Energy equations for gas-liquid turbulent flows with phase change

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A thesis
submitted in partial fulfillment of the
requirements for the degree of

Master of Science in Aeronautics & Astronautics

University of Washington

2024

Reading Committee:
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Program Authorized to Offer Degree:
Aeronautics and Astronautics

University of Washington

Abstract

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In this work, we derive the equations of turbulence kinetic energy (TKE), mean-flow kinetic energy (MFKE) and mean internal energy (MIE) for gas-liquid interfacial flows with phase change for the first time. We present the equations for a fully compressible flow in gas or liquid phase which allows us to study the effects of global volume changes away from the interface. We then take the interface into account and present the energy equations for the gas-liquid interfacial flow allowing for local volume changes at the gas-liquid interface. These equations allow us to explain the pathways of energy exchange between the gas and liquid phase while highlighting the effects of local volume changes at the interface due to phase change. We explain the role of the phase change in the energy interactions through phase change free energy. We relate the rate of change of phase change free energy to the powers of mass flux, momentum flux and heat flux due to phase change. This link allows us to explain how evaporation or condensation affects the temporal evolution of energies for compressible gas-liquid flows with phase change. We rewrite these energy equations for droplet-laden homogeneous shear turbulence with phase change (DLHST-PC), and droplet-laden decaying homogeneous isotropic turbulence with phase change (DLHIT-PC). In addition, we explain the role of interfacial surface energy through the power of surface tension.

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ACKNOWLEDGMENTS

Firstly, I thank Professor Antonino Ferrante for recognizing the potential of this work and providing me with the freedom and guidance to question, explore, and expand what began as a month-long venture into a 50-page thesis. His ability to spot errors in my math and his candid, constructive criticism, comparing them to the errors made by his daughter, motivated me to strive for higher standards in my mathematical work. I'd like to believe that I've absorbed some of his curiosity and passion for understanding the beauty of turbulence.

Secondly, I thank Dr. Pablo Trefftz-Posada for being an amazing mentor and collaborator. I deeply appreciate the patience and time he has generously dedicated to me over the past two years, starting from the moment I expressed my desire to do well in the CFM lab.

Thirdly, I thank Professor Michelle DiBenedetto for her invaluable input regarding my thesis. The foundational math I knew at the outset of this mathematically intensive work largely stemmed from her course notes on incompressible fluid mechanics and turbulence.

I'd also like to express my sincere gratitude to the professors, friends and mentors I have made during my time at the Indian Institute of Technology Goa, SankhyaSutra Labs Ltd. and the University of Washington.

Most importantly, I'd like to thank my mother, my little brothers, aunts, uncles, my little cousins and my grand parents who never complained about not seeing me in person for two straight years. They were content with the few minutes I could spend on a video call during the weekends that I found free from coursework, research or teaching assistant duties.

DEDICATION

To my father

Chapter 1

INTRODUCTION

Phase change is a key phenomenon in gas-liquid flows, especially in the presence of turbulence. In spray combustion systems, the fuel droplet vaporization rate is the controlling factor for combustion (Sirignano, 1983). The liquid droplet vaporization rate depends particularly on the turbulence characteristics of the gas phase (Birouk & Gökalp, 2006). Condensation of water vapor in air to liquid droplets, taking place at the droplet scale, introduces effects that drive cloud-scale motion (Grabowski & Wang, 2013). In order to understand the effect of phase change in turbulent flows, it is important to study its role in the modulation of energy budget of such flows.

Historically, studying the energy budget equations, such as the turbulence kinetic energy (TKE) equation, has provided insight into the flow physics. For example, TKE equation for single phase incompressible decaying homogeneous isotropic turbulence led to the power law of dissipation of TKE, e.g. Pope (2000, p. 158). Analogous set of equations for droplet-laden decaying homogeneous isotropic turbulence were derived by Dodd & Ferrante (2016). Physical mechanisms like droplet deformation, coalescence and breakup were explained using the power of surface tension which appears as budget term in the TKE equation. This formalism was extended to droplet-laden homogeneous shear turbulence by Trefftz-Posada & Ferrante (2023) and the analysis of the TKE budget equation led to the discovery of new physical mechanisms like droplet-catching up and droplet-shearing. In this work, we derive analogous equations for gas-liquid turbulent flows with phase change.

A preliminary analysis using the conservation of mass leads to the conclusion that phase change leads to local volume changes at the interface. Evaporation leads to local expansion and condensation leads to local compression at the interface. In addition, if the gas-liquid system is confined to a constant total volume, the local compression (expansion) at the interface must be compensated by global expansion (compression) in the individual phases. Thus, in order to study the effect of phase change, we must study the effect of compressibility.

TKE equations for single-phase compressible turbulent flows were derived and the TKE pathways were presented where compressibility effects were extensively discussed by Lele (1994) and Wilcox (2006, ch. 5). A set of governing equations to describe the motion of two fluids separated by an interface, involving surface-tension effects and allowing phase change were presented by Joseph & Renardy (1993, ch. 1). These were applied to two-fluids problem with phase change between inclined walls in Joseph & Renardy (1993, ch. IV.10). However, in the problem explored, the gas phase was considered to be incompressible and the energy interactions due with phase change were not discussed. We make use of the governing equations by Joseph & Renardy (1993, ch. 1) and follow methodology employed by Lele (1994) to derive the energy equations for fully compressible gas-liquid turbulent flows with phase change.

This work is organized in the following way: in chapter 2, we present the governing equations for compressible gas-liquid flows with phase change and analyze how they reduce to a set of jump conditions in the limit of the interface. In chapter 3, we derive the energy equations for compressible flow in a gas or liquid phase and highlight the effects of compressibility in the pathways of energy exchange. We then take the interface into account and with the help of jump conditions, derive the energy equations for compressible gas-liquid flows with phase change. We present the pathway schematic highlighting the effect of phase change in the modulation of TKE, MFKE and MIE. In addition, we also rewrite these general set of equations for droplet-laden homogeneous shear turbulence with phase change (DLHST-PC) and droplet-laden decaying homogeneous isotropic turbulence with phase change (DLHIT-PC).

Chapter 2

MATHEMATICAL FORMULATION

In Sec. 2.1, we present the set of governing equations for compressible gas-liquid flows with phase change. In Sec. 2.2, we reduce these equations to jump conditions across the interface.

We consider the flow of a gas-liquid system separated by a common interface across which phase change is allowed. The geometry of the control volume we consider, which is adapted from (Joseph & Renardy, 1993, p. 20) and (Dodd & Ferrante, 2016, Appendix A) is illustrated in Fig. 2.1. The volume, $\mathcal{V}(t)$ is a material volume, i.e., fluid elements cannot cross its boundaries. $\mathcal{V}(t)$ consists of two volumes of fluid, $\mathcal{V}_g(t)$ (gas phase) and $\mathcal{V}_l(t)$ (liquid phase) such that

$$\mathcal{V} = \mathcal{V}_g \cup \mathcal{V}_l \tag{2.1}$$

The volumes $\mathcal{V}_g(t)$ and $\mathcal{V}_l(t)$ are separated by an interface $\Sigma(t)$, across which phase change could occur, and are bounded by the surfaces $\partial\mathcal{V}_g(t)$ and $\partial\mathcal{V}_l(t)$, respectively, such that

$$\Sigma = \partial\mathcal{V}_g \cap \partial\mathcal{V}_l \tag{2.2}$$

and the boundary of $\mathcal{V}(t)$ is $\partial\mathcal{V}(t)$, such that

$$\partial\mathcal{V} = \partial\mathcal{V}_g \cup \partial\mathcal{V}_l - \Sigma = (\partial\mathcal{V}_g - \Sigma) \cup (\partial\mathcal{V}_l - \Sigma) \tag{2.3}$$

$\partial\Sigma(t)$ is a contact line between Σ and $\partial\mathcal{V}$ satisfying

$$\partial\Sigma = \Sigma \cap \partial\mathcal{V} \tag{2.4}$$

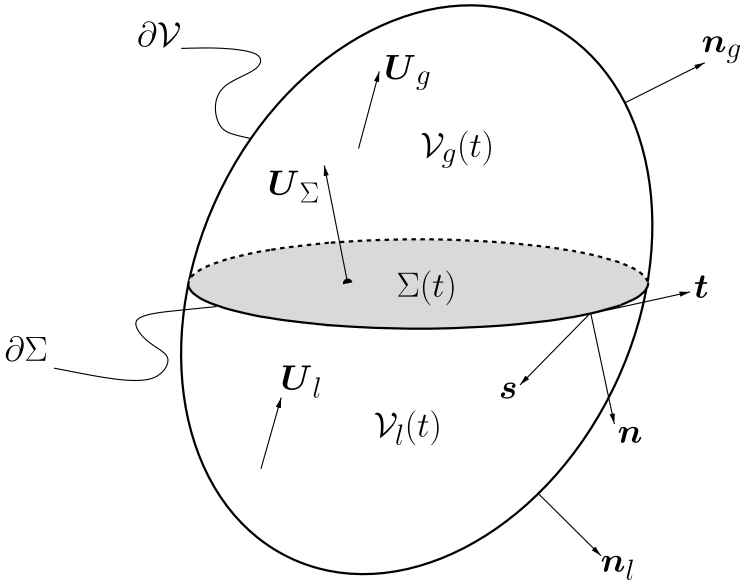

Figure 2.1: Volume $\mathcal{V}(t)$ containing a gas phase $\mathcal{V}_g(t)$ and a liquid phase $\mathcal{V}_l(t)$ separated by an interface $\Sigma(t)$ across which phase change could occur.

$\boldsymbol{n}_g$, $\boldsymbol{n}_l$ and $\boldsymbol{n}$ are the unit vector normals to surfaces $\partial\mathcal{V}_g$, $\partial\mathcal{V}_l$ and Σ respectively. $\boldsymbol{n}$ is directed from $\mathcal{V}_g$ to $\mathcal{V}_l$ and as a consequence,

$$\boldsymbol{n} = \begin{cases} \boldsymbol{n}_g & \forall \boldsymbol{x} \in (\Sigma \cap \partial\mathcal{V}_g) \\ -\boldsymbol{n}_l & \forall \boldsymbol{x} \in (\Sigma \cap \partial\mathcal{V}_l) \end{cases} \tag{2.5}$$

$\boldsymbol{t}$ is a unit vector tangent to $\partial\Sigma$ and $\boldsymbol{s}$ is a unit vector perpendicular to $\partial\Sigma$ that is pointed out of Σ such that $\boldsymbol{s}$, $\boldsymbol{n}$ and $\boldsymbol{t}$ form an orthonormal set (*e.g.*, $\boldsymbol{s} = \boldsymbol{n} \times \boldsymbol{t}$). If Σ forms a closed surface, then, $\partial\Sigma$ ceases to exist and both $\boldsymbol{s}$, $\boldsymbol{t}$ form an orthonormal pair of tangent vectors on the curved surface of Σ.

2.1 Governing equations

In this section, we present the integral formulation of a set of equations governing compressible flow in a gas-liquid system separated by an interface across which phase change occurs, considering surface-tension and body force effects, in the conservative form.

2.1.1 Conservation of mass

The principle of conservation of mass states that the mass of the material volume $\mathcal{V}$ does not change

$$\frac{d}{dt}\int_{\mathcal{V}} \rho d\mathcal{V} = 0 \tag{2.6}$$

where $\rho = \rho(\boldsymbol{x}, t)$ is the density field with a finite discontinuity across Σ such that

$$\rho = \begin{cases} \rho_g & \forall \boldsymbol{x} \in \mathcal{V}_g \\ \rho_l & \forall \boldsymbol{x} \in \mathcal{V}_l \end{cases} \tag{2.7}$$

where ρ_g and ρ_l are the density fields of gas and liquid phases respectively.

2.1.2 Conservation of momentum

The conservation equation for the linear momentum of fluid in $\mathcal{V}$ as presented in (Joseph & Renardy, 1993, p. 22) is given as,

$$\frac{d}{dt}\int_{\mathcal{V}} \rho \boldsymbol{U} d\mathcal{V} = \int_{\partial\mathcal{V}} \boldsymbol{\tau}\boldsymbol{n} d\mathcal{A} + \oint_{\partial\Sigma} \sigma \boldsymbol{s} dl + \int_{\mathcal{V}} \rho \boldsymbol{f_b} d\mathcal{V} \tag{2.8}$$

On the left-hand of Eq. (2.8) is the rate of change of momentum in $\mathcal{V}$, and on the right-hand side, the three terms represent the surface forces due to fluid stresses, surface-tension forces, and body forces in the presence of a body force field $\boldsymbol{f_b}$ (e.g., $\boldsymbol{f_b} = \boldsymbol{g}$, in the presence of gravitational field), respectively. $\boldsymbol{U} = \boldsymbol{U}(\boldsymbol{x}, t)$ is the velocity field which given by

$$\boldsymbol{U} = \begin{cases} \boldsymbol{U}_g & \forall \boldsymbol{x} \in \mathcal{V}_g \\ \boldsymbol{U}_l & \forall \boldsymbol{x} \in \mathcal{V}_l \\ \boldsymbol{U}_\Sigma & \forall \boldsymbol{x} \in \Sigma \end{cases} \tag{2.9}$$

where $\boldsymbol{U}_g$ and $\boldsymbol{U}_l$ are the velocity fields in the $\mathcal{V}_g$ and $\mathcal{V}_l$, respectively and $\boldsymbol{U}_\Sigma$ is the velocity of the interface. We note here that the interface Σ might move with a different velocity compared to the gas or liquid when there is phase change. $\boldsymbol{\tau}$ is the fluid stress tensor, which can be divided into pressure and viscous stress as,

$$\boldsymbol{\tau} = -p\mathsf{I} + \mathsf{T} \tag{2.10}$$

where $p = p(\boldsymbol{x}, t)$ is the pressure field, I is an identity matrix, -$p\mathsf{I}$ is the pressure stress tensor and T is the viscous stress tensor. σ is the surface-tension coefficient of Σ. We note that the line-integral along $\partial\Sigma$ in the surface-tension force term can be converted into a surface integral using the divergence theorem for curved surfaces as,

$$\oint_{\partial\Sigma} \sigma \boldsymbol{s} dl = \int_{\Sigma} \nabla_s \sigma d\mathcal{A} + \int_{\Sigma} \kappa\sigma\boldsymbol{n} dl \tag{2.11}$$

where ∇_s is the surface gradient defined by,

$$\nabla_s \equiv \nabla - \boldsymbol{n}\left(\boldsymbol{n} \cdot \nabla\right) \tag{2.12}$$

and κ is the curvature of the interface Σ.

2.1.3 Conservation of energy

The conservation equation for the total energy of the fluid in $\mathcal{V}$ as presented in (Joseph & Renardy, 1993, p. 23) is given as,

$$\begin{aligned} \frac{d}{dt}\int_{\mathcal{V}} \rho \left(e + \frac{1}{2}\boldsymbol{U}\cdot\boldsymbol{U}\right) d\mathcal{V} + \frac{d}{dt}\int_{\Sigma} \sigma d\mathcal{A} = \int_{\partial\mathcal{V}} (\boldsymbol{\tau}\boldsymbol{n}) \cdot \boldsymbol{U} d\mathcal{A} + \oint_{\partial\Sigma} (\sigma\boldsymbol{s}) \cdot \boldsymbol{U}_{\Sigma} dl \\ - \int_{\partial\mathcal{V}} \boldsymbol{q} \cdot \boldsymbol{n} d\mathcal{A} + \int_{\mathcal{V}} \rho \boldsymbol{f_b} \cdot \boldsymbol{U} d\mathcal{V} \end{aligned} \tag{2.13}$$

On the left-hand side of Eq. (2.13) the first two terms express the rate of change of specific internal energy plus kinetic energy, respectively and the third term denotes the rate of change of stored energy of the surface, and on the right-hand side, the terms represent the power of the traction forces on the boundary, power of the surface-tension, rate of conductive heat transfer due to the conductive heat flux $\boldsymbol{q}$ and the power of the body force, respectively.

The transport equation for surface energy is derived by (Joseph & Renardy, 1993, p. 23) using the analogue of Reynolds transport theorem for surfaces given by (Aris, 1989, p.230):

$$\frac{d}{dt}\int_{\Sigma}\sigma d\mathcal{A} = \int_{\Sigma}\left(\frac{\partial\sigma}{\partial t} - \boldsymbol{U}_{\Sigma}\cdot\nabla\sigma - \sigma\kappa\boldsymbol{U}_{\Sigma}\cdot\boldsymbol{n}\right)d\mathcal{A} + \oint_{\partial\Sigma}(\sigma\boldsymbol{s})\cdot\boldsymbol{U}_{\Sigma}dl \tag{2.14}$$

For a constant surface tension, the Eq. (2.14) reduces to,

$$\frac{d}{dt}\int_{\Sigma}\sigma d\mathcal{A} = \int_{\Sigma}\sigma\kappa\boldsymbol{U}_{\Sigma}\cdot\boldsymbol{n}d\mathcal{A} + \oint_{\partial\Sigma}(\sigma\boldsymbol{s})\cdot\boldsymbol{U}_{\Sigma}dl \tag{2.15}$$

We note that in assuming a constant surface tension, we are ignoring the Marangoni effects caused due to local surface tension gradients as a result of temperature gradients across Σ.

2.2 Jump conditions at the gas-liquid interface

In this section, we adopt the methodology given in (Joseph & Renardy, 1993, ch. 1.2) to derive two identities useful in deriving the energy equations in further sections. Let $\mathcal{F}(\boldsymbol{x}, t)$ be a smooth field with a simple discontinuity across Σ and smooth derivatives on either side of Σ such that

$$\mathcal{F} = \begin{cases} \mathcal{F}_g & \forall\boldsymbol{x}\in\mathcal{V}_g \\ \mathcal{F}_l & \forall\boldsymbol{x}\in\mathcal{V}_l \end{cases} \tag{2.16}$$

where $\mathcal{F}_g$ and $\mathcal{F}_l$ are the values of $\mathcal{F}$ in gas and liquid phases respectively. Consider the limits in which $\mathcal{V}(t)$ is collapsed onto Σ, while Σ is held fixed. We define the jump in f across Σ as,

$$[[f]] = \mathcal{F}_g - \mathcal{F}_l \tag{2.17}$$

2.2.1 Jump identity 1

We integrate $\mathcal{F}\boldsymbol{n}$ over $\partial\mathcal{V}$ and use Eq. (2.3), and the additive property of integrals, to obtain,

$$\int_{\partial\mathcal{V}}\mathcal{F}\boldsymbol{n}d\mathcal{A} = \int_{\partial\mathcal{V}_g-\Sigma}\mathcal{F}_g\boldsymbol{n}_g d\mathcal{A} + \int_{\partial\mathcal{V}_l-\Sigma}\mathcal{F}_l\boldsymbol{n}_l d\mathcal{A} \tag{2.18}$$

We further divide the right-hand side of Eq. (2.18), into regions of $\partial\mathcal{V}_g$, $\partial\mathcal{V}_l$ and Σ as,

$$\int_{\partial\mathcal{V}} \mathcal{F}\boldsymbol{n}d\mathcal{A} = \int_{\partial\mathcal{V}g} \mathcal{F}_g\boldsymbol{n}_g d\mathcal{A} - \int_{\Sigma} \mathcal{F}_g\boldsymbol{n}_g d\mathcal{A} + \int_{\partial\mathcal{V}l} \mathcal{F}_l\boldsymbol{n}_l d\mathcal{A} - \int_{\Sigma} \mathcal{F}_l\boldsymbol{n}_l d\mathcal{A} \tag{2.19}$$

We group the terms with Σ as integral using the definition of $\boldsymbol{n}$ given in Eq. (2.5) as,

$$\int_{\partial\mathcal{V}} \mathcal{F}\boldsymbol{n}d\mathcal{A} = \int_{\partial\mathcal{V}g} \mathcal{F}_g\boldsymbol{n}_g d\mathcal{A} + \int_{\partial\mathcal{V}l} \mathcal{F}_l\boldsymbol{n}_l d\mathcal{A} - \int_{\Sigma} \left(\mathcal{F}_g - \mathcal{F}_l\right)\boldsymbol{n}d\mathcal{A} \tag{2.20}$$

We rewrite Eq. (2.20) using the definition of jump condition from Eq. (2.17) as the final jump identify 1:

$$\int_{\partial\mathcal{V}} \mathcal{F}\boldsymbol{n}d\mathcal{A} = \int_{\partial\mathcal{V}g} \mathcal{F}_g\boldsymbol{n}_g d\mathcal{A} + \int_{\partial\mathcal{V}l} \mathcal{F}_l\boldsymbol{n}_l d\mathcal{A} - \int_{\Sigma} [[\mathcal{F}]]\,\boldsymbol{n}d\mathcal{A} \tag{2.21}$$

2.2.2 Jump identity 2

We use the Reynolds transport theorem to find the rate of accumulation of $\mathcal{F}$ in $\mathcal{V}$ as,

$$\frac{d}{dt}\int_{\mathcal{V}} \mathcal{F}d\mathcal{V} = \int_{\mathcal{V}} \left(\frac{\partial\mathcal{F}}{\partial t} + \nabla\cdot\left(\mathcal{F}\boldsymbol{U}\right)\right)d\mathcal{V} \tag{2.22}$$

We divide the right-hand side of Eq. (2.22) into $\mathcal{V}_g$ and $\mathcal{V}_l$ portions using Eq. (2.1),

$$\frac{d}{dt}\int_{\mathcal{V}} \mathcal{F}d\mathcal{V} = \int_{\mathcal{V}g} \left(\frac{\partial\mathcal{F}}{\partial t} + \nabla\cdot\left(\mathcal{F}\boldsymbol{U}\right)\right)d\mathcal{V} + \int_{\mathcal{V}l} \left(\frac{\partial\mathcal{F}}{\partial t} + \nabla\cdot\left(\mathcal{F}\boldsymbol{U}\right)\right)d\mathcal{V} \tag{2.23}$$

We convert the volume integrals in the right-hand side of Eq. (2.23) to surface integrals using the divergence theorem as,

$$\frac{d}{dt}\int_{\mathcal{V}} \mathcal{F}d\mathcal{V} = \int_{\mathcal{V}g} \frac{\partial\mathcal{F}}{\partial t}d\mathcal{V} + \int_{\partial\mathcal{V}g} \mathcal{F}\boldsymbol{U}\cdot\boldsymbol{n}_g d\mathcal{A} + \int_{\mathcal{V}l} \frac{\partial\mathcal{F}}{\partial t}d\mathcal{V} + \int_{\partial\mathcal{V}l} \mathcal{F}\boldsymbol{U}\cdot\boldsymbol{n}_l d\mathcal{A} \tag{2.24}$$

We realize that $\mathcal{F}\boldsymbol{U}$ is no longer a simple discontinuity from Eq. (2.9) and Eq. (2.16), and we cannot simplify the surface integral terms in Eq. (2.24) using Eq. (2.21). We simplify it using the definition of $\boldsymbol{U}$ given in Eq. (2.9) as,

$$\int_{\partial\mathcal{V}g} \mathcal{F}\boldsymbol{U}\cdot\boldsymbol{n}_g d\mathcal{A} = \int_{\partial\mathcal{V}_g-\Sigma} \mathcal{F}_g\boldsymbol{U}_g\cdot\boldsymbol{n}_g d\mathcal{A} + \int_{\Sigma} \mathcal{F}_g\boldsymbol{U}_\Sigma\cdot\boldsymbol{n}_g d\mathcal{A} \tag{2.25a}$$

$$\int_{\partial\mathcal{V}l} \mathcal{F}\boldsymbol{U}\cdot\boldsymbol{n}_l d\mathcal{A} = \int_{\partial\mathcal{V}_l-\Sigma} \mathcal{F}_l\boldsymbol{U}_l\cdot\boldsymbol{n}_l d\mathcal{A} + \int_{\Sigma} \mathcal{F}_l\boldsymbol{U}_\Sigma\cdot\boldsymbol{n}_l d\mathcal{A} \tag{2.25b}$$

