb{N}$-graded free algebra on generators $x_1, \ldots, x_g$ with $\deg x_i = d_i$. The graded F-module $k = F/F_{\geq 1}$ has a graded resolution

$$0 \to \sum_{i=1}^{g} F(-d_i) \xrightarrow{\begin{pmatrix} x_1 & \cdots & x_g \end{pmatrix}\cdot} F \to k \to 0, \tag{4.1-1}$$

so the Hilbert series of F is $H_F(t) = f(t)^{-1}$ where

$$f(t) := 1 - \sum_{i=1}^{g} t^{d_i}.$$

Let d be the maximum of the degrees of $x_1, \ldots, x_g$. We write $f(t) = 1 - \sum_{i=1}^{d} n_i t^i$ where n_i is the number of generators of degree i.

4.2 F satisfies C3

4.2.1 F satisfies C1

By definition, F is finitely generated and connected-graded. Since the right ideals of F are free F-modules, F is coherent. By the resolution (4.1-1), F has global dimension equal to one. Hence F satisfies C1.

4.2.2 F satisfies C2

Let $a_i := \dim_k F_i$. Since the degrees of $x_1, \ldots, x_g$ are relatively prime, $a_i \geq 1$ for all $i \gg 0$.

Since $f(0) = 1$, $f(1) = 1 - g \leq 0$ and $f(t)$ is decreasing for $t \geq 0$, f has one positive real root, say

$$\theta := \text{the positive real root of } f,$$

and $0 < \theta \leq 1$.

By the following result which will be proved in §4.4, F satisfies C2.

Proposition 4.2.1. *The root θ of f is simple and $\theta < |\lambda|$ for every other root λ of f.*

4.2.3 F satisfies C3

Lemma 4.2.2. *Let $p \in \mathbb{Z}[t, t^{-1}]$. If $p(\theta) > 0$ then there exists an $M \in \mathsf{gr}\, F$ such that $q_M(t) - p(t) \in (f)$.*

Proof. Write $p(t) = \sum_{i=-s}^{s} p_i t^i$. If $\{p_i\} \subset \mathbb{N}$ then $p(t) = q_M(t)$ for $M = \sum_{i=-s}^{s} F(-i)^{p_i}$.

Suppose $\{p_i\} \not\subset \mathbb{N}$. Define integers b_j for $j \geq -s$ by the requirement that

$$\sum_{j=-s}^{\infty} b_j t^j := p(t) H_F(t).$$

Therefore

$$p(t) = f(t) \sum_{j=-s}^{\infty} b_j t^j = \left(1 - \sum_{i=1}^{d} n_i t^i\right) \sum_{j=-s}^{\infty} b_j t^j.$$

Equating coefficients gives

$$p_i = b_i - \sum_{j=1}^{d} n_j b_{i-j} \tag{4.2-1}$$

for all $i \geq -s$ with the convention that $p_i = 0$ for $i > s$ and $b_j = 0$ for $j < -s$.

Since $a_j \neq 0$ for $j \gg 0$,

$$\lim_{j\to\infty} \left(\frac{b_j}{a_j}\right) = \lim_{j\to\infty} \left(\sum_{i=-s}^{s} \left(\frac{a_{j-i}}{a_j}\right) p_i\right) = \sum_{i=-s}^{s} p_i \theta^i = p(\theta) > 0.$$

There is therefore an integer $m \geq s$ such that b_j is a nonnegative integer for all $j \geq m+1-d$.

We fix such an m.

We will complete the proof by showing that the Laurent polynomial

$$q(t) := p(t) - \left(\sum_{i=-s}^{m} b_i t^i\right) f(t)$$

is $q_M(t)$ for a suitable $M \in \mathsf{gr}(F)$. Define

$$r_i := \sum_{j=i-m}^{d} n_j b_{i-j}$$

for $m+1 \leq i \leq m+d$. By the choice of m, r_i is a nonnegative integer for all i. By (4.2-1),

$$\begin{aligned}
q(t) &= p(t) - \left(\sum_{i=-s}^{m} b_i t^i\right)\left(1 - \sum_{j=1}^{d} n_j t^j\right) \\
&= p(t) - \sum_{i=-s}^{m} \left[b_i - \sum_{j=1}^{d} n_j b_{i-j}\right] t^i + \sum_{i=m+1}^{m+d} \left[\sum_{j=i-m}^{d} n_j b_{i-j}\right] t^i \\
&= p(t) - p(t) + \sum_{i=m+1}^{m+d} r_i t^i \\
&= \sum_{i=m+1}^{m+d} r_i t^i.
\end{aligned}$$

Thus $q(t) = q_M(t)$ for $M = \sum_{i=m+1}^{m+d} F(-i)^{r_i}$. $\square$

Thus F satisfies C3.

4.3 The Grothendieck group of $\mathsf{qgr}(F)$

We make $\mathbb{Z}[t^{\pm 1}]/(f)$ an ordered abelian group by defining

$$\left(\frac{\mathbb{Z}[t,t^{-1}]}{(f)}\right)_{\geq 0} := \{\overline{p} \mid p(\theta) > 0\} \cup \{0\} \tag{4.3-1}$$

where $\overline{p}$ denotes the image of the Laurent polynomial p in $\mathbb{Z}[t,t^{-1}]/(f)$. The order structure on $\mathbb{Z}[\theta]$ is inherited from its embedding in $\mathbb{R}$.

Theorem 4.3.1. *Let F be the algebra discussed in §4.1. The Grothendieck group $K_0(\mathsf{qgr}\, F)$ is isomorphic as an ordered abelian group to*

$$\frac{\mathbb{Z}[t,t^{-1}]}{(f)}$$

via the map $[\pi^* M] \mapsto \overline{q_M(t)}$. If f is irreducible, $K_0(\mathsf{qgr}\, F)$ is isomorphic as an ordered abelian group to $\mathbb{Z}[\theta]$ via the map $[\pi^* M] \mapsto q_M(\theta)$.

Furthermore, under the isomorphism(s), the functor $\mathcal{M} \mapsto \mathcal{M}(1)$ corresponds to multiplication by t^{-1} and θ^{-1}.

Proof. By §4.2.1, 4.2.2 and 4.2.3, F satisfies C3. The result now follows from Theorem 3.3.1. $\square$

4.4 The proof of Proposition 4.2.1

4.4.1 The idea of the proof

We will associate to F a particular finite directed graph G. An $\mathsf{incidence\ matrix}$ for G is a square matrix whose rows and columns are labelled by the vertices of G and whose uv-entry is the number of arrows from v to u. The characteristic polynomial of G is

$$p_G(t) := \det(tI - M)$$

where M is an incidence matrix for G. We will show that $p_G(t) = t^\ell f(1/t)$ where $\ell = d_1 + \cdots + d_g$. We also show that M is primitive, i.e., all entries of M^n are positive for $n \gg 0$. We then apply the Perron-Frobenius theorem which says that a primitive matrix has a positive real eigenvalue of multiplicity 1, ρ say, with the property that $|\lambda| < \rho$ for all other eigenvalues λ. But the non-zero eigenvalues of M are the reciprocals of the roots of $f(t)$. Since we already know that $f(t)$ has only one positive real root, namely θ, $\rho = \theta^{-1}$. Hence θ is a simple root of $f(t)$ and $|\lambda| > \theta$ for every other root λ of $f(t)$.

4.4.2 The associated graph and its characteristic polynomial

We will use Theorem 4.4.1 to compute the characteristic polynomial of the directed graph G. First we need some notation.

