Shellable Simplicial Spheres: Enumeration and Reconstruction

Yirong Yang

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Reading Committee:

Isabella Novik, Chair

Gaku Liu

Rekha R. Thomas

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Abstract

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Yirong Yang

Chair of the Supervisory Committee:
Isabella Novik
Department of Mathematics

This thesis studies shellable simplicial spheres, which are triangulations of topological spheres with a special combinatorial decomposition property.

In Chapter 3 we present enumeration results for shellable simplicial spheres. Nevo, Santos, and Wilson constructed $2^{\Omega(n^k)}$ combinatorially distinct simplicial $(2k-1)$-spheres with n vertices. We prove that all spheres produced by one of their constructions are shellable. Combining this with earlier results of Kalai, Lee, and Benedetti–Ziegler, we conclude that for all $d \geq 3$ there exist $2^{\Theta(n^{\lceil d/2 \rceil})}$ shellable simplicial d-spheres with n vertices.

In Chapter 4 we show that the facet-ridge graph of a shellable simplicial sphere uniquely determines its entire combinatorial structure. This generalizes the celebrated results of Blind–Mani and Kalai on reconstructing simple polytopes from their graphs. Our proof uses the notion of good acyclic orientations from Kalai's proof together with k-systems introduced by Joswig, Kaibel, and Körner.

In Chapter 5 we study flag shellable simplicial spheres arising as independence complexes of Gorenstein planar graphs with no induced cycles of length divisible by three. We analyze transformations among these flag spheres using edge subdivisions and contractions and prove that their h-polynomials are real-rooted.

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Dedication

To my mother, 蓉蓉,

and her parents, 飞 and 桢.

Chapter 1

Introduction

This thesis studies **shellable simplicial spheres**. These are **simplicial complexes** that satisfy a certain combinatorial decomposition property and whose **geometric realizations** are homeomorphic to spheres. Within mathematics, simplicial complexes arise in almost every area of study, including combinatorics and closely related fields such as computational topology, commutative algebra, and discrete geometry. The interdisciplinary nature of simplicial complexes makes it worthwhile to study the connections between their properties. Outside of mathematics, simplicial complexes have found real-world applications, especially in 3D graphics [PH97] and topological data analysis [EH08]. Graph theory has been a staple in many sciences, helping to develop algorithms and analyze networks. As generalizations of graphs, simplicial complexes can model higher-dimensional datasets and systems, such as protein interaction networks [ER18] and crystalline structures [JCC^{+}21].

Simplicial spheres

A **simplicial** (or **triangulated**) manifold is a simplicial complex whose **geometric realization** is homeomorphic to a manifold. The study of simplicial spheres–triangulations of one of the most fundamental topological spaces–spans several areas of mathematics. In discrete geometry, simplicial spheres can be seen as generalizations of simplicial **polytopes**. The **face rings** of simplicial spheres are **Gorenstein** (and thus **Cohen–Macaulay**), making them central objects in commutative algebra. The f**-numbers** (also called **face numbers**) of a simplicial complex count the number of faces of each dimension in the complex, and one sometimes works with related invariants called the h- and g-**numbers**. When the complex in question is a simplicial sphere, these invariants exhibit rich patterns and are widely studied in combinatorics, see for example the Upper Bound Theorem [Sta75], Dehn–Sommerville relations [Deh06, Som27], and the g-theorem [Adi18, PP20, APP21, KX23]. For more results about face numbers, particularly those related to the **neighborliness** (which describes the density of the lower-dimensional skeleta) of the simplicial sphere, we refer the reader to the survey [Nov23]. In addition to neighborly, one can place many other adjectives before "simplicial spheres," that give rise to interesting classes worth studying. Among these are **centrally symmetric** simplicial spheres and **flag** simplicial spheres. We recommend [Nov19] and [Zhe19] for further background on these respectively. We will also discuss flag spheres in more detail in Chapter 5.

The enumeration of simplicial spheres has been of interest for many decades. Results of this kind not only offer insight into the degree of combinatorial complexity possible when the topology is fixed, but also shed light on the rarity of simplicial spheres with certain properties, as we will see later. The minimal number of vertices needed to triangulate a d-dimensional sphere is $d+2$. In this case, the only triangulation possible is the boundary of a $(d+1)$-dimensional simplex. However, as the number of vertices is allowed to grow, the number of combinatorially non-equivalent triangulations of spheres drastically increases. To describe this growth more precisely, for a fixed d, let $s(d,n)$ denote the number of combinatorially distinct d-dimensional simplicial spheres on n vertices. The current best lower bound on $s(d,n)$ for $d \geq 3$, as n grows to infinity, is $2^{\Omega(n^{\lceil d/2 \rceil})}$ [Kal88a, PZ04, NSW16]. In contrast, there are only $2^{\Theta(n \log n)}$ combinatorially distinct d-polytopes with n vertices for $d \geq 4$ [GP86, Alo86]. This shows that simplicial spheres with additional geometric structures are asymptotically rare. For enumeration results on centrally symmetric and/or neighborly spheres, see [NZ20, NZ22, NZ24].

Shellability

A **pure** d-dimensional simplicial complex is **shellable** if it can be built by attaching its facets one at a time so that, at each step, the intersection of the new facet with the existing complex is pure $(d-1)$-dimensional. Shellability is the most studied decomposability property of simplicial complexes.

Shellability can also be defined for pure **polyhedral** complexes, and the notion has been further extended to nonpure complexes and posets [BW96].

Shellability gained its attention partly due to the famous theorem that the boundaries of polytopes are all shellable [BM71], and now it is widely studied across discrete geometry and topological combinatorics. Shellable complexes are easy to work with due to the inductive nature of their construction. In contrast, deciding whether a complex is shellable is NP-hard [GPP$^+$19], even if we know it is a simplicial ball [PT24].

While the h-numbers of a simplicial complex are not necessarily nonnegative in general, they are when the complex is shellable. In this case, the h-numbers count the **minimal new faces** of each dimension that appear during the shelling process. We describe one application of this fact: the **Ehrhart h^*-vector** of an integral polytope coincides with the h-vector of any unimodular triangulation of the polytope [Sta80] (which are always shellable). Therefore, keeping track of the shelling order, one can obtain combinatorial interpretations of the Ehrhart h^*-vector for certain integral polytopes, such as in [Li12].

Shellable simplicial complexes also exhibit desirable topological properties such as being Cohen–Macaulay and homotopy-equivalent to a bouquet of spheres. Moreover, if a triangulated pseudomanifold (with or without boundary) is shellable, then it must be a **piecewise linear** (PL) ball or sphere [DK74]. On the other hand, shellability itself is not a topological property. For example, while the simplex and its boundary are shellable, there exist nonshellable balls and spheres in all dimensions greater than two [HZ00].

For all the reasons mentioned above, it is often useful to determine whether a given complex is shellable, ranging from finding an explicit shelling for a specific complex to identifying obstructions to shellability for broader classes of complexes. For instance, it is a simple observation that any shellable simplicial ball must have a **free facet** (see, for example, [Zie98]). For simplicial spheres, however, the obstructions are much harder to characterize (see Section 6.3 for more details). It was shown in [HZ00] that a simplicial sphere is not shellable if its 1**-skeleton** contains a nontrivial knot with “few edges.”

Summary of the thesis

This thesis is organized as follows. In **Chapter 2**, we review basic definitions related to simplicial complexes. Additional background specific to each chapter will be provided in the corresponding chapters.

In **Chapter 3**, we settle the enumeration problem for shellable simplicial spheres. Specifically, we prove that for all $d \geq 3$, there are $2^{\Theta(n^{\lceil d/2 \rceil})}$ combinatorially distinct shellable simplicial d-spheres with n vertices. This is done by showing that the spheres in [NSW16, Construction 3] are all shellable; this is the construction that gives the current best lower bound for odd-dimensional spheres. The asymptotic bound is then computed by combining this with the result that the $2^{\Omega(n^{\lfloor d/2 \rfloor})}$ many spheres constructed in [Kal88a] are shellable [Lee00], the Upper Bound Theorem [Sta75], and another asymptotic result on **locally constructible** spheres in terms of the number of facets [BZ11].

A simplicial sphere is **polytopal** if it is combinatorially isomorphic to the boundary of some simplicial polytope. As discussed above, polytopal spheres are all shellable. On the other hand, there exist shellable but nonpolytopal spheres (see for example [Bar73]), as well as nonshellable simplicial spheres [HZ00]. Therefore, we have the following chain of containments between classes of simplicial spheres: $\{\text{polytopal spheres}\} \subsetneq \{\text{shellable spheres}\} \subsetneq \{\text{simplicial spheres}\}$. We have seen that, asymptotically, most simplicial spheres are not polytopal. The result in Chapter 3 tells us that polytopal spheres are rare even among shellable spheres.

In **Chapter 4**, we prove that any shellable d-dimensional simplicial sphere can be reconstructed from its **facet-ridge graph**, which encodes the incidence relationship between its d-dimensional faces (**facets**) and $(d-1)$-dimensional faces (**ridges**). A celebrated result by Blind and Mani [BML87] and then Kalai [Kal88b] shows that the facet-ridge graph of a simplicial polytope determines the combinatorial type of the polytope. This result is generalized to the asymptotically larger class of shellable simplicial spheres in Chapter 4.

The proof utilizes the notion of **good acyclic orientations** from [Kal88b]. A sphere is shellable if and only if there exists a good acyclic orientation on its facet-ridge graph. Given a face of the sphere, the **star** of the face is the subcomplex generated by all facets containing that face. Kalai's proof relies on the fact that if the sphere is polytopal, then every star is the beginning of some shelling of the sphere. The stars can thus be recovered by checking the initial sets of all good acyclic orientations.

For shellable but nonpolytopal spheres, however, one cannot necessarily find a shelling starting with an arbitrary star. Instead of using many good acyclic orientions as in the case of polytopal spheres, we will show that any one such orientation would suffice to reconstruct the sphere. From this orientation, we can partially recover the face lattice of the shellable sphere by listing all its faces as they are added to the complex during shelling. Then, for every minimal new face at each shelling step, we determine exactly which facets contain it. This is done using the language of k**-frames** and k**-systems** introduced in [JKK02]. We finish the proof by induction on the dimension of the sphere. Here one relies on the well-known fact that **links** of shellable spheres are also shellable spheres.

In **Chapter 5**, we switch gears and present a class of shellable simplicial spheres arising from a different context. Specifically, we investigate the independence complexes of graphs. These complexes are flag simplicial complexes, that is, all of their **minimal nonfaces** have size two. Flag simplicial complexes are studied extensively in combinatorial commutative algebra because of the special structure of their nonface ideals (also known in the literature as the **Stanley–Reisner ideals**). To this end, flag **homology spheres** are of particular interest since the Stanley–Reisner rings of homology spheres have another interesting property: they are Gorenstein.

A graph is **ternary** if it has no induced cycle of length divisible by three. This condition imposes strong restrictions on the combinatorics and topology of its independence complex. We determine when such complexes are homology spheres and show that, when the graph is planar, the complex is in fact a polytopal simplicial sphere. We also show that the transformations among these flag spheres using edge subdivisions and contractions can be modeled by the Hasse diagram of the partition refinement poset. In addition, we prove that their h**-polynomials** are products of Delannoy polynomials and thus **real-rooted**.

We conclude with further remarks and related open problems in **Chapter 6**. This includes discussions on the possibility of improving the lower bound for $s(d,n)$, on generalizing the result from Chapter 4 to other classes of simplicial spheres, as well as on **constructibility** and **partitionability** of simplicial spheres, both of which are relaxations of shellability.

Chapters 3, 4, and 5 are based on research papers [Yan24], [Yan25], and [BDH$^+$25], respectively.

Chapter 2

Preliminaries

2.1 Simplicial complexes

Let V be a finite set. An **abstract simplicial complex** Δ on V is a finite collection of subsets of V such that $\{v\} \in \Delta$ for every $v \in V$ and if $\sigma \in \Delta$ and $\tau \subseteq \sigma$, then $\tau \in \Delta$. The elements of Δ are called **faces**. The **dimension** of a face $\sigma \in \Delta$ is $\dim \sigma = |\sigma| - 1$. We define the dimension of the empty face $\varnothing \in \Delta$ to be -1. Conventionally, we call the 0-dimensional faces **vertices**, and the 1-dimensional faces **edges**. The i-dimensional faces are usually referred to as i-faces. We often omit the usual set notation for faces if it does not cause ambiguity.

The dimension of Δ is $\dim \Delta = \max\{\dim \sigma : \sigma \in \Delta\}$. We say that Δ is **pure** if all of its maximal faces with respect to inclusion have dimension $\dim \Delta$. In this case, the maximal faces are called **facets**, and faces of dimension $\dim \Delta - 1$ are called **ridges**. For $0 \leq i \leq \dim \Delta$, the ***i*-skeleton** of Δ is the simplicial complex $\{\sigma \in \Delta : \dim \sigma \leq i\}$.

The **abstract simplex** on V, denoted $\overline{V}$, is the collection of all subsets of V. To specify a simplicial complex, it suffices to specify its maximal faces $M_1, \ldots, M_n$. We say that the complex is **generated** by these maximal faces, and write it as $\overline{M}_1 \cup \cdots \cup \overline{M}_n$. Starting from the next chapter, we blur the difference between a face $\sigma \in \Delta$ and the simplex $\overline{\sigma} \subseteq \Delta$ and denote both as σ by abuse of notation.

2.2 Polytopes

A **polytope** P is the convex hull of finitely many points in a Euclidean space. Its **dimension** is given by the dimension of the affine hull of P. An important example is the **geometric d-simplex**, defined as the convex hull of $d+1$ affinely independent points.

Let $P \subset \mathbb{R}^d$ be a d-dimensional polytope. A hyperplane $H = \{x \in \mathbb{R}^d : a \cdot x = b\}$ is called a **supporting hyperplane** of P if P is contained in one of the two closed half-spaces determined by H. The proper **faces** of P are the intersections of P with its supporting hyperplanes. We also count P and $\varnothing$ as faces of P. One can check that the faces of P are themselves polytopes. Similar to the situation with simplicial complexes, we call the 0-, 1-, $(d-2)$-, and $(d-1)$- dimensional faces vertices, edges, ridges, and facets, respectively. The polytope P is **simplicial** if all of its facets are geometric simplices.

2.3 Geometric realizations and polyhedral complexes

A **polyhedral complex** C is a collection of polytopes such that

- if $P \in C$ and Q is a face of P, then $Q \in C$, and
- if $P, Q \in C$, then $P \cap Q$ is a common face of P and Q.

The union of all of these polytopes is usually denoted by $\|C\|$, but we will not distinguish between C and $\|C\|$.

If all polytopes of a polyhedral complex are geometric simplices, then we refer to such a complex as a **geometric simplicial complex**. The collection of vertex sets of faces in a geometric simplicial complex provides an abstract simplicial complex. Conversely, an abstract simplicial complex Δ gives rise to a geometric simplicial complex via its **geometric realization** $\|\Delta\|$, constructed as follows. For each maximal face $\sigma \in \Delta$, build a geometric $(|\sigma|-1)$-simplex with vertices labeled by elements in σ. Glue the simplices in a way that every two simplices are identified along their common (possibly empty) face. From now on, we will not distinguish between abstract and geometric simplicial complexes.

Many notions we used for simplicial complexes, such as faces, dimension, vertices, ridges, facets, and purity, could be easily extended to the generality of polyhedral complexes.

A **triangulation** of a polyhedral complex C is a simplicial complex Δ such that

- the geometric realization of Δ coincides with C, and
- every face of $\|\Delta\|$ is contained in a polytope of C.

The **boundary complex** ∂P of a polytope P is the polyhedral complex containing all faces of P except P itself.

2.4 Pseudomanifolds, spheres and balls

Let Δ be a $(d-1)$-dimensional simplicial complex.

We say that Δ is a **pseudomanifold** if all of the following conditions are satisfied:

1. Δ is pure;
2. Given any two facets F_1, F_2 of Δ, there exists a sequence of facets $G_0, \ldots, G_s$ such that $G_0 = F_1$, $G_s = F_2$ and $|G_i \cap G_{i+1}| = d-1$ for all $i \in [0, s-1]$. In other words, Δ is **strongly connected**, and;
3. Every ridge of Δ is contained in at most two facets of Δ.

In this case, the **boundary** of Δ, denoted $\partial\Delta$, is the subcomplex of Δ generated by all ridges that are contained in only one facet. If $\partial\Delta$ is empty, then we say Δ is a pseudomanifold without boundary. A face of $\Delta \setminus \partial\Delta$ is in the **interior** of Δ; such faces are called **interior faces**.

We call Δ a **simplicial** $(d-1)$**-sphere** (resp. a **simplicial** $(d-1)$**-ball**) if its geometric realization is homeomorphic to a topological $(d-1)$-sphere (resp. $(d-1)$-ball). Simplicial spheres and balls are examples of pseudomanifolds (without and with boundary, respectively). A simplicial $(d-1)$-sphere is said to be **polytopal** if it is combinatorially isomorphic to the boundary complex of a simplicial d-polytope. Minimal examples of nonpolytopal simplicial spheres include the Barnette sphere [Bar73] and the Grünbaum–Sreedharan sphere [GS67]; both are 3-dimensional and have 8 vertices.

In analogy with simplicial spheres and balls, we say that a polyhedral complex is a **polyhedral** $(d-1)$**-sphere** (resp. **polyhedral** $(d-1)$**-ball**) if it is homeomorphic to a $(d-1)$-sphere (resp. $(d-1)$-ball). Given a polyhedral $(d-1)$-ball C, its **boundary complex** ∂C is the subcomplex of C whose facets are the ridges of C that are contained in exactly one facet of C.

2.5 Face numbers

Let X be a simplicial complex or a polyhedral complex of dimension $d-1$, or a polytope of dimension d. We denote by $f_i(X)$ the number of i-dimensional faces of X. The **f-vector** of X is the sequence $f(X) = (f_{-1}(X), \ldots, f_{d-1}(X))$.

Let $M(t) = M_d(t) = (t, t^2, t^3, \ldots, t^d)$ be the d-th **moment curve**, and let $n \geq d+1$. The d-dimensional **cyclic polytope** on n vertices is defined to be

$$C_d(n) = \operatorname{conv}\{M_d(t_1), \ldots, M_d(t_n)\},$$

where $t_1 < t_2 < \cdots < t_n$ are real numbers. This polytope is simplicial and its combinatorial type is independent of the choice of $t_1, t_2, \ldots, t_n$. One of the many reasons for the importance of this polytope is that it maximizes all face numbers among polytopes and simplicial spheres with the same parameters.

Theorem 2.5.1 (Upper Bound Theorem [Sta75]). *Let Δ be a $(d-1)$-dimensional simplicial sphere with n vertices. Then $f_i(\Delta) \leq f_i(C_d(n))$ for all $1 \leq i \leq d-1$.*

The Upper Bound Theorem was first proved for all polytopes by McMullen [McM70]. Note that by a standard trick of "pulling vertices," it suffices to prove the theorem for simplicial polytopes to conclude it for all polytopes.

2.6 Stars, links, and joins

Let Δ be a $(d-1)$-dimensional simplicial complex and σ be a face of Δ. The **star** and **link** of σ in Δ are respectively defined to be the subcomplexes

$$\operatorname{st}_\Delta \sigma = \{\tau \in \Delta : \tau \cup \sigma \in \Delta\} \text{ and } \operatorname{lk}_\Delta \sigma = \{\tau \in \Delta : \tau \cap \sigma = \varnothing, \tau \cup \sigma \in \Delta\}.$$

In particular, if Δ is pure, then $\operatorname{lk}_\Delta \sigma$ is a pure $(d - \dim \sigma - 2)$-dimensional simplicial complex whose set of facets is $\{F \setminus \sigma : F \text{ is a facet of } \operatorname{st}_\Delta \sigma\}$.

If Δ and Γ are simplicial complexes on disjoint vertex sets V and V', then the **join** of Δ and Γ is the simplicial complex $\Delta * \Gamma = \{F \cup G : F \in \Delta,\ G \in \Gamma\}$. When one of the the complexes, say Γ, has only a single vertex v, we call $\Delta * \overline{\{v\}}$ (or simply $\Delta * v$) the **cone** over Δ with apex v. More generally, if $\Gamma = \overline{V'}$, then the join is the cone over Δ with the simplex Γ.

2.7 Shellings

For two integers n_1, n_2 such that $n_1 < n_2$, define $[n_1, n_2] := \{n_1, \ldots, n_2\}$; when $n_1 > 0$, we also define $[n_1] := \{1, \ldots, n_1\}$. There are many equivalent definitions of shellability, see for example [Zie95, Chapter 8]. In this thesis, we use the following ones:

Let Δ be a pure $(d-1)$-dimensional simplicial complex with n facets. A **shelling** of Δ is a total ordering $F_1, \ldots, F_n$ of the facets of Δ that satisfies any of the following equivalent conditions:

1. For every $i \in [2, n]$ and for every $j < i$, there exists some $m < i$ with the property that $F_i \cap F_m$ is a $(d-2)$-face containing $F_i \cap F_j$.
2. For every $i \in [2, n]$, the simplicial complex $\overline{F}_i \cap (\overline{F}_1 \cup \cdots \cup \overline{F}_{i-1})$ is pure and $(d-2)$-dimensional.
3. For every $i \in [n]$, the family of sets $\{\sigma \subseteq F_i : \sigma \nsubseteq F_j \text{ for all } j < i\}$ ordered by inclusion has a unique minimal element $\mathcal{R}(F_i)$. We call $\mathcal{R}(F_i)$ the **restriction face** or the **minimal new face** of F_i, and $\mathcal{R}$ the **restriction map** of this shelling.

If such an ordering exists, then we say that Δ is **shellable**.

Every shelling of Δ induces a shelling of $\mathrm{st}_\Delta\, \sigma$ and $\mathrm{lk}_\Delta\, \sigma$. Moreover, a join $\Delta * \Gamma$ is shellable if and only if both Δ and Γ are shellable.

Chapter 3

Enumeration of shellable spheres

3.1 Asymptotic results

The goal of this chapter is to establish the asymptotics of the number of shellable D-spheres with N vertices, as N grows to infinity. To achieve this, we show that the spheres produced in [NSW16, Construction 3] by Nevo, Santos, and Wilson are shellable.

It follows from Steinitz's theorem (see [Zie95, Chapter 4]) that all simplicial 2-spheres can be realized as the boundary complexes of 3-polytopes. However, in higher dimensions, there are many more simplicial spheres than the boundaries of polytopes. Let $s(D,N)$ denote the number of combinatorially distinct D-spheres with N vertices. For $D \geq 4$, Kalai [Kal88a] proved that $s(D,N) \geq 2^{\Omega(N^{\lfloor D/2 \rfloor})}$. Pfeifle and Ziegler [PZ04] then complemented Kalai's result by showing $s(3,N) \geq 2^{\Omega(N^{5/4})}$. Later, Nevo, Santos, and Wilson [NSW16] improved the lower bound of $s(D,N)$ for odd $D \geq 3$ to $2^{\Omega(N^{\lceil D/2 \rceil})}$. In constrast to these bounds, we know from works by Goodman and Pollack [GP86] as well as Alon [Alo86] that there are only $2^{\Theta(N \log N)}$ combinatorially distinct D-polytopes with N vertices for $D \geq 4$. See also [PPS23] for the current best lower bound.

Recall the result by Bruggesser and Mani [BM71] that the boundary complexes of simplicial polytopes are always shellable. This naturally leads to the study of shellable spheres. How many shellable spheres are there? How does this number compare to the number of polytopes? These questions were partially answered by Lee's proof in [Lee00] that Kalai's spheres in [Kal88a] are all shellable. Let $s_{\text{shell}}(D,N)$ denote the number of shellable D-spheres with N vertices. Lee's result implies that

$$s_{\text{shell}}(D,N) \geq 2^{\Omega(N^{\lfloor D/2 \rfloor})}. \tag{3.1}$$

What about the Nevo–Santos–Wilson spheres? The main result of this chapter is:

Theorem 3.1.1. *The Nevo–Santos–Wilson spheres in* [NSW16, Construction 3] *are all shellable.*

On the other hand, Benedetti and Ziegler [BZ11] proved that for $D \geq 2$, the number of combinatorially distinct locally constructible (LC) D-spheres with M facets grows not faster than $2^{D^2 M}$. In addition, they proved that shellable spheres are LC. Meanwhile, the Upper Bound Theorem for simplicial spheres by Stanley [Sta75] asserts that a simplicial D-sphere with N vertices has at most $O(N^{\lceil D/2 \rceil})$ facets. We make the following observation by combining Theorem 3.1.1 with Benedetti and Ziegler's results, the bounds in (3.1), and the Upper Bound Theorem for simplicial spheres.

Corollary 3.1.2.

$$s_{\text{shell}}(D,N) = 2^{\Theta(N^{\lceil D/2 \rceil})} \textit{ for all } D \geq 3.$$

The structure of this chapter is as follows. Several key definitions and facts related to Nevo, Santos, and Wilson's construction are provided in Section 3.2. Sections 3.3 and 3.4 contain a detailed proof of the shellability of the Nevo–Santos–Wilson spheres. Detailed computations leading to Corollary 3.1.2 can be found at the end of Section 3.4.

3.2 The Nevo–Santos–Wilson spheres

In this section, we present the key facts about [NSW16, Construction 3] as well as introduce some new definitions and notation. We closely follow Nevo, Santos, and Wilson's paper [NSW16] and state their results without proof. The reader is encouraged to check their paper for more details.

A **path** of length $n-1$ is a 1-dimensional pure simplicial complex on the vertex set $\{v_1, \ldots, v_n\}$ whose facets are $\{v_i, v_{i+1}\}$ for $i \in [n-1]$. We denote this path as $P(v_1, \ldots, v_n)$.

Remark. A word about notation: the construction below is based on the join of d paths, each of length $n-1$. This join is a $(2d-1)$-dimensional complex. We let $D := 2d-1$ and mention right away that each sphere produced in [NSW16, Construction 3] is D-dimensional and has $N = dn + \lceil d(n-1)/(d+2) \rceil + 1$ vertices.

Let B be the join of d paths of length $n-1$. Suppose $d \geq 2$ and $n \geq 3$ for nontrivial results. For each $\ell \in [d]$, we denote the ℓ-th path by $P(1^\ell, \ldots, n^\ell)$. Then each $(2d-1)$-simplex σ in the join B is of the form

$$\sigma = \{i_1^1, i_1 + 1^1, \ldots, i_d^d, i_d + 1^d\}.$$

Notation. For simplicity, we omit parentheses and commas in the notation for simplices when the meaning is unambiguous. Thus σ becomes $i_1^1 i_1 + 1^1 \cdots i_d^d i_d + 1^d$. On the other hand, σ can be uniquely represented by a d-tuple of indices $(i_1, \ldots, i_d) \in [n-1] \times \cdots \times [n-1]$, or $(i_1^1, \ldots, i_d^d)$ when we would like to clarify the coordinates with the superscripts. We use these notations interchangeably. Let $\sum \sigma$ denote $i_1 + \cdots + i_d$, the **index sum** of σ. For the rest of the chapter, we reserve the letter σ for the $(2d-1)$-simplices of B.

Each $\sigma \in B$ can be visualized as a d-cube in the d-dimensional grid with side length $n-1$ that represents B. The left of Figure 3.1 illustrates the case of $d = 3$ and $n = 5$.

The join B is a simplicial $(2d-1)$-ball. For each $k \in [\lceil d(n-1)/(d+2) \rceil]$, define B_k to be the union of all $\sigma \in B$ such that $\sum \sigma \in [(k-1)(d+2), k(d+2) - 1]$. This is a $(2d-1)$-ball contained in B [NSW16, Lemma 5.2]. Note that $B = \bigcup_k B_k$. Figure 3.1 depicts the union for the case of $d = 3$ and $n = 5$.

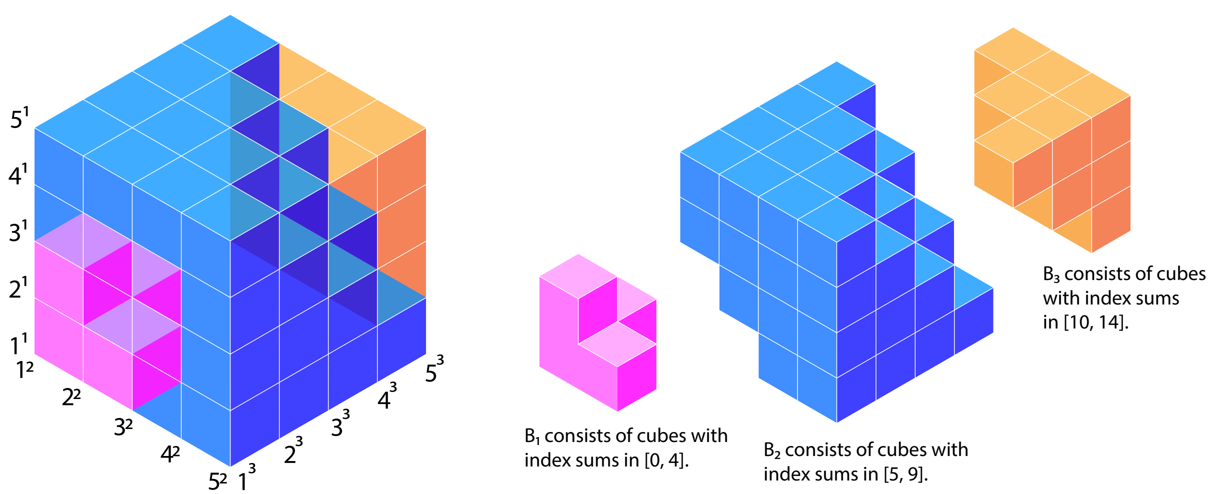


Figure 3.1: A grid representing B for $d = 3$ and $n = 5$. It is the union of balls B_k for $k \in [3]$.

The idea of [NSW16, Construction 3] is to replace each B_k with a new ball $\widetilde{B_k}$ with the same boundary. For each k, a new vertex o_k is introduced as follows (assuming B_k is not a simplex):

- Consider the collections of $(2d-1)$-simplices:

$$L_k := \left\{\sigma \in B_k : \sum \sigma = (k-1)(d+2)\right\} \text{ and } U_k := \left\{\sigma \in B_k : \sum \sigma = k(d+2) - 1\right\}.$$

  We call them respectively the **lower diagonal** and **upper diagonal** of B_k. The corresponding cubes for $k = 3$ when $d = 2$ and $n = 8$ are indicated in Figure 3.2.

