# On an inverse problem for fractional connection Laplacians

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**Abstract**

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Classical inverse problems seek to determine the unknown coefficients of a PDE from boundary or local measurements of solutions. In the past few years, there has been a sharp increase in attention paid to inverse problems for fractional Laplacians and their associated nonlocal equations. While most of this research takes place on $\mathbb{R}^n$, recently [FGKU21] showed that the Riemannian metric on a closed manifold is uniquely determined by local Riemannian structure and a source-to-solution map for the fractional Laplace-Beltrami operator.

Our paper [Chi22] generalizes this result by considering instead a fractional operator $P^s$, $0 < s < 1$, for connection Laplacian $P := \nabla^*\nabla + A$ on a smooth Hermitian vector bundle over a closed, connected Riemannian manifold of dimension $n \geq 2$. Assuming local knowledge of the metric, Hermitian bundle, connection, potential, and source-to-solution map associated with $P^s$, we show that all of these geometric structures are determined globally up to gauge invariance and isometry.

This thesis shares content with our preprint [Chi22], which is currently under revision. New additions include a more detailed discussion on fractional inverse problems and a significantly expanded exposition of some of the more technical tools involved.

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## DEDICATION

To Mom,

for all of her sacrifices and life lessons to get me to where I am today.

# Chapter 1

# INTRODUCTION

A classical inverse problem is the Calderón problem [Cal80], one form of which asks whether the metric on a compact Riemannian manifold-with-boundary $(M, g)$ can be determined up to boundary-fixing isometries from the Dirichlet-to-Neumann (DN) map $\Lambda : f \to (\partial_\nu u)|_{\partial M}$ of the boundary value problem

$$\begin{cases} \Delta_g u = 0 \text{ on } M, \\ u = f \text{ on } \partial M. \end{cases}$$

Physically, in dimension $n \geq 3$, this problem corresponds to determining an anisotropic conductivity modelled by $g$ from applying a voltage $f$ to $\partial M$ and measuring the resultant current flux across the boundary. Despite the simplicity of the setup, this problem has only been resolved in dimension $n = 2$ [LU01], up to an inevitable conformal invariance. The fact that $g$ is uniquely determined up to conformal class from the DN map in dimension two subsequently played a key role in the proof of boundary rigidity in the same dimension by [PU05]; this showed that the boundary distance function $d_g(x, y)|_{\partial M \times \partial M}$ determines $g$ uniquely for a certain class of simple metrics.

Meanwhile, a fractional analog of this Calderón problem encounters no such dimensional difficulties. We postpone precise definitions for later. First proposed by [Fei21], one considers the fractional Laplace-Beltrami operator $\Delta_g^s$, $0 < s < 1$, on a closed Riemannian manifold and asks if the fractional source-to-solution map

$$C_0^\infty(U) \ni f \mapsto \Delta_g^{-s} f|_U \in C^\infty(U)$$

along with the local Riemannian structure of $U \subset M$ determines $g$ up to isometry. This was answered in the affirmative by [FGKU21] in all dimensions $n \geq 2$.

We observe that the Laplace-Beltrami operator is a special case of a connection Laplacian with zero potential for the trivial line bundle $\mathbb{C} \to M$. It is then natural to consider a connection Laplacian $P_g := \nabla^* \nabla + A$ on a Hermitian bundle $E \to M$, where $A \in \operatorname{End}(E)$ is symmetric. Its local expression in coordinates

$$P_g = -g^{ij} \operatorname{Id}_E \partial_i \partial_j + b^j \partial_j + c$$

for $\operatorname{End}(E)$-valued $b^j, c$ shows the link between lower order coefficients of the differential operator and the additional geometric structures besides the metric. Determination of the coefficients of this differential operator is then equivalent to determining the metric, connection, and $\operatorname{End}(E)$-potential. Such geometric structures appear frequently in theoretical physics as gauge theories, and inverse problems for connection Laplacians arise in testable physical phenomena, such as the quantum mechanical Aharonov-Bohm effect [GT11]. Work of [GT11] and [AGTU13] prove cases when the connection and potential for $P_g$ can be determined in $n = 2$ from standard boundary Cauchy data, up to natural gauge invariance.

Our paper [Chi22] shows that in the fractional case, local measurements associated with a source-to-solution map for the fractional connection Laplacian $P_g^s$ along with the local geometric structure over a neighborhood $U \subset M$ fully determine the geometric structures globally over a closed manifold $M$ in any dimension $n \geq 2$, up to a natural isomorphism. The proof relies on heat kernel estimates to transform the fractional inverse problem into an inverse problem for a wave equation, where unique continuation properties derived from an appropriate Carleman estimate can be exploited.

### *1.1 Outline*

In Chapter 2, we give a standard definition and results for the fractional Laplacian in $\mathbb{R}^n$. We also recount recent inverse problems results for related nonlocal operators in $\mathbb{R}^n$. The following chapter defines the fractional connection Laplacian on closed manifolds with boundary, which sets up our main result mentioned above. We also include a discussion on future directions of research. Before jumping into the proof of our result, we give some background on the bundle-valued wave equation on closed manifolds in Chapter 4. We then spend significant time giving an exposition on the local unique continuation principle for such wave equations in Chapter 5. Armed with these facts, we transform our fractional inverse problem into a more standard inverse problem for a linear wave equation in Chapter 6. Chapter 7 is a brief digression into standard tools linking wave equation properties to metric geometry, and finally Chapter 8 uses these tools to prove our main result.

# Chapter 2

# THE FRACTIONAL LAPLACIAN ON $\mathbb{R}^N$

## 2.1 Notation

Our convention for the Fourier transform is

$$\hat{f}(\xi) = \mathcal{F}(f)(\xi) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx$$

with inverse

$$\mathcal{F}^{-1}(f)(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{+ix\cdot\xi} f(\xi)\, d\xi$$

Plancherel's theorem then has the form $(f, g)_{L^2} = (2\pi)^{-n}(\hat{f}, \hat{g})_{L^2}$. We use the geometers' convention for the Laplacian: $\Delta := -\sum_{j=1}^{n} \partial_j^2 = -\operatorname{div}\nabla$

The $L^2$-Sobolev spaces of order $s \in \mathbb{R}$ are given by

$$H^s(\mathbb{R}^n) := \{f \in \mathcal{S}' : \|f\|_{H^s} < \infty\},$$

where the norm $\|\cdot\|_{H^s}$ is induced by the inner product

$$(f, g)_{H^s} := \int_{\mathbb{R}^n} (1 + |\xi|^2)^s \hat{f}(\xi)\, \overline{\hat{g}}(\xi)\, d\xi = \left((1 + |\xi|^2)^{s/2}\hat{f}, (1 + |\xi|^2)^{s/2}\hat{g}\right)_{L^2}.$$

## 2.2 Definitions and the Caffarelli-Silvestre Extension

The fractional Laplacian $(\Delta)^s$, $0 < s < 1$, on $\mathbb{R}^n$ has several equivalent formulations, each the subject of active research in analysis from several perspectives such as PDEs, probability, and inverse problems. While we give only a brief overview here, the interested reader should consult [Gar17, Kwa17] for more thorough accounts and references.

Our preferred definition comes from Fourier analysis. Recall that standard properties of the Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ give $\Delta f = \mathcal{F}^{-1}(|\xi|^2\hat{f})$, from which we see

$\Delta$ is a Fourier multiplier with symbol $|\xi|^2 = \sum_j \xi_j^2$, extending to a bounded operator from $H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. We then arrive at a definition of $\Delta^s : H^{2s}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ given by

$$\Delta^s := \mathcal{F}^{-1}|\xi|^{2s}\mathcal{F}.$$

We observe that unlike the usual Laplacian, the symbol of the fractional Laplacian is not polynomial in $\xi$, hence it is not a differential operator. By a classic result of Peetre [Pee59, Pee60], this means the fractional Laplacian is a nonlocal operator: $\operatorname{supp}(\Delta^s u) \not\subset \operatorname{supp}(u)$ in general.

An important reformulation of $\Delta^s$ in terms of a local, albeit degenerate, operator was given in the seminal paper of Caffarelli and Silvestre [CS07]. We give a modification of their Fourier analysis proof to avoid variational arguments. In what follows, we assume everything is smooth enough with sufficient decay for the computations to make sense; a more precise regularity statement can be found in [Kwa17, Thm. 1.1(j)].

Caffarelli and Silvestre consider the following PDE on the upper half space $(x, y) \in \mathbb{R}^n \times \mathbb{R}_{>0}$,

$$\begin{cases} -\Delta_x u + \frac{1-2s}{y}\partial_y u + \partial_y^2 u = 0 \text{ on } \mathbb{R}^n \times \mathbb{R}_{>0}\,, \\ \lim_{y\to 0^+} u(x,y) = f(x). \end{cases} \tag{2.1}$$

By taking the Fourier transform in the $x$-variable, one transforms the first equation into an ODE in $y$ for fixed $\xi$:

$$\frac{d^2}{dy^2}\hat{u}(\xi, y) + \frac{1-2s}{y}\frac{d}{dy}\hat{u}(\xi, y) - |\xi|^2\hat{u}(\xi, y) = 0. \tag{2.2}$$

The solution to

$$\begin{cases} \phi''(t) + \frac{1-2s}{t}\phi'(t) - \phi(t) = 0 \text{ on } \mathbb{R}_{>0} \\ \lim_{t\to 0^+} \phi(t) = 1 \\ \lim_{t\to\infty} \phi(t) = 0, \end{cases} \tag{2.3}$$

is given by $\phi(t) = ct^s K_s(t)$, where $K_s$ is a modified Bessel function of the second type [AS48, Ch. 10]. This is obtained by transforming (2.3) into the modified Bessel equation by the substitution $\phi \mapsto t^s\phi$ [CL55, Ch. 4, Prob. 9]. It follows by rescaling that $\hat{u}(\xi, y) = \phi(y|\xi|)\hat{f}(\xi)$ solves (2.2). We observe that any solution to (2.1) also solves $\operatorname{div}(y^{1-2s}\nabla u) = 0$ on $\mathbb{R}^n \times \mathbb{R}_{>0}$, and $\frac{d}{dt}(t^s K_s(t)) \sim t^{2s-1}$ near 0 by a property of modified Bessel functions. This allows us to integrate by parts and apply Plancherel to find

$$\begin{aligned}
-\int_{\mathbb{R}^n} f(x) \lim_{y\to 0^+} \left(y^{1-2s}\partial_y u\right)\, dx &= \iint_{\mathbb{R}^n\times\mathbb{R}_{>0}} \langle \nabla u, \nabla u\rangle y^{1-2s}\, dxdy \\
&\quad + \iint_{\mathbb{R}^n\times\mathbb{R}_{>0}} u \operatorname{div}(y^{1-2s}\nabla u)\, dxdy \\
&= \int_{\mathbb{R}_{>0}} \int_{\mathbb{R}^n} \left(u\Delta_x u + (\partial_y u)^2\right) y^{1-2s}\, dx\, dy \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}_{>0}} \int_{\mathbb{R}^n} \left(|\xi|^2 \hat{u}(\xi)^2 + (\partial_y \hat{u})^2\right) y^{1-2s}\, d\xi\, dy \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}_{>0}} \hat{f}(\xi)^2 |\xi|^2 \left(\phi(y|\xi|)^2 + \phi'(y|\xi|)^2\right) y^{1-2s}\, dy\, d\xi \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\xi|^{2s} \hat{f}(\xi)^2 \left(\int_{\mathbb{R}_{>0}} \left(\phi(y)^2 + \phi'(y)^2\right) y^{1-2s}\, dy\right) d\xi \qquad (2.4) \\
&= \frac{1}{(2\pi)^n} J[\phi] \int_{\mathbb{R}^n} \hat{f}(\xi) \left(|\xi|^{2s} \hat{f}(\xi)\right) d\xi \\
&= J[\phi] \int_{\mathbb{R}^n} f(x) \Delta_x^s f(x)\, dx,
\end{aligned}$$

where $J[\phi]$ denotes the integral in brackets in (2.4). To verify $J[\phi] < \infty$, we used the exponential decay of $K_s$. Alternatively, this would have followed by observing $\phi$ is a minimizer of the functional $J[\,\cdot\,]$ with Euler-Lagrange equation (2.3), acting on the Sobolev space $H^1$ with weight $t^{1-2s}$ [Tur00, Ch. 1], [Eva10, Ch. 8].

Polarizing the above, we find

$$-\int_{\mathbb{R}^n} \varphi(x) \lim_{y\to 0^+} \left(y^{1-2s}\partial_y u\right)\, dx = J[\phi] \int_{\mathbb{R}^n} \varphi(x) \Delta_x^s f(x)\, dx$$

for any $\varphi \in C_0^\infty(\mathbb{R}^n)$, and we conclude

$$\Delta_x^s f(x) = -J[\phi] \lim_{y\to 0^+} \left(y^{1-2s}\partial_y u\right) \qquad (2.5)$$

for pairs $u$ and $f$ satisfying (2.1).

This localization of the fractional Laplacian allows one to import classical PDE techniques to study equations involving nonlocal operators. Examples of such applications include regularity theory of nonlocal obstacle problems [CSS08] and nonlocal anologues of minimal surfaces [CRS10]. Other important examples are the proofs of unique continuation principles for $\Delta^s$ given in [Rül15] and [GSU20, Thm. 1.2], which are vital tools in nonlocal inverse problems as we see shortly.

### 2.3 Fractional Inverse Problems on $\mathbb{R}^n$

A major motivation for studying both forward and inverse problems for nonlocal equations comes from the probabilistic interpretation of the fractional Laplacian, where $\Delta^s$ is viewed as the infinitesimal generator of a $2s$-stable Lévy process. These are stochastic processes with jumps and are a generalization of Brownian motion: we refer to [KI99, Kwa17] for details. Such processes manifest scientifically as anomalous diffusion in physics [BG90] as well as a proposed models of foraging patterns in biology [VBH+99, HWQ+12].

This abundance of phenomena given by the fractional Laplacian and related fractional operators has attracted significant interest from an inverse problems perspective. Inverse problems for the fractional Laplacian in Euclidean space were first considered by Ghosh, Salo, and Uhlmann [GSU20]. The authors showed how a potential function $q$ that is bounded on $U \Subset \mathbb{R}^n$ is uniquely determined by a certain DN map $\Lambda_q$ on the exterior region $\mathbb{R}^n \setminus \overline{U}$ associated with the nonlocal equation

$$\begin{cases} (\Delta^s + q)u = 0 \ \text{ on } U, \\ u = f \ \text{ on } \mathbb{R}^n \setminus \overline{U}. \end{cases} \tag{2.6}$$

More precisely, [GSU20, Thm. 1.1] gives the stronger result of unique determination of $q$ from DN measurements just on arbitrary subsets of $\mathbb{R}^n \setminus \overline{U}$.

Though the DN map is defined in general in terms of a bilinear form on suitable function spaces, it coincides with the map given by $f \mapsto \Delta^s u|_{\mathbb{R}^n \setminus \overline{U}}$ for sufficiently regular $U$, $f$, and $q$. As mentioned in [GSU20, Sec. 3], when $q = 0$, $\Lambda_q f$ is the cost of maintaining the exterior value $f$ in steady state nonlocal diffusion.

One of the key tools in their proof is a density result that is reliant on a unique continuation principle for $\Delta^s$. Indeed, after showing

$$0 = ((\Lambda_1 - \Lambda_2)f_1, f_2) = \int_U (q_1 - q_2)u_1 u_2 \, dx, \tag{2.7}$$

where $\Lambda_j$ denotes the DN map associated with potential $q_j$ and the left hand side is a distributional pairing, it suffices to show the set of solutions $u \in H^s(\mathbb{R}^n)$ to (2.6) for $f \in C_0^\infty(\mathbb{R}^n \setminus \overline{U})$ is dense in $L^2(U)$ upon restriction. Let $r : f \to u|_U$ denote this map; see [GSU20, Lem. 4.1] for the precise Sobolev regularity conditions. The authors compute the formal adjoint of $r$ to find that for any $v \in L^2(U)$ with $(rf, v)_{L^2(U)} = 0$, we have $(\Delta^s \varphi, f)_{L^2(\mathbb{R}^n)} = 0$ for $\varphi \in H^s(\mathbb{R}^n)$ solving

$$(\Delta^s + q)\varphi = v \ \text{ in } U$$

with $\operatorname{supp}(\varphi) \subset \overline{U}$. The unique continuation principle states that if $\varphi|_W = 0$ and $\Delta^s \varphi|_W = 0$, then $\varphi = 0$ on all of $\mathbb{R}^n$. It follows that $v = 0$ and the image of $r(C_0^\infty(\mathbb{R}^n \setminus \overline{U}))$ is dense in $L^2(U)$ as required. For any $h \in L^2(U)$, one chooses $rf_1 = u_1|_U, rf_2 = u_2|_U$ in (2.7) to be arbitrarily close in $L^2$ to $h$ and 1 respectively; it follows that $q_1 = q_2$ on $U$.

By extending the identity (2.7), the work of [RS20] extended the uniqueness result for lower regularity potentials $q$. Additionally, the authors provide a new stability estimate of the form $\|q_1 - q_2\| \leq C\|\Lambda_1 - \Lambda_2\|'$ between appropriate spaces; this proof relies on a quantitative unique continuation principle and the Caffarelli-Silvestre extension.

Instead of requiring exterior DN data for all $f \in C_0^\infty(\mathbb{R}^n \setminus \overline{U})$, [GRSU20] showed just a single exterior source $f$ and exterior measurement $\Lambda_q f$ suffices to determine the potential $q$. Significantly, the authors also prove an explicit reconstruction procedure for $q$ from this single measurement. This is based on a Tikhonov regularization scheme for the fractional equation, combined with a modified unique continuation principle for sets of positive measure to deal with lower regularity potentials. This is analogous to the reconstruction procedure in classical inverse problems for PDE [KT12]. In [GRSU20, Sec. 3], the authors give two further fractional regularization schemes that could be applied.

There has also been progress in determination of the coefficients of general lower order perturbations of $\Delta^s$ beyond just the potential. [CLR20] considers a fractional equation with first order drift term of the form $\Delta^s + b\nabla + q$. In the similar setup of (2.6) for this operator, they prove that the DN map uniquely determines both $b$ and $q$. The authors also show unique determination from finite number of exterior measurements along with a logarithmic stability result. More generally, the unique determination of lower order coefficients was shown to hold in [CMRU22] for operators of the form $\Delta^{s'} + P(x, D)$, where $s' \in \mathbb{R}_{>1} \setminus \mathbb{N}$ and $P(x, D)$ is a linear differential operator of order less than $s'$.

Inverse problems for lower order nonlocal perturbations of the fractional Laplacian have also been considered, as in [BGU21]. The authors consider the operator $\Delta^s + \Delta_U^{t/2} b \Delta_U^{t/2} + q$ with $0 < t < s < 1$, where $\Delta_U^{t/2}$ is defined using the singular integral formulation of the fractional Laplacian [Kwa17, Thm. 1.1(e)] and is the infinitesimal generator of a $t$-stable censored Lévy process. Roughly speaking, this is a stochastic process with jumps that is restricted to within $U$. [BGU21] shows that finitely many suitably interpreted exterior DN measurements are sufficient to uniquely determine $b$ and $q$.

All of the results mentioned above deal with the determination of unknown coefficients of order less than the leading order $s$. One can also consider the problem of recovering the unknown fractional principal symbol. In [GLX17], the Laplacian $\Delta$ is replaced by the elliptic operator $P := -\operatorname{div}(A(x)\nabla)$, where $A(x)$ is a smooth, bounded, positive definite matrix. The fractional operator $P^s$ is then defined using a heat semigroup, similar to the construction for $\Delta^s$ in [Kwa17, Thm. 1.1(c)]. For (2.6) with the operator $P^s + q$, [GLX17] shows the DN map given by the restriction of $f \mapsto P^s u$ to the exterior region uniquely determines $A$ and $q$. Their methods rely on an adaptation of the Caffarelli-Silvestre extension given in [ST10] combined with Almgren's frequency function to prove the necessary unique continuation principle.

Instead of directly using unique continuation principles to directly prove unique determination, an alternative is to reduce the problem to a local inverse problem and appeal to existing inverse problem results. This strategy was first pursued in the work of [GU21], where one considers a second-order positive, symmetric, elliptic operator

$$P := -a^{ij}(x)\partial_{ij} + b^j\partial_j + c$$

with $a^{ij} = \delta^{ij}$ outside $U \subset \mathbb{R}^n$ and defines $P^s$, $0 < s < 1$, using the heat semigroup. Fix any two bounded open sets $W, W' \subset \mathbb{R}^n \setminus \overline{U}$, and consider the nonlocal exterior Cauchy data for $P^s$ to be the collection of pairs $\{u|_W, P^s u|_{W'}\}$ for $u \in H^2(\mathbb{R}^n)$ solving $P^s u = 0$ on $U$ with $\mathrm{supp}(u) \subset \overline{U} \cup \overline{W}$. Then authors show if this Cauchy data uniquely determines the local DN data, that is, the pairs $(f, (\partial_\nu)v|_{\partial U})$ that solve $Pv = 0$ on $U$ with $v|_{\partial U} = f$. This nonlocal-to-local reduction relies on properties of the heat semigroup and kernel estimates.

Most recently as of the writing of this thesis, [CGRU23] accomplishes the same nonlocal-to-local reduction via an extension to the upper half-space as in [CS07, ST10] instead of using heat semigroup arguments. This reframes the nonlocal Cauchy data by means of a Caffarelli-Silvestre DN map of the type (2.5). Moreover, [CGRU23] gives an explicit reconstruction procedure for the unknown coefficients using this extension. The author also provides proof of why this procedure could not be reversed: one cannot expect to reduce an inverse problem for local DN data to one involving nonlocal data.

# Chapter 3

# THE FRACTIONAL LAPLACIAN ON MANIFOLDS

Unlike the situation on $\mathbb{R}^n$, an arbitrary Riemannian manifold $(M, g)$ is not equipped with global coordinates or a Fourier transform with which to define a fractional Laplacian like we do above. Instead, as we describe below, one uses a spectral decomposition to define a fractional version of a Laplacian. This approach also allows one to define fractional powers of connection Laplacians on general vector bundles, not just $\Delta_g^s$ for the scalar-valued Laplace-Beltrami operator $\Delta_g$. From a functional analytic perspective, the Fourier multiplier definition on $\mathbb{R}^n$ and our spectral definition on $M$ both present the respective fractional operators as multiplication operators with respect to projection-valued measures once a self-adjoint extension of $\Delta$ has been fixed [RS80, RS75].

## *3.1 Fractional connection Laplacians and source-to-solution maps*

Let $(M, g)$ be a smooth, closed Riemannian manifold equipped with a smooth Hermitian vector bundle $(E, \langle \cdot\, , \,\cdot \rangle_E)$. Recall that if $\tilde{E}$ is an arbitrary tensor product $E^{\otimes m} \otimes T^{k,l}M$, then the appropriate tensor product of $\langle \cdot\, , \,\cdot \rangle_E$ and $g$ together determine a bundle metric $\langle \cdot\, , \,\cdot \rangle_{\tilde{E}}$ on $\tilde{E}$. This induces an $L^2$-product on smooth sections $C^\infty(\tilde{E})$ given by

$$(u, v)_{L^2(\tilde{E})} := \int_M \langle u, v \rangle_{\tilde{E}} \, dV_g,$$

and $L^2(M; \tilde{E})$ is the completion of $C^\infty(M; \tilde{E})$ under the induced norm. If $U \subset M$ is an open set, we denote the compactly supported smooth sections of $\tilde{E}$ inside $U$ by $C_0^\infty(U; \tilde{E})$. We will drop the reference to the bundle $\tilde{E}$ whenever it is contextually clear.

Consider any smooth connection $\nabla : C^\infty(M; E) \to C^\infty(M; E \otimes T^*M)$ compatible with the Hermitian structure on $E$, that is $d\langle u\, , \, v \rangle = \langle \nabla u\, , \, v \rangle + \langle u\, , \, \nabla v \rangle$. The Sobolev spaces

$H^k(M;E)$ of nonnegative integer order $k$ are given by the completion of $C^\infty(M;E)$ under the norm induced by the inner product

$$(u,v)_{H^k} := (u,v)_{L^2} + (\nabla u, \nabla v)_{L^2} + \cdots + (\nabla^k u, \nabla^k v)_{L^2}.$$

Since $M$ is compact, the spaces $L^2$ and $H^k$ as defined do not depend on $g$, $\langle \cdot\,, \cdot \rangle_E$, or $\nabla$, though the inner products do.

With respect to $(\cdot, \cdot)_{L^2}$, the connection $\nabla$ has formal adjoint $\nabla^*$ in the sense that $(\nabla u, v)_{L^2} = (u, \nabla^* v)_{L^2}$ for $u \in C^\infty(M;E)$, $v \in C^\infty(M; E \otimes T^*M)$. Their composition is the *bundle Laplacian* $\nabla^*\nabla = -\operatorname{tr}_g \nabla^2 =: \Delta^E$.

In this case, the minimal domain of $\Delta^E$ is $H^2(M;E)$, given by the Friedrichs extension. The maximal domain, characterized by $\{u \in L^2(M;E) : \Delta^E u \in L^2(M;E)\}$, is also $H^2(M;E)$ by elliptic regularity from the elliptic parametrix construction and the fact that we have a Fredholm mapping $\Delta^E : H^2(M;E) \to L^2(M;E)$ [Gil18, Sec. 1.3]. Because the maximal and minimal domains coincide, there is a unique self-adjoint extension $\Delta^E : \mathcal{D}(\Delta^E) \subset L^2(M;E) \to L^2(M;E)$ with domain $\mathcal{D}(\Delta^E) = H^2(M;E)$ [Gru08, Ch. 4.1].

Taking a symmetric potential $A \in L^\infty(M; \operatorname{End}(E))$ in the sense that

$$(u,\, Av)_{L^2} = (Au,\, v)_{L^2},$$

we now consider the *(generalized) connection Laplacian*

$$P := \Delta^E + A \tag{3.1}$$

which is also self-adjoint with domain $\mathcal{D}(P) = \mathcal{D}(\Delta^E) = H^2(M;E)$. The spectral theorem [RS80, Ch. 8] yields an orthogonal decomposition $L^2 = \bigoplus_{k=1}^\infty V_k$ in terms of the spectrum $-c = \lambda_1 < \lambda_2 < \cdots \to +\infty$, $c \geq 0$, of $P$ (which is discrete), where each $V_k := \operatorname{Ker}(P - \lambda_k)$ is the finite dimensional eigenspace for eigenvalue $\lambda_k$. In terms of the $L^2$-projectors $\pi_k : L^2 \to V_k$, we define:

**Definition 1.** *Let $P$ as in* (3.1) *be nonnegative (i.e. $\lambda_1 \geq 0$). The fractional connection*

*Laplacian* $P^s : \mathcal{D}(P^s) \subset L^2 \to L^2$ *for* $0 < s < 1$ *is given by*

$$P^s u := \sum_{k\geq 1} \lambda_k^s \pi_k u \tag{3.2}$$

*with domain* $\mathcal{D}(P^s) = H^{2s} := \{u \in L^2 : \sum_{k\geq 1} \lambda_k^{2s} \|u\|_{L^2}^2 < \infty\}$.

