BROWNIAN MOTION IN A BROWNIAN CRACK

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Abstract. Let D be the Wiener sausage of width ε around two-sided Brownian motion. The components of 2-dimensional reflected Brownian motion in D converge to 1-dimensional Brownian motion and iterated Brownian motion, resp., as ε goes to 0.

1. Introduction. Our paper is concerned with a model for a diffusion in a crack. This should not be confused with the "crack diffusion model" introduced by Chudnovsky and Kunin (1987) which proposes that cracks have the shape of a diffusion path. The standard Brownian motion is the simplest of models proposed by Chudnovsky and Kunin. An obvious candidate for a "diffusion in a Brownian crack" is the "iterated Brownian motion" or IBM (we will define IBM later in the introduction). The term IBM has been coined in Burdzy (1993) but the idea is older than that. See Burdzy (1993, 1994) and Khoshnevisan and Lewis (1997) for the review of literature and results on IBM. The last paper is the only article known to us which considers the problem of diffusion in a crack; however the results in Khoshnevisan and Lewis (1997) have considerably different nature from our own results.

There are many papers devoted to diffusions on fractal sets, see, e.g., Barlow (1990), Barlow and Bass (1993, 1997) and references therein. Diffusions on fractals are often constructed by an approximation process, i.e., they are first constructed on an appropriate ε-enlargement of the fractal set and then a limit is obtained as the width ε goes to 0. This procedure justifies the claims of applicability of such models as the real sets are likely to be "fat" approximations to ideal fractals. The purpose of this article is to provide a similar justification for the use of IBM as a model for the Brownian motion in a Brownian crack.

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We will show that the two components of the reflected 2-dimensional Brownian motion in a Wiener sausage of width ε converge to the usual Brownian motion and iterated Brownian motion, resp., when $\varepsilon \to 0$.

It is perhaps appropriate to comment on the possible applications of our main result. The standard interpretation of the Brownian motion with reflection is that of a diffusing particle which is trapped in a set, i.e., crack. However, the transition probabilities for reflected Brownian motion represent also the solution for the heat equation with the Neumann boundary conditions. Our Theorem 1 may be interpreted as saying that inside very narrow cracks, the solution of the Neumann heat equation is well approximated by the solution of the heat equation on the straight line which is projected back onto the crack. We leave the proof of this claim to the reader.

We proceed with the rigorous statement of our main result. We define "two-sided" Brownian motion by

$$X^1(t) = \begin{cases} X^+(t) & \text{if } t \geq 0, \\ X^-(-t) & \text{if } t < 0, \end{cases}$$

where X^+ and X^- are independent standard Brownian motions starting from 0. For $\varepsilon > 0$ and a continuous function $\eta : \mathbf{R} \to \mathbf{R}$ let

$$D = D(\eta, \varepsilon) = \{(x, y) \in \mathbf{R}^2 : y \in (\eta(x) - \varepsilon, \eta(x) + \varepsilon)\}.$$

Note that D is open for every continuous function η. Let $\mathcal{Y}_t = \mathcal{Y}_t^\varepsilon$, $\mathcal{Y}_0 = (0, 0)$, be the reflected 2-dimensional Brownian motion in $D(X^1, \varepsilon)$ with normal vector of reflection (the construction of such a process is discussed in Section 2).

We will identify $\mathbf{R}^2$ with the complex plane $\mathbf{C}$ and switch between real and complex notation.

Informally speaking, our main result is that $\operatorname{Re} \mathcal{Y}(c(\varepsilon)\varepsilon^{-2}t)$ converges in distribution to a Brownian motion independent of X^1, where the constants $c(\varepsilon)$ are uniformly bounded away from 0 and ∞ for all $\varepsilon > 0$.

Suppose that X^2 is a standard Brownian motion independent of X^1. The process $\{X(t) \stackrel{\text{df}}{=} X^1(X^2(t)), t \geq 0\}$ is called an "iterated Brownian motion" (IBM). Let ϱ be a metric on $C[0, \infty)$ corresponding to the topology of uniform convergence on compact intervals.

Theorem 1. *One can construct X^1, X^2 and $\mathcal{Y}^\varepsilon$ for every $\varepsilon > 0$ on a common probability space so that the following holds. There exist $c_1, c_2 \in (0,\infty)$ and $c(\varepsilon) \in (c_1, c_2)$ such that the processes $\{\mathrm{Re}\,\mathcal{Y}^\varepsilon(c(\varepsilon)\varepsilon^{-2}t), t \geq 0\}$ converge in metric ϱ to $\{X^2_t, t \geq 0\}$ in probability as $\varepsilon \to 0$. It follows easily that the processes $\{\mathrm{Im}\,\mathcal{Y}^\varepsilon(c(\varepsilon)\varepsilon^{-2}t), t \geq 0\}$ converge in metric ϱ to the iterated Brownian motion $\{X_t, t \geq 0\}$ in probability as $\varepsilon \to 0$.*

We would like to state a few open problems.

(i) There are alternative models for cracks, see, e.g., Kunin and Gorelik (1991). For which processes X^1 besides Brownian motion a result analogous to Theorem 1 holds?

(ii) Can one construct $\mathcal{Y}^\varepsilon$ so that the convergence in Theorem 1 holds in almost sure sense rather than in probability?

(iii) Let $B(x,r) = \{y \in \mathbf{R}^2 : |x-y| < r\}$ and $\widetilde{D}(X^1,\varepsilon) = \bigcup_{t\in\mathbf{R}} B((t, X^1_t), \varepsilon)$. Does Theorem 1 hold for $\widetilde{D}$ in place of D?

(iv) Can Brownian motion on the Sierpiński gasket be constructed as a limit of reflected Brownian motions on ε-enlargements of the state space? The standard construction of this process uses random walk approximations, see, e.g., Barlow and Perkins (1988), Goldstein (1987) and Kusuoka (1987).

Let us outline the main idea of the proof of Theorem 1. Although the Wiener sausage has a constant width ε in the vertical direction, the width is not constant from the point of view of the reflected Brownian motion $\mathcal{Y}$. Large increments of X^1 over short intervals produce narrow spots along the crack. We will relate the effective width of the crack to the size of X^1 increments and show that the narrow parts of the crack appear with large regularity due to the independence of X^1 increments. This gives the proof the flavor of a random homogenization problem. Our main technical tool will be the Riemann mapping as the reflected Brownian motion is invariant under conformal mappings.

The proof of Theorem 1 consists of a large number of lemmas. The first few lemmas deal with non-random domains $D(\eta,\varepsilon)$ and their distortion under conformal mappings onto the strip $\{z \in \mathbf{C} : \mathrm{Im}\, z \in (-1,1)\}$. The next set of lemmas provides more information of the same kind, but more accurate and more specific as this time we examine the distortion of random Wiener sausages $D(X^1,\varepsilon)$. The lemmas mentioned above provide information on the hitting probabilities for $\mathcal{Y}^\varepsilon$ which is analogous to the scale function for diffusions on the line. Next, we obtain some information analogous to the speed measure. This is done in another set of lemmas analyzing how much time the process spends in small sections of $D(X^1, \varepsilon)$. Finally, a few more lemmas and the final piece of the proof put the estimates together to show that the process $\mathcal{Y}^\varepsilon$ behaves very much like the simple symmetric random walk on a certain scale.

We would like to point out that the usually convenient "Brownian scaling" arguments cannot be applied in many of our proofs. The Brownian scaling requires a different scaling of the time and state space coordinates. In our model time and space for X^1 represent two directions in space for $\mathcal{Y}$. In other words, the versions of Brownian scaling for X^1 and $\mathcal{Y}$ are incompatible.

The paper has two more sections. Section 2 contains a sketch of the construction of the reflected Brownian motion $\mathcal{Y}$. The lemmas comprising the proof of Theorem 1 are presented in Section 3.

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2. Preliminaries. First we will sketch a construction of the reflected Brownian motion $\mathcal{Y}$ in D. We start by introducing some notation.

Since we will deal with one- and two-dimensional Brownian motions, we will sometimes write "1-D" or "2-D Brownian motion" to clarify the statements.

The following definitions apply to $D(\eta, \varepsilon)$ for any η, not necessarily X^1. The resulting objects will depend on η, of course. Let $D_* = \{z \in \mathbf{C} : \operatorname{Im} z \in (-1, 1)\}$. The Carathéodory prime end boundaries of D and D_* contain points at $-\infty$ and $+\infty$ which are defined in the obvious way. Let f be the (unique) analytic one-to-one function mapping D_* onto D such that $f(\infty) = \infty, f(-\infty) = -\infty$ and $f((0, -1)) = (0, -\varepsilon)$.

We owe the following remarks on the construction of $\mathcal{Y}$ to Zhenqing Chen.

It is elementary to construct a reflected Brownian motion (RBM) $\mathcal{Y}^*$ in D_*. We will argue that $f(\mathcal{Y}^*_t)$ is an RBM in D, up to a time-change.

First, the RBM on a simply connected Jordan domain D can be characterized as the continuous Markov process associated with $(H^1(D), E)$ in $L^2(D, dz)$, which is a regular Dirichlet form on $\bar{D}$, where $E(f,g) = 1/2 \int_D \nabla f \nabla g dx$ (see Remark 1 to Theorem 2.3 of Chen (1992)). We note that if $(H^1(D), E)$ is regular on $\bar{D}$ then the associated process is the unique continuous strong Markov process in $\bar{D}$ that is reversible with respect to the Lebesgue measure, behaves like planar Brownian motion in D and spends zero time on the boundary.

The Dirichlet form for the process $f(\mathcal{Y}^*_t)$ under the reference measure $|f'(z)|^2 dz$ is $(H^1(f^{-1}(D)), E)$. Therefore $f(\mathcal{Y}^*_t)$ is a time change of the RBM on D. The last assertion follows from a Dirichlet form characterization of time-changed processes due to Silverstein and Fitzsimmons (see Theorem 8.2 and 8.5 of Silverstein (1974); the proof seems to contain a gap; see Fitzsimmons (1989) for a correct proof; see also Theorem 6.2.1 in Fukushima, Oshima and Takeda (1994)).

We will denote the time-change by κ, i.e., $\mathcal{Y}(\kappa(t)) = f(\mathcal{Y}^*(t))$. Unless indicated otherwise we will assume that $\mathcal{Y}^*_0 = (0,0)$.

The above argument provides a construction of an RBM in $D(\eta, \varepsilon)$ for fixed (non-random) η. However, we need a construction in the case when η is random, i.e., $D = D(X^1, \varepsilon)$.

Let $\{S_n, -\infty \leq n \leq \infty\}$ be two-sided simple random walk on integers and let $\{S_t, -\infty \leq t \leq \infty\}$ be a continuous extension of S_n to all reals which is linear on all intervals of the form $[j, j+1]$. Next we renormalize S_t to obtain processes $S_{kt}/\sqrt{k}$ which converge to X^1 in distribution as $k \to \infty$. Let U^k_t agree with $S_{kt}/\sqrt{k}$ on $[-k, k]$ and let U^k_t be constant on intervals $(-\infty, -k]$ and $[k, \infty)$. Note that the number of different paths of U^k_t is finite. Fix an $\varepsilon > 0$. We construct an RBM $\mathcal{Y}^k$ in $D(U^k_t, \varepsilon)$ by repeating the construction outlined above for every possible path of U^k_t. We can view $(U^k, \mathcal{Y}^k)$ as a random element with values in $C(-\infty, \infty) \times C[0, \infty)^2$. It is possible to show that when η_k converge to η uniformly on compact subsets of $\mathbf{R}$ then RBM's in $D(\eta_k, \varepsilon)$ starting from the same point converge in distribution to an RBM in $D(\eta, \varepsilon)$. It follows that when $k \to \infty$ then $(U^k, \mathcal{Y}^k)$ converges in distribution to a process whose first coordinate has the distribution of X^1, i.e., two-sided Brownian motion. The distribution of the second component under the limiting measure is that of reflected Brownian motion in $D(X^1, \varepsilon)$.

The above discussion also serves as a construction of a regular conditional probability for $\mathcal{Y}_t$ given X^1. Alternatively, it is possible to show that the law of the reflected Brownian motion in $D(\eta, \varepsilon)$ is measurable as a function of η using the fact that the mapping f is a continuous function of η. We leave the proof to the reader.

We will use letters P and E to denote probabilities and expectations for RBM in D when the function X^1 is fixed. In other words, P and E denote the conditional distribution and expectation given X^1. They will be also used in the situations when we consider a domain $D(\eta, \varepsilon)$ with non-random η. We will use $\mathcal{P}$ and $\mathcal{E}$ to denote the distribution and expectation corresponding to the probability space on which X^1 and $\mathcal{Y}$ are defined. In other words, $\mathcal{E}$ applied to a random element gives us a number while E applied to a random element results in a random variable measurable with respect to X^1.

We will write $P^z_{\mathcal{Y}}$ to denote the distribution of the process $\mathcal{Y}$ starting from $\mathcal{Y}_0 = z$. Analogous notation will be used for other processes. We will write P^z if the process is clear from the context.

For the convenience of the reader we state here a version of a lemma proved in Burdzy, Toby and Williams (1989) which is easily applicable in our argument below. The notation of the original statement was tailored for the original application and may be hard to understand in the present context.

Lemma 1. (Burdzy, Toby and Williams (1989)) *Suppose that functions $h(x,y), g(x,y)$ and $h_1(x,y)$ are defined on product spaces $W_1 \times W_2$, $W_2 \times W_3$ and $W_1 \times W_3$, resp. Assume that for some constant $c_1, c_2 \in (0,1)$ the functions satisfy for all $x, y, x_1, x_2, y_1, y_2, z_1, z_2$,*

$$h_1(x,y) = \int_{W_2} h(x,z) g(z,y) dz,$$

$$\frac{h(x_1,z_1)}{h(x_1,z_2)} \geq \frac{h(x_2,z_1)}{h(x_2,z_2)}(1-c_1),$$

and

$$\frac{g(z_1,y_1)}{g(z_1,y_2)} \geq c_2 \frac{g(z_2,y_1)}{g(z_2,y_2)}.$$

Then

$$\frac{h_1(x_1,y_1)}{h_1(x_1,y_2)} \geq \frac{h_1(x_2,y_1)}{h_1(x_2,y_2)}(1-c_1+c_2^2 c_1).$$

Proof. All we have to do is to translate the original statement to our present notation. The symbols on the left hand side of the arrow appeared in Lemma 6.1 of Burdzy, Toby and Williams (1989). The symbols on the right hand side of the arrow are used in the present version of the lemma.

$$b \longmapsto 1, \qquad U \longmapsto \emptyset, \qquad k \longmapsto x_1, \qquad 3-k \longmapsto x_2, \qquad v \longmapsto z_1, \qquad w \longmapsto z_2,$$
$$x \longmapsto y_1, \qquad y \longmapsto y_2, \qquad f \longmapsto h, \qquad g \longmapsto g, \qquad h \longmapsto h_1. \quad \square$$

For a set A, $\tau(A)$ and $T(A)$ will denote the exit time from and the hitting time of a set A for 2-D Brownian motion or reflected Brownian motion.

3. Proof of the main result. The proof of Theorem 1 will consist of several lemmas.

Recall the definitions of D_* and f from Section 2. Let $M_*(a,b) = \{z \in D_* : a \leq \operatorname{Re} z \leq b\}$, $M_*(a) = M_*(a,a)$, $M(a,b) = f(M_*(a,b))$ and $M(a) = f(M_*(a))$. It is easy to show that f and f^{-1} can be extended in a continuous fashion to the closures of their domains.

The closure of the part of D between $M(a)$ and $M(a+1)$ will be called a *cell* and denoted $C(a)$. The closure of the part of D between the lines $\{z : \operatorname{Re} z = a\}$ and $\{z : \operatorname{Re} z = b\}$ will be called a *worm* and denoted $W(a,b)$. Let $W_*(a,b) = f^{-1}(W(a,b))$. The degenerate worm $W(a,a)$ will be denoted $W(a)$, i.e., $W(a) = \{z \in D : \operatorname{Re} z = a\}$.

The first three lemmas in this section deal with domains $D(\eta, \varepsilon)$ for arbitrary continuous functions η.

Lemma 2. *There exists $c < \infty$ independent of η and f such that the diameter of the cell $C(a)$ is less than $c\varepsilon$ for every a. In particular, the diameter of $M(a)$ is less than $c\varepsilon$ for every a.*

Proof. It is easy to see that it will suffice to prove the lemma for $a = 0$. Moreover, we will assume that $\varepsilon = 1$ because the general case follows by scaling.

We start by proving the second assertion of the lemma.

Let $\partial^- M_*(a,b)$ be the part of $\partial M_*(a,b)$ on the line $\{\operatorname{Im} z = -1\}$ and let $\partial^- M(a,b) = f(\partial^- M_*(a,b))$. Consider a point $z \in M_*(0)$ with $\operatorname{Im} z \le 0$. 2-D Brownian motion starting from z has no less than $1/4$ probability of exiting D_* through $\partial^- M_*(-\infty, 0)$. By conformal invariance, Brownian motion starting from $f(z)$ has no less than $1/4$ probability of exiting D through $\partial^- M(-\infty,0)$. Find large $c_1 < \infty$ so that the probability that Brownian motion starting from $f(z)$ will make a closed loop around $B(f(z),2)$ before exiting $B(f(z),c_1)$ is greater than $7/8$. Suppose for a moment that $\operatorname{Re} f(z) \ge 0$ and the distance from $f(z)$ to $(0,-1)$ is greater than c_1+2. Then $\partial B(f(z),c_1)$ does not intersect $W(0)$, and so a closed loop around $B(f(z),2)$ which does not exit $B(f(z),c_1)$ has to intersect $\partial D \setminus \partial^- M(-\infty,0)$ before hitting $\partial^- M(-\infty,0)$. In order to avoid contradiction we have to conclude that $|f(z)-f((0,-1))| = |f(z)-(0,-1)| \le c_1+2$ in the case $\operatorname{Re} f(z) \ge 0$. The case $\operatorname{Re} f(z) \le 0$ can be treated in the similar way and we conclude that the diameter of $f(\{z \in M_*(0) : \operatorname{Im} z \le 0\})$ cannot be greater than $2(c_1+2)$. By symmetry, the diameter of the other part of $M(0)$ is bounded by the same constant and so the diameter of $M(0)$ cannot be greater than $4(c_1+2)$.

