Constructing stability conditions on nodal curves from subdivisions of Lawrence polytopes

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Abstract

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A stability condition on a nodal curve X is the assignment of integers to the nontrivial biconnected subcurves of X satisfying some desired properties; equivalently, it is an assignment of integers to the biconnected subsets of $V(G)$, where G is the graph dual to X. In this thesis, we study the combinatorics of these stability conditions, and provide a construction of these stability conditions from subdivisions of the Lawrence polytope of the cographic matroid $\mathcal{M}^*(G)$ using the theory of single-element extensions of oriented matroids. We also discuss connections to chip-firing on graphs through the theory of generalized break divisors.

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DEDICATION

In memory of Jessie, the dog who was always by my side

Chapter 1

INTRODUCTION

Combinatorial algebraic geometry studies the way combinatorics and algebraic geometry interact, often using one field to provide insight into the other. One problem in the intersection of these fields is the study of compactified Jacobians of nodal curves. We are particularly interested in the construction of compactified Jacobians using *stability conditions*. A *stability condition* $\mathfrak{n}$ on a nodal curve X can be viewed as an assignment of integers $\mathfrak{n}_W$ to the biconnected subsets W of the graph G dual to X as in, for example, [27], [28], or [35]. There exists a smaller family of compactified Jacobians, called *classical compactified Jacobians*, which have been constructed by several different authors such as Oda and Seshadri [25], Simpson [31], Caporaso [11], and Esteves [17]. In this thesis, we work with the graph theoretic construction in [25] through numerical polarizations, or real-valued divisors of integer degree, on the dual graph G. We wish to understand the combinatorics of stability conditions, especially how to enumerate them, their structure, and how to distinguish which ones are classical.

Our combinatorial motivation is to study representatives of divisors on G of degree g, especially representatives that are constructed from the spanning trees of G. The canonical choice of representatives are the *integral break divisors*, which are constructed in the following way: if T is a spanning tree of G and $\mathcal{O}_\mathcal{E}$ is an orientation of the edges $\mathcal{E}$ not in T, the integral break divisor $\mathcal{D}(T, \mathcal{O}_\mathcal{E})$ is the degree $g = |E(G)| - |V(G)| + 1$ divisor on G resulting from placing a chip at the head of oriented edge in $\mathcal{O}_\mathcal{E}$. Every degree g divisor on G is linearly equivalent to a unique integral break divisor and the number of integral break divisors in the number of spanning trees of G [1, Theorem 1.3]. We can modify this construction to study non-canonical choices of representatives. We can consider some function I that assigns a divisor $I(T)$ of degree 0 to each spanning tree T of G. Then, a generalized break divisor is a divisor of the form $\mathcal{D}(T, \mathcal{O}_{\mathcal{E}}) + I(T)$. For a fixed function I_G, we denote the set of all generalized break divisors of the form $\mathcal{D}(T, \mathcal{O}_{\mathcal{E}})$ by $\mathcal{BD}_I(G)$. For every function I, $|\mathcal{BD}_I(G)| \geq |\mathcal{ST}(G)|$, with $I \cong 0$ recovering the integral break divisors [37, Theorem 3.5.1]. One question is to describe all functions I such that every divisor of degree g on G is linearly equivalent to a unique divisor in $\mathcal{BD}_I(G)$. Such functions I are in one-to-one correspondence with (nondegenerate) stability conditions on the graph G [35, Theorem 1.20].

To describe these I_G, we turn to a family of bijections constructed by Ding [16] between the bases of a regular matroid $\mathcal{M}$, and orientations of $\mathcal{M}$ up to *circuit* and *cocircuit* reversal. This family of bijections includes the Backman-Baker-Yuen (BBY) bijection or "geometric bijections" constructed in [3] and the Bernardi bijection restricted to spanning trees constructed in [7]. These bijections arise from *triangulations* and *dissections* of the *Lawrence polytopes* $\mathcal{P}$ and $\vec{P}^*$ corresponding respectively to the regular matroid $\mathcal{M}$ and its dual $\mathcal{M}^*$. When $\mathcal{M}$ is a graphic matroid, each triangulation and dissection of $\mathcal{P}$, respectively $\mathcal{P}^*$, corresponds to a choice of an external, respectively internal, orientation of each of the spanning trees of G, called *triangulating* and *dissecting atlases* [16, Theorem 1.28].

If we expand our focus to the construction of smoothable but not necessarily fine compactified Jacobians, we consider stability conditions $\mathfrak{n}$, where some biconnected subsets $W \subset V(G)$ are degenerate. A *degeneracy set* is a set $\mathcal{D}$ of biconnected subsets that is closed under taking complements and biconnected unions of disjoint biconnected subsets. A degeneracy set can be defined without a reference stability condition, and it is unknown if all degeneracy sets can be realized by a stability condition. We demonstrate that these degeneracy sets correspond to single-element extensions of the unoriented graphic matroid $M(G)$, through Crapo's characterization of single-element extensions [14]. We can also generalize our combinatorial motivation to study divisors induced by certain spanning forests of G. We extend the construction of triangulating atlases to subdivisions of the Lawrence polytope, using the known bijection between subdivisions of the Lawrence polytope of the oriented matroid $\mathcal{M}$ and single-element liftings of the dual matroid $\mathcal{M}^*$ [29, Theorem 4.14], which were completely characterized by Las Vergnas [34].

The contents of this thesis are organized as follows. In Chapter 2, we introduce graphs and graphic matroids, both unoriented and oriented. We also introduce the theory of chip-firing on graphs and cycle-cocycle reversal classes of graph orientations. We give an overview of polytopes and their subdivisions, and introduce the Lawrence polytope of an oriented matroid. Finally, we define V-stability on nodal curves in terms of the dual graph of the curve. In Chapter 3, we introduce the theory of generalized break divisors and their connection to nondegenerate stability conditions. We also cover the use of dissections and triangulations of Lawrence polytopes to construct bijections between bases of a matroid and its circuit-cocircuit reversal classes, as shown in [16]. We prove that triangulations of the Lawrence polytope of the cographic matroid $\mathcal{M}^*(G)$ correspond to nondegenerate stability conditions on the graph G, and the regular triangulations of the Lawrence polytope correspond to classical stability conditions. The contents of Chapter 3 can also be found in [15]. In Chapter 4, we connect degeneracy sets of the graph G to single-element extensions of the unoriented matroid $M(G)$ and the oriented matroid $\mathcal{M}(G)$, using the characterizations of Crapo (see [14]) and Las Vergnas (see [34]). Finally, we generalize the results of Chapter 3 to construct degenerate stability conditions on the graph from subdivisions of the Lawrence polytope of $\mathcal{M}^*(G)$.

Chapter 2

PRELIMINARIES

2.1 Graphs

Let $G = (V, E)$ be a loopless connected graph. The *genus* of G is $g = |E(G)| - |V(G)| + 1$. A *subgraph* of G is a graph $G' = (V', E')$, where $V' \subseteq V(G)$ and $E' \subseteq E(G)$, and a subgraph G' is *spanning* if $V' = V(G)$. If $W \subseteq V(G)$, the *induced subgraph* $G[W]$ is the subgraph of G containing all edges between any vertices in W. For disjoint subsets W_1 and W_2 of $V(G)$, $E(W_1, W_2)$ is the set of edges connecting a vertex in W_1 to a vertex in W_2. If the induced subgraphs $G[W]$ and $G[W^c]$ are both connected, then we say W is *biconnected*. A *cycle* is a walk along the graph G that starts and ends at the same vertex without using any edge or any vertex twice, and a *cocycle*, or *bond*, is a minimal subset of edges E' such that $G \backslash E'$ has two connected components. A *spanning tree* T of G is a connected spanning subgraph of G that does not have any cycles. The collection of all spanning trees of a graph G is denoted by $\mathcal{ST}(G)$. Given a spanning tree T and an edge $e \in E(G \backslash T)$, the *fundamental cycle* $C(T, e)$ is the unique cycle in $T \cup \{e\}$. If $e \in T$, $T \backslash e$ has two connected components with vertices W and W^c. The *fundamental cocycle* $C^*(T, e)$ is the unique cocycle partitioning $V(G)$ into W and W^c.

A *divisor* D on a graph G is an assignment of integers $D(v)$ to the vertices of G. The *degree* of a divisor D is $\deg(D) = \sum_{v \in V(G)} D(v)$. We let $\mathrm{Div}^d(G)$ be the collection of all divisors of degree d on G. A *chip-firing move* takes a divisor D to a divisor D' by either *firing at* v, where v sends a chip to each vertex adjacent to it, or *borrowing at* v, where v receives a chip from each vertex adjacent to it. If D' is reached from D through a sequence of chip-firing moves, we say that D and D' are *linearly (or chip-firing) equivalent*. The equivalence class of a divisor D is denoted by $[D]$. If D is degree k, then $[D] \in \mathrm{Pic}^k(G)$, the set of all divisors

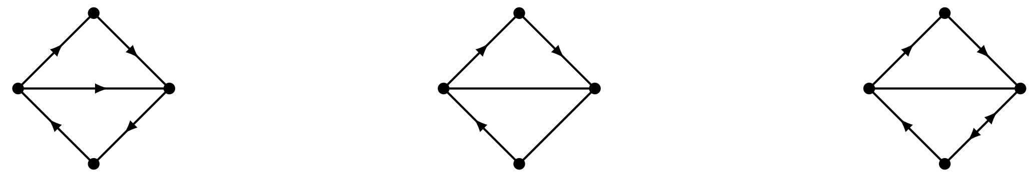


Figure 2.1: An orientation, partial orientation, and fourientation of a graph G.

of degree k of G modulo chip-firing equivalence.

If $e = vw$ is an edge in $E(G)$, we can orient e by assigning it a direction of towards v or away from v. An *orientation* $\mathcal{O}$ of a graph is a choice of direction for every edge of G. The *indegree* of $\mathcal{O}$ at a vertex v is the number of edges incident to v pointing towards v. Given an orientation $\mathcal{O}$, we can associate a divisor $D_{\mathcal{O}}$ to it by setting $D_{\mathcal{O}}(v) = \text{indegree}_{\mathcal{O}}(v) - 1$ for each $v \in V(G)$. The degree of $D_{\mathcal{O}}$ is always $g - 1$. If $D = D_{\mathcal{O}}$ for some orientation $\mathcal{O}$, we say that D is *orientable*. A *partial orientation* of G may have some unoriented edges. In addition, we will consider *fourientations* of a graph, which were introduced in [4]. A fourientation generalizes orientations by allowing unoriented edges and edges oriented in both directions (bioriented), in addition to one-way oriented edges. We can also define the divisor $D_{\mathcal{O}}$ for a fourientation $\mathcal{O}$, where bioriented edges do contribute to the indegree of a vertex and unoriented edges do not.

Given an orientation $\mathcal{O}$ on a graph G, if $\mathcal{O}$ has a directed cycle $\vec{C}$ or cocycle $\vec{C^*}$, a *cycle reversal* replaces $\vec{C}$ with the opposite directed cycle $-\vec{C}$ in $\mathcal{O}$, and a *cocycle reversal* replaces $\vec{C^*}$ with the opposite directed cocycle $-\vec{C^*}$ in $\mathcal{O}$. Gioan introduced the *cycle-cocycle reversal system* in [20], where two orientations are considered to be equivalent if one can be reached from the other through a sequence of cycle and cocycle reversals. If G is a connected graph, there are $|\mathcal{ST}(G)|$ such equivalence classes. For partial orientations, Backman in [2] introduced the generalized cycle-cocycle reversal system, which allows for *edge pivots*: if $e = vw$ is oriented away from v, and v is connected to another vertex y by an unoriented edge e', e becomes unoriented and e' is oriented away from v.

Cycle-cocycle reversal systems and chip-firing are connected, as shown in the following result.

Theorem 2.1.1 ([2, Theorem 3.5]). *Two partial orientations $\mathcal{O}$ and $\mathcal{O}'$ are equivalent in the generalized cycle-cocycle reversal system if and only if $D_{\mathcal{O}}$ is linearly equivalent to $D_{\mathcal{O}'}$.*

2.2 Matroids

Let E be a finite set, which we call the *ground set.* Let $\mathcal{I}$ be a family of subsets of E, called *independent sets*, that satisfy the following axioms:

1. $\emptyset \in \mathcal{I}$
2. If $I \in \mathcal{I}$ and $J \subset I \subset E$, then $J \in \mathcal{I}$.
3. If $I, J \in \mathcal{I}$ and $|I| > |J|$, then there exists $x \in I \backslash J$ such that $J \cup \{x\} \in \mathcal{I}$.

Then, $M = (E, \mathcal{I})$ is a *matroid.* If $e \in E$, the *deletion* of e from M is the matroid $M \backslash e = (E \backslash \{e\}, \mathcal{I}')$, where $\mathcal{I}' = \{I : I \in \mathcal{I} \text{ and } e \notin I\}$, or the *contraction* of e $M/e = (E \backslash \{e\}, \mathcal{I}'')$, where $\mathcal{I}'' = \{I \backslash \{e\} : I \in \mathcal{I} \text{ and } e \in I\}$. A *minor* of M is any matroid obtained from M through a sequence of deletions and contractions.

We are interested in *graphic matroids*, which are matroids associated from an undirected graph G. Let $M(G)$ denote the (unoriented) graphic matroid associated to G. The ground set E of $M(G)$ is the set of edges of the graph $E(G)$, and the independent sets are the subgraphs of G that do not contain a cycle. The largest (with respect to inclusion) independent sets of a matroid are called *bases*. By the third axiom, any two bases have the same size, which we call the *rank* of M. The bases of $M(G)$ are the spanning trees of G, and the rank of $M(G)$ is $r(M(G)) = |V(G)| - 1$. A matroid is *loopless* if there does not exist $e \in E$ such that $e \notin \mathcal{I}$.

A *circuit* C of M is a minimally dependent subset of the ground set E; therefore the circuits of $M(G)$ are the subsets corresponding to simple cycles of G. A matroid M has a dual M^*, whose basis elements B^* are the complements of the basis elements B of M in E.

The *cocircuits* C^* of M are the circuits of the dual matroid M^*. In the graphic matroid case, the cocircuits of $M(G)$ correspond to the minimal (with respect to inclusion) cuts of the graph. The rank of $M^*(G)$ is the genus $g(G)$ of the graph G. If G is planar, then the dual $M^*(G)$ is the graphic matroid of the planar dual of G; however is G is not planar, there is no graph interpretation of $M^*(G)$. For a basis B and $e \notin B$, the *fundamental circuit* $C(B, e)$ is the unique circuit contained in $B \cup \{e\}$, and for $e \in B$ the *fundamental cocircuit* $C^*(B, e)$ is the unique cocircuit in $(E \backslash B) \cup \{e\}$.

Graphic matroids are an example of *regular matroids*, which are matroids that can be defined from a totally unimodular matrix U. The dimensions of U are $r \times n$, where r is the rank of the matroid and $n = |E|$. For a graphic matroid $M(G)$, the matrix U is the oriented incidence matrix of G with a row removed. For any loopless regular matroid M that is represented by a totally unimodular matrix U, U can be written as $U = [I_r|L]$. Then, the dual matroid M^* can be represented by the matrix $U^* = [-L^T|I_{n-r}]$, as shown in Theorem 2.2.8 of [26].

Another combinatorial structure we are interested in are *oriented matroids*, which have been used to study objects like vector configurations, hyperplane arrangements, and polytopes. Like matroids, oriented matroids are defined on a set of edges E. An *arc* is an oriented edge $\vec{e}$. A *signed subset* of E is a set X such that $X \in \{-, 0, +\}^E$, where X_e is the sign of the edge e. The *support* of a signed set X is $\underline{X} = \{e|X_e \neq 0\}$.

One way to construct an oriented matroid on a set E is through defining its *signed circuits*.

Definition 2.2.1. [9, Definition 3.2.1] A collection of $\mathcal{C}$ of signed subsets of a set E is the set of *signed circuits* of an *oriented matroid* on E if and only if it satisfies the following axioms:

1. $\emptyset \notin \mathcal{C}$,

2. if $X \in \mathcal{C}$, then $-X \in \mathcal{C}$,

3. for all $X, Y \in \mathcal{C}$, if $\underline{X} \subset \underline{Y}$, then $X = Y$ or $X = -Y$,

4. for all $X, Y \in \mathcal{C}$, $X \neq -Y$, and $e \in X^+ \cap Y^-$ there is a $Z \in \mathcal{C}$ such that $Z^+ \subseteq (X^+ \cup Y^+) \backslash \{e\}$ and $Z^- \subseteq (X^- \cup Y^-) \backslash \{e\}$.

For an oriented matroid $\mathcal{M}$, we denote the underlying unoriented matroid by $\underline{\mathcal{M}}$. An unoriented matroid M is *orientable* is there exists an oriented matroid $\mathcal{M}$ such that $\underline{\mathcal{M}} = M$. For an orientable matroid M, the support of the signed circuits of $\mathcal{M}$ are exactly the circuits of M.

We can also define minors of oriented matroids using the set of signed circuits. Consider $e \in E$. Then, the deletion of e from the oriented matroid $\mathcal{M}$, denoted $\mathcal{M} \backslash e$, is defined by the signed circuits $\mathcal{C}' = \{\vec{C} \in \mathcal{C} | \underline{\vec{C}} \subseteq E \backslash \{e\}\}$, and the contraction of e in the oriented matroid $\mathcal{M}$, denoted $\mathcal{M}/e$, is defined by its signed circuits, which are the inclusion-wise minimal elements in $\{\vec{C} \backslash \{e\} : \vec{C} \in \mathcal{C}\}$.

When G is a directed graph, we can view $M(G)$ and $M^*(G)$ as the oriented matroids $\mathcal{M}(G)$ and $\mathcal{M}^*(G)$ in the following way: let G be a directed graph with an orientation $\mathcal{O}$ and arcs E. Then, for each cycle C of G, let $\vec{C}$ denote a choice of orientation of C, where every arc in $\vec{C}$ either agrees with the arc in $\mathcal{O}$ or is oriented in the opposite direction in $\mathcal{O}$. We can then view $\vec{C}$ as a signed subset of E, where

$$\vec{C}_e = \begin{cases} + & \text{if } \vec{e} \in \vec{C} \text{ and } \vec{e} \in \mathcal{O} \\ - & \text{if } \vec{e} \in \vec{C} \text{ and } -\vec{e} \in \mathcal{O} \\ 0 & \text{otherwise.} \end{cases}$$

Every signed cycle $\vec{C}$ can then be described by its positive part $\vec{C}_+ : \{e : \vec{C}_e = +\}$ and its negative part $\vec{C}_- = \{e : \vec{C}_e = -\}$. We can write any signed cycle $\vec{C}$ as $\vec{C} = (\vec{C}_+, \vec{C}_-)$. The collection of signed cycles of G satisfy the circuit axioms in Definition 2.2.1, so the collection of signed cycles are the set of (signed) circuits of the oriented matroid $\mathcal{M}(G)$. The collection of all signed cocycles also yields the set of signed cocircuits of the oriented matroid $\mathcal{M}(G)$, and we denote signed cocycles by $\vec{C^*}$. In [16], the definition of fourientation is also extended to regular matroids, where a fourientation is a subset of the set of all arcs of $\mathcal{M}$; as in the graph case, $e \in E$ can be bioriented, unoriented, or one-way oriented. For a fourientation $\vec{F}$, $-\vec{F}$ is the fourientation with all one-way oriented arcs reversed, while $\vec{F}^c$ reverses one-way oriented arcs and switches unoriented and bioriented arcs.