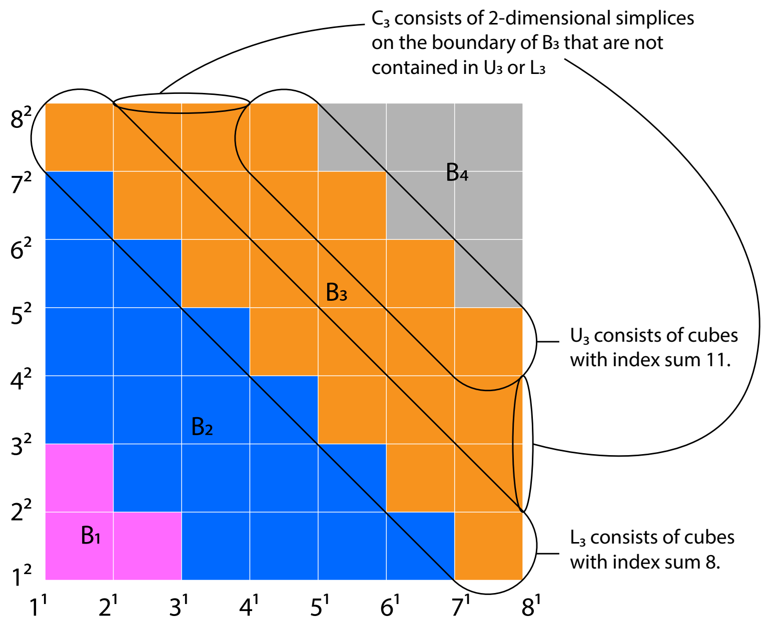


Figure 3.2: A grid representing B for $d = 2$ and $n = 8$.

For each $\sigma \in L_k \cup U_k$, define $D_\sigma := \sigma \cap \partial B_k$. Let C_σ be a new polyhedral cell whose boundary complex is $D_\sigma \cup (\partial D_\sigma * o_k)$. Let F_σ be the only subset of $V(D_\sigma)$ not in D_σ but such that all of its proper subsets are in D_σ. We call F_σ the **missing face** of D_σ. Let $G_\sigma := \sigma \setminus F_\sigma$. There are two ways to triangulate C_σ without introducing new vertices or changing the boundary of C_σ:

$$T_{\sigma,1} = F_\sigma * \partial(G_\sigma * o_k) \text{ and } T_{\sigma,2} = \partial F_\sigma * (G_\sigma * o_k).$$

We use $T_{\sigma,j}$ to mean either one of the two triangulations.

- Let C_k be the set of all $(2d-2)$-simplices on the boundary of B_k not contained in any $\sigma \in L_k \cup U_k$. We call C_k the **connecting path** of B_k. The corresponding edges for $k = 3$ when $d = 2$ and $n = 8$ are indicated in Figure 3.2.

Let $\widetilde{B_k} := \bigcup_{\sigma \in L_k \cup U_k} T_{\sigma,j} \cup \bigcup_{\tau \in C_k} \tau * o_k$. In this way, $2^{|L_k \cup U_k|}$ different labeled balls $\widetilde{B_k}$ have been created, because there are two choices of $T_{\sigma,j}$ for each $\sigma \in L_k \cup U_k$. Let $\widetilde{B} := \bigcup_k \widetilde{B_k}$. From $\widetilde{B}$, we obtain a $(2d-1)$-sphere $\widetilde{B} \cup (\partial\widetilde{B} * o)$ by introducing a new vertex o and taking a cone over the boundary of $\widetilde{B}$.

As shown in [NSW16, Corollary 5.5], this construction yields $2^{\Omega(N^d)}$ combinatorially distinct $(2d-1)$-spheres with N vertices. Indeed, there are at least $2^{\sum_k |L_k \cup U_k|} > 2^{2N^d/3d^{d+1}}$ labeled N-vertex triangulations $\widetilde{B} \cup (\partial\widetilde{B} * o)$ of the $(2d-1)$-sphere. Since $N! = 2^{O(N \log N)}$, dividing by $N!$ does not change the asymptotic order of the bound.

We end this section by providing a few explicit descriptions of B and the balls B_k. These will come in handy for the proofs later.

Lemma 3.2.1. *The set of facets of ∂B is*

$$\bigcup_{\sigma=(i_1,\dots,i_d)\in B} (\{\sigma \setminus i_\ell + 1^\ell : i_\ell = 1\} \cup \{\sigma \setminus i_\ell^\ell : i_\ell = n-1\}).$$

Proof. Any $\tau := \sigma \setminus i_\ell + 1^\ell$ is contained in $(i_1^1, \dots, i_\ell - 1^\ell, \dots, i_d^d)$. Therefore, $\tau \in \partial B$ if and only if $i_\ell = 1$. Similarly, any $\tau := \sigma \setminus i_\ell^\ell$ is contained in $(i_1^1, \dots, i_\ell + 1^\ell, \dots, i_d^d)$. Therefore, $\tau \in \partial B$ if and only if $i_\ell = n-1$. □

Lemma 3.2.2. *The set of facets of ∂B_k is*

$$\bigcup_{\sigma=(i_1,\dots,i_d)\in L_k} \{\sigma \setminus i_\ell + 1^\ell : \ell \in [d]\} \cup \bigcup_{\sigma=(i_1,\dots,i_d)\in U_k} \{\sigma \setminus i_\ell^\ell : \ell \in [d]\} \cup C_k,$$

and

$$C_k = \bigcup_{\sigma=(i_1,\dots,i_d)\in B_k \setminus (L_k \cup U_k)} (\{\sigma \setminus i_\ell + 1^\ell : i_\ell = 1\} \cup \{\sigma \setminus i_\ell^\ell : i_\ell = n-1\}).$$

Proof. The proof is similar to that of Lemma 3.2.2. Recall that $\sigma \in B_k$ if and only if $\sum \sigma \in [(k-1)(d+2), k(d+2)-1]$. Therefore, $\sigma' := (i_1^1, \dots, i_\ell - 1^\ell, \dots, i_d^d)$ does not belong to B_k if and only if either $i_\ell = 1$ or $\sigma \in L_k$, which forces $\sum \sigma' = \sum \sigma - 1 < (k-1)(d+2)$. Analogously, $\sigma' :=$

$(i_1^1, \ldots, i_\ell + 1^\ell, \ldots, i_d^d)$ does not belong to B_k if and only if either $i_\ell = n-1$ or $\sigma \in U_k$, which forces $\sum \sigma' = \sum \sigma + 1 > k(d+2) - 1$.

This also implies that if $\tau \in C_k$, then since $\tau \subset \sigma$ for some σ such that $(k-1)(d+2) < \sum \sigma < k(d+2) - 1$, τ must be of the form $\sigma \setminus i_\ell + 1^\ell$, $i_\ell = 1$ or $\sigma \setminus i_\ell^\ell$, $i_\ell = n-1$. □

Lemma 3.2.3. *Let $\sigma = (i_1, \ldots, i_d) \in L_k \cup U_k$. Then its missing face F_σ is*

$$\{i_\ell^\ell : i_\ell = n-1\} \cup \{i_\ell + 1^\ell : \ell \in [d]\} \text{ if } \sigma \in L_k, \text{ and } \{i_\ell + 1^\ell : i_\ell = 1\} \cup \{i_\ell^\ell : \ell \in [d]\} \text{ if } \sigma \in U_k.$$

Proof. This follows from the definition of F_σ and Lemmas 3.2.1 and 3.2.2. □

3.3 The shellability of $\widetilde{B}$

3.3.1 Overview

To prove that the Nevo–Santos–Wilson spheres from [NSW16, Construction 3] are shellable, we first prove the shellability of the new ball $\widetilde{B}$ obtained by modifying the join of paths B. This is where the major work lies. We start by by outlining the ideas behind and the structure of the proof.

Recall that, to better visualize the join B of d paths of length $n-1$, we identify B with a d-dimensional grid with side length $n-1$, where each d-cube in the grid represents a $(2d-1)$-simplex in B. We can examine the d-dimensional grid by "layers." Each layer is a $(d-1)$-dimensional grid representing a join of $d-1$ paths of length $n-1$:

$$B = \bigcup_{i_1 \in [n-1]} i_1^1 i_1 + 1^1 * \mathrm{lk}_B\, i_1^1 i_1 + 1^1.$$

We can further divide each $(d-1)$-dimensional grid $\mathrm{lk}_B\, i_1^1 i_1 + 1^1$ into layers of $(d-2)$-dimensional grids, and so on. Roughly speaking, to modify B, we can modify each layer of the grid and stack the new layers together, with some caveats to be addressed soon. This allows us to prove the shellability of $\widetilde{B}$ by induction on d. We illustrate the process in the following example before formalizing it.

Example 3.3.1. Return to the example for $d = 3$ and $n = 5$. On the left of Figure 3.3, the balls B_1, B_2, and B_3 are colored differently. Each B_k consists of cubes with index sums in $[5(k-1), 5k-1]$.

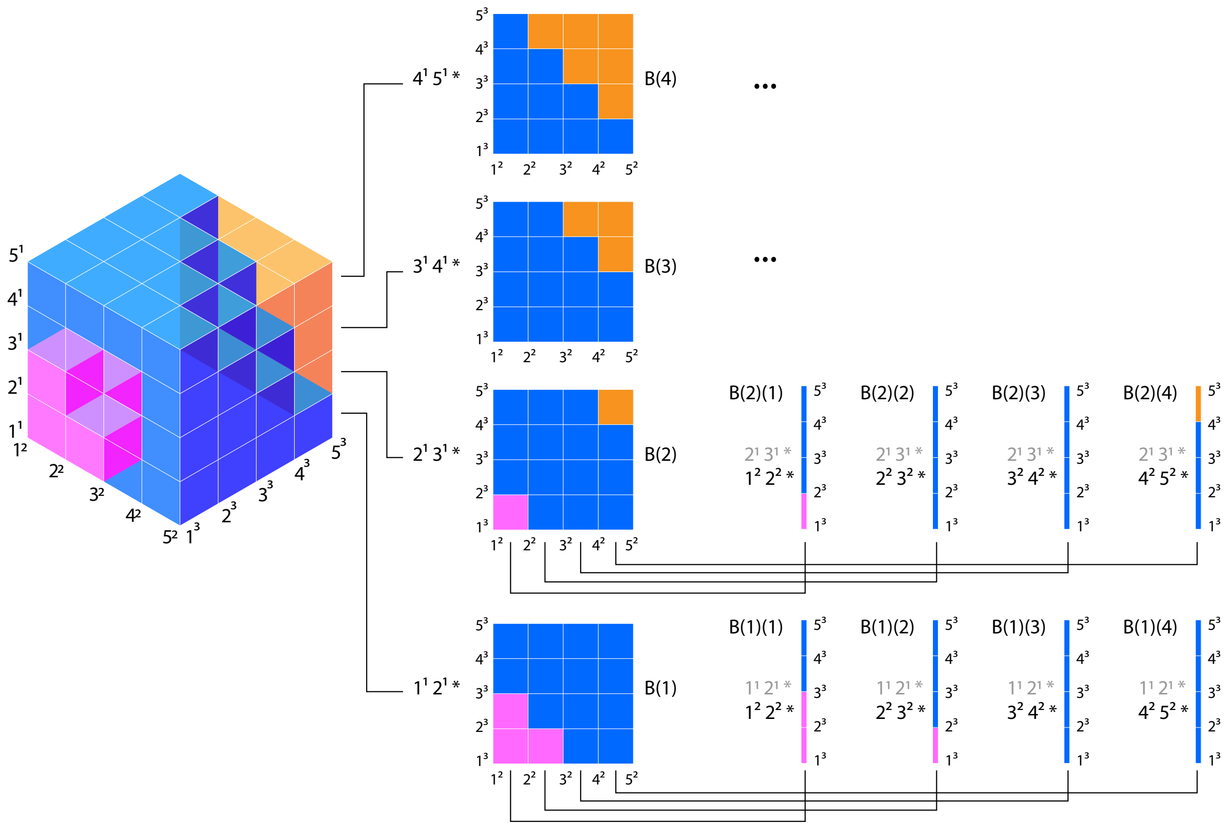


Figure 3.3: Dividing B into layers recursively.

We divide the 3-dimensional grid into 4 layers of 2-dimensional grids, as shown in the middle column. Let $B(i_1)$ denote the i_1-th layer, i.e., $\mathrm{lk}_B\, i_1^1 i_1+1^1$. Moreover, let

$$B(i_1)_k := \{\tau \in \mathrm{lk}_B\, i_1^1 i_1+1^1 : i_1^1 i_1+1^1 \cup \tau \in B_k\}.$$

Then each $B(i_1)_k$ consists of all 2-cubes with index sums in $[5(k-1)-i_1, 5k-1-i_1]$. It is colored the same as the corresponding B_k in the 3-dimensional grid.

Repeat the process with the 2-dimensional grids. As shown on the right, we divide the link of $i_1^1 i_1+1^1$ into 4 layers of 1-dimensional grids $B(i_1)(i_2)$. We have reached the base case. Each $B(i_1)(i_2)_k$ consists of all 1-cubes with index sums in $[5(k-1)-i_1-i_2, 5k-1-i_1-i_2]$.

Let $\widetilde{B(i_1)} := \mathrm{lk}_{\widetilde{B}}\, i_1^1 i_1+1^1$. Later, we will show that $\widetilde{B(i_1)}$ can be obtained from $B(i_1)$ following the same rules of modifying B. Assuming each $\widetilde{B(i_1)}$ is shellable, then $i_1^1 i_1+1^1 * \widetilde{B(i_1)}$ is also shellable. The plan is to concatenate the shellings of $i_1^1 i_1+1^1 * \widetilde{B(i_1)}$ together: start with a shelling of $1^1 2^1 * \widetilde{B(1)}$, then attach a shelling of $2^1 3^1 * \widetilde{B(2)}$, and so on.

There is one problem: some facets of $\widetilde{B}$ do not belong to any $i_1^1 i_1+1^1 * \widetilde{B(i_1)}$. For example, consider $\sigma = (2,2,1) = 2^1 3^1 2^2 3^2 1^3 2^3 \in L_2$. Suppose it is replaced by the triangulation $T_{\sigma,1}$, then one resulting facet is $\sigma \setminus 2^1 \cup o_2$. Another example is $1^1 1^2 2^2 1^3 2^3 o_1$, with $1^1 1^2 2^2 1^3 2^3 \in C_1$. We will characterize these extra facets in Lemma 3.3.6 and insert them between the shellings of $i_1^1 i_1+1^1 * \widetilde{B(i_1)}$.

Let $d \geq 2$, $n \geq 3$, and B be the join of d paths of length $n-1$. We extend the notation in Example 3.3.1 to the general case. Let $m \in [d]$. Consider the join of m paths

$$B(i_1)\cdots(i_{d-m}) := \mathrm{lk}_B\{i_1^1, i_1+1^1, \ldots, i_{d-m}^{d-m}, i_{d-m}+1^{d-m}\}.$$

When $m = d$, this is just B. For $k \in [\lceil d(n-1)/(d+2)\rceil]$, let $B(i_1)\cdots(i_{d-m})_k$ denote the union of all m-cubes $(i_{d-m+1}, \ldots, i_d)$ with index sums in

$$I_m := \left[(k-1)(d+2) - \sum_{\ell=1}^{d-m} i_\ell,\ k(d+2) - 1 - \sum_{\ell=1}^{d-m} i_\ell\right].$$

Then $B(i_1)\cdots(i_{d-m}) = \bigcup_k B(i_1)\cdots(i_{d-m})_k$. Just like in B, the lower and upper diagonal cubes of $B(i_1)\cdots(i_{d-m})_k$ are respectively those whose index sums attain the minimum and maximum of I_m. The connecting path consists of the $(2m-2)$-simplices on the boundary of $B(i_1)\cdots(i_{d-m})_k$ that are not contained in any lower or upper diagonal cubes. Lastly, let

$$B(i_1)\widetilde{\cdots}(i_{d-m}) := \mathrm{lk}_{\widetilde{B}}\{i_1^1, i_1+1^1, \ldots, i_{d-m}^{d-m}, i_{d-m}+1^{d-m}\}.$$

Based on the inductive idea from Example 3.3.1, we would like to prove the statement below, which directly implies the shellability of $\widetilde{B}$.

Proposition 3.3.2. *Let $m \in [d]$. Then $B(i_1)\widetilde{\cdots}(i_{d-m})$ is shellable for all $i_1, \ldots, i_{d-m} \in [n-1]$.*

The base case of $m = 1$ is immediately true. Since $\widetilde{B}$ is a ball, the link $B(i_1)\widetilde{\cdots}(i_{d-1})$ is a connected 1-dimensional simplicial complex, which is automatically shellable.

Remark 3.3.3. For the inductive step, we need to prove the following: let $m \in [2,d]$ and $i_1, \ldots, i_{d-m} \in [n-1]$, assuming $B(i_1)\widetilde{\cdots}(i_{d-m+1})$ is shellable for all $i_{d-m+1} \in [n-1]$, then $B(i_1)\widetilde{\cdots}(i_{d-m})$ is shellable. For cleanliness of the write-up, we provide the proof for the shellability of $\widetilde{B}$ assuming $\widetilde{B(i_1)}$ is shellable for all $i_1 \in [n-1]$. The proof for the general case is analogous. The fact that I_m is "shifted" does not matter in the proof; all we need is that the difference of the endpoints of the interval is greater than the dimension of the grid. To accommodate this, we use the relaxed assumption that the difference between the index sum of the upper diagonal cubes and that of the lower diagonal cubes in B_k is greater than d (instead of being equal to $d+1$).

In short, we aim to prove the next proposition in the rest of Section 3.3.

Proposition 3.3.4. *Suppose $\widetilde{B(i_1)}$ is shellable for all $i_1 \in [n-1]$. Then $\widetilde{B}$ is shellable.*

We claim that the order of facets below is a shelling of $\widetilde{B}$.

Start with any shelling order of $1^1 2^1 * \widetilde{B(1)}$.
Then list in any order facets of the form $\sigma \setminus 2^1 \cup o_k$ for $\sigma \in L_k \cup U_k \cap \mathrm{st}_B\, 1^1 2^1$ for all k.
Then list in any order facets of the form $\sigma \setminus 1^1 \cup o_k$ for $\sigma \in L_k \cup U_k \cap \mathrm{st}_B\, 1^1 2^1$ for all k.
Repeat for all $k \in [\lceil d(n-1)/(d+2) \rceil]$ **as** k **increases**:
List the facets of the form $\sigma \setminus 2^1 \cup o_k$ for $\sigma \in (B_k \setminus (L_k \cup U_k)) \cap \mathrm{st}_B\, 1^1 2^1$
in the order $\prec$ defined in Section 3.3.3.
Repeat for all $2 \leq i_1 \leq n-1$ **as** i_1 **increases**:
List in any order facets of the form $\sigma \setminus i_1 + 1^1 \cup o_k$ for $\sigma \in L_k \cup U_k \cap \mathrm{st}_B\, i_1^1 i_1 + 1^1$ for all k.
Follow this by any shelling order of $i_1^1 i_1 + 1^1 * \widetilde{B(i_1)}$.
Then list in any order facets of the form $\sigma \setminus i_1^1 \cup o_k$ for $\sigma \in L_k \cup U_k \cap \mathrm{st}_B\, i_1^1 i_1 + 1^1$ for all k.
Repeat for all $k \in [\lceil d(n-1)/(d+2) \rceil]$ **as** k **increases**:
List the facets of the form $\sigma \setminus n-1^1 \cup o_k$ for $\sigma \in (B_k \setminus (L_k \cup U_k)) \cap \mathrm{st}_B\, n-1^1 n^1$
in the order $\prec$ defined in Section 3.3.3.

Table 3.1: A shelling of $\widetilde{B}$.

Here, we do not claim that $\widetilde{B}$ must contain $\sigma \setminus i_1 + 1^1 \cup o_k$ or $\sigma \setminus i_1^1 \cup o_k$ for every k and every

$\sigma \in L_k \cup U_k \cap \operatorname{st}_B i_1^1 i_1 + 1^1$. In fact, whether it is a facet depends on the choice of $T_{\sigma,j}$. We merely say that if it is a facet of $\widetilde{B}$, it should be listed in the corresponding slot in this order.

In Section 3.3.2, we analyze the facets of $\widetilde{B}$ that are not contained in $\bigcup_{i_1 \in [n-1]} i_1^1 i_1 + 1^1 * \widetilde{B(i_1)}$, showing that Table 3.1 is indeed the list of all facets of $\widetilde{B}$. In Section 3.3.3, we define a certain order $\prec$ for a specific subset of these facets. We leave the actual proof of the fact that Table 3.1 provides a shelling of $\widetilde{B}$ to Section 3.3.4.

3.3.2 Facets between layers

We first establish that, essentially, the links $B(i_1)$ are modified the same way as B is.

Lemma 3.3.5. *Let $i_1 \in [n-1]$. Then $\widetilde{B(i_1)}$ can be obtained from $B(i_1)$ following the same steps of changing B to $\widetilde{B}$ in* [NSW16, Construction 3].

More precisely, if $\sigma \in \operatorname{st}_B i_1^1 i_1 + 1^1$ is replaced by the triangulation $T_{\sigma,j}$, then $\sigma \setminus i_1^1 i_1 + 1^1 \in B(i_1)$ is replaced by $T_{\sigma \setminus i_1^1 i_1 + 1^1,\, j}$. If a $(2d-2)$-simplex $\tau \in C_k \cap \operatorname{st}_B i_1^1 i_1 + 1^1$ is coned with o_k, then $\tau \setminus i_1^1 i_1 + 1^1$ in the connecting path of $B(i_1)_k$ is coned with o_k.

Proof. Let $\sigma \in L_k \cup U_k \cap \operatorname{st}_B i_1^1 i_1 + 1^1$. Recall from Section 3.2 that $D_\sigma = \sigma \cap \partial B_k$, F_σ is the missing face of D_σ, and $G_\sigma = \sigma \setminus F_\sigma$. We extend these notations to $\sigma \setminus i_1^1 i_1 + 1^1$. By simple set-theoretic computations, one can check that

$$D_{\sigma \setminus i_1^1 i_1 + 1^1} = (\sigma \setminus i_1^1 i_1 + 1^1) \cap \partial B(i_1)_k = \operatorname{lk}_{D_\sigma} i_1^1 i_1 + 1^1,$$
$$F_{\sigma \setminus i_1^1 i_1 + 1^1} = F_\sigma \setminus i_1^1 i_1 + 1^1, \text{ and } G_{\sigma \setminus i_1^1 i_1 + 1^1} = G_\sigma \setminus i_1^1 i_1 + 1^1.$$

Let $X = F_\sigma \cap i_1^1 i_1 + 1^1$ and $Y = i_1^1 i_1 + 1^1 \setminus X$. Then $F_{\sigma \setminus i_1^1 i_1 + 1^1} = \operatorname{lk}_{F_\sigma} X$ and $G_{\sigma \setminus i_1^1 i_1 + 1^1} = \operatorname{lk}_{G_\sigma} Y$. Observe that the triangulation

$$\begin{aligned} T_{\sigma \setminus i_1^1 i_1 + 1^1,\, 1} &= F_{\sigma \setminus i_1^1 i_1 + 1^1} * \partial(G_{\sigma \setminus i_1^1 i_1 + 1^1} * o_k) \\ &= \operatorname{lk}_{F_\sigma} X * \partial(\operatorname{lk}_{G_\sigma} Y * o_k) = \operatorname{lk}_{F_\sigma} X * (\partial \operatorname{lk}_{G_\sigma} Y * o_k \cup \operatorname{lk}_{G_\sigma} Y * \partial o_k) \\ &= \operatorname{lk}_{F_\sigma} X * \operatorname{lk}_{\partial G_\sigma} Y * o_k \cup \operatorname{lk}_{F_\sigma} X * \operatorname{lk}_{G_\sigma} Y = \operatorname{lk}_{F_\sigma * \partial G_\sigma * o_k}(X \sqcup Y) \cup \operatorname{lk}_{F_\sigma * G_\sigma}(X \sqcup Y) \\ &= \operatorname{lk}_{T_{\sigma,1}} i_1^1 i_1 + 1^1. \end{aligned}$$

Similarly, $T_{\sigma \setminus i_1^1 i_1 + 1^1,\, 2} = \operatorname{lk}_{T_{\sigma,2}} i_1^1 i_1 + 1^1$.

Finally, for $\tau \in C_k \cap \operatorname{st}_B i_1^1 i_1 + 1^1$, $\tau \setminus i_1^1 i_1 + 1^1 * o_k = \operatorname{lk}_{\tau * o_k} i_1^1 i_1 + 1^1$. □

With this, we can characterize the facets of $\widetilde{B}$ that are "between" the layers.

Lemma 3.3.6. *Let F be a facet of $\widetilde{B} \setminus \bigcup_{i_1 \in [n-1]} i_1^1 i_1 + 1^1 * \widetilde{B(i_1)}$ that arises from $\sigma = (i_1, \ldots, i_d) \in B_k \cap \mathrm{st}_B\, i_1^1 i_1 + 1^1$ for some k.*

1. *If $\sigma \in L_k \cup U_k \cap \mathrm{st}_B\, i_1^1 i_1 + 1^1$, then F takes the following forms depending on F_σ and the triangulation $T_{\sigma,j}$:*

$T_{\sigma,j}$ \ $F_\sigma \cap i_1^1 i_1 + 1^1$	i_1^1	$i_1 + 1^1$	$\varnothing$	$i_1^1 i_1 + 1^1$
$T_{\sigma,1}$	$\sigma \setminus i_1 + 1^1 \cup o_k$	$\sigma \setminus i_1^1 \cup o_k$	$\sigma \setminus i_1 + 1^1 \cup o_k$, $\sigma \setminus i_1^1 \cup o_k$	Does not exist
$T_{\sigma,2}$	$\sigma \setminus i_1^1 \cup o_k$	$\sigma \setminus i_1 + 1^1 \cup o_k$	Does not exist	$\sigma \setminus i_1 + 1^1 \cup o_k$, $\sigma \setminus i_1^1 \cup o_k$

2. *If $\sigma \in (B_k \setminus (L_k \cup U_k)) \cap \mathrm{st}_B\, i_1^1 i_1 + 1^1$, then $F = \sigma \setminus i_1 + 1^1 \cup o_k$ when $i_1 = 1$ and $F = \sigma \setminus i_1^1 \cup o_k$ when $i_1 = n - 1$.*

Proof. Case 1 is immediate from Lemma 3.3.5. The facets of $T_{\sigma,j} \setminus \left(i_1^1 i_1 + 1^1 * T_{\sigma \setminus i_1^1 i_1 + 1^1,\, j}\right)$ are those with exactly one of i_1^1 and $i_1 + 1^1$. The specific cases follow from the formulas $T_{\sigma,1} = F_\sigma * \partial(G_\sigma * o_k)$ and $T_{\sigma,2} = \partial F_\sigma * (G_\sigma * o_k)$.

For case 2, by Lemma 3.2.2, $C_k = \bigcup_{\sigma \in B_k \setminus (L_k \cup U_k)} (\{\sigma \setminus i_\ell + 1^\ell : i_\ell = 1\} \cup \{\sigma \setminus i_\ell^\ell : i_\ell = n-1\})$. Therefore, the only facets in $\bigcup_{\tau \in C_k} \tau * o_k$ that are not covered by $\bigcup_{i_1 \in [n-1]} i_1^1 i_1 + 1^1 * B(i_1)$ are those listed above. $\square$

3.3.3 The total order $\prec$

The goal of this section is to define an order $\prec$ for each of

$$\begin{aligned}&\{\sigma \setminus 2^1 \cup o_k : \sigma \in B_k \setminus (L_k \cup U_k) \cap \mathrm{st}_B\, 1^1 2^1\}\\ =&\left\{1^1 \cup \tau \cup o_k : \tau \in \mathrm{lk}_B\, 1^1 2^1, \sum \tau \in [(k-1)(d+2),\ k(d+2)-3]\right\}\end{aligned}$$

and

$$\begin{aligned}&\{\sigma \setminus n-1^1 \cup o_k : \sigma \in B_k \setminus (L_k \cup U_k) \cap \mathrm{st}_B\, n-1^1 n^1\}\\ =&\left\{n^1 \cup \tau \cup o_k : \tau \in \mathrm{lk}_B\, n-1^1 n^1, \sum \tau \in [(k-1)(d+2)+1-n,\ k(d+2)-2-n]\right\}.\end{aligned}$$

Let n_1, n_2 be positive integers. We consider the **lexicographic order** on the elements of $[n_1]^{n_2}$. For $A, B \in [n_1]^{n_2}$, define $A <_{\text{lex}} B$ if the leftmost nonzero coordinate of $B - A$ is positive.

Notation. For a set S and a total order $<$ on S, let $S_<$ denote the ordered set of elements in S with respect to $<$, and let $S_{<^{-1}}$ denote the ordered set of elements in S with respect to $<$ in reverse.

Consider $T := \{\tau = (i_2, \ldots, i_d) : \sum \tau \in [(k-1)(d+2)+1-a,\ k(d+2)-2-a]\}$ for a fixed k and $a \in \{1, n\}$. We define a total order $\prec$ on T, which can then be extended to the two sets above. We begin by visualizing this order through an example.

Example 3.3.7. Consider an example for $d = 4, n = 6, k = 2, a = 1$. The pile of 3-cubes at the top left corner of Figure 3.4 is $B(1)_2$. The gray cubes are from the lower and upper diagonals of $B(1)_2$. The cubes in $T = \{\tau = (i_2, \ldots, i_d) : \sum \tau \in [6, 9]\}$ are transparent with black outlines.

The order $\prec$ on T needs to be somewhat involved for the following reason. We plan to insert this order (having added 1^1 and o_k to each τ) after a shelling of $1^1 2^1 * \widetilde{B(1)}$ and some extra facets. Although we are not shelling the gray cubes but their related triangulations, we can still visualize the rough locations of the triangulations by looking at these original cubes. Supposing that the gray cubes are already in place as part of a bigger complex, we want to fill in the transparent cubes in such a way that each new cube intersects the existing complex nicely. Therefore, although T is in fact shellable, merely inserting a shelling order of T would not suffice.

The top right corner shows T only. First, put $\{\tau : \sum \tau = 9\}$ in lexicographic order. The 19 cubes are in blue and labeled by this order. We use the idea of "layers" again and construct the rest of the order recursively.

