

©Copyright 2020

Sean T. Griffin

Combinatorics and Representation Theory of
Rank Varieties, Springer Fibers, and
Hyperplane Arrangements

Sean T. Griffin

A dissertation
submitted in partial fulfillment of the
requirements for the degree of

Doctor of Philosophy

University of Washington

2020

Reading Committee:

Sara C. Billey, Chair

Isabella Novik

Julia Pevtsova

Program Authorized to Offer Degree:
Mathematics

University of Washington

Abstract

Combinatorics and Representation Theory of
Rank Varieties, Springer Fibers, and
Hyperplane Arrangements

Sean T. Griffin

Chair of the Supervisory Committee:
Professor Sara C. Billey
Mathematics

This thesis is dedicated to applications of symmetric function theory to problems in combinatorics, representation theory, and geometry. Crucial to our applications is the Frobenius characteristic map from Algebraic Combinatorics, which associates a symmetric function to each finite-dimensional symmetric group module.

First, we introduce a family of quotient rings $R_{n,\lambda,s}$ that have the structure of graded symmetric group modules. This family of rings simultaneously generalizes the cohomology rings of Springer fibers studied by Garsia and Procesi and the generalized coinvariant rings of Haglund, Rhoades, and Shimozono. We use techniques developed by Garsia and Procesi to prove formulas for the graded Frobenius characteristic of $R_{n,\lambda,s}$, generalizing previous formulas for Springer fibers and generalized coinvariant rings. We then apply our results to Eisenbud-Saltman rank varieties.

Second, we present joint work with Gessel and Tewari in which we prove conjectures of Gessel relating a multivariate generating function G encoding labeled binary trees to symmetric group representations. We prove these conjectures by expanding G positively in terms of ribbon Schur symmetric functions. We then connect specializations of G to symmetric group actions on hyperplane arrangements.

TABLE OF CONTENTS

	Page
List of Figures	ii
Chapter 1: Introduction	1
Chapter 2: Background	8
Chapter 3: Garsia-Procesi modules and generalized coinvariant rings	22
3.1 Frobenius characteristic of $R_{n,\lambda,s}$	34
3.2 Skewing formulas and exact sequences for $R_{n,\lambda,s}$	50
3.3 Inversions and diagonal inversions	63
3.4 Applications to rank varieties	84
Chapter 4: Labeled binary trees and hyperplane arrangements	92
4.1 Binary trees	99
4.2 Two functional equations for G	103
4.3 Marked and augmented interlacing partitions	109
4.4 S_n -modules from deformations of Coxeter arrangements	114
4.5 Local binary search trees and γ -nonnegativity	133
Chapter 5: A bijective approach to labeled tree enumeration	136
5.1 A bijective proof of Theorem 4.0.3	136
5.2 Proofs of Theorem 4.0.4 and Theorem 4.0.6	165
Bibliography	170

LIST OF FIGURES

Figure Number	Page
2.1 On the left, the Young diagram of $\lambda = (3, 2, 1, 1, 1)$. On the right, the Young diagram of $\lambda' = (5, 2, 1)$	9
2.2 The conjugate diagram $\text{dg}'(4, 5, 3, 0, 1)$	10
2.3 Two examples of attacking pairs in $\text{dg}'(4, 5, 2, 0, 3)$	19
2.4 A column-increasing filling $\sigma \in \text{SCI}_{10, \alpha, 5}$ where $\alpha = (4, 5, 3, 0, 1)$	19
2.5 All standard column-increasing fillings in $\text{SCI}_{5, \lambda, 2}$ where $\lambda = (3, 2)$	20
3.1 Two examples of the path Q constructed from the path P in the proof of Lemma 3.1.5 for the case where $a = 5$ and $b = 8$. On the left, we have $\gamma = (0, 1, 2, 0, 1, 3, 2, 3, 4, 5, 4, 6, 7)$. On the right, $\gamma = (0, 1, 0, 1, 2, 3, 2, 4, 3, 4, 5, 6, 7)$	39
3.2 The standard extended column-increasing filling associated to $(1, 3, 5, 7, 8 \mid 2, 9, 10 \mid 4, 6) \in \mathcal{OP}_{10, (3, 2), 3}$, where the basement cells are bolded.	64
3.3 An extended column-increasing filling in $\text{ECI}_{11, (2, 3, 0, 1), 4}$. We have labeled the rows and columns for the reader's aid.	65
3.4 An example of the insertion algorithm for $\iota^\alpha(c_n, \dots, c_1)$ for $n = 9$, $\alpha = (3, 2, 0)$, $s = 3$, $(c_n, \dots, c_1) = (0, 0, 2, 1, 1, 0, 0, 1, 0)$, and $\gamma = (2, 2, 1, 2, 1, 1)$	74
3.5 An example of the insertion algorithm for $\iota_d^\alpha(c_n, \dots, c_1)$ when $n = 9$, $\alpha = (3, 2, 0)$, $s = 3$, and $(d_n, \dots, d_1) = (0, 0, 2, 1, 1, 0, 0, 1, 0)$	77
4.1 The nodes are labeled according to preorder on the left and according to inorder on the right.	101
4.2 A tree T with canopy $\text{can}(T) = \text{UDUDDUD}$	102
4.3 On the left, a tree with weight $\text{wt}(T) = \bar{\lambda}^3 \lambda^1 \bar{\rho}^3 \rho^1$. On the right, the inorder standardization $\text{sort}(T)$	103
4.4 An example of a marked tree on 9 nodes with nodes v_1, v_2, v_3, v_6 , and v_8 in inorder marked.	106
4.5 The arc diagram of the interlacing partition $\pi = 1\ 2\ 3\ 7\ 8\ 11/4\ 6/5/9\ 10$, with nodes v_2, v_4 , and v_8 marked.	111
4.6 An augmented interlacing partition π^* with $w(\pi^*) = \text{DUDDUDDUU}$	114

4.7	On the left is the arrangement \mathcal{I}_3 (projected onto $x_1 + x_2 + x_3 = 0$). On the right is a depiction of the S_3 -action on $\text{Regions}(\mathcal{I}_3)$ with regions of the same color belonging to the same orbit (see the electronic version).	115
4.8	On the left is a binary tree T , and on the right is its corresponding noncrossing partition $\text{nc}(T)$	117
4.9	An example of a local binary search (LBS) tree on 9 nodes.	123
4.10	An example of an augmented interlacing partition on 8 nodes with all nodes unmarked, and its corresponding elements of $\text{NC}(8)$ and \mathcal{T}_8	124
4.11	The Bernardi trees on 3 nodes.	126
4.12	Regions of \mathcal{L}_3 indexed by trees in \mathcal{T}_3^B	127
4.13	On the left, a tree $T \in \mathcal{T}_9^B$. On the right, $\sigma(T)$ where $\sigma = (38) \in S_9$	128
5.1	Determining the weight of a labeled edge.	138
5.2	A path-tree sequence S	139
5.3	The 12 possibilities for $\text{sort}(T)$ for T with $\text{can}(T) = DU$	144
5.4	The 12 possible standardizations of the path-tree sequences with canopy DU	144
5.5	A summary of the map $\Phi_{3,DU}$ defined on standard trees.	145
5.6	A schematic diagram for distinguished triples (v_i, v_j, v_k)	148
5.7	On the left, a tree $T \in \mathcal{T}_{16}$ with its roof and attic highlighted. On the right, a table listing the distinguished triples (v_i, v_j, v_k) , where in each row the distinguished ancestor of the node v_j is starred.	149
5.8	The six cases where v_j is life-sustaining.	150
5.9	The six cases where v_j is prunable.	151
5.10	A tree $T \in \mathcal{T}_{16}$ with canopy $DUDUUUDUUDUUDUUD$, partitioned using the antichain $\{v_2, v_4, v_{12}\}$	155
5.11	A tree $T \in \mathcal{T}_{16}^\ell$ whose prunable nodes are highlighted in green with $\text{can}(T) = DUDUUUDUUDUUDUUD$	157
5.12	The image under $\Phi_{n,\nu}$ of the tree in Figure 5.11.	157
5.13	The Ω -tree corresponding to the path-tree sequence in Figure 5.12.	160
5.14	The inverse bijection $\Psi_{n,\nu}$ applied to the path-tree sequence S from Figure 5.12.	162
5.15	The augmented interlacing partition $\Theta_{n,\nu}(S)$ for S in Figure 5.12.	166

ACKNOWLEDGMENTS

I am deeply indebted to my advisor, Sara Billey, in many ways. This work would not have been possible without her guidance, insight, and detailed feedback. I am truly grateful for all of the things she did that contributed to my graduate school experience, including introducing me to her colleagues at conferences, advocating for me, giving me honest and sincere advice, and so much more. I would not be where I am today without her excellent mentorship.

I would also like to thank Jake Levinson, Steve Mitchell, and Vasu Tewari, who have taught me so much about both math and life.

Many thanks to Ira Gessel and Alexander Woo for interesting conversations and for sharing interesting problems on labeled binary trees and spanning line arrangements, respectively.

I am also grateful to Jarod Alper, Sergi Elizalde, Erik Carlsson, Adriano Garsia, Maria Monks Gillespie, Jim Haglund, Zach Hamaker, Monty McGovern, Brendan Pawlowski, Martha Precup, Brendon Rhoades, Mark Shimozono, Nathan Williams, Andy Wilson, for many helpful conversations and useful references. Thanks also to the remaining members of my committee Paul Beame, Julia Pevtsova, and Isabella Novik for valuable conversations and interesting questions that helped me improve this thesis.

I would like to thank Swati Padmanabhan for her overwhelming support and encouragement, and for many pieces of helpful advice throughout graduate school. Thanks also to Jacob Richey for our many adventures and misadventures.

Further thanks go to the faculty, staff, and graduate students in the University of Washington Math Department for creating a welcoming and collaborative environment. I will also miss all of my friends from Combinatorics Seminar, AWM, our premier ultimate frisbee team

Unit Disc, and the many attendees of the math department's regular tea times and happy hours.

Finally, I am grateful to my parents and siblings for their unwavering love and support. I would not have made it this far without them.

DEDICATION

To my parents, who taught me the value of hard work.

To my grandparents, who taught me kindness and patience.

Chapter 1

INTRODUCTION

Coinvariant rings are a family of graded rings with rich connections to the combinatorics of the symmetric group, invariant theory, and enumerative algebraic geometry. Let S_n be the symmetric group of permutations of the integers $1, \dots, n$, and let $\mathbf{x}_n = \{x_1, \dots, x_n\}$ be a set of n commuting variables. The group S_n then acts on the polynomial ring in n variables $\mathbb{Z}[\mathbf{x}_n]$ by permuting the x_i , and this action respects the natural grading on the polynomial ring by total degree in the variables \mathbf{x}_n . Let $I_n = \langle \mathbb{Z}[\mathbf{x}_n]_+^{S_n} \rangle$ be the ideal generated by all nonconstant homogeneous polynomials that are invariant under the action of S_n . The *coinvariant ring* is defined to be the quotient

$$R_n := \frac{\mathbb{Z}[\mathbf{x}_n]}{I_n}, \quad (1.0.1)$$

which inherits the structure of a graded S_n -module. It is a classical theorem of Chevalley that the coinvariant algebra is isomorphic to the regular representation.

Theorem 1.0.1 (Chevalley [12]). *There is an isomorphism of S_n -modules*

$$R_n \cong \mathbb{Z}S_n. \quad (1.0.2)$$

Furthermore, there is an explicit basis consisting of monomials, which witnesses the fact that R_n is a free \mathbb{Z} -module of rank $n!$.

Theorem 1.0.2 (Artin [1]). *The set of staircase monomials,*

$$\mathcal{A}_n := \{x_1^{i_1} \cdots x_n^{i_n} : i_j \leq j - 1 \text{ for all } 1 \leq j \leq n\} \quad (1.0.3)$$

represents a \mathbb{Z} -basis of the coinvariant ring R_n .

Given a permutation $\pi \in S_n$, we say an index $i < n$ is a *descent* of π if $\pi(i) > \pi(i + 1)$. The *major index statistic* on permutations is the quantity $\text{maj}(\pi)$ computed as the sum of all i such that i is a descent of π . Recall that the *Hilbert series* of a graded free \mathbb{Z} -module is the generating function recording the ranks of the graded pieces. It can be seen from Theorem 1.0.2 that the Hilbert series of R_n has the following beautiful formula,

$$\text{Hilb}_q(R_n) = (1 + q)(1 + q + q^2) \cdots (1 + q + \cdots + q^{n-1}) = \sum_{\pi \in S_n} q^{\text{maj}(\pi)}, \quad (1.0.4)$$

where the second equality follows from an identity of MacMahon [57].

Lusztig (unpublished) and Stanley [83] identified the graded pieces of the coinvariant ring as symmetric group modules using symmetric functions. A *symmetric function* is a formal power series in the infinite set of commuting variables $\mathbf{x} = \{x_1, x_2, \dots\}$ that is invariant under any permutation of the variables. It is a basic fact [89] that the vector space of symmetric functions with coefficients in \mathbb{Q} has dimension equal to the number of partitions λ of n . One of the most important bases of this vector space is the set of *Schur functions* $s_\lambda(\mathbf{x})$. Recall that the irreducible representations of the symmetric group over \mathbb{Q} are also indexed by integer partitions of n . Denote by S^λ the irreducible representation of S_n indexed by the partition λ of n , which we refer to as the *Specht module* indexed by λ . The *Frobenius characteristic map* Frob is the map from representations of S_n to symmetric functions of degree n that sends

$$V = \bigoplus_{\lambda \vdash n} (S^\lambda)^{\oplus a_\lambda} \quad (1.0.5)$$

to $\text{Frob}(V) := \sum_{\lambda \vdash n} a_\lambda s_\lambda(\mathbf{x})$, where the sums are over all integer partitions λ of n . The Frobenius characteristic map can be extended to a map on graded representations $V = \bigoplus_{i \geq 0} V_i$ of S_n by defining $\text{Frob}_q(V) := \sum_{i \geq 0} q^i \text{Frob}(V_i)$.

Given an integer partition $\lambda = (\lambda_1, \dots, \lambda_\ell)$ of n , we can visualize it by its *Young diagram*,

which consists of rows of boxes called *cells*. We adhere to the French style of drawing Young diagrams, where we draw λ_i many boxes in the i th row from the bottom. A *standard Young tableau* of λ is a labeling of the cells of the Young diagram using each of the positive integers $1, \dots, n$ exactly once, such that the labeling increases up each column and along each row from left to right. The *major index statistic* $\text{maj}(T)$ of a tableau T is defined to be the sum of all i such that $i + 1$ appears in a row higher than i in T . We have the following theorem of Lusztig and Stanley that identifies the graded Frobenius characteristic of $R_n^{\mathbb{Q}} = R_n \otimes \mathbb{Q}$.

Theorem 1.0.3 (Lusztig-Stanley [83]). *The graded Frobenius characteristic of the coinvariant ring is*

$$\text{Frob}_q(R_n^{\mathbb{Q}}) = \sum_{\substack{\lambda \vdash n, \\ T \in \text{SYT}(\lambda)}} q^{\text{maj}(T)} s_{\lambda}(\mathbf{x}), \quad (1.0.6)$$

where the sum is over all partitions λ of n and all standard Young tableaux of shape λ .

The coinvariant ring has further importance to the geometry of flag varieties. A *complete flag* of \mathbb{C}^n is a sequence of nested vector subspaces

$$V_{\bullet} = (0 \subset V_1 \subset V_2 \subset \dots \subset V_n = \mathbb{C}^n), \quad (1.0.7)$$

such that $\dim_{\mathbb{C}} V_i = i$ for all i , and the *complete flag variety* $\text{Fl}(n)$ is the space of all complete flags of \mathbb{C}^n . Borel showed that the coinvariant ring R_n is isomorphic to the cohomology ring of the complete flag variety.

Theorem 1.0.4 (Borel [10]). *There is an isomorphism of graded rings*

$$R_n \cong H^*(\text{Fl}(n); \mathbb{Z}), \quad (1.0.8)$$

where the i th graded piece of R_n is mapped to the $(2i)$ th cohomology group.

Another important family of varieties are the *Springer fibers*, whose cohomology rings

give a geometric construction of the Specht modules S^λ . Given a nilpotent $n \times n$ matrix X , the Springer fiber of X is

$$\mathcal{B}^X := \{V_\bullet \in \text{Fl}(n) : XV_i \subseteq V_{i-1} \text{ for all } i\}. \quad (1.0.9)$$

The Springer fibers get their name from the fact that they are the fibers of the Springer resolution of the cone of nilpotent matrices. These varieties have importance in Geometric Representation Theory, because their cohomology groups provide a geometric construction of the Specht modules S^λ . Let λ be the Jordan type of X and $n(\lambda) = \sum_i (i-1)\lambda_i$. Then there is an action of S_n on the rational cohomology ring of \mathcal{B}^X , originally constructed by Springer [27], such that $H^{2n(\lambda)}(\mathcal{B}^X; \mathbb{Q}) \cong S^\lambda$. It should be noted that the action on the cohomology of a Springer fiber discussed in this paper differs from the one originally constructed by Springer by twisting with the sign representation. As a refinement of this result, the graded Frobenius characteristic of the cohomology ring $H^*(\mathcal{B}^X; \mathbb{Q})$ is the dual Hall-Littlewood symmetric function [56] up to a change of variables,

$$\text{Frob}_q(H^*(\mathcal{B}^X; \mathbb{Q})) = q^{2n(\lambda)} Q'_\lambda(\mathbf{x}; q^{-2}). \quad (1.0.10)$$

In their seminal paper, Garsia and Procesi [29] showed how to prove (1.0.10) with only Algebraic Combinatorics starting from a presentation of the cohomology ring due to Tanisaki [90].

In the case when $X = 0$, the Springer fiber is simply the complete flag variety $\mathcal{B}^X = \text{Fl}(n)$, so the cohomology of the Springer fiber is the coinvariant ring. Furthermore, it can be shown that the right-hand side of (1.0.10) specializes to the Lusztig-Stanley formula, Theorem 1.0.3. Hence, Garsia and Procesi's results can be seen as a generalization of the work on coinvariant rings. See [13] for more background on Springer fibers and [92] for more details on their combinatorial properties.

In the 90s and early 2000s, many of these results were extended to the setting of the *diagonal coinvariant ring*, which is the quotient of the polynomial ring $\mathbb{Z}[x_1, \dots, x_n, y_1, \dots, y_n]$

by the ideal of positive degree homogenous elements invariant under the action of S_n acting diagonally by permutations of the variables x_1, \dots, x_n and y_1, \dots, y_n simultaneously. In [28], Garsia and Haiman conjectured that a certain S_n -module, defined in terms of derivatives of a Vandermonde determinant in the variables $x_1, \dots, x_n, y_1, \dots, y_n$, gives a representation-theoretic interpretation for the Macdonald symmetric function $\tilde{H}_\mu(\mathbf{x}; q, t)$, a symmetric function whose $t = 0$ specialization is $\tilde{H}_\mu(\mathbf{x}; q) = q^{n(\mu)} Q'_\mu(\mathbf{x}; q^{-1})$. Haiman proved this conjecture by finding a remarkable connection between these modules and the isospectral Hilbert scheme of n points in the complex plane [47, 48].

Another family of graded rings that generalize the coinvariant rings are the rings $R_{n,k}$ of Haglund, Rhoades, and Shimozono [45], depending on two positive integers $k \leq n$. To define the ring $R_{n,k}$, first let $e_d(\mathbf{x}_n)$ be the *elementary symmetric polynomial of degree d* in the variables \mathbf{x}_n , defined by

$$e_d(\mathbf{x}_n) = \sum_{1 \leq i_1 < \dots < i_d \leq n} x_{i_1} x_{i_2} \cdots x_{i_d}. \quad (1.0.11)$$

It can be shown that the ideal I_n in the definition of the coinvariant ring is generated by the elementary symmetric polynomials, $I_n = \langle e_1(\mathbf{x}_n), \dots, e_n(\mathbf{x}_n) \rangle$. Haglund, Rhoades, and Shimozono introduced a variant of this ideal depending on both k and n ,

$$I_{n,k} := \langle x_1^k, x_2^k, \dots, x_n^k, e_n(\mathbf{x}_n), e_{n-1}(\mathbf{x}_n), \dots, e_{n-k+1}(\mathbf{x}_n) \rangle \subseteq \mathbb{Q}[\mathbf{x}_n]. \quad (1.0.12)$$

They defined the *generalized coinvariant ring* $R_{n,k}$ to be the quotient ring $R_{n,k} := \mathbb{Q}[\mathbf{x}_n]/I_{n,k}$. Since $I_{n,k}$ is homogeneous and stable under the action of S_n , the quotient ring $R_{n,k}$ has the structure of a graded S_n -module. When $k = n$, it can be shown that $I_n = I_{n,n}$ [45, Section 1], and hence the generalized coinvariant ring specializes to the usual coinvariant ring when $n = k$.