A simple cycle in G is a directed path that begins and ends at the same vertex and does not pass through any vertex more than once. We introduce the notation for an arbitrary directed graph G:

1. $v(G)$:=the number of vertices in G;
2. $c(G)$:=the number of connected components in G;
3. $Z(G) := \{$simple cycles in $G\}$;
4. $\overline{Z}(G) := \{$subgraphs of G that are a disjoint union of simple cycles$\}$.

Theorem 4.4.1. *[3, Theorem 1.2] Let G be a directed graph with ℓ vertices. Then*

$$p_G(t) = t^\ell + c_1 t^{\ell-1} + \cdots + c_{\ell-1} t + c_\ell$$

where

$$c_i := \sum_{\substack{Q \in \overline{Z}(G) \\ v(Q)=i}} (-1)^{c(Q)}.$$

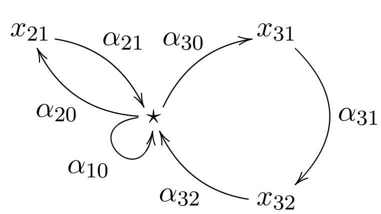
The x_is are labelled so that $\deg(x_1) \leq \cdots \leq \deg(x_g)$.

The free algebra F is the path algebra (as an ungraded algebra) of the quiver with one vertex $\star$ and g loops from $\star$ to $\star$ labelled $x_1, \ldots, x_g$. We replace each loop x_i by $d_i' := \deg(x_i) - 1 = d_i - 1$ vertices labelled $x_{i1}, \ldots, x_{id_i'}$ and arrows

$$\star \xrightarrow{\alpha_{i0}} x_{i1} \xrightarrow{\alpha_{i1}} \cdots \qquad \cdots \longrightarrow x_{id_i'} \xrightarrow{\alpha_{id_i'}} \star$$

The graph G obtained by this procedure is the graph associated to F in [7].

Example 4.4.2. If $g = 3$ and $d_i = i$ then G is



Proposition 4.4.3. *Let $\ell = v(G)$. The characteristic polynomial of G is $t^\ell f(1/t)$.*

Proof. Any two simple cycles in G share the vertex $\star$, so $Z(G) = \overline{Z}(G)$. The number of simple cycles of length i in G is equal to n_i, the number of generators of degree i in F. By

Theorem 4.4.1,

$$\begin{aligned} p_G(t) &= t^\ell + c_1 t^{\ell-1} + \cdots + c_{\ell-1} t + c_\ell \\ &= t^\ell - n_1 t^{\ell-1} + \cdots - n_d t^{\ell-d} \\ &= t^\ell (1 - n_1 t^{-1} - \cdots - n_d t^{-d}) \\ &= t^\ell f(1/t), \end{aligned}$$

as claimed. □

Proposition 4.4.4. *Let M be an incidence matrix for G. Then every entry in M^n is non-zero for $n \gg 0$.*

Proof. If u and v are vertices in G, then there is a directed path in G from u to v: in the language of [8, Defn. 4.2.2], M is irreducible.

The period of a vertex v in G is the greatest common divisor of the non-trivial directed paths that begin and end at v. The period of G is the greatest common divisor of the periods of its vertices. Since there is a directed path of length $d_i = \deg(x_i)$ from x_{i0} to itself, the period of G divides $\gcd\{d_1, \ldots, d_g\}$ which is 1. The period of G is therefore 1. Thus, in the language of [8, Defn. 4.5.2], M is aperiodic and therefore primitive [8, Defn. 4.5.7]. Hence [8, Thm. 4.5.8] applies to M, and gives the result claimed. □

The Perron-Frobenius theorem [6, Thm. 1, p.64] therefore applies to M giving the following result.

Corollary 4.4.5. *The characteristic polynomial for G has a unique eigenvalue of maximal modulus and that eigenvalue is simple and real.*

As explained at the start of §4.4.1, Proposition 4.2.1 follows from Proposition 4.4.3 and Corollary 4.4.5. Hence F satisfies C2.

4.5 A second way to compute $K_0(\mathsf{qgr}\, F)$

We denote by S the ordered abelian group $\mathbb{Z}[t^{\pm 1}]/(f(t))$ with order structure defined by (4.3-1). Since $t^{-1} = \sum_{i=1}^{d} n_i t^{i-1}$ in S, we may disregard negative powers of t in S. That is, $S = \mathbb{Z}[t]/(f(t))$.

We have two ways to compute $K_0(\mathsf{qgr}\, F)$ as an ordered abelian group. By Theorem 4.3.1, $K_0(\mathsf{qgr}\, F) \cong S$.

A second way is as follows. By [7], $\mathsf{qgr}\, F \equiv \mathsf{qgr}\, kG$. By [14], there exists an ultramatricial algebra U such that $\mathsf{qgr}\, kG \equiv \mathsf{Mod} U$. Thus we can present $K_0(\mathsf{qgr}\, F)$ as a direct limit of ordered abelian groups.

Can we describe an isomophism between the ordered abelian groups obtained by each method? The fact that G does not have a nonsingular incidence matrix in general complicates the search for such an isomorphism. Thus we define a quiver Q with all arrows in degree one and a nonsingular incidence matrix and prove that

1. $\mathsf{qgr}\, F \equiv \mathsf{qgr}\, kQ$ and

2. $K_0(\mathsf{qgr}\, F) \cong K_0(\mathsf{qgr}\, kQ)$ as ordered abelian groups.

By Elliott's classification of ultramatricial algebras [4], (1) and (2) are equivalent statements, but we will prove each directly.

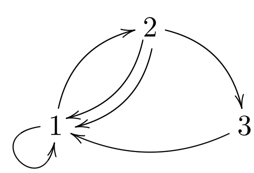
4.5.1 The second quiver

Let Q be the quiver with vertices $1, 2, \ldots, d$ and arrows

- $i \to i + 1$ for all $1 \le i \le d - 1$ and
- $d_i \to 1$ for all $1 \le i \le g$.

Let kQ be the path algebra of Q. We grade kQ by placing all arrows in degree one.

Example 4.5.1. If $g = 4$ and $d_1 = 1$, $d_2 = d_3 = 2$ and $d_4 = 3$, then Q is the quiver



The quiver Q has an incidence matrix

$$M = \begin{pmatrix} n_1 & 1 & 0 & \cdots & 0 \\ n_2 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \\ n_{d-1} & 0 & 0 & \cdots & 1 \\ n_d & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

The characteristic polynomial of M is

$$\begin{aligned} \det(tI - M) &= t^d - n_1 t^{d-1} - \cdots - n_{d-1} t - n_d \\ &= t^d f(1/t). \end{aligned}$$

4.5.2 An equivalence of categories

By an equivalence similar to the one presented in [7], $\mathsf{qgr}\, F \equiv \mathsf{qgr}\, kQ$. We illustrate this equivalence through an example.