The proof of the first assertion follows along similar lines. Consider a point z in the interval $I \stackrel{\text{df}}{=} \{z : \operatorname{Im} z = 0, 0 \le \operatorname{Re} z \le 1\}$. There exists $p > 0$ such that Brownian motion starting from z will hit $M_*(0)$ before hitting ∂D_* with probability greater than p for all $z \in I$. By conformal invariance, Brownian motion starting from $f(z)$ will hit $M(0)$ before hitting ∂D with probability greater than p. Find large $c_2 < \infty$ so that the probability that Brownian motion starting from $f(z)$ will make a closed loop around $B(f(z),2)$ before exiting $B(f(z),c_2)$ is greater than $1-p/2$. If the distance from $f(z)$ to $M(0)$ is greater than c_2 then such a loop would intersect ∂D before hitting $M(0)$. This would contradict our previous bound on hitting $M(0)$ so we conclude that the distance from $f(z)$ to $M(0)$ is bounded by c_2. Since the diameter of $M(0)$ is bounded by $4(c_1+2)$, we see that $4(c_1+2)+2c_2$ is a bound for the diameter of $f(I)$. The cell $C(0)$ is the union of sets $M(a)$ for $0 \le a \le 1$, the diameter of each of these sets is bounded by $4(c_1+2)$ and each of these sets intersects $f(I)$. Hence, the diameter of $C(0)$ is bounded by $12(c_1+2)+2c_2$. □

Lemma 3. *For every $\delta > 0$ there exists $0 < c < \infty$ with the following property. Suppose* *that* $\eta_1, \eta_2 : \mathbf{R} \to \mathbf{R}$ *are continuous and* $\eta_1(x) = \eta_2(x)$ *for all* $x \geq -c\varepsilon$. *Then* $M^{\eta_2}(-\infty, a-\delta) \subset M^{\eta_1}(-\infty, a)$ *for all* $a \geq 0$.

Proof. We will assume that $\varepsilon = 1$. The general case follows by scaling.

Step 1. First we show that for an arbitrary $b > -\infty$ we can find $c_1 > -\infty$ (depending on b but independent of η) so that $W(-\infty, c_1) \subset M(-\infty, b)$.

Suppose that $W_*(c_2)$ intersects $M_*(b_1)$ for some $b_1 > b$ and $c_2 \in \mathbf{R}$. Planar Brownian motion starting from $(0,0)$ can hit the line segment $I = \{z : \operatorname{Im} z = 0, b-2 < \operatorname{Re} z < b-1\}$ before exiting D_* with probability $p_0 = p_0(b) > 0$. A Brownian particle starting from any point of I can first make a closed loop around I and then exit D_* through the upper part of the boundary before hitting $M_*(b-3)$ or $M_*(b)$ with probability $p_1 > 0$. The same estimate holds for a closed loop around I exiting through the lower part of the boundary. Since $b_1 > b$ and $W_*(c_2) \cap M_*(b_1) \neq \emptyset$, either a closed loop around I exiting D_* through the upper part of ∂D_* must cross $W_*(-\infty, c_2)$ or this is true for the analogous path exiting through the lower part of ∂D_*. We see that the 2-D Brownian motion starting from $(0,0)$ can hit $W_*(-\infty, c_2)$ before exiting D_* with probability greater than $p_0 p_1 > 0$.

By conformal invariance of planar Brownian motion, it will suffice to show that for sufficiently large negative c_2, Brownian motion starting from $f(0,0)$ cannot hit $W(-\infty, c_2)$ before exiting D with probability greater than $p_0 p_1/2$.

By Lemma 2 the diameter of $M(0)$ is bounded by $c_3 < \infty$. Since $f(0,0) \in M(0)$ and $M(0) \cap W(0) \neq \emptyset$, we have $\operatorname{Re} f(0,0) > -c_3$. Find large $c_4 < \infty$ so that the probability that Brownian motion starting from $f(0,0)$ will make a closed loop around $B(f(0,0), 2)$ before exiting $B(f(0,0), c_4)$ is greater than $1 - p_0 p_1/2$. If such a loop is made, the process exits D before hitting $W(-\infty, -c_3 - c_4)$. It follows that Brownian motion starting from $f(0,0)$ cannot hit $W(-\infty, -c_3 - c_4)$ before exiting D with probability greater than $p_0 p_1/2$. This completes the proof of Step 1.

Step 2. In this step, we will prove the lemma for $a = 0$. Recall the notation $\partial^- M(a,b)$ and $\partial^- M_*(a,b)$ from the proof of Lemma 2.

Reflected Brownian motion (RBM) in D_* starting from any point of $M_*(0)$ has a fifty-fifty chance of hitting $\partial^- M_*(-\infty, 0)$ before hitting $\partial^- M_*(0, \infty)$. By conformal invariance, RBM in $D(\eta_1, 1)$ starting from any point of $M(0)$ can hit $\partial^- M^{\eta_1}(-\infty, 0)$ before hitting $\partial^- M^{\eta_1}(0, \infty)$ with probability 1/2.

It follows from Lemma 2 that $M^\eta(0) \subset W^\eta(c_0, \infty)$ and $W_*^\eta(c_0, \infty) \subset M_*^\eta(c', \infty)$ for some $c_0, c' > -\infty$ and all η. An argument similar to that in the proof of Lemma 2 or Step 1 of this proof easily shows that for any $z \in M_*^\eta(c', \infty)$, and so for any $z \in W_*^\eta(c_0, \infty)$, the probability that RBM in D_* starting from z can hit $M_*^\eta(b)$ before hitting $\partial^- M_*^\eta(0, \infty)$ is bounded from above by $p_2(b)$ such that $p_2(b) \to 0$ when $b \to -\infty$.

Fix an arbitrarily small $\delta > 0$. It is easy to see that for all $v \in M_*(-\delta)$ we have

$$P_{\mathcal{Y}_*}^v(T(\partial^- M_*(-\infty, 0)) < T(\partial^- M_*(0, \infty))) > 1/2 + p_3, \tag{1}$$

where $p_3 = p_3(\delta) > 0$. Next find b such that $p_2(b) < p_3(\delta)$. Find $c_1 < c_0$ (independent of η_2) such that $W^{\eta_2}(-\infty, c_1) \subset M^{\eta_2}(-\infty, b)$, as in Step 1. Suppose that η_1 and η_2 agree on the interval $[c_1, \infty)$. This assumption and the fact that $M^{\eta_1}(0) \subset W^{\eta_1}(c_0, \infty)$ yield $M^{\eta_1}(0) \subset W^{\eta_2}(c_0, \infty)$. The conformal invariance of RBM implies that for the RBM $\mathcal{Y}_*$ in D_*, RBM $\mathcal{Y}$ in $D(\eta_2, 1)$ and any $z \in M^{\eta_1}(0)$ and $v = f_{\eta_2}^{-1}(z)$ we have

$$\begin{aligned}
&P_{\mathcal{Y}_*}^v(T(\partial^- M_*(-\infty, 0)) < T(\partial^- M_*(0, \infty))) \\
&\quad = P_{\mathcal{Y}}^z(T(\partial^- M^{\eta_2}(-\infty, 0)) < T(\partial^- M^{\eta_2}(0, \infty))) \\
&\quad \le P_{\mathcal{Y}}^z(T(\partial^- M^{\eta_2}(-\infty, 0)) < T(\partial^- M^{\eta_2}(0, \infty)) < T(W^{\eta_2}(c_1))) \\
&\qquad + P_{\mathcal{Y}}^z(T(W^{\eta_2}(c_1)) < T(\partial^- M^{\eta_2}(0, \infty))).
\end{aligned}$$

Since $f_{\eta_1}((0, -1)) = (0, -1)$, and the same holds for f_{η_2}, we have $\partial^- M^{\eta_2}(-\infty, 0) = \partial^- M^{\eta_1}(-\infty, 0)$ and $\partial^- M^{\eta_2}(0, \infty) = \partial^- M^{\eta_1}(0, \infty)$. It follows that the last displayed formula is equal to

$$\begin{aligned}
&P_{\mathcal{Y}}^z(T(\partial^- M^{\eta_1}(-\infty, 0)) < T(\partial^- M^{\eta_1}(0, \infty)) < T(W^{\eta_2}(c_1))) \\
&\qquad + P_{\mathcal{Y}}^z(T(W^{\eta_2}(c_1)) < T(\partial^- M^{\eta_2}(0, \infty))).
\end{aligned}$$

Since $W^{\eta_2}(-\infty, c_1) \subset M^{\eta_2}(-\infty, b)$, this is less or equal to

$$\begin{aligned}
&P_{\mathcal{Y}}^z(T(\partial^- M^{\eta_1}(-\infty, 0)) < T(\partial^- M^{\eta_1}(0, \infty)) < T(W^{\eta_2}(c_1))) \\
&\qquad + P_{\mathcal{Y}}^z(T(M^{\eta_2}(b)) < T(\partial^- M^{\eta_2}(0, \infty))).
\end{aligned}$$

Let $\mathcal{Y}'$ denote the RBM in $D(\eta_1, 1)$. In view of our assumption that η_1 and η_2 agree on the interval $[c_1, \infty)$, the last quantity is equal to

$$\begin{aligned}
&P^z_{\mathcal{Y}'}(T(\partial^- M^{\eta_1}(-\infty,0)) < T(\partial^- M^{\eta_1}(0,\infty)) < T(W^{\eta_2}(c_1))) \\
&\qquad + P^z_{\mathcal{Y}}(T(M^{\eta_2}(b)) < T(\partial^- M^{\eta_2}(0,\infty))) \\
&\quad \le P^z_{\mathcal{Y}'}(T(\partial^- M^{\eta_1}(-\infty,0)) < T(\partial^- M^{\eta_1}(0,\infty))) \\
&\qquad + P^z_{\mathcal{Y}}(T(M^{\eta_2}(b)) < T(\partial^- M^{\eta_2}(0,\infty))) \\
&\quad = 1/2 + P^z_{\mathcal{Y}}(T(M^{\eta_2}(b)) < T(\partial^- M^{\eta_2}(0,\infty))).
\end{aligned}$$

We have $z \in M^{\eta_1}(0) \subset W^{\eta_2}(c_0, \infty)$, so the last probability is bounded by $p_2(b)$. By retracing our steps we obtain the following inequality

$$P^v_{\mathcal{Y}_*}(T(\partial^- M_*(-\infty,0)) < T(\partial^- M_*(0,\infty))) \le 1/2 + p_2(b) < 1/2 + p_3(\delta).$$

This and (1) imply that $f^{-1}_{\eta_2}(M^{\eta_1}(0))$ lies totally to the right of $M_*(-\delta)$. This is equivalent to $M^{\eta_2}(-\infty, -\delta) \subset M^{\eta_1}(-\infty, 0)$. We have proved the lemma for $a = 0$.

Step 3. We will extend the result to all $a \ge 0$ in this step.

The mapping $g \stackrel{\text{df}}{=} f^{-1}_{\eta_2} \circ f_{\eta_1}$ is a one-to-one analytic function from D_* onto itself. We have proved that $g(M_*(0,\infty)) \subset M_*(-\delta, \infty)$. In order to finish the proof of the lemma it will suffice to show that $g(M_*(a,\infty)) \subset M_*(a-\delta, \infty)$ for every $a \ge 0$.

Let $h(z) \stackrel{\text{df}}{=} g(z) + \delta$. It will suffice to show that $h(M_*(a,\infty)) \subset M_*(a,\infty)$ for $a \ge 0$. We already know that $h(M_*(0,\infty)) \subset M_*(0,\infty)$. Consider any two points $z \in M_*(a)$ and $v \in M_*(-\infty, a) \setminus M_*(a)$. It will suffice to show that $v \ne h(z)$. Let u be such that $\operatorname{Re} u = \operatorname{Re} z$ and $\operatorname{Im} u = \operatorname{Im} v$. Suppose for a moment that $\operatorname{Im} v \ge \operatorname{Im} z$. Consider a planar Brownian motion $\mathcal{Y}$ starting from z. By comparing, path by path, the hitting times of $\partial^- M_*(0,\infty)$ and $\partial M_*(0,\infty)$ for the processes $\mathcal{Y}$ and $\mathcal{Y} + (u-z)$, and then for the processes $\mathcal{Y} + (u-z)$ and $\mathcal{Y} + (u-v)$, we arrive at the following inequalities,

$$\begin{aligned}
P^z(T(\partial^- M_*(0,\infty)) \le T(\partial M_*(0,\infty))) &\ge P^u(T(\partial^- M_*(0,\infty)) \le T(\partial M_*(0,\infty))) \\
&> P^v(T(\partial^- M_*(0,\infty)) \le T(\partial M_*(0,\infty))).
\end{aligned}$$

Since $h(M_*(0,\infty)) \subset M_*(0,\infty)$ and $h(\partial^- M_*(0,\infty)) \subset \partial^- M_*(0,\infty)$,

$$\begin{aligned}
P^v(T(\partial^- M_*(0,\infty)) \le T(\partial M_*(0,\infty))) &\ge P^v(T(h(\partial^- M_*(0,\infty))) \le T(\partial M_*(0,\infty))) \\
&\ge P^v(T(h(\partial^- M_*(0,\infty))) \le T(h(\partial M_*(0,\infty)))) \\
&= P^{h^{-1}(v)}(T(\partial^- M_*(0,\infty)) \le T(\partial M_*(0,\infty))).
\end{aligned}$$

Hence

$$P^z(T(\partial^- M_*(0,\infty)) \le T(\partial M_*(0,\infty))) > P^{h^{-1}(v)}(T(\partial^- M_*(0,\infty)) \le T(\partial M_*(0,\infty)))$$

and so $v \neq h(z)$. The case $\operatorname{Im} v \le \operatorname{Im} z$ can be treated in the same way. This completes the proof of Step 3 and of the entire lemma. □

Recall the definitions of cells and worms from the beginning of this section. Let $K^-(a_1, a_2)$ be the number of cells $C(k)$, $k \in \mathbf{Z}$, which lie in the worm $W(a_1, a_2)$ and let $K^+(a_1, a_2)$ be the number of cells which intersect the same worm.

The cell count is relative to the conformal mapping establishing equivalence of D_* and D. We will say that a conformal mapping $f : D_* \to D$ is *admissible* if $f(-\infty) = -\infty$ and $f(\infty) = \infty$.

Lemma 4. *There exists $c_1 < \infty$ with the following property. Suppose that $a_1 < a_2$ and $\eta_1, \eta_2 : \mathbf{R} \to \mathbf{R}$ are continuous functions such that $\eta_1(x) = \eta_2(x)$ for all $x \in [a_1 - c_1\varepsilon, a_2 + c_1\varepsilon]$. Then $K^-_{\eta_2}(a_1, a_2) \ge K^-_{\eta_1}(a_1, a_2) - 12$ and $K^+_{\eta_2}(a_1, a_2) \ge K^+_{\eta_1}(a_1, a_2) - 12$ where the cell counts are relative to any admissible mappings.*

Proof. Let c_2 be such that the cell diameter is bounded by $c_2\varepsilon$ (see Lemma 2). According to Lemma 3 we can find c_3 so large that if $\eta_3(x) = \eta_1(x)$ for $x > -c_3\varepsilon$ then $M^{\eta_3}(-\infty, a - 1) \subset M^{\eta_1}(-\infty, a)$ for $a \ge 0$ and similarly with the roles of η_1 and η_3 reversed.

We can assume without loss of generality that $\eta_1(a_1 - c_2\varepsilon - c_3\varepsilon) = \eta_2(a_1 - c_2\varepsilon - c_3\varepsilon)$. Let

$$\eta_3(x) = \begin{cases} \eta_2(x) & \text{for } x \le a_1 - c_2\varepsilon - c_3\varepsilon, \\ \eta_1(x) & \text{otherwise.} \end{cases}$$

The conformal functions f used to define conformal equivalence of D_* and $D(\eta, \varepsilon)$ in the proof of Lemma 3 have the property that $f((0, -1)) = (0, -\varepsilon)$, by assumption. In order to be able to apply Lemma 3 we introduce functions $f_n : D_* \to D(\eta_n, \varepsilon)$ with $f_n(\infty) = \infty$, $f(-\infty) = -\infty$ and $f_n((0, -1)) = (a_1 - c_2\varepsilon, \eta_n(a_1 - c_2\varepsilon) - \varepsilon)$. Lemma 3 now applies with a suitable shift. Let j_0 and j_1 be the smallest and largest integers k with the property $M(k) \cap W(a_1, a_2) \neq \emptyset$. Then $M(j_0, j_k) \subset W(a_1 - c_2\varepsilon, a_2 + c_2\varepsilon)$. Now we use Lemma 3 to see that

$$M^{\eta_1}(-\infty, j_0) \subset M^{\eta_3}(-\infty, j_0 + 1)$$

and

$$M^{\eta_3}(-\infty, j_1 - 1) \subset M^{\eta_1}(-\infty, j_1)$$

assuming that these sets are defined relative to f_n's. This implies that $K^+_{\eta_3}(a_1, a_2) \geq K^+_{\eta_1}(a_1, a_2) - 2$.

It is easy to see that by switching from the mapping f_n to any admissible mapping we change the number $K^+_{\eta_n}(a_1, a_2)$ by at most 2. Hence, $K^+_{\eta_3}(a_1, a_2) \geq K^+_{\eta_1}(a_1, a_2) - 6$ with the usual choice of conformal mappings.

The analogous estimate applies to η_3 and η_2, by the symmetry of the real axis. Thus $K^+_{\eta_2}(a_1, a_2) \geq K^+_{\eta_1}(a_1, a_2) - 12$.

The proof of the inequality $K^-_{\eta_3}(a_1, a_2) \geq K^-_{\eta_1}(a_1, a_2) - 12$ is completely analogous.

□

Before stating our next lemma we introduce some notation. Recall that $\{X^1(t), -\infty < t < \infty\}$ is a one-dimensional Brownian motion with $X^1(0) = 0$. Suppose $\varepsilon > 0$ and let $S_0 = 0$,

$$S_k = (S_{k-1} + \varepsilon^2/4) \wedge \inf\{t > S_{k-1} : |X^1(S_{k-1}) - X^1(t)| \geq \varepsilon/4\}, \qquad k \geq 1,$$

$$S_k = (S_{k-1} - \varepsilon^2/4) \vee \sup\{t < S_{k+1} : |X^1(S_{k+1}) - X^1(t)| \geq \varepsilon/4\}, \qquad k \leq -1.$$

Suppose that $b > 2\varepsilon^2$. Let $J = J(b, \varepsilon)$ be the smallest k such that $S_k \geq b$ and let

$$N_m = N_m(\varepsilon) = \sum_{k=0}^{m+1} (S_k - S_{k-1})^{-1}.$$

Lemma 5. *We have* $\mathcal{E} N_{J(b)}(\varepsilon)^k \leq c(k) b^k \varepsilon^{-4k}$ *for* $k \geq 1$.