Haglund, Rhoades, and Shimozono proved that their rings give a representation-theoretic meaning to ordered set partitions. An *ordered set partition of $[n]$* is a partitioning of the set

$[n]$ into an ordered list of subsets B_1, \dots, B_k . We denote such an ordered set partition by $(B_1|B_2|\dots|B_k)$. Let $\mathcal{OP}_{n,k}$ be the collection of ordered set partitions of $[n]$ into k nonempty blocks. The group S_n acts on $\mathcal{OP}_{n,k}$ by permuting the letters $1, 2, \dots, n$. Haglund, Rhoades, and Shimozono [45] proved that $R_{n,k}$ is isomorphic as an S_n -module to the module $\mathbb{Q}\mathcal{OP}_{n,k}$ corresponding to the S_n action on $\mathcal{OP}_{n,k}$ via permuting the letters $1, \dots, n$. Even more remarkably, they proved that $\text{Frob}_q(R_{n,k})$ coincides with the symmetric function in the well-known Delta Conjecture from Algebraic Combinatorics when $t = 0$, up to a minor twist.

The goal of Chapter 3 of this thesis is to unify the combinatorics and representation theory of the cohomology rings of Springer fibers and the generalized coinvariant rings $R_{n,k}$. We do this by introducing a larger family of rings $R_{n,\lambda,s}$ that includes the cohomology ring of a Springer fiber and $R_{n,k}$ as special cases. We then generalize many of the previous results on these two families of rings to $R_{n,\lambda,s}$. We also give a geometric interpretation of these rings in terms of Eisenbud-Saltman rank varieties. In particular, we apply our results on $R_{n,\lambda,s}$ to give a formula for the Hilbert series of the coordinate ring of the scheme-theoretic intersection of a rank variety with diagonal matrices.

Another class of combinatorial objects with strong connections to the representation theory of the symmetric group are *hyperplane arrangements*, which are finite collections of hyperplanes in \mathbb{R}^n . One important example of a hyperplane arrangement is the *braid arrangement*, consisting of the set of hyperplanes $x_i = x_j$ for $1 \leq i < j \leq n$ in \mathbb{R}^n . The hyperplanes in a hyperplane arrangement cut \mathbb{R}^n into *regions*, which are the connected components of the complement of the union of the hyperplanes in the arrangement. It is not hard to verify that the number of regions for the braid arrangement are in bijection with permutations. The symmetric group S_n then acts on the hyperplanes in this arrangement by permuting the coordinates. We then get an action of S_n on the regions of the braid arrangements, whose corresponding S_n -module is the regular representation.

Another hyperplane arrangement, which fits into the theory of diagonal coinvariant rings, is the Shi arrangement consisting of the hyperplanes in the braid arrangement along with hyperplanes $x_i - x_j = 1$ for $1 \leq i < j \leq n$. Observe that the set of hyperplanes in the Shi

arrangement are not invariant under permutations of coordinates. However, there is a hidden action of S_n on the regions of the Shi arrangement which utilizes the bijection between *parking functions* and regions of the Shi arrangement due to Pak and Stanley [86]. The corresponding S_n -module is the well-studied parking function representation PF_n . Haiman [46] proved that the S_n -module structure of the diagonal coinvariant ring is simply the parking function representation tensored with the sign representation. Furthermore, the graded Hilbert series is given by the inversion generating function on certain labeled trees.

The subject of Chapter 4 of this thesis is on joint work with Gessel and Tewari [35, 36] in which we find connections between descent statistics on labeled binary trees and symmetric group actions on hyperplane arrangements. In particular, we study a multivariate generating function G first defined by Gessel encoding ascent-descent statistics on labeled plane binary trees. We prove that certain specializations of G are the Frobenius characteristics of symmetric group actions on hyperplane arrangements.

Finally, in Chapter 5 we prove formulas for the multivariate generating function G in terms of ribbon Schur functions. As a corollary, we see that G is a positive sum of Schur symmetric functions. One of our main tools is an extension of the Push-Glide bijection of Préville-Ratelle and Viennot [69] between the set of binary trees and the set of path-tree sequences.

Chapter 2

BACKGROUND

2.0.1 Partitions and compositions

A *partition of n into ℓ parts* is a weakly decreasing sequence of positive integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ such that $\sum_{i=1}^{\ell} \lambda_i = n$. We sometimes write $\lambda \vdash n$ or $|\lambda| = n$ to denote that λ has *size n* . Let $\ell(\lambda) := \ell$ be the *length* of λ . Let $\text{Par}(n, s)$ be the set of partitions of n into at most s parts. For example, we have

$$\text{Par}(4, 2) = \{(4), (3, 1), (2, 2)\}. \quad (2.0.1)$$

Given integers a and b with $b \geq 0$, we denote by (a^b) the partition (a, a, \dots, a) where a appears b many times. When $b = 0$, then (a^b) is the empty partition \emptyset .

Given two partitions λ and μ of n , we say that λ is *dominated by* μ , denoted by $\lambda \leq_{\text{dom}} \mu$, if and only if for all $i \leq \ell(\lambda)$, we have that $\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$. The partial ordering \leq_{dom} on the partitions of n is called *dominance order*. For $\lambda \vdash n$, we draw its *Young diagram* as rows of boxes, called *cells*, with λ_i cells in row i . We follow the French convention where the rows are numbered from bottom to top. We also number the columns of the diagram from left to right. Let λ' be the *conjugate partition* of λ , which is the partition of n whose i th entry records the number of cells in the i th column of the Young diagram of λ . See Figure 2.1 for the Young diagrams of $\lambda = (3, 2, 1, 1, 1)$ and its conjugate $\lambda' = (5, 2, 1)$.

Recall from Chapter 1 that $n(\lambda)$ is the statistic

$$n(\lambda) := \sum_{i \geq 1} (i-1)\lambda_i. \quad (2.0.2)$$

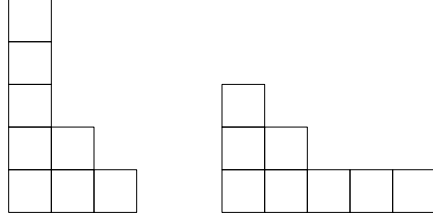


Figure 2.1: On the left, the Young diagram of $\lambda = (3, 2, 1, 1, 1)$. On the right, the Young diagram of $\lambda' = (5, 2, 1)$.

Alternatively, the statistic $n(\lambda)$ can be computed as follows. Starting with the French Young diagram, label all of the entries in the i th row from the bottom with $i - 1$. Then $n(\lambda)$ is the sum of the labels. If we instead sum the labels in each column we easily see that

$$n(\lambda) = \sum_{i=1}^{\lambda_1} \binom{\lambda'_i}{2}. \quad (2.0.3)$$

Given $\lambda \vdash n$ and an index $0 \leq i < \ell(\lambda)$, let $\lambda^{(i)}$ be the partition of $k - 1$ obtained by sorting the parts of

$$(\lambda_1, \dots, \lambda_i, \lambda_{i+1} - 1, \lambda_{i+2}, \dots, \lambda_{\ell(\lambda)}) \quad (2.0.4)$$

into decreasing order. We say that $\lambda^{(i)}$ is the i th reduction of λ . Note that our $\lambda^{(i)}$ is the same as $\lambda^{(i+1)}$ defined in [29]. For example, if $\lambda = (6, 6, 4, 4, 1)$, then $\lambda^{(2)} = (6, 6, 4, 3, 1)$. Observe that if j is maximal such that $i < \lambda'_j$, then

$$\lambda^{(i)} = (\lambda_1, \dots, \lambda_{\lambda'_j} - 1, \dots, \lambda_{\ell}). \quad (2.0.5)$$

A finite ordered list of nonnegative integers $\alpha = (\alpha_1, \dots, \alpha_{\ell})$ such that $\sum_{i=1}^{\ell} \alpha_i = n$ is called a *composition* of n . If $\alpha_i > 0$ for all i , we say that it is a *strong composition* of n , and denote this by $\alpha \vDash n$. If α is a composition of n , we say that the *size* of α is n . We call α_i the *parts* of α and denote the number of parts of α by $\ell(\alpha)$, also called the *length* of

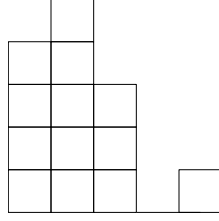


Figure 2.2: The conjugate diagram $\text{dg}'(4, 5, 3, 0, 1)$.

α . Let $\text{Comp}(n, s)$ be the set of compositions of n of length s . Given $\alpha \in \text{Comp}(n, s)$, the *conjugate diagram*, denoted by $\text{dg}'(\alpha)$, is the diagram consisting of α_i cells in the i th column from the left. For $\lambda \in \text{Par}(n, s)$, then $\text{dg}'(\lambda)$ is the Young diagram of λ' . If $\alpha_i = 0$ for some i , we simply draw a horizontal bar at the bottom of column i to signify an empty column of cells. See Figure 2.2 for the conjugate diagram of $\alpha = (4, 5, 3, 0, 1)$. We index the cells (i, j) of the conjugate diagram of α in Cartesian coordinates starting with $(1, 1)$ in the lower left corner, so that (i, j) is the cell in the i th column and j th row. In this thesis, we will never draw the left-justified version of the diagram of a composition as in [43]. These conventions follow those in [40].

Let $\text{sort}(\alpha) \in \text{Par}(n, s)$ be the partition obtained by sorting the entries of α from greatest to smallest and then deleting trailing 0s. We denote by $\text{trunc}(\alpha) := (\alpha_1, \dots, \alpha_{n-1})$ the *truncation* of α of its last entry. Let $\text{rev}(\alpha) := (\alpha_s, \alpha_{s-1}, \dots, \alpha_1)$ be the reversed composition.

If $\alpha \in \text{Comp}(n, s)$ and $\beta \in \text{Comp}(m, s)$, we say that α is *contained in* β , denoted by $\alpha \subseteq \beta$, if $\alpha_i \leq \beta_i$ for all $i \leq s$. If $\alpha \subseteq \beta$, let $\text{dg}'(\beta)/\text{dg}'(\alpha)$ be the set difference of the conjugate diagrams of β and α . When $\alpha = \mu' \subseteq \beta = \lambda'$ for partitions μ and λ , this is the *skew diagram* λ/μ .

We define two operations on compositions. Given compositions $\alpha = (\alpha_1, \dots, \alpha_\ell)$ and $\beta = (\beta_1, \dots, \beta_m)$, we define the *concatenation* of α and β , denoted by $\alpha \bullet \beta$, to be the composition $(\alpha_1, \dots, \alpha_\ell, \beta_1, \dots, \beta_m)$. The *near-concatenation* of α and β , denoted by $\alpha \odot \beta$, is defined to be the composition $(\alpha_1, \dots, \alpha_{\ell-1}, \alpha_\ell + \beta_1, \beta_2, \dots, \beta_m)$. For example, if $\alpha = (2, 1, 3)$ and $\beta = (4, 1)$, then $\alpha \bullet \beta = (2, 1, 3, 4, 1)$ while $\alpha \odot \beta = (2, 1, 7, 1)$.

Recall the well-known bijection between strong compositions $\alpha = (\alpha_1, \dots, \alpha_\ell)$ of n and subsets $S \subseteq [n-1]$ given by $S = \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \dots + \alpha_{\ell-1}\}$. We denote the set corresponding to $\alpha \vDash n$ by $\text{set}(\alpha)$, and in the opposite direction, given $S \subseteq [n-1]$, we denote the corresponding strong composition of size n by $\text{comp}(S)$. The inclusion order on subsets induces a natural poset structure on the set of strong compositions of size n . More precisely, given $\alpha, \beta \vDash n$, we say that $\alpha \preceq \beta$ if and only if $\text{set}(\beta) \subseteq \text{set}(\alpha)$, and call \preceq the *refinement order* on compositions. For instance, consider $\alpha = (1, 2, 4, 2, 3, 2, 1)$ and $\beta = (3, 4, 7, 1)$, both compositions of size 15. Then $\text{set}(\beta) = \{3, 7, 14\}$ and $\text{set}(\alpha) = \{1, 3, 7, 9, 12, 14\}$. Clearly we have that $\text{set}(\beta) \subseteq \text{set}(\alpha)$, and therefore $\alpha \preceq \beta$. We denote this poset on strong compositions of size n by Comp_n^{\preceq} and refer to it as the *composition poset*. Given strong compositions $\alpha, \gamma \vDash n$ such that $\alpha \preceq \gamma$, we denote the interval in Comp_n^{\preceq} comprising compositions β satisfying $\alpha \preceq \beta \preceq \gamma$ by $[\alpha, \gamma]$.

2.0.2 Words and permutations

Denote the symmetric group of permutations of $1, \dots, n$ by S_n . For $1 \leq i < n$, let $s_i \in S_n$ be the adjacent transposition $s_i = (i, i+1)$. For a composition of length n , let $s_i(\alpha)$ be the composition obtained by swapping the i th and $(i+1)$ th entries of α .

Let \mathbb{P} be the set of positive integers. Let \mathbb{P}^+ be the set of nonempty words on \mathbb{P} , which is the set of finite sequences of positive integers with positive length. If w is a word with letters w_1, w_2, \dots, w_n , we write $w = w_1 \cdots w_n$. To w , we associate the monomial $\mathbf{x}^w = x_{w_1} x_{w_2} \cdots x_{w_n}$. We denote the set of words in n letters by \mathbb{P}^n . An *ascent* of w is an index $1 \leq i \leq n-1$ such that $w_i \leq w_{i+1}$. A *descent* of w is an index $1 \leq i \leq n-1$ such that $w_i > w_{i+1}$. Let the *descent set* of w be $\text{Des}(w) := \{1 \leq i \leq n-1 : w_i > w_{i+1}\}$. An *inversion* in w is a pair of indices $1 \leq i < j \leq n$ such that $w_i > w_j$. We denote the set of inversions of w by $\text{Inv}(w)$. Therefore, we have $i \in \text{Des}(w)$ if and only if $(i, i+1) \in \text{Inv}(w)$. Let $\text{asc}(w)$, $\text{des}(w)$, and $\text{inv}(w)$ be the number of ascents, descents, and inversions in w , respectively. The *inverse descent set* of w is defined by $\text{iDes}(w) := \text{Des}(w^{-1})$.

The *standardization* of w , denoted by $\text{std}(w)$, is the permutation in \mathfrak{S}_n obtained by

replacing the entries of w with $1, 2, \dots, n$, keeping the same relative order, where repeated letters are considered as increasing from left to right. For example, the standardization of 112123 is 124356. For $w \in \mathbb{P}^n$, the standardization $\sigma = \text{sort}(w)$ has the key property that $\text{Inv}(\sigma) = \text{Inv}(w)$.

2.0.3 Symmetric functions

A *symmetric function* is a formal power series in the variables \mathbf{x} which is invariant under swapping any two of the variables. It is well known that the set of symmetric functions with coefficients in \mathbb{Q} , denoted by Λ , forms a ring under the usual operations of addition and multiplication. For notions related to the ring of symmetric functions that are not made explicit here, we refer the reader to [56, 87].

We denote by h_n the n th complete homogeneous symmetric function, which is the sum over all monomials in $\mathbf{x} := \{x_1, x_2, \dots\}$ of degree n . The n th elementary symmetric function, denoted by e_n , is the sum over all squarefree monomials of degree n . For $\lambda \vdash n$ of length k , let $h_\lambda := h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_k}$ be the complete homogeneous symmetric function indexed by λ . Similarly, let $e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_k}$ be the elementary symmetric function indexed by λ . Let $\alpha \vDash n$ of length k . For convenience, we occasionally write h_α to mean $h_{\alpha_1} h_{\alpha_2} \cdots h_{\alpha_k}$, and similarly for e_α . Observe that $h_\alpha = h_{\text{sort}(\alpha)}$ and $e_\alpha = e_{\text{sort}(\alpha)}$. Let

$$H(z) := \sum_{n \geq 0} h_n z^n = \prod_{i \geq 1} \frac{1}{1 - x_i z}, \quad (2.0.6)$$

$$E(z) := \sum_{n \geq 0} e_n z^n = \prod_{i \geq 1} (1 + x_i z). \quad (2.0.7)$$

Furthermore, let ω be the automorphism of Λ such that

$$\omega(e_\lambda) = h_\lambda, \quad \omega(h_\lambda) = e_\lambda, \quad \omega(s_\lambda) = s_{\lambda'}. \quad (2.0.8)$$

Given a partition $\lambda \vdash n$, a *standard Young tableau* (henceforth SYT) of *shape* λ is a filling of the boxes of the Young diagram of λ with integers from $[n]$ using each label exactly

once, such that the labeling increases from left to right along rows and up columns. We denote the set of all SYT of shape λ by $\text{SYT}(\lambda)$. For $T \in \text{SYT}(\lambda)$, we define $\text{Des}(T)$ to be the set of all $1 \leq i \leq n-1$ such that i belongs to a row below $i+1$ in T .

Given partitions $\lambda \supseteq \mu$, a *semi-standard Young tableau* of shape λ/μ is a filling of the boxes of the Young diagram of λ/μ with positive integers such that the labeling weakly increases from left to right along rows and strictly increases up columns. Let $\text{SSYT}(\lambda/\mu)$ be the set of semi-standard Young tableaux of shape λ/μ . Given such a tableau T , let \mathbf{x}^T be the monomial defined as the product over all i of $x_i^{n_i}$, where n_i is the number of times i appears in T . The *skew Schur function* $s_{\lambda/\mu}$ is the sum over all \mathbf{x}^T for T a semi-standard Young tableau of shape λ/μ . In the case when μ is the empty partition, define $s_\lambda := s_{\lambda/\mu}$ to be the *Schur function* indexed by λ .

Most of the results in Chapter 4 will be in terms of *ribbon Schur functions*. These symmetric functions are special instances of skew Schur functions indexed by skew shapes that are *ribbons*, which are connected skew shapes that do not contain a 2×2 box. We can associate a composition to a ribbon by counting the number of boxes in every row of the ribbon from top to bottom. This association allows us to consider ribbon Schur functions as being indexed by strong compositions. We refer to the ribbon Schur function indexed by a strong composition α as r_α . If $\alpha \models n$, we can alternatively define r_α as

$$r_\alpha = \sum_{\substack{w \in \mathbb{P}^n, \\ \text{Des}(w) = \text{set}(\alpha)}} \mathbf{x}^w. \quad (2.0.9)$$

A useful property of ribbon Schur functions is that the product of two ribbon Schur functions is a sum of two ribbon Schur functions,

$$r_\alpha r_\beta = r_{\alpha \bullet \beta} + r_{\alpha \odot \beta}, \quad (2.0.10)$$

which follows easily from (2.0.9). More generally, suppose $\alpha^{(1)}, \dots, \alpha^{(m)}$ are compositions such that $\alpha^{(i)} \models n_i$ and $n_1 + \dots + n_m = n$. Letting $\beta := \alpha^{(1)} \bullet \dots \bullet \alpha^{(m)}$ and $\delta := \alpha^{(1)} \odot \dots \odot \alpha^{(m)}$,

we have that

$$r_{\alpha(1)}r_{\alpha(2)}\cdots r_{\alpha(m)} = \sum_{\gamma \in [\beta, \delta]} r_{\gamma}, \quad (2.0.11)$$

where $[\beta, \delta]$ is the interval between β and δ in Comp_n^{\leq} .

The following proposition gives a positive expansion of a ribbon Schur function in the basis of Schur functions. This is a special case of the expansion of a skew Schur function into Schur functions.

Proposition 2.0.1. [95, Equation 2.2.4] *Given a strong composition $\alpha \vDash n$, we have*

$$r_{\alpha} = \sum_{\lambda \vdash n} b_{\lambda, \alpha} s_{\lambda},$$

where $b_{\lambda, \alpha}$ is the number of $T \in \text{SYT}(\lambda)$ satisfying $\text{comp}(\text{Des}(T)) = \alpha$. In particular, ribbon Schur functions are Schur positive.

Let ex denote the homomorphism from Λ to $\mathbb{Q}[[x]]$ mapping h_n to $x^n/n!$ [87, Section 7.8]. This homomorphism is known as the *exponential specialization* and has the property that the coefficient of $x^n/n!$ in the image $\text{ex}(f)$ of a symmetric function f is the coefficient of $x_1 \cdots x_n$ in f [87, Proposition 7.8.4].

Recall that the *Eulerian polynomial* is defined to be

$$A_n(t) = \sum_{\pi \in S_n} t^{\text{des}(\pi)}. \quad (2.0.12)$$

They were introduced by Leonhard Euler in 1749 in his study of the Dirichlet eta function. It is well known that the Eulerian polynomials satisfy the following useful identity,

$$\sum_{n \geq 0} A_n(t) \frac{x^n}{n!} = \frac{t-1}{t - e^{(t-1)x}}. \quad (2.0.13)$$

See [18, 66] for more details.