Oriented regular matroids also have a *circuit-cocircuit reversal system*, introduced in [21], where an orientation $\mathcal{O}$ of $\mathcal{M}$ is equivalent to another orientation $\mathcal{O}'$ if $\mathcal{O}'$ is reachable from $\mathcal{O}$ through a sequence of circuit and cocircuit reversals. The number of circuit-cocircuit reversal classes is the number of bases of the matroid. The following lemma is helpful when studying these classes.

Lemma 2.2.2 ([16, Lemma 2.7])**.** *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two orientations in the same circuit-cocircuit reversal class of $\mathcal{M}$. Then $\mathcal{O}_2$ can be obtained by reversing signed circuits and signed cocircuits in $\mathcal{O}_1$ that do not share any edges.*

Choosing an orientation $\vec{C}$ for every circuit C forms a *circuit signature* σ, where $\sigma(C) = \vec{C}$. A *cocircuit signature* σ^* is defined similarly. We can identify $\sigma(C)$ with a vector in $\mathbb{R}^{|E|}$ with respect to a reference orientation of $\mathcal{M}$ in the following way: if $\vec{e}$ is the orientation of e in the reference orientation, we set the entry corresponding to e be zero if $e \notin C$, 1 if the arc $\vec{e} \in \vec{C}$, and -1 if the arc $-\vec{e} \in \vec{C}$. A circuit signature is *acyclic* if for nonnegative reals a_C such that $\sum_C a_C\sigma(C) = 0$, we must have $a_C = 0$ for all circuits C. Acyclic cocircuit signatures are defined similarly.

2.3 Polytopes and subdivisions

A *polytope* $\mathcal{P}$ is the convex hull of a finite set of points V in $\mathbb{R}^n$. A *face* of a polytope is its intersection with any hyperplane that does not cross the relative interior of P. Faces of dimension 0 are called *vertices*. A d-dimensional *simplex* is a polytope of dimension d with exactly $d+1$ vertices.

A *subdivision* $\mathcal{S}$ of a polytope $\mathcal{P} \subset \mathbb{R}^d$ is a collection of polytopes, called *cells*, whose union is $\mathcal{P}$, and any two distinct cells intersect at a (possibly empty) common face. A *triangulation* is a subdivision where every cell is a d-dimensional simplex. A *dissection* of $\mathcal{P}$

is a collection of d-dimensional simplices whose union is $\mathcal{P}$, and any two distinct maximal simplices must have a disjoint interior (but are not required to intersect at a common face).

We denote the set of all subdivisions of a polytope $\mathcal{P}$ as $\mathrm{Subdiv}(\mathcal{P})$. We recall the following definitions from [24]. A subdivision $\mathcal{S}$ is a *refinement* of the subdivision $\mathcal{S}'$ if for every cell $c \in \mathcal{S}$, there exists a cell $c' \in \mathcal{S}'$ such that $c \subseteq c'$. Refinement induces a partial order on $\mathrm{Subdiv}(\mathcal{P})$. The minimal elements of the poset $(\mathrm{Subdiv}(\mathcal{P}), \preccurlyeq)$ are the triangulations of $\mathcal{P}$, and the unique maximal element is given by the trivial subdivision of $\mathcal{P}$.

One special kind of subdivision is a *regular subdivision*. We recall the following definition from [30]. Let V be the vertices of a polytope $\mathcal{P} \subset \mathbb{R}^d$, and let $\omega : V \to \mathbb{R}$ be a function that lifts V to $\mathbb{R}^{d+1}$ by sending V to

$$V_\omega \coloneqq \{(v, \omega(v)) : v \in V\}.$$

A *lower facet* of $\mathrm{conv}(V_\omega)$ is a facet whose supporting hyperplane lies below the interior of $\mathrm{conv}(V_\omega)$. Then, project the lower facets of $\mathrm{conv}(V_\omega)$ back onto $\mathcal{P}$. If all of the lower facets are simplices, then their projections form the triangulation $\mathcal{T}_\omega$ of $\mathcal{P}$, which we call a regular triangulation; otherwise, the projections form a subdivision of $\mathcal{P}$, which we call a regular subdivision. Every subdivision of a polytope can be refined to a triangulation of that polytope; furthermore, every regular subdivision of a polytope can be refined to a regular triangulation of that polytope [24, Corollary 2.3.18].

Starting from one triangulation of $\mathcal{P}$, we may be able to find another through a *flip operation*. A circuit C of a polytope $\mathcal{P}$ is a minimally affinely dependent set of the vertices V of $\mathcal{P}$; in other words, if $C = \{c_1, ..., c_k\}$ is a minimal set of vertices such that $\lambda_1 c_1 + \cdots + \lambda_k c_k = 0$ with $\sum_i \lambda_i = 0$, and all λ_i are nonzero, then C is a circuit. Therefore, we can view C as a signed circuit $C = (C_+, C_-)$, where $C_+ = \{c_i : \lambda_i > 0\}$ and $C_- = \{c_i : \lambda_i < 0\}$. The circuit C (as a point configuration) has two triangulations: $\mathcal{T}_+^C = \{C \backslash \{c_i\} : c_i \in C_+\}$ and $\mathcal{T}_-^C = \{C \backslash \{c_i\} : c_i \in C_-\}$. We also define the *link* of $\tau \subseteq V$ with respect to a triangulation $\mathcal{T}$ as

$$\mathrm{link}_{\mathcal{T}}(\tau) = \{p \subseteq V : p \cap \tau = \emptyset, p \cup \tau \in \mathcal{T}\}.$$

Definition 2.3.1 ([30, Definition 1.14])**.** Let $\mathcal{T}_1$ be a triangulation of the polytope $\mathcal{P}$. Assume $\mathcal{T}_1$ contains $\mathcal{T}_+^C$ for a circuit C of the vertices V of $\mathcal{P}$. Suppose all cells $\tau \in \mathcal{T}_+^C$ have the same link L in $\mathcal{T}_1$. The circuit C *supports a flip* in $\mathcal{T}_1$ and the triangulation $\mathcal{T}_2$ of $\mathcal{P}$ is obtained from $\mathcal{T}_1$ by this flip:

$$\mathcal{T}_2 \coloneqq \mathcal{T}_1 \backslash \{p \cup \tau : p \in L, \tau \in \mathcal{T}_+^C\} \cup \{p \cup \tau : p \in L, \tau \in \mathcal{T}_-^C\}$$

The *flip graph* of triangulations of a polytope $\mathcal{P}$ is constructed by having the vertices correspond to triangulations, where two triangulations are connected by an edge if they differ by a flip. As is well-known (see for example [8]), the flip graph of regular triangulations is the 1-skeleton of the secondary polytope of $\mathcal{P}$, so it is connected.

2.4 Lawrence polytopes

Every oriented matroid $\mathcal{M}$ of rank r on n elements can be lifted to an associated oriented matroid $\Lambda(\mathcal{M})$ of rank $n+r$ on $2n$ elements through a construction of Jim Lawrence. One reference is [6]. If $\mathcal{M}$ is a loopless matroid and can be realized by a matrix $M_{r\times n}$, the *Lawrence matrix* of $\mathcal{M}$ is given by

$$\Lambda(\mathcal{M}) = \begin{pmatrix} M_{r\times n} & \mathbf{0} \\ I_{n\times n} & I_{n\times n} \end{pmatrix}.$$

We label the columns of the Lawrence matrix as $P_1, ..., P_n, P_{-1}, ..., P_{-n}$. The *Lawrence polytope* $\mathcal{P} \subset \mathbb{R}^{n+r}$ of $\mathcal{M}$ is the convex hull of the points $P_1, ..., P_n, P_{-1}, ..., P_{-n}$. In Section 3.3, we work with the triangulations of the Lawrence polytope of the cographic oriented matroid $\mathcal{M}^*(G)$, and in Chapter 4, we expand to considering subdivisions of the Lawrence polytope of $\mathcal{M}^*(G)$ and their bijection with single-element extensions of the graphic oriented matroid $\mathcal{M}(G)$.

2.5 Stability conditions on nodal curves

Given a nodal curve X and its dual graph G, a question that one can ask is how to construct smoothable compactified Jacobians of X. In the fine case, the question was answered by

Pagani and Tommasi in [28] and by Viviani in [35] through the study of stability conditions on the graph G. For smoothable but not fine compacitified Jacobians, the question was answered by Fava, Pagani, and Viviani in [18].

Definition 2.5.1 ([18, Definition 2.1])**.** Let BCon(G) denote the set of all biconnected and nontrivial $W \subset V(G)$. A *V-stability condition* $\mathfrak{n}$ of degree $d \in \mathbb{Z}$ on G is an assignment of integers

$$\begin{aligned} \mathfrak{n} : \mathrm{BCon(G)} &\to \mathbb{Z} \\ W &\mapsto \mathfrak{n}_W \end{aligned}$$

satisfying the following properties: For any $W \in \mathrm{BCon}(G)$, we have

$$\mathfrak{n}_W + \mathfrak{n}_{W^c} + |E(W, W^c)| - d \in \{0, 1\},$$

where any W such that $\mathfrak{n}_W + \mathfrak{n}_{W^c} + |E(W, W^c)| - d = 0$ are called $\mathfrak{n}$-*degenerate*, and for any pairwise disjoint W_1, W_2, and W_3 in BCon(G) such that $W_1 \cup W_2 \cup W_3 = V(G)$ satisfy the following two properties:

1. for $i \neq j \neq k$, if W_i and W_j are $\mathfrak{n}$-degenerate, then W_k is $\mathfrak{n}$-degenerate;

2. the following equation is satisfied:

$$\sum_{i=1}^{3} \mathfrak{n}_{W_i} + \sum_{1 \leq i < j \leq 3} |E(W_i, W_j)| - d = \begin{cases} \{1, 2\} & W_i \text{ is } \mathfrak{n}\text{-nondegenerate for all } i \\ \{1\} & \exists! i \text{ such that } W_i \text{ is } \mathfrak{n}\text{-degenerate} \\ \{0\} & \text{if } W_i \text{ is } \mathfrak{n}\text{-degenerate for all } i. \end{cases} \tag{2.1}$$

The *degeneracy set* of a stability condition $\mathfrak{n}$, denoted $\mathcal{D}(\mathfrak{n})$ is the set of all $W \in \mathrm{BCon}(G)$ such that W is $\mathfrak{n}$-degenerate. The degree of the stability condition is denoted by $|\mathfrak{n}|$.

A *numerical polarization* $\phi : V(G) \to \mathbb{R}$ is a real-valued divisor on a graph G such that the degree $\deg(\phi) = \sum_{v \in V(G)} \phi(v)$ is an integer d. For $W \in \mathrm{BCon}(G)$, we let $\phi_W = \sum_{v \in W} \phi(v)$.

A numerical polarization ϕ is *generic* or *nondegenerate* if

$$\sum_{v\in W}\phi(v)-\frac{|E(W,W^c)|}{2}=\phi_W-\frac{|E(W,W^c)|}{2}\notin\mathbb{Z}$$

for all $W\in\mathrm{BCon}(G)$. Given a numerical polarization ϕ, we can construct the V-stability condition $\mathfrak{n}(\phi)$, where each $\mathfrak{n}(\phi)_W=\left\lceil\phi_W-\frac{|E(W,W^c)|}{2}\right\rceil$. We say such a V-stability condition is *classical*.

As discussed in Section 1.3 of [25] (see also [28] and [35]), one can construct an infinite arrangement of hyperplanes in the affine space of numerical polarizations of degree d supported on $V(G)$, $V^d(G)$, defined as

$$\mathcal{A}_G^d\coloneqq\left\{\phi_W-\frac{|E(W,W^c)|}{2}=n\right\}_{W\subset V(G),n\in\mathbb{Z}}$$

where W again ranges over the biconnected subsets of $V(G)$. This yields a wall and chamber decomposition of $V^d(G)$. A numerical polarization $\phi\in V^d(G)$ is generic (or nondegenerate) if ϕ does not lie on any wall, and from the construction of $\mathfrak{n}(\phi)$, two generic polarizations ϕ and ϕ' are in the same chamber if and only if $\mathfrak{n}(\phi)=\mathfrak{n}(\phi')$.

We focus on *nondegenerate* stability conditions in Chapter 3, which are stability conditions satisfying $\mathcal{D}(\mathfrak{n})=\emptyset$. We expand on degenerate stability conditions and the combinatorics of degeneracy sets in Chapter 4.

Chapter 3

GENERALIZED BREAK DIVISORS AND TRIANGULATIONS OF LAWRENCE POLYTOPES

The content of this chapter can be found in [15].

3.1 Generalized break divisors

Let T be a spanning tree of G, and denote the complement of T in G by $\mathcal{E}_T$. Let $\mathcal{O}_{\mathcal{E}_T}$ be a choice of orientation for the edges in $\mathcal{E}_T$. leaving the edges in T unoriented. By dropping a chip at the head of each oriented edge in $\mathcal{O}_{\mathcal{E}_T}$, we get the *(integral) break divisor* $\mathcal{D}(T, \mathcal{O}_{\mathcal{E}_T})$. Theorem 1.1 of [1] shows that the integral break divisors form a complete set of chip-firing representatives.

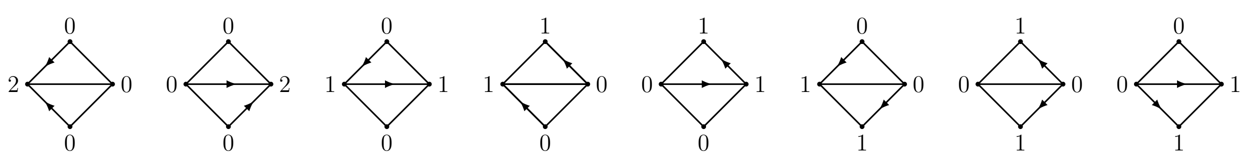


Figure 3.1: Break divisors of the graph G, with an example $\mathcal{O}_{\mathcal{E}_T}$ for each.

In [37], Yuen, motivated by the work of Kass and Pagani in [23], studied the notion of *generalized break divisors*. Let $I_G : \mathcal{ST}(G) \to \mathrm{Div}^0(G)$, be any function that assigns a degree zero divisor to every spanning tree T of G.

Definition 3.1.1. For a function $I_G : \mathcal{ST}(G) \to \mathrm{Div}^0(G)$, the set of *generalized break divisors with respect to* I_G is defined as

$$\mathcal{BD}_{I_G}(G) := \{\mathcal{D}(T, \mathcal{O}_{\mathcal{E}_T}) + I_G(T) | T \in \mathcal{ST}(G) \text{ and } \mathcal{O}_{\mathcal{E}_T} \text{ is an orientation of edges } \mathcal{E}_T\}.$$

If $I_G \cong 0$, $\mathcal{BD}_{I_G}(G)$ is the set of integral break divisors of G, which we denote by $\mathcal{BD}_0(G)$. For a function I_G, the number of generalized break divisors is always bounded below by $|\mathcal{ST}(G)|$.

Theorem 3.1.2. *[37, Theorem 3.5.1] For any function I_G, $|\mathcal{BD}_{I_G}(G)| \geq |\mathcal{ST}(G)|$. Equality holds when $I_G \cong 0$.*

3.2 Nondegenerate stability conditions

In this chapter, we assume that all stability conditions on the graph G are nondegenerate and of degree $g(G)$. We provide the following equivalent definition.

Definition 3.2.1 ([35, Definition 1.4])**.** Let BCon(G) denote the set of all biconnected and nontrivial $W \subset V(G)$. A *nondegenerate V-stability condition* $\mathfrak{n}$ of degree $d \in \mathbb{Z}$ on G is an assignment of integers

$$\begin{aligned} \mathfrak{n} : \mathrm{BCon(G)} &\to \mathbb{Z} \\ W &\mapsto \mathfrak{n}_W \end{aligned}$$

satisfying the following properties: for any $W \in \mathrm{BCon}(G)$, we have

$$\mathfrak{n}_W + \mathfrak{n}_{W^c} = d + 1 - |E(W, W^c)|;$$

and any disjoint W_1, W_2 such that W_1, W_2 and $W_1 \cup W_2$ are all biconnected and nontrivial satisfy

$$-1 \leq \mathfrak{n}_{W_1 \cup W_2} - \mathfrak{n}_{W_1} - \mathfrak{n}_{W_2} - |E(W_1, W_2)| \leq 0.$$

The following theorem connects nondegenerate V-stability conditions and sets of generalized break divisors.

Theorem 3.2.2 ([35, Theorem 1.20])**.** *Let G be a connected graph. There is a bijection*

$$\{V\textit{-stability conditions } \mathfrak{n}\} \leftrightarrow \{\mathcal{BD}_{I_{\mathfrak{n}}}(G) \textit{ such that } |\mathcal{BD}_{I_{\mathfrak{n}}}(G)| = |\mathcal{ST}(G)|\}$$

where $I_{\mathfrak{n}}$ *is defined as follows: for* $T \in \mathcal{ST}(G)$ *and* $e \in T$, $T \backslash e$ *is the disjoint union of connected components* $T[W_e]$ *and* $T[W_e^c]$, *where* W_e *and* W_e^c *are biconnected subsets of* $V(G)$. *Then* $I_{\mathfrak{n}}(T)$ *is the unique element of* $\mathrm{Div}^0(G)$ *such that*

$$I_{\mathfrak{n}}(T)_{W_e} = \mathfrak{n}_{W_e} - g(G[W_e]) \text{ and } I_{\mathfrak{n}}(T)_{W_e^c} = \mathfrak{n}_{W_e^c} - g(G[W_e^c]),$$

where $I_{\mathfrak{n}}(T)_{W_e} = -I_{\mathfrak{n}}(T)_{W_e^c} = \sum_{v \in W_e} I_{\mathfrak{n}}(T)(v)$. *Additionally,* $|\mathcal{BD}_{I_G}(G)| = |\mathcal{ST}(G)|$ *is equivalent to* $\mathcal{BD}_{I_G}(G)$ *being a complete set of chip-firing representatives.*

As a result, we refer to the $\mathcal{BD}_{I_G}(G)$ satisfying $|\mathcal{BD}_{I_G}(G)| = |\mathcal{ST}(G)|$ as stability conditions. If $\mathcal{BD}_{I_G}(G) = \mathcal{BD}_{I_{\mathfrak{n}(\phi)}}(G)$ for a generic numerical polarization ϕ, we say $\mathcal{BD}_{I_G}(G)$ is classical.

3.3 Bijections between signatures, atlases, and Lawrence polytopes

In this section, we recall some of the bijections in [16]. Let $\mathcal{M}$ be an oriented regular matroid. Let B be a basis of $\underline{\mathcal{M}}$. We call the edges in B *internal* and the edges not in B *external*. An *externally (resp. internally) oriented basis* $\vec{B}$ (resp. $\vec{B^*}$) is a fourientation where all the internal (resp. external) edges are bioriented and all the external (resp. internal) edges are one-way oriented. An *external (resp. internal) atlas* $\mathcal{A}$ (resp. $\mathcal{A}^*$) of $\mathcal{M}$ is a collection of externally (resp. internally) oriented bases $\vec{B}$ such that each basis of $\underline{\mathcal{M}}$ appears exactly once. External and internal atlases can be constructed from circuit and cocircuit signatures, respectively. Given a circuit signature σ, $\vec{B} \in \mathcal{A}_\sigma$ is oriented such that if $e \notin B$, e is oriented to be compatible with the signed circuit $\sigma(C(B,e))$. Similarly, given a cocircuit signature σ^*, $\vec{B^*} \in \mathcal{A}^*_{\sigma^*}$ is oriented such that if $e \in B$, e is oriented to be compatible with the signed cocircuit $\sigma^*(C^*(B,e))$.