We add and subtract $\mathcal{F}_g\boldsymbol{U}_g \cdot \boldsymbol{n}_g$ and $\mathcal{F}_l\boldsymbol{U}_l \cdot \boldsymbol{n}_l$ to the right-hand side of Eq. (2.25a) and Eq. (2.25b) respectively to obtain,

$$\begin{aligned}\int_{\partial\mathcal{V}_g} \mathcal{F}\boldsymbol{U} \cdot \boldsymbol{n}_g d\mathcal{A} = &\int_{\partial\mathcal{V}_g-\Sigma} \mathcal{F}_g\boldsymbol{U}_g \cdot \boldsymbol{n}_g d\mathcal{A} + \int_{\Sigma} \mathcal{F}_g\boldsymbol{U}_g \cdot \boldsymbol{n}_g d\mathcal{A} \\ &- \int_{\Sigma} \mathcal{F}_g\boldsymbol{U}_\Sigma \cdot \boldsymbol{n}_g d\mathcal{A} + \int_{\Sigma} \mathcal{F}_g\boldsymbol{U}_\Sigma \cdot \boldsymbol{n}_g d\mathcal{A} \end{aligned} \tag{2.26a}$$

$$\begin{aligned}\int_{\partial\mathcal{V}l} \mathcal{F}\boldsymbol{U} \cdot \boldsymbol{n}_l d\mathcal{A} = &\int_{\partial\mathcal{V}l-\Sigma} \mathcal{F}_l\boldsymbol{U}_l \cdot \boldsymbol{n}_l d\mathcal{A} + \int_{\Sigma} \mathcal{F}_l\boldsymbol{U}_l \cdot \boldsymbol{n}_l d\mathcal{A} \\ &- \int_{\Sigma} \mathcal{F}_l\boldsymbol{U}_l \cdot \boldsymbol{n}_g d\mathcal{A} + \int_{\Sigma} \mathcal{F}_l\boldsymbol{U}_\Sigma \cdot \boldsymbol{n}_l d\mathcal{A} \end{aligned} \tag{2.26b}$$

We group the first two terms and last two terms on the right-hand side of Eq. (2.26a) and Eq. (2.26b) to obtain,

$$\int_{\partial\mathcal{V}_g} \mathcal{F}\boldsymbol{U} \cdot \boldsymbol{n}_g d\mathcal{A} = \int_{\partial\mathcal{V}_g} \mathcal{F}_g\boldsymbol{U}_g \cdot \boldsymbol{n}_g d\mathcal{A} + \int_{\Sigma} \mathcal{F}_g \left(\boldsymbol{U}_\Sigma - \boldsymbol{U}_g\right) \cdot \boldsymbol{n}_g d\mathcal{A} \tag{2.27a}$$

$$\int_{\partial\mathcal{V}l} \mathcal{F}\boldsymbol{U} \cdot \boldsymbol{n}_l d\mathcal{A} = \int_{\partial\mathcal{V}_l} \mathcal{F}_l\boldsymbol{U}_l \cdot \boldsymbol{n}_l d\mathcal{A} + \int_{\Sigma} \mathcal{F}_l \left(\boldsymbol{U}_\Sigma - \boldsymbol{U}_l\right) \cdot \boldsymbol{n}_l d\mathcal{A} \tag{2.27b}$$

We add Eq. (2.27a) and Eq. (2.27b), and simplify using Eq. (2.5) and Eq. (2.17) as,

$$\begin{aligned}\int_{\partial\mathcal{V}g} \mathcal{F}\boldsymbol{U} \cdot \boldsymbol{n}_g d\mathcal{A} + \int_{\partial\mathcal{V}l} \mathcal{F}\boldsymbol{U} \cdot \boldsymbol{n}_l d\mathcal{A} = &\int_{\partial\mathcal{V}_g} \mathcal{F}_g\boldsymbol{U}_g \cdot \boldsymbol{n}_g d\mathcal{A} + \int_{\partial\mathcal{V}_l} \mathcal{F}_l\boldsymbol{U}_l \cdot \boldsymbol{n}_l d\mathcal{A} \\ &+ \int_{\Sigma} [[\mathcal{F}\left(\boldsymbol{U}_\Sigma - \boldsymbol{U}\right)]] \cdot \boldsymbol{n} d\mathcal{A} \end{aligned} \tag{2.28}$$

We rewrite Eq. (2.24) using Eq. (2.28) as,

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{V}} \mathcal{F} d\mathcal{V} = &\int_{\mathcal{V}_g} \frac{\partial \mathcal{F}}{\partial t} d\mathcal{V} + \int_{\partial\mathcal{V}g} \mathcal{F}_g\boldsymbol{U}_g \cdot \boldsymbol{n}_g d\mathcal{A} + \int_{\mathcal{V}l} \frac{\partial \mathcal{F}}{\partial t} d\mathcal{V} + \int_{\partial\mathcal{V}l} \mathcal{F}_l\boldsymbol{U}_l \cdot \boldsymbol{n}_l d\mathcal{A} \\ &+ \int_{\Sigma} [[\mathcal{F}\left(\boldsymbol{U}_\Sigma - \boldsymbol{U}\right)]] \cdot \boldsymbol{n} d\mathcal{A} \end{aligned} \tag{2.29}$$

We use the divergence theorem and convert the surface integrals in Eq. (2.29) to volume integrals to obtain the jump identify 2:

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{V}} \mathcal{F} d\mathcal{V} = &\int_{\mathcal{V}g} \left(\frac{\partial \mathcal{F}_g}{\partial t} + \nabla \cdot \left(\mathcal{F}_g\boldsymbol{U}_g\right)\right) d\mathcal{V} + \int_{\mathcal{V}l} \left(\frac{\partial \mathcal{F}_l}{\partial t} + \nabla \cdot \left(\mathcal{F}_l\boldsymbol{U}_l\right)\right) d\mathcal{V} \\ &+ \int_{\Sigma} [[\mathcal{F}(\boldsymbol{U}_\Sigma - \boldsymbol{U})]] \cdot \boldsymbol{n} d\mathcal{A} \end{aligned} \tag{2.30}$$

2.2.3 Jump condition for mass

We rewrite the jump identity 2 from Eq. (2.30) for mass, i.e., $\mathcal{F} = \rho$ as,

$$\frac{d}{dt}\int_{\mathcal{V}} \rho d\mathcal{V} = \int_{\mathcal{V}g} \left(\frac{\partial \rho_g}{\partial t} + \nabla \cdot \left(\rho_g \boldsymbol{U}_g\right)\right) d\mathcal{V} + \int_{\mathcal{V}l} \left(\frac{\partial \rho_l}{\partial t} + \nabla \cdot \left(\rho_l \boldsymbol{U}_l\right)\right) d\mathcal{V} + \\ + \int_{\Sigma} [[\rho \left(\boldsymbol{U}_\Sigma - \boldsymbol{U}\right)]] \cdot \boldsymbol{n} d\mathcal{A} \tag{2.31}$$

We reduce the left-hand side of Eq. (2.31) to 0 using mass conservation equation given in Eq. (2.6). and argue that since $\partial\mathcal{V}_g$, $\partial\mathcal{V}_l$ and Σ are chosen arbitrarily, for Eq. (2.31) to be valid for every choice, the respective integrands on the right-hand side of Eq. (2.31) must both be 0. Hence, we can say that in the bulk of the gas and liquid phases,

$$\int_{\mathcal{V}g} \left(\frac{\partial \rho_g}{\partial t} + \nabla \cdot \left(\rho_g \boldsymbol{U}_g\right)\right) d\mathcal{V} = 0 \tag{2.32a}$$

$$\int_{\mathcal{V}l} \left(\frac{\partial \rho_l}{\partial t} + \nabla \cdot \left(\rho_l \boldsymbol{U}_l\right)\right) d\mathcal{V} = 0 \tag{2.32b}$$

and at the interface, the conservation of mass reduces to a jump condition for mass across Σ as,

$$\int_{\Sigma} [[\rho \left(\boldsymbol{U}_\Sigma - \boldsymbol{U}\right)]] \cdot \boldsymbol{n} d\mathcal{A} = 0 \tag{2.33}$$

We examine Eq. (2.33) using the definition of jump condition given in Eq. (2.17) as,

$$\int_{\Sigma} \rho_g \left(\boldsymbol{U}_\Sigma - \boldsymbol{U}_g\right) \cdot \boldsymbol{n} d\mathcal{A} = \int_{\Sigma} \rho_l \left(\boldsymbol{U}_\Sigma - \boldsymbol{U}_l\right) \cdot \boldsymbol{n} d\mathcal{A} \tag{2.34}$$

We realize that the each side of Eq. (2.34) corresponds to the mass flux due to phase-change, denoted by $\dot{m}''$ such that,

$$\dot{m}'' \equiv \rho_g \left(\boldsymbol{U}_\Sigma - \boldsymbol{U}_g\right) \cdot \boldsymbol{n} = \rho_l \left(\boldsymbol{U}_\Sigma - \boldsymbol{U}_l\right) \cdot \boldsymbol{n} \tag{2.35}$$

In addition, for viscous fluids under standard operating conditions, it is an experimentally observed fact that two fluids do not slip, and therefore, the velocity is continuous across the interface such that,

$$\boldsymbol{U}_g \cdot \boldsymbol{t} = \boldsymbol{U}_\Sigma \cdot \boldsymbol{t} = \boldsymbol{U}_l \cdot \boldsymbol{t} \tag{2.36a}$$

$$\boldsymbol{U}_g \cdot \boldsymbol{s} = \boldsymbol{U}_\Sigma \cdot \boldsymbol{s} = \boldsymbol{U}_l \cdot \boldsymbol{s} \tag{2.36b}$$

We can derive the expression for the normal component of the velocity of Σ by rearranging the terms in Eq. (2.35),

$$\boldsymbol{U}_\Sigma \cdot \boldsymbol{n} = \frac{1}{2}\left\{\dot{m}''\left(\frac{1}{\rho_g}+\frac{1}{\rho_l}\right)+\left(\boldsymbol{U}_g+\boldsymbol{U}_l\right)\cdot\boldsymbol{n}\right\} \tag{2.37}$$

2.2.4 Jump identity 2: auxiliary formulation

In this section, we provide an auxiliary formulation of jump identity 2 from Eq. (2.30) by simplifying it using the conservation of mass from Eq. (2.6). This formulation will be useful in deriving the jump conditions for momentum and energy provided in Sec. 2.2.5 and Sec. 2.2.6, respectively, in a concise manner. We first write the expression for rate of change of $\rho\mathcal{F}$ in $\mathcal{V}$ by rewriting jump identify 2 from Eq. (2.30) for $\mathcal{F} = \rho\mathcal{F}$ as,

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{V}}\rho\mathcal{F}d\mathcal{V} = & \int_{\mathcal{V}_g}\left(\frac{\partial\rho_g\mathcal{F}_g}{\partial t}+\nabla\cdot\,\rho_g\mathcal{F}_g\boldsymbol{U}_g\right)d\mathcal{V}+\int_{\mathcal{V}_l}\left(\frac{\partial\rho_l\mathcal{F}_l}{\partial t}+\nabla\cdot\,\rho_l\mathcal{F}_l\boldsymbol{U}_l\right)d\mathcal{V}\\ & +\int_\Sigma [[\rho\mathcal{F}\left(\boldsymbol{U}_\Sigma-\boldsymbol{U}\right)]]\cdot\boldsymbol{n}d\mathcal{A}\end{aligned} \tag{2.38}$$

We can now write Eq. (2.38) in non-conservative form using Eq. (2.6) as,

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{V}}\rho\mathcal{F}d\mathcal{V} = & \int_{\mathcal{V}_g}\rho_g\left(\frac{\partial\mathcal{F}_g}{\partial t}+\boldsymbol{U}_g\cdot\nabla\mathcal{F}_g\right)d\mathcal{V}+\int_{\mathcal{V}_l}\rho_l\left(\frac{\partial\mathcal{F}_l}{\partial t}+\boldsymbol{U}_l\cdot\nabla\mathcal{F}_l\right)d\mathcal{V}\\ & +\int_\Sigma [[\rho\mathcal{F}\left(\boldsymbol{U}_\Sigma-\boldsymbol{U}\right)]]\cdot\boldsymbol{n}d\mathcal{A}\end{aligned} \tag{2.39}$$

We define the material derivative as,

$$\frac{D\mathcal{F}}{Dt} \equiv \frac{\partial\mathcal{F}}{\partial t}+\boldsymbol{U}\cdot\nabla\mathcal{F} \tag{2.40}$$

and simplify the right-hand side of Eq. (2.39) using Eq. (2.35) as,

$$[[\rho\mathcal{F}\left(\boldsymbol{U}_\Sigma-\boldsymbol{U}\right)]]\cdot\boldsymbol{n} = \dot{m}''\,[[\mathcal{F}]] \tag{2.41}$$

We rewrite Eq. (2.39) using Eq. (2.40) and Eq. (2.41) to obtain the auxiliary form of jump identity 2:

$$\frac{d}{dt}\int_{\mathcal{V}}\rho\mathcal{F}d\mathcal{V} = \int_{\mathcal{V}_g}\rho_g\frac{D\mathcal{F}_g}{Dt}d\mathcal{V}+\int_{\mathcal{V}_l}\rho_l\frac{D\mathcal{F}_l}{Dt}d\mathcal{V}+\int_\Sigma\dot{m}''\,[[\mathcal{F}]]\,d\mathcal{A} \tag{2.42}$$

We use this auxiliary form of jump condition 2 to derive the jump conditions for momentum and energy in Sec. 2.2.5 and Sec. 2.2.6, respectively.

2.2.5 Jump condition for momentum

We simplify the conservation of momentum equation given in Eq. (2.8), using Eq. (2.11) as,

$$\frac{d}{dt}\int_{\mathcal{V}} \rho \boldsymbol{U} d\mathcal{V} = \int_{\partial\mathcal{V}} \boldsymbol{\tau}\boldsymbol{n} d\mathcal{A} + \int_{\Sigma} \kappa\sigma\boldsymbol{n} d\mathcal{A} + \int_{\mathcal{V}} \rho \boldsymbol{f_b} d\mathcal{V} \tag{2.43}$$

We modify the surface integral term in Eq. (2.43) using Eq. (2.21), with $\mathcal{F} = \boldsymbol{\tau}$ as,

$$\int_{\partial\mathcal{V}} \boldsymbol{\tau}\boldsymbol{n} d\mathcal{A} = \int_{\partial\mathcal{V}g} \boldsymbol{\tau}_g\boldsymbol{n}_g d\mathcal{A} + \int_{\partial\mathcal{V}l} \boldsymbol{\tau}_l\boldsymbol{n}_l d\mathcal{A} - \int_{\Sigma} [[\boldsymbol{\tau}]]\, \boldsymbol{n} d\mathcal{A} \tag{2.44}$$

We rewrite Eq. (2.43) using Eq. (2.44) as,

$$\frac{d}{dt}\int_{\mathcal{V}} \rho \boldsymbol{U} d\mathcal{V} = \int_{\partial\mathcal{V}g} \boldsymbol{\tau}_g\boldsymbol{n}_g d\mathcal{A} + \int_{\partial\mathcal{V}l} \boldsymbol{\tau}_l\boldsymbol{n}_l d\mathcal{A} - \int_{\Sigma} [[\boldsymbol{\tau}]]\, \boldsymbol{n} d\mathcal{A} + \int_{\Sigma} \kappa\sigma\boldsymbol{n} d\mathcal{A} + \int_{\mathcal{V}} \rho \boldsymbol{f_b} d\mathcal{V} \tag{2.45}$$

We convert the surface integral terms on the right-hand side of Eq. (2.45) using divergence theorem as,

$$\frac{d}{dt}\int_{\mathcal{V}} \rho \boldsymbol{U} d\mathcal{V} = \int_{\mathcal{V}g} \nabla\cdot\boldsymbol{\tau}_g d\mathcal{V} + \int_{\mathcal{V}l} \nabla\cdot\boldsymbol{\tau}_l d\mathcal{V} + \int_{\Sigma} [[\boldsymbol{\tau}]]\, \boldsymbol{n} d\mathcal{A} + \int_{\Sigma} \kappa\sigma\boldsymbol{n} d\mathcal{A} + \int_{\mathcal{V}} \rho \boldsymbol{f_b} d\mathcal{V} \tag{2.46}$$

We modify the left-hand side of Eq. (2.46), using the auxiliary form of jump identity 2 from Eq. (2.42) for momentum, i.e., $\mathcal{F} = \boldsymbol{U}$ as,

$$\frac{d}{dt}\int_{\mathcal{V}} \rho \boldsymbol{U} d\mathcal{V} = \int_{\mathcal{V}g} \rho_g \frac{D\boldsymbol{U}_g}{Dt} d\mathcal{V} + \int_{\mathcal{V}l} \rho_l \frac{D\boldsymbol{U}_l}{Dt} d\mathcal{V} - \int_{\Sigma} \dot{m}''\, [[\boldsymbol{U}]]\, d\mathcal{A} \tag{2.47}$$

We subtract Eq. (2.45) from Eq. (2.47), and group the terms with common integrals as,

$$\begin{aligned} 0 = &\int_{\mathcal{V}g} \left(\rho_g \frac{D\boldsymbol{U}_g}{Dt} - \nabla\cdot\boldsymbol{\tau}_g - \rho_g\boldsymbol{f_b}\right) d\mathcal{V} + \int_{\mathcal{V}l} \left(\rho_l \frac{D\boldsymbol{U}_l}{Dt} - \nabla\cdot\boldsymbol{\tau}_l - \rho_l\boldsymbol{f_b}\right) d\mathcal{V} \\ &- \int_{\Sigma} \left(\dot{m}''\, [[\boldsymbol{U}]] - \kappa\sigma\boldsymbol{n} - [[\boldsymbol{\tau}]]\, \boldsymbol{n}\right) d\mathcal{A} \end{aligned} \tag{2.48}$$

We argue that as $\partial\mathcal{V}_g$, $\partial\mathcal{V}_l$ and Σ are chosen arbitrarily, for Eq. (2.48) to be valid for every choice, the respective integrands must be 0. Hence, we can say that, in the bulk of the gas and liquid phases,

$$\int_{\mathcal{V}_g} \rho_g \frac{D\boldsymbol{U}_g}{Dt} d\mathcal{V} = \int_{\mathcal{V}_g} \nabla \cdot \boldsymbol{\tau}_g d\mathcal{V} + \int_{\mathcal{V}_g} \rho_g \boldsymbol{f_b} d\mathcal{V} \tag{2.49a}$$

$$\int_{\mathcal{V}_l} \rho_l \frac{D\boldsymbol{U}_l}{Dt} d\mathcal{V} = \int_{\mathcal{V}_l} \nabla \cdot \boldsymbol{\tau}_l d\mathcal{V} + \int_{\mathcal{V}_l} \rho_l \boldsymbol{f_b} d\mathcal{V} \tag{2.49b}$$

and at the interface, conservation of momentum reduces to a jump condition for momentum across Σ as,

$$\int_{\Sigma} [[\boldsymbol{\tau}]] \boldsymbol{n} d\mathcal{A} = \int_{\Sigma} \kappa\sigma\boldsymbol{n} d\mathcal{A} - \int_{\Sigma} \dot{m}'' [[\boldsymbol{U}]] d\mathcal{A} \tag{2.50}$$

We simplify the second term on the right-hand side of Eq. (2.50) using Eq. (2.35) as,

$$\begin{aligned}
-\dot{m}'' [[\boldsymbol{U}]] &= -\dot{m}'' (\boldsymbol{U}_g - \boldsymbol{U}_l) \\
&= \dot{m}'' ((\boldsymbol{U}_\Sigma - \boldsymbol{U}_g) - (\boldsymbol{U}_\Sigma - \boldsymbol{U}_l)) \\
&= \dot{m}'' (((\boldsymbol{U}_\Sigma - \boldsymbol{U}_g) \cdot \boldsymbol{n}) \boldsymbol{n} - ((\boldsymbol{U}_\Sigma - \boldsymbol{U}_l) \cdot \boldsymbol{n}) \boldsymbol{n}) \\
&= \dot{m}'' \left(\frac{\dot{m}''}{\rho_g} \boldsymbol{n} - \frac{\dot{m}''}{\rho_l} \boldsymbol{n} \right) \\
&= (\dot{m}'')^2 \left(\frac{1}{\rho_g} - \frac{1}{\rho_l} \right) \boldsymbol{n}
\end{aligned} \tag{2.51}$$

We rewrite the final result of Eq. (2.51) as,

$$-\dot{m}'' [[\boldsymbol{U}]] = (\dot{m}'')^2 \left(\frac{1}{\rho_g} - \frac{1}{\rho_l} \right) \boldsymbol{n} \tag{2.52}$$

We combine Eq. (2.50) and Eq. (2.52) to present the jump condition for momentum across Σ as,

$$\int_{\Sigma} [[\boldsymbol{\tau}]] \boldsymbol{n} d\mathcal{A} = \int_{\Sigma} \kappa\sigma\boldsymbol{n} d\mathcal{A} + \int_{\Sigma} (\dot{m}'')^2 \left(\frac{1}{\rho_g} - \frac{1}{\rho_l} \right) \boldsymbol{n} d\mathcal{A} \tag{2.53}$$

We convert the surface integrals on the right-hand side of Eq. (2.53) to volume integrals as,

$$\int_{\Sigma} \kappa\sigma\boldsymbol{n} d\mathcal{A} = \int_{\mathcal{V}} \boldsymbol{f}_\sigma d\mathcal{V} \tag{2.54}$$

where $\boldsymbol{f}_\sigma \equiv \kappa\sigma\boldsymbol{n}\delta_\Sigma$ is the surface tension force and,

$$\int_{\Sigma} (\dot{m}'')^2 \left(\frac{1}{\rho_g} - \frac{1}{\rho_l} \right) \boldsymbol{n} = \int_{\mathcal{V}} \boldsymbol{f}_{\dot{m}} d\mathcal{V} \tag{2.55}$$

where $\boldsymbol{f}_{\dot{m}} \equiv (\dot{m}'')^2 \left(\frac{1}{\rho_g} - \frac{1}{\rho_l}\right) \boldsymbol{n}\delta_\Sigma$ is the force on Σ due to mass flux caused by phase change, and we define δ_Σ as

$$\delta_\Sigma = \begin{cases} 0 \ \ \forall \boldsymbol{x} \in \mathcal{V}_g \\ 0 \ \ \forall \boldsymbol{x} \in \mathcal{V}_l \\ 1 \ \ \forall \boldsymbol{x} \in \Sigma \end{cases} \tag{2.56}$$

We rewrite the jump condition for momentum from Eq. (2.53) using the definitions of force due to surface tension from Eq. (2.54) and force due to phase change from Eq. (2.55) as,

$$\int_\Sigma [[\boldsymbol{\tau}]]\, \boldsymbol{n} d\mathcal{A} = \int_\mathcal{V} \boldsymbol{f}_\sigma d\mathcal{V} + \int_\mathcal{V} \boldsymbol{f}_{\dot{m}} d\mathcal{V} \tag{2.57}$$