We use a 2-parameter weighted power series analogue of the Eulerian polynomial,

$$A(\mathbf{x}; s, t) := \sum_{w \in \mathbb{P}^+} s^{\text{asc}(w)} t^{\text{des}(w)} \mathbf{x}^w. \quad (2.0.14)$$

By Equation (2.0.9),

$$A(\mathbf{x}; s, t) = \sum_{n \geq 1} \sum_{\alpha \neq n} s^{n-\ell(\alpha)} t^{\ell(\alpha)-1} r_\alpha. \quad (2.0.15)$$

By the homogenized version of a result of MacMahon [58, Vol. 1, p. 186], we have that

$$A(\mathbf{x}; s, t) = \frac{\sum_{n \geq 1} (s-t)^{n-1} h_n}{1-t \sum_{n \geq 1} (s-t)^{n-1} h_n} = \frac{H(s-t) - 1}{s-tH(s-t)}. \quad (2.0.16)$$

Let $\Lambda_{\mathbb{Q}(q)}$ be the ring of symmetric functions with coefficients in the field $\mathbb{Q}(q)$ of rational functions in q . Let $\langle \cdot, \cdot \rangle$ be the *Hall inner product* on symmetric functions with the property that

$$\langle s_\lambda(\mathbf{x}), s_\mu(\mathbf{x}) \rangle = \delta_{\lambda, \mu}, \quad (2.0.17)$$

where $\delta_{\lambda, \mu}$ takes the value 1 if $\lambda = \mu$, and 0 otherwise. Given a symmetric function $F(\mathbf{x})$, let $F(\mathbf{x})^\perp$ be the linear operator on Λ which is adjoint to multiplication by $F(\mathbf{x})$ with respect to the Hall inner product. Precisely, given a symmetric function $G(\mathbf{x})$, then $F(\mathbf{x})^\perp G(\mathbf{x})$ is the unique symmetric function such that

$$\langle F(\mathbf{x})^\perp G(\mathbf{x}), H(\mathbf{x}) \rangle = \langle G(\mathbf{x}), F(\mathbf{x})H(\mathbf{x}) \rangle \quad (2.0.18)$$

for all symmetric functions $H(\mathbf{x})$. We primarily work with the operators $e_j(\mathbf{x})^\perp$, which we refer to as the *jth skewing operator*.

2.0.4 Frobenius characteristic

Symmetric functions have a close connection to representations of the symmetric group via the Frobenius characteristic map. Given $\lambda \vdash n$, let S^λ be the irreducible S_n -module indexed by λ . Given a finite-dimensional vector space V over \mathbb{Q} which has the structure of an S_n -module, it decomposes as a direct sum as

$$V \cong \bigoplus_{\lambda \vdash n} (S^\lambda)^{c_\lambda} \quad (2.0.19)$$

for some nonnegative integers c_λ . The *Frobenius characteristic* of V is defined to be the symmetric function

$$\text{Frob}(V) = \sum_{\lambda \vdash n} c_\lambda s_\lambda(\mathbf{x}). \quad (2.0.20)$$

Given a graded S_n -module $V = \bigoplus_{i=0}^m V_i$ with finite-dimensional direct summands V_i , the *graded Frobenius characteristic* of V is defined to be

$$\text{Frob}_q(V) = \sum_{i=0}^m \text{Frob}(V_i) q^i \in \Lambda_{\mathbb{Q}(q)}. \quad (2.0.21)$$

The *Hilbert series* of V is defined to be

$$\text{Hilb}_q(V) = \sum_{i=0}^m \dim_{\mathbb{Q}}(V_i) q^i. \quad (2.0.22)$$

The graded Frobenius characteristic and the Hilbert series of V are related by

$$\text{Hilb}_q(V) = \langle h_{(1^n)}, \text{Frob}_q(V) \rangle = [x_1 \cdots x_n] \text{Frob}_q(V), \quad (2.0.23)$$

where if $F(\mathbf{x}; q) \in \Lambda_{\mathbb{Q}(q)}$, then $[x_1 \cdots x_n] F(\mathbf{x}; q)$ denotes the coefficient of the monomial $x_1 \cdots x_n$ in $F(\mathbf{x}; q)$.

We sometimes expand symmetric functions in terms of *Gessel's fundamental quasisym-*

metric functions, defined as follows. Given a subset $D \subseteq [n - 1]$, $F_{n,D}(\mathbf{x})$ is defined by

$$F_{n,D}(\mathbf{x}) := \sum \mathbf{x}^w, \quad (2.0.24)$$

where the sum is over all words $w \in \mathbb{P}^n$ with $1 \leq w_1 \leq w_2 \leq \cdots \leq w_n$ such that $w_i < w_{i+1}$ for all $i \in D$. Alternatively, Gessel's fundamental quasisymmetric function [30] can be written as

$$F_{n,D}(\mathbf{x}) = \sum_{\substack{w \in \mathbb{P}^n, \\ \text{std}(w) = \pi}} \mathbf{x}^w, \quad (2.0.25)$$

where $\pi \in S_n$ is a fixed permutation such that $\text{iDes}(\pi) = D$. We also have the *monomial quasisymmetric function*,

$$M_{n,D}(\mathbf{x}) := \sum \mathbf{x}^w, \quad (2.0.26)$$

where the sum is over all $w \in \mathbb{P}^n$ with $1 \leq w_1 \leq w_2 \leq \cdots \leq w_n$ such that $w_i < w_{i+1}$ for all $i \in D$ and $w_i = w_{i+1}$ for all $i \in [n - 1] \setminus D$.

For $1 \leq j \leq n$, let $S_{n-j} \times S_j \subseteq S_n$ be the subgroup of S_n of permutations which permute $1, \dots, n - j$ among themselves and permute $n - j + 1, \dots, n$ among themselves. Let $\mathbb{Q}S_n$ be the group algebra of S_n , and let $\epsilon_j \in \mathbb{Q}S_n$ be the idempotent element

$$\epsilon_j = \frac{1}{j!} \sum_{\pi \in S_{\{n-j+1, \dots, n\}}} \text{sgn}(\pi) \pi. \quad (2.0.27)$$

If V is an S_n -module, then $\epsilon_j V$ is an S_{n-j} -module. It is well known (see e.g. [45, Equation 6.20]) that

$$\text{Frob}_q(\epsilon_j V) = e_j(\mathbf{x})^\perp \text{Frob}_q(V). \quad (2.0.28)$$

Furthermore, the symmetric function $\text{Frob}_q(V)$ is uniquely defined by its images under the $e_j(\mathbf{x})^\perp$ operators for $j > 1$.

Lemma 2.0.2 ([29],[45] Lemma 3.6). *Let $F(\mathbf{x})$ and $G(\mathbf{x})$ be symmetric functions with equal constant terms. We have that $F(\mathbf{x}) = G(\mathbf{x})$ if and only if $e_j(\mathbf{x})^\perp F(\mathbf{x}) = e_j(\mathbf{x})^\perp G(\mathbf{x})$ for all $j \geq 1$.*

2.0.5 Hall-Littlewood symmetric functions

The algebra $\Lambda_{\mathbb{Q}(q)}$ of symmetric functions has a basis given by the *Hall-Littlewood symmetric functions* $P_\lambda(\mathbf{x}; q)$ which have the property that

$$s_\lambda(\mathbf{x}) = \sum_{\mu \vdash n} K_{\lambda, \mu}(q) P_\mu(\mathbf{x}; q), \quad (2.0.29)$$

where $K_{\lambda, \mu}(q)$ is the *Kostka-Foulkes polynomial*, see [56]. The *dual Hall-Littlewood symmetric functions* $Q'_\lambda(\mathbf{x}; q)$ are given by

$$Q'_\mu(\mathbf{x}; q) = \sum_{\lambda \vdash n} K_{\lambda, \mu}(q) s_\lambda(\mathbf{x}). \quad (2.0.30)$$

The degree of $Q'_\lambda(\mathbf{x}; q)$ is given by $\deg(Q'_\lambda(\mathbf{x}; q)) = n(\lambda)$, defined in (2.0.2). The reversal of these symmetric functions are sometimes denoted by $\tilde{H}_\lambda(\mathbf{x}; q) := \text{rev}_q(Q'_\lambda(\mathbf{x}; q)) = q^{n(\lambda)} Q'_\lambda(\mathbf{x}; 1/q)$.

Theorem 2.0.3 ([29, 80]). *The Frobenius characteristic of the S_n -module R_λ is the reversal of the dual Hall-Littlewood symmetric function,*

$$\text{Frob}_q(R_\lambda) = \tilde{H}_\lambda(\mathbf{x}; q) = \text{rev}_q(Q'_\lambda(\mathbf{x}; q)). \quad (2.0.31)$$

Next, we recall the $t = 0$ specialization of the Haglund-Haiman-Loehr formula for Macdonald polynomials, which gives an expansion for $\tilde{H}_\lambda(\mathbf{x}; q) = \text{rev}_q(Q'_\lambda(\mathbf{x}; q))$ in terms of inversions in labelings of the Young diagram of λ' . Given $\alpha \in \text{Comp}(n, s)$, a pair of cells

3		2		1		2		1	
4	1	4	1	4	2	3	1	3	2
5	2	5	3	5	3	5	4	5	4
1		2		1		1		1	
2	3	3	1	3	2	2	3	2	4
5	4	4	5	4	5	4	5	3	5

Figure 2.5: All standard column-increasing fillings in $\text{SCI}_{5,\lambda,2}$ where $\lambda = (3, 2)$.

set of column-increasing fillings of $\text{dg}'(\alpha)$, and let $\text{SCI}_{n,\alpha,s}$ be the subset of $\text{CI}_{n,\alpha,s}$ of standard column-increasing fillings. See Figure 2.5 for all fillings in $\text{SCI}_{5,\lambda,2}$ where $\lambda = (3, 2)$.

Define a *diagonal inversion* of σ to be an attacking pair $((i, j), (i', j'))$ of cells of $\text{dg}'(\alpha)$ with (i, j) appearing earlier in the reading order, such that $\sigma_{i,j} > \sigma_{i',j'}$. Denote by $\text{dinv}(\sigma)$ the number of diagonal inversions of σ . Letting σ be the filling in Figure 2.4, then the cells $(1, 3)$ and $(3, 3)$ form a diagonal inversion in σ , and the cells $(2, 3)$ and $(1, 2)$ also form a diagonal inversion in σ . The reader can check that $\text{dinv}(\sigma) = 6$.

The diagonal inversion statistic defined above appears in the following corollary of the beautiful fundamental quasisymmetric function expansion for Macdonald symmetric functions proven by Haglund, Haiman, and Loehr [41]. Setting $t = 0$ in [41, Equation 36], we have the following expansion for the reversal of the dual Hall-Littlewood function in our notation.

Corollary 2.0.4. *For $\lambda \vdash n$, we have*

$$\text{Frob}_q(R_\lambda) = \text{rev}_q(Q'_\lambda(\mathbf{x}; q)) = \sum_{\sigma \in \text{SCI}_{n,\lambda,\ell(\lambda)}} q^{\text{dinv}(\sigma)} F_{n, \text{iDes}(\text{rw}(\sigma))}(\mathbf{x}). \quad (2.0.32)$$

A family of symmetric functions generalizing the dual Hall-Littlewood functions are the *LLT polynomials* introduced by Lascoux, Leclerc, and Thibon [55]. We use here a variant introduced in [42]. Let $\lambda \supseteq \mu$, and let $\nu = \lambda/\mu$ be the skew diagram. Let $c = (i, j)$ be the

cell of ν in column i and row j . The *content* of u is $c(u) := j - i$.

Given a tuple of skew diagrams $\boldsymbol{\nu} = (\nu^{(1)}, \dots, \nu^{(m)})$, let $\text{SSYT}(\boldsymbol{\nu}) := \text{SSYT}(\nu^{(1)}) \times \text{SSYT}(\nu^{(2)}) \times \dots \times \text{SSYT}(\nu^{(m)})$. Given $\mathbf{T} = (T^{(1)}, \dots, T^{(m)}) \in \text{SSYT}(\boldsymbol{\nu})$, let $\mathbf{x}^{\mathbf{T}} = \mathbf{x}^{T^{(1)}} \dots \mathbf{x}^{T^{(m)}}$. Given $u = (i, j)$ a cell in $T^{(a)}$ and $v = (i', j')$ a cell of $T^{(b)}$, we say (u, v) form an *inversion* if we have the inequality of entries $T_u^{(a)} > T_v^{(b)}$, and either $a < b$ and $c(u) = c(v)$, or $a > b$ and $c(u) = c(v) + 1$. Denote by $\text{inv}(\mathbf{T})$ the number of inversions of \mathbf{T} . The *LLT polynomial indexed by $\boldsymbol{\nu}$* is

$$G_{\boldsymbol{\nu}}(\mathbf{x}; q) := \sum_{\mathbf{T} \in \text{SSYT}(\boldsymbol{\nu})} q^{\text{inv}(\mathbf{T})} \mathbf{x}^{\mathbf{T}}. \quad (2.0.33)$$

We need the following theorem, which has several algebraic and combinatorial proofs.

Theorem 2.0.5 ([41, 42, 55]). *The LLT polynomial $G_{\boldsymbol{\nu}}(\mathbf{x}; q)$ is symmetric in the \mathbf{x} variables.*

Chapter 3

GARSIA-PROCESI MODULES AND GENERALIZED COINVARIANT RINGS

The goal of this chapter is to unify the representation theory and combinatorics of the *generalized coinvariant algebras* $R_{n,k}$ introduced by Haglund, Rhoades, and Shimozono [45], and the singular cohomology rings R_λ of the *Springer fibers* introduced by T. A. Springer [80, 81]. On the one hand, the generalized coinvariant algebras are graded modules of the symmetric group whose combinatorics are controlled by ordered set partitions. On the other hand, the cohomology rings of Springer fibers are graded modules of the symmetric group whose combinatorics are controlled by tabloids. We introduce a family of rings $R_{n,\lambda,s}$ which are graded modules of the symmetric group whose combinatorics are controlled by (n, λ, s) -ordered set partitions. We recover the rings $R_{n,k}$ and R_λ as special cases of our rings. Furthermore, we show that the rings $R_{n,\lambda,s}$ have connections to the geometry of rank varieties defined by Eisenbud and Saltman [23]. These rank varieties are not to be confused with the rank varieties of Billey and Coskun [9]. In particular, we obtain a formula for the Hilbert series and graded Frobenius characteristic of $\mathbb{Q}[\overline{\mathcal{O}}_{n,\lambda} \cap \mathfrak{t}]$, the coordinate ring of the scheme-theoretic intersection of a rank variety with diagonal matrices. The contents of this chapter are based on work in [38, 39].

Let us recall the generalized coinvariant algebras $R_{n,k}$. Fix positive integers $k \leq n$, and let $\mathbf{x}_n = \{x_1, \dots, x_n\}$ be a set of n commuting variables. Let $\mathbb{Q}[\mathbf{x}_n]$ be the polynomial ring on the variables \mathbf{x}_n with rational coefficients, and let S_n be the symmetric group of permutations of $1, 2, \dots, n$. We consider $\mathbb{Q}[\mathbf{x}_n]$ as an S_n -module, where S_n acts by permuting the variables. For $1 \leq d \leq n$, let $e_d(\mathbf{x}_n)$ be the *elementary symmetric polynomial of degree d* in the variables

\mathbf{x}_n , defined by $e_d(\mathbf{x}_n) = \sum_{1 \leq i_1 < \dots < i_d \leq n} x_{i_1} x_{i_2} \cdots x_{i_d}$. The ideal $I_{n,k}$ is defined to be

$$I_{n,k} := \langle x_1^k, x_2^k, \dots, x_n^k, e_n(\mathbf{x}_n), e_{n-1}(\mathbf{x}_n), \dots, e_{n-k+1}(\mathbf{x}_n) \rangle \subseteq \mathbb{Q}[\mathbf{x}_n]. \quad (3.0.1)$$

Haglund, Rhoades and Shimozono defined the *generalized coinvariant algebra* $R_{n,k}$ to be the quotient ring $R_{n,k} := \mathbb{Q}[\mathbf{x}_n]/I_{n,k}$. Since $I_{n,k}$ is homogeneous and stable under the action of S_n , the quotient ring $R_{n,k}$ has the structure of a graded S_n -module. When $k = n$, then it can be shown that (see [45, Section 1])

$$I_{n,n} = \langle e_1(\mathbf{x}_n), \dots, e_n(\mathbf{x}_n) \rangle = \langle \mathbb{Q}[\mathbf{x}_n]_+^{S_n} \rangle, \quad (3.0.2)$$

which is the ideal generated by the homogeneous positive degree invariants of $\mathbb{Q}[\mathbf{x}_n]$. Hence, $R_{n,n}$ is the well-known *coinvariant algebra*.

We also recall some standard terminology in order to state our main results. A *weak ordered set partition of $[n]$* is a partitioning of the set $[n]$ into an ordered list of subsets B_1, \dots, B_k , where we allow B_i to be empty in general. We denote such an ordered set partition by $(B_1|B_2|\cdots|B_k)$. Let $\mathcal{OP}_{n,k}$ be the collection of ordered set partitions of $[n]$ into k nonempty blocks. The size of $\mathcal{OP}_{n,k}$ is easy to compute in terms of Stirling numbers of the second kind,

$$|\mathcal{OP}_{n,k}| = k! \cdot \text{Stir}(n, k). \quad (3.0.3)$$

The group S_n acts on $\mathcal{OP}_{n,k}$ by permuting the letters $1, 2, \dots, n$. Define the usual q -analogues of numbers, factorials, and multinomial coefficients,

$$[n]_q := 1 + q + \cdots + q^{n-1}, \quad [n]!_q := [n]_q [n-1]_q \cdots [1]_q, \quad (3.0.4)$$

$$\begin{bmatrix} n \\ a_1, \dots, a_r \end{bmatrix}_q := \frac{[n]!_q}{[a_1]!_q \cdots [a_r]!_q}, \quad \begin{bmatrix} n \\ a \end{bmatrix}_q := \frac{[n]!_q}{[a]!_q [n-a]!_q}. \quad (3.0.5)$$

Let $\mathbb{Q}[[\mathbf{x}]]$ be the formal power series ring over the rational numbers in the variables \mathbf{x} .

Given $f \in \mathbb{Q}[[\mathbf{x}]]\langle q \rangle$, let $f = a_0 + a_1q + \cdots + a_nq^n$ be its expansion as a polynomial in q with coefficients in $\mathbb{Q}[[\mathbf{x}]]$. Define $\text{rev}_q(f) = a_n + a_{n-1}q + \cdots + a_0q^n$.

Given two sequences of nonnegative integers (a_1, \dots, a_r) and (b_1, \dots, b_s) , a *shuffle* of these two sequences is an interleaving (c_1, \dots, c_{r+s}) of the two sequences such that the a_i appear in order from left to right and the b_i appear in order from left to right. An (n, k) -*staircase* is a shuffle of the sequence $(0, 1, \dots, k-1)$ and the sequence $((k-1)^{n-k})$ consisting of $k-1$ repeated $n-k$ many times [45].

Haglund, Rhoades, and Shimozono proved that $R_{n,k}$ has the following properties which generalize the well-known properties of the coinvariant algebra [45].

- The dimension of $R_{n,k}$ is given by $\dim_{\mathbb{Q}}(R_{n,k}) = |\mathcal{OP}_{n,k}| = k! \cdot \text{Stir}(n, k)$. The Hilbert series is

$$\text{Hilb}_q(R_{n,k}) = \text{rev}_q([k]!_q \cdot \text{Stir}_q(n, k)) = \sum_{\sigma \in \mathcal{OP}_{n,k}} q^{\text{coinv}(\sigma)}, \quad (3.0.6)$$

where $\text{Stir}_q(n, k)$ is a well-known q -analogue of the Stirling number of the second kind, and where coinv is the coinversion statistic on ordered set partitions, respectively. See [97] for more details on ordered set partition statistics.

- The set of monomials

$$\mathcal{A}_{n,k} = \{x_1^{a_1} \cdots x_n^{a_n} : (a_1, \dots, a_n) \text{ is component-wise } \leq \text{some } (n, k)\text{-staircase}\} \quad (3.0.7)$$

represents a basis of $R_{n,k}$, generalizing the Artin basis of the coinvariant algebra. As a consequence, we have $|\mathcal{A}_{n,k}| = |\mathcal{OP}_{n,k}|$.

- As S_n -modules,

$$R_{n,k} \cong_{S_n} \mathbb{Q}\mathcal{OP}_{n,k}, \quad (3.0.8)$$

where $\mathbb{Q}\mathcal{OP}_{n,k}$ is the vector space over \mathbb{Q} whose basis is indexed by $\mathcal{OP}_{n,k}$ and whose S_n -module structure is induced from the natural action of S_n on $\mathcal{OP}_{n,k}$ permuting the letters $1, \dots, n$.

- The graded S_n -module structure of $R_{n,k}$ can be expressed in terms of the dual Hall-Littlewood functions $Q'_\mu(\mathbf{x}; q)$ as follows,

$$\text{Frob}_q(R_{n,k}) = \text{rev}_q \left[\sum_{\mu} q^{\sum_{i=1}^k (i-1)(\mu_i-1)} \left[\begin{matrix} k \\ m_1(\mu), \dots, m_n(\mu) \end{matrix} \right]_q Q'_\mu(\mathbf{x}; q) \right], \quad (3.0.9)$$

where the sum is over partitions μ of n into k parts.

