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WONG, Chun Wai Carto



# Smoothness of Loewner Slits

WONG, Chun Wai Carlo

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Reading Committee:

Steffen Rohde, Chair

Donald E. Marshall

Boris Solomyak

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**Abstract**

Smoothness of Loewner Slits

WONG, Chun Wai Carlo

Chair of the Supervisory Committee:  
Professor Steffen Rohde  
Mathematics

In this dissertation, we show that the chordal Loewner differential equation with  $C^\beta$  driving function generates a  $C^{\beta+\frac{1}{2}}$  slit for  $\frac{1}{2} < \beta \leq 2$ , except when  $\beta = \frac{3}{2}$  the slit is only proved to be weakly  $C^{1,1}$ .



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## GLOSSARY

$\lesssim$ :  $a(\varepsilon) \lesssim b(\varepsilon)$  means  $a(\varepsilon) \leq Cb(\varepsilon)$  for some constant  $C > 0$  (independent of  $\varepsilon$ ).

$\asymp$ :  $a(\varepsilon) \asymp b(\varepsilon)$  means  $a(\varepsilon) \lesssim b(\varepsilon)$  and  $b(\varepsilon) \lesssim a(\varepsilon)$ .

$\simeq$ :  $a(\varepsilon) \simeq b(\varepsilon)$  means  $\frac{a(\varepsilon)}{b(\varepsilon)}$  has a positive and finite limit as  $\varepsilon \rightarrow 0$ .

$\mathbb{D}$ : the open unit disk  $\{z \in \mathbb{C}: |z| < 1\}$

$\mathbb{H}$ : the upper half-plane  $\{z \in \mathbb{C}: \text{Im } z > 0\}$

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## DEDICATION

to my dear wife, Anita

## Chapter 1

**BACKGROUND**

In the last decade, there has been a growing interest in studying the *Schramm-Loewner Evolution* (SLE), which can be thought of as a conformally invariant random curve in a simply connected planar domain  $D$  (usually taken to be the upper half-plane  $\mathbb{H}$  or the open unit disk  $\mathbb{D}$ ) from a boundary point  $a \in \partial D$  to  $b \in \overline{D}$ . Depending on whether  $b \in \partial D$  or  $b \in D$ , these are classified as the chordal or radial cases. Discrete lattice versions of SLEs with various parameters are proved or conjectured to describe many discrete models in statistical mechanics as the mesh of the underlying lattice tends to zero. These models include loop-erased random walk and uniform spanning tree [27], [17], spin and FK Ising model [4], [32], self-avoiding walk [22], [6], contour line of Gaussian free field [29], [30], harmonic explorer [28] and critical percolation [31], [33].

We use the critical percolation model as an example to illustrate why the Loewner differential equation, which is a classical tool in complex analysis, comes into play in the studies of discrete lattice models. Figure 1.1 shows a discrete hexagonal lattice domain  $D_\delta$  which approximates a parallelogram domain  $D = [0, 1] + e^{i\pi/3}[0, 1] \subset \mathbb{C}$ . The quantity  $\delta > 0$  is the circumradius of the hexagons. We will let  $\delta$  tend to zero later. On the boundary of  $D_\delta$  the bottom and the right hexagons are colored grey, while the remaining boundary hexagons are colored white. For each of the interior hexagons, we flip an independent fair coin to decide its color, i.e. each has probability  $p = \frac{1}{2}$  of being grey. The word “critical” refers to the fact that  $p$  is chosen to be equal to  $p_c = \frac{1}{2}$ , which is known as the critical probability for the hexagonal lattice (see [10] for the general background of percolation theory and the definition of critical probability on other lattices).

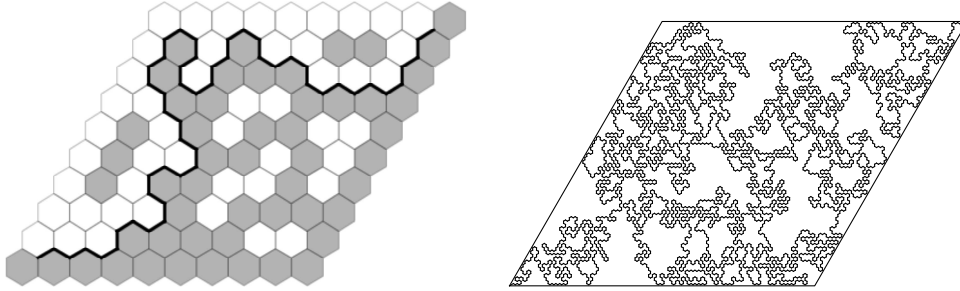


Figure 1.1: On the left side is a discrete domain consists of  $10 \times 10$  hexagons and a critical percolation interface (the dark curve). On the right side is a critical percolation interface on a finer  $80 \times 80$  hexagonal domain. The lattice is made invisible and the boundary of the limit domain  $D$  is drawn.

We are particularly interested in the *critical interface* (also known as the *critical percolation exploration path*)  $\gamma_\delta$  which is, up to reparametrization, determined by the following two conditions.

- (i) The curve  $\gamma_\delta$  starts at  $z_0 + \delta e^{i\frac{\pi}{6}}$ , travels along the sides of the hexagons and ends at  $z_1 - \delta e^{i\frac{\pi}{6}}$ , where  $z_0 \in \mathbb{C}$  is the center of the bottom-left hexagon and  $z_1 \in \mathbb{C}$  is the center of the top-right hexagon.
- (ii) The left side of  $\gamma_\delta$  is touching only white hexagons and the right side is touching only grey hexagons.

One can see that  $\gamma_\delta$  exists and is uniquely determined by the configuration. On the right side of Figure 1.1 is a sample of  $\gamma_\delta$  on a finer invisible lattice together with  $\partial D$  (the boundary of the limit domain). In this example,  $D$  is a parallelogram with two marked points  $0$  (the bottom-left corner) and  $1 + e^{i\frac{\pi}{3}}$  (the top-right corner) in  $\partial D$ . In general, suppose  $D \subset \mathbb{C}$  is a bounded simply connected domain with two distinct marked points  $a, b \in \partial D$ . (When  $\partial D$  is not smooth, we impose a technical assumption that both  $a$  and  $b$  are the impressions of two distinct degenerate prime ends. See [23].) For any such triple  $(D, a, b)$ , we can approximate  $D$  by discrete hexagonal lattice domains  $D_\delta$ . Among the set of boundary hexagons of  $D_\delta$ , let  $H_0$  be a boundary hexagon with minimal distance from  $a$  and

$H_1$  be a boundary hexagon with minimal distance from  $b$ . The boundary hexagons from  $H_0$  to  $H_1$  counterclockwise, including  $H_0$  and  $H_1$ , are colored grey. The remaining boundary hexagons are colored white. Similar to the case of a parallelogram, we independently color the interior hexagons white or grey with equal probability. Then we can construct a critical percolation path  $\gamma_\delta$  in a similar fashion. Physicists predicted that the law of  $\gamma_\delta$  converges, under an appropriate notion of weak convergence, to a probability measure  $P_{D,a,b}$  on the space of curves in  $\overline{D}$  from  $a$  to  $b$ . (See [1], [7] and the references therein.) Moreover,  $P_{D,a,b}$  was believed to satisfy the following two properties.

- (Conformal Invariance) If  $f: D \rightarrow f(D)$  is a conformal map, then

$$f_*P_{D,a,b} = P_{f(D),f(a),f(b)}.$$

- (Domain Markov Property) Let  $\gamma$  be a random curve whose law is  $P_{\mathbb{H},0,\infty}$ . Conditioned on an initial subarc  $\gamma([0, t])$ , the law of the remaining part of  $\gamma$  is equal to  $P_{H_t, \gamma(t), \infty}$ , where  $H_t$  is the unbounded component of  $\mathbb{H} \setminus \gamma([0, t])$ .

In the physics literature, it is widely believed that many planar lattice models at the critical temperature satisfy the conformal invariance condition in the scaling limit. Most physics arguments only give a non-rigorous justification to a considerably weaker form of conformal invariance called the Möbius invariance, and sometimes those arguments only consider the special case of rectangular domains. To the best of the author's knowledge, the first mathematical proof of the full conformal invariance of the critical percolation model was due to S. Smirnov [32]. Avoiding the technicalities such as what notion of convergence is being considered, a simplified version of Smirnov's result is the following.

**Theorem 1.0.1.** *As  $\delta \rightarrow 0$ , the law of the critical percolation exploration path  $\gamma_\delta$  converges to a probability measure  $P_{D,a,b}$  which satisfies the conformal invariance and domain Markov properties.*

The limit is known as SLE<sub>6</sub>, which has an independent definition (to be given below) that does not depend on the lattice. We remark that Smirnov's proof only applies to the

hexagonal lattice, although it is widely believed that Theorem 1.0.1 holds for several other lattices, such as the square lattice.

The SLE process in the upper half-plane  $\mathbb{H}$  from 0 to  $\infty$  can be defined using the forward chordal *Loewner differential equation* (1.1) and is defined for other domains using conformal invariance. Given a continuous function  $\lambda: [0, T] \rightarrow \mathbb{R}$ , consider the initial value problem

$$\begin{cases} \partial_t g_t(z) = \frac{2}{g_t(z) - \lambda(t)} \\ g_0(z) = z \end{cases} \quad (1.1)$$

for  $z \in \mathbb{H}$ . For  $t \geq 0$ , let

$$H_t := \{z \in \mathbb{H}: g_s(z) \neq \lambda(s) \text{ for } s \in [0, t]\}$$

be the set of points  $z$  in the upper half-plane that have still “survived” up to time  $t$ . By the word “survived” we mean the solution of the initial value problem (1.1) is defined up to time  $t$ . Intuitively, a particle starts at  $z$  and moves according to (1.1) until it hits  $\lambda(t)$ , which is also moving along the real line. We say that  $z$  is “killed” at time  $t$  if  $\lim_{s \rightarrow t^-} g_s(z) = \lambda(t)$ . After this time,  $g_t(z)$  is not defined. Some points  $z \in \mathbb{H}$  may survive forever. The set  $K_t := \mathbb{H} \setminus H_t$  is called the *hull* generated by  $\lambda(t)$ . In other words,  $K_t$  is the set of points  $z \in \mathbb{H}$  that are killed at or before time  $t$ . From the definition one can show that  $K_t$  is strictly increasing in  $t$  and  $K_0 = \emptyset$ . We call  $\lambda(t)$  the *driving function* of the hull  $K_t$ . If there is a slit (see Definition 3.1.1)  $\gamma: [0, \infty) \rightarrow \overline{\mathbb{H}}$  such that  $K_t = \gamma([0, t])$  for all  $t \geq 0$ , we say that the driving function  $\lambda(t)$  generates a slit.

We can now give a definition of SLE.

**Definition 1.0.2.** For each  $\kappa \geq 0$ ,  $\text{SLE}_\kappa$  is the random family of hulls  $\{K_t\}$  generated by the Loewner differential equation (1.1) with  $\lambda(t) = \sqrt{\kappa}B_t$ , where  $\{B_t: t \geq 0\}$  is the standard one-dimensional Brownian motion.

Depending on the parameter  $\kappa$ ,  $\text{SLE}_\kappa$  may or may not be a simple curve. In both cases, the *trace* of  $\text{SLE}_\kappa$  is defined by

$$\gamma_\kappa(t) := \lim_{y \rightarrow 0^+} g_t^{-1}(\sqrt{\kappa}B_t + iy).$$

It was proved by Schramm and Rohde [24] that  $\gamma_\kappa(t)$  exists for all  $t \geq 0$  and is almost surely a curve (i.e. continuous in  $t$ ). More precisely,

$$\mathbb{P}[\text{the trace } \gamma_\kappa(t) \text{ of SLE}_\kappa \text{ exists as a curve}] = 1 \quad (\kappa \neq 8). \quad (1.2)$$

The statement was later proved to be true also for  $\kappa = 8$ , by Lawler, Schramm and Werner using the uniform spanning tree model [17]. It is a natural question asked by many researchers, and is supported by computer simulation, that the trace  $\gamma_\kappa(t)$  is also continuous in  $\kappa$  almost surely. In a recent paper [12], it was proved that almost surely the trace of  $\text{SLE}_\kappa$  simultaneously exists for all  $\kappa$  in certain range. More precisely,

$$\mathbb{P}[\text{the trace } \gamma_\kappa(t) \text{ of SLE}_\kappa \text{ exists as a curve for all } \kappa \in [0, \kappa_0) \cup (\kappa_\infty, \infty)] = 1, \quad (1.3)$$

where  $\kappa_0 = 8(2 - \sqrt{3}) \approx 2.1$  and  $\kappa_\infty = 8(2 + \sqrt{3}) \approx 29.9$ . We remark that (1.3) is not an immediate consequence of (1.2) and requires some extra work. Not only do the curves  $\gamma_\kappa(t)$  simultaneously exist, the map  $\kappa \mapsto \gamma_\kappa$  is proved to be almost surely continuous as a function from  $[0, \kappa_0) \cup (\kappa_\infty, \infty)$  to  $C([0, 1])$ .