Proof. First we will assume that $\varepsilon = 1$. Since $S_k - S_{k-1}$ is bounded by $1/4$, we have for $0 < \lambda < 1/2$,

$$\mathcal{E}\exp(-\lambda(S_k - S_{k-1})) \leq \mathcal{E}(1 - c_1\lambda(S_k - S_{k-1})) = 1 - c_1\lambda\mathcal{E}(S_k - S_{k-1}) \leq \exp(-c_2\lambda).$$

We see that for $\lambda \in (0, 1/2)$ and $n \geq 2$,

$$\begin{aligned}\mathcal{P}(J \geq n) &= \mathcal{P}(S_{n-1} < b) \leq e^{\lambda b}\mathcal{E}\exp(-\lambda S_{n-1}) \\ &= e^{\lambda b}\Big(\mathcal{E}\exp(-\lambda(S_k - S_{k-1}))\Big)^{n-1} \leq \exp(\lambda(b - (n-1)c_2)).\end{aligned}$$

Let $a = n/b$ and $\lambda = 1/b$ to see that for $a \geq 1/b$,

$$\mathcal{P}\big(J \geq ab\big) \leq c_3 \exp(-c_2 a). \tag{2}$$

For all $x > 4$,

$$\begin{aligned}
\mathcal{P}(1/(S_k - S_{k-1}) > x) &= \mathcal{P}(S_k - S_{k-1} < 1/x) \\
&\leq 2\mathcal{P}(\sup_{0 \leq t \leq 1/x} X(S_{k-1}+t) - X(S_{k-1}) > 1/4) \\
&\leq 4\mathcal{P}(X(S_{k-1}+1/x) - X(S_{k-1}) > 1/4) \\
&= 4\mathcal{P}(X(S_{k-1}+1) - X(S_{k-1}) > \sqrt{x}/4) \\
&\leq \frac{4}{\sqrt{2\pi x}} \exp(-x/32).
\end{aligned}$$

Therefore, $\mathcal{E}\exp(1/64(S_k - S_{k-1})) \leq e^{c_4}$ with $c_4 < \infty$. Hence for $m \geq 1$,

$$\begin{aligned}
\mathcal{P}\big(N_m \geq x) &\leq e^{-x/64}\mathcal{E}\exp(N_m/64) \leq e^{-x/64}(\mathcal{E}\exp(1/64(S_k - S_{k-1})))^m \\
&\leq e^{-x/64}e^{mc_4} = e^{-x/64+mc_4}.
\end{aligned} \tag{3}$$

Putting (2) and (3) together, for $a \geq 2/b$ and $y > 0$,

$$\begin{aligned}
\mathcal{P}\big(N_J \geq yb\big) &\leq \mathcal{P}\big(J \geq ab\big) + \mathcal{P}\big(N_{ab} \geq yb\big) \\
&\leq c_3 e^{-c_2 a} + e^{-yb/64 + c_4 ba}.
\end{aligned}$$

Let $a = (128c_4)^{-1}y$. Since a has to be bigger than $2/b$ and $b > 2$, the following estimate holds for $y > 128c_4$,

$$\mathcal{P}\big(N_J/b \geq y\big) \leq c_3 e^{-c_5 y} + e^{-yb/128} \leq c_3 e^{-c_5 y} + e^{-y/64}.$$

We conclude that for all $k, b \geq 1$,

$$\mathcal{E}\big(N_J\big)^k \leq c(k) b^k.$$

By Brownian scaling, the distribution of $N_{J(b)}(\varepsilon)$ is the same as $\varepsilon^{-2}N_{J(b\varepsilon^{-2})}(1)$. Hence,

$$\mathcal{E}N_{J(b)}(\varepsilon)^k = \varepsilon^{-2k}\mathcal{E}N_{J(b\varepsilon^{-2})}(1)^k \leq c(k)b^k\varepsilon^{-4k}. \quad \square$$

Let c_0 be the constant defined in Lemma 2, i.e., c_0 is such that the cell diameter is bounded by $c_0\varepsilon$. Recall that $K^-(a_1,a_2)$ $(K^+(a_1,a_2))$ is the number of cells $C(k)$, $k \in \mathbf{Z}$, which lie inside (intersect) the worm $W(a_1,a_2)$.

Lemma 6. *For $D = D(X^1,\varepsilon)$, $k \geq 1$ and sufficiently small $\varepsilon < (a_2-a_1)/(3c_0)$, we have,*

(i) $\mathcal{E}K^-(a_1,a_2) \geq c_1(a_2-a_1)\varepsilon^{-3}$,

(ii) $\mathcal{E}K^-(a_1,a_2)^{-k} \leq c(k)(a_2-a_1)^{-k}\varepsilon^{3k}$,

(iii) $\mathcal{E}K^+(a_1,a_2)^k \leq c(k)(a_2-a_1)^k\varepsilon^{-3k}$.

Proof. (i) Find $p>0$ with the following property. If z_1 and z_2 lie on the line $L \stackrel{\text{df}}{=} \{z : \operatorname{Im} z = 0\}$ and $|z_1-z_2| \leq 2$ then 2-D Brownian motion starting from z_1 can make a closed loop around z_2 before exiting D_* with probability greater than p. Let $c_2 < \infty$ be such that 2-D Brownian motion starting from z has more than $1-p/2$ chance of making a closed loop around $B(z,\varepsilon^2)$ before exiting $B(z,c_2\varepsilon^2)$. Note that c_2 may be chosen so that it does not depend on $\varepsilon>0$ and $z \in \mathbf{C}$.

Let $t_k = a_1 + c_0\varepsilon + k\varepsilon^2$ for $k \geq 0$. If the event

$$\mathcal{C}_k \stackrel{\text{df}}{=} \{X^1(t_{k+1}) > X^1(t_k) + 3\varepsilon\}$$

holds we let $b_j^k = X^1(t_k) + \varepsilon + jc_2\varepsilon^2$, $1 \leq j \leq 1/(c_2\varepsilon) - 2$. Let

$$\Gamma_j^k = \{z \in D : \operatorname{Im} z = b_j^k, \operatorname{Re} z \in (t_k, t_{k+1})\}.$$

Since $t_{k+1} - t_k = \varepsilon^2$, the diameter of Γ_j^k does not exceed ε^2. Choose a point z_j^k which belongs to $f(L) \cap \Gamma_j^k$. 2-D Brownian motion starting from z_j^k can make a closed loop around $B(z_j^k,\varepsilon^2)$ before exiting $B(z_j^k, c_2\varepsilon^2)$ with probability greater than $1-p/2$. Since the diameter of Γ_j^k is bounded by ε^2, the chance of making a closed loop around z_m^n before exiting D is less than $p/2$ assuming $(n,m) \neq (k,j)$. By conformal invariance, the probability that 2-D Brownian motion starting from $f^{-1}(z_j^k)$ will make a closed loop around $f^{-1}(z_m^n)$ before exiting D_* is less than p. Hence, $|f^{-1}(z_j^k) - f^{-1}(z_m^n)| > 2$ and so z_j^k and z_m^n belong to different cells. Therefore $K^+(a_1+c_0\varepsilon, a_2-c_0\varepsilon)$ is at least as big as the number of different points z_j^k for $0 \leq k \leq (a_2-a_1-2c_0\varepsilon)/\varepsilon^2 - 2$. Note that for small

ε we have $(a_2 - a_1 - 2c_0\varepsilon)/\varepsilon^2 - 2 \geq c_3(a_2 - a_1)/\varepsilon^2$. Since cell diameter is bounded by $c_0\varepsilon$ we have $K^-(a_1, a_2) \geq K^+(a_1 + c_0\varepsilon, a_2 - c_0\varepsilon)$ and so

$$K^-(a_1, a_2) \geq K^+(a_1 + c_0\varepsilon, a_2 - c_0\varepsilon) \geq \sum_{k=0}^{c_3(a_2-a_1)/\varepsilon^2} (1/(c_2\varepsilon) - 2)\mathbf{1}_{\mathcal{C}_k}.$$

Note that $\mathcal{P}(\mathcal{C}_k) = p_1$ for some absolute constant $p_1 > 0$ and so for small ε,

$$\mathcal{E}K^-(a_1, a_2) \geq p_1(1/(c_2\varepsilon) - 2)c_3(a_2 - a_1)\varepsilon^{-2} \geq c_4(a_2 - a_1)\varepsilon^{-3}.$$

(ii) Let $U_k = (1/(c_2\varepsilon) - 2)\mathbf{1}_{\mathcal{C}_k}$, $V_n = \sum_{k=1}^n U_k$ and $n_0 = c_3(a_2 - a_1)/\varepsilon^2$. Let us assume that ε is small so that $U_k > (c_5/\varepsilon)\mathbf{1}_{\mathcal{C}_k}$. Since U_k's are independent we have for $\lambda > 0$,

$$\mathcal{E}\exp(-\lambda V_n) = \Big(\mathcal{E}\exp(-\lambda U_k)\Big)^n \leq p_1^n \exp(-nc_5\lambda/\varepsilon) = \exp(nc_6 - nc_5\lambda/\varepsilon).$$

Hence, for $\lambda > 0$, $u \geq 2/(c_3c_5)$,

$$\begin{aligned}
\mathcal{P}((a_2 - a_1)\varepsilon^{-3}/V_{n_0} \geq u) &= \mathcal{P}(V_{n_0} \leq u^{-1}(a_2 - a_1)\varepsilon^{-3}) \\
&\leq \exp(\lambda u^{-1}(a_2 - a_1)\varepsilon^{-3})\mathcal{E}\exp(-\lambda V_{n_0}) \\
&\leq \exp(\lambda u^{-1}(a_2 - a_1)\varepsilon^{-3} + n_0c_6 - n_0c_5\lambda/\varepsilon) \\
&= \exp((a_2 - a_1)\varepsilon^{-2}(\lambda(u\varepsilon)^{-1} + c_3c_6 - c_3c_5\lambda/\varepsilon)) \\
&\leq \exp\Big((a_2 - a_1)\varepsilon^{-2}(c_3c_6 - c_3c_5\lambda/(2\varepsilon))\Big).
\end{aligned}$$

Now take

$$\lambda = \frac{2\varepsilon}{c_3c_5}\left(\frac{u}{(a_2 - a_1)\varepsilon^{-2}} + c_3c_6\right).$$

Then

$$(a_2 - a_1)\varepsilon^{-2}(c_3c_6 - c_3c_5\lambda/(2\varepsilon)) = -u$$

and so

$$\mathcal{P}((a_2 - a_1)\varepsilon^{-3}/V_{n_0} \geq u) \leq \exp(-u).$$

This implies that for $k \geq 1$,

$$\mathcal{E}K^-(a_1, a_2)^{-k} \leq \mathcal{E}V_{n_0}^{-k} \leq c_7(a_2 - a_1)^{-k}\varepsilon^{3k}.$$

(iii) Fix some $p_0 < 1$ such that for every $a \in \mathbf{R}$ and $z \in M_*(a)$, the probability that planar Brownian motion starting from z will hit $M_*(a-1) \cup M_*(a+1)$ before exiting from D_* is less than p_0. By conformal invariance of Brownian motion, the probability that Brownian motion starting from $z \in M(a)$ will hit $M(a-1) \cup M(a+1)$ before exiting from D is less than p_0.

We choose $\rho > 0$ so small that the probability that Brownian motion starting from any point of $B(x, r\rho)$ will make a closed loop around $B(x, r\rho)$ inside $B(x,r) \setminus B(x, r\rho)$ before exiting $B(x,r)$ is greater than p_0. Suppose that two Jordan arcs V_1 and V_2 have endpoints outside $B(x,r)$ and they both intersect $B(x, r\rho)$. Then Brownian motion starting from any point of $V_1 \cap B(x, r\rho)$ will hit V_2 before exiting $B(x,r)$ with probability greater than p_0. It follows that if $B(x,r) \subset D$ then either $M(a)$ or $M(a+1)$ must be disjoint from $B(x, r\rho)$.

We slightly modify the definition of S_k's considered in Lemma 4 by setting $S_0 = a_1 - c_0\varepsilon$. The rest of the definition remains unchanged, i.e.,

$$S_k = (S_{k-1} + \varepsilon^2/4) \wedge \inf\{t > S_{k-1} : |X^1(S_{k-1}) - X^1(t)| \geq \varepsilon/4\}, \qquad k \geq 1,$$

$$S_k = (S_{k+1} - \varepsilon^2/4) \vee \sup\{t < S_{k+1} : |X^1(S_{k+1}) - X^1(t)| \geq \varepsilon/4\}, \qquad k \leq -1.$$

We will connect points $(S_k, X^1(S_k))$, $k \geq 1$, by chains of balls. The chain of balls $B(x_j^k, r_j^k)$, $1 \leq j \leq m_k$, connecting $(S_k, X^1(S_k))$ and $(S_{k+1}, X^1(S_{k+1}))$ will have centers on the line segments

$$\{z : \operatorname{Im} z = X^1(S_k), S_k \leq \operatorname{Re} z \leq (S_k + S_{k+1})/2\},$$

$$\{z : \operatorname{Re} z = (S_k + S_{k+1})/2, \operatorname{Im} z \in [X^1(S_k), X^1(S_{k+1})]\},$$

$$\{z : \operatorname{Im} z = X^1(S_{k+1}), (S_k + S_{k+1})/2 \leq \operatorname{Re} z \leq S_{k+1}\}.$$

Here $[X^1(S_k), X^1(S_{k+1})]$ denotes the interval with endpoints $X^1(S_k)$ and $X^1(S_{k+1})$, not necessarily in this order since $X^1(S_{k+1})$ may be smaller than $X^1(S_k)$. Let J be the smallest k such that $S_k \geq a_2 + c_0\varepsilon$. It is elementary to check that we can choose the balls so that

(i) $B(x_j^k, r_j^k) \subset D$,

(ii) the set $\bigcup_{1 \leq j \leq m_k} B(x_j^k, r_j^k \rho)$ is connected and contains $(S_k, X^1(S_k))$ and $(S_{k+1}, X^1(S_{k+1}))$,

(iii) the radii r_j^k, $1 \leq j \leq m_k$, are not smaller than $\min_{m=k,k+1,k+2}(S_m - S_{m-1})/2$ and so

(iv) m_k is not greater than $(4\varepsilon/\rho)\max_{m=k,k+1,k+2} 1/(S_m - S_{m-1})$.

The total number $\widetilde{m}$ of balls needed to connect all points $(S_k, X^1(S_k)), 1 \le k \le J$, is bounded by

$$\sum_{k=1}^{J} m_k \le \frac{12\varepsilon}{\rho} \sum_{k=0}^{J+1} (S_k - S_{k-1})^{-1}.$$

In the notation of Lemma 5,

$$\widetilde{m} \le c_8 \varepsilon N_J(a_2 - a_1 + 2c_0\varepsilon, \varepsilon).$$

Since cell diameter is bounded by $c_0\varepsilon$ (Lemma 1), none of cells which touch $W(a_1, a_2)$ can extend beyond $W(a_1 - c_0\varepsilon, a_2 + c_0\varepsilon)$. Every curve $M(a)$ which lies totally inside $W(a_1 - c_0\varepsilon, a_2 + c_0\varepsilon)$ must intersect at least one ball $B(x_j^k, r_j^k \rho)$, $1 \le k \le J$, $1 \le j \le m_k$. Since $M(a)$ and $M(a+1)$ cannot intersect the same ball $B(x_j^k, r_j^k\rho)$ it follows that $K^+(a_1, a_2)$ is bounded by $c_8\varepsilon N_J(a_2 - a_1 + 2c_0\varepsilon, \varepsilon)$. Note that $a_2 - a_1 + 2c_0\varepsilon \le c_9(a_2 - a_1)$ for small ε. Hence by Lemma 5, for all $k \ge 1$,

$$\mathcal{E}[K^+(a_1, a_2)]^k \le \mathcal{E} c_8^k \varepsilon^k N_J^k(c_9(a_2 - a_1), \varepsilon) \le c_8^k \varepsilon^k c_9^k (a_2 - a_1)^k \varepsilon^{-4k} = c_{10}(a_2 - a_1)^k \varepsilon^{-3k}. \quad \square$$

Lemma 7. *There exist constants* $0 < c_1 < c_2 < \infty$ *and* $c(\varepsilon) \in (c_1, c_2)$ *such that for every fixed* $d > 0$, *both* $c(\varepsilon)\varepsilon^3 K^+(0, d)$ *and* $c(\varepsilon)\varepsilon^3 K^-(0, d)$ *converge in probability to* d *as* $\varepsilon \to 0$.

Remark 1. Since the functions $d \to K^+(0, d)$ and $d \to K^-(0, d)$ are non-decreasing we immediately obtain the following corollary: The functions $d \to \varepsilon^3 c(\varepsilon) K^+(0, d)$ and $d \to \varepsilon^3 c(\varepsilon) K^-(0, d)$ converge to identity in the metric of uniform convergence on compact intervals, in probability as $\varepsilon \to 0$.

Proof. Let c_3 be the constant defined in Lemma 4. Suppose that $\alpha > c_3$ is a large positive constant; we will choose a value for α later in the proof. Let $a_k = k(2c_3 + \alpha)\varepsilon$ for integer k. Let η_k be the continuous function which is equal to X^1 on $[a_k, a_{k+1}]$ and constant on $(-\infty, a_k]$ and $[a_{k+1}, \infty)$. Let $D_k = D(\eta_k, \varepsilon)$. For every k find a conformal mapping $f_k : D_* \to D_k$ so that $f_k((0, -1)) = (a_k, X^1(a_k))$, $f(-\infty) = -\infty$ and $f(\infty) = \infty$. Let

$\widetilde{K}_k^- = K^-(a_k + c_3\varepsilon, a_{k+1} - c_3\varepsilon)$ be defined relative to D_k and f_k. Then random variables $\widetilde{K}_k^-$ are independent and identically distributed. According to Lemma 6,

$$\mathcal{E}\widetilde{K}_k^- \geq c_4\varepsilon^{-2}\alpha,$$
$$\mathcal{E}\widetilde{K}_k^- \leq c_5\varepsilon^{-2}\alpha,$$
$$\mathcal{E}(\widetilde{K}_k^-)^2 \leq c_6\varepsilon^{-4}\alpha^2. \tag{4}$$

Hence, $\mathcal{E}\widetilde{K}_k^- = c_7(\varepsilon)\varepsilon^{-2}\alpha$ where $c_7(\varepsilon)$ are uniformly bounded away from 0 and ∞ for all ε.