- The S_n -module $R_{n,k}$ is related to the *Delta Conjecture* of Haglund, Remmel, and Wilson [44]. Precisely, Haglund, Rhoades, and Shimozono [45] proved that

$$\text{Frob}_q(R_{n,k}) = (\text{rev}_q \circ \omega) C_{n,k}(\mathbf{x}; q), \quad (3.0.10)$$

where $C_{n,k}(\mathbf{x}; q)$ is the expression in the Delta Conjecture at $t = 0$.

- More recently, Pawlowski and Rhoades [65] proved that $R_{n,k}$ is isomorphic to the rational cohomology ring for the space of spanning line arrangements $X_{n,k}$. Their result also holds with integral coefficients.

Next, we describe certain quotient rings coming from the geometry of *Springer fibers*. Let $\lambda \vdash n$, and let the conjugate partition be $\lambda' = (\lambda'_1 \geq \lambda'_2 \geq \dots \geq \lambda'_n \geq 0)$. Here, we are padding the conjugate partition by 0s to make it length n . Let $p_m^n(\lambda) := \lambda'_n + \lambda'_{n-1} + \dots + \lambda'_{n-m+1}$ for $1 \leq m \leq n$. Given a subset of variables $S \subseteq \mathbf{x}_n$ and a positive integer d , define $e_d(S)$ to be the sum over all squarefree monomials of degree d in the set of variables S . For example, we have $e_2(\{x_1, x_3, x_5\}) = x_1x_3 + x_1x_5 + x_3x_5$. The *Tanisaki ideal* I_λ is defined by

$$I_\lambda := \langle e_d(S) : S \subseteq \mathbf{x}_n, d > |S| - p_{|S|}^n(\lambda) \rangle, \quad (3.0.11)$$

and the ring R_λ is defined by

$$R_\lambda := \mathbb{Q}[\mathbf{x}_n]/I_\lambda. \quad (3.0.12)$$

By work of De Concini and Procesi [17], the ring R_λ is isomorphic to the singular cohomology ring with rational coefficients of the *Springer fiber* corresponding to a nilpotent matrix with Jordan type λ . This particular presentation for the cohomology ring in terms of partial elementary symmetric polynomials is due to Tanisaki [90]. When $\lambda = (1^n)$, the Springer fiber corresponding to (1^n) is the *complete flag variety* whose cohomology ring is the coinvariant algebra. Indeed, we have $R_{(1^n)} = \mathbb{Q}[\mathbf{x}_n]/\langle e_1(\mathbf{x}_n), \dots, e_n(\mathbf{x}_n) \rangle$, which is the coinvariant algebra. See [13] for more background on Springer fibers and [92] for more details on their combinatorial properties.

We refer to the graded S_n -module R_λ as the *Garsia-Procesi module* based on their seminal work in [29] on the S_n -module structure of R_λ . The ring R_λ has the following properties.

- The dimension of R_λ is the multinomial coefficient

$$\dim_{\mathbb{Q}}(R_\lambda) = \binom{n}{\lambda_1, \dots, \lambda_\ell}, \quad (3.0.13)$$

where $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell > 0)$. The Hilbert series of R_λ is given by the generating function for the *cocharge* statistic on a certain set of words, see [29, Remark 1.2]. Alternatively, we have the following characterization of the Hilbert series which follows from work of Haglund, Haiman, and Loehr [41],

$$\text{Hilb}_q(R_\lambda) = \sum_{\sigma} q^{\text{inv}(\sigma)}, \quad (3.0.14)$$

where the sum is over standard fillings of the Young diagram of λ' which increase down each column, and inv is the number of attacking pairs which form an inversion of σ .

- There is a monomial basis \mathcal{A}_λ of R_λ which specializes to the Artin basis of the coin-

variant algebra when $\lambda = (1^n)$. In [29], this basis is denoted by $\mathcal{B}(\lambda)$.

- As S_n -modules, we have

$$R_\lambda \cong_{S_n} \mathbb{Q}(S_n/S_{\lambda_1} \times \cdots \times S_{\lambda_k}), \quad (3.0.15)$$

where $S_{\lambda_1} \times \cdots \times S_{\lambda_k}$ is the Young subgroup of S_n permuting $1, \dots, \lambda_1$ among themselves, $\lambda_1 + 1, \dots, \lambda_1 + \lambda_2$ among themselves, and so on. Equivalently, R_λ is isomorphic to the S_n -module given by the action of S_n on tabloids of shape λ .

- Springer [80, 81] proved that the top degree component of R_λ is isomorphic to the irreducible representation S^λ . Thus, the cohomology ring of a Springer fiber R_λ gives a geometric construction of the irreducible representation S^λ of S_n .
- The graded S_n -module structure of R_λ is given by the reversal of the dual Hall-Littlewood function,

$$\text{Frob}_q(R_\lambda) = \text{rev}_q(Q'_\lambda(\mathbf{x}; q)). \quad (3.0.16)$$

- If $\lambda, \mu \vdash n$ such that $\lambda \leq_{\text{dom}} \mu$, we have the monotonicity property

$$[s_\nu] \text{Frob}_q(R_\lambda) \geq [s_\nu] \text{Frob}_q(R_\mu), \quad (3.0.17)$$

for all $\nu \vdash n$, where $[s_\nu]f$ stands for the coefficient of s_ν in the Schur function expansion of f , and the inequality is a coefficient-wise comparison of two polynomials in q .

Fix positive integers $k \leq n$, a partition λ of k , and an integer $s \geq \ell(\lambda)$, where $\ell(\lambda)$ is the length of λ . Let the conjugate of λ be $\lambda' = (\lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_n \geq 0)$, where we pad the conjugate partition by 0s to make it length n , and define $p_m^n(\lambda) := \lambda'_n + \lambda'_{n-1} + \cdots + \lambda'_{n-m+1}$ for $1 \leq m \leq n$. We introduce the ring $R_{n,\lambda,s}$, defined as follows.

Definition 3.0.1. Define the ideal $I_{n,\lambda,s}$ and quotient ring $R_{n,\lambda,s}$ by

$$I_{n,\lambda,s} := \langle x_i^s : 1 \leq i \leq n \rangle + \langle e_d(S) : S \subseteq \mathbf{x}_n, d > |S| - p_{|S|}^n(\lambda) \rangle, \quad (3.0.18)$$

$$R_{n,\lambda,s} := \mathbb{Q}[\mathbf{x}_n]/I_{n,\lambda,s}. \quad (3.0.19)$$

Since the ideal $I_{n,\lambda,s}$ is generated by homogeneous polynomials, it is a homogeneous ideal. Furthermore, since the generating set is closed under the action of S_n , the ideal $I_{n,\lambda,s}$ is closed under the action of S_n . Therefore, the quotient ring $R_{n,\lambda,s}$ inherits the structure of a graded S_n -module. For example, the ideal $I_{6,(3,2),3}$ is generated by the set of homogeneous polynomials

$$\{x_1^3, \dots, x_6^3\} \cup \{e_2(\mathbf{x}_6), e_3(\mathbf{x}_6), e_4(\mathbf{x}_6), e_5(\mathbf{x}_6), e_6(\mathbf{x}_6)\} \cup \{e_3(S) \mid S \subseteq \mathbf{x}_6, |S| = 5\} \quad (3.0.20)$$

$$\cup \{e_4(S) \mid S \subseteq \mathbf{x}_6, |S| = 5\} \cup \{e_5(S) \mid S \subseteq \mathbf{x}_6, |S| = 5\} \cup \{e_4(S) \mid S \subseteq \mathbf{x}_6, |S| = 4\}, \quad (3.0.21)$$

which is closed under the action of S_6 .

The generalized coinvariant algebras $R_{n,k}$ and the rings R_λ are special cases of the rings $R_{n,\lambda,s}$. We have

$$R_{n,k} = R_{n,(1^k),k} \quad \text{for } k \leq n, \quad (3.0.22)$$

$$R_\lambda = R_{n,\lambda,\ell(\lambda)} \quad \text{for } \lambda \vdash n, \quad (3.0.23)$$

where (3.0.22) follows from Definition 3.0.1. See Remark 3.1.4 for the justification of (3.0.23). As a bonus, we also have $R_{n,k,s} = R_{n,(1^s),k}$, where $R_{n,k,s}$ is the ring defined in [45, Section 6].

Let a (n, λ, s) -ordered set partition be a weak ordered set partition $(B_1 | \dots | B_s)$ of $[n]$ into s blocks such that $|B_i| \geq \lambda_i$ for $i \leq \ell(\lambda)$. Here, we allow B_i to be empty for $\ell(\lambda) < i \leq s$. Let $\mathcal{OP}_{n,\lambda,s}$ be the set of (n, λ, s) -ordered set partitions. The group S_n acts on $\mathcal{OP}_{n,\lambda,s}$ by permuting the letters $1, 2, \dots, n$.

We prove the following properties of $R_{n,\lambda,s}$, generalizing many of the properties of $R_{n,k}$

and R_λ . All terminology not defined here is defined in the section corresponding to each theorem.

- The dimension of $R_{n,\lambda,s}$ is given by $\dim_{\mathbb{Q}}(R_{n,\lambda,s}) = |\mathcal{OP}_{n,\lambda,s}|$ (see Theorem 3.1.20). We give a formula for the Hilbert polynomial (see Corollary 3.3.14),

$$\text{Hilb}_q(R_{n,\lambda,s}) = \sum_{\varphi \in \text{SECI}_{n,\lambda,s}} q^{\text{dinv}(\varphi)}. \quad (3.0.24)$$

The indexing set $\text{SECI}_{n,\lambda,s}$ in the sum on the right-hand side of (3.0.24) is a set of *standard extended column-increasing fillings*, which is in bijection with $\mathcal{OP}_{n,\lambda,s}$. See Subsection 3.3.1 for the definition of a standard extended column-increasing filling.

- An alternative formula for the Hilbert series was obtained by Rhoades, Yu, and Zhao on ordered set partitions [71, Corollary 4.9] involving a statistic they call *coinv*. They also characterize the harmonic spaces corresponding to the ideals $I_{n,\lambda,s}$. We define a statistic *inv* on $\text{SECI}_{n,\lambda,s}$ which is similar to, but not the same as, their *coinv* statistic. We prove that the *inv* statistic also gives a formula for the Hilbert series (see Corollary 3.3.14),

$$\text{Hilb}_q(R_{n,\lambda,s}) = \sum_{\varphi \in \text{SECI}_{n,\lambda,s}} q^{\text{inv}(\varphi)}. \quad (3.0.25)$$

The formula (3.0.25) follows naturally as a corollary of our graded Frobenius characteristic formula. Since [71] uses our work to prove their results, we have been careful to give independent proofs.

- In Section 3.1, we define an (n, λ, s) -*staircase* as a shuffle of a certain set of compositions depending on n , λ , and s . We have that

$$\mathcal{A}_{n,\lambda,s} = \{x_1^{a_1} \cdots x_n^{a_n} : (a_1, \dots, a_n) \text{ is component-wise } \leq \text{ some } (n, \lambda, s)\text{-staircase}\} \quad (3.0.26)$$

represents a monomial basis of $R_{n,\lambda,s}$ (see Theorem 3.1.17).

- We have the following chain of equalities and S_n -module isomorphisms (see Theorem 3.1.1, Corollary 3.1.2, and Theorem 3.1.20)

$$R_{n,\lambda,s} = \frac{\mathbb{Q}[\mathbf{x}_n]}{\text{gr } I(X_{n,\lambda,s})} \cong_{S_n} \mathbb{Q}X_{n,\lambda,s} \cong_{S_n} \mathbb{Q}\mathcal{OP}_{n,\lambda,s}, \quad (3.0.27)$$

where $X_{n,\lambda,s}$ is a finite set of points in \mathbb{Q}^n which is stable under the S_n -action permuting coordinates.

- In the case when $s > \ell(\lambda)$, we prove that the top degree component of $R_{n,\lambda,s}$ is isomorphic to the induction of the irreducible S_k -module S^λ up to an S_n module (see Corollary 3.3.15),

$$\text{Ind}_{S_k}^{\uparrow S_n} S^\lambda. \quad (3.0.28)$$

- The graded Frobenius characteristic can be expressed as a sum of monomials (see Theorem 3.3.13), or equivalently in terms of Gessel's fundamental quasisymmetric functions (see Corollary 3.3.11),

$$\text{Frob}_q(R_{n,\lambda,s}) = \sum_{\varphi \in \text{ECI}_{n,\lambda,s}} q^{\text{inv}(\varphi)} \mathbf{x}^\varphi = \sum_{\varphi \in \text{SECI}_{n,\lambda,s}} q^{\text{inv}(\varphi)} F_{n, \text{iDes}(\text{rw}(\varphi))}(\mathbf{x}), \quad (3.0.29)$$

$$= \sum_{\varphi \in \text{ECI}_{n,\lambda,s}} q^{\text{dinv}(\varphi)} \mathbf{x}^\varphi = \sum_{\varphi \in \text{SECI}_{n,\lambda,s}} q^{\text{dinv}(\varphi)} F_{n, \text{iDes}(\text{rw}(\varphi))}(\mathbf{x}), \quad (3.0.30)$$

where \mathbf{x}^φ is the monomial whose powers record the number of occurrences of each label in φ .

- In forthcoming work, we prove $\text{Frob}_q(R_{n,\lambda,s})$ can be expressed in terms of the dual

Hall-Littlewood functions,

$$\text{Frob}_q(R_{n,\lambda,s}) = \text{rev}_q \left[\sum_{\substack{\mu \in \text{Par}(n,s), \\ \mu \supseteq \lambda}} q^{n(\mu,\lambda)} \prod_{i \geq 0} \left[\begin{matrix} \mu'_i - \lambda'_{i+1} \\ \mu'_i - \mu'_{i+1} \end{matrix} \right]_q Q'_\mu(\mathbf{x}; q) \right], \quad (3.0.31)$$

where $\mu'_0 := s$, and $n(\mu, \lambda) := \sum_{i \geq 1} \binom{\mu'_i - \lambda'_i}{2}$. The proof of this formula has been moved to a forthcoming article, since the techniques for proving it are different than the techniques used to prove (3.0.29).

- Let $h \leq k \leq n$ be positive integers, let $\lambda \in \text{Par}(h, s)$, and let $\mu \in \text{Par}(k, s)$ such that either $h = k$ and $\lambda \leq_{\text{dom}} \mu$ or $h < k$ and $\lambda \subseteq \mu$. We have the monotonicity property (see Theorem 3.1.19)

$$[s_\nu] \text{Frob}_q(R_{n,\lambda,s}) \geq [s_\nu] \text{Frob}_q(R_{n,\mu,s}), \quad (3.0.32)$$

for all $\nu \vdash n$.

- The definition of the ring $R_{n,\lambda,s}$ was originally motivated by a geometry question posed to the author by Alexander Woo involving an analogue of Springer fibers in the setting of spanning line arrangements. The elementary symmetric polynomials in $I_{n,\lambda,s}$ represent relations discovered by the author involving Chern classes in the cohomology ring of this variety. There were many technical difficulties in trying to work out the cohomology ring of the varieties posed by Woo, and this work was never completed. However, study of the ring $R_{n,\lambda,s}$ has led the author to introduce a new family of varieties $Y_{n,\lambda,s}$ that directly generalizes the Springer fibers. In forthcoming joint work with Levinson and Woo, we prove that

$$H^*(Y_{n,\lambda,s}; \mathbb{Q}) \cong R_{n,\lambda,s}. \quad (3.0.33)$$

By Corollary 3.3.15 mentioned above, in the case $s > \ell(\lambda)$ have the following extension of Springer's theorem,

$$H^{\text{top}}(Y_{n,\lambda,s}; \mathbb{Q}) \cong \text{Ind} \uparrow_{S_k}^{S_n} S^\lambda. \quad (3.0.34)$$

Unlike the spanning line arrangement varieties of Pawlowski and Rhoades, the varieties $Y_{n,\lambda,s}$ are compact and singular. Hence, the variety $Y_{n,(1^k),k}$ provides a compact and singular variety whose cohomology ring gives a geometric representation for the $t = 0$ case of the Delta Conjecture.

One of our main tools for proving these results is Theorem 3.1.1 which identifies the ring $R_{n,\lambda,s}$ as the associated graded ring of the coordinate ring of a finite set of points in \mathbb{Q}^n . This identification extends results of Garsia and Procesi [29, Proposition 3.1, Remark 3.1] and Haglund, Rhoades, and Shimozono [45, Equation 4.28]. See also the work of Kraft [54, Proof of Proposition 4] from 1981 for a proof in the case of R_λ using associated cones.

Haglund, Rhoades, and Shimozono use Gröbner bases to prove their results. In particular, they find Gröbner bases of the ideals $I_{n,k}$ in terms of *Demazure characters*. To the author's knowledge, such explicit Gröbner bases for the ideals I_λ are not known. Therefore, different techniques are required to prove our results. Indeed, we prove the above results without the use of Gröbner bases using techniques similar to those of Garsia and Procesi. In particular, we use a straightening algorithm which expresses any element of $R_{n,\lambda,s}$ in terms of our monomial basis $\mathcal{A}_{n,\lambda,s}$. It is an open problem to find explicit Gröbner bases for the ideals $I_{n,\lambda,s}$.

Let $\lambda \vdash n$, and let O_λ be the conjugacy class of nilpotent $n \times n$ matrices over \mathbb{Q} whose Jordan blocks are of sizes recorded by λ . Let \overline{O}_λ be its closure in the space of $n \times n$ matrices. Let \mathfrak{t} be the set of diagonal matrices. De Concini and Procesi [17], extending work of Kostant [53] on the coinvariant algebras, proved that R_λ is isomorphic to the coordinate ring of the scheme-theoretic intersection $\overline{O}_\lambda \cap \mathfrak{t}$.

We connect the rings $R_{n,\lambda,s}$ to a generalization of these scheme-theoretic intersections as

follows. Given $k \leq n$ and $\lambda \vdash k$, define $I_{n,\lambda}$ to be the ideal

$$I_{n,\lambda} := \langle e_d(S) : S \subseteq \mathbf{x}_n, d > |S| - p_{|S|}^n(\lambda) \rangle. \quad (3.0.35)$$

Define the quotient ring $R_{n,\lambda} := \mathbb{Q}[\mathbf{x}_n]/I_{n,\lambda}$. When $k < n$, the ring $R_{n,\lambda}$ has positive Krull dimension, and hence it is infinite-dimensional as a \mathbb{Q} -vector space. Observe that for fixed n , λ , and d , the d th degree components of $R_{n,\lambda,s}$ stabilize to the d th degree component of $R_{n,\lambda}$ as $s \rightarrow \infty$.

In Section 3.4, using work of Weyman [96] we prove that $R_{n,\lambda}$ is isomorphic to the coordinate ring of the scheme-theoretic intersection $\overline{O}_{n,\lambda'} \cap \mathfrak{t}$, where $\overline{O}_{n,\lambda}$ is the rank variety of Eisenbud and Saltman [23] (see Corollary 3.4.4),

$$R_{n,\lambda} \cong \mathbb{Q}[\overline{O}_{n,\lambda'} \cap \mathfrak{t}]. \quad (3.0.36)$$

We use our results on the finite-dimensional rings $R_{n,\lambda,s}$ to find monomial bases of these coordinate rings by allowing s to approach infinity in our combinatorial formulas. We also prove the following formula for the graded Frobenius characteristic of these coordinate rings (see Theorem 3.4.6 and Corollary 3.4.7),

$$\text{Frob}_q(\mathbb{Q}[\overline{O}_{n,\lambda'} \cap \mathfrak{t}]) = \sum_{\varphi \in \text{ECI}_{n,\lambda}} q^{\text{inv}(\varphi)} \mathbf{x}^\varphi = \sum_{\varphi \in \text{SECI}_{n,\lambda}} q^{\text{inv}(\varphi)} F_{n, \text{iDes}(\text{rw}(\varphi))}(\mathbf{x}). \quad (3.0.37)$$

In [14, Question 5.3.1], Church, Ellenberg, and Farb ask for the dimensions of the graded pieces of the ring of polynomial functions on a certain scheme supported on a rank variety. We solve a related question by giving a formula for the Hilbert series of $\mathbb{Q}[\overline{O}_{n,\lambda} \cap \mathfrak{t}]$ (see Corollary 3.4.8),

$$\text{Hilb}_q(\mathbb{Q}[\overline{O}_{n,\lambda} \cap \mathfrak{t}]) = \sum_{\varphi \in \text{SECI}_{n,\lambda}} q^{\text{inv}(\varphi)}. \quad (3.0.38)$$

The rest of the chapter is structured as follows. In Section 3.1, we construct a monomial

basis of $R_{n,\lambda,s}$ and prove that $R_{n,\lambda,s}$ is isomorphic to $\mathbb{Q}\mathcal{OP}_{n,\lambda,s}$ as an S_n -module. In Section 3.2, we give algebraic tools for analyzing the graded Frobenius characteristic of $R_{n,\lambda,s}$, including a skewing formula. We also prove that the rings $R_{n,\lambda,s}$ fit into certain exact sequences. In Section 3.3, we define the inversion and diagonal inversion statistics and use them to provide formulas for the graded Frobenius characteristic of $R_{n,\lambda,s}$. In Section 3.4, we relate the rings $R_{n,\lambda}$ to the geometry of the rank varieties of Eisenbud and Saltman. We then prove our formula for the graded Frobenius characteristic of the scheme-theoretic intersection of a rank variety with diagonal matrices.