2.2.6 *Jump condition for energy*

We simplify the conservation of energy equation given in Eq. (2.13) using the transport equation for surface energy given in Eq. (2.15) by assuming constant surface tension as,

$$\begin{aligned}\frac{d}{dt}\int_\mathcal{V} \rho \left(e + \frac{1}{2}\boldsymbol{U}\cdot\boldsymbol{U}\right) d\mathcal{V} = \int_{\partial\mathcal{V}} (\boldsymbol{\tau n})\cdot\boldsymbol{U} d\mathcal{A} + \int_\Sigma \sigma\kappa \boldsymbol{U}_\Sigma \cdot \boldsymbol{n} d\mathcal{A} \\ - \int_{\partial\mathcal{V}} \boldsymbol{q}\cdot\boldsymbol{n} d\mathcal{A} + \int_\mathcal{V} \rho \boldsymbol{f_b}\cdot\boldsymbol{U} d\mathcal{V}\end{aligned} \tag{2.58}$$

We simplify the surface integral term in Eq. (2.58) by realizing that $\boldsymbol{\tau}$ is a symmetric tensor and hence,

$$(\boldsymbol{\tau n})\cdot\boldsymbol{U} = \left(\boldsymbol{\tau}^T\boldsymbol{U}\right)\cdot\boldsymbol{n} = (\boldsymbol{\tau U})\cdot\boldsymbol{n} \tag{2.59}$$

We split the total power of fluid stresses into gas, liquid and interface components using jump identity 1 from Eq. (2.21), with $\mathcal{F} = \boldsymbol{\tau U}$ as,

$$\int_{\partial\mathcal{V}} (\boldsymbol{\tau U})\cdot\boldsymbol{n} d\mathcal{A} = \int_{\partial\mathcal{V}g} (\boldsymbol{\tau}_g\boldsymbol{U}_g)\cdot\boldsymbol{n}_g d\mathcal{A} + \int_{\partial\mathcal{V}l} (\boldsymbol{\tau}_l\boldsymbol{U}_l)\cdot\boldsymbol{n}_l d\mathcal{A} - \int_\Sigma [[\boldsymbol{\tau U}]]\cdot\boldsymbol{n} d\mathcal{A} + \tag{2.60}$$

We split the total heat conducted in a similar way using jump identity 1 from Eq. (2.21), with $\mathcal{F} = \boldsymbol{q}$ as,

$$\int_{\partial\mathcal{V}} \boldsymbol{q}\cdot\boldsymbol{n} d\mathcal{A} = \int_{\partial\mathcal{V}_g} \boldsymbol{q}_g\cdot\boldsymbol{n}_g d\mathcal{A} + \int_{\partial\mathcal{V}l} \boldsymbol{q}_l\cdot\boldsymbol{n}_l d\mathcal{A} - \int_\Sigma [[\boldsymbol{q}]]\cdot\boldsymbol{n} d\mathcal{A} \tag{2.61}$$

We simplify Eq. (2.58) using Eq. (2.60) and Eq. (2.61) and grouping the terms with common integral as,

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{V}}\rho(e+\frac{1}{2}\boldsymbol{U}\cdot\boldsymbol{U})d\mathcal{V} &= \int_{\partial\mathcal{V}_g}\left(\boldsymbol{\tau}_g\boldsymbol{U}_g-\boldsymbol{q}_g\right)\cdot\boldsymbol{n}_g d\mathcal{A}+\int_{\partial\mathcal{V}l}\left(\boldsymbol{\tau}_l\boldsymbol{U}_l-\boldsymbol{q}_l\right)\cdot\boldsymbol{n}_l d\mathcal{A}\\ &+\int_{\Sigma}\left(\kappa\sigma\boldsymbol{U}_\Sigma+[[\boldsymbol{q}]]-[[\boldsymbol{\tau}\boldsymbol{U}]]\right)\cdot\boldsymbol{n}d\mathcal{A}+\int_{\mathcal{V}}\rho\boldsymbol{f_b}\cdot\boldsymbol{U}d\mathcal{V} \qquad (2.62)\end{aligned}$$

We convert the surface integral terms in Eq. (2.62) to volume integrals using the divergence theorem as,

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{V}}\rho(e+\frac{1}{2}\boldsymbol{U}\cdot\boldsymbol{U})d\mathcal{V} &= \int_{\mathcal{V}_g}\left(\nabla\cdot\left(\boldsymbol{\tau}_g\boldsymbol{U}_g-\boldsymbol{q}_g\right)+\rho_g\boldsymbol{f_b}\cdot\boldsymbol{U}_g\right)d\mathcal{V}\\ &+\int_{\mathcal{V}l}\left(\nabla\cdot\left(\boldsymbol{\tau}_l\boldsymbol{U}_l-\boldsymbol{q}_l\right)+\rho_l\boldsymbol{f_b}\cdot\boldsymbol{U}_l\right)d\mathcal{V}\\ &+\int_{\Sigma}\left(\kappa\sigma\boldsymbol{U}_\Sigma+[[\boldsymbol{q}]]-[[\boldsymbol{\tau}\boldsymbol{U}]]\right)\cdot\boldsymbol{n}d\mathcal{A} \qquad (2.63)\end{aligned}$$

We split the left-hand side of Eq. (2.63) using the auxiliary form of the jump identity 2 from Eq. (2.30), with $\mathcal{F}=\rho(e+\frac{1}{2}\boldsymbol{U}\cdot\boldsymbol{U})$ as,

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{V}}\rho\left(e+\frac{1}{2}\boldsymbol{U}\cdot\boldsymbol{U}\right)d\mathcal{V} &= \int_{\mathcal{V}_g}\rho_g\frac{D}{Dt}\left(e+\frac{1}{2}\boldsymbol{U}\cdot\boldsymbol{U}\right)d\mathcal{V}\\ &+\int_{\mathcal{V}_l}\rho_l\frac{D}{Dt}\left(e+\frac{1}{2}\boldsymbol{U}\cdot\boldsymbol{U}\right)d\mathcal{V}\\ &+\int_{\Sigma}\dot{m}''\left[\left[e+\frac{1}{2}\boldsymbol{U}\cdot\boldsymbol{U}\right]\right]d\mathcal{A} \qquad (2.64)\end{aligned}$$

We subtract Eq. (2.63) from Eq. (2.64), and group the terms with common integral limits to obtain,

$$\begin{aligned}0 = &\int_{\mathcal{V}_g}\left(\rho_g\frac{D}{Dt}\left(e+\frac{1}{2}\boldsymbol{U}\cdot\boldsymbol{U}\right)-\nabla\cdot\left(\boldsymbol{\tau}_g\boldsymbol{U}_g-\boldsymbol{q}_g\right)-\rho_g\boldsymbol{f_b}\cdot\boldsymbol{U}_g\right)d\mathcal{V}\\ &+\int_{\mathcal{V}_l}\left(\rho_l\frac{D}{Dt}\left(e+\frac{1}{2}\boldsymbol{U}\cdot\boldsymbol{U}\right)-\nabla\cdot\left(\boldsymbol{\tau}_l\boldsymbol{U}_l-\boldsymbol{q}_l\right)-\rho_l\boldsymbol{f_b}\cdot\boldsymbol{U}_l\right)d\mathcal{V}\\ &+\int_{\Sigma}\left(\dot{m}''\left[\left[e+\frac{1}{2}\boldsymbol{U}\cdot\boldsymbol{U}\right]\right]-\left(\kappa\sigma\boldsymbol{U}_\Sigma+[[\boldsymbol{q}]]-[[\boldsymbol{\tau}\boldsymbol{U}]]\right)\cdot\boldsymbol{n}\right)d\mathcal{A} \qquad (2.65)\end{aligned}$$

We argue that since $\partial\mathcal{V}_g$, $\partial\mathcal{V}_l$ and Σ are chosen arbitrarily, for Eq. (2.65) to be valid for every choice, the respective integrands must be 0. Hence, we can say that, in the bulk of the gas and liquid phases,

$$\int_{\mathcal{V}_g} \rho_g \frac{D}{Dt}\left(e + \frac{1}{2}\boldsymbol{U}\cdot\boldsymbol{U}\right) d\mathcal{V} = \int_{\mathcal{V}_g} \left\{\nabla\cdot\,\left(\boldsymbol{\tau}_g\boldsymbol{U}_g - \boldsymbol{q}_g\right) + \rho_g \boldsymbol{f_b}\cdot\boldsymbol{U}_g\right\} d\mathcal{V} \tag{2.66a}$$

$$\int_{\mathcal{V}l} \rho_l \frac{D}{Dt}\left(e + \frac{1}{2}\boldsymbol{U}\cdot\boldsymbol{U}\right) d\mathcal{V} = \int_{\mathcal{V}l} \left\{\nabla\cdot\,\left(\boldsymbol{\tau}_l\boldsymbol{U}_l - \boldsymbol{q}_l\right) + \rho_l \boldsymbol{f_b}\cdot\boldsymbol{U}_l\right\} d\mathcal{V} \tag{2.66b}$$

and at the interface, the conservation of energy reduces to a jump condition for energy across Σ as,

$$\int_{\Sigma} \dot{m}'' \left[\left[e + \frac{1}{2}\boldsymbol{U}\cdot\boldsymbol{U}\right]\right] d\mathcal{A} = \int_{\Sigma} \left(\kappa\sigma\boldsymbol{U}_\Sigma + [[\boldsymbol{q}]] - [[\boldsymbol{\tau}\boldsymbol{U}]]\right)\cdot\boldsymbol{n}d\mathcal{A} \tag{2.67}$$

2.2.7 Summary of jump conditions

In this section, we summarize the balance equations and jump conditions for mass, momentum and energy for convenience. We decomposed the governing equations of fluid flow from Sec. 2.1 into balance equations in the bulk of the fluid volume $\mathcal{V}_f$, and jump conditions across the interface Σ in Sec. 2.2.3, Sec. 2.2.5 and Sec. 2.2.6. The subscript f stands for fluid and refers to either the gas or the liquid phase:

$$f = \begin{cases} g & \forall\,\boldsymbol{x}\in\mathcal{V}_g \\ l & \forall\,\boldsymbol{x}\in\mathcal{V}_l \end{cases} \tag{2.68}$$

The balance equation for mass in $\mathcal{V}_f$ and jump condition for mass across Σ are:

$$\int_{\mathcal{V}_f} \left(\frac{\partial\rho_f}{\partial t} + \nabla\cdot\,\left(\rho_f\boldsymbol{U}_f\right)\right) d\mathcal{V} = 0;\;\; \forall\boldsymbol{x}\in\mathcal{V}_f \tag{2.69a}$$

$$\int_{\Sigma} [[\rho\left(\boldsymbol{U}_\Sigma - \boldsymbol{U}\right)]]\cdot\boldsymbol{n}d\mathcal{A} = 0;\;\; \forall\boldsymbol{x}\in\Sigma \tag{2.69b}$$

The balance equation for momentum in $\mathcal{V}_f$ and jump condition for momentum across Σ are:

$$\int_{\mathcal{V}_f} \rho_f \frac{D\boldsymbol{U}_f}{Dt} d\mathcal{V} = \int_{\mathcal{V}_f} \nabla\cdot\,\boldsymbol{\tau}_f d\mathcal{V} + \int_{\mathcal{V}_f} \rho_f \boldsymbol{f_b} d\mathcal{V};\quad \forall\boldsymbol{x}\in\mathcal{V}_f \tag{2.70a}$$

$$\int_{\Sigma} [[\boldsymbol{\tau}]]\,\boldsymbol{n}d\mathcal{A} = \int_{\Sigma} \kappa\sigma\boldsymbol{n}d\mathcal{A} + \int_{\Sigma} \left(\dot{m}''\right)^2 \left(\frac{1}{\rho_g} - \frac{1}{\rho_l}\right)\boldsymbol{n}d\mathcal{A};\quad \forall\boldsymbol{x}\in\Sigma \tag{2.70b}$$

The balance equation for energy in $\mathcal{V}_f$ and jump condition for energy across Σ are:

$$\int_{\mathcal{V}_f} \rho_f \frac{D}{Dt}\left(e_f + \frac{1}{2}\boldsymbol{U}_f \cdot \boldsymbol{U}_f\right) d\mathcal{V} = \int_{\mathcal{V}_f} \left\{\nabla \cdot \left(\boldsymbol{\tau}_f \boldsymbol{U}_f - \boldsymbol{q}_f\right) + \rho_f \boldsymbol{f_b} \cdot \boldsymbol{U}_f\right\} d\mathcal{V}; \quad \forall \boldsymbol{x} \in \mathcal{V}_f \tag{2.71a}$$

$$\int_{\Sigma} \dot{m}'' \left[\left[e + \frac{1}{2}\boldsymbol{U} \cdot \boldsymbol{U}\right]\right] d\mathcal{A} = \int_{\Sigma} \left(\kappa\sigma \boldsymbol{U}_\Sigma + [[\boldsymbol{q}]] - [[\boldsymbol{\tau}\boldsymbol{U}]]\right) \cdot \boldsymbol{n} d\mathcal{A}; \quad \forall \boldsymbol{x} \in \Sigma \tag{2.71b}$$

Chapter 3

ENERGY EQUATIONS FOR GAS-LIQUID TURBULENT FLOWS WITH PHASE CHANGE

In this section, we derive the balance equation for turbulence kinetic energy (TKE), mean-flow kinetic energy (MFKE) and mean internal energy (MIE) for gas-liquid flows with phase change. We then explain the interaction between TKE, MFKE and MIE with the help of pathway schematics. In Sec. 3.1, we derive the compressible formulation of TKE, MFKE and MIE for individual gas or liquid phases, followed by a discussion on energy pathways in those individual phases. In Sec. 3.2, we derive the balance equations of TKE, MFKE and MIE for gas-liquid interfacial flows with phase change. We also rewrite these equations for droplet-laden homogeneous shear turbulence (DLHST-PC) and droplet-laden decaying homogeneous isotropic turbulence (DLHIT-PC) in Sec. 3.3 and Sec. 3.4, respectively.

We start with the momentum balance equation for individual phases (gas or liquid) from Eq. (2.49). If we proceed in the same way as Dodd & Ferrante (2016), where Reynolds averaging was performed, we cannot remove the density fluctuation terms from the equations and this will lead to the presence of complicated terms involving triple correlation of density fluctuations and velocity fluctuations in the energy equations. Hence, we resort to Favre-averaging as presented by Favre (1965), Lele (1994) and Wilcox (2006) which is a mass-averaging technique used to remove density fluctuations from the governing equations defined by,

$$\widetilde{\boldsymbol{U}} \equiv \frac{\langle \rho \boldsymbol{U} \rangle}{\langle \rho \rangle} \tag{3.1}$$

where $\langle ... \rangle$ indicates Reynolds averaging and we consider volume averaging for our purpose:

$$\langle ... \rangle \equiv \frac{1}{\mathcal{V}} \int_{\mathcal{V}} (...) \, d\mathcal{V} \tag{3.2}$$

We define Reynolds averaging over a volume $\mathcal{V}_f$ of fluid f as,

$$\langle ... \rangle_f \equiv \frac{1}{\mathcal{V}_f} \int_{\mathcal{V}_f} (...) \, d\mathcal{V} \tag{3.3}$$

We decompose density (ρ), fluid stress ($\boldsymbol{\tau}$) and as a consequence, pressure (p) and viscous stress ($\mathbf{T}$) into their Reynolds-averaged mean and fluctuating quantities:

$$\rho = \overline{\rho} + \rho' \tag{3.4a}$$

$$\boldsymbol{\tau} = \overline{\boldsymbol{\tau}} + \boldsymbol{\tau}' \tag{3.4b}$$

$$p = \overline{p} + p' \tag{3.4c}$$

$$\mathbf{T} = \overline{\mathbf{T}} + \mathbf{T}' \tag{3.4d}$$

We decompose velocity ($\boldsymbol{U}$) into Favre-averaged mean and fluctuating velocities:

$$\boldsymbol{U} = \widetilde{\boldsymbol{U}} + \boldsymbol{u}'' \tag{3.5}$$

We highlight a property of Favre-averaging by taking a Favre-average using Eq. (3.1) on both sides of the Eq. (3.5) as,

$$\widetilde{\boldsymbol{U}} = \widetilde{\widetilde{\boldsymbol{U}}} + \widetilde{\boldsymbol{u}''} = \widetilde{\boldsymbol{U}} + \widetilde{\boldsymbol{u}''} \tag{3.6}$$

We infer from Eq. (3.6) that $\widetilde{\boldsymbol{u}''} = 0$ which is explicitly stated using the definition of Favre-averaging given in Eq. (3.1) as,

$$\widetilde{\boldsymbol{u}''} = \frac{\langle \rho \boldsymbol{u}'' \rangle}{\langle \rho \rangle} = 0 \implies \langle \rho \boldsymbol{u}'' \rangle = 0 \tag{3.7}$$

The kinetic energy of the fluid (per unit mass) is:

$$E(\boldsymbol{x}, t) \equiv \frac{1}{2} \boldsymbol{U} \cdot \boldsymbol{U} \tag{3.8}$$

E can be decomposed into two parts:

$$E(\boldsymbol{x}, t) = \widetilde{E}(\boldsymbol{x}, t) + k(\boldsymbol{x}, t) \tag{3.9}$$

where $\widetilde{E}$ is the the Favre-averaged mean-flow kinetic energy (per unit mass) (MFKE)

$$\widetilde{E} \equiv \frac{\left\langle \rho \frac{1}{2} \widetilde{\boldsymbol{U}} \cdot \widetilde{\boldsymbol{U}} \right\rangle}{\langle \rho \rangle} = \frac{1}{2} \widetilde{\boldsymbol{U}} \cdot \widetilde{\boldsymbol{U}} \tag{3.10}$$

and k is the turbulence kinetic energy (per unit mass) (TKE)

$$k \equiv \frac{\left\langle \rho \frac{1}{2} \boldsymbol{u}'' \cdot \boldsymbol{u}'' \right\rangle}{\langle \rho \rangle} \tag{3.11}$$

We decompose the internal energy per unit mass (e) into its Favre-averaged mean and fluctuating quantities:

$$e = \widetilde{e} + e'' \tag{3.12}$$

where $\widetilde{e}$ is the mean internal energy (per unit mass) or (MIE) and e'' is the fluctuating part of IE. We present an argument similar to the one presented from Eq. (3.6) to Eq. (3.7) for e and state that,

$$\langle \rho e'' \rangle = 0 \tag{3.13}$$

In the following sections, we derive the equations for TKE (k), MFKE ($\widetilde{E}$) and MIE ($\widetilde{e}$) for various flow scenarios.

3.1 Compressible flow (gas or liquid phase)

In this section, we derive the TKE, MFKE and MIE evolution equations for a compressible flow in a gas or liquid phase. We start with the conservation of mass for each phase given by Eq. (2.32). We drop the integrals $\mathcal{V}_g$ and $\mathcal{V}_l$ as they are arbitrary and re-write it as,

$$\frac{\partial \rho_f}{\partial t} + \nabla \cdot (\rho_f \boldsymbol{U}_f) = 0 \tag{3.14}$$

We write the momentum balance equation for each phase given by Eq. (2.49), by dropping the integrals as,

$$\rho_f \frac{D\boldsymbol{U}_f}{Dt} = \nabla \cdot \boldsymbol{\tau}_f + \rho_f \boldsymbol{f_b} \tag{3.15}$$

We write Eq. (3.15) in the conservative form using Eq. (3.14) and the definition of material derivative in Eq. (2.40) as,

$$\frac{\partial}{\partial t}\left(\rho_f \boldsymbol{U}_f\right) + \nabla\cdot\left(\rho_f \boldsymbol{U}_f \boldsymbol{U}_f\right) = \nabla\cdot\boldsymbol{\tau}_f + \rho_f \boldsymbol{f_b} \tag{3.16}$$

3.1.1 Turbulence kinetic energy (TKE) for compressible flow (gas or liquid phase), k_f

We take a dot product of Eq. (3.16) with $\boldsymbol{u}''_f$

$$\boldsymbol{u}''_f\cdot\left(\frac{\partial}{\partial t}\left(\rho_f \boldsymbol{U}_f\right) + \nabla\cdot\left(\rho_f \boldsymbol{U}_f \boldsymbol{U}_f\right)\right) = \boldsymbol{u}''_f\cdot\left(\nabla\cdot\boldsymbol{\tau}_f + \rho_f \boldsymbol{f_b}\right) \tag{3.17}$$

We integrate Eq. (3.17) over $\mathcal{V}_f$ and divide by $\mathcal{V}_f$ to obtain,

$$\frac{1}{\mathcal{V}_f}\int_{\mathcal{V}_f}\boldsymbol{u}''_f\cdot\left(\frac{\partial}{\partial t}\left(\rho_f \boldsymbol{U}_f\right) + \nabla\cdot\left(\rho_f \boldsymbol{U}_f \boldsymbol{U}_f\right)\right)d\mathcal{V} = \frac{1}{\mathcal{V}_f}\int_{\mathcal{V}_f}\boldsymbol{u}''_f\cdot\left(\nabla\cdot\boldsymbol{\tau}_f + \rho_f \boldsymbol{f_b}\right)d\mathcal{V} \tag{3.18}$$

We use the definition of Reynolds averaging given in Eq. (3.3) to rewrite Eq. (3.18) as,

$$\left\langle\boldsymbol{u}''_f\cdot\left(\frac{\partial}{\partial t}\left(\rho_f \boldsymbol{U}_f\right) + \nabla\cdot\left(\rho_f \boldsymbol{U}_f \boldsymbol{U}_f\right)\right)\right\rangle_f = \left\langle\boldsymbol{u}''_f\cdot\left(\nabla\cdot\boldsymbol{\tau}_f + \rho_f \boldsymbol{f_b}\right)\right\rangle_f \tag{3.19}$$

We use the fact that both dot product and Reynolds-averaging are linear operators to rewrite Eq. (3.19) as,

$$\left\langle\boldsymbol{u}''_f\cdot\left(\frac{\partial}{\partial t}\left(\rho_f \boldsymbol{U}_f\right)\right)\right\rangle_f + \left\langle\boldsymbol{u}''_f\cdot\left(\nabla\cdot\rho_f \boldsymbol{U}_f \boldsymbol{U}_f\right)\right\rangle_f = \left\langle\boldsymbol{u}''_f\cdot\left(\nabla\cdot\boldsymbol{\tau}_f\right)\right\rangle_f + \left\langle\rho_f \boldsymbol{f_b}\cdot\boldsymbol{u}''\right\rangle_f \tag{3.20}$$

We simplify each term in Eq. (3.20) separately using Eq. (3.5), Eq. (3.11), and properties of Favre-averaging given in Eq. (3.7). We interchange certain terms between their vector notation and inidicial notation for convenience and in doing so, we make use of the following shorthand notation,

$$\frac{\partial\left(ABCD\right)}{\partial t} = \partial_t\, ABCD \tag{3.21a}$$

$$\nabla\left(ABCD\right) = \partial_j\, ABCD \tag{3.21b}$$

where A, B, C, D are different variables and all the terms that are under the standard operators like time-derivative or gradient or divergence are written to the right of the operators such as ∂_t or ∂_j.