It was observed by Schramm [27] that the conformal invariance and domain Markov properties together characterize the law of  $\text{SLE}_\kappa$ . This is known as the Schramm's principle. Roughly speaking, these two conditions imply that the driving process  $\lambda(t)$  has stationary independent increments, forcing  $\lambda(t)$  to be a Brownian motion with drift. The drift term can be shown to be zero by a simple scaling argument. Since many discrete planar models at the critical temperature are widely believed to satisfy the conformal invariance and domain Markov properties in the scaling limit,  $\text{SLE}_\kappa$  is the most natural candidate. In fact, many models have now been proved to have a scaling limit equal to  $\text{SLE}_\kappa$  for some  $\kappa$ .

Many path properties of SLE are well-understood. The Hausdorff dimension of the  $\text{SLE}_\kappa$  trace  $\gamma_\kappa$  is known to be  $1 + \frac{\kappa}{8}$  for  $\kappa \leq 8$  (see [24] and [3]). The optimal Hölder continuity exponent of  $\gamma_\kappa$  has been found and proved in [20] and [11]. The proof uses the fact that the driving function  $\sqrt{\kappa}B_t$  is in  $\text{Lip}(\frac{1}{2} - \varepsilon)$  for all  $\varepsilon > 0$  but is not in  $\text{Lip}(\frac{1}{2})$ . Except for the trivial case  $\kappa = 0$ , the trace  $\gamma_\kappa$  is almost surely not Lipschitz. When the driving function  $\lambda$  is deterministic, depending on the class  $\lambda$  belongs to, the slit  $\gamma$  may be a fractal curve (such as the case where  $\lambda$  is a sample of the Brownian motion) or a smooth curve. It is not

fully understood what driving functions generate smooth curves. In fact, there is no known necessary and sufficient condition, in terms of the regularity of  $\lambda$ , for which  $\lambda$  generates a slit.

In 2005, Rohde, Marshall [24] and Joan Lind [19] found a sufficient condition, in terms of the  $\text{Lip}(\frac{1}{2})$ -norm of  $\lambda$ , for  $\lambda$  to generate a slit. Their condition is not necessary because they also proved that the slit satisfies an additional condition which we do not discuss here. The precise statement of their result is stated in chapter 2. As a follow-up of their work, this dissertation studies the smoothness of  $\gamma$  when the driving function  $\lambda$  is more regular (smooth) than  $\text{Lip}(\frac{1}{2})$ . We begin with the case  $\lambda \in \text{Lip}(\frac{1}{2} + \delta)$  for some  $0 < \delta \leq \frac{1}{2}$  and will prove that the slit  $\gamma$  is in  $C^{1,\delta}$ . The case  $\lambda \in C^{1,\alpha}$  ( $0 < \alpha \leq 1$ ) will also be discussed in the remaining sections.

## Chapter 2

## INTRODUCTION

The Loewner differential equation is a classical tool in complex analysis which has been successfully applied to various extremal problems, including the famous de Branges theorem (see [5]). In recent years, Schramm-Loewner Evolution (SLE) has been extensively studied by mathematicians and physicists. One can think of SLE as a random curve in the upper half-plane, which is generated via the Loewner differential equation with a random driving function. Currently, some natural questions on the deterministic side of SLE are still open. In this paper, we investigate the smoothness of slits generated by  $C^\beta$  driving functions.

Given a slit (defined in section 3.1)  $\gamma: [0, T] \rightarrow \overline{\mathbb{H}}$  in the upper half-plane  $\mathbb{H} = \{z \in \mathbb{C}: \text{Im } z > 0\}$ , the region  $H_t := \mathbb{H} \setminus \gamma([0, t])$  is simply connected for each  $t$ . There is a unique conformal map  $g_t: H_t \rightarrow \mathbb{H}$  satisfying the hydrodynamic normalization

$$g_t(z) = z + \frac{a_1(t)}{z} + \frac{a_2(t)}{z^2} + \dots$$

as  $z \rightarrow \infty$ . Figure 2.1 illustrates the situation. The coefficient  $a_1(t)$  is called the *half-plane capacity* of the set  $K_t = \gamma([0, t])$ . See [26] for a geometric interpretation of half-plane capacity and its relation to conformal radius, and see [15] for a probabilistic approach. Although it is not immediate from the definition, it is routine to show that  $a_1(t)$  is a continuous and strictly increasing real-valued function with  $a_1(0) = 0$ . If the slit is reparametrized so that  $a_1(t) = 2t$ , then  $g_t(z)$  is differentiable in  $t$  and satisfies the *chordal Loewner differential equation*

$$\begin{cases} \frac{\partial}{\partial t} g(t, z) = \frac{2}{g(t, z) - \lambda(t)} \\ g(0, z) = z \end{cases} \quad (2.1)$$

for  $z \in H_t$ , where  $\lambda: [0, T] \rightarrow \mathbb{R}$  is a continuous function called the *driving function* for the slit. Moreover,  $\lambda(t) = g_t(\gamma(t))$  is the image of the tip under the conformal map.

The foregoing procedure can be reversed. Suppose we are given some continuous function

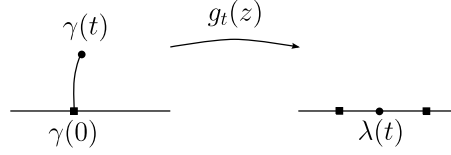


Figure 2.1: A slit  $\gamma(t)$  and its driving function  $\lambda(t)$  are related by a conformal map.

$\lambda: [0, T] \rightarrow \mathbb{R}$ . For each  $t \in [0, T]$ , let  $H_t$  be the set of points  $z \in \mathbb{H}$  for which the solution of (2.1) is well-defined up to time  $t$ , i.e.  $g(s, z) \neq \lambda(s)$  for  $s \in [0, t]$ . One can show that  $H_t$  is a simply connected region and  $z \mapsto g(t, z)$  maps  $H_t$  conformally onto  $\mathbb{H}$  and satisfies the hydrodynamic normalization. The set  $K_t := \mathbb{H} \setminus H_t$  is in general not a slit. Kufarev [14] constructed an example, in the classical (radial) setting, for which a continuous driving function does not generate a slit. The example can also be found in [8, section 3.4]. Even if the driving function is in  $\text{Lip}(\frac{1}{2})$ , also known as  $\frac{1}{2}$ -Holder continuous, the set  $K_t$  may not be locally connected (see [21] for an example).

Throughout this dissertation, we assume  $\lambda: [0, T] \rightarrow \mathbb{R}$  is  $\text{Lip}(\frac{1}{2})$ , i.e.

$$\|\lambda\|_{\text{Lip}(\frac{1}{2})} := \sup_{t_1 \neq t_2 \in [0, T]} \frac{|\lambda(t_1) - \lambda(t_2)|}{|t_1 - t_2|^{\frac{1}{2}}} < \infty.$$

In 2005, Marshall and Rohde [21] showed<sup>1</sup> that there is an absolute constant  $c_0 > 0$  so that for any  $\lambda: [0, T] \rightarrow \mathbb{R}$  with  $\|\lambda\|_{\text{Lip}(\frac{1}{2})} < c_0$ , the Loewner equation (2.1) generates a quasi-slit<sup>2</sup> in the upper half-plane  $\mathbb{H}$  and the slit meets  $\mathbb{R}$  non-tangentially. Lind [19] proved that all these statements hold for  $c_0 = 4$ , and this constant is the largest possible. In an unpublished paper [25] Steffen Rohde, Huy Vo Tran and Michel Zinsmeister give a sufficient condition for the driving function to generate a rectifiable curve.

What more can we say if  $\lambda: [0, T] \rightarrow \mathbb{R}$  is more regular (smooth) than  $\text{Lip}(\frac{1}{2})$ ? In [2, page 59] it was proved that if  $\lambda: [0, T] \rightarrow \mathbb{R}$  has bounded first derivative then its slit is  $C^1$ . (The original statement was for the radial version. Here we formulate it in the chordal setting.) As of the writing of this dissertation and up to the author's knowledge, it is the only

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<sup>1</sup>The theorems in [21] were proved in the radial case. In a private communication, Don Marshall translated (with rigorous proof) the results to the chordal case.

<sup>2</sup>By definition, a *quasi-slit* is a slit satisfying the Ahlfors three-point condition, i.e. there is a constant  $L < \infty$  such that for all points  $z_1, z_2, z_3$  on the slit in that order,  $|z_1 - z_2| + |z_2 - z_3| \leq L|z_1 - z_3|$ .

result in the literature concerning the smoothness of a slit generated by a driving function more regular than  $\text{Lip}(\frac{1}{2})$ . Marshall and Rohde [21] implicitly suggest the following.

**Main Statement** (heuristic version).  $\lambda \in C^\beta \Rightarrow \gamma \in C^{\beta+\frac{1}{2}}$  for  $\beta > \frac{1}{2}$ .

In this dissertation, we will prove this statement for  $\frac{1}{2} < \beta \leq 2$ , except when  $\beta = \frac{3}{2}$  the slit  $\gamma$  is only proved to be weakly  $C^{1,1}$ . The precise statements are in Theorem 3.3.8, Theorem 3.4.2 and Theorem 3.5.2, corresponding to the cases  $\beta \in (\frac{1}{2}, 1]$ ,  $\beta \in (1, \frac{3}{2}]$  and  $\beta \in (\frac{3}{2}, 2]$ . One of the key ingredients of our method is the Lipschitz continuity (Theorem 3.2.4 below) of the map  $\lambda \mapsto \gamma^\lambda$ , which was previously only known to be continuous [18, Theorem 4.1]. Another ingredient is an integral representation of  $\gamma'(t)$ , see Corollary 3.3.3.

**Theorem 3.2.4** (Lipschitz continuity). *Suppose  $\lambda_1, \lambda_2: [0, T] \rightarrow \mathbb{R}$  satisfy  $\|\lambda_j\|_{\text{Lip}(\frac{1}{2})} \leq 1$  for  $j = 1, 2$ . Then  $\|\gamma^{\lambda_1} - \gamma^{\lambda_2}\|_\infty \leq c \|\lambda_1 - \lambda_2\|_\infty$ , where  $c > 0$  is an absolute constant.*

There is another natural and interesting question which we won't discuss in this paper but we mention it for the sake of completeness. If we know that a slit  $\gamma$  is  $C^n$  for some positive integer  $n$ , how smooth is its driving function? Earle and Epstein [9] answered this question in 2001 for the radial case (and the chordal case follows). Suppose  $0 \in \Omega \subseteq \mathbb{C}$  is a simply connected region and  $\gamma: (0, T] \rightarrow \Omega$  is a slit avoiding the origin with base point  $\gamma(0) \in \partial\Omega$ . Let  $R(t)$  be the conformal radius of  $\Omega_t := \Omega \setminus \gamma((0, T])$  with respect to the origin. Earle and Epstein showed that if  $\gamma$  is  $C^n$  with non-vanishing first derivative on  $(0, T]$  for some integer  $n \geq 2$ , then the radial capacity  $a(t) := -\log R(t)$  is  $C^{n-1}$  on  $(0, T]$ . Moreover, if the slit is reparametrized so that  $a(t) = a(0) + t$ , then its driving function  $\lambda$  is  $C^{n-1}$  on  $(a(0), a(T)]$ . See [9] for the precise definitions and statements. In the same paper, it was also proved that real analytic slits generate real analytic driving functions.

Chapter 3  
PROOFS

**3.1 Definitions, notations and preliminaries**

In this dissertation, the lowercase  $c$  is reserved to denote an absolute constant which may vary even in a single chain of equalities.

**Definition 3.1.1.** A *slit* in  $\mathbb{H}$  is a simple curve  $\gamma: [0, T] \rightarrow \overline{\mathbb{H}}$  with  $\gamma(0) \in \mathbb{R}$  and  $\gamma(t) \in \mathbb{H}$  for  $0 < t \leq T$ .

All driving functions  $\lambda: [0, T] \rightarrow \mathbb{R}$  in this dissertation satisfy

$$\|\lambda\|_{\text{Lip}(\frac{1}{2})} := \sup_{t_1 \neq t_2 \in [0, T]} \frac{|\lambda(t_1) - \lambda(t_2)|}{|t_1 - t_2|^{\frac{1}{2}}} < 4$$

(at least locally) and therefore generate slits by [21] and [19]. We will use the following notations frequently.

**Notation 3.1.2.**

(i) The  $\text{Lip}(\frac{1}{2})$ -norm<sup>1</sup> of  $\lambda: [0, T] \rightarrow \mathbb{R}$  is denoted by

$$\|\lambda\|_{\text{Lip}(\frac{1}{2}, [0, T])} := \sup_{t_1 \neq t_2 \in [0, T]} \frac{|\lambda(t_1) - \lambda(t_2)|}{|t_1 - t_2|^{\frac{1}{2}}}.$$

Usually, we write  $\|\lambda\|_{\text{Lip}(\frac{1}{2})}$  instead of  $\|\lambda\|_{\text{Lip}(\frac{1}{2}, [0, T])}$ .

(ii) For positive integer  $n \in \mathbb{N}$  and  $0 < \alpha \leq 1$ , the  $C^{n, \alpha}$ -norm of  $\lambda: [0, T] \rightarrow \mathbb{R}$  is

$$\|\lambda\|_{C^{n, \alpha}} = \|\lambda\|_{C^{n, \alpha}([0, T])} := \sum_{k=0}^n \sup_{t \in [0, T]} |\lambda^{(k)}(t)| + \sup_{t_1 \neq t_2 \in [0, T]} \frac{|\lambda^{(n)}(t_1) - \lambda^{(n)}(t_2)|}{|t_1 - t_2|^\alpha}.$$

For a slit  $\gamma$ , its  $C^{n, \alpha}$ -norm  $\|\gamma\|_{C^{n, \alpha}}$  is defined similarly. If  $\beta > 1$  is not an integer, the notation  $C^\beta$  refers to  $C^{[\beta], \beta - [\beta]}$ , where  $[\beta]$  is the integer part of  $\beta$ . For example,  $C^{\frac{3}{2}}$  is the same as  $C^{1, \frac{1}{2}}$ .