3.1 The Frobenius characteristic of $R_{n,\lambda,s}$

In this section, we identify $R_{n,\lambda,s}$ as a symmetric group module. Our main strategy, used by Garsia-Procesi [29] and formalized by Haglund-Rhoades-Shimozono [45, Section 4.1], is to show that $R_{n,\lambda,s}$ is the associated graded ring of the coordinate ring of a finite set of points in \mathbb{Q}^n . We then prove Theorem 3.1.17, which identifies a monomial basis of $R_{n,\lambda,s}$.

3.1.1 Associated graded rings and point orbits

Throughout this section, we fix positive integers $k \leq n$ and s , and a partition $\lambda \in \text{Par}(k, s)$. Fix s distinct rational numbers $\alpha_1, \dots, \alpha_s \in \mathbb{Q}$. Let $X_{n,\lambda,s}$ be the set of points $p = (p_1, \dots, p_n) \in \mathbb{Q}^n$ such that for each $1 \leq i \leq n$, $p_i = \alpha_j$ for some j , and for each $1 \leq i \leq s$, α_i appears as a coordinate in p at least λ_i many times. The *defining ideal* of $X_{n,\lambda,s}$ is

$$I(X_{n,\lambda,s}) := \{f \in \mathbb{Q}[\mathbf{x}_n] : f(p) = 0 \text{ for all } p \in X_{n,\lambda,s}\} \subseteq \mathbb{Q}[\mathbf{x}_n]. \quad (3.1.1)$$

The quotient ring $\mathbb{Q}[\mathbf{x}_n]/I(X_{n,\lambda,s})$ is the *coordinate ring* of the set $X_{n,\lambda,s}$. It is isomorphic to the ring of polynomial functions $X_{n,\lambda,s} \rightarrow \mathbb{Q}$. See [16] for more background on the defining ideal and the coordinate ring of a variety.

For a degree d polynomial $f = f_d + f_{d-1} + \dots + f_0 \in \mathbb{Q}[\mathbf{x}_n]$ where f_i is the degree i homogenous summand of f , define $\tau(f) = f_d$ to be the top homogenous component of f .

For example, if $f = x_1^2x_2 + 2x_1^2x_3 + x_2x_3 + x_1 + 3$, then $\tau(f) = x_1^2x_2 + 2x_1^2x_3$.

The *associated graded ideal* of $I(X_{n,\lambda,s})$ with respect to the filtration by degree is

$$\text{gr } I(X_{n,\lambda,s}) = \langle \tau(f) : f \in I(X_{n,\lambda,s}) \rangle, \quad (3.1.2)$$

which is a homogenous ideal since each of the generators is homogeneous. See [22] for more details.

It is well known that the corresponding quotient ring $\mathbb{Q}[\mathbf{x}_n]/\text{gr } I(X_{n,\lambda,s})$ is isomorphic to the associated graded ring of $\mathbb{Q}[\mathbf{x}_n]/I(X_{n,\lambda,s})$ with respect to the filtration by degree. This is true more generally for the associated graded ideal of any ideal in $\mathbb{Q}[\mathbf{x}_n]$. See, e.g. [29, Remark 3.1], for a proof of this fact in the case of the ideal $I(X_{n,\lambda,s})$ when $k = n$. However, the proof easily extends to any ideal.

Since $X_{n,\lambda,s}$ is a finite set, we have

$$|X_{n,\lambda,s}| = \dim_{\mathbb{Q}} \frac{\mathbb{Q}[\mathbf{x}_n]}{I(X_{n,\lambda,s})} = \dim_{\mathbb{Q}} \frac{\mathbb{Q}[\mathbf{x}_n]}{\text{gr } I(X_{n,\lambda,s})}, \quad (3.1.3)$$

where all dimensions are as \mathbb{Q} -vector spaces. Since $X_{n,\lambda,s}$ is stable under the action of S_n given by permuting coordinates, we have an S_n -action on the rings $\mathbb{Q}[\mathbf{x}_n]/I(X_{n,\lambda,s})$ and $\mathbb{Q}[\mathbf{x}_n]/\text{gr } I(X_{n,\lambda,s})$ given by permuting the variables \mathbf{x}_n . As S_n -modules,

$$\mathbb{Q}X_{n,\lambda,s} \cong_{S_n} \frac{\mathbb{Q}[\mathbf{x}_n]}{I(X_{n,\lambda,s})} \cong_{S_n} \frac{\mathbb{Q}[\mathbf{x}_n]}{\text{gr } I(X_{n,\lambda,s})}, \quad (3.1.4)$$

where $\mathbb{Q}X_{n,\lambda,s}$ is the S_n -module of formal \mathbb{Q} -linear combinations of points in $X_{n,\lambda,s}$. See [45, Section 4.1] for more details.

Theorem 3.1.1. *We have $I_{n,\lambda,s} = \text{gr } I(X_{n,\lambda,s})$. Hence, we have the equality of rings*

$$R_{n,\lambda,s} = \frac{\mathbb{Q}[\mathbf{x}_n]}{\text{gr } I(X_{n,\lambda,s})}. \quad (3.1.5)$$

Corollary 3.1.2. *As S_n -modules, $R_{n,\lambda,s} \cong_{S_n} \mathbb{Q}X_{n,\lambda,s}$.*

Proof. Combine Theorem 3.1.1 with (3.1.4). \square

We prove Theorem 3.1.1 in three parts. Recall the definition of $I_{n,\lambda,s}$ in Definition 3.0.1. First, we show that all of the generators of $I_{n,\lambda,s}$ are in $\text{gr } I(X_{n,\lambda,s})$ in Lemma 3.1.3 by adapting the proof of Garsia-Procesi [29, Proposition 3.1]. Second, we find a monomial spanning set of the quotient $R_{n,\lambda,s} = \mathbb{Q}[\mathbf{x}_n]/I_{n,\lambda,s}$ of size $|X_{n,\lambda,s}|$. Finally, we finish the proof of Theorem 3.1.1 using a dimension counting argument. As a consequence, we see that our monomial spanning set is a monomial basis of $R_{n,\lambda,s}$.

Lemma 3.1.3. *We have the containment of ideals*

$$I_{n,\lambda,s} \subseteq \text{gr } I(X_{n,\lambda,s}).$$

Proof. First we show that $x_i^s \in \text{gr } I(X_{n,\lambda,s})$ for all i . For any $p \in X_{n,\lambda,s}$, the coordinates of p are in the set $\{\alpha_1, \dots, \alpha_s\}$. Therefore, for each i , the polynomial function

$$(x_i - \alpha_1)(x_i - \alpha_2) \cdots (x_i - \alpha_s) \tag{3.1.6}$$

is in $I(X_{n,\lambda,s})$. Since the top degree component of (3.1.6) is x_i^s , we have $x_i^s \in \text{gr } I(X_{n,\lambda,s})$.

Second, we show that for any d and $S \subseteq \mathbf{x}_n$ such that $|S| \geq d > |S| - p_{|S|}^n(\lambda)$, we have $e_d(S) \in \text{gr } I(X_{n,\lambda,s})$. For $m \leq n$, let $\mathbf{x}_m := \{x_1, \dots, x_m\}$. Since the ideal $\text{gr } I(X_{n,\lambda,s})$ is closed under the action of S_n , it suffices to prove that $e_d(\mathbf{x}_m) \in \text{gr } I(X_{n,\lambda,s})$ for d and m such that $m \geq d > m - p_m^n(\lambda)$.

Observe that $p_m^n(\lambda)$ is the number of cells of λ weakly to the right of column $n - m + 1$. For each $i \leq s$, let $c_{i,m}$ be the number of cells of the Young diagram of λ which are in the i th row and are weakly to the right of column $n - m + 1$. Observe that for any $p \in X_{n,\lambda,s}$, at least $c_{i,m}$ many α_i 's must appear among the coordinates p_1, \dots, p_m . Therefore, $\prod_{i=1}^m (t + p_i)$

is divisible by $\prod_{i=1}^s (t + \alpha_i)^{c_{i,m}}$. Hence, there exists a polynomial $z(t)$ such that

$$\prod_{i=1}^m (t + p_i) = z(t) \prod_{i=1}^s (t + \alpha_i)^{c_{i,m}}. \quad (3.1.7)$$

Let $\mathbf{y}_s = \{y_1, \dots, y_s\}$ be a second set of indeterminates. To show that $e_d(\mathbf{x}_m) \in \text{gr } I(X_{n,\lambda,s})$ for $d > m - p_m^n(\lambda)$, we consider the more general division problem of dividing $\prod_{i=1}^m (t + x_i)$ by $\prod_{i=1}^s (t + y_i)^{c_{i,m}}$ as polynomials in t with coefficients in $\mathbb{Q}[\mathbf{x}_m, \mathbf{y}_s]$. Since $\sum_i c_{i,m} = p_m^n(\lambda)$, the remainder upon dividing $\prod_{i=1}^m (t + x_i)$ by $\prod_{i=1}^s (t + y_i)^{c_{i,m}}$ as polynomials in t will be degree at most $p_m^n(\lambda) - 1$ in t . Therefore, there exist polynomials $q(t, \mathbf{x}_m, \mathbf{y}_s)$ and $r_d(\mathbf{x}_m, \mathbf{y}_s)$ for $0 \leq d \leq p_m^n(\lambda) - 1$ such that

$$\prod_{i=1}^m (t + x_i) = q(t, \mathbf{x}_m, \mathbf{y}_s) \prod_{i=1}^s (t + y_i)^{c_{i,m}} + \sum_{d=0}^{p_m^n(\lambda)-1} r_d(\mathbf{x}_m, \mathbf{y}_s) t^d. \quad (3.1.8)$$

Observe $r_d(\mathbf{x}_m, \mathbf{y}_s)$ is homogeneous as a polynomial in $\mathbb{Q}[\mathbf{x}_m, \mathbf{y}_s]$, so $r_d(\mathbf{x}_m, 0^s)$ is the top degree component of $r_d(\mathbf{x}_m, \mathbf{y}_s)$ as a polynomial in \mathbf{x}_m with coefficients in $\mathbb{Q}[\mathbf{y}_s]$. Hence, $r_d(\mathbf{x}_m, 0^s) = \tau(r_d(\mathbf{x}_m, \alpha_1, \dots, \alpha_s))$. Plugging $y_i = 0$ into (3.1.8) for $1 \leq i \leq s$, we have

$$\prod_{i=1}^m (t + x_i) = \sum_{i=0}^m e_{m-i}(\mathbf{x}_m) t^i = q(t, \mathbf{x}_m, 0^s) t^{p_m^n(\lambda)} + \sum_{d=0}^{p_m^n(\lambda)-1} r_d(\mathbf{x}_m, 0^s) t^d. \quad (3.1.9)$$

Hence, for $0 \leq d \leq p_m^n(\lambda) - 1$, we have $e_{m-d}(\mathbf{x}_m) = r_d(\mathbf{x}_m, 0^s)$.

By (3.1.7), we have $r_d(p_1, \dots, p_m, \alpha_1, \dots, \alpha_s) = 0$ for all $p \in X_{n,\lambda,s}$ and $0 \leq d \leq p_m^n(\lambda) - 1$, and hence $r_d(\mathbf{x}_m, \alpha_1, \alpha_2, \dots, \alpha_s) \in I(X_{n,\lambda,s})$. Hence, we have

$$e_{m-d}(\mathbf{x}_m) = r_d(\mathbf{x}_m, 0^s) = \tau(r_d(\mathbf{x}_m, \alpha_1, \alpha_2, \dots, \alpha_s)) \in \text{gr } I(X_{n,\lambda,s}) \quad (3.1.10)$$

for all $0 \leq d \leq p_m^n(\lambda) - 1$. Replacing d with $m - d$ yields $e_d(\mathbf{x}_m) \in \text{gr } I(X_{n,\lambda,s})$ for all $m \geq d \geq m - p_m^n(\lambda) + 1$. Hence, all of the generators of $I_{n,\lambda,s}$ are in $\text{gr } I(X_{n,\lambda,s})$, so $I_{n,\lambda,s} \subseteq \text{gr } I(X_{n,\lambda,s})$. \square

Remark 3.1.4. When $k = n$, by [29, Remark 3.1] we have $I_\lambda = \text{gr } I(X_{n,\lambda,\ell(\lambda)})$, which contains $x_i^{\ell(\lambda)}$ by the proof of Lemma 3.1.3. Hence, we have $R_{n,\lambda,s} = R_\lambda$ for all $s \geq \ell(\lambda)$ in this case.

3.1.2 Shuffles and (n, λ, s) -staircases

Before we proceed, we prove a couple of combinatorial lemmata concerning shuffles of compositions. We use these lemmata to construct a monomial basis of $R_{n,\lambda,s}$ in the next subsection.

Lemma 3.1.5. *Let $1 \leq a \leq b$, and let $n = a + b$. Let γ be a shuffle of the compositions $(0, 1, \dots, a - 1)$ and $(0, 1, \dots, b - 1)$. Then there exists a shuffle δ of $(0, 1, \dots, a - 1)$ and $(0, 1, \dots, b - 1)$ such that $\delta_n = a - 1$ and $\text{trunc}(\gamma) \subseteq \text{trunc}(\delta)$.*

Proof. If $\gamma_n = a - 1$, then we may take $\delta = \gamma$, and we are done. Therefore, we assume $\gamma_n = b - 1$ for the remainder of the proof.

Let P be the labeled lattice path whose corresponding shuffle is γ as follows. Suppose we have a lattice path in the plane starting at $(0, 0)$ and ending at (a, b) and taking only east steps $E = (1, 0)$ and north steps $N = (0, 1)$. Label the i th east step with $i - 1$ for $i \leq a$, and label the i th north step of the path with $i - 1$ for all $i \leq b$. To such a labeled lattice path, we associate the shuffle of $(0, 1, \dots, a - 1)$ and $(0, 1, \dots, b - 1)$ obtained by reading off the labels of the steps from left to right.

We construct a lattice path Q starting at $(0, 0)$ and ending at (a, b) which stays weakly above P as follows. See Figure 3.1 for examples of the path Q we construct from the path P . First, suppose that the last east step of P lies weakly above the diagonal $y = x$, and suppose it is the m th step of P . Then the starting point of the last east step is at $(a - 1, m - a)$, and we must have $m - a > a - 1$. Define the first $m - 1$ steps of Q to be identical to the first $m - 1$ steps of P . Define the rest of the path Q to be $N^{n-m}E$. Since the m th step in P is the last east step of P , then Q stays weakly above P and ends at (a, b) . Furthermore, the label of the i th step of Q is greater than or equal to the i th step of P for $i < n$. Letting δ be the composition read from the labels of Q from left to right, then we have $\text{trunc}(\gamma) \subseteq \text{trunc}(\delta)$.

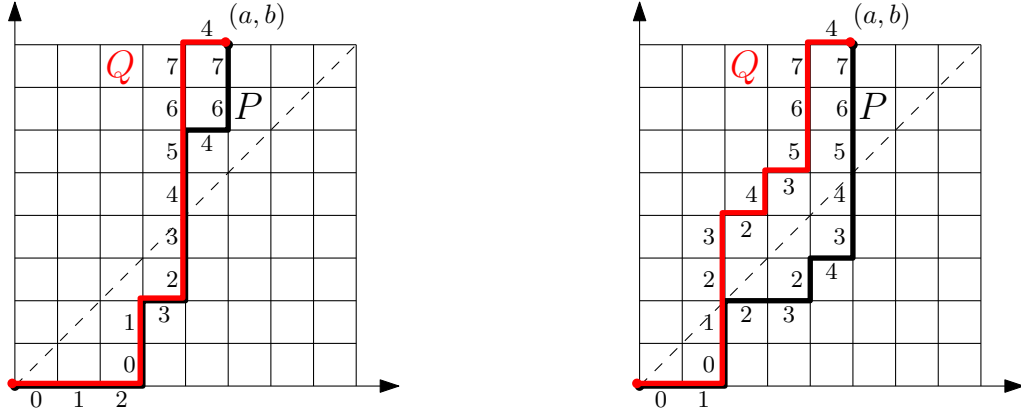


Figure 3.1: Two examples of the path Q constructed from the path P in the proof of Lemma 3.1.5 for the case where $a = 5$ and $b = 8$. On the left, we have $\gamma = (0, 1, 2, 0, 1, 3, 2, 3, 4, 5, 4, 6, 7)$. On the right, $\gamma = (0, 1, 0, 1, 2, 3, 2, 4, 3, 4, 5, 6, 7)$.

Furthermore, since Q ends in an east step we have $\delta_n = a - 1$.

Second, suppose the last east step of P lies below the diagonal $y = x$. In this case, the path P must cross the point (a, a) and then end with $b - a$ many north steps. In addition, P must touch the diagonal $y = x$ in at least one other point. Suppose the point (c, c) is the second to last time P touches the diagonal. We define Q in three segments, as follows. Define the first segment of Q to be identical to the subpath of P between the points $(0, 0)$ and (c, c) . Let P' be the subpath of P from the point (c, c) to the point $(a, a - 1)$. Define the second segment of Q to be the reflection of P' across the diagonal $y = x$, which is a path from (c, c) to $(a - 1, a)$. Define the third segment of Q to be $N^{b-a}E$, so that Q ends at (a, b) . By construction, Q lies weakly above P . Observe that the label of the i th step of P equals the label of the i th step of Q for all $i \leq 2a - 1$. Furthermore, the label of the i th step of Q is greater than or equal to the i th step of P for $i < n$. Letting δ be the composition read from the labels of Q from left to right, we have $\delta_n = a - 1$ and $\text{trunc}(\gamma) \subseteq \text{trunc}(\delta)$. \square

Definition 3.1.6. For $1 \leq j \leq \lambda_1$, let $\beta^j(\lambda) = (0, 1, \dots, \lambda'_j - 1)$. An (n, λ, s) -staircase is a

shuffle $\gamma = (\gamma_1, \dots, \gamma_n)$ of the compositions $\beta^1(\lambda), \beta^2(\lambda), \dots, \beta^{\lambda_1}(\lambda)$, and $((s-1)^{n-k})$. Let

$$\mathcal{C}_{n,\lambda,s} := \{\alpha = (\alpha_1, \dots, \alpha_n) : \alpha \subseteq \gamma \text{ for some } (n, \lambda, s)\text{-staircase } \gamma\}. \quad (3.1.11)$$

Observe that when $\lambda = (1^k) \in \text{Par}(k, k)$ and $s = k$, then $\beta^1(1^k) = (0, 1, \dots, k-1)$. Hence, an $(n, (1^k), k)$ -staircase is an (n, k) -staircase, as defined in [45] and in the Introduction.

Lemma 3.1.7. *Let $\alpha \in \mathcal{C}_{n,\lambda,s}$, and suppose $\alpha_n < \ell(\lambda)$. Then $\text{trunc}(\alpha) \in \mathcal{C}_{n-1,\lambda^{(\alpha_n)},s}$.*

Proof. Since $\alpha \in \mathcal{C}_{n,\lambda,s}$, then $\alpha \subseteq \gamma$, for some (n, λ, s) -staircase γ . Let j be maximal such that $\alpha_n < \lambda'_j$. It suffices to prove that $\text{trunc}(\alpha)$ is contained in a shuffle of the compositions

$$\beta^1(\lambda), \dots, \text{trunc}(\beta^j(\lambda)), \dots, \beta^{\lambda_1}(\lambda), \text{ and } ((s-1)^{n-k}). \quad (3.1.12)$$

We have two cases: either $\gamma_n = s-1$, or $\gamma_n = \lambda'_h - 1$ for some h .

In the first case when $\gamma_n = s-1$, let q be the index such that $\gamma_q = \lambda'_j - 1$, corresponding to the last part of the composition $\beta^j(\lambda)$ in the shuffle γ . Let $\bar{\gamma}$ be the composition obtained by swapping the q th and n th entries of γ . Then $\bar{\gamma}$ is still an (n, λ, s) -staircase. Furthermore, since $\alpha_q \leq \gamma_q = \lambda'_j - 1 \leq s-1$, we have $\text{trunc}(\alpha) \subseteq \text{trunc}(\bar{\gamma})$, where $\text{trunc}(\bar{\gamma})$ is a shuffle of the compositions listed in (3.1.12), hence $\text{trunc}(\alpha) \in \mathcal{C}_{n-1,\lambda^{(\alpha_n)},s}$.