We note that this spectral decomposition also allows us to write this fractional operator in terms of the resolvent of $P$, and a result of [See67] shows that $P^s$ has principal symbol $|\xi|_g^{2s}\,\mathrm{Id}$. Therefore, once again by the result of Peetre, we see $P^s$ is a nonlocal operator.

Turning our attention to the restriction of $P^s$ to $\mathcal{D}(P^s) \cap \mathrm{Ker}(P)^\perp$ in the case that $P$ has a zero eigenvalue, we see that $P^s$ has inverse

$$P^{-s} f := \sum_{k\geq 1} \lambda_k^{-s} \pi_k f, \quad 0 < s < 1, \tag{3.3}$$

where the sum is taken over the nonzero eigenvalues. That is, if $f \in \mathrm{Ker}(P)^\perp$, then $u = P^{-s} f$ is the unique solution to $P^s u = f$ on $M$. This motivates the following:

**Definition 2.** *The local fractional source-to-solution map* $\mathcal{L}_{P,U}^{\mathrm{frac}} : C_0^\infty(U; E) \cap \mathrm{Ker}(P)^\perp \to C^\infty(U; E)$ *associated with* $P^s$ *over an open set* $U \subset M$ *is given by* $f \mapsto P^{-s} f|_U$.

We now examine how $\mathcal{L}_{P,U}^{\mathrm{frac}}$ behaves under pullback. Consider smooth, closed, connected Riemannian manifolds equipped with respective Hermitian vector bundles, compatible connections, and symmetric potentials:

$$\mathcal{M}_i := (M_i, g_i, E_i, \langle \cdot\,, \cdot \rangle_{E_i}, \nabla^{E_i}, A^{E_i}),\ i = 1, 2. \tag{3.4}$$

**Definition 3.** *Suppose we have a smooth vector bundle isomorphism*

$$\begin{array}{ccc} E_2 & \xrightarrow{\ \Psi\ } & E_1 \\ {\scriptstyle \pi_2}\big\downarrow & & \big\downarrow{\scriptstyle \pi_1} \\ M_2 & \xrightarrow{\ \psi\ } & M_1 \end{array} \tag{3.5}$$

*We say* $\Psi : \mathcal{M}_2 \to \mathcal{M}_1$ *is a structure-preserving isomorphism if the following conditions are satisfied:*

1. *The map* $\psi$ *is an isometry:* $g_2 = \psi^* g_1$.

2. *The Hermitian metric* $\langle \cdot\,, \cdot \rangle_{E_2} = \Psi^* \langle \cdot\,, \cdot \rangle_{E_1}$*, where* $\Psi^* \langle u_2\,, \tilde{u}_2 \rangle_{E_1} = \langle u_1\,, \tilde{u}_1 \rangle_{E_1}$ *for* $u_2 = \Psi^* u_1$*,* $\tilde{u}_2 = \Psi^* \tilde{u}_1$*. Note that the sections* $u_2 \in C^\infty(M_2; E_2)$ *are in bijective correspondence with* $u_1 \in C^\infty(M_1; E_1)$ *by the pullback* $u_2 = \Psi^* u_1 := \Psi^{-1} \circ u_1 \circ \psi$*.*

3. *The* $\langle \cdot\,, \cdot \rangle_{E_2}$*-compatible connection is* $\nabla^{E_2} = \Psi^* \nabla^{E_1}$*, with the pullback connection acting on* $u \in C^\infty(M_2; E_2)$*,* $\nu \in C^\infty(TM_2)$ *by*

$$(\Psi^* \nabla^{E_2})_\nu u := \Psi^* (\nabla^{E_1}_{d\psi(\nu)} (\Psi^{-1^*} u)).$$

4. *The symmetric potential satisfies* $A^{E_2} = \Psi^* A^{E_1} := \Psi^* A^{E_1} \Psi^{-1*}$*.*

We note that a structure-preserving isomorphism naturally arises from any diffeomorphism $\psi : M_2 \to M_1$ by defining $g_2 := \psi^* g_1$, $E_2 := \psi^* E_1$, and pulling back the remaining structures using $\Psi$; in this case, the bundle isomorphism $\Psi$ restricts to the identity on each fibre.

Letting $P_i := \Delta^{E_i} + A^{E_i}$ for $i = 1, 2$, we find $P_2(\Psi^* u) = \Psi^*(P_1 u)$ for $u \in C^\infty(M_1; E_1)$ and furthermore $\operatorname{Ker}(P_2 - \lambda_k) = \Psi^* \operatorname{Ker}(P_1 - \lambda_k)$ by looking at pullbacks of $L^2$-eigensections. For sections not in $\operatorname{Ker}(P_1)$, we therefore find $\Psi^*(P_1^{-s} u) = P_2^{-s}(\Psi^* u)$. Moreover, restricting to $f \in C_0^\infty(U_1; E_1) \cap \operatorname{Ker}(P_1)^\perp$, setting $U_2 = \psi^{-1}(U_1)$ and letting $\mathcal{L}_i^{\text{frac}} = \mathcal{L}_{P_i, U_i}^{\text{frac}}$ for $i = 1, 2$, we have

$$\Psi^* \mathcal{L}_1^{\text{frac}} f = \mathcal{L}_2^{\text{frac}} \Psi^* f. \tag{3.6}$$

### 3.2 Main result

Our discussion above implies that (3.6) holds whenever $\Psi : \mathcal{M}_2 \to \mathcal{M}_1$ is a structure-preserving isomorphism. This motivates our main theorem, which is the following rigidity result:

**Theorem 1.** *Let $\mathcal{M}_1, \mathcal{M}_2$ be smooth, closed, connected Riemannian manifolds of dimension $n \geq 2$ equipped with respective Hermitian vector bundles, compatible connections, and symmetric potentials as in* (3.4) *such that $P_1, P_2 \geq 0$. For $i = 1, 2$, consider open sets $U_i \subset M_i$ such that there is a structure-preserving isomorphism $\mathcal{M}_1|_{U_1} \cong \mathcal{M}_2|_{U_2}$:*

$$\begin{array}{ccc} E_2 & \xrightarrow{\tilde{\Psi}} & E_1 \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ U_2 & \xrightarrow{\tilde{\psi}} & U_1 \end{array} \tag{3.7}$$

*If $\tilde{\Psi}^* \mathcal{L}_1^{\mathrm{frac}} = \mathcal{L}_2^{\mathrm{frac}} \tilde{\Psi}^*$ on $C_0^\infty(U_1; E_1) \cap \mathrm{Ker}(P_1)^\perp$, then there exists a global structure-preserving isomorphism $\Psi : \mathcal{M}_2 \to \mathcal{M}_1$ with $\Psi|_{U_2} = \tilde{\Psi}$ and $\psi|_{U_2} = \tilde{\psi}$.*

Any symmetric Laplace-type operator on $E \to M$ locally of form

$$P = -g^{ij} \,\mathrm{Id}_E\, \partial_i \partial_j + b^j \partial_j + c, \quad \text{for } b^j, c \in C^\infty(U; \mathrm{End}(E)), \tag{3.8}$$

is given by $\Delta^E + A$ for a suitably chosen connection on $E$ [BGV03, Ch. 2], hence our result can also be interpreted as recovering the coefficients of (3.8) up to gauge for symmetric Laplace-type operators from local fractional source-to-solution data.

### 3.3 Previous work

A generalization of the Caffarelli-Silvestre extension for the fractional Laplacians on manifolds was given by [CdMG11] in their study of conformally covariant powers of the Laplacian. Specifically, the authors consider a *conformally compact* Einstein manifold $(X^{n+1}, g^+)$, $n \geq 2$, where $X \subset \overline{X}$ is the interior of a compact manifold-with-boundary, with Riemannian metric $g^+$ satisfying $\mathrm{Ric}(g^+) = -ng^+$ and the additional property that there exists a boundary defining function $r$ for $:= \partial X = \{r = 0\}$ such that $\overline{g} := r^2 g^+$ extends smoothly to $\overline{X}$. Upon fixing a special defining function $\rho$ such that $g^+ = \frac{d\rho^2 + g_\rho}{\rho^2}$ near $\partial X$ with $\tilde{g} := g_{\rho=0}$ conformal to $\overline{g}|_M$, one can define *conformally covariant powers* of the Laplacian $P_s[g^+, \tilde{g}]$, $s \notin \mathbb{N}$, which have principal symbol $|\xi|_{\tilde{g}}^{2s}$, by solving a certain eigenvalue problem for $\Delta_{g^+}$ in $X$

[GZ03]. In the classic example of the Poincaré ball $X = \mathbb{B}^{n+1}$ with metric $g^+ := \frac{4}{(1-|x|^2)^2}\,dx^2$, which is isometric to the upper halfspace model of hyperbolic space $\mathbb{H}^{n+1}$, it is shown by [CdMG11] that $P_s = \Delta^s$ on $\mathbb{R}^n$ and the construction of $P_s$ matches that of $\Delta^s$ using the Caffarelli-Silvestre extension. Moreover, $P_s$ is also defined for higher non-integer powers $s$, and it is shown that these higher powers also match higher powers of $\Delta$ defined by iterating the Caffarelli-Silvestre extension.

Research in fractional inverse problems on manifolds is currently less developed than in $\mathbb{R}^n$. The work of [GU21] can be interpreted as the unique determination of an unknown metric $g$ on $U \subset \mathbb{R}^n$ from exterior DN measurements when $g$ is known to be Euclidean outside $U$. Our result generalizes that of [FGKU21], which concerned the trivial line bundle $\mathbb{C} \to M$ with Euclidean connection and $A = 0$ so that $P$ is the Laplace-Beltrami operator and there is only the metric $g$ to recover from the source-to-solution map. Recent work of [QU22] considered a similar problem for fractional Dirac operators, where unique determination of geometric structures also includes the Clifford structure of the bundle.

### *3.4 Future work*

Similar to recent progress in the Euclidean case, one may consider the problems of stability, explicit reconstruction, and determination of geometric structures from finite measurements from the source-to-solution map. Many of these further results in the Euclidean setting rely on a fractional unique continuation principle, and such a fundamental property would be a highly useful tool if it could be proved in certain geometric settings.

A more immediate question is if our unique determination results can be extended to noncompact manifolds. This seems possible because many of the tools we exploited in the closed manifold case still persist in more general settings. Though the spectrum associated with a Laplace-type operator may no longer be discrete, we can once again define fractional Laplacians by an integral representation via the spectral calculus, and a source-to-solution map over a bounded open set still makes sense. For complete manifolds with, say, nonnegative Ricci curvature, exponential heat kernel bounds for Laplace-type operators are known by work of [LY86].

Thus far, we have only considered unique determination using a source-to-solution map over an open set, whereas the fractional inverse problems considered in $\mathbb{R}^n$ mostly consider an exterior nonlocal DN map. One might therefore consider a nonlocal exterior DN map associated with $P_g^s + q = 0$ on a bounded subset $U \subset M$. The complications of this problem include deciding on the correct functions spaces and boundary trace map to define such DN operators. For this, one can look at the $\mu$-transmission conditions on manifolds [Hör07] that have been studied by [Gru15].

If an exterior DN map can be suitably defined on manifolds, one can try to reduce the fractional inverse problem into a classical one in the sense of [GU21]. When $M$ is a surface, the unique determination results of [GT11, AGTU13] from classic boundary Cauchy data on $U \subset M$ can hopefully be used. However, this reduction approach also requires the investigation of the relationship between $P_g^s$ appearing in the $\mu$-transmission theory on $M$ with the local operator $P_g|_U$.

# Chapter 4
# THE BUNDLE-VALUED WAVE EQUATION

In this chapter, we recall some common properties of linear wave equations we need for our inverse problem, adapted to closed manifolds equipped with a vector bundle $E \to M$. We assume $P > 0$ without loss of generality, since we disregard the kernel of $P \geq 0$ in our assumptions for our main result. Geometric computations like integration by parts and an area variation formula are found at the end of this chapter.

## 4.1 Existence and uniqueness of solutions

Consider the wave equation

$$\begin{cases} (\partial_t^2 + P)u = f \text{ on } (0,T) \times M, \\ u|_{\{t=0\}} = \varphi,\ \partial_t u|_{\{t=0\}} = \psi. \end{cases} \tag{4.1}$$

We first assume smooth data, with $f \in C_0^\infty((0,T) \times M; E)$, $\phi, \psi \in C^\infty(M; E)$. For any $F \in L^2(M; E)$, we use the spectral decomposition of $P = \Delta_E + A > 0$ to write $\hat{F}(k) := (F, \phi_k)_{L^2(M;E)}$. Along each $\lambda_k$ eigenspace, the equation (4.1) becomes an ODE

$$\begin{cases} \frac{d^2}{dt^2}\hat{u}(t,k) + \lambda_k \hat{u}(t,k) = \hat{f}(t,k) \text{ on } (0,T) \\ \hat{u}(0,k) = \hat{\varphi}(k),\ \frac{d}{dt}\hat{u}(0,k) = \hat{\psi}(k) \end{cases} \tag{4.2}$$

with unique solution

$$\hat{u}(t,k) = \cos(t\sqrt{\lambda_k})\hat{\varphi}(k) + \frac{\sin t\sqrt{\lambda_k}}{\sqrt{\lambda_k}}\hat{\psi}(k) + \int_0^t \frac{\sin s\sqrt{\lambda_k}}{\sqrt{\lambda_k}}\hat{f}(t-s,k)\, ds. \tag{4.3}$$

Let

$$u(t,x) = \sum_k \hat{u}(t,k)\phi_k = \cos(t\sqrt{P})\varphi + \frac{\sin t\sqrt{P}}{\sqrt{P}}\psi + \int_0^t \frac{\sin s\sqrt{P}}{\sqrt{P}} f(t-s,x)\, ds. \tag{4.4}$$

Then for $\|u\|_{C(0,T;L^2(M,E))} := \sup_{(0,T)} \|u(t,\cdot)\|_{L^2(M;E)}$, we have

$$\begin{cases} \|u\|_{C(0,T;L^2(M,E))} & \leq \|\varphi\|_{L^2} + \left\| \frac{1}{\sqrt{P}}\psi \right\|_{L^2} + T \left\| \frac{1}{\sqrt{P}} f \right\|_{C(0,T;L^2(M,E))} \\ \|\partial_t u\|_{C(0,T;L^2(M,E))} & \leq \|\sqrt{P}\varphi\|_{L^2} + \|\psi\|_{L^2} + \left\| \frac{1}{\sqrt{P}} f \right\|_{C(0,T;L^2(M,E))} \end{cases}$$

Therefore, by density, we have a solution $u \in C\left((0,T); H^1(M;E)\right) \cap C^1\left((0,T); L^2(M;E)\right)$ given by (4.4) that uniquely solves the wave equation (4.1) for sources $f \in L^2((0,T)\times M; E)$ and initial data $\varphi \in H^1(M;E), \psi \in L^2(M;E)$.

### *4.2 Finite speed of propagation*

To abbreviate notation, we write $\mathbf{M} := \mathbb{R} \times M$, equipped with the Riemannian metric $\mathbf{g} := dt^2 \oplus g$ and volume form $d\mathbf{V}$. For $T > 0$, consider the geodesic ball

$$B(p,T) := \{q \in M : d_g(p,q) < T\}$$

and the backwards cone

$$C(p,T) := \{(t,q) \in (0,T) \times M : d_g(p,q) < T - t\}.$$

We show the following:

**Proposition 2** (Finite speed of propagation)**.** *Let $f \in L^2(\boldsymbol{M}; E)$, and suppose $u$ is a solution to*

$$\begin{cases} (\partial_t^2 + P)u = f \quad \text{on } (0,\infty) \times M, \\ f|_{C(p,T)} = 0, \\ u|_{B(p,T)\times\{t=0\}} = \partial_t u|_{B(p,T)\times\{t=0\}} = 0. \end{cases}$$

*Then $u \equiv 0$ on $C(p,T)$.*

Consider the region $\Omega(\bar{t}) \subset C(p,T)$ bounded by $B(p,T)$ and the set

$$\{(x,t) : t = \bar{t}(T - |x-p|)\},\ 0 < \bar{t} < 1.$$

On the geodesic ball $B(p,T)$, we take polar normal coordinates $(r,\theta)$ [Lee18, 6.45] centered at $p = \{r = 0\}$. For $0 \le t < \bar{t}$, define the time slices $\Sigma(t) := (\{t\} \times M) \cap \Omega(\bar{t})$ which together foliate $\Omega(\bar{t})$. This family is parametrized by the map $F : [0,\bar{t}) \times B(p,T) \to \mathbf{M}$, $(t,r,\theta) \mapsto (t, (T - t/\bar{t})r, \theta)$ in polar coordinates on $B(p,T)$. In an abuse of notation, we define the energy

$$E(t) := \frac{1}{2}\int_{\Sigma(t)} |\partial_t u|^2 + |\nabla u|^2 + |u|^2 \, d\Sigma(t).$$

where the integral is taken to mean

$$\frac{1}{2}\int_{B(p,T)} F(t)^* \left(|\partial_t u|^2 + |\nabla u|^2 + |u|^2 \, d\mathbf{V}\right).$$

We calculate the derivative:

$$\frac{d}{dt}E(t) = \frac{1}{2}\int_{\Sigma(t)} \frac{d}{dt}\left(|\partial_t u|^2 + |\nabla u|^2 + |u|^2\right) d\Sigma(t) \tag{4.5}$$

$$+ \frac{1}{2}\int_{\Sigma(t)} |\partial_t u|^2 + |\nabla u|^2 + |u|^2 \frac{d}{dt} d\Sigma(t). \tag{4.6}$$

Upon further calculation and using integration by parts (4.7) with $f := \partial_t u$ and $X := \nabla u$, we see (4.5) becomes

$$\begin{aligned}
\frac{1}{2}\int_{\Sigma(t)} \frac{d}{dt}\left(|\partial_t u|^2 + |\nabla u|^2 + |u|^2\right) d\Sigma(t) &= \int_{\Sigma(t)} \Re\langle \partial_t u, \partial_t^2 u + u\rangle + \Re\langle \nabla\partial_t u, \nabla u\rangle \, d\Sigma(t) \\
&= \int_{\Sigma(t)} \Re\langle \partial_t u, \partial_t^2 u + \Delta u + u\rangle \, d\Sigma(t) \\
&+ \int_{\partial\Sigma(t)} \Re\langle \partial_t u, \nabla_{\partial_r} u\rangle \, d\partial\Sigma(t)
\end{aligned}$$

where $\partial_r$ is the outward pointing normal on $\partial\Sigma(t)$. Because $|\partial_r| = 1$, we have the bound

$$\Re\langle \partial_t u, \nabla_{\partial_r} u\rangle \, d\partial\Sigma(t) \le \frac{1}{2}\int_{\partial\Sigma(t)} |\partial_t u|^2 + |\nabla u|^2 \, d\partial\Sigma(t)$$

For (4.6), we observe $\partial_t F = \partial_t - t/\bar{t}\partial_r$ and apply the first variation formula (4.10). We then have

$$\frac{1}{2}\int_{\Sigma(t)} |\partial_t u|^2 + |\nabla u|^2 + |u|^2 \frac{d}{dt} d\Sigma(t) = -\frac{1}{2\bar{t}}\int_{\partial\Sigma(t)} |\partial_t u|^2 + |\nabla u|^2 + |u|^2 \, d\partial\Sigma(t).$$

Combining our calculations for (4.5) and (4.6), we see

$$\begin{aligned}
\frac{d}{dt}E(t) &= \int_{\Sigma(t)} \Re\langle \partial_t u, \partial_t^2 u + \Delta u + u\rangle \, d\Sigma(t) + \int_{\partial\Sigma(t)} \Re\langle \partial_t u, \nabla_{\partial_r} u\rangle \\
&\quad - \frac{1}{2\bar{t}}\left(|\partial_t u|^2 + |\nabla u|^2 + |u|^2\right) \, d\partial\Sigma(t) \\
&\leq \int_{\Sigma(t)} \Re\langle \partial_t u, \partial_t^2 u + \Delta u + u\rangle \, d\Sigma(t) - \frac{1}{2\bar{t}}\int_{\partial\Sigma(t)} |u|^2 \, d\partial\Sigma(t) \\
&\quad + \left(\frac{1}{2} - \frac{1}{2\bar{t}}\right)\int_{\partial\Sigma(t)} |\partial_t u|^2 + |\nabla u|^2 \, d\partial\Sigma(t) \\
&\leq \int_{\Sigma(t)} \Re\langle \partial_t u, (1-A)u\rangle \, d\Sigma(t),
\end{aligned}$$

where in the last inequality we used the fact that $1/\bar{t} > 1$ and $f$ is supported outside of $C(T,p) \supset \Sigma(t)$. We note that because $u(0) = 0$, on $\Sigma(t)$, we have

$$\begin{aligned}
\Re\langle \partial_t u, (1-A)u\rangle &= \Re\left\langle \partial_t u, \int_0^t \partial_s((1-A)u(s)) \, ds\right\rangle \\
&\leq \sup_{\Omega'} |1-A| \, \Re\langle \partial_t u, u\rangle,
\end{aligned}$$

so we conclude

$$\begin{aligned}
\frac{d}{dt}E(t) &\leq C\int_{\Sigma(t)} \langle \partial_t u, u\rangle \, d\Sigma(t) \\
&\leq C \cdot \frac{1}{2}\int_{\Sigma(t)} |\partial_t u|^2 + |u|^2 \, d\Sigma(t) \\
&\leq C \cdot E(t).
\end{aligned}$$

Because $E(0) = 0$, this implies $E(t) = 0$ for $0 < t < \bar{t}$ and $u$ vanishes on $\Omega'$. Taking $\bar{t} \to 1$, we find $u$ vanishes on $C(T,p)$.

### 4.3 Some geometric computations

We recall the following integration by parts formula; see [Lee18, Problem 5-16] when $\nabla$ is the Levi-Civita connection:

**Lemma 3** (Integration by parts/Divergence theorem)**.** *If $\Omega \subset M$ is a bounded $C^1$ open set, with $X \in C^1(\Omega; E \otimes T^*M), f \in C^1(\Omega; E)$, then*

$$\int_\Omega \langle f, \operatorname{tr}_g \nabla X \rangle_E \, dV_\Omega = -\int_\Omega \langle \nabla f, X \rangle \, dV_\Omega + \int_{\partial\Omega} \langle f, \operatorname{tr}(X \otimes N) \rangle_E \, dS. \tag{4.7}$$

*Proof.* To see this, first consider the product rule with $f \in C^1(\Omega; \mathbb{C})$

$$\operatorname{tr}_g \nabla^E(fX) = \operatorname{tr}_g(df \otimes X) + f \operatorname{tr}_g(\nabla^E X),$$

from which we find the bundle-valued formula

$$\begin{aligned} \int_\Omega f \operatorname{tr}_g(\nabla X) \, dV_\Omega &= \int_\Omega \operatorname{tr}_g \nabla^E(fX) - \operatorname{tr}_g(df \otimes X) \, dV_\Omega \\ &= -\int_\Omega \operatorname{tr}_g(df \otimes X) \, dV_\Omega + \int_{\partial\Omega} f \operatorname{tr}(X \otimes N) \, dS, \end{aligned}$$

where $N$ is the unit outward normal vector of $\partial\Omega$ and $dS$ is the induced volume form on $\partial\Omega$. Choosing a local orthonormal frame for $E$, we can apply the above to the component functions of $f \in C^1(\Omega; E)$ to get (4.7). □

Next, we give a first variation formula for embedded submanifolds, although we only need this for the case of hypersurfaces. Proofs can be found in e.g. [CM11].

We consider an orientable $k$-dimensional embedded submanifold-with-boundary $\Sigma \subset (M, g)$, with outward pointing unit normal $\nu$ on $\partial\Sigma$. Consider a variation $F : \Sigma \times (-\epsilon, \epsilon) \to M$, which is a smooth map where for each $t \in (-\epsilon, \epsilon)$, $\Sigma_t := F(\Sigma, t)$ is an embedding and $\Sigma_0 = \Sigma$; note that we do not necessarily require $\partial\Sigma$ to be fixed. The variation field of $F$ is the vector field $\partial_t F(p, 0)$ along $\Sigma$; given a smooth vector field $V$ along $\Sigma$, we can construct an associated variation with $\partial_t F(p, 0) = V_p$ using the exponential map. In an abuse of notation, we will sometimes abbreviate $\partial_t = \partial_t F$, $\partial_i = \partial_i F$, and $\nabla_i \partial_j = \nabla_{\partial_i F} \partial_j F$. Where convenient, the time derivative will also be denoted $(\cdot)'$.

The metric $g$ pulls back to a metric on $\Sigma$ via $F(\cdot, t)$, and we denote the induced volume form on $\Sigma$ by $dV_t$. We then have

$$Vol(\Sigma_t) = \int_\Sigma dV_t\,,$$

where the left hand side is $Area(\Sigma_t)$ when $k = 2$.

Given a variation $F$ as above, we first want to express the derivative

$$\frac{d}{dt}\Big|_{t=0} Vol(\Sigma_t)$$

in terms of $H$. Due to compactness, we can differentiate under the integral $\int_\Sigma d/dt|_{t=0}\, dV_t$. In local coordinates on $\Sigma$, the induced metric on $\Sigma_t$ is $g_t(x) = (g_t(x))_{ij} dx^i dx^j$, where $(g_t)_{ij} = \langle \partial_i F\,,\, \partial_j F\rangle$, and $dV_t = \sqrt{\det g_t}\, dx^1 \wedge \cdots \wedge dx^k$,

To calculate $d/dt\, dV_t$, we first observe that the map $t \mapsto g_t$ at $p$ in coordinates defines a smooth path in $GL(k, \mathbb{R})$. Recall that for any $\mathbb{M} \in GL(k, \mathbb{R})$, the Leibniz formula implies

$$\frac{d}{dt}\Big|_{t=0} \det(I + t\mathbb{M}) = \operatorname{tr} \mathbb{M}$$

We then have:

$$\frac{d}{dt} \det\, g_t = d(\det)_{g_t} g_t' = \frac{d}{ds}\Big|_{s=0} \det(g_t(I + s g_t^{-1} g_t')) = \operatorname{tr}\,(g_t^{-1} g_t') \det g_t,$$

from which it follows

$$\frac{d}{dt}\sqrt{\det\, g_t} = \frac{1}{2}\, \operatorname{tr}(g_t^{-1} g_t') \sqrt{\det\, g_t}. \tag{4.8}$$

Choose normal coordinates $(x^i)_{i=1}^k$ with respect to the induced metric $g_0$ on $\Sigma_0$ centered at $p$; in these coordinates, we know that $(g_0)^{ij}(p) = \delta^{ij}$. Recall that the Levi-Civita connection $\nabla$ of $g$ is symmetric, hence $\nabla_t \partial_i - \nabla_i \partial_t = [\partial_t, \partial_i] = 0$. Using this and the metric compatibility of $\nabla$, we find

$$\operatorname{tr}(g_0^{-1} g_0') = \sum_{i=1}^k \frac{d}{dt}\Big|_{t=0} \langle \partial_i\,,\, \partial_i\rangle = 2\sum_{i=1}^k \langle D_t \partial_i\,,\, \partial_i\rangle = 2\sum_{i=1}^k \langle \nabla_i \partial_t\,,\, \partial_i\rangle = 2\operatorname{div}_\Sigma \partial_t F.$$

Here, for any $X \in \Gamma(TM)$, $\mathrm{div}_\Sigma X$ is defined by taking the trace of the map $Y \mapsto \nabla^{\|}_Y X$, $Y \in T\Sigma$. We therefore find (4.8) becomes

$$\frac{d}{dt}\Big|_{t=0} \sqrt{\det g_t} = \sqrt{\det g_0}\ \mathrm{div}_\Sigma\, \partial_t F.$$

Since we can find normal coordinates at any $p \in \Sigma$, we conclude that $d/dt|_{t=0}\, dV_t = \mathrm{div}_\Sigma \partial_t F\, dV_0$.