The first term one the left-hand side of Eq. (3.20) is simplified as,

$$\begin{aligned}
\left\langle \boldsymbol{u}''_f \cdot \left(\frac{\partial}{\partial t} \left(\rho_f \boldsymbol{U}_f \right) \right) \right\rangle_f &= \left\langle \boldsymbol{u}'' \cdot \left(\frac{\partial}{\partial t} \left(\rho \boldsymbol{U} \right) \right) \right\rangle_f \\
&= \langle u''_i \; \partial_t \rho U_i \rangle_f \\
&= \left\langle u''_i \; \partial_t \rho (\widetilde{U}_i + u''_i) \right\rangle_f \\
&= \left\langle u''_i \; \partial_t \rho \widetilde{U}_i + u''_i \; \partial_t \rho u''_i \right\rangle_f \\
&= \left\langle u''_i \widetilde{U}_i \; \partial_t \rho + u''_i \rho \; \partial_t \widetilde{U}_i \right\rangle_f + \langle u''_i \; \partial_t \rho u''_i \rangle_f \\
&= \left\langle u''_i \widetilde{U}_i \; \partial_t \rho \right\rangle_f + \underbrace{\langle u''_i \rho \rangle_f}_{=0} \partial_t \widetilde{U}_i + \left\langle \partial_t \rho \frac{1}{2} u''_i u''_i \right\rangle_f \\
&= \left\langle u''_i \widetilde{U}_i \; \partial_t \rho \right\rangle_f + \; \partial_t \left(\overline{\rho}_f k_f \right) \qquad (3.22)
\end{aligned}$$

We rewrite the final result of Eq. (3.22) as,

$$\left\langle \boldsymbol{u}''_f \cdot \frac{\partial}{\partial t} \left(\rho_f \boldsymbol{U}_f \right) \right\rangle_f = \left\langle u''_i \widetilde{U}_i \partial_t \rho \right\rangle_f + \partial_t \left(\overline{\rho}_f k_f \right) \qquad (3.23)$$

where k_f is the TKE of the fluid f defined as,

$$k_f \equiv \frac{\langle \rho \frac{1}{2} u''_i u''_i \rangle_f}{\overline{\rho}_f} \qquad (3.24)$$

The second term on the left-hand side of Eq. (3.20) is simplified as,

$$
\begin{aligned}
\left\langle \boldsymbol{u}''_f \cdot \nabla \cdot \left(\rho_f \boldsymbol{U}_f \boldsymbol{U}_f\right)\right\rangle_f &= \left\langle \boldsymbol{u}'' \cdot \nabla \cdot \left(\rho \boldsymbol{U} \boldsymbol{U}\right)\right\rangle_f \\
&= \left\langle u_i''\ \partial_j \rho U_i U_j \right\rangle_f \\
&= \left\langle u_i''\ \partial_j \rho \left(\widetilde{U}_i + u_i''\right) U_j \right\rangle_f \\
&= \left\langle u_i''\ \partial_j \rho \widetilde{U}_i U_j + u_i''\ \partial_j \rho u_i'' U_j \right\rangle_f \\
&= \left\langle u_i'' \widetilde{U}_i\ \partial_j \rho U_j + u_i'' \rho U_j\ \partial_j \widetilde{U}_i +\ \partial_j \rho \frac{1}{2} u_i'' u_i'' U_j \right\rangle_f \\
&= \left\langle u_i'' \widetilde{U}_i\ \partial_j \rho U_j \right\rangle_f + \left\langle u_i'' \rho U_j\ \partial_j \widetilde{U}_i \right\rangle_f + \left\langle\ \partial_j \rho \frac{1}{2} u_i'' u_i'' U_j \right\rangle_f \\
&= \left\langle u_i'' \widetilde{U}_i\ \partial_j \rho U_j \right\rangle_f + \left\langle u_i'' \rho \widetilde{U}_j\ \partial_j \widetilde{U}_i \right\rangle_f + \left\langle u_i'' \rho u_j''\ \partial_j \widetilde{U}_i \right\rangle_f \\
&\qquad + \left\langle\ \partial_j \rho \frac{1}{2} u_i'' u_i'' \widetilde{U}_j \right\rangle_f + \left\langle\ \partial_j \rho \frac{1}{2} u_i'' u_i'' u_j'' \right\rangle_f \\
&= \left\langle u_i'' \widetilde{U}_i\ \partial_j \rho U_j \right\rangle_f + \underbrace{\left\langle u_i'' \rho \right\rangle_f}_{=0} \widetilde{U}_j\ \partial_j \widetilde{U}_i + \left\langle \rho u_i'' u_j'' \right\rangle_f\ \partial_j \widetilde{U}_i \\
&\qquad +\ \partial_j \left\langle \rho \frac{1}{2} u_i'' u_i'' \widetilde{U}_j \right\rangle_f +\ \partial_j \left\langle \rho \frac{1}{2} u_i'' u_i'' u_j'' \right\rangle_f \\
&= \left\langle u_i'' \widetilde{U}_i\ \partial_j \rho U_j \right\rangle_f - \mathcal{P}_f + \nabla \cdot \left(\overline{\rho}_f k_f \widetilde{U}_f\right) + \mathcal{T}''_{u'',f} \qquad (3.25)
\end{aligned}
$$

We rewrite the final result of Eq. (3.25) as,

$$\left\langle \boldsymbol{u}''_f \cdot \nabla \cdot \left(\rho_f \boldsymbol{U}_f \boldsymbol{U}_f\right)\right\rangle_f = \left\langle u_i'' \widetilde{U}_i \partial_j \rho U_j \right\rangle_f - \mathcal{P}_f + \nabla \cdot \left(\overline{\rho}_f k_f \widetilde{U}_f\right) + \mathcal{T}''_{u'',f} \tag{3.26}$$

where, $\mathcal{P}_f$ is the production in fluid f given by:

$$\mathcal{P}_f \equiv -\left\langle \rho u_i'' u_j'' \partial_j \widetilde{U}_i \right\rangle_f \tag{3.27}$$

and $\mathcal{T}''_{u'',f}$ is the turbulence transport of k_f given by:

$$\mathcal{T}''_{u'',f} \equiv \partial_j \left\langle \rho \frac{1}{2} u_i'' u_i'' u_j'' \right\rangle_f \tag{3.28}$$

We simplify the left-hand side of Eq. (3.20) using Eq. (3.23) and Eq. (3.26) as,

$$\begin{aligned} \left\langle \boldsymbol{u}''_f \cdot \left(\frac{\partial}{\partial t}\left(\rho_f \boldsymbol{U}_f\right)\right)\right\rangle_f + \left\langle \boldsymbol{u}''_f \cdot \left(\nabla \cdot \, \rho_f \boldsymbol{U}_f \boldsymbol{U}_f\right)\right\rangle_f = \frac{\partial}{\partial t}\left(\overline{\rho} k_f\right) + \nabla \cdot \, \left(\overline{\rho} k_f \widetilde{\boldsymbol{U}}_f\right) \\ - \mathcal{P}_f + \mathcal{T}''_{u'',f} \end{aligned} \tag{3.29}$$

The right-hand side of Eq. (3.20) is simplified as,

$$\begin{aligned} \left\langle (\nabla \cdot \, \boldsymbol{\tau}) \cdot \boldsymbol{u}''_f \right\rangle_f + \left\langle \rho_f \boldsymbol{f_b} \cdot \boldsymbol{u}''_f \right\rangle_f &= \left\langle (\nabla \cdot \, \overline{\boldsymbol{\tau}} + \boldsymbol{\tau}') \cdot \boldsymbol{u}''_f \right\rangle_f + \left\langle \rho_f \boldsymbol{f_b} \cdot \boldsymbol{u}'' \right\rangle_f \\ &= \left\langle (\nabla \cdot \, \overline{\boldsymbol{\tau}}) \cdot \boldsymbol{u}''_f \right\rangle_f + \left\langle (\nabla \cdot \, \boldsymbol{\tau}') \cdot \boldsymbol{u}''_f \right\rangle_f + \left\langle \rho_f \boldsymbol{f_b} \cdot \boldsymbol{u}'' \right\rangle_f \end{aligned} \tag{3.30}$$

The first term on the right-hand side of Eq. (3.30) is simplified as,

$$\begin{aligned} \left\langle (\nabla \cdot \, \overline{\boldsymbol{\tau}}) \cdot \boldsymbol{u}''_f \right\rangle_f &= \left\langle \left(\nabla \cdot \, \left(-\overline{p}\mathbf{I} + \overline{\mathbf{T}}\right)_f\right) \cdot \boldsymbol{u}''_f \right\rangle \\ &= \left\langle (-\nabla \cdot \, \overline{p}\mathbf{I}) \cdot \boldsymbol{u}''_f + \left(\nabla \cdot \, \overline{\mathbf{T}}\right) \cdot \boldsymbol{u}''_f \right\rangle_f \\ &= \left\langle (-\nabla \cdot \, \overline{p}\mathbf{I}) \cdot \boldsymbol{u}''_f \right\rangle_f + \left\langle \left(\nabla \cdot \, \overline{\mathbf{T}}_f\right) \cdot \boldsymbol{u}''_f \right\rangle_f \\ &= (-\nabla \overline{p}) \cdot \left\langle \boldsymbol{u}''_f \right\rangle_f + \left(\nabla \cdot \overline{\mathbf{T}}\right) \cdot \left\langle \boldsymbol{u}''_f \right\rangle_f \\ &= \Pi_{p,f} + \Pi_{\nu,f} \end{aligned} \tag{3.31}$$

We rewrite the final result of Eq. (3.31) as,

$$\left\langle (\nabla \cdot \, \overline{\boldsymbol{\tau}}) \cdot \boldsymbol{u}''_f \right\rangle = \Pi_{p,f} + \Pi_{\nu,f} \tag{3.32}$$

where $\Pi_{p,f}$ and $\Pi_{\nu,f}$ are the artifacts of Favre-averaging and in-principle can have either sign (Lele, 1994), and are given by:

$$\Pi_{p,f} \equiv (-\nabla \overline{p}) \cdot \left\langle \boldsymbol{u}''_f \right\rangle_f \tag{3.33a}$$

$$\Pi_{\nu,f} \equiv \left(\nabla \cdot \, \overline{\mathbf{T}}\right) \cdot \left\langle \boldsymbol{u}''_f \right\rangle_f \tag{3.33b}$$

where $\Pi_{p,f}$ has been called the enthalpic production of TKE (Lele, 1994) or simply rate of pressure work (Wilcox, 2006). $\Pi_{\nu,f}$ can analogously be called rate of reversible viscous work.

The second term on the right-hand side of Eq. (3.30) is simplified as,

$$\begin{aligned}
\left\langle (\nabla \cdot \boldsymbol{\tau}') \cdot \boldsymbol{u}''_f \right\rangle_f &= \left\langle \nabla \cdot \left(\boldsymbol{\tau}' \boldsymbol{u}''_f \right) - \boldsymbol{\tau}' : \nabla \boldsymbol{u}''_f \right\rangle_f \\
&= \nabla \cdot \left\langle \boldsymbol{\tau}' \boldsymbol{u}''_f \right\rangle - \left\langle \boldsymbol{\tau}' : \nabla \boldsymbol{u}''_f \right\rangle_f \\
&= \nabla \cdot \left\langle \left(-p' \mathbf{\mathsf{I}} + \mathbf{\mathsf{T}}' \right) \boldsymbol{u}''_f \right\rangle_f - \left\langle \left(-p' \mathbf{\mathsf{I}} + \mathbf{\mathsf{T}}' \right) : \nabla \boldsymbol{u}''_f \right\rangle_f \\
&= -\nabla \cdot \left\langle p' \mathbf{\mathsf{I}} \boldsymbol{u}''_f \right\rangle_f + \nabla \cdot \left\langle \mathbf{\mathsf{T}}' \boldsymbol{u}''_f \right\rangle_f + \left\langle p' \mathbf{\mathsf{I}} : \nabla \boldsymbol{u}'' \right\rangle_f - \left\langle \mathbf{\mathsf{T}}' : \nabla \boldsymbol{u}''_f \right\rangle_f \\
&= \mathcal{T}''_{p,f} + \mathcal{T}''_{\nu,f} + \Pi_{\Delta'',f} - \varepsilon''_f
\end{aligned} \tag{3.34}$$

We rewrite the final result of Eq. (3.34) as,

$$\left\langle (\nabla \cdot \boldsymbol{\tau}') \cdot \boldsymbol{u}''_f \right\rangle_f = \mathcal{T}''_{p,f} + \mathcal{T}''_{\nu,f} + \Pi_{\Delta'',f} - \varepsilon''_f \tag{3.35}$$

where $\mathcal{T}''_{p,f}$ and $\mathcal{T}''_{\nu,f}$ are the fluctuating pressure power and fluctuating viscous power of the fluid f respectively given by:

$$\mathcal{T}''_{p,f} \equiv -\nabla \cdot \left\langle p' \mathbf{\mathsf{I}} \boldsymbol{u}'' \right\rangle_f \tag{3.36a}$$

$$\mathcal{T}''_{\nu,f} \equiv \nabla \cdot \left\langle \mathbf{\mathsf{T}}' \boldsymbol{u}'' \right\rangle_f \tag{3.36b}$$

$\Pi_{\Delta'',f}$ is the rate of work done by pressure fluctuations due to the simultaneous fluctuations in dilatation, or simply called fluctuating pressure dilation (Lele, 1994). It can have either positive or negative sign and the value is given by:

$$\Pi_{\Delta'',f} \equiv \left\langle p' \nabla \cdot \boldsymbol{u}'' \right\rangle_f \tag{3.37}$$

ε''_f is the rate of viscous dissipation of k_f which is always negative and the value is given by:

$$\varepsilon''_f \equiv \left\langle \mathbf{\mathsf{T}}' : \nabla \boldsymbol{u}'' \right\rangle_f \tag{3.38}$$

The last term on the right-hand side of Eq. (3.30) is the fluctuating part of the power of body force given by:

$$\Gamma''_f \equiv \left\langle \rho_f \boldsymbol{f_b} \cdot \boldsymbol{u}''_f \right\rangle \tag{3.39}$$

We note here that if the body force field $\boldsymbol{f_b} = \boldsymbol{g}$, the gravitational field which is spatially uniform, Γ''_f reduces to 0 using Eq. (3.7) as,

$$\Gamma''_f \equiv \left\langle \rho_f \boldsymbol{g} \cdot \boldsymbol{u}''_f \right\rangle = \boldsymbol{g} \cdot \underbrace{\left\langle \rho_f \boldsymbol{u}''_f \right\rangle}_{=0} = 0 \tag{3.40}$$

Thus, body forces like gravity have no contribution to the modulation of k_f. We combine Eq. (3.29), Eq. (3.32), Eq. (3.35), and rearrange, to arrive at the TKE equation for compressible flow in gas or liquid phase,

$$\frac{\partial}{\partial t}\left(\overline{\rho}_f k_f\right) + \nabla \cdot \left(\overline{\rho_f} k_f \widetilde{\boldsymbol{U}}_f\right) + \mathcal{T}''_{u'',f} = \mathcal{P}_f - \varepsilon''_f + \Pi_{\Delta'',f} + \mathcal{T}''_{p,f} + \mathcal{T}''_{\nu,f} + \Pi_{p,f} + \Pi_{\nu,f} + \Gamma''_f \tag{3.41}$$

We want to be able to write the Eq. (3.41) in a non-conservative form. To do this, we make use of the Eq. (3.14) and perform Reynolds averaging to obtain an equation for conservation of mean density:

$$\left\langle \frac{\partial \rho_f}{\partial t} + \nabla \cdot \left(\rho_f \boldsymbol{U}_f\right) \right\rangle_f = \frac{\partial \left\langle \rho_f \right\rangle_f}{\partial t} + \nabla \cdot \left(\left\langle \rho_f \right\rangle_f \widetilde{\boldsymbol{U}}_f\right) = 0 \tag{3.42}$$

We use Eq. (3.42), to write the compressible TKE equation in a non-conservative form:

$$\overline{\rho}_f \left(\frac{\partial k_f}{\partial t} + \widetilde{\boldsymbol{U}}_f \cdot \nabla k_f\right) + \mathcal{T}''_{u'',f} = \mathcal{P}_f - \varepsilon''_f + \Pi_{\Delta'',f} + \mathcal{T}''_{p,f} + \mathcal{T}''_{\nu,f} + \Pi_{p,f} + \Pi_{\nu,f} + \Gamma''_f \tag{3.43}$$

We define the material derivative operator with Favre-averaged mean-flow velocity as,

$$\frac{\widetilde{D}}{\widetilde{D}t} \equiv \frac{\partial}{\partial t} + \widetilde{\boldsymbol{U}} \cdot \nabla \tag{3.44}$$

We use Eq. (3.44) to present the non-conservative form of the TKE equation for a compressible flow:

$$\overline{\rho}_f \frac{\widetilde{D}k}{\widetilde{D}t} + \mathcal{T}''_{u'',f} = \mathcal{P}_f - \varepsilon''_f + \Pi_{\Delta'',f} + \mathcal{T}''_{p,f} + \mathcal{T}''_{\nu,f} + \Pi_{p,f} + \Pi_{\nu,f} + \Gamma''_f \tag{3.45}$$

3.1.2 Mean-flow kinetic energy (MFKE) for compressible flow (gas or liquid phase), $\widetilde{E}_f$

We take a dot product of Eq. (3.16) with $\widetilde{\boldsymbol{U}}_f$ to obtain,

$$\widetilde{\boldsymbol{U}}_f \cdot \left(\frac{\partial}{\partial t}\left(\rho_f \boldsymbol{U}_f\right) + \nabla \cdot \left(\rho_f \boldsymbol{U}_f \boldsymbol{U}_f\right)\right) = \widetilde{\boldsymbol{U}}_f \cdot \left(\nabla \cdot \boldsymbol{\tau} + \rho_f \boldsymbol{f_b}\right) \tag{3.46}$$

We integrate Eq. (3.46) over $\mathcal{V}_f$ and divide by $\mathcal{V}_f$ to obtain,

$$\frac{1}{\mathcal{V}_f}\int_{\mathcal{V}_f} \widetilde{\boldsymbol{U}}_f \cdot \left(\frac{\partial}{\partial t}\left(\rho_f \boldsymbol{U}_f\right) + \nabla\cdot\left(\rho_f \boldsymbol{U}_f \boldsymbol{U}_f\right)\right) d\mathcal{V} = \frac{1}{\mathcal{V}_f}\int_{\mathcal{V}_f} \widetilde{\boldsymbol{U}}_f \cdot \left(\nabla\cdot\boldsymbol{\tau} + \rho_f \boldsymbol{f_b}\right) d\mathcal{V} \tag{3.47}$$

We use the definition of Reynolds averaging given in Eq. (3.3) to rewrite Eq. (3.47) as,

$$\left\langle \widetilde{\boldsymbol{U}}_f \cdot \left(\frac{\partial}{\partial t}\left(\rho_f \boldsymbol{U}_f\right) + \nabla\cdot\left(\rho_f \boldsymbol{U}_f \boldsymbol{U}_f\right)\right)\right\rangle_f = \left\langle \widetilde{\boldsymbol{U}}_f \cdot \left(\nabla\cdot\boldsymbol{\tau} + \rho_f \boldsymbol{f_b}\right)\right\rangle_f \tag{3.48}$$

We use the fact that both dot product and $\langle ... \rangle$ are linear operators to rewrite Eq. (3.19) as,

$$\left\langle \widetilde{\boldsymbol{U}}_f \cdot \left(\frac{\partial}{\partial t}\left(\rho_f \boldsymbol{U}_f\right)\right)\right\rangle_f + \left\langle \widetilde{\boldsymbol{U}}_f \cdot \left(\nabla\cdot\rho_f \boldsymbol{U}_f \boldsymbol{U}_f\right)\right\rangle_f = \left\langle \widetilde{\boldsymbol{U}}_f \cdot \left(\nabla\cdot\boldsymbol{\tau}\right)\right\rangle_f + \left\langle \widetilde{\boldsymbol{U}}_f \cdot \rho_f \boldsymbol{f_b}\right\rangle_f \tag{3.49}$$

We simplify each term in Eq. (3.49) separately using Eq. (3.5), Eq. (3.11), and properties of Favre-averaging given in Eq. (3.7). The first term one the left-hand side of Eq. (3.20) is simplified as,

$$\begin{aligned}
\left\langle \widetilde{\boldsymbol{U}}_f \cdot \frac{\partial}{\partial t}\left(\rho_f \boldsymbol{U}_f\right)\right\rangle_f &= \left\langle \widetilde{\boldsymbol{U}} \cdot \frac{\partial}{\partial t}\left(\rho \boldsymbol{U}\right)\right\rangle_f \\
&= \left\langle \widetilde{U}_i \partial_t \rho U_i \right\rangle_f \\
&= \left\langle \widetilde{U}_i \partial_t \rho \left(\widetilde{U}_i u_i''\right)\right\rangle_f \\
&= \left\langle \widetilde{U}_i \partial_t \rho \widetilde{U}_i + \widetilde{U}_i \partial_t \rho u_i'' \right\rangle_f \\
&= \left\langle \partial_t \rho \frac{1}{2} \widetilde{U}_i \widetilde{U}_i \right\rangle_f + \left\langle \widetilde{U}_i \partial_t \rho u_i'' \right\rangle_f \\
&= \partial_t \left\langle \rho \frac{1}{2} \widetilde{U}_i \widetilde{U}_i \right\rangle_f + \widetilde{U}_i \partial_t \underbrace{\langle \rho u_i'' \rangle_f}_{=0} \\
&= \frac{\partial}{\partial t}\left(\overline{\rho}_f \widetilde{E}_f\right)
\end{aligned} \tag{3.50}$$

We rewrite the final result of Eq. (3.50) as,

$$\left\langle \widetilde{\boldsymbol{U}}_f \cdot \frac{\partial}{\partial t}\left(\rho_f \boldsymbol{U}_f\right)\right\rangle_f = \frac{\partial}{\partial t}\left(\overline{\rho}_f \widetilde{E}_f\right) \tag{3.51}$$

where $\widetilde{E}_f$ is the kinetic energy of the mean-flow, (MFKE) in fluid f given by:

$$\widetilde{E}_f \equiv \frac{1}{2}\widetilde{\boldsymbol{U}}_f \cdot \widetilde{\boldsymbol{U}}_f \tag{3.52}$$

The second term on the left-hand side of Eq. (3.49) is simplified as,

$$\begin{aligned}
\left\langle \widetilde{\boldsymbol{U}}_f \cdot \{\nabla \cdot (\rho_f \boldsymbol{U}_f \boldsymbol{U}_f)\} \right\rangle_f &= \left\langle \widetilde{\boldsymbol{U}} \cdot \{\nabla \cdot (\rho \boldsymbol{U}\boldsymbol{U})\} \right\rangle_f \\
&= \left\langle \widetilde{U}_i \ \partial_j \rho U_i U_j \right\rangle_f \\
&= \left\langle \widetilde{U}_i \ \partial_j \rho \left(\widetilde{U}_i + u_i''\right) U_j \right\rangle_f \\
&= \left\langle \widetilde{U}_i \ \partial_j \rho \widetilde{U}_i U_j + \widetilde{U}_i \ \partial_j \rho u_i'' U_j \right\rangle_f \\
&= \left\langle \ \partial_j \rho \frac{1}{2} \widetilde{U}_i \widetilde{U}_i U_j \right\rangle_f + \left\langle \ \partial_j \rho \widetilde{U}_i u_i'' U_j \right\rangle_f - \left\langle \rho u_i'' U_j \ \partial_j \widetilde{U}_i \right\rangle_f \\
&= \ \partial_j \left\langle \rho \frac{1}{2} \widetilde{U}_i \widetilde{U}_i U_j \right\rangle_f + \left\langle \ \partial_j \rho \widetilde{U}_i u_i'' \widetilde{U}_j \right\rangle_f + \left\langle \ \partial_j \rho \widetilde{U}_i u_i'' u_j'' \right\rangle_f \\
&\qquad - \left\langle \rho u_i'' \widetilde{U}_j \ \partial_j \widetilde{U}_i \right\rangle_f - \left\langle \rho u_i'' u_j'' \ \partial_j \widetilde{U}_i \right\rangle_f \\
&= \ \partial_j \left\langle \rho \frac{1}{2} \widetilde{U}_i \widetilde{U}_i \widetilde{U}_j \right\rangle_f + \ \partial_j \frac{1}{2} \widetilde{U}_i \widetilde{U}_i \underbrace{\langle \rho u_j'' \rangle_f}_{=0} + \partial_j \underbrace{\langle \rho u_i'' \rangle_f}_{=0} \widetilde{U}_i \widetilde{U}_j \\
&\qquad + \left\langle \ \partial_j \rho \widetilde{U}_i u_i'' u_j'' \right\rangle_f - \underbrace{\langle \rho u_i'' \rangle_f}_{=0} \widetilde{U}_j \ \partial_j \widetilde{U}_i - \langle \rho u_i'' u_j'' \rangle_f \ \partial_j \widetilde{U}_i \\
&= \nabla \cdot \left(\overline{\rho}_f \widetilde{E} \widetilde{\boldsymbol{U}}_f\right) + \widetilde{\mathcal{T}}_{u'',f} + \mathcal{P}_f
\end{aligned} \tag{3.53}$$