In the second case, we have $\gamma_n = \lambda'_h - 1$ for some h . Since $\alpha_n \leq \gamma_n = \lambda'_h - 1$, then $\alpha_n < \lambda'_h$, so we must have that $h \leq j$ by maximality of j . Let δ be the restriction of the composition γ to the parts corresponding to $\beta^h(\lambda)$ and $\beta^j(\lambda)$ in the shuffle, so that $\delta_{\lambda'_h + \lambda'_j} = \lambda'_h - 1$. By Lemma 3.1.5 with $a = \lambda'_j$ and $b = \lambda'_h$, there exists a shuffle ϵ of $\beta^h(\lambda)$ and $\beta^j(\lambda)$ such that $\epsilon_{\lambda'_h + \lambda'_j} = \lambda'_j - 1$ and $\text{trunc}(\delta) \subseteq \text{trunc}(\epsilon)$. Let $\bar{\gamma}$ be the composition obtained by replacing the parts of γ corresponding to δ with the parts of the composition ϵ , in order from left to right. Then $\text{trunc}(\bar{\gamma})$ is a shuffle of the compositions in (3.1.12). Furthermore, since $\text{trunc}(\alpha) \subseteq \text{trunc}(\gamma)$ as compositions of length n and $\text{trunc}(\delta) \subseteq \text{trunc}(\epsilon)$ as compositions of length $\lambda'_h + \lambda'_j$, we have $\text{trunc}(\alpha) \subseteq \text{trunc}(\gamma) \subseteq \text{trunc}(\bar{\gamma})$, hence $\text{trunc}(\alpha) \in \mathcal{C}_{n-1,\lambda^{(\alpha_n)},s}$. \square

For a collection \mathcal{C} of compositions and an integer i , let us denote by $\mathcal{C} \bullet (i)$ the collection

of compositions $\alpha \bullet (i)$ for $\alpha \in \mathcal{C}$. The symbol \cup denotes a union of sets which are pairwise disjoint.

Lemma 3.1.8. *We have the following decomposition of the set $\mathcal{C}_{n,\lambda,s}$,*

$$\mathcal{C}_{n,\lambda,s} = \bigcup_{i=0}^{\ell(\lambda)-1} \mathcal{C}_{n-1,\lambda^{(i)},s} \bullet (i) \cup \bigcup_{i=\ell(\lambda)}^{s-1} \mathcal{C}_{n-1,\lambda,s} \bullet (i), \quad (3.1.13)$$

where on the right-hand side we interpret $\mathcal{C}_{n-1,\lambda,s} = \emptyset$ if $n = |\lambda|$.

Proof. Observe that the right-hand side of (3.1.13) is indeed a disjoint union of sets, since each set contains compositions with a distinct last coordinate. Given $\alpha \in \mathcal{C}_{n-1,\lambda^{(i)},s}$ for $i < \ell(\lambda)$ with $\alpha = (\alpha_1, \dots, \alpha_{n-1})$, then $\alpha \subseteq \beta$ for some $(n-1, \lambda^{(i)}, s)$ -staircase β by definition. Let j be maximal such that $i < \lambda'_j$. Then we have $\alpha \bullet (i) \subseteq \beta \bullet (\lambda'_j - 1)$, where $\beta \bullet (\lambda'_j - 1)$ is an (n, λ, s) -staircase. Hence, we have $\alpha \bullet (i) \in \mathcal{C}_{n,\lambda,s}$. Given $\alpha \in \mathcal{C}_{n-1,\lambda,s}$ and $\ell(\lambda) \leq i \leq s-1$, then $\alpha \subseteq \beta$ for some $(n-1, \lambda, s)$ -staircase β by definition. Then we have $\alpha \bullet (i) \subseteq \beta \bullet (s-1)$, where $\beta \bullet (s-1)$ is an (n, λ, s) -staircase. Hence, we have $\alpha \bullet (i) \in \mathcal{C}_{n,\lambda,s}$. Therefore, the disjoint union on the right-hand side of (3.1.13) is contained in the left-hand side as sets.

Let $\alpha \in \mathcal{C}_{n,\lambda,s}$. By definition of $\mathcal{C}_{n,\lambda,s}$, there is a shuffle $\beta = (\beta_1, \dots, \beta_n)$ of $\beta^1(\lambda), \dots, \beta^{\lambda_1}(\lambda)$, and $((s-1)^{n-k})$ such that $\alpha \subseteq \beta$. We have $\text{trunc}(\alpha) \subseteq \text{trunc}(\beta)$ and $\alpha_n \leq \beta_n$.

If $\alpha_n \geq \ell(\lambda)$, since each part of the composition $\beta^j(\lambda)$ is at most $\ell(\lambda) - 1$, it must be that $\beta_n = s-1$. Therefore, $\text{trunc}(\beta)$ is an $(n-1, \lambda, s)$ -staircase. Since $\text{trunc}(\alpha) \subseteq \text{trunc}(\beta)$, then $\text{trunc}(\alpha) \in \mathcal{C}_{n-1,\lambda,s}$, so that $\alpha = \text{trunc}(\alpha) \bullet (\alpha_n) \in \mathcal{C}_{n-1,\lambda,s} \bullet (\alpha_n)$.

If $\alpha_n < \ell(\lambda)$, then by Lemma 3.1.7, we have $\text{trunc}(\alpha) \in \mathcal{C}_{n-1,\lambda^{(\alpha_n)},s}$. Therefore, we have $\alpha = \text{trunc}(\alpha) \bullet (\alpha_n) \in \mathcal{C}_{n-1,\lambda^{(i)},s} \bullet (i)$ for $i = \alpha_n$. Hence, we have the equality of sets (3.1.13). \square

Lemma 3.1.9. *Let $h \leq k \leq n$ and s be positive integers. Let $\lambda \in \text{Par}(h, s)$, and let $\mu \in \text{Par}(k, s)$. If $h = k$ and $\lambda \leq_{\text{dom}} \mu$, or if $h < k$ and $\lambda \subseteq \mu$, then $\mathcal{C}_{n,\mu,s} \subseteq \mathcal{C}_{n,\lambda,s}$.*

Proof. It suffices to show that every (n, μ, s) -staircase is contained in some (n, λ, s) -staircase. First, suppose $h = k$ and $\lambda \leq_{\text{dom}} \mu$. It suffices to consider the case when μ covers λ in dominance order. In this case, the Young diagrams of μ and λ differ in only two columns. Suppose these two columns of μ are lengths b and a from left to right, so that $a \leq b$ and the two columns of λ are lengths $b + 1$ and $a - 1$ from left to right. Given β a shuffle of the compositions $(0, 1, \dots, b - 1)$ and $(0, 1, \dots, a - 1)$, then by Lemma 3.1.5 there exists a shuffle δ of $(0, 1, \dots, b - 1)$ and $(0, 1, \dots, a - 1)$ such that $\delta_{a+b} = a - 1$ and $\text{trunc}(\beta) \subseteq \text{trunc}(\delta)$. Let $\gamma = \text{trunc}(\delta) \bullet (b)$. Then γ is a shuffle of $(0, 1, \dots, b)$ and $(0, 1, \dots, a - 2)$ such that $\beta \subseteq \gamma$. Therefore, any (n, μ, s) -staircase is contained in a (n, λ, s) -staircase, hence $\mathcal{C}_{n, \mu, s} \subseteq \mathcal{C}_{n, \lambda, s}$.

Second, suppose $h < k$ and $\lambda \subseteq \mu$. It suffices to consider the case when $k = h + 1$, hence when the Young diagrams of μ and λ only differ by one box. In this case, we have $\mu'_j = \lambda'_j + 1$ for some j . Given any (n, μ, s) -staircase β , replace the copy of $\mu'_j - 1$ in β corresponding to the last entry of $\beta^j(\mu)$ with an $s - 1$. Then the resulting composition is an (n, λ, s) -staircase containing β , hence $\mathcal{C}_{n, \mu, s} \subseteq \mathcal{C}_{n, \lambda, s}$. \square

3.1.3 Monomial basis of $R_{n, \lambda, s}$

To each weak composition α of length n , we associate a monomial

$$\mathbf{x}_n^\alpha := \prod_{i=1}^n x_i^{\alpha_i}. \quad (3.1.14)$$

Let $\mathcal{A}_{n, \lambda, s}$ be the following set of monomials in $\mathbb{Q}[\mathbf{x}_n]$,

$$\mathcal{A}_{n, \lambda, s} := \{\mathbf{x}_n^\alpha : \alpha \in \mathcal{C}_{n, \lambda, s}\}. \quad (3.1.15)$$

Remark 3.1.10. Observe that if $n = k$, then $\mathcal{A}_{n, \lambda, s} = \mathcal{A}_{n, \lambda, \ell(\lambda)}$. This is consistent with the fact that $I_{n, \lambda, s} = I_{n, \lambda, \ell(\lambda)} = I_\lambda$ in this case.

Given a monomial \mathbf{x}_n^α and a set of monomials \mathcal{A} , we denote by $\mathbf{x}_n^\alpha \mathcal{A}$ the set $\{\mathbf{x}_n^\alpha \mathbf{x}_n^\beta : x^\beta \in \mathcal{A}\}$. We have the following recursion for the sets of monomials $\mathcal{A}_{n, \lambda, s}$, which is an immediate

corollary of Lemma 3.1.8.

Corollary 3.1.11. *We have the following decomposition of the set $\mathcal{A}_{n,\lambda,s}$,*

$$\mathcal{A}_{n,\lambda,s} = \bigcup_{i=0}^{\ell(\lambda)-1} x_n^i \mathcal{A}_{n-1,\lambda^{(i)},s} \cup \bigcup_{i=\ell(\lambda)}^{s-1} x_n^i \mathcal{A}_{n-1,\lambda,s}. \quad (3.1.16)$$

Example 3.1.12. Let $\lambda \in \text{Par}(k, s)$. We can obtain the set $\mathcal{A}_{n,\lambda,s}$ by iteratively applying the recursion in Corollary 3.1.11. We have that

$$\begin{aligned} \mathcal{A}_{4,(2,1),3} = \{ & 1, x_1, x_1^2, x_2, x_2^2, x_3, x_1x_3, x_1^2x_3, x_2x_3, x_2^2x_3, x_3^2, \\ & x_2x_3^2, x_4, x_1x_4, x_1^2x_4, x_2x_4, x_2^2x_4, x_3x_4, x_3^2x_4, x_4^2, x_2x_4^2, x_3x_4^2 \}. \end{aligned} \quad (3.1.17)$$

Lemma 3.1.13. *Let d and m be positive integers. Let i be a nonnegative integer, and let $S \subseteq \mathbf{x}_{n-1}$ with $m = |S|$. We have that*

$$x_n^i e_d(S) \in x_n^{i+1} \mathbb{Q}[\mathbf{x}_n] + I_{n,\lambda,s} \quad (3.1.18)$$

in the following cases,

(a) $d > m + 1 - p_{m+1}^n(\lambda)$,

(b) $d = m + 1 - p_{m+1}^n(\lambda)$ and $d + i > m - p_m^n(\lambda)$,

(c) $i < \ell(\lambda)$ and $e_d(S)$ is in the generating set of $I_{n-1,\lambda^{(i)},s}$,

(d) $\ell(\lambda) \leq i \leq s - 1$, $k < n$, and $e_d(S)$ is in the generating set of $I_{n-1,\lambda,s}$.

Proof. In case (a), by our hypothesis and Definition 3.0.1, we have $e_d(S \cup \{x_n\}) \in I_{n,\lambda,s}$.

Hence, we have

$$e_d(S) = -x_n e_{d-1}(S) + e_d(S \cup \{x_n\}) \equiv -x_n e_{d-1}(S) \pmod{I_{n,\lambda,s}}, \quad (3.1.19)$$

$$x_n^i e_d(S) \equiv -x_n^{i+1} e_{d-1}(S) \pmod{I_{n,\lambda,s}}, \quad (3.1.20)$$

so (3.1.18) holds.

In case (b), we assume $d = m + 1 - p_{m+1}^n(\lambda)$. For $u > 1$, we have $d + u > m + 1 - p_{m+1}^n(\lambda)$, so $e_{d+u}(S \cup \{x_n\}) \in I_{n,\lambda,s}$. Furthermore, since $d + i > m - p_m^n(\lambda)$ by assumption, we have $e_{d+i}(S) \in I_{n,\lambda,s}$. Consider the identity

$$(1 - x_n^i t^i) \prod_{x_j \in S} (1 - x_j t) = (1 + x_n t + \cdots + x_n^{i-1} t^{i-1}) \prod_{x_j \in S \cup \{x_n\}} (1 - x_j t). \quad (3.1.21)$$

The coefficient of t^{d+i} on the left-hand side of (3.1.21) is

$$(-1)^{d+i} e_{d+i}(S) + (-1)^{d+1} x_n^i e_d(S) \equiv (-1)^{d+1} x_n^i e_d(S) \pmod{I_{n,\lambda,s}}, \quad (3.1.22)$$

while the coefficient of t^{d+i} on the right-hand side of (3.1.21) is in $I_{n,\lambda,s}$ by the fact that $e_{d+u}(S \cup \{x_n\}) \in I_{n,\lambda,s}$ for all $u > 0$. Therefore, we have $x_n^i e_d(S) \in I_{n,\lambda,s}$, so (3.1.18) holds.

In case (c), we have $i < \ell(\lambda)$ and $d > m - p_m^{n-1}(\lambda^{(i)})$. Let j be maximal such that $i < \lambda'_j$, so the Young diagram of $\lambda^{(i)}$ is obtained from the Young diagram of λ by deleting a cell from the j th column from the left. If $n - m \leq j$, then $p_m^{n-1}(\lambda^{(i)}) = p_{m+1}^n(\lambda) - 1$. Combining this with the inequality $d > m - p_m^{n-1}(\lambda^{(i)})$, we have $d > m + 1 - p_{m+1}^n(\lambda)$, and we are done by case (a).

If on the other hand we have $n - m > j$, then $p_m^{n-1}(\lambda^{(i)}) = p_{m+1}^n(\lambda)$. Since $d > m - p_m^{n-1}(\lambda^{(i)})$, we have $d \geq m + 1 - p_m^{n-1}(\lambda^{(i)})$. If $d > m + 1 - p_m^{n-1}(\lambda^{(i)}) = m + 1 - p_{m+1}^n(\lambda)$, then we are again done by case (a). If we have $d = m + 1 - p_m^{n-1}(\lambda^{(i)}) = m + 1 - p_{m+1}^n(\lambda)$. Furthermore,

$$p_{m+1}^n(\lambda) = p_m^n(\lambda) + \lambda'_{n-m} < p_m^n(\lambda) + i + 1, \quad (3.1.23)$$

which follows from the maximality of j and our assumption that $n - m > j$. Combining our

assumption that $d = m + 1 - p_{m+1}^n(\lambda)$ with (3.1.23), we have

$$d + i > m - p_m^n(\lambda). \quad (3.1.24)$$

Hence, we are done by case (b).

In case (d), we have $d > m - p_m^{n-1}(\lambda)$. Since $k < n$ by assumption, we have that $\lambda'_n = 0$, so $p_{m+1}^n(\lambda) = p_m^{n-1}(\lambda)$. Hence, we have $d > m - p_{m+1}^n(\lambda)$. If $d > m + 1 - p_{m+1}^n(\lambda)$, then we are done by case (a). Otherwise, we have $d = m + 1 - p_{m+1}^n(\lambda)$. Furthermore, since $\lambda'_{n-m} \leq \ell(\lambda) < i + 1$, then (3.1.23) continues to hold, and combining it with the equality $d = m + 1 - p_{m+1}^n(\lambda)$, we obtain $d + i > m - p_m^n(\lambda)$. Hence, we are done by case (b). \square

Lemma 3.1.14. *The set $\mathcal{A}_{n,\lambda,s}$ represents a \mathbb{Q} -spanning set of $R_{n,\lambda,s}$.*

Proof. We proceed by induction on n . When $n = 1$, either $\lambda = (1)$ or $\lambda = \emptyset$. In the case when $\lambda = (1)$, then $R_{1,\lambda,s} = \mathbb{Q}[\mathbf{x}_1]/\langle x_1 \rangle$ and $\mathcal{A}_{1,\lambda,s} = \{1\}$. In the case when $\lambda = \emptyset$, then $R_{1,\lambda,s} = \mathbb{Q}[x_1]/(x_1^s)$ and $\mathcal{A}_{1,\lambda,s} = \{1, x_1, \dots, x_1^{s-1}\}$. Therefore, the statement holds for $n = 1$.

Assume $n > 1$. Suppose by way of induction that $\mathcal{A}_{m,\mu,s}$ is a \mathbb{Q} -spanning set of $R_{m,\mu,s}$ for all μ and $m < n$. Let $\lambda \in \text{Par}(k, s)$ for some $k \leq n$. Since $x_n^s \in I_{n,\lambda,s}$, we have an isomorphism of \mathbb{Q} -vector spaces,

$$R_{n,\lambda,s} \cong \bigoplus_{i=0}^{s-1} x_n^i R_{n,\lambda,s} / x_n^{i+1} R_{n,\lambda,s}. \quad (3.1.25)$$

Therefore, it suffices to show that every polynomial of the form $x_n^i p(x_1, \dots, x_{n-1})$ is congruent to a polynomial in $\text{span}_{\mathbb{Q}}(\mathcal{A}_{n,\lambda,s})$ modulo $x_n^{i+1} \mathbb{Q}[\mathbf{x}_n] + I_{n,\lambda,s}$. We have two cases: either $0 \leq i < \ell(\lambda)$ or $\ell(\lambda) \leq i \leq s - 1$.

In the first case when $0 \leq i < \ell(\lambda)$, by our inductive hypothesis we have that $\mathcal{A}_{n-1,\lambda^{(i)},s}$ is a \mathbb{Q} -spanning set of $R_{n-1,\lambda^{(i)},s}$, so

$$p(x_1, \dots, x_{n-1}) = \sum_{\mathbf{x}_{n-1}^\alpha \in \mathcal{A}_{n-1,\lambda^{(i)},s}} c_\alpha \mathbf{x}_{n-1}^\alpha + \sum_{e_d(S) \in I_{n-1,\lambda^{(i)},s}} A(d, S) e_d(S) + \sum_{j=1}^{n-1} A_j x_j^s, \quad (3.1.26)$$

for some constants $c_\alpha \in \mathbb{Q}$ and some polynomials $A(d, S)$ and A_j in $\mathbb{Q}[\mathbf{x}_{n-1}]$, where the second sum is over all generators of $I_{n-1, \lambda^{(i)}, s}$ of the form $e_d(S)$. Hence,

$$x_n^i p(x_1, \dots, x_{n-1}) = \sum_{\mathbf{x}_{n-1}^\alpha \in \mathcal{A}_{n-1, \lambda^{(i)}, s}} c_\alpha x_n^i \mathbf{x}_{n-1}^\alpha + \sum_{e_d(S) \in I_{n-1, \lambda^{(i)}, s}} A(d, S) x_n^i e_d(S) + \sum_{j=1}^{n-1} A_j x_n^i x_j^s. \quad (3.1.27)$$

Observe that $x_n^i x_j^s \in I_{n, \lambda, s}$ and that $x_n^i e_d(S) \in x_n^i \mathbb{Q}[\mathbf{x}_n] + I_{n, \lambda, s}$ by Lemma 3.1.13(a). Furthermore, $x_n^i \mathbf{x}_{n-1}^\alpha \in \mathcal{A}_{n, \lambda, s}$ for all $\mathbf{x}_{n-1}^\alpha \in \mathcal{A}_{n-1, \lambda^{(i)}, s}$ by Corollary 3.1.11. Therefore, $x_n^i p$ is congruent to a polynomial in $\text{span}_{\mathbb{Q}}(\mathcal{A}_{n, \lambda, s})$ modulo $x_n^{i+1} \mathbb{Q}[\mathbf{x}_n] + I_{n, \lambda, s}$, and we are done.