For the bijections in [16], we want two types of atlases: dissecting and triangulating. Triangulating and dissecting atlases are constructed from triangulations and dissections of the corresponding Lawrence polytopes. Triangulations of Lawrence polytopes always have the same number of simplices [29, Proposition 4.12], equal to the number of bases of $\mathcal{M}$.

Theorem 3.3.1 ([16, Theorem 1.28])**.** *Let $\mathcal{P}$ be the Lawrence polytope of the loopless matroid $\mathcal{M}$. The map*

$$\chi : \{\textit{triangulations (resp. dissections) } \mathcal{T} \textit{ of } \mathcal{P}\} \to \{\textit{triangulating (resp. dissecting) external atlases of } \mathcal{M}\}$$

is a bijection, where for a triangulation (or dissection) $\mathcal{T}$ of $\mathcal{P}$, each simplex $\tau \in \mathcal{T}$ corresponds to an externally oriented basis of $\mathcal{M}$. If we replace the Lawrence polytope $\mathcal{P}$ with $\mathcal{P}^*$, χ with χ^*, and "external" with "internal" then the statement also holds.

Circuit and cocircuit signatures can also be triangulating.

Definition 3.3.2 ([16, Definition 1.14])**.** A circuit signature σ is said to be *triangulating* if for any $\vec{B} \in \mathcal{A}_\sigma$ and any signed circuit $\vec{C} \subseteq \vec{B}$, $\vec{C}$ is in the signature σ. A cocircuit signature σ^* is said to be *triangulating* if for any $\vec{B^*} \in \mathcal{A}^*_{\sigma^*}$ and any signed cocircuit $\vec{C^*} \subseteq \vec{B^*}$, $\vec{C^*}$ is in the signature σ^*.

Lemma 3.4 in [16] states that acyclic signatures are always triangulating.

Theorem 3.3.3 ([16, Theorem 1.16])**.** *The following maps are bijections.*

$$\alpha : \{\textit{triangulating circuit sig. of } \mathcal{M}\} \to \{\textit{triangulating external atlases of } \mathcal{M}\}$$
$$\sigma \mapsto \mathcal{A}_\sigma$$
$$\alpha^* : \{\textit{triangulating cocircuit sig. of } \mathcal{M}\} \to \{\textit{triangulating internal atlases of } \mathcal{M}\}$$
$$\sigma^* \mapsto \mathcal{A}^*_{\sigma^*}$$

One example of a triangulating internal atlas is shown in Figure 3.2. Restricting to

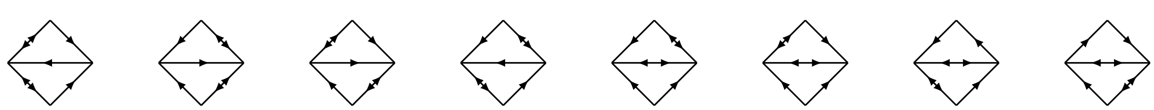


Figure 3.2: Triangulating internal atlas of $\mathcal{M}(G)$

regular triangulations gives another useful bijection.

Theorem 3.3.4 ([16, Theorem 1.29])**.** *The restriction of the bijection $\chi^{-1} \circ \alpha$ to the set of acyclic circuit signatures of $\mathcal{M}$ is bijective onto the set of regular triangulations of $\mathcal{P}$. Dually, the restriction of the bijection $(\chi^*)^{-1} \circ \alpha^*$ to the set of acyclic cocircuit signatures of $\mathcal{M}$ is bijective onto the set of regular triangulations of $\mathcal{P}^*$, where α and α^* are the bijections defined in Theorem 3.3.3.*

The final bijection we present from [16] demonstrates the importance of dissecting and triangulating atlases.

Theorem 3.3.5 ([16, Theorem 1.7])**.** *Given a pair of dissecting atlases $(\mathcal{A}, \mathcal{A}^*)$ of a regular matroid $\mathcal{M}$, if at least one of the atlases is triangulating, then the map*

$$\bar{f}_{\mathcal{A},\mathcal{A}^*} : \{\textit{bases of } \mathcal{M}\} \to \{\textit{circuit-cocircuit reversal classes of } \mathcal{M}\}$$
$$B \mapsto [\vec{B} \cap \vec{B^*}]$$

is bijective, where $[\vec{B} \cap \vec{B^*}]$ is the circuit-cocircuit reversal class containing the orientation $\vec{B} \cap \vec{B^*}$.

3.4 Triangulations and stability conditions

Our main result is to connect the bijections in Theorems 3.3.1 and 3.3.3 to nondegenerate stability conditions. Recall that given an orientation $\mathcal{O}$, the divisor $D_{\mathcal{O}}$ is given by $D_{\mathcal{O}}(v) = \text{indegree}(v) - 1$ for all $v \in V(G)$.

Recall that for a basis element B, the intersection $\vec{B} \cap \vec{B^*}$ of an externally oriented basis $\vec{B}$ and an internally oriented basis $\vec{B^*}$, which are fourientations of B, is some orientation of B. Additionally, recall that for the fourientation $\vec{B^*}$, $-(\vec{B^*})^c$ replaces the bioriented edges of $\vec{B^*}$ with unoriented edges. Finally, recall that the bases of the graphic matroid $\mathcal{M}(G)$ are the spanning trees of G.

Theorem 3.4.1. *For a graph G, consider the graphic matroid $\mathcal{M}(G)$. Let q be a vertex of G. Let $\mathcal{A}$ and $\mathcal{A}^*$ be a pair of dissecting atlases with $\mathcal{A}^* = \mathcal{A}^*_{\sigma^*}$ for some triangulating cocycle* *signature σ^*. Then, the set $\{D_{\vec{T}\cap\vec{T^*}} + (q) : T \in \mathcal{ST}(G)$ and $\vec{T^*} \in \mathcal{A}^*_{\sigma^*}\}$ is a set of pairwise linearly inequivalent generalized break divisors $\mathcal{BD}_{I_G^{\sigma^*}}(G)$, where*

$$I_G^{\sigma^*} : \mathcal{ST}(G) \to \mathrm{Div}^0(G)$$
$$T \mapsto D_{-(\vec{T^*})^c} + (q)$$

and $\mathcal{BD}_{I_G^{\sigma^*}}(G)$ is a complete set of chip-firing representatives.

Proof. First, we define $I_G^{\sigma^*}(T) = D_{-(\vec{T^*})^c} + (q)$ for each spanning tree T of $\mathcal{ST}(G)$. We have $\deg(I_G^{\sigma^*}(T)) = 0$, so $\mathcal{BD}_{I_G^{\sigma^*}}(G)$ is a set of generalized break divisors. The external atlas $\mathcal{A}$ orients the external edges of each spanning tree T, which corresponds to placing a chip at the head of each one-way oriented edge, corresponding to some break divisor $\mathcal{D}(T, \mathcal{O}_{\mathcal{E}_T})$, where $\mathcal{E}_T = E(G\backslash T)$. Therefore, for each $D_{\vec{T}\cap\vec{T^*}} + (q)$, we have

$$D_{\vec{T}\cap\vec{T^*}} + (q) = \mathcal{D}(T, \mathcal{O}_{\mathcal{E}_T}) + D_{-(\vec{T^*})^c} + (q) = \mathcal{D}(T, \mathcal{O}_{\mathcal{E}_T}) + I_G^{\sigma^*}(T) \tag{3.1}$$

so the set $\{D_{\vec{T}\cap\vec{T^*}} + (q) : T \in \mathcal{ST}(G)\}$ is contained in $\mathcal{BD}_{I_G^{\sigma^*}}(G)$.

Then, we wish to show that any two divisors $D_{\vec{T_i}\cap\vec{T_i^*}} + (q)$ and $D_{\vec{T_j}\cap\vec{T_j^*}} + (q)$ are linearly inequivalent. It suffices to show $D_{\vec{T_i}\cap\vec{T_i^*}}$ and $D_{\vec{T_j}\cap\vec{T_j^*}}$ are linearly inequivalent. The bijection $\bar{f}_{\mathcal{A},\mathcal{A}^*_{\sigma^*}}$ from Theorem 3.3.5 sends T_i and T_j to $\vec{T_i}\cap\vec{T_i^*}$ and $\vec{T_j}\cap\vec{T_j^*}$, which are representatives of different cycle-cocycle reversal classes; therefore, the orientations $\vec{T_i}\cap\vec{T_i^*}$ and $\vec{T_j}\cap\vec{T_j^*}$ are not equivalent under cycle and cocycle reversal. If $D_{\vec{T_i}\cap\vec{T_i^*}} \sim D_{\vec{T_j}\cap\vec{T_j^*}}$, then by Theorem 2.1.1 the orientations $\vec{T_i}\cap\vec{T_i^*}$ and $\vec{T_j}\cap\vec{T_j^*}$ are in the same cycle-cocycle reversal class, a contradiction.

Therefore, $\{D_{\vec{T}\cap\vec{T^*}} + (q) : T \in \mathcal{ST}(G)\}$ are all pairwise linearly inequivalent. Because there are $|\mathcal{ST}(G)|$ divisors in $\{D_{\vec{T}\cap\vec{T^*}} + (q) : T \in \mathcal{ST}(G)\}$, the $D_{\vec{T}\cap\vec{T^*}} + (q)$ form a complete set of representatives of divisors in $\mathrm{Div}^g(G)$ under chip-firing.

Finally, we show that $|\mathcal{BD}_{I_G^{\sigma^*}}(G)| = |\mathcal{ST}(G)|$. Consider a divisor $D \in \mathcal{BD}_{I_G^{\sigma^*}}(G)$ which is not of the form $D_{\vec{T}\cap\vec{T^*}} + (q)$ for any spanning tree $T \in \mathcal{ST}(G)$. There exists a spanning tree $T \in \mathcal{ST}(G)$ and an external orientation $\mathcal{O}_T$ of T not in $\mathcal{A}$ such that

$$D = I_G^{\sigma^*}(T) + \mathcal{D}(T, \mathcal{O}_{\mathcal{E}_T}),$$

where $\mathcal{O}_{\mathcal{E}_T}$ is the orientation of edges $\mathcal{E}_T$ induced by the orientation $\mathcal{O}_T$. Since $\vec{T^*} \cap \mathcal{O}_T$ is an orientation of G, there exists a spanning tree $T' \in \mathcal{ST}(G)$ such that $\vec{T^*} \cap \mathcal{O}_T$ is a member of the cycle-cocycle reversal class $[\vec{T'} \cap \vec{T'^*}]$.

By assumption $\vec{T'} \cap \vec{T'^*}$ is not the same orientation as $\vec{T^*} \cap \mathcal{O}_T$, so by Lemma 2.2.2 the two orientations are related by reversals of disjoint signed cycles and cocycles. Cycle reversal does not change the associated divisor, so we focus on cocycles. Say $\vec{T^*} \cap \mathcal{O}_T$ and $\vec{T'^*} \cap \vec{T'}$ differ by a sequence of disjoint cocycle reversals. Let $\vec{C^*}$ be a signed cocycle of $\vec{T'} \cap \vec{T'^*}$ such that $-\vec{C^*}$ is in $\vec{T^*} \cap \mathcal{O}_T$. Therefore, $\vec{C^*}$ is in the cocycle signature σ^*, since σ^* is triangulating. However if $-\vec{C^*}$ is in $\vec{T^*} \cap \mathcal{O}_T$, $-\vec{C^*}$ is in $\vec{T^*}$, which contradicts σ^* being triangulating. Therefore, $\vec{T'} \cap \vec{T'^*}$ and $\vec{T^*} \cap \mathcal{O}_T$ can only differ up to cycle reversal, and $D_{\vec{T^*} \cap \mathcal{O}_T} = D_{\vec{T'^*} \cap \vec{T'}}$ for some spanning tree $T' \in \mathcal{ST}(G)$. $\square$

Example 3.4.2. The first row of Figure 3.3 details the construction of $I_G^{\sigma^*}$ from the triangulating internal atlas $\mathcal{A}^*_{\sigma^*}$ in Figure 3.2. The divisors in the second row are exactly $\mathcal{BD}_{I_G^{\sigma^*}}(G)$, which are constructed by adding the break divisors $\mathcal{D}(T, \mathcal{O}_{\mathcal{E}_T})$ in Figure 3.1 to $I_G^{\sigma^*}(T)$.

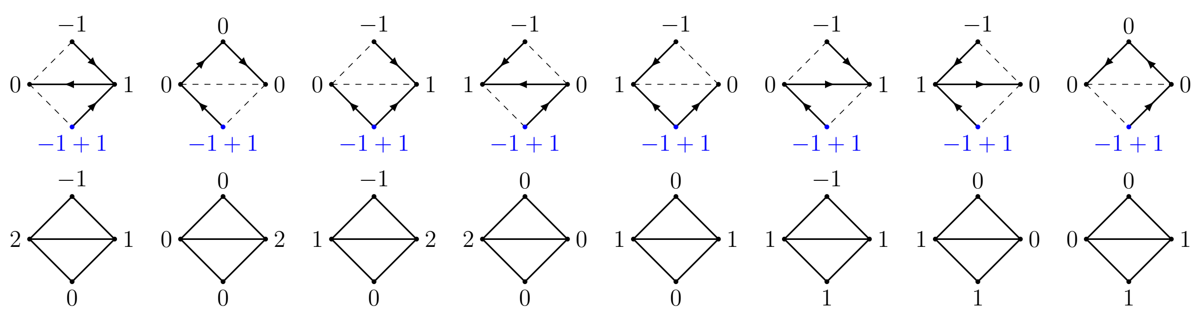
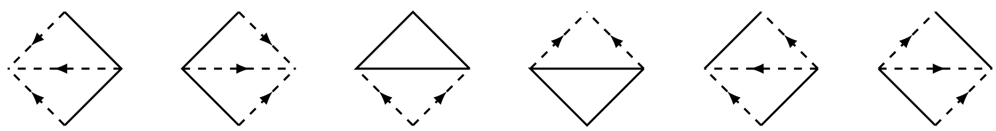


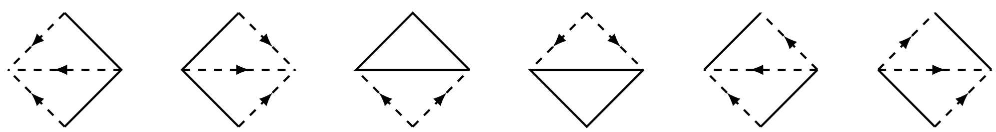
Figure 3.3: Construction of $I_G^{\sigma^*}$ and the generalized break divisors in $\mathcal{BD}_{I_G^{\sigma^*}}$, with q being the lower blue vertex.

Let σ_1^* and σ_2^* be two triangulating cocycle signatures which, by Theorem 3.4.1, correspond to the functions $I_G^{\sigma_1^*}$ and $I_G^{\sigma_2^*}$. To better understand stability conditions of G, we wish to relate $I_G^{\sigma_1^*}$ and $I_G^{\sigma_2^*}$.

Definition 3.4.3. Suppose σ_1^* and σ_2^* are two triangulating cocycle signatures. Then, we say they differ by a *cocycle flip* if there exists a unique cocycle C^* such that $\vec{C^*} \in \sigma_1^*$ and $-\vec{C^*} \in \sigma_2^*$.



(a) Cocycle signature σ_1^*



(b) Cocycle signature σ_2^*

Figure 3.4: Two triangulating cocycle signatures that differ by a cocycle flip in the fourth cocycle.

The following lemma follows immediately from the definition of being triangulating.

Lemma 3.4.4. *If σ_1^* and σ_2^* are two triangulating cocycle signatures that differ by a cocycle flip at C^*, then, for each basis B such that $\sigma_1^*(C^*) \subseteq \vec{B^*} \in \mathcal{A}^*_{\sigma_1^*}$, B only has one internal edge in C^*.*

Proof. Let $\vec{B^{*,1}} \in \mathcal{A}^*_{\sigma_1^*}$ and $\vec{B^{*,2}} \in \mathcal{A}^*_{\sigma_2^*}$. Assume for contradiction there exists a basis B such that $\sigma_1^*(C^*) \subseteq \vec{B^{*,1}}$ and C^* is not a fundamental cocycle of B. Then, $\vec{B^{*,1}} = \vec{B^{*,2}}$ and $\sigma_1^*(C^*) \subseteq \vec{B^{*,2}}$, but this contradicts σ_2^* being triangulating. $\square$

In fact, using cocycle flips to move between triangulating cocycle signatures is the same as using flips to move between triangulations of the Lawrence polytope $\mathcal{P}^*$.

Lemma 3.4.5. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be triangulations of the Lawrence polytope $\mathcal{P}^*$. Let σ_1^* be the cocycle signature such that $\chi^*(\mathcal{T}_1) = \mathcal{A}^*_{\sigma_1^*}$, and define σ_2^* similarly. Then, $\mathcal{T}_1$ and $\mathcal{T}_2$ differ by a flip if and only if σ_1^* and σ_2^* differ by a cocycle flip.*

Proof. First, assume that $\mathcal{T}_1$ and $\mathcal{T}_2$ differ by a flip. Therefore, there exists a circuit C of the dual matroid $\mathcal{M}^*$ such that (without loss of generality) $\mathcal{T}_+^C$ is contained in $\mathcal{T}_1$ and $\operatorname{link}_{\mathcal{T}_1}(\tau)$ is the same for every cell τ in $\mathcal{T}_+^C$.

We then note that $\mathcal{T}_2$ can be defined as

$$\mathcal{T}_2 \coloneqq \mathcal{T}_1 \backslash \{p \cup \tau : p \in \operatorname{link}_{\mathcal{T}_1}(\tau), \tau \in \mathcal{T}_+^C\} \cup \{p \cup \tau : p \in \operatorname{link}_{\mathcal{T}_1}(\tau), \tau \in \mathcal{T}_-^C\}.$$

By duality, the signed circuit C of $\mathcal{M}^*$ corresponds to the signed cocircuit $C^* = (C_+^*, C_-^*)$ of $\mathcal{M}(G)$. Recall the map χ^* from Theorem 3.3.1. For each $\tau \in \mathcal{T}_+^C$, the image of $\chi^*(\tau)$ is a fourientation of the subgraph formed by the edges of the cocycle C^*, where every edge is bioriented except for a single one-way oriented edge. Call that edge e_τ. For each τ, that edge e_τ must have the same orientation, since each τ is constructed by $c_i \in C_+$ from C. We give the cocycle this orientation and denote it by $\vec{C^*}$.

Since σ_1^* is triangulating, we must have $\vec{C^*} = \sigma_1^*(C^*)$. The bistellar flip replaces C_+ with C_-, and therefore $\sigma_2^*(C^*) = -\vec{C^*}$, as desired.