We rewrite the final result of Eq. (3.53) as,

$$\left\langle \widetilde{\boldsymbol{U}}_f \cdot \{\nabla \cdot (\rho_f \boldsymbol{U}_f \boldsymbol{U}_f)\} \right\rangle_f = \nabla \cdot \left(\overline{\rho}_f \widetilde{E} \widetilde{\boldsymbol{U}}_f\right) + \widetilde{\mathcal{T}}_{u'',f} + \mathcal{P}_f \tag{3.54}$$

where $\mathcal{P}_f$ is the production of k_f as defined in Eq. (3.27) and $\widetilde{\mathcal{T}}_{u'',f}$ is the turbulence transport of $\widetilde{E}_f$ given by:

$$\widetilde{\mathcal{T}}_{u'',f} \equiv \partial_j \left\langle \rho \widetilde{U}_i u_i'' u_j'' \right\rangle_f \tag{3.55}$$

The right-hand side of Eq. (3.49) is simplified as,

$$\begin{aligned}
\left\langle \widetilde{\boldsymbol{U}} \cdot (\nabla \cdot \boldsymbol{\tau}) \right\rangle_f + \left\langle \widetilde{\boldsymbol{U}}_f \cdot \rho_f \boldsymbol{f_b} \right\rangle_f &= \left\langle \widetilde{\boldsymbol{U}} \cdot (\nabla \cdot \overline{\boldsymbol{\tau}} + \boldsymbol{\tau}') \right\rangle_f + \widetilde{\Gamma}_f \\
&= \left\langle \widetilde{\boldsymbol{U}} \cdot (\nabla \cdot \overline{\boldsymbol{\tau}}) \right\rangle_f + \widetilde{\boldsymbol{U}} \cdot (\nabla \cdot \underbrace{\langle \boldsymbol{\tau}' \rangle}_{=0}) + \widetilde{\Gamma}_f \\
&= \left\langle \nabla \cdot \left(\overline{\boldsymbol{\tau}} \widetilde{\boldsymbol{U}} \right) - \overline{\boldsymbol{\tau}} : \nabla \widetilde{\boldsymbol{U}} \right\rangle_f + \widetilde{\Gamma}_f \\
&= \left\langle \nabla \cdot \left(\overline{(-p\mathbf{I} + \mathbf{T})} \widetilde{\boldsymbol{U}} \right) \right\rangle_f - \left\langle \overline{(-p\mathbf{I} + \mathbf{T})} : \nabla \widetilde{\boldsymbol{U}} \right\rangle_f + \widetilde{\Gamma}_f \\
&= \nabla \cdot \left\langle -\overline{p}\mathbf{I} \widetilde{\boldsymbol{U}} \right\rangle_f + \nabla \cdot \left\langle \overline{\mathbf{T}} \widetilde{\boldsymbol{U}} \right\rangle_f + \overline{p}\mathbf{I} : \nabla \widetilde{\boldsymbol{U}} - \overline{\mathbf{T}} : \nabla \widetilde{\boldsymbol{U}} + \widetilde{\Gamma}_f \\
&= \widetilde{\mathcal{T}}_{p,f} + \widetilde{\mathcal{T}}_{\nu,f} + \Pi_{\widetilde{\Delta},f} - \widetilde{\varepsilon}_f + \widetilde{\Gamma}_f \qquad (3.56)
\end{aligned}$$

We rewrite the final result of Eq. (3.56) as,

$$\left\langle \widetilde{\boldsymbol{U}} \cdot (\nabla \cdot \boldsymbol{\tau}) \right\rangle_f = \widetilde{\mathcal{T}}_{p,f} + \widetilde{\mathcal{T}}_{\nu,f} + \Pi_{\widetilde{\Delta},f} - \widetilde{\varepsilon}_f + \widetilde{\Gamma}_f \qquad (3.57)$$

where $\widetilde{\mathcal{T}}_{p,f}$ and $\widetilde{\mathcal{T}}_{\nu,f}$ are the mean pressure power and mean viscous power of the fluid f respectively given by:

$$\widetilde{\mathcal{T}}_{p,f} \equiv -\nabla \cdot \left\langle \overline{p}\mathbf{I} \widetilde{\boldsymbol{U}} \right\rangle_f \qquad (3.58a)$$

$$\widetilde{\mathcal{T}}_{\nu,f} \equiv \nabla \cdot \left\langle \overline{\mathbf{T}} \widetilde{\boldsymbol{U}} \right\rangle_f \qquad (3.58b)$$

$\Pi_{\widetilde{\Delta},f}$ is the rate of work done by mean pressure due to the simultaneous mean dilatation, or simply called mean pressure dilation. It can have either positive or negative sign and the value is given by:

$$\Pi_{\widetilde{\Delta},f} \equiv \overline{p}_f \nabla \cdot \widetilde{\boldsymbol{U}}_f \qquad (3.59)$$

$\widetilde{\varepsilon}_f$ is the rate of viscous dissipation of $\widetilde{E}_f$ which is always negative and the value is given by:

$$\widetilde{\varepsilon}_f \equiv \overline{\mathbf{T}}_f : \nabla \widetilde{\boldsymbol{U}}_f \qquad (3.60)$$

The last term on the right-hand side of Eq. (3.57) is the mean part of power of body force given by

$$\widetilde{\Gamma}_f \equiv \overline{\rho}_f \boldsymbol{f_b} \cdot \widetilde{\boldsymbol{U}}_f \qquad (3.61)$$

We note here that $\widetilde{\Gamma}_f$ does not reduce to 0, even if $\boldsymbol{f_b}$ is spatially uniform (e.g., $\boldsymbol{f_b} = \boldsymbol{g}$), unlike Γ'' as discussed in Eq. (3.40). We combine, Eq. (3.51), Eq. (3.54), Eq. (3.57), and rearrange to obtain the MFKE equation for compressible flow in gas or liquid phase:

$$\frac{\partial}{\partial t}\left(\overline{\rho}\widetilde{E}\right) + \nabla \cdot \left(\overline{\rho}\widetilde{E}\right) + \widetilde{\mathcal{T}}_{u'',f} = -\mathcal{P}_f - \widetilde{\varepsilon}_f + \Pi_{\widetilde{\Delta},f} + \widetilde{\mathcal{T}}_{p,f} + \widetilde{\mathcal{T}}_{\nu,f} + \widetilde{\Gamma}_f \tag{3.62}$$

We use Eq. (3.42), and Eq. (3.44) to obtain the non-conservative form of the MFKE equation for compressible flow in gas or liquid phase:

$$\overline{\rho}_f \frac{\widetilde{D}\widetilde{E}}{\widetilde{D}t} + \widetilde{\mathcal{T}}_{u'',f} = -\mathcal{P}_f - \widetilde{\varepsilon}_f + \Pi_{\widetilde{\Delta},f} + \widetilde{\mathcal{T}}_{p,f} + \widetilde{\mathcal{T}}_{\nu,f} + \widetilde{\Gamma}_f \tag{3.63}$$

3.1.3 Mean internal energy (MIE) for compressible flow (gas or liquid phase), e_f

We take a dot product of Eq. (3.15) with $\boldsymbol{U}_f$ to obtain the balance equation for kinetic energy of fluid f as,

$$\boldsymbol{U}_f \cdot \left(\rho_f \frac{D\boldsymbol{U}_f}{Dt}\right) = \boldsymbol{U}_f \cdot (\nabla \cdot \boldsymbol{\tau}_f) + \boldsymbol{U}_f \cdot (\rho_f \boldsymbol{f_b}) \tag{3.64}$$

We simplify both sides of Eq. (3.64) using the product rule and the definition of material derivative given in Eq. (2.40) as,

$$\rho_f \left(\frac{\partial}{\partial t}\left(\frac{1}{2}\boldsymbol{U}_f \cdot \boldsymbol{U}_f\right) + \boldsymbol{U}_f \cdot \nabla \left(\frac{1}{2}\boldsymbol{U}_f \cdot \boldsymbol{U}_f\right)\right) = \nabla \cdot (\boldsymbol{\tau}_f \boldsymbol{U}_f) - \boldsymbol{\tau}_f : \nabla \boldsymbol{U}_f + \boldsymbol{U}_f \cdot (\rho_f \boldsymbol{f_b}) \tag{3.65}$$

We subtract Eq. (3.65) from the total energy balance equation for fluid f given in Eq. (2.66) to obtain the equation for the internal energy balance in fluid f

$$\rho_f \frac{De_f}{Dt} = \boldsymbol{\tau} : \nabla \boldsymbol{U}_f - \nabla \cdot \boldsymbol{q} \tag{3.66}$$

We write Eq. (3.66) in the conservative form using Eq. (2.6), and Eq. (2.40) as,

$$\frac{\partial}{\partial t}(\rho_f e_f) + \nabla \cdot (\rho_f e_f \boldsymbol{U}_f) = \boldsymbol{\tau} : \nabla \boldsymbol{U}_f - \nabla \cdot \boldsymbol{q} \tag{3.67}$$

We integrate Eq. (3.67) over $\mathcal{V}_f$ and divide by $\mathcal{V}_f$ to obtain,

$$\frac{1}{\mathcal{V}_f}\int_{\mathcal{V}_f} \frac{\partial}{\partial t}(\rho_f e_f)\, d\mathcal{V} + \nabla \cdot (\rho_f e_f \boldsymbol{U}_f)\, d\mathcal{V} = \frac{1}{\mathcal{V}_f}\int_{\mathcal{V}_f} \boldsymbol{\tau} : \nabla \boldsymbol{U}_f d\mathcal{V} - \frac{1}{\mathcal{V}_f}\int_{\mathcal{V}_f} \nabla \cdot \boldsymbol{q} d\mathcal{V} \tag{3.68}$$

We rewrite Eq. (3.68) using the definition of Reynolds averaging in Eq. (3.3) as,

$$\left\langle \frac{\partial}{\partial t}\left(\rho_f e_f\right)\right\rangle_f + \left\langle \nabla \cdot \left(\rho_f e_f \boldsymbol{U}_f\right)\right\rangle_f = \left\langle \boldsymbol{\tau} : \nabla \boldsymbol{U}_f \right\rangle_f - \left\langle \nabla \cdot \boldsymbol{q} \right\rangle_f \tag{3.69}$$

We simplify each term in Eq. (3.69) separately using Eq. (3.12), and Favre-averaging properties given in Eq. (3.7) and Eq. (3.13). The first term on the left-hand side of Eq. (3.69) is simplified as,

$$\left\langle \frac{\partial}{\partial t}\left(\rho_f e_f\right)\right\rangle_f = \frac{\partial}{\partial t}\left\langle \rho_f e_f \right\rangle_f = \frac{\partial}{\partial t}\left(\overline{\rho}_f \widetilde{e_f}\right) \tag{3.70}$$

where $\widetilde{e}_f$ is the Favre-averaged mean internal energy, (MIE) of the fluid f given by

$$\widetilde{e}_f = \frac{\left\langle \rho_f e_f \right\rangle_f}{\left\langle \rho_f \right\rangle_f} \tag{3.71}$$

The second term on the left-hand side of Eq. (3.69) is simplified as,

$$\begin{aligned}
\left\langle \nabla \cdot \left(\rho_f e_f \boldsymbol{U}_f\right)\right\rangle_f &= \nabla \cdot \left\langle \rho e \boldsymbol{U} \right\rangle_f \\
&= \nabla \cdot \left\langle \rho\left(\widetilde{e} + e''\right)\left(\widetilde{\boldsymbol{U}} + \boldsymbol{u}''\right)\right\rangle_f \\
&= \nabla \cdot \left\langle \rho \widetilde{e} \widetilde{\boldsymbol{U}}_f + \rho \widetilde{e} \boldsymbol{u}'' + \rho e'' \widetilde{\boldsymbol{U}} + \rho e'' \boldsymbol{u}'' \right\rangle_f \\
&= \nabla \cdot \left\langle \rho \widetilde{e} \widetilde{\boldsymbol{U}} \right\rangle_f + \nabla \cdot \left(\underbrace{\left\langle \rho \boldsymbol{u}'' \right\rangle_f}_{=0} \widetilde{e}\right) + \nabla \cdot \left(\underbrace{\left\langle \rho e'' \right\rangle_f}_{=0} \widetilde{\boldsymbol{U}}\right) + \nabla \cdot \left\langle \rho e'' \boldsymbol{u}'' \right\rangle_f \\
&= \nabla \cdot \left\langle \rho_f \widetilde{e_f} \widetilde{\boldsymbol{U}}_f \right\rangle_f + \mathcal{T}^*_{u'',f}
\end{aligned} \tag{3.72}$$

We rewrite the final result of Eq. (3.72) as,

$$\left\langle \nabla \cdot \left(\rho_f e_f \boldsymbol{U}_f\right)\right\rangle_f = \nabla \cdot \left\langle \rho_f \widetilde{e_f} \widetilde{\boldsymbol{U}}_f \right\rangle_f + \mathcal{T}^*_{u'',f} \tag{3.73}$$

where $\mathcal{T}^*_{u'',f}$ is the turbulence transport of $\widetilde{e}_f$ given by:

$$\mathcal{T}^*_{u'',f} \equiv \partial_j \left\langle \rho e'' u''_j \right\rangle_f \tag{3.74}$$

The first term on the right-hand side of Eq. (3.69) is simplified as,

$$\begin{aligned}
\langle \boldsymbol{\tau} : \nabla \boldsymbol{U}_f \rangle_f &= \langle \boldsymbol{\tau} : \nabla \boldsymbol{U} \rangle_f \\
&= \langle \tau_{ij} \partial_j U_i \rangle_f \\
&= \left\langle \left(\overline{\tau}_{ij} + \tau'_{ij} \right) \partial_j \left(\widetilde{U}_i + u''_i \right) \right\rangle_f \\
&= \left\langle \overline{\tau}_{ij} \partial_j \widetilde{U}_i \right\rangle_f + \langle \overline{\tau}_{ij} \partial_j u''_i \rangle_f + \underbrace{\langle \tau'_{ij} \partial_j \rangle_f}_{=0} \widetilde{U}_i + \langle \tau'_{ij} \partial_j u''_i \rangle_f \\
&= \left(\overline{\tau}_{ij} \partial_j \widetilde{U}_i \right)_f + \langle \partial_j \overline{\tau}_{ij} u''_i \rangle_f - \langle u''_i \partial_j \overline{\tau}_{ij} \rangle_f + \langle \tau'_{ij} \partial_j u''_i \rangle_f
\end{aligned} \tag{3.75}$$

We rewrite the final result of Eq. (3.75) as,

$$\langle \boldsymbol{\tau} : \nabla \boldsymbol{U}_f \rangle_f = \left(\overline{\tau}_{ij} \partial_j \widetilde{U}_i \right)_f + \langle \partial_j \overline{\tau}_{ij} u''_i \rangle_f - \langle u''_i \partial_j \overline{\tau}_{ij} \rangle_f + \langle \tau'_{ij} \partial_j u''_i \rangle_f \tag{3.76}$$

The first term on the right-hand side of Eq. (3.76) is simplified in Eq. (3.56) as,

$$\left(\overline{\tau}_{ij} \partial_j \widetilde{U}_i \right)_f = -\Pi_{\widetilde{\Delta},f} + \widetilde{\varepsilon}_f \tag{3.77}$$

where $\Pi_{\widetilde{\Delta},f}$ and $\widetilde{\varepsilon}_f$ are the dilatation and dissipation due to mean velocity and are defined in Eq. (3.59) and Eq. (3.60) respectively. The second term on the right-hand side of Eq. (3.76) is simplified as,

$$\begin{aligned}
\langle \partial_j \overline{\tau}_{ij} u''_i \rangle_f &= \left\langle \partial_j \left(-\overline{p} I_{ij} + \overline{T}_{ij} \right) u''_i \right\rangle_f \\
&= - \langle \partial_j \overline{p} I_{ij} u''_i \rangle_f + \left\langle \partial_j \overline{T}_{ij} u''_i \right\rangle_f \\
&= -\partial_i \langle \overline{p} u''_i \rangle_f + \partial_j \left\langle \overline{T}_{ij} u''_i \right\rangle_f \\
&= \mathcal{T}^*_{p,f} + \mathcal{T}^*_{\nu,f}
\end{aligned} \tag{3.78}$$

We rewrite the final result of Eq. (3.79) as,

$$\langle \partial_j \overline{\tau}_{ij} u''_i \rangle_f = \mathcal{T}^*_{p,f} + \mathcal{T}^*_{\nu,f} \tag{3.79}$$

where $\mathcal{T}^*_{p,f}$ and $\mathcal{T}^*_{\nu,f}$ are the powers of mean pressure and mean viscous stress due to fluctuating velocity and are given by:

$$\mathcal{T}^*_{p,f} \equiv -\partial_i \langle \overline{p} u''_i \rangle_f \tag{3.80a}$$

$$\mathcal{T}^*_{\nu,f} \equiv \partial_j \left\langle \overline{T}_{ij} u''_i \right\rangle_f \tag{3.80b}$$

The third term on the right-hand side of Eq. (3.76) is simplified in Eq. (3.32) as,

$$\langle u_i'' \partial_j \overline{\tau}_{ij} \rangle_f = \Pi_{p,f} + \Pi_{v,f} \tag{3.81}$$

where $\Pi_{p,f}$ and $\Pi_{\nu,f}$ are the reversible work done by pressure and viscosity respectively and are defined in Eq. (3.33b). The fourth term on the right-hand side of Eq. (3.76) is simplified in Eq. (3.34) as,

$$\left\langle \tau_{ij}' \partial_j u_i'' \right\rangle_f = -\Pi_{\Delta'';f} + \varepsilon_f'' \tag{3.82}$$

where $\Pi_{\Delta'';f}$ and ε_f'' are the dilatation and dissipation due to fluctuating velocity and are defined in Eq. (3.37) and Eq. (3.38)respectively. The last term on the right-hand side of Eq. (3.76) is simplified as,

$$-\left\langle \nabla \cdot \boldsymbol{q} \right\rangle_f = \nabla \cdot \overline{\boldsymbol{q}}_f \equiv \dot{\mathcal{Q}}_f \tag{3.83}$$

where $\dot{\mathcal{Q}}_f$ is the rate at which heat is transferred through thermal conduction in fluid f. We use Eq. (3.70), Eq. (3.73), Eq. (3.76) and Eq. (3.83), and present the MIE equation for compressible flow in individual gas or liquid phases:

$$\begin{aligned} \frac{\partial}{\partial t}\overline{\rho}_f \widetilde{e}_f + \nabla \cdot \left\langle \rho_f \widetilde{e}_f \widetilde{\boldsymbol{U}}_f \right\rangle_f + \mathcal{T}^*_{u'';f} = \varepsilon_f'' + \widetilde{\varepsilon}_f - \Pi_{\widetilde{\Delta},f} - \Pi_{\Delta'';f} \\ + \mathcal{T}^*_{p,f} + \mathcal{T}^*_{v,f} - \Pi_{p,f} - \Pi_{v,f} + \dot{\mathcal{Q}}_f \end{aligned} \tag{3.84}$$

We use Eq. (3.42), and Eq. (3.44), to present the non-conservative form of the MIE equation for compressible flow in the gas or liquid phase:

$$\overline{\rho}_f \frac{\widetilde{D}\widetilde{e}}{\widetilde{D}t} + \mathcal{T}^*_{u'';f} = \varepsilon_f'' + \widetilde{\varepsilon}_f - \Pi_{\widetilde{\Delta},f} - \Pi_{\Delta'';f} + \mathcal{T}^*_{p,f} + \mathcal{T}^*_{v,f} - \Pi_{p,f} - \Pi_{\nu,f} + \dot{\mathcal{Q}}_f \tag{3.85}$$

3.1.4 Energy pathways for compressible flow (gas or liquid phase)

For clarity, TKE, MFKE, MIE equations for compressible flow in the gas or liquid phase are given together as,

$$\bar{\rho}_f \frac{\widetilde{D}k}{\widetilde{D}t} + \mathcal{T}''_{u'',f} = \mathcal{P}_f - \varepsilon''_f + \Pi_{\Delta'',f} + \mathcal{T}''_{p,f} + \mathcal{T}''_{\nu,f} + \Pi_{p,f} + \Pi_{\nu,f} + \Gamma''_f \tag{3.86a}$$

$$\bar{\rho}_f \frac{\widetilde{D}\widetilde{E}}{\widetilde{D}t} + \widetilde{\mathcal{T}}_{u'',f} = -\mathcal{P}_f - \widetilde{\varepsilon}_f + \Pi_{\widetilde{\Delta},f} + \widetilde{\mathcal{T}}_{p,f} + \widetilde{\mathcal{T}}_{\nu,f} + \widetilde{\Gamma}_f \tag{3.86b}$$

$$\bar{\rho}_f \frac{\widetilde{D}\widetilde{e}}{\widetilde{D}t} + \mathcal{T}^*_{u'',f} = \varepsilon''_f + \widetilde{\varepsilon}_f - \Pi_{\widetilde{\Delta},f} - \Pi_{\Delta'',f} + \mathcal{T}^*_{p,f} + \mathcal{T}^*_{v,f} - \Pi_{p,f} - \Pi_{\nu,f} + \dot{\mathcal{Q}}_f \tag{3.86c}$$

We present a schematic showcasing the pathways of energy exchange between k_f, $\widetilde{E}_f$, $\widetilde{e}_f$, Θ and Φ in Fig. 3.1.

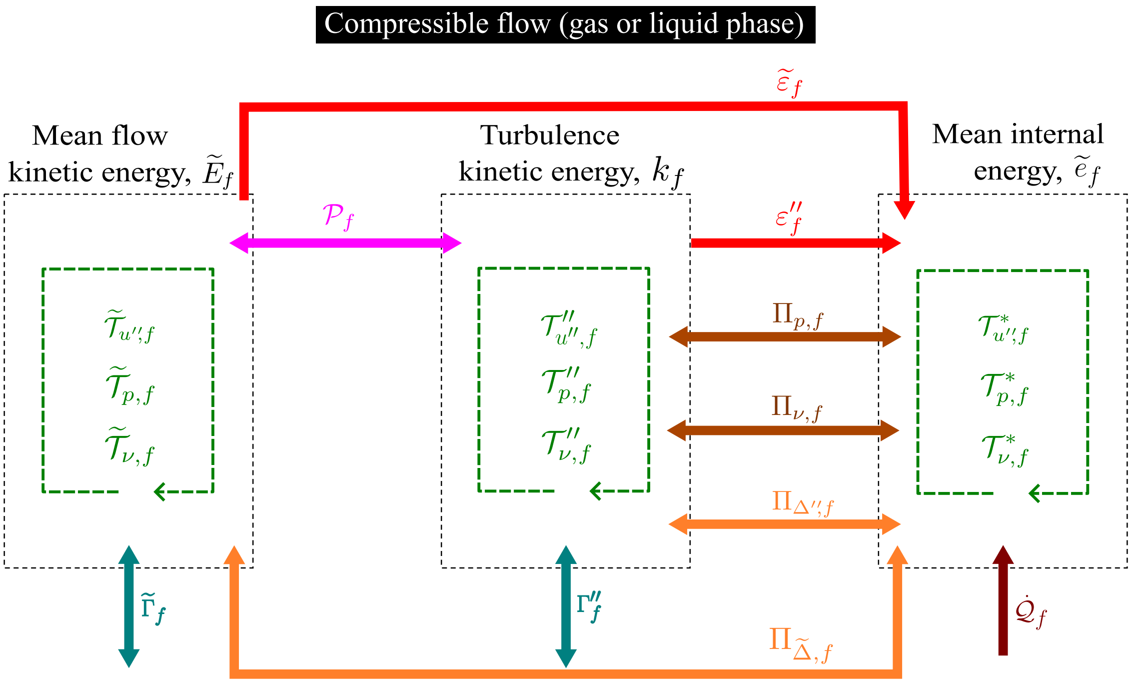

Figure 3.1: Schematic showing the pathways of energy exchange for a compressible flow (gas or liquid phase), which summarizes Eq. (3.86).