Let $D = D(X^1, \varepsilon)$. We let $K_k^- = K^-(a_k + c_3\varepsilon, a_{k+1} - c_3\varepsilon)$ be defined relative to D and the usual conformal mapping $f : D_* \to D$. By Lemma 4, $|\widetilde{K}_k^- - K_k^-| \leq 12$ for every k.

Fix some $d > 0$ and let k_0 be the largest integer such that $a_{k_0} \leq d$. Then

$$|k_0 - d/(\varepsilon(2c_3 + \alpha))| < 2.$$

Note that $K^-(0, d) \geq \sum_{k=1}^{k_0} K_k^-$. Recall that $\widetilde{K}_k^-$'s are independent and so $\mathcal{E}\sum_{k=1}^{k_0}\widetilde{K}_k^- = \sum_{k=1}^{k_0}\mathcal{E}\widetilde{K}_k^-$ and $\mathrm{Var}\sum_{k=1}^{k_0}\widetilde{K}_k^- = \sum_{k=1}^{k_0}\mathrm{Var}\widetilde{K}_k^-$. We use Chebyshev's inequality and (4) to see that for arbitrary $\delta, p > 0$ one can find $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$,

$$\begin{aligned}
&\mathcal{P}(K^-(0,d) > (1-\delta)c_7(\varepsilon)\varepsilon^{-2}\alpha[d/(\varepsilon(2c_3+\alpha)) - 2]) \qquad (5)\\
&\geq \mathcal{P}\Big(\sum_{k=1}^{k_0} K_k^- > (1-\delta)c_7(\varepsilon)\varepsilon^{-2}\alpha[d/(\varepsilon(2c_3+\alpha)) - 2]\Big)\\
&\geq \mathcal{P}\Big(\sum_{k=1}^{k_0} \widetilde{K}_k^- > (1-\delta)c_7(\varepsilon)\varepsilon^{-2}\alpha[d/(\varepsilon(2c_3+\alpha)) - 2] + 12k_0\Big)\\
&\geq \mathcal{P}\Big(\sum_{k=1}^{k_0} \widetilde{K}_k^- > (1-\delta)c_7(\varepsilon)\varepsilon^{-2}\alpha[d/(\varepsilon(2c_3+\alpha)) - 2] + 12[d/(\varepsilon(2c_1+\alpha)) + 2]\Big)\\
&\geq \mathcal{P}\Big(\sum_{k=1}^{k_0} \widetilde{K}_k^- > (1-\delta/2)c_7(\varepsilon)\varepsilon^{-2}\alpha[d/(\varepsilon(2c_3+\alpha)) - 2]\Big) > 1 - p.
\end{aligned}$$

In the same way we obtain

$$\mathcal{P}\Big(\sum_{k=-1}^{k_0+1} K_k^- < (1+\delta/2)c_7(\varepsilon)\varepsilon^{-2}\alpha[d/(\varepsilon(2c_3+\alpha)) + 2]\Big) > 1 - p/2.$$

The maximum cell diameter is bounded by $c_8\varepsilon$ according to Lemma 2. A cell must either lie inside $\bigcup_k W(a_k+c_3\varepsilon, a_{k+1}-c_3\varepsilon)$ or intersect $\bigcup_k W(a_k-c_3\varepsilon-c_8\varepsilon, a_k+c_3\varepsilon+c_8\varepsilon)$. It follows that

$$K^+(0,d) \le \sum_{k=-1}^{k_0+1} K_k^- + \sum_{k=-1}^{k_0+1} K_k^+$$

where $K_k^+ = K^+(a_k - c_3\varepsilon - c_8\varepsilon, a_k + c_3\varepsilon + c_8\varepsilon)$ and the cells are counted in D relative to the usual conformal mapping. By Lemma 6,

$$\mathcal{E}K_k^+ \le c_9 2(c_3+c_8)\varepsilon^{-2} = c_{10}\varepsilon^{-2}$$

and so,

$$\mathcal{P}\left(\sum_{k=-1}^{k_0+1} K_k^+ \ge (2/p)[d/(\varepsilon(2c_3+\alpha))+2]c_{10}\varepsilon^{-2}\right) \le p/2.$$

Now choose $\alpha = \alpha(\delta,p)$ sufficiently large so that for all $\varepsilon < \varepsilon_0$,

$$(2/p)[d/(\varepsilon(2c_3+\alpha))+2]c_{10} < (\delta/2)c_7(\varepsilon)\alpha[d/(\varepsilon(2c_3+\alpha))-2].$$

Then for sufficiently small $\varepsilon > 0$,

$$\begin{aligned}
&\mathcal{P}(K^+(0,d) < (1+\delta)c_7(\varepsilon)\varepsilon^{-2}\alpha[d/(\varepsilon(2c_3+\alpha))-2]) \\
&\ge \mathcal{P}\left(\sum_{k=-1}^{k_0+1} K_k^- + \sum_{k=-1}^{k_0+1} K_k^+ < (1+\delta)c_7(\varepsilon)\varepsilon^{-2}\alpha[d/(\varepsilon(2c_3+\alpha))-2]\right) \\
&\ge \mathcal{P}\left(\sum_{k=-1}^{k_0+1} K_k^- < (1+\delta/2)c_7(\varepsilon)\varepsilon^{-2}\alpha[d/(\varepsilon(2c_3+\alpha))+2]\right) \\
&\qquad - \mathcal{P}\left(\sum_{k=-1}^{k_0+1} K_k^+ \ge (2/p)[d/(\varepsilon(2c_3+\alpha))+2]c_{10}\varepsilon^{-2}\right) \\
&\ge 1 - p/2 - p/2 = 1-p.
\end{aligned}$$

Since δ and p can be taken arbitrarily small and α can be taken arbitrarily large, a comparison of (5) with the last inequality easily implies the lemma. $\square$

Lemma 8. *There exists $c < \infty$ such that for all $b > 0$ and small ε,*

$$\mathcal{E}\left(\inf_{x\in W(0)} E^x(\tau(W(-b,b)) \mid T(W(-b)) < T(W(b)))\right) \ge cb^2\varepsilon^{-2}.$$

Proof. Fix some $b > 0$. Let $G^*_A(x,y)$ be Green's function for the reflected Brownian motion (RBM) in D_* killed on exiting $A \subset D_*$. We define in an analogous way $G_A(x,y)$ to be Green's functions for RBM in D killed on exiting $A \subset D$. The Green function is conformal invariant so

$$G_{W(-b,b)}(f(x), f(y)) = G^*_{W_*(-b,b)}(x,y)$$

for $x, y \in W_*(-b,b)$.

Lemma 7 implies that for any $p < 1$ and $c_1 > 0$ we can find small ε_0 such that for $\varepsilon < \varepsilon_0$ we have

$$\begin{aligned}(1-c_1)K^-(0,b)/2 < K^-(0,b/2) < (1+c_1)K^-(0,b)/2,\\ (1-c_1)K^-(-b,0)/2 < K^-(-b/2,0) < (1+c_1)K^-(-b,0)/2,\\ (1-c_1)K^-(-b,0) < K^-(0,b) < (1+c_1)K^-(-b,0),\end{aligned} \tag{6}$$

with probability greater than p.

Standard estimates show that $G^*_{M_*(j_1,j_2)}(x,y) > c_2(j_2 - j_1)$ for $x, y \in (j_1 + (j_2 - j_1)/8, j_2 - (j_2 - j_1)/8)$.

Suppose that $x \in W_*(0)$. Lemma 2 and (6) easily imply that for small ε, $\operatorname{Re} x$ belongs to the interval with endpoints

$$(-K^-(-b,0)) + (K^-(0,b) - (-K^-(-b,0)))/8$$

and

$$K^-(0,b) + (K^-(0,b) - (-K^-(-b,0)))/8$$

with $\mathcal{P}$-probability greater than p. If this event occurs and (6) holds then

$$G^*_{M_*(-K^-(-b,0),K^-(0,b))}(x,y) > c_2(K^-(0,b) + K^-(-b,0)) > c_3K^-(0,b)$$

for $y \in M_*(-K^-(-b/2,0), K^-(0,b/2))$. By conformal invariance,

$$G_{M(-K^-(-b,0),K^-(0,b))}(f(x), y) > c_3K^-(0,b)$$

for $y \in M(-K^-(-b/2,0), K^-(0,b/2))$. Since $M(-K^-(-b,0), K^-(0,b)) \subset W(-b,b)$, we have for $z \in W(0)$,

$$G_{W(-b,b)}(z,y) > G_{M(-K^-(-b,0),K^-(0,b))}(z,y) > c_3 K^-(0,b)$$

for $y \in M(-K^-(0,b)/4, K^-(0,b)/4) \subset W(-b/2,b/2)$, for small ε with probability greater than p. Since the area of $W(-b/2,b/2)$ is equal to $b\varepsilon$ we obtain for $x \in W(0)$ with $\mathcal{P}$-probability greater than p,

$$\begin{aligned} E^x \tau(W(-b,b)) &= \int_{W(-b,b)} G_{W(-b,b)}(x,y)dy \\ &\geq \int_{W(-b/2,b/2)} G_{W(-b,b)}(x,y)dy \geq b\varepsilon c_3 K^-(0,b). \end{aligned} \tag{7}$$

Next we derive a similar estimate for the conditioned process. Assume that (6) is true. Then the process starting from a point of $W(b/2)$ has at least 1/8 probability of hitting $W(-b)$ before hitting $W(b)$ and the probability is even greater for the process starting from $W(-b/2)$. The probability of hitting $W(-b/2)$ before hitting $W(b/2)$ for the process starting from $W(0)$ is very close to 1/2. The Bayes' theorem now implies that for the process starting from a point of $W(0)$ and conditioned to hit $W(-b)$ before hitting $W(b)$, the chance of hitting $W(b/2)$ before hitting $W(-b/2)$ is between 1/16 and 15/16. We have therefore, assuming (6) and using (7),

$$E^x(\tau(W(-b,b)) \mid T(W(-b)) < T(W(b))) \geq 1/16 E^x \tau(W(-b/2,b/2)) \geq b\varepsilon c_4 K^-(0,b/2).$$

Let $\mathcal{A}$ denote the event in (6). Using Lemma 6 and assuming that p is large we obtain,

$$\mathcal{E}\mathbf{1}_{\mathcal{A}^c} K^-(0,b/2) \leq (1-p)^{1/2}\Big(\mathcal{E}\mathbf{1}_{\mathcal{A}^c} K^-(0,b/2)^2\Big)^{1/2} \leq (1-p)^{1/2} c_6 b\varepsilon^{-3} \leq 1/2\mathcal{E}K^-(0,b/2).$$

Again by Lemma 6,

$$\mathcal{E} \inf_{x\in W(0)} E^x(\tau(W(-b,b)) \mid T(W(-b)) < T(W(b))) \geq \mathcal{E}\mathbf{1}_{\mathcal{A}} b\varepsilon c_4 K^-(0,b/2) \geq c_7 b^2 \varepsilon^{-2}. \quad \square$$

Lemma 9. *There exists $c_1 < \infty$ such that for any continuous function $\eta : \mathbf{R} \to \mathbf{R}$ and any $a \in \mathbf{R}$, $\varepsilon > 0$ and $d > 0$, if $D = D(\eta, \varepsilon)$, $x \in M(a)$ and the diameter of $M(a-1, a+1)$*

is not greater than d, then

$$E^x T(M(a-1)\cup M(a+1)) \le c_1 d^2,$$

$$E^x T(M(a-1)\cup M(a+1))^2 \le c_1 d^4,$$

$$E^x[T(M(a-1)\cup M(a+1)) \mid T(M(a-1)) < T(M(a+1))] \le c_1 d^2,$$

$$E^x[T(M(a-1)\cup M(a+1))^2 \mid T(M(a-1)) < T(M(a+1))] \le c_1 d^4.$$

Proof. First we deal with the case $d=1$.

Consider a point $x \in M(a)$. As a consequence of the assumption that the diameter of $M(a-1,a+1)$ is not greater than $d=1$, the set $M(a-1,a+1)$ lies below or on the line $Q=\{y\in \mathbf{C} : \operatorname{Im} y = \operatorname{Im} x + 1\}$. Let $L = f(\{y \in D_* : \operatorname{Im} y = 0\})$. Let V be the union of $M(a-1)$, $M(a+1)$ and the part of L between $M(a-1)$ and $M(a+1)$.

Without loss of generality assume that x lies on or below L. The vertical component of the reflected Brownian motion $\mathcal{Y}$ in D starting from x will have only a nonnegative drift (singular drift on the boundary) until $\mathcal{Y}$ hits the upper part of the boundary of D. However, the process $\mathcal{Y}$ has to hit V before hitting the upper part of ∂D. Let S be the hitting time of V. Since V lies below Q, there are constants $t_0<\infty$ and $p_0>0$ independent of η and ε and such that the process $\mathcal{Y}$ can hit V before t_0 with probability greater than p_0. At time S the process $\mathcal{Y}$ can be either at a point of $M(a-1)\cup M(a+1)$ or a point of L.

Suppose $\mathcal{Y}(S)\in L$. It is easy to see that the reflected Brownian motion in D_* starting from any point z of $\{y \in D_* : \operatorname{Im} y = 0\}$ between $M_*(a-1)$ and $M_*(a+1)$ can hit $M_*(a-1)\cup M_*(a+1)$ before hitting the boundary of D_* with probability greater than $p_1>0$ where p_1 is independent of z or a. By conformal invariance, reflected Brownian motion in D starting from any point z of $L\cap M(a-1,a+1)$ can hit $M(a-1)\cup M(a+1)$ before hitting the boundary of D with probability greater than p_1.

Since the diameter of $M(a-1,a+1)$ is bounded by 1, there exists $t_1<\infty$ such that 2-D Brownian motion starting from any point of $M(a-1,a+1)$ will exit this set before time t_1 with probability greater than $1-p_1/2$. It follows that the reflected Brownian motion in D starting from any point z of $L\cap M(a-1,a+1)$ can hit $M(a-1)\cup M(a+1)$ before time t_1 and before hitting the boundary of D with probability greater than $p_1/2$.

Let $t_2 = t_0 + t_1$. Our argument so far has shown that the process $\mathcal{Y}$ starting from x will hit $M(a-1) \cup M(a+1)$ an hence exit the set $M(a-1, a+1)$ before time t_2 with probability greater than $p_2 \stackrel{\text{df}}{=} p_0 p_1/2$.

By the repeated application of the Markov property at times t_2^k we see that the chance that $\mathcal{Y}$ does not hit $M(a-1) \cup M(a+1)$ before t_2^k is less that $(1-p_2)^k$. It follows that the distribution of the hitting time of $M(a-1) \cup M(a+1)$ for reflected Brownian motion in D starting from any point of $M(a)$ has an exponential tail and so it has all finite moments. This proves the first two formulae of the lemma in the case $d = 1$.

The probability that the reflected Brownian motion in D starting from a point of $M(a)$ will hit $M(a-1)$ before hitting $M(a+1)$ is equal to $1/2$. This and the first two formulae imply the last two formulae when $d = 1$.

Note that the estimates are independent of η and ε and so we can apply them for all d with appropriate scaling. $\square$

Let ν_a^* denote the uniform probability distribution on $M_*(a)$ and let $\nu_a = \nu_a^* \circ f^{-1}$. For a planar set A, its area will be denoted $|A|$.

Lemma 10. *There exists $c_1 < \infty$ such that for all a,*

$$E^{\nu_a}(T(M(a-1)) \mid T(M(a-1)) < T(M(a+1))) \le c_1 |M(a-1, a+1)|.$$

Proof. Let $G^*_{M_*(a-1,a+1)}(\nu_a^*, y)$ be the Green function for the reflected Brownian motion in D_* with the initial distribution ν_a^* and killed upon exiting $M_*(a-1, a+1)$. By analogy, $G_{M(a-1,a+1)}(\nu_a, y)$ will denote the Green function for the reflected Brownian motion in D with the initial distribution ν_a and killed upon exiting $M(a-1, a+1)$. It is elementary to see that

$$G^*_{M_*(a-1,a+1)}(\nu_a^*, y) = c_2 |1 - \operatorname{Re} y/a|$$

for $y \in M_*(a-1, a+1)$. In particular, $G^*_{M_*(a-1,a+1)}(\nu_a^*, y) \le c_2 < \infty$ for all y. By the conformal invariance of the Green function, $G_{M(a-1,a+1)}(\nu_a, f(y)) = G^*_{M_*(a-1,a+1)}(\nu_a^*, y)$.

Thus $G_{M(a-1,a+1)}(\nu_a, x) \le c_2$ for all x and so

$$E^{\nu_a} T(M(a-1) \cup M(a+1)) = \int_{M(a-1,a+1)} G_{M(a-1,a+1)}(\nu_a, x) dx \le c_2 |M(a-1, a+1)|.$$

The event $\{T(M(a-1)) < T(M(a+1))\}$ has probability 1/2 so

$$E^{\nu_a}(T(M(a-1)) \mid T(M(a-1)) < T(M(a+1))) \le 2c_2 |M(a-1, a+1)|. \quad \square$$

Lemma 11. *There exists $c < \infty$ such that for all $a, b > 0$ and small ε,*

$$\mathcal{E}\Big(\sup_{x \in W(0)} E^x(\tau(W(-a,b) \mid T(W(-a)) < T(W(b))) \Big)^2$$
$$\le \mathcal{E} \sup_{x \in W(0)} E^x(\tau(W(-a,b))^2 \mid T(W(-a)) < T(W(b))) \le ca(a+b)^3 \varepsilon^{-4}.$$

Proof. *Step 1.* The estimates in Step 1 of the proof do not depend on X^1. In other words, they are valid in $D(\eta, \varepsilon)$ for any continuous function η.

Let $\mathcal{Y}^*_t$ be the reflected Brownian motion in D_*, $\mathcal{Y}^*(0) = x_0 \in M_*(0)$, let $T^*_0 = 0$, let d_k be defined by $\mathrm{Re}\, \mathcal{Y}^*(T^*_k) = d_k$ and let

$$T^*_k = \inf\{t > T^*_{k-1} : \mathcal{Y}^*_t \in M_*(d_{k-1} - 1) \cup M_*(d_{k-1} + 1)\}, \qquad k \ge 1.$$

Note $\{d_k\}$ is simple symmetric random walk.