Example 4.5.2. If $g = 3$ and $d_1 = 1$, $d_2 = d_3 = 2$ then Q is

$$\begin{tikzcd} 1 \arrow[loop left] \arrow[r, bend left] & 2 \arrow[l, bend left] \arrow[l, bend left=50] \end{tikzcd} .$$

View F as the path algebra of the quiver

$$\begin{tikzcd} \star \arrow[loop above, "a_2"] \arrow[loop left, "a_1"'] \arrow[loop right, "a_3"] \end{tikzcd}$$

with $\deg(a_1) = 1$, $\deg(a_2) = \deg(a_3) = 2$. The functor $\mathsf{gr}\, F \to \mathsf{gr}\, kQ$ given by

$$\begin{tikzcd} V \arrow[loop above, "g"] \arrow[loop left, "f"'] \arrow[loop right, "h"] \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} V \arrow[loop left, "f"'] \arrow[rr, bend left, "\mathrm{id}"] & & V(-1) \arrow[ll, bend left, "h"'] \arrow[ll, bend left=50, "g"] \end{tikzcd}$$

and the functor $\mathsf{gr}\, kQ \to \mathsf{gr}\, F$ given by

$$\begin{tikzcd} V \arrow[loop left, "f"'] \arrow[rr, bend left, "m"] & & W \arrow[ll, bend left, "h"'] \arrow[ll, bend left=50, "g"] \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} V \arrow[loop above, "g \circ m"] \arrow[loop left, "f"'] \arrow[loop right, "h \circ m"] \end{tikzcd}$$

descend to a equivalence $\mathsf{qgr}\, F \equiv \mathsf{qgr}\, kQ$.

4.5.3 The Grothendieck group

Hence

$$K_0(\mathsf{qgr}\, F) \cong \varinjlim \left(\mathbb{Z}^d \xrightarrow{M\cdot} \mathbb{Z}^d \xrightarrow{M\cdot} \cdots \right)$$

where each $\mathbb{Z}^d$ is an ordered abelian group in the standard way, i.e. $(\mathbb{Z}^d)_{\geq 0} = \mathbb{N}^d$. The limit is

$$\begin{tikzcd} \mathbb{Z}^d \arrow[r, "M\cdot"] \arrow[dr, "1"'] & \mathbb{Z}^d \arrow[r, "M\cdot"] \arrow[d, "t"] & \mathbb{Z}^d \arrow[r, "M\cdot"] \arrow[dl, "t^2"] & \cdots \\ & \dfrac{\mathbb{Z}^d[t]}{(e_i - tMe_i)} & & \end{tikzcd}$$

Where $e_i \in \mathbb{Z}^d$ is the column vector with one in the i-th entry and zeros in all other entries. We write $T = \mathbb{Z}^d[t]/(e_i - tMe_i)$. The positive cone of T is

$$T_{\geq 0} = \left\{ \overline{p(t)} \mid p(t) \in \mathbb{N}^d[t] \right\}. \tag{4.5-1}$$

The relations $e_i = tMe_i$ in T tell us that in T,

$$e_i = e_1 t^{i-1} \text{ for all } 1 \leq i \leq d$$

and

$$e_1 = t \sum_{i=1}^{d} n_i e_i.$$

4.5.4 An isomorphism of Grothendieck groups

By the equivalence of categories $\mathsf{qgr}\, F \equiv \mathsf{qgr}\, kQ$, S and T are isomorphic ordered abelian groups. In this subsection we find an explicit isomorphism.

For ease of reading, we suppress the overline notation for quotient groups. For example, $p(t) \in S$ denotes both a polynomial $p(t) \in \mathbb{Z}[t]$ and the image of that polynomial in S.

Both S and T are $\mathbb{Z}[t]$-modules.

Proposition 4.5.3. *The $\mathbb{Z}[t]$-module homomorphisms*

$$\varphi : T \to S, \quad e_i \mapsto t^{i-1}$$

and

$$\psi : S \to T, \quad 1 \mapsto e_1$$

are well-defined and mutually inverse.

Proof. To show φ is well-defined, we must show that $\varphi(e_i - tMe_i) \in (f(t))$ for all $1 \leq i \leq d$.

For $i = 1$,

$$\begin{aligned} \varphi(e_1 - tMe_1) &= 1 - t\left(\sum_{i=1}^{d} n_i t^{i-1}\right) \\ &= 1 - \sum_{i=1}^{d} n_i t^i = f(t). \end{aligned}$$

For $2 \leq i \leq d$,

$$\varphi(e_i - tMe_i) = t^{i-1} - t(t^{i-2}) = 0.$$

Hence φ is well-defined.

To show that ψ is well defined, we must show that $\psi(f(t)) \in (e_i - tMe_i)$. Since

$$\begin{aligned} \psi(f(t)) &= e_1 - \sum_{i=1}^{d} n_i e_1 t^i \\ &= e_1 - t\sum_{i=1}^{d} n_i e_i \\ &= e_1 - tMe_1, \end{aligned}$$

ψ is well-defined.

The composition $\varphi \circ \psi : S \to S$ sends 1 to 1 and therefore is the identity. Since $e_i = e_1 t^{i-1}$ in T for all $1 \leq i \leq d$, the composition

$$\psi \circ \varphi : T \to T, \quad e_i \mapsto e_1 t^{i-1}$$

is also the identity. $\square$

The following result is proved in the proof of Lemma 4.2.2.

Proposition 4.5.4. *If $p(t) \in \mathbb{Z}[t^{\pm 1}]$ such that $p(\theta) > 0$, then there exists an $n \in \mathbb{N}$ such that*

$$\left(\frac{p(t)}{f(t)}\right)_{\geq n} = \frac{q(t)}{f(t)}$$

with $q(t) \in \mathbb{N}[t]$.

It remains to show that φ and ψ respect the orderings of S and T, for which the next result is key.

Proposition 4.5.5. *If $p(t) \in \mathbb{Z}[t]$ then*

$$\left(\frac{p(t)}{f(t)}\right)_{\geq 1} = \frac{t\varphi M\psi(p(t))}{f(t)} + tg(t)$$

for some $g(t) \in \mathbb{Z}[t]$.

Proof. Write $p(t) = \sum_{i=0}^{m} p_i t^i$. We may assume, by taking high coefficients to be zero, that $m \geq d$.

Let $q(t) = \sum_{i=1}^{d} p_{i-1}e_i + \sum_{i=d}^{m} p_i e_2 t^{i-1} \in \mathbb{Z}^d[t]$. Since $\varphi(q(t)) = p(t)$, $q(t) - \psi(p(t)) \in (e_i - tMe_i)$. By the relations in T, $tMq(t) - tM\psi(p(t)) \in (e_i - tMe_i)$, so $\varphi tMq(t) - \varphi tM\psi(p(t)) \in (f(t))$. Since φ is a $\mathbb{Z}[t]$-module homomorphism, $t\varphi Mq(t) - t\varphi M\psi(p(t)) \in (f(t))$.

Now

$$Mq(t) = p_0 \sum_{i=1}^{d} n_i e_i + \sum_{i=2}^{d} p_{i-1}e_{i-1} + \sum_{i=d}^{m} p_i e_1 t^{i-1}$$

so

$$\begin{aligned} t\varphi M q(t) &= p_0 \sum_{i=1}^{d} n_i t^i + \sum_{i=2}^{d} p_{i-1} t^{i-1} + \sum_{i=d}^{m} p_i t^i \\ &= \sum_{i=1}^{d} (p_0 n_i + p_i) t^i + \sum_{i=d+1}^{m} p_i t^i \\ &= p(t) - p(0) f(t). \end{aligned}$$

Hence

$$\begin{aligned} \left(\frac{p(t)}{f(t)}\right)_{\geq 1} &= \frac{p(t) - p(0) f(t)}{f(t)} \\ &= \frac{t\varphi M q(t)}{f(t)} \\ &= \frac{t\varphi M \psi(p(t)) + t g(t) f(t)}{f(t)} \\ &= \frac{t\varphi M \psi(p(t))}{f(t)} + t g(t) \end{aligned}$$

for some $g(t) \in \mathbb{Z}[t]$. □

Corollary 4.5.6. *If $p(t) \in \mathbb{Z}[t]$ and $n \in \mathbb{N}$ then*

$$\left(\frac{p(t)}{f(t)}\right)_{\geq n} = \frac{t^n \varphi M^n \psi(p(t))}{f(t)} + t^n g(t)$$

for some $g(t) \in \mathbb{Z}[t]$.