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<sup>1</sup>Strictly speaking,  $\|\lambda\|_{\text{Lip}(\frac{1}{2})}$  is only a semi-norm.

(iii)  $\gamma^\lambda: [0, T] \rightarrow \overline{\mathbb{H}}$  denotes the slit generated by  $\lambda: [0, T] \rightarrow \mathbb{R}$ . When no confusion can occur, we write  $\gamma$  instead of  $\gamma^\lambda$  for the sake of notation. The *base* of  $\gamma$  is  $\gamma(0) = \lambda(0) \in \mathbb{R}$ .

(iv) For each  $t \in [0, T]$ ,  $g_t: H_t \rightarrow \mathbb{H}$  denotes the (unique) conformal map from  $H_t = \mathbb{H} \setminus \gamma([0, t])$  onto the upper half-plane  $\mathbb{H}$  satisfying the normalization

$$g_t(z) = z + \frac{a_1(t)}{z} + \dots$$

as  $z \rightarrow \infty$ . All slits in this paper are parametrized by half-plane capacity, i.e.  $a_1(t) = 2t$ . Alternative notations such as  $g(t, z)$  or  $g_t^\lambda(z)$  may be used interchangeably.

(v)  $f_t: \mathbb{H} \rightarrow H_t$  is the inverse function of  $g_t$ , i.e.  $g_t(f_t(z)) = z$  for all  $z \in \mathbb{H}$ . We sometimes write  $f(t, z)$  or  $f_t^\lambda(z)$ .

(vi) For  $0 \leq s < t \leq T$ , we define  $\gamma_s(t) := g_s(\gamma(t)) - \lambda(s)$ . To be flexible it may also be written as  $\gamma(s, t)$  or  $\gamma_s^\lambda(t)$ . The normalized version of  $\gamma(s, t)$  is  $\tau(s, t) := \frac{\gamma(s, t)}{\sqrt{t-s}}$ . Note that  $\tau(s, t) = 2i$  if and only if  $\lambda$  is constant on  $[s, t]$ .

(vii) In section 3.3, we introduce the notation

$$L(s) = L^\lambda(s) = \int_0^s \left[ \frac{1}{2} + \frac{2}{\tau(s-u, s)^2} \right] \frac{du}{u}$$

and show that  $\gamma'(s) = \frac{i}{\sqrt{s}} e^{L(s)}$  under appropriate assumptions.

**Definition 3.1.3.**

(i) We say that  $\lambda: [0, T] \rightarrow \mathbb{R}$  satisfies the  $\sigma$ -*Lip*( $\frac{1}{2}$ ) *condition* if  $\sigma \geq 0$  and

$$|\lambda(t_1) - \lambda(t_2)| \leq \sigma |t_1 - t_2|^{\frac{1}{2}}$$

for all  $t_1, t_2 \in [0, T]$ .

(ii) We say that  $\lambda: [0, T] \rightarrow \mathbb{R}$  satisfies the  $(M, T, \delta)$ -*Lip*( $\frac{1}{2} + \delta$ ) *condition* if  $\|\lambda\|_{\text{Lip}(\frac{1}{2})} \leq 1$  and

$$|\lambda(t_1) - \lambda(t_2)| \leq M |t_1 - t_2|^{\frac{1}{2} + \delta}$$

for all  $t_1, t_2 \in [0, T]$ , where  $M, T, \delta > 0$ .

- (iii) We say that  $\lambda: [0, T] \rightarrow \mathbb{R}$  satisfies the  $(M, T, n, \alpha)$ - $C^{n, \alpha}$  condition if  $\|\lambda\|_{\text{Lip}(\frac{1}{2})} \leq 1$ ,  $\lambda \in C^{n, \alpha}$  on  $[0, T]$  and  $\|\lambda\|_{C^{n, \alpha}} \leq M$ , where  $n \in \mathbb{N}$ ,  $0 < \alpha \leq 1$  and  $M > 0$ .

We will not consider  $\text{Lip}(\frac{1}{2} + \delta)$  driving functions until section 3.3. As a remark on the terminologies, careful readers may see that in (i) we do not make explicit reference to  $T$  but we do in (ii). This is because  $\text{Lip}(\frac{1}{2} + \delta)$ -norm is not invariant under Brownian scaling<sup>2</sup>. The terminologies reflect that all quantitative estimates in section 3.3 depend only on  $M$ ,  $T$  and  $\delta$ , while in section 3.2 our estimates are mostly in terms of  $\sigma$ .

In this paper, we use the diagram in Figure 3.1 to represent a situation that  $\gamma(t)$  and  $\lambda(t)$  are related by the Loewner equation.

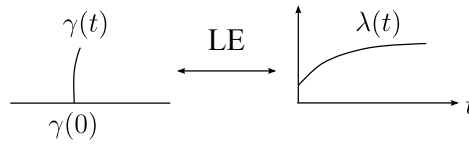


Figure 3.1:  $\lambda(t)$  is the driving function for  $\gamma(t)$ .

For any continuous driving function  $\lambda: [0, T] \rightarrow \mathbb{R}$ , the solution  $g_t(z)$  of (2.1) satisfies

$$\log g'_t(z) = - \int_0^t \frac{2}{[g_u(z) - \lambda(u)]^2} du \quad (3.1)$$

$$g''_t(z) = 4g'_t(z) \int_0^t \frac{g'_u(z)}{[g_u(z) - \lambda(u)]^3} du \quad (3.2)$$

for all  $z \in H_t$ . Equality (3.1) can be derived easily if we differentiate (2.1) with respect to  $z$ , which gives

$$\frac{\partial}{\partial t} g'_t(z) = - \frac{2g'_t(z)}{[g_t(z) - \lambda(t)]^2}.$$

To prove (3.2), we differentiate (3.1) with respect to  $z$ . We comment that (3.1) can be used to estimate the size of  $|g'_s(\gamma(s + \varepsilon))|$  as  $\varepsilon \downarrow 0$ , and this kind of estimate is crucial in our work as well as in other SLE problems. Equality (3.2) will be useful if one wants to obtain second derivative estimates near the tip.

---

<sup>2</sup>For any  $r > 0$ , the function  $\tilde{\lambda}(t) = r^{-2}\lambda(rt)$  satisfies  $\|\lambda\|_{\text{Lip}(\frac{1}{2})} = \|\tilde{\lambda}\|_{\text{Lip}(\frac{1}{2})}$ . The analog statement is not true for  $\text{Lip}(\frac{1}{2} + \delta)$ -norm

Equalities (3.1) and (3.2) hold for any continuous driving function  $\lambda: [0, T] \rightarrow \mathbb{R}$ . So far we haven't made any smoothness assumption on  $\lambda$ . We are going to do it in the coming sections.

### 3.2 When $\lambda \in \mathbf{Lip}(\frac{1}{2})$

We begin by stating some useful facts.

**Fact 3.2.1.** Suppose  $\lambda: [0, T] \rightarrow \mathbb{R}$  satisfies  $\|\lambda\|_{\mathbf{Lip}(\frac{1}{2})} < 4$ .

- (a) (Scaling property) If we define  $\tilde{\lambda}: [0, 1] \rightarrow \mathbb{R}$  by  $\tilde{\lambda}(s) := \frac{1}{\sqrt{T}} [\lambda(sT) - \lambda(0)]$ , then  $\|\tilde{\lambda}\|_{\mathbf{Lip}(\frac{1}{2})} = \|\lambda\|_{\mathbf{Lip}(\frac{1}{2})}$  and for all  $s \in [0, 1]$ ,

$$\gamma^{\tilde{\lambda}}(s) = \frac{1}{\sqrt{T}} \left[ \gamma^\lambda(sT) - \gamma^\lambda(0) \right].$$

For example, suppose a slit  $\gamma$  is parametrized by half-plane capacity and  $\lambda$  is the driving function for  $\gamma$ . The half-plane capacity reparametrization of the slit  $3\gamma(t)$  is  $\tilde{\gamma}(t) = 3\gamma(\frac{t}{9})$ . The scaling property says that the driving function for  $\tilde{\gamma}$  is  $\tilde{\lambda}(t) = 3\lambda(\frac{t}{9})$ .

- (b) (Stationary property) For any  $s \in (0, T)$ , the *time shift*  $\lambda_s: [0, T - s] \rightarrow \mathbb{R}$  of  $\lambda$  is the function  $\lambda_s(u) := \lambda(s + u)$ . The corresponding slit is

$$\gamma^{\lambda_s}(u) = g_s(\gamma(s + u)).$$

See Figure 3.2 for an illustration.

- (c) For any  $t \in [0, T]$ ,

$$\gamma^\lambda(t) = \lim_{y \downarrow 0} f_t(\lambda(t) + iy) = \lambda(t) - \int_0^t \frac{2}{\gamma(t-u, t)} du. \quad (3.3)$$

The scaling property is extremely useful; in many situations it suffices to work only on the case  $T = 1$ . The proofs of the scaling property and stationary property are elementary exercises. A proof of (3.3) is given below. We remark that the first equality in (3.3) is a non-trivial result for SLE curves, whose driving functions are random and almost surely not  $\mathbf{Lip}(\frac{1}{2})$  (see [24], [17]).

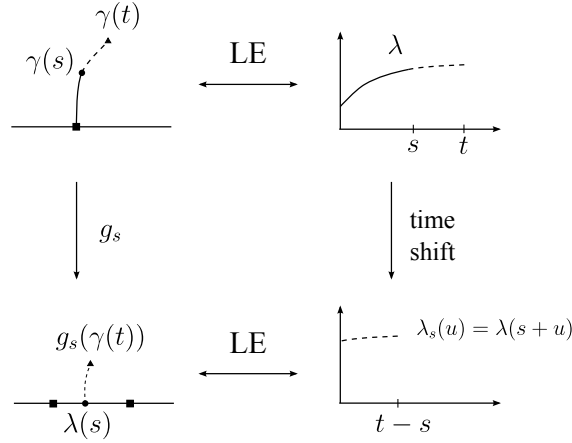


Figure 3.2: The stationary property states that the above diagram commutes.

*Proof of (3.3).* We know from [21] that  $\gamma([0, t])$  is a slit (in particular, locally connected), it follows from the Caratheodory continuity theorem (see, for example, [23, Theorem 2.1]) that the conformal map  $f_t: \mathbb{H} \rightarrow H_t$  is continuous at the boundary point  $\lambda(t)$ . This proves the first equality in (3.3). The second equality is an immediate consequence of the Loewner differential equation (2.1) and the fundamental theorem of calculus applied to the function  $u \mapsto g_u(\gamma(t))$  ( $0 \leq u \leq t$ ).  $\square$

For  $0 \leq \sigma < 4$ , let  $X_\sigma$  be the space of all (continuous) functions  $\lambda: [0, 1] \rightarrow \mathbb{R}$  satisfying  $\lambda(0) = 0$  and  $\|\lambda\|_{\text{Lip}(\frac{1}{2})} \leq \sigma$ . Under the supremum norm  $\|\cdot\|_\infty$ , the metric space  $X_\sigma$  is compact. It is known [18, Theorem 4.1] that the map  $X_\sigma \rightarrow \mathbb{H}$  defined by  $\lambda \mapsto \gamma^\lambda(1)$  is continuous. It follows that  $E_\sigma = \{\gamma^\lambda(1): \lambda \in X_\sigma\}$  is a compact subset of  $\mathbb{H}$ .

By the scaling property,

$$\gamma^\lambda(t) \in \sqrt{t} E_\sigma$$

for any  $0 \leq t \leq 1$ ,  $\lambda \in X_\sigma$ ,  $0 \leq \sigma < 4$ . On the other hand, it is easy to show (using compactness argument) that  $E_\sigma$  shrinks to a singleton  $\{2i\}$  as  $\sigma \rightarrow 0$ . Our first question is: at what rate does the diameter of  $E_\sigma$  go to zero?

**Lemma 3.2.2.** *Suppose  $\lambda: [0, 1] \rightarrow \mathbb{R}$  satisfies the  $\sigma$ -Lip( $\frac{1}{2}$ ) condition with  $0 \leq \sigma < 4$  and*

$\lambda(0) = 0$ . Then

$$\left| \operatorname{Re} \gamma^\lambda(1) \right| \leq \sigma \quad \text{and} \quad 4 - \sigma^2 \leq [\operatorname{Im} \gamma^\lambda(1)]^2 \leq 4.$$

In particular,  $\operatorname{diam}(E_\sigma) \leq c\sigma$  for all  $0 \leq \sigma < 4$ , where  $c > 0$  is an absolute constant.

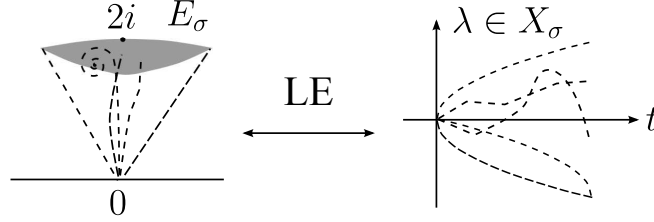


Figure 3.3: A sketch of the compact set  $E_\sigma$  for  $\sigma = 1$ . When  $\sigma$  is close to zero, the set  $E_\sigma$  becomes very thin by Lemma 3.2.2.