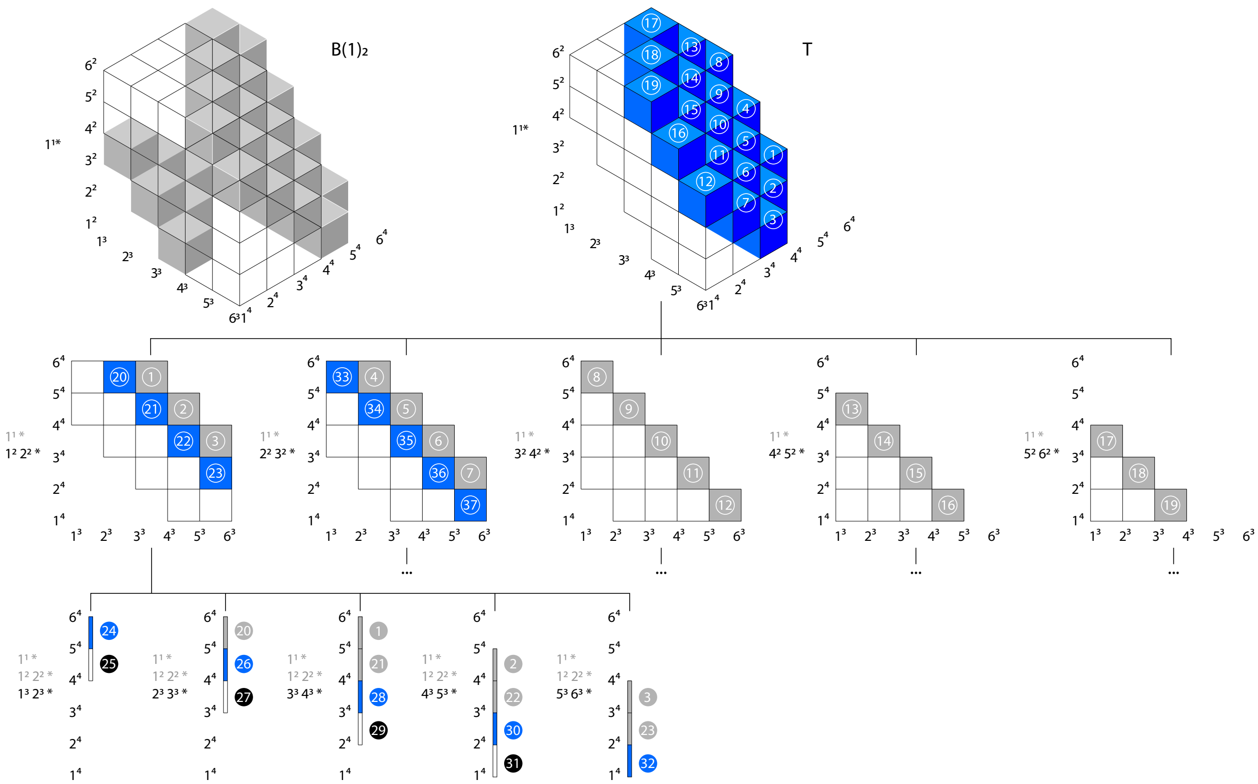


Figure 3.4: Order $\prec$ on $T = \{\tau \in \mathrm{lk}_B\, 1^1 2^1 : \sum \tau \in [6,9]\}$.

On the second row, T is divided into 5 layers, $T \cap \mathrm{st}_{B(1)}\, i_2^2 i_2 + 1^2$ for $i_2 \in [5]$. In each 2-dimensional layer, the cubes already ordered (1 through 19) are now in gray. Starting from $T \cap \mathrm{st}_{B(1)}\, 1^2 2^2$, arrange $\{\tau : \sum \tau = 8\}$ in lexicographic order. The corresponding cubes are now in blue and labeled 20 through 23.

Next, divide $T \cap \mathrm{st}_{B(1)}\, 1^2 2^2$ into $T \cap \mathrm{st}_{B(1)}\, 1^2 2^2 i_3^3 i_3 + 1^3$ for $i_3 \in [5]$, shown on the last row. The cubes 1 through 23 are now gray. Starting from $T \cap \mathrm{st}_{B(1)}\, 1^2 2^2 1^3 2^3$, put $\{\tau : \sum \tau = 7\}$ in lexicographic order. There is only one such cube, which we label as 24. Then we add the last cube in that layer to our order. Repeat the procedure for all other $T \cap \mathrm{st}_{B(1)}\, 1^2 2^2 i_3^3 i_3 + 1^3$. After this, return to work on $T \cap \mathrm{st}_{B(1)}\, 2^2 3^2$: first order the cubes with index sum 8, then divide it into layers $T \cap \mathrm{st}_{B(1)}\, 2^2 3^2 i_3^3 i_3 + 1^3$, and so on.

We can view the picture as a tree rooted at T. The order is given by traversing the tree from top to bottom and left to right.

We now formalize this order and provide a simple algorithm to compare any two cubes in T. We first address the weakened assumption given in Remark 3.3.3. That is, the difference between the index sum of the upper diagonal cubes and that of the lower diagonal cubes in B_k is greater than d. Therefore, we assume $T = \{\tau : \sum \tau \in [t, t']\}$ for some t, t' such that $t' - t \geq d - 1$.

For $\tau \in T$, define $I(\tau) := t + d - \sum \tau$. Then for $\tau \in T$ such that $\sum \tau - t \geq d - 1$, $I(\tau) \leq 1$. For $\tau \in T$ such that $\sum \tau - t < d - 1$, $I(\tau) \in [2, d]$. In particular, the smaller $I(\tau)$ is, the closer τ is to the "upper diagonal" of T.

Let $i_1 \in \{1, n-1\}$. For $m \in [2, d]$ and $i_2, \ldots, i_m \in [n-1]$, define $T_m(i_2, \ldots, i_m)$ to be an ordered set of cubes in $\{\tau \in T : I(\tau) \in [2, d]\} \cap \mathrm{st}_{B(i_1)}\, i_2^2 i_2 + 1^2 \cdots i_m^m i_m + 1^m$ as follows:

- If $m = d$, then $T_m(i_2, \ldots, i_m)$ either contains only $\tau = i_2^2 i_2 + 1^2 \cdots i_m^m i_m + 1^m$ when $I(\tau) = m = d$, or is empty when $I(\tau) \neq d$.
- If $m < d$, then let

$$T_m(i_2, \ldots, i_m) := (\{\tau \in T : I(\tau) = m\} \cap \mathrm{st}_{B(i_1)}\, i_2^2 i_2 + 1^2 \cdots i_m^m i_m + 1^m)_{<_{\mathrm{lex}}}$$
$$\cup\, T_{m+1}(i_2, \ldots, i_m, 1) \cup \cdots \cup T_{m+1}(i_2, \ldots, i_m, n-1).$$

We concatenate the ordered sets together:

$$T_{\prec} := \{\tau \in T : I(\tau) \leq 1\}_{<_{\mathrm{lex}}^{-1}} \cup T_2(1) \cup \cdots \cup T_2(n-1).$$

The lemma below provides an algorithm to compare two cubes in $\{\tau \in T : I(\tau) \in [2, d]\}$.

Lemma 3.3.8. *The order on $T(1,2) \cup \cdots \cup T(n-1, 2)$ defined above is equivalent to the following: let $\tau = (i_2, \ldots, i_d), \tau' = (i'_2, \ldots, i'_d) \in \{\tau \in T : \sum \tau - t < d - 1\}$.*

- *If $I(\tau) = I(\tau')$, then τ precedes τ' if and only if $\tau <_{\mathrm{lex}} \tau'$.*
- *If $I(\tau) < I(\tau')$, then let m be the smallest in $[2, d]$ such that $i_m \neq i'_m$. Let $\ell = \min\{m, I(\tau)\}$. In this case, τ precedes τ' if and only if $i_\ell \leq i'_\ell$.*

Proof. This follows by construction. □

Finally, we extend $\prec$ on T to $\{1^1 \cup \tau \cup o_k : \tau \in T\}$ and $\{n^1 \cup \tau \cup o_k : \tau \in T\}$ respectively and keep the notation $\prec$ for the two sets.

3.3.4 Minimal new faces of the shelling

So far, we have explained the geometric intuition behind the shelling of $\widetilde{B}$ proposed in Table 3.1. In Lemma 3.3.6, we characterized all facets of $\partial\widetilde{B}$ that are not contained in $\bigcup_{i_1\in[n-1]} i_1^1 i_1+1^1 * \widetilde{B(i_1)}$. In Section 3.3.3, we defined an order $\prec$ on $\{\sigma \setminus 2^1 \cup o_k : \sigma \in B_k \setminus (L_k \cup U_k) \cap \mathrm{st}_B\, 1^1 2^1\}$ and $\{\sigma \setminus n-1^1 \cup o_k : \sigma \in B_k \setminus (L_k \cup U_k) \cap \mathrm{st}_B\, n-1^1 n^1\}$ for every k.

In this section, we prove that the order of facets of $\widetilde{B}$ in Table 3.1 is a shelling of $\widetilde{B}$. Recall the hypothesis of Proposition 3.3.4 that $\widetilde{B}_i$ is shellable for every $i_1 \in [n-1]$. Therefore, there is nothing to check for $1^1 2^1 * \widetilde{B(1)}$ at the beginning.

We focus on the rest of the facets $F \in \widetilde{B}$. It suffices to find for every F an $X(F) \subset F$ such that

- $X(F)$ is not contained in any facet preceding F in Table 3.1 ($X(F)$ is a new face),
- for every $x \in X(F)$, $F \setminus x$ is contained in a facet F' preceding F ($X(F)$ is minimal).

Indeed, suppose G is a facet preceding F such that $\dim(F \cap G) < D-1$. Then $F \cap G \neq X(F)$ because $F \cap G$ is not a new face. Take any $x \in X(F) \setminus (F \cap G)$. There is a facet F' preceding F such that the $F \setminus x \subseteq F'$. Thus the ridge $F \cap F' = F \setminus x$ contains $F \cap G$. The face $X(F)$ is called the **minimal new face** of F in the shelling. We find the minimal new faces of the facets of $\widetilde{B} \setminus 1^1 2^1 * \widetilde{B(1)}$ in Lemmas 3.3.9-3.3.11 below.

Lemma 3.3.9. *Let F be a facet of $i_1^1 i_1+1^1 * \widetilde{B(i_1)}$ such that $i_1 \in [2, n-1]$. Then $X(F) = i_1+1^1 \cup Y(F \setminus i_1^1 i_1+1^1)$, where $Y(F \setminus i_1^1 i_1+1^1)$ is the minimal new face of $F \setminus i_1^1 i_1+1^1$ in the shelling of $\widetilde{B(i_1)}$.*

Proof. No facet F' of $i_1^1 i_1+1^1 * \widetilde{B(i_1)}$ can contain $X(F)$, otherwise $Y(F) \subset F' \setminus i_1^1 i_1+1^1$ contradicts $Y(F)$ being new. Any other facets preceding F miss i_1+1^1, so $X(F)$ is a new face.

If $x \in Y(F \setminus i_1^1 i_1+1^1)$, then $F \setminus x \subset F'$ for some facet F' such that $F' \setminus i_1^1 i_1+1^1$ precedes $F \setminus i_1^1 i_1+1^1$ in the shelling of $\widetilde{B(i_1)}$. It remains to show that $F \setminus i_1+1^1 \subset F'$ for some F' preceding F. We gather the case discussions in the table below for clarity. Suppose $\sigma = (i_1, \cdots, i_d)$.

<table>
<tr><td rowspan="3">$\sigma \in L_k$</td><td rowspan="2">$T_{\sigma,1}$</td><td>$F = \sigma$</td><td>Let $\sigma' := (i_1 - 1^1, \dots, i_d^d) \in U_{k-1}$.
Take $F' = \sigma'$ for $T_{\sigma',1}$,
$F' = \sigma' \setminus i_1 - 1^1 \cup o_k$ for $T_{\sigma',2}$.</td></tr>
<tr><td>$F = \sigma \setminus i_\ell^\ell \cup o_k$,
$i_\ell \neq n-1$</td><td>Let $\sigma' := (i_1 - 1^1, \dots, i_\ell + 1^\ell, \dots, i_d^d) \in L_k$.
Take $F' = \sigma' \setminus i_1 - 1^1 \cup o_k$ for $T_{\sigma',1}$,
$F' = \sigma' \setminus i_\ell + 2^\ell \cup o_k$ for $T_{\sigma',2}$.</td></tr>
<tr><td>$T_{\sigma,2}$</td><td colspan="2">Take $F' = \sigma \setminus i_1 + 1^1 \cup o_k$.</td></tr>
<tr><td rowspan="3">$\sigma \in U_k$</td><td>$T_{\sigma,1}$</td><td colspan="2">Take $F' = \sigma \setminus i_1 + 1^1 \cup o_k$.</td></tr>
<tr><td rowspan="2">$T_{\sigma,2}$</td><td>$F = \sigma \setminus i_\ell + 1^\ell \cup o_k$,
$i_\ell = 1$</td><td>Let $\tau := i_1 - 1^1 i_1^1 \cdots i_\ell^\ell \cdots i_d^d i_d + 1^d \in C_k$.
Take $F' = \tau \cup o_k$.</td></tr>
<tr><td>$F = \sigma \setminus i_\ell^\ell \cup o_k$</td><td>If $i_\ell = n-1$, then let
$\tau := i_1 - 1^1 i_1^1 \cdots i_\ell + 1^\ell \cdots i_d^d i_d + 1^d \in C_k$.
Take $F' = \tau \cup o_k$.
If $i_\ell \neq n-1$, then let
$\sigma' := (i_1 - 1^1, \dots, i_\ell + 1^\ell, \dots, i_d^d) \in U_k$.
Take $F' = \sigma' \setminus i_\ell + 2^\ell \cup o_k$ for $T_{\sigma',1}$,
$F' = \sigma' \setminus i_1 - 1^1 \cup o_k$ for $T_{\sigma',2}$.</td></tr>
<tr><td rowspan="2" colspan="2">$\sigma \in B_k \setminus (L_k \cup U_k)$</td><td>$F = \sigma \setminus i_\ell + 1^\ell \cup o_k$,
$i_\ell = 1$</td><td>Let $\sigma' := (i_1 - 1^1, \dots, i_d^d)$.
If $\sigma \in L_k$, then take $F' = \sigma' \setminus i_1 - 1^1 \cup o_k$ for $T_{\sigma',1}$,
$F' = \sigma \setminus i_\ell + 1^\ell \cup o_k$ for $T_{\sigma',2}$.
Otherwise, take $F' = \sigma' \setminus i_\ell + 1^\ell \cup o_k \in C_k * o_k$.</td></tr>
<tr><td>$F = \sigma \setminus i_\ell^\ell \cup o_k$,
$i_\ell = n-1$</td><td>Let $\sigma' := (i_1 - 1^1, \dots, i_d^d)$.
If $\sigma \in L_k$, then take $F' = \sigma' \setminus i_1 - 1^1 \cup o_k$ for $T_{\sigma',1}$,
$F' = \sigma \setminus i_\ell^\ell \cup o_k$ for $T_{\sigma',2}$.
Otherwise, take $F' = \sigma' \setminus i_\ell^\ell \cup o_k \in C_k * o_k$.</td></tr>
</table>

$\square$

Lemma 3.3.10. *Let F be a facet in case 1 of Lemma 3.3.6 such that F comes after $i_1^1 i_1 + 1^1 * \widetilde{B(i_1)}$, so that F is either $\sigma \setminus i_1^1 \cup o_k$ with $i_1 \in [n-1]$ or $\sigma \setminus i_1 + 1^1 \cup o_k$ with $i_1 = 1$ for some $\sigma = (i_1, \dots, i_d) \in L_k \cup U_k \cap \mathrm{st}_B\, i_1^1 i_1 + 1^1$. Then $X(F)$ is the minimal new face of F when we restrict the order in Table 3.1 to $T_{\sigma,j}$.*

Proof. The minimality of $X(F)$ follows from its minimality in the restricted order to $T_{\sigma,j}$.

We now show that $X(F)$ is a new face. Recall that $T_{\sigma,1} = F_\sigma * \partial(G_\sigma * o_k)$ and $T_{\sigma,2} = \partial F_\sigma * (G_\sigma * o_k)$. If $F = \sigma \setminus i_1^1 \cup o_k$, then $T_{\sigma,j} \setminus F$ precedes F. So $X(F) = G_\sigma * o_k \setminus i_1^1$ for $T_{\sigma,1}$ and $X(F) = F_\sigma \setminus i_1^1$ for $T_{\sigma,2}$. If $F = \sigma \setminus i_1 + 1^1 \cup o_k$ with $i_1 = 1$, then all facets of $T_{\sigma,j} \setminus F$ except $\sigma \setminus i_1^1 \cup o_k$ precede F. So $X(F) = G_\sigma * o_k \setminus i_1^1 i_1 + 1^1$ for $T_{\sigma,1}$ and $X(F) = F_\sigma \setminus i_1^1 i_1 + 1^1$ for $T_{\sigma,2}$.

Suppose for contradiction that $X(F)$ is contained in a facet $F' \notin T_{\sigma,j}$ preceding F. Suppose that F' arises from $\sigma' = (i'_1, \ldots, i'_d) \neq \sigma$. Then by Table 3.1, $i'_1 \leq i_1$. Using the explicit expressions of F_σ from Lemma 3.2.3, we finish the rest of the proof in the table below.

$\sigma \in L_k$	$T_{\sigma,1}$	$G_\sigma = \{i_\ell^\ell : i_\ell \neq n-1\}$	$F' \supset X(F) = G_\sigma * o_k \setminus i_1^1$ implies $\sum \sigma' < \sum \sigma$. But then $\sigma' \notin B_k$, contradicting $o_k \notin F'$.
	$T_{\sigma,2}$	$F_\sigma = \{i_\ell^\ell : i_\ell = n-1\} \cup \{i_\ell + 1^\ell : \ell \in [d]\}$	$F' \supset X(F) \in \{F_\sigma \setminus i_1^1, F_\sigma \setminus i_1^1 i_1 + 1^1\}$ implies $\sum \sigma' - \sum \sigma \leq d-1$. So $\sigma' \in B_k \setminus (L_k \cup U_k)$. $\sigma' \setminus i_1^1 \cup o_k$ comes after F, which leaves $F' = \sigma' \setminus i_\ell + 1^\ell \cup o_k$ for some $i_\ell + 1 = n-1$. But $i_\ell + 1^\ell \in X(F)$.
$\sigma \in U_k$	$T_{\sigma,1}$	$G_\sigma = \{i_\ell + 1^\ell : i_\ell \neq 1\}$	Not applicable since $F = \sigma \setminus i_1 + 1 \cup o_k$ comes before $i_1^1 i_1 + 1^1 * \widetilde{B(i_1)}$ for all $i_1 \neq 1$.
	$T_{\sigma,2}$	$F_\sigma = \{i_\ell + 1^\ell : i_\ell = 1\} \cup \{i_\ell^\ell : \ell \in [d]\}$	$F' \supset X(F) \in \{F_\sigma \setminus i_1^1, F_\sigma \setminus i_1^1 i_1 + 1^1\}$, implies $\sum \sigma - \sum \sigma' \leq d-1$. So $\sigma' \in B_k \setminus (L_k \cup U_k)$. $\sigma' \setminus i_1^1 \cup o_k$ comes after F, which leaves $F' = \sigma' \setminus i_\ell^\ell \cup o_k$ for some $i_\ell - 1 = 1$. But $i_\ell^\ell \in X(F)$.

□

Lemma 3.3.11. *Let F be a facet in case 1 of Lemma 3.3.6 such that F comes before $i_1^1 i_1 + 1^1 * \widetilde{B(i_1)}$, so that $F = \sigma \setminus i_1 + 1^1 \cup o_k$ with $i_1 \neq 1$ for some $\sigma = (i_1, \ldots, i_d) \in L_k \cup U_k \cap \mathrm{st}_B\, i_1^1 i_1 + 1^1$. Let $Y(F)$ be the minimal face of F that is not in $T_{\sigma,j} \setminus F$. Then $X(F) = (F \setminus Y(F)) \setminus i_1^1$ if $\sigma \in L_k$ and $X(F) = F \setminus Y(F)$ if*

$\sigma \in U_k$.

Proof. The triangulations where F can arise are $T_{\sigma,2}$ for $\sigma \in L_k$ and $T_{\sigma,1}$ for $\sigma \in U_k$.

We first show that $X(F)$ is a new face. Suppose for contradiction that $X(F)$ is contained in a facet F' preceding F. Suppose that F' arises from $\sigma' = (i'_1, \ldots, i'_d) \neq \sigma$.

If $F \in T_{\sigma,2}$ for $\sigma \in L_k$, then $X(F) = G_\sigma * o_k \setminus i_1^1 = \{i_\ell^\ell : \ell \in [2,d],\ i_\ell \neq n-1\} \cup o_k$. But this means $\sum \sigma' < \sum \sigma$, so $\sigma' \notin B_k$, contradicting $o_k \in F'$.

If $F \in T_{\sigma,1}$ for $\sigma \in U_k$, then $X(F) = F_\sigma = \{i_\ell^\ell : \ell \in [d]\} \cup \{i_\ell + 1^\ell : \ell \in [2,d],\ i_\ell = 1\}$. This means $\sum \sigma - \sum \sigma' \leq d$, so $\sigma' \in B_k \setminus (L_k \cup U_k)$. But then F' must be missing either i_ℓ^ℓ for some $i_\ell = n-1$ or $i_\ell + 1^\ell$ for some $i_\ell = 1$.

Next, for every $x \in X(F)$, we find a facet F' preceding F containing $F \setminus x$. The argument for the case of $F \in T_{\sigma,2}$ for $\sigma \in L_k$ is identical to the case of $T_{\sigma,1}$, $\sigma \in L_k$ in the table of Lemma 3.3.9. The argument for the case of $F \in T_{\sigma,1}$ for $\sigma \in U_k$ is identical to the case of $T_{\sigma,2}$, $\sigma \in U_k$ in the table of Lemma 3.3.9. $\square$

Lemma 3.3.12. *Let $F = a \cup \tau \cup o_k$ be a facet in case 2 of Lemma 3.3.6 that arises from $\sigma = (i_1, \ldots, i_d)$. Recall the definition of $I(\tau)$ from Section 3.3.3. Then*

$$X(F) = (a \cap n^1) \cup \bigcup_{\ell \in [2,d]} \{i_\ell^\ell : i_\ell = n-1 \text{ or } \ell \geq I(\tau)\} \cup \bigcup_{\ell \in [2,d]} \{i_\ell + 1^\ell : i_\ell = 1 \text{ or } \ell \leq I(\tau)\}.$$

Proof. Suppose F' is a facet containing $X(F)$ and F' arises from some $\sigma' = (i'_1, \ldots, i'_d)$. Then $X(F) \subset \sigma'$. Because $\sigma \in B_k \setminus (L_k \cup U_k)$, if $\sigma' = \sigma$, then F' misses at least an i_ℓ^ℓ for some $i_\ell = n-1$ or an $i_\ell + 1^\ell$ for some $i_\ell = 1$. Therefore, $\sigma' \neq \sigma$. Moreover, if $i_1 = 1$ and $i'_1 = n-1$, then F' comes after F. So assume $i_1 = i'_1$ for the discussion.

If $I(\tau) \leq 1$, then $X(F) = (a \cap n^1) \cup \{i_\ell : \ell \geq 2\} \cup \{i_\ell + 1^\ell : i_\ell = 1, \ell \geq 2\}$. For all $\ell \in [2,d]$, $i_\ell - i'_\ell \in \{0,1\}$. So $\sum \sigma - \sum \sigma' \leq d-1$. This means σ' is also in $B_k \setminus (L_k \cup U_k)$, and $F' = a \cup \tau' \cup o_k$. If $I(\tau') \in [2,d]$, then $\tau \prec \tau'$. If $I(\tau') \leq 1$, then again $\tau \prec \tau'$ because $\tau' <_{\text{lex}} \tau$. Therefore, $F \prec F'$.

For the rest of the proof, suppose $I(\tau) \in [2,d]$. For every $\ell \in [I(\tau)+1, d]$, $i'_\ell - i_\ell \in \{-1, 0\}$. And for every $\ell \in [2, I(\tau)-1]$, $i'_\ell - i_\ell \in \{0,1\}$. Thus

$$\sum \sigma' - \sum \sigma \in [I(\tau) - d, I(\tau) - 2].$$

This means $\sigma' \in B_k \setminus (L_k \cup U_k)$, and $F' = a \cup \tau' \cup o_k$ for some $I(\tau') \in [2,d]$. We compare τ and τ' with respect to $\prec$.

First suppose $I(\tau') < I(\tau)$. Note that $I(\tau) - I(\tau')$ is not larger than the number of $\ell \in [2, I(\tau)]$ such that $i'_\ell - i_\ell = 1$. Let $\ell_{\min}$ be the smallest of such ℓ, then $\ell_{\min} \le I(\tau')$. By the second bullet point of Lemma 3.3.8, this implies $\tau \prec \tau'$.

Suppose alternatively that $I(\tau') > I(\tau)$. Let $\ell_{\min} = \min\{\ell \in [2, d] : i'_\ell \neq i_\ell\}$. If $\ell_{\min} < I(\tau)$, then it must be that $i'_{\ell_{\min}} > i_{\ell_{\min}}$. Or $\ell_{\min} > I(\tau)$. In both cases, $\tau \prec \tau'$ follows again from the second bullet point of Lemma 3.3.8.

Finally, suppose $I(\tau') = I(\tau)$. This means there must be some $\ell < I(\tau')$ such that $i'_\ell - i_\ell = 1$, so that $\tau <_{\text{lex}} \tau'$. By the first bullet point of Lemma 3.3.8, $\tau \prec \tau'$.

Therefore, F' always comes after F in the order, proving that $X(F)$ is a new face. To show the minimality of $X(F)$, we find a preceding facet F' containing $F \setminus x$ for each $x \in X(F)$.

If $x = a = n^1$, then take $F' = F \setminus a \cup 1^1$.

If $x = i_\ell^\ell$ for some $\ell \ge 2$, $i_\ell = n - 1$, then take $F' = \sigma \setminus i_\ell^\ell \cup o_k \in i_1^1 i_1 + 1^1 * \widetilde{B(i_1)}$.

If $x = i_\ell^\ell$ for some $\ell \ge 2$, $\ell \ge I(\tau)$, and $i_\ell \ne n-1$, then consider $\sigma' = (i_1^1, \ldots, i_\ell + 1^\ell, \ldots, i_d^d)$. If $\sigma' \in U_k$, then take $F' = \sigma' \setminus i_\ell + 2^\ell \cup o_k$ for $T_{\sigma',1}$ and $F' = \sigma' \setminus i_1^1 i_1 + 1^1 \cup a o_k$ for $T_{\sigma',2}$. If $\sigma' \in B_k \setminus (L_k \cup U_k)$, then take $F' = \sigma' \setminus i_1^1 i_1 + 1^1 \cup a o_k$. Since $\sum \sigma' = 1 + \sum \sigma$, $F' = a \cup \tau' \cup o_k$ for some $I(\tau') < I(\tau)$. Since $I(\tau) \le \ell$, $I(\tau') < \ell$. Moreover, $F <_{\text{lex}} F'$. Hence $F' \prec F$ always holds.

If $x = i_\ell + 1^\ell$ for some $\ell \ge 2$, $i_\ell = 1$, then take $F' = \sigma \setminus i_\ell + 1^\ell \cup o_k \in i_1^1 i_1 + 1^1 * \widetilde{B(i_1)}$.

If $x = i_\ell + 1^\ell$ for some $\ell \ge 2$, $\ell \le I(\tau)$, and $i_\ell \ne 1$, then consider $\sigma' = (i_1^1, \ldots, i_\ell - 1^\ell, \ldots, i_d^d)$. If $\sigma' \in L_k$, then take $F' = \sigma' \setminus i_\ell - 1^\ell \cup o_k$ for $T_{\sigma',1}$ and $F' = \sigma' \setminus i_1^1 i_1 + 1^1 \cup a o_k$ for $T_{\sigma',2}$. If $\sigma' \in B_k \setminus (L_k \cup U_k)$, then take $F' = \sigma' \setminus i_1^1 i_1 + 1^1 \cup a o_k$. Since $\sum \sigma' = -1 + \sum \sigma$, $F' = a \cup \tau' \cup o_k$ for some $I(\tau') > I(\tau) \ge \ell \ge 2$. Thus $F' \prec F$. $\square$

3.4 The shellability of the sphere $\widetilde{B} \cup (\partial \widetilde{B} * o)$

Observe that $\partial \widetilde{B} = \partial B$ except when $d + 2$ divides $d(n-1)$. See Figure 3.5 for an example of $d = 2, n = 7$. In this case, the "uppermost right" cube $(n-1, \ldots, n-1) =: \sigma_{\max}$ is not included in any B_k. As a result, the ridges $\sigma_{\max} \setminus n - 1^m$ for $m \in [d]$ are not facets of $\partial \widetilde{B}$. Instead, $\partial \widetilde{B}$ intersects the uppermost right cube at $\bigcup_{m \in [d]} (\sigma_{\max} \setminus n^m)$.

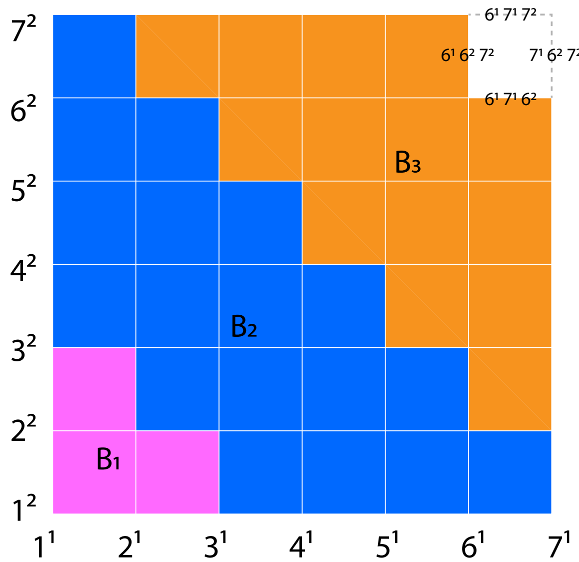


Figure 3.5: A grid representing B for $d = 2$ and $n = 7$. The facets $6^1 7^1 7^2$ and $7^1 6^2 7^2$ of ∂B are not included in $\partial\widetilde{B}$. The latter boundary contains instead $6^1 6^2 7^2$ and $6^1 7^1 6^2$.

Given any face of B, we can identify each of its vertices a^m with $a + (m-1)n$. For example, $7^1 6^2 7^2$ can be expressed as 7 13 14. When we compare two facets of ∂B with respect to lexicographic order below, we mean comparing them using this identification. Moreover, for a facet τ of ∂B, let $\sigma(\tau)$ be the unique facet of B containing τ.

To prove the shellability of $\partial\widetilde{B}$ in both scenarios, we first find a shelling for ∂B that ends with $\{\sigma_{\max} \setminus n-1^m : m \in [d]\}$, then replace it with $\{\sigma_{\max} \setminus n^m : m \in [d]\}_{<_{\text{lex}}}$ and prove that the order remains a shelling.

Define a total order $\prec_\partial$ on the set of facets of ∂B as follows: for any two facets τ, τ' of ∂B, write $\tau' \prec_\partial \tau$ if and only if

- $\sigma(\tau') = \sigma(\tau)$ and $\tau' <_{\text{lex}} \tau$, or
- $\sigma(\tau') <_{\text{lex}} \sigma(\tau)$.