Proof. We induct on n. The result holds for $n = 1$ by Proposition 4.5.5.

If the result holds for $N \in \mathbb{N}$, then

$$\begin{aligned}
\left(\frac{p(t)}{f(t)}\right)_{\geq N+1} &= \left(\left(\frac{p(t)}{f(t)}\right)_{\geq N}\right)_{\geq N+1} \\
&= \left(\frac{t^N \varphi M^N \psi(p(t))}{f(t)} + t^N g(t)\right)_{\geq N+1} \\
&= t^N \left(\frac{\varphi M^N \psi(p(t))}{f(t)} + g(t)\right)_{\geq 1} \\
&= t^N \left(\frac{t\varphi M \psi \varphi M^N \psi(p(t))}{f(t)} + t\varphi M \psi(g(t)) + th(t)\right) \\
&= \frac{t^{N+1} \varphi M^{N+1} \psi(p(t))}{f(t)} + t^{N+1}(\varphi M \psi(g(t)) + h(t)).
\end{aligned}$$

The result follows. □

Lemma 4.5.7. *The $\mathbb{Z}[t]$-module map*

$$\varphi : \frac{\mathbb{Z}^d[t]}{(e_i - tMe_i)} \to \frac{\mathbb{Z}[t]}{(f(t))}, \quad e_i \mapsto t^{i-1}$$

is an isomorphism of ordered abelian groups.

Proof. By Proposition 4.5.3, $\varphi : T \to S$ is an isomorphism of abelian groups (in fact, of $\mathbb{Z}[t]$-modules). It remains to show that $\varphi(T_{\geq 0}) \subseteq S_{\geq 0}$ and $\psi(S_{\geq 0}) \subseteq T_{\geq 0}$.

If $q(t) \in T_{\geq 0}$, i.e. $q(t) \in \mathbb{N}^d[t]$, then $p(t) = \varphi(q(t)) \in \mathbb{N}[t]$. If $p(\theta) = 0$ then $q(t) = 0$, otherwise $p(\theta) > 0$. Hence $p(t) \in S_{\geq 0}$.

Suppose $p(t) \in S_{\geq 0}$. If $p(t) = 0$ then $\psi(p(t)) = 0 \in T_{\geq 0}$. If $p(\theta) > 0$ then for some n,

$$\left(\frac{p(t)}{f(t)}\right)_{\geq n} = \frac{q(t)}{f(t)}$$

for some $q(t) \in \mathbb{N}[t]$ by Proposition 4.5.4. By Corollary 4.5.6,

$$q(t) = t^n \varphi M^n \psi(p(t)) + t^n g(t) f(t)$$

for some $g(t) \in \mathbb{Z}[t]$. Since $\psi(q(t)) \in \mathbb{N}^d[t]$ and

$$\psi(q(t)) = t^n M^n \psi(p(t)) = \psi(p(t))$$

in T, $\psi(p(t)) \in T_{\geq 0}$. $\square$

4.6 Examples

4.6.1

If $F = k\langle x_1, \ldots, x_g \rangle$ with $\deg(x_i) = 1$ for all i, then $f(t) = 1 - gt$ is irreducible, so

$$K_0(\mathsf{qgr}\, F) \cong \mathbb{Z}[1/g] \subseteq \mathbb{R}$$

as ordered abelian groups by Theorem 3.3.1.

4.6.2

Let $F = k\langle x_1, x_2, x_3 \rangle$ with $\deg(x_1) = 1$ and $\deg(x_2) = \deg(x_3) = 2$. In this case, $f(t) = 1 - t - 2t^2 = (1 + t)(1 - 2t)$ and $\theta = 1/2$. The map

$$\frac{\mathbb{Z}[t, t^{-1}]}{(f)} \to \mathbb{Z} \oplus \mathbb{Z}[1/2], \quad g \mapsto (g(-1), g(1/2))$$

is an isomorphism of abelian groups. By Theorem 3.3.1, $K_0(\mathsf{qgr}\, F) \cong \mathbb{Z} \oplus \mathbb{Z}[1/2]$ as ordered abelian groups where $(\mathbb{Z} \oplus \mathbb{Z}[1/2])_{\geq 0} = (\mathbb{Z} \oplus \mathbb{Z}[1/2]_{>0}) \cup \{0\}$.

4.6.3

Let $F = k\langle x_1, x_2 \rangle$ with $\deg(x_i) = i$. Then $K_0(\mathsf{qgr}(F)) \cong \mathbb{Z}[\frac{1}{2}(1 + \sqrt{5})] \subseteq \mathbb{R}$ since $f(t) = 1 - t - t^2$ is irreducible and $\theta = \frac{1}{2}(-1 + \sqrt{5})$. This ordered group shows up as the Grothendieck group of categories associated to Penrose tilings in [2, Sect II.3] and [15] and as the Grothendieck group of $\mathsf{qgr}(A)$ where $A = k\langle x_1, x_2, x_3\rangle/(x_1x_3 + x_2^2 + x_3x_1)$ with $\deg(x_1) = \deg(x_2) = \deg(x_3) = 1$ in chapter 5.

Chapter 5

THE CASE B IS A REGULAR ALGEBRA OF DIMENSION 2

In this chapter, we use the methods of Chapter 3 to compute $K_0(\mathsf{qgr}\, A)$ as an ordered abelian group where A is a regular algebra of global dimension 2.

5.1 Regular algebras of global dimension 2

Let k be a field and $A = \bigoplus_{n\geq 0} A_n$ an $\mathbb{N}$-graded k-algebra such that $A_0 = k$. The left and right global dimensions of A are the same and equal the projective dimension of the A-module $k := A/A_{\geq 1}$. We say A is $\mathsf{regular}$ if it has finite global dimension, n say, and

$$\operatorname{Ext}^j_A(k, A) \cong \begin{cases} k & \text{if } j = n \\ \\ 0 & \text{if } j \neq n. \end{cases}$$

Zhang [17, Theorem 0.1] proved that A is regular of global dimension 2 if and only if it is isomorphic to some

$$A := \frac{k\langle x_1, \ldots, x_g\rangle}{(b)} \tag{5.1-1}$$

where $g \geq 2$, the x_i's can be labelled so that $\deg(x_i) + \deg(x_{g+1-i}) =: d$ is the same for all i, and σ is a graded k-algebra automorphism of the free algebra $k\langle x_1, \ldots, x_g\rangle$, and $b = \sum_{i=1}^{g} x_i \sigma(x_{g+1-i})$.

Because A is regular of global dimension two, the minimal projective resolution of ${}_A k$ is

$$0 \longrightarrow A(-d) \xrightarrow{\alpha} \bigoplus_{i=1}^{g} A(-\deg(x_i)) \xrightarrow{\beta} A \longrightarrow k \longrightarrow 0 \tag{5.1-2}$$

where $d = \deg(x_i) + \deg(x_{g+1-i})$, α is right multiplication by $(x_g, \ldots, x_1)$, and β is right multiplication by $(\sigma(x_1), \sigma(x_2), \ldots, \sigma(x_g))^{\mathsf{T}}$. The Hilbert series for A is therefore

$$H_A(t) := \sum_{n=0}^{\infty} \dim_k(A_n)t^n = \frac{1}{f(t)}$$

where

$$f(t) := t^d - \sum_{i=1}^{g} t^{\deg(x_i)} + 1. \tag{5.1-3}$$

For the rest of the chapter, A denotes the algebra in (5.1-1) where the degrees of the generators and the relation b have the properties stated after (5.1-1). We will also assume that $g \geq 3$ (the case $g = 2$ is well-understood) and without loss of generality, that the greatest common divisor of the degrees of the generators x_i is one.