Now, assume that σ_1^* and σ_2^* differ by a cocycle flip. Let C^* be that cocycle, and let C be the circuit dual to it in $\mathcal{M}^*$. Because σ_1^* is triangulating, we assume without loss of generality that C_+ is contained in $\mathcal{T}_1$. Let $L = \{(\chi^*)^{-1}(\vec{B^*})\backslash\tau | \vec{C^*} \subseteq \vec{B^*}, \vec{B^*} \in \mathcal{A}_{\sigma_1^*}^*\}$. We wish to show that $L = \operatorname{link}_{\mathcal{T}_1}(\tau)$ for every $\tau \in \mathcal{T}_+^C$.

If $p \in \operatorname{link}_{\mathcal{T}_1}(\tau)$, we have $p \cup \tau \in \mathcal{T}_1$ by definition of the link. Therefore, $\chi^*(p \cup \tau) = \vec{B^*}$ by Theorem 3.3.1. We then have $\vec{C^*} \subseteq \chi^*(\tau) \subseteq \vec{B^*}$, and therefore $p \in L$. For the reverse inclusion assume $\ell \in L$. Then, $\ell = (\chi^*)^{-1}(\vec{B^*})\backslash\tau$ for some internally oriented basis $\vec{B^*}$ of $\mathcal{M}(G)$. By definition $\ell \cap \tau = \emptyset$, and additionally $\chi^*(\ell) \cup \chi^*(\tau) = \vec{B^*}$ which is in the internal atlas $\mathcal{A}_{\sigma_1^*}^*$. Therefore, $\ell \cup \tau \in \mathcal{T}_1$ and $\ell \in \operatorname{link}_{\mathcal{T}_1}(\tau)$.

Therefore, C supports a bistellar flip in $\mathcal{T}_1$, and the result of that flip is $\mathcal{T}_2$. $\square$

We can then consider the function $I_G^{\sigma_2^*}$ to be *reachable* from $I_G^{\sigma_1^*}$ if σ_2^* and σ_1^* are connected by a sequence of cocycle flips. We recall that the flip graph restricted to regular triangulations is connected.

Corollary 3.4.6. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be regular triangulations of $\mathcal{P}^*$, and σ_1^* and σ_2^* be the cocycle signatures such that $\chi^*(\mathcal{T}_1) = \mathcal{A}^*_{\sigma_1^*}$ and $\chi^*(\mathcal{T}_2) = \mathcal{A}^*_{\sigma_2^*}$. Then, σ_1^* is reachable from σ_2^*.*

The first step towards constructing classical stability conditions from regular triangulations of $\mathcal{P}^*$ is to demonstrate that $\mathcal{BD}_0(G)$ is classical. We use the following lemma.

Lemma 3.4.7. *Let W be a subset of $V(G)$. Then, $\sum_{v\in W} \deg(v) = |E(W, W^c)| + 2|E(G[W])|$.*

We then consider the generic polarization ϕ_{pcan} defined in Example 5.4 [13] or in Example 8.10 of [28], given by

$$\phi_{pcan}(v) = \frac{g(G) + |V(G)|}{2(g(G) + |V(G)|) - 2} \cdot \deg(v) - 1.$$

Lemma 3.4.8. *The V-stability condition $\mathfrak{n}$ associated to ϕ_{pcan} is*

$$\mathfrak{n}(\phi_{pcan}) = \{\mathfrak{n}(\phi_{pcan})_W = g(G[W]) \text{ for any biconnected and nontrivial } W\}.$$

Proof. First, we rewrite ϕ_{pcan} using $g(G) = |E(G)| - |V(G)| + 1$. Then,

$$\phi_{pcan}(v) = \frac{|E(G)| + 1}{2|E(G)|} \cdot \deg(v) - 1. \tag{3.2}$$

Using Lemma 3.4.7 and equation (3.2), for biconnected $W \subset V(G)$ we have

$$\phi_{pcan}(W) = \sum_{v\in W} \phi_{pcan}(v) = -|W| + \frac{|E(G)| + 1}{2|E(G)|} (|E(W, W^c)| + 2|E(G[W])|)$$

Substituting this back into the definition of $\mathfrak{n}(\phi_{pcan})_W$, direct calculation gives

$$\mathfrak{n}(\phi_{pcan})_W = \left\lceil \phi_{pcan}(W) - \frac{|E(W, W^c)|}{2} \right\rceil = \left\lceil \frac{1}{2|E(G)|} \sum_{v\in W} \deg(v) \right\rceil + |E(G[W])| - |W|$$

By the Handshake Lemma, $\sum_{v\in W} \deg(v) < 2|E(G)|$, so $\mathfrak{n}(\phi_{pcan})_W = g(G[W])$. □

Theorem 3.2.2 and Lemma 3.4.8 yield the following corollary.

Corollary 3.4.9. *Let $\mathcal{BD}_0(G)$ denote the break divisors of graph G. Then, there exists a generic numerical polarization ϕ such that $\mathcal{BD}_0(G) = \mathcal{BD}_{I_{\mathfrak{n}(\phi)}}(G)$.*

For the final result, we use a particular triangulating internal atlas. First, fix $q \in V(G)$ and consider each spanning tree T of G as a rooted tree with the root at q. For the internal atlas $\mathcal{A}^*_q$, orient the internal edges of T away from q. As shown in [37] (see also [16]), this atlas is triangulating and corresponds to acyclic cocycle signature σ^*_q. Additionally, $(I_G^{\sigma^*_q} + (q))(T) = \vec{0}$ for every spanning tree T. By Theorem 3.3.4, $((\chi^*)^{-1} \circ \alpha^*)(\sigma^*_q)$ is a regular triangulation of the Lawrence polytope $\mathcal{P}^*$. Therefore, any other acyclic cocycle signature of $\mathcal{M}(G)$ is reachable from σ^*_q.

Theorem 3.4.10. *If σ^* is an acyclic cocycle signature of the graph G, then $\mathcal{BD}^{\sigma^*}_{I_G}$ is a classical stability condition.*

Proof. For some $q \in V(G)$, let σ^*_q be the acyclic cocycle signature such that $I_G^{\sigma^*_q} = \vec{0}$. Let $\mathfrak{n} = \mathfrak{n}(\phi_{pcan})$. We induct on the number of cocycle flips k needed to reach σ^* from σ^*_q. If $\sigma^*_q = \sigma^*$, Corollary 3.4.9 states that the break divisors are a classical stability condition, so $I_G^{\sigma^*_q} = I_{\mathfrak{n}}$. If $k = 1$, let C^*_1 be the cocycle such that $\vec{C^*_1} \in \sigma^*_q$ and $-\vec{C^*_1} \in \sigma^*$. Let W_1 be the biconnected subset of $V(G)$ such that $C^*_1 = E(W_1, W_1^c)$. Let $\phi = \phi_{pcan}$ and recall the wall and chamber decomposition of $V^d(G)$ induced by the hyperplane arrangement $\mathcal{A}^d_G$.

Then, ϕ lives in the chamber cut out by the hyperplanes $\left\{\phi_W - \frac{|E(W,W^c)|}{2} = g(G[W])\right\}$ and $\left\{\phi_W - \frac{|E(W,W^c)|}{2} = g(G[W]) - 1\right\}$ for each biconnected $W \subset V(G)$.

Assume that $\vec{C^*_1} \in \sigma^*_q$ points towards W_1. Consider a generic polarization ϕ' in the chamber cut out by the same hyperplanes

$$\left\{\phi_W - \frac{|E(W,W^c)|}{2} = g(G[W])\right\} \bigcup \left\{\phi_W - \frac{|E(W,W^c)|}{2} = g(G[W]) - 1\right\}$$

for biconnected $W \subset V(G), W \neq W_1, W_1^c$, as well as

$$\left\{\phi_{W_1} - \frac{|E(W_1,W_1^c)|}{2} = g(G[W_1]) - 1\right\} \bigcup \left\{\phi_{W_1^c} - \frac{|E(W_1,W_1^c)|}{2} = g(G[W_1^c])\right\}$$

from the starting chamber, with the choice of new hyperplanes

$$\left\{\phi_{W_1} - \frac{|E(W_1,W_1^c)|}{2} = g(G[W_1]) - 2\right\} \bigcup \left\{\phi_{W_1^c} - \frac{|E(W_1,W_1^c)|}{2} = g(G[W_1^c]) + 1\right\}.$$

Therefore, $\mathfrak{n}(\phi')_{w_1} = \mathfrak{n}(\phi)_{W_1} - 1$, $\mathfrak{n}(\phi')_{W_1^c} = \mathfrak{n}(\phi)_{W_1^c} + 1$, and $\mathfrak{n}(\phi')_W = \mathfrak{n}(\phi)_W$ for all other biconnected $W \subset V(G)$. With $\mathfrak{n}' = \mathfrak{n}(\phi')$, we can construct $I_{\mathfrak{n}'}$ as described in Theorem 3.2.2, and compare with $I_G^{\sigma^*}$. First, consider T such that C_1^* is not a fundamental cocycle of T. Then, $\vec{T^*}$ does not change, so $I^{\sigma^*}(T) = I_G^{\sigma_q^*}(T) = \vec{0}$. Because C_1^* is not a fundamental cocycle of T, there is no edge $e \in T$ such that $T \backslash e = T[W_1] \sqcup T[W_1^c]$, so $I_{\mathfrak{n}'}(T) = I_{\mathfrak{n}}(T) = \vec{0}$.

Now, consider T such that C_1^* is a fundamental cocycle of T. Let $\vec{T^*} \in \mathcal{A}^*_{\sigma_q^*}$ and $\vec{T^{*\prime}} \in \mathcal{A}^*_{\sigma^*}$, and let e be the edge such that $C^*(T,e) = C_1^*$. Therefore, $\vec{e} \in \vec{T^*}$ and $-\vec{e} \in \vec{T^{*\prime}}$. If $\vec{e} = (v, v')$, we then have $I_G^{\sigma^*}(T)(v) = I_G^{\sigma_q^*}(T)(v) - 1$ and $I_G^{\sigma^*}(T)(v') = I_G^{\sigma_q^*}(T)(v') + 1$, since the indegree of v decreases by 1 and the indegree of v' increases by 1. Equivalently by construction of $\mathfrak{n}' = \mathfrak{n}(\phi')$, $I_{\mathfrak{n}'}(T)_{W_1} = I_{\mathfrak{n}}(T)_{W_1} - 1$ and $I_{\mathfrak{n}'}(T)_{W_1^c} = I_{\mathfrak{n}}(T)_{W_1^c} + 1$.

The uniqueness of $I_{\mathfrak{n}'}$ guarantees $I_{\mathfrak{n}'} = I_G^{\sigma^*}$, as desired. Therefore, $\mathcal{BD}_{I_G^{\sigma^*}}(G)$ is a classical stability condition.

Now, assume the inductive hypothesis holds for all $0 \leq k \leq K$, and assume σ^* is reachable from σ_q^* through a sequence of $K+1$ cocycle flips. Let $C_1^*, ..., C_{K+1}^*$ be the sequence of cocycles flipped, and assume σ_K^* is the acyclic cocycle signature reached by the sequence $C_1^*, ..., C_K^*$ of cocycle flips. By the inductive hypothesis, $\mathcal{BD}_{I_G^{\sigma_K^*}}(G)$ is a classical stability condition. Assume that $W_{K+1} \subset V(G)$ such that $C_{K+1}^* = E(W_{K+1}, W_{K+1}^c)$ and $\vec{C_{K+1}^*}$ is oriented towards W_{K+1}. If ϕ is the generic numerical polarization such that $I_{\mathfrak{n}(\phi)} = I_G^{\sigma_K^*}$, we can then select a generic numerical polarization ϕ' such that $\mathfrak{n}(\phi')_{W_{K+1}} = \mathfrak{n}(\phi)_{W_{K+1}} - 1$, $\mathfrak{n}(\phi')_{W_{K+1}^c} = \mathfrak{n}(\phi)_{W_{K+1}^c} + 1$, and $\mathfrak{n}(\phi')_W = \mathfrak{n}(\phi)_W$ for all other biconnected $W \subset V(G)$. By following the same procedure as the $k = 1$ case, we see that $I_{\mathfrak{n}(\phi')} = I_G^{\sigma^*}$, as desired. $\square$

Chapter 4

STABILITY CONDITIONS FROM SUBDIVISIONS OF LAWRENCE POLYTOPES

In this chapter, we define single-element extensions of unoriented and oriented matroids. We prove that $\mathcal{D}$ is a degeneracy set of a graph G if and only if it corresponds to a single-element extension of the unoriented graphic matroid of G (Lemma 4.2.4). We also show that every subdivision of the Lawrence polytope $\Lambda(\mathcal{M}^*(G))$ yields a stability condition (Theorem 4.4.10). However, because not all matroids are orientable, not every stability condition can correspond to subdivision of $\Lambda(\mathcal{M}^*(G))$.

4.1 Single-element extensions of matroids and subdivisions of Lawrence polytopes

In this section, we define *single-element extensions* and *liftings* of oriented and unoriented matroids.

Let $\mathcal{M}$ be a matroid on a ground set E. Then, a *single-element extension* of $\mathcal{M}$ by some element p is a matroid $\mathcal{M}'$ on the ground set $E \cup \{p\}$ such that $\mathcal{M}' \backslash p = \mathcal{M}$. Dually, a *single-element lifting* of $\mathcal{M}$ is a matroid $\tilde{\mathcal{M}}$ on the ground set $E \cup \{p\}$ such that $\tilde{M}/p = \mathcal{M}$. We note that single-element extensions of $\mathcal{M}$ are dual to single-element liftings of $\mathcal{M}^*$, so we only provide background on extensions in this section. A single-element extension is *nontrivial* if $r(\mathcal{M}') = r(\mathcal{M})$, and we assume all of the extensions we consider to be nontrivial.

In both the unoriented and oriented case, extensions are characterized by how they extend the cocircuits of starting matroid. First, we present the characterization of single-element extensions of unoriented matroids from [14].

Proposition 4.1.1 ([14],[9, Proposition 7.1.2])**.** *Let M be a matroid on the ground set E,* *and denote the set of cocircuits of* M *by* $C^*(M) \subseteq \{0,1\}^E$*, given by their incidence vectors. Let* $M' = M \cup \{p\}$ *be an extension of* M *by a new element* p*. Let* $C^*(M')$ *be the cocircuits of* M'*. Then for every cocircuit* $C^* \in C^*(M)$*, there is a unique cocircuit of* M' *contained in* $C^* \cup \{p\}$*, that is, there is a unique function*

$$\sigma^* : C^*(M) \to \{0,1\}$$

such that $\{(C^*, \sigma^*(C^*)) : C^* \in C^*(M)\} \subseteq C^*(M')$*.*

Furthermore, all the cocircuits of M' *can be explicitly described using* σ^**. Thus,* M' *is uniquely determined by* σ^**.*

However, not every function σ^* corresponds to a single-element extension of a matroid M. Two sets $A, B \subseteq E$ form a *modular pair* of M if $r(A) + r(B) = r(A \cup B) + r(A \cap B)$. Two cocircuits C_1^* and C_2^* of M are a *modular pair of cocircuits* if they are a modular pair of the dual matroid M^*.

Theorem 4.1.2 ([14],[9, Theorem 7.1.3])**.** *Let* M *be a matroid. Then, for a function* $\sigma^* : C^*(M) \to \{0,1\}$*, which assigns a signature* $\sigma^*(C^*) \in \{0,1\}$ *to every cocircuit* C^* *of* M*, the following are equivalent:*

1. *There exists a single-element extension* M' *of* M *by an element* p *such that* $\{(C^*, \sigma^*(C^*)) : C^* \in C^*(M)\} \subseteq C^*(M')$*.*

2. σ^* *defines a single-element extension on every rank* 2 *contraction of* M*.*

3. *If a modular pair of cocircuits has signature* 0*, then so has every other cocircuit in their union.*

Therefore, we can describe single-element extensions by the cocircuits C^* with signature $\sigma^*(C^*) = 0$. When σ^* is the signature of a single-element extension of M as in Theorem 4.1.2, the collection $\{C^* \in C^*(M) : \sigma^*(C^*) = 0\}$ is a *linear subclass of cocircuits*. The set of linear subclasses of cocircuits of M, when partially ordered by inclusion, form the *extension* *lattice* of M [12]. For graphic matroids, the single-element extensions can be characterized by cobiased graphs [33].

Now, consider an oriented matroid $\mathcal{M}$. Single-element extensions of $\mathcal{M}$ are characterized by how they extend the signed cocircuits of $\mathcal{M}$.

Proposition 4.1.3 ([34], [9, Proposition 7.1.4])**.** *Let $\mathcal{M}$ be an oriented matroid on the ground set E, and denote the signed cocircuits of $\mathcal{M}$ by $\mathcal{C}^*(\mathcal{M})$. Let $\mathcal{M}'$ be a single-element extension of $\mathcal{M}$. Then for every signed cocircuit $\vec{C^*} \in \mathcal{C}^*(\mathcal{M})$, there is a unique way to extend $\vec{C^*}$ to a cocircuit of $\mathcal{M}'$: there is a unique function*

$$\sigma^* : \mathcal{C}^*(\mathcal{M}) \to \{+, -, 0\}$$

such that $(\vec{C^*}, \sigma^*(\vec{C^*}))$ is a signed cocircuit of $\mathcal{M}'$. Furthermore, this σ^* satisfies $\sigma^*(-\vec{C^*}) = -\sigma^*(\vec{C^*})$. Finally, $\mathcal{M}'$ is uniquely determined by σ^*.

Every single-element extension of $\mathcal{M}$ defines a single-element extension of $\underline{\mathcal{M}}$. If σ^* determines an extension of an oriented matroid $\mathcal{M}$, we say that σ^* is a localization. Localizations are characterized by their behavior on the rank two contractions of $\mathcal{M}$. Oriented matroids of rank 2 can always be represented by a central hyperplane arrangement, where each line corresponds to a cocircuit of the matroid.

Theorem 4.1.4 ([34], [9, Theorem 7.1.8])**.** *Let $\mathcal{M}$ be an oriented matroid, and $\sigma^* : \mathcal{C}^*(\mathcal{M}) \to \{+, -, 0\}$ a signature satisfying $\sigma^*(-\vec{C^*}) = -\sigma^*(\vec{C^*})$ for all $\vec{C^*} \in \mathcal{C}^*(\mathcal{M})$ Then, the following statements are equivalent:*

1. *σ^* is a localization: there exists a single-element extension $\mathcal{M}'$ of $\mathcal{M}$ such that*

$$\{(\vec{C^*}, \sigma^*(\vec{C^*})) : \vec{C^*} \in \mathcal{C}^*(\mathcal{M})\} \subseteq \mathcal{C}^*(\mathcal{M}').$$

2. *σ^* defines a single-element extension on every contraction of $\mathcal{M}$ of rank 2.*

3. *The signature σ^* produces none of the three excluded subconfigurations (minors) of rank 2 on three elements, as given in Figure 4.1. Each ray corresponds to a cocircuit of the*

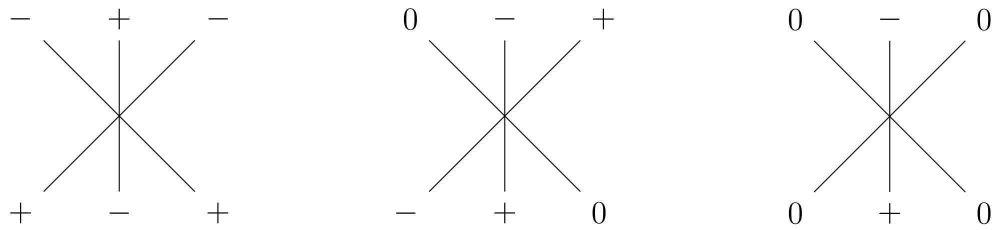


Figure 4.1: Forbidden subconfigurations for signatures of cocircuits

rank 2 minor. The signs on each ray from the origin represent the sign of the new element p in $\mathcal{M}' = \mathcal{M} \cup \{p\}$.