Lemma 3.4.1. *The order $\prec_\partial$ gives a shelling of ∂B.*

Proof. Let $\tau = i_1^1 i_1 + 1^1 \cdots a \cdots i_d^d i_d + 1^d$ where $a = 1^m$ or $a = n^m$. We claim that

$$X := (a \cap n^m) \cup \{i_\ell + 1^\ell : \ell < m, i_\ell \neq 1\} \cup \{i_\ell + 1^\ell : \ell > m\} \cup \{i_\ell^\ell : \ell > m, i_\ell = n-1\}$$

is a minimal new face of τ with respect to $\prec_\partial$. Any facet τ' such that $\sigma(\tau') <_{\text{lex}} \sigma(\tau)$ must miss at least one $i_\ell + 1^\ell$, so $X \notin \tau'$. On the other hand, if $\sigma(\tau') = \sigma(\tau)$ and $\tau' <_{\text{lex}} \tau$, then τ' must be $\sigma(\tau) \setminus i_\ell + 1^\ell$ where $i_\ell = 1$ or $\sigma(\tau) \setminus i_\ell^\ell$ where $i_\ell = n-1$ for some $\ell > m$. Thus τ' cannot contain X either. Lastly, for every $x \in X$, we find a $\tau' \prec_\partial \tau$ containing $\tau \setminus x$ in the table below:

x	τ' containing $\tau \setminus x$	Justification
$i_\ell + 1^\ell, \ell \in [d], i_\ell \neq 1$	$\tau \setminus i_\ell + 1^\ell \cup i_\ell - 1^\ell$	$\sigma(\tau') <_{\text{lex}} \sigma(\tau)$
$i_\ell + 1^\ell, \ell > m, i_\ell = 1$	$\tau \setminus i_\ell + 1^\ell \cup n^m$	$\sigma(\tau') = \sigma(\tau)$ and $\tau' <_{\text{lex}} \tau$
$i_\ell^\ell : \ell > m, i_\ell = n-1$	$\tau \setminus i_\ell^\ell \cup n^m$	$\sigma(\tau') = \sigma(\tau)$ and $\tau' <_{\text{lex}} \tau$

□

Proposition 3.4.2. *The boundary of $\widetilde{B}$ is shellable.*

Proof. If $d+2$ does not divide $d(n-1)$, then $\partial\widetilde{B} = \partial B$, so it is shellable by Lemma 3.4.1. Suppose $(d+2)$ divides $d(n-1)$. Then

$$\partial\widetilde{B} = \partial B \setminus \bigcup_{m\in[d]} (\sigma_{\max} \setminus n-1^m) \cup \bigcup_{m\in[d]} (\sigma_{\max} \setminus n^m)\,.$$

Observe that $\{\sigma_{\max} \setminus n-1^m : m \in [d]\}$ is terminal with respect to $\prec_\partial$. Remove these facets from the order and attach $\{\sigma_{\max} \setminus n^m : m \in [d]\}_{<_{\text{lex}}}$.

To see that this is a shelling of $\partial\widetilde{B}$, take $\tau = \sigma_{\max} \setminus n^m$. Then $X := \{n-1^\ell : \ell \in [d]\} \cup \{n^\ell : \ell > m\}$ is the minimal new face of τ. Indeed, any facet $\tau' \not\subset \sigma_{\max}$ misses at least one $n-1^\ell$. If $\tau' \subset \sigma_{\max}$ and $\{n^\ell : l > m\} \subset \tau'$, then $\tau <_{\text{lex}} \tau'$.

Finally, for any $\ell \in [d]$, $\tau \setminus n-1^\ell \cup n-2^m$ is a facet from $\partial B \setminus \bigcup_{m\in[d]} (\sigma_{\max} \setminus n-1^m)$ containing $\tau \setminus n-1^\ell$. For $\ell > m$, $\tau \setminus n^\ell \cup n^m \subset \sigma_{\max}$ contains $\tau \setminus n^\ell$ and is lexicographically smaller than τ. This proves the minimality of X. □

Therefore, $\partial\widetilde{B} * o$ is also shellable. Simply attaching a shelling of $\partial\widetilde{B} * o$ to a shelling of $\widetilde{B}$ gives a shelling of $\widetilde{B} \cup \partial\widetilde{B} * o$. This proves Theorem 3.1.1.

Combining Theorem 3.1.1 with the lower bound in (3.1), we obtain

$$s_{\text{shell}}(D, N) \geq 2^{\Omega(N^{\lceil D/2 \rceil})} \text{ for all } D \geq 3.$$

On the other hand, we can compute an upper bound for $s_{\text{shell}}(D,N)$. As mentioned towards the end of Section 3.1, this uses the result of Benedetti and Ziegler [BZ11] on the number of LC D-spheres with M facets along with the Upper Bound Theorem [Sta96a] for simplicial spheres:

$$s_{\text{shell}}(D,N) \leq \sum_{M=1}^{O(N^{\lceil D/2 \rceil})} 2^{D^2 M} = \frac{2^{D^2}(2^{D^2 O(N^{\lceil D/2 \rceil})} - 1)}{2^{D^2} - 1} = 2^{O(N^{\lceil D/2 \rceil})}.$$

This immediately leads to Corollary 3.1.2: the number of combinatorially distinct shellable D-spheres with N vertices is asympotically given by

$$s_{\text{shell}}(D,N) = 2^{\Theta(N^{\lceil D/2 \rceil})} \text{ for all } D \geq 3.$$

Chapter 4

Reconstruction of shellable spheres

4.1 Reconstruction problems

How much partial information is needed to reconstruct a triangulated manifold? There have been studies for when this partial information is given by face incidences or lower-dimensional skeleta [Dan84, Hin25]. In discrete geometry, the problem of reconstructing (not necessarily simplicial) polytopes is also of particular interest, see [Bay18]. Such reconstruction may rely not only on combinatorial data but also on geometric information, for instance the space of affine stresses on the polytope [NZ23].

In the 1980s, Blind and Mani [BML87] proved an unpublished conjecture of Perles asserting that the graph of a simple polytope determines the entire combinatorial structure of the polytope. Only a year later, Kalai [Kal88b] gave a much simpler proof of this result by presenting an elegant reconstruction algorithm. Kalai's algorithm requires exponential computation time in the size of the graph. In 2009, Friedman [Fri09] provided a polynomial time algorithm using the polynomial certificates of Joswig, Kaibel, and Körner [JKK02].

Given a triangulated d-dimensional manifold, its **facet-ridge graph** encodes the incidence relationship between its d-dimensional faces (**facets**) and $(d-1)$-dimensional faces (**ridges**). Not all manifolds are reconstructible from their facet-ridge graphs. Counterexamples have been found for the torus and the projective plane [CD25]. If we allow manifolds with boundary, then even the simplest ones–simplicial balls–fail to be reconstructed.

Stated in the dual form, Blind and Mani's result asserts that the combinatorial structure of a simplicial polytope is determined by its facet-ridge graph. This naturally leads to the following question (asked in [BML87] and later formulated as a conjecture by Kalai [Kal88b]):

Problem 4.1.1. Is every simplicial sphere completely determined by its facet-ridge graph?

Recently, Ceballos and Doolittle [CD25] gave a positive answer to this question for the family of spherical subword complexes; these complexes are strongly shellable. The goal of this chapter is to establish the same result for a much larger class of spheres:

Theorem 4.1.2. *The combinatorial structure of a shellable simplicial sphere is determined by its facet-ridge graph.*

This is significant progress from the result for polytopal spheres, given that most shellable spheres are not polytopal.

The structure of the chapter is as follows. In Section 4.2, we provide additional definitions and classical results needed for this chapter. In Sections 4.3 and 4.4, we review good acyclic orientations, k-frames, and k-systems and establish new results about these objects that are crucial for the proof of Theorem 4.1.2. Of particular notice are Proposition 4.3.2, which provides the starting point of our construction, as well as Proposition 4.4.1 and Lemma 4.4.7, which translate the reconstruction problem into the language of k-frames and k-systems. In Section 4.5, we prove a few lemmas that eventually lead to the proof of Theorem 4.1.2. An outline of the main proof can be found at the beginning of Section 4.5.

4.2 Additional background

For the rest of this chapter, we assume that all simplicial complexes are pure. We will also often omit the adjective “simplicial” when referring to complexes, spheres, and balls.

4.2.1 Facet-ridge graphs

Let Δ be a $(d-1)$-dimensional simplicial complex. Its **facet-ridge graph** is the graph whose vertices represent the facets of Δ and where two vertices form an edge if and only if the corresponding facets share a ridge. Recall the definition of a pseudomanifold from Section 2.4. Note that the second condition in that definition is equivalent to the facet-ridge graph being connected. In particular, the facet-ridge graph of a simplicial $(d-1)$-sphere is connected and d-regular.

4.2.2 Partitionability

We say that Δ is **partitionable** if there exists a **partitioning** $\Delta = \bigsqcup_{i=1}^{n}[\sigma_i, T_i]$, where the T_i's are the facets of Δ and the intervals $[\sigma_i, T_i] = \{\sigma \in \Delta : \sigma_i \subseteq \sigma \subseteq T_i\}$ for $1 \leq i \leq n$ are pairwise disjoint. Partitionable complexes are not necessarily shellable. On the other hand, if $T_1, \ldots, T_n$ is a shelling of Δ and $\mathcal{R}$ is the corresponding restriction map, then $\bigsqcup_{i=1}^{n}[\mathcal{R}(T_i), T_i]$ is a partitioning of Δ.

4.2.3 Basic properties of shellable pseudomanifolds

If Δ is a pseudomanifold with more than one facet, then a facet of Δ is **free** if its intersection with $\partial\Delta$ is a simplicial $(d-2)$-ball. The definition of a shelling illustrates how to build a complex by attaching one facet at a time in a "nice" way. When considering a shellable pseudomanifold with boundary, it is sometimes more convenient and intuitive to consider how to take the complex apart by consecutively removing free facets, due to the result below:

Proposition 4.2.1. [Zie98, Proposition 2.4(iii)] *If Δ is a pseudomanifold and T is a free facet of Δ, then the geometric realization of the complex given by the union of all other facets of Δ is homeomorphic to $\|\Delta\|$.*

The following result about shellable pseudomanifolds is well-known and easy to prove. The first part is originally due to Danaraj and Klee [DK74] (and can also be proved using Proposition 4.2.1). A version of the second part appears in [Zie95, Chapter 8].

Proposition 4.2.2. *Let Δ be a shellable simplicial complex.*

1. *If Δ is a pseudomanifold, then Δ is either a ball or a sphere.*
2. *If Δ is a sphere, then the reverse of any shelling of Δ is also a shelling of Δ. The restriction face of the last facet in the shelling is the facet itself.*

Combining both parts leads to the following observation.

Corollary 4.2.3. *If Δ is a sphere with shelling $T_1, \ldots, T_n$, then for every $1 < i \leq n$, the subcomplex $\overline{T}_i \cup \cdots \cup \overline{T}_n$ is a shellable ball.*

Moreover, since the links and stars of shellable simplicial complexes are shellable, and since all links of a simplicial sphere are pseudomanifolds, we conclude:

Corollary 4.2.4. *Every link of a shellable simplicial sphere is a shellable simplicial sphere.*

Example 4.2.5. (Simplicial 1-spheres) A simplicial 1-sphere Δ with n vertices is an n-cycle. Its facets are the edges of the cycle, and an ordering $e_1, \ldots, e_n$ of these edges is a shelling of Δ if and only if for every $1 \leq i \leq n$, the union $e_1 \cup \cdots \cup e_i$ is a connected graph. Moreover, the facet-ridge graph G of Δ is isomorphic to Δ. It follows immediately that Theorem 4.1.2 holds for (shellable) simplicial 1-spheres. Figure 4.1 depicts Δ (in black) and G (in red) when $n = 5$. We use $v_i v_j$ to denote the edge whose endpoints are the vertices v_i and v_j.

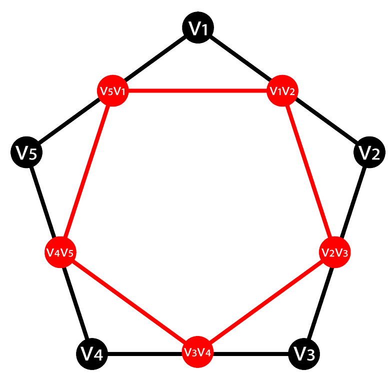


Figure 4.1: A simplicial 1-sphere with 5 vertices and its facet-ridge graph.

4.3 Good acyclic orientations

Throughout this section, let $d \geq 3$ and let G be the facet-ridge graph of a simplicial $(d-1)$-sphere Δ. The goal of this section is to define and utilize good acyclic orientations of G to obtain a shelling and a corresponding partitioning of Δ. This is the first step of our reconstruction of Δ.

For any subset W of the vertex set $V(G)$ of G, let $G[W]$ denote the subgraph of G induced by W. For a $(d-k-1)$-face $\sigma \in \Delta$, let $\mathcal{V}_k^{\Delta}(\sigma)$ denote the set of vertices of G corresponding to the facets of $\mathrm{st}_{\Delta}\,\sigma$. Observe that $G[\mathcal{V}_k^{\Delta}(\sigma)]$ is the facet-ridge graph of $\mathrm{lk}_{\Delta}\,\sigma$ as well as of $\mathrm{st}_{\Delta}\,\sigma$. In particular, if Δ is shellable, then $\operatorname{lk}_\Delta \sigma$ is a simplicial $(k-1)$-sphere by Corollary 4.2.4, and thus $G[\mathcal{V}_k^\Delta(\sigma)]$ is k-regular.

An orientation $\mathcal{O}$ of G is **good** if for every $0 \leq k \leq d$ and $\sigma \in \Delta$ of dimension $d-k-1$, $\mathcal{O}$ induces exactly one sink on $G[\mathcal{V}_k^\Delta(\sigma)]$. In this chapter, we focus on the good orientations that are acyclic. We invoke two simple but important facts about acyclic graphs: first, any acyclic orientation of a graph induces at least one sink on any of its induced subgraphs; second, in an acyclic graph there exists a directed path from any vertex to one of its sinks.

Notation. For the rest of the chapter, we use the uppercase T to represent a facet of Δ, and the lowercase t to represent the corresponding vertex in $V(G)$.

We now relate the existence of good acyclic orientations to shellability of the simplicial complex.

Proposition 4.3.1. *If Δ is shellable, then G has a good acyclic orientation.*

Proof. Suppose $T_1, \ldots, T_n$ is a shelling of Δ. For every pair of T_i, T_j such that $i < j$ and $T_i \cap T_j$ is a ridge, orient the edge $t_i t_j$ from t_j to t_i. The resulting orientation of G must be acyclic with the unique sink t_1. Furthermore, since the shelling of Δ induces a shelling on every star of Δ, this orientation is good. □

The converse of Proposition 4.3.1 also holds. The crucial beginning step of our construction is the following result.

Proposition 4.3.2. *Let $\mathcal{O}$ be a good acyclic orientation of G. Assume $|V(G)| = n$. Let t_1 be the unique sink of G with respect to $\mathcal{O}$. Now define $t_2, \ldots, t_n$ recursively by taking t_i to be a sink of $G[V(G) \setminus \{t_1, \ldots, t_{i-1}\})]$ for each $2 \leq i \leq n$. Then $T_1, \ldots, T_n$ is a shelling of Δ.*

This proposition follows easily from [HM08, Theorem 3]. For completeness, we outline the proof using the terminology of our chapter.

Proof. Let $t_j \xrightarrow{\mathcal{O}} t_i$ denote the directed edge from t_j to t_i with respect to $\mathcal{O}$. For every $1 \leq i \leq n$, define

$$\mathcal{R}^{\mathcal{O}}(T_i) := \bigcap_{j:\, t_j \xrightarrow{\mathcal{O}} t_i} (T_i \cap T_j) = T_i \cap \left(\bigcap_{j:\, t_j \xrightarrow{\mathcal{O}} t_i} T_j \right).$$

Since $\mathcal{O}$ is a good orientation, $\bigsqcup_{i=1}^{n}[\mathcal{R}^{\mathcal{O}}(T_i), T_i]$ is a partitioning of Δ. Associate with this partitioning a (different) digraph $G_{\mathcal{R}^{\mathcal{O}}}$ with the vertex set $V(G)$ as follows: draw an edge directed from t_i to t_j if and only if $\mathcal{R}^{\mathcal{O}}(T_j) \subseteq T_i$. It can be shown that $\mathcal{O}$ being an acyclic orientation of G implies that $G_{\mathcal{R}^{\mathcal{O}}}$ is acyclic. The linear extensions of $G_{\mathcal{R}^{\mathcal{O}}}$ are shellings of Δ with the restriction map $\mathcal{R}^{\mathcal{O}}$. For more details on this part of the proof, the reader is encouraged to check [HM08].

It remains to show that the ordering $t_1, \ldots, t_n$ defined in the statement of the proposition is a linear extension of $G_{\mathcal{R}^{\mathcal{O}}}$. That is, if $i < j$, then $\mathcal{R}^{\mathcal{O}}(T_j) \not\subseteq T_i$. Let $1 \leq j \leq n$ and assume $\dim \mathcal{R}^{\mathcal{O}}(T_j) = d - k - 1$. The face $\mathcal{R}^{\mathcal{O}}(T_j)$ is the intersection of k distinct ridges $T_j \cap T_\ell$ in T_j where $t_\ell \xrightarrow{\mathcal{O}} t_j$ is a directed edge in G, so t_j has indegree k in G. Therefore, t_j is the unique sink of the k-regular graph $G[\mathcal{V}_k^{\Delta}(\mathcal{R}^{\mathcal{O}}(T_j))]$. Therefore, if $i < j$, then $t_i \notin \mathcal{V}_k^{\Delta}(\mathcal{R}^{\mathcal{O}}(T_j))$. $\square$

The following example elucidates the recursive defining steps described in the statement of Proposition 4.3.2.

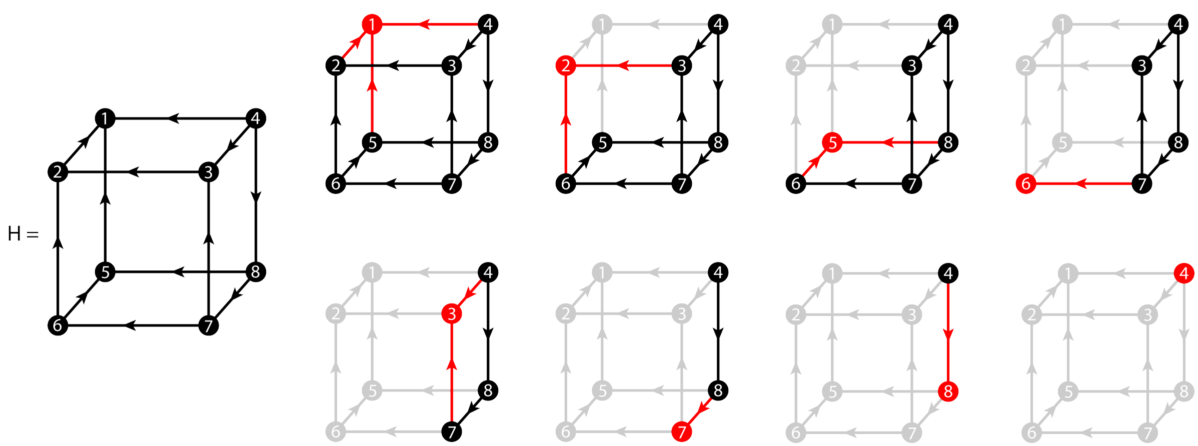


Figure 4.2: The facet-ridge graph of the boundary of the octahedron equipped with a good acyclic orientation, and an ordering of its vertices satisfying the hypothesis of Proposition 4.3.2.

Example 4.3.3. Let Γ be the boundary of the octahedron and let H be its facet-ridge graph, labeled and equipped with a good acyclic orientation as indicated in Figure 4.2. The eight graphs on the right are the induced subgraphs $H[V(H) \setminus \{t_1, \ldots, t_{i-1}\})]$ for $1 \leq i \leq 8$; the chosen sink t_i and the edges that it is incident to at each step are highlighted in red. Note that except for the first step, the choice of sink at each step is not necessarily unique. The particular choices made here give rise to the ordering $1, 2, 5, 6, 3, 7, 8, 4$. By Proposition 4.3.2, this ordering of $V(H)$ corresponds to a shelling of Γ.

Suppose that Δ is shellable. Proposition 4.3.1 then implies that G has a good acyclic orientation. To be able to use Proposition 4.3.2 with only the information of the facet-ridge graph G, it is essential to characterize the good acyclic orientations of G intrinsically. For the case of simple d-polytopes, Kalai [Kal88b] provided an ingenious method to find these good acyclic orientations. Here we extend Kalai's method to the case of shellable simplicial $(d-1)$-spheres. Let $\mathcal{O}$ be an acyclic orientation of G. Let $h_k^{\mathcal{O}}$ be the number of vertices of G with indegree k with respect to $\mathcal{O}$. Define

$$f^{\mathcal{O}} := h_0^{\mathcal{O}} + 2h_1^{\mathcal{O}} + \cdots + 2^k h_k^{\mathcal{O}} + \cdots + 2^d h_d^{\mathcal{O}}.$$

Recall our hypothesis that Δ is a shellable simplicial $(d-1)$-sphere with the facet-ridge graph G. Let $f(\Delta)$ denote the total number of faces of Δ.

Proposition 4.3.4. *Let $M = \min\{f^{\mathcal{O}} : \mathcal{O}$ is an acyclic orientation of $G\}$. Then $M = f(\Delta)$, and $\mathcal{O}$ is good if and only if $f^{\mathcal{O}} = M$.*

Proof. Let t be a vertex of G of indegree k with respect to $\mathcal{O}$. For any i edges directed into t, denote the other endpoints of these edges by $t^1, \ldots, t^i$. Then t is a sink of $G[\mathcal{V}_i^{\Delta}(T \cap T^1 \cap \cdots \cap T^i)]$. This means t is a sink in 2^k subgraphs of G induced by the stars of Δ. On the other hand, since $\mathcal{O}$ is acyclic, for each $(d-i-1)$-face of Δ, the subgraph induced by its star has at least one sink t' with neighbors $t'^1, \ldots, t'^i$, so this subgraph is $G[\mathcal{V}_i^{\Delta}(T' \cap T'^1 \cap \cdots \cap T'^i)]$. It follows that $f^{\mathcal{O}} \geq f(\Delta)$ and $\mathcal{O}$ is good if and only if $f^{\mathcal{O}} = f(\Delta) = M$. $\square$

Consequently, to distinguish the good acyclic orientations from all other orientations of G, it suffices to compute $f^{\mathcal{O}}$ for each acyclic orientation $\mathcal{O}$ of G and pick out those orientations that attain the minimum.

4.4 k-frames and k-systems

Throughout this section, let $d \geq 3$ and let G be the facet-ridge graph of a shellable simplicial $(d-1)$-sphere Δ. Recall that we use T to represent a facet of Δ, and t to represent the corresponding vertex in $V(G)$. The definitions of k-frames and k-systems were introduced in [JKK02].

Let $0 \leq k \leq d$. A k**-frame** of G is a (not necessarily induced) subgraph of G isomorphic to the complete bipartite graph $K_{1,k}$; the vertex of degree k in this subgraph is called the **root** of the k-frame. A 0-frame is just a vertex, and a 1-frame is an edge. In Example 4.3.3, the union of the two (undirected) edges $37, 67$ is a 2-frame of H rooted at 7. We denote by $\langle t, t^1, \ldots, t^k \rangle$ the k-frame with root t and $t^1, \ldots, t^k$ the k neighbors of t.

Every k-frame $\langle t, t^1, \ldots, t^k \rangle$ of G determines a $(d-k-1)$-face $T \cap T^1 \cap \cdots \cap T^k$ of Δ. Moreover, if σ is a $(d-k-1)$-face of Δ, then the k-frames giving rise to σ are exactly all of the k-frames contained in the k-regular graph $G[\mathcal{V}_k^{\Delta}(\sigma)]$. Therefore, for $1 \leq k \leq d$, the $(d-k-1)$-faces of Δ are **not** in one-to-one correspondence with the k-frames of G. To achieve a bijection, fix a good acyclic orientation $\mathcal{O}$ of G. Then $\mathcal{O}$ induces a unique sink on $G[\mathcal{V}_k^{\Delta}(\sigma)]$. We can thus consider the k-frame rooted at that sink. This observation leads to the proposition below.

Proposition 4.4.1. *Let $\mathcal{O}$ be a good acyclic orientation of G and let $0 \leq k \leq d$. Then the set of $(d-k-1)$-faces of Δ is in bijection with the set of k-frames of G whose roots have indegree k in their respective k-frames with respect to $\mathcal{O}$.*

Therefore, knowing the facet-ridge graph G of Δ (and nothing else), we can uniquely express every face $\sigma \in \Delta$ with a certain $(d - \dim \sigma - 1)$-frame of G. This gives rise to the following definition.

Definition 4.4.2. Let $\mathcal{O}$ be a good acyclic orientation of G. Let σ be a $(d-k-1)$-face of Δ and let $\langle t, t^1, \ldots, t^k \rangle$ be the unique k-frame representing σ such that the root has indegree k in the k-frame. Then we say that $\langle t, t^1, \ldots, t^k \rangle$ is the **principal** k**-frame** representing σ (with respect to $\mathcal{O}$).

Observe that the root t of the principal frame representing a face σ corresponds exactly to the unique facet T such that $\sigma \in [\mathcal{R}^{\mathcal{O}}(T), T]$ as defined in Proposition 4.3.2. Therefore, we can readily list all faces of Δ: given a good acyclic orientation $\mathcal{O}$ (which can be found using Proposition 4.3.4), we use Hasse diagrams to represent the Boolean intervals $[\mathcal{R}^{\mathcal{O}}(T_i), T_i]$ and represent the faces in each interval by the principal frames.

Example 4.4.3. (Example 4.3.3 continued) In Example 4.3.3, we found an ordering $1, 2, 5, 6, 3, 7, 8, 4$ of $V(H)$ that corresponds to a shelling of Γ. The resulting partitioning consists of the intervals in

Figure 4.3. For $0 \leq k \leq 3$, every $(2-k)$-face of Γ is represented by its principal k-frame. The roots of these k-frames are highlighted in red.

Consider the 0-face represented by $\langle 1, 2, 4 \rangle$ in the first interval. We know that it is contained in the facets of Γ corresponding to the vertices 1, 2, and 4. However, with only the facet-ridge graph at our disposal, we do not yet know which additional facets contain this face. On the other hand, for the faces in gray shaded boxes, which are the facets, ridges, and the empty face of Γ, we know exactly which facets contain them.

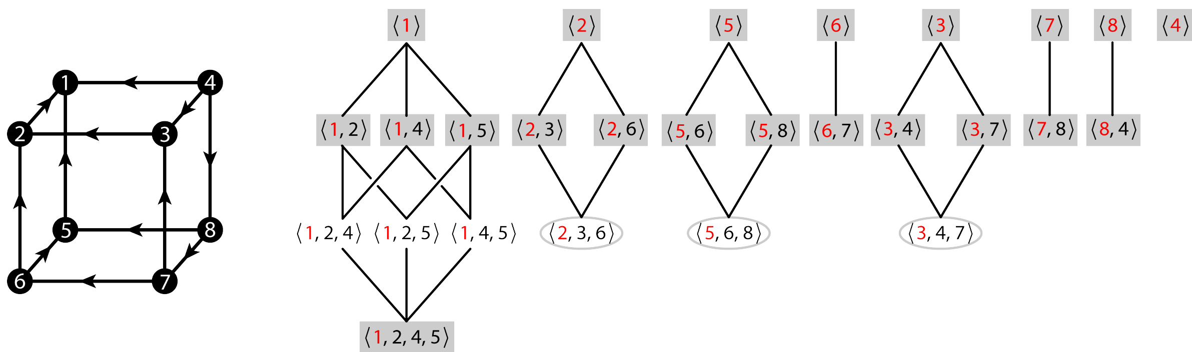


Figure 4.3: A partitioning of the boundary of the octahedron derived from a good acyclic orientation of its facet-ridge graph.

A set $\mathcal{S}$ of subsets of $V(G)$ is a k**-system** of the (undirected) graph G if for every $S \in \mathcal{S}$, the induced subgraph $G[S]$ is k-regular and every vertex set of a k-frame of G is contained in a unique set from $\mathcal{S}$. In Example 4.4.3, $\{\{1,2,3,4\}, \{1,2,5,6\}, \{1,4,5,8\}, \{2,3,6,7\}, \{3,4,7,8\}, \{5,6,7,8\}\}$ is a 2-system of H. Note that the definition of a k-system of G depends only on G and k.

Notation. For a k-system $\mathcal{S}$ and a k-frame $\langle t, t^1, \ldots, t^k \rangle$, let $\mathcal{S}(\langle t, t^1, \ldots, t^k \rangle)$ denote the unique set from $\mathcal{S}$ that contains $\{t, t^1, \ldots, t^k\}$. If additionally a good acyclic orientation $\mathcal{O}$ of G is specified, and $\langle t, t^1, \ldots, t^k \rangle$ is the principal k-frame representing a $(d-k-1)$-face $\sigma \in \Delta$, then we use the symbol $\mathcal{S}(\sigma)$ interchangeably with $\mathcal{S}(\langle t, t^1, \ldots, t^k \rangle)$.

Remark. Reconstructing Δ is equivalent to recognizing the facets of $\mathrm{st}_\Delta\, \sigma$ for all $(d-k-1)$-faces $\sigma \in \Delta$ for all $0 \leq k \leq d$. As mentioned in Example 4.4.3, the facet-ridge graph G immediately tells us the facets of the stars of all $(d-1)$-, $(d-2)$-, and (-1)-faces of Δ. This is also reflected by the fact that G has a unique k-system for $k \in \{0, 1, d\}$. Therefore, for the rest of the chapter, we mainly focus on the k-frames and k-systems for $2 \leq k \leq d-1$.

Let $2 \leq k \leq d-1$. So far, we have set up a correspondence between the $(d-k-1)$-faces of Δ and the k-frames of G. Lemma 4.4.7 below will establish that the set of all stars of the $(d-k-1)$-faces corresponds to a k-system of G. Therefore, to recover these stars, we only need to search among the k-systems. The next two definitions describe the particular k-systems that we wish to examine.

Definition 4.4.4. Let $\mathcal{S}_k$ be a k-system of G. Then $\mathcal{S}_k$ is **star-like** if every good acyclic orientation $\mathcal{O}$ induces exactly one sink on $G[S]$ for every $S \in \mathcal{S}_k$.

Before introducing another definition, we make the following observation.

Lemma 4.4.5. *Let $\mathcal{S}_k$ be a star-like k-system and fix a good acyclic orientation $\mathcal{O}$ of G. Then $\mathcal{S}_k = \{\mathcal{S}_k(\sigma) : \dim \sigma = d-k-1\}$. In particular, two star-like k-systems are identical if and only if they agree on all principal k-frames representing the $(d-k-1)$-faces of Δ.*

Proof. If σ and σ' are distinct $(d-k-1)$-faces of Δ, then $\mathcal{S}_k(\sigma) \neq \mathcal{S}_k(\sigma')$. This is because the root of the principal k-frame representing σ (σ', respectively) is the unique sink of $G[\mathcal{S}_k(\sigma)]$ ($G[\mathcal{S}_k(\sigma')]$, respectively). On the other hand, for every $S \in \mathcal{S}$, the unique sink t of $G[S]$ and its k-neighbors in $G[S]$ form a principal k-frame representing some $(d-k-1)$-face σ. $\square$

Definition 4.4.6. Let $\{\mathcal{S}_k\}_{2 \leq k \leq d-1}$ be a family of k-systems of G. Then $\{\mathcal{S}_k\}_{2 \leq k \leq d-1}$ is **compatible** if for every $(k-1)$-frame Φ_{k-1} that is a subset of a k-frame Φ_k, the inclusion $\mathcal{S}_{k-1}(\Phi_{k-1}) \subseteq \mathcal{S}_k(\Phi_k)$ holds.

For $2 \leq k \leq d-1$, define

$$\mathcal{V}_k^\Delta := \{\mathcal{V}_k^\Delta(\sigma) : \dim \sigma = d-k-1\}.$$

This set corresponds to the stars of all $(d-k-1)$-faces of Δ.