5.2 *A satisfies C3*

5.2.1 *A satisfies C1*

By definition, A is finitely generated and connected-graded. Because A is defined by a single homogeneous quadratic relation, A is coherent by [11, Theorem 1.2]. By [17, Theorem 0.1], A has global dimension equal to two. Hence A satisfies C1.

5.2.2 *A satisfies C2*

Let $a_i := \dim_k A_i$. Because A is a domain [17, Thm. 0.2] and 1 is the greatest common divisor of the degrees of its generators, $a_i \geq 1$ for all $i \gg 0$.

Descartes' rule of signs implies that $f(t)$ has either 0 or 2 positive real roots. The hypothesis that $g \geq 3$ implies $f(1) < 0$. Since $f(0) > 0$, we conclude that $f(t)$ has two positive roots, $\theta^{-1} > 1$ and $\theta \in (0, 1)$, say.

By the following result proved in §5.4, F satisfies C2.

Proposition 5.2.1. *The root θ of f is simple and $\theta < |\lambda|$ for λ any other root of f.*

5.2.3 A satisfies C3

Lemma 5.2.2. *Let $p \in \mathbb{Z}[t, t^{-1}]$. If $p(\theta) > 0$, then there is an M in $\mathsf{gr}(A)$ such that $q_M(t) - p(t) \in (f)$.*

Proof. It suffices to show that $t^s q_M(t) - t^s p(t) \in (f)$ for some integer s. Since $q_{M(-s)}(t) = t^s q_M(t)$, we can, and will, assume $p(t) \in \mathbb{Z}[t]$.

Write $p(t) = \sum_{i=0}^{s} p_i t^i$. Define integers b_j, $j \geq 0$, by the requirement that

$$\sum_{j=0}^{\infty} b_j t^j := p(t) H_A(t). \tag{5.2-1}$$

Therefore

$$p(t) = f(t) \sum_{j=0}^{\infty} b_j t^j = \left(1 - \sum_{\ell=1}^{d-1} n_\ell t^\ell + t^d\right) \sum_{j=0}^{\infty} b_j t^j.$$

Equating coefficients gives

$$p_i = b_i + b_{i-d} - \sum_{\ell=1}^{d-1} n_\ell b_{i-\ell} \tag{5.2-2}$$

for all $i \geq 0$ with the convention that $p_i = 0$ for $i > s$ and $b_j = 0$ for $j < 0$.

Since $a_j \neq 0$ for $j \gg 0$,

$$\lim_{j\to\infty} \left(\frac{b_j}{a_j}\right) \;=\; \lim_{j\to\infty} \left(\sum_{i=0}^{s} \left(\frac{a_{j-i}}{a_j}\right) p_i \right) \;=\; \sum_{i=0}^{s} p_i \theta^i \;=\; p(\theta) \;>\; 0.$$

Therefore

$$\lim_{j\to\infty} \left(\frac{b_j}{b_{j+1}}\right) \;=\; \lim_{j\to\infty} \left(\frac{b_j}{a_j} \frac{a_{j+1}}{b_{j+1}} \frac{a_j}{a_{j+1}} \right) \;=\; p(\theta) p(\theta)^{-1} \theta \;=\; \theta.$$

There is therefore an integer $m \geq s$ such that $\{b_j\}_{j \geq m+1-d}$ is a strictly increasing sequence of positive integers. We fix such an m.

We will complete the proof by showing that the Laurent polynomial

$$q(t) := p(t) - \left(\sum_{i=0}^{m} b_i t^i\right) f(t)$$

is $q_M(t)$ for a suitable $M \in \mathsf{gr}(A)$. Before beginning the proof we define

$$r_i := \sum_{\ell=i-m}^{d-1} n_\ell b_{i-\ell} - b_{i-d} \tag{5.2-3}$$

for $m+1 \leq i \leq m+d$. To start the proof, we note that $q(t)$ is equal to

$$p(t) - \left(\sum_{i=0}^{m} b_i t^i\right)\left(1 - \sum_{i=1}^{d-1} n_i t^i + t^d\right)$$

which equals

$$p(t) - \sum_{i=0}^{m}\left[b_i - \sum_{\ell=1}^{d-1} n_\ell b_{i-\ell} + b_{i-d}\right] t^i + \sum_{i=m+1}^{m+d}\left[\sum_{\ell=i-m}^{d-1} n_\ell b_{i-\ell} - b_{i-d}\right] t^i.$$

By (5.2-2), the left-hand sum is $p(t)$ so

$$q(t) = \sum_{i=m+1}^{m+d} r_i t^i.$$

Suppose $\deg(x_i) = 1$ for all $i = 1, \ldots, g$. Then

$$q(t) \;=\; r_{m+1}t^{m+1} + r_{m+2}t^{m+2} \;=\; at^{m+1} + b_m(1-t)t^{m+1}$$

where $a = (g-1)b_m - b_{m-2} \geq 0$. Thus, $q(t) = q_M(t)$ where $M = M'(-m-1)$ and

$$M' \;=\; A^a \;\oplus\; \left(\frac{A}{x_1 A}\right)^{b_m}.$$

Suppose $\deg(x_i) \neq 1$ for some i. Then $d_1 \neq d_g$ and Lemmas 5.2.4 and 5.2.5 below show that $q(t) = q_M(t)$ for some $M \in \mathsf{gr}(A)$. □

5.2.4 Technical lemmas

The next three lemmas complete the proof of Lemma 5.2.2 when $d_1 \neq d_g$ so are proved under that hypothesis.

Lemma 5.2.3. *For each integer i between $m + d_1 + 1$ and $m + d_g$,*

$$\sum_{\ell=i-m}^{d-1} n_\ell b_{i-\ell} \;\geq\; b_{i-d}.$$

Proof. Since n_ℓ is the number of the generators $x_1, \dots, x_g$ having degree ℓ, $n_\ell \geq 0$ for all ℓ between $i - m$ and $d - 1$. Since $i - m$ is between $d_1 + 1$ and d_g, the only ℓ's between $i - m$ and $d - 1$ for which n_ℓ is non-zero are $d_2, \dots, d_g$. If $\ell = d_j$, then $n_\ell b_{i-\ell} = n_{d_j} b_{i-d_j}$; but $i - d_j \geq i - d \geq m + 1 - d$ so $b_{i-d_j} \geq b_{i-d}$. The result follows. □

Lemma 5.2.4. *There is $N \in \mathsf{gr}(A)$ such that*

$$q_N(t) = \sum_{i=m+d_1+1}^{m+d_g} r_i t^i.$$

Proof. By definition,

$$r_i = -b_{i-d} + \sum_{\ell=i-m}^{d-1} n_\ell b_{i-\ell}.$$

By Lemma 5.2.3, $r_i \geq 0$ for all i between $m + d_1 + 1$ and $m + d_g$. The module

$$N := \bigoplus_{i=m+d_1+1}^{m+d_g} A^{r_i}(-i).$$

satisfies the conclusion of the lemma. □

Lemma 5.2.5. *There is $L \in \mathsf{gr}(A)$ such that*

$$q_L(t) \;=\; \sum_{i=m+1}^{m+d_1} r_i t^i \;+\; \sum_{i=m+d_g+1}^{m+d} r_i t^i.$$