Finally, we note that not every unoriented extension of an orientable matroid corresponds to an oriented extension of that oriented matroid.

Remark 4.1.5. The signatures in Theorems 4.1.2 and 4.1.4 are referred to as cocircuit signatures in [9], but to avoid confusion we refer to them as a signature of an extension. The *(partial) cocircuit signature* of a single-element extension is the set

$$\{\vec{C^*} : \sigma^*(\vec{C^*}) = +\} \sqcup \{C^* : \sigma^*(\vec{C^*}) = \sigma^*(-\vec{C^*}) = 0\}.$$

Recall from Section 2.4 that for any oriented matroid $\mathcal{M}$, we can lift $\mathcal{M}$ to an oriented matroid $\Lambda(\mathcal{M})$. If $\mathcal{M}$ can be realized by a matrix $M_{r\times n}$, then $\Lambda(\mathcal{M})$ is the Lawrence polytope of $\mathcal{M}$. Each column of $M_{r\times n}$ corresponds to an arc $\vec{e}$ of $\mathcal{M}$. The vertices of $\Lambda(\mathcal{M})$ are then the set $E \cup \overline{E}$, where E is the ground set E of $\mathcal{M}$ and $\overline{E} = \{-\vec{e} : \vec{e} \in E\}$.

As an oriented matroid, $\Lambda(\mathcal{M})$ inherits its circuits, cocircuits, and bases from $\mathcal{M}$. If B is an unsigned subset of E and $A \subseteq B$, the *reorientation* of B at A is ${}_AB := (B\backslash A) \cup \overline{A}$. If $X = (X_+, X_-)$, and $Y \subseteq X_+ \cup X_-$ is a signed subset of E, the *reorientation* of X at Y is ${}_YX := ((X_+\backslash Y \cup \overline{(X_- \cap A)}), (X_-\backslash A, \overline{(X_+ \cap A)})).$

Lemma 4.1.6 ([29, Lemma 4.11])**.** *Let $\Lambda(\mathcal{M})$ be the Lawrence polytope associated with an oriented matroid $\mathcal{M}$.*

1. *The set of circuits of* $\Lambda(\mathcal{M})$ *is*

$$\mathcal{C}(\Lambda(\mathcal{M})) := \{(C_+ \cup \overline{C_-}, C_- \cup \overline{C_+}) : (C_+, C_-) \in \mathcal{C}(\mathcal{M})\}.$$

2. *The set of cocircuits of* $\Lambda(\mathcal{M})$ *is*

$$\mathcal{C}^*(\Lambda(\mathcal{M})) := \{{}_A\vec{C^*} : \vec{C^*} \in \mathcal{C}^*(\mathcal{M}), A \subseteq \vec{C}^*_+ \cup \vec{C}^*_-\} \cup \{(\{\vec{e}, -\vec{e}\}, \emptyset), (\emptyset, \{\vec{e}, -\vec{e}\}) : e \text{ is not a coloop}\}.$$

3. *The set of bases of* $\Lambda(\mathcal{M})$ *is*

$$\mathcal{B}(\Lambda(\mathcal{M})) := \{{}_A(E \backslash B) \cup B \cup \overline{B} : B \in \mathcal{B}(\mathcal{M}), A \subseteq E \backslash B\}.$$

Through the relationship between circuits of $\Lambda(\mathcal{M})$, the circuits of $\mathcal{M}$, and the cocircuits of $\mathcal{M}^*$, we can identify extensions of $\mathcal{M}^*$ with subdivisions of $\Lambda(\mathcal{M})$ using the following lemma.

Lemma 4.1.7 ([29, Lemma 4.13(i)-(iii)])**.** *Let* $\Lambda(\mathcal{M})$ *be the Lawrence polytope associated to an oriented matroid* $\mathcal{M}$*. Let* $\mathcal{S}$ *be a subdivision of* $\Lambda(\mathcal{M})$*.*

1. *The support of every signed circuit of* $\Lambda(\mathcal{M})$ *is a face of* $\Lambda(\mathcal{M})$*.*

2. *Let* F *be a face of* $\Lambda(\mathcal{M})$ *which is the support of a circuit* $\vec{C}$*. Let* k *be the rank of* F*. Consider the "restriction" of* $\mathcal{S}$ *to* F *defined as:*

$$\mathcal{S}_F := \{\gamma \cap F : \gamma \in \mathcal{S}, \mathit{rank}(\gamma \cap F) = k\}.$$

   Then, $\mathcal{S}_F$ *is either the trivial subdivision* $\{F\}$ *of the restricted oriented matroid* $\Lambda(\mathcal{M})(F) := \Lambda(\mathcal{M}) \backslash F^c$ *or one of the triangulations* $\mathcal{T}^+_{\vec{C}} := \{(\vec{C}_+ \cup \vec{C}_-) \backslash \{e\} : e \in \vec{C}_+\}$ *or* $\mathcal{T}^-_{\vec{C}} := \{(\vec{C}_+ \cup \vec{C}_-) \backslash \{e\} : e \in \vec{C}_-\}$ *of the face* F*.*

3. *The signature of* $\Lambda(\mathcal{M})^*$ *defined by* $\vec{C}(p) = 0$ *if* $\mathcal{S}_C = \{C\}$ *and* $\vec{C}(p) = +$ *(resp.* $\vec{C}(p) = -$*) if* $\mathcal{S}_C = \mathcal{T}^+_{\vec{C}}$ *(resp.* $\mathcal{S}_C = \mathcal{T}^-_{\vec{C}}$*) is the signature of an extension* $\Lambda(\mathcal{M})^* \cup p$ *of* $\Lambda(\mathcal{M})^*$*.*

We note that in Lemma 4.1.7(2), the triangulations $\mathcal{T}^+_{\vec{C}}$ and $\mathcal{T}^-_{\vec{C}}$ are triangulations of the oriented matroid $\Lambda(\mathcal{M})(F)$, as defined in Chapter 2 of [29].

Theorem 4.1.8 ([29, Theorem 4.14])**.** *Let $\Lambda(\mathcal{M})$ be the Lawrence polytope associated to an oriented matroid $\mathcal{M}$. Then, there is a natural bijection between the extensions of $\mathcal{M}^*$ and the subdivisions of $\Lambda(\mathcal{M})$.*

4.2 Degeneracy sets and orientability

In Section 2.5, we defined degeneracy sets of stability conditions, but a degeneracy set can be defined without a reference stability condition. We recall the following definition from [18].

Definition 4.2.1. A *degeneracy set* for a graph G is a subset $\mathcal{D} \subseteq \mathrm{BCon}(G)$ satisfying the following properties:

1. If $W \in \mathcal{D}$, then $W^c \in \mathcal{D}$;
2. for all $W_1, W_2 \in \mathrm{BCon}(G)$ such that $W_1 \cap W_2 = \emptyset$ and $W_1 \cup W_2 \in \mathrm{BCon}(G)$, if $W_1, W_2 \in \mathcal{D}$, then $W_1 \cup W_2 \in \mathcal{D}$.

For a degeneracy set, it can be useful to consider the cuts of G corresponding the the biconnected subsets in $\mathcal{D}$. We define $C^*(\mathcal{D}) := \{E(W, W^c) : W \in \mathcal{D}\}$.

Lemma 4.2.2. *Let W_1 and W_2 be biconnected subsets of $V(G)$ such that $W_1 \cap W_2 = \emptyset$ and either $W_1 \cup W_2$ is biconnected and nontrivial, or $(W_1 \cup W_2)^c$ is connected and nontrivial. Then, the cuts $C_1^* = E(W_1, W_1^c)$ and $C_2^* = E(W_2, W_2^c)$ are a modular pair of cocircuits of $M(G)$.*

Proof. Let r^* be the rank function of cographic matroid M^*; for any subset S, we have $r^*(S) = r(E \backslash S) + |S| - r(M(G))$. We recall that C_1^* and C_2^* are a modular pair of cocircuits of $M(G)$ if they are a modular pair of circuits for $M^*(G)$. By duality, C_1^* and C_2^* are circuits of $M^*(G)$, so $r^*(C_1^*) = |C_1^*| - 1$ and $r^*(C_2^*) = |C_2^*| - 1$. The intersection $C_1^* \cap C_2^*$ is the set of edges $E(W_1, W_2)$, and since $(W_1 \cup W_2)^c$ is biconnected, the subgraph $G \backslash (E(W_1, W_2))$ is connected, and $r(E \backslash E(W_1, W_2)) = |V(G)| - 1$. Therefore, $r^*(C_1^* \cap C_2^*) = |E(W_1, W_2)|$.

Since W_1 and W_2 are disjoint and biconnected, $G \backslash (C_1^* \cup C_2^*)$ has 3 connected components. Therefore, $r^*(C_1^* \cup C_2^*) = |C_1^*| + |C_2^*| - |E(W_1, W_2)| - 2$.

We then see $r^*(C_1^*) + r^*(C_2^*) = r^*(C_1^* \cap C_2^*) + r^*(C_1^* \cup C_2^*)$, so C_1^* and C_2^* is a modular pair of cocircuits of $M(G)$. □

Lemma 4.2.3. *Consider biconnected, disjoint subsets W_1 and W_2 such that $W_1 \cup W_2$ is a biconnected, nontrivial subset of $V(G)$. Then, there exists a rank 2 minor of $\mathcal{M}(G)$, which we call $\mathcal{N}$, such that $\mathcal{C}^*(\mathcal{N}) = \{E(W_1, W_1^c), E(W_2, W_2^c), E(W_1 \cup W_2, (W_1 \cup W_2)^c)\}$.*

Proof. Let $E' = E(G[W_1]) \sqcup E(G[W_2]) \sqcup E(G[(W_1 \cup W_2)^c])$. Each of these sets is disjoint because W_1, W_2, and $(W_1 \cup W_2)^c$ are all disjoint from each other. The contraction G/E' is a three vertex graph with edges $E(G/E') = E(W_1, W_1^c) \cup E(W_2, W_2^c)$; therefore, $\mathcal{M}(G)/E'$ is a rank 2 minor of $\mathcal{M}(G)$. Identify the vertices of G/E' as W_1, W_2, and $(W_1 \cup W_2)^c$. Then, the cuts of G/E' are exactly $E(W_1, W_1^c), E(W_2, W_2^c)$, and $E(W_1 \cup W_2, (W_1 \cup W_2)^c)$. □

Lemma 4.2.4. *$\mathcal{D}$ is a degeneracy set for a graph G if and only if there exists a single-element extension of M by an element p such that $\{C^* \in C^*(M) : \sigma^*(C^*) = 0\} = C^*(\mathcal{D})$.*

Proof. First, let $\mathcal{D}$ be a degeneracy set for a graph G. Then, we define a signature σ^* such that $\sigma^*(C^*) = 0$ if and only if $C^* \in C^*(\mathcal{D})$. We then consider modular pairs C_1^*, C_2^* of cocircuits such that $\sigma^*(C_1^*) = \sigma^*(C_2^*) = 0$. For such a pair, we have $C_1^*, C_2^* \in C^*(\mathcal{D})$. Let W_1, W_2 be the biconnected sets in $\mathcal{D}$ such that $C_1^* = E(W_1, W_1^c)$, $C_2^* = E(W_2, W_2^c)$. We note that $W_1 \cap W_2 = \emptyset$ in order for C_1^*, C_2^* to be a modular pair, and $W_1 \cup W_2 \subset V(G)$ since $C_1^* \neq C_2^*$. Then, consider $C_3^* \subset C_1^* \cup C_2^*$. By the minimality of cocircuits, $C_3^* = E(W_1 \cup W_2, (W_1 \cup W_2)^c)$. Because $\mathcal{D}$ is a degeneracy set, $W_1 \cup W_2 \in \mathcal{D}$, and $\sigma^*(C_3^*) = 0$, by definition. By Theorem 4.1.2, there exists a single-element extension of $M(G)$ such that σ^* is the signature of that extension.

Now, consider some extension M' of $M(G)$ by an element p, with signature σ^*. Let $\mathcal{D}(M') = \{W, W^c : \sigma^*(E(W, W^c)) = 0\}$. The first condition of Definition 4.2.1 is immediately satisfied, and by Theorem 4.1.2(3), the second condition is satisfied. Therefore $\mathcal{D}(M')$ is a degeneracy set. □

Corollary 4.2.5. *If $\mathcal{D}$ is a degeneracy set for a graph G, then $C^*(\mathcal{D})$ is a linear subclass of cocircuits.*

A graphic matroid $M(G)$ may have single-element extensions $M'(G)$ that are not orientable. We say a degeneracy set $\mathcal{D}$ is *orientable* if the single-element extension corresponding to $\mathcal{D}$ is an orientable matroid.

The set of all degeneracy sets of a graph G is denoted by $\mathrm{Deg}(G)$. While Deg(G) and the set of linear subclasses of cocircuits of $M(G)$ are the same, the poset structure on Deg(G) is stronger than inclusion.

Definition 4.2.6 ([19, Definition 4.18])**.** Let $\mathcal{D}_1, \mathcal{D}_2 \in \mathrm{Deg}(G)$. We say that $\mathcal{D}_1 \geq \mathcal{D}_2$ if $\mathcal{D}_1 \subseteq \mathcal{D}_2$ and there exists a subset $\mathcal{E} \subset \mathcal{D}_2 \backslash \mathcal{D}_1$ such that the following conditions hold:

1. $\mathcal{D}_2 \backslash \mathcal{D}_1 = \mathcal{E} \bigsqcup \mathcal{E}^c$, where $\mathcal{E}^c = \{Y^c : Y \in \mathcal{E}\}$,

2. for any $Z_1, Z_2 \in \mathcal{D}_2 \backslash \mathcal{D}_1$ such that $Z_1 \cap Z_2 = \emptyset$ and $Z_1 \cup Z_2 \in \mathcal{D}_1$, either Z_1 or Z_2 is in $\mathcal{E}$, but not both.

3. for any $Z_1, Z_2, Z_3 \in \mathcal{D}_2 \backslash \mathcal{D}_1$ such that Z_1, Z_2, Z_3 are all pairwise disjoint, and $V(G) = Z_1 \cup Z_2 \cup Z_3$, $\mathcal{E}$ contains exactly 1 or 2 of the Z_i.

We let $\mathcal{O}\mathrm{Deg}(G)$ denote the set of all orientable degeneracy sets with the above poset structure.

4.3 Degenerate stability conditions

In this section, we connect the degenerate stability conditions on a graph G defined in Definition 2.5.1 to functions on spanning forests of G. We recall the following definitions and results from Sections 2 and 3 of [18].

Definition 4.3.1. Let $\mathcal{D}$ be a degeneracy set for a graph G, and let $\mathrm{Con}(G)$ denote all $W \subset V(G)$ such that $G[W]$ is connected. The *extended degeneracy subset* associated to $\mathcal{D}$ is the subset $\hat{\mathcal{D}} \subseteq \mathrm{Con}(G)$ defined as

$$\hat{\mathcal{D}} := \{V \in \mathrm{Con}(G) : G[V^c] = \bigsqcup_{i \in I} G[W_i] \text{ with all } W_i \in \mathcal{D}\}.$$

Recall that a subgraph G' of G is a graph $G' = (V', E')$ such that $V' \subseteq V(G)$ and $E' \subseteq E(G)$, and G' is spanning if $V' = V(G)$. For a degeneracy set $\mathcal{D}$, a connected subgraph G' is $\mathcal{D}$-*admissible* if $V' \in \hat{\mathcal{D}}$. If G' is a spanning subgraph, but not necessarily connected, it admits a decomposition $G' = \bigsqcup_{i \in I} G'_i$ into connected components. A spanning subgraph G' is $\mathcal{D}$-*admissible* if $V(G'_i) \in \hat{\mathcal{D}}$ for all $i \in I$. The poset of $\mathcal{D}$-admissible spanning subgraphs of G, ordered by inclusion, is denoted by $\mathcal{SS}_{\mathcal{D}}(G)$. We are especially interested in the poset of $\mathcal{D}$-admissible spanning forests of G, denoted by $\mathcal{SF}_{\mathcal{D}}(G)$, ordered by inclusion. The $\mathcal{D}$-*complexity* of G is $c_{\mathcal{D}}(G) = |\mathcal{SF}_{\mathcal{D}}(G)|$. Finally, we denote the minimal elements of $\mathcal{SS}_{\mathcal{D}}(G)$ by $m\mathcal{SS}_{\mathcal{D}}(G)$, and note that $m\mathcal{SS}_{\mathcal{D}}(G) \subseteq \mathcal{SF}_{\mathcal{D}}(G)$.

We can restrict degeneracy sets and extended degeneracy sets of a graph G to $\mathcal{D}$-admissible connected subgraphs.

Lemma 4.3.2 ([18, Lemma 3.5(2)])**.** *Let $\mathcal{D}$ be a degeneracy set for G, and let G' be a $\mathcal{D}$-admissible connected subgraph. The restriction of $\mathcal{D}$ to G' is $\mathcal{D}(G') := \{Z \in BCon(G') : Z \in \hat{\mathcal{D}}\}$ and its associated degeneracy set is $\hat{\mathcal{D}}(G') := \{Z \in Con(G') : Z \in \hat{\mathcal{D}}\}$.*

We can then generalize functions on spanning trees and associated divisors, as defined in Section 3.1, to functions on minimal $\mathcal{D}$-admissible spanning forests of G. For a forest F, let $b_0(F)$ denote the number of connected components of F.

Definition 4.3.3. Let $\mathcal{D}$ be a degeneracy set for a graph G. Let $I : m\mathcal{SS}_{\mathcal{D}}(G) \to \mathrm{Div}(G)$ be any function that assigns a degree $d - g(G) - b_0(F) + 1$ divisor $I(F)$ to every $F \in m\mathcal{SS}_{\mathcal{D}}(G)$.

For any spanning forest F of G, let $\mathcal{O}_{\mathcal{E}_F}$ be a choice of orientation of the edges in $\mathcal{E}_F$. By dropping a chip at the head of each oriented edge in $\mathcal{O}_{\mathcal{E}_F}$, we get a divisor, which we denote by $\mathcal{D}(F, \mathcal{O}_{\mathcal{E}_F})$.

Definition 4.3.4. For a $\mathcal{D}$-forest function I, the set of *generalized break divisors with respect to the $\mathcal{D}$-forest function I* is defined as

$$\mathcal{BD}_I(G) \coloneqq \{\mathcal{D}(F, \mathcal{O}_{\mathcal{E}_F}) + I(F) : F \in m\mathcal{SS}_{\mathcal{D}}(G) \text{ and } \mathcal{O}_{\mathcal{E}_F} \text{ is an orientation of edges } \mathcal{E}_F\}.$$

Theorem 3.2.2 generalizes to the degenerate case.