In the second case, we have $\ell(\lambda) \leq i \leq s-1$. If $n = k$, then $x_n^i \in I_\lambda = I_{n, \lambda, s}$ by Remark 3.1.4, so $x_n^i p \in I_{n, \lambda, s}$. Otherwise, we have $k < n$, and by our inductive hypothesis we have that $\mathcal{A}_{n-1, \lambda, s}$ is a \mathbb{Q} -spanning set of $R_{n-1, \lambda, s}$. A similar application of Lemma 3.1.13(d) completes the induction. Hence, $\mathcal{A}_{n, \lambda, s}$ is a \mathbb{Q} -spanning set of $R_{n, \lambda, s}$. \square

Lemma 3.1.15. *We have the equality of cardinalities $|X_{n, \lambda, s}| = |\mathcal{A}_{n, \lambda, s}|$.*

Proof. Recall that $X_{n, \lambda, s}$ is the set of points $p = (p_1, \dots, p_n) \in \mathbb{Q}^n$ such that for each $1 \leq i \leq n$, $p_i = \alpha_j$ for some j , and for each $1 \leq i \leq s$, α_i appears as a coordinate in p at least λ_i many times. Observe that the size of $X_{n, \lambda, s}$ does not depend on our choice of the distinct rational numbers α_i . The statement of the lemma holds when $n = 1$. Indeed, if $\lambda = (1)$ then $X_{1, \lambda, s} = \{(\alpha_1)\}$ and $\mathcal{A}_{1, \lambda, s} = \{1\}$. Otherwise, we have that $\lambda = \emptyset$, in which case $X_{1, \lambda, s} = \{(\alpha_1), (\alpha_2), \dots, (\alpha_s)\}$ and $\mathcal{A}_{1, \lambda, s} = \{1, x_1, \dots, x_1^{s-1}\}$, which are equinumerous. In light of Corollary 3.1.11, it suffices to prove that the cardinalities $|X_{n, \lambda, s}|$ satisfy the same recursion as $|\mathcal{A}_{n, \lambda, s}|$, namely that

$$|X_{n, \lambda, s}| = \sum_{i=0}^{\ell(\lambda)-1} |X_{n-1, \lambda^{(i)}, s}| + \sum_{i=\ell(\lambda)}^{s-1} |X_{n-1, \lambda, s}|. \quad (3.1.28)$$

The identity (3.1.28) follows by observing that for $\ell(\lambda) \leq i \leq s-1$, the set $X_{n-1, \lambda, s}$ is in

bijection with the set

$$\{p \in X_{n,\lambda,s} : (p_1, \dots, p_{n-1}) \in X_{n-1,\lambda,s}, p_n = \alpha_{i+1}\} \quad (3.1.29)$$

and for $0 \leq i < \ell(\lambda)$, the set $X_{n-1,\lambda^{(i)},s}$ is in bijection with the set

$$\{p \in X_{n,\lambda,s} : (p_1, \dots, p_{n-1}) \in X_{n-1,\lambda^{(i)},s}, p_n = \alpha_{i+1}\}, \quad (3.1.30)$$

which completes the proof. \square

Remark 3.1.16. The proof of Lemma 3.1.15 naturally leads to a recursively constructed bijection between the sets $X_{n,\lambda,s}$ and $\mathcal{A}_{n,\lambda,s}$, though we will have no use for this bijection.

Proof of Theorem 3.1.1. Recall that by Lemma 3.1.3, we have the containment of ideals $I_{n,\lambda,s} \subseteq \text{gr } I(X_{n,\lambda,s})$. Combining this with (3.1.3), we have

$$\dim_{\mathbb{Q}}(R_{n,\lambda,s}) \geq \dim_{\mathbb{Q}} \frac{\mathbb{Q}[\mathbf{x}_n]}{\text{gr } I(X_{n,\lambda,s})} = |X_{n,\lambda,s}|. \quad (3.1.31)$$

Furthermore, by Lemma 3.1.14 and Lemma 3.1.15, we have

$$|X_{n,\lambda,s}| = |\mathcal{A}_{n,\lambda,s}| \geq \dim_{\mathbb{Q}}(R_{n,\lambda,s}). \quad (3.1.32)$$

Stringing together (3.1.31) and (3.1.32), we see that all inequalities must be equalities. In particular, we have the equality

$$\dim_{\mathbb{Q}}(R_{n,\lambda,s}) = \dim_{\mathbb{Q}} \frac{\mathbb{Q}[\mathbf{x}_n]}{\text{gr } I(X_{n,\lambda,s})}. \quad (3.1.33)$$

As a consequence, we also have $I_{n,\lambda,s} = \text{gr } I(X_{n,\lambda,s})$, hence $R_{n,\lambda,s} = \mathbb{Q}[\mathbf{x}_n]/\text{gr } I(X_{n,\lambda,s})$. \square

Theorem 3.1.17. *The set of monomials $\mathcal{A}_{n,\lambda,s}$ represents a basis of $R_{n,\lambda,s}$.*

Proof. By the proof of Theorem 3.1.1, we have that $|\mathcal{A}_{n,\lambda,s}| = \dim_{\mathbb{Q}}(R_{n,\lambda,s})$. By Lemma 3.1.14, $\mathcal{A}_{n,\lambda,s}$ is a \mathbb{Q} -spanning set of $R_{n,\lambda,s}$. Hence, $\mathcal{A}_{n,\lambda,s}$ is a basis of $R_{n,\lambda,s}$. \square

Lemma 3.1.18. *Let $h \leq k \leq n$ be positive integers, let $\lambda \in \text{Par}(h, s)$, and let $\mu \in \text{Par}(k, s)$. If $h = k$ and $\lambda \leq_{\text{dom}} \mu$, or if $h < k$ and $\lambda \subseteq \mu$, then $\mathcal{A}_{n,\mu,s} \subseteq \mathcal{A}_{n,\lambda,s}$ and $I_{n,\lambda,s} \subseteq I_{n,\mu,s}$.*

Proof. If $h = k$ and $\lambda \leq_{\text{dom}} \mu$ or if $h < k$ and $\lambda \subseteq \mu$, then $p_m^n(\lambda) \leq p_m^n(\mu)$ for all m . Therefore, the generating set of $I_{n,\lambda,s}$ is contained in the generating set of $I_{n,\mu,s}$, so $I_{n,\lambda,s} \subseteq I_{n,\mu,s}$. By Lemma 3.1.9, we have $\mathcal{C}_{n,\mu,s} \subseteq \mathcal{C}_{n,\lambda,s}$, so the containment $\mathcal{A}_{n,\mu,s} \subseteq \mathcal{A}_{n,\lambda,s}$ follows. \square

As a consequence of the containment of ideals in Lemma 3.1.18, we have a monotonicity property of the multiplicities of irreducible representations contained in the ring $R_{n,\lambda,s}$. We state it next in terms of graded Frobenius characteristics.

Theorem 3.1.19. *Let $h \leq k \leq n$ be positive integers, let $\lambda \in \text{Par}(h, s)$, and let $\mu \in \text{Par}(k, s)$ such that either $h = k$ and $\lambda \leq_{\text{dom}} \mu$, or $h < k$ and $\lambda \subseteq \mu$. For each partition $\nu \vdash n$, we have the inequality*

$$[s_\nu] \text{Frob}_q(R_{n,\lambda,s}) \geq [s_\nu] \text{Frob}_q(R_{n,\mu,s}), \quad (3.1.34)$$

where $[s_\nu]f$ stands for the coefficient of s_ν in the Schur function expansion of f , and the inequality is a coefficient-wise comparison of two polynomials in q .

3.1.4 Ordered set partitions

In this subsection, we relate the S_n -module $R_{n,\lambda,s}$ to an action on (n, λ, s) -ordered set partitions. We then find the Frobenius characteristic of $R_{n,\lambda,s}$.

Theorem 3.1.20. *We have that $\dim_{\mathbb{Q}}(R_{n,\lambda,s}) = |\mathcal{OP}_{n,\lambda,s}|$. Furthermore, we have the isomorphism of S_n -modules*

$$R_{n,\lambda,s} \cong_{S_n} \mathbb{Q} \mathcal{OP}_{n,\lambda,s}. \quad (3.1.35)$$

Proof. By Corollary 3.1.2, we have that $R_{n,\lambda,s} \cong_{S_n} \mathbb{Q} X_{n,\lambda,s}$ as S_n -modules. Therefore, it suffices to show there is an S_n -equivariant bijection between $X_{n,\lambda,s}$ and $\mathcal{OP}_{n,\lambda,s}$. Define a

map $\varphi : X_{n,\lambda,s} \rightarrow \mathcal{OP}_{n,\lambda,s}$ as follows. Given $p \in X_{n,\lambda,s}$, define $\varphi(p) = (B_1 | \cdots | B_s)$ where B_i is the set of indices $1 \leq j \leq n$ such that $p_j = \alpha_i$. By the definition of $X_{n,\lambda,s}$, we have that α_i appears at least λ_i many times as a coordinate in p . Therefore, we have $|B_i| \geq \lambda_i$ for all $i \leq s$, so $\varphi(p) \in \mathcal{OP}_{n,\lambda,s}$. The map φ is clearly the desired S_n -equivariant bijection, which completes the proof. \square

Corollary 3.1.21. *The Frobenius characteristic of $R_{n,\lambda,s}$ is*

$$\text{Frob}(R_{n,\lambda,s}) = \sum_{\substack{\mu \in \text{Par}(n,s), \\ \mu \supseteq \lambda}} h_\mu \prod_{i \geq 0} \binom{\mu'_i - \lambda'_{i+1}}{\mu'_i - \mu'_{i+1}}, \quad (3.1.36)$$

where $\mu'_0 := s$.

Proof. By Theorem 3.1.20, we have $R_{n,\lambda,s} \cong_{S_n} \mathbb{Q}\mathcal{OP}_{n,\lambda,s}$. We can partition the set $\mathcal{OP}_{n,\lambda,s}$ into S_n -orbits, where an orbit is determined by the tuple of block sizes $\alpha = (|B_1|, \dots, |B_s|)$. This correspondence sets up a bijection between the set of S_n -orbits of $\mathcal{OP}_{n,\lambda,s}$ and the set of all weak compositions $\alpha = (\alpha_1, \dots, \alpha_s)$ of n such that $\alpha \supseteq \lambda$. Given such a composition α , let \mathcal{O} be the corresponding S_n orbit of $\mathcal{OP}_{n,\lambda,s}$. Letting $\mu = \text{sort}(\alpha)$, then \mathcal{O} is isomorphic as an S_n -module to the set of tabloids with μ_i boxes in the i th row. Hence, the Frobenius characteristic of the action of S_n on the submodule $\mathbb{Q}\mathcal{O}$ of $\mathbb{Q}\mathcal{OP}_{n,\lambda,s}$ is equal to h_μ .

Since $\mathbb{Q}\mathcal{OP}_{n,\lambda,s}$ is the direct sum over all $\mathbb{Q}\mathcal{O}$ where \mathcal{O} is an S_n -orbit, we have

$$\text{Frob}(R_{n,\lambda,s}) = \sum_{\substack{\mu \in \text{Par}(n,s), \\ \mu \supseteq \lambda}} a_{\lambda,\mu}^{(s)} h_\mu, \quad (3.1.37)$$

where $a_{\lambda,\mu}^{(s)}$ is the number of $\alpha \in \text{Comp}(n,s)$ such that $\alpha \supseteq \lambda$ and $\text{sort}(\alpha) = \mu$. It is then an easy exercise to verify $a_{\lambda,\mu}^{(s)}$ is equal to the coefficient of h_μ in the right-hand side of (3.1.36). \square

3.2 Skewing formulas and exact sequences for $R_{n,\lambda,s}$

In this section, we develop algebraic tools for analyzing the graded Frobenius characteristic of $\text{Frob}_q(R_{n,\lambda,s})$. Our main tool is a recursive formula for the image of $\text{Frob}_q(R_{n,\lambda,s})$ under the skewing operator $e_j(\mathbf{x})^\perp$ from Subsection 2.0.3. We also show that the rings $R_{n,\lambda,s}$ fit into certain exact sequences.

3.2.1 Skewing formula

Fix j, k, s , and n positive integers with $k \leq n$, and fix $\lambda \in \text{Par}(k, s)$ throughout the subsection. In order to simplify notation, let $z_i := x_{n-j+i}$ for $1 \leq i \leq j$ and $\mathbf{z}_j := \{z_1, \dots, z_j\} = \{x_{n-j+1}, \dots, x_n\}$. Given a polynomial $f(\mathbf{x}_{n-j}, \mathbf{z}_j)$, then $\sigma \in S_{n-j}$ acts on the \mathbf{x}_{n-j} variables, and ϵ_j acts on the \mathbf{z}_j variables. We have the following definitions, which we need for our formulas for $e_j^\perp \text{Frob}_q(R_{n,\lambda,s})$.

Definition 3.2.1. Let $I = (i_1, \dots, i_j) \in [0, s-1]^j$. Construct a partition $\lambda^{(I)}$ recursively using the reduction operations defined in (2.0.4) as follows. Let $\lambda_{(j)} := \lambda$, and for $1 \leq h \leq j$, let

$$\lambda_{(h-1)} := \begin{cases} \lambda_{(h)}^{(i_h)} & \text{if } i_h < \ell(\lambda_{(h)}), \\ \lambda_{(h)} & \text{if } i_h \geq \ell(\lambda_{(h)}). \end{cases}$$

Define $\lambda^{(I)} := \lambda_{(0)}$.

Let \mathcal{I}_s^j be the set of increasing sequences $(i_1 < \dots < i_j)$ of nonnegative integers with $i_j < s$. For $I \in \mathcal{I}_s^j$, let m be maximal such that $i_m < \ell(\lambda)$. In this case, observe that $\lambda^{(I)} \in \text{Par}(k-m, s)$ is the partition obtained from λ by deleting one box from the end of the rows $i_1 + 1, \dots, i_m + 1$ and then sorting the rows. Further observe that for $I = (i)$, if $0 \leq i < \ell(\lambda)$, then $\lambda^{(I)} = \lambda^{(i)}$, and if $\ell(\lambda) \leq i < s$, then $\lambda^{(I)} = \lambda$.

Given a sequence $I = (i_1, \dots, i_j)$ of distinct nonnegative integers which is not necessarily increasing, let $\mathbf{z}_j^I := z_1^{i_1} \dots z_j^{i_j}$. Furthermore, let $\Sigma(I) := i_1 + \dots + i_j$, and let $\text{sort}_{\leq}(I)$ be the increasing sequence obtained by sorting the entries of I .

Theorem 3.2.2. *We have*

$$e_j^\perp \text{Frob}_q(R_{n,\lambda,s}) = \text{Frob}_q(\epsilon_j R_{n,\lambda,s}) = \sum_{I \in \mathcal{I}_s^j} q^{\Sigma(I)} \text{Frob}_q(R_{n-j,\lambda^{(I)},s}). \quad (3.2.1)$$

The first equality in Theorem 3.2.2 follows immediately by (2.0.28), so it suffices to prove the second equality.

Define

$$V_r := \bigoplus_{\substack{I \in \mathcal{I}_s^j, \\ \Sigma(I)=r}} R_{n-j,\lambda^{(I)},s} \otimes \mathbb{Q}\{\epsilon_j \mathbf{z}_j^I\}. \quad (3.2.2)$$

The vector space V_r has the structure of an S_{n-j} -module, where S_{n-j} acts on the first factor of each tensor product. It is also graded, where the degree of a nonzero simple tensor $f \otimes \epsilon_j \mathbf{z}_j^I$ is $\deg(f) + \Sigma(I)$. Observe that if we could directly show that $\epsilon_j R_{n,\lambda,s} \cong \bigoplus_{r \geq 0} V_r$ as graded S_{n-j} -modules, then the second equality in Theorem 3.2.2 would follow. A natural choice for this isomorphism would be the map from $\bigoplus_{r \geq 0} V_r$ to $\epsilon_j R_{n,\lambda,s}$ induced by multiplication, if it is well-defined. Unfortunately, this map is not well-defined in general, as the next example illustrates.

Example 3.2.3. Let $n = 4$, $\lambda = (1^3)$, $s = 3$, and $j = 2$. In order to have a well-defined map $V_r \rightarrow \epsilon_j R_{n,\lambda,s}$ induced by multiplication, then in particular we would need a well-defined map $R_{n-j,\lambda^{(I)},s} \otimes \mathbb{Q}\{\epsilon_j \mathbf{z}_j^I\} \rightarrow \epsilon_j R_{n,\lambda,s}$ for each $I \in \mathcal{I}_s^j$ induced by multiplication. Letting $I = (0, 1)$, then we would need a map

$$R_{2,(1),3} \otimes \mathbb{Q}\{x_3 - x_4\} \rightarrow \epsilon_2 R_{4,(1^3),3}. \quad (3.2.3)$$

However, in order for this map to be induced from multiplication, we would need the con-

tainment $I_{2,(1),3} \cdot (x_3 - x_4) \subseteq \epsilon_2 I_{4,(1^3),3}$. In particular, we would need

$$e_2(x_1, x_2)(x_3 - x_4) = x_1 x_2 (x_3 - x_4) \in \epsilon_2 I_{4,(1^3),3}, \quad (3.2.4)$$

which is not true. However, observe that we do have

$$x_1 x_2 (x_3 - x_4) + x_1 (x_3^2 - x_4^2) + x_2 (x_3^2 - x_4^2) + x_3 x_4 (x_3 - x_4) \in \epsilon_2 I_{4,(1^3),3}, \quad (3.2.5)$$

which contains $x_1 x_2 (x_3 - x_4)$ as a term. Further observe that all other terms in this element have higher degree in the variables $z_1 = x_3$ and $z_2 = x_4$. This suggests that we may be able to define a map from V_r induced by multiplication if we first filter the codomain $\epsilon_j R_{n,\lambda,s}$ by total degree in the \mathbf{z}_j variables.

Indeed, we prove the second equality in Theorem 3.2.2 by filtering $\epsilon_j R_{n,\lambda,s}$ by total degree in the $\mathbf{z}_j = \{x_{n-j+1}, \dots, x_n\}$ variables and then constructing an explicit isomorphism of S_{n-j} -modules which corresponds to the equality of symmetric functions in Theorem 3.2.2. Before giving the proof, we need a few lemmata. Let $S_{\{n-j+1, \dots, n\}}$ be the subgroup of S_n consisting of permutations of the letters $n - j + 1, \dots, n$.

Lemma 3.2.4. *Given a finite set Y with an S_n -action, let y_1, \dots, y_m be a distinct set of representatives of the $S_{\{n-j+1, \dots, n\}}$ -orbits of Y . Then $\dim_{\mathbb{Q}}(\epsilon_j \mathbb{Q}Y)$ is equal to the number of y_i which have trivial $S_{\{n-j+1, \dots, n\}}$ -stabilizer.*

Proof. We have a direct sum decomposition

$$\epsilon_j \mathbb{Q}Y = \bigoplus_{i=1}^m \epsilon_j \mathbb{Q}S_{\{n-j+1, \dots, n\}} y_i. \quad (3.2.6)$$

Observe that if y and y' are in the same $S_{\{n-j+1, \dots, n\}}$ -orbit, then $\epsilon_j y = \pm \epsilon_j y'$. Furthermore, if y has nontrivial stabilizer under the $S_{\{n-j+1, \dots, n\}}$ -action, then $\epsilon_j y = 0$. Hence, $\epsilon_j \mathbb{Q}S_{\{n-j+1, \dots, n\}} y_i$ is either one-dimensional if y_i has trivial $S_{\{n-j+1, \dots, n\}}$ -stabilizer or zero-dimensional otherwise. Hence, $\dim_{\mathbb{Q}}(\epsilon_j \mathbb{Q}Y)$ is equal to the number of y_i which have trivial $S_{\{n-j+1, \dots, n\}}$ -stabilizer. \square

Lemma 3.2.5. *We have*

$$\dim_{\mathbb{Q}}(\epsilon_j R_{n,\lambda,s}) = \dim_{\mathbb{Q}}(\epsilon_j \mathbb{Q} \mathcal{OP}_{n,\lambda,s}) = \sum_{I \in \mathcal{I}_s^j} |\mathcal{OP}_{n-j,\lambda^{(I)},s}|. \quad (3.2.7)$$

Proof. The first equality in (3.2.7) follows by Theorem 3.1.20, so it suffices to prove the second equality. Given $\sigma \in \mathcal{OP}_{n,\lambda,s}$, we may find a unique representative of its $S_{\{n-j+1,\dots,n\}}$ -orbit by sorting the letters $n-j+1, \dots, n$ so that they appear in order from left to right in $\sigma = (B_1|B_2|\dots|B_s)$. By Lemma 3.2.4 applied to $Y = \mathcal{OP}_{n,\lambda,s}$, we have that $\dim_{\mathbb{Q}}(\epsilon_j \mathbb{Q} \mathcal{OP}_{n,\lambda,s})$ is the number of such representatives with trivial $S_{\{n-j+1,\dots,n\}}$ -stabilizer. Since $\sigma \in \mathcal{OP}_{n,\lambda,s}$ has trivial $S_{\{n-j+1,\dots,n\}}$ -stabilizer if and only if $n-j+1, \dots, n$ are in distinct blocks of σ , then $\dim_{\mathbb{Q}}(\epsilon_j \mathbb{Q} \mathcal{OP}_{n,\lambda,s})$ is the number of $\sigma \in \mathcal{OP}_{n,\lambda,s}$ such that $n-j+1, \dots, n$ are in distinct blocks and appear in order from left to right.