Proof. Because $d_1 + d_g = d$,

$$\sum_{i=m+1}^{m+d_1} r_i t^i \;+\; \sum_{i=m+d_g+1}^{m+d} r_i t^i \;=\; \sum_{i=m+1}^{m+d_1} (r_i + r_{i+d_g} t^{d_g}) t^i.$$

However, $n_\ell = 0$ for all $\ell \geq d_g + 1$ so, when $m+1 \leq i \leq m+d_1$,

$$r_i + r_{i+d_g} t^{d_g} \;=\; r_i - b_{i+d_g-d} t^{d_g} \;=\; r_i - b_{i-d_1} + b_{i-d_1}(1 - t^{d_g}),$$

We must therefore show there is $L \in \mathsf{gr}(A)$ such that

$$q_L(t) \;=\; \sum_{i=m+1}^{m+d_1} (r_i - b_{i-d_1}) t^i + \sum_{i=m+1}^{m+d_1} b_{i-d_1}(1 - t^{d_g}) t^i$$

Since $t^i = q_{A(-i)}(t)$ and $(1 - t^{d_g})t^i = q_{(A/x_gA)(-i)}(t)$, $q(t)$ equals $q_L(t)$ where

$$L \;=\; \left(\bigoplus_{i=m+1}^{m+d_1} A^{r_i - b_{i-d_1}}(-i) \right) \oplus \left(\bigoplus_{i=m+1}^{m+d_1} \left(\frac{A}{x_g A} \right)^{b_{i-d_1}} (-i) \right)$$

provided the coefficients $r_i - b_{i-d_1}$ and b_{i-d_1} are non-negative. Since $i - d_1 \geq m+1-d$, $b_{i-d_1} > 0$.

If $m+1 \leq i \leq m+d_1$, then

$$r_i \;\geq\; n_{d_1} b_{i-d_1} + n_{d_g} b_{i-d_g} - b_{i-d} \;\geq\; b_{i-d_1}$$

so $r_i - b_{i-d_1} \geq 0$. □

5.3 The Grothendieck group of $\mathsf{qgr}(A)$

We make $\mathbb{Z}[t^{\pm 1}]/(f)$ an ordered abelian group by defining

$$\left(\frac{\mathbb{Z}[t, t^{-1}]}{(f)} \right)_{\geq 0} := \{ \overline{p} \mid p(\theta) > 0 \} \cup \{0\}$$

where $\overline{p}$ denotes the image of the Laurent polynomial p in $\mathbb{Z}[t, t^{-1}]/(f)$. The order structure on $\mathbb{Z}[\theta]$ is inherited from its embedding in $\mathbb{R}$.

Theorem 5.3.1. *Let A be the algebra discussed in §5.1. The Grothendieck group $K_0(\mathsf{qgr}\, A)$ is isomorphic as an ordered abelian group to*

$$\frac{\mathbb{Z}[t, t^{-1}]}{(f)}$$

via the map $[\pi^* M] \mapsto \overline{q_M(t)}$. If f is irreducible, $K_0(\mathsf{qgr}\, A)$ is isomorphic as an ordered abelian group to $\mathbb{Z}[\theta]$ via the map $[\pi^* M] \mapsto q_M(\theta)$.

Furthermore, under the isomorphism(s), the functor $\mathcal{M} \mapsto \mathcal{M}(1)$ corresponds to multiplication by t^{-1} and θ^{-1}.

Proof. By §5.2.1, 5.2.2 and 5.2.3, F satisfies C3. The result now follows from Theorem 3.3.1. $\square$

5.4 The proof of Proposition 5.2.1

5.4.1 The idea of the proof

The idea of the proof of Proposition 5.2.1 is similar to the idea of the proof of Proposition 4.2.1 described in §4.4.1. Namely, we will associate to A a particular finite directed graph G and show that the characteristic polynomial of G is $t^{\ell-d} f(t)$ where ℓ is the sum of the degrees of the generators x_i. We also show that M is primitive, i.e., all entries of M^n are positive for $n \gg 0$. We then apply the Perron-Frobenius theorem which says that a primitive matrix has a positive real eigenvalue of multiplicity 1, ρ say, with the property that $|\lambda| < \rho$ for all other eigenvalues λ. But the non-zero eigenvalues of M are the roots of $f(t)$. Since we already know that $f(t)$ has only two positive real roots, $\theta < 1$ and $\theta^{-1} > 1$, $\rho = \theta^{-1}$. Since the coefficient of t^i in $f(t)$ is the same as that of t^{d-i}, $f(t) = t^d f(t^{-1})$. Thus

$f(\lambda) = 0$ if and only if $f(\lambda^{-1}) = 0$. Hence θ^{-1} is the unique root of $f(t)$ having largest modulus, so θ is the unique root of $f(t)$ having smallest modulus.

The x_is are labelled so that $\deg(x_1) \leq \cdots \leq \deg(x_g)$.

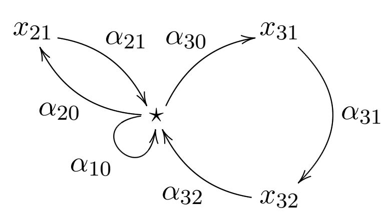
The free algebra on $k\langle x_1, \ldots, x_g\rangle$ is the path algebra of the quiver with one vertex $\star$ and g loops from $\star$ to $\star$ labelled $x_1, \ldots, x_g$. We replace each loop x_i by $d_i' := \deg(x_i) - 1 = d_i - 1$ vertices labelled $x_{i1}, \ldots, x_{id_i'}$ and arrows

$$\star \xrightarrow{\alpha_{i0}} x_{i1} \xrightarrow{\alpha_{i1}} \cdots \qquad\qquad \cdots \longrightarrow x_{id_i'} \xrightarrow{\alpha_{id_i'}} \star$$

The graph obtained by this procedure is the graph associated to $k\langle x_1, \ldots, x_g\rangle$ in 4.4.2.

5.4.2 Example

If A is generated by x_1, x_2, x_3 and $\deg(x_i) = i$, the associated graph is

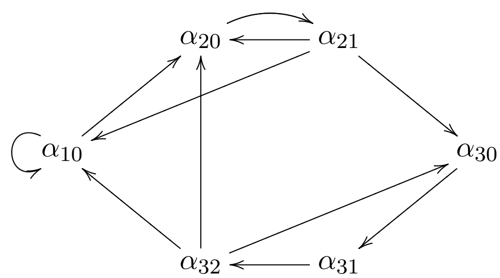


5.4.3 The second graph associated to A

We now form a second directed graph, the vertices of which are the arrows in the previous graph. In the second graph there is an arrow from vertex u to vertex v if in the first graph the arrow u can be followed by the arrow v, **except** we do not include an arrow $\alpha_{1d_1'} \to \alpha_{g0}$.

We write G, or $G(A)$, for the second graph associated to A.

The second graph associated to Example 5.4.2 is



Note the absence of an arrow from α_{10} to α_{30}.