Theorem 4.3.5 ([18, Theorem 3.32(1)])**.** *Let G be a connected graph. For any degeneracy set $\mathcal{D}$ of G, the map*

$$\{V\text{-stability conditions on } G\} \leftrightarrow \{\mathcal{BD}_I(G) \text{ such that } |\mathcal{BD}_I(G)| = c_{\mathcal{D}}(G)\}$$

is a bijection, in the following way: every V-stability condition $\mathfrak{n}$ with degeneracy set $\mathcal{D}(\mathfrak{n})$ corresponds to a $\mathcal{D}(\mathfrak{n})$-forest function I such that $|\mathcal{BD}_I(G)| = c_{\mathcal{D}(\mathfrak{n})}(G)$.

For future reference, we provide the construction of a $\mathcal{D}$-forest function from a stability condition $\mathfrak{n}$. First, every stability condition $\mathfrak{n}$ can be extended to all subsets of $V(G)$ in the following way:

Definition 4.3.6. The *extended V-function* associated to a V-stability condition $\mathfrak{n}$ of degree d is defined as follows:

1. If $W \subseteq V(G)$ is connected and $G[W^c] = \bigsqcup_{i=1}^{k} G[Z_i]$ is the decomposition of $G[W^c]$ into connected components, we define
$$\mathfrak{n}_W = d - \sum_{i=1}^{k} \mathfrak{n}_{Z_i} - |E(W, W^c)| + |\{Z_i : Z_i \notin \mathcal{D}(\mathfrak{n})\}|.$$

2. If $W \subseteq V(G)$ is not connected, and $\bigsqcup_{j=1}^{h} G[W_j]$ is the decomposition of $G[W]$ into connected components, we set
$$\mathfrak{n}_W = \sum_{j=1}^{h} \mathfrak{n}_{W_j}.$$

We can also restrict a stability condition on a graph G to a stability condition on a connected subgraph G'.

Lemma 4.3.7 ([18, Lemma-Definition 2.16]). *Let $\mathfrak{n}$ be a V-stability condition of degree d on a connected graph G. Let $G' = (V', E')$ be a connected, $\mathcal{D}(\mathfrak{n})$-admissible subgraph of G. We can write $G' = G[V']\backslash X$ for some $X \subseteq E(G)$.*

Then, the restriction of $\mathfrak{n}$ to G' is a V-stability condition $\mathfrak{n}(G')$ on G' defined by

$$\mathfrak{n}(G')_W := \mathfrak{n}_W - |X \cap E(G[W])|$$

for any $W \in BCon(G')$.

We then have the following:

1. $|\mathfrak{n}(G')| = \mathfrak{n}_{V(G')} - |X|$;

2. $\hat{\mathcal{D}}(\mathfrak{n}(G')) = \hat{\mathcal{D}}(\mathfrak{n}) \cap Con(G')$.

Finally, we use the previous lemma to construct $\mathcal{D}$-forest functions from stability conditions $\mathfrak{n}$.

Lemma 4.3.8 ([18, Lemma-Definition 3.30]). *Let $\mathfrak{n}$ be a V-stability condition of degree d on G. Consider $F \in m\mathcal{SS}_{\mathcal{D}(\mathfrak{n})}(G)$ and let $F = \bigsqcup_{i=1}^{k} F_i$ be its decomposition into connected components. For any edge $e \in F$, let F_{i_e} denote the connected component of F containing e, and consider the decomposition of $F\backslash\{e\}$ given by*

$$F\backslash\{e\} = F[W_e] \cup F[V(F_{i_e})\backslash W_e] \bigsqcup_{i \neq i_e} F_i,$$

where $F[W_e]$ and $F[V(F_{i_e})\backslash W_e]$ are both connected. The $\mathcal{D}(\mathfrak{n})$-forest function of degree d associated to $\mathfrak{n}$ is the function

$$I_{\mathfrak{n}} : m\mathcal{SS}_{\mathcal{D}}(G) \to Div(G),$$

where $I_{\mathfrak{n}}(F)$ is the unique divisor on G satisfying

$$\begin{cases} I_{\mathfrak{n}}(F)_{W_e} = \mathfrak{n}(F_{i_e})_{W_e} \\ I_{\mathfrak{n}}(F)_{V(F_{i_e})\backslash W_e} = \mathfrak{n}(F_{i_e})_{V(F_{i_e})\backslash W_e} \\ I_{\mathfrak{n}}(F)_{V(F_i)} = |\mathfrak{n}(F_i)|. \end{cases} \tag{4.1}$$

Finally, we are interested in $\mathcal{D}$-forest functions I such that every $I(F)$ is an orientable divisor. We call such an I an *orientable* $\mathcal{D}$-forest function I. Note that every function I constructed from a triangulation of a Lawrence polytope, as constructed in Theorem 3.4.1, is orientable.

4.4 Constructing stability conditions from subdivisions

Theorem 3.4.1 shows that every triangulation of the Lawrence polytope corresponds to a nondegenerate stability condition. In this section, we generalize this result by showing that every subdivision $\mathcal{S}$ of the Lawrence polytope $\Lambda(\mathcal{M}^*(G))$ yields a V-stability condition $\mathfrak{n}^{\mathcal{S}}$.

First, for every subdivision $\mathcal{S}$, we construct a degeneracy set $\mathcal{D}_{\mathcal{S}}$.

Proposition 4.4.1. *Every subdivision $\mathcal{S}$ of $\Lambda(\mathcal{M}^*(G))$ corresponds to some orientable degeneracy set $\mathcal{D}_S \subseteq BCon(G)$.*

Proof. Let $\mathcal{S}$ be a subdivision of $\Lambda(\mathcal{M}^*(G))$. By Theorem 4.1.8, the subdivision $\mathcal{S}$ corresponds to a single-element extension of $\mathcal{M}(G)$. Let σ^* be the localization describing that extension of $\mathcal{M}(G)$, as defined in Theorem 4.1.4. Consider the set $\mathcal{D}_{\mathcal{S}} = \{W \in \mathrm{BCon(G)} : \sigma^*(E(W, W^c)) = 0\}$.

We wish to show that $\mathcal{D}_{\mathcal{S}}$ is a degeneracy set of the graph G. First, it is trivial to see that if $W \in \mathcal{D}_{\mathcal{S}}$, $W^c \in \mathcal{D}_{\mathcal{S}}$. We then check the second condition of Definition 4.2.1. Consider biconnected subsets W_1 and W_2 in $\mathcal{D}_S$ such that $W_1 \cap W_2 = \emptyset$, and $W_1 \cup W_2$ is biconnected and nontrivial. Note that if G has less than 3 vertices, there does not exist such a W_1 and W_2, and this condition is trivially satisfied. We then assume G has at least three vertices. Then, for the cocircuits $C_1^* = E(W_1, W_1^c)$ and $C_2^* = E(W_2, W_2^c)$, we have $\sigma^*(C_1^*) = \sigma^*(C_2^*) = 0$. Let $C_3^* = E(W_1 \cup W_2, (W_1 \cup W_2)^c)$, which is also a cocircuit of $\mathcal{M}(G)$. By Lemma 4.2.3, we can consider the rank two minor $\mathcal{N} = \mathcal{M}(G)/E'$, where $E' = E(G[W_1]) \sqcup E(G[W_2]) \sqcup E(G[(W_1 \cup W_2)^c])$. The minor $\mathcal{N}$ can be reduced to having three edges by deletion. Since σ^* is a localization, by Theorem 4.1.4, it cannot produce one of the three excluded subconfigurations in Figure 4.1. If $\sigma^*(C_3^*) = \pm$, we have the third forbidden subconfiguration in Figure 4.1. Therefore, we must have $\sigma^*(C_3^*) = 0$, and $W_1 \cup W_2 \in \mathcal{D}_{\mathcal{S}}$, as desired. $\square$

As is done for triangulations in Theorem 3.3.1, we then wish to show that every subdivision of the Lawrence polytope $\Lambda(\mathcal{M}^*(G))$ corresponds to a collection of certain internally oriented spanning forests of G, which we call a *subdividing atlas*. We extend the map χ^* defined in Theorem 3.3.1 to a map on subdivisions of Lawrence polytopes in the natural way.

Lemma 4.4.2. *Let $\mathcal{S}$ be some subdivision of $\Lambda(\mathcal{M}^*)$. Then, for each cell $c \in \mathcal{S}$, we have that $\chi^*(c)$ is the internal orientation of some forest F of G.*

Proof. Recall that every subdivision can be refined to a triangulation of $\Lambda(\mathcal{M}^*)$. Say that triangulation is $\mathcal{T}$. Then, c is the union of some simplices in $\mathcal{T}$, which each correspond to the internal orientation of a spanning tree. The union of such orientations will be some internally oriented forest F of G. $\square$

The following lemma translates Lemma 4.1.7 into the language of this thesis.

Lemma 4.4.3. *Consider a subdivision $\mathcal{S}$ of $\Lambda(\mathcal{M}^*(G))$. Consider a signed cocircuit $\vec{C^*}$ of $\mathcal{M}(G)$, and let $F_{\vec{C^*}}$ be the face of $\Lambda(\mathcal{M}^*(G))$ corresponding to $\vec{C^*}$. Let c be a cell of $\mathcal{S}$. Then, $c \cap F_{\vec{C^*}}$ is a nontrivial cell of the restriction of $\mathcal{S}$ to $F_{\vec{C^*}}$ if and only if $\vec{C^*} \subseteq \chi^*(c)$, C^* is a fundamental cut of a spanning tree T such that $T \geq_{\mathcal{SS}_{\mathcal{D}_{\mathcal{S}}}(G)} \underline{\chi^*(c)}$, and $\sigma^*_{\mathcal{S}}(\vec{C^*}) = +$.*

Proof. Let $\mathcal{T}$ be a triangulation of $\Lambda(\mathcal{M}^*(G))$ that refines $\mathcal{S}$. For the reverse direction, by Lemma 4.4.5, $\chi^*(c)$ is an internally oriented forest $F \in m\mathcal{SS}_{\mathcal{D}_{\mathcal{S}}}(G)$. Let T be a spanning tree such that $T \geq F$, which has C^* as a fundamental cocircuit. Because $\mathcal{T}$ refines $\mathcal{S}$, we also have $\sigma^*_{\mathcal{T}}(\vec{C^*}) = +$, and $\vec{C^*} \subseteq \vec{T^*} = \chi^*(\tau)$ for some $\tau \in \mathcal{T}$. It is clear from Theorem 3.3.1 that if $\vec{C^*} \subseteq \chi^*(\tau) = \vec{T^*}$ and C^* is a fundamental cut of T that the rank of $\tau \cap F_{\vec{C^*}}$ is the rank of $F_{\vec{C^*}}$. Additionally, because $\mathcal{T}$ is a triangulation, $\tau \cap F_{\vec{C^*}}$ is a simplex and the restriction of $\mathcal{T}$ to $F_{\vec{C^*}}$ is nontrivial. The proof follows from $\tau \cap F_{\vec{C^*}} = c \cap F_{\vec{C^*}}$.

For the forward direction, denote the restriction of $\mathcal{S}$ to $F_{\vec{C^*}}$ by $\mathcal{S}_{\vec{C^*}}$. By nontriviality, we have that $\sigma_{\mathcal{S}}(\vec{C^*}) = \pm$ and $c \cap F_{\vec{C^*}}$ is a simplex. Assume without loss of generality that

$\sigma^*_{\mathcal{S}}(\vec{C}^*) = +$. By Lemma 4.1.7, $c \cap F_{\vec{C}^*}$ is a cell of the triangulation $\mathcal{T}^+_{\vec{C}^*}$ of $\Lambda(\mathcal{M}^*(G))(F_{\vec{C}^*})$. Therefore, $c \cap F_{\vec{C}^*}$ is of the form $E(\Lambda(\mathcal{M}^*(G))(F_{\vec{C}^*}))\backslash e$ for some $e \in (\vec{C}^*)_+$. We then have $\vec{C}^* \subseteq \chi^*(c \cap F_{\vec{C}^*}) \subseteq \chi^*(c)$. Additionally, since $\chi^*(c \cap F_{\vec{C}^*})$ only has one one-way oriented edge, any spanning tree T such that $T \geq_{\mathcal{SS}_{\mathcal{D}_{\mathcal{S}}}(G)} \underline{\chi^*(c)}$ also only contains $e \in C^*$, and therefore C^* is a fundamental cocircuit of T. $\square$

Lemma 4.4.4. *Let $\mathcal{S}$ be a subdivision of $\Lambda(\mathcal{M}^*(G))$. Then, for every cell $c \in \mathcal{S}$, $\chi^*(c)$ is an internally oriented $\mathcal{D}_{\mathcal{S}}$-admissible spanning forest of G.*

Proof. By Lemma 4.4.2, $\chi^*(c)$ is an internally oriented spanning forest $\vec{F}^*$ of G. The forest F has a decomposition $F = \bigsqcup_{i \in I} F_i$ into connected compositions. If F is connected, then F is a spanning tree.

Recall that F is $\mathcal{D}_{\mathcal{S}}$-admissible if $V(F_i) \in \hat{\mathcal{D}}_{\mathcal{S}}$, where $\hat{\mathcal{D}}_{\mathcal{S}}$ is the extended degeneracy set defined in Definition 4.3.1. We consider $V(F_i)$. Every spanning forest F of G is a subgraph of a spanning tree of G. Recall that removing edges from a spanning tree T partitions $V(G)$ into two connected components. In this way, we can view $G[V(F_i)^c]$ as the disjoint union of $G[W_j]$ for $j \in J$, where W_j is a biconnected subset of $V(G)$. Let $C^*_j = E(W_j, W^c_j)$. Then, if we show that every W_j is in $\mathcal{D}_{\mathcal{S}}$, we have $V(F_i) \in \hat{\mathcal{D}}_{\mathcal{S}}$. If F is a spanning tree, then $V(F) = V(G) \in \hat{\mathcal{D}}_{\mathcal{S}}$, as desired, so we now assume F is not a spanning tree.

Let $F_{C^*_j}$ denote the face of $\Lambda(\mathcal{M}^*(G))$ corresponding to the cocircuit C^*_j of $\mathcal{M}(G)$, as described in Lemma 4.1.7(2). For every cut C^*_j of G, the edges of C^*_j are bioriented in $\vec{F}^*$. Therefore, the intersection of the cell c with the face $F_{C^*_j}$ is the face $F_{C^*_j}$ for every $j \in J$. As a result, the induced subdivision $\mathcal{S}_{F_{C^*_j}}$ is the trivial subdivision for every $j \in J$. By Lemma 4.1.7(3), the signature of $\mathcal{M}(G)$, σ^*, defined by $\mathcal{S}$ satisfies $\sigma^*(C^*_j) = 0$ for all $j \in J$. By definition of $\mathcal{D}_{\mathcal{S}}$, we have $W_j \in \mathcal{D}_{\mathcal{S}}$ for every $j \in J$. Since every $V(F_i) \in \hat{\mathcal{D}}_{\mathcal{S}}$, we have that F is $\mathcal{D}_{\mathcal{S}}$-admissible, as desired. $\square$

Lemma 4.4.5. *Let $\mathcal{S}$ be a subdivision of $\Lambda(\mathcal{M}^*(G))$. Then, for every cell $c \in \mathcal{S}$, $\chi^*(c)$ is an internally oriented forest $\vec{F}^*$ for $F \in m\mathcal{SS}_{\mathcal{D}_{\mathcal{S}}}(G)$.*

Proof. From Lemma 4.4.4, let $\underline{\chi^*(c)}$ denote the $\mathcal{D}_\mathcal{S}$-admissible spanning subgraph. For contradiction, assume that $\underline{\chi^*(c)}$ is not minimal in the poset $\mathcal{SS}_{\mathcal{D}_\mathcal{S}}(G)$. Therefore, there exists some minimal $\mathcal{D}_\mathcal{S}$-admissible subgraph H such that $\underline{\chi^*(c)} > H$. Additionally, let T be a spanning tree in $\mathcal{SS}_{\mathcal{D}_\mathcal{S}}(G)$ such that $T \geq \underline{\chi^*(c)} > H$. Such a spanning tree exists because all spanning trees are $\mathcal{D}$-admissible for any degeneracy set $\mathcal{D}$.

Since T is a connected $\mathcal{D}_\mathcal{S}$-admissible subgraph of G, we can consider the restriction of $\mathcal{D}_\mathcal{S}$ to T, as defined in Lemma 4.3.2. We have

$$\mathcal{D}_\mathcal{S}(T) = \{Z \in \mathrm{BCon}(T) : Z \in \hat{\mathcal{D}}_\mathcal{S}\}.$$

Since $\mathrm{BCon}(T) \subseteq \mathrm{BCon}(G)$, if $Z \in \mathrm{BCon}(T)$ and $Z \in \hat{\mathcal{D}}_\mathcal{S}$, Z is also biconnected in G and $Z \in \mathcal{D}_\mathcal{S}$. Therefore,

$$\mathcal{D}_\mathcal{S}(T) = \{Z \in \mathrm{BCon}(T) : Z \in \mathcal{D}_\mathcal{S}\}.$$

We note that both $\underline{\chi^*(c)}$ and H are also $\mathcal{D}_\mathcal{S}(T)$-admissible.

Consider some edge e in $\underline{\chi^*(c)}$ but not in H. Then, there exists a connected component of H, called H_e, such that e is incident to some vertex in H. Since H is $\mathcal{D}_\mathcal{S}(T)$-admissible, we have $V(H_e) \in \hat{\mathcal{D}}_\mathcal{S}(T)$, so $G[(V(H_e))^c] = \sqcup_{i \in I} G[V_i]$, where all V_i are in $\mathcal{D}_\mathcal{S}(T)$. There exists a unique V_i in this decomposition such that $e \in E(V_i, V_i^c)$, and by definition $V_i \in \mathcal{D}_\mathcal{S}(T)$, so $V_i \in \mathcal{D}_\mathcal{S}$. Let C_i^* be the cut $E(V_i, V_i^c)$ of G. Then, C_i^* is the fundamental cut $C^*(T, e)$. Consider the face of the Lawrence polytope $\Lambda(\mathcal{M}^*(G))$ corresponding to the the cocircuit C_i^*, which we denote by $F_{C_i^*}$. Since $\underline{\chi^*(c)}$ contains the edge e, we have that $\chi^*(c) \cap F_{C^*}$ is a cell of the triangulation $\mathcal{T}_{C^*}^+$ or $\mathcal{T}_{C^*}^-$ of the restricted oriented matroid $\Lambda(\mathcal{M}^*(G))(F_{C^*})$. Therefore, the subdivision S orients the cocircuit C^*, and by definition $V_i \notin \mathcal{D}_S$, a contradiction. □

We can describe the cells of the subdivision more explicitly in terms of the refining triangulation.