3.2 Compressible gas-liquid flows with phase change

In this section, we derive the TKE, MFKE and MIE equations for compressible gas-liquid turbulent flows with phase change. We define the volume fraction ϕ_v as,

$$\phi_v = \frac{\mathcal{V}_l}{\mathcal{V}} \tag{3.87}$$

We use Eq. (3.87), Eq. (2.1), and Eq. (3.2) to define the following identity for a field $\mathcal{F}$:

$$\begin{aligned}
\langle \mathcal{F} \rangle &= \frac{1}{\mathcal{V}} \int_{\mathcal{V}} \mathcal{F} d\mathcal{V} \\
&= \frac{1}{\mathcal{V}} \int_{\mathcal{V}_g} \mathcal{F}_g d\mathcal{V} + \frac{1}{\mathcal{V}} \int_{\mathcal{V}l} \mathcal{F}_l d\mathcal{V} \\
&= \frac{\mathcal{V}_g}{\mathcal{V}} \frac{1}{\mathcal{V}_g} \int_{\mathcal{V}_g} \mathcal{F}_g d\mathcal{V} + \frac{\mathcal{V}_l}{\mathcal{V}} \frac{1}{\mathcal{V}_l} \int_{\mathcal{V}l} \mathcal{F}_l d\mathcal{V} \\
&= (1 - \phi_v) \langle \mathcal{F} \rangle_g + \phi_v \langle \mathcal{F} \rangle_l
\end{aligned} \tag{3.88}$$

We rewrite the final result of Eq. (3.88) as,

$$\langle \mathcal{F} \rangle = \frac{1}{\mathcal{V}} \int_{\mathcal{V}} \mathcal{F} d\mathcal{V} = (1 - \phi_v) \langle \mathcal{F} \rangle_g + \phi_v \langle \mathcal{F} \rangle_l \tag{3.89}$$

3.2.1 TKE for compressible gas-liquid flow with phase change, k

We integrate Eq. (3.17) over $\mathcal{V}_g$ and $\mathcal{V}_l$ respectively, use the identity $\boldsymbol{u}'' \cdot (\nabla \cdot \boldsymbol{\tau}) = \nabla \cdot (\boldsymbol{u}''\boldsymbol{\tau}) - \boldsymbol{\tau} : \nabla \boldsymbol{u}''$ and the divergence theorem to obtain,

$$\begin{aligned}
\int_{\mathcal{V}_g} \boldsymbol{u}_g'' \cdot \left(\frac{\partial}{\partial t} (\rho_g \boldsymbol{U}_g) + \nabla \cdot (\rho_g \boldsymbol{U}_g \boldsymbol{U}_g) \right) d\mathcal{V} = & \int_{\partial \mathcal{V}_g} \left(\boldsymbol{u}_g'' \boldsymbol{\tau} \right) \cdot \boldsymbol{n}_g d\mathcal{A} - \int_{\mathcal{V}_g} \boldsymbol{\tau} : \nabla \boldsymbol{u}_g'' d\mathcal{V} \\
& + \int_{\mathcal{V}_g} \rho_g \boldsymbol{f_b} \cdot \boldsymbol{u}_g'' d\mathcal{V}
\end{aligned} \tag{3.90a}$$

$$\begin{aligned}
\int_{\mathcal{V}l} \boldsymbol{u}_l'' \cdot \left(\frac{\partial}{\partial t} (\rho_l \boldsymbol{U}_l) + \nabla \cdot (\rho_l \boldsymbol{U}_l \boldsymbol{U}_l) \right) d\mathcal{V} = & \int_{\partial \mathcal{V}_l} (\boldsymbol{u}_l'' \boldsymbol{\tau}) \cdot \boldsymbol{n}_l d\mathcal{A} - \int_{\mathcal{V}l} \boldsymbol{\tau} : \nabla \boldsymbol{u}_l'' d\mathcal{V} \\
& + \int_{\mathcal{V}l} \rho_l \boldsymbol{f_b} \cdot \boldsymbol{u}_l'' d\mathcal{V}
\end{aligned} \tag{3.90b}$$

We add Eq. (3.90a) and Eq. (3.90b) and rearrange to get,

$$\int_{\mathcal{V}} \boldsymbol{u}'' \cdot \left(\frac{\partial}{\partial t}(\rho \boldsymbol{U}) + \nabla \cdot (\rho \boldsymbol{U}\boldsymbol{U}) \right) d\mathcal{V} = -\int_{\mathcal{V}} \boldsymbol{\tau} : \nabla \boldsymbol{u}'' d\mathcal{V} + \int_{\mathcal{V}} \rho \boldsymbol{f_b} \cdot \boldsymbol{u}'' d\mathcal{V} \\ \int_{\partial \mathcal{V}_g} \left(\boldsymbol{u}_g'' \boldsymbol{\tau} \right) \cdot \boldsymbol{n}_g d\mathcal{A} + \int_{\partial \mathcal{V}_l} \left(\boldsymbol{u}_l'' \boldsymbol{\tau} \right) \cdot \boldsymbol{n}_l d\mathcal{A} \quad (3.91)$$

We treat the surface integral terms in Eq. (3.91) by using the jump identity 1 from Eq. (2.21) with $\mathcal{F} = \boldsymbol{u}''\boldsymbol{\tau}$ as,

$$\int_{\partial \mathcal{V}} (\boldsymbol{u}''\boldsymbol{\tau}) \cdot \boldsymbol{n} d\mathcal{A} = \int_{\partial \mathcal{V}_g} \left(\boldsymbol{u}_g'' \boldsymbol{\tau} \right) \cdot \boldsymbol{n}_g d\mathcal{A} + \int_{\partial \mathcal{V}_l} \left(\boldsymbol{u}_l'' \boldsymbol{\tau} \right) \cdot \boldsymbol{n}_l d\mathcal{A} - \int_{\Sigma} [[\boldsymbol{u}''\boldsymbol{\tau}]] \cdot \boldsymbol{n} d\mathcal{A} \quad (3.92)$$

We use Eq. (3.92) to simplify Eq. (3.91) as,

$$\int_{\mathcal{V}} \boldsymbol{u}'' \cdot \left(\frac{\partial}{\partial t}(\rho \boldsymbol{U}) + \nabla \cdot (\rho \boldsymbol{U}\boldsymbol{U}) \right) d\mathcal{V} = -\int_{\mathcal{V}} \boldsymbol{\tau} : \nabla \boldsymbol{u}'' d\mathcal{V} + \int_{\mathcal{V}} \rho \boldsymbol{f_b} \cdot \boldsymbol{u}'' \\ \int_{\partial \mathcal{V}} (\boldsymbol{u}''\boldsymbol{\tau}) \cdot \boldsymbol{n} d\mathcal{A} + \int_{\Sigma} [[\boldsymbol{u}''\boldsymbol{\tau}]] \cdot \boldsymbol{n} d\mathcal{A} \quad (3.93)$$

We convert the surface integral terms on the right-hand side of Eq. (3.93) to volume integrals using the divergence theorem, and divide by $\mathcal{V}$ to obtain,

$$\frac{1}{\mathcal{V}} \int_{\mathcal{V}} \boldsymbol{u}'' \cdot \left(\frac{\partial}{\partial t}(\rho \boldsymbol{U}) + \nabla \cdot (\rho \boldsymbol{U}\boldsymbol{U}) \right) d\mathcal{V} = -\frac{1}{\mathcal{V}} \int_{\mathcal{V}} \boldsymbol{\tau} : \nabla \boldsymbol{u}'' d\mathcal{V} + \frac{1}{\mathcal{V}} \int_{\mathcal{V}} \rho \boldsymbol{f_b} \cdot \boldsymbol{u}'' d\mathcal{V} \\ \frac{1}{\mathcal{V}} \int_{\mathcal{V}} \nabla \cdot (\boldsymbol{u}''\boldsymbol{\tau}) \, d\mathcal{V} + \frac{1}{\mathcal{V}} \int_{\Sigma} [[\boldsymbol{u}''\boldsymbol{\tau}]] \cdot \boldsymbol{n} d\mathcal{A} \quad (3.94)$$

We use the definition of Reynolds averaging given in Eq. (3.2) to write Eq. (3.94) as,

$$\left\langle \boldsymbol{u}'' \cdot \left(\frac{\partial}{\partial t}(\rho \boldsymbol{U}) + \nabla \cdot (\rho \boldsymbol{U}\boldsymbol{U}) \right) \right\rangle = -\left\langle \boldsymbol{\tau} : \nabla \boldsymbol{u}'' \right\rangle + \left\langle \rho \boldsymbol{f_b} \cdot \boldsymbol{u}'' \right\rangle + \left\langle \nabla \cdot (\boldsymbol{u}''\boldsymbol{\tau}) \right\rangle \\ + \frac{1}{\mathcal{V}} \int_{\Sigma} [[\boldsymbol{u}''\boldsymbol{\tau}]] \cdot \boldsymbol{n} d\mathcal{A} \quad (3.95)$$

We simplify every term in Eq. (3.95) similar to the way we simplified Eq. (3.20), except for the jump term, and present the simplified equation as,

$$\overline{\rho} \frac{\widetilde{D} k}{\widetilde{D} t} + \mathcal{T}_{u''}'' = \mathcal{P} - \varepsilon'' + \Pi_{\Delta''} + \Gamma'' + \mathcal{T}_p'' + \mathcal{T}_\nu'' + \Pi_p + \Pi_\nu + \frac{1}{\mathcal{V}} \int_{\Sigma} [[\boldsymbol{u}''\boldsymbol{\tau}]] \cdot \boldsymbol{n} d\mathcal{A} \quad (3.96)$$

We define every term in Eq. (3.96) and relate it to its gas and liquid components using Eq. (3.89) as following. k is the TKE of the gas-liquid system and it is defined as,

$$k \equiv \frac{\left\langle \rho \frac{1}{2} u_i'' u_i'' \right\rangle}{\langle \rho \rangle} \tag{3.97}$$

The rate of change of TKE is related to its gas and liquid components using Eq. (3.89) as,

$$\overline{\rho} \frac{\widetilde{D} k}{\widetilde{D} t} = \left(1 - \phi_v\right) \overline{\rho}_g \frac{\widetilde{D} k_g}{\widetilde{D} t} + \phi_v \overline{\rho}_l \frac{\widetilde{D} k_l}{\widetilde{D} t} \tag{3.98}$$

$\mathcal{P}$ is the production of TKE which acts as a source of k. It is defined and related to $\mathcal{P}_g$ and $\mathcal{P}_l$ as,

$$\mathcal{P} \equiv - \left\langle \rho u_i'' u_j'' \right\rangle \partial_j \widetilde{U}_i = \left(1 - \phi_v\right) \mathcal{P}_g + \phi_v \mathcal{P}_l \tag{3.99}$$

ε'' is the rate of viscous dissipation of k, which acts as a sink. It is defined and related to ε_g'' and ε_l'' as,

$$\varepsilon'' \equiv \left\langle T_{ij}' \partial_j u_i'' \right\rangle = \left(1 - \phi_v\right) \varepsilon_g'' + \phi_v \varepsilon_l'' \tag{3.100}$$

$\Pi_{\Delta''}$ is the fluctuating pressure-dilatation correlation term which could act as a source or sink of k. It is defined and related to $\Pi_{\Delta'',g}$ and $\Pi_{\Delta'',l}$ as,

$$\Pi_{\Delta''} \equiv \langle p' \partial_i u_i'' \rangle = \left(1 - \phi_v\right) \Pi_{\Delta'',g} + \phi_v \Pi_{\Delta'',l} \tag{3.101}$$

Γ'' is the fluctuating component of the power of body force for the gas-liquid system and can act as a source or sink of k. It is defined and related to Γ_g'' and Γ_l'' as,

$$\Gamma'' \equiv \langle \rho g_i u_i'' \rangle = \left(1 - \phi_v\right) \Gamma_g'' + \phi_v \Gamma_l'' \tag{3.102}$$

$\mathcal{T}_p''$ and $\mathcal{T}_\nu''$ are the fluctuating component of pressure power and fluctuating component of viscous power of the gas-liquid system respectively and are defined as,

$$\mathcal{T}_p'' \equiv -\partial_i \langle p' u_i'' \rangle \tag{3.103a}$$

$$\mathcal{T}_\nu'' \equiv \partial_j \left\langle T_{ij}' u_i'' \right\rangle \tag{3.103b}$$

Π_p is the enthalpic production of k in the gas-liquid system or simply, the rate of pressure work and could act as a source or sink of k. Π_ν can analogously be called rate of reversible viscous work and could act as a source or sink of k. They are defined as,

$$\Pi_p \equiv (-\partial_i \overline{p}) \langle u_i'' \rangle \tag{3.104a}$$

$$\Pi_\nu \equiv \left(\partial_j \overline{T}_{ij}\right) \langle u_i'' \rangle \tag{3.104b}$$

The jump term in Eq. (3.96) is further simplified using Eq. (2.17) and Eq. (2.59), and Eq. (2.53) as,

$$\begin{aligned}
[[\boldsymbol{u}''\boldsymbol{\tau}]] \cdot \boldsymbol{n} &= \left(\boldsymbol{u}_g''\boldsymbol{\tau}\right) \cdot \boldsymbol{n}_g - \left(\boldsymbol{u}_l''\boldsymbol{\tau}\right) \cdot \boldsymbol{n}_l \\
&= \left(\boldsymbol{\tau}^T \boldsymbol{n}_g\right) \cdot \boldsymbol{u}_g'' - \left(\boldsymbol{\tau}^T \boldsymbol{n}_l\right) \cdot \boldsymbol{u}_l'' \\
&= \left(\boldsymbol{\tau} \boldsymbol{n}_g\right) \cdot \boldsymbol{u}_g'' - \left(\boldsymbol{\tau} \boldsymbol{n}_l\right) \cdot \boldsymbol{u}_l'' \\
&= \left(\boldsymbol{\tau} \boldsymbol{n}_g\right) \cdot \left(\boldsymbol{u}_\Sigma'' + \boldsymbol{u}_g'' - \boldsymbol{u}_\Sigma''\right) - \left(\boldsymbol{\tau} \boldsymbol{n}_l\right) \cdot \left(\boldsymbol{u}_\Sigma'' + \boldsymbol{u}_l'' - \boldsymbol{u}_\Sigma''\right) \\
&= \left(\boldsymbol{\tau} \boldsymbol{n}_g\right) \cdot \boldsymbol{u}_\Sigma'' - \left(\boldsymbol{\tau} \boldsymbol{n}_l\right) \cdot \boldsymbol{u}_\Sigma'' + \left(\boldsymbol{\tau} \boldsymbol{n}_g\right) \cdot \left(\boldsymbol{u}_g'' - \boldsymbol{u}_\Sigma''\right) - \left(\boldsymbol{\tau} \boldsymbol{n}_l\right) \cdot \left(\boldsymbol{u}_l'' - \boldsymbol{u}_\Sigma''\right) \\
&= [[\boldsymbol{\tau}]]\, \boldsymbol{n} \cdot \boldsymbol{u}_\Sigma'' - [[(\boldsymbol{\tau}\boldsymbol{n}) \cdot (\boldsymbol{u}_\Sigma'' - \boldsymbol{u}'')]] \\
&= \kappa\sigma \boldsymbol{n} \cdot \boldsymbol{u}_\Sigma'' + (\dot{m}'')^2 \left(\frac{1}{\rho_g} - \frac{1}{\rho_l}\right) \boldsymbol{n} \cdot \boldsymbol{u}_\Sigma'' - [[(\boldsymbol{\tau}\boldsymbol{n}) \cdot (\boldsymbol{u}_\Sigma'' - \boldsymbol{u}'')]]
\end{aligned} \tag{3.105}$$

We rewrite the final result of Eq. (3.105) as,

$$[[\boldsymbol{u}''\boldsymbol{\tau}]] \cdot \boldsymbol{n} = \kappa\sigma \boldsymbol{n} \cdot \boldsymbol{u}_\Sigma'' + (\dot{m}'')^2 \left(\frac{1}{\rho_g} - \frac{1}{\rho_l}\right) \boldsymbol{n} \cdot \boldsymbol{u}_\Sigma'' - [[(\boldsymbol{\tau}\boldsymbol{n}) \cdot (\boldsymbol{u}_\Sigma'' - \boldsymbol{u}'')]] \tag{3.106}$$

We integrate the terms in Eq. (3.106) over Σ and divide with $\mathcal{V}$ to obtain,

$$\begin{aligned}
\frac{1}{\mathcal{V}} \int_\Sigma [[\boldsymbol{u}''\boldsymbol{\tau}]] \cdot \boldsymbol{n} d\mathcal{A} = \frac{1}{\mathcal{V}} \int_\Sigma \kappa\sigma \boldsymbol{n} \cdot \boldsymbol{u}_\Sigma'' d\mathcal{A} + \frac{1}{\mathcal{V}} \int_\Sigma (\dot{m}'')^2 \left(\frac{1}{\rho_g} - \frac{1}{\rho_l}\right) \boldsymbol{n} \cdot \boldsymbol{u}_\Sigma'' d\mathcal{A} \\
- \frac{1}{\mathcal{V}} \int_\Sigma [[(\boldsymbol{\tau}\boldsymbol{n}) \cdot (\boldsymbol{u}_\Sigma'' - \boldsymbol{u}'')]]\, d\mathcal{A}
\end{aligned} \tag{3.107}$$

We transform the first two surface integrals on the right-hand side of Eq. (3.107) to volume integral terms as,

$$\frac{1}{\mathcal{V}} \int_\Sigma \kappa\sigma \boldsymbol{n} \cdot \boldsymbol{u}_\Sigma'' d\mathcal{A} = \frac{1}{\mathcal{V}} \int_\mathcal{V} \boldsymbol{f}_\sigma \cdot \boldsymbol{u}_\Sigma'' d\mathcal{V} = \langle \boldsymbol{f}_\sigma \cdot \boldsymbol{u}_\Sigma'' \rangle \equiv \Psi_\sigma'' \tag{3.108}$$

where $\boldsymbol{f}_\sigma$ is the force due to surface tension and Ψ''_σ is the power of surface tension due to fluctuating velocity. It can act as a source or sink or k.

$$\frac{1}{\mathcal{V}}\int_\Sigma (\dot{m}'')^2\left(\frac{1}{\rho_g}-\frac{1}{\rho_l}\right)\boldsymbol{n}\cdot\boldsymbol{u}''d\mathcal{A}=\frac{1}{\mathcal{V}}\int_\mathcal{V}\boldsymbol{f}_{\dot{m}}\cdot\boldsymbol{u}''_\Sigma d\mathcal{V}=\langle\boldsymbol{f}_{\dot{m}}\cdot\boldsymbol{u}''_\Sigma\rangle\equiv\Psi''_{\dot{m}} \tag{3.109}$$

where $\boldsymbol{f}_{\dot{m}}$ is the force due to mass flux and $\Psi''_{\dot{m}}$ is the power of mass flux due to fluctuating velocity. It can act as the source of k in case of evaporation and sink in case of condensation. The third term on the right-hand side of Eq. (3.107) is is the power of momentum flux due to fluctuating velocity which is the fluctuating part of the compressible work done by the fluid stress tensor due of phase change and is defined as,

$$\Psi''_{\dot{I}}\equiv-\frac{1}{\mathcal{V}}\int_\Sigma\left[\left[(\boldsymbol{\tau}\boldsymbol{n})\cdot(\boldsymbol{u}''_\Sigma-\boldsymbol{u}'')\right]\right]d\mathcal{A} \tag{3.110}$$

It can act as the source of k in case of evaporation and sink in case of condensation. We combine Eq. (3.93), Eq. (3.108), Eq. (3.109) and Eq. (3.110) to obtain,

$$\frac{1}{\mathcal{V}}\int_\Sigma[[\boldsymbol{u}''\boldsymbol{\tau}]]\cdot\boldsymbol{n}d\mathcal{A}=\Psi''_\sigma+\Psi''_{\dot{m}}+\Psi''_{\dot{I}} \tag{3.111}$$

We use Eq. (2.21) and Eq. (3.89) to relate $\mathcal{T}''_p$, $\mathcal{T}''_\nu$, Π_p and Π_ν to their gas and liquid components as,

$$\begin{aligned}\mathcal{T}''_p+\mathcal{T}''_\nu+\Pi_p+\Pi_\nu+\Psi''_\sigma+\Psi''_{\dot{m}}+\Psi''_{\dot{I}}=(1-\phi_v)\left(\mathcal{T}''_{p,g}+\mathcal{T}''_{\nu,g}+\Pi_{p,g}+\Pi_{\nu,g}\right)\\+\phi_v\left(\mathcal{T}''_{p,l}+\mathcal{T}''_{\nu,l}+\Pi_{p,l}+\Pi_{\nu,l}\right)\end{aligned} \tag{3.112}$$

We use Eq. (3.111) and rewrite Eq. (3.96) as the TKE evolution equation for a compressible gas-liquid flow with phase change:

$$\bar{\rho}\frac{\widetilde{D}k}{\widetilde{D}t}+\mathcal{T}''_{u''}=\mathcal{P}-\varepsilon''+\Pi_{\Delta''}+\Gamma''+\mathcal{T}''_p+\mathcal{T}''_\nu+\Pi_p+\Pi_\nu+\Psi''_\sigma+\Psi''_{\dot{m}}+\Psi''_{\dot{I}} \tag{3.113}$$

3.2.2 MFKE for compressible gas-liquid flow with phase change, $\widetilde{E}$

We use a methodology similar to the one used in reducing Eq. (3.90) to Eq. (3.95) by replacing $\boldsymbol{u}''$ with $\widetilde{\boldsymbol{U}}$ to obtain the equation for MFKE of the gas-liquid system:

$$\left\langle \widetilde{\boldsymbol{U}} \cdot \left(\frac{\partial}{\partial t} (\rho \boldsymbol{U}) + \nabla \cdot (\rho \boldsymbol{U} \boldsymbol{U}) \right) \right\rangle = - \left\langle \boldsymbol{\tau} : \nabla \widetilde{\boldsymbol{U}} \right\rangle + \left\langle \rho \boldsymbol{f_b} \cdot \widetilde{\boldsymbol{U}} \right\rangle + \left\langle \nabla \cdot \left(\widetilde{\boldsymbol{U}} \boldsymbol{\tau} \right) \right\rangle + \frac{1}{\mathcal{V}} \int_{\Sigma} \left[\left[\widetilde{\boldsymbol{U}} \boldsymbol{\tau} \right] \right] \cdot \boldsymbol{n} d\mathcal{A} \tag{3.114}$$

We simplify every term in Eq. (3.114) similar to the way we simplified Eq. (3.49), except for the jump term, and present the simplified equation as,

$$\overline{\rho} \frac{\widetilde{D} \widetilde{E}}{\widetilde{D} t} + \widetilde{\mathcal{T}}_{\widetilde{U}} = -\mathcal{P} - \widetilde{\varepsilon} + \Pi_{\widetilde{\Delta}} + \widetilde{\Gamma} + \widetilde{\mathcal{T}}_p + \widetilde{\mathcal{T}}_\nu + \frac{1}{\mathcal{V}} \int_{\Sigma} \left[\left[\widetilde{\boldsymbol{U}} \boldsymbol{\tau} \right] \right] \cdot \boldsymbol{n} d\mathcal{A} \tag{3.115}$$