Proposition 5.4.1. *If u and v are vertices in G, there is a directed path starting at u and ending at v.*

Proof. There is a directed path $\alpha_{i0} \to \alpha_{i1} \to \cdots \to \alpha_{id_i'} \to \alpha_{i0}$ so the result is true if $u = \alpha_{ij}$ and $v = \alpha_{ik}$. There are also arrows

$$\alpha_{1d_1'} \to \alpha_{20}, \quad \alpha_{2d_2'} \to \alpha_{30}, \quad \ldots \quad \alpha_{g-1,d_{g-1}'} \to \alpha_{g0}, \quad \alpha_{gd_g'} \to \alpha_{10}$$

so the result is true if $u = \alpha_{i_1j_1}$ and $v = \alpha_{i_2j_2}$. □

Proposition 5.4.2. *Let M be an incidence matrix for G. Then every entry in M^n is non-zero for $n \gg 0$.*

Proof. In the language of [8, Defn. 4.2.2], Proposition 5.4.1 says that M is irreducible.

The period of a vertex v in G is the greatest common divisor of the non-trivial directed paths that begin and end at v. The period of G is the greatest common divisor of the periods of its vertices. Since there is a directed path of length $d_i = \deg(x_i)$ from α_{i0} to itself, the period of G divides $\gcd\{d_1, \ldots, d_g\}$ which is 1. The period of G is therefore 1. Thus, in the language of [8, Defn. 4.5.2], M is aperiodic and therefore primitive [8, Defn. 4.5.7]. Hence [8, Thm. 4.5.8] applies to M, and gives the result claimed. □

The Perron-Frobenius theorem [6, Thm. 1, p.64] therefore applies to M giving the following result.

Corollary 5.4.3. *The characteristic polynomial for G has a unique eigenvalue of maximal modulus and that eigenvalue is simple and real.*

Our next goal, achieved in Proposition 5.4.8, is to show that $p_G(t) = t^{\ell-d}f(t)$ for a suitable ℓ.

5.4.4 Other graphs associated to A

We now write $\mathcal{X} := \{x_1, \ldots, x_g\}$ and define the directed graph $\widehat{\mathcal{X}}$ by declaring that its vertex set is $\mathcal{X}$ and there is an arrow $x_i \to x_j$ for all $(x_i, x_j) \in \mathcal{X}^2 - \{(x_1, x_g)\}$. For each non-empty subset $X \subset \mathcal{X}$ let $\widehat{X}$ be the full subgraph of $\widehat{\mathcal{X}}$ with vertex set X.

If $g = 4$, then

$$\{x_1, x_2, x_3\}\widehat{\ } \quad = \quad \begin{tikzcd} & x_2 \arrow[loop above] \arrow[dl, bend left=10] \arrow[dr, bend left=10] & \\ x_1 \arrow[loop left] \arrow[ur, bend left=10] \arrow[rr, bend left=10] & & x_3 \arrow[loop right] \arrow[ul, bend left=10] \arrow[ll, bend left=10] \end{tikzcd}$$

and

$$\{x_1, x_2, x_4\}\widehat{\ } \quad = \quad \begin{tikzcd} & x_2 \arrow[loop above] \arrow[dl, bend left=10] \arrow[dr, bend left=10] & \\ x_1 \arrow[loop left] \arrow[ur, bend left=10] & & x_4 \arrow[loop right] \arrow[ul, bend left=10] \arrow[ll, bend left=10] \end{tikzcd}$$

Lemma 5.4.4. *Let $X \subset \{x_1, \dots, x_g\}$. The constant term in the characteristic polynomial for $\widehat{X}$ is*

$$p_{\widehat{X}}(0) = \begin{cases} 1 & \textit{if } X = \{x_1, x_g\} \\ -1 & \textit{if } |X| = 1 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Let M be an incidence matrix of $\widehat{X}$. Then the constant term in the characteristic polynomial for $\widehat{X}$ is $p_{\widehat{X}}(0) = (-1)^{|X|}\det(M)$.

If $|X| = 1$, then $\widehat{X}$ consists of one vertex with a single loop so $M = (1)$ whence $p_{\widehat{X}}(0) = -1$.

If $X = \{x_1, x_g\}$, then $\widehat{X}$ has vertices x_1 and x_g, an arrow from x_g to x_1, and a loop at each vertex. Hence $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ is an incidence matrix for $\widehat{X}$ and the constant term is 1.

If $|X| = 2$ and $X \neq \{x_1, x_g\}$, then the incidence matrix for $\widehat{X}$ is $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ so the constant term is 0.

Suppose $|X| \geq 3$. If $\{x_1, x_g\} \subseteq X$, then M has a single off-diagonal 0 and all its other entries are 1; in particular, M is singular so the constant term is 0. If $\{x_1, x_g\} \not\subseteq X$, then every entry in M is 1 so M is singular and the constant term is 0. □

5.4.5 The paths $\beta_1, \dots, \beta_g$ in G

For each $1 \leq i \leq g$, let β_i be the path

$$\alpha_{i0} \to \alpha_{i1} \to \cdots \to \alpha_{id'_i}.$$

In example 5.4.2, β_1 is the trivial path at vertex α_{10}, β_2 is the arrow $\alpha_{20} \longrightarrow \alpha_{21}$, and β_3 is the path $\alpha_{30} \longrightarrow \alpha_{31} \longrightarrow \alpha_{32}$.

Proposition 5.4.5. *Let $i_1, \ldots, i_m$ be pairwise distinct elements of $\{1, \ldots, g\}$ such that $(1, g) \notin \{(i_m, i_1), (i_1, i_2), \ldots, (i_{m-1}, i_m)\}$. Then there is a simple cycle in G of the form*

$$\beta_{i_1} \to \beta_{i_2} \to \cdots \to \beta_{i_m} \to \alpha_{i_1 0} \tag{5.4-1}$$

and every simple cycle in G is of this form, up to choice of starting point.

Proof. Let $r, s \in \{1, \ldots, g\}$ and assume $r \neq s$. If $(r, s) \neq (1, g)$, then there is an arrow from $\alpha_{rd'_r}$, the vertex at which β_r ends, to α_{s0}, the vertex at which β_s starts; hence there is a path "traverse β_r then traverse β_s"; we denote this path by $\beta_r \to \beta_s$. It follows that there is a path of the form (5.4-1).

Let p be a simple cycle in G. A simple cycle passes through a vertex α_{ij} if and only if it passes through α_{i0}. Every simple cycle that passes through α_{i0} contains β_i as a subpath because there is a unique arrow starting at α_{ij} for all $j = 0, \ldots, d'_i - 1$. Hence p is of the form (5.4-1). □

Lemma 5.4.6. *There is a bijection $\Phi : Z(\widehat{\mathcal{X}}) \to Z(G)$ defined by*

$$\Phi(x_{i_1} \to \cdots \to x_{i_m} \to x_{i_1}) \ := \ \beta_{i_1} \to \cdots \to \beta_{i_m} \to \alpha_{i_1 0} \tag{5.4-2}$$

whose inverse is

$$\Phi^{-1}(\beta_{i_1} \to \cdots \to \beta_{i_m} \to \alpha_{i_1 0}) := x_{i_1} \to \cdots \to x_{i_m} \to x_{i_1}. \tag{5.4-3}$$

Proof. We need only check that Φ and Ψ are well-defined. Because $\widehat{\mathcal{X}}$ does not contain an arrow $x_1 \to x_g$ and G does not contain an arrow $\alpha_{1d'_1} \to \alpha_{g0}$, the right-hand sides of (5.4-2) and (5.4-3) are simple cycles. □

The next result is obvious.