*Proof.* Write  $\gamma(t) = \gamma^\lambda(t)$  for the sake of notation. The estimate  $|\operatorname{Re} \gamma(1)| \leq \sigma$  follows from the following simple observation. If  $a \leq \lambda(t) \leq b$  for all  $t \in [0, 1]$ ,  $z_0 \in \mathbb{H}$  and  $\operatorname{Re}(z_0) > b$  (respectively  $\operatorname{Re}(z_0) < a$ ), then the Loewner differential equation (2.1) implies that  $g_t(z_0)$  is defined and satisfies  $\operatorname{Re} g_t(z_0) > b$  (respectively  $\operatorname{Re} g_t(z_0) < a$ ) for all  $t \in [0, 1]$ .

To estimate  $\operatorname{Im} \gamma(1)$ , we use the fact that

$$\gamma(1) = \lim_{y \downarrow 0} h_1(\lambda(1) + iy),$$

where  $h_t(z)$  is the solution to the initial value problem

$$\begin{cases} \dot{h}_t(z) = -\frac{2}{h_t(z) - \lambda(1-t)} \\ h_0(z) = z \end{cases} \quad (3.4)$$

Fix any  $y > 0$  and write  $h_t(\lambda(1) + iy) = x_t + iy_t$  ( $x_t, y_t \in \mathbb{R}$ ). Let  $A_t = (x_t - \lambda(1-t))^2$  and  $B_t = y_t^2$ . By scaling, or by our argument in the beginning of the proof,  $A_t \leq \sigma^2 t$  for all  $0 \leq t \leq 1$ . Comparing the imaginary parts of (3.4), we have

$$\dot{B}_t = \frac{4B_t}{A_t + B_t}.$$

Since  $A_t \geq 0$ ,  $\dot{B}_t \leq 4$  and therefore  $B_1 \leq 4 + y$ . We have  $\operatorname{Im} h_t(\lambda(1) + iy) \leq \sqrt{4 + y}$ . Letting  $y \downarrow 0$  gives  $\operatorname{Im} \gamma(1) \leq 2$ .

For the lower bound of  $B_1$ , we assume without loss of generality that  $a := 4 - \sigma^2 > 0$ . (Otherwise, the lower bound is trivial.) Suppose for the sake of contradiction that  $B_t < at$  for some  $t \in [0, 1)$ . Let  $T = \inf\{t \geq 0: B_t < at\}$ . We have  $0 < T < 1$ ,  $B_T = aT$  and, for  $0 \leq t \leq T$ ,

$$\dot{B}_t = \frac{4B_t}{A_t + B_t} \geq \frac{4at}{\sigma^2 t + at} = a.$$

This shows that  $B_T \geq B_0 + aT > aT$ , which is a contradiction. We have proved that  $B_1 \geq 4 - \sigma^2$ .  $\square$

Consider the example  $\lambda(t) = \sigma(1 - \sqrt{1-t})$ . When  $0 < \sigma < 4$ , this driving function satisfies the  $\sigma$ -Lip( $\frac{1}{2}$ ) condition and generates a logarithmic spiral with tip

$$\gamma(1) = \frac{\sigma}{2} + i\sqrt{4 - \frac{\sigma^2}{4}}.$$

(See [13] for the computation and [18] for a more conceptual approach.) This example together with Lemma 3.2.2 show

$$c_1\sigma \leq \text{diam}(E_\sigma) \leq c_2\sigma.$$

for all  $0 < \sigma < 4$ , where  $c_1, c_2 > 0$  are absolute constants.

The compactness of  $E_\sigma$  has a simple geometric consequence. If  $\lambda(0) = 0$  and  $\sigma = \|\lambda\|_{\text{Lip}(\frac{1}{2})} < 4$ , by scaling we see that  $\gamma(t) \in \sqrt{t}E_\sigma$  and the slit  $\gamma$  is contained in a cone whose angle depends on  $\sigma$ . If  $\lambda \in \text{Lip}(\frac{1}{2} + \delta)$ , one has  $\gamma(t) \in \sqrt{t}E_{\sigma(t)}$  with  $\sigma(t) \lesssim t^\delta$  as  $t \rightarrow 0$ . The slit grows vertically. See Figure 3.4.

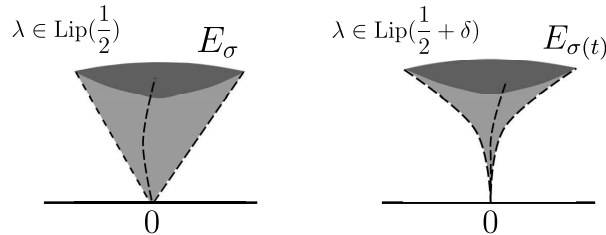


Figure 3.4: One important difference between slits of Lip( $\frac{1}{2}$ ) and Lip( $\frac{1}{2} + \delta$ ) driving functions.

In section 3.3, we will show that

$$\gamma'(t) = \lim_{\varepsilon \downarrow 0} \frac{i}{\sqrt{\varepsilon} g'_{t-\varepsilon}(\gamma(t))} = \lim_{\varepsilon \downarrow 0} \frac{i}{\sqrt{\varepsilon} g'_t(\gamma(t+\varepsilon))}$$

under an appropriate smoothness assumption on  $\lambda$ . We will see that  $|g'_t(\gamma(t+\varepsilon))| \simeq \varepsilon^{-\frac{1}{2}}$  as  $\varepsilon \downarrow 0$  is more or less equivalent to the existence of  $\gamma'(t)$ . Of course, in this section we are still in the  $\text{Lip}(\frac{1}{2})$  case and do not expect  $\gamma(t)$  to be differentiable. The next lemma says that  $|g'_{t-\varepsilon}(\gamma(t))| \asymp \varepsilon^{-\frac{1}{2}+O(\sigma)}$ , with an error term in the exponent.

**Lemma 3.2.3.** *If  $\lambda: [0, T] \rightarrow \mathbb{R}$  satisfies the  $\sigma$ -Lip( $\frac{1}{2}$ ) condition for some  $\sigma \in [0, 1]$ , then for any  $0 < s < t \leq T$ ,*

$$\left(\frac{t}{t-s}\right)^{\frac{1}{2}-c\sigma} \leq |g'_s(\gamma(t))| \leq \left(\frac{t}{t-s}\right)^{\frac{1}{2}+c\sigma},$$

where  $c > 0$  is an absolute constant.

*Proof.* Without loss of generality, we assume  $s = 1$  and  $\lambda(0) = 0$ . Let  $w = \gamma(t)$ . Then (3.1) gives

$$\log g'_1(w) = \int_0^1 -\frac{2}{\tau(u, t)^2} \frac{du}{t-u},$$

where  $\tau(u, t) = \frac{g_u(w) - \lambda(u)}{\sqrt{t-u}}$  was defined in Notation (vi) in section 3.1. If the driving function  $\lambda$  is identically zero,  $\tau^\lambda(u, t)$  becomes  $\tau^0(u, t) \equiv 2i$  and the above equality reduces to

$$\frac{1}{2} \log \frac{t}{t-1} = \int_0^1 \frac{1}{2} \frac{du}{t-u}.$$

Subtracting the two equalities gives

$$\log g'_1(w) - \frac{1}{2} \log \frac{t}{t-1} = - \int_0^1 \left[ \frac{1}{2} + \frac{2}{\tau(u, t)^2} \right] \frac{du}{t-u}. \quad (3.5)$$

By Lemma 3.2.2,  $\left| \frac{1}{2} + \frac{2}{\tau(u, t)^2} \right| \leq c\sigma$ , where  $c > 0$  is an absolute constant. Here we have implicitly used the condition  $\sigma \leq 1$ , which guarantees that  $\tau(u, t)$  stays in a fixed compact set  $E_1 \subseteq \mathbb{H}$ . The absolute constant  $c$  in our last estimate is related to the derivative bound of the map  $z \mapsto \frac{2}{z^2}$  on the compact set  $E_1$ .

Finally, equation (3.5) gives

$$\left| \log g'_1(w) - \frac{1}{2} \log \frac{t}{t-1} \right| \leq \int_0^1 \frac{c\sigma}{t-u} du = c\sigma \log \frac{t}{t-1}.$$

□

Lemma 3.2.2 and the following Theorem 3.2.4 will serve as two fundamental tools for the rest of this paper.

**Theorem 3.2.4** (Lipschitz continuity). *Suppose  $\lambda, \tilde{\lambda}: [0, T] \rightarrow \mathbb{R}$  satisfy the  $\sigma$ -Lip( $\frac{1}{2}$ ) condition for  $\sigma = 1$ . Then,  $\|\gamma^\lambda - \gamma^{\tilde{\lambda}}\|_\infty \leq c \|\lambda - \tilde{\lambda}\|_\infty$ , where  $c > 0$  is an absolute constant.*

Fix any  $T > 0$ . Let  $\tilde{X}_\sigma$  be the space of all (continuous)  $\lambda: [0, T] \rightarrow \mathbb{R}$  with  $\|\lambda\|_{\text{Lip}(\frac{1}{2})} \leq \sigma$ . Recently, Joan Lind, Don Marshall and Steffen Rohde proved [18, Theorem 4.1] that the map  $\lambda \mapsto \gamma^\lambda$  is a continuous map from  $(\tilde{X}_\sigma, \|\cdot\|_\infty)$  into  $(C([0, T]), \|\cdot\|_\infty)$  for every  $0 \leq \sigma < 4$ . Their proof uses the theory of quasi-conformal maps. When  $\sigma \leq 1$ , Theorem 3.2.4 says the map is Lipschitz continuous.

For  $\sigma = 1$ , the slit  $\gamma^\lambda$  of  $\lambda \in X_\sigma$  is contained in the cone  $V = \{z \in \mathbb{H}: \frac{\pi}{4} < \arg(z) < \frac{3\pi}{4}\}$ . Theorem 3.2.4 remains true (with a larger absolute constant  $c$ ) if the constant 1 is replaced by a slightly larger number where the slit is still contained in  $V$ . We do not know whether Theorem 3.2.4 holds for  $\sigma = 4 - \varepsilon$  when  $\varepsilon > 0$  is small.

*Proof.* By scaling we can assume  $T = 1$ . Let  $\varepsilon := \sup_{0 \leq t \leq 1} |\lambda(t) - \tilde{\lambda}(t)|$ . Without loss of generality, we can further assume  $\lambda(1) = \tilde{\lambda}(1)$ . (If not, translate one of the slits by  $\lambda(1) - \tilde{\lambda}(1)$ , which has absolute value at most  $\varepsilon$ .) We extend  $\lambda$  so that  $\lambda(t) = \lambda(1)$  for all  $t \geq 1$ . Fix any small  $\delta > 0$ . The tip  $\gamma^\lambda(1 + \delta)$  is equal to  $h_1$ , where  $h_t: [0, 1] \rightarrow \mathbb{C}$  is the solution of the backward Loewner differential equation

$$\begin{cases} \partial_t h_t = -\frac{2}{h_t - \xi(t)} \\ h_0 = \xi(0) + 2i\sqrt{\delta} \end{cases}$$

and  $\xi(t) := \lambda(1 - t)$ . Similarly, we extend  $\tilde{\lambda}$ , define  $\tilde{\xi}(t)$ ,  $\tilde{h}_t$  and let  $Y(t) = h_t - \tilde{h}_t$ . Note that  $Y(1) = \gamma^\lambda(1 + \delta) - \gamma^{\tilde{\lambda}}(1 + \delta)$  and  $Y(0) = 0$  since  $\lambda(1) = \tilde{\lambda}(1)$ . By direct computation,

$$\partial_t Y(t) = A(t) \left[ Y(t) + \left( \tilde{\xi}(t) - \xi(t) \right) \right], \quad (3.6)$$

where  $A(t) = 2(h_t - \xi(t))^{-1}(\tilde{h}_t - \tilde{\xi}(t))^{-1}$ . We view (3.6) as a first order linear ODE in  $Y(t)$  and solve it using the method of integrating factor. Let  $\mu(t) = \exp\left(-\int_0^t A(s) ds\right)$ . One has  $\frac{d}{dt} [\mu(t)Y(t)] = \mu(t)A(t) \left( \tilde{\xi}(t) - \xi(t) \right)$  and

$$Y(1) = \int_0^1 \frac{\mu(s)}{\mu(1)} A(s) \left( \tilde{\xi}(s) - \xi(s) \right) ds.$$

We know  $\left| \tilde{\xi}(s) - \xi(s) \right| \leq \varepsilon$  for all  $s \in [0, 1]$ . To complete the proof, it remains to estimate the size of the integrating factor  $\mu(t)$ .