Lemma 4.4.7. *For every $2 \leq k \leq d-1$, $\mathcal{V}_k^\Delta$ is a k-system of G. Moreover, $\mathcal{V}_k^\Delta$ is star-like for every k and the family $\{\mathcal{V}_k^\Delta\}_{2 \leq k \leq d-1}$ is compatible.*

Proof. As previously discussed, the induced subgraphs on the elements of $\mathcal{V}_k^\Delta$ are k-regular. Let $\langle t, t^1, \ldots, t^k \rangle$ be a k-frame representing a face σ. Then $\dim \sigma = d - k - 1$ and $t, t^1, \ldots, t^k \in \mathcal{V}_k^\Delta(\sigma)$. Suppose in addition that $t, t^1, \ldots, t^k \in \mathcal{V}_k^\Delta(\sigma')$ for some $(d-k-1)$-face σ'. Then $\sigma' \subseteq T \cap T_1 \cap \cdots \cap T_k = \sigma$, so $\sigma' = \sigma$. Therefore, $\mathcal{V}_k^\Delta$ is a k-system of G. That $\mathcal{V}_k^\Delta$ is star-like follows from the definition of good acyclic orientations. Finally, $\{\mathcal{V}_k^\Delta\}_{2 \le k \le d-1}$ is compatible because the star of a $(d-k)$-face is contained in the star of any of its $(d-k-1)$-faces. $\square$

If G is equipped with a good acyclic orientation, then for every principal k-frame $\langle t, t^1, \ldots, t^k \rangle$, the vertices of $\mathcal{V}_k^\Delta(\langle t, t^1, \ldots, t^k \rangle)$ correspond to the facets of $\mathrm{st}_\Delta(T \cap T^1 \cap \cdots \cap T^k)$. Therefore, determining the facets of the stars of the $(d-k-1)$-faces of Δ for all $2 \le k \le d-1$ is equivalent to finding $\{\mathcal{V}_k^\Delta\}_{2 \le k \le d-1}$.

4.5 Proof of reconstruction

We assume that $d \ge 3$ throughout this section. Let G be the facet-ridge graph of a shellable simplicial $(d-1)$-sphere Δ. Our goal is to recover the combinatorial structure of Δ.

We summarize the key observations from Sections 4.3 and 4.4 and outline the main steps of the proof:

- Proposition 4.3.4 allows us to identify all good acyclic orientations of G.
- Given a good acyclic orientation of G, Proposition 4.4.1 provides a bijection between $\{(d-k-1)-$faces of $\Delta\}$ and $\{k$-frames of G with a root of indegree k in their respective k-frames with respect to $\mathcal{O}\}$, for every $0 \le k \le d$.
- This bijection turns the problem of reconstructing Δ into determining $\{\mathcal{V}_k^\Delta\}_{2 \le k \le d-1}$.
- By Lemma 4.4.7, $\{\mathcal{V}_k^\Delta\}_{2 \le k \le d-1}$ is a compatible family of star-like k-systems of G.
- This section is devoted to proving that every compatible family $\{\mathcal{S}_k\}_{2 \le k \le d-1}$ of star-like k-systems of G is identical to $\{\mathcal{V}_k^\Delta\}_{2 \le k \le d-1}$.
- We can identify all star-like k-systems by inspecting all k-systems of G. Next, considering all such k-systems for all $2 \le k \le d-1$, we can identify the **unique** compatible family of star-like k-systems, namely $\{\mathcal{V}_k^\Delta\}_{2 \le k \le d-1}$, hence the reconstruction is complete.

We now begin to prove that every compatible family $\{\mathcal{S}_k\}_{2 \le k \le d-1}$ of star-like k-systems of G is identical to $\{\mathcal{V}_k^\Delta\}_{2\le k\le d-1}$. By Lemma 4.4.5, when we have fixed a good acyclic orientation $\mathcal{O}$ of G, this is equivalent to showing that

$$\mathcal{S}_k(\sigma) = \mathcal{V}_k^\Delta(\sigma) \tag{4.1}$$

for every $2 \le k \le d-1$ and every $(d-k-1)$-face $\sigma \in \Delta$. The plan to do so is as follows. Let $T_1, \ldots, T_n$ be a shelling of Δ induced by $\mathcal{O}$ (see Proposition 4.3.2). We first prove that (4.1) holds for every σ that is a restriction face with respect to this shelling; see Lemma 4.5.1. Next, assuming (4.1) holds for all $\sigma \nsubseteq T_1$, we prove that (4.1) also holds for all $\sigma \subseteq T_1$; see Lemma 4.5.2. We then combine both lemmas and complete the proof of Theorem 4.1.2 using induction on d.

Lemma 4.5.1. *Let $\mathcal{O}$ be a good acyclic orientation of G and let $T_1, \ldots, T_n$ be a shelling induced by $\mathcal{O}$ with restriction faces $\rho_i := \mathcal{R}^{\mathcal{O}}(T_i)$ of dimension $\dim \rho_i = d - k_i - 1$ for $1 \le i \le n$. Let $\{\mathcal{S}_k\}_{2\le k\le d-1}$ be a compatible family of star-like k-systems of G. Then $\mathcal{S}_{k_i}(\rho_i) = \mathcal{V}_{k_i}^\Delta(\rho_i)$ for every $1 \le i \le n$.*

We first prove Claim 4.5.1.1 to establish equality between two unions of graphs. To do so, we use the construction of the induced shelling and properties of initial sets of directed graphs. Then by induction, we show that since these unions are equal, the individual graphs have to be pairwise equal for every i. The key step of this induction is proved in Claim 4.5.1.2.

Claim 4.5.1.1. For every $1 \le i \le n$,

$$G[\mathcal{S}_{k_i}(\rho_i)] \cup \cdots \cup G[\mathcal{S}_{k_n}(\rho_n)] = G[\mathcal{V}_{k_i}^\Delta(\rho_i)] \cup \cdots \cup G[\mathcal{V}_{k_n}^\Delta(\rho_n)]. \tag{4.2}$$

Proof of Claim 4.5.1.1. Let $1 \le i \le n$. Let $G_{\mathcal{S},i}$ and $G_{\mathcal{V}^\Delta,i}$ denote the graphs on either side of (4.2) respectively. We first show that

$$G_{\mathcal{V}^\Delta,i} = G[V(G) \setminus \{t_1, \ldots, t_{i-1}\}].$$

By the proof of Proposition 4.3.2, if $t \in \mathcal{V}_{k_j}^\Delta(\rho_j)$ for any $i \le j \le n$, then T appears after T_j in the shelling. Therefore, the two graphs have the same vertex set. Moreover, if $t_\ell \xrightarrow{\mathcal{O}} t_j$ is a directed edge in G with $\ell > j \ge i$, then $t_\ell \in \mathcal{V}_{k_j}^\Delta(\rho_j)$, so the edge is also contained in $G[\mathcal{V}_{k_j}^\Delta(\rho_j)]$.

Let $\Phi_{k_i}(t_i)$ denote the principal k_i-frame representing ρ_i. It follows from the above that

$$G_{\mathcal{V}^\Delta,i} = \Phi_{k_i}(t_i) \cup G_{\mathcal{V}^\Delta,i+1}.$$

We now prove that $G_{\mathcal{S},i} = G_{\mathcal{V}^\Delta,i}$ for every $1 \le i \le n$; we do so by backward induction on i. The base case is to show that $G[\mathcal{S}_{k_n}(\rho_n)] = G[\mathcal{V}^\Delta_{k_n}(\rho_n)]$. The restriction face of the last facet in any shelling of a sphere is the facet itself, so $k_n = 0$ and $G[\mathcal{S}_{k_n}(\rho_n)] = G[\mathcal{V}^\Delta_{k_n}(\rho_n)] = \{t_n\}$. Let $1 \le i \le n-1$ and assume inductively that $G_{\mathcal{S},i+1} = G_{\mathcal{V}^\Delta,i+1}$. Then

$$G_{\mathcal{S},i} = G[\mathcal{S}_{k_i}(\rho_i)] \cup G_{\mathcal{V}^\Delta,i+1} \supseteq \Phi_{k_i}(t_i) \cup G_{\mathcal{V}^\Delta,i+1} = G_{\mathcal{V}^\Delta,i}.$$

On the other hand, observe that $V(G) \setminus \{t_1, \ldots, t_{i-1}\}$ is initial in G. Since $\mathcal{S}_{k_i}$ is star-like, the sink t_i of $G[\mathcal{S}_{k_i}(\rho_i)]$ is unique. Thus there is a directed path from every vertex in $\mathcal{S}_{k_i}(\rho_i)$ to t_i. This implies $G[\mathcal{S}_{k_i}(\rho_i)] \subseteq G[V(G) \setminus \{t_1, \ldots, t_{i-1}\}]$. Thus

$$G_{\mathcal{S},i} = G[\mathcal{S}_{k_i}(\rho_i)] \cup G_{\mathcal{V}^\Delta,i+1} \subseteq G_{\mathcal{V}^\Delta,i}.$$

This completes the induction and (4.2) follows. □

Next, we will show a way to reduce $G_{\mathcal{V}^\Delta,i}$ to its subgraph $G[\mathcal{V}^\Delta_{k_i}(\rho_i)]$ using only the information about $\{G[\mathcal{V}^\Delta_{k_j}(\rho_j)]\}_{i+1 \le j \le n}$. Specifically, we would like to remove a sequence of vertices $t_{j_1}, \ldots, t_{j_m}$ from $G_{\mathcal{V}^\Delta,i}$ such that for every $1 \le \ell \le m$, the vertex t_{j_ℓ} is **free** in $G_{\ell-1} := G[V(G_{\mathcal{V}^\Delta,i}) \setminus \{t_{j_1}, \ldots, t_{j_{\ell-1}}\}]$ (with $G_0 := G_{\mathcal{V}^\Delta,i}$). That is, there exists some $i+1 \le j \le n$ and a principal k-frame $\Phi_k(t_j)$ (where $k \le k_j$) rooted at t_j such that

(A1) $G[\mathcal{V}^\Delta_k(\Phi_k(t_j))] \subseteq G_{\ell-1}$, and

(A2) t_{j_ℓ} and all its neighbors in $G_{\ell-1}$ are contained in $\mathcal{V}^\Delta_k(\Phi_k(t_j))$.

Observe that (A1) and (A2) together imply that $\deg_{G_{j_{\ell-1}}} t_{j_\ell} = k$. See Figure 4.4 for an illustration of a free vertex.

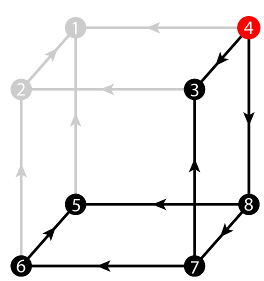


Figure 4.4: Using the notation of Example 4.3.3, The vertex 4 is free in $H[V(H) \setminus \{1,2\}]$ because $4, 3, 8 \in \mathcal{V}^\Gamma_2(\langle 3,4,7 \rangle)$.

Claim 4.5.1.2. Suppose that there is no free vertex in G_m. Then $G_m = G[\mathcal{V}^{\Delta}_{k_i}(\rho_i)]$.

Proof of Claim 4.5.1.2. This is better seen by analyzing the subcomplexes of Δ corresponding to $G_0, \ldots, G_m$. For $1 \le \ell \le m$, let $B_{\ell-1}$ denote the complex with the facet set $\{T_i, \ldots, T_n\} \setminus \{T_{j_1}, \ldots, T_{j_{\ell-1}}\}$ (with $B_0 := \overline{T}_i \cup \cdots \cup \overline{T}_n$). Then t_{j_ℓ} being free in $G_{\ell-1}$ is equivalent to the following: there exists some $i+1 \le j \le n$ and a face $\tilde{\rho}_j \in [\rho_j, T_j]$ such that

(C1) $\mathrm{st}_\Delta\, \tilde{\rho}_j \subseteq B_{\ell-1}$, and

(C2) $\mathrm{st}_\Delta\, \tilde{\rho}_j$ contains T_{j_ℓ} and all facets of $B_{\ell-1}$ that share a ridge with T_{j_ℓ} (for convenience, we refer to these facets as **neighbors** of T_{j_ℓ}).

By Corollary 4.2.3, B_0 is a ball whose boundary coincides with the boundary of $\overline{T}_1 \cup \cdots \cup \overline{T}_{i-1}$. Thus the interior faces of B_0 are exactly $\bigsqcup_{j=i}^{n} [\rho_j, T_j]$, the new faces added to Δ by $T_i, \ldots, T_n$. In particular, $B_0 = \bigcup_{j=i}^{n} \mathrm{st}_\Delta\, \rho_j$.

Suppose $B_{\ell-1}$ is a ball and t_{j_ℓ} is free in $G_{\ell-1}$. Then T_{j_ℓ} must be a free facet of $B_{\ell-1}$, because (C1) and (C2) imply that T_{j_ℓ} intersects $B_{\ell-1}$ at its set of ridges $\{T_{i_j} \setminus \{v\} : v \in \tilde{\rho}_j\}$. By Proposition 4.2.1, B_ℓ is also a ball. We can then consider the chain of $(d-1)$-balls $B_\ell \subset B_{\ell-1} \subset \cdots \subset B_0$. Interior faces of the smaller balls must also be in the interior of the larger balls. Let $\tilde{\rho}_j \in [\rho_j, T_j]$ be an interior face of B_ℓ. Suppose $\mathrm{st}_\Delta\, \tilde{\rho}_j \subseteq B_{\ell-1}$. Then $T_{i_j} \notin \mathrm{st}_\Delta\, \tilde{\rho}_j$ since all remaining faces of $\overline{T}_{i_j}$ in B_ℓ are now on the boundary of B_ℓ. Therefore, $\mathrm{st}_\Delta\, \tilde{\rho}_j \subseteq B_\ell$. We can thus express B_ℓ as a union of the stars (in Δ) of its interior faces.

Now suppose $\mathrm{st}_\Delta\, \rho_i \subseteq B_{\ell-1}$. We will show that either we can find a $T_{j_\ell} \in B_{\ell-1}$, or $B_{\ell-1} = \mathrm{st}_\Delta\, \rho_i$. Let $J = \{i+1, \ldots, n\} \setminus \{j_1, \ldots, j_{\ell-1}\}$. By the above, we can write $B_{\ell-1} = \mathrm{st}_\Delta\, \rho_i \cup \bigcup_{j \in J} \mathrm{st}_\Delta\, \tilde{\rho}_j$, where $\tilde{\rho}_j$ is the smallest face in $[\rho_j, T_j]$ that is an interior face of $B_{\ell-1}$ for every $j \in J$. Let $j \in \{i\} \cup J$ be the largest index such that $\mathrm{st}_\Delta\, \tilde{\rho}_{j'} \subseteq \mathrm{st}_\Delta\, \tilde{\rho}_j$ ($= \mathrm{st}_\Delta\, \rho_i$ if $j = i$) for all $j < j' \in J$.

Case 1. Suppose $j = i$, then we are done.

Case 2. Suppose $j \ne i$ and there exists a $T \in \mathrm{st}_\Delta\, \tilde{\rho}_j$ such that T and all its neighbors in $B_{\ell-1}$ are in $\mathrm{st}_\Delta\, \tilde{\rho}_j$, then take T to be T_{j_ℓ}.

Case 3. Suppose $j \ne i$ and for every $T \in \mathrm{st}_\Delta\, \tilde{\rho}_j$, not all its neighbors are contained in $\mathrm{st}_\Delta\, \tilde{\rho}_j$. Let $T \in \mathrm{st}_\Delta\, \tilde{\rho}_j$ and T' be a neighbor of T such that $T' \notin \mathrm{st}_\Delta\, \tilde{\rho}_j$. Then $T \cap T'$ is an interior ridge of $B_{\ell-1}$, which means $T \cap T' \supseteq \tilde{\rho}_{j'}$ for some $j' \in \{i\} \cup J$. Therefore, $T, T' \in \mathrm{st}_\Delta\, \tilde{\rho}_{j'}$. Since $T' \in \mathrm{st}_\Delta\, \tilde{\rho}_{j'} \setminus \mathrm{st}_\Delta\, \tilde{\rho}_j$, it must be that $j' < j$ by the maximality of j. Therefore, we can rewrite $B_{\ell-1}$ as $\mathrm{st}_\Delta\, \rho_i \cup \bigcup_{j' \in J'} \mathrm{st}_\Delta\, \tilde{\rho}_{j'}$, where $J' = J \setminus \{j, \ldots, n\}$. In this case, repeat the argument of finding the largest index. We will eventually find a T_{j_ℓ} (Case 2) or write $B_{\ell-1} = \mathrm{st}_\Delta \rho_i$ (Case 1). $\square$

Proof of Lemma 4.5.1. To show that $\mathcal{S}_{k_i}(\rho_i) = \mathcal{V}^\Delta_{k_i}(\rho_i)$ for every $1 \leq i \leq n$, we again use backward induction on i. The base case $\mathcal{S}_{k_n}(\rho_n) = \mathcal{V}^\Delta_{k_n}(\rho_n)$ holds by Claim 4.5.1.1. Let $1 \leq i \leq n-1$ and assume inductively that $\mathcal{S}_{k_j}(\rho_j) = \mathcal{V}^\Delta_{k_j}(\rho_j)$ for every $i+1 \leq j \leq n$.

By Claim 4.5.1.2, we can obtain $G[\mathcal{V}^\Delta_{k_i}(\rho_i)]$ from $G_{\mathcal{V}^\Delta,i}$ by deleting free vertices $t_{j_1}, \ldots, t_{j_m}$. By Claim 4.5.1.1, $G[\mathcal{S}_{k_i}(\rho_i)] \subseteq G_{\mathcal{S},i} = G_{\mathcal{V}^\Delta,i} = G_0$. We will show in the next paragraph that $t_{j_\ell} \notin \mathcal{S}_{k_i}(\rho_i)$ for every $1 \leq \ell \leq m$. Equivalently, $G[\mathcal{S}_{k_i}(\rho_i)] \subseteq G_{\ell-1}$ for every $1 \leq \ell \leq m$. From this, we can conclude that

$$G[\mathcal{S}_{k_i}(\rho_i)] \subseteq G_m = G[\mathcal{V}^\Delta_{k_i}(\rho_i)].$$

However, $G[\mathcal{S}_{k_i}(\rho_i)]$ is a k_i-regular graph and $G[\mathcal{V}^\Delta_{k_i}(\rho_i)]$ is a connected k_i-regular graph, so they must be equal.

Suppose inductively that $G[\mathcal{S}_{k_i}(\rho_i)] \subseteq G_{\ell-1}$. Let $k := \deg_{G_{j_{\ell-1}}} t_{j_\ell}$. If $k < k_i$, then $t_{j_\ell} \notin \mathcal{S}_{k_i}(\rho_i)$ because $G[\mathcal{S}_{k_i}(\rho_i)]$ is k_i-regular. If $k \geq k_i$, by (A2) the k-frame rooted at t_{i_j} in $G_{\ell-1}$ is contained in $G[\mathcal{V}^\Delta_k(\Phi_k(t_j))] \subseteq G[\mathcal{V}^\Delta_{k_j}(\rho_j)] = G[\mathcal{S}_{k_j}(\rho_j)]$ for some $i+1 \leq j \leq n$. By compatibility of $\{\mathcal{S}_k\}$, if $t_{i_j} \in \mathcal{S}_{k_i}(\rho_i)$, then $G[\mathcal{S}_{k_i}(\rho_i)] \subseteq G[\mathcal{S}_{k_j}(\rho_j)]$. But that is impossible since $t_i \notin \mathcal{V}^\Delta_{k_j}(\rho_j) = \mathcal{S}_{k_j}(\rho_j)$. Therefore, $t_{i_j} \notin \mathcal{S}_{k_i}(\rho_i)$ and $G[\mathcal{S}_{k_i}(\rho_i)] \subseteq G_\ell$. $\square$

Lemma 4.5.2. *Let $\mathcal{O}$ be a good acyclic orientation of G and let t_1 be the unique sink of G. Let $\{\mathcal{S}_k\}_{2\leq k\leq d-1}$ be a compatible family of star-like k-systems such that*

$$\mathcal{S}_k(\tau) = \mathcal{V}^\Delta_k(\tau) \text{ for every } 2 \leq k \leq d-1 \text{ and every } (d-k-1)\text{-face } \tau \nsubseteq T_1.$$

Then $\mathcal{S}_k(\sigma) = \mathcal{V}^\Delta_k(\sigma)$ for every $2 \leq k \leq d-1$ and every $(d-k-1)$-face $\sigma \subseteq T_1$.

Recall our assumption that $d \geq 3$. Let $2 \leq k \leq d-1$ and let $\sigma_1, \ldots, \sigma_{\binom{d}{k}}$ be the $(d-k-1)$-faces contained in T_1. The plan is to prove the lemma by induction on k. We divide the proof into three parts, starting with Claims 4.5.2.1 and 4.5.2.2.

Claim 4.5.2.1. The following equality between two graphs holds:

$$G[\mathcal{S}_k(\sigma_1)] \cup \cdots \cup G[\mathcal{S}_k(\sigma_{\binom{d}{k}})] = G[\mathcal{V}^\Delta_k(\sigma_1)] \cup \cdots \cup G[\mathcal{V}^\Delta_k(\sigma_{\binom{d}{k}})]. \tag{4.3}$$

Proof of Claim 4.5.2.1. This follows from Lemma 4.4.5. Let Φ_k be a k-frame contained in $G[\mathcal{S}_k(\sigma_i)]$ for some $1 \leq i \leq \binom{d}{k}$. Then the vertex set of Φ_k is not contained in any $\mathcal{S}_k(\tau)$ such that $\tau \nsubseteq T_1$, and therefore not contained in $\mathcal{V}_k^\Delta(\tau)$ by hypothesis. Therefore, $\Phi_k \subseteq G[\mathcal{V}_k^\Delta(\sigma_j)]$ for some $1 \leq j \leq \binom{d}{k}$. The argument works in reverse by swapping "$\mathcal{S}$" with "$\mathcal{V}^\Delta$." This proves the equality in (4.3). □

Let U denote the union $\mathcal{S}_k(\sigma_1) \cup \cdots \cup \mathcal{S}_k(\sigma_{\binom{d}{k}}) = \mathcal{V}_k^\Delta(\sigma_1) \cup \cdots \cup \mathcal{V}_k^\Delta(\sigma_{\binom{d}{k}})$ and let G_U denote the graph in (4.3). The next claim will serve as the base case for induction.

Claim 4.5.2.2. Suppose $k = 2$. Then $\mathcal{S}_2(\sigma_i) = \mathcal{V}_2^\Delta(\sigma_i)$ for $1 \leq i \leq \binom{d}{2}$.

Proof of Claim 4.5.2.2. In this case, $\{G[\mathcal{S}_2(\sigma_i)]\}_{1 \leq i \leq \binom{d}{2}}$ and $\{G[\mathcal{V}_2^\Delta(\sigma_i)]\}_{1 \leq i \leq \binom{d}{2}}$ are sets of induced cycles of G, and G_U is the union of each set of these cycles.

We show that the cycles in $\{G[\mathcal{V}_2^\Delta(\sigma_i)]\}_{1 \leq i \leq \binom{d}{2}}$ only intersect at t_1 and its neighbors. Suppose $t \in G[\mathcal{V}_2^\Delta(\sigma_i)] \cap G[\mathcal{V}_2^\Delta(\sigma_j)]$ for some $1 \leq i \neq j \leq \binom{d}{2}$ (note that $\binom{d}{2} \geq 3$ because $d \geq 3$). Then T is in the star of two $(d-3)$-faces contained in T_1. This implies $\dim(T \cap T_1) \geq d - 2$, so t is either t_1 or a neighbor of t_1. Therefore, a vertex $t \neq t_1$ has degree 2 in G_U if and only if t is not a neighbor of t_1.

To prove that $\mathcal{S}_2(\sigma_i) = \mathcal{V}_2^\Delta(\sigma_i)$ for $1 \leq i \leq \binom{d}{2}$, it suffices to show that if t has degree 2 in G_U, then t is in the cycles $G[\mathcal{S}_2(\sigma_i)]$ and $G[\mathcal{V}_2^\Delta(\sigma_i)]$ for the same i. Let t be a vertex of degree 2 in G_U and let t^1 and t^2 denote the two neighbors of t in G_U. Then there is a unique (undirected) path $p_1 = t\, t^1\, v_1 \, \cdots \, v_m \, t_1^j$ in G_U and a unique path $p_2 = t\, t^2\, w_1 \, \cdots \, w_{m'} \, t_1^\ell$ in G_U such that all inner vertices of p_1 and p_2 have degree 2 in G_U while t_1^j and t_1^ℓ have degrees larger than 2. See Figure 4.5 for an illustration, where G_U is in black. Then the respective cycles in $\{G[\mathcal{S}_2(\sigma_i)]\}_{1 \leq i \leq \binom{d}{2}}$ and $\{G[\mathcal{V}_2^\Delta(\sigma_i)]\}_{1 \leq i \leq \binom{d}{2}}$ containing t must also contain p_1 and p_2. In other words, $t \in G[\mathcal{S}_2(\langle t_1, t_1^j, t_1^\ell \rangle)] \cap G[\mathcal{V}_2^\Delta(\langle t_1, t_1^j, t_1^\ell \rangle)]$. □

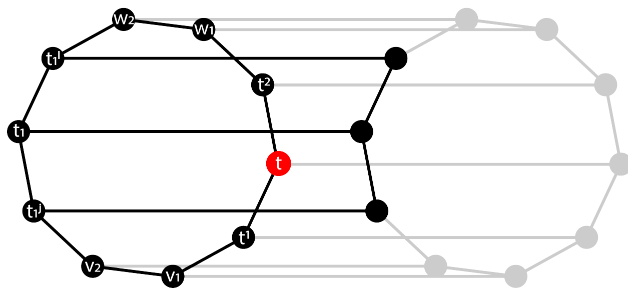


Figure 4.5: A degree 2 vertex $t \in G_U$ and the paths p_1 and p_2.

Proof of Lemma 4.5.2. Assume inductively that $\mathcal{S}_{k-1}(\sigma) = \mathcal{V}^{\Delta}_{k-1}(\sigma)$ for every $(d-k)$-face $\sigma \subseteq T_1$. This, combined with the hypothesis in the statement of the lemma and Proposition 4.4.1, implies that $\mathcal{S}_{k-1} = \mathcal{V}^{\Delta}_{k-1}$.

Let $1 \leq i \leq \binom{d}{k}$ and let $\langle t, t^1, \ldots, t^k \rangle$ be a k-frame of $G[\mathcal{S}_k(\sigma_i)]$. Consider a neighbor t^j of t. Let $\Phi_k(t^j)$ denote the k-frame rooted at t^j in $G[\mathcal{S}_k(\sigma_i)]$. Furthermore, for $1 \leq \ell \leq k$ such that $\ell \neq j$, let $\Phi^{\ell}_{k-1}(t^j)$ denote the $(k-1)$-frame rooted at t^j in $G[\mathcal{S}_{k-1}(\langle t, t^1, \ldots, t^{\ell-1}, t^{\ell+1}, \ldots, t^k \rangle)]$. The $(k-1)$-frames $\Phi^{\ell}_{k-1}(t^j)$ must all be distinct, so their union $\bigcup_{1 \leq \ell \leq k, \ell \neq j} \Phi^{\ell}_{k-1}(t^j)$ is a k-frame. Since $\{\mathcal{S}_k\}$ is compatible, this union is contained in $G[\mathcal{S}_k(\sigma_i)]$. Therefore,

$$\Phi_k(t^j) = \bigcup_{1 \leq \ell \leq k,\ \ell \neq j} \Phi^{\ell}_{k-1}(t^j).$$

In other words, when $\langle t, t^1, \ldots, t^k \rangle$ is known to be contained in $G[\mathcal{S}_k(\sigma_i)]$, for every $1 \leq j \leq k$, the neighbors of t^j in $G[\mathcal{S}_k(\sigma_i)]$ are completely determined by $\mathcal{S}_{k-1}$. Therefore, starting from the principal k-frame representing σ_i, we can determine the neighbors of all its vertices in $G[\mathcal{S}_k(\sigma_i)]$. Repeating this process allows us to determine the entire $G[\mathcal{S}_k(\sigma_i)]$ using $\mathcal{S}_{k-1}$.

By applying the same argument to $\mathcal{V}^{\Delta}_k$, we can conclude that $G[\mathcal{V}^{\Delta}_k(\sigma_i)]$ is determined by $\mathcal{V}^{\Delta}_{k-1}$. Since $\mathcal{S}_{k-1} = \mathcal{V}^{\Delta}_{k-1}$, it follows that $G[\mathcal{S}_k(\sigma_i)] = G[\mathcal{V}^{\Delta}_k(\sigma_i)]$, completing the induction. □

The following example provides a preview of how we combine Lemmas 4.5.1 and 4.5.2 to prove the uniqueness of the compatible families of star-like k-systems. It is also the base case for our proof by induction of Theorem 4.1.2.

Example 4.5.3. (Simplicial 2-spheres) Suppose $d = 3$, so that Δ is a shellable simplicial 2-sphere. We will reconstruct Δ from G with the tools we developed throughout the chapter. First, apply

Proposition 4.3.4 to find a good acyclic orientation $\mathcal{O}$ of G. Then Proposition 4.3.2 recovers a shelling of Δ induced by $\mathcal{O}$ and the corresponding partitioning. We would like to show that every compatible family $\{\mathcal{S}_k\}_{2\leq k\leq d-1}$ of k-systems is identical to $\{\mathcal{V}_k^{\Delta}\}_{2\leq k\leq d-1}$. Since $d=3$, this is reduced to showing that $\mathcal{S}_2(\sigma)=\mathcal{V}_2^{\Delta}(\sigma)$ for all 0-faces $\sigma\in\Delta$.

Let σ be a 0-face of Δ. Let T_1 denote the first facet in the shelling. Since Δ is 2-dimensional, the intervals other than $[\mathcal{R}^{\mathcal{O}}(T_1), T_1]$ are of height at most 2. Therefore, either σ is contained in T_1 or σ is a restriction face of one of the later facets (in the case of Example 4.4.3, such faces are circled in Figure 4.3). Hence $\mathcal{S}_2(\sigma)=\mathcal{V}_2^{\Delta}(\sigma)$ by either Lemma 4.5.2 or Lemma 4.5.1.

The proof of the main theorem requires only one extra step than that for the case of $d=3$ in Example 4.5.3. Recall that Theorem 4.1.2 asserts that any shellable $(d-1)$-sphere Δ can be reconstructed from its facet-ridge graph. We assume that $d\geq 3$ as the case of $d=2$ has been addressed by Example 4.2.5.

Proof of Theorem 4.1.2. To prove the statement, we show by induction on d that if Δ is a shellable simplicial $(d-1)$-sphere and $\{\mathcal{S}_k\}_{2\leq k\leq d-1}$ is a compatible family of star-like k-systems of the facet-ridge graph of Δ, then $\{\mathcal{S}_k\}_{2\leq k\leq d-1}$ coincides with $\{\mathcal{V}_k^{\Delta}\}_{2\leq k\leq d-1}$. The base case of $d=3$ has been proved in Example 4.5.3. Suppose now that the assertion holds for all shellable simplicial spheres of dimension less than $d-1$.