We define every term in Eq. (3.115) and relate it to its gas and liquid components using Eq. (3.89) as following. $\widetilde{E}$ is the MFKE of the gas-liquid system and it is defined as,

$$\widetilde{E} \equiv \frac{1}{2} \widetilde{U}_i \widetilde{U}_i \tag{3.116}$$

The rate of change of MFKE is related to its gas and liquid components using Eq. (3.89) as,

$$\overline{\rho} \frac{\widetilde{D} \widetilde{E}}{\widetilde{D} t} = (1 - \phi_v) \overline{\rho}_g \frac{\widetilde{D} \widetilde{E}_g}{\widetilde{D} t} + \phi_v \overline{\rho}_l \frac{\widetilde{D} \widetilde{E}_l}{\widetilde{D} t} \tag{3.117}$$

$\mathcal{P}$ is the production of k and is defined already in Eq. (3.99). $\widetilde{\varepsilon}$ is the rate of viscous dissipation of $\widetilde{E}$ which acts as a sink. It is defined and related to $\widetilde{\varepsilon}_g$ and $\widetilde{\varepsilon}_l$ as,

$$\widetilde{\varepsilon} \equiv \overline{T}_{ij} \partial_j \widetilde{U}_i = (1 - \phi_v) \widetilde{\varepsilon}_g + \phi_v \widetilde{\varepsilon}_l \tag{3.118}$$

$\Pi_{\widetilde{\Delta}}$ is the mean component of pressure-dilatation correlation term which could act as a source or sink of $\widetilde{E}$. It is defined and related to $\Pi_{\widetilde{\Delta},g}$ and $\Pi_{\widetilde{\Delta},l}$ as,

$$\Pi_{\widetilde{\Delta}} \equiv \overline{p} \partial_i \widetilde{U}_i = (1 - \phi_v) \Pi_{\widetilde{\Delta},g} + \phi_v \Pi_{\widetilde{\Delta},l} \tag{3.119}$$

$\widetilde{\Gamma}$ is the mean component of power of body force for the gas-liquid system and can act as a source or sink of $\widetilde{E}$. It is defined and related to $\widetilde{\Gamma}_g$ and $\widetilde{\Gamma}_l$ as,

$$\widetilde{\Gamma} \equiv \rho g_i \widetilde{U}_i = (1 - \phi_v) \widetilde{\Gamma}_g + \phi_v \widetilde{\Gamma}_l \tag{3.120}$$

$\widetilde{\mathcal{T}}_p$ and $\widetilde{\mathcal{T}}_\nu$ are the fluctuating component of pressure power and fluctuating component of viscous power of the gas-liquid system respectively and are defined as,

$$\widetilde{\mathcal{T}}_p \equiv -\partial_i \left(\overline{p} \widetilde{U}_i \right) \tag{3.121a}$$

$$\widetilde{\mathcal{T}}_\nu \equiv \partial_j \left(\overline{T}_{ij} \widetilde{U}_i \right) \tag{3.121b}$$

The jump term in Eq. (3.115) is further simplified similar to the way Eq. (3.96) was reduced to Eq. (3.122), by replacing $\boldsymbol{u}''$ with $\widetilde{\boldsymbol{U}}$. We present the final form of the jump term as,

$$\frac{1}{\mathcal{V}} \int_\Sigma \left[\left[\widetilde{\boldsymbol{U}} \boldsymbol{\tau} \right] \right] \cdot \boldsymbol{n} d\mathcal{A} = \widetilde{\Psi}_\sigma + \widetilde{\Psi}_{\dot{m}} + \widetilde{\Psi}_{\dot{I}} \tag{3.122}$$

where $\widetilde{\Psi}_\sigma$ is the power of surface tension force, $\boldsymbol{f}_\sigma$, due to mean velocity $\widetilde{\boldsymbol{U}}$. It can act as a source or sink of $\widetilde{E}$

$$\frac{1}{\mathcal{V}} \int_\Sigma \kappa \sigma \boldsymbol{n} \cdot \widetilde{\boldsymbol{U}}_\Sigma d\mathcal{A} = \frac{1}{\mathcal{V}} \int_\mathcal{V} \boldsymbol{f}_\sigma \cdot \widetilde{\boldsymbol{U}}_\Sigma d\mathcal{V} = \left\langle \boldsymbol{f}_\sigma \cdot \widetilde{\boldsymbol{U}}_\Sigma \right\rangle \equiv \widetilde{\Psi}_\sigma \tag{3.123}$$

$\widetilde{\Psi}_{\dot{m}}$ is the power of force caused by mass flux due to phase change, $\boldsymbol{f}_{\dot{m}}$, and mean velocity $\widetilde{\boldsymbol{U}}$. It can act as a source or sink of $\widetilde{E}$ It can act as the source of $\widetilde{E}$ in case of evaporation and sink in case of condensation.

$$\frac{1}{\mathcal{V}} \int_\Sigma (\dot{m}'')^2 \left(\frac{1}{\rho_g} - \frac{1}{\rho_l} \right) \boldsymbol{n} \cdot \widetilde{\boldsymbol{U}} d\mathcal{A} = \frac{1}{\mathcal{V}} \int_\mathcal{V} \boldsymbol{f}_{\dot{m}} \cdot \widetilde{\boldsymbol{U}}_\Sigma d\mathcal{V} = \left\langle \boldsymbol{f}_{\dot{m}} \cdot \widetilde{\boldsymbol{U}}_\Sigma \right\rangle \equiv \widetilde{\Psi}_{\dot{m}} \tag{3.124}$$

$\widetilde{\Psi}_{\dot{I}}$ is the power of momentum flux due to mean velocity which is the mean part of the compressible work done by the fluid stress tensor due of phase change and is defined as,

$$\widetilde{\Psi}_{\dot{I}} \equiv -\frac{1}{\mathcal{V}} \int_\Sigma \left[\left[(\boldsymbol{\tau n}) \cdot \left(\widetilde{\boldsymbol{U}}_\Sigma - \widetilde{\boldsymbol{U}} \right) \right] \right] d\mathcal{A} \tag{3.125}$$

It can act as the source of $\widetilde{E}$ in case of evaporation and sink in case of condensation. We use Eq. (2.21) and Eq. (3.89) to relate $\widetilde{\mathcal{T}}_p$, $\widetilde{\mathcal{T}}_\nu$ to their gas and liquid components as,

$$\begin{aligned} \widetilde{\mathcal{T}}_p + \widetilde{\mathcal{T}}_\nu + \widetilde{\Psi}_\sigma + \widetilde{\Psi}_{\dot{m}} + \widetilde{\Psi}_{\dot{I}} = (1 - \phi_v) \left(\widetilde{\mathcal{T}}_{p,g} + \widetilde{\mathcal{T}}_{\nu,g} \right) \\ + \phi_v \left(\widetilde{\mathcal{T}}_{p,l} + \widetilde{\mathcal{T}}_{\nu,l} \right) \end{aligned} \tag{3.126}$$

We use Eq. (3.122) and rewrite Eq. (3.115) as the MFKE evolution equation for a compressible gas-liquid flow with phase change:

$$\overline{\rho}\frac{\widetilde{D}\widetilde{E}}{\widetilde{D}t} + \widetilde{\mathcal{T}}_{u''} = -\mathcal{P} - \widetilde{\varepsilon} + \Pi_{\widetilde{\Delta}} + \widetilde{\Gamma} + \widetilde{\mathcal{T}}_p + \widetilde{\mathcal{T}}_\nu + \widetilde{\Psi}_\sigma + \widetilde{\Psi}_{\dot{m}} + \widetilde{\Psi}_{\dot{I}} \tag{3.127}$$

3.2.3 MIE for compressible gas-liquid flow with phase change, $\widetilde{e}$

We start by integrating the internal energy balance equation obtained in Eq. (3.67) over $\mathcal{V}_g$ and $\mathcal{V}_l$ to obtain,

$$\int_{\mathcal{V}_g} \left(\frac{\partial}{\partial t}(\rho_g e_g) + \nabla\cdot(\rho_g e_g \boldsymbol{U}_g)\right) d\mathcal{V} = \int_{\mathcal{V}_g} \boldsymbol{\tau} : \nabla \boldsymbol{U}_g d\mathcal{A} - \int_{\mathcal{V}_g} \nabla\cdot \boldsymbol{q} d\mathcal{A} \tag{3.128a}$$

$$\int_{\mathcal{V}_l} \left(\frac{\partial}{\partial t}(\rho_l e_l) + \nabla\cdot(\rho_l e_l \boldsymbol{U}_l)\right) d\mathcal{V} = \int_{\mathcal{V}_l} \boldsymbol{\tau} : \nabla \boldsymbol{U}_l d\mathcal{A} - \int_{\mathcal{V}_l} \nabla\cdot \boldsymbol{q} d\mathcal{A} \tag{3.128b}$$

We use the divergence theorem to convert the volume integral on the heat flux to a surface integral term to account for conduction through the interface as,

$$\int_{\mathcal{V}_g} \left(\frac{\partial}{\partial t}(\rho_g e_g) + \nabla\cdot(\rho_g e_g \boldsymbol{U}_g)\right) d\mathcal{V} = \int_{\mathcal{V}_g} \boldsymbol{\tau} : \nabla \boldsymbol{U}_g d\mathcal{A} - \int_{\partial\mathcal{V}_g} \boldsymbol{q}\cdot\boldsymbol{n}_g d\mathcal{A} \tag{3.129a}$$

$$\int_{\mathcal{V}_l} \left(\frac{\partial}{\partial t}(\rho_l e_l) + \nabla\cdot(\rho_l e_l \boldsymbol{U}_l)\right) d\mathcal{V} = \int_{\mathcal{V}_l} \boldsymbol{\tau} : \nabla \boldsymbol{U}_l d\mathcal{A} - \int_{\partial\mathcal{V}_l} \boldsymbol{q}\cdot\boldsymbol{n}_l d\mathcal{A} \tag{3.129b}$$

We add both the sub-equations in Eq. (3.129) to obtain,

$$\int_{\mathcal{V}} \left(\frac{\partial}{\partial t}(\rho e) + \nabla\cdot(\rho e \boldsymbol{U})\right) d\mathcal{V} = \int_{\mathcal{V}} \boldsymbol{\tau} : \nabla \boldsymbol{U} d\mathcal{A} - \int_{\partial\mathcal{V}_g} \boldsymbol{q}\cdot\boldsymbol{n}_g d\mathcal{A} - \int_{\partial\mathcal{V}_l} \boldsymbol{q}\cdot\boldsymbol{n}_l d\mathcal{A} \tag{3.130}$$

where the last two terms on the right-hand side of Eq. (3.130) are the rate of heat conduction into the gas and liquid phases respectively. We use Eq. (2.61) to simplify Eq. (3.130) as,

$$\int_{\mathcal{V}} \left(\frac{\partial}{\partial t}(\rho e) + \nabla\cdot(\rho e \boldsymbol{U})\right) d\mathcal{V} = \int_{\mathcal{V}} \boldsymbol{\tau} : \nabla \boldsymbol{U} d\mathcal{A} - \int_{\partial\mathcal{V}} \boldsymbol{q}\cdot\boldsymbol{n} d\mathcal{A} - \int_{\Sigma} [[\boldsymbol{q}]]\cdot\boldsymbol{n} d\mathcal{A} \tag{3.131}$$

We use the divergence theorem to convert the surface integral term to a volume integral, and divide by $\mathcal{V}$ to write Eq. (3.131) as,

$$\frac{1}{\mathcal{V}}\int_{\mathcal{V}} \left(\frac{\partial}{\partial t}(\rho e) + \nabla\cdot(\rho e \boldsymbol{U})\right) d\mathcal{V} = \frac{1}{\mathcal{V}}\int_{\mathcal{V}} \boldsymbol{\tau} : \nabla \boldsymbol{U} d\mathcal{A} - \frac{1}{\mathcal{V}}\int_{\mathcal{V}} \nabla\cdot \boldsymbol{q} d\mathcal{V} - \frac{1}{\mathcal{V}}\int_{\Sigma} [[\boldsymbol{q}]]\cdot\boldsymbol{n} d\mathcal{A} \tag{3.132}$$

We use the definition of Reynolds averaging given in Eq. (3.2) to write Eq. (3.132) as,

$$\left\langle \frac{\partial}{\partial t}(\rho e) + \nabla\cdot\,(\rho e \boldsymbol{U}) \right\rangle = \ \langle \boldsymbol{\tau} : \nabla \boldsymbol{U} \rangle - \langle \nabla\cdot\, \boldsymbol{q} \rangle - \frac{1}{\mathcal{V}} \int_{\Sigma} [[\boldsymbol{q}]] \cdot \boldsymbol{n} d\mathcal{A} \tag{3.133}$$

We simplify every term in Eq. (3.133) similar to the way Eq. (3.69) was reduced to Eq. (3.85), except for the jump term and write it as,

$$\overline{\rho}\frac{\widetilde{D}e}{\widetilde{D}t} + \mathcal{T}^*_{u''} = \varepsilon'' + \widetilde{\varepsilon} - \Pi_{\Delta''} - \Pi_{\widetilde{\Delta}} + \mathcal{T}^*_p + \mathcal{T}^*_\nu - \Pi_p - \Pi_\nu + \dot{\mathcal{Q}} + \dot{\mathcal{Q}}_\Sigma \tag{3.134}$$

where $\dot{\mathcal{Q}}$ is the net rate at which a portion of the total heat that is conducted into the gas-liquid system is used to cause temperature gradients in the gas-liquid system, and it is defined as,

$$\dot{\mathcal{Q}} \equiv - \int_{\mathcal{V}} \nabla\cdot\, \boldsymbol{q} d\mathcal{V} = - \int_{\partial\mathcal{V}} \boldsymbol{q} \cdot \boldsymbol{n} d\mathcal{A} \tag{3.135}$$

and $\dot{\mathcal{Q}}_\Sigma$ is the rate at which heat is being conducted from one phase to the other through Σ to cause a phase change and we call it the power of heat flux. $\dot{\mathcal{Q}}_\Sigma$ acts as a source of $\widetilde{e}$ in case of evaporation and as a sink in case of condensation. We define $\dot{\mathcal{Q}}_\Sigma$ as,

$$\dot{\mathcal{Q}}_\Sigma \equiv - \int_{\Sigma} [[\boldsymbol{q}]] \cdot \boldsymbol{n} d\mathcal{A} \tag{3.136}$$

We find that $\dot{\mathcal{Q}}$, $\dot{\mathcal{Q}}_\Sigma$ can be related to $\dot{\mathcal{Q}}_g$ and $\dot{\mathcal{Q}}_l$ using Eq. (3.89) as,

$$\dot{\mathcal{Q}} + \dot{\mathcal{Q}}_\Sigma = (1 - \phi_v)\,\dot{\mathcal{Q}}_g + \phi_v \dot{\mathcal{Q}}_l \tag{3.137}$$

$\dot{\mathcal{Q}}_\Sigma$ can be further analyzed using the jump condition for energy given in Eq. (2.67) which is re-arranged as,

$$- \int_{\Sigma} [[\boldsymbol{q}]] \cdot \boldsymbol{n} d\mathcal{A} = - \int_{\Sigma} [[\boldsymbol{\tau} \boldsymbol{U}]] \cdot \boldsymbol{n} d\mathcal{A} + \int_{\Sigma} \kappa\sigma \boldsymbol{n} \cdot \boldsymbol{U}_\Sigma d\mathcal{A} - \ \int_{\Sigma} \dot{m}'' \left[\left[(e + \frac{1}{2}\boldsymbol{U}\cdot\boldsymbol{U}) \right]\right] d\mathcal{A} \tag{3.138}$$

We realize that the second term on right-hand side of Eq. (3.139) is the power of surface tension Ψ_σ. We add Eq. (3.111) and Eq. (3.122) and realize that,

$$\frac{1}{\mathcal{V}} \int_{\Sigma} [[\boldsymbol{\tau} \boldsymbol{U}]] \cdot \boldsymbol{n} d\mathcal{A} = \Psi_\sigma + \Psi_{\dot{m}} + \Psi_{\dot{I}} \tag{3.139}$$

We use Eq. (3.139) to simplify Eq. (3.138) as,

$$-\int_{\Sigma}[[\boldsymbol{q}]]\cdot\boldsymbol{n}d\mathcal{A} = -\Psi_{\dot{m}} - \Psi_{\dot{I}} - \int_{\Sigma}\dot{m}''\left[\left[(e+\frac{1}{2}\boldsymbol{U}\cdot\boldsymbol{U})\right]\right]d\mathcal{A} \tag{3.140}$$

We rearrange the terms in Eq. (3.140) to obatin,

$$\int_{\Sigma}\dot{m}''\left[\left[(e+\frac{1}{2}\boldsymbol{U}\cdot\boldsymbol{U})\right]\right]d\mathcal{A} = -\Psi_{\dot{m}} - \Psi_{\dot{I}} - \dot{\mathcal{Q}}_{\Sigma}, \tag{3.141}$$

where the first term on the left-hand side of Eq. (3.141) is the rate of change of internal energy of the species due to phase change, and the second term on the left-hand side is the rate of change of kinetic energy of the species during phase change. The terms on the right-hand side of Eq. (3.141) are the powers of mass flux, momentum flux and heat flux respectively. We infer that the left-hand side of Eq. (3.141) is the net rate of change of total energy of the gas-liquid system due to phase change and we define it as Φ, the phase change potential. Φ can also be thought as the rate of change of the free energy due to phase change and is defined as,

$$\Phi \equiv \int_{\Sigma}\dot{m}''\left[\left[(e+\frac{1}{2}\boldsymbol{U}\cdot\boldsymbol{U})\right]\right]d\mathcal{A} \tag{3.142}$$

We use Eq. (3.141), Eq. (3.142) and Eq. (3.136), to write an equation for Φ:

$$\Phi = -\Psi_{\dot{m}} - \Psi_{\dot{I}} - \dot{\mathcal{Q}}_{\Sigma} \tag{3.143}$$

3.2.4 Energy pathways for compressible gas-liquid flow with phase change

For clarity, equations for TKE, MFKE, MIE for compressible gas-liquid flow with phase change are given together as,

$$\overline{\rho}\frac{\widetilde{D}k}{\widetilde{D}t} + \mathcal{T}''_{u''} = \mathcal{P} - \varepsilon'' + \Pi_{\Delta''} + \Gamma'' + \mathcal{T}''_{p} + \mathcal{T}''_{\nu} + \Pi_{p} + \Pi_{\nu} + \Psi''_{\sigma} + \Psi''_{\dot{m}} + \Psi''_{\dot{I}} \tag{3.144a}$$

$$\overline{\rho}\frac{\widetilde{D}\widetilde{E}}{\widetilde{D}t} + \widetilde{\mathcal{T}}_{u''} = -\mathcal{P} - \widetilde{\varepsilon} + \Pi_{\widetilde{\Delta}} + \widetilde{\Gamma} + \widetilde{\mathcal{T}}_{p} + \widetilde{\mathcal{T}}_{\nu} + \widetilde{\Psi}_{\sigma} + \widetilde{\Psi}_{\dot{m}} + \widetilde{\Psi}_{\dot{I}} \tag{3.144b}$$

$$\overline{\rho}\frac{\widetilde{D}e}{\widetilde{D}t} + \mathcal{T}^{*}_{u''} = \varepsilon'' + \widetilde{\varepsilon} - \Pi_{\Delta''} - \Pi_{\widetilde{\Delta}} + \mathcal{T}^{*}_{p} + \mathcal{T}^{*}_{\nu} - \Pi_{p} - \Pi_{\nu} + \dot{\mathcal{Q}} + \dot{\mathcal{Q}}_{\Sigma} \tag{3.144c}$$

TKE and MFKE are modulated due to surface deformations of the gas-liquid interface, Σ. This modulated energy is stored or expended as the interfacial surface energy, denoted by Θ and is equivalent to the power of surface tension,

$$\Theta \equiv -\widetilde{\Psi}_{\sigma} - \Psi''_{\sigma} \tag{3.145}$$

Phase change free-energy acts as a source or sink of TKE, MFKE and IE depending on the direction of phase change and is re-written from Eq. (3.143) as,

$$\Phi = -\Psi_{\dot{m}} - \Psi_{\dot{I}} - \dot{\mathcal{Q}}_{\Sigma} \tag{3.146}$$

We present a schematic showcasing the pathways of energy exchange between k, $\widetilde{E}$, $\widetilde{e}$, Θ and Φ in Fig. 3.2.

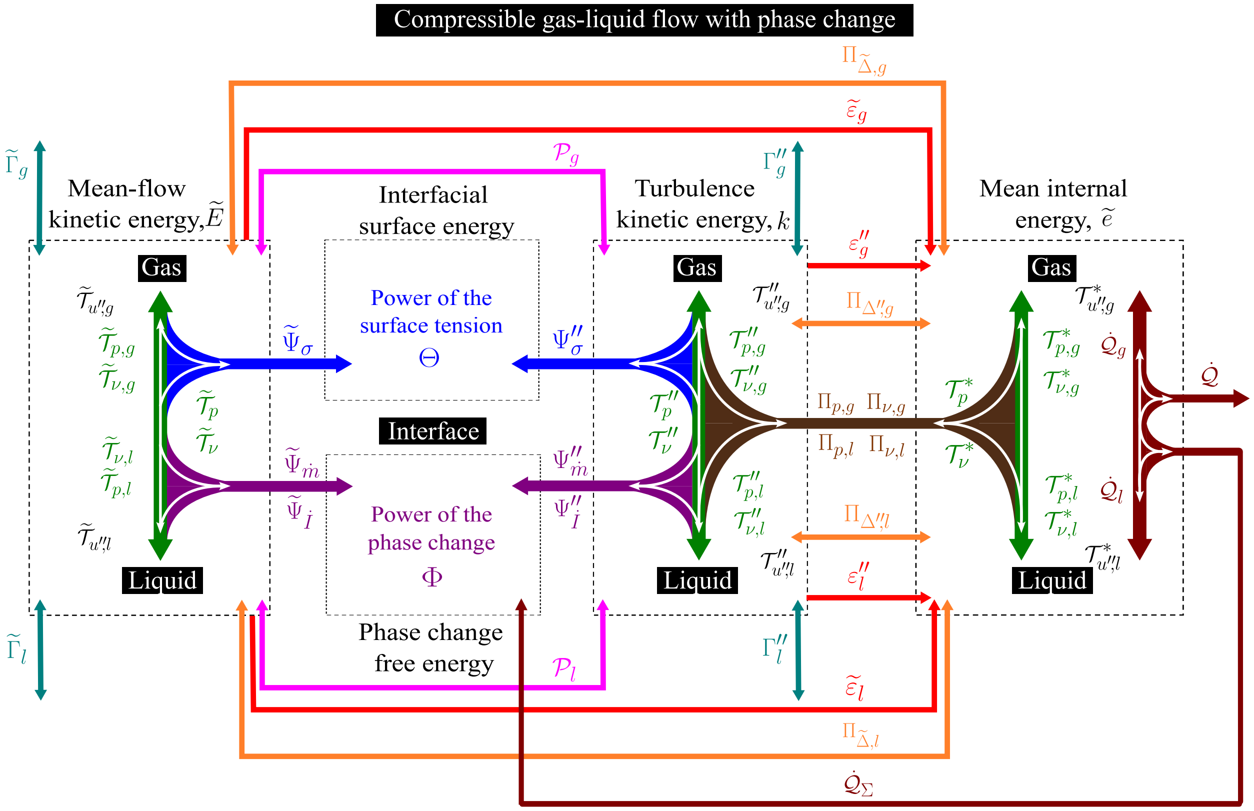

Figure 3.2: Schematic showing the pathways for energy exchange for compressible gas-liquid flow with phase change, which summarizes Eq. (3.144), Eq. (3.145) and Eq. (3.146).