Fix some sequence $(j_0, j_1, j_2, \dots)$ of integers such that $j_0 = 0$ and $|j_k - j_{k-1}| = 1$ for all k. We will denote the event $\{d_k = j_k, \forall k\}$ by $\mathcal{J}_*$.

Note that if $|j - k| = 1$ then $\hat{k} = 2j - k$ is different from k and $|\hat{k} - j| = 1$.

For j and k such that $|j - k| = 1$ we define $g_{j,k}(x, y)$ for $x \in M_*(j)$ and $y \in M_*(k)$ by

$$g_{j,k}(x, y) dy = P^x(\mathcal{Y}^*(T(M_*(k))) \in dy \mid T(M_*(k)) < T(M_*(2j - k))).$$

Recall that ν^*_a denotes the uniform probability distribution on $M_*(a)$. It is easy to see that there exists $c_1 < \infty$ such that for all j, k, such that $|j - k| = 1$ and all $x_1, x_2 \in M_*(j)$, $y \in M_*(k)$,

$$\frac{g_{j,k}(x_1, y)}{g_{j,k}(x_2, y)} < c_1. \tag{8}$$

It follows that for all $x_1, x_2 \in M_*(j_k)$ and $y \in M_*(j_{k+1})$,

$$\frac{P^{x_0}(\mathcal{Y}^*(T^*_{k+1}) \in dy \mid \mathcal{J}_*, \mathcal{Y}^*(T^*_k) = x_1)}{P^{x_0}(\mathcal{Y}^*(T^*_{k+1}) \in dy \mid \mathcal{J}_*, \mathcal{Y}^*(T^*_k) = x_2)} < c_1$$

and so

$$P^{x_0}(\mathcal{Y}^*(T^*_{k+1}) \in dy \mid \mathcal{J}_*, \mathcal{Y}^*(T^*_k) = x_1) < c_2 \nu^*_{j_{k+1}}(dy). \tag{9}$$

We now introduce a few objects for the reflected Brownian motion in D which are completely analogous to those defined for $\mathcal{Y}^*$. Recall that $\mathcal{Y}_t$ is the reflected Brownian motion in D. Let $\mathcal{Y}(0) = y_0 = f(x_0) \in M(0)$, let $T_0 = 0$, let d_k be defined by $\operatorname{Re} \mathcal{Y}(T_k) = d_k$ and let

$$T_k = \inf\{t > T_{k-1} : \mathcal{Y}_t \in M(d_{k-1} - 1) \cup M(d_{k-1} + 1)\}, \qquad k \geq 1.$$

The d_k's are the same for $\mathcal{Y}$ and $\mathcal{Y}^*$ because $\mathcal{Y}$ is an appropriate time-change of $f(\mathcal{Y}^*)$. The event $\{d_k = j_k, \forall k\}$ will be called $\mathcal{J}$ in this new context. We obtain from (9),

$$P^{y_0}(\mathcal{Y}(T_{k+1}) \in dy \mid \mathcal{J}, \mathcal{Y}(T_k) = x_1) < c_2 \nu_{j_{k+1}}(dy), \tag{10}$$

where $\nu_a = \nu^*_a \circ f^{-1}$. Hence for all $x \in M(j_k)$ and $m \geq k+1$,

$$\begin{aligned} &E^{y_0}(T_{m+1} - T_m \mid \mathcal{J}, \mathcal{Y}(T_k) = x) \\ &\qquad \leq c_2 E^{\nu_{j_m}}(T(M(j_{m+1})) \mid T(M(j_{m+1})) < T(M(2j_m - j_{m+1}))). \end{aligned}$$

Lemma 10 yields

$$E^{y_0}(T_{m+1} - T_m \mid \mathcal{J}, \mathcal{Y}(T_k) = x_1) \leq c_3 |M(j_m - 1, j_m + 1)|$$

and so

$$E^{y_0}((T_{m+1} - T_m)(T_k - T_{k-1}) \mid \mathcal{J}, \mathcal{F}(T_k)) \leq c_3 |M(j_m - 1, j_m + 1)|(T_k - T_{k-1}).$$

If we remove the conditioning on $\mathcal{F}(T_k)$, apply again (10) and Lemma 10, we obtain the following estimate for all $m \geq k+1$ and $k \geq 2$,

$$E^{y_0}((T_{m+1} - T_m)(T_k - T_{k-1}) \mid \mathcal{J}) \leq c_4 |M(j_m - 1, j_m + 1)||M(j_{k-1} - 1, j_{k-1} + 1)|. \tag{11}$$

By Lemma 9 for all $m \geq 0$,

$$E^{y_0}((T_{m+1} - T_m)^2 \mid \mathcal{J}) \leq c_5 \mathrm{diam}(M(j_m - 1, j_m + 1))^4$$

and so, by Cauchy-Schwarz inequality, for any $k \geq 1$, $m \geq 0$,

$$\begin{aligned} &E^{y_0}((T_{m+1} - T_m)(T_k - T_{k-1}) \mid \mathcal{J}) \\ &\quad \leq c_6 \mathrm{diam}(M(j_m - 1, j_m + 1))^2 \mathrm{diam}(M(j_{k-1} - 1, j_{k-1} + 1))^2. \end{aligned}$$

This and Lemma 2 imply

$$E^{y_0}((T_{m+1} - T_m)(T_k - T_{k-1}) \mid \mathcal{J}) \leq c_7 \varepsilon^4. \tag{12}$$

Fix some integers $I_a < 0$ and $I_b > 0$ and let k_0 be the smallest integer such that $j_{k_0} = I_a$ or $j_{k_0} = I_b$. Let $\mathcal{N}_k$ be the number of m such that $m \leq k_0$ and $k = j_m$. Let $\mathcal{N} = \max_k \mathcal{N}_k$. We use (11) and (12) to derive the following estimate,

$$\begin{aligned} E^{y_0}(T_{k_0}^2 \mid \mathcal{J}) &= E^{y_0}\left(\left(\sum_{k=0}^{k_0-1} T_{k+1} - T_k \right)^2 \mid \mathcal{J} \right) \\ &\leq E^{y_0}\Bigg(\sum_{\substack{k,m=1 \\ |k-m| \geq 2}}^{k_0-1} (T_{m+1} - T_m)(T_{k+1} - T_k) \mid \mathcal{J} \Bigg) \\ &\quad + E^{y_0}\Bigg(\sum_{\substack{k,m=0 \\ |k-m| \leq 1}}^{k_0-1} (T_{m+1} - T_m)(T_{k+1} - T_k) \mid \mathcal{J} \Bigg) \\ &\quad + E^{y_0}\left(2 \sum_{m=0}^{k_0-1} (T_{m+1} - T_m)(T_1 - T_0) \mid \mathcal{J} \right) \\ &\leq \sum_{k,m=I_a}^{I_b} c_4 \mathcal{N}_m |M(m-1, m+1)| \mathcal{N}_k |M(k-2, k)| + \sum_{k=0}^{k_0-1} c_8 \varepsilon^4 \\ &\leq c_4^2 \Bigg(\sum_{m=I_a}^{I_b} \mathcal{N} |M(m-1, m+1)| \Bigg)^2 + c_8 k_0 \varepsilon^4 \\ &\leq c_9 \mathcal{N}^2 \Bigg| \bigcup_{m=I_a}^{I_b} C(m) \Bigg|^2 + c_8 k_0 \varepsilon^4. \end{aligned}$$

Now we remove conditioning on $\mathcal{J}$. The quantities $\mathcal{N}$ and k_0 become random variables defined relative to the random sequence $\{d_k\}$ in place of $\{j_k\}$. We recall that $\{d_k\}$ is a simple random walk on integers and so $Ek_0 \le c_{10}|I_a I_b|$ and $E\mathcal{N}^2 \le c_{11}|I_a I_b|$. This implies that

$$E^{y_0} T_{k_0}^2 \le c_{12}|I_a I_b| \left| \bigcup_{m=I_a}^{I_b} C(m) \right|^2 + c_{13}|I_a I_b|\varepsilon^4.$$

Before we go to the next step of the proof we note that the last estimate applies not only to $y_0 \in M(0)$ but to $y_0 \in W(0)$ as well. To see this, note that if $y_0 \in W(0)$ then $y_0 \in M(k_1)$ where k_1 is not necessarily an integer. Then T_{k_0} represents the exit time from $M(k_1 - I_a, k_1 + I_b)$.

Step 2. Fix some $a, b > 0$ and apply the last result with $I_a = -K^+(-a, 0)$ and $I_b = K^+(0, b)$. With this choice of I_a and I_b we have $\tau(W(-a, b)) \le T_{k_0}$ for $y_0 \in W(0)$. By Lemma 2, cells which intersect the worm $W(-a, b)$ cannot extend beyond $W(-a - c_{14}\varepsilon, b + c_{14}\varepsilon)$. Hence the area of $\bigcup_{m=I_a}^{I_b} C(m)$ is bounded by $c_{15}(a+b)\varepsilon$. This yields for small ε,

$$E^{y_0} T_{k_0}^2 \le c_{16}|I_a I_b|(a+b)^2\varepsilon^2 + c_{13}|I_a I_b|\varepsilon^4 \le c_{17}|I_a I_b|(a+b)^2\varepsilon^2.$$

The probability of $\{T(W(-a)) < T(W(b))\}$ is bounded below by

$$K^-(0, b)/(K^+(-a, 0) + K^+(0, b)).$$

Thus

$$\begin{aligned}
&E^{y_0}(\tau(W(-a, b))^2 \mid T(W(-a)) < T(W(b))) \\
&\quad \le E^{y_0}\Big(\tau(W(-a, b))^2 (K^+(-a, 0) + K^+(0, b))/K^-(0, b)\Big) \\
&\quad \le E^{y_0}\Big(T_{k_0}^2 (K^+(-a, 0) + K^+(0, b))/K^-(0, b)\Big) \\
&\quad \le c_{17}|I_a I_b|(a+b)^2\varepsilon^2\Big((K^+(-a, 0) + K^+(0, b))/K^-(0, b)\Big) \\
&\quad \le c_{18}K^+(-a, 0)K^+(0, b)(a+b)^2\varepsilon^2\Big((K^+(-a, 0) + K^+(0, b))/K^-(0, b)\Big) \\
&\quad = c_{18}(a+b)^2\varepsilon^2\left(\frac{K^+(-a, 0)^2 K^+(0, b)}{K^-(0, b)} + \frac{K^+(-a, 0)K^+(0, b)^2}{K^-(0, b)}\right).
\end{aligned}$$

Lemma 6 implies that

$$\begin{aligned}
\mathcal{E}\frac{K^+(-a, 0)^2 K^+(0, b)}{K^-(0, b)} &\le \Big(\mathcal{E}K^+(-a, 0)^6\Big)^{1/3}\Big(\mathcal{E}K^+(0, b)^3\Big)^{1/3}\Big(\mathcal{E}K^-(0, b)^{-3}\Big)^{1/3} \\
&\le (c_{19}a^6\varepsilon^{-18})^{1/3}(c_{20}b^3\varepsilon^{-9})^{1/3}(c_{21}b^{-3}\varepsilon^9)^{1/3} = c_{22}a^2\varepsilon^{-6}.
\end{aligned}$$

We obtain in a similar way

$$\mathcal{E}\frac{K^+(-a,0)K^+(0,b)^2}{K^-(0,b)} \le c_{23}ab\varepsilon^{-6}.$$

Hence

$$\begin{aligned}\mathcal{E}\Big(\sup_{y_0\in W(0)} &E^{y_0}(\tau(W(-a,b)) \mid T(W(-a)) < T(W(b)))\Big)^2\\ &\le \mathcal{E}\sup_{x\in W(0)} E^x(\tau(W(-a,b)^2 \mid T(W(-a)) < T(W(b)))\\ &\le c_{18}(a+b)^2\varepsilon^2(c_{22}a^2\varepsilon^{-6} + c_{23}ab\varepsilon^{-6}) \le c_{24}a(a+b)^3\varepsilon^{-4}. \quad \square\end{aligned}$$

The following definitions are similar to those used in the proof of Lemma 11 but not identical to them. Suppose that $b > 0$. Let $\mathcal{Y}_t^*$ be the reflected Brownian motion in D_*, $\mathcal{Y}^*(0) = x \in W_*(0)$, let $T_0^* = 0$, let d_k be defined by $\operatorname{Re}\mathcal{Y}^*(T_k^*) = d_k$ and let

$$T_k^* = \inf\{t > T_{k-1}^* : \mathcal{Y}_t^* \in W_*((d_{k-1}-1)b) \cup W_*((d_{k-1}+1)b)\}, \qquad k \ge 1.$$

Fix some sequence $(j_0, j_1, j_2, \dots)$ of integers such that $j_0 = 0$ and $|j_k - j_{k-1}| = 1$ for all k. We will denote the event $\{d_k = j_k, \forall k\}$ by $\mathcal{J}_*$. The corresponding definitions for the reflected Brownian motion in D are as follows. Let $\mathcal{Y}_t$ be the reflected Brownian motion in D, $\mathcal{Y}(0) = x \in W(0)$, let $T_0 = 0$, let d_k be defined by $\operatorname{Re}\mathcal{Y}(T_k) = d_k$ and let

$$T_k = \inf\{t > T_{k-1} : \mathcal{Y}_t \in W((d_{k-1}-1)b) \cup W((d_{k-1}+1)b)\}, \qquad k \ge 1.$$

Recall that the process $\mathcal{Y}$ is constructed as a time-change of $f(\mathcal{Y}^*)$ and so d_k's are the same for $\mathcal{Y}$ and $\mathcal{Y}^*$. The event $\mathcal{J}_* = \{d_k = j_k, \forall k\}$ will be also called $\mathcal{J}$.

Let $h_k(x,y)$ be defined for $x \in W(0)$, $y \in W(j_k b)$ by

$$h_k(x,y)dy = P^x(\mathcal{Y}(T_k) \in dy \mid \mathcal{J}).$$

Lemma 12. *There exist $\varepsilon_0, c_1, c_2 \in (0,1)$ such that*

$$h_k(x_1,y) \ge h_k(x_2,y)(1 - c_1c_2^k)$$

and

$$h_k(x_1,y) \le h_k(x_2,y)(1+c_1c_2^k)$$

for $x_1,x_2 \in W(0)$ *and* $y \in W(j_k b)$ *provided* $\varepsilon < \varepsilon_0$.

Proof. Recall that if $|j-k|=1$ then $\hat{k} \stackrel{\text{df}}{=} 2j-k$ is different from k and $|\hat{k}-j|=1$.

For j and k such that $|j-k|=1$ we define $g^*_{j,k}(x,y)$ for $x \in W_*(jb)$ and $y \in W_*(kb)$ by

$$g^*_{j,k}(x,y)dy = P^x(\mathcal{Y}_*(T(W_*(kb))) \in dy \mid T(W_*(kb)) < T(W_*((2j-k)b))).$$

For small ε, the sets $W_*(jb)$ and $W_*(kb)$ are separated by at least two cells in D_*, by Lemma 2. This and (8) easily imply that there exists $c_1 < \infty$ such that for all j,k, such that $|j-k|=1$ and all $x_1,x_2 \in W_*(jb)$, $y \in W_*(kb)$,

$$\frac{g^*_{j,k}(x_1,y)}{g^*_{j,k}(x_2,y)} < c_1. \tag{13}$$

Let $h^*_k(x,y)$ be defined for $x \in W_*(0)$, $y \in W_*(j_k b)$ by

$$h^*_k(x,y)dy = P^x(\mathcal{Y}_*(T^*_k) \in dy \mid \mathcal{J}_*).$$

By the strong Markov property,

$$h^*_{k+1}(x,y) = \int_{W_*(j_k b)} h^*_k(x,z) g^*_{j_k,j_{k+1}}(z,y)dz. \tag{14}$$

We will prove by induction on k that there exist constants $0 < c_1, c_2 < 1$ such that for all $x_1,x_2 \in W_*(0)$ and $y_1,y_2 \in W_*(j_k b)$,

$$\frac{h^*_k(x_1,y_1)}{h^*_k(x_1,y_2)} \ge \frac{h^*_k(x_2,y_1)}{h^*_k(x_2,y_2)}(1-c_1c_2^k). \tag{15}$$

It is elementary to verify that the inequality holds for $k=1$ and some $c_1,c_2 \in (0,1)$ using (8) and the fact that $W_*(0)$ and $W_*(j_1 b)$ are separated by at least two cells for small ε.

We proceed with the proof of the induction step. Assume that (15) holds for k. We have from (13),

$$\frac{g^*_{j_k,j_{k+1}}(x_1,y_1)}{g^*_{j_k,j_{k+1}}(x_1,y_2)} \ge c_3 \frac{g^*_{j_k,j_{k+1}}(x_2,y_1)}{g^*_{j_k,j_{k+1}}(x_2,y_2)} \tag{16}$$

for some constant $c_3 > 0$, all $k \geq 1$, $x_1, x_2 \in W_*(j_k b)$ and $y_1, y_2 \in W_*(j_{k+1}b)$. We can adjust c_1 and c_2 in (15) for the case $k = 1$ so that $c_2 \geq 1 - c_3^2$. Lemma 1 implies in view of (14), (15) and (16) that

$$\frac{h^*_{k+1}(x_1, y_1)}{h^*_{k+1}(x_1, y_2)} \geq \frac{h^*_{k+1}(x_2, y_1)}{h^*_{k+1}(x_2, y_2)}[(1 - c_1 c_2^k) + c_3^2 c_1 c_2^k] \geq \frac{h^*_{k+1}(x_2, y_1)}{h^*_{k+1}(x_2, y_2)}(1 - c_1 c_2^{k+1})$$

for $x_1, x_2 \in W_*(0)$ and $y_1, y_2 \in W_*(j_{k+1}b)$. This completes the proof of the induction step. We conclude that (15) holds for all k.

Formula (15) easily implies

$$h^*_k(x_1, y) \geq h^*_k(x_2, y)(1 - c_4 c_5^k) \tag{17}$$

and

$$h^*_k(x_1, y) \leq h^*_k(x_2, y)(1 + c_4 c_5^k) \tag{18}$$

for $x_1, x_2 \in W_*(0)$ and $y \in W_*(j_k b)$, with $c_4, c_5 \in (0, 1)$.