Lemma 4.4.6. *Let $\mathcal{S}$ be a subdivision of the Lawrence polytope $\Lambda(\mathcal{M}^*(G))$, and let χ^* be the map between cells of $\mathcal{S}$ to internal orientations of $F \in m\mathcal{SS}_{\mathcal{D}_\mathcal{S}}(G)$. Then, for each $F \in m\mathcal{SS}_{\mathcal{D}_\mathcal{S}}(G)$, there exists a cell c_F of $\mathcal{S}$ such that $\chi^*(c_F) = \vec{F}^*$.*

Proof. Consider a forest $F \in m\mathcal{SS}_{\mathcal{D}_\mathcal{S}}(G)$. Define

$$T(F) \coloneqq \{T \in \mathcal{ST}(G) : T \geq F \text{ in } \mathcal{SS}_{\mathcal{D}_\mathcal{S}}(G)\}.$$

Then, $E(F) = \cap_{T \in T(F)} E(T)$. Consider a triangulation $\mathcal{T}$ of $\Lambda(\mathcal{M}^*(G))$ that refines $\mathcal{S}$, and consider a $T \in T(F)$. Then for edges in $E(T) \backslash E(F)$, the fundamental cocircuits $C^*(T, e)$ satisfy $\sigma^*_\mathcal{S}(\pm\vec{C^*}) = 0$. Let $\tau \in \mathcal{T}$ be the maximal simplex of $\mathcal{T}$ such that $\chi^*(\tau) = \vec{T^*}$. We cannot have $\tau \in \mathcal{S}$, because $\tau \cap F_{\vec{C^*}(T,e)}$ is a cell of the restricted triangulation of $F_{\vec{C^*}(T,e)}$ as described in Lemma 4.4.3. Therefore, there exists a cell c of $\mathcal{S}$ such that $c \cap F_{\vec{C^*}(T,e)} = F_{\vec{C^*}(T,e)}$, and $\tau \subset c$. From Lemma 4.4.5, $\chi^*(c) = \vec{F^*_T}$ for some forest $F_T \in m\mathcal{SS}_{\mathcal{D}_\mathcal{S}}(G)$, but since $m\mathcal{SS}_{\mathcal{D}_\mathcal{S}}(G)$ is ordered with respect to inclusion, we have $F_T = F$, as desired. We can then take $c_F = \cup_{T \in T(F)} (\chi^*)^{-1}(\vec{T^*})$, where each $(\chi^*)^{-1}(\vec{T^*})$ is a maximal simplex of $\mathcal{T}$. □

Theorem 4.4.7. *Let $\mathcal{S}$ be a subdivision of $\Lambda(\mathcal{M}^*(G))$. Then, the set $\{\chi^*(c) : c \in \mathcal{S}\}$ is a collection of internal orientations $\{\vec{F^*} : F \in m\mathcal{SS}_{\mathcal{D}_\mathcal{S}}(G)\}$.*

Proof. From Lemma 4.4.5, for every cell c of $\mathcal{S}$, $\chi^*(c)$ is an internally oriented forest $\vec{F^*}$ for $F \in m\mathcal{SS}_{\mathcal{D}_\mathcal{S}}(G)$. Each $F \in m\mathcal{SS}_{\mathcal{D}_\mathcal{S}}(G)$ receives an orientation by Lemma 4.4.6. □

As we do for triangulations, one could take the subdividing atlas in Theorem 4.4.7, construct a $\mathcal{D}_\mathcal{S}$-forest function I, and count the number of divisors in $\mathcal{BD}_I(G)$. What makes this strategy difficult for subdivisions is that $c_{\mathcal{D}_\mathcal{S}}(G)$ does not always have a nice combinatorial interpretation. Instead, for a subdivision $\mathcal{S}$, we use a refining triangulation $\mathcal{T}$ to construct a stability condition, and then show the $\mathcal{D}$-forest function corresponding to that stability condition is equivalent to the one constructed directly from the subdividing atlas.

We first prove the following lemma.

Lemma 4.4.8. *Let $\mathcal{T}$ be a triangulation of the Lawrence polytope $\Lambda(\mathcal{M}^*(G))$, and consider the stability condition $\mathfrak{n}^\mathcal{T}$ constructed in Theorem 3.4.1. Consider pairwise disjoint, biconnected sets W_1, W_2, W_3 such that $W_1 \cup W_2 \cup W_3 = V(G)$. Then, either $I_{W_i} = -1$, $I_{W_j} = -1$, and $I_{W_k} = 0$ or $I_{W_i} = -1$, $I_{W_j} = 0$, and $I_{W_k} = 0$ for $\{i, j, k\} = \{1, 2, 3\}$.*

Proof. Because $\mathfrak{n}^{\mathcal{T}}$ is nondegenerate, by Definition 2.5.1, we have

$$\sum_{i=1}^{3} \mathfrak{n}^{\mathcal{T}}_{W_i} + \sum_{1 \leq i < j \leq 3} |E(W_i, W_j)| - d \in \{1, 2\}. \tag{4.2}$$

Without lifting at q as described in Theorem 3.4.1, we have $d = g(G) - 1$. By Theorem 3.2.2, we have $\mathfrak{n}^{\mathcal{T}}_W = I_W + g(G[W])$ for every $W \in \mathrm{BCon}(G)$. We observe that

$$\sum_{i=1}^{3} g(G[W_i]) = \sum_{i=1}^{3}(|E(G[W_i])| - |W_i| + 1) = |E(G)| - \sum_{1 \leq i < j \leq 3} E|(W_i, W_j)| + 3 - |V(G)| \tag{4.3}$$

so Equation 4.2 becomes $I_{W_1} + I_{W_2} + I_{W_3} \in \{-1, -2\}$. We recall from Theorem 3.4.1 that each $I_{W_i} \in \{0, -1\}$, so either $I_{W_i} = -1$, $I_{W_j} = -1$, and $I_{W_k} = 0$ or $I_{W_i} = -1$, $I_{W_j} = 0$, and $I_{W_k} = 0$. $\square$

We can also consider the extension of $\mathcal{M}(G)$ corresponding to a triangulation $\mathcal{T}$ of $\Lambda(\mathcal{M}^*(G))$, and the localization $\sigma^*_{\mathcal{T}}$ of that extension. By Lemma 4.2.3, there exists a rank 2 minor of $\mathcal{M}(G)$, called $\mathcal{N}$, which is the graphic matroid of the graph minor in Figure 4.2.

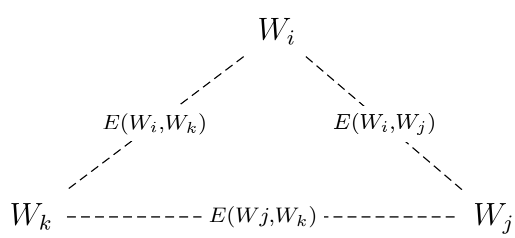
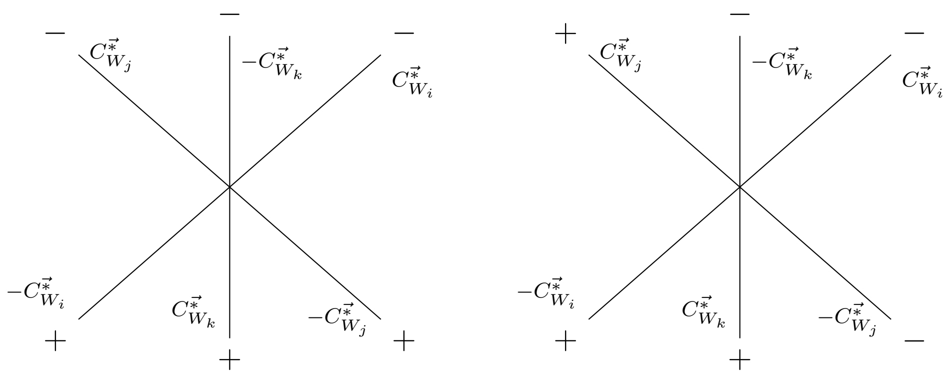


Figure 4.2: Minor of G with cuts $E(W_i, W_i^c), E(W_j, W_j^c), E(W_k, W_k^c)$

We can then consider the extension defined by $\sigma^*_{\mathcal{T}}$ on $\mathcal{N}$.

Lemma 4.4.9. *Let $\sigma^*_{\mathcal{T}}$ be a localization of the extension of $\mathcal{M}(G)$ corresponding to the triangulation $\mathcal{T}$. Let I be the function on $\mathcal{ST}(G)$ constructed from the triangulation $\mathcal{T}$, as described in Theorem 3.4.1. Let $\mathcal{N}$ be the minor of $\mathcal{M}(G)$ with cocircuits $E(W_i, W_i^c)$, $E(W_j, W_j^c)$, and $E(W_k, W_k^c)$, and let $\vec{C^*_W}$ be the cocircuit $E(W, W^c)$ oriented towards W. Then, $\sigma^*_{\mathcal{T}}$ yields one of two configurations in Figure 4.3 on the minor $\mathcal{N}$.*



(a) Configuration A (b) Configuration B

Figure 4.3: Possible configurations

Proof. We consider the two possibilities for $I_{W_i}, I_{W_j}, I_{W_k}$ outlined in Lemma 4.4.8. It follows from the definition of localizations that if $I_{W_\ell} = -1$, then $\sigma^*_{\mathcal{T}}(-C^{\vec{*}}_{W_\ell}) = +$, and if $I_{W_\ell} = 0$, then $\sigma^*_{\mathcal{T}}(C^{\vec{*}}_{W_\ell}) = +$.

First, consider if $I_{W_i} = I_{W_j} = -1$ and $I_{W_k} = 0$. Then, $\sigma^*_{\mathcal{T}}(-C^{\vec{*}}_{W_i}) = \sigma^*_{\mathcal{T}}(-C^{\vec{*}}_{W_j}) = +$, and $\sigma^*_{\mathcal{T}}(-C^{\vec{*}}_{W_k}) = -$. This corresponds to Configuration A in Figure 4.3; for example, the orientation of the edges $E(W_i, W_j)$ in $-C^{\vec{*}}_{W_i}$ in opposite from the orientation of those edges in $-C^{\vec{*}}_{W_j}$. If instead $I_{W_i} = -1$ and $I_{W_j} = I_{W_k} = 0$, we have Configuration B. $\square$

Theorem 4.4.10. *Let $\mathcal{S}$ be a subdivision of the Lawrence polytope $\Lambda(\mathcal{M}^*(G))$. Let $\mathcal{T}$ be a triangulation of $\Lambda(\mathcal{M}^*(G))$ that refines $\mathcal{S}$, and consider the associated stability condition $\mathfrak{n}^{\mathcal{T}}$ described in Theorem 3.4.1. Let $\sigma^*_{\mathcal{T}}$ be the localization of the extension of $\mathcal{M}(G)$ corresponding to $\mathcal{T}$, and let $\sigma^*_{\mathcal{S}}$ be the localization of the extension of $\mathcal{M}(G)$ corresponding to $\mathcal{S}$. Let $C^{\vec{*}}_W$ be the cocircuit $E(W, W^c)$ of $\mathcal{M}(G)$ oriented towards W.*

Define $\mathfrak{n}^{\mathcal{S}} : BCon(G) \to \mathbb{Z}$ by

$$\mathfrak{n}^{\mathcal{S}}_W = \begin{cases} \mathfrak{n}^{\mathcal{T}}_W - 1 & W \in \mathcal{D}_{\mathcal{S}} \text{ and } \sigma^*_{\mathcal{T}}(C^{\vec{*}}_W) = +, \\ \mathfrak{n}^{\mathcal{T}}_W & \text{otherwise.} \end{cases}$$

Then, $\mathfrak{n}^{\mathcal{S}}$ *is a degenerate V-stability condition of degree* $g(G)-1$ *such that* $\mathcal{D}(\mathfrak{n}^{\mathcal{S}}) = \mathcal{D}_{\mathcal{S}}$*.*

Proof. First, we check that

$$\mathfrak{n}^{\mathcal{S}}_W + \mathfrak{n}^{\mathcal{S}}_{W^c} + |E(W, W^c)| - (g(G)-1) \in \{0,1\}.$$

Since $\mathfrak{n}^{\mathcal{T}}$ is a nondegenerate stability condition, we have

$$\mathfrak{n}^{\mathcal{T}}_W + \mathfrak{n}^{\mathcal{T}}_{W^c} + |E(W, W^c)| - g(G) + 1 = 1$$

for every $W \in \mathrm{BCon}(G)$. If $W \notin \mathcal{D}_{\mathcal{S}}$, then by definition $W^c \notin \mathcal{D}_{\mathcal{S}}$, so we have $\mathfrak{n}^{\mathcal{S}}_W = \mathfrak{n}^{\mathcal{T}}_W$ and $\mathfrak{n}^{\mathcal{S}}_{W^c} = \mathfrak{n}^{\mathcal{T}}_{W^c}$, so the desired equality holds. If $W \in \mathcal{D}_{\mathcal{S}}$, then either $\mathfrak{n}^{\mathcal{S}}_W = \mathfrak{n}^{\mathcal{T}}_W - 1$ or $\mathfrak{n}^{\mathcal{S}}_{W^c} = \mathfrak{n}^{\mathcal{T}}_{W^c} - 1$, and the desired equality holds. By construction, $\mathcal{D}(\mathfrak{n}^{\mathcal{S}}) = \mathcal{D}_{\mathcal{S}}$.

Now, consider biconnected, pairwise disjoint subsets W_1, W_2, W_3 such that $W_1 \cup W_2 \cup W_3 = V(G)$. If W_i and W_j are $\mathfrak{n}^{\mathcal{S}}$-degenerate, both W_i and W_j are in $\mathcal{D}_{\mathcal{S}}$. It follows from the definition of a degeneracy set that $W_k = (W_i \cup W_j)^c \in \mathcal{D}_{\mathcal{S}}$, and therefore W_k is $\mathfrak{n}^{\mathcal{S}}$-degenerate. Finally, we show that

$$\sum_{i=1}^{3} \mathfrak{n}^{\mathcal{S}}_{W_i} + \sum_{1\le i<j\le 3} |E(W_i, W_j)| - g(G) + 1 = \begin{cases} \{1,2\} & W_i \text{ is } \mathfrak{n}^{\mathcal{S}}\text{-nondegenerate for all } i \\ \{1\} & \exists! i \text{ such that } W_i \text{ is } \mathfrak{n}^{\mathcal{S}}\text{-degenerate} \\ \{0\} & \text{if } W_i \text{ is } \mathfrak{n}^{\mathcal{S}}\text{-degenerate for all } i. \end{cases} \tag{4.4}$$

In the first case, we have $\mathfrak{n}^{\mathcal{S}}_{W_i} = \mathfrak{n}^{\mathcal{T}}_{W_i}$ for $i = 1, 2, 3$, and the condition is satisfied because $\mathfrak{n}^{\mathcal{T}}$ is a stability condition. In the last case, we have $\mathfrak{n}^{\mathcal{S}}_{W_i} = -1 + g(G[W_i])$ for all i, so using Equation 4.3, we see Equation 4.4 is 0. Finally, we consider when there exists a unique i such that W_i is $\mathfrak{n}^{\mathcal{S}}$-degenerate. Without loss of generality, assume W_1 is the unique $\mathfrak{n}^{\mathcal{S}}$-degenerate set. Let I be the function on spanning trees constructed from the triangulation $\mathcal{T}$, as detailed in Theorem 3.4.1. Then, Lemma 4.4.8 gives us two possibilities for $I_{W_i}, I_{W_j}, I_{W_k}$, where $\{i,j,k\} = \{1,2,3\}$:

1. $I_{W_i} = -1, I_{W_j} = -1$, $I_{W_k} = 0$;

2. $I_{W_i} = -1, I_{W_j} = 0, I_{W_k} = 0$.

For the first option, we have

$$\sum_{i=1}^{3} \mathfrak{n}^{\mathcal{T}}_{W_i} + \sum_{1\le i<j\le 3} |E(W_i, W_j)| - g(G) + 1 = 1.$$

Therefore, we wish to show that $\mathfrak{n}^{\mathcal{S}}_{W_i} = \mathfrak{n}^{\mathcal{T}}_{W_i}$ for each $i \in \{1,2,3\}$. By Lemma 4.4.9, the localization $\sigma^*_{\mathcal{T}}$ yields Configuration A on the minor of $\mathcal{M}(G)$ corresponding to contracting all edges $E(G)\backslash(E(W_i,W_j) \cup E(W_i,W_k) \cup E(W_j,W_k))$. If W_k is $\mathfrak{n}^{\mathcal{S}}$-degenerate, then Configuration A becomes the second forbidden configuration in Figure 4.1; therefore, either W_i or W_j is $\mathfrak{n}^{\mathcal{S}}$-degenerate. In either case, we have $\sigma^*_{\mathcal{T}}(\vec{C^*_{W_i}}) = \sigma^*_{\mathcal{T}}(\vec{C^*_{W_j}}) = -$, so $\mathfrak{n}^{\mathcal{S}}_{W_i} = \mathfrak{n}^{\mathcal{T}}_{W_i}$ for each $i \in \{1,2,3\}$. For the second option, we have

$$\sum_{i=1}^{3} \mathfrak{n}^{\mathcal{T}}_{W_i} + \sum_{1\le i<j\le 3} |E(W_i, W_j)| - g(G) + 1 = 2.$$

By Lemma 4.4.9, the localization $\sigma^*_{\mathcal{T}}$ yields Configuration B on the minor of $\mathcal{M}(G)$ corresponding to contracting all edges $E(G)\backslash(E(W_i,W_j) \cup E(W_i,W_k) \cup E(W_j,W_k))$. If W_i is $\mathfrak{n}^{\mathcal{S}}$-degenerate, then Configuration B becomes the second forbidden configuration in Figure 4.1; therefore, exactly one of W_j or W_k is $\mathfrak{n}^{\mathcal{S}}$-degenerate. Without loss of generality, assume W_j is $\mathfrak{n}^{\mathcal{S}}$-degenerate. We have $\sigma^*_{\mathcal{T}}(\vec{C^*_{W_j}}) = +$, so $\mathfrak{n}^{\mathcal{S}}_{W_j} = \mathfrak{n}^{\mathcal{T}}_{W_j} - 1$ and the desired equality holds. Therefore, $\mathfrak{n}^{\mathcal{S}}$ is a stability condition with degeneracy set $\mathcal{D}_{\mathcal{S}}$. □

Recall from Lemma 4.3.8 that for any degenerate stability condition $\mathfrak{n}$ with degeneracy set $\mathcal{D}(\mathfrak{n})$, we can construct a $\mathcal{D}(\mathfrak{n})$-forest function $I_{\mathfrak{n}}$ such that $|\mathcal{BD}_{I_{\mathfrak{n}}}(G)| = c_{\mathcal{D}(\mathfrak{n})}(G)$. For the stability condition $\mathfrak{n}^{\mathcal{S}}$, we can explicitly describe $I_{\mathfrak{n}^{\mathcal{S}}}$.