To express this in terms of the mean curvature vector $H$, let $(N_\ell)_{\ell=1}^{n-k}$ be an orthonormal frame for the normal bundle $N\Sigma$ in a neighbourhood of $p$. Decomposing $\partial_t F$ into normal and tangential components, we calculate using normal coordinates at $p$:

$$\begin{aligned}
\mathrm{div}_\Sigma \partial_t &= \mathrm{div}_\Sigma \left( \sum_{\ell=1}^{n-k} \langle \partial_t\, ,\, N_\ell \rangle N_\ell \right) + \mathrm{div}_\Sigma (\partial_t)^{\|} \\
&= \sum_{i=1}^{k} \sum_{\ell=1}^{n-k} \langle \nabla_i \langle \partial_t\, ,\, N_\ell \rangle N_\ell\, ,\, \partial_i \rangle + \mathrm{div}_\Sigma (\partial_t)^{\|} \\
&= \sum_{i=1}^{k} \sum_{\ell=1}^{n-k} \langle \partial_t\, ,\, N_\ell \rangle \langle \nabla_i N_\ell\, ,\, \partial_i \rangle + \mathrm{div}_\Sigma (\partial_t)^{\|} \\
&= -\sum_{i=1}^{k} \sum_{\ell=1}^{n-k} \langle \partial_t\, ,\, N_\ell \rangle \langle \nabla_i \partial_i\, ,\, N_\ell \rangle + \mathrm{div}_\Sigma (\partial_t)^{\|} \\
&= -\sum_{i=1}^{k} \langle (\partial_t)^N\, ,\, A(\partial_i, \partial_i) \rangle + \mathrm{div}_\Sigma (\partial_t)^{\|} \\
&= -\langle \partial_t F\, ,\, H \rangle + \mathrm{div}_\Sigma (\partial_t F)^{\|}.
\end{aligned}$$

The third equality above follows from $\langle N_\ell\, ,\, \partial_i F \rangle \equiv 0$ and the fourth from metric compatibility.

We note that when $X$ is a smooth vector field on a neighbourhood of $p$ in $M$ that is tangent to $\Sigma$, then $\mathrm{div}_\Sigma X$ is the usual divergence on $\Sigma$. Our previous formula (4.7) then gives:

**Proposition 4** (First Variation Formula)**.** *Let $X$ be a variation field on $\Sigma$, with variation $F(\Sigma, t) = \Sigma_t$. Then*

$$\frac{d}{dt}\Big|_{t=0} Vol(\Sigma_t) = \int_\Sigma \mathrm{div}_\Sigma\ X\, dV_0 = -\int_\Sigma \langle X\, ,\, H \rangle\, dV_0 + \int_{\partial\Sigma} \langle X\, ,\, \nu \rangle\, dS_0 \tag{4.9}$$

*where $dS_0$ is the induced volume form on the boundary. In particular, when $\Sigma$ is minimal (i.e. $H \equiv 0$), we have*

$$\frac{d}{dt}\Big|_{t=0} Vol(\Sigma_t) = \int_{\partial\Sigma} \langle X \, , \, \nu \rangle \, dS_0 \tag{4.10}$$

# Chapter 5

# UNIQUE CONTINUATION FOR THE WAVE EQUATION

In this chapter, we give a detailed exposition of the local unique continuation principle (UCP) for the bundle-valued wave equation via Carleman estimates given in [Tat95, EINT02]. We closely follow [KKL01, Ch. 2.5], which gives an exposition of the same result for the scalar-valued case. However, we have reorganized the order of presentation, included details, and added more geometric intuition where possible.

## *5.1 Notation*

We will work on the manifold $\mathbf{M} := \mathbb{R} \times M$, equipped with either the Riemannian metric $\mathbf{g} := dt^2 \oplus g$ or the Lorentzian metric $\mathbf{h} := -dt^2 \oplus g$. The isomorphisms between $T^*\mathbf{M}$ and $T\mathbf{M}$ for raising and lowering indices with respect to $\mathbf{h}$ are both denoted $\mathbf{J}$; its meaning will be clear depending on context. The notation $\mathbf{J}_0$ refers to the same isomorphism at the (co)tangent space to the point $y = 0$ in local coordinates. The exterior differential on $\mathbf{M}$ is denoted $\mathbf{d}$, while the unbolded $d$ is the exterior differential on $M$.

A Hermitian vector bundle $(E, \langle\cdot,\cdot\rangle_E)$ over $M$ pulls back along the projection $\mathbb{R}\times M \to M$, and we will refer to this pullback bundle over $\mathbf{M}$ by $E$ as well. If $E \to M$ is equipped with a compatible connection $\nabla^E$, then the pullback bundle $E \to \mathbf{M}$ can be equipped with either the pullback connection, denoted $\nabla_M$, or the connection $\nabla := \partial_t + \nabla_M$. The formal adjoint of an operator $B$ on $\mathbf{M}$ with respect to $\mathbf{g}$ is denoted $B^*$, whereas the formal adjoint taken with respect to the (indefinite) $L^2$-inner product induced by $\mathbf{h}$ is denoted $B^T$.

If $(U, x)$ is a chart on $M$, we relabel the coordinates on the product neighborhood $\mathbb{R}\times U \subset \mathbf{M}$ to be $y := (y^0, y') := (t, x)$. We adopt Einstein summation in all of our calculations, where bold indices $\mathbf{j}$ correspond to $\mathbf{M}$; normal font indices $j$ correspond to $M$; and Greek indices

$\alpha$ correspond to $E$. It will be convenient to have the notation

$$\begin{cases} D_j & := -i\partial_j \\ \mathbf{D_j} & := -i\partial_{\mathbf{j}} \\ D^\nabla & := -i\nabla_M \\ \mathbf{D}^\nabla & := -i\nabla. \end{cases}$$

These conventions mean $D_j = \mathbf{D}_j$, $D_j^\nabla = \mathbf{D}_j^\nabla$, and $(D^\nabla)^* = +\operatorname{tr}_g D^\nabla$.

Fixing $y$, we may choose coordinates on the ball $B(y,\delta)$ such that $y = 0$. We consider our bundle wave operator

$$\mathbf{P}(y,\mathbf{D}) := \partial_0^2 + (\Delta^E + A) = -D_0^2 + (D^\nabla)^*(D^\nabla) = (\mathbf{D}^\nabla)^T\mathbf{D}^\nabla =: \Box \tag{5.1}$$

where $\Delta^E = \nabla_M^*\nabla_M = -\operatorname{tr}_g(\nabla_M)^2 = \operatorname{tr}_g(D^\nabla)^2$ is the bundle Laplacian on $E$. We may assume $B(y,\delta)$ is a trivializing neighborhood and admits an orthonormal frame $(E_\alpha)$, with dual frame $(\mathcal{E}^\alpha)$ and connection 1-forms $\omega = \omega_{j\alpha}^\beta := \langle \nabla_j E_\alpha, E_\beta\rangle$. The full symbol can be decomposed into the principal (second-order) symbol and a lower order symbol:

$$\begin{aligned} \mathbf{P}(y,\xi) &= p(y,\xi) + p_1(y,\xi) \\ p(y,\xi) &:= \left(-(\xi_0)^2 + g^{ij}(y')\xi_i\xi_j\right)\operatorname{Id}_E \\ p_1(y,\xi) &:= -i\left(\frac{1}{2}g^{ij}g^{kl}\partial_j g_{kl} + \partial_j g^{ij}\right)\xi_i \operatorname{Id}_E - i(g^{ij}\omega_{j\alpha}^\gamma\xi_i)\,\mathcal{E}^\alpha\otimes E_\gamma \\ &\quad - \left(\frac{1}{2}g^{ij}g^{kl}\partial_j g_{kl}\omega_{i\alpha}^\gamma + \partial_j g^{ij}\omega_{i\alpha}^\gamma + g^{ij}\partial_j\omega_{i\alpha}^\gamma + g^{ij}\omega_{i\alpha}^\eta\omega_{j\eta}^\gamma + A_\alpha^\gamma\right)\mathcal{E}^\alpha\otimes E_\gamma \end{aligned}$$

We will freely abuse notation when it comes to the local $L^2$ and $H^1$ spaces on the trivializing neighborhood $B(y,\delta)$. Depending on context, the integration will either be taken with respect to the Lebesgue measure $dy$ in coordinates or the Riemannian density $d\mathbf{V}$ in local coordinates. These are equivalent locally. The only time we wish to point out the distinction is in the section proving the Carleman inequality, where estimation of $\mathbf{p}_{\Re}(y,\mathbf{D},\tau)$ involving formally self-adjoint operators is to be taken with respect to the local Lebesgue measure $dy$ in coordinates.

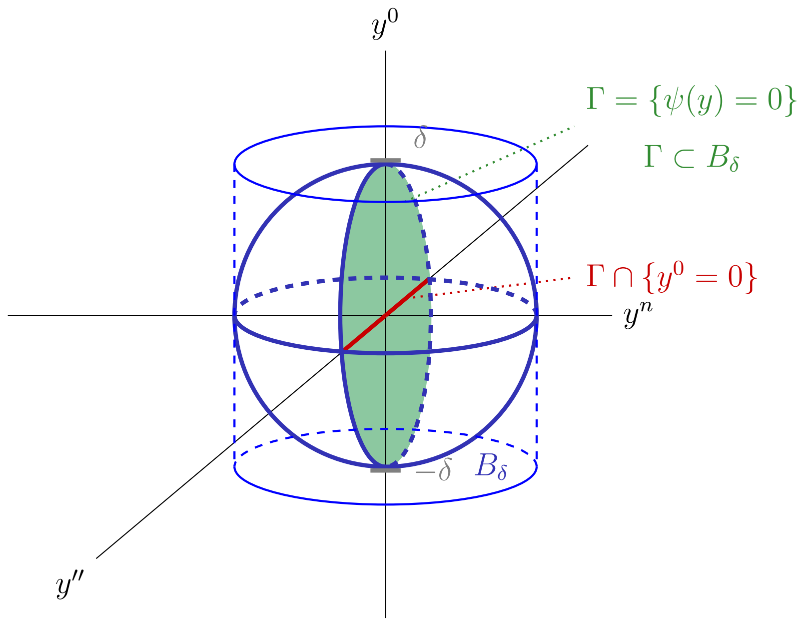


Figure 5.1: Local setup of UCP

### *5.2 UCP Statement and Outline*

We begin with the statement of local unique continuation:

**Theorem 5** (Tataru's UCP)**.** *Suppose* $u \in H^1(\mathbf{M}; E)$ *is a solution to* $\mathbf{P}(y, D)u = 0$ *in* $B(y, \delta)$*. For a local defining function* $\psi$*, let* $\Gamma = \{\psi(y) = 0\}$ *be a hypersurface in* $B(y, \delta)$ *satisfying the following:*

1. *The surface is noncharacteristic, that is,* $p(y, \boldsymbol{d}\psi) \neq 0$ *on* $\Gamma$*, and*

2. *The support of* $u$ *vanishes on one side of* $\Gamma$*, that is,* $\operatorname{supp}(u) \subset \{\psi(y) \leq 0\}$.

*It follows that* $u$ *must vanish in a neighborhood of* $\Gamma$*.*

For the proof, we work in local coordinates on a ball $B(0, \delta)$ centered at $y = 0 \in \Gamma$ and show that given the hypotheses of Theorem 5, we must have $u = 0$ in some neighborhood $U \subset B(0, \delta)$ of 0. See Figure 5.1. We rely on three main tools, to be elaborated on in the following sections:

1. We are motivated by the observation that if $\|e^{\tau\phi}u\|_{L^2} < C$ uniformly for all $\tau > \tau_0$ for some $\tau_0 > 0$, then $\text{supp}(u) \subset \{\phi(y) \leq 0\}$.

   Define the operator

$$Q^{\phi}_{\epsilon,\tau} := e^{(-\epsilon/2\tau)D_0^2} e^{\tau\phi(y)}, \tag{5.2}$$

   where $e^{(-\epsilon/2\tau)D_0^2}$ acts as a Fourier multiplier in the $y^0$ variable and $\phi \in C^\infty(\mathbb{R})$. Let $u \in L^2(\mathbb{R}^{n+1})$ have compact support, and fix $\epsilon > 0$. We claim that if $\|Q^{\phi}_{\epsilon,\tau}u\|_{L^2} \leq C$ uniformly for all $\tau > \tau_0 > 0$, then $\text{supp}(u) \subset \{\phi(y) \leq 0\}$.

2. In $B(0, \delta)$, we can replace $\psi$ with a second-order polynomial $\phi$ satisfying certain *pseudoconvexity* conditions with respect to $p(0, \xi)$, to be explained later. In particular, $\phi$ can be chosen such that $\text{supp}(u) \subset \{\phi < 0\} \cup \{0\}$.

3. For reasons that will shortly become apparent in the proof of UCP, we need the following technical result, known as a *Carleman estimate*, given in [Tat95] for scalar-valued $\mathbf{P}$ and [EINT02] for the system that concerns us.

   **Proposition 6** (Carleman estimate)**.** *Suppose $B(0, \kappa/16) \subset B(0, \delta)$, $\boldsymbol{P}$ is the wave operator above, and $Q^{\phi}_{\epsilon,\tau}$ with pseudoconvex $\phi$ is as defined before. If $v \in H^1_0(B(0, \kappa/16)$ such that $\boldsymbol{P}(y, \boldsymbol{D})v \in L^2(B(0, \kappa/16))$, then there exists $\tau_0(\delta, \kappa), c = c(\delta, \kappa) > 0$ such that*

$$\|Q^{\phi}_{\epsilon,\tau}v\|_{1;\tau} \leq c\tau^{-1/2}\|Q^{\phi}_{\epsilon,\tau}\boldsymbol{P}(y, \boldsymbol{D})v\|_{L^2} + ce^{-\frac{\tau\kappa^2}{16\epsilon}}\|e^{\tau\phi}v\|_{1;\tau}$$

   *for all $\tau > \tau_0$, where we use the weighted $L^2$-Sobolev norm $\|v\|^2_{1;\tau} := \|\boldsymbol{D}v\|^2_{L^2} + \tau^2\|v\|^2_{L^2}$.*

We first prove UCP assuming these three tools before proceeding to elaborate on the preceding technicalities in the remainder of this chapter.

*Proof.* On $B(0, \delta)$, we choose a smooth cutoff function $\chi \in C_0^\infty(B(0, \kappa/16))$ with $\chi = 1$ on $B(0, \rho)$ for $0 < \rho < \kappa/16 < \delta$. From (1), it suffices to show that for our pseudoconvex $\phi$

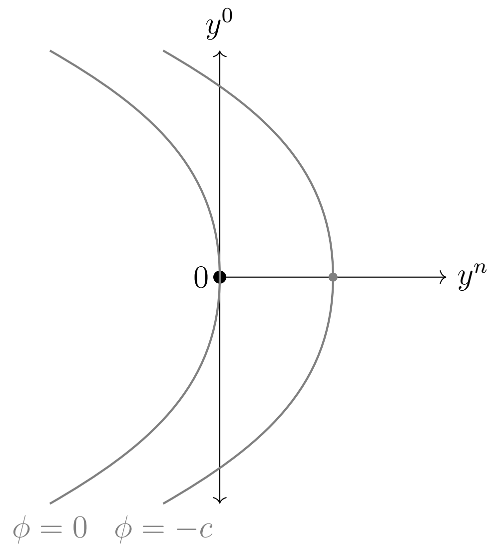


Figure 5.2: Level sets of $\phi$

chosen in (2), there exists $c_0 > 0$ such that for all $\tau > \tau_0$,

$$e^{c_0\tau}\|Q^{\phi}_{\epsilon,\tau}(\chi u)\|_{L^2} = \|Q^{\phi+c_0}_{\epsilon,\tau}(\chi u)\|_{L^2} \leq C.$$

The Carleman estimate (3) applied to $v = \chi u$ gives $\tau_0, c$ such that, for $\tau > \tau_0$,

$$\|Q^{\phi}_{\epsilon,\tau}\chi u\|_{1;\tau} \leq c\tau^{-1/2}\|Q^{\phi}_{\epsilon,\tau}\mathbf{P}(y,\mathbf{D})\chi u\|_{L^2} + ce^{-\frac{\tau\kappa^2}{16\epsilon}}\|e^{\tau\phi}\chi u\|_{1;\tau}. \tag{5.3}$$

To estimate the first term on the right in (5.3), we see that our choice of $\phi$ gives $\operatorname{supp}\mathbf{P}(y,\mathbf{D})(\chi u) \subset \{\phi < 0\} \setminus B(0,\rho) \subset \{\phi \leq -c'\}$, $c' > 0$, since $\mathbf{P}u = 0$ in $B(0,\rho)$. By Plancherel, we have

$$\begin{aligned}\|Q^{\phi}_{\epsilon,\tau}\mathbf{P}(y,\mathbf{D})\chi u\|_{L^2} &= C\|e^{(-\epsilon\xi_0^2/2\tau)}\mathcal{F}(e^{\tau\phi}\mathbf{P}\chi u)\|_{L^2} \\ &\leq Ce^{-\tau c'}\|\mathbf{P}\chi u\|_{L^2} \\ &\leq Ce^{-\tau c'}\|u\|_{H^1},\end{aligned} \tag{5.4}$$

where the last inequality follows from $\chi\mathbf{P}(y,\mathbf{D})u = 0$, so $\mathbf{P}(y,\mathbf{D})\chi u$ sees at most one derivative of $u$.

The second term on the right in (5.3) can be bounded by

$$\|e^{\tau\phi}\chi u\|_{1;\tau}^2 \leq \tau^2\|\chi u\|_{L^2}^2 + C\tau^2\|\chi u\|_{L^2}^2 + \|\mathbf{D}(\chi u)\|_{L^2}^2 \leq C\tau^2\|u\|_{H^1}^2 \tag{5.5}$$

for sufficiently large $\tau$. Combining (5.4) and (5.5) yields

$$\|Q_{\epsilon,\tau}^{\phi}\chi u\|_{L^2} \leq C\|Q_{\epsilon,\tau}^{\phi}\chi u\|_{1;\tau} \leq C(\tau^{-1/2}e^{-\tau c'} + \tau e^{-\frac{\tau\kappa^2}{16\epsilon}})\|u\|_{H^1} \leq Ce^{-c_0\tau}\|u\|_{H^1}$$

for all $\tau > \tau_0$. We conclude $\operatorname{supp}(u) \subset \{\phi \leq -c_0\}$, hence $u = 0$ in a neighborhood of 0. See Figure 5.2. □

### 5.3 *Time smoothing operator with exponential weight*

Our first tool concerns the operator $Q_{\epsilon,\tau}^{\phi}$. As mentioned in step (1) of our outline previously, we show

**Lemma 7.** *Let $u \in L^2(\mathbb{R}^{n+1})$ have compact support, and fix $\epsilon > 0$. If $\|Q_{\epsilon,\tau}^{\phi}u\|_{L^2} \leq C$ uniformly for all $\tau > \tau_0 > 0$, then* $\operatorname{supp}(u) \subset \{\phi(y) \leq 0\}$.

We will need the Phragmén-Lindelöf principle, which can be found in any standard complex analysis text, e.g. [Rud74]:

**Lemma 8** (Phragmén-Lindelöf Principle)**.** *Let $S_\alpha := \{z \in \mathbb{C} \setminus 0 : a < \arg(z) < b,\, b - a = \pi/\alpha\}$ be a sector of aperture $\pi/\alpha$. If an analytic function $F(z)$ is bounded on $\partial\overline{S_\alpha}$ and satisfies the growth condition*

$$|F(z)| \leq Ce^{c|z|^{\alpha'}},\ \alpha' < \alpha, \tag{5.6}$$

*then $F(z)$ is bounded on $S_\alpha$.*

*Proof of Lemma 7.* Consider $f \in S(\mathbb{R}^{n+1})$ where

$$S(\mathbb{R}^{n+1}) := \{f \in \mathcal{S}(\mathbb{R}^{n+1}) : \hat{f} \in C_0^\infty(\mathbb{R}^{n+1})\}. \tag{5.7}$$

We then define a compactly supported distribution $\phi_*(fu) \in \mathcal{E}'(\mathbb{R})$ given by

$$\langle\phi_*(fu), \psi\rangle := (fu, \overline{\psi\circ\phi})_{L^2(\mathbb{R}^{n+1})},\ \psi \in C^\infty(\mathbb{R}^{n+1}).$$

Observe that this vanishes whenever $\psi$ is supported in the complement of $\phi(\text{supp}(u))$, hence $\text{supp}(\phi_*(fu)) \subset \phi(\text{supp}(u))$, which is indeed compact by our choice of $u$.

We may then take the Fourier transform [Hör03]

$$\widehat{\phi_*(fu)} := \langle \phi_*(fu), e^{i\tau\phi(\cdot)} \rangle = (u, \overline{f} e^{i\tau\phi(\cdot)})_{L^2(\mathbb{R}^{n+1})}$$

for any $\tau \in \mathbb{C}$. Owing to the compact support of $u$, it is easy to verify $\widehat{\phi_*(fu)}(\tau)$ is analytic in $\tau$ by differentiating under the integral, and furthermore $\widehat{\phi_*(fu)}(\tau)$ is uniformly bounded in when $\tau \in \mathbb{R}$. If $\tau > 1$,we also see from the compact support of $\hat{f}$ that

$$\begin{aligned}
|\widehat{\phi_*(fu)}(i\tau)| &= |(e^{-\tau\phi}u, \overline{f})_{L^2(\mathbb{R}^{n+1})}| \\
&= |(e^{(-\epsilon/2\tau)\xi_0^2} \mathcal{F} e^{-\tau\phi} u, e^{(+\epsilon/2\tau)\xi_0^2} \mathcal{F}\overline{f})_{L^2(\mathbb{R}^{n+1})}| \\
&= |(\mathcal{F}(Q^{\phi}_{\epsilon,\tau} u), e^{(+\epsilon/2\tau)\xi_0^2} \mathcal{F}\overline{f})_{L^2(\mathbb{R}^{n+1})}| \\
&\leq C \|Q^{\phi}_{\epsilon,\tau} u\|_{L^2} \|e^{(+\epsilon/2\tau)\xi_0^2} \hat{f}\|_{L^2} < C'.
\end{aligned}$$

Let $c := \max_{supp(u)} |\phi|$. For any $\tau \in \mathbb{C}$, we calculate directly to find

$$|\widehat{\phi_*(fu)}(\tau)| \leq e^{c|\tau|} \|uf\|_{L^2} = Ce^{c|\tau|}.$$

By our above estimates, $\widehat{\phi_*(fu)}(\tau)$ satifies the hypotheses for Phragmén-Lindelöf (5.6) on $S_2$ corresponding to both the first and second quadrants, with $\alpha' = 1 < 2$ in both quadrants. We conclude that $\widehat{\phi_*(fu)}(\tau)$ is bounded on the upper half-plane.

The Paley-Wiener-Schwartz Theorem [Hör03, Thm. 7.3.1] implies $\phi_*(fu)$ must be supported in $\mathbb{R}_{\leq 0}$: that is,

$$\langle \phi_*(fu), \psi \rangle = \int_{\text{supp}(u)} fu(y)(\psi \circ \phi)(y)\, dy = 0$$

for all $\psi \in C_0^\infty(\mathbb{R}_{>0})$.

Due to the density of $C_0^\infty$ in $L^2$ and Plancherel, we see that $S(\mathbb{R}^{n+1})$ is dense in $L^2$, hence $u(y)(\psi \circ \phi)(y) = 0$ on $\text{supp}(u)$. Choosing $\psi$ to be any mollified characteristic function for an interval $I \subset \mathbb{R}_{>0}$, we conclude $\text{supp}(u) \subset \{\phi \leq 0\}$. □

### 5.4 Pseudoconvexity

We now take a look at step (2) of our outline. Let $\mathbf{p}(y,\xi,\tau) := p(y,\xi + i\tau\mathbf{d}\phi(y))$. As we will see in the subsequent section, this symbol naturally appears upon conjugating our operator $\mathbf{P}(y,\mathbf{D})$ by $e^{\tau\phi}$; such a conjugation is needed for the subsequent Carleman estimate. In what follows, our definition of the Poisson bracket is

$$\{p,q\} := \sum_{\mathbf{j}} \partial_{y^{\mathbf{j}}} p \, \partial_{\xi_{\mathbf{j}}} q - \partial_{\xi_{\mathbf{j}}} p \, \partial_{y^{\mathbf{j}}} q.$$

Note that this is the same convention as [Lee18, Tat95] but opposite that of [Zwo12].

**Definition 4** (Pseudoconvexity)**.** *We say a second order polynomial $\phi$ is pseudoconvex with respect to $p(y,\xi)$ at 0 if the following conditions are satisfied:*

1. $\mathrm{supp}(u) \subset \{\psi \leq 0\} \subset \{\phi < 0\} \cup \{0\}$;
2. $\{\psi = 0\} \cap \{\phi = 0\} = 0$;
3. *On the set*
$$\{\xi_0 = 0, \tau > 0, \mathbf{p}(0,\xi,\tau) = 0\}, \tag{5.8}$$
$\mathbf{p}$ *satisfies*
$$\frac{\{\mathbf{p}(0,\xi,\tau), \overline{\mathbf{p}}(0,\xi,\tau)\}}{8\tau i} > 0 \tag{5.9}$$

While this is a straightforward analytic condition to check, the geometric picture is not immediately clear. We will first clarify this before proceeding to construct such pseudoconvex $\phi$ for a noncharacteristic surface $\Gamma$.