Now we will translate these estimates into the language of reflected Brownian motion in D. Recall the definition of T_k's and $\mathcal{J}$ given before the lemma. The function $h_k(x, y)$ has been defined for $x \in W(0)$, $y \in W(j_k b)$ by

$$h_k(x, y)dy = P^x(\mathcal{Y}(T_k) \in dy \mid \mathcal{J}).$$

The conformal invariance of reflected Brownian motion allows us to deduce from (17) and (18) that for $x_1, x_2 \in W(0)$ and $y \in W(j_k b)$,

$$h_k(x_1, y) \geq h_k(x_2, y)(1 - c_4 c_5^k)$$

and

$$h_k(x_1, y) \leq h_k(x_2, y)(1 + c_4 c_5^k)$$

with $c_4, c_5 \in (0, 1)$. □

Remark 2. We list two straighforward consequences of the last lemma.
(i) An application of the strong Markov property shows that for $m < k$, $x \in W(0)$, $z_1, z_2 \in W(j_m b)$, $y \in W(j_k b)$,

$$P^x(\mathcal{Y}(T_k) \in dy \mid \mathcal{Y}(T_m) \in dz_1, \mathcal{J}) \geq (1 - c_1 c_2^{k-m}) P^x(\mathcal{Y}(T_k) \in dy \mid \mathcal{Y}(T_m) \in dz_2, \mathcal{J}) \tag{19}$$

and

$$P^x(\mathcal{Y}(T_k) \in dy \mid \mathcal{Y}(T_m) \in dz_1, \mathcal{J}) \leq (1 + c_1 c_2^{k-m}) P^x(\mathcal{Y}(T_k) \in dy \mid \mathcal{Y}(T_m) \in dz_2, \mathcal{J}). \quad (20)$$

(ii) By averaging over appropriate sequences $\{j_k\}$ we obtain the following estimate. For every $a, b > 0$ and $\delta > 0$ we can find small $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$, $x_1, x_2 \in W(0)$ and $y \in W(-a)$,

$$\begin{aligned}
&(1-\delta) P^{x_1}(\mathcal{Y}(\tau(W(-a,b)) \in dy \mid T(W(-a)) < T(W(b)))) \\
&\qquad \leq P^{x_2}(\mathcal{Y}(\tau(W(-a,b)) \in dy \mid T(W(-a)) < T(W(b))) \\
&\qquad \leq (1+\delta) P^{x_1}(\mathcal{Y}(\tau(W(-a,b)) \in dy \mid T(W(-a)) < T(W(b))).
\end{aligned}$$

Lemma 13. *For $b > 0$ and an integer $N_1 > 1$, let $b_j = bj/N_1$. There exist $0 < c_1 < c_2 < \infty$ and $c(\varepsilon) \in (c_1, c_2)$ independent of b with the following property. For every $\delta > 0$ and $p > 0$ there exists $N_2 < \infty$ such that the following inequalities hold with probability greater than $1 - p$ when $N_1 > N_2$ and $\varepsilon < \varepsilon_0(N_1)$,*

$$\left| 1 - c(\varepsilon)\varepsilon^2 N_1 b^{-2} \sum_{j=0}^{N_1-1} \sup_{x \in W(b_j)} E^x\Big(\tau(W(b_{j-1}, b_{j+1})) \mid T(W(b_{j+1})) < T(W(b_{j-1}))\Big) \right| < \delta,$$

$$\left| 1 - c(\varepsilon)\varepsilon^2 N_1 b^{-2} \sum_{j=0}^{N_1-1} \sup_{x \in W(b_j)} E^x\Big(\tau(W(b_{j-1}, b_{j+1})) \mid T(W(b_{j+1})) > T(W(b_{j-1}))\Big) \right| < \delta,$$

$$\left| 1 - c(\varepsilon)\varepsilon^2 N_1 b^{-2} \sum_{j=0}^{N_1-1} \inf_{x \in W(b_j)} E^x\Big(\tau(W(b_{j-1}, b_{j+1})) \mid T(W(b_{j+1})) < T(W(b_{j-1}))\Big) \right| < \delta,$$

$$\left| 1 - c(\varepsilon)\varepsilon^2 N_1 b^{-2} \sum_{j=0}^{N_1-1} \inf_{x \in W(b_j)} E^x\Big(\tau(W(b_{j-1}, b_{j+1})) \mid T(W(b_{j+1})) > T(W(b_{j-1}))\Big) \right| < \delta.$$

Proof. Fix some $d > 0$. Suppose $a, \delta_1, \delta_2 > 0$ are small. For $x \in W(0)$,

$$\begin{aligned}
&E^x(\tau(W(-d,d)) \mid T(W(-d)) < T(W(d))) \\
&\quad = E^x(\tau(W(-a,d)) \mid T(W(-a)) < T(W(d))) \\
&\qquad + E^x(\tau(W(-d,d)) - \tau(W(-a,d)) \mid T(W(-d)) < T(W(d))).
\end{aligned}$$

Remark 2 (ii) and the strong Markov property imply that the Radon-Nikodym derivative for P^{x_1} and P^{x_2} distributions of the post-$\tau(W(-a,d))$ process is bounded below and above by $1-\delta_1$ and $1+\delta_1$ for all $x_1, x_2 \in W(0)$ provided ε is small. Suppose $x_1 = (0,0)$. We use Lemma 11 to see that

$$
\begin{aligned}
&(1-\delta_1)\mathcal{E}E^{x_1}(\tau(W(-d,d)) - \tau(W(-a,d)) \mid T(W(-d)) < T(W(d))) \\
&\quad \le \mathcal{E} \inf_{x\in W(0)} E^x(\tau(W(-d,d)) - \tau(W(-a,d)) \mid T(W(-d)) < T(W(d))) \\
&\quad \le \mathcal{E} \inf_{x\in W(0)} E^x(\tau(W(-d,d) \mid T(W(-d)) < T(W(d))) \\
&\quad \le \mathcal{E} \sup_{x\in W(0)} E^x(\tau(W(-d,d) \mid T(W(-d)) < T(W(d))) \\
&\quad \le \mathcal{E} \sup_{x\in W(0)} E^x(\tau(W(-a,d)) \mid T(W(-a)) < T(W(d))) \\
&\qquad + \mathcal{E} \sup_{x\in W(0)} (\tau(W(-d,d)) - \tau(W(-a,d)) \mid T(W(-d)) < T(W(d))) \\
&\quad \le c_3 a^{1/2}(a+d)^{3/2}\varepsilon^{-2} \\
&\qquad + (1+\delta_1)\mathcal{E}E^{x_1}(\tau(W(-d,d)) - \tau(W(-a,d)) \mid T(W(-d)) < T(W(d))).
\end{aligned}
$$

Since a and δ_1 can be taken arbitrarily small, the following estimate holds for any $\delta_2 > 0$ provided ε is sufficiently small,

$$
\begin{aligned}
&\mathcal{E} \sup_{x\in W(0)} E^x(\tau(W(-d,d)) \mid T(W(-d)) < T(W(d))) \\
&\quad \le (1+\delta_2)\mathcal{E} \inf_{x\in W(0)} E^x(\tau(W(-d,d)) \mid T(W(-d)) < T(W(d))).
\end{aligned} \tag{21}
$$

Lemma 11 applied with $a = b$ shows that

$$
\mathcal{E}\Big[\sup_{x\in W(b_j)} E^x(\tau(W(b_{j-1}, b_{j+1})) \mid T(W(b_{j-1})) < T(W(b_{j+1})))\Big]^2 \le c_3(b/N_1)^4\varepsilon^{-4}.
$$

Lemmas 8 and 11 imply that the random variables

$$
\sup_{x\in W(b_j)} E^x\tau(W(b_{j-1}, b_{j+1})) \mid T(W(b_{j-1})) < T(W(b_{j+1})))
$$

are i.i.d. with the mean bounded below by $c_3(b/N_1)^2\varepsilon^{-2}$ and variance bounded above by $c_4(b/N_1)^4\varepsilon^{-4}$. Hence the mean is bounded above by $c_5(b/N_1)^2\varepsilon^{-2}$. The first formula of the lemma now follows easily from the Chebyshev inequality. The second formula follows by symmetry. The last two formulae follow in a similar way. The only thing to be checked is that the normalizing constants $c(\varepsilon)$ may be chosen the same for the the last two formulae as for the first ones. This, however, can be deduced from (21) applied with $d = b/N_1$ since δ_2 can be taken arbitrarily small. $\square$

We recall some definitions stated before Lemma 12. For the reflected Brownian motion $\mathcal{Y}$ in D with $\mathcal{Y}(0) = x \in W(0)$, we let $T_0 = 0$, let d_k be defined by $\operatorname{Re}\mathcal{Y}(T_k) = d_k$ and let

$$T_k = \inf\{t > T_{k-1} : \mathcal{Y}_t \in W((d_{k-1} - 1)b) \cup W((d_{k-1} + 1)b)\}, \qquad k \geq 1.$$

We consider a sequence $(j_0, j_1, j_2, \ldots)$ of integers such that $j_0 = 0$ and $|j_k - j_{k-1}| = 1$ for all k. The event $\{d_k = j_k, \forall k\}$ is denoted by $\mathcal{J}$.

Lemma 14. *There exists $c < \infty$ such that for all $b > 0$, integers $n > 1$ and sufficiently small $\varepsilon > 0$,*

$$\mathcal{E} E^x((T_n - E^x(T_n \mid \mathcal{J}))^2 \mid \mathcal{J}) \leq c n b^4 \varepsilon^{-4}.$$

Proof. We start with an estimate for the covariance of $T_m - T_{m-1}$ and $T_k - T_{k-1}$ when $k - 1 > m + 1$. Remark 2 (i) implies that

$$E^x(T_k - T_{k-1} \mid \mathcal{Y}(T_{m+1}) = y, \mathcal{J}) \geq (1 - c_1 c_2^{k-m}) E^x(T_k - T_{k-1} \mid \mathcal{J})$$

and

$$E^x(T_k - T_{k-1} \mid \mathcal{Y}(T_{m+1}) = y, \mathcal{J}) \leq (1 + c_1 c_2^{k-m}) E^x(T_k - T_{k-1} \mid \mathcal{J}).$$

These inequalities imply that

$$E^x(|\Delta_{k-1}T - E^x(\Delta_{k-1}T \mid \mathcal{J})| \mid \mathcal{Y}(T_{m+1}) = y, \mathcal{J}) \leq c_1 c_2^{k-m} E^x(\Delta_{k-1}T \mid \mathcal{J}) \qquad (22)$$

where $\Delta_k T = T_{k+1} - T_k$. For small ε, the sets $W(j_{m+1}b)$ and $W(j_m b)$ are separated by at least two cells, by Lemma 2. Some standard arguments then show that the ratio of densities of $\mathcal{Y}(T_m)$ under conditional distributions given $\{\mathcal{Y}(T_{m+1}) = y_1\} \cap \mathcal{J}$ or given $\{\mathcal{Y}(T_{m+1}) = y_2\} \cap \mathcal{J}$ is bounded above and below by strictly positive finite constants which do not depend on $y_1, y_2 \in W(j_{m+1}b)$. It follows that for every $y \in W(j_{m+1}b)$,

$$E^x(\Delta_{m-1}T \mid \mathcal{Y}(T_{m+1}) = y, \mathcal{J}) \leq c_3 E^x(\Delta_{m-1}T \mid \mathcal{J}).$$

This, (22) and an application of the strong Markov property at T_{m+1} yield

$$\begin{aligned}
&E^x(|(\Delta_{k-1}T - E^x(\Delta_{k-1}T \mid \mathcal{J}))(\Delta_{m-1}T - E^x(\Delta_{m-1}T \mid \mathcal{J}))| \mid \mathcal{J}) \\
&\quad \le c_4 c_2^{k-m} E^x(\Delta_{k-1}T \mid \mathcal{J}) E^x(\Delta_{m-1}T \mid \mathcal{J}).
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&E^x((T_n - E^x(T_n \mid \mathcal{J}))^2 \mid \mathcal{J}) = E^x\left(\left(\sum_{k=0}^{n-1}[\Delta_k T - E^x(\Delta_k T \mid \mathcal{J})]\right)^2 \mid \mathcal{J}\right) \\
&\le E^x\left(2\sum_{m=0}^{n-1}\sum_{k=m}^{(m+2)\wedge(n-1)} |(\Delta_k T - E^x(\Delta_k T \mid \mathcal{J}))(\Delta_m T - E^x(\Delta_m T \mid \mathcal{J}))| \mid \mathcal{J}\right) \\
&+ E^x\left(2\sum_{m=0}^{n-1}\sum_{k=m+3}^{n-1} |(\Delta_k T - E^x(\Delta_k T \mid \mathcal{J}))(\Delta_m T - E^x(\Delta_m T \mid \mathcal{J}))| \mid \mathcal{J}\right) \\
&\le 2\sum_{m=0}^{n-1}\sum_{k=m}^{(m+2)\wedge(n-1)} \\
&\qquad \big(E^x([\Delta_k T - E^x(\Delta_k T \mid \mathcal{J})]^2 \mid \mathcal{J})\big)^{1/2}\big(E^x([\Delta_m T - E^x(\Delta_m T \mid \mathcal{J})]^2 \mid \mathcal{J})\big)^{1/2} \\
&+ 2\sum_{m=0}^{n-1}\sum_{k=m+3}^{n-1} c_4 c_2^{k-m} E^x(\Delta_k T \mid \mathcal{J}) E^x(\Delta_m T \mid \mathcal{J}) \\
&\stackrel{\text{df}}{=} \Xi_1 + \Xi_2.
\end{aligned}$$

In order to estimate $\mathcal{E}\Xi_1$ we apply Lemma 11 to see that

$$\begin{aligned}
&\mathcal{E}\Big(E^x([\Delta_k T - E^x(\Delta_k T \mid \mathcal{J})]^2 \mid \mathcal{J})\Big)^{1/2}\Big(E^x([\Delta_m T - E^x(\Delta_m T \mid \mathcal{J})]^2 \mid \mathcal{J})\Big)^{1/2} \\
&\le \Big(\mathcal{E}\big(E^x([\Delta_k T - E^x(\Delta_k T \mid \mathcal{J})]^2 \mid \mathcal{J})\big)\Big)^{1/2}\Big(\mathcal{E}\big(E^x([\Delta_m T - E^x(\Delta_m T \mid \mathcal{J})]^2 \mid \mathcal{J})\big)\Big)^{1/2} \\
&\le (c_1 b^4 \varepsilon^{-4})^{1/2}(c_1 b^4 \varepsilon^{-4})^{1/2} = c_1 b^4 \varepsilon^{-4}.
\end{aligned}$$

We obtain

$$\begin{aligned}
\mathcal{E}\Xi_1 &= \mathcal{E}2\sum_{m=0}^{n-1}\sum_{k=m+1}^{(m+2)\wedge(n-1)} \\
&\quad \big(E^x([\Delta_k T - E^x(\Delta_k T \mid \mathcal{J})]^2 \mid \mathcal{J})\big)^{1/2}\big(E^x([\Delta_m T - E^x(\Delta_m T \mid \mathcal{J})]^2 \mid \mathcal{J})\big)^{1/2} \\
&\le 2\sum_{m=0}^{n-1}\sum_{k=m+1}^{(m+2)\wedge(n-1)} c_1 b^4 \varepsilon^{-4} \le c_2 n b^4 \varepsilon^{-4}.
\end{aligned}$$

Next we estimate $\mathcal{E}\Xi_2$. We use Lemma 11 again to obtain

$$\begin{aligned}\mathcal{E}E^x(\Delta_k T \mid \mathcal{J})E^x(\Delta_m T \mid \mathcal{J}) &\le \Big(\mathcal{E}[E^x(\Delta_k T \mid \mathcal{J})]^2\Big)^{1/2}\Big(\mathcal{E}[E^x(\Delta_m T \mid \mathcal{J})]^2\Big)^{1/2}\\ &\le (c_1 b^4\varepsilon^{-4})^{1/2}(c_1 b^4\varepsilon^{-4})^{1/2} = c_1 b^4\varepsilon^{-4}\end{aligned}$$

and so

$$\begin{aligned}\mathcal{E}\Xi_2 &= \mathcal{E}2\sum_{m=0}^{n-1}\sum_{k=m+3}^{n-1} c_4 c_2^{k-m} E^x(\Delta_k T \mid \mathcal{J})E^x(\Delta_m T \mid \mathcal{J})\\ &\le 2\sum_{m=0}^{n-1}\sum_{k=m+3}^{n-1} c_4 c_2^{k-m} c_5 b^4\varepsilon^{-4} \le c_6 n b^4 \varepsilon^{-4}.\end{aligned}$$

We conclude that

$$\mathcal{E}E^x((T_n - E^x(T_n \mid \mathcal{J}))^2 \mid \mathcal{J}) \le \mathcal{E}\Xi_1 + \mathcal{E}\Xi_2 \le c_7 n b^4 \varepsilon^{-4}. \quad \square$$

We will need two estimates involving downcrossings of 1-dimensional Brownian motion. For convenience we will state the next lemma in terms of a special Brownian motion, namely $\operatorname{Re}\mathcal{Y}^*$. Let $U^*(x, x+a, t)$ be the number of crossings from x to $x+a$ by the process $\operatorname{Re}\mathcal{Y}^*$ before time t.

Lemma 15. *(i) For every $\delta, p > 0$ there exist $\zeta_0, \gamma_0 > 0$ and $M_0 < \infty$ such that if $\zeta < \zeta_0$, $\gamma < \gamma_0$, $M > M_0$, and for some random time S both events*

$$\left\{\sum_{j=-\infty}^{\infty} U^*((j-\gamma)\zeta, (j+1+\gamma)\zeta, S) \le M\right\}$$

and

$$\left\{\sum_{j=-\infty}^{\infty} U^*((j+\gamma)\zeta, (j+1-\gamma)\zeta, S) \ge M\right\}$$

hold with probability greater than $1-p/4$ then $|1 - S/(M\zeta^2)| < \delta$ with probability greater than $1-p$.

(ii) Let

$$\mathcal{Q}_n = \min_{nN_1 \le j < (n+1)N_1} U^*((j-\gamma)\zeta/N_1, (j+1+\gamma)\zeta/N_1, (1-\eta)M\zeta^2).$$

For every $\delta, p > 0$ there exist $M_0, \eta_0, \gamma_0 > 0$ and $N_2 < \infty$ such that for any $\zeta > 0$, $M > M_0$, $\eta < \eta_0$, $\gamma < \gamma_0$ and $N_1 > N_2$ we have

$$\begin{aligned}(1-\delta)MN_1^2 &\le N_1 \sum_{n=-\infty}^{\infty} \mathcal{Q}_n \\ &\le \sum_{j=-\infty}^{\infty} U^*((j+\gamma)\zeta/N_1, (j+1-\gamma)\zeta/N_1, (1+\eta)M\zeta^2) \le (1+\delta)MN_1^2\end{aligned}$$

with probability greater than $1-p$.