We carry the usual notations and assumptions from Example 4.5.3. If τ is a $(d-k-1)$-face that is neither contained in the first facet in the shelling T_1 nor a restriction face of any of the other facets, then there is a facet $T\neq T_1$ such that $\mathcal{R}^{\mathcal{O}}(T)$ is a proper subset of τ. By Lemma 4.5.1, $\mathcal{S}_{d-\dim\mathcal{R}^{\mathcal{O}}(T)-1}(\mathcal{R}^{\mathcal{O}}(T))=\mathcal{V}^{\Delta}_{d-\dim\mathcal{R}^{\mathcal{O}}(T)-1}(\mathcal{R}^{\mathcal{O}}(T))$. Since $\operatorname{lk}_{\Delta}\mathcal{R}^{\mathcal{O}}(T)$ is a shellable simplicial sphere of dimension less than $d-1$, it follows from the inductive hypothesis that $\mathcal{S}_k(\tau)=\mathcal{V}_k^{\Delta}(\tau)$. Therefore, $\{\mathcal{S}_k\}_{2\leq k\leq d-1}$ satisfies the hypothesis of Lemma 4.5.2. Finally, if σ is a $(d-k-1)$-face in $[\mathcal{R}^{\mathcal{O}}(T_1), T_1]$, then $\mathcal{S}_k(\sigma)=\mathcal{V}_k^{\Delta}(\sigma)$ by Lemma 4.5.2.

We have thus determined for all faces σ of Δ the facets of $\operatorname{st}_{\Delta}\sigma$, and therefore the entire combinatorial structure of Δ. $\square$

Chapter 5

Flag spheres arising from planar ternary graphs

5.1 Independence complexes of ternary graphs

In this chapter, we explore the algebraic, combinatorial, geometric, and topological properties of the independence complexes of **ternary** graphs, graphs that do not contain induced cycles of length divisible by three. Independence complexes of graphs are *flag complexes*, complexes in which the minimal nonfaces are sets of two vertices. Topological properties of simplicial complexes are translated into algebraic properties of the associated Stanley–Reisner ring. For instance, it is known that the Stanley–Reisner ring of a homology sphere is **Gorenstein**; see [Sta96a].

It turns out that the Gorenstein property for independence complexes is strongly related to properties of the graphs. A property of independent sets in graphs, introduced by Staples [Sta79b] and further studied by Pinter [Pin95], turns out to be important; the graphs with this property are called 1-well-covered, and the set of such graphs is denoted W_2. Hoang and Trung [HT16] proved that for triangle-free graphs without isolated vertices, a graph G is in W_2 if and only if the independence complex of G is Gorenstein. Trung [Tru18] determined the complete list of connected planar graphs in W_2 (and hence with Gorenstein independence complexes); the list contains exactly one graph for each independence number. These graphs (with one exception) are in fact ternary.

Kim [Kim22] proved that a graph is ternary if and only if the independence complex of every induced subgraph is contractible or homotopy equivalent to a sphere. (This resolved a conjecture by Engström [Eng20].) This led Faridi and Holleben [FH23] to ask the question of the connection between homotopy type and homology of these independence complexes. For our purposes we phrase the question as follows.

Question 5.1.1. Let G be a ternary graph without isolated vertices, $\operatorname{Ind}(G)$ its independence complex, and $d-1$ the dimension of $\operatorname{Ind}(G)$. Are the following two statements equivalent?

1. $\operatorname{Ind}(G)$ is homotopy equivalent to a $(d-1)$-dimensional sphere.
2. $\operatorname{Ind}(G)$ is a homology sphere.

In this chapter, we give an affirmative answer to this question.

Theorem 5.1.2. *Let G be a ternary graph without isolated vertices. Then $\operatorname{Ind}(G)$ is a homology sphere if and only if $\operatorname{Ind}(G)$ is homotopy equivalent to S^d, where $d = \dim \operatorname{Ind}(G)$.*

Moreover, using the characterization of planar homology spheres from [Tru18], we are able to provide further details on the independence complex of planar graphs. In general, for $d \geq 4$, deciding if a simplicial $(d-1)$-sphere can be realized as the boundary of a d-polytope is a very hard problem.

Our result below shows that being planar and ternary imposes not only strong topological restrictions on the independence complexes but also strong geometric ones.

Theorem 5.1.3. *Let G be a planar ternary graph such that $\operatorname{Ind}(G)$ is a homology sphere. Then $\operatorname{Ind}(G)$ is combinatorially equivalent to the boundary of a simplicial polytope and is vertex decomposable.*

Remark 5.1.4. Theorems 5.1.2 and 5.1.3 together imply that if G is a planar ternary graph whose independence complex is $(d-1)$-dimensional, then $\operatorname{Ind}(G)$ is homotopy equivalent to a $(d-1)$-dimensional sphere if and only if $\operatorname{Ind}(G)$ is the boundary of a vertex decomposable simplicial polytope. Moreover, the only nonternary planar graph G for which $\operatorname{Ind}(G)$ is a homology sphere is the graph of the triangular prism, for which $\operatorname{Ind}(G)$ is the hexagon.

By a result of Lutz and Nevo [LN16], for a fixed d, the following graph T_d is connected: let the vertices of T_d be the set of all d-dimensional flag PL spheres, and connect two vertices by an edge if and only if the corresponding spheres differ by an edge subdivision. Studying T_d is one approach to the difficult task of recognizing flag PL spheres. Moreover, the graph may help compute combinatorial data, such as the f- and h-vectors, and provide insights into the extremal properties of entire classes of flag spheres.

In this chapter, we give a description of the induced subgraph of the Lutz–Nevo graph T_d on the set of $\mathrm{Ind}(G)$ (as vertices) where G is planar and ternary:

Theorem 5.1.5 (Partition refinement and the Lutz–Nevo graph). *Let $\mathcal{P}_{d+1}$ denote the Hasse diagram of the partition refinement poset of partitions of $d+1$. Then $\mathcal{P}_d$ is an induced subgraph of T_d for every d.*

We then focus on combinatorial properties of the flag spheres in Theorem 5.1.3. We begin by giving a combinatorial interpretation for their h-vectors, and as a consequence, using results from [WZC19, Mun19] we prove the following.

Theorem 5.1.6 (h-vectors and Delannoy polynomials). *Let G be a planar graph such that $\mathrm{Ind}(G)$ is a homology sphere. Then the h-polynomial $h(\mathrm{Ind}(G), t)$ of $\mathrm{Ind}(G)$ is real-rooted. If additionally G is ternary and connected, then $h(\mathrm{Ind}(G), t)$ is a Delannoy polynomial.*

The chapter is structured as follows. In Section 5.2 we recall important definitions and related results. In Section 5.3, we investigate ternary graphs and their independence complexes, proving Theorem 5.1.2. We introduce and delve into the properties of the graph family $\{G_m\}$, which gives all triangle-free connected planar Gorenstein graphs without isolated vertices. The section ends with a proof of Theorem 5.1.3. In Section 5.4, we further examine the effect of edge subdivisions and contractions on the flag spheres arising from G_m and prove Theorem 5.1.5. Section 5.5 is dedicated to proving Theorem 5.1.6. Along the way, we give an explicit formula for the h-polynomial of $\mathrm{Ind}(G_m)$ and find a connection between the h-vector and the Delannoy numbers. We conclude this chapter by posing some open questions and giving future research directions that might be of interest in Section 5.6.

5.2 Additional background

5.2.1 Graphs and independence complexes

1-dimensional simplicial complexes are exactly **simple graphs** (or just "graphs" from now on). Given a graph G, an **independent set** of G is a collection of vertices S of G such that there is no edge in G between elements in S. The **independence number**, denoted $\alpha(G)$, is the size of the largest independent set of G. The collection of independent sets of G forms a simplicial complex called the **independence complex** of G, which we denote by $\mathrm{Ind}(G)$. Simplicial complexes that arise as independence complexes of graphs are called **flag complexes** and can be characterized as complexes Δ where the minimal sets (with respect to inclusion) not in Δ have size two, in other words, every minimal nonface is a missing edge. Figure 5.1 illustrates two graphs and their independence complexes.

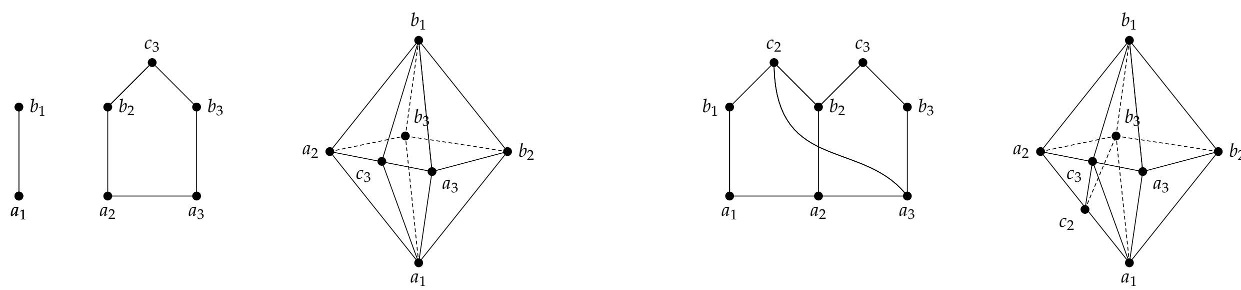


Figure 5.1: Two graphs whose independence complexes are spheres

The **induced subcomplex** of Δ on vertex set A is the complex $\Delta[A]$ on vertex set A with faces $\{\sigma \in \Delta : \sigma \subseteq A\}$. In the case of a graph G with vertex set V, we denote the induced subgraph of G on the set $V \setminus A$ as $G - A$.

Remark 5.2.1 (Well-covered graphs and pure flag complexes). A graph G is **well-covered** if all its maximal independent sets have the same cardinality. Since the maximal independent sets of G correspond to the facets of $\mathrm{Ind}(G)$, this is equivalent to saying $\mathrm{Ind}(G)$ is pure of dimension $\alpha(G) - 1$. A well-covered graph is called **1-well-covered** if the deletion of any vertex leaves a graph that is also well-covered. In 1979, Staples [Sta79b] introduced the class of W_n graphs, which consists of graphs where every n disjoint independent sets are contained in n disjoint maximum independent sets. Levit and Mandrescu [LM19, Theorem 2.2] then showed that 1-well-covered graphs without isolated vertices are exactly the graphs in the class W_2.

5.2.2 Subcomplexes of independence complexes

If Δ is a simplicial complex and v is a vertex of Δ, we denote by $\mathrm{del}_\Delta(v) = \{\tau \in \Delta \colon v \notin \tau\}$ the **deletion** of v in Δ. Note that $\mathrm{lk}_\Delta(v) = \mathrm{del}_\Delta(v) \cap \mathrm{st}_\Delta(v)$ and $\Delta = \mathrm{del}_\Delta(v) \cup \mathrm{st}_\Delta(v)$. A pure simplicial complex Δ is **vertex decomposable** if it is a simplex (including $\varnothing$) or if there is a vertex $v \in \Delta$ such that $\mathrm{lk}_\Delta(v)$ and $\mathrm{del}_\Delta(v)$ are both vertex decomposable simplicial complexes.

For a vertex v of a graph G with vertex set V and edge set E, we denote by $N(v) = \{u \in V \colon \{u, v\} \in E\}$ the **neighborhood of** v and by $N[V] = v \cup N(v)$ the **closed neighborhood of** v. We will often use the following observations:

$$\mathrm{lk}_{\mathrm{Ind}(G)}(v) = \mathrm{Ind}(G - N[v]), \quad \mathrm{del}_{\mathrm{Ind}(G)}(v) = \mathrm{Ind}(G - v) \quad \text{and} \quad \mathrm{st}_{\mathrm{Ind}(G)}(v) = \mathrm{Ind}(G - N(v)).$$

Moreover, if G and H are disjoint graphs, then one can check by the definition of independent set that

$$\mathrm{Ind}(G \sqcup H) = \mathrm{Ind}(G) * \mathrm{Ind}(H).$$

5.2.3 Cohen–Macaulay complexes, Gorenstein graphs, and homology spheres

A simplicial complex Δ is **Cohen–Macaulay** over a field $\mathbb{K}$ if for every $\sigma \in \Delta$

$$\tilde{H}_i(\mathrm{lk}_\Delta(\sigma); \mathbb{K}) = 0 \quad \text{for} \quad i < \dim \mathrm{lk}_\Delta(\sigma),$$

where $\tilde{H}_i(\Gamma; \mathbb{K})$ denotes the reduced i-th simplicial homology group of Γ over $\mathbb{K}$. If moreover Δ satisfies $\tilde{H}_{\dim \mathrm{lk}_\Delta(\sigma)}(\mathrm{lk}_\Delta(\sigma); \mathbb{K}) \cong \mathbb{K}$ for every $\sigma \in \Delta$, then we say Δ is a **$\mathbb{K}$-homology sphere**. Simplicial spheres are necessarily homology spheres. The Poincaré homology sphere is a homology sphere but not a simplicial sphere.

If $\Delta = \mathrm{st}_\Delta(\sigma)$ for some $\sigma \in \Delta$ and $\mathrm{lk}_\Delta(\sigma)$ is a homology sphere, then we say Δ is **Gorenstein**. A graph G for which $\mathrm{Ind}(G)$ is Gorenstein is called a **Gorenstein graph** (for more details see [Sta96a, II Theorem 5.1]).

It is well known that a Cohen–Macaulay complex Δ is pure and strongly connected. Therefore, a Cohen–Macaulay complex Δ can only fail to be a pseudomanifold if Δ has a ridge that is contained in more than two facets.

Remark 5.2.2. It turns out that the class of W_2 graphs (see Remark 5.2.1) plays a key role in the study of flag homology spheres. It was shown in [HMT15, Proposition 3.7] that a triangle-free graph G without isolated vertices has a Gorenstein independence complex if and only if G is a W_2 graph.

5.2.4 PL spheres and stellar moves.

A simplicial $(d-1)$-sphere is **piecewise linear (PL)** if it is PL homeomorphic to the boundary of a d-simplex. Let Δ be a simplicial complex and σ be a face of Δ. Then the **stellar subdivision** of Δ at σ is given by

1. first, introducing a new vertex a in the interior of σ, and
2. next, replacing $\mathrm{st}_\Delta(\sigma)$ in Δ with $a * \partial\sigma * \mathrm{lk}_\Delta(\sigma)$.

The inverse operation is called a **stellar weld**. When σ is an edge of Δ, these operations are also called **edge subdivision** (see also Theorem 5.3.12) and **edge contraction**.

A classical result by Alexander [Ale30] states that two simplicial complexes are PL homeomorphic if and only if they can be connected by a sequence of stellar moves. More recently, Lutz and Nevo [LN16] showed that two flag simplicial spheres are PL homeomorphic if and only if they can be connected by a sequence of edge subdivisions and contractions such that all the complexes in the sequence are flag. The barycentric subdivision of the double suspension of the Poincaré homology sphere is an example of a non-PL flag sphere.

5.2.5 f-, h-, γ-vectors and γ-positivity

For a simplicial complex Δ, the f-polynomial is the generating function of its f-vector $f(\Delta)$. The **h-vector of** Δ is the sequence $h(\Delta) = (h_0(\Delta), \ldots, h_d(\Delta))$ (where $d = \dim\Delta + 1$) obtained from $f(\Delta)$ via the polynomial relations

$$\sum_{i=0}^{d} h_i(\Delta) t^{d-i} = \sum_{i=0}^{d} f_{i-1}(\Delta)(t-1)^{d-i}.$$

The **f-polynomial** and **h-polynomial** of Δ are respectively $h(\Delta, t) := \sum_{i=0}^{d} h_i(\Delta) t^i$ and $f(\Delta, t) := \sum_{i=0}^{d} f_{i-1}(\Delta) t^i$. It is well-known that

$$f(\Delta * \Gamma, t) = f(\Delta, t) \cdot f(\Gamma, t) \quad \text{and} \quad h(\Delta * \Gamma, t) = h(\Delta, t) \cdot h(\Gamma, t).$$

If it is clear from the context, we may omit Δ in the previous definitions.

When Δ is a $(d-1)$-dimensional homology sphere, $h(\Delta)$ satisfies the Dehn–Sommerville relations: $h_i = h_{d-i}$ for $0 \leq i \leq d$. It is then possible to express $h(\Delta, t)$ as $h(\Delta, t) = \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i t^i (1+t)^{d-2i}$. The **$\gamma$-vector** of Δ is the sequence $\gamma(\Delta) = (\gamma_0, \ldots, \gamma_{\lfloor d/2 \rfloor})$. If $\gamma_i \geq 0$ for every i, we say $h(\Delta, t)$ is **γ-positive** (or simply Δ is γ-positive). In 2005, Gal [Gal05] conjectured that every flag homology sphere is γ-positive. As is pointed out in [Gal05, Remark 3.1.1] (see also [Brä04, Lemma 4.1]), if every root of the polynomial $h(\Delta, t)$ is real, then $h(\Delta, t)$ is γ-positive.

5.3 Flag spheres arising from ternary graphs

In 1978, Lovász [Lov78] proved Kneser's conjecture by showing connections between topological invariants of certain spaces arising from a graph G and the chromatic number of G. Since then, many topological obstructions to graph coloring have been shown. One particular example can be found in the class of **ternary graphs**, that is, graphs without induced cycles of length divisible by three. Two conjectures of Kalai and Meshulam [Kalb, Conjectures 1,2] imply there exists a constant C, such that the chromatic number of every ternary graph G is bounded above by C. The bound on chromatic numbers of ternary graphs was proven by Bonamy, Charbit, and Thomassé in [BCT14], while the following topological characterization was proven by Kim.

Theorem 5.3.1 ([Kim22, Theorem 1.1])**.** *A graph G is ternary if and only if the independence complex of every induced subgraph of G is either contractible or homotopy equivalent to a sphere.*

In this section, we take a closer look at ternary graphs and their independence complexes. We prove Theorem 5.1.2 characterizing when the independence complex of a ternary graph is a homology sphere. Then we turn our attention to a family of graphs $\{G_m\}$, which together with the graph R_3 give all planar Gorenstein graphs [Tru18]. We show that each G_m is ternary and provide some additional observations about their structure. Finally, we prove that the independence complex of each G_m can be realized as the boundary of a polytope.

For a ternary graph G and a sequence of vertices $v_1, \ldots, v_s$ of G, a direct consequence of Theorem 5.3.1 is that by recursively taking links (or deletions) of the v_i, the resulting complex is either contractible or homotopy equivalent to a sphere. This property should be contrasted with the definition of homology spheres. On the one hand, subcomplexes of homology spheres obtained by taking links always have the homology of a sphere of the highest possible dimension. On the other hand, at the homology level, the condition from Theorem 5.3.1 includes more subcomplexes than just the ones obtained from links, but it does not require the spheres to be of the highest possible dimension. In [FH23] the authors introduce the class of **spherical complexes** as the class of complexes satisfying the topological condition arising from Theorem 5.3.1. The algebraic perspective from [FH23] led the authors to conjecture a characterization of Gorenstein spherical complexes, which we state for ternary graphs. We give an affirmative answer to [FH23, Question 6.5] below.

Lemma 5.3.2. *Let G be a ternary graph and $d = \dim \operatorname{Ind}(G)$. If $\operatorname{Ind}(G)$ is homotopy equivalent to a d-dimensional sphere, then $\operatorname{Ind}(G)$ is Cohen–Macaulay.*

Proof. Let $\Delta = \operatorname{Ind}(G)$. By [FH23, Theorem 3.6], we know that for any $\sigma \in \Delta$ either $\operatorname{lk}_\Delta(\sigma)$ has trivial homology or $\operatorname{lk}_\Delta(\sigma)$ has the homology of a d'-dimensional sphere, where

$$d - |\sigma| \leq d' \leq d.$$

In particular, $\tilde{H}_i(\operatorname{lk}_\Delta(\sigma); \mathbb{K}) = 0$ for $i < d - |\sigma|$, and since $\dim \operatorname{lk}_\Delta(\sigma) \leq d - |\sigma|$, we conclude Δ is Cohen–Macaulay. $\square$

Lemma 5.3.3. *Let G be a ternary graph. If $\operatorname{Ind}(G)$ is Cohen–Macaulay, then $\operatorname{Ind}(G)$ is a pseudomanifold.*

Proof. Since $\operatorname{Ind}(G)$ is Cohen–Macaulay, we conclude it is pure and strongly connected. Let σ be a $(d-1)$-dimensional face of $\operatorname{Ind}(G)$. Then $\operatorname{lk}_{\operatorname{Ind}(G)}(\sigma) = \operatorname{Ind}(G - N[\sigma])$, and since $G - N[\sigma]$ is also a ternary graph, we conclude the 0-dimensional complex $\operatorname{lk}_{\operatorname{Ind}(G)}(\sigma)$ is either contractible or homotopy equivalent to S^0. More specifically, $\operatorname{lk}_{\operatorname{Ind}(G)}(\sigma)$ is either one point or two points. In other words, there are at most two vertices of $\operatorname{Ind}(G)$ such that $v \notin \sigma$ and $\sigma \cup v \in \operatorname{Ind}(G)$. This property, together with the fact that $\operatorname{Ind}(G)$ is pure and strongly connected, implies $\operatorname{Ind}(G)$ is a (Cohen–Macaulay) pseudomanifold. $\square$

Theorem 5.3.4. (Theorem 5.1.2) *Let G be a ternary graph without isolated vertices. Then $\operatorname{Ind}(G)$ is homology sphere if and only if $\operatorname{Ind}(G)$ is homotopy equivalent to S^d, where $d = \dim \operatorname{Ind}(G)$.*

Proof. Let G be a ternary graph without isolated vertices. Assume $\mathrm{Ind}(G)$ is Gorenstein. Since G does not have isolated vertices, $\mathrm{Ind}(G)$ is not a cone. Then $\mathrm{Ind}(G)$ has the same homology as S^d by [Sta96a, Theorem II.5.1]. Theorem 5.3.1 then implies $\mathrm{Ind}(G)$ is homotopy equivalent to S^d.

Assume now that $\mathrm{Ind}(G)$ is homotopy equivalent to S^d, where $d = \dim \mathrm{Ind}(G)$. By Theorem 5.3.2 we know $\mathrm{Ind}(G)$ is Cohen–Macaulay, hence by Theorem 5.3.3 we conclude $\mathrm{Ind}(G)$ is a Cohen–Macaulay pseudomanifold.

We now show that $\mathrm{Ind}(G)$ is a Cohen–Macaulay pseudomanifold without boundary. Since $\mathrm{Ind}(G)$ is Cohen–Macaulay and homotopy equivalent to S^d, $\tilde{H}_d(\mathrm{Ind}(G); \mathbb{Z}_2) = \mathbb{Z}_2$. Consider the corresponding chain complex

$$0 \longrightarrow C_d \xrightarrow{\partial_d} C_{d-1} \xrightarrow{\partial_{d-1}} \cdots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \longrightarrow 0.$$

Since $\tilde{H}_d$ is nontrivial, there exists a nonempty set $\{F_i\}$ of facets of $\mathrm{Ind}(G)$ such that $\partial_d(\sum F_i) = 0$. Let Δ be the subcomplex of $\mathrm{Ind}(G)$ generated by $\{F_i\}$. Since $\partial_d(\sum F_i) = 0$, every ridge of Δ is contained in an even number of facets, and thus it must be contained in exactly two since $\mathrm{Ind}(G)$ is a pseudomanifold. Thus, Δ is a d-dimensional pseudomanifold without boundary, forcing $\Delta = \mathrm{Ind}(G)$.

Finally, since $\mathrm{Ind}(G)$ is a Cohen–Macaulay orientable pseudomanifold, [Sta96a, Theorem II.5.1] directly implies $\mathrm{Ind}(G)$ is Gorenstein. □

Remark 5.3.5. Although we state Theorem 5.1.2 for ternary graphs, as it fits the context of this chapter, the same proof works for arbitrary spherical complexes (see [FH23, Definition 2.3]).

In [Pin95] a recursive construction of planar 1-well-covered graphs, G_m, $m \in \mathbb{N}$, of girth four is given. The base cases are $G_1 = K_2$ and $G_2 = C_5$ with vertices a_1, a_2, b_1, b_2, and c_2. See Figure 5.1 for G_3. The graph G_{m+1} is constructed from G_m by adding vertices $a_{m+1}, b_{m+1}, c_{m+1}$ and edges $a_m a_{m+1}, b_m c_{m+1}, c_m a_{m+1}, a_{m+1} b_{m+1}, b_{m+1} c_{m+1}$. The result is the set of graphs $G_m = (V_m, E_m)$, where

$$\begin{aligned} V_m &= V(G_m) = V(G_{m-1}) \cup \{a_m, b_m, c_m\} \quad \text{and} \\ E_m &= E(G_m) = E(G_{m-1}) \cup \{a_{m-1}a_m, b_{m-1}c_m, c_{m-1}a_m, a_m b_m, b_m c_m\}. \end{aligned}$$

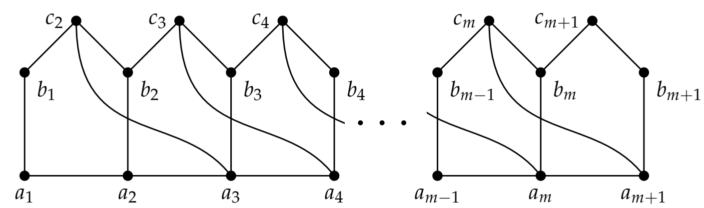


Figure 5.2: G_m

In [HT16] this construction is shown to produce Gorenstein graphs, and in [Tru18] it is shown that these are all the triangle-free, connected, planar (see Figure 5.3), Gorenstein graphs without isolated vertices.

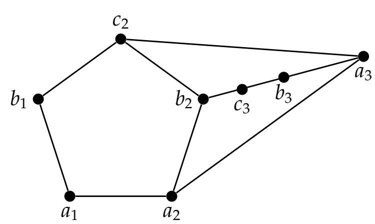


Figure 5.3: G_3 is planar

More generally, the following holds.

Theorem 5.3.6 ([Tru18, Theorem 3.7])**.** *If a planar graph G is Gorenstein, then G is a disjoint union of graphs of the form G_m and R_3, where the independence complex of R_3 is the 6-cycle C_6.*

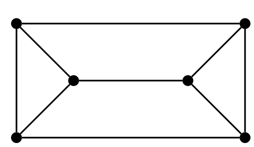
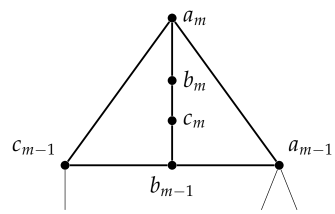


Figure 5.4: The graph R_3

We now establish some more properties of G_m for any $m \in \mathbb{N}$, unless otherwise stated. Recall that it was proven in [Tru18] that G_m is triangle-free. In fact, we can show the following stronger statement.

Lemma 5.3.7. *G_m is ternary for all m.*

Proof. We proceed by induction on m. Recall that G_1 is isomorphic to a single edge K_2, and G_2 is isomorphic to a 5-cycle C_5. Thus, we immediately see that G_1 and G_2 are ternary. Now assume that G_{m-1} is ternary for some $m \geq 3$. Recall that $V(G_m) = V(G_{m-1}) \cup \{a_m, b_m, c_m\}$ and $E(G_m) = E(G_{m-1}) \cup \{a_{m-1}a_m, b_{m-1}c_m, c_{m-1}a_m, a_m b_m, b_m c_m\}$.



Note that all induced cycles in G_{m-1} are induced cycles in G_m. As G_{m-1} is ternary, none of these induced cycles have length $3k$ for any $k \in \mathbb{N}$. Additionally, G_m has induced 5-cycles $(a_{m-1}, b_{m-1}, c_m, b_m, a_m, a_{m-1})$ and $(b_{m-1}, c_{m-1}, a_m, b_m, c_m, b_{m-1})$, and an induced 4-cycle $(a_{m-1}, b_{m-1}, c_{m-1}, a_m, a_{m-1})$. Finally, we can build induced cycles in G_m from induced cycles in G_{m-1} as follows. Observe that as $\deg_{G_{m-1}}(b_{m-1}) = 2$, a cycle in G_{m-1} contains the edge $a_{m-1}b_{m-1}$ if and only if it contains the edge $b_{m-1}c_{m-1}$. Let C be such a cycle in G_{m-1}, in particular, $\text{len}(C) \neq 3k$ for any $k \in \mathbb{N}$. Replacing the path $(a_{m-1}, b_{m-1}, c_{m-1})$ in C by the path (a_{m-1}, a_m, c_{m-1}) results in an induced cycle C' in G_m of length $\text{len}(C') = \text{len}(C) \neq 3k$. We can now conclude that none of the induced cycles in G_m have length $3k$ for any $k \in \mathbb{N}$, and thus G_m is ternary as desired. $\square$

Remark 5.3.8. In view of Theorems 5.3.6 and 5.3.7, we will often state our results for planar ternary graphs. For result we state with this assumption, there is a less clean version including the graph R_3, which is the only nonternary connected planar Gorenstein graph. For the sake of brevity, we omit these modified versions.

Having excluded the existence of induced cycles of length divisible by three, we now investigate the existence and interactions of other cycles in G_m.

Proposition 5.3.9. *Let $m \geq 1$. Then G_m has the following properties:*

(a) Let C_4^1 and C_4^2 be two 4-cycles in G_m. Then either $C_4^1 = C_4^2$ or $E(C_4^1) \cap E(C_4^2) = \varnothing$.

(b) *G_m has no 6-cycles.*

(c) *Let C_4 and C_5 be a 4- and 5-cycle, respectively, in G_m. Then either $E(C_4) \cap E(C_5) = \varnothing$ or they intersect in a 2-path.*

Proof.

(a) By construction of G_m, all of its 4-cycles are of the form $(a_{t-1}, a_t, c_{t-1}, b_{t-1}, a_{t-1})$ for some $t \geq 3$. Thus, if C_4^1 and C_4^2 are two distinct 4-cycles, they are edge-disjoint.

(b) Suppose G_m had a 6-cycle, say $(x_1, x_2, \ldots, x_6, x_1)$. As G_m is ternary, there exists an edge $x_i x_{i+3 \mod 6}$ for some $i \in [6]$. However, this gives two distinct 4-cycles sharing one edge, which do not exist by part (a). Hence, G_m has no 6-cycles.

(c) Recall that every 4-cycle C_4 in G_m is of the form $(a_{t-1}, a_t, c_{t-1}, b_{t-1}, a_{t-1})$ for some $t \geq 3$. By construction of G_m, every 5-cycle C_5 is of the form $(a_{t-1}, a_t, b_t, c_t, b_{t-1}, a_{t-1})$ for some $t \geq 2$, $(a_{t-2}, a_{t-1}, a_t, c_{t-1}, b_{t-2}, a_{t-2})$ for some $t \geq 3$, $(a_t, b_t, c_t, b_{t-1}, c_{t-1}, a_t)$ for some $t \geq 3$, or $(a_{t-1}, a_t, c_{t-1}, b_{t-2}, c_{t-2}, a_{t-1})$ for some $t \geq 4$. If $E(C_4) \cap E(C_5) \neq \varnothing$, it is now straightforward to see that the intersection is a 2-path.