#### *5.4.1 Geometric meaning*

We have the property

$$\{p,\overline{p}\} = 2i \sum_{\mathbf{j}} \partial_{y^{\mathbf{j}}} \Im p \, \partial_{\xi_{\mathbf{j}}} \Re p - \partial_{\xi_{\mathbf{j}}} \Im p \, \partial_{y^{\mathbf{j}}} \Re p = 2i \{\Im p, \Re p\}\,,$$

from which a direct calculation shows

$$\begin{aligned}\mathbf{p}(y,\xi,\tau) &= \langle \xi + i\tau\mathbf{d}\phi(y), \xi + i\tau\mathbf{d}\phi(y)\rangle_{\mathbf{h}} \\ &= \big(\langle \xi, \mathbf{J}_0\xi\rangle - \tau^2\langle \mathbf{d}\phi, \mathbf{J}\mathbf{d}\phi\rangle\big) + 2\tau i\langle \xi, J\mathbf{d}\phi\rangle.\end{aligned}$$

We take coordinates at $y = 0$ such that $y' = (y'', y^n)$ are Fermi coordinates with $y''$ normal coordinates centered at $0 \in \Gamma \cap \{y^0 = 0\}$. By another calculation, we find

$$\begin{aligned}\partial_{y^{\mathbf{j}}}\mathbf{p}(0,\xi,\tau) &= -\tau^2\nabla^2\phi(0)(\partial_{y^{\mathbf{j}}}, \mathbf{J}\mathbf{d}\phi(0)) + 2\tau i\nabla^2\phi(0)(\partial_{y^{\mathbf{j}}}, \mathbf{J}_0\xi) \\ \partial_{\xi_{\mathbf{j}}}\mathbf{p}(0,\xi,\tau) &= 2dy^{\mathbf{j}}(\mathbf{J}_0\xi) + 2\tau i dy^{\mathbf{j}}(\mathbf{J}\mathbf{d}\phi(0)).\end{aligned}$$

Putting the above together yields

$$\{\mathbf{p}(0,\xi,\tau), \overline{\mathbf{p}}(0,\xi,\tau)\} = 2i\left(4\tau\nabla^2\phi(0)(\mathbf{J}_0\xi, \mathbf{J}_0\xi) + 4\tau^3\nabla^2\phi(0)(\mathbf{J}\mathbf{d}\phi(0), \mathbf{J}\mathbf{d}\phi(0))\right),$$

from which we see

$$\Pi_\phi(\xi,\tau) := \frac{1}{8\tau i}\{\mathbf{p}(0,\xi,\tau), \overline{\mathbf{p}}(0,\xi,\tau)\} = \nabla^2\phi(0)(\mathbf{J}_0\xi, \mathbf{J}_0\xi) + \tau^2\nabla^2\phi(0)(\mathbf{J}\mathbf{d}\phi(0), \mathbf{J}\mathbf{d}\phi(0)). \tag{5.10}$$

When $\xi_0 = 0$, in our chosen normal coordinates, we look at the real and imaginary parts in (5.10) to find that $\mathbf{p}(0,\xi,\tau)$ is equivalent to

$$\begin{cases}\|\xi'\|_g^2 + \tau^2(\partial_0\phi)^2 &= \tau^2\|d\phi(0)\|_g^2 \\ \langle \xi', d\phi(0)\rangle_g = 0.\end{cases}$$

Since $\mathbf{d}\phi(0) = \partial_0\phi(0)dy^0 + \partial_n\phi(0)dy^n$, we find that the the set (5.8) is given by

$$\begin{cases}\xi &= (0, \xi'', 0) \\ \|\xi''\|_g^2 &= \tau^2\left(\|d\phi(0)\|_g^2 - (\partial_0\phi(0))^2\right), \ \tau > 0\end{cases} \tag{5.11}$$

Note that this is the empty set when $\Gamma$ is spacelike, whereas when $\Gamma$ is timelike, this is a $\tau$-family of ellipses along the codimension two subspace spanned by $dy''$.

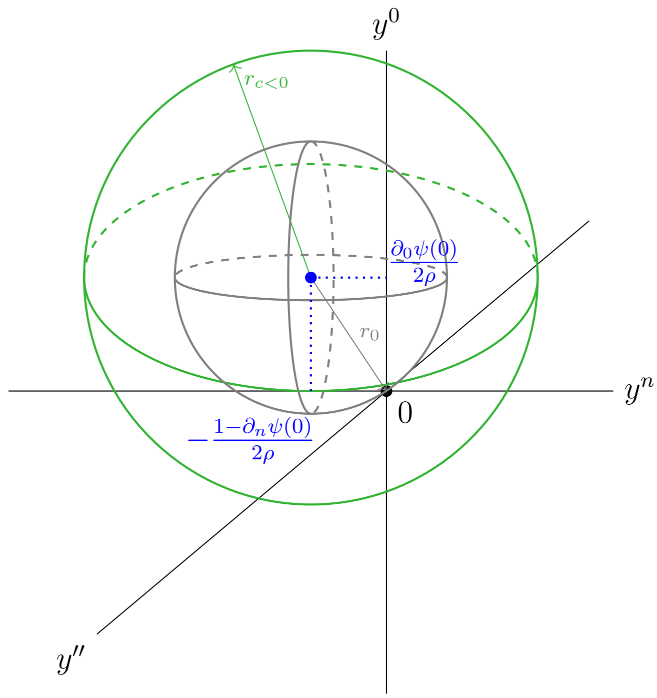


Figure 5.3: Spherical level sets

The condition $\Pi_\phi(\xi,\tau) > 0$ on such as set is equivalent to $\Pi_\phi(\xi'') > 0$ on $|\xi''| = 1$ for

$$\Pi_\phi(\xi) := \nabla^2\phi(0)(\mathbf{J}_0\xi, \mathbf{J}_0\xi) + \frac{\nabla^2\phi(0)(\mathbf{Jd}\phi(0), \mathbf{Jd}\phi(0))}{\langle \mathbf{d}\phi(0), \mathbf{Jd}\phi(0)\rangle} > 0. \tag{5.12}$$

Therefore, we find

**Proposition 9.** *Condition (3) of pseudoconvexity for $\phi$ is equivalent to the strict positivity of the bilinear form in* (5.12) *whenever $\Gamma$ is timelike near* 0*.*

### *5.4.2 Construction of pseudoconvex weight*

To construct such a $\phi$, we first find a second order polynomial satisfying conditions (1) and (2). Let

$$\tilde{\psi}(y) := \partial_0\psi(0)y^0 + \partial_n\psi(0)y^n - \rho|y|^2,\ \rho > 0.$$

Observe that the level sets $\{\tilde{\psi}(y) = c\}$ are given by spheres

$$\left\{ \left(y^n + \frac{1 - \partial_n \psi(0)}{2\rho}\right)^2 + \left(y^0 - \frac{\partial_0 \psi(0)}{2\rho}\right)^2 + |y''|^2 = r_c^2 \right\},$$

$$r_c^2 := \frac{(1 - \partial_n \psi(0))^2 + \partial_0 \psi(0)^2}{4\rho^2} - c/\rho.$$

See Figure 5.3. By Taylor's theorem, we can choose $\rho \geq 0$ large enough such that

$$\tilde{\psi}(y) = \partial_0 \psi(0) y^0 + \partial_n \psi(0) y^n - \rho |y|^2 < \psi(y),\ y \neq 0$$

so that $\{\psi \leq 0\} \subset \{\tilde{\psi} < 0\} \cup \{0\}$, thereby satisfying (1) and (2).

If $\Gamma$ is spacelike, condition (3) is trivially satisfied as remarked earlier, and we set $\phi = \tilde{\psi}$. Otherwise, if $\Gamma$ is timelike, let us consider $\tilde{\phi} := \lambda \tilde{\psi} + (\lambda \tilde{\psi})^2$. We calculate

$$\mathbf{d}\tilde{\phi}(0) = \lambda \mathbf{d}\tilde{\psi}(0),\ \nabla^2 \tilde{\psi}(0) = \lambda \nabla^2 \tilde{\psi}(0) + 2\lambda^2 \mathbf{d}\tilde{\psi}(0) \otimes \mathbf{d}\tilde{\psi}(0).$$

It follows by more calculation that

$$\begin{aligned} \Pi_{\tilde{\phi}}(\xi'') = \lambda &\left( \nabla^2 \tilde{\psi}(\xi'', \xi'') + \frac{\nabla^2 \tilde{\psi}(0)(\mathbf{J}\mathbf{d}\tilde{\psi}(0), \mathbf{J}\mathbf{d}\tilde{\psi}(0))}{\langle \mathbf{d}\tilde{\psi}(0), \mathbf{d}\tilde{\psi}(0) \rangle_{\mathbf{h}}} \right) \\ &+ 2\lambda^2 \left( \langle \mathbf{d}\tilde{\psi}(0), \xi'' \rangle_g^2 + \langle \mathbf{d}\tilde{\psi}(0), \mathbf{d}\tilde{\psi}(0) \rangle_{\mathbf{h}} \right) \\ &\geq \lambda \left( \min_{|\xi''|=1} \Pi_{\tilde{\psi}}(\xi'') + 2\lambda \langle \mathbf{d}\tilde{\psi}(0), \mathbf{d}\tilde{\psi}(0) \rangle_{\mathbf{h}} \right), \end{aligned}$$

where we get to the last line by observing $\mathbf{d}\tilde{\psi}(0) = \mathbf{d}\psi(0)$. By choosing $\lambda$ large enough, we see that $\Pi_{\tilde{\phi}}(\xi'') \geq C > 0$. By Taylor's theorem, we can assume $B(0, \delta)$ is small enough such that

$$\phi(y) := \sum_{|\alpha| \leq 2} \frac{\partial_\alpha \tilde{\phi}(0)}{\alpha!} y^\alpha - \rho' |y|^2,\ \rho' = \rho(\delta) > 0$$

satisfies $\phi(y) \leq \tilde{\phi}(y) \leq \tilde{\psi}(y) \leq 0$, where strict equality holds only at $y = 0$, and moreover $\Pi_\phi = \Pi_{\tilde{\phi}} + O(\rho') > C/2 > 0$. We see then that $\phi$ as constructed is pseudoconvex at 0 with respect to $p(y, \xi)$.

It turns out that when we conjugate $\mathbf{P}(y,\mathbf{D})$ by $Q^{\phi}_{\epsilon,\tau}$ for our proof of the Carleman estamate, we will need to consider certain perturbations of $\mathbf{p}(y,\xi,\tau)$. For small $\epsilon > 0$, consider the symbol $p(x,\xi - \epsilon\nabla^2\phi(0)(\xi_0,\cdot))$ where $\xi_0 = (\xi_0, 0)$. We claim $\phi$ remains pseudoconvex at 0 with respect to this symbol. Indeed, conditions (1)-(2) remain the same. Observe that being either spacelike or timelike is an open condition, hence persists for small $\epsilon$. In the latter case, a calculation similar to the preceding shows that condition (3) still holds.

## 5.5 *Conjugating the symbol*

As mentioned earlier, we will eventually need estimates that involve conjugating our operator by $Q^{\phi}_{\epsilon,\tau}$, hence we need the symbols of the conjugated operators. We show this in two steps:

### *5.5.1 Conjugation by exponential weight*

Our first goal is to recover the form of the conjugated operator $\mathbf{P}(y,D,\tau) = e^{\tau\phi}\mathbf{P}(y,D)e^{-\tau\phi}$ so that $e^{\tau\phi}\mathbf{P}(y,\mathbf{D}) = \mathbf{P}(y,\mathbf{D},\tau)e^{\tau\phi}$. We record the following computation:

**Lemma 10.** *Let $f,\phi \in C^\infty(M;\mathbb{R}), X \in C^\infty(M;E), Y \in C^\infty(M; E\otimes T^*M)$. The following identities hold:*

$$\nabla_M(fX) = df \otimes X + f\nabla_M X,$$
$$\nabla^*_M(fY) = -\operatorname{tr}_g \nabla_M(fY) = -\operatorname{tr}_g(df\otimes Y) + f\nabla^*_M Y = \left((df\otimes)^* + f\nabla^*_M\right)Y.$$

This is used to show:

**Proposition 11.** *Let $v \in H^1(\mathbb{R}^{n+1})$ with $\operatorname{supp}(v) \subset B(0,\delta)$. We have $e^{\tau\phi}\mathbf{P}(y,\boldsymbol{D})v = \boldsymbol{P}(y,\boldsymbol{D},\tau)e^{\tau\phi}v$ for*

$$\begin{aligned}\mathbf{P}(y,\boldsymbol{D},\tau) = \mathbf{P}(y,\boldsymbol{D}+i\tau\mathbf{d}\phi) &= -(D_0 + i\tau\partial_0\phi dy^0\otimes)^2 + \left(D^\nabla + i\tau d\phi\otimes\right)^*\left(D^\nabla + i\tau d\phi\otimes\right) + A\\ &= \left(\boldsymbol{D}^\nabla + i\tau\boldsymbol{d}\phi\otimes\right)^T\left(\boldsymbol{D}^\nabla + i\tau\boldsymbol{d}\phi\otimes\right) + A\\ &= \boldsymbol{P}_{\Re}(y,\boldsymbol{D},\tau) + i\boldsymbol{P}_{\Im}(y,\boldsymbol{D},\tau)\end{aligned}$$

*where*

$$\boldsymbol{P}_{\Re}(y, \boldsymbol{D}, \tau) = \Box - \tau^2 \langle \boldsymbol{d}\phi, \boldsymbol{d}\phi \rangle_{\boldsymbol{h}} + A$$
$$\boldsymbol{P}_{\Im}(y, \boldsymbol{D}, \tau) = \tau \operatorname{tr}_{\boldsymbol{h}} \left( \boldsymbol{d}\phi \otimes \boldsymbol{D}^{\nabla} + \boldsymbol{D}^{\nabla} (\boldsymbol{d}\phi \otimes \cdot) \right).$$

*Proof.* Using Lemma 10, we calculate

$$\nabla(e^{-\tau\phi} u) = e^{-\tau\phi}(\nabla - \tau d\phi \otimes) u,$$

and so

$$\begin{aligned} e^{\tau\phi} \nabla^* \nabla e^{-\tau\phi} u &= e^{\tau\phi} \left( (d(e^{-\tau\phi}) \otimes)^* + e^{-\tau\phi} \nabla^* \right) (\nabla - \tau d\phi \otimes) u \\ &= (\nabla - \tau d\phi \otimes)^* (\nabla - \tau d\phi \otimes) u. \end{aligned}$$

Following the same but more straightforward calculation for $\partial_0^2 = -\partial_0^* \partial_0$, we conclude the proof. □

Observe that $\mathbf{p}(y, \xi, \tau)$ from our section on pseudoconvexity is the principal symbol in $\xi, \tau$ of $\mathbf{P}(y, \mathbf{D}, \tau)$, that is, the portion of the full symbol that is homogeneous of order 2 in $\xi, \tau$. By our expressions above, we have the decomposition of this principal symbol into real and imaginary parts:

$$\mathbf{p}(y, \xi, \tau) = \mathbf{p}_{\Re}(y, \xi, \tau) + i \mathbf{p}_{\Im}(y, \xi, \tau) \tag{5.13}$$
$$\mathbf{p}_{\Re}(y, \xi, \tau) := \Re \mathbf{p}(y, \xi, \tau) = \langle \xi, \xi \rangle_{\mathbf{h}} - \tau^2 \langle \mathbf{d}\phi, \mathbf{d}\phi \rangle_{\mathbf{h}} \tag{5.14}$$
$$\mathbf{p}_{\Im}(y, \xi, \tau) := \Im \mathbf{p}(y, \xi, \tau) = 2\tau \langle \mathbf{d}\phi, \xi \rangle_{\mathbf{h}} \tag{5.15}$$

### *5.5.2 Conjugating by $Q^{\phi}_{\epsilon,\tau}$*

**Proposition 12.** *Let $v \in C_0^\infty(B(0, \delta))$. We have the identity*

$$Q^{\phi}_{\epsilon,\tau} \boldsymbol{P}(y, \boldsymbol{D}) v = \boldsymbol{P}(y, \boldsymbol{D} - \epsilon \nabla^2 \phi(0)(D_0, \cdot), \tau) Q^{\phi}_{\epsilon,\tau} v$$

*Proof.* Upon substituting $u = e^{\tau\phi}v$, it suffices by Proposition 11 to show

$$e^{-\frac{\epsilon D_0^2}{2\tau}}\mathbf{P}(y,\mathbf{D},\tau)u = \mathbf{P}(y,\mathbf{D}-\epsilon\nabla^2\phi(0)(D_0,\cdot)+i\tau\mathbf{d}\phi(y))e^{-\frac{\epsilon D_0^2}{2\tau}}u.$$

We first note that our choice of $\phi$ as a second order polynomial mean we can express $\mathbf{d}\phi(y) = \mathbf{d}\phi(0) + \nabla^2_{\mathbf{ij}}\phi(0)y^{\mathbf{i}}\mathbf{d}y^{\mathbf{j}}$ on $B(0,\delta)$. This means the coefficients of $\mathbf{P}(y,\mathbf{D},\tau)$ are either smooth functions of $y'$ or products of such functions with polynomials in $y^0$. Because $e^{-\frac{\epsilon D_0^2}{2\tau}}$ commutes with smooth functions of $y'$ as well as all derivatives $D_{\mathbf{j}}$ as seen easily by writing out the Fourier multiplier, we only need to check how $e^{-\frac{\epsilon D_0^2}{2\tau}}$ commutes with powers of $y^0$.

Pick any $u \in \mathcal{S}(\mathbb{R})$. By the product rule, we have

$$D_{\xi_0}(e^{-\frac{\epsilon\xi_0^2}{2\tau}}\hat{u}) = \frac{i\epsilon\xi_0}{\tau}e^{-\frac{\epsilon\xi_0^2}{2\tau}}\hat{u} + e^{-\frac{\epsilon\xi_0^2}{2\tau}}D_{\xi_0}\hat{u}.$$

Taking the inverse Fourier transform, we find

$$-y^0e^{-\frac{\epsilon D_0^2}{2\tau}}u = \frac{i\epsilon D_0}{\tau}e^{-\frac{\epsilon D_0^2}{2\tau}}u - e^{-\frac{\epsilon D_0^2}{2\tau}}y^0u,$$

which we rearrange to get the identity

$$(y^0)^k = \left(e^{+\frac{\epsilon D_0^2}{2\tau}}\left(y^0+\frac{i\epsilon D_0}{\tau}\right)e^{-\frac{\epsilon D_0^2}{2\tau}}\right)^k = e^{+\frac{\epsilon D_0^2}{2\tau}}\left(y^0+\frac{i\epsilon D_0}{\tau}\right)^k e^{-\frac{\epsilon D_0^2}{2\tau}},\ k\in\mathbb{N}_+$$

Direct calculation shows

$$\begin{aligned}
e^{-\frac{\epsilon D_0^2}{2\tau}}(D_0+i\tau d\phi_0(y)) &= e^{-\frac{\epsilon D_0^2}{2\tau}}\left(D_0+i\tau\left(d\phi_0(0)+\nabla^2_{0\mathbf{j}}\phi(0)y^{\mathbf{j}}\right)\right)\\
&= \left(D_0+i\tau\left(d\phi_0(0)+\nabla^2_{00}\phi(0)\left(y^0+\frac{i\epsilon D_0}{\tau}\right)+\nabla^2_{0j}\phi(0)y^j\right)\right)e^{-\frac{\epsilon D_0^2}{2\tau}}\\
&= \left(D_0+i\tau d\phi_0(y)-\epsilon\nabla^2_{00}\phi(0)D_0\right)e^{-\frac{\epsilon D_0^2}{2\tau}}
\end{aligned}$$

Combined with our discussion above, our claim follows. □

### 5.6 Some Remainder Estimates

Also necessary in the proof of the main Carleman estimate are $L^2$ estimates of certain remainder terms. We collect all of these in this section for convenience.

**Lemma 13** (Remainder estimates)**.** *Let $\mathcal{R}_k^j$ denote differential operators of order $\leq k$ in $\tau, D$ with top degree coefficients $O(|y|^j)$, and suppose $w \in C_0^\infty(B(0,\delta))$.*

*The following estimates hold for large enough $\tau$:*

1. $|(\mathcal{R}_0^1 w, w)_{L^2}| \leq C\delta \|w\|_{L^2}^2$

2. $|(\mathcal{R}_1^0 w, w)_{L^2}| \leq \frac{C}{\tau}\|w\|_{1;\tau}^2$

3. $|(\mathcal{R}_2^0 w, w)_{L^2}| \leq C\left(1+\frac{1}{\tau}\right)\|w\|_{1;\tau}^2$

4. $|(\mathcal{R}_2^1 w, w)_{L^2}| \leq C\left(\delta+\frac{1}{\tau}\right)\|w\|_{1;\tau}^2$ .

*Proof.* The proof of (1) is straightfoward. For (2), we have

$$\begin{aligned}|(\mathcal{R}_1^0 w, w)_{L^2}| &\leq C(\tau\|w\|_{L^2}^2 + \|Dw\|_{L^2}\|w\|_{L^2}) \leq C\left(\tau\|w\|_{L^2}^2 + \frac{1}{\tau}\|Dw\|_{L^2}^2 + \tau\|w\|_{L^2}^2\right)\\ &\leq \frac{C}{\tau}\|w\|_{1;\tau}^2,\end{aligned}$$

while for (3) we have

$$\begin{aligned}|(\mathcal{R}_2^0 w, w)_{L^2}| &\leq C(\|Dw\|_{L^2}^2 + \tau^2\|w\|_{L^2}^2 + \tau\|Dw\|_{L^2}\|w\|_{L^2}) + C\|Dw\|_{L^2}\|w\|_{L^2}\\ &\leq C\left(\|Dw\|_{L^2}^2 + \tau^2\|w\|_{L^2}^2 + \tau\left(\frac{1}{\tau}\|Dw\|_{L^2}^2 + \tau\|w\|_{L^2}^2\right)\right)\\ &\quad + C\tau\|w\|_{L^2}^2 + \frac{C}{\tau}\|Dw\|_{L^2}^2\\ &\leq C\left(1+\frac{1}{\tau}\right)\|w\|_{1;\tau}^2.\end{aligned}$$

Finally, (4) follows from

$$\begin{aligned}|(\mathcal{R}_2^1 w, w)_{L^2}| &\leq C\delta(\|Dw\|_{L^2}^2 + \tau^2\|w\|_{L^2}^2 + \tau\|Dw\|_{L^2}\|w\|_{L^2}) + C\|Dw\|_{L^2}\|w\|_{L^2}\\ &\leq C\delta\left(\|Dw\|_{L^2}^2 + \tau^2\|w\|_{L^2}^2 + \tau\left(\frac{1}{\tau}\|Dw\|_{L^2}^2 + \tau\|w\|_{L^2}^2\right)\right)\\ &\quad + C\tau\|w\|_{L^2}^2 + \frac{C}{\tau}\|Dw\|_{L^2}^2\\ &\leq C\left(\delta+\frac{1}{\tau}\right)\|w\|_{1;\tau}^2.\end{aligned}$$

□

### 5.7 Proof of the Carleman estimate

We now prove the Carleman estimate in Proposition 6, which we recall is the following:

If $v \in H_0^1(B(0,\kappa/16)$ such that $\mathbf{P}(y,\mathbf{D})v \in L^2(B(0,\kappa/16))$, then there exists $\tau_0(\delta,\kappa), c = c(\delta,\kappa) > 0$ such that

$$\|Q^{\phi}_{\epsilon,\tau}v\|_{1;\tau} \leq c\tau^{-1/2}\|Q^{\phi}_{\epsilon,\tau}\mathbf{P}(y,\mathbf{D})v\|_{L^2} + ce^{-\frac{\tau\kappa^2}{16\epsilon}}\|e^{\tau\phi}v\|_{1;\tau} \tag{5.16}$$

for all $\tau > \tau_0$, where we use the weighted $L^2$-Sobolev norm $\|v\|^2_{1;\tau} := \|\mathbf{D}v\|^2_{L^2} + \tau^2\|v\|^2_{L^2}$.

Suppose first that $u$ is smooth, and consider the following smooth cutoffs: Take $\chi \in C_0^\infty(\mathbb{R})$ with $0 \leq \chi \leq 1$, $\chi = 1$ on $(-1,1)$ and $\text{supp}(\chi) \subset (-2,2)$. For any $0 < \kappa \leq 1$, define $\chi_k(s) := \chi(s/\kappa)$, $\tilde{\chi}_\kappa := 1 - \chi_\kappa$.

We proceed to decompose

$$Q^{\phi}_{\epsilon,\tau}u := \tilde{\chi}_\kappa(y^0)Q^{\phi}_{\epsilon,\tau}u + \chi_\kappa(y^0)Q^{\phi}_{\epsilon,\tau}u \tag{5.17}$$

and estimate the two terms on the right of (5.17) separately.