Proof. (i) Fix arbitrarily small $\delta, p > 0$. Let L_t^a denote the local time of the Brownian motion $\operatorname{Re}\mathcal{Y}^*$ at the level a at time t. See Karatzas and Shreve (1988) for an introduction to the theory of local time. By Brownian scaling, the distributions of L_t^a and $c^{-1}L_{tc^2}^{ac}$ are the same. We will use the following inequality of Bass (1987),

$$\mathcal{P}\left(\sup_{a\in\mathbf{R}, t\in[0,T]}\left|U^*(a, a+\varepsilon, t) - \frac{1}{\varepsilon}L_t^a\right| \ge \lambda\sqrt{T/\varepsilon}\right) \le e^{-c_1\lambda}. \tag{23}$$

Let $S_1 = (1-\delta)M\zeta^2$ and suppose that j_0 is a large integer whose value will be chosen later in the proof. The following eqalities hold in the sense of distributions,

$$\begin{aligned}&\left(\sum_{j=-\infty}^{\infty} U^*((j+\gamma)\zeta, (j+1-\gamma)\zeta, S_1)\right) - M \qquad (24)\\ &\quad= \sum_{|j|>j_0} U^*((j+\gamma)\zeta, (j+1-\gamma)\zeta, S_1) \\ &\qquad+ \sum_{|j|\le j_0} U^*((j+\gamma)\zeta, (j+1-\gamma)\zeta, S_1) - \sum_{|j|\le j_0} \frac{1}{\zeta(1-2\gamma)} L_{S_1}^{(j+\gamma)\zeta} \\ &\qquad+ \left(\sum_{|j|\le j_0} \frac{1}{\zeta(1-2\gamma)} L_{S_1}^{(j+\gamma)\zeta}\right) - M \\ &\quad= \sum_{|j|>j_0} U^*((j+\gamma)\zeta, (j+1-\gamma)\zeta, S_1) \\ &\qquad+ \sum_{|j|\le j_0}\left(U^*((j+\gamma)\zeta, (j+1-\gamma)\zeta, S_1) - \frac{1}{\zeta(1-2\gamma)} L_{S_1}^{(j+\gamma)\zeta}\right) \\ &\qquad+ \left(\frac{(1-\delta)M}{1-2\gamma}\sum_{|j|\le j_0} \frac{1}{\sqrt{(1-\delta)M}} L_1^{(j+\gamma)/\sqrt{(1-\delta)M}}\right) - \frac{1-\delta^2}{1-2\gamma}M - \frac{\delta^2-2\gamma}{1-2\gamma}M.\end{aligned}$$

Suppose that γ is so small that $(\delta^2 - 2\gamma)M/(1-2\gamma) > (\delta^2/2)M$. Let $j_0 = c_1\sqrt{M}$ with c_1 so large that the probability that the Brownian motion hits $(j_0+1-\gamma)\zeta$ or $(j_0+\gamma)\zeta$ before time S_1 is less than $p/16$. With this choice of j_0, the sum

$$\sum_{|j|>j_0} U^*((j+\gamma)\zeta, (j+1-\gamma)\zeta, S_1)$$

is equal to 0 with probability exceeding $1-p/16$. Recall that $a \to L_1^a$ is almost surely continuous and has a finite support, and $\int_{-\infty}^{\infty} L_1^a da = 1$. Hence, we can increase the value of c_1, if necessary, and choose sufficiently large M so that

$$\sum_{|j|\le j_0} \frac{1}{\sqrt{(1-\delta)M}} L_1^{(j+\gamma)/\sqrt{(1-\delta)M}} \le (1+\delta)\int_{-\infty}^{\infty} L_1^a da = (1+\delta),$$

with probability greater than $1-p/16$. Then

$$\left(\frac{(1-\delta)M}{1-2\gamma}\sum_{|j|\le j_0}\frac{1}{\sqrt{(1-\delta)M}}L_1^{(j+\gamma)/\sqrt{(1-\delta)M}}\right) - \frac{1-\delta^2}{1-2\gamma}M < 0$$

with probability greater than $1-p/16$.

Let $\lambda = c_2/\sqrt{\zeta}$. First choose c_2 so small that

$$2j_0\lambda\sqrt{S_1/(\zeta(1-2\gamma))} = 2c_1\sqrt{M}(c_2/\sqrt{\zeta})\sqrt{(1-\delta)M\zeta^2/(\zeta(1-2\gamma))} \le (\delta^2/4)M.$$

Then assume that ζ is so small that $e^{-c\lambda} \le p/16$. In view of (23) we have

$$\begin{aligned}\sum_{|j|\le j_0}\left(U^*((j+\gamma)\zeta,(j+1-\gamma)\zeta,S_1) - \frac{1}{\zeta(1-2\gamma)}L_{S_1}^{(j+\gamma)\zeta}\right) &\le 2j_0\lambda\sqrt{S_1/(\zeta(1-2\gamma))}\\ &\le (\delta^2/4)M,\end{aligned}$$

with probability greater than $1-e^{-c\lambda} \ge 1-p/16$. Combining (24) with the estimates following it we see that

$$\left(\sum_{j=-\infty}^{\infty} U^*((j+\gamma)\zeta,(j+1-\gamma)\zeta,S_1)\right) - M < 0,$$

with probability greater than $1-3p/16$. The function $s \to U^*((j+\gamma)\zeta,(j+1-\gamma)\zeta,s)$ is non-decreasing, so if we assume that

$$\left(\sum_{j=-\infty}^{\infty} U^*((j+\gamma)\zeta,(j+1-\gamma)\zeta,S)\right) - M < 0,$$

for some random variable S with probability less than $p/4$ then it follows that $S \geq S_1 = (1-\delta)M\zeta^2$ with probability greater than $1 - 3p/16 - p/4 > 1 - p/2$. We can prove in a completely analogous way that $S \leq (1+\delta)M\zeta^2$ with probability greater than $1 - p/2$. This easily implies part (i) of the lemma.

(ii) We leave the proof of part (ii) of the lemma to the reader. The proof proceeds along the same lines as the proof of part (i), i.e., it uses an approximation of the number of upcrossings by the local time and the continuity of the local time as a function of the space variable. $\square$

Suppose $b > 0$ and $N_1 > 1$ is an integer. Let $T_0 = 0$, let d_k be defined by $\operatorname{Re} \mathcal{Y}(T_k) = d_k$ and let

$$T_k = \inf\{t > S_{k-1} : \mathcal{Y}_t \in W((d_{k-1} - 1)b) \cup W((d_{k-1} + 1)b)\}, \qquad k \geq 1.$$

Let $R = [1/b^2]$ and $b_j = jb/N_1$. Let $U(x, x + a, t)$ denote the number of crossings from the level x to $x + a$ by the process $\operatorname{Re} \mathcal{Y}$ before time t and $\mathcal{N}_j = U(b_j, b_{j+1}, T_R)$. Let

$$\mathcal{Q}_n = \min_{nN_1 \leq j < (n+1)N_1} \mathcal{N}_j$$

and $\mathcal{N}_j^+ = \mathcal{N}_j - \mathcal{Q}_{n(j)}$ where $n(j)$ is an integer such that $n(j)N_1 \leq j < (n(j) + 1)N_1$.

Lemma 16. *For every $p, \delta > 0$ there exist $\varepsilon_0, b_0 > 0$ and $N_2 < \infty$ such that if $\varepsilon < \varepsilon_0$, $b \leq b_0$ and $N_1 \geq N_2$ then with probability greater than $1 - p$,*

$$(1-\delta)(N_1/b)^2 \leq N_1 \sum_{n=-\infty}^{\infty} \mathcal{Q}_n \leq \sum_{j=-\infty}^{\infty} \mathcal{N}_j \leq (1+\delta)(N_1/b)^2$$

and so

$$\sum_{j=-\infty}^{\infty} \mathcal{N}_j^+ \leq \delta(N_1/b)^2.$$

Proof. Find large M_0 as in Lemma 15 (i) with δ replaced by $\eta = \eta_0/2$ of Lemma 15 (ii). Then suppose that $b < M_0^{-1/2}/2$.

Let N_0 be so large that if $\mathcal{Y}^*_0 \in M_*(0)$ then the process $\operatorname{Re}\mathcal{Y}^*$ will cross from $-a$ to a more than R times before hitting $-N_0 a/4$ or $N_0 a/4$ with probability greater than $1-p$ (N_0 does not depend on a by scaling). By Lemma 7, if ε is sufficiently small, $K^-(-N_0 b, 0) > N_0 K^+(-b,b)/4$ and $K^-(0, N_0 b) > N_0 K^+(-b,b)/4$ with probability greater than $1-p$. The conformal invariance of reflected Brownian motion then implies that for $x \in W(0)$,

$$T_R < T(W(-N_0 b) \cup W(N_0 b)) \tag{25}$$

with $\mathcal{P}^x$-probability greater than $1-2p$.

Suppose that N_1 is greater than N_2 in Lemma 15 (ii). Let $c(\varepsilon)$ be the constants from Lemma 7, let $\xi = bc^{-1}(\varepsilon)\varepsilon^{-3}N_1^{-1}$ and recall that $b_j = jb/N_1$. Suppose that $\gamma > 0$ is less than both constants γ_0 of parts (i) and (ii) of Lemma 15. Let $c_1 < \infty$ be so large that the diameter of a cell is bounded by $c_1\varepsilon$, as in Lemma 2. We invoke Lemma 7 to see that there exists $\varepsilon_0 > 0$ so small that all of the following inequalities

$$\begin{aligned}
|\varepsilon^3 c(\varepsilon) K^+(-c_1\varepsilon, b_j + c_1\varepsilon) - b_j| &< \gamma b/N_1, \qquad 1 \le j \le 2N_0N_1,\\
|\varepsilon^3 c(\varepsilon) K^-(c_1\varepsilon, b_j - c_1\varepsilon) - b_j| &< \gamma b/N_1, \qquad 1 \le j \le 2N_0N_1,\\
|\varepsilon^3 c(\varepsilon) K^+(-b_j - c_1\varepsilon, c_1\varepsilon) - b_j| &< \gamma b/N_1, \qquad 1 \le j \le 2N_0N_1,\\
|\varepsilon^3 c(\varepsilon) K^-(-b_j + c_1\varepsilon, -c_1\varepsilon) - b_j| &< \gamma b/N_1, \qquad 1 \le j \le 2N_0N_1,
\end{aligned} \tag{26}$$

hold simultaneously with probability greater than $1-p$ provided $\varepsilon < \varepsilon_0$. Inequalities (26) imply that for all $j = -N_0N_1, \dots, N_0N_1$,

(i) $M((j-\gamma)\xi)$ lies to the left of $W(b_j)$, and

(ii) $M((j+\gamma)\xi)$ lies to the right of $W(b_j)$.

Let $\zeta = bc^{-1}(\varepsilon)\varepsilon^{-3} = \xi N_1$. If (25), (i) and (ii) hold then an upcrossing of $((j-\gamma)\zeta, (j+1+\gamma)\zeta)$ by $\operatorname{Re}\mathcal{Y}^*$ must correspond to an upcrossing of $(jb, (j+1)b)$ by $\operatorname{Re}\mathcal{Y}$. Vice versa, an upper bound for the number of upcrossings of $(jb,(j+1)b)$ by $\operatorname{Re}\mathcal{Y}$ is provided by the number of upcrossings of $((j+\gamma)\zeta, (j+1-\gamma)\zeta)$ by $\operatorname{Re}\mathcal{Y}^*$. Recall that κ denotes the time change for $f(\mathcal{Y}^*)$, i.e., $\mathcal{Y}(\kappa(t)) = f(\mathcal{Y}^*(t))$. Let $S = \kappa^{-1}(T_R)$. We have

$$\sum_{j=-\infty}^{\infty} U^*((j-\gamma)\zeta, (j+1+\gamma)\zeta, S) \le R$$

and

$$\sum_{j=-\infty}^{\infty} U^*((j+\gamma)\zeta, (j+1-\gamma)\zeta, S) \geq R.$$

Lemma 15 (i) implies that with probability greater than $1-4p$,

$$(1-\eta)R\zeta^2 \leq S \leq (1+\eta)R\zeta^2. \tag{27}$$

We apply similar analysis in order to compare the crossings of intervals $((j-\gamma)\xi, (j+1+\gamma)\xi)$. An upcrossing of $((j-\gamma)\xi, (j+1+\gamma)\xi)$ by $\operatorname{Re}\mathcal{Y}^*$ must correspond to an upcrossing of (b_j, b_{j+1}) by $\operatorname{Re}\mathcal{Y}$, assuming (26). The number of upcrossings of (b_j, b_{j+1}) by $\operatorname{Re}\mathcal{Y}$ does not exceed the number of upcrossings of $((j+\gamma)\xi, (j+1-\gamma)\xi)$ by $\operatorname{Re}\mathcal{Y}^*$. This, (27) and Lemma 15 (ii) yield

$$\begin{aligned}
(1-\delta)RN_1^2 &\leq N_1 \sum_{n=-\infty}^{\infty} \min_{nN_1 \leq j < (n+1)N_1} U^*((j-\gamma)\zeta/N_1, (j+1+\gamma)\zeta/N_1, (1-\eta)R\zeta^2) \\
&\leq N_1 \sum_{n=-\infty}^{\infty} \min_{nN_1 \leq j < (n+1)N_1} U(b_j, b_{j+1}, T_R) \\
&\leq \sum_{j=-\infty}^{\infty} U(b_j, b_{j+1}, T_R) \\
&\leq \sum_{j=-\infty}^{\infty} U^*((j+\gamma)\zeta/N_1, (j+1-\gamma)\zeta/N_1, (1+\eta)R\zeta^2) \leq (1+\delta)RN_1^2.
\end{aligned}$$

Since (27) holds with probability greater than $1-4p$, the probability of the event in the last formula is not less than $1-5p$ provided b and ε are small and N_1 is large. □

Recall T_R from the last lemma.

Lemma 17. *Let $R = R(t) = [t/b^2]$. There exist $0 < c_1 < c_2 < \infty$ and $c_3(\varepsilon) \in (c_1, c_2)$, $\varepsilon > 0$, with the following property. For every $t > 0$ and $\delta > 0$ we can find $b_0 > 0$ and $\varepsilon_0 > 0$ such that*

$$\mathcal{P}(|c_3(\varepsilon)\varepsilon^2 T_R - t| > \delta) < p$$

provided $b < b_0$ and $\varepsilon < \varepsilon_0$.

Proof. We will only consider the case $t = 1$. Take any $p, \delta_1 \in (0,1)$.

Choose $b > 0$ and N_1 which satisfy Lemma 16. We will impose more conditions on these numbers later in the proof. First of all, we will assume that $b^2/p < \delta_1$. Take any $\delta > 0$ with $\delta < \delta/p < \delta_1$. Recall a large integer N_0 from the proof of Lemma 16 so that we have

$$T_R < T(W(-N_0 b) \cup W(N_0 b)) \tag{28}$$

for $x \in W(0)$ with $\mathcal{P}^x$-probability greater than $1 - 2p$.

Recall that for N_1 we write $b_j = jb/N_1$ and $b_j^m = mb + b_j$. According to Lemma 13 we can find large N_1, small $\varepsilon_0 = \varepsilon_0(N_1) > 0$, constants $0 < c^- < c^+ < \infty$ and $c_4(\varepsilon) \in (c^-, c^+)$ such that

$$\begin{aligned}
&\left|1 - c_4(\varepsilon)\varepsilon^2 N_1 b^{-2} \sum_{j=0}^{N_1-1} \sup_{x \in W(b_j^m)} E^x\Big(\tau(W(b_{j-1}^m, b_{j+1}^m)) \mid T(W(b_{j+1}^m)) < T(W(b_{j-1}^m))\Big)\right| < \delta,\\
&\left|1 - c_4(\varepsilon)\varepsilon^2 N_1 b^{-2} \sum_{j=0}^{N_1-1} \sup_{x \in W(b_j^m)} E^x\Big(\tau(W(b_{j-1}^m, b_{j+1}^m)) \mid T(W(b_{j+1}^m)) > T(W(b_{j-1}^m))\Big)\right| < \delta,\\
&\left|1 - c_4(\varepsilon)\varepsilon^2 N_1 b^{-2} \sum_{j=0}^{N_1-1} \inf_{x \in W(b_j^m)} E^x\Big(\tau(W(b_{j-1}^m, b_{j+1}^m)) \mid T(W(b_{j+1}^m)) < T(W(b_{j-1}^m))\Big)\right| < \delta,\\
&\left|1 - c_4(\varepsilon)\varepsilon^2 N_1 b^{-2} \sum_{j=0}^{N_1-1} \inf_{x \in W(b_j^m)} E^x\Big(\tau(W(b_{j-1}^m, b_{j+1}^m)) \mid T(W(b_{j+1}^m)) > T(W(b_{j-1}^m))\Big)\right| < \delta,
\end{aligned} \tag{29}$$

simultaneously for all $m \in (-N_0, N_0)$ with probability greater than $1 - p$ when $\varepsilon < \varepsilon_0$.

Recall the following notation introduced before Lemma 16. We write $R = [1/b^2]$. The number of crossings from the level x to $x + a$ by the process $\mathrm{Re}\,\mathcal{Y}$ before time t is denoted $U(x, x + a, t)$ and we let $\mathcal{N}_j = U(b_j, b_{j+1}, T_R)$. Let

$$\mathcal{Q}_n = \min_{nN_1 \le j < (n+1)N_1} \mathcal{N}_j$$

and let $\mathcal{N}_j^+ = \mathcal{N}_j - \mathcal{Q}_{n(j)}$ where $n(j)$ is an integer such that $n(j)N_1 \le j < (n(j) + 1)N_1$. Then Lemma 16 shows that for small ε,

$$(1 - \delta)(N_1/b)^2 \le N_1 \sum_{n=-\infty}^{\infty} \mathcal{Q}_n \le \sum_{j=-\infty}^{\infty} \mathcal{N}_j \le (1 + \delta)(N_1/b)^2 \tag{30}$$

and so

$$\sum_{j=-\infty}^{\infty} \mathcal{N}_j^+ \leq \delta(N_1/b)^2 \tag{31}$$

with probability greater than $1-p$.