Notice that  $A(t) \in \frac{1}{t+\delta}K$  for all  $t \in [0, 1]$ , where

$$K = \left\{ \frac{2}{z_1 z_2} : z_1, z_2 \in E_1 \right\}$$

and  $E_1$  is the compact set defined right before Lemma 3.2.2 (see Figure 3.3). For convenience of the readers, we recall the definition

$$E_1 := \left\{ \gamma^\lambda(1) : \lambda(0) = 0 \text{ and } \|\lambda\|_{\text{Lip}(\frac{1}{2}, [0, 1])} \leq 1 \right\}.$$

By Lemma 3.2.2 and the assumption  $\sigma = 1$ ,  $K$  is contained in the left half-plane  $\{z \in \mathbb{C} : \text{Re}(z) < 0\}$ . Let

$$\beta = \inf\{-\text{Re}(z) : z \in K\} > 0.$$

Since  $A(t) \in \frac{1}{t+\delta}K$ , we have  $-\text{Re} A(t) \geq \frac{\beta}{t+\delta}$  for all  $t \in [0, 1]$ . For any  $s \in [0, 1]$ ,

$$\frac{\mu(s)}{\mu(1)} = \exp \left[ \int_s^1 A(u) du \right] \quad \text{and} \quad \left| \frac{\mu(s)}{\mu(1)} \right| \leq (s + \delta)^\beta.$$

Finally,

$$\left| \gamma^\lambda(1 + \delta) - \tilde{\gamma}^\lambda(1 + \delta) \right| = |Y(1)| \leq \varepsilon \int_0^1 \left| \frac{\mu(s)}{\mu(1)} \right| \cdot |A(s)| ds \leq c\varepsilon \int_0^1 (s + \delta)^{-1+\beta} ds.$$

where  $c = \sup_{z \in K} |z| < \infty$  is an absolute constant. The result follows by letting  $\delta \rightarrow 0$ .  $\square$

### 3.3 When $\lambda \in \text{Lip}(\frac{1}{2} + \delta)$ with $0 < \delta \leq \frac{1}{2}$

In this section,  $\lambda : [0, T] \rightarrow \mathbb{R}$  satisfies the  $(M, T, \delta)$ - $\text{Lip}(\frac{1}{2} + \delta)$  condition, i.e.  $\|\lambda\|_{\text{Lip}(\frac{1}{2})} \leq 1$  and

$$|\lambda(t_1) - \lambda(t_2)| \leq M |t_1 - t_2|^{\frac{1}{2} + \delta}$$

for all  $t_1, t_2 \in [0, T]$ , where  $M, T, \delta > 0$ . The extra smoothness allows us to improve the exponent in Lemma 3.2.3.

**Lemma 3.3.1.** *If  $\lambda : [0, T] \rightarrow \mathbb{R}$  satisfies the  $(M, T, \delta)$ - $\text{Lip}(\frac{1}{2} + \delta)$  condition for some  $0 < \delta \leq \frac{1}{2}$ , then for any  $0 < s < t \leq T$ ,*

$$\frac{1}{C} \leq \sqrt{\frac{t-s}{t}} |g'_s(\gamma(t))| \leq C, \tag{3.7}$$

where  $C = C(M, T, \delta) > 0$ . Moreover, for all  $s \in (0, T)$ , the limit

$$\lim_{\varepsilon \downarrow 0} \sqrt{\varepsilon} g'_s(\gamma(s + \varepsilon)) = \sqrt{s} \exp \left\{ - \int_0^s \left[ \frac{1}{2u} + \frac{2}{\gamma(s-u, s)^2} \right] du \right\}$$

exists and is nonzero.

*Proof.* As in the proof of Lemma 3.2.3,

$$\log g'_s(\gamma(t)) - \frac{1}{2} \log \frac{t}{t-s} = - \int_0^s \left[ \frac{1}{2} + \frac{2}{\tau(u, t)^2} \right] \frac{du}{t-u}.$$

The  $(M, T, \delta)$ -Lip $(\frac{1}{2} + \delta)$  condition implies  $\|\lambda\|_{\text{Lip}(\frac{1}{2}, [u, t])} \leq M(t-u)^\delta$ . Lemma 3.2.2 gives an estimate of our integral kernel:

$$\left| \frac{1}{2} + \frac{2}{\tau(u, t)^2} \right| \leq c |\tau(u, t) - 2i| \leq cM(t-u)^\delta$$

for some absolute constant  $c > 0$ . (Again, we have implicitly used the condition  $\|\lambda\|_{\text{Lip}(\frac{1}{2})} \leq 1$ , which guarantees that  $\tau^\lambda(u, t)$  stays in a fixed compact set  $E_1 \subseteq \mathbb{H}$ .) Therefore,

$$\left| \log g'_s(\gamma(t)) - \frac{1}{2} \log \frac{t}{t-s} \right| \leq cM \int_0^s (t-u)^{\delta-1} du = \frac{cM}{\delta} (t^\delta - (t-s)^\delta) \leq \frac{cMs^\delta}{\delta}$$

and (3.7) follows. Taking  $t = s + \varepsilon$  with  $\varepsilon > 0$  gives

$$\log \frac{\sqrt{\varepsilon} g'_s(\gamma(s + \varepsilon))}{\sqrt{s + \varepsilon}} = - \int_0^s \left[ \frac{1}{2} + \frac{2}{\tau(u, s + \varepsilon)^2} \right] \frac{du}{s + \varepsilon - u}.$$

The existence of  $\lim_{\varepsilon \downarrow 0} \sqrt{\varepsilon} g'_s(\gamma(s + \varepsilon))$  follows from the Lebesgue dominated convergence theorem.  $\square$

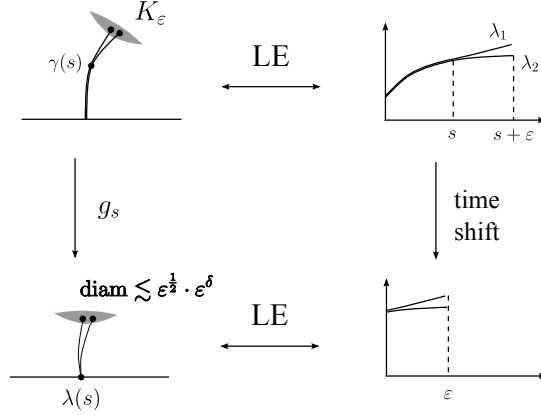
**Lemma 3.3.2.** *Let  $\lambda_1, \lambda_2: [0, T] \rightarrow \mathbb{R}$  satisfy the  $(M, T, \delta)$ -Lip $(\frac{1}{2} + \delta)$  condition for some  $0 < \delta \leq \frac{1}{2}$ . Suppose  $\lambda_1 = \lambda_2$  on  $[0, s]$  for some  $s \in (0, T)$ . Then, for any  $\varepsilon \in (0, T - s)$ ,*

$$\left| \gamma^{\lambda_1}(s + \varepsilon) - \gamma^{\lambda_2}(s + \varepsilon) \right| \leq C\varepsilon^{1+\delta},$$

where  $C = C(M, T, \delta, s) > 0$ .

*Proof.* With  $M, T, \delta$  fixed, let  $X$  be the space of all functions  $\lambda: [0, T] \rightarrow \mathbb{R}$  satisfying the  $(M, T, \delta)$ -Lip $(\frac{1}{2} + \delta)$  condition and  $\lambda = \lambda_1$  on  $[0, s]$ . For each  $\varepsilon \in (0, T - s)$ , consider the compact set

$$K_\varepsilon = \left\{ \gamma^\lambda(s + \varepsilon) \in \mathbb{H} : \lambda \in X \right\}.$$



It suffices to show  $\text{diam}(K_\epsilon) \leq C\epsilon^{1+\delta}$ . Let  $g_s = g_s^{\lambda_j}$  be the hydrodynamically normalized conformal map from  $\mathbb{H} \setminus \gamma^{\lambda_j}([0, s])$  onto  $\mathbb{H}$ , and let  $f_s = g_s^{-1}$ . Note that

$$\text{diam}(K_\epsilon) \leq \text{diam}(g_s(K_\epsilon)) \cdot \sup_{z \in E} |f'_s(z)|,$$

where  $E$  is the convex hull of  $g_s(K_\epsilon)$ . By Lemma 3.2.2,  $\text{diam}(g_s(K_\epsilon)) \leq cM\epsilon^{\frac{1}{2}+\delta}$ . On the other hand, Lemma 3.3.1 implies

$$\sup_{z \in g_s(K_\epsilon)} |f'_s(z)| \leq C\sqrt{\epsilon}, \quad (3.8)$$

where  $C = C(M, T, \delta, s) > 0$  does not depend on  $\epsilon$ . If we replace  $g_s(K_\epsilon)$  by its convex hull, the supremum in (3.8) can only increase by a bounded factor, by Koebe distortion theorem (see [23]) and the fact that the hyperbolic diameter of  $g_s(K_\epsilon)$  is bounded above by some absolute constant  $c_1$ . Actually, we can take  $c_1$  to be the hyperbolic diameter of the set  $E_1$  in Figure 3.3.  $\square$

**Corollary 3.3.3.** *Suppose  $\lambda: [0, T] \rightarrow \mathbb{R}$  satisfies the  $(M, T, \delta)$ -Lip( $\frac{1}{2} + \delta$ ) condition for some  $0 < \delta \leq \frac{1}{2}$ , then  $\gamma = \gamma^\lambda$  is differentiable on  $(0, T)$  and*

$$\gamma'(s) = \frac{i}{\sqrt{s}} \exp \left\{ \int_0^s \left[ \frac{1}{2u} + \frac{2}{\gamma(s-u, s)^2} \right] du \right\} \quad (3.9)$$

for all  $s \in (0, T)$ . At  $s = T$ , the left derivative  $\gamma'_-(T)$  exists and is given by the same formula.

*Proof.* It suffices to show that the right derivative  $\gamma'_+(s)$  exists for  $s \in (0, T)$  and is given by (3.9). (Using Theorem 3.2.4, it is not hard to see that (3.9) is continuous in  $s$ , and it is an exercise to show that any right differentiable function on an open interval with continuous right derivative is in fact differentiable. A proof of this elementary fact can be found in [16, Lemma 4.3].)

Fix any  $s \in (0, T)$ . We may assume without loss of generality that  $\lambda(t) = \lambda(s)$  for all  $t \in [s, T]$ , because modifying  $\lambda$  this way does not change the right derivative  $\gamma'_+(s)$ , by Lemma 3.3.2. We have  $\gamma(s + \varepsilon) = f_s(\lambda(s) + 2i\sqrt{\varepsilon})$  and therefore

$$\gamma(s + \varepsilon) - \gamma(s) = \int_0^\varepsilon \frac{if'_s(\lambda(s) + 2i\sqrt{u})}{\sqrt{u}} du$$

for all  $\varepsilon \in (0, T - s]$ . By Lemma 3.3.1, the integrand is continuous at  $u = 0$ . It follows that  $\gamma'_+(s)$  exists and is given by

$$\gamma'_+(s) = \lim_{\varepsilon \downarrow 0} \frac{if'_s(\lambda(s) + 2i\sqrt{\varepsilon})}{\sqrt{\varepsilon}} = \frac{i}{\sqrt{s}} \exp \left\{ \int_0^s \left[ \frac{1}{2u} + \frac{2}{\gamma(s-u, s)^2} \right] du \right\}.$$

□

By formula (3.9), proving the smoothness of  $\gamma$  is equivalent to proving the smoothness of the integral, which we call  $L(s)$  from now on.

**Notation 3.3.4.** Let

$$L(s) = L^\lambda(s) = \int_0^s \left[ \frac{1}{2u} + \frac{2}{\gamma(s-u, s)^2} \right] du.$$

This integral makes sense provided that  $\lambda: [0, T] \rightarrow \mathbb{R}$  satisfies the  $(M, T, \delta)$ -Lip( $\frac{1}{2} + \delta$ ) condition for some  $0 < \delta \leq \frac{1}{2}$ . In the coming sections, when we impose more regularity assumptions on  $\lambda$ , we will keep this notation.

In the proof of Lemma 3.3.1, we have implicitly proved an upper bound of  $|L(s)|$ . By (3.9), controlling the size of  $|L(s)|$  gives an upper bound of  $\left| \gamma'(s) - \frac{i}{\sqrt{s}} \right|$ . This estimate will be useful later and we now explicitly state it.

**Lemma 3.3.5.** *Under the  $(M, T, \delta)$ -Lip( $\frac{1}{2} + \delta$ ) condition with  $0 < \delta \leq \frac{1}{2}$ ,*

$$|L(s)| \leq \frac{cMs^\delta}{\delta}$$

for all  $s \in [0, T]$ , where  $c > 0$  is an absolute constant.