#### 5.7.1 Estimates for the $\tilde{\chi}_\kappa$ cutoff term

For the first term, we start with a lemma:

**Lemma 14.** *Let $\epsilon, \kappa > 0$, $w \in C_0^\infty((-\kappa/4,\kappa/4))$, and $\tau > 0$. We have*

$$\|\tilde{\chi}_\kappa(y^0)e^{(-\epsilon/2\tau)D_0^2}w\|_{L^2} \leq ce^{-(\tau\kappa^2/4\epsilon)}\|w\|_{L^2} \tag{5.18}$$

*Proof of lemma.* Observing that $\text{supp}(\tilde{\chi}) \subset \{|y^0| > \kappa\}$, we have

$$\|\tilde{\chi}_\kappa(y^0)e^{(-\epsilon/2\tau)D_0^2}w\|^2_{L^2} \leq \int_{|y^0|>\kappa} \frac{y^0}{\kappa}\left|e^{(-\epsilon/2\tau)D_0^2}w\right|^2 dy^0 \tag{5.19}$$

At a fixed $|y^0| > \kappa$, we write out the Fourier multiplier and interchange the order of integration to find

$$\begin{aligned}
|e^{(-\epsilon/2\tau)D_0^2}w(y^0)| &= \frac{1}{2\pi}\left|\iint_{\mathbb{R}^2} e^{i(y^0-x)\xi_0}e^{-\frac{\epsilon\xi_0^2}{2\tau}}w(x)\,dx\,d\xi_0\right| \\
&= \sqrt{\frac{2\tau}{\epsilon}}\left|\int_{\mathbb{R}} \mathcal{F}^{-1}\left(e^{-|\cdot|^2}\right)\left(\sqrt{\frac{2\tau}{\epsilon}}(y^0-x)\right)w(x)\,dx\right| \\
&= \sqrt{\frac{\tau}{2\epsilon}}\left|\int_{|x|<\kappa/4} e^{-\frac{\tau}{2\epsilon}(y^0-x)^2}w(x)\,dx\right| \\
&\le c\sqrt{\frac{\tau}{\epsilon}}\,\|w\|_{L^2}\left(\int_{|x|<\kappa/4} e^{-\frac{\tau}{\epsilon}(y^0-x)^2}\,dx\right)^{1/2} \\
&\le c\sqrt{\frac{\tau}{\epsilon}}\,\|w\|_{L^2}\left(\int_{|x|<\kappa/4} e^{-(9(y^0)^2\tau/16\epsilon)}\,dx\right)^{1/2} \qquad (5.20)\\
&\le c\sqrt{\frac{\kappa\tau}{\epsilon}}\|w\|_{L^2}e^{-(\tau(y^0)^2/4\epsilon)}
\end{aligned}$$

where we used $|y^0-x|\ge\frac{3}{4}|y^0|$ to get (5.20). Thus, we find

$$\begin{aligned}
\|\tilde{\chi}_\kappa(y^0)e^{(-\epsilon/2\tau)D_0^2}w\|_{L^2}^2 &\le c\frac{\tau}{\epsilon}\|w\|_{L^2}^2\int_{|y^0|>\kappa} y^0e^{-(\tau(y^0)^2/2\epsilon)}\,dy^0 \\
&= c\|w\|_{L^2}^2\int_{|y^0|>\kappa}\frac{d}{dy^0}e^{-(\tau(y^0)^2/2\epsilon)}\,dy^0 \\
&\le c\|w\|_{L^2}^2e^{-(\tau\kappa^2/2\epsilon)}
\end{aligned}$$

as required. □

Because $\mathrm{supp}(u)\subset B(0,\kappa/8)$, we can apply this lemma to find

$$\|\tilde{\chi}_\kappa(y^0)Q_{\epsilon,\tau}^\phi u\|_{L^2}\le ce^{-(\tau\kappa^2/4\epsilon)}\|e^{\tau\phi}u\|_{L^2} \qquad (5.21)$$

Observing that $D_0\tilde{\chi}_\kappa(y^0)=0$ for $\kappa<y^0<2\kappa$ and $D_0\chi_\kappa(y^0)=\kappa^{-1}D_0\chi(y^0/\kappa)$, we have derivative estimates given in the form

$$\begin{aligned}
\|D_{\mathbf{j}}(\tilde{\chi}_\kappa(y^0)Q_{\epsilon,\tau}^\phi u)\|_{L^2} &\le \|D_{\mathbf{j}}\tilde{\chi}_\kappa\|_{L^\infty}\,\|\tilde{\chi}_{\kappa/2}Q_{\epsilon,\tau}^\phi u\|_{L^2}+\|\tilde{\chi}_{\kappa/2}e^{(-\epsilon/2\tau)D_0^2}D_{\mathbf{j}}(e^{\tau\phi}u)\|_{L^2} \\
&\le c\kappa^{-1}e^{-(\tau\kappa^2/16\epsilon)}\|e^{\tau\phi}u\|_{L^2}+ce^{-(\tau\kappa^2/4\epsilon)}\|e^{\tau\phi}u\|_{1;\tau}. \qquad (5.22)
\end{aligned}$$

Combining (5.21) and (5.22), we find that for all large $\tau>\tau_0(\kappa,\epsilon)$, we have

$$\|\tilde{\chi}_\kappa(y^0)Q_{\epsilon,\tau}^\phi u\|_{1;\tau}\le c\kappa^{-1}e^{-(\tau\kappa^2/16\epsilon)}\|e^{\tau\phi}u\|_{1;\tau}. \qquad (5.23)$$

### *5.7.2 A Gårding-type inequality*

Estimating the $\chi_\kappa$ cutoff term in (5.17) requires a Gårding-type inequality, which we prove in this section:

**Proposition 15.** *Let $w \in C_0^\infty(B(0,\delta))$.If $\phi$ is a pseudoconvex second-order polynomial at $y = 0$, then there exist $C > 0$, $\delta > 0$, and $\tau_0 > 0$ such that*

$$\|w\|_{1;\tau}^2 \leq C\left(\frac{\|\boldsymbol{p}(y,\boldsymbol{D},\tau)w\|_{L^2}^2}{\tau} + \|D_0 w\|_{L^2}^2\right) \tag{5.24}$$

*for $\tau > \tau_0$.*

*Proof.* Instead of (5.24), it suffices to prove that there exist $C_1, C_2, C_3 > 0$ such that

$$\|w\|_{1;\tau}^2 \leq C_1\frac{\|\mathbf{p}(y,\mathbf{D},\tau)w\|_{L^2}^2}{\tau} + C_2\frac{\|\mathbf{p}(y,\mathbf{D},\tau)w\|_{L^2}^2}{\tau^2} + C_3\|D_0 w\|_{L^2}^2. \tag{5.25}$$

To show this, we being by considering the *self-adjoint* quantization of the symbols (5.13)–(5.15) with respect to $L^2(B(0,\delta), dy)$ given by

$$\mathbf{p}(y,\mathbf{D},\tau) = \mathbf{p}_\Re(y,\mathbf{D},\tau) + i\mathbf{p}_\Im(y,\mathbf{D},\tau) \tag{5.26}$$

$$\mathbf{p}_\Re(y,\mathbf{D},\tau) = \Box - \tau^2\langle \mathbf{d}\phi, \mathbf{d}\phi\rangle_{\mathbf{h}} \tag{5.27}$$

$$\mathbf{p}_\Im(y,\mathbf{D},\tau) = \tau\,\mathrm{tr}_{\mathbf{h}}\left(\mathbf{d}\phi \otimes \mathbf{D} + \mathbf{D}(\mathbf{d}\phi \otimes \cdot)\right). \tag{5.28}$$

Following [KKL01], this seems to be a somewhat *ad hoc* quantization, but see the note at the end of the section for its relation to the more standard self-adjoint Weyl quantization.

The full operator can be decomposed as $\mathbf{P}(y,\mathbf{D},\tau) = \mathbf{p}(y,\mathbf{D},\tau) + \mathbf{P}_1(y,\mathbf{D},\tau)$, where $\mathbf{P}_1(y,\mathbf{D},\tau)$ is first order in $D,\tau$.

Because $\mathbf{p}_\Re(y,\mathbf{D},\tau), \mathbf{p}_\Im(y,\mathbf{D},\tau)$ are formally self adjoint, we have

$$\|\mathbf{p}(y,\mathbf{D},\tau)w\|_{L^2}^2 = \|\mathbf{p}_\Re w\|_{L^2}^2 + \|\mathbf{p}_\Im w\|_{L^2}^2 + i([\mathbf{p}_\Re, \mathbf{p}_\Im]w, w)_{L^2}. \tag{5.29}$$

We now derive estimates for the right hand side of (5.29) to apply to (5.25). These are collected in the following series of lemmas:

**Lemma 16.** *For estimates associated with* $i\langle[\boldsymbol{p}_{\Re}, \boldsymbol{p}_{\Im}]w, w\rangle$*, we have*

$$\left|\left([\boldsymbol{p}_{\Re}, \boldsymbol{p}_{\Im}]w, w\right)_{L^2}\right| \leq C\tau\|w\|_{1;\tau}^2 \tag{5.30}$$

*and*

$$|\left(([\boldsymbol{p}_{\Re}, \boldsymbol{p}_{\Im}] - [\boldsymbol{p}_{\Re}^0, \boldsymbol{p}_{\Im}^0])w, w\right)_{L^2}| \leq C\tau\left(\delta + \frac{1}{\tau}\right)\|w\|_{1;\tau}^2 \tag{5.31}$$

*for*

$$\boldsymbol{p}_{\Re}^0(y,\xi,\tau) = \Re\mathbf{p}(0,\xi,\tau) - 2\tau^2\langle\nabla^2\phi(0)(y,\cdot), \boldsymbol{d}\phi(0)\rangle_{\boldsymbol{h}(0)} \tag{5.32}$$

$$\boldsymbol{p}_{\Im}^0(y,\xi,\tau) = \Im\mathbf{p}(0,y,\xi) + 2\tau\langle\nabla^2\phi(0)(y,\cdot), \xi\rangle_{\boldsymbol{h}(0)}. \tag{5.33}$$

*Proof of Lemma 16.* We estimate (5.31) first. Remember that we fix normal coordinates at $y = 0$ and an orthonormal frame for $E$, so first derivatives of the metric and the connection forms vanish at 0.

The first order Taylor approximations in $y$ of the symbols $\mathbf{p}_{\Re}, \mathbf{p}_{\Im}$ are given by

$$\mathbf{p}_{\Re}^0(y,\xi,\tau) = \left(-(\xi_0)^2 + \sum_j(\xi_j)^2\right) - \tau^2\left(\langle\mathbf{d}\phi(0), \mathbf{d}\phi(0)\rangle_{\mathbf{h}(0)} + 2\langle\nabla^2\phi(0)(y,\cdot), \mathbf{d}\phi(0)\rangle_{\mathbf{h}(0)}\right)$$

$$\mathbf{p}_{\Im}^0(y,\xi,\tau) = 2\tau\left(\langle\mathbf{d}\phi(0), \xi\rangle_{\mathbf{h}(0)} + \langle\nabla^2\phi(0)(y,\cdot), \xi\rangle_{\mathbf{h}(0)}\right),$$

from which we get the symbols (5.32) and (5.33).

The operators $\mathbf{p}_{\Re}^0(y,\mathbf{D},\tau), \mathbf{p}_{\Im}^0(y,\mathbf{D},\tau)$ are given by the standard quantization, following [KKL01]. However, looking at the symbols above, $\mathbf{p}_{\Re}^0(y,\mathbf{D},\tau)$ coincides with the self-adjoint quantization while $\mathbf{p}_{\Im}^0(y,\mathbf{D},\tau)$ differs by a multiplication by $c\tau$; the latter commutes with differentiation, hence we may just use the Weyl symbol calculus for estimates of $[\mathbf{p}_{\Re}^0(y,\mathbf{D},\tau), \mathbf{p}_{\Im}^0(y,\mathbf{D},\tau)]$.

Recall [Zwo12, Ch. 4] that the commutator has principal symbol $i\{\mathbf{p}_{\Re}^0, \mathbf{p}_{\Im}^0\}$. A calculation then shows $i[\mathbf{p}^0_{\Re}, \mathbf{p}^0_{\Im}]$ has principal symbol

$$\begin{aligned}
-\{\mathbf{p}^0_{\Re}, \mathbf{p}^0_{\Im}\} = &-\{\Re\mathbf{p}, \Im\mathbf{p}\} + 2\tau^2\{\langle\nabla^2\phi(0)(y,\cdot), \mathbf{d}\phi(0)\rangle_{\mathbf{h}(0)}, \Im\mathbf{p}\} \\
&- 2\tau\{\Re\mathbf{p}, \langle\nabla^2\phi(0)(y,\cdot), \xi\rangle_{\mathbf{h}(0)}\} \\
&+ 4\tau^3\{\langle\nabla^2\phi(0)(y,\cdot), \mathbf{d}\phi(0)\rangle_{\mathbf{h}(0)}, \langle\nabla^2\phi(0)(y,\cdot), \xi\rangle_{\mathbf{h}(0)}\} \\
=& 4\tau\Pi_\phi(\xi,\tau) + 4\tau^3\nabla^2\phi(0)(\mathbf{J}\mathbf{d}\phi(0), \mathbf{J}\mathbf{d}\phi(0)) + 4\tau\nabla^2\phi(0)(J\xi, J\xi) \\
&+ 4\tau^3 O(|y|, |x|^0) \\
=& 8\tau\Pi_\phi(\xi,\tau) + \tau^3 O(|y|^1, |\xi|^0)
\end{aligned}$$

We wish to estimate $[\mathbf{p}_{\Re}, \mathbf{p}_{\Im}]$ in terms of $[\mathbf{p}^0_{\Re}, \mathbf{p}^0_{\Im}]$. Rewrite the brackets

$$\begin{aligned}
[\mathbf{p}_{\Re}, \mathbf{p}_{\Im}] - [\mathbf{p}^0_{\Re}, \mathbf{p}^0_{\Im}] &= [\mathbf{p}_{\Re}, \mathbf{p}_{\Im}] + [\mathbf{p}^0_{\Re}, \mathbf{p}_{\Im}] - [\mathbf{p}^0_{\Re}, \mathbf{p}_{\Im}] - [\mathbf{p}^0_{\Re}, \mathbf{p}^0_{\Im}] \\
&= \tau\left(\left[\mathbf{p}_{\Re} - \mathbf{p}^0_{\Re}, \frac{\mathbf{p}_{\Im}}{\tau}\right] + \left[\mathbf{p}^0_{\Re}, \frac{\mathbf{p}_{\Im} - \mathbf{p}^0_{\Im}}{\tau}\right]\right).
\end{aligned}$$

Note that for $y$ near 0, our symbols satisfy

$$\begin{cases}
\mathbf{p}_{\Re}, \mathbf{p}^0_{\Re} & = O(|y|^0, |\xi|^2) + \tau^2 O(|y|^0) \\
\mathbf{p}_{\Re} - \mathbf{p}^0_{\Re} & = O(|y|^2, |\xi|^2) + \tau^2 O(|y|^2) \\
\mathbf{p}_{\Im}/\tau, \mathbf{p}^0_{\Im}/\tau & = O(|y|^0, |\xi|^1) \\
\frac{\mathbf{p}_{\Im} - \mathbf{p}^0_{\Im}}{\tau} & = O(|y|^2, |\xi|^1),
\end{cases}$$

and we further calculate, schematically,

$$\begin{cases}
\partial_y(\mathbf{p}_{\Re} - \mathbf{p}^0_{\Re}) & = O(|y|^1, |\xi|^2) + \tau^2 O(|y|^1) \\
\partial_\xi(\mathbf{p}_{\Re} - \mathbf{p}^0_{\Re}) & = O(|y|^2, |\xi|^1) \\
\partial_y\left(\frac{\mathbf{p}_{\Im}}{\tau}\right) & = O(|y|^0, |\xi|^1) \\
\partial_\xi\left(\frac{\mathbf{p}_{\Im}}{\tau}\right) & = O(|y|^0, |\xi|^0)
\end{cases}
\qquad
\begin{cases}
\partial_y \mathbf{p}^0_{\Re} & = \tau^2 O(|y|^0, |\xi|^0) \\
\partial_\xi \mathbf{p}^0_{\Re} & = O(|y|^0, |\xi|^1) \\
\partial_y\left(\frac{\mathbf{p}_{\Im} - \mathbf{p}^0_{\Im}}{\tau}\right) & = O(|y|^1, |\xi|^1) \\
\partial_\xi\left(\frac{\mathbf{p}_{\Im} - \mathbf{p}^0_{\Im}}{\tau}\right) & = O(|y|^2)
\end{cases}$$

to get the Poisson brackets

$$\begin{aligned}
\left\{\mathbf{p}_{\Re} - \mathbf{p}^0_{\Re}, \frac{\mathbf{p}_{\Im}}{\tau}\right\} &= O(|y|^1, |\xi|^2) + \tau^2 O(|y|^1) \\
\left\{\mathbf{p}^0_{\Re}, \frac{\mathbf{p}_{\Im} - \mathbf{p}^0_{\Im}}{\tau}\right\} &= O(|y|^1, |\xi|^2) + \tau^2 O(|y|^2).
\end{aligned}$$

We conclude that $[\mathbf{p}_{\Re}, \mathbf{p}_{\Im}] - [\mathbf{p}^0_{\Re}, \mathbf{p}^0_{\Im}] = \tau(\mathcal{R}^1_2 + \mathcal{R}^0_1)$. By our remainder estimates, we have

$$\left|\left(([\mathbf{p}_{\Re}, \mathbf{p}_{\Im}] - [\mathbf{p}^0_{\Re}, \mathbf{p}^0_{\Im}])w, w\right)_{L^2}\right| \leq C\tau\left(\delta + \frac{1}{\tau}\right)\|w\|^2_{1;\tau}$$

For (5.30), we find

$$\begin{cases} \partial_y \mathbf{p}_{\Re} & = O(|y|^0, |\xi|^2) + \tau^2 O(|y|^0) \\ \partial_\xi \mathbf{p}_{\Re} & = O(|y|^0, |\xi|^1), \end{cases}$$

hence $[\mathbf{p}_{\Re}, \mathbf{p}_{\Im}] = \tau(\mathcal{R}^0_2 + \mathcal{R}^0_1)$ and

$$\left| ([\mathbf{p}_{\Re}, \mathbf{p}_{\Im}]w, w)_{L^2} \right| \leq C\tau\|w\|^2_{1;\tau}.$$

□

**Lemma 17.** *For the $\boldsymbol{p}_{\Im}$ term in* (5.29), *we have*

$$\left| \|\boldsymbol{p}_{\Im} w\|^2_{L^2} - \|\boldsymbol{p}^c_{\Im} w\|^2_{L^2} \right| \leq C\tau^2\left(\delta + \frac{1}{\tau}\right)\|w\|^2_{1;\tau}$$

*for*

$$\boldsymbol{p}^c_{\Im} := 2\tau\langle \boldsymbol{d}\phi(0), \boldsymbol{D}\rangle_{h(0)} = 2\tau(\partial_0\phi(0)D_0 - \partial_n\phi(0)D_n),$$

.

*Proof of Lemma 17.* Note that $\mathbf{p}^c_{\Im}$ is formally self-adjoint. Rewriting

$$\|\mathbf{p}_{\Im} w\|^2_{L^2} - \|\mathbf{p}^c_{\Im} w\|^2_{L^2} = \tau^2\left(\left(\frac{(\mathbf{p}_{\Im} - \mathbf{p}^c_{\Im})w}{\tau}, \frac{\mathbf{p}_{\Im} w}{\tau}\right) + \left(\frac{\mathbf{p}^c_{\Im} w}{\tau}, \frac{(\mathbf{p}_{\Im} - \mathbf{p}^c_{\Im})w}{\tau}\right)\right),$$

we estimate the right hand side by first noting

$$\frac{\mathbf{p}_{\Im}}{\tau}, \frac{\mathbf{p}^c_{\Im}}{\tau} \in \mathcal{R}^0_1, \quad \frac{\mathbf{p}_{\Im} - \mathbf{p}^c_{\Im}}{\tau} \in \mathcal{R}^1_1 + \mathcal{R}^0_0,$$

where $\mathcal{R}^1_1$ comes from the first derivatives of the metric vanishing at $y = 0$. We then integrate by parts in the right hand side while observing

$$\frac{\mathbf{p}_{\Im}}{\tau}\frac{\mathbf{p}_{\Im} - \mathbf{p}^c_{\Im}}{\tau}, \frac{\mathbf{p}^c_{\Im}}{\tau}\frac{\mathbf{p}_{\Im} - \mathbf{p}^c_{\Im}}{\tau} \in \mathcal{R}^1_2$$

to conclude. □

**Lemma 18.** *For the $\boldsymbol{p}_{\Re}$ term in (5.29), we have*

$$\frac{1}{\tau^2}\|\boldsymbol{p}_{\Re}w\|^2_{L^2} = \frac{1}{\tau^2}\|\Box w\|^2_{L^2} - 2\langle \boldsymbol{d}\phi(0), \boldsymbol{d}\phi(0)\rangle_{\boldsymbol{h}(0)}(\boldsymbol{J}_0\boldsymbol{D}w, \boldsymbol{D}w)_{L^2} + \tau^2\langle \boldsymbol{d}\phi(0), \boldsymbol{d}\phi(0)\rangle^2_{\boldsymbol{h}(0)}\|w\|^2_{L^2} + \left((\mathcal{R}^1_2 + \mathcal{R}^0_1)w, w\right)_{L^2}$$

*with*

$$\left|\left((\mathcal{R}^1_2 + \mathcal{R}^0_1)w, w\right)_{L^2}\right| \leq C\left(\delta + \frac{1}{\tau}\right)\|w\|^2_{1;\tau}.$$

*Proof of Lemma 18.* We write

$$\begin{aligned}\frac{1}{\tau^2}\|\mathbf{p}_{\Re}w\|^2_{L^2} &= \frac{1}{\tau^2}\|\Box w - \tau^2\langle \mathbf{d}\phi, \mathbf{d}\phi\rangle_{\mathbf{h}}\|^2_{L^2} \\ &= \frac{1}{\tau^2}\|\Box w\|^2_{L^2} - 2\Re\left(\Box w, \langle \mathbf{d}\phi, \mathbf{d}\phi\rangle_{\mathbf{h}}w\right)_{L^2} + \tau^2\|\langle \mathbf{d}\phi, \mathbf{d}\phi\rangle_{\mathbf{h}}w\|^2_{L^2}.\end{aligned}$$

By expanding the polynomial $\phi$ at 0, we have

$$\begin{aligned}2\Re\left(\Box w, \langle \mathbf{d}\phi, \mathbf{d}\phi\rangle_h w\right)_{L^2} = 2\Re\Big(&\left(\Box w, \langle \mathbf{d}\phi(0), \mathbf{d}\phi(0)\rangle_{\mathbf{h}}w\right)_{L^2} \\ &+ 2\left(\Box w, \langle \mathbf{d}\phi(0), \nabla^2\phi(0)(y,\cdot)\rangle_{\mathbf{h}}w\right)_{L^2} + \left(\Box w, \langle \nabla^2\phi(0)(y,\cdot), \nabla^2\phi(0)(y,\cdot)\rangle_{\mathbf{h}}w\right)_{L^2}\Big).\end{aligned}$$

Considering a second order expansion of $\mathbf{h}$ at 0 and noting the vanishing of first derivatives once again, we integrate by parts to find

$$2\Re\left(\Box w, \langle \mathbf{d}\phi(0), \mathbf{d}\phi(0)\rangle_{\mathbf{h}}w\right)_{L^2} = 2\langle \mathbf{d}\phi(0), \mathbf{d}\phi(0)\rangle_{\mathbf{h}(0)}(\mathbf{J}_0\mathbf{D}w, \mathbf{D}w)_{L^2} + \left((\mathcal{R}^1_2 + \mathcal{R}^0_1)w, w\right)_{L^2}.$$

Similar reasoning applies to find the order of the operators for the remaining terms. $\Box$

We now use (5.29) to substitute into the right hand side of (5.25). For $C_1, C_2, C_3 > 0$ we have

$$\begin{aligned}&C_1\frac{\|\mathbf{p}(y,\mathbf{D},\tau)w\|^2}{\tau} + C_2\frac{\|\mathbf{p}(y,\mathbf{D},\tau)w\|^2}{\tau^2} + C_3\|D_0w\|^2 \\ &\quad\geq C_1\frac{\langle i[\mathbf{p}_{\Re}, \mathbf{p}_{\Im}]w, w\rangle}{\tau} + C_2\left(\frac{\|\mathbf{p}_{\Re}w\|^2}{\tau^2} + \frac{\|\mathbf{p}_{\Im}w\|^2}{\tau^2}\right) + C_3\|D_0w\|^2 + C_2\frac{\langle i[\mathbf{p}_{\Re}, \mathbf{p}_{\Im}]w, w\rangle}{\tau^2}.\end{aligned}$$

By Lemmas 16, 17, and 18, this expression is bounded below by

$$\begin{aligned}&8C_1\left(\Pi_\phi(\mathbf{D},\tau)w,w\right)_{L^2}+\left(\mathcal{R}_2^1+\mathcal{R}_1^0 w,w\right)_{L^2}\\&+C_4\Big(\frac{\|\Box w\|^2_{L^2}-2\tau^2\langle\mathbf{d}\phi(0),\mathbf{d}\phi(0)\rangle_{\mathbf{h}(0)}(\mathbf{J}_0\mathbf{D}w,\mathbf{D}w)_{L^2}}{\tau^2}+\tau^2\langle\mathbf{d}\phi(0),\mathbf{d}\phi(0)\rangle^2_{\mathbf{h}(0)}\|w\|^2_{L^2}\Big)\\&+\left((\mathcal{R}_2^1+\mathcal{R}_1^0)w,w\right)_{L^2}+C_2\frac{\|\mathbf{p}^c_{\Im}w\|^2}{\tau^2}+\left(\mathcal{R}_2^1 w,w\right)_{L^2}+C_3\|D_0w\|^2+C_2\frac{((\mathcal{R}_2^0+\mathcal{R}_1^0)w,w)_{L^2}}{\tau}\end{aligned}$$

with $C_4 < C_2$. Cleaning this up, we have

$$\begin{aligned}&C_1\frac{\|\mathbf{p}(y,\mathbf{D},\tau)w\|^2}{\tau}+C_2\frac{\|\mathbf{p}(y,\mathbf{D},\tau)w\|^2}{\tau^2}+C_3\|D_0w\|^2\\&\quad\geq 8C_1\left(\Pi_\phi(\mathbf{D},\tau)w,w\right)_{L^2}+C_2\frac{\|\mathbf{p}^c_{\Im}w\|^2}{\tau^2}+C_3\|D_0w\|^2+(\mathcal{R}w,w)_{L^2}\\&\quad+C_4\Big(\frac{\|\Box w\|^2_{L^2}-2\tau^2\langle\mathbf{d}\phi(0),\mathbf{d}\phi(0)\rangle_{\mathbf{h}(0)}(\mathbf{J}_0\mathbf{D}w,\mathbf{D}w)_{L^2}}{\tau^2}+\tau^2\langle\mathbf{d}\phi(0),\mathbf{d}\phi(0)\rangle^2_{\mathbf{h}(0)}\|w\|^2_{L^2}\Big)\end{aligned}\tag{5.34}$$

with $|(\mathcal{R}w,w)_{L^2}|\leq C\left(\delta+\frac{1}{\tau}\right)\|w\|^2_{1;\tau}$.