□

We now turn our attention to induced paths in G_m. We find that unless two vertices are close to each other, there exist multiple induced paths of different lengths between them.

Lemma 5.3.10. *Let $x, y \in V(G_m)$. If $\operatorname{dist}(x, y) \geq 3$, then there exist induced paths of length 0, 1, and 2 mod 3 between x and y in G_m.*

Proof. In the following, all paths are induced paths unless otherwise stated. We start by finding paths of different lengths between a_k and a_{k+l}, and between b_k and b_{k+l}. These paths are a crucial step in finding the desired paths claimed in the statement of the lemma. Let $k \in [m]$ and $l \geq 0$ such that $k + l \leq m$. A **type–$(1, a)$–path** is a path of length l of the form $(a_k, a_{k+1}, \ldots, a_{k+l})$, where $l \geq 1$. A **type–$(2, a)$–path** is a path of length $l + 1$ of the form $(a_k, b_k, c_{k+1}, a_{k+2}, a_{k+3}, \ldots, a_{k+l})$, where $l \geq 2$. A **type–$(3, a)$–path** is a path of length $l + 2$ of the form $(a_k, b_k, c_{k+1}, b_{k+1}, c_{k+2}, a_{k+3}, a_{k+4}, \ldots, a_{k+l})$, where $l \geq 3$. Observe that these three paths immediately give the desired paths if $x, y \in \{a_i\}_{i \in [m]}$. Moreover, if $x = a_i$ and $y = b_j$, we obtain the desired paths as follows. Note that since $\operatorname{dist}(x, y) \geq 3$, we may disregard $j \in \{i-2, i-1, i, i+1\}$. If $j \geq i + 3$, concatenating a type–(t, a)–path for

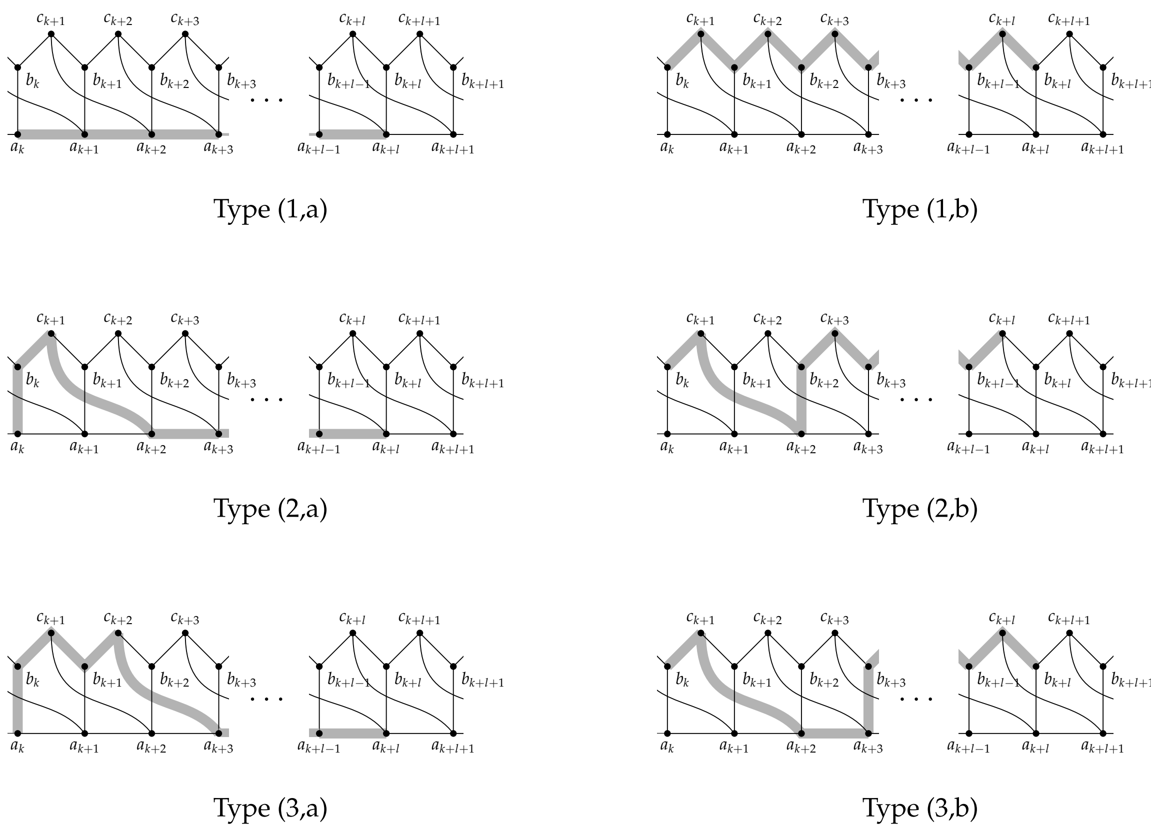


Figure 5.5: Induced paths of length $0, 1,$ and $2 \mod 3$ in G_m.

$t \in [3]$ with the edge $a_j b_j$ gives the desired paths. Similarly, if $j \le i-4$, concatenating the path (b_j, a_j, a_{j+1}) with a type–(t,a)–path for $t \in [3]$ gives the desired paths. For the remaining two cases, we refer to Table 5.1.

If $x = a_i$ and $y = c_j$, we obtain the desired paths as follows. Again, since $\text{dist}(x,y) \ge 3$, we may disregard $j \in \{i-2, i-1, i, i+1\}$. If $j \ge i+3$, concatenating a type–(t,a)–path for $t \in [3]$ with the path (a_j, b_j, c_j) gives the desired paths. Similarly, if $j \le i-4$, concatenating the edge $c_j a_{j+1}$ with a type–(t,a)–path for $t \in [3]$ gives the desired paths. For the remaining two cases, we refer to Table 5.1.

x	y	$0 \mod 3$	$1 \mod 3$	$2 \mod 3$
a_i	b_{i+2}	$(a_i, a_{i+1}, a_{i+2}, b_{i+2})$	$(a_i, b_i, c_{i+1}, a_{i+2}, b_{i+2})$	$(a_i, b_i, c_{i+1}, b_{i+1}, c_{i+2}, b_{i+2})$
a_i	b_{i-3}	$(b_{i-3}, c_{i-2}, a_{i-1}, a_i)$	$(b_{i-3}, a_{i-3}, a_{i-2}, a_{i-1}, a_i)$	$(b_{i-3}, a_{i-3}, a_{i-2}, b_{i-2}, c_{i-1}, a_i)$
a_i	c_{i+2}	$(a_i, a_{i+1}, b_{i+1}, c_{i+2})$	$(a_i, b_i, c_{i+1}, b_{i+1}, c_{i+2})$	$(a_i, b_i, c_{i+1}, a_{i+2}, b_{i+2}, c_{i+2})$
a_i	c_{i-3}	$(c_{i-3}, a_{i-2}, a_{i-1}, a_i)$	$(c_{i-3}, b_{i-3}, c_{i-2}, a_{i-1}, a_i)$	$(c_{i-3}, b_{i-3}, a_{i-3}, a_{i-2}, a_{i-1}, a_i)$

Table 5.1: Short paths a_i

Similarly, we can define a **type–$(1,b)$–path** as a path of length $2l$ of the form $(b_k, c_{k+1}, b_{k+1}, \dots, c_{k+l}, b_{k+l})$, where $l \ge 1$. A **type–$(2,b)$–path** is a path of length $2l-1$ of the form $(b_k, c_{k+1}, a_{k+2}, b_{k+2}, c_{k+3}, b_{k+3}, \dots, c_{k+l}, b_{k+l})$, where $l \ge 2$. A **type–$(3,b)$–path** is a path of length $2l-2$ of the form $(b_k, c_{k+1}, a_{k+2}, a_{k+3}, b_{k+3}, c_{k+4}, b_{k+4}, \dots, c_{k+l}, b_{k+l})$, where $l \ge 3$. These immediately give the desired paths if $x = b_i, y = b_j$, and $|i-j| \ge 3$. As $\text{dist}(x,y) \ge 3$, we may disregard pairs $i, j \in [m]$ with $|i-j| \le 1$. If $|i-j| = 2$, assume without loss of generality that $i < j$, and we refer to Table 5.2.

If $x = b_i$ and $y = c_j$, we obtain the desired paths as follows. Note that since $\text{dist}(x,y) \ge 3$, we may disregard $j \in \{i-1, i, i+1\}$. If $j \ge i+4$, concatenating a type–(t,b)–path for $t \in [3]$ with the edge $b_j c_j$ gives the desired paths. If $j \le i-3$, concatenating the edge $c_j b_j$ with a type–(t,b)–path for $t \in [3]$ gives the desired paths. For the remaining three cases, we refer to Table 5.2.

x	y	0 mod 3	1 mod 3	2 mod 3
b_i	b_{i+2}	$(b_i, c_{i+1}, a_{i+2}, b_{i+2})$	$(b_i, c_{i+1}, b_{i+1}, c_{i+2}, b_{i+2})$	$(b_i, a_i, a_{i+1}, b_{i+1}, c_{i+2}, b_{i+2})$
b_i	c_{i+3}	$(b_i, a_i, a_{i+1}, a_{i+2}, a_{i+3}, b_{i+3}, c_{i+3})$	$(b_i, c_{i+1}, a_{i+2}, b_{i+2}, c_{i+3})$	$(b_i, c_{i+1}, b_{i+1}, c_{i+2}, b_{i+2}, c_{i+3})$
b_i	c_{i+2}	$(b_i, c_{i+1}, b_{i+2}, c_{i+2})$	$(b_i, a_i, a_{i+1}, b_{i+1}, c_{i+2})$	$(b_i, a_i, a_{i+1}, a_{i+2}, b_{i+2}, c_{i+2})$
b_i	c_{i-2}	$(c_{i-2}, a_{i-1}, a_i, b_i)$	$(c_{i-2}, b_{i-2}, c_{i-1}, a_i b_{,i})$	$(c_{i-2}, b_{i-2}, c_{i-1}, b_{i-1}, c_i, b_i)$

Table 5.2: Short paths b_i

Finally, if $x = c_i, y = c_j$, assume without loss generality that $i < j$. As $\mathrm{dist}(x, y) \geq 3$, we may disregard pairs $i, j \in [m]$ with $j \leq i + 1$. If $j \geq i + 4$, concatenating the edge $c_i b_i$, a type–(t, b)–path for $t \in [3]$, and the edge $b_{j-1} c_j$ gives the desired paths. For the remaining two cases, we refer to Table 5.3.

x	y	0 mod 3	1 mod 3	2 mod 3
c_i	c_{i+3}	$(c_i, b_i, c_{i+1}, b_{i+1}, c_{i+2}, b_{i+2}, c_{i+3})$	$(c_i, a_{i+1}, a_{i+2}, b_{i+2}, c_{i+2})$	$(c_i, a_{i+2}, a_{i+2}, a_{i+3}, b_{i+3}, c_{i+3})$
c_i	c_{i+2}	$(c_i, a_{i+1}, b_{i+1}, c_{i+2})$	$(c_i, b_i, c_{i+1}, b_{i+1}, c_{i+2})$	$(c_i, b_i, c_{i+1}, a_{i+2}, b_{i+2}, c_{i+2})$

Table 5.3: Short paths c_i

□

The recursive construction of G_m allows us to recursively describe the independent sets of G_m. We can moreover specify when such independent sets are maximal.

Proposition 5.3.11. *Every independent set A of G_m has one of the following forms:*

1. *$A \cap \{a_m, b_m, c_m\} = \varnothing$ and A is an independent set in G_k for some $k < m$.*
2. *$A \cap \{a_m, b_m, c_m\} = \{b_m\}$ and $A \setminus \{b_m\}$ is an independent set in G_k for some $k < m$.*
3. *$A \cap \{a_m, b_m, c_m\} = \{a_m, c_m\}$ and $A \setminus \{a_m, c_m\}$ is an independent set in G_k for some $k \leq m - 2$. In particular, $A \cap \{a_{m-1}, b_{m-1}, c_{m-1}\} = \varnothing$.*
4. *$A \cap \{a_m, b_m, c_m\} = \{a_m\}$ and $A \setminus \{a_m\}$ is an independent set in G_k for some $k < m$ disjoint from $\{a_{m-1}, c_{m-1}\}$.*
5. *$A \cap \{a_m, b_m, c_m\} = \{c_m\}$ and $A \setminus \{c_m\}$ is an independent set in G_k for some $k < m$ disjoint from $\{b_{m-1}\}$.*

Moreover, the independent sets from (2), (4), and (5) are maximal if $A \setminus \{b_m\}$, $A \setminus \{a_m\}$, and $A \setminus \{c_m\}$ are maximal independent sets in G_{m-1}, respectively. Finally, the independent set from (3) is maximal if $A \setminus \{a_m, c_m\}$ is a maximal independent set in G_{m-2}.

Proof. The independent sets (1) - (5) immediately follow from the construction of G_m. To obtain the maximality statements, recall that a maximal independent set in G_m has cardinality m. □

Definition 5.3.12. Let Δ be a simplicial complex and $\{x, y\}$ an edge of Δ. The **edge subdivision** of Δ at $\{x, y\}$ is the simplicial complex Δ' with vertex set $V(\Delta) \sqcup \{a\}$ and faces:

- σ, for $\sigma \in \Delta$ such that $\{x, y\} \not\subset \sigma$
- $\sigma \setminus \{x\} \cup \{a\}$, for $\sigma \in \Delta$ such that $\{x, y\} \subset \sigma$
- $\sigma \setminus \{y\} \cup \{a\}$, for $\sigma \in \Delta$ such that $\{x, y\} \subset \sigma$

Remark 5.3.13. Edge subdivisions preserve flagness of the complex. We will focus on performing edge subdivisions on Δ when $\Delta = \mathrm{Ind}(G)$ for some graph G. We often keep track of the effect of the edge subdivision on G. Observe that G is the complement of the 1-skeleton of Δ. Therefore, using the notation from the definition above, $\Delta' = \mathrm{Ind}(G')$ where G' is a graph with vertex set $V(G) \sqcup \{a\}$ and edges

- xy
- e, for every $e \in E(G)$
- az, for every $z \in N_G(x) \cup N_G(y)$

In Figure 5.1, the right-hand-side sphere is obtained by performing an edge subdivision at the edge a_1a_2 on the left-hand-side sphere with a new vertex c_3. On the left-hand-side graph, $N(a_1) \cup N(a_2) = \{b_1, b_2, a_3\}$. Therefore, the edge subdivision is reflected on the graph as adding a vertex c_3 and edges $a_1a_2, c_2b_1, c_2b_2, c_2a_3$.

A special polytope that will play an important role in the following results in this section is the m-dimensional **crosspolytope**, which is the convex hull of m mutually orthogonal line segments in $\mathbb{R}^m$ intersecting at a point. Moreover, any polytope combinatorially equivalent to a crosspolytope is called a crosspolytope. The independence complex of the graph of m disjoint edges (mK_2) is the boundary of a crosspolytope.

Theorem 5.3.14. *For each $m \geq 1$ the simplicial complex* $\mathrm{Ind}(G_m)$ *is obtained by a sequence of $m-1$ edge subdivisions of the boundary of the m-dimensional crosspolytope.*

Proof. Let Q_1^m be the boundary of the m-dimensional crosspolytope. It is a flag complex. Thus, it is the independence complex of the graph that is the complement of its 1-skeleton. Fix m, and let S_1 be this complementary graph. Note that S_1 is mK_2, which is mG_1. Let the vertex set of S_1 be $\{b_i : 1 \leq i \leq m\} \cup \{c_i : 1 \leq i \leq m\}$, with edge set $\{b_i c_i : 1 \leq i \leq m\}$. Now we perform a sequence of edge subdivisions on Q_1^m and keep track of the effect on the complements of the 1-skeletons, which we will denote $S_2, S_3, \ldots, S_m$. Subdivide edge $b_1 c_2$ of Q_1^m with new vertex a_2. Call the new complex Q_2^m, and the complement of its 1-skeleton S_2. The graph S_2 has edges $b_i c_i$ and $b_1 c_2$, $a_2 c_1$, and $a_2 b_2$. Thus S_2 is a pentagon along with $m-2$ disjoint edges, that is, S_2 is the disjoint union of G_2 and $m-2$ copies of G_1. Suppose we have formed Q_i^m and S_i, $2 \leq i \leq m-2$, and we know that S_i is the disjoint union of G_i and $m-i$ copies of G_1. Then subdivide the edge $b_i c_{i+1}$ with new vertex a_{i+1}. Call the new complex Q_{i+1}^m and the complement of its 1-skeleton S_{i+1}. S_{i+1} has the edges of S_i (including $b_{i+1} c_{i+1}$), and edges $b_i c_{i+1}$ and $a_{i+1} c_i$, $a_{i+1} b_{i+1}$, and $a_i a_{i+1}$. Thus S_{i+1} is the disjoint union of G_{i+1} and $m-i-1$ copies of G_1. Finally, when $i = m-1$, $S_m = G_m$. So G_m is the complement of the 1-skeleton of Q_m^m. Thus, Q_m^m, the simplicial complex resulting from subdividing $m-1$ edges of the boundary of the m-dimensional crosspolytope, is the independence complex of G_m. □

In order to prove Theorem 5.1.3, we establish the two corollaries below, which follow from [Gal05, Proposition 2.4.2] and [PB80, Theorem 2.7], respectively.

Corollary 5.3.15. *For $m \geq 1$,* $\mathrm{Ind}(G_m)$ *is the boundary of an m-dimensional simplicial polytope.*

Corollary 5.3.16. *For $m \geq 1$,* $\mathrm{Ind}(G_m)$ *is vertex decomposable.*

Theorem 5.1.3 now quickly follows from Theorem 5.3.6 and the two previous corollaries.

Corollary 5.3.17 (Theorem 5.1.3)**.** *Let G be a planar ternary graph such that* $\mathrm{Ind}(G)$ *is a homology sphere. Then* $\mathrm{Ind}(G)$ *is combinatorially equivalent to the boundary of a simplicial polytope and is vertex decomposable.*

Proof. Let G be a planar ternary graph such that $\mathrm{Ind}(G)$ is a homology sphere. Then G is Gorenstein, so by Theorem 5.3.6, G is the disjoint union of G_m graphs. Thus $\mathrm{Ind}(G)$ is the join of the independence complexes $\mathrm{Ind}(G_m)$, each of which is the boundary of a simplicial polytope and is vertex decomposable. These properties are preserved under the join operation, so $\mathrm{Ind}(G)$ is the boundary of a simplicial polytope and is vertex decomposable. □

5.4 An induced subgraph of the Lutz–Nevo graph

Theorem 5.3.14 demonstrates how to obtain $\mathrm{Ind}(G_m)$ from the boundary of the crosspolytope by edge subdivisions. As a result, independence complexes of disjoint union of G_m's are polytopal simplicial spheres. Given two such spheres $\mathrm{Ind}(G)$ and $\mathrm{Ind}(G')$ of the same dimension, we now show how to obtain one from the other by edge subdivisions and contractions. For example, Figure 5.1 illustrates an edge subdivision that turns $\mathrm{Ind}(G_1 \sqcup G_2)$ into $\mathrm{Ind}(G_3)$. These observations allow us to characterize the subgraph of the Lutz–Nevo graph induced on the set of vertices representing these spheres.

Theorem 5.4.1 ([LN16, Theorem 1.2])**.** *Two flag simplicial complexes are PL homeomorphic if and only if they can be connected by a sequence of edge contractions and subdivisions such that every complex in the sequence is flag.*

In view of Theorem 5.4.1 the goal of this section is to study the following graph.

Definition 5.4.2. Let V_d be the set of all isomorphism classes of d-dimensional flag PL spheres for some fixed d, and E_d be the collection of pairs $\{\Delta_1, \Delta_2\}$ of elements of V_d such that $\{\Delta_1, \Delta_2\} \in E_d$ if and only if Δ_2 can be obtained from Δ_1 by either contracting an edge, or subdividing an edge. For an integer $d \geq 0$, the **Lutz–Nevo graph** T_d is the graph with vertex set V_d and edge set E_d.

Lemma 5.4.3 ([LN16])**.** *Let Δ be a flag complex. The complex Δ' obtained by contracting an edge e of Δ is flag if and only if e is not part of an induced square of Δ.*

A **partition** $\lambda = (m_1, \ldots, m_s)$ of n is an ordered nonincreasing list of integers such that $m_1 + \cdots + m_s = n$.

Definition 5.4.4 (Partition refinement graph). A partition λ of n *refines* a partition μ of n if each part of μ is the sum of parts of λ. The *partition refinement graph*, $\mathcal{P}_n$, is the graph of the Hasse diagram of the refinement poset on the integer partitions of n.

The goal of this section is to prove that $\mathcal{P}_n$ is isomorphic to an induced subgraph of T_{n-1}.

Definition 5.4.5. Denote by H_{n-1} the induced subgraph of T_{n-1} where the vertices are given by the independence complexes of planar ternary graphs with independence number n.

Lemma 5.4.6. *There is a bijection between vertices of $V(H_{n-1})$ and $V(\mathcal{P}_n)$.*

Proof. If the independence complex of a planar ternary graph G is an $(n-1)$-dimensional sphere, then by [Tru18]

$$G = G_{m_1} \sqcup G_{m_2} \sqcup \cdots \sqcup G_{m_s}$$

where $m_1 + \cdots + m_s = n$. Hence the bijection from $V(\mathcal{P}_n)$ to $V(H_{n-1})$ is given by

$$(m_1, \ldots, m_s) \mapsto G_{m_1} \sqcup G_{m_2} \sqcup \cdots \sqcup G_{m_s}.$$

□

Lemma 5.4.7. *Let $\operatorname{Ind}(G)$ be a vertex of H_{n-1}, and xy an edge of $\operatorname{Ind}(G)$ where x and y are vertices of the same connected component in G. Then the sphere given by subdividing the edge xy is not a vertex of H_{n-1}.*

Proof. Subdividing the edge xy of $\operatorname{Ind}(G_m)$ adds the edge xy to G_m (among other edges / vertices). There are two cases:

1. If the distance between x and y is two, then the new graph has a triangle, and hence is not ternary.
2. If the distance between x and y is at least three, by Theorem 5.3.10 there exists an induced path between x and y of length 2 mod 3. In particular, this path together with the new edge xy gives us an induced cycle of length $3k$ and hence the graph is not ternary.

□

Lemma 5.4.8. *Let $G = G_{m_1} \sqcup G_{m_2} \sqcup \cdots \sqcup G_{m_s}$ and $\Delta = \operatorname{Ind}(G)$. Suppose xy is an edge of Δ, where x is a vertex of G_{m_i} and y is a vertex of G_{m_j}. Let Δ' be obtained from Δ by an edge subdivision of xy, and let G' be the graph with $\operatorname{Ind}(G') = \Delta'$. Then G' has a connected component with subgraphs G_{m_i} and G_{m_j}, and this component has independence number $m_i + m_j$.*

Proof. Let Δ' be as given in the statement of the lemma, with new vertex a created by the edge subdivision. By [LN16, Lemma 5.2], Δ' is a flag complex, and so it is the independence complex of some graph G'. The faces of Δ' are those faces of Δ not containing both x and y, and the faces $\sigma \setminus \{x\} \cup \{a\}$ and $\sigma \setminus \{y\} \cup \{a\}$ for each $\sigma \in \Delta$ containing $\{x, y\}$. The edges of G' are the nonedges of Δ'; these are the nonedges of Δ, the pair xy and some edges containing a. Note that for any vertex u of G not in $V(G_{m_i} \cup G_{m_j})$, $ux \in \Delta$, so $ua \in \Delta'$, so ua is not an edge of G'. Thus, the subgraph G'' induced by the vertex set $V(G_{m_i} \cup G_{m_j}) \cup \{a\}$ is a connected component of G' containing as subgraphs G_{m_i} and G_{m_j}. If A_i is a maximal independent set of G_{m_i} containing x, and A_j is a maximal independent set of G_{m_j} containing y, then $A_i \cup A_j \setminus \{\mathrm{x}\} \cup \{a\}$ is an independent set of size $m_i + m_j$ in G'. Moreover, this set must be a maximal independent set in G'', because every other vertex of $G_{m_i} \cup G_{m_j}$ is adjacent to some vertex of $A_i \cup A_j$. □

Theorem 5.4.9. (Theorem 5.1.5) *The graph H_{n-1} is isomorphic to $\mathcal{P}_n$. In particular, $\mathcal{P}_n$ is an induced subgraph of T_{n-1} for every n.*

Proof. We know there is a bijection between the vertices of H_{n-1} and $\mathcal{P}_n$; hence we just need to show that this bijection preserves edges.

Let $G = G_{m_1} \sqcup G_{m_2} \sqcup \cdots \sqcup G_{m_s}$, $x \in G_{m_1}$, $y \in G_{m_2}$, and G' the graph obtained by subdividing the edge xy of $\mathrm{Ind}(G)$. Recall from Section 5.2.4 that stellar moves preserve homeomorphism type, so $\mathrm{Ind}(G')$ is a sphere. Theorem 5.4.8 says that if G' is planar and ternary, then $G' \cong G_{m_1+m_2} \sqcup G_{m_3} \sqcup \cdots \sqcup G_{m_s}$. First suppose x and y are vertices of degree two in their respective connected components. Observe that up to isomorphism, the choice of x and y is unique. We first show that G' is planar. Denote by Q_i for $i \in \{1, 2\}$ the path of length three in G_{m_i} such that all interior vertices have degree two and $x \in V(Q_1)$, $y \in V(Q_2)$. For each $i \in \{1, 2\}$, we can pick a planar representation of G_{m_i} such that Q_i is on the boundary cycle of the outer face. Now it is easy to see that G' is indeed planar. By the characterization from Theorem 5.3.6, the connected components of G' are either of the form G_m, or are the complement of a hexagon. Since the new connected component has $|V(G_{m_1})| + |V(G_{m_2})| + 1 \neq 6$ vertices, we conclude the new graph must also be ternary. In particular, G' is a planar ternary graph and hence $\mathrm{Ind}(G') \in H_{n-1}$.

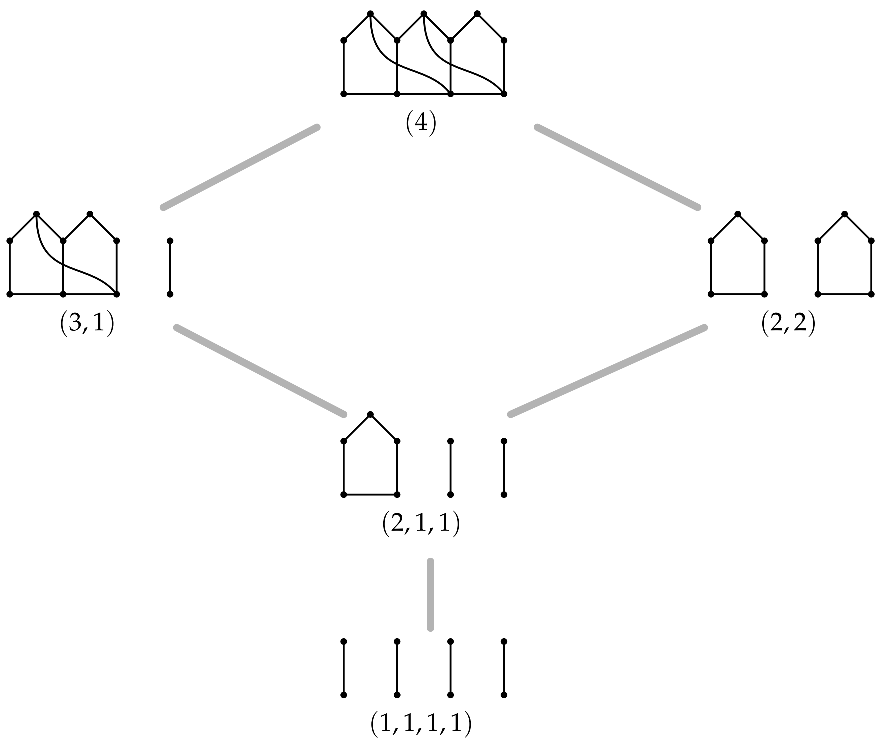


Figure 5.6: A visualization of Theorem 5.4.9 for $n = 4$

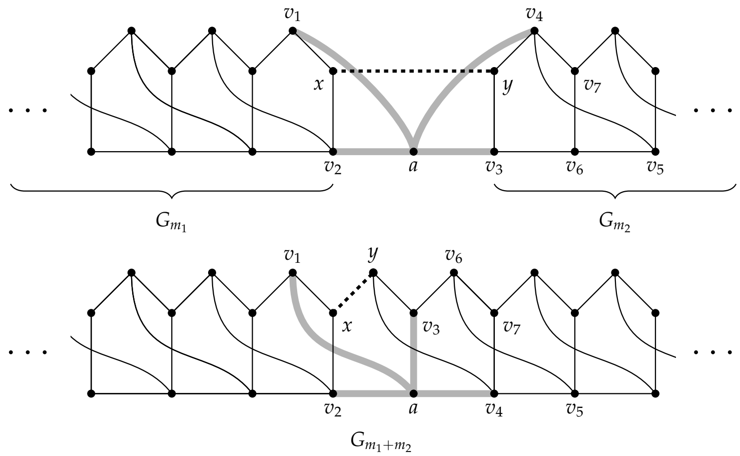


Figure 5.7: Isomorphism between $G_{m_1+m_2}$ and graph obtained after subdividing the edge xy (connecting two disconnected G_m). The dotted edge is being subdivided, and the gray edges are the new ones in the new graph.

Finally, to see that these are the only edges in H_{n-1}, note that when subdividing an edge xy of $\mathrm{Ind}(G)$, where x and y are in different connected components of G, the new vertex a will have degree $\deg x + \deg y$. In particular, if the degree of either x or y is strictly bigger than 2, then $\deg a > 4$. This implies the new connected component is not of the form G_m for any m, and hence by Theorem 5.3.6 the new graph is not a vertex of H_{n-1}. $\square$

5.5 The f, h and γ-vectors of planar flag spheres

Recall that Gal's conjecture [Gal05] posits that all flag homology spheres are γ-positive. Moreover, a real-rooted h-polynomial implies γ-positivity. Gal's conjecture was proposed after he found counterexamples to the stronger conjecture that the h-polynomial of flag spheres were real rooted. By [Gal05, Section 2.4], flag simplicial spheres obtained by successive edge subdivisions from the crosspolytope are γ-positive. Therefore, Theorem 5.3.14 implies that $\mathrm{Ind}(G_m)$ is γ-positive. In this section, we investigate further the f-vector, h-vector and γ-vector of $\mathrm{Ind}(G_m)$, and prove the stronger result that a flag homology sphere whose 1-skeleton is complementary to a planar graph has a real-rooted h-polynomial.

For $m \geq 1$ and $0 \leq i \leq m-1$, let $f_i(m) := f_i(\mathrm{Ind}(G_m))$. When needed, we use $f_m(m) = 0$.