We will show that  $L \in \text{Lip}(\delta)$ . For any  $s \in (0, T)$  and  $\varepsilon \in [0, T - s]$ ,

$$\begin{aligned} L(s + \varepsilon) - L(s) &= \int_0^s \left[ \frac{2}{\tau(s + \varepsilon - u, s + \varepsilon)^2} - \frac{2}{\tau(s - u, s)^2} \right] \frac{du}{u} + \\ &\quad \int_s^{s+\varepsilon} \left[ \frac{1}{2} + \frac{2}{\tau(s + \varepsilon - u, s + \varepsilon)^2} \right] \frac{du}{u}. \end{aligned} \quad (3.10)$$

Since  $\|\lambda\|_{\text{Lip}(\frac{1}{2}, [a, b])} \leq M(b - a)^\delta$ , Lemma 3.2.2 gives  $|\tau(s + \varepsilon - u, s + \varepsilon) - 2i| \leq cMu^\delta$  and therefore

$$\int_s^{s+\varepsilon} \left| \frac{1}{2} + \frac{2}{\tau(s + \varepsilon - u, s + \varepsilon)^2} \right| \frac{du}{u} \leq \int_s^{s+\varepsilon} cMu^{\delta-1} du \leq \frac{cM}{\delta} \varepsilon^\delta. \quad (3.11)$$

The second integral in (3.10) is under control. The first integral can be estimated using the quantity

$$\omega(s, u, \varepsilon) := \sup_{0 \leq v \leq u} |\lambda(s + \varepsilon - v) - \lambda(s + \varepsilon - u) - \lambda(s - v) + \lambda(s - u)|. \quad (3.12)$$

Note that  $\omega(s, u, \varepsilon)$  can be expressed as  $\|\lambda_1 - \lambda_2\|_\infty$ , where  $\lambda_1, \lambda_2: [0, u] \rightarrow \mathbb{R}$  are driving functions whose tips are  $\gamma(s + \varepsilon - u, s + \varepsilon)$  and  $\gamma(s - u, s)$ . Theorem 3.2.4 implies

$$|\gamma(s + \varepsilon - u, s + \varepsilon) - \gamma(s - u, s)| \leq c\omega(s, u, \varepsilon)$$

for some absolute constant  $c > 0$ . This gives an estimate of the first integral in (3.10). We have proved the following lemma.

**Lemma 3.3.6.** *If  $\lambda: [0, T] \rightarrow \mathbb{R}$  satisfies the  $(M, T, \delta)$ -Lip( $\frac{1}{2} + \delta$ ) condition for some  $0 < \delta \leq \frac{1}{2}$ , then for any  $0 \leq s < s + \varepsilon \leq T$ ,*

$$\begin{aligned} |L(s + \varepsilon) - L(s)| &\leq c \int_0^s u^{-\frac{3}{2}} \omega(s, u, \varepsilon) du + \frac{cM}{\delta} [(s + \varepsilon)^\delta - s^\delta] \\ &\leq c \int_0^s u^{-\frac{3}{2}} \omega(s, u, \varepsilon) du + \frac{cM\varepsilon^\delta}{\delta}, \end{aligned} \quad (3.13)$$

where  $c > 0$  is an absolute constant and  $\omega(s, u, \varepsilon)$  is defined in (3.12).

The first inequality in (3.13) will be used in section 3.4, and we use the second estimate in this section. When  $\delta = \frac{1}{2}$ , we will see soon (3.13) gives  $|L(s + \varepsilon) - L(s)| = O(\sqrt{\varepsilon})$  as  $\varepsilon \downarrow 0$ . We achieve this by controlling the size of  $\omega(s, u, \varepsilon)$ . The estimate depends on the regularity of  $\lambda$ . In this section,  $\lambda \in \text{Lip}(\frac{1}{2} + \delta)$  and the following estimate of  $\omega(s, u, \varepsilon)$  is what we should expect. The estimate will be improved in section 3.4 under the assumption  $\lambda \in C^{1, \delta}$ .

**Lemma 3.3.7.** *Let  $\lambda: [0, T] \rightarrow \mathbb{R}$  satisfy the  $(M, T, \delta)$ -Lip( $\frac{1}{2} + \delta$ ) condition. For any  $0 \leq s < s + \varepsilon \leq T$  and  $0 < u < s$ ,*

$$\omega(s, u, \varepsilon) \leq \begin{cases} 2Mu^{\frac{1}{2}+\delta}, & u \leq \varepsilon \\ 2M\varepsilon^{\frac{1}{2}+\delta}, & u \geq \varepsilon \end{cases}$$

and  $|L(s + \varepsilon) - L(s)| \leq C\varepsilon^\delta$ , where  $C = C(M, T, \delta) > 0$ .

*Proof.* When  $u \leq \varepsilon$ ,

$$\begin{aligned} & |\lambda(s + \varepsilon - v) - \lambda(s + \varepsilon - u) - \lambda(s - v) + \lambda(s - u)| \\ & \leq |\lambda(s + \varepsilon - v) - \lambda(s + \varepsilon - u)| + |\lambda(s - v) - \lambda(s - u)| \\ & \leq 2Mu^{\frac{1}{2}+\delta} \end{aligned}$$

for any  $0 \leq v \leq u$ . When  $u \geq \varepsilon$ , we rearrange terms as

$$\begin{aligned} & |\lambda(s + \varepsilon - v) - \lambda(s + \varepsilon - u) - \lambda(s - v) + \lambda(s - u)| \\ & \leq |\lambda(s + \varepsilon - v) - \lambda(s - v)| + |\lambda(s + \varepsilon - u) - \lambda(s - u)| \\ & \leq 2M\varepsilon^{\frac{1}{2}+\delta} \end{aligned}$$

for any  $0 \leq v \leq u$ . We have proved the desired estimates of  $\omega(s, u, \varepsilon)$ .

To prove  $|L(s + \varepsilon) - L(s)| \leq C\varepsilon^\delta$ , we split the integral in (3.13):

$$\left| \int_0^s u^{-\frac{3}{2}} \omega(s, u, \varepsilon) du \right| \leq \int_0^\varepsilon 2Mu^{-1+\delta} du + \int_\varepsilon^s 2Mu^{-\frac{3}{2}} \varepsilon^{\frac{1}{2}+\delta} du \leq C\varepsilon^\delta$$

□

We have all the ingredients for proving our first main result.

**Theorem 3.3.8.** *Let  $\lambda: [0, T] \rightarrow \mathbb{R}$  be such that*

$$|\lambda(t_1) - \lambda(t_2)| \leq M |t_1 - t_2|^{\frac{1}{2}+\delta}$$

for all  $t_1, t_2 \in [0, T]$ , where  $M > 0$  and  $\delta \in (0, \frac{1}{2}]$  are constants. Then

(a)  $\Gamma(t) := \gamma(t^2)$  is  $C^{1,\delta}$  regular<sup>3</sup> on  $[0, \sqrt{T}]$ ; and

---

<sup>3</sup>whose derivative is nonzero everywhere

(b) the slit  $\gamma(t)$  grows vertically at  $t = 0$ .

With an extra assumption that  $\|\lambda\|_{\text{Lip}(\frac{1}{2})} \leq 1$ , these statements are quantitative:

$$\|\Gamma\|_{C^{1,\delta}([0,T])} \leq N \quad \text{and} \quad \inf_{t \in [0,T]} |\Gamma'(t)| \geq \frac{1}{N}, \quad (3.14)$$

where  $N = N(M, T, \delta) > 0$  depends only on  $M$ ,  $T$  and  $\delta$ . Furthermore,

$$|\Gamma'(t) - 2i| \leq N t^{2\delta} \quad (0 < t \leq \sqrt{T}). \quad (3.15)$$

*Proof of Theorem 3.3.8.* We first assume  $\|\lambda\|_{\text{Lip}(\frac{1}{2})} \leq 1$ . By Corollary 3.3.3,  $\Gamma(t) = \gamma(t^2)$  is differentiable and  $\Gamma'(t) = 2ie^{L(t^2)}$ . With this formula of  $\Gamma'(t)$ , we claim that  $\Gamma'(t)$  is  $\text{Lip}(\delta)$ . To see this, we first note from Lemma 3.3.5 that  $\sup_{0 \leq t \leq \sqrt{T}} |L(t^2)| \leq R$  for some constant  $R = \frac{cMT^\delta}{\delta}$  depending only on  $M$ ,  $T$  and  $\delta$ . This tells us

$$|\Gamma'(t_1) - \Gamma'(t_2)| \leq 2 \left( \sup_{|z| \leq R} |e^z| \right) |L(t_1^2) - L(t_2^2)| \leq C |t_1 - t_2|^\delta$$

for some  $C = C(M, T, \delta) > 0$  by Lemma 3.3.7. This proves (a). The estimates (3.14) and (3.15) follow from Lemma 3.3.5 and other estimates we have proved. For example,

$$|\Gamma'(t)| = 2 \left| e^{L(t^2)} \right| = 2e^{\text{Re} L(t^2)} \geq 2e^{-R}$$

and (3.15) can be derived from

$$|\Gamma'(t) - 2i| = 2 \left| e^{L(t^2)} - 1 \right| \leq 2 \left( \sup_{|z| \leq R} |e^z| \right) |L(t^2)|.$$

If  $\|\lambda\|_{\text{Lip}(\frac{1}{2})} > 1$ , we pick a partition  $0 = t_0 < t_1 < \dots < t_n = T$  for which  $M(t_{j+2} - t_j)^\delta < 1$  for all  $j = 0, 1, \dots, n-2$ . This guarantees that  $\|\lambda\|_{\text{Lip}(\frac{1}{2}, [t_j, t_{j+2}])} < 1$  for all  $j$ . It suffices to show

(i)  $\Gamma(t) = \gamma(t^2)$  is  $C^{1,\delta}$  regular on  $[t_0, t_2]$ ; and

(ii)  $\gamma(t)$  is  $C^{1,\delta}$  regular on  $[t_j, t_{j+2}]$  for  $j = 1, 2, \dots, n-2$ .

(i) follows from the argument of the previous paragraph, since we are back to the case  $\|\lambda\|_{\text{Lip}(\frac{1}{2})} \leq 1$ . To show (ii), we pick  $\varepsilon > 0$  for which  $M(t_{j+2} - t_j + \varepsilon)^\delta \leq 1$ . Again, by our earlier argument for the case  $\|\lambda\|_{\text{Lip}(\frac{1}{2})} \leq 1$ , the function

$$u \mapsto \gamma(t_j - \varepsilon, t_j + u)$$

is  $C^{1,\delta}$  regular on  $[0, t_{j+2} - t_j]$ , and therefore

$$u \mapsto \gamma(t_j + u) = f_{t_j - \varepsilon}(\lambda(t_j - \varepsilon) + \gamma(t_j - \varepsilon, t_j + u))$$

is also  $C^{1,\delta}$  regular on  $[0, t_{j+2} - t_j]$ . □

### 3.4 When $\lambda \in C^{1,\delta}$ with $0 < \delta \leq \frac{1}{2}$

In this section, our driving function  $\lambda: [0, T] \rightarrow \mathbb{R}$  satisfies the  $(M, T, 1, \delta)$ - $C^{1,\delta}$  condition with  $0 < \delta \leq \frac{1}{2}$ . That is to say,  $\|\lambda\|_{\text{Lip}(\frac{1}{2})} \leq 1$ ,  $\lambda \in C^1([0, T])$  and

$$\|\lambda\|_{C^{1,\delta}} = \sup_{t \in [0, T]} |\lambda(t)| + \sup_{t \in [0, T]} |\lambda'(t)| + \sup_{t_1 \neq t_2 \in [0, T]} \frac{|\lambda'(t_1) - \lambda'(t_2)|}{|t_1 - t_2|^\delta} \leq M.$$

Our goal is to improve the estimate  $|L(s + \varepsilon) - L(s)| = O(\varepsilon^\delta)$  given in section 3.3, and we are expecting  $O(\varepsilon^{\frac{1}{2} + \delta})$ . Any  $C^{1,\delta}$  function is Lipschitz, i.e.  $\text{Lip}(\frac{1}{2} + \frac{1}{2})$ . Applying Lemma 3.3.5 and the first inequality in (3.13) with  $\delta = \frac{1}{2}$  yields

$$|L(s)| \leq cM\sqrt{s} \tag{3.16}$$

and

$$|L(s + \varepsilon) - L(s)| \leq c \int_0^s u^{-\frac{3}{2}} \omega(s, u, \varepsilon) du + \frac{cM\varepsilon}{\sqrt{s}}, \tag{3.17}$$

for  $0 < s < s + \varepsilon \leq T$ . We now improve the estimate of  $\omega(s, u, \varepsilon)$  in Lemma 3.3.7 to the following. Recall the definition

$$\omega(s, u, \varepsilon) := \sup_{0 \leq v \leq u} |\lambda(s + \varepsilon - v) - \lambda(s + \varepsilon - u) - \lambda(s - v) + \lambda(s - u)|.$$

**Lemma 3.4.1.** *Let  $\lambda: [0, T] \rightarrow \mathbb{R}$  satisfy the  $(M, T, 1, \alpha)$ - $C^{1,\alpha}$  condition with  $0 < \alpha \leq 1$ .*

*For any  $0 \leq s < s + \varepsilon \leq T$  and  $0 < u < s$ ,*

$$\omega(s, u, \varepsilon) \leq \begin{cases} M\varepsilon^\alpha u, & u \leq \varepsilon \\ Mu^\alpha \varepsilon, & u \geq \varepsilon \end{cases}$$

If  $\alpha = \delta < \frac{1}{2}$ , for any  $0 < s < s + \varepsilon \leq T$ , we have

$$|L(s + \varepsilon) - L(s)| \leq \frac{cM}{1 - 2\delta} \cdot \left( \varepsilon^{\frac{1}{2} + \delta} + \frac{\varepsilon}{\sqrt{s}} \right), \quad (3.18)$$

where  $c > 0$  is an absolute constant. When  $\delta = \frac{1}{2}$ ,

$$|L(s + \varepsilon) - L(s)| \leq cM\varepsilon \cdot \left[ 1 + \log^+ \left( \frac{s}{\varepsilon} \right) + \frac{1}{\sqrt{s}} \right], \quad (3.19)$$

where  $\log^+ x = \max\{\log(x), 0\}$ .