First, we examine the term on the right hand side of (5.34) given by

$$\begin{aligned}I_1&:=C_2\frac{\|\mathbf{p}^c_{\Im}w\|^2}{\tau^2}+C_3\|D_0w\|^2\\&=\left(4C_2(\partial_0\phi(0))^2+C_3\right)\|D_0w\|^2_{L^2}+4C_2(\partial_n\phi(0))^2\|D_nw\|^2_{L^2}\\&\quad-8C_2\partial_0\phi(0)\partial_n\phi(0)\Re(D_0w,D_nw).\end{aligned}$$

As this quadratic form is positive definite, we conclude there exists $C_5 > 0$ such that

$$I_1\geq C_5\left(\|D_0w\|^2_{L^2}+\|D_nw\|^2_{L^2}\right)$$

Next, we look at the term

$$\begin{aligned}I_2&:=8C_1\left(\Pi_\phi(\mathbf{D},\tau)w,w\right)_{L^2}\\&\quad+C_4\Big(\frac{\|\Box w\|^2_{L^2}-2\tau^2\langle\mathbf{d}\phi(0),\mathbf{d}\phi(0)\rangle_{\mathbf{h}(0)}(\mathbf{J}_0\mathbf{D}w,\mathbf{D}w)_{L^2}}{\tau^2}+\tau^2\langle\mathbf{d}\phi(0),\mathbf{d}\phi(0)\rangle^2_{\mathbf{h}(0)}\|w\|^2_{L^2}\Big)\\&=8C_1\left(\int\nabla^2\phi(0)(\mathbf{J}_0\mathbf{D}w,\overline{\mathbf{J}_0\mathbf{D}w})\,dy+\tau^2\nabla^2\phi(0)(\mathbf{J}\mathbf{d}\phi(0),\mathbf{J}\mathbf{d}\phi(0))\|w\|^2_{L^2}\right)\\&\quad+C_4\Big(\frac{\|\Box w\|^2_{L^2}-2\tau^2\langle\mathbf{d}\phi(0),\mathbf{d}\phi(0)\rangle_{\mathbf{h}(0)}(\mathbf{J}_0\mathbf{D}w,\mathbf{D}w)_{L^2}}{\tau^2}+\tau^2\langle\mathbf{d}\phi(0),\mathbf{d}\phi(0)\rangle^2_{\mathbf{h}(0)}\|w\|^2_{L^2}\Big)\end{aligned}$$

By an abuse of notation, we write $D_{\mathbf{j}} = (0, \ldots, D_{\mathbf{j}}, 0, \ldots)$. Due to our construction of $\phi$, we see that $\nabla^2\phi(0)(\mathbf{J}_0\mathbf{D}, \mathbf{J}_0\mathbf{D})$ contains derivatives that are second order in purely $y^0$ or $y^n$, or mixed derivatives between one of $y^0, y^n$ and $y''$. For the latter, we use

$$2\Re \int D''w\overline{D_{\mathbf{j}}w}\, dy \geq -c\epsilon\|D''w\|_{L^2}^2 - \frac{c}{\epsilon}\|D_{\mathbf{j}}w\|_{L^2}^2$$

to get

$$\int \nabla^2\phi(0)(\mathbf{J}_0\mathbf{D}w, \overline{\mathbf{J}_0\mathbf{D}w})\, dy \geq \int \nabla^2\phi(0)(\mathbf{J}_0\mathbf{D}''w, \overline{\mathbf{J}_0\mathbf{D}''w})\, dy - \epsilon\|D''w\|_{L^2}^2 - K_\epsilon\left(\|D_0 w\|_{L^2}^2 + \|D_n w\|_{L^2}^2\right)$$

for arbitrary $\epsilon > 0$ with $K_\epsilon > 0$ inversely depending on $\epsilon$. Letting

$$\gamma := \min_{|\xi''|=1} \nabla^2\phi(0)(\xi'', \xi''),\ \eta := \nabla^2\phi(0)(\mathbf{J}\mathbf{d}\phi(0), \mathbf{J}\mathbf{d}\phi(0)),\ \sigma := \langle \mathbf{d}\phi(0), \mathbf{J}\mathbf{d}\phi(0)\rangle,$$

we find

$$\begin{aligned} I_2 \geq\ & 8C_1\left((\gamma - \epsilon)\|D''w\|_{L^2}^2 - K_\epsilon\left(\|D_0\|_{L^2}^2 + \|D_n w\|_{L^2}^2\right) + \tau^2\eta\|w\|_{L^2}^2\right) \\ & + C_4\Big(\frac{\|\Box w\|_{L^2}^2}{\tau^2} - 2\sigma(\mathbf{J}_0\mathbf{D}w, \mathbf{D}w)_{L^2} + \tau^2\sigma^2\|w\|_{L^2}^2\Big). \end{aligned}$$

Using the relation

$$\begin{aligned} \left\|(\Box - \tau^2\left(\sigma - \frac{C_6\gamma}{2} + \lambda\right)w\right\|_{L^2}^2 =\ & \|\Box w\|_{L^2}^2 + \left(\tau^2\left(\sigma - \frac{C_6\gamma}{2} + \lambda\right)\right)^2\|w\|_{L^2}^2 \\ & - \tau^2\left(2\sigma - C_6\gamma + 2\lambda\right)(\Box w, w)_{L^2} \end{aligned}$$

with $\lambda > 0$, $C_6 := 8C_1/C_4$, we find

$$\begin{aligned} I_2 \geq C_4\Bigg(& C_6(\gamma - \epsilon)\|D''w\|_{L^2}^2 - K_\epsilon\left(\|D_0\|_{L^2}^2 + \|D_n w\|_{L^2}^2\right) - 2\sigma(\mathbf{J}_0\mathbf{D}w, \mathbf{D}w)_{L^2} \\ & + \tau^{-2}\left\|\left(\Box - \tau^2\left(\sigma - \frac{C_6\gamma}{2} + \lambda\right)\right)w\right\|_{L^2}^2 + \left(2\sigma - C_6\gamma + 2\lambda\right)(\Box w, w)_{L^2} \\ & + \tau^2\left(\sigma^2 + C_6\eta - \left(\sigma - \frac{C_6\gamma}{2} + \lambda\right)^2\right)\|w\|_{L^2}^2\Bigg). \end{aligned}$$

We have

$$(\Box w, w)_{L^2} = (\mathbf{J}_0\mathbf{D}w, \mathbf{D}w)_{L^2} + \left((\mathcal{R}_2^2 + \mathcal{R}_1^1)w, w\right)_{L^2},$$

so using our bound for $R_2^1$ remainder and expanding the $((\Box + V)w, w)_{L^2}$ term yields

$$\begin{aligned}
I_2 &\geq C_4\Bigg( (-C_6\gamma + 2\lambda)(\mathbf{J}_0\mathbf{D}w, \mathbf{D}w)_{L^2} + C_6(\gamma - \epsilon)\|D''w\|_{L^2}^2 - K_\epsilon\left(\|D_0\|_{L^2}^2 + \|D_n w\|_{L^2}^2\right) \\
&\quad + \tau^2\left(C_6(\eta + \sigma\gamma) - \left(\frac{C_6\gamma}{2}\right)^2 - \lambda^2 - 2\sigma\lambda + C_6\gamma\lambda\right)\|w\|_{L^2}^2 - c_0\left(\delta + \frac{1}{\tau}\right)\|w\|_{1;\tau}^2\Bigg) \\
&\geq C_4\Bigg( (2\lambda - C_6\epsilon)\|D''w\|_{L^2}^2 - K'_\epsilon\left(\|D_0 w\|_{L^2}^2 + \|D_n w\|_{L^2}^2\right) \\
&\quad + \tau^2\left(C_6(\eta + \sigma\gamma) - \left(\frac{C_6\gamma}{2}\right)^2 - \lambda^2 - 2\sigma\lambda + C_6\gamma\lambda\right)\|w\|_{L^2}^2 - c_0\left(\delta + \frac{1}{\tau}\right)\|w\|_{1;\tau}^2\Bigg) \\
&=: I'_2,
\end{aligned}$$

where in the second inequality we used $(\mathbf{J}_0\mathbf{D}w, \mathbf{D}w)_{L^2} = \|D''w\|_{L^2}^2 + \|D_n w\|_{L^2}^2 - \|D_0 w\|_{L^2}^2$ and $K'_\epsilon := K_\epsilon + C_6|\gamma| + 2\lambda$. Because $\Gamma$ is timelike, we have $\sigma > 0$, hence pseudoconvexity implies $\eta + \sigma\gamma > 0$. We let $C_6 < 2(\eta + \sigma\gamma)/\gamma^2$. For small enough $\lambda > 0$ and $0 < \epsilon < \lambda/C_6$, we have

$$\begin{aligned}
I'_2 &\geq C_4\, c(C_6, \lambda)\left(\|D''w\|_{L^2}^2 + \tau^2\|w\|_{L^2}^2\right) - C_4 K'_\epsilon\left(\|D_0 w\|_{L^2}^2 + \|D_n w\|_{L^2}^2\right) \\
&\quad - C_4 c_0\left(\delta + \frac{1}{\tau}\right)\|w\|_{1;\tau}^2.
\end{aligned}$$

Using these estimates for $I_1, I_2$ in (5.34), we have

$$\begin{aligned}
&C_1\frac{\|\mathbf{p}(y, \mathbf{D}, \tau)w\|^2}{\tau} + C_2\frac{\|\mathbf{p}(y, \mathbf{D}, \tau)w\|^2}{\tau^2} + C_3\|D_0 w\|^2 \\
&\quad\geq C_4\, c(C_6, \lambda)\left(\|D''w\|_{L^2}^2 + \tau^2\|w\|_{L^2}^2\right) + (C_5 - C_4 K'_\epsilon)\left(\|D_0 w\|_{L^2}^2 + \|D_n w\|_{L^2}^2\right) \\
&\quad - C_4 c_0\left(\delta + \frac{1}{\tau}\right)\|w\|_{1;\tau}^2.
\end{aligned}$$

Observe the first two terms in the lower bound is a decomposition of $\|\cdot\|_{1;\tau}^2$. With $\epsilon, \lambda, C_6$ fixed already, we may choose $C_4, C_5$ such that

$$C_1\frac{\|\mathbf{p}(y,\mathbf{D},\tau)w\|^2}{\tau} + C_2\frac{\|\mathbf{p}(y,\mathbf{D},\tau)w\|^2}{\tau^2} + C_3\|D_0 w\|^2 \geq \left(2 - c_0\left(\delta + \frac{1}{\tau}\right)\right)\|w\|_{1;\tau}^2.$$

We have $C_1 = 8C_4C_6$; choosing $C_2 > C_4$ then determines $C_5$. The proof of (5.25) concludes by taking $\delta > 0$ small enough and $\tau_0 > 0$ large enough. □

*5.7.3 Estimates for the $\chi_\kappa$ cutoff term*

We proceed to estimate the remaining term in (5.17). Because pseudoconvexity of $\phi$ persists for $p(y, \xi - \epsilon\nabla^2\phi(0)(\xi_0, \cdot))$ for small $\epsilon$, the Gårding-type estimate given by Proposition 15 holds and we have

$$\|\chi_\kappa Q_{\epsilon,\tau}^\phi u\|_{1;\tau}^2 \leq c\|D_0(\chi_\kappa Q_{\epsilon,\tau}^\phi u)\|_{L^2}^2 + \frac{c}{\tau}\left\|\mathbf{p}(y, \mathbf{D} - \epsilon\nabla^2\phi(0)(D_0,\cdot) + i\tau\mathbf{d}\phi(y))(\chi_\kappa Q_{\epsilon,\tau}^\phi u)\right\|_{L^2}^2$$

Writing $\mathbf{p}(y,\mathbf{D},\tau) = \mathbf{P}(y,\mathbf{D},\tau) - \mathbf{P}_1(y,\mathbf{D},\tau)$ and noting that the first-order operator $\mathbf{P}_1$ in $\mathbf{D}, \tau$ satisfies

$$\left\|\mathbf{P}_1(y, \mathbf{D} - \epsilon\nabla^2\phi(0)(D_0,\cdot) + i\tau\mathbf{d}\phi(y))(\chi_\kappa Q_{\epsilon,\tau}^\phi u)\right\|_{L^2}^2 \leq c\|\chi_\kappa Q_{\epsilon,\tau}^\phi u\|_{1;\tau}^2$$

gives

$$\begin{aligned}\|Q_{\epsilon,\tau}^\phi u\|_{1;\tau}^2 \leq &\frac{1}{\tau}\|D_0(\chi_\kappa Q_{\epsilon,\tau}^\phi u)\|_{1;\tau}^2 + c\|D_0(\chi_\kappa Q_{\epsilon,\tau}^\phi u)\|_{L^2}^2 \\ &+ \frac{c}{\tau}\left\|\mathbf{P}(y, \mathbf{D} - \epsilon\nabla^2\phi(0)(D_0,\cdot) + i\tau\mathbf{d}\phi(y))(\chi_\kappa Q_{\epsilon,\tau}^\phi u)\right\|_{L^2}^2,\end{aligned}$$

yielding

$$\|Q_{\epsilon,\tau}^\phi u\|_{1;\tau}^2 \leq c\|D_0(\chi_\kappa Q_{\epsilon,\tau}^\phi u)\|_{L^2}^2 + \frac{c}{\tau}\left\|\mathbf{P}(y, \mathbf{D} - \epsilon\nabla^2\phi(0)(D_0,\cdot) + i\tau\mathbf{d}\phi(y))(\chi_\kappa Q_{\epsilon,\tau}^\phi u)\right\|_{L^2}^2$$

for sufficiently large $\tau$. For the latter term, we commute $\mathbf{P}$ and $\chi_\kappa$ to obtain

$$\begin{aligned}&\mathbf{P}(y, \mathbf{D} - \epsilon\nabla^2\phi(0)(D_0,\cdot) + i\tau\mathbf{d}\phi(y))(\chi_\kappa Q_{\epsilon,\tau}^\phi u) = \\ &\qquad\chi_\kappa \mathbf{P}(y, \mathbf{D} - \epsilon\nabla^2\phi(0)(D_0,\cdot) + i\tau\mathbf{d}\phi(y))(Q_{\epsilon,\tau}^\phi u) + \mathcal{R}_1^0 Q_{\epsilon,\tau}^\phi u \\ &\qquad= \chi_\kappa Q_{\epsilon,\tau}^\phi \mathbf{P}(y,\mathbf{D})u + \mathcal{R}_1^0 Q_{\epsilon,\tau}^\phi u\end{aligned}$$

Decomposing $Q^{\phi}_{\epsilon,\tau}u$ using $\chi_\kappa, \tilde{\chi}_\kappa$ and applying the $L^2$ estimates for the term containing $\tilde{\chi}_\kappa$ gives

$$\|\mathcal{R}_1^0 Q^{\phi}_{\epsilon,\tau}u\|_{L^2} \leq c_3\|\chi_\kappa \mathcal{R}_1^0 Q^{\phi}_{\epsilon,\tau}u\|_{1;\tau} + c\kappa^{-1}e^{-(\tau\kappa^2/16\epsilon)}\|e^{\tau\phi}u\|_{1;\tau}$$

The same decomposition applied to the first term yields

$$\|D_0\chi_\kappa Q^{\phi}_{\epsilon,\tau}u\|_{L^2} \leq \|D_0 Q^{\phi}_{\epsilon,\tau}u\|_{L^2} + c\kappa^{-1}e^{-(\tau\kappa^2/16\epsilon)}\|e^{\tau\phi}u\|_{1;\tau} + \frac{c_4}{\tau}\|\chi_\kappa Q^{\phi}_{\epsilon,\tau}u\|_{1;\tau}$$

These two estimates give

$$\begin{aligned}\|\chi_\kappa Q^{\phi}_{\epsilon,\tau}u\|_{1;\tau} \leq \frac{c}{\tau^{1/2}}\left(\left\|Q^{\phi}_{\epsilon,\tau}\mathbf{P}(y,\mathbf{D})u\right\|_{L^2} + c_3\left\|\chi_\kappa Q^{\phi}_{\epsilon,\tau}u\right\|_{1;\tau}\right)\\ + c\|D_0 Q^{\phi}_{\epsilon,\tau}u\|_{L^2} + c\kappa^{-1}e^{-(\tau\kappa^2/16\epsilon)}\|e^{\tau\phi}u\|_{1;\tau}\end{aligned}$$

which for large $\tau$ gives

$$\|\chi_\kappa Q^{\phi}_{\epsilon,\tau}u\|_{1;\tau} \leq \frac{c}{\tau^{1/2}}\left\|Q^{\phi}_{\epsilon,\tau}\mathbf{P}(y,\mathbf{D})u\right\|_{L^2} + c\kappa^{-1}e^{-(\tau\kappa^2/16\epsilon)}\|e^{\tau\phi}u\|_{1;\tau} + c_5\|D_0 Q^{\phi}_{\epsilon,\tau}u\|_{L^2}$$

To estimate $\|D_0 Q^{\phi}_{\epsilon,\tau}u\|_{L^2}$, need the following lemma:

**Lemma 19.** *For $\epsilon, \kappa > 0$ and any $w \in C_0^\infty(\mathbb{R})$, we have*

$$\left\|(\epsilon D_0/\tau)e^{-\frac{\epsilon D_0^2}{2\tau}}w\right\|_{L^2} \leq 2\kappa\left\|e^{-\frac{\epsilon D_0^2}{2\tau}}w\right\|_{L^2} + \left(e^{-\frac{\tau\kappa^2}{4\epsilon}}\right)\|w\|_{1;\tau}$$

*Proof of lemma.* We first decompose $e^{-\frac{\epsilon D_0^2}{2\tau}}w$ in terms of the Fourier multipliers given by $\chi_\kappa(\epsilon D_0/\tau)$, $\tilde{\chi}_\kappa(\epsilon D_0/\tau)$. By properties of the cutoff $\chi_\kappa$, we have

$$\left\|(\epsilon D_0/\tau)\chi_\kappa(\epsilon D_0/\tau)e^{-\frac{\epsilon D_0^2}{2\tau}}w\right\|_{L^2} \leq 2\kappa\left\|e^{-\frac{\epsilon D_0^2}{2\tau}}w\right\|_{L^2}$$

Note that $\tilde{\chi}_\kappa(\epsilon\xi_0/\tau)$ is only supported on the set $\{|\epsilon\xi_0| \geq |\tau\kappa|\} = \{\epsilon|\xi_0|/\tau \geq \kappa\}$, on which we have

$$\frac{\tau\kappa^2}{4\epsilon} \leq \kappa|\xi_0|\left(\frac{1}{2} - \frac{1}{4}\right) \leq \frac{\epsilon\xi_0^2}{2\tau} - \frac{\tau\kappa^2}{4\epsilon}.$$

Exponentiating, we have for fixed $\kappa, \epsilon$:

$$\left|\frac{\epsilon\xi_0}{\tau}\right| \frac{\tilde{\chi}_\kappa(\epsilon\xi_0/\tau)}{(\xi_0^2+\tau^2)^{1/2}} \leq e^{\frac{\kappa^2\tau}{4\epsilon}} \leq e^{\frac{\epsilon\xi_0^2}{2\tau} - \frac{\tau\kappa^2}{4\epsilon}}$$

for all $\xi_0$ and sufficiently large $\tau$. Rearranging the above and applying Plancherel, we find

$$\left\|(\epsilon D_0/\tau)\tilde{\chi}_\kappa(\epsilon D_0/\tau)e^{-\frac{\epsilon D_0^2}{2\tau}} w\right\|_{L^2} \leq e^{-\frac{\tau\kappa^2}{4\epsilon}} \|w\|_{1;\tau}.$$

□

Applying this lemma, we find

$$\|D_0 Q^\phi_{\epsilon,\tau} u\|_{L^2} \leq \frac{c\kappa\tau}{\epsilon}\|Q^\phi_{\epsilon,\tau}u\|_{L^2} + \frac{c\tau}{\epsilon}e^{-\frac{\tau\kappa^2}{4\epsilon}}\|e^{\tau\phi}u\|_{1;\tau},$$

hence for small enough $\kappa$ we have

$$\|\chi_\kappa Q^\phi_{\epsilon,\tau}u\|_{1;\tau} \leq \frac{c}{\tau^{1/2}}\left\|Q^\phi_{\epsilon,\tau}\mathbf{P}(y,\mathbf{D})u\right\|_{L^2} + c\kappa^{-1}e^{-(\tau\kappa^2/16\epsilon)}\|e^{\tau\phi}u\|_{1;\tau} + \frac{\tau}{2}\|Q^\phi_{\epsilon,\tau}u\|_{L^2}. \quad (5.35)$$

### *5.7.4 Combining cutoff estimates and smooth approximation*

Absorbing the rightmost term of (5.35) into the left hand side, we combine our estimates of the $\chi_\kappa, \tilde{\chi}_\kappa$ terms for (5.17) to arrive at the Carleman estimate (5.16) for $u \in C_0^\infty(B(0,\kappa/8))$.

For the full Carleman estimate, we approximate with smooth functions. For $\rho < \kappa/8$, let $u \in H_0^1(B(0,\rho))$ with $\mathbf{P}(y,\mathbf{D})u \in L^2(B(0,\rho))$. Let $\psi_s \in C_0^\infty(\mathbb{R})$ be the family of standard mollifiers in $y^0$, and denote $u_s(y^0,y') := u * \psi_s \in C_0^\infty(\mathbb{R}_{y^0}; H_0^1(B'(0',\rho)))$. Because the coefficients of $\mathbf{P}$ are independent of $y^0$, standard properties of mollification give $P(y,\mathbf{D})u_s = P(y,\mathbf{D})u * \psi_s \in C_0^\infty(\mathbb{R}_{y^0}; L^2(B'(0',\rho)))$, and due to the ellipticity of $\mathbf{P}(y,\mathbf{D}) - \partial_0^2$ we have $u_s \in C_0^\infty(\mathbb{R}_{y^0}; H^2(B'(0',\rho)))$ by the elliptic Gårding inequality. Since $\text{supp}(u_s) \subset B(0,\kappa/8)$, we may approximate $u_s$ by $\tilde{u}_s \in C_0^\infty(B(0,\kappa/8))$ in $C_0^\infty(\mathbb{R}_{y^0}; H^2(B'(0',\rho)))$, with $\tilde{u}_s$ satisfying the Carleman estimate. With

$$\begin{cases} \lim_{s\to\infty} P(y,\mathbf{D})u_s & = P(y,\mathbf{D})u \text{ in } L^2 \\ \lim_{s\to\infty} u_s & = u \text{ in } H_0^1 \end{cases}$$

we see (5.16) holds by continuity, which proves the full Carleman estimate Proposition 6.

### 5.8 A note on self-adjoint quantization

For our commutation calculations above, we used the Weyl calculus applied to a different self-adjoint quantization of the symbol. Here, we observe the difference between the two quantizations. Consider the symbol $a(x,\xi) := g^{ij}(x)\xi_i\xi_j$. The standard quantization associates to this the operator $a(x,D) := g^{ij}D_iD_j$. The Weyl quantization on the other hand is defined to be

$$a^W(x,D)u(x) := \frac{1}{(2\pi)^n}\iint_{\xi,y} e^{i(x-y)\xi} g^{ij}(\frac{x+y}{2})\xi_i\xi_j u(y). \tag{5.36}$$

We now compare this to the standard divergence form operator $D_j(g^{ij}D_i)$, which is formally self-adjoint in a familiar way.

By Fourier inversion and related properties, we have:

$$\begin{aligned}
D_j g^{ij}(x) D_i u(x) &= \frac{1}{(2\pi)^n}\int_\xi e^{ix\xi}\xi_j \mathcal{F}(g^{ij}D_i u)(\xi)\\
&= \frac{1}{(2\pi)^{2n}}\int_\xi e^{ix\xi}\xi_j \left(\mathcal{F}(g^{ij})\star\mathcal{F}(D_i u)\right)(\xi)\\
&= \frac{1}{(2\pi)^{2n}}\int_{\xi,\eta} e^{ix\xi}\xi_j \widehat{g^{ij}}(\xi-\eta)\widehat{D_i u}(\eta)\\
&= \frac{1}{(2\pi)^{2n}}\int_{\xi,\eta} e^{ix\xi}\xi_j\eta_i \widehat{g^{ij}}(\xi-\eta)\widehat{u}(\eta)\\
&= \frac{1}{(2\pi)^{2n}}\int_{\xi,\eta,y,z} e^{ix\xi}\xi_j\eta_i e^{-iz(\xi-\eta)} g^{ij}(z) e^{-iy\eta} u(y)\\
&= \frac{1}{(2\pi)^{2n}}\int_{\xi,\eta,y,z} e^{i(x\xi+z(\eta-\xi)-y\eta)}\xi_j\eta_i g^{ij}(z)u(y)
\end{aligned}$$

We do a change of variable $\xi' := \xi+\eta$, $\eta' := \xi-\eta$. This means $\xi = (1/2)(\xi'+\eta')$, $\eta = (1/2)(\xi'-\eta')$, and $d\xi\, d\eta = (1/2)^n d\xi'\, d\eta'$. The above becomes

$$\frac{1}{(2\pi)^{2n}}\frac{1}{2^{n+2}}\int_{\xi',\eta',y,z} e^{i(\xi'\frac{x-y}{2}+\eta'\frac{x+y}{2}-\eta' z)}(\xi'_i\xi'_j-\eta'_j\eta'_i)g^{ij}(z)u(y)$$

For the first term in brackets, we integrate first $dz$ then $\eta'$ to find:

$$\begin{aligned}\frac{1}{(2\pi)^{2n}}\frac{1}{2^{n+2}}\int_{\xi',\eta',y,z} & e^{i(\xi'\frac{x-y}{2}+\eta'\frac{x+y}{2}-\eta' z)}\xi'_i\xi'_j g^{ij}(z)u(y) \\ &= \frac{1}{(2\pi)^n}\frac{1}{2^{n+2}}\int_{\xi',y} e^{i\xi'\frac{x-y}{2}}\xi'_i\xi'_j g^{ij}(\frac{x+y}{2})u(y) \\ &= \frac{1}{(2\pi)^n}\int_{\xi,y} e^{i(x-y)\xi} g^{ij}(\frac{x+y}{2})\xi_i\xi_j u(y).\end{aligned}$$

The same procedure for the other term yields

$$\begin{aligned}\frac{1}{(2\pi)^{2n}}\frac{1}{2^{n+2}}\int_{\xi',\eta',y,z} & e^{i(\xi'\frac{x-y}{2}+\eta'\frac{x+y}{2}-\eta' z)}\eta'_i\eta'_j g^{ij}(z)u(y) \\ &= \frac{1}{(2\pi)^n}\frac{1}{2^{n+2}}\int_{\xi',y} e^{i\xi'\frac{x-y}{2}} D^2_{ij} g^{ij}(\frac{x+y}{2})u(y) \\ &= \frac{1}{(2\pi)^n}\frac{1}{4}\int_{\xi,y} e^{i(x-y)\xi} D^2_{ij} g^{ij}(\frac{x+y}{2})u(y) \\ &= \frac{1}{4}\mathcal{F}^{-1}\mathcal{F}\left(D^2_{ij}g^{ij}\left(\frac{x+(\cdot)}{2}\right)u(\cdot)\right)(x) \\ &= \frac{1}{4}D^2_{ij}g^{ij}\,(x)\,u(x),\end{aligned}$$

so we conclude that

$$D_j g^{ij}(x) D_i u(x) = a^W(x,D)u - \frac{1}{4}D^2_{ij}g^{ij}\,(x)\,u(x).$$

# Chapter 6

# TRANSFORMATION INTO A LINEAR WAVE INVERSE PROBLEM

## 6.1 Local identification of the heat kernel

On $\mathcal{M}$ and associated $P$, we start with an alternative expression for (3.3) in terms of the Gamma function

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t}\, dt$$

via spectral mapping. A substitution $t \mapsto t/\lambda$ yields

$$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-t\lambda}\, dt,$$

and therefore

$$P^{-s} u = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-tP} u\, dt, \ \ 0 < s < 1, \ \ 0 < s < 1, u \in L^2(M; E), \tag{6.1}$$

where $e^{-tP}$ is the heat semigroup generated by $P$. Recall that for $f \in L^2(M; E)$, the heat semigroup gives the smooth solution to the heat equation

$$\begin{cases} (\partial_t + P) u(t, x) = 0 \ \text{ on } (0, \infty) \times M, \\ u|_{t=0} = f. \end{cases} \tag{6.2}$$

by setting $u(t, x) := e^{-tP} f \in C^\infty((0, \infty) \times M; E)$. This follows by the spectral calculus and expressing the semigroup in terms of the resolvent of $P$, just as in the scalar setting: see e.g. [Gil18, Lem 1.7.5] for the bundle valued case. Furthermore, the heat semigroup $e^{-tP}$ satisfies the following:

**Lemma 20.** *Let $n = \dim(M) \geq 2$ be the dimension of the underlying manifold of $\mathcal{M}$.*

*1.* $e^{-tP}(x,y)$ *satisfies the pointwise bound*

$$|e^{-tP}(x,y)| < Ct^{-(n+1)}e^{-d_g(x,y)^2/t} \text{ for } 0 < t < 1; \tag{6.3}$$

*2. In terms of the spectral decomposition, we have*

$$e^{-tP}f(t,x) = \int_M e^{-tP}(x,y)f(y)\,dV(y) = \sum_\ell e^{-t\lambda_\ell}\pi_\ell f(x) \text{ for } t > 0. \tag{6.4}$$

The first inequality can be found in [Lud19, Thm 3.5], while the decomposition can be found in [Gil18, Lem 1.6.5]. These properties are crucial in showing the following:

**Proposition 21.** *Let* $E \to M$ *be a smooth vector bundle over a closed manifold, and consider geometric structures* $(g_i, \langle \cdot\,,\,\cdot\rangle_i, \nabla_i, A_i)$*,* $i = 1, 2$*, on this bundle and manifold that coincide over an open set* $U \subset M$*. Consider the local source-to-solutions maps* $\mathcal{L}_i^{\text{frac}} := \mathcal{L}_{P_i,U}^{\text{frac}}$ *associated with these structures. If* $\mathcal{L}_1^{\text{frac}} = \mathcal{L}_2^{\text{frac}}$*, then we have the local equality of heat kernels*

$$e^{-tP_1}(x,y) = e^{-tP_2}(x,y) \text{ for } (t,x,y) \in (0,\infty) \times U \times U. \tag{6.5}$$

*Proof.* For integer $k > 0$, we have $P^k f := P_1^k f = P_2^k f \in C_0^\infty(U;E)$ for $f \in C_0^\infty(U;E)$, hence $\mathcal{L}_1^{\text{frac}} P^k f(x) = \mathcal{L}_2^{\text{frac}} P^k f(x)$ for $x \in U$. Consider open sets $\Omega_1, \Omega_2 \Subset U$ whose closures are disjoint, and choose $f \in C_0^\infty(\Omega_1;E)$. Expressing the fractional power by (6.1), we find for $x \in U$

$$\int_0^\infty t^{s-1}(e^{-tP_1} - e^{-tP_2})P^k f(x)\,dt = 0. \tag{6.6}$$

Since $e^{-tP_i}$ is generated by $P_i$ and $e^{-tP_i}f$ is smooth, we have

$$e^{-tP_i}P^k f = P^k e^{-tP_i} f = (-\partial_t)^k e^{-tP_i} f, \tag{6.7}$$

on $U$, and (6.6) implies

$$\int_0^\infty t^{s-1}\partial_t^k(e^{-tP_1} - e^{-tP_2})f(x)\,dt = 0. \tag{6.8}$$

Consider any point $x \in \Omega_2$. We proceed to integrate (6.8) by parts $k$ times to find

$$\int_0^\infty t^{-(k+1)} \frac{(e^{-tP_1} - e^{-tP_2})f(x)}{t^{-s}}\, dt = 0. \tag{6.9}$$

This follows because the boundary terms in the integration by parts disappear, as seen from first using the heat kernel to write

$$(-\partial_t)^m (e^{-tP_1} - e^{-tP_2})f(x) = \int_{\Omega_1} (e^{-tP_1}(x,y) - e^{-tP_2}(x,y))P^m f(y)\, dV(y),\ 0 \le m < k. \tag{6.10}$$

At zero, using (6.3) and $d_g(\Omega_1, \Omega_2) > 0$, we have the decay

$$\begin{aligned} \lim_{t\to 0^+} \int_{\Omega_1} \left| t^{s-(k-m)} (e^{-tP_1}(x,y) - e^{-tP_2}(x,y)) P^m f(y) \right| dV(y) \\ \le \lim_{t\to 0^+} C t^{-(k+n+1)} e^{-d_g(\Omega_1,\Omega_2)^2/t} = 0. \end{aligned} \tag{6.11}$$

At infinity, the decay follows from noting $(k-m) \ge 1$ and (6.4):

$$\begin{aligned} \lim_{t\to\infty} \left| \int_{\Omega_1} t^{s-(k-m)} (e^{-tP_1}(x,y) - e^{-tP_2}(x,y)) P^m f(y)\, dV(y) \right| \\ \le \lim_{t\to\infty} t^{s-1} \left| \sum_\ell e^{-t\lambda_\ell^{(1)}} \lambda_\ell^{(1)m} \pi_\ell^{(1)} f(x) - \sum_\ell e^{-t\lambda_\ell^{(2)}} \lambda_\ell^{(2)m} \pi_\ell^{(2)} f(x) \right| = 0. \end{aligned} \tag{6.12}$$

Substituting $t \mapsto 1/t$ in the above and re-indexing $k$ shows that

$$\int_0^\infty t^k \frac{(e^{-\frac{1}{t}P_1} - e^{-\frac{1}{t}P_2})f(x)}{t^s}\, dt = 0 \tag{6.13}$$

on $\Omega_2$ for all $k \ge 0$. Let

$$\varphi(t) := \chi_{(0,\infty)} \left( \frac{(e^{-\frac{1}{t}P_1} - e^{-\frac{1}{t}P_2})f(x)}{t^s} \right). \tag{6.14}$$

The exponential bound in (6.3) implies that the Fourier transform $\widehat{\varphi}(\xi)$ is holomorphic for $\mathrm{Im}(\xi) < c$, $c > 0$, and (6.13) implies that $\widehat{\varphi}(\xi)$ vanishes to infinite order at $\xi = 0$. We conclude that $(e^{-tP_1} - e^{-tP_2})f(x) = 0$ for $t > 0$ and $x \in \Omega_2$. In fact, by unique continuation of the heat equation (6.2), we have $(e^{-tP_1} - e^{-tP_2})f(x) \equiv 0$ on all of $(0,\infty) \times \tilde{\Omega}_2$ (see [Lin90]), where $\tilde{\Omega}_2$ is the component of $U$ containing $\Omega_2$. Since this holds for any $f \in C_0^\infty(\Omega_1; E)$, we see that $e^{-tP_1}(x,y) = e^{-tP_2}(x,y)$ on $(0,\infty) \times U \times U$ by varying $\Omega_1$ and $\Omega_2$. $\square$

We adapt this proposition to the setting of Theorem 1 by pulling back $P_1$ and the kernel $e^{-tP_1}(x,y)$ locally on $U_2$ to find:

**Proposition 22.** *Let $\mathcal{M}_1, \mathcal{M}_2$ satisfy the assumptions of Theorem 1 with local structure-preserving isomorphism $\tilde{\Psi}$ as in* (3.7)*. If $\tilde{\Psi}^*\mathcal{L}_1^{\mathrm{frac}} = \mathcal{L}_2^{\mathrm{frac}}\tilde{\Psi}^*$ on $C_0^\infty(U_1; E_1) \cap \mathrm{Ker}(P_1)^\perp$, then we have the local equality of heat kernels*

$$\tilde{\Psi}^* e^{-tP_1}(x,y) = e^{-tP_2}(x,y) \ \textit{ for } (t,x,y) \in (0,\infty) \times U_2 \times U_2. \tag{6.15}$$

*Proof.* We first observe that the local structure-preserving isomorphism $\tilde{\Psi}$ induces a structure-preserving isomorphism of external tensor products $\tilde{\Psi} : (E_2 \boxtimes E_2^*)|_{U_2\times U_2} \to (E_1 \boxtimes E_1^*)|_{U_1\times U_1}$, where $E^* \cong E$ is the dual bundle to $E$ with isomorphism given by the Hermitian bundle metric. Therefore, it makes sense to use (2) of Definition 3 to pull back the kernel $e^{tP_1}(\cdot,\cdot) \in C^\infty((0,\infty)\times U_1 \times U_1; E_1 \boxtimes E_1^*)$ to a smooth kernel $\tilde{\Psi}^* e^{-tP_1}(\cdot,\cdot) \in C^\infty((0,\infty)\times U_2\times U_2; E_2\boxtimes E_2^*)$. (Note that $\tilde{\Psi}^* e^{-tP_1}(\cdot,\cdot)$ is *not* the restriction of the kernel of the heat semigroup $e^{-t\tilde{\Psi}^* P_1}$, as the latter would not make sense because $\tilde{\Psi}$ is not defined globally over $M_2$.)

For $f \in C_0^\infty(U_1; E_1)$ and integer $k \geq 0$, we find

$$\begin{aligned} \mathcal{L}_1^{\mathrm{frac}} P_1^k f &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-tP_1} P_1^k f \, dt \,\Big|_{U_1} \\ &= \frac{(-1)^k}{\Gamma(s)} \int_0^\infty t^{s-1} \partial_t^k (e^{-tP_1} f) \, dt \,\Big|_{U_1} \\ &= \frac{(-1)^k}{\Gamma(s)} \int_0^\infty t^{s-1} \partial_t^k \left( \int_{U_1} e^{-tP_1}(\cdot, \tilde{y}) f(\tilde{y}) dV_1(\tilde{y}) \right) dt \,\Big|_{U_1}, \end{aligned}$$

so under pullback we have

$$\tilde{\Psi}^*(\mathcal{L}_1^{\mathrm{frac}} P_1^k f) = \frac{(-1)^k}{\Gamma(s)} \int_0^\infty t^{s-1} \partial_t^k \left( \int_{U_2} \tilde{\Psi}^* e^{-tP_1}(\cdot, y) \tilde{\Psi}^* f(y) dV_2(y) \right) dt \,\Big|_{U_2}. \tag{6.16}$$

We also see

$$\mathcal{L}_2^{\mathrm{frac}} \tilde{\Psi}^*(P_1^k f) = \frac{(-1)^k}{\Gamma(s)} \int_0^\infty t^{s-1} \partial_t^k \left( \int_{U_2} e^{-tP_2}(\cdot, y) \tilde{\Psi}^* f(y) dV_2(y) \right) dt \,\Big|_{U_2} \tag{6.17}$$

where we used the fact that $\tilde{\Psi}^* P_1 = P_2 \tilde{\Psi}^*$ on $U_2$. Because $\tilde{\Psi}^*\mathcal{L}_1^{\mathrm{frac}} = \mathcal{L}_2^{\mathrm{frac}}\tilde{\Psi}^*$, we combine (6.16) and (6.17) to find the equivalent of (6.8),

$$\int_0^\infty t^{s-1} \partial_t^k \left( \int_{U_2} (\tilde{\Psi}^* e^{-tP_1}(x,y) - e^{-tP_2}(x,y)) \tilde{\Psi}^* f(y) dV_2(y) \right) dt = 0 \tag{6.18}$$

for all $x \in U_2$.

Fixing $\Omega, \Omega' \Subset U_2$ with disjoint closures, we consider sections $\tilde{\Psi}^* f \in C_0^\infty(\Omega; E_2)$, so (6.18) is true for $x \in \Omega'$. We claim the analog of (6.9) holds, that is:

$$\int_0^\infty t^{-(k+1)} \left( \frac{1}{t^{-s}} \int_\Omega (\tilde{\Psi}^* e^{-tP_1}(x,y) - e^{-tP_2}(x,y)) \tilde{\Psi}^* f(y) dV_2(y) \right) dt = 0. \tag{6.19}$$

This follows from integration by parts as before. The boundary term $t \to 0^+$ vanishes as in (6.11), with the pointwise bound (6.3) persisting for $\tilde{\Psi}^* e^{-tP_1}(x,y)$ because $\tilde{\Psi}$ is an isomorphism between finite rank bundles over a compact region. We observe

$$\begin{aligned}
(-\partial_t)^m \int_\Omega \tilde{\Psi}^* e^{-tP_1}(x,y) \tilde{\Psi}^* f(y)\, dV_2(y) &= \tilde{\Psi}^*\Big|_x \left( (-\partial_t)^m \int_{\tilde{\psi}(\Omega)} e^{-tP_1}(\cdot, \tilde{y}) f(\tilde{y})\, dV_1(\tilde{y}) \right) \\
&= \tilde{\Psi}^*\Big|_x (-\partial_t)^m e^{-tP_1} f(\cdot) \\
&= \tilde{\Psi}^*\Big|_x e^{-tP_1} P_1^m f(\cdot) \\
&= \tilde{\Psi}^*\Big|_x \left( \sum_\ell e^{-t\lambda_\ell^{(1)}} \lambda_\ell^{(1)m} \pi_\ell^{(1)} f(\cdot) \right)
\end{aligned}$$

Since $\tilde{\Psi}^*|_x$ is just a linear isomorphism of finite rank vector spaces, the boundary term $t \to \infty$ vanishes as in (6.12) . Thus, (6.19) holds.

Changing the variable $t \to 1/t$ in (6.19) and substituting the term in brackets into the bracketed term in (6.14), our proposition follows from the proof of Proposition 21. $\square$

### 6.2 The wave source-to-solution map

Having shown that $\mathcal{L}_{P,U}^{\text{frac}}$ determines the heat kernel of $P$ locally, we now show that knowledge of the heat kernel gives the source-to-solution map for a wave equation. Namely, consider

$$\begin{cases} (\partial_t^2 + P) w^f = f \ \text{ on } (0,\infty) \times M, \\ w^f|_{\{t=0\}} = \partial_t w^f|_{\{t=0\}} = 0. \end{cases} \tag{6.20}$$

For smooth sections $f \in C^\infty((0,\infty) \times M; E)$, the unique solution to (6.20) is given by

$$w^f(t,x) = \int_0^t G(t-s, P) f(s,x)\, ds, \tag{6.21}$$

where $G(s,P)$ is the wave kernel defined via the functional calculus with

$$G(s,\lambda) := \sum_{k=0}^{\infty} \frac{s^{2k+1}\lambda^k}{(2k+1)!} = \begin{cases} \sin(s\sqrt{\lambda})/\sqrt{\lambda} \text{ for } \lambda > 0, \\ \sinh(s\sqrt{-\lambda})/\sqrt{-\lambda} \text{ for } \lambda < 0. \end{cases} \tag{6.22}$$

This can be found in e.g. [Lud19, Sec. 2 and Thm 2.5].

**Definition 5.** *The local wave source-to-solution map* $\mathcal{L}^{wave}_{P,U} : C_0^\infty((0,\infty)\times U; E) \to C^\infty((0,\infty)\times U; E)$ *is given by* $\mathcal{L}^{wave}_{P,U} f := w^f|_{(0,\infty)\times U}$*, with* $w^f$ *as in* (6.21).

The link between $\mathcal{L}^{\mathrm{frac}}_{P,U}$ and $\mathcal{L}^{wave}_{P,U}$ is given by Proposition 21 and the following:

**Proposition 23.** *Let* $E \to M$*,* $(g_i, \langle\cdot\,,\,\cdot\rangle_i, \nabla_i, A_i)$ *be as in Proposition 21, and let* $\mathcal{L}^{wave}_i := \mathcal{L}^{wave}_{P_i,U}$ *for* $U \subset M$*. Equality of the heat kernels* $e^{-tP_1}(x,y) = e^{-tP_2}(x,y)$ *on* $(0,\infty)\times U \times U$ *implies that* $\mathcal{L}^{wave}_1 = \mathcal{L}^{wave}_2$*.*

*Proof.* By (6.21), it suffices to show that $G(s,P_1)u(x) = G(s,P_2)u(x)$ for all $u \in C_0^\infty(U;E)$, $x \in U$. To do so, we exploit a bundle-valued version of the Kannai transmutation formula as used in [FGKU21]. By Theorem 2.1 in [Lud19], we have

$$e^{-tP}u = \frac{1}{(4\pi t)^{1/2}} \int_{-\infty}^{\infty} e^{-s^2/4t} G'(s,P)u\, ds$$

Because $e^{-s^2}G'(s,\lambda) \to 0$ as $s \to \pm\infty$, we integrate by parts to find

$$e^{-tP}u = \frac{1}{4\pi^{1/2}t^{3/2}} \int_0^{\infty} e^{-s/4t} G(\sqrt{s},P)u\, ds.$$

For smooth sections $u \in C_0^\infty(U;E)$, we have $e^{-tP_1}u = e^{-tP_2}u$ on $U$ by assumption. Therefore, using the Laplace transform $\mathcal{L}$, we find on $U$:

$$\mathcal{L}(G(\cdot,P_1)u)(1/4t) = \int_0^\infty e^{-s/4t}G(\sqrt{s},P_1)u\,ds = \int_0^\infty e^{-s/4t}G(\sqrt{s},P_2)u\,ds = \mathcal{L}(G(\cdot,P_2)u)(1/4t).$$

Taking the inverse Laplace transform gives $G(s,P_1)u(x) = G(s,P_2)u(x)$ for $s>0$, $x \in U$. □

Once again, computing with the local pullback $\tilde{\Psi}^*$ yields the following:

**Proposition 24.** *Let $\mathcal{M}_1$, $\mathcal{M}_2$ satisfy the assumptions of Theorem 1 with local structure-preserving isomorphism $\tilde{\Psi}$ as in* (3.7)*. Consider $\mathcal{L}_i^{wave} := \mathcal{L}_{P_i,U}^{wave}$ for $i = 1, 2$. If the local equality of heat kernels* (6.15) *holds, then $\tilde{\Psi}^*\mathcal{L}_1^{wave} = \mathcal{L}_2^{wave}\tilde{\Psi}^*$.*

*Proof.* From (6.15), we see that for any $u \in C_0^\infty(U_1; E_1)$ and $x \in U_2$, we have

$$\begin{aligned} e^{-tP_2}\tilde{\Psi}^*u(x) &= \int_{U_2} e^{-tP_2}(x,y)\tilde{\Psi}^*u(y)\,dV_2(y) = \int_{U_2} \tilde{\Psi}^*e^{-tP_1}(x,y)\tilde{\Psi}^*u(y)\,dV_2(y) \\ &= \tilde{\Psi}^*\Big|_x \left(\int_{U_1} e^{-tP_2}(\cdot,\tilde{y})u(\tilde{y})\,dV_1(\tilde{y})\right) = \tilde{\Psi}^*\Big|_x e^{-tP_1}u, \end{aligned}$$

so by the transmutation formula we also have

$$\int_0^\infty e^{-s/4t}G(\sqrt{s},P_2)\tilde{\Psi}^*u\,ds\Big|_x = \tilde{\Psi}^*\Big|_x \int_0^\infty e^{-s/4t}G(\sqrt{s},P_1)u\,ds.$$

Inverting the Laplace transform gives $G(s,P_2)\tilde{\Psi}^*u = \tilde{\Psi}^*G(s,P_1)u$ for $s > 0$, $x \in U_2$ as needed. □

# Chapter 7

# WAVE PROPAGATION AND GEOMETRY

We have now reduced our fractional inverse problem to one involving a linear wave equation and its associated $\mathcal{L}^{wave}_{P,U}$ for which we have techniques used in [HLOS18],[KOP18]. We proceed to recount these and other necessary tools in this section.

## 7.1 Approximate controllability from unique continuation

We recall the finite speed of propagation and unique continuation properties we proved earlier:

**Proposition 25.** *(Finite speed of propagation) Let $f \in L^2(\mathbb{R} \times M; E)$, and suppose $u$ is a solution to*

$$\begin{cases} (\partial_t^2 + P)u = f \ \ \text{on } (0,\infty) \times M, \\ f|_{C(p,T)} = 0, \\ u|_{B(p,T)\times\{t=0\}} = \partial_t u|_{B(p,T)\times\{t=0\}} = 0. \end{cases} \tag{7.1}$$

*Then $u \equiv 0$ on $C(p,T)$.*

**Proposition 26.** *(Unique continuation) Suppose $U \subset M$ is a bounded open set. If $u \in C_0^\infty(\mathbb{R} \times M; E)$ solves $(\partial_t^2 + P)u = 0$ in $(0, 2T) \times M(T,U)$ and $u|_{(0,2T)\times U} \equiv 0$, then $u \equiv 0$ on $C(T,U)$.*

*Proof.* The local unique continuation property implies $u \equiv 0$ on $C(T,U)$ by [ET12, Thm 1.1]. □

These two propositions above combine to yield two important lemmas.

**Lemma 27.** *(Approximate controllability) Let $U \subset M$ be a bounded open set. For any $T > \epsilon > 0$, the set of solutions $\mathcal{W}_T := \{w^f(T,\cdot) : f \in C_0^\infty((T-\epsilon,T)\times U;E),\ w^f$ solves (6.20)$\}$ is dense in $L^2(M(\epsilon,U);E)$.*

*Proof.* This follows standard arguments found in e.g. [HLOS18, Thm 11]. We show that $\mathcal{W}_T^\perp = \{0\}$ in $L^2(M(\epsilon,U);E)$, from which the claim follows. Let $\phi \in \mathcal{W}_T^\perp \subset L^2(M(\epsilon,U);E)$ and $u \in C^\infty((0,\infty)\times M)$ be the solution in $(0,T)\times M$ to

$$\begin{cases} (\partial_t^2 + P)u = 0 \\ u|_{t=T} = 0, \partial_t u|_{t=T} = \phi. \end{cases} \tag{7.2}$$

For any $\psi \in \mathcal{C}_0^\infty((T-\epsilon,T)\times U;E)$, we have

$$(u,\psi)_{L^2((0,T)\times M;E)} = (u,(\partial_t^2+P)w^\psi)_{L^2((0,T)\times M;E)} = ((\partial_t^2+P)u,w^\psi)_{L^2((0,T)\times M;E)} = 0,$$

where the boundary terms at $t=0,T$ vanish in part due to $(\phi, w^\psi(T,\cdot))_{L^2} = 0$ by assumption. (In the case that $M$ is not closed but merely complete, we further use Proposition 25 in the above calculation.) By density, we conclude $u=0$ on $(T-\epsilon,T]\times U$.

Using the odd reflection $-u(2T-t,x)$, we also construct another solution to (7.2) on $(T,2T)\times M$ with the same initial conditions at $t=T$. These two solutions combine to give a solution $\tilde{u}$ to (7.2) on $(0,2T)\times M$ satisfying $\tilde{u}=0$ on $(T-\epsilon,T+\epsilon)\times U$. After translating in time, Proposition 26 implies $\tilde{u}=0$ on

$$\{(t,p) \in (T-\epsilon,T+\epsilon)\times M : d_g(p,U) < \min(t-(T-\epsilon),(T+\epsilon)-t)\}$$

and in particular $\partial_t\tilde{u}(T,\cdot)|_{M(\epsilon,U)} = \phi|_{M(\epsilon,U)} = 0$ as needed. □

We now define $\mathcal{F}(T,U) := \{f \in C_0^\infty(\mathbb{R}\times M;E) : supp(f) \subset (0,T)\times U\}$ and consider the time averaging operator $J\phi(t) := \frac{1}{2}\int_t^{2T-t}\phi(s)\,ds$.

**Lemma 28.** *(Blagovestchenskii identity) Let $U \subset M$ be a bounded open set and $T > 0$. For any $f,h \in \mathcal{F}(2T,U)$, we have:*

$$(w^f(T,\cdot),\ w^h(T,\cdot))_{L^2} = (f,\ (J\mathcal{L}_{P,U}^{wave} - \mathcal{L}_{P,U}^{wave*}J)h)_{L^2((0,T)\times M;E)}$$

*Proof.* This argument is standard and can be found in e.g. [HLOS18, Thm 12]. Observe that

$$(\partial_t^2 w^f(t,\cdot),\, w^h(s,\cdot))_{L^2} = (f(t,\cdot),\, \mathcal{L}_{P,U}^{wave} h(s,\cdot))_{L^2} - (Pw^f(t,\cdot),\, w^h(s,\cdot))_{L^2}$$

because

$$\operatorname{supp}(f(t,\cdot)) \subset U.$$

Consider $W \in C^\infty((0,\infty)^2;\mathbb{R})$ given by $W(t,s) := (w^f(t,\cdot),\, w^h(s,\cdot))_{L^2}$ and $F(t,s) := (\partial_t^2 - \partial_s^2)W(t,s)$. By our above observation, we find that $W$ solves the wave equation

$$\begin{cases} (\partial_t^2 - \partial_s^2)W = F \ \text{ on } (0,2T)\times(0,\infty), \\ W|_{\{t=0\}} = \partial_t W|_{\{t=0\}} = 0, \end{cases}$$

hence is given by

$$W(t,s) = \frac{1}{2}\int_0^t \int_\tau^{2T-\tau} F(\tau,y)\, dy d\tau.$$

Changing variables yields $W(T,T) = (f,\, J\mathcal{L}_{P,U}^{wave} h)_{L^2} - (\mathcal{L}_{P,U}^{wave} f,\, Jh)_{L^2}$ as needed. □

### 7.2 Cut times and geodesically transported neighborhoods

We review some basic properties of cut times which will be of use in metric reconstruction, and we further construct special neighborhoods that are adapted for our proof of bundle reconstruction. In this subsection, $(M,g)$ is a complete, connected, smooth Riemannian manifold.

Recall that the cut time $t^* : SM \to (0,\infty]$ is a continuous function on the unit tangent bundle given by $t^*(x,v) := \inf\{t > 0 : \gamma_{x,v}([0,t]) \text{ is length minimizing}\}$, where $\gamma_{x,v}$ is the unit speed geodesic with initial point and direction $(x,v)$; see e.g. [Lee18, Prop 10.32]. The following lemma finds $t^*(x,v)$ in terms of containment of balls:

**Lemma 29.** *Let $x,y \in M$ and $s := d_g(x,y) > 0$. Suppose $\gamma_{x,v}$ is a geodesic starting at $x$ whose restriction to the interval $[0,s]$ is the length minimizer joining $x$ to $y$. Let $\mathcal{I}(x,y)$ be the collection of times where the following containment is possible:*

$$\mathcal{I}(x,y) := \{r > s : \ \textit{there exists } \varepsilon(r) > 0 \textit{ such that } B(y, r-s+\varepsilon(r)) \subset B(x,r)\}.$$

*Then the cut time* $t^*(x,v) = \inf \mathcal{I}(x,y)$

*Proof.* Let $r \in \mathcal{I}(x,y)$ and $z := \gamma_{x,v}(r+\delta)$ for any $\delta < \varepsilon(r)$. Since $d_g(y,z) \leq \ell(\gamma_{x,v}([s,r+\delta]))$, we find $z \in B(y, r-s+\varepsilon(r)) \subset B(x,r)$, hence $\gamma_{x,v}$ is not length minimizing from $x$ to $z$. Therefore, $t^*(x,v) < r+\delta$, and we conclude $t^*(x,v) \leq \inf \mathcal{I}(x,y)$. Conversely, let $r' = t^*(x,v)+\delta$. In order to show $r' \in \mathcal{I}(x,y)$, it suffices to show $\partial B(y, r'-s) \subset B(x,r')$. Let $z' \in \partial B(y,r'-s)$, so $d_g(x,z') \leq r'$. Observe that if equality is achieved, then $z' = \gamma_{x,v}(r')$ because any other broken geodesic passing through $y$ can be shortened by smoothing. However, this contradicts $r' > t^*(x,v)$, so we must have $d_g(x,z') < r'$, thereby concluding the proof. □

The cut locus of $x$ is given by $\mathrm{Cut}(x) := \{q = \gamma_{x,v}(t^*(x,v))$, and the injectivity domain of the exponential map at $x$ is given by $\mathcal{D}^{\mathrm{inj}}(x) := \{sv \in T_xM : s < t^*(x,v)\} \subset T_xM$. A proof of the following can be found in [Lee18, Thm 10.32]:

**Lemma 30.** *For any* $x \in M$*, we have the following:*

*(a)* $\mathrm{Cut}(x) \subset M$ *has measure zero.*

*(b) The restricted exponential map* $\exp_x : \mathcal{D}^{\mathrm{inj}}(x) \to M \setminus \mathrm{Cut}(x)$ *is a diffeomorphism.*

For points within $\mathrm{Cut}(x)$, we construct the following pairs of open sets that shrink to points as $k \to \infty$ by transporting polar neighborhoods of $x$ along distance minimizing geodesic segments.