3.3 Droplet-laden homogeneous shear turbulence with phase change

In this section, we reduce the general form of the energy equations derived in Sec. 3.2 for a specific case of turbulent gas-liquid flow with phase change: droplet-laden homogeneous shear turbulence with phase change (DLHST-PC). We realize that due to the homogeneous nature of the flow, averaged quantities are uniform in space but could vary in time. We first reduce the material derivative operator with Favre-averaged mean velocity given in Eq. (3.44) to a time derivative as,

$$\begin{aligned}\overline{\rho}\frac{\widetilde{D}\left\langle \mathcal{F}\right\rangle}{\widetilde{D}t} &= \overline{\rho}\left(\frac{\partial\left\langle \mathcal{F}\right\rangle}{\partial t}+\widetilde{\boldsymbol{U}}\cdot\nabla\left\langle \mathcal{F}\right\rangle\right)\\ &= \frac{\partial\left(\overline{\rho}\left\langle \mathcal{F}\right\rangle\right)}{\partial t}+\underbrace{\nabla\cdot\left\langle\overline{\rho}\widetilde{\boldsymbol{U}}\mathcal{F}\right\rangle}_{=0}\\ &= \frac{\mathrm{d}\left(\overline{\rho}\left\langle \mathcal{F}\right\rangle\right)}{\mathrm{d}t}\end{aligned} \tag{3.147}$$

We rewrite the final result of Eq. (3.147) as,

$$\overline{\rho}\frac{\widetilde{D}\left\langle \mathcal{F}\right\rangle}{\widetilde{D}t} = \frac{\mathrm{d}\left(\overline{\rho}\left\langle \mathcal{F}\right\rangle\right)}{\mathrm{d}t} \tag{3.148}$$

In case of N_d droplets, $\mathcal{V}_l$ consists of N_d droplet volumes $\mathcal{V}_l^{(n)}$. Similarly, the interface Σ is composed of the interfaces of all the droplets $\Sigma^{(n)}$ such that,

$$\mathcal{V}_l = \sum_{n=0}^{N_d}\mathcal{V}_l^{(n)} \tag{3.149a}$$

$$\Sigma = \sum_{n=0}^{N_d}\Sigma^{(n)} \tag{3.149b}$$

We define the Reynolds averaging of each droplet by $\langle ...\rangle_l^{(n)}$ and relate it to $\langle ...\rangle_l$ as,

$$\langle ...\rangle_l = \sum_{n=0}^{N_d}\langle ...\rangle_l^{(n)} \tag{3.150}$$

In Sec. 3.3 and Sec. 3.4, when we refer to a quantity averaged over $\mathcal{V}_l$, we refer to the summation of average over volume of N_d droplets, each with volume $\mathcal{V}_l^{(n)}$ such that,

$$\langle \mathcal{F}\rangle_l = \sum_{n=0}^{N_d}\langle \mathcal{F}\rangle_l^{(n)} \tag{3.151}$$

3.3.1 TKE of DLHST-PC, k

We use Eq. (3.148) to simplify the first term of Eq. (3.113) as,

$$\overline{\rho}\frac{\widetilde{D}k}{\widetilde{D}t} = \frac{\mathrm{d}\left(\overline{\rho}k\right)}{\mathrm{d}t} \tag{3.152}$$

We realize that all the transport terms in Eq. (3.113) reduce to 0 due to homogeneous nature of the flow as,

$$\mathcal{T}''_{u''} - \mathcal{T}''_{p} - \mathcal{T}''_{\nu} \equiv \underbrace{\nabla \cdot \left\langle \overline{\rho}\boldsymbol{u}'' \cdot \boldsymbol{u}''\widetilde{\boldsymbol{U}} \right\rangle}_{=0} + \underbrace{\nabla \cdot \left\langle p'\mathbf{I}\boldsymbol{u}'' \right\rangle}_{=0} - \underbrace{\nabla \cdot \left\langle \mathbf{T}'\boldsymbol{u}'' \right\rangle}_{=0} = 0 \tag{3.153}$$

We also realize that the rate of pressure work and the rate of viscous work terms reduce to 0 due to the fact that mean pressure and mean viscous stress are invariant in space:

$$\begin{aligned} \Pi_p &\equiv \underbrace{\left(-\nabla\overline{p}\right)}_{=0} \cdot \left\langle \boldsymbol{u}''_f \right\rangle = 0 \\ \Pi_\nu &\equiv \underbrace{\left(\nabla \cdot \overline{\mathbf{T}}\right)}_{=0} \cdot \left\langle \boldsymbol{u}''_f \right\rangle = 0 \end{aligned} \tag{3.154}$$

In addition, we neglect the effects of body forces, which reduces Γ'' to 0. We use the results of Eq. (3.152), Eq. (3.153) and Eq. (3.154) to reduce Eq. (3.113) to an evolution equation of TKE for DLHST-PC:

$$\frac{\mathrm{d}\left(\overline{\rho}k\right)}{\mathrm{d}t} = \mathcal{P} - \varepsilon'' + \Pi_{\Delta''} + \Psi''_{\sigma} + \Psi''_{\dot{m}} + \Psi''_{\dot{I}} \tag{3.155}$$

The terms in Eq. (3.155) from left to right are, the rate of change of TKE of the gas-liquid system under homogeneous shear turbulence and allowing phase change, production of TKE, dissipation rate of TKE, fluctuating part of the compressible work done by the fluctuating pressure, power of surface tension forces, power of mass flux due to phase change and power of momentum flux due to phase change.

3.3.2 MFKE of DLHST-PC, $\widetilde{E}$

In a homogeneous shear turbulence, mean-flow gradient is prescribed and maintained constant, which allows for the continuous production of TKE and thus, prevents the turbulence from decaying. In other words, the MFKE of DLHST-PC is maintained constant. Hence, an explicit equation for MFKE is not required. However, we discuss the status of the terms in the right-hand side of Eq. (3.127) in order to understand the nature of interaction of MFKE with TKE and MIE. We use Eq. (3.148) to simplify the first term of Eq. (3.127) to a time-derivative of MFKE which reduces to 0 as,

$$\overline{\rho}\frac{\widetilde{D}\widetilde{E}}{\widetilde{D}t} = \frac{\mathrm{d}\left(\overline{\rho}\widetilde{E}\right)}{\mathrm{d}t} = 0 \tag{3.156}$$

We find that, analogous to Eq. (3.153), all the transport terms in Eq. (3.127) reduce to 0 as,

$$\widetilde{\mathcal{T}}_{u''} - \widetilde{\mathcal{T}}_p - \widetilde{\mathcal{T}}_\nu \equiv \underbrace{\nabla \cdot \left\langle \rho \left(\widetilde{\boldsymbol{U}} \cdot \boldsymbol{u}''\right) \boldsymbol{u}'' \right\rangle}_{=0} + \underbrace{\nabla \cdot \left\langle \overline{p}\mathbf{I}\widetilde{\boldsymbol{U}} \right\rangle}_{=0} - \underbrace{\nabla \cdot \left\langle \overline{\mathbf{T}}\widetilde{\boldsymbol{U}} \right\rangle}_{=0} = 0 \tag{3.157}$$

In addition, we find that the mean part of the pressure-dilatation correlation reduces to 0 as,

$$\Pi_{\widetilde{\Delta}} = \overline{p}\underbrace{\nabla \cdot \widetilde{\boldsymbol{U}}}_{=0} = 0 \tag{3.158}$$

We infer from Eq. (3.158) that there is no mean effect of compressibility due to phase change, which supports our understanding that even though there are local compressibility effects present in the gas-liquid system due to phase-change, there is no net-volume change in $\mathcal{V}$. This is due to the fact that the expansion due to evaporation at the interface is counter-acted by the compression away from the interface, i.e., in the bulk of the gas and liquid phases, and in the case of condensation, local compression at the interface is counter-acted by the expansion away from the interface. In addition, we neglect the effects of body forces, which reduces $\widetilde{\Gamma}$ to 0.

3.3.3 MIE of DLHST-PC, $\widetilde{e}$

We find that analogous to Eq. (3.152) and Eq. (3.156), the first term on the left-hand side of Eq. (3.134) reduces to a time derivative of MIE as,

$$\overline{\rho}\frac{\widetilde{D}\widetilde{e}}{\widetilde{D}t} = \frac{\mathrm{d}\left(\overline{\rho}\widetilde{e}\right)}{\mathrm{d}t} \tag{3.159}$$

We also find that analogous to Eq. (3.153) and Eq. (3.157), the transport terms in Eq. (3.134) reduce to 0 as,

$$\mathcal{T}^{*}_{u''} - \mathcal{T}^{*}_{p} - \mathcal{T}^{*}_{\nu} \equiv \underbrace{\nabla \cdot \left\langle \rho \widetilde{e} \boldsymbol{u}'' \right\rangle}_{=0} + \underbrace{\nabla \cdot \left\langle \overline{p} \mathbf{I} \boldsymbol{u}''_{i} \right\rangle}_{=0} - \underbrace{\nabla \cdot \left\langle \overline{\mathbf{T}} \boldsymbol{u}'' \right\rangle}_{=0} = 0 \tag{3.160}$$

In addition, we find that the net heat conducted in the system reduces to 0 as,

$$\dot{\mathcal{Q}} = \nabla \cdot \left\langle \boldsymbol{q} \right\rangle = 0 \tag{3.161}$$

We use the results of Eq. (3.159), Eq. (3.160), Eq. (3.161), Eq. (3.158), and Eq. (3.154) to reduce Eq. (3.134) to an evolution equation of MIE for DLHST-PC:

$$\frac{\mathrm{d}\left(\overline{\rho}\widetilde{e}\right)}{\mathrm{d}t} = \varepsilon'' + \widetilde{\varepsilon} - \Pi_{\Delta''} + \dot{\mathcal{Q}}_{\Sigma} \tag{3.162}$$

3.3.4 Energy pathways for DLHST-PC

For clarity, equations for TKE, MIE for DLHST-PC are given together as,

$$\frac{\mathrm{d}\left(\overline{\rho}k\right)}{\mathrm{d}t} = \mathcal{P} - \varepsilon'' + \Pi_{\Delta''} + \Psi''_{\sigma} + \Psi''_{\dot{m}} + \Psi''_{\dot{I}} \tag{3.163}$$

$$\frac{\mathrm{d}\left(\overline{\rho}\widetilde{e}\right)}{\mathrm{d}t} = \varepsilon'' + \widetilde{\varepsilon} - \Pi_{\Delta''} + \dot{\mathcal{Q}}_{\Sigma} \tag{3.164}$$

In addition, the equation for interfacial surface energy from Eq. (3.145) is re-written as,

$$\Theta \equiv -\widetilde{\Psi}_{\sigma} - \Psi''_{\sigma} \tag{3.165}$$

and the equation for phase change free-energy from Eq. (3.146) is re-written as,

$$\Phi = -\Psi_{\dot{m}} - \Psi_{\dot{I}} - \dot{\mathcal{Q}}_{\Sigma} \tag{3.166}$$

We present a schematic showcasing the pathways of energy exchange between k, $\widetilde{E}$, $\widetilde{e}$, Θ and Φ for DLHST-PC in Fig. 3.3.

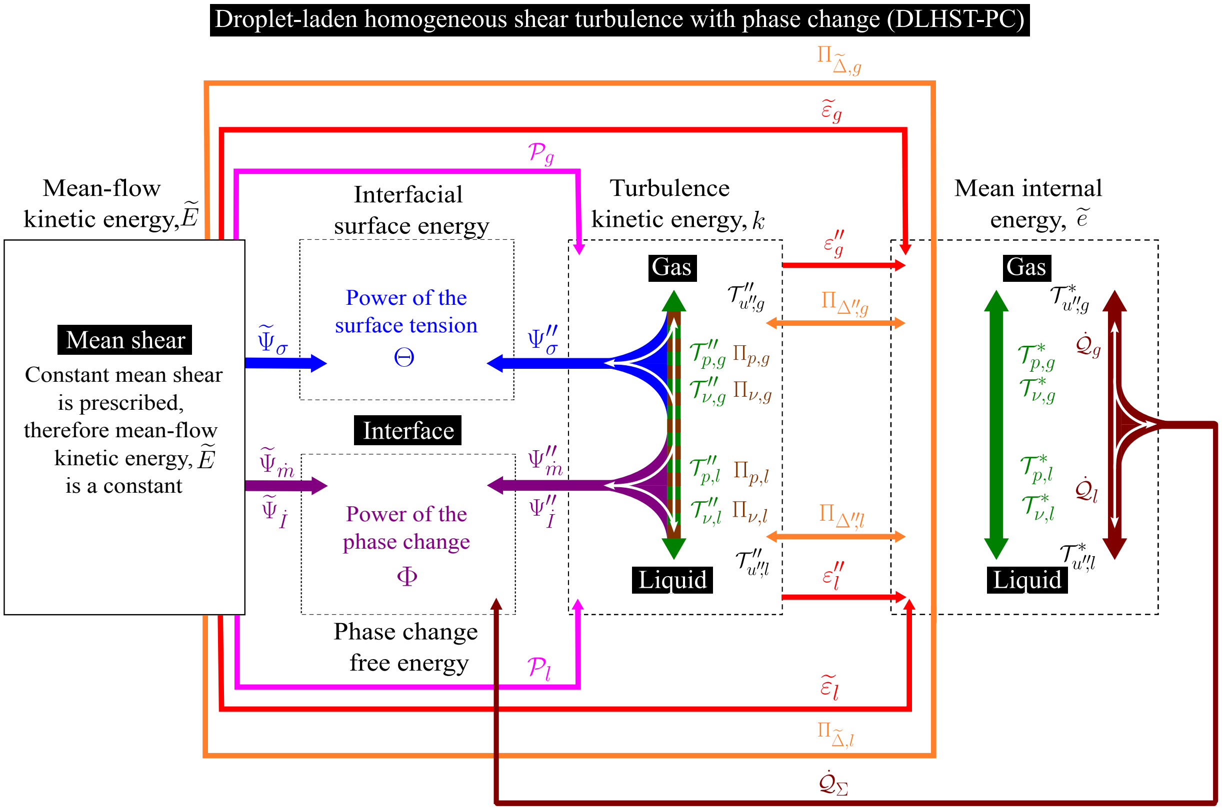

Figure 3.3: Schematic showing the pathways of energy exchange for DLHST-PC, which summarizes Eq. (3.164), Eq. (3.165) and Eq. (3.166).

3.4 *Droplet-laden decaying homogeneous isotropic turbulence with phase change*

In this section, we reduce the energy equations obtained in Sec. 3.3, to the case of droplet-laden decaying homogeneous isotropic turbulence with phase change (DLHIT-PC). In DLHIT-PC, there is no mean-shear to sustain turbulence and hence, turbulence decays with time. In addition, for a periodic boxed turbulence, there is no mean velocity component $\widetilde{\boldsymbol{U}}$.

3.4.1 TKE of DLHIT-PC, k

The production term $\mathcal{P}$ in Eq. (3.155) reduces to 0 as,

$$\mathcal{P} = -\left\langle \rho u''_i u''_j \right\rangle \underbrace{\partial_j \widetilde{U}_i}_{=0} = 0 \tag{3.167}$$

We use Eq. (3.167) to reduce the TKE equation for DLHST-PC given in Eq. (3.155) to an evolution equation of TKE for DLHIT-PC:

$$\frac{\mathrm{d}\,(\overline{\rho}k)}{\mathrm{d}t} = -\varepsilon'' + \Pi_{\Delta''} + \Psi''_{\sigma} + \Psi''_{\dot{m}} + \Psi''_{\dot{I}} \tag{3.168}$$

3.4.2 TKE of DLHIT-PC, $\widetilde{e}$

We reduce the dissipation of mean velocity term in Eq. (3.162) to 0 as,

$$\widetilde{\varepsilon} = \overline{T}_{ij}(\underbrace{\partial_j \widetilde{U}_i}_{=0}) = 0 \tag{3.169}$$

We use Eq. (3.169) to reduce the MIE equation for DLHST-PC given in Eq. (3.155) to an evolution equation of MIE for DLHIT-PC as,

$$\frac{\mathrm{d}\,(\overline{\rho}\widetilde{e})}{\mathrm{d}t} = \varepsilon'' - \Pi_{\Delta''} + \dot{\mathcal{Q}}_{\Sigma} \tag{3.170}$$

In addition, the mean power of surface tension in Eq. (3.165) reduces to 0 as,

$$\widetilde{\Psi}_{\sigma} = \left\langle \boldsymbol{f}_{\sigma} \cdot \underbrace{\widetilde{\boldsymbol{U}}_{\Sigma}}_{=0} \right\rangle = 0, \tag{3.171}$$

and the equation for the rate of change of surface energy reduces given in Eq. (3.172) to:

$$\Theta = -\Psi''_{\sigma} \tag{3.172}$$

In a similar way, the mean parts of powers of mass flux and momentum flux terms in Eq. (3.166) reduce to 0 as,

$$\widetilde{\Psi}_{\dot{m}} = \left\langle \boldsymbol{f}_{\dot{m}} \cdot \underbrace{\widetilde{\boldsymbol{U}}_{\Sigma}}_{=0} \right\rangle = 0, \tag{3.173a}$$

$$\widetilde{\Psi}_{\dot{I}} = -\frac{1}{\mathcal{V}} \int_{\Sigma} \left[\left[(\boldsymbol{\tau}\boldsymbol{n}) \cdot \left(\underbrace{\widetilde{\boldsymbol{U}}_{\Sigma}}_{=0} - \underbrace{\widetilde{\boldsymbol{U}}}_{=0} \right) \right]\right] = 0, \tag{3.173b}$$

and the equation for rate of change of phase change free energy given in Eq. (3.166) reduces to:

$$\Phi = -\Psi''_{\dot{m}} - \Psi''_{\dot{I}} - \dot{\mathcal{Q}}_{\Sigma} \tag{3.174}$$

3.4.3 Energy pathways for DLHIT-PC

For clarity, equations for TKE, MIE for DLHIT-PC are given together as,

$$\frac{\mathrm{d}\left(\overline{\rho}k\right)}{\mathrm{d}t} = -\varepsilon'' + \Pi_{\Delta''} + \Psi''_{\sigma} + \Psi''_{\dot{m}} + \Psi''_{\dot{I}} \tag{3.175}$$

$$\frac{\mathrm{d}\left(\overline{\rho}\widetilde{e}\right)}{\mathrm{d}t} = \varepsilon'' - \Pi_{\Delta''} + \dot{\mathcal{Q}}_{\Sigma} \tag{3.176}$$

In addition, the equation for interfacial surface energy from Eq. (3.172) is re-written as,

$$\Theta \equiv -\Psi''_{\sigma} \tag{3.177}$$

and the equation for phase change free-energy from Eq. (3.174) is re-written as,

$$\Phi = -\Psi''_{\dot{m}} - \Psi''_{\dot{I}} - \dot{\mathcal{Q}}_{\Sigma} \tag{3.178}$$

We present a schematic showcasing the pathways of energy exchange between k, $\widetilde{e}$, Θ and Φ for DLHIT-PC in Fig. 3.4.

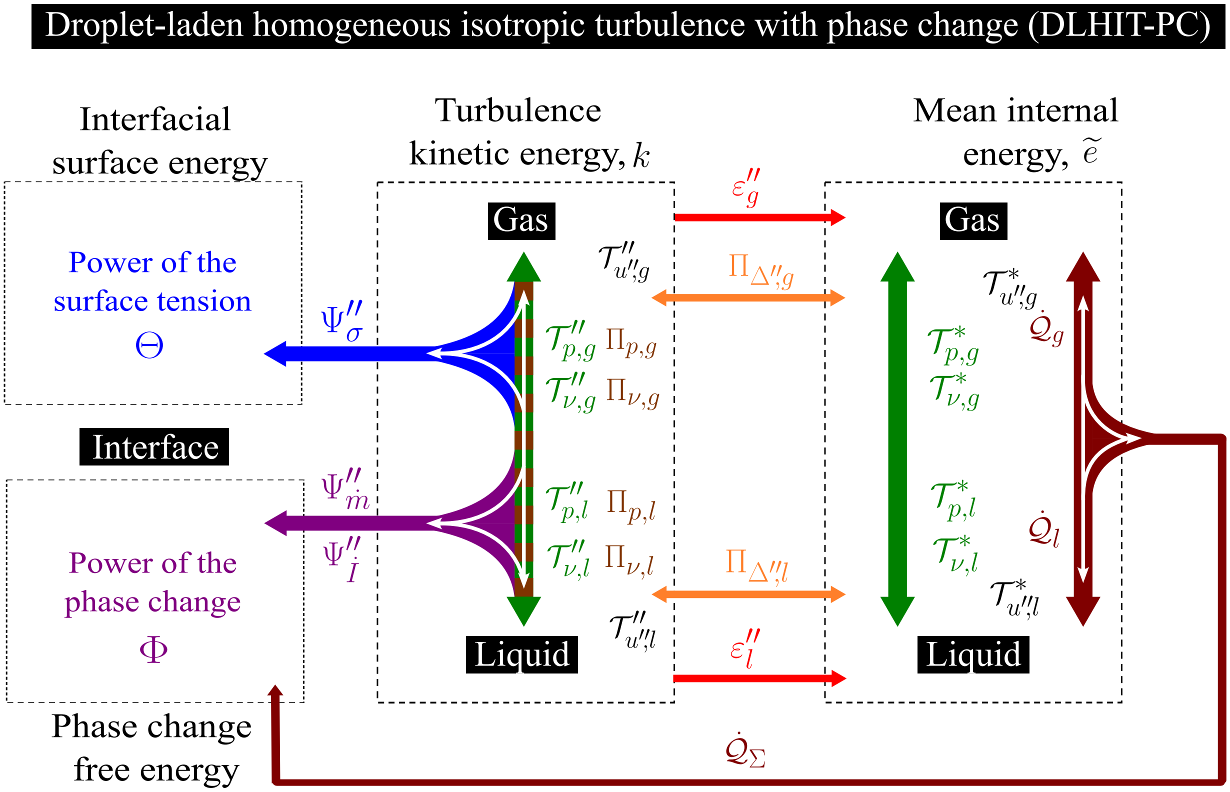

Figure 3.4: Schematic showing the pathways of energy exchange for DLHIT-PC, which summarizes Eq. (3.176), Eq. (3.177) and Eq. (3.178).

Chapter 4

SUMMARY

In this work, we started from the governing equations of mass, momentum and energy for a compressible gas-liquid flow considering phase change, surface tension and body force effects. We decomposed these equations into balance equations in the bulk of the gas or liquid phase and jump conditions at the gas-liquid interface. We used these balance equations, jump conditions and employed a Favre-averaging technique to derive the fundamental evolution equations of turbulence kinetic energy (TKE), mean-flow kinetic energy (MFKE), mean internal energy (MIE) and explained the pathways of energy exchange for

- compressible flow (gas or liquid phases) in
    - Eq. (3.45), Eq. (3.63), Eq. (3.85), and Fig. 3.1,
- compressible gas-liquid flow with phase change, for the first time, in
    - Eq. (3.113), Eq. (3.127), Eq. (3.134), and Fig. 3.2,
- droplet-laden homogeneous shear turbulence with phase change in
    - Eq. (3.155), Eq. (3.162), and Fig. 3.3, and
- droplet-laden decaying homogeneous isotropic turbulence with phase change in
    - Eq. (3.168), Eq. (3.170), and Fig. 3.4.

Furthermore we, explained the role of phase change through phase change free energy and related it to the powers of mass flux, momentum flux and heat flux in Eq. (3.143).

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