*Proof.* The equalities

$$\begin{aligned} & \lambda(s + \varepsilon - v) - \lambda(s + \varepsilon - u) - \lambda(s - v) + \lambda(s - u) \\ &= \int_{s-u}^{s-v} \lambda'(w + \varepsilon) - \lambda'(w) dw \\ &= \int_{s-u}^{s+\varepsilon-u} \lambda'(w + u - v) - \lambda'(w) dw \end{aligned}$$

hold for all  $v \in [0, u]$ . Since  $\lambda' \in \text{Lip}(\alpha)$ , we deduce the desired estimate of  $\omega(s, u, \varepsilon)$ . If  $\alpha = \delta \in (0, \frac{1}{2})$ , equation (3.17) and our estimates of  $\omega(s, u, \varepsilon)$  give the following. (We assume  $s > \varepsilon$  in the computation below. When  $s \leq \varepsilon$ , the integral  $\int_\varepsilon^s$  is not present and our estimate still holds.)

$$\begin{aligned} |L(s + \varepsilon) - L(s)| &\leq cM \int_0^\varepsilon u^{-\frac{3}{2}} \varepsilon^\delta u du + cM \int_\varepsilon^s u^{-\frac{3}{2}} u^\delta \varepsilon du + \frac{cM\varepsilon}{\sqrt{s}} \\ &\leq cM\varepsilon^{\frac{1}{2} + \delta} + \frac{cM\varepsilon^{\frac{1}{2} + \delta}}{\frac{1}{2} - \delta} + \frac{cM\varepsilon}{\sqrt{s}} \end{aligned}$$

This proves (3.18). When  $\delta = \frac{1}{2}$ , the second term in the last expression should be replaced by  $cM\varepsilon \log^+ \frac{s}{\varepsilon}$ , and (3.19) follows.  $\square$

Combining Theorem 3.3.8 and Lemma 3.4.1, we have the following theorem.

**Theorem 3.4.2.** *Suppose  $\lambda: [0, T] \rightarrow \mathbb{R}$  satisfies the  $(M, T, 1, \delta)$ - $C^{1, \delta}$  condition with  $0 < \delta < \frac{1}{2}$ . Then the curve  $\Gamma(t) := \gamma(t^2)$  is  $C^{1, \frac{1}{2} + \delta}$  regular on  $[0, \sqrt{T}]$ . In fact,*

$$\|\Gamma\|_{C^{1, \frac{1}{2} + \delta}([0, T])} \leq N \quad \text{and} \quad \inf_{t \in [0, T]} |\Gamma'(t)| \geq \frac{1}{N},$$

where  $N = N(M, T, \delta) > 0$ . When  $\delta = \frac{1}{2}$ ,  $\Gamma'(t)$  is weakly Lipschitz in the sense that

$$|\Gamma'(t_1) - \Gamma'(t_2)| \leq N |t_1 - t_2| \max \left\{ 1, \log \frac{1}{|t_1 - t_2|} \right\} \quad (0 \leq t_1 < t_2 \leq \sqrt{T}), \quad (3.20)$$

where  $N = N(M, T) > 0$ .

We do not know whether or not  $\limsup_{\delta \uparrow \frac{1}{2}} N(M, T, \delta) < \infty$ . The (non-optimal) constant  $C = \frac{eM}{1-2\delta}$  in inequality (3.18) blows up when  $\delta \uparrow \frac{1}{2}$ . When  $\delta = \frac{1}{2}$ , inequality (3.19) implies  $|\Gamma'(s + \varepsilon) - \Gamma'(s)| = O(\varepsilon \log^+ \frac{1}{\varepsilon})$  as  $\varepsilon \downarrow 0$ . In particular,  $\Gamma(t)$  is weakly  $C^{1,1}$  in the sense that it is  $C^{1,\alpha}$  for every  $\alpha < 1$ .

*Proof.* We know from Theorem 3.3.8 that  $\Gamma \in C^1$ . Since  $\Gamma'(s) = 2ie^{L(s^2)}$ , all we need to show is that  $s \mapsto L(s^2)$  is  $\text{Lip}(\frac{1}{2} + \delta)$ . Suppose  $0 < s < s + \varepsilon \leq \sqrt{T}$ . If  $s \leq \varepsilon$ , then (3.16) gives

$$|L((s + \varepsilon)^2) - L(s^2)| \leq |L((s + \varepsilon)^2)| + |L(s^2)| \leq C\varepsilon$$

for some constant  $C > 0$ . If  $s > \varepsilon$ , Lemma 3.4.1 implies

$$\begin{aligned} |L((s + \varepsilon)^2) - L(s^2)| &\leq C(2s\varepsilon + \varepsilon^2)^{\frac{1}{2} + \delta} + \frac{C(2s\varepsilon + \varepsilon^2)}{s} \\ &\leq C(2s + \varepsilon)^{\frac{1}{2} + \delta} \varepsilon^{\frac{1}{2} + \delta} + 3C\varepsilon \\ &\leq C\varepsilon^{\frac{1}{2} + \delta}. \end{aligned}$$

Suppose  $\delta = \frac{1}{2}$ . To prove (3.20), it suffices to show

$$|L((s + \varepsilon)^2) - L(s^2)| \leq C\varepsilon(1 + |\log \varepsilon|)$$

for  $0 \leq s < s + \varepsilon \leq \sqrt{T}$  and some constant  $C > 0$ . As before, when  $s \leq 2\varepsilon$  the desired estimate follows from (3.16). When  $s > 2\varepsilon$ , (3.19) implies

$$\begin{aligned} |L((s + \varepsilon)^2) - L(s^2)| &\leq C(2s + \varepsilon)\varepsilon \left( 1 + \log^+ \frac{s^2}{(2s + \varepsilon)\varepsilon} + \frac{1}{s} \right) \\ &\leq Cs\varepsilon \left( 1 + \log \frac{s}{2\varepsilon} + \frac{1}{s} \right) \\ &\leq C\varepsilon(1 + |\log \varepsilon|) \end{aligned}$$

where  $C = C(M, T) > 0$ . □

**Corollary 3.4.3.** *Under the assumptions of Theorem 3.4.2, the slit  $\gamma(t) = \gamma^\lambda(t)$  is  $C^{1, \frac{1}{2} + \delta}$  regular on  $[a, T]$  (or weakly  $C^{1,1}$  when  $\delta = \frac{1}{2}$ ) for every  $a > 0$ . When  $0 < \delta < \frac{1}{2}$ ,*

$$\|\gamma\|_{C^{1, \frac{1}{2} + \delta}([a, T])} \leq N \quad \text{and} \quad \inf_{t \in [a, T]} |\gamma'(t)| \geq \frac{1}{N},$$

where  $N = N(M, T, \delta, a) > 0$ .

**3.5** When  $\lambda \in C^{1, \frac{1}{2} + \delta}$  with  $0 < \delta \leq \frac{1}{2}$

In this section,  $\lambda: [0, T] \rightarrow \mathbb{R}$  satisfies the  $(M, T, n, \alpha)$ - $C^{n, \alpha}$  condition for  $n = 1$  and  $\alpha = \frac{1}{2} + \delta$ , where  $0 < \delta \leq \frac{1}{2}$ . That is to say,  $\|\lambda\|_{\text{Lip}(\frac{1}{2})} \leq 1$  and

$$\|\lambda\|_{C^{1, \frac{1}{2} + \delta}} = \sup_{t \in [0, T]} |\lambda(t)| + \sup_{t \in [0, T]} |\lambda'(t)| + \sup_{t_1 \neq t_2 \in [0, T]} \frac{|\lambda'(t_1) - \lambda'(t_2)|}{|t_1 - t_2|^{\frac{1}{2} + \delta}} \leq M.$$

Our goal is to show  $\gamma \in C^{2, \delta}$  on  $[a, T]$  for every  $a > 0$ , which is equivalent to proving  $L \in C^{1, \delta}$  on the same interval.

Since  $\lambda \in C^{1, \frac{1}{2} + \delta}$ , it is in particular  $C^{1, \frac{1}{2}}$  and we know from Lemma 3.4.1 that  $L^\lambda(s)$  is weakly Lipschitz on  $[a, T]$  for every  $a > 0$ . We claim that  $L(s)$  is differentiable on  $(0, T]$  and  $L'(s) \in \text{Lip}(\delta, [a, T])$ . By (3.10), at least formally one has

$$\begin{aligned} L'(s) &= \frac{1}{2s} + \frac{2}{\gamma(s)^2} + \int_0^s \partial_s \left[ \frac{2}{\gamma(s-u, s)^2} \right] du \\ &= \frac{1}{2s} + \frac{2}{\gamma(s)^2} - 4 \int_0^s \frac{\partial_s \gamma(s-u, s)}{\gamma(s-u, s)^3} du. \end{aligned} \quad (3.21)$$

To see that this formula is valid, we must show that  $\partial_s \left[ \frac{2}{\gamma(s-u, s)^2} \right]$  is integrable over  $u \in [0, s]$ .

**Lemma 3.5.1.** *Let  $\lambda: [0, T] \rightarrow \mathbb{R}$  satisfy the  $(M, T, 1, \alpha)$ - $C^{1, \alpha}$  condition with  $0 < \alpha \leq 1$ . For any  $0 < u \leq s < s + \varepsilon \leq T$ , we have*

$$|\gamma(s + \varepsilon - u, s + \varepsilon) - \gamma(s - u, s)| \leq cM \min(u\varepsilon^\alpha, \varepsilon u^\alpha) \quad (3.22)$$

$$|\partial_s \gamma(s - u, s)| \leq cMu^\alpha \quad (3.23)$$

$$|\partial_s \gamma(s + \varepsilon - u, s + \varepsilon) - \partial_s \gamma(s - u, s)| \leq C\varepsilon^\alpha, \quad (3.24)$$

where  $C = C(M, T) > 0$  and  $c > 0$  is an absolute constant. When  $\alpha = \frac{1}{2} + \delta$  with  $0 < \delta \leq \frac{1}{2}$ ,  $L(s)$  is differentiable for  $s \in (0, T]$  and  $L'(s)$  is given by (3.21). Moreover,  $L'(s) \in \text{Lip}(\delta)$  on  $[a, T]$  for every  $a > 0$ .

*Proof.* By Theorem 3.2.4,  $|\gamma(s + \varepsilon - u, s + \varepsilon) - \gamma(s - u, s)| \leq c\omega(s, u, \varepsilon)$ . Inequality (3.22) follows immediately from Lemma 3.4.1, and it implies

$$\left| \frac{\gamma(s + \varepsilon - u, s + \varepsilon) - \gamma(s - u, s)}{\varepsilon} \right| \leq cMu^\alpha.$$

Letting  $\varepsilon \rightarrow 0$  gives (3.23). To prove (3.24), we differentiate  $\gamma(s-u, s) = g_{s-u}(\gamma(s)) - \lambda(s-u)$ :

$$\begin{aligned}\partial_s \gamma(s-u, s) &= \frac{2}{g_{s-u}(\gamma(s)) - \lambda(s-u)} + g'_{s-u}(\gamma(s)) \gamma'(s) - \lambda'(s-u) \\ &= \frac{2}{\gamma(s-u, s)} + \gamma'_{s-u}(s) - \lambda'(s-u)\end{aligned}$$

The last term  $\lambda'(s-u)$  is  $\text{Lip}(\alpha)$  in  $s$  by assumption. The term  $\frac{2}{\gamma(s-u, s)}$  is also  $\text{Lip}(\alpha)$  in  $s$  by (3.22):

$$\begin{aligned}\left| \frac{2}{\gamma(s+\varepsilon-u, s+\varepsilon)} - \frac{2}{\gamma(s-u, s)} \right| &= \frac{2|\gamma(s+\varepsilon-u, s+\varepsilon) - \gamma(s-u, s)|}{|\gamma(s+\varepsilon-u, s+\varepsilon)| \cdot |\gamma(s-u, s)|} \\ &\leq \frac{cMu\varepsilon^\alpha}{\sqrt{u} \cdot \sqrt{u}} \\ &= cM\varepsilon^\alpha.\end{aligned}$$

The remaining term  $\gamma'_{s-u}(s)$  is given by  $\gamma'_{s-u}(s) = \frac{i}{\sqrt{u}} \exp L(s-u, s)$ , where

$$L^\lambda(s-u, s) = L(s-u, s) := \int_0^u \left[ \frac{1}{2v} + \frac{2}{\gamma(s-v, s)^2} \right] dv.$$

Note that  $L^\lambda(s-u, s) = L^{\tilde{\lambda}}(u)$ , where  $\tilde{\lambda}(t) = \lambda(s-u+t)$  is a time shift of  $\lambda$ . From Lemma 3.3.5 we know that  $|L(s-u, s)| \leq cM\sqrt{u} \leq cMT$ . On the ball  $\{z \in \mathbb{C} : |z| \leq cMT\}$  the function  $e^z$  has bounded derivative, so

$$\begin{aligned}|\gamma'_{s+\varepsilon-u}(s+\varepsilon) - \gamma'_{s-u}(s)| &= \frac{1}{\sqrt{u}} \left| e^{L(s+\varepsilon-u, s+\varepsilon)} - e^{L(s-u, s)} \right| \\ &\leq \frac{C}{\sqrt{u}} |L(s+\varepsilon-u, s+\varepsilon) - L(s-u, s)|\end{aligned}\tag{3.25}$$

where  $C = e^{cMT}$ . On the other hand,

$$\begin{aligned}|L(s+\varepsilon-u, s+\varepsilon) - L(s-u, s)| &\leq \int_0^u \left| \frac{2}{\gamma(s+\varepsilon-v, s+\varepsilon)^2} - \frac{2}{\gamma(s-v, s)^2} \right| dv \\ &\leq c \int_0^u v^{-\frac{3}{2}} \omega(s, v, \varepsilon) dv \\ &\leq c \int_0^u v^{-\frac{3}{2}} Mv\varepsilon^\alpha dv \\ &\leq cM\sqrt{u}\varepsilon^\alpha\end{aligned}\tag{3.26}$$

By (3.25) and (3.26),  $|\gamma'_{s+\varepsilon-u}(s+\varepsilon) - \gamma'_{s-u}(s)| \leq C\varepsilon^\alpha$  with  $C = C(M, T) > 0$ , and (3.24) holds.