**Lemma 31.** *Suppose* $y = \gamma_{x,v}(s)$ *for some* $0 < s < t^*(x,v)$*, and let* $\pi : E \to M$ *be a smooth vector bundle. For sufficiently large* $k > k^* > 0$*, there are neighborhoods* $(X_k)_{k \geq k^*}, (Y_k)_{k \geq k^*}$ *of* $x, y$ *respectively such that*

1. $\lim_{k\to\infty} \mathrm{Diam}(X_k) = 0 = \lim_{k\to\infty} \mathrm{Diam}(Y_k)$*;*

2. $d_g(X_k, Y_k) < s + 1/k$*;*

*3. Both $E|_{X_k}$ and $E|_{Y_k}$ are trivial.*

*Proof.* By extending the geodesic segment $\gamma_{x,v}([0,s])$ past $x$, we can find $(x',v') \in SM$ such that $x = \gamma_{x',v'}(r')$ and $y = \gamma_{x',v'}(s+r')$ with $s + r' < t^*(x',v') = t^*(x,v)$. Taking $r' > 0$ small enough so that $x$ is contained within a polar normal chart $(\mathcal{O}', (r, \theta^j))$ centered at $x'$, we see $x = (r', \theta^j)$ in these polar normal coordinates. By continuity of $t^*$, we see that for small enough $\epsilon, \delta > 0$,

$$\mathcal{U}(x,\epsilon,\delta) := (r' - \epsilon, r' + \epsilon) \times (\theta^j - \delta, \theta^j + \delta) \subset \mathcal{O}' \tag{7.3}$$

is a neighborhood of $x$ such that

$$\mathcal{V}(y,\epsilon,\delta) := \exp_{x'}(s\mathcal{U}(x,\epsilon,\delta)) \subset M \setminus \mathrm{Cut}(x'), \tag{7.4}$$

where we regarded $\mathcal{U}(x,\epsilon,\delta)$ as a subset of $T_{x'}M$ in (7.4). Observe that with this construction, we can also turn $\mathcal{V}(y,\epsilon,\delta)$ into a coordinate chart by using the same polar coordinates associated with $\mathcal{O}'$ on $\mathcal{U}(x,\epsilon,\delta)$ and translating the radial coordinate by $s$. This corresponds to how every point in $\mathcal{V}(y,\epsilon,\delta)$ is joined to a unique point in $\mathcal{U}(x,\epsilon,\delta)$ via a length $s$ segment of a radial geodesic emanating from $x'$. Moreover, both (7.3) and (7.4) are contractible, hence any vector bundle over those respective sets are trivial.

Choosing $k > 0$ large enough such that $1/k = \epsilon = \delta$ satisfies the above allows us to construct $X_k$ using (7.3) and $Y_k$ using (7.4). □

# Chapter 8

# PROOF OF THEOREM

## 8.1 Recovery of the metric

We adapt the method of [HLOS18] to our bundle-valued setting to determine the metric $g$. Let $S(x, \varepsilon, \tau) := (T - \tau, T) \times B(x, \varepsilon)$. Due to finite speed of propagation and approximate controllability of the wave equation, we have the following characterization of containment of balls using solutions to (6.20):

**Lemma 32.** *For $x, y, z \in M$ and $\tau_x, \tau_y, \tau_z > \delta > 0$, the following conditions are equivalent:*

1. $B(x, \tau_x) \subsetneq B(y, \tau_y) \cup B(z, \tau_z)$

2. *For all $f \in C_0^\infty(S(x, \delta, \tau_x - \delta); E)$, there is a sequence $(f_j)_{j=1}^\infty \subset C_0^\infty(S(y, \delta, \tau_y - \delta) \cup S(z, \delta, \tau_z - \delta); E)$ whose respective solutions to* (6.20) *approximate the solution for $f$, that is:*

$$\lim_{j \to \infty} \|w^f(T, \cdot) - w^{f_j}(T, \cdot)\|_{L^2(M;E)} = 0$$

*Proof.* For the forward implication, choose $f \in C_0^\infty(S(x, \delta, \tau_x - \delta); E)$, so in particular the restriction to backward cones $f|_{C(p,T)} = 0$ for all points satisfying $d_g(p, B(x, \delta)) \geq \tau_x - \delta$. Lemma 25 then yields $supp(w^f(T, \cdot)) \subset B(x, \tau_x) \subset B(y, \tau_y) \cup B(z, \tau_z)$, the latter containment following by assumption. Let $\chi_y$ be the characteristic function for $B(y, \tau_y)$, and rewrite $w^f(T, \cdot) = \chi_y w^f(T, \cdot) + (1 - \chi_y) w^f(T, \cdot)$. We see $\chi_y w^f(T, \cdot) \in L^2(B(y, \tau_y); E) = L^2(M(\tau_y - \delta, B(y, \delta)); E)$, hence Lemma 27 implies $\chi_y w^f(T, \cdot)$ can be approximated in $L^2(B(y, \tau_y); E)$ by $w^{\phi_j^{(y)}}(T, \cdot)$ for $\phi_j^{(y)} \in S(y, \delta, \tau_y - \delta)$. Similarly, one can find a suitable sequence approximating $(1 - \chi_y) w^f(T, \cdot)$, and the sequence constructed from their sum gives the result.

Conversely, suppose (1) does not hold and take a nonempty open set $U \subset B(x, \tau_x) \setminus B(y, \tau_y) \cup B(z, \tau_z)$. By Lemma 25, $w^{\tilde{f}}(T, \cdot) = 0$ on $U$ for all $\tilde{f} \in C_0^\infty(S(y, \delta, \tau_y - \delta) \cup S(z, \delta, \tau_z - \delta); E)$, whereas we can find $f \in C_0^\infty(S(x, \delta, \tau_x - \delta); E)$ such that $\|w^f(T, \cdot)\|_{L^2} > 0$ by Lemma 27. Therefore, (2) also does not hold. $\square$

Our approach requires the restricted distance function $d_g|_{U \times U}$. Observe that while we have knowledge of $(U, g|_U)$, we do not assume $U \subset M$ is geodesically convex, nor do we initially have the information to shrink to such an appropriate subset of $U$. Instead, we use properties of the wave equation to prove the following:

**Proposition 33.** *The local structure $(U, g|_U)$ and the wave source-to-solution data $\mathcal{L}_{P,U}^{wave}$ determine the local distance map $d_g(\cdot, \cdot)|_{U \times U}$.*

*Proof.* For $x, y \in U$ and $k > 0$, consider the sets

$$D(x, y, k) := \{t > 0 : \text{there exists } f \in C_0^\infty((0, \infty) \times B_0(x, 1/k)) \text{ such that } \operatorname{supp}(w^f(t, \cdot)) \cap B_0(y, 1/k) \neq \varnothing\}.$$

Note that for any $t > 0$, the wave solutions $w^h(t, \cdot)$ for sources $h \in C_0^\infty((t-\delta, t) \times B(y, (1/k) - \delta; E))$, $0 < \delta < t$, are dense in $L^2(B(y, 1/k); E)$ by Lemma 27. The sets $D(x, y, k)$ are determined from $\mathcal{L}_{P,U}^{wave}$ by evaluating $(w^f(t, \cdot), w^h(t, \cdot))_{L^2} = (f, (J\mathcal{L}_{P,U}^{wave} - \mathcal{L}_{P,U}^{wave*} J)h)_{L^2}$, given in Lemma 28. Using Lemma 27 and similar reasoning as in the proof of Lemma 32, we find $d_g(x, y) = \liminf_{k \to \infty} D(x, y, k)$. $\square$

**Proposition 34.** *Let $U \subset M$ be a nonempty open set. The restricted Riemannian structure $(U, g|_U, d_g|_{U \times U})$ and the wave source-to-solution data $\mathcal{L}_{P,U}^{wave}$ determines on $U$ the family of distance functions*

$$\mathcal{R}_U(M) := \{d_g(p, \cdot)|_U : p \in M\}.$$

*Proof.* We first note that because $U$ is open and nonempty, any point in $M$ can be reached by a unit speed geodesic emanating from $U$ before its cut time. Indeed, if this were not true, $U$ would be contained in the cut locus of a point, which has measure zero, leading to a contradiction. Therefore, it suffices to show that we can determine $t^*$ on $SM|_U$ and, further, find $d(p,\cdot)|_U$ for any $p = \gamma_{x,v}(r')$ with $(x,v) \in SM|_U$, $r' < t^*(x,v)$.

To this end, for any $(x,v) \in SM|_U$ we choose $s > 0$ small enough such that $\gamma_{x,v}([0,s]) \subset U$. Letting $y := \gamma_{x,v}(s)$, we see $\mathcal{L}^{wave}_{P,U}$ determines $\mathcal{I}(x,y)$ via Lemma 28, polarization, and Condition 2 of Lemma 32. Therefore, we know $t^*(x,v)$ by the characterization of Lemma 29.

For $z \in U$, we consider the set

$$\mathcal{I}(x,y,z) := \{r > 0 : \text{ there exists } \varepsilon(r) > 0 \text{ such that } B(y, r' - s + \varepsilon(r)) \subset B(x,r') \cup B(z,r)\}.$$

We claim that $d(p,z) = \inf \mathcal{I}(x,y,z)$. Indeed, $d(p,x) = r'$, so for $r \in \mathcal{I}(x,y,z)$, we have $p \in B(z,r)$ since $p \in B(y, r'-s+\varepsilon(r))$; thus, $d(p,z) \le \inf \mathcal{I}(x,y,z)$. Now, let $\tilde{r} := d(p,z) + \delta$ for some $\delta > 0$. If $\tilde{r} \notin \mathcal{I}(x,y,z)$, then by passing to a subsequence we may find points

$$p_k \in B(y, r' - s + 1/k) \setminus (B(x,r') \cup B(z,\tilde{r})) \subset \overline{B(y, r' - s + 1)}$$

such that $p_k \to p' \in \partial B(y, r' - s)$. Observe that $d(x,p') \le r'$, with equality achieved if and only if $p = p'$ by a smoothing argument as in the proof of Lemma 29. The equality $p = p'$ and the fact $\tilde{r} > d(p,z)$ would contradict the fact that our limiting sequence stays outside $B(z,\tilde{r})$; however, $d(x,p') < r'$ would imply our limiting sequence does not stay outside $B(x,r')$. Therefore, we find $\inf \mathcal{I}(x,y,z) \le d(p,z)$ and our claim follows.

Because $\mathcal{I}(x,y,z)$ can be determined from our initial data by similar methods as those used for $\mathcal{I}(x,y)$ earlier, we are able to deduce $d(p,z)$ and ultimately determine $\mathcal{R}_U(M)$. □

**Proposition 35.** *Let $\mathcal{M}_1$, $\mathcal{M}_2$ satisfy the assumptions of Theorem 1 with local structure-preserving isomorphism $\tilde{\Psi}$ as in (3.7). If $\tilde{\Psi}^* \mathcal{L}_1^{wave} = \mathcal{L}_2^{wave} \tilde{\Psi}^*$, then there is an isometry $\psi : M_2 \to M_1$ with $\psi|_{U_2} = \tilde{\psi}$.*

*Proof.* Recall $U_1 = \tilde{\psi}(U_2)$. We are able to determine

$$\mathcal{R}_i(M_i) := \{d_{g_i}(p,\cdot)|_{\overline{U_i}} : p \in M_i\} \text{ for } i = 1,2 \tag{8.1}$$

by our previous proposition, since each function $d_{g_i}(p,\cdot)|_{U_i}$ continuously extends to $\overline{U}_i$. As shown in [HLOS18, Thm 8], the map

$$R_i : M_i \ni p \mapsto d_{g_i}(p,\cdot)|_{\overline{U_i}} \in \mathcal{R}_i(M)$$

is a diffeomorphism when $\mathcal{R}_i(M)$ is equipped with a smooth structure constructed explicitly using the injectivity domain of points in $\overline{U}_i$, which is known from $\mathcal{L}_i^{wave}$ using Proposition 34. Equipping $\mathcal{R}_i(M)$ with the pullback of $g_i$ using $R_i^{-1}$ makes $R_i$ an isometry, and by Proposition 5 in [HLOS18] the coordinate representation of this metric on $\mathcal{R}_i(M)$ is determined by Proposition 34. Because $\tilde{\Psi}^*\mathcal{L}_1^{wave} = \mathcal{L}_2^{wave}\tilde{\Psi}^*$, the map $\psi_R : \mathcal{R}_1(M_1) \to \mathcal{R}_2(M_2)$ defined using the pullback of distance functions by $\tilde{\psi}$ is an isometry. Setting $\psi := R_1^{-1}\psi_R^{-1}R_2$ yields a global isometry with the required properties. □

By Propositions 22, 24, and 35, the existence of $\psi$ in Theorem 1 follows.

### *8.2 Recovery of the bundle and operator*

Recovery of our remaining geometric quantities follows the procedure of [KOP18] in the setting of closed manifolds.

**Lemma 36.** *Let $U_1, U_2 \subset M$ be open sets and $\epsilon_1, \epsilon_2 > 0$. Suppose there exists a sequence $(f_j)_{k=1}^{\infty} \subset C_0^{\infty}((T-\epsilon_1, T) \times U_1)$ such that:*

1. *The sequence of solutions $(w^{f_j}(T,\cdot))_{j=1}^{\infty}$ converges weakly to $\phi \in L^2(M;E)$;*

2. *For all $h \in C_0^{\infty}((T-\epsilon_2, T) \times U_2)$, we have $\lim_{j\to\infty}(w^{f_j}(T,\cdot),\, w^h(T,\cdot))_{L^2} = 0$.*

*Then supp($\phi$) $\subset M(\epsilon_1, U_1) \setminus M(\epsilon_2, U_2)^{int}$.*

*Proof.* By Lemma 25, each $w^{j_j}(T,\cdot)$ has support in $M(\epsilon_1, U_1)$, hence (1) implies the same for the support of $\phi$. Using Lemma 27, we can approximate any $\psi \in C_0^{\infty}(M(\epsilon_2, U_2)^{int}; E)$ in $L^2$ by $(w^{h_\ell}(T,\cdot))_{\ell=1}^{\infty}$, $h_\ell \in C_0^{\infty}((T-\epsilon_2, T) \times U_2)$. Therefore,

$$(\phi,\, \psi)_{L^2} = \lim_{\ell\to\infty}(\phi,\, w^{h_\ell}(T,\cdot))_{L^2} = \lim_{j,\ell\to\infty}(w^{f_j}(T,\cdot),\, w^{h_\ell}(T,\cdot))_{L^2} = 0$$

by condition (2), so $\phi$ is also supported in the complement of $M(\epsilon_2, U_2)^{int}$. □

**Lemma 37.** *Consider an open set $U \subset M$. For any $T > 0$ and point $p \in M$ such that $d_g(p, U) < T$, there exists a tuple $(h_\ell)_{\ell=1}^{rk(E)} \subset \mathcal{F}(2T, U)$ such that $(w^{h_\ell}(T,\cdot))_{\ell=1}^{rk(E)}$ is a smooth local frame for $E$ in a neighborhood $x \in V_x \subset M$ and is orthonormal at $x$.*

*Proof.* It suffices to show $w^h(T,x)$, $h \in \mathcal{F}(T,U)$, spans $E_x$; our claims then follow by Gram-Schmidt and smoothness of $w^h(T,\cdot)$. To this end, we show that if $e \in E_x$ is such that $\langle e\,,\, w^h(T,x)\rangle_E = 0$, then $e = 0$. If we let $\psi := e\delta_x$ in (7.2), our assertion follows from the same reasoning as the proof of Lemma 27. □

The following two lemmas identify certain double sequences of sections with vectors in the fibre $E_y$, where $y$ may not be contained in the support of the sequence. For convenience, we define $s_k := s + 1/k$ where $\gamma_{x,v}(s) = y$, $s < t^*(x,v)$. In the case $s > 0$, we construct $X_k$, $Y_k$ as in (7.3) and (7.4); otherwise, set $Y_k = X_k$ as defined in (7.3). Observe that in either case, $Y_k \subset M(s_k, X_k)$.

**Lemma 38.** *Let $U \subset M$ be open, and consider $y = \gamma_{x,v}(s) \in M$ for $(x,v) \in SM|_U$, $s < t^*(x,v)$. Suppose we have a double sequence of sections $\Phi^y := (f_{jk})_{j,k=1}^{\infty}$ with $f_{jk} \in C_0^\infty((T-s_k,T)\times X_k;E)$ such that:*

1. *For each $k$, $(w^{f_{jk}}(T,\cdot))_{j=1}^{\infty}$ converges weakly in $L^2(M;E)$ to a section $u_k$ supported in $Y_k$;*

2. *There is a uniform $C > 0$ such that $\|w^{f_{jk}}(T,\cdot)\|_{L^2} < C(Vol(Y_k))^{-1/2}$;*

3. *For any $h \in \mathcal{F}(2T,U)$, $\lim_{k\to\infty}\lim_{j\to\infty}(w^{jk}(T,\cdot),\, w^h(T,\cdot))_{L^2}$ exists.*

*Then there is a vector $e(\Phi^y) \in E_y$ depending on $\Phi^y$ such that for all $\phi \in C^\infty(M;E)$,*

$$\lim_{k\to\infty}\lim_{j\to\infty}(w^{f_{jk}}(T,\cdot),\, \phi)_{L^2} = \langle e(\Phi^y)\,,\, \phi\rangle_E. \tag{8.2}$$

*Proof.* By taking $k$ large enough, we may assume without loss of generality that $X_k \subset U$ and $Y_k$ is contained within a fixed coordinate chart $(W,\tilde{y})$ with $\tilde{y} = 0$ at $y$. Let $b_\ell(\cdot) := w^{h_\ell}(T,\cdot)$ denote the sections of the local frame for $E$ constructed using Lemma 37 which are orthonormal at $E_y$. Any $\phi \in C^\infty(M;E)$ can be written locally using a linearization as

$$\phi(\tilde{y}) = \sum_{\ell=1}^{rk(E)} \langle \phi\,,\, b_\ell\rangle_E b_\ell = c^\ell b_\ell(\tilde{y}) + \tilde{y}^i\psi_i(\tilde{y})$$

where $\tilde{\psi}_k \in C^\infty(W;E)$, $1 \leq i \leq n$, and $c^\ell \in \mathbb{C}$. Taking the inner product with the weak limit in (1) yields

$$(u_k,\, \phi)_{L^2} = \overline{c^\ell}(u_k,\, b_\ell)_{L^2} + r_k$$

where

$$\begin{aligned} |r_k| &< n \max_i \|\psi_i\|_{C(W)} \operatorname{Diam}(Y_k) \int_{Y_k} |u_k|\, dV \\ &< n \max_i \|\psi_i\|_{C(W)} \operatorname{Diam}(Y_k) |(Vol(Y_k))^{1/2} \|u_k\|_{L^2}. \end{aligned}$$

Since $Y_k \to y$ by construction, (2) implies $r_k \to 0$, and (3) implies we can set $a^\ell := \lim_{k\to\infty}(u_k,\, b_\ell)_{L^2}$. Letting $e^S_y := a^\ell b_\ell(y)$, we find

$$\lim_{k\to\infty}\lim_{j\to\infty}(w^{f_{jk}}(T,\cdot),\, \phi)_{L^2} = \lim_{k\to\infty}(u_k,\, \phi)_{L^2} = \sum_{\ell=1}^{rk(E)} \overline{c^\ell} a^\ell = \langle e(\Phi^y)\,,\, \phi\rangle_E.$$

□

Conversely, we also have:

**Lemma 39.** *Let $U \subset M$ be open, and consider $y = \gamma_{x,v}(s) \in M$ for $(x,v) \in SM|_U$, $s < t^*(x,v)$. For any $e_y \in E_y$, there is a double sequence $\Phi^y = (f_{jk})_{j,k=1}^\infty$ with $f_{jk} \in C_0^\infty((T - s_k, T) \times X_k; E)$ satisfying conditions (1)–(3) of Lemma 38, with $e(\Phi^y) = e_y$.*

*Proof.* For any smooth section $\tilde{e} \in C^\infty(M;E)$ with $\tilde{e}(y) = e_y$, we construct

$$u_k \in Vol(Y_k)^{-1}\chi_k\tilde{e} \in L^2(M(s_k, X_k); E)$$

with $\chi_k$ the indicator function for $Y_k$. Using Lemma 27 for each $k$, we find our double sequence $\Phi^y$ to approximate each $u_k$ strongly, and Condition (1) follows. Condition (2) is also immediate with $C = \|\tilde{e}\|_{L^\infty}$, and Condition (3) and (8.2) follow by direct calculation. □

**Proposition 40.** *Let $\mathcal{M}$ be as in (3.4). If we know the Riemannian structure $(M,g)$, the local Hermitian bundle $(E, \langle\cdot\,,\,\cdot\rangle_E)|_U$, and $\mathcal{L}^{wave}_{P,U}$, then we can determine the structures $(E, \langle\cdot\,,\,\cdot\rangle_E, \nabla, A)$ globally on $M$.*

*Proof.* Shrinking $U$ as necessary, we assume $U$ is a coordinate ball that trivializes $E$. Observe that due to compactness of $M$ and Lemma 30, we can cover $E$ with finitely many charts of the form

$$\mathcal{A} = ({}^{(0)}\mathcal{A}, {}^{(1)}\mathcal{A}, \ldots, {}^{(N)}\mathcal{A}) := (U, {}^{(1)}\mathcal{V}, \ldots, {}^{(N)}\mathcal{V}),$$

with ${}^{(j)}\mathcal{V}$ constructed using (7.4). Therefore, $E$ is trivial over each ${}^{(j)}\mathcal{A}$. For each $y \in {}^{(j)}\mathcal{A}$, we can use $\mathcal{L}^{wave}_{P,U}$ to detect sequences $(\Phi^y_\alpha)_\alpha$ satisfying the conditions of Lemma 38 by Lemmas 27, 28, and 36. We are able to test for smoothness of $y \mapsto e(\Phi^y)$ by checking the smoothness of $y \mapsto \langle e(\Phi^y)\,,\, w^h(T,\cdot)\rangle_E$ for $h \in \mathcal{F}(2T,U)$, per Condition 3 of Lemma 38 and Lemma 37. Further, due to the existence of an orthonormal frame $(e_\ell)_{\ell=1}^{rk(E)}$ over ${}^{(j)}\mathcal{A}$ and Lemma 39, we can select for sequences $(\Phi^y_\ell)_{\ell=1}^{rk(E)}$ such that $e_\ell(y) = e(\Phi^y_\ell)$ by verifying

$$\lim_{k\to\infty}\lim_{j\to\infty} (\chi_{{}^{(j)}\mathcal{A}} e_\kappa(\Phi^y),\, w^{f^y_{jk,\ell}}(T,\cdot))_{L^2} = \langle e_\kappa(y)\,,\, e_\ell(y)\rangle_E = \delta_{\kappa\ell}. \tag{8.3}$$

The Hermitian metric $\langle\cdot\,,\,\cdot\rangle_E$ on this trivialization over ${}^{(j)}\mathcal{A}$ is then $\langle v\,,\,w\rangle_E = \delta_{\kappa\ell} a^\kappa \overline{c^\ell}$ for $v = a^\kappa e_\kappa$, $w = c^\ell e_\ell$. Given a similar orthonormal frame $(\tilde{e}_\ell)_{\ell=1}^{rk(E)}$ and associated sequence over ${}^{(k)}\mathcal{A}$, we apply the method of (8.3) to calculate $\langle e_\kappa\,,\,\tilde{e}_\ell\rangle_E$ to find the transition functions $\tau_{jk} : {}^{(j)}\mathcal{A} \cap {}^{(k)}\mathcal{A} \to GL(rk(E),\mathbb{C})$.

Having determined the Hermitian bundle structure $(E, \langle\cdot\,,\,\cdot\rangle_E) \to M$, we now show how we can recover the connection $\nabla$ and potential $A$. From our reasoning above, we know the components of $w^h(T,\cdot)$ for $h \in \mathcal{F}(2T,U)$ on ${}^{(j)}\mathcal{A}$, and by a time translation we know the components of $w^h(t,\cdot)$ in the same chart, for $t \in (0,T)$ and $h \in \mathcal{F}(T,U)$. We therefore know the left hand side of

$$-(\partial_t^2 w^h(T,\cdot),\, \phi)_{L^2} = (P w^h(T,\cdot),\, \phi)_{L^2} = (w^h(T,\cdot),\, P\phi)_{L^2}. \tag{8.4}$$

for any $\phi \in C_0^\infty({}^{(j)}\mathcal{A}; E)$, $h \in C_0^\infty((0,T)\times U; E)$. By density of $w^h(T,\cdot)$ in $L^2({}^{(j)}\mathcal{A}; E)$, we deduce $P\phi$ from (8.4).

Since $\nabla = d + B$ for skew-Hermitian $B \in C^\infty(M; E \otimes E^* \otimes T^*M)$, we have:

$$(P - A)\phi = \nabla^*\nabla\phi = d^*d\phi - 2\,\mathrm{tr}_g(B \otimes d\phi) + (d^*B)\phi - \mathrm{tr}_g(B \otimes B\phi) \tag{8.5}$$

where the contractions are between the $E^* \otimes T^*M$ and $E \otimes T^*M$ entries. Locally on ${}^{(j)}\mathcal{A}$, if we choose $\phi = \phi^\ell e_\ell$ such that $\phi(0) = 0$, $\partial_k \phi^\ell(0) = 1$, then combined with our knowledge of $g$ we may calculate $d^* d\phi$ and find

$$2\,\mathrm{tr}(B \otimes d\phi) = 2g^{ik} e_\ell(B_i)$$

from (8.5). This yields $B$ and therefore the connection $\nabla$. Subtracting $P\phi$ from (8.5) then gives $A$, as needed. □

By Propositions 22, 24, 35, and 40, we have recovered the full structure-preserving isomorphism $\Psi$.

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