Proposition 5.5.1. *For each $m \geq 1$, $f_0(m) = 3m-1$. For each $m \geq 2$, $f_1(m) = (9m^2 - 19m + 12)/2$. For $m \geq 3$ and $2 \leq i \leq m-1$,*

$$f_i(m) = 2f_{i-1}(m-1) + f_i(m-1) + f_{i-2}(m-2) + f_{i-1}(m-2).$$

Proof. First note that $G_1 = K_2$, and $\mathrm{Ind}(G_1) = 2K_1$, so $f_0(1) = 2$. Also, $f_1(m)$ is the number of edges of $\mathrm{Ind}(G_m)$, that is, the number of pairs of vertices in G_m that do not form an edge. So for $m \geq 2$, $f_1(m) = \binom{3m-1}{2} - 5(m-1) = (9m^2 - 19m + 12)/2$. Now assume $m \geq 3$ and $2 \leq i \leq m-1$.

Refer to the recursive construction of G_m for the vertices and edges of G_m not in G_{m-1}. Note that $G_m - (N[a_m] \cup N[c_m]) = G_m - \{a_{m-1}, b_{m-1}, c_{m-1}, a_m, b_m, c_m\} = G_{m-2}$.

Using the characterization of independent sets of G_m from Proposition 5.3.11, we count the number of independent sets of size $i+1$ for each of the five cases: Case (1) gives exactly $f_i(m-1)$ subsets, Case (2) gives $f_{i-1}(m-1)$ subsets, and Case (3) gives exactly $f_{i-2}(m-2)$.

For Cases (4) and (5), note that every independent set of size i of G_{m-1} intersects $\{a_{m-1}, b_{m-1}, c_{m-1}\}$ in either $a_{m-1}, b_{m-1}, c_{m-1}, \{a_{m-1}, c_{m-1}\}$ or $\emptyset$. We now show that the number of these intersections is $f_{i-1}(m-1) + f_{i-1}(m-2)$. To see this, first consider independent sets J of size i (dimension $i-1$) in G_{m-1}. If $J \cap \{a_{m-1}, b_{m-1}, c_{m-1}\} = \emptyset$, extend J to a set of case 4 by adding a_m. If $J \cap \{a_{m-1}, b_{m-1}, c_{m-1}\} \neq \emptyset$, then J can be extended to a unique set of case 4 or 5 by adding a_m (if $b_{m-1} \in J$) or c_m (if a_{m-1} and/or c_{m-1} is in J). This gives all independent sets of Case (4) or (5) except the sets A for which $A \cap \{a_{m-1}, b_{m-1}, c_{m-1}, a_m, b_m, c_m\} = c_m$. The number of these is $f_{i-1}(G_{m-2})$.

Counting the elements in each partition we have

$$\begin{aligned} f_i(m) &= f_i(m-1) + f_{i-1}(m-1) + f_{i-2}(m-2) + f_{i-1}(m-1) + f_{i-1}(m-2) \\ &= 2f_{i-1}(m-1) + f_i(m-1) + f_{i-2}(m-2) + f_{i-1}(m-2). \end{aligned}$$

□

We now define the Delannoy polynomials mentioned in Theorem 5.1.6.

Definition 5.5.2. The *Delannoy number $D(m,k)$ is the number of lattice paths from $(0,0)$ to (m,k) using steps $(1,0)$, $(0,1)$, and $(1,1)$.*

Following [WZC19], let $d(m,k) = D(m-k,k)$, and for fixed m, let $d_m(t)$ be the generating function $d_m(t) = \sum_{k=0}^{m} d(m,k)t^k$. It is known (see for example [OEI25, A008288] and [WZC19, Section 2]) that $d(m,k)$ satisfies the recursion $d(m,0) = d(m,m) = 1$ for $m \geq 0$ and, for $m > 1$ and $0 < k < m$,

$$d(m,k) = d(m-1,k) + d(m-1,k-1) + d(m-2,k-1).$$

In 2019, Wang, Zheng and Chen [WZC19] showed that Delannoy polynomials are real-rooted.

Theorem 5.5.3 ([WZC19, Theorem 2.1])**.** *The zeros of the Delannoy polynomial* $\sum_{k=0}^{m} d(m,k)t^k$ *are negative real numbers.*

We are now ready to tackle Theorem 5.1.6. The first part of the theorem is stated for all planar graphs whose independence complexes are homology spheres. Let G be such a graph and assume further it is connected. Then recall from Theorem 5.3.8 that G is either a G_m or R_3. The case of $G = G_m$ is stated as the second part of Theorem 5.1.6 and proved as Theorem 5.5.4. When $G = R_3$, the h-polynomial of $\mathrm{Ind}(G) = C_6$ is $t^2 + 4t + 1$, which is real-rooted.

Theorem 5.5.4. *Let G be a planar ternary connected graph such that* $\mathrm{Ind}(G)$ *is an $(m-1)$-dimensional homology sphere. Then* $h_k(\mathrm{Ind}(G)) = d(m,k)$.

Proof. We may assume $G = G_m$ by the previous discussion. Write $f_i(m) = f_i(\mathrm{Ind}(G_m))$, $h_k(m) = h_k(\mathrm{Ind}(G_m))$, and $h(m,t) = \sum_{k=0}^{m} h_k(m)t^{m-k} = \sum_{i=0}^{m} f_{i-1}(m)(t-1)^{m-i}$. Proposition 5.5.1 gives

$$f_{i-1}(m) = 2f_{i-2}(m-1) + f_{i-1}(m-1) + f_{i-3}(m-2) + f_{i-2}(m-2).$$

So

$$\begin{aligned}
h(m,t) &= \sum_{i=0}^{m} f_{i-1}(m)(t-1)^{m-i} \\
&= 2\sum_{i=0}^{m} f_{i-2}(m-1)(t-1)^{m-i} + \sum_{i=0}^{m} f_{i-1}(m-1)(t-1)^{m-i} \\
&\quad + \sum_{i=0}^{m} f_{i-3}(m-2)(t-1)^{m-i} + \sum_{i=0}^{m} f_{i-2}(m-2)(t-1)^{m-i} \\
&= 2\sum_{i=0}^{m-1} f_{i-1}(m-1)(t-1)^{m-1-i} + (t-1)\sum_{i=0}^{m-1} f_{i-1}(m-1)(t-1)^{m-1-i} \\
&\quad + \sum_{i=0}^{m-2} f_{i-1}(m-2)(t-1)^{m-2-i} + (t-1)\sum_{i=0}^{m-2} f_{i-1}(m-2)(t-1)^{m-2-i} \\
&= (t+1)h(m-1,t) + th(m-2,t).
\end{aligned}$$

Equating coefficients of t^{m-k}, $h_k(m) = h_k(m-1) + h_{k-1}(m-1) + h_{k-1}(m-2)$. This is the same recursion as for $d(m,k)$. Since $(h_0(1), h_1(1)) = (1,1) = (d(1,0), d(1,1))$ and $(h_0(2), h_1(2), h_2(2)) = (1,3,1) = (d(2,0), d(2,1), d(3,1))$, $h_k(m) = d(m,k)$ for all $m \geq 1$, $0 \leq k \leq m$. □

We immediately conclude the following from Theorems 5.5.3 and 5.5.4.

Corollary 5.5.5. *The zeros of the h-polynomial of* $\operatorname{Ind}(G_m)$ *are negative real numbers.*

To remove the requirement of connectedness, recall from Sections 5.2.2 and 5.2.5 that for any pair of graphs G and H,

$$h(\operatorname{Ind}(G \sqcup H), t) = h(\operatorname{Ind}(G) * \operatorname{Ind}(H), t) = h(\operatorname{Ind}(G), t) \cdot h(\operatorname{Ind}(H), t).$$

The first part of Theorem 5.1.6 then follows.

Corollary 5.5.6. *If G is a planar graph such that* $\operatorname{Ind}(G)$ *is a homology sphere, then the zeros of* $h(\operatorname{Ind}(G), t)$ *are negative real numbers.*

5.6 Future work

Recall that Trung characterized Gorenstein planar graphs [Tru18], all of which are ternary except the graph R_3. Pinter constructed Gorenstein nonplanar graphs in [Pin97], but the ternary property of such graphs was not explored. In particular, many of the graphs introduced in [Pin97] contain an induced subgraph isomorphic to G_m for some m, see for example [Pin97, Figure 4, Figure 10]. This naturally leads to the question of finding nonplanar ternary graphs that are Gorenstein.

We now provide one such example below. Start with a disconnected graph $G^0 = H_1 \sqcup H_2 \sqcup H_3$ with $H_i \cong C_5$ for $i \in [3]$. Consider vertices $1 \in V(H_1)$ and $6 \in V(H_2)$. We obtain a graph G^1 by subdividing the edge $\{1,6\}$ in $\operatorname{Ind}(G)$. Next, consider vertices $7 \in N_{H_2}(6)$ and $11 \in V(H_3)$. We obtain the graph G^2 pictured below by subdividing the edge $\{7,11\}$ in $\operatorname{Ind}(G^1)$.

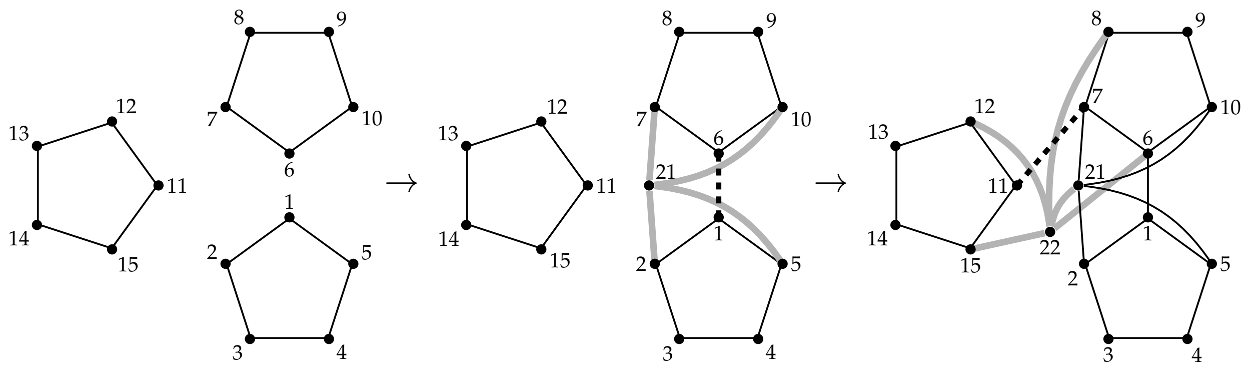


Figure 5.8: Constructing a nonplanar ternary Gorenstein graph

It is quick to check that all induced cycles in G^2 have length four or five. G^2 is nonplanar as the induced subgraph of G^2 on vertices $\{6, 7, 8, 9, 10, 21, 22\}$ is homeomorphic to $K_{3,3}$. Lastly, to see that G^2 is a Gorenstein graph, notice that $\mathrm{Ind}(G)$ is homeomorphic to $\mathrm{Ind}(G^2)$ since we obtain $\mathrm{Ind}(G^2)$ from $\mathrm{Ind}(G)$ by subdividing edges, and by Theorem 5.1.2 $\mathrm{Ind}(G)$ is homotopy equivalent to $S^{\dim \mathrm{Ind}(G)}$. The result then follows since $\dim \mathrm{Ind}(G) = \dim \mathrm{Ind}(G^2)$.

We can use a generalization of this construction to obtain Gorenstein graphs, and we suspect that many of these are ternary. Additionally, under certain conditions we can guarantee the output of this construction to be nonplanar. The general construction looks as follows.

1. Set $k = 0$.
2. Let $G^k \cong \bigsqcup_{i=1}^n H_i$ with $H_i \cong G_{m_i}$ for all i and $n \geq 2$.
3. Pick two vertices $v_i \in V(H_i)$ and $v_j \in V(H_j)$ such that v_i, v_j are in different components of G^k.
4. Subdivide the edge $v_i v_j$ in $\mathrm{Ind}(G^k)$ and call the resulting graph G^{k+1}.
5. If G^{k+1} is connected, stop the procedure. Else, either stop the procedure or set $k = k + 1$ and return to (3).

Observe that we used a special case of this in the proof of Theorem 5.4.9. In particular, as long as we choose the vertices v_i and v_j such that $\deg v_i = \deg v_j = 2$, the graph G^k is isomorphic to a disjoint union of G_m, that is, it is a planar ternary Gorenstein graph.

We can actually show that G^k is nonplanar if and only if we at least once pick the vertex v_i such that $\deg v_i \geq 3$: Note that the forward direction is given by the above observation. For the other direction, let k be largest such that at each prior step $\deg v_i = \deg v_j = 2$. Hence, we currently have $G^k \cong \bigsqcup_{i=1}^n H_i$ with $H_i \cong G_{m_i}$ for all i. Now assume $\deg v_i \geq 3$ and let $N_{v_i} \supseteq \{x_1, x_2, x_3\}$. Calling the new vertex in G^{k+1} y, the induced subgraph on vertices $\{y, x_1, x_2, x_3, v_i\}$ is isomorphic to $K_{2,3}$. By the structure of G_m we can additionally find a vertex $z \in V(H_i)$ such that for each $\ell \in [3]$ there exist pairwise disjoint paths from x_ℓ to z in H_i avoiding v_i. Hence, G^{k+1} contains a subgraph homeomorphic to $K_{3,3}$ and is therefore nonplanar.

Question 5.6.1. If G^k is nonplanar, under which conditions is it also ternary?

Remark 5.6.2. We note that given a graph G that is the output of the construction stated above starting from a graph G^0 with N connected components, it is possible to define a new graph W with N vertices corresponding to each connected component of G^0, and edges corresponding to pairs uv such that at some point in the construction, the vertices picked are from connected components u and v respectively. We believe that if W is a tree, then the graph G is ternary.

Question 5.6.3. How do the conditions for being nonplanar or ternary change if we just require v_i and v_j to be in different components of G^k, but not necessarily vertices of H_i?

Chapter 6

Concluding remarks and future directions

6.1 Improving the lower bound on the number of spheres

Recall that $s(d,n)$ (resp. $s_{\text{shell}}(d,n)$) denotes the number of combinatorially distinct simplicial d-spheres (resp. shellable simplicial d-spheres) on n vertices. The best known constructions yield $s(d,n) \geq 2^{\Omega(n^{\lceil d/2 \rceil})}$ for $d \geq 3$. On the other hand, as computed in [Kal88a], the Upper Bound Theorem and the Dehn–Sommerville relations together imply that $s(d,n) \leq 2^{O(n^{\lceil d/2 \rceil}\log n)}$. Therefore, there is still a difference of $\log n$ in the exponent between the lower and upper bounds for $s(d,n)$. It remains open whether this gap can be closed further. In Chapter 3, we have established that $s_{\text{shell}}(d,n) = 2^{\Theta(n^{\lceil d/2 \rceil})}$ for all $d \geq 3$. In particular, the current best constructions remain within the class of shellable spheres. The reason may be that these constructions all involve building numerous complexes from a single prototypical well-understood complex that is very geometrical. The squeezed balls in [Kal88a] come from the cyclic polytope, and the Nevo–Santos–Wilson construction starts with a join of paths. To improve the current best lower bound for $s(d,n)$, the spheres that one needs to construct cannot all be shellable (or **constructible**, as we will see in Section 6.3), and it may be necessary to investigate less geometric constructions.

6.2 Reconstructing different classes of spheres

It remains an open problem whether Theorem 4.1.2 holds for other interesting classes of simplicial spheres.

Partitionable spheres

One natural relaxation of shellability is partitionability (though, at the moment, we do not know if there exist simplicial spheres that are not partitionable, see Section 6.4 for more details). However, the method we employ in Chapter 4 cannot be applied to simplicial spheres that are partitionable but not shellable. While partitionability of Δ implies the existence of a good orientation of G, only shellability of Δ guarantees that there is an acyclic one. Therefore, without the shellability assumption, Proposition 4.3.4 will not work as an intrinsic way to identify the good orientations of G.

Balanced spheres and flag spheres

A d-dimensional simplicial complex is **balanced** if its 1-skeleton is d-colorable in the graph-theoretic sense. The facet-ridge graph of a balanced sphere is bipartite [Jos02]. There may also be additional combinatorial constraints that one can exploit for the reconstruction of flag spheres, since the minimal nonfaces of a flag sphere all have size two.

The classes of flag spheres, balanced spheres, and shellable spheres intersect (for example, the boundary of the **cross polytope** is in all three classes), but none of them contains the others. It is easy to find shellable spheres that are flag but not balanced, balanced but not flag, and neither flag nor balanced.

We note that the **barycentric subdivision** of any simplicial complex is both balanced and flag, and PL homeomorphic to the original complex. The renowned Double Suspension Theorem by Cannon [Can79] states that the double suspension S^{d+2} of any homology d-sphere H^d is homeomorphic to a $(d+2)$-sphere. In particular, if H^d is not a topological sphere, then the sphere S^{d+2} is non-PL, since H^d appears as a link of a vertex in S^{d+2}, while all links of a PL sphere are PL. For explicit triangulations of such spheres arising from the Poincaré homology sphere, which is a homology sphere but not an honest topological sphere, see [BL00]. The barycentric subdivision of these non-PL simplicial spheres are thus non-PL. As shellable spheres must be PL spheres, these flag and balanced spheres can never be shellable. It is significantly harder to construct nonshellable but flag or balanced PL spheres, primarily due to the scarcity of explicit examples of nonshellable PL spheres, as we will see in Section 6.3.

Weaker reconstructions

It is even unknown whether the facet-ridge graph determines the f-vector of a nonshellable sphere (or more generally, of any Cohen–Macaulay pseudomanifold). It was stated as Conjecture 2 in [Kala] that the facet-ridge graph of a d-dimensional Cohen–Macaulay simplicial complex Δ determines the f-vector of Δ. We note that without the pseudomanifold assumption on Δ, this conjecture is false, as shown by the following simple example.

Let $\Delta = \overline{123} \cup \overline{234} \cup \overline{134}$ (the boundary of a tetrahedron minus one triangle) and $\Gamma = \overline{123} \cup \overline{124} \cup \overline{125}$ (a "book" of three triangles). Since the ridge 12 is contained in all three facets, Γ is not a pseudomanifold. Both complexes are easily seen to be shellable and therefore Cohen–Macaulay. Moreover, both have the 3-cycle K_3 as their facet-ridge graph. However, $f(\Delta) = (1, 4, 6, 3)$ and $f(\Gamma) = (1, 5, 7, 3)$.

6.3 Constructible but nonshellable spheres

A d-dimensional pure simplicial complex Δ is **constructible** if either Δ is a simplex, or $\Delta = \Delta_1 \cup \Delta_2$, where Δ_1, Δ_2 are two d-dimensional constructible complexes such that $\Delta_1 \cap \Delta_2$ is a $(d-1)$-dimensional constructible complex. For extensive studies of constructible complexes, especially constructible spheres and balls, see [Hac00].

It is easy to check that shellability implies constructibility. Moreover, constructibility implies Cohen–Macaulayness. Constructible pseudomanifolds must be either balls or spheres (this is a more general version of Proposition 4.2.2(1)), and they are necessarily PL. For all $d \geq 3$, there exist examples of nonconstructible d-balls and d-spheres [HZ00]. There also exist constructible but nonshellable balls (see for example [Rud58, DK74, Zie98, Lut04b]). However, currently, the only obstruction discovered to shellability of PL spheres also obstructs constructibility [HZ00], and the following problem remains open.

Problem 6.3.1. Does there exist a constructible but nonshellable sphere?

This problem is challenging in many ways. In general, explicit constructions of nonshellable PL spheres are scarce. The smallest known example is $S^3_{13,56}$ (introduced in [Lut04a]; see [Lut] for the complete list of facets), a 3-dimensional sphere with 13 vertices and 56 facets.

Modifying $S^3_{13,56}$

One can try to modify $S^3_{13,56}$ to obtain an "in-between" sphere that is still nonshellable but constructible. Because the embedded trefoil consisting of the three edges $12, 23, 13$ obstructs constructibility, any such modification must alter this 3-cycle in the 1-skeleton. As one such attempt, the author found that the sphere with 55 facets

$$S^3_{13,55} := S^3_{13,56} \setminus \left(\overline{\{1,3,5,8\}} \cup \overline{\{1,3,5,11\}} \cup \overline{\{1,3,8,11\}}\right) \cup \left(\overline{\{1,5,8,11\}} \cup \overline{\{3,5,8,11\}}\right),$$

which is obtained from $S^3_{13,56}$ by a single bistellar flip, is shellable. A shelling is listed below (read column by column):

$\{2,3,4,7\}$	$\{2,7,8,11\}$	$\{5,7,9,13\}$	$\{5,6,10,11\}$	$\{1,2,6,9\}$	$\{3,8,9,11\}$
$\{2,3,4,10\}$	$\{2,7,11,13\}$	$\{1,5,8,10\}$	$\{1,10,11,13\}$	$\{1,2,6,12\}$	$\{2,8,9,11\}$
$\{2,3,7,10\}$	$\{3,4,6,7\}$	$\{1,7,8,10\}$	$\{6,10,11,13\}$	$\{1,2,9,12\}$	$\{3,8,9,12\}$
$\{2,4,5,7\}$	$\{3,4,6,10\}$	$\{1,7,8,11\}$	$\{5,6,9,11\}$	$\{1,9,10,12\}$	$\{2,8,9,12\}$
$\{2,4,5,10\}$	$\{3,6,7,12\}$	$\{1,5,8,11\}$	$\{2,6,11,13\}$	$\{3,9,10,12\}$	$\{3,5,8,13\}$
$\{2,5,7,13\}$	$\{3,7,10,12\}$	$\{1,7,10,12\}$	$\{2,6,9,11\}$	$\{3,9,10,13\}$	$\{2,8,12,13\}$
$\{2,5,8,10\}$	$\{4,5,6,7\}$	$\{1,7,11,13\}$	$\{1,6,7,9\}$	$\{3,5,9,13\}$	$\{3,8,12,13\}$
$\{2,5,8,13\}$	$\{4,5,6,10\}$	$\{1,7,9,13\}$	$\{1,6,7,12\}$	$\{3,5,9,11\}$	$\{3,6,10,13\}$
$\{2,7,8,10\}$	$\{5,6,7,9\}$	$\{1,5,10,11\}$	$\{1,9,10,13\}$	$\{3,5,8,11\}$	$\{3,6,12,13\}$
					$\{2,6,12,13\}$

Limitations of asymptotic and ball-related approaches

Using the argument for Corollary 3.1.2, it can also be derived that the number of constructible spheres is asymptotically the same as the number of shellable spheres. This rules out the possibility of a nonexplicit approach to Problem 6.3.1 by showing different asymptotics.

The available method for obtaining constructible but nonshellable balls does not apply to spheres. The method relies on the fact that every shellable contractible complex must have a free facet. However, spheres are not contractible. Attempts have also been made in the past to construct spheres by coning over the boundaries of the aforementioned constructible but nonshellable balls, but the resulting spheres all turn out to be shellable [Kam04].

Recoverability from the facet-ridge graph

Theorem 3.1.1 offers the possibility of addressing Problem 6.3.1 from another angle:

Problem 6.3.2. Does there exist a constructible simplicial sphere whose combinatorial type cannot be recovered from its facet-ridge graph?

One possible approach is to utilize the "nonrecoverability" of balls. It is easy to find nonisomorphic balls with isomorphic facet-ridge graphs, even pairs with isomorphic boundaries. Gluing such balls along their common boundary might yield the desired spheres. Alternatively, one can try to "thicken" the "nonrecoverable" tori and projective planes from [CD25] and embed them into spheres in a way that preserves both the difference between the complexes and the isomorphism between their facet-ridge graphs.

6.4 Partitionability of spheres

Let Δ be a partitionable simplicial complex with a partitioning $\Delta = \bigsqcup_{i=1}^{n} [\sigma_i, F_i]$. Then its h-vector has a simple combinatorial interpretation: $h_k(\Delta) = \#\{j : |\sigma_j| = k\}$ [Sta96b, Proposition III.2.3]. A longstanding conjecture (often refered to as the Partitionability Conjecture) by Stanley [Sta79a] states that every Cohen–Macaulay simplicial complex is partitionable. Had the conjecture been true, it would have provided a combinatorial interpretation of the h-numbers of Cohen–Macaulay complexes, which are always nonnegative. However, it was disproven in 2016 [DGKM16]. The counterexample produced is not a pseudomanifold, and it remains unknown whether all simplicial balls and spheres, which are always Cohen–Macaulay, are partitionable.

For spheres, a natural place to look next is among nonshellable examples, since shellability implies partitionability. However, the only sufficiently small example to experiment with, $S^3_{13,56}$, turns out to be partitionable. The author obtained a partitioning by first taking the partitioning induced by the shelling of $S^3_{13,55}$ listed above, and then modifying the partitioning intervals involving the facets affected by the bistellar flip. Specifically, we

- remove intervals $[\{5,11\},\{1,5,8,11\}]$ and $[\{3,8\},\{3,5,8,11\}]$;
- add intervals $[\{1,3\},\{1,3,5,8\}]$, $[\{5,11\},\{1,3,5,11\}]$, and $[\{3,11\},\{1,3,8,11\}]$;
- adjust intervals $[\{3,8,13\},\{3,5,8,13\}]$ and $[\{3,11\},\{3,5,9,11\}]$ to $[\{3,8\},\{3,5,8,13\}]$ and $[\{3,9,11\},\{3,5,9,11\}]$, respectively.

We conclude with several observations on partitionable complexes involving links, stars, and joins that generalize analogous properties for shellable complexes, in the hope of aiding future work on partitionable complexes.

Proposition 6.4.1. *All links of a partitionable simplicial complex are partitionable.*

Proof. This follows immediately from the definition of signability of posets [KO96]. Alternatively, let Δ be a simplicial complex with a partitioning $\bigsqcup_i p_i$, and let $\sigma \in \Delta$. For each i, let $p'_i = \{\tau \setminus \sigma : \sigma \subseteq \tau \in p_i\}$. Then $\bigsqcup_i p'_i$ is a partitioning of $\operatorname{lk}_\Delta \sigma$. $\square$

Proposition 6.4.2. *Let Γ and Δ be simplicial complexes. Then $\Gamma * \Delta$ is partitionable if and only if both Γ and Δ are partitionable.*

Proof. The only if direction follows from Proposition 6.4.1: let F be any facet of Γ and G be any facet of Δ. Then $\Delta = \operatorname{lk}_{\Gamma*\Delta} F$ and $\Gamma = \operatorname{lk}_{\Gamma*\Delta} G$.

Now suppose both Γ and Δ are partitionable. Let $F_1, \ldots, F_n$ denote the facets of Γ and $\bigsqcup_i [R(F_i), F_i]$, $i \in [n]$ be a partitioning of Γ. Similarly, let $G_1, \ldots, G_m$ denote the facets of Δ and $\bigsqcup_i [R(G_i), G_i]$, $i \in [m]$ be a partitioning of Δ. The facets of $\Gamma * \Delta$ are $F_i \cup G_j$ for $i \in [n]$ and $j \in [m]$. We will show that $\bigsqcup_{i,j} [R(F_i) \cup R(G_j), F_i \cup G_j]$ is a partitioning of $\Gamma * \Delta$.

To see that $\bigsqcup_{i,j} [R(F_i) \cup R(G_j), F_i \cup G_j]$ covers $\Gamma * \Delta$, consider an arbitrary face $f \cup g \in \Gamma * \Delta$ with $f \in \Gamma$ and $g \in \Delta$. Then there exist $i \in [n]$ and $j \in [m]$ such that $f \in [R(F_i), F_i]$ and $g \in [R(G_i), G_i]$.

Then clearly $R(F_i) \cup R(G_i) \subseteq f \cup g \subseteq F_i \cup G_i$.

To see that the intervals are indeed disjoint, suppose $f \cup g \in [R(F_i) \cup R(G_j), F_i \cup G_j] \cap [R(F_i') \cup R(G_j'), F_i' \cup G_j']$. Then $f \in [R(F_i), F_i] \cap [R(F_i'), F_i']$ and $g \in [R(G_j), G_j] \cap [R(G_j'), G_j']$. This forces $F_i = F_i'$ and $G_j = G_j'$ by partitionability of Γ and Δ. □

Proposition 6.4.3. *All stars of a partitionable simplicial complex are partitionable.*

Proof. This follows from Propositions 6.4.1 and 6.4.2. □

Proposition 6.4.4. *Stellar subdivisions preserve partitionability.*

Proof. Let Δ be a partitionable simplicial complex with a partitioning $\bigsqcup_i [R(F_i), F_i]$. Let σ be a nonempty face of Δ and consider the stellar subdivision at σ by introducing a new vertex a in the interior of σ. We would like to show that $\Delta' := \Delta \setminus \mathrm{st}_\Delta \sigma \cup (a * \partial\sigma * \mathrm{lk}_\Delta \sigma)$ is partitionable. The intersection of st_Δ and $a * \partial\sigma * \mathrm{lk}_\Delta \sigma$ is $\partial\sigma * \mathrm{lk}_\Delta \sigma$.

Suppose that an interval $[R(F_i), F_i]$ contains facets $f_1, \ldots, f_k$ of $\partial\sigma * \mathrm{lk}_\Delta \sigma$. Observe that

$$[R(F_i), F_i]_{f_1,\ldots,f_k} := \{\tau \in [R(F_i), F_i] : \tau \not\subseteq f_1 \cup \cdots \cup f_k\}$$

(i.e., $[R(F_i), F_i]_{f_1,\ldots,f_k}$ is obtained from $[R(F_i), F_i]$ by removing the order ideal generated by $f_1, \ldots, f_k$ in the interval) is also a boolean interval. Therefore, for every facet $F_i \notin \mathrm{st}_\Delta \sigma$, let $[R(F_i), F_i]' = [R(F_i), F_i]_{f_1,\ldots,f_k}$ if F_i contains facets $f_1, \ldots, f_k$ of $\partial\sigma * \mathrm{lk}_\Delta \sigma$, and $[R(F_i), F_i]' = [R(F_i), F_i]$ otherwise.

By Propositions 6.4.1 and 6.4.2, $a * \partial\sigma * \mathrm{lk}_\Delta \sigma$ is partitionable (since $a * \partial\sigma$ is shellable), say with a partitioning $\bigsqcup_j p_j$, then

$$\bigsqcup_{F_i \notin \mathrm{st}_\Delta \sigma} [R(F_i), F_i]' \sqcup \bigsqcup_j p_j$$

is a partitioning of Δ'. □

On the other hand, stellar welds do not preserve partitionability. To see this, consider the non-partitionable complex C_3 constructed in [DGKM16, Theorem 3.5]. It is obtained by gluing three copies of a shellable 3-ball X along a shellable 2-ball $A \subseteq \partial X$. By [Lic99, Theorem 3.6], there exists a sequence of stellar moves at interval faces that transforms X into $v * \partial X$, where v is a new vertex. Let C_3' be the complex obtained by gluing three copies of $v * \partial X$ along A. In particular, C_3' can be obtained from C_3 by performing the same sequence of stellar moves on all three copies of X.

Moreover, C_3' is shellable. Indeed, all 2-dimensional spheres are extendably shellable [DK78], which means there exists a shelling of ∂X that starts with all facets of A, or ends with all facets of A. This gives a shelling of C_3', and hence C_3' is partitionable. Since C_3' is partitionable while C_3 is not, we conclude that stellar welds do not preserve partitionability, as we have shown that stellar subdivisions do.

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