Let $\widetilde{\mathcal{N}}_j = U(b_j, b_{j-1}, T_R)$ and define $\widetilde{\mathcal{N}}_j^+$ and $\widetilde{\mathcal{Q}}_n$ relative to $\widetilde{\mathcal{N}}_j$ just like $\mathcal{N}_j^+$ and $\mathcal{Q}_n$ have been defined for $\mathcal{N}_j$. It is clear that the inequalities (30) and (31) hold also for $\widetilde{\mathcal{N}}_j$, $\widetilde{\mathcal{N}}_j^+$ and $\widetilde{\mathcal{Q}}_n$ with probability greater than $1-p$.

Suppose $(j_0, j_1, j_2, \dots)$ is a sequence of integers such that $j_0 = 0$ and $|j_k - j_{k-1}| = 1$ for all k. We will denote the event $\{d_k = j_k, \forall k\}$ by $\mathcal{J}$. By conditioning on $\mathcal{J}$ and X^1 we have

$$\begin{aligned}
&\mathcal{E}(T_R \mid \mathcal{J}, X^1) \qquad (32)\\
&\geq \mathcal{E}\Bigg(\sum_{j=-\infty}^{\infty} \mathcal{N}_j \inf_{x\in W(b_j)} E^x \tau(W(b_{j-1}, b_{j+1}) \mid T(W(b_{j+1})) < T(W(b_{j-1}))) \mid \mathcal{J}, X^1 \Bigg)\\
&+ \mathcal{E}\Bigg(\sum_{j=-\infty}^{\infty} \widetilde{\mathcal{N}}_j \inf_{x\in W(b_j)} E^x \tau(W(b_{j-1}, b_{j+1}) \mid T(W(b_{j+1})) > T(W(b_{j-1}))) \mid \mathcal{J}, X^1 \Bigg),
\end{aligned}$$

and similarly

$$\begin{aligned}
&\mathcal{E}(T_R \mid \mathcal{J}, X^1) \qquad (33)\\
&\leq \mathcal{E}\Bigg(\sum_{j=-\infty}^{\infty} \mathcal{N}_j \sup_{x\in W(b_j)} E^x \tau(W(b_{j-1}, b_{j+1}) \mid T(W(b_{j+1})) < T(W(b_{j-1}))) \mid \mathcal{J}, X^1 \Bigg)\\
&+ \mathcal{E}\Bigg(\sum_{j=-\infty}^{\infty} \widetilde{\mathcal{N}}_j \sup_{x\in W(b_j)} E^x \tau(W(b_{j-1}, b_{j+1}) \mid T(W(b_{j+1})) > T(W(b_{j-1}))) \mid \mathcal{J}, X^1 \Bigg).
\end{aligned}$$

Assume that (29) and (30) hold. Then

$$\begin{aligned}
&\sum_{j=-\infty}^{\infty} \mathcal{N}_j \inf_{x\in W(b_j)} E^x \tau(W(b_{j-1}, b_{j+1}) \mid T(W(b_{j+1})) < T(W(b_{j-1})))\\
&\geq \sum_{n=-\infty}^{\infty} \mathcal{Q}_n \sum_{nN_1 \leq j < (n+1)N_1} \inf_{x\in W(b_j)} E^x \tau(W(b_{j-1}, b_{j+1}) \mid T(W(b_{j+1})) < T(W(b_{j-1})))\\
&\geq \sum_{n=-\infty}^{\infty} \mathcal{Q}_n (1-\delta) c_4^{-1}(\varepsilon)\varepsilon^{-2} b^2 N_1^{-1} \geq (1-\delta)^2 c_4^{-1}(\varepsilon)\varepsilon^{-2}.
\end{aligned}$$

A similar estimate can be obtained for $\widetilde{\mathcal{N}}_j$'s. Note that in order to derive the last estimate we had to make assumptions (28), (29) and (30). They all hold simultaneously with probability greater than $1-4p$. Hence, in view of (32),

$$\mathcal{E}(T_R \mid \mathcal{J}, X^1) \geq 2(1-\delta)^2 c_4^{-1}(\varepsilon)\varepsilon^{-2} \geq 2(1-\delta_1)^2 c_4^{-1}(\varepsilon)\varepsilon^{-2} \tag{34}$$

with probability greater than $1-4p$.

Next we derive the opposite inequality. We have

$$\begin{aligned}
&\sum_{j=-\infty}^{\infty} \mathcal{N}_j \sup_{x\in W(b_j)} E^x \tau(W(b_{j-1}, b_{j+1}) \mid T(W(b_{j+1})) < T(W(b_{j-1}))) \\
&\leq \sum_{n=-\infty}^{\infty} \mathcal{Q}_n \sum_{nN_1 \leq j < (n+1)N_1} \sup_{x\in W(b_j)} E^x \tau(W(b_{j-1}, b_{j+1}) \mid T(W(b_{j+1})) < T(W(b_{j-1}))) \\
&\quad + \sum_{j=-\infty}^{\infty} \mathcal{N}_j^+ \sup_j \sup_{x\in W(b_j)} E^x \tau(W(b_{j-1}, b_{j+1}) \mid T(W(b_{j+1})) < T(W(b_{j-1}))) \\
&\leq \sum_{n=-\infty}^{\infty} \mathcal{Q}_n (1+\delta) c_4^{-1}(\varepsilon)\varepsilon^{-2} b^2 N_1^{-1} \\
&\quad + \sum_{j=-\infty}^{\infty} \mathcal{N}_j^+ \sup_j \sup_{x\in W(b_j)} E^x \tau(W(b_{j-1}, b_{j+1}) \mid T(W(b_{j+1})) < T(W(b_{j-1}))) \\
&\leq (1+\delta)^2 c_4^{-1}(\varepsilon)\varepsilon^{-2} \\
&\quad + \sum_{j=-\infty}^{\infty} \mathcal{N}_j^+ \sup_j \sup_{x\in W(b_j)} E^x \tau(W(b_{j-1}, b_{j+1}) \mid T(W(b_{j+1})) < T(W(b_{j-1}))).
\end{aligned}$$

The expectation of the last sum may be estimated using Lemma 11 and (31) as follows,

$$\begin{aligned}
&\mathcal{E}\Bigg(\sum_{j=-\infty}^{\infty} \mathcal{N}_j^+ \sup_j \sup_{x\in W(b_j)} E^x \tau(W(b_{j-1}, b_{j+1}) \mid T(W(b_{j+1})) < T(W(b_{j-1}))) \mid \mathcal{J}, \mathcal{N} \Bigg) \\
&\quad \leq \sum_{j=-\infty}^{\infty} \mathcal{N}_j^+ c_5 (b/N_1)^2 \varepsilon^{-2} \leq \delta (N_1/b)^2 c_{11} (b/N_1)^2 \varepsilon^{-2} = c_5 \delta \varepsilon^{-2}.
\end{aligned}$$

Hence

$$\sum_{j=-\infty}^{\infty} \mathcal{N}_j^+ \sup_j \sup_{x\in W(b_j)} E^x \tau(W(b_{j-1}, b_{j+1}) \mid T(W(b_{j+1})) < T(W(b_{j-1}))) \leq c_5 p^{-1} \delta \varepsilon^{-2}$$

with probability greater than $1-p$. We obtain

$$\sum_{j=-\infty}^{\infty} \mathcal{N}_j \sup_{x\in W(b_j)} E^x \tau(W(b_{j-1},b_{j+1}) \mid T(W(b_{j+1})) < T(W(b_{j-1})))$$
$$\leq (1+c_6 p^{-1}\delta) c_4^{-1}(\varepsilon)\varepsilon^{-2}.$$

Recall that $\delta/p < \delta_1$. We again double the estimate to account for the term with $\widetilde{\mathcal{N}}_j$'s in (33) and we obtain

$$\mathcal{E}(T_R \mid \mathcal{J}, X^1) \leq 2(1+c_6 p^{-1}\delta) c_4^{-1}(\varepsilon)\varepsilon^{-2} \leq 2(1+c_6\delta_1) c_4^{-1}(\varepsilon)\varepsilon^{-2} \tag{35}$$

with probability greater than $1-5p$.

Recall that $R = [1/b^2]$ and $b^2/p < \delta_1$. We obtain from Lemma 14

$$\mathcal{E}E^x((T_R - E^x(T_R \mid \mathcal{J}))^2 \mid \mathcal{J}) \leq c_7 R b^4 \varepsilon^{-4} \leq c_8 b^2 \varepsilon^{-4}.$$

It follows that

$$\mathcal{E}^x((T_R - E^x(T_R \mid \mathcal{J}))^2 \mid \mathcal{J}) \leq (1/p) c_8 b^2 \varepsilon^{-4} \leq c_8 \delta_1 \varepsilon^{-4} \tag{36}$$

with probability greater than $1-p$. Recall that δ_1 and p may be chosen arbitrarily close to 0. The Chebyshev inequality and (34)-(36) show that $(1/2)c_4(\varepsilon)\varepsilon^2 T_R$ converges in probability to 1 as $\varepsilon \to 0$. $\square$

Recall that $\kappa(t)$ is the time-change which turns $f(\mathcal{Y}^*)$ into a reflected Brownian motion in D, i.e., $\mathcal{Y}(\kappa(t)) = f(\mathcal{Y}^*_t)$.

Corollary 1. *There exist $0 < c_1 < c_2 < \infty$ and $c_3(\varepsilon), c_4(\varepsilon) \in (c_1, c_2)$ such that for every fixed $t > 0$ the random variable $c_3(\varepsilon)\varepsilon^2\kappa(c_4^{-2}(\varepsilon)\varepsilon^{-6}t)$ converges in probability to t as $\varepsilon \to 0$.*

Proof. We will discuss only the case $t = 1$. Fix arbitrarily small $\delta, p > 0$.

Estimate (27) in the proof of Lemma 16 shows that for some $c_4(\varepsilon)$ and arbitrarily small $\eta > 0$ we have

$$\kappa((1-\delta)c_4^{-2}(\varepsilon)\varepsilon^{-6}) \leq T_R \leq \kappa((1+\delta)c_4^{-2}(\varepsilon)\varepsilon^{-6})$$

with probability greater than $1-p$ provided ε is small. A similar argument shows that

$$\kappa(c_4^{-2}(\varepsilon)\varepsilon^{-6}) \leq (1-\delta)^{-1}T_R \tag{37}$$

and

$$(1+\delta)^{-1}T_R \leq \kappa(c_4^{-2}(\varepsilon)\varepsilon^{-6}) \tag{38}$$

with probability greater than $1-p$ provided ε is small. By Lemma 17, for some $c_3(\varepsilon)$, the quantities $c_3(\varepsilon)\varepsilon^2(1-\delta)^{-1}T_R$ and $c_3(\varepsilon)\varepsilon^2(1+\delta)^{-1}T_R$ converge to $(1-\delta)^{-1}$ and $(1+\delta)^{-1}$, resp. This and (37)-(38) prove the corollary. $\square$

Proof of Theorem 1. Fix arbitrarily small $\delta, p > 0$. Recall that $\mathcal{Y}^*$ denotes a reflected Brownian motion in D_*. Then $\operatorname{Re}\mathcal{Y}^*$ is a standard 1-D Brownian motion. Let $c_1 \in (1,\infty)$ be so large that for the 1-dimensional Brownian motion $\operatorname{Re}\mathcal{Y}^*_t$ starting from 0 and any $t > 0$ we have $P(\sup_{0\leq s\leq t} |\operatorname{Re}\mathcal{Y}^*_s| \geq c_1\sqrt{t}) < p/2$. Choose a large integer N_0 such that $1/N_0 < \delta$. Lemma 2 defines a constant c_2 such that the diameter of a cell is bounded by $c_2\varepsilon$.

Remark 1 following Lemma 7 and Corollary 1 imply that there exist constants $c_3(\varepsilon)$ and $c_4(\varepsilon)$, all uniformly bounded away from 0 and ∞ for all ε and such that all of the following statements hold simultaneously with probability greater than $1-p$, assuming $\varepsilon < \varepsilon_0$,

(i) for all $j = -2N_0, \dots, 2N_0$,

$$|c_3(\varepsilon)\varepsilon^2\kappa(c_4^{-2}(\varepsilon)\varepsilon^{-6}j/N_0) - j/N_0| < \delta;$$

(ii) $\sup_{0\leq s\leq 2c_4^{-2}(\varepsilon)\varepsilon^{-6}} |\operatorname{Re}\mathcal{Y}^*_s| < c_1\sqrt{2}c_4^{-1}(\varepsilon)\varepsilon^{-3}$;

(iii) for all $b \in [0, 2c_1]$ we have

$$|\varepsilon^3 c_4(\varepsilon)K^-(0, b-c_2\varepsilon) - b| < \delta \qquad \text{and} \qquad |\varepsilon^3 c_4(\varepsilon)K^+(0, b+c_2\varepsilon) - b| < \delta;$$

(iv) for all $b \in [-2c_1, 0]$,

$$|\varepsilon^3 c_4(\varepsilon)K^+(b-c_2\varepsilon, 0) - b| < \delta \qquad \text{and} \qquad |\varepsilon^3 c_4(\varepsilon)K^-(b+c_2\varepsilon, 0) - b| < \delta.$$

Let $s=\kappa(t)$. Then $f(\mathcal{Y}^*_t)=\mathcal{Y}(\kappa(t))=\mathcal{Y}_s$. Now consider the following transformation of the trajectories $(t,\mathrm{Re}\,\mathcal{Y}^*_t)$ and $(s,\mathrm{Re}\,\mathcal{Y}_s)$,

$$(t,\mathrm{Re}\,\mathcal{Y}^*_t)\longmapsto(c_4^2(\varepsilon)\varepsilon^6 t, c_4(\varepsilon)\varepsilon^3\mathrm{Re}\,\mathcal{Y}^*_t)\stackrel{\mathrm{df}}{=}(\psi(t),\Psi(\mathrm{Re}\,\mathcal{Y}^*_t)),$$
$$(s,\mathrm{Re}\,\mathcal{Y}_s)\longmapsto(c_3(\varepsilon)\varepsilon^2 s,\mathrm{Re}\,\mathcal{Y}_s)\stackrel{\mathrm{df}}{=}(\varphi(s),\mathrm{Re}\,\mathcal{Y}_s).$$

Assume that all statements (i)-(iv) hold true. We see from (i) that

$$|\psi(t)-\varphi(s)|=|\psi(t)-\varphi(\kappa(t))|=|c_4^2(\varepsilon)\varepsilon^6 t-c_3(\varepsilon)\varepsilon^2\kappa(t)|<\delta$$

for all t of the form $c_4^{-2}(\varepsilon)\varepsilon^{-6}j/N_0$, $j=-2N_0,\dots,2N_0$. Since $1/N_0<\delta$ and $\varphi(\kappa(t))$ is non-decreasing, we have

$$|\psi(t)-\varphi(s)|\le 2\delta$$

for all $t\in[-2c_4^{-2}(\varepsilon)\varepsilon^{-6},2c_4^{-2}(\varepsilon)\varepsilon^{-6}]$. It follows that

$$|\psi(t)-\varphi(s)|\le 2\delta \tag{39}$$

if $\psi(t)\in[-1,1]$ or $\varphi(s)\in[-1,1]$.

Property (iii) implies that $K^+(0,b+c_2\varepsilon)<(b+\delta)c_4^{-1}(\varepsilon)\varepsilon^{-3}$ and so $K^+(0,b-\delta+c_2\varepsilon)<bc_4^{-1}(\varepsilon)\varepsilon^{-3}$ for $b<3c_2/2$. For similar reasons, $K^-(0,b+\delta-c_2\varepsilon)>bc_4^{-1}(\varepsilon)\varepsilon^{-3}$. Since the diameter of a cell is bounded by $c_2\varepsilon$ we must have $M(bc_4^{-1}\varepsilon^{-3})\in W(b-\delta,b+\delta)$. Property (iv) yields the same conclusion for negative b bounded below by $-3c_2/2$. Consider a $t\in(0,2c_4^{-2}\varepsilon^{-6})$. Then, according to (ii) there exists $b\in(-c_2\sqrt{2},c_2\sqrt{2})$ such that $\mathrm{Re}\,\mathcal{Y}^*_t=bc_4^{-1}(\varepsilon)\varepsilon^{-3}$. Since $f(\mathcal{Y}^*_t)\in M(bc_4^{-1}(\varepsilon)\varepsilon^{-3})$ we have $|\mathrm{Re}\,f(\mathcal{Y}^*_t)-b|\le\delta$. This implies that

$$|\mathrm{Re}\,\mathcal{Y}_{\kappa(t)}-\Psi(\mathrm{Re}\,\mathcal{Y}^*_t)|=|\mathrm{Re}\,f(\mathcal{Y}^*_t)-b|\le\delta. \tag{40}$$

We have shown in (39) and (40) that for small ε there exists, with probability greater than $1-p$, a transformation of paths

$$(\psi(t),\Psi(\mathrm{Re}\,\mathcal{Y}^*_t))\longmapsto(\varphi(s),\mathrm{Re}\,\mathcal{Y}_s)=(\varphi(\kappa(t)),\mathrm{Re}\,\mathcal{Y}(\kappa(t)))\stackrel{\mathrm{df}}{=}(\theta(\psi(t)),\Theta(\Psi(\mathrm{Re}\,\mathcal{Y}^*_t)))$$

with the property that θ is increasing, $|\theta(u)-u|<2\delta$ and $|\Theta(r)-r|<\delta$ for $u\in[0,1]$, $r\in\mathbf{R}$. A similar argument would give a transformation which would work on any finite interval. Note that $(\psi(t), \Psi(\mathrm{Re}\,\mathcal{Y}^*_t))$ is the trajectory of a Brownian motion independent of X^1. Hence, if $\alpha(\varepsilon) = c_4(\varepsilon)\varepsilon^3$ then the process $t \to \alpha \mathrm{Re}\,\mathcal{Y}^*_{t/\alpha^2}$ is a standard one-dimensional Brownian motion. If we fix a one-dimesional Brownian motion X^2 and then for a fixed ε define $\mathrm{Re}\,\mathcal{Y}^*_t$ to be $\alpha^{-1} X^2_{\alpha^2 t}$, taking the vertical component $\mathrm{Im}\,\mathcal{Y}^*$ to be an independent reflected Brownian motion in $[-1, 1]$ (independent of ε) then the preceding work shows that $t \to \mathrm{Re}\,\mathcal{Y}_{c_3^{-1}(\varepsilon)\varepsilon^{-2}t}$ converges in probability to X^2, uniformly on compact intervals. $\square$

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