Lemma 4.4.11. *Let $\mathcal{S}$ be a subdivision of $\Lambda(\mathcal{M}^*(G))$, and consider the stability condition $\mathfrak{n}^{\mathcal{S}}$, as defined in Theorem 4.4.10, with degeneracy set $\mathcal{D}_{\mathcal{S}}$. For $e \in F$, recall that $F\backslash e =$*

$F[W_e] \sqcup F[V(F_{i_e})\backslash W_e] \sqcup_{i \neq i_e} F_i$, where F_{i_e} is the connected component of F containing e.

$$\begin{cases} I_{\mathfrak{n}^{\mathcal{S}}}(F)_{W_e} \in \{0,-1\} \\ I_{\mathfrak{n}^{\mathcal{S}}}(F)_{V(F_{i_e})\backslash W_e} \in \{0,-1\}\backslash I_{\mathfrak{n}^{\mathcal{S}}}(F)_{W_e} \\ I_{\mathfrak{n}^{\mathcal{S}}}(F)_{V(F_i)} = -1. \end{cases}$$

Proof. Recall from Lemma 4.3.8 that $I_{\mathfrak{n}^{\mathcal{S}}}(F)$ is the unique divisor on G satisfying

$$\begin{cases} I_{\mathfrak{n}^{\mathcal{S}}}(F)_{W_e} = \mathfrak{n}^{\mathcal{S}}(F_{i_e})_{W_e} \\ I_{\mathfrak{n}^{\mathcal{S}}}(F)_{V(F_{i_e})\backslash W_e} = \mathfrak{n}^{\mathcal{S}}(F_{i_e})_{V(F_{i_e})\backslash W_e} \\ I_{\mathfrak{n}^{\mathcal{S}}}(F)_{V(F_i)} = |\mathfrak{n}^{\mathcal{S}}(F_i)|. \end{cases}$$

We first consider $I_{\mathfrak{n}^{\mathcal{S}}}(F)_{V(F_i)} = |\mathfrak{n}^{\mathcal{S}}(F_i)|$. Because $F \in m\mathcal{S}\mathcal{S}_{\mathcal{D}_{\mathcal{S}}}(G)$, each F_i is a connected, $\mathcal{D}_{\mathcal{S}}$-admissible subgraph of G. Let $E_i \subset E(G)$ be the edges such that $F_i = G[V(F_i)]\backslash E_i$. We note that $|E_i| = g(G[V(F_i)])$, because E_i is the complement of the spanning tree F_i of $G[V(F_i)]$. From Lemma 4.3.7, we have $|\mathfrak{n}^{\mathcal{S}}(F_i)| = \mathfrak{n}^{\mathcal{S}}_{V(F_i)} - |E_i|$. We then wish to determine $\mathfrak{n}^{\mathcal{S}}_{V(F_i)}$.

Since F_i is connected, $V(F_i)$ is a connected subset of $V(G)$. We can extend $\mathfrak{n}^{\mathcal{S}}$ to be a function on connected subsets of $V(G)$, as defined in Definition 4.3.6. Let $G[V(F_i)^c] = \bigsqcup_{i \in I} G[Z_i]$ be the decomposition of $G[V(F_i)^c]$ into biconnected subsets Z_i. We define

$$n^{\mathcal{S}}_{V(F_i)} = g(G) - 1 - \sum_{i \in I} \mathfrak{n}^{\mathcal{S}}_{Z_i} - |E(V(F_i), V(F_i)^c)| + |\{Z_i | Z_i \notin \mathcal{D}_{\mathcal{S}}\}|. \tag{4.5}$$

Because F_i is a connected component of F, and F is $\mathcal{D}_{\mathcal{S}}$-admissible, $V(F_i) \in \hat{\mathcal{D}}_{\mathcal{S}}$, and every Z_i in the decomposition of $G[V(F_i)^c]$ is in $\mathcal{D}_{\mathcal{S}}$. Therefore, $|\{Z_i | Z_i \notin \mathcal{D}_{\mathcal{S}}\}| = 0$. By the definition of $\mathfrak{n}^{\mathcal{S}}$ in Theorem 4.4.10, for each Z_i in the decomposition, we have $\mathfrak{n}^{\mathcal{S}}_{Z_i} = -1 + g(G[Z_i]) = |E(G[Z_i])| - |Z_i|$. Using the decomposition $G[V(F_i)^c] = \bigsqcup_{i \in I} G[Z_i]$, we have

$$\sum_{i \in I} |Z_i| = |V(F_i)^c| = |V(G)| - |V(F_i)|$$

$$\sum_{i \in I} |E(G[Z_i])| = |E(G)| - |E(V(F_i), V(F_i)^c)| - |E(G[V(F_i)])|.$$

After plugging into Equation 4.5 and simplifying, we have $n^{\mathcal{S}}_{V(F_i)} = |E(G[V(F_i)])| - |V(F_i)|$, so $|\mathfrak{n}^{\mathcal{S}}(F_i)| = -1$.

Now, we consider $I_{\mathfrak{n}^{\mathcal{S}}}(F)_{W_e} = \mathfrak{n}^{\mathcal{S}}(F_{i_e})_{W_e}$. Again, W_e is a connected subset of G, so $G[W_e^c] = \sqcup_{i \in I} G[Z_i]$ for biconnected subsets Z_i of $V(G)$. We first examine $\mathfrak{n}^{\mathcal{S}}_{W_e}$. As before,

$$n^{\mathcal{S}}_{W_e} = g(G) - 1 - \sum_{i \in I} \mathfrak{n}^{\mathcal{S}}_{Z_i} - |E(W_e, W_e^c)| + |\{Z_i | Z_i \notin \mathcal{D}_{\mathcal{S}}\}|. \tag{4.6}$$

W_e is not the vertex set of a connected component of F, since it is contained in $V(F_{i_e})$. Since F is minimal in $\mathcal{SS}_{\mathcal{D}_{\mathcal{S}}}(G)$, we also have $W_e \notin \hat{\mathcal{D}}_{\mathcal{S}}$. Therefore, for the decomposition of $G[W_e^c]$ into disjoint $G[Z_i]$ where all Z_i are biconnected, there exists a Z_e such that $Z_e \notin \mathcal{D}_{\mathcal{S}}$. Additionally, because all of the $G[Z_i]$ are disjoint, Z_e is the unique biconnected subset in the decomposition of $G[W_e^c]$ such that $e \in E(Z_e, Z_e^c)$. Consider the localization $\sigma^*_{\mathcal{S}}$ of the extension of $\mathcal{M}(G)$ corresponding to $\mathcal{S}$. From Theorem 4.4.10, we have

$$\mathfrak{n}^{\mathcal{S}}_{Z_i} = \begin{cases} g(G[Z_i]) & \text{if } Z_i \notin \mathcal{D}_{\mathcal{S}} \text{ and } \sigma^*_{\mathcal{S}}(\vec{C^*_{Z_i}}) = + \\ -1 + g(G[Z_i]) & \text{otherwise.} \end{cases}$$

This allows us to rewrite Equation 4.6 as

$$\begin{aligned} \sum_{i \in I} \mathfrak{n}^{\mathcal{S}}_{Z_i} &= \sum_{i \in I} g(G[Z_i]) - |\{Z_i | Z_i \in \mathcal{D}_{\mathcal{S}}\}| - |\{Z_i | Z_i \notin \mathcal{D}_{\mathcal{S}} \text{ and } \sigma^*_{\mathcal{S}}(\vec{C^*_{Z_i}}) = -\}| \\ &= |I| + |E(G)| - |E(W_e, W_e^c)| - |E(G[W_e])| - |V(G)| + |W_e| - |\{Z_i | Z_i \in \mathcal{D}_{\mathcal{S}}\}| \\ &\qquad - |\{Z_i | Z_i \notin \mathcal{D}_S \text{ and } \sigma^*_{\mathcal{S}}(\vec{C^*_{Z_i}}) = -\}| \\ &= |\{Z_i | Z_i \notin \mathcal{D}_S \text{ and } \sigma^*_{\mathcal{S}}(\vec{C^*_{Z_i}}) = +\}| + g(G) - 1 - |E(W_e, W_e^c)| - |E(G[W_e])| + |W_e|. \end{aligned}$$

Plugging back into our expression for $\mathfrak{n}^{\mathcal{S}}_{W_e}$, we have

$$\begin{aligned} \mathfrak{n}^{S}_{W_e} &= |\{Z_i | Z_i \notin \mathcal{D}_{\mathcal{S}} \text{ and } \sigma^*_{\mathcal{S}}(\vec{C^*_{Z_i}}) = -\}| + |E(G[W_e])| - |W_e| \\ &= |\{Z_i | Z_i \notin \mathcal{D}_S \text{ and } \sigma^*_{\mathcal{S}}(\vec{C^*_{Z_i}}) = -\}| + g(G[W_e]) - 1. \end{aligned}$$

Let $E_{i_e} \subset E(G)$ such that $F_{i_e} = G[V(F_{i_e})] \backslash E_{i_e}$. Then, $|E_{i_e} \cap E(G[W_e])| = g(G[W_e])$. We then have

$$\begin{aligned}\mathfrak{n}(F_{i_e})_{W_e} &= \mathfrak{n}_{W_e} - |E_{i_e} \cap E(G[W_e])| \\ &= |\{Z_i | Z_i \notin \mathcal{D}_S \text{ and } \sigma^*_{\mathcal{S}}(\vec{C^*_{Z_i}}) = -\}| - 1.\end{aligned}$$

We know $Z_e \notin \mathcal{D}_{\mathcal{S}}$. Then, for all other $Z_i \neq Z_e$, we have that $E(Z_i, Z_i^c) \subseteq E(G)\backslash E(F_{i_e})$. Assume that $G[Z_i] \subseteq G[V(F_{i_e}^c)]$. Since $V(F_{i_e})$ is $\mathcal{D}_{\mathcal{S}}$-admissible, we have a decomposition $G[V(F_{i_e})^c] = \bigsqcup_{j \in J} G[W_j]$ where each $W_j \in \mathcal{D}_{\mathcal{S}}$. If there exists a W_j such that $Z_i = W_j$, then Z_i is $\mathcal{D}_{\mathcal{S}}$-degenerate, as desired. Because Z_i is biconnected, the other possibility is there exists W_j such that $Z_i \subset W_j$. However, in this case $G[W_i]$ would also be in the decomposition of $G[W_e^c]$, so such a Z_i cannot exist.

Therefore, $|\{Z_i | Z_i \notin \mathcal{D}_{\mathcal{S}} \text{ and } \sigma^*_{\mathcal{S}}(\vec{C^*_{Z_i}}) = -\}|$ is either 0 or 1, depending on the orientation of $C^*_{Z_e}$, so $I_{\mathfrak{n}^{\mathcal{S}}}(F)_{W_e} \in \{0, -1\}$. By the same calculation, we have that $I_{\mathfrak{n}^{\mathcal{S}}}(F)_{V(F_{i_e})\backslash W_e} \in \{0, -1\}\backslash I_{\mathfrak{n}^{\mathcal{S}}}(F)_{W_e}$, as desired. □

Corollary 4.4.12. *Let $\mathcal{S}$ be a subdivision of $\Lambda(\mathcal{M}^*(G))$, and $\mathfrak{n}^{\mathcal{S}}$ be the associated stability condition with degeneracy set $\mathcal{D}_{\mathcal{S}}$. Then, the $\mathcal{D}_{\mathcal{S}}$-forest function $I_{\mathfrak{n}^{\mathcal{S}}}$ is orientable.*

Proof. The degree of an orientable divisor on a tree is -1; because $I_{\mathfrak{n}^{\mathcal{S}}}(F)_{V(F_i)} = -1$ for every connected component F_i of F, this requirement is met.

From $I_{\mathfrak{n}^{\mathcal{S}}}(F)$, we construct an orientation $\mathcal{O}$ of F in the following natural way: consider an edge e of F, and let F_{i_e} denote the connected component of F containing e. Then, if $I(F)_{W_{i_e}} = 0$, we orient e towards W_{i_e}; otherwise, we orient e away from W_{i_e}. By construction, $D_{\mathcal{O}}$ satisfies the same conditions as $I_{\mathfrak{n}^{\mathcal{S}}}(F)$, and by uniqueness $D_{\mathcal{O}} = I_{\mathfrak{n}^{\mathcal{S}}}(F)$. □

Lemma 4.4.11 and Corollary 4.4.12 allow us to generalize Theorem 3.4.1 to subdivisions of the Lawrence polytope $\Lambda(\mathcal{M}^*(G))$. We show this agrees from constructing a $\mathcal{D}_{\mathcal{S}}$-forest function directly from the subdivision.

Theorem 4.4.13. *Let $\mathcal{S}$ be a subdivision of the Lawrence polytope $\Lambda(\mathcal{M}^*(G))$. Consider the map χ^* from cells of the subdivision $\mathcal{S}$ to internal orientations $\vec{F}^*$ for each $F \in m\mathcal{S}\mathcal{S}_{\mathcal{D}_{\mathcal{S}}}(G)$.*

Then, define the $\mathcal{D}_{\mathcal{S}}$-forest function $I_{\mathcal{S}}$ by

$$\begin{aligned} I_{\mathcal{S}} : m\mathcal{SS}_{\mathcal{D}_{\mathcal{S}}}(G) &\to \mathrm{Div}^{-b_0(F)}(G) \\ F &\mapsto D_{-(\vec{F^*})^c} \end{aligned}$$

and $\mathcal{BD}_{I_{\mathcal{S}}}(G)$ satisfies $|\mathcal{BD}_{I_{\mathcal{S}}}(G)| = c_{\mathcal{D}_{\mathcal{S}}}(G)$.

Proof. From Theorem 4.4.7, the subdivision $\mathcal{S}$ gives an internal orientation $\vec{F^*}$ of every $F \in m\mathcal{SS}_{\mathcal{D}_{\mathcal{S}}}(G)$. Recall that $T(F)$ is the set of all spanning trees of G such that $T \geq F$ in $m\mathcal{SS}_{\mathcal{D}_{\mathcal{S}}}(G)$. Then, $\vec{F^*} = \cup_{T \in T(F)} \vec{T^*}$. The one-way oriented edges in $\vec{F^*}$ are described by $\sigma^*_{\mathcal{S}}$ as in Corollary 4.4.12, and therefore $I_{\mathcal{S}}$ is the $\mathcal{D}_{\mathcal{S}}$-forest function $I_{\mathfrak{n}^{\mathcal{S}}}$ above. $\square$

Example 4.4.14. Let $\mathcal{S}_{\text{triv}}$ be the trivial subdivision of $\Lambda(\mathcal{M}^*(G))$, where the only cell is the Lawrence polytope itself. Then $\chi^*(\mathcal{S}_{\text{triv}})$ is the fourientation $\vec{G^*}$ where every edge of G is bioriented. The localization $\sigma^*_{\mathcal{S}}$ sends every signed corcircuit to 0. Therefore, the degeneracy subset of $\mathcal{S}_{\text{triv}}$ is all biconnected subsets of G. We then have $I_{\mathcal{S}}(G) = D_{-(\vec{G^*})^c} = \begin{bmatrix} -1 & -1 & \cdots & -1 \end{bmatrix}^T \in \mathbb{Z}^{|V(G)|}$. The set $\mathcal{BD}_{I_{\mathcal{S}}}(G)$ is all orientable divisors $D_{\mathcal{O}}$ on G. By Lemma 40 of [7], $|\mathcal{BD}_{I_{\mathcal{S}}}(G)| = |\text{spanning forests of } G|$. The function $I_{\mathcal{S}}$ corresponds to the maximally degenerate stability condition $\mathfrak{n}^{\deg}$ defined in Lemma 5.2 of [18].

It is currently unknown if every degeneracy set of a graph G is the degeneracy set of some stability condition [19, Question 4.23]. We recall that a degeneracy set $\mathcal{D}$ is orientable if the single-element extension corresponding to $\mathcal{D}$ is an orientable matroid. The bijection between subdivisions of $\Lambda(\mathcal{M}^*(G))$ and extensions of $\mathcal{M}(G)$ yields the following corollary.

Corollary 4.4.15. *Let VStab(G) be the set of all V-stability conditions on G. Then the map VStab$(G) \to \mathcal{O}$Deg(G) is surjective.*

We can also generalize Theorem 3.4.10 to subdivisions of the Lawrence polytope $\Lambda(\mathcal{M}^*(G))$.

Corollary 4.4.16. *Let $\mathcal{S}$ be a regular subdivision of the Lawrence polytope $\Lambda(\mathcal{M}^*(G))$. The stability condition $\mathfrak{n}^{\mathcal{S}}$ is classical.*

Proof. Recall that any regular subdivision of the Lawrence polytope $\Lambda(\mathcal{M}^*(G))$ can be refined to a regular triangulation $\mathcal{T}$ of $\Lambda(\mathcal{M}^*(G))$. From Theorem 3.4.10, the triangulation $\mathcal{T}$ corresponds to a classical stability condition $\mathfrak{n}^{\mathcal{T}} = \mathfrak{n}(\phi^{\mathcal{T}})$, where $\phi^{\mathcal{T}}$ is a generic numerical polarization on G. From Theorem 3.4.10, $\phi^{\mathcal{T}}$ lives in a chamber in $V^g(G)$ defined by the hyperplanes

$$\left\{\phi^{\mathcal{T}}_W - \frac{|E(W,W^c)|}{2} = \mathfrak{n}^{\mathcal{T}}_W\right\} \bigcup \left\{\phi^{\mathcal{T}}_W - \frac{|E(W,W^c)|}{2} = \mathfrak{n}^{\mathcal{T}}_W - 1\right\}.$$

Consider a polarization $\phi^{\mathcal{S}}$ that lives in between the hyperplanes

$$\left\{\phi^{\mathcal{T}}_W - \frac{|E(W,W^c)|}{2} = \mathfrak{n}^{\mathcal{T}}_W = \mathfrak{n}^{\mathcal{S}}_W\right\} \bigcup \left\{\phi^{\mathcal{T}}_W - \frac{|E(W,W^c)|}{2} = \mathfrak{n}^{\mathcal{T}}_W - 1 = \mathfrak{n}^{\mathcal{S}}_W - 1\right\}$$

for $W \notin \mathcal{D}_{\mathcal{S}}$, and on the hyperplanes

$$\left\{\phi^{\mathcal{T}}_W - \frac{|E(W,W^c)|}{2} = \mathfrak{n}^{\mathcal{T}}_W - 1 = \mathfrak{n}^{\mathcal{S}}_W\right\}$$

if $W \in \mathcal{D}_{\mathcal{S}}$ and $\sigma^*_{\mathcal{S}}(\vec{C^*_W}) = +$, and on the hyperplanes

$$\left\{\phi^{\mathcal{T}}_W - \frac{|E(W,W^c)|}{2} = \mathfrak{n}^{\mathcal{T}}_W = \mathfrak{n}^{\mathcal{S}}_W\right\}$$

for the remaining $W \in \mathcal{D}_{\mathcal{S}}$. One can check that if there exists $W_1, W_2 \in \mathcal{D}_{\mathcal{S}}$ such that $W_1 \cap W_2 = \emptyset$ and $W_1 \cup W_2$ is also biconnected, then $\phi^{\mathcal{S}}_{W_1\cup W_2} - \frac{|E(W_1\cup W_2,(W_1\cup W_2)^c)|}{2}$ is also an integer. It follows that $\mathfrak{n}(\phi^{\mathcal{S}}) = \mathfrak{n}^{\mathcal{S}}$, and $\mathfrak{n}^{\mathcal{S}}$ is classical. $\square$

However, not every degenerate classical stability condition $\mathfrak{n}$ corresponds to a orientable degeneracy set.

Example 4.4.17. We consider the following stability condition associated to thenumerical polarization in Example 5 in Section 8 of [25]. Consider K_4, the complete graph on 4 vertices. We label those vertices by $V(K_4) = \{v_1, v_2, v_3, v_4\}$. We consider the polarization ϕ on G given by $\phi(v_1) = \phi(v_4) = 1$, and $\phi(v_2) = \phi(v_3) = 0$. The stability condition $\mathfrak{n}(\phi)$ has degeneracy set $\mathcal{D}(\mathfrak{n}(\phi)) = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4, v_3v_4\}$. From Lemma 4.2.4, $\mathcal{D}(\mathfrak{n}(\phi))$ corresponds to a linear subclass of cocircuits of $M(K_4)$, and therefore corresponds to a single-element extension of $M(K_4)$. However, this extension of $M(K_4)$ yields a matroid that is not orientable [32, Example 3.3].

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