If  $\frac{1}{2} < \alpha \leq 1$ , we will show  $L(s)$  is differentiable on  $(0, T]$  and  $L'(s)$  is given by (3.21). By extending  $\lambda$ , we can assume without loss of generality that  $0 < s < T$ . For small  $\varepsilon > 0$ ,

$$\begin{aligned} \frac{L(s+\varepsilon) - L(s)}{\varepsilon} &= \frac{1}{\varepsilon} \int_s^{s+\varepsilon} \left[ \frac{1}{2u} + \frac{2}{\gamma(s+\varepsilon-u, s+\varepsilon)^2} \right] du + \\ &\quad \int_0^s \frac{1}{\varepsilon} \left[ \frac{2}{\gamma(s+\varepsilon-u, s+\varepsilon)^2} - \frac{2}{\gamma(s-u, s)^2} \right] du \end{aligned}$$

It is not hard to see that the integrand in the last term is dominated by  $Cu^{\alpha-\frac{3}{2}}$ , which is integrable since  $\alpha > \frac{1}{2}$ . By Lebesgue dominated convergence theorem, the second integral converges as  $\varepsilon \downarrow 0$ . Convergence of the first integral follows from continuity and does not require  $\alpha > \frac{1}{2}$ . Since  $L(s)$  has a continuous right derivative, it is differentiable on  $(0, T)$ .  $\square$

We now prove the main result in this section.

**Theorem 3.5.2.** *Suppose  $\lambda: [0, T] \rightarrow \mathbb{R}$  satisfies the  $(M, T, 1, \alpha)$ - $C^{1, \alpha}$  condition with  $\alpha = \frac{1}{2} + \delta$  and  $0 < \delta \leq \frac{1}{2}$ . Then the slit  $\gamma(t) = \gamma^\lambda(t)$  is  $C^{2, \delta}$  regular on  $[a, T]$  for every  $a > 0$ . The statement is quantitative in the sense that*

$$\|\gamma\|_{C^{2, \delta}([a, T])} \leq N \quad \text{and} \quad \inf_{t \in [a, T]} |\gamma'(t)| \geq \frac{1}{N},$$

where  $N = N(M, T, \delta, a) > 0$  depends only on  $M, T, \delta$  and  $a$ .

*Proof.* Any  $C^{1, \frac{1}{2} + \delta}$  driving function  $\lambda$  is in particular  $C^{1, \frac{1}{2}}$ . Applying Theorem 3.4.2, we know that  $\gamma$  is weakly  $C^{1, 1}$  regular on every  $[a, T]$ . All we need to show is that  $\gamma''$  exists and is  $\text{Lip}(\delta)$  on  $[a, T]$ .

By Lemma 3.5.1,  $L$  is differentiable and therefore  $\gamma''$  exists on  $(0, T]$  and is given by

$$\gamma''(s) = \frac{2\gamma'(s)}{\gamma(s)^2} - 4\gamma'(s)Q(s), \quad (3.27)$$

where

$$Q(s) := \int_0^s \frac{\partial_s \gamma(s-u, s)}{\gamma(s-u, s)^3} du.$$

The first term  $\frac{2\gamma'(s)}{\gamma(s)^2}$  in (3.27) is  $\text{Lip}(\delta)$  on  $[a, T]$  because both  $\gamma'$  and  $\gamma$  are  $\text{Lip}(\delta)$  and the size of the denominator  $|\gamma(s)|^2 \asymp s$  is bounded below by positive constant. It remains to

prove  $Q \in \text{Lip}(\delta, [0, T])$ . The integral kernel of  $Q$  has the form  $K(x, y) = \frac{x}{y^3}$ . We have

$$\begin{aligned} |Q(s + \varepsilon) - Q(s)| &\leq \int_0^s |K(x + \Delta x, y + \Delta y) - K(x, y)| \, du + \\ &\int_s^{s+\varepsilon} |K(x + \Delta x, y + \Delta y)| \, du, \end{aligned} \quad (3.28)$$

where  $x = \partial_s \gamma(s - u, s)$ ,  $y = \gamma(s - u, s)$ ,  $\Delta x = \partial_s \gamma(s + \varepsilon - u, s + \varepsilon) - \partial_s \gamma(s - u, s)$  and  $\Delta y = \gamma(s + \varepsilon - u, s + \varepsilon) - \gamma(s - u, s)$ . By Lemma 3.5.1 and scaling,

$$|K(x + \Delta x, y + \Delta y)| \leq \frac{cMu^{\frac{1}{2}+\delta}}{u^{\frac{3}{2}}} = cMu^{-1+\delta}$$

and the last term in (3.28) is bounded by

$$\int_s^{s+\varepsilon} |K(x + \Delta x, y + \Delta y)| \, du \leq \frac{cM}{\delta} \left[ (s + \varepsilon)^\delta - s^\delta \right] \leq \frac{cM}{\delta} \varepsilon^\delta,$$

whenever  $0 \leq s < s + \varepsilon \leq T$ . On the other hand, we split the first integral in (3.28) into two terms and handle them separately. If  $s \leq \varepsilon$ , by triangle inequality and Lemma 3.5.1,

$$|K(x + \Delta x, y + \Delta y) - K(x, y)| \leq |K(x + \Delta x, y + \Delta y)| + |K(x, y)| \leq cMu^{-1+\delta}.$$

Integrating gives

$$\int_0^\varepsilon |K(x + \Delta x, y + \Delta y) - K(x, y)| \, du \leq \frac{cM}{\delta} \varepsilon^\delta.$$

If  $s \geq \varepsilon$ , we still need to estimate the integral from  $\varepsilon$  to  $s$ .

$$\begin{aligned} &|K(x + \Delta x, y + \Delta y) - K(x, y)| \\ &\leq |K(x + \Delta x, y) - K(x, y)| + |K(x + \Delta x, y + \Delta y) - K(x + \Delta x, y)| \\ &\leq |\Delta x| \cdot \sup |\partial_x K| + |\Delta y| \cdot \sup |\partial_y K| \\ &\leq C\varepsilon^{\frac{1}{2}+\delta} u^{-\frac{3}{2}} + C\varepsilon u^{-1+2\delta} \end{aligned}$$

by Lemma 3.5.1 again. Integrating gives

$$\int_\varepsilon^s |K(x + \Delta x, y + \Delta y) - K(x, y)| \, du \leq C\varepsilon^\delta$$

with  $C = C(M, T, \delta) > 0$ . □

For  $\lambda(t) \in C^{1, \frac{1}{2} + \delta}$  with  $0 < \delta \leq \frac{1}{2}$ , Theorem 3.5.2 shows that  $\gamma(t)$  is  $C^{2, \delta}([a, T])$  for every  $a > 0$ . Certainly  $\|\gamma\|_{C^1([a, T])} \rightarrow \infty$  as  $a \downarrow 0$ . To obtain smoothness up to  $t = 0$ , one has to reparametrize the slit, and a natural candidate is  $\Gamma(t) = \gamma(t^2)$ . We do not know whether  $\Gamma(t)$  is  $C^{2, \delta}$  up to  $t = 0$ , but assuming this we have a quadratic approximation

$$\Gamma(t) = \gamma(t^2) = 2it + \frac{2}{3}\lambda'(0)t^2 + O(t^{2+\delta}) \quad (3.29)$$

as  $t \rightarrow 0$ . The heuristic reason is that on a small interval close to the origin any  $C^{1, \alpha}$  driving function  $\lambda(t)$  can be approximated by a fixed linear function  $\lambda'(0)t$ . For driving functions of the form  $\lambda(t) = at$  ( $a > 0$ ) the quadratic approximation of  $\gamma(t^2)$  can be explicitly computed.

**Example 3.5.3.** Let  $\lambda(t) = t$ . Since a linear function is invariant under time shift,  $\gamma(s - u, s) = \gamma(u)$  does not depend on  $s$ , and we have

$$L(s) = \int_0^s \left[ \frac{1}{2u} + \frac{2}{\gamma(u)^2} \right] du \quad \text{and} \quad L'(s) = \frac{1}{2s} + \frac{2}{\gamma(s)^2}.$$

For this case it is possible to explicitly compute the series expansion of  $L(s)$  near  $s = 0$ . The reader may refer to [13] for the computation. As  $s \rightarrow 0$ , one has  $\gamma(s) = 2i\sqrt{s} + \frac{2}{3}s + O(s^{\frac{3}{2}})$  and  $L'(s) = -\frac{i}{3\sqrt{s}} + O(1)$ . Note that  $[L(s^2)]' = 2sL'(s^2) \rightarrow -\frac{2i}{3}$ . The function  $s \mapsto L(s^2)$  is  $C^1$  up to  $s = 0$ . Since  $\Gamma'(s) = 2i \exp L(s^2)$ , the curve  $\Gamma(s) = \gamma(s^2)$  is  $C^2$  up to  $s = 0$  and has a quadratic approximation  $\Gamma(s) = 2is + \frac{2}{3}s^2 + O(s^3)$ . Note that this agrees with (3.29).

For any constant  $b > 0$ , the driving function  $\lambda_b(t) = bt$  can be obtained from  $\lambda_1(t) = t$  and a Brownian scaling:  $\lambda_b(t) = \frac{1}{b}\lambda_1(b^2t)$ . The computation in the above example gives

$$\gamma^{\lambda_b}(s) = \frac{1}{b}\gamma^{\lambda_1}(b^2s) = 2i\sqrt{s} + \frac{2b}{3}s + O(s^{\frac{3}{2}})$$

as  $s \rightarrow 0$ . We have just verified (3.29) for all driving functions of the form  $\lambda_b(t) = bt$  with  $b \in \mathbb{R}$ . (The case  $b < 0$  follows from symmetry.)

**Proposition 3.5.4.** *Suppose  $\lambda: [0, T] \rightarrow \mathbb{R}$  satisfies the  $(M, T, 1, \alpha)$ - $C^{1, \alpha}$  condition with  $\alpha = \frac{1}{2} + \delta$  and  $0 < \delta \leq \frac{1}{2}$ . Then  $\Gamma(t) = \gamma(t^2)$  is twice differentiable everywhere on  $[0, \sqrt{T}]$  and  $\Gamma''(0) = \frac{4}{3}\lambda'(0)$ .*

*Proof.* We already know  $\Gamma(t)$  is  $C^2$  on  $(0, \sqrt{T}]$  (Theorem 3.5.2) and still need to show the existence of  $\Gamma''(0)$ . By comparing  $\lambda(t)$  with the linear driving function  $\tilde{\lambda}(t) = \lambda'(0)t$ , we

will show that  $s \mapsto L^\lambda(s^2)$  is differentiable at  $s = 0$ . To simplify the notations, we write  $\tilde{L}(\cdot) = L^{\tilde{\lambda}}(\cdot)$  and  $\tilde{\tau}(\cdot, \cdot) = \tau^{\tilde{\lambda}}(\cdot, \cdot)$ . Notice that

$$\begin{aligned} \left| L(s^2) - \tilde{L}(s^2) \right| &\leq \int_0^{s^2} \left| \frac{2}{\tau(s^2 - u, s^2)^2} - \frac{2}{\tilde{\tau}(s^2 - u, s^2)^2} \right| \frac{du}{u} \\ &\leq c \int_0^{s^2} \left| \tau(s^2 - u, s^2) - \tilde{\tau}(s^2 - u, s^2) \right| \frac{du}{u} \end{aligned}$$

Using the condition  $\lambda \in C^{1, \frac{1}{2} + \delta}$ , we can estimate the  $\|\cdot\|_\infty$  distance between the two driving functions which generate  $\tau(s^2 - u, s^2)$  and  $\tilde{\tau}(s^2 - u, s^2)$ . The Lipschitz continuity Theorem 3.2.4 implies that

$$\left| L(s^2) - \tilde{L}(s^2) \right| = O(s^{1+2\alpha}).$$

For the purpose of computing  $\lim_{s \rightarrow 0} \frac{L(s^2)}{s}$ , we can replace  $\lambda(t)$  by  $\lambda'(0)t$  without affecting the existence of the limit and its value. Since we are able to compute this limit for linear driving functions, it follows that

$$\left. \frac{dL(s^2)}{ds} \right|_{s=0} = \lim_{s \rightarrow 0} \frac{L(s^2)}{s} = \lim_{s \rightarrow 0} \frac{\tilde{L}(s^2)}{s} = -\frac{2i}{3} \lambda'(0).$$

From the formula  $\Gamma'(s) = 2i \exp L(s^2)$  and the above computation, we have  $\Gamma''(0) = \frac{4}{3} \lambda'(0)$ . □

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