Proposition 5.4.7. *The function* Φ *extends to a bijection* $\overline{\Phi} : \overline{Z}(\widehat{\mathcal{X}}) \to \overline{Z}(G)$ *defined by*

$$\overline{\Phi}(E_1 \sqcup \cdots \sqcup E_m) \ := \ \Phi(E_1) \sqcup \cdots \sqcup \Phi(E_m)$$

for disjoint simple cycles $E_1, \ldots, E_m$ *in* $\widehat{\mathcal{X}}$*. Furthermore,* $c(E) = c(\overline{\Phi}(E))$ *for all* $E \in \overline{Z}(\widehat{\mathcal{X}})$*.*

The support of a subgraph Q of G is

$$\mathrm{Supp}(Q) := \{x_i \mid \beta_i \text{ is a path in } Q\}.$$

For each non-empty subset $X \subset \{x_1, \cdots, x_g\}$ let

$$\overline{Z}(G, X) := \{Q \in \overline{Z}(G) \mid \mathrm{Supp}(Q) = X\}$$

and let $d(X) = \sum_{x \in X} \deg(x)$.

Proposition 5.4.8. *Let* $\ell = \sum_{i=1}^{g} \deg(x_i)$*. The characteristic polynomial of* G *is* $t^{\ell - d} f(t)$*.*

Proof. The characteristic polynomial of G is

$$p_G(t) = t^\ell + c_1 t^{\ell-1} + \cdots + c_{\ell-1} t + c_\ell$$

where $\ell = v(G) = \sum_{i=1}^{g} d_i$ and

$$c_i \ = \sum_{\substack{Q \in \overline{Z}(G) \\ v(Q) = i}} (-1)^{c(Q)} \ = \sum_{\substack{X \subset \mathcal{X} \\ d(X) = i}} \left(\sum_{Q \in \overline{Z}(G,X)} (-1)^{c(Q)} \right). \qquad (5.4\text{-}4)$$

Since $\overline{Z}(G, X) = \{\overline{\Phi}(E) \mid E \in \overline{Z}(\widehat{X}) \ \& \ v(E) = d(X)\}$ we have

$$\sum_{Q \in \overline{Z}(G,X)} (-1)^{c(Q)} \ = \sum_{\substack{E \in \overline{Z}(\widehat{X}) \\ v(E) = |X|}} (-1)^{c(E)}. \qquad (5.4\text{-}5)$$

Since $v(\widehat{X}) = |X|$, the right-hand side of (5.4-5) is $p_{\widehat{X}}(0)$. Hence by Lemma 5.4.4,

$$c_i = \begin{cases} 1 & \text{if } i = d_1 + d_g = d, \\ -n_i & \text{if } 1 \leq i \leq d_g, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $p_G(t) = t^\ell - n_1 t^{\ell-1} - \cdots - n_{d-1} t^{\ell-d+1} + t^{\ell-d} = t^{\ell-d} f(t)$, as claimed. $\square$

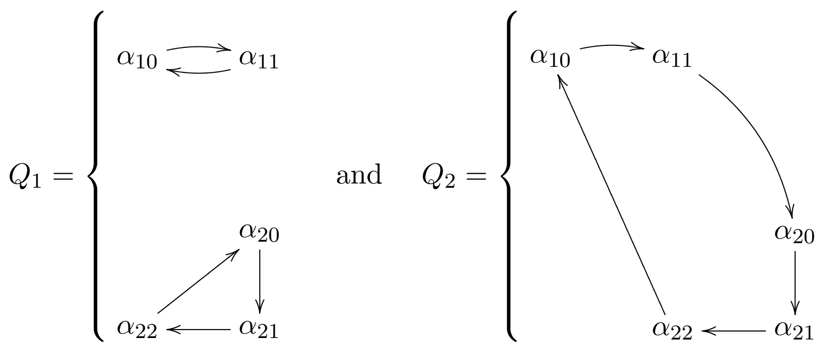
As explained at the end of §5.4.1, Proposition 5.2.1 follows from Proposition 5.4.8 and Corollary 5.4.3.

Example

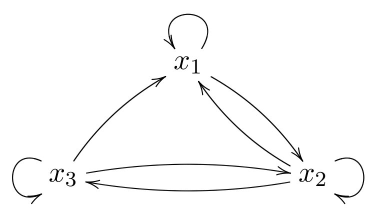
In order to clarify some of the technicalities in this section, we will compute the coefficient c_5 in $p_G(t) = t^9 + c_1 t^8 + \cdots + c_8 t + c_9$ where G is the second graph associated to the algebra $A = k\langle x_1, x_2, x_3 \rangle/(b)$ where $\deg(x_i) = i + 1$. First, G is

$$\begin{tikzcd}
 & \alpha_{10} \arrow[r, bend left] & \alpha_{11} \arrow[l, bend left] \arrow[dr, bend left] & \\
\alpha_{33} \arrow[ur, bend left] \arrow[r] \arrow[rrr, bend left] & \alpha_{30} \arrow[d] & & \alpha_{20} \arrow[d] \\
\alpha_{32} \arrow[u] & \alpha_{31} \arrow[l] & \alpha_{22} \arrow[l] \arrow[ul, to=1-2] \arrow[ur] & \alpha_{21} \arrow[l]
\end{tikzcd}$$

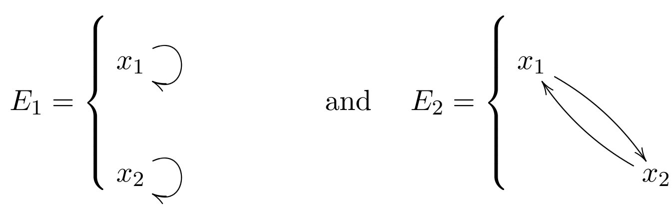
There are two subgraphs of G that have exactly five vertices and are disjoint unions of simple cycles, namely



The only subset X of $\mathcal{X} = \{x_1, x_2, x_3\}$ such that $d(X) = 5$ is $X = \{x_1, x_2\}$. The graph $\widehat{\mathcal{X}}$ is



Since $Q_1 = \overline{\Phi}(E_1)$ and $Q_2 = \overline{\Phi}(E_2)$ where



equations (5.4-4) and (5.4-5) give

$$
\begin{aligned}
c_5 &= (-1)^{c(Q_1)} + (-1)^{c(Q_2)} \\
&= (-1)^{c(E_1)} + (-1)^{c(E_2)} \\
&= 1 - 1 \\
&= 0.
\end{aligned}
$$

5.5 Examples

5.5.1

When A is generated by $g \geq 3$ elements of degree one, $f(t)$ is the irreducible polynomial $1 - gt + t^2$ so

$$K_0(\mathsf{qgr}(A)) \;\cong\; \mathbb{Z}\left[\frac{g - \sqrt{g^2 - 4}}{2}\right] \;\subset\; \mathbb{R}$$

as ordered abelian groups.

5.5.2 Non-irreducible f

Suppose $g = 4$, $d_1 = d_2 = 1$ and $d_3 = d_4 = 2$. Then $f(t) = 1 - 2t - 2t^2 + t^3 = (1+t)(1-3t+t^2)$ and $\theta = \frac{1}{2}(3 - \sqrt{5})$. The map

$$\frac{\mathbb{Z}[t, t^{-1}]}{(f)} \to \mathbb{Z} \oplus \mathbb{Z}[\theta], \quad \overline{p} \mapsto (p(-1), p(\theta))$$

is an isomorphism of abelian groups. The image of the positive cone under that isomorphism $K_0(\mathsf{qgr}(A)) \longrightarrow \mathbb{Z} \oplus \mathbb{Z}[\theta]$ is $(\mathbb{Z} \oplus \mathbb{Z}[\theta]_{\geq 0}) \cup \{0\}$.

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