Define a bijection between the set of $\sigma \in \mathcal{OP}_{n,\lambda,s}$ such that $n-j+1, \dots, n$ are in distinct blocks and appear in order from left to right and the formal disjoint union of sets

$$\bigsqcup_{I \in \mathcal{I}_s^j} \mathcal{OP}_{n-j,\lambda^{(I)},s} \quad (3.2.8)$$

as follows. Given such a σ , let $I = (i_1 < \dots < i_j) \in \mathcal{I}_s^j$ be such that $n-j+1, \dots, n$ are in $B_{i_1+1}, \dots, B_{i_j+1}$, respectively. Map σ to the ordered set partition obtained by removing $n-j+1, \dots, n$ from σ , considered as an element of $\mathcal{OP}_{n-j,\lambda^{(I)},s}$ in the formal disjoint union (3.2.8). This defines a bijection. Indeed, given $\sigma' = (B'_1|\dots|B'_s) \in \mathcal{OP}_{n-j,\lambda^{(I)},s}$ for $I = (i_1 < \dots < i_j) \in \mathcal{I}_s^j$, then the corresponding σ may be recovered by adding $n-j+h$ to block B'_{i_h+1} for $1 \leq h \leq j$. Hence, the second equality in (3.2.7) follows. \square

Lemma 3.2.6. *Let $H \in [0, s-1]^j$ with distinct entries, and let $I = \text{sort}_{\leq}(H)$. We have $\lambda^{(I)} \leq_{\text{dom}} \lambda^{(H)}$.*

Proof. Recall that we draw Young diagrams according to the French convention. If $s = \ell(\lambda)$, then there is a bijection between the cells of the skew Young diagram $\lambda/\lambda^{(H)}$ and the cells

of $\lambda/\lambda^{(I)}$ such that each cell of $\lambda/\lambda^{(H)}$ is weakly above and to the left of its corresponding cell of $\lambda/\lambda^{(I)}$. It follows that $\lambda^{(I)} \leq_{\text{dom}} \lambda^{(H)}$. See [29, pp. 128-130] for the full proof of the case when $s = \ell(\lambda)$. In the case when $s > \ell(\lambda)$, let m be maximal such that $i_m < \ell(\lambda)$, and let H' be the subsequence of H consisting of elements less than $\ell(\lambda)$. Then we have

$$\lambda^{(I)} = \lambda^{(i_1, \dots, i_m)} \leq_{\text{dom}} \lambda^{(H')} = \lambda^{(H)}, \quad (3.2.9)$$

where the inequality in the middle follows from the first case, since $H' \in [0, \ell(\lambda) - 1]^m$ and $(i_1, \dots, i_m) = \text{sort}_{\leq}(H')$. \square

Remark 3.2.7. The alternating polynomial $\epsilon_j \mathbf{z}_j^I$ is equal to a scalar multiple of a generalized Vandermonde determinant, denoted by $\Delta_{i_1, i_2, \dots, i_j} = j! \det(z_p^{i_q})_{p, q=1, \dots, j}$ in [29].

Lemma 3.2.8. *The collection of polynomials $\mathcal{A}_{n, \lambda, s}^{j\text{-alt}}$ defined by*

$$\mathcal{A}_{n, \lambda, s}^{j\text{-alt}} := \bigcup_{I \in \mathcal{I}_s^j} \mathcal{A}_{n-j, \lambda^{(I)}, s} \cdot \epsilon_j \mathbf{z}_j^I \quad (3.2.10)$$

represents a basis of $\epsilon_j R_{n, \lambda, s}$, where $\mathcal{A}_{n, \lambda, s}$ is as defined in (3.1.15).

Proof. By Theorem 3.1.17, we have $|\mathcal{A}_{n, \lambda, s}| = |\mathcal{OP}_{n, \lambda, s}|$. By Lemma 3.2.5, we have

$$|\mathcal{A}_{n, \lambda, s}^{j\text{-alt}}| = \sum_{I \in \mathcal{I}_s^j} |\mathcal{A}_{n-j, \lambda^{(I)}, s}| = \sum_{I \in \mathcal{I}_s^j} |\mathcal{OP}_{n-j, \lambda^{(I)}, s}| = \dim_{\mathbb{Q}}(\epsilon_j \mathbb{Q} \mathcal{OP}_{n, \lambda, s}) = \dim_{\mathbb{Q}}(\epsilon_j R_{n, \lambda, s}). \quad (3.2.11)$$

Hence, it suffices to prove $\mathcal{A}_{n, \lambda, s}^{j\text{-alt}}$ spans $\epsilon_j R_{n, \lambda, s}$.

Recall $R_{n, \lambda, s}$ is spanned by $\mathcal{A}_{n, \lambda, s}$, so $\epsilon_j R_{n, \lambda, s}$ is spanned by $\epsilon_j \mathcal{A}_{n, \lambda, s}$. Applying the recursion in Corollary 3.1.11 for $\mathcal{A}_{n, \lambda, s}$ exactly j many times shows $\mathcal{A}_{n, \lambda, s}$ can be partitioned as

$$\mathcal{A}_{n, \lambda, s} = \bigcup_{H \in [0, s-1]^j} \mathcal{A}_{n-j, \lambda^{(H)}, s} \cdot \mathbf{z}_j^H. \quad (3.2.12)$$

Applying ϵ_j to both sides of (3.2.12), we see that $\epsilon_j R_{n,\lambda,s}$ is spanned by the union

$$\bigcup_{H \in [0, s-1]^j} \mathcal{A}_{n-j, \lambda^{(H)}, s} \cdot \epsilon_j \mathbf{z}_j^H. \quad (3.2.13)$$

Given $H \in [0, s-1]^j$, if any two entries of H are equal, then we have $\epsilon_j \mathbf{z}_j^H = 0$ by Remark 3.2.7. Therefore, the union (3.2.13) is equal to the same union restricted to those H with distinct entries. In order to prove $\mathcal{A}_{n,\lambda,s}^{j\text{-alt}}$ spans $\epsilon_j R_{n,\lambda,s}$, it suffices to prove that for any $H \in [0, s-1]^j$ with distinct entries and any $\mathbf{x}_{n-j}^\alpha \in \mathcal{A}_{n-j, \lambda^{(H)}, s}$, we have that $\mathbf{x}_{n-j}^\alpha \cdot \epsilon_j \mathbf{z}_j^H$ is in the span of $\mathcal{A}_{n,\lambda,s}^{j\text{-alt}}$.

Indeed, let $I = \text{sort}_{\leq}(H) \in \mathcal{I}_s^j$. By Lemma 3.2.6, we have $\lambda^{(I)} \leq_{\text{dom}} \lambda^{(H)}$, so by Lemma 3.1.18 we have $\mathcal{A}_{n-j, \lambda^{(H)}, s} \subseteq \mathcal{A}_{n-j, \lambda^{(I)}, s}$. Hence, we have $\mathbf{x}_{n-j}^\alpha \in \mathcal{A}_{n-j, \lambda^{(I)}, s}$. Furthermore, since H is a permutation of I , we have $\epsilon_j \mathbf{z}_j^H = \pm \epsilon_j \mathbf{z}_j^I$, so $\mathbf{x}_{n-j}^\alpha \cdot \epsilon_j \mathbf{z}_j^H$ is in the span of $\mathcal{A}_{n,\lambda,s}^{j\text{-alt}}$, and the proof is complete. \square

Given a list of distinct variables $W = (w_1, \dots, w_v) \subseteq \mathbf{z}_j$ and any subset $U = \{u_1 < \dots < u_{|U|}\} \subset [v]$, let $W_U := (w_{u_1}, \dots, w_{u_{|U|}})$. Given a tuple $I \in \mathcal{I}_s^v$, let $I_U := (i_{u_1} < \dots < i_{u_{|U|}}) \in \mathcal{I}_s^{|U|}$. For any tuple $H = (h_1 < \dots < h_{|U|}) \in \mathcal{I}_s^{|U|}$, define the monomial $W_U^H := w_{u_1}^{h_1} \dots w_{u_{|U|}}^{h_{|U|}}$. Recall the following lemma on elementary symmetric polynomials.

Lemma 3.2.9 ([29], Lemma 6.1). *Let $C \subseteq \mathbf{x}_n$ and $W = (w_1, \dots, w_v) \subseteq \mathbf{z}_j$ be disjoint sets of variables, where W comes with a total ordering. Let $I = (i_1 < \dots < i_v) \in \mathcal{I}_s^v$. For any $d \geq 1$, we have*

$$e_d(C) \cdot (-w_1)^{i_1} \dots (-w_v)^{i_v} = \sum_{U \subseteq [v]} (-1)^{|U|} \sum_H (-1)^{\Sigma(H)} e_{d+\Sigma(I)-\Sigma(H)}(C \cup W_U) \cdot W_U^H, \quad (3.2.14)$$

where the inner sum on the right-hand side is over all tuples $H = (h_1, \dots, h_{|U|})$ of nonnegative integers, such that $h_p < i_{u_p}$ for $1 \leq p \leq |U|$.

Lemma 3.2.10. *Let $I \in \mathcal{I}_s^j$, and let $e_d(S)$ be a generator of $I_{n-j, \lambda^{(I)}, s}$ for some d and $S \subseteq \mathbf{x}_{n-j}$. We have $e_d(S \cup \mathbf{z}_a) \cdot \mathbf{z}_j^I \in I_{n,\lambda,s}$, where $a = p_{|S|+j}^n(\lambda) - p_{|S|}^{n-j}(\lambda^{(I)})$ and $\mathbf{z}_a = \{z_1, \dots, z_a\}$.*

Proof. Let $m = |S|$. Then $a = p_{m+j}^n(\lambda) - p_m^{n-j}(\lambda^{(I)})$, which is the number of boxes weakly to the right of column $n - (m + j) + 1$ of the skew diagram $\lambda/\lambda^{(I)}$. In order to show $e_d(S \cup \mathbf{z}_a) \cdot \mathbf{z}_j^I \in I_{n,\lambda,s}$, it suffices to prove that $e_d(S \cup \mathbf{z}_a) \cdot z_{a+1}^{i_{a+1}} \cdots z_j^{i_j} \in I_{n,\lambda,s}$, where the product $z_{a+1}^{i_{a+1}} \cdots z_j^{i_j}$ is taken to be 1 in the case $a = j$. By our assumption that $e_d(S)$ is a generator of $I_{n-j,\lambda^{(I)},s}$, we have

$$d > m - p_m^{n-j}(\lambda^{(I)}). \quad (3.2.15)$$

In the case $a = j$, then by combining (3.2.15) and $p_m^{n-j}(\lambda^{(I)}) = p_{m+j}^n(\lambda) - a$, we have $d > m + a - p_{m+a}^n(\lambda)$. Hence, $e_d(S \cup \mathbf{z}_a) \in I_{n,\lambda,s}$ so $e_d(S \cup \mathbf{z}_a) \mathbf{z}_j^I \in I_{n,\lambda,s}$, and we are done.

In the case $a < j$, then applying Lemma 3.2.9 with $C = S \cup \mathbf{z}_a$ and $W = (z_{a+1}, \dots, z_j)$, we have that

$$e_d(S \cup \mathbf{z}_a) z_{a+1}^{i_{a+1}} \cdots z_j^{i_j} \quad (3.2.16)$$

is an alternating sum of terms of the form

$$e_{d+\Sigma(I)-\Sigma(H)}(S \cup \mathbf{z}_a \cup \mathbf{z}_U) \cdot \mathbf{z}_U^H, \quad (3.2.17)$$

where $U \subseteq \{a+1, \dots, j\}$ and $H = (h_1, \dots, h_{|U|}) \in \mathcal{I}_s^{|U|}$ such that $h_p < i_{u_p}$ for $1 \leq p \leq |U|$. To complete the proof, we show that in each case $e_{d+\Sigma(I)-\Sigma(H)}(S \cup \mathbf{z}_a \cup \mathbf{z}_U) \in I_{n,\lambda,s}$, or equivalently that

$$d + \Sigma(I) - \Sigma(H) > m + a + |U| - p_{m+a+|U|}^n(\lambda). \quad (3.2.18)$$

We claim that $i_{a+1} \geq \lambda'_{n-(m+j)+1}$. If $|\lambda/\lambda^{(I)}| = a$, then by the definition of $\lambda^{(I)}$, we see that $|\lambda/\lambda^{(I)}|$ is the number of elements of I which are strictly less than $\ell(\lambda)$. Under the assumption that $|\lambda/\lambda^{(I)}| = a$, we have $i_{a+1} \geq \ell(\lambda) \geq \lambda'_{n-(m+j)+1}$. Otherwise, we have $|\lambda/\lambda^{(I)}| \geq a+1$, so by the definition of a , the $(a+1)$ th box of $\lambda/\lambda^{(I)}$ from the right must

be in a column strictly to the left of column $n - (m + j) + 1$ of λ . Therefore, the inequality $i_{a+1} \geq \lambda'_{n-(m+j)+1}$ continues to hold.

We have the string of inequalities

$$i_j > \cdots > i_{a+2} > i_{a+1} \geq \lambda'_{n-(m+j)+1} \geq \lambda'_{n-(m+j-1)+1} \geq \cdots \quad (3.2.19)$$

Therefore, we have

$$\sum_{t \in [a+1, j] \setminus U} i_t \geq \sum_{t=m+a+|U|+1}^{m+j} \lambda'_{n-t+1} = p_{m+j}^n(\lambda) - p_{m+a+|U|}^n(\lambda), \quad (3.2.20)$$

since both sides of the inequality sum over $j - a - |U|$ many terms. By our assumption that $i_{u_p} > h_p$ for $1 \leq p \leq |U|$, we have

$$\sum_{t \in U} i_t \geq \Sigma(H) + |U|. \quad (3.2.21)$$

Recalling (3.2.15), we have

$$d > m - p_m^{n-j}(\lambda^{(I)}) = m + a - p_{m+j}^n(\lambda). \quad (3.2.22)$$

The inequality (3.2.18) then follows by combining (3.2.20), (3.2.21), and (3.2.22) with

$$\Sigma(I) \geq \sum_{t \in [a+1, j] \setminus U} i_t + \sum_{t \in U} i_t, \quad (3.2.23)$$

which completes the proof. \square

Lemma 3.2.11. *With the same hypotheses as Lemma 3.2.10, we have*

$$e_d(S) \cdot \epsilon_j \mathbf{z}_j^I \in \text{span}_{\mathbb{Q}} \left(\bigcup_{\substack{H \in \mathcal{I}_s^j \\ \Sigma(H) \geq \Sigma(I)+1}} \mathbb{Q}[\mathbf{x}_{n-j}] \cdot \epsilon_j \mathbf{z}_j^H \right) + \epsilon_j I_{n, \lambda, s}. \quad (3.2.24)$$

Proof. We have that

$$e_d(S) = e_d(S \cup \mathbf{z}_a) - \sum_{t=1}^d e_{d-t}(S) \cdot e_t(\mathbf{z}_a), \quad (3.2.25)$$

so multiplying both sides by \mathbf{z}_j^I and applying ϵ_j , we have

$$e_d(S) \cdot \epsilon_j \mathbf{z}_j^I = \epsilon_j(e_d(S \cup \mathbf{z}_a) \cdot \mathbf{z}_j^I) - \sum_{t=1}^d e_{d-t}(S) \cdot \epsilon_j(e_t(\mathbf{z}_a) \cdot \mathbf{z}_j^I). \quad (3.2.26)$$

By Lemma 3.2.10, we have $e_d(S \cup \mathbf{z}_a) \cdot \mathbf{z}_j^I \in I_{n,\lambda,s}$. Therefore, by (3.2.26), it suffices to prove that each polynomial of the form $e_{d-t}(S) \cdot \epsilon_j(e_t(\mathbf{z}_a) \cdot \mathbf{z}_j^I)$ is in the set on the right-hand side of (3.2.24).

For each $t \geq 1$, consider the expansion of $e_t(\mathbf{z}_a) \cdot \mathbf{z}_j^I$ on the right-hand side of (3.2.26) into monomials. Each term in the expansion is of one of the following types: (1) a monomial in \mathbf{z}_j such that two of the exponents agree, (2) a monomial in \mathbf{z}_j whose largest exponent is s , or (3) a monomial of the form $\pi \cdot \mathbf{z}_j^H$ for some $\pi \in S_{\{n-j+1, \dots, n\}}$ and some $H \in \mathcal{I}_s^j$ such that $\Sigma(H) \geq \Sigma(I) + 1$. Monomials of the first type are sent to 0 by the operator ϵ_j , monomials of the second type are elements of $I_{n,\lambda,s}$, and for a monomial of the third type, we have $\epsilon_j(\pi \cdot \mathbf{z}_j^H) = \pm \epsilon_j \mathbf{z}_j^H$. Therefore, each term in the sum on the right-hand side of (3.2.26) is in the set on the right-hand side of (3.2.24). Hence, $e_d(S) \cdot \epsilon_j \mathbf{z}_j^I$ is in the set as well. \square

Lemma 3.2.12. *We have an isomorphism of graded S_{n-j} -modules*

$$\epsilon_j R_{n,\lambda,s} \cong \bigoplus_{r \geq 0} V_r, \quad (3.2.27)$$

where V_r is defined in (3.2.2).

Proof. Given $r \geq 0$, let

$$U_r := \bigoplus_{\substack{I \in \mathcal{I}_s^j, \\ \Sigma(I)=r}} \mathbb{Q}[\mathbf{x}_{n-j}] \otimes \mathbb{Q}\{\epsilon_j \mathbf{z}_j^I\}, \quad (3.2.28)$$

$$W_{\geq r} := \text{span}_{\mathbb{Q}} \left(\bigcup_{\substack{I \in \mathcal{I}_s^j, \\ \Sigma(I) \geq r}} \mathbb{Q}[\mathbf{x}_{n-j}] \cdot \epsilon_j \mathbf{z}_j^I \right) \subseteq \epsilon_j R_{n,\lambda,s}, \quad (3.2.29)$$

$$W_r := W_{\geq r} / W_{\geq r+1}. \quad (3.2.30)$$

By Lemma 3.2.8, we have $W_{\geq 0} = \epsilon_j R_{n,\lambda,s}$, so the subspaces $W_{\geq r}$ of $\epsilon_j R_{n,\lambda,s}$ form a descending filtration of the space $\epsilon_j R_{n,\lambda,s}$. Observe that the spaces U_r , V_r , $W_{\geq r}$, and W_r are $\mathbb{Q}[\mathbf{x}_{n-j}]$ -modules. Furthermore, each of them has the structure of a graded S_{n-j} -module, where S_{n-j} acts on the variables \mathbf{x}_{n-j} and the grading is by total degree in the variables $\mathbf{x}_n = \mathbf{x}_{n-j} \cup \mathbf{z}_j$. Since $W_{\geq r}$ is a filtration of $\epsilon_j R_{n,\lambda,s}$ which respects the graded S_{n-j} -module structure, we have that $\epsilon_j R_{n,\lambda,s} \cong \bigoplus_{r \geq 0} W_r$ as graded S_{n-j} -modules. Therefore, it suffices to prove that $V_r \cong W_r$ as graded S_{n-j} -modules.

For $r \geq 0$, let

$$\tilde{\mu}_r : U_r \rightarrow W_r \quad (3.2.31)$$

be the map induced by sending $f(\mathbf{x}_{n-j}) \otimes \epsilon_j \mathbf{z}_j^I$ to the product $f(\mathbf{x}_{n-j}) \cdot \epsilon_j \mathbf{z}_j^I$. Then $\tilde{\mu}_r$ is a homomorphism of $\mathbb{Q}[\mathbf{x}_{n-j}]$ -modules, as well as a homomorphism of graded S_{n-j} -modules. For each $I \in \mathcal{I}_s^j$ such that $\Sigma(I) = r$ and each generator of $I_{n-j,\lambda^{(I)},s}$ of the form $e_d(S)$, we have that $\tilde{\mu}_r(e_d(S) \otimes \epsilon_j \mathbf{z}_j^I) = 0$ by Lemma 3.2.11. We also have that $\tilde{\mu}_r(x_i^s \otimes \epsilon_j \mathbf{z}_j^I) = 0$ for all $i \leq n-j$ since $x_i^s \cdot \epsilon_j \mathbf{z}_j^I = \epsilon_j(x_i^s \cdot \mathbf{z}_j^I) \in \epsilon_j I_{n,\lambda,s}$. Since $\tilde{\mu}_r$ is a homomorphism of $\mathbb{Q}[\mathbf{x}_{n-j}]$ -modules, the map $\tilde{\mu}_r$ descends to an S_{n-j} -module homomorphism $\mu_r : V_r \rightarrow W_r$.

By Theorem 3.1.17, the set $\mathcal{A}_{n-j,\lambda^{(I)},s}$ is a basis of $R_{n-j,\lambda^{(I)},s}$, so

$$\bigcup_{\substack{I \in \mathcal{I}_s^j, \\ \Sigma(I)=r}} \mathcal{A}_{n-j,\lambda^{(I)},s} \otimes \epsilon_j \mathbf{z}_j^I \quad (3.2.32)$$

represents a basis of V_r . Furthermore, by Lemma 3.2.8 we have that $\mathcal{A}_{n,\lambda,s}^{j\text{-alt}}$ represents a basis of $\epsilon_j R_{n,\lambda,s}$, hence

$$\bigcup_{\substack{I \in \mathcal{I}_s^j, \\ \Sigma(I)=r}} \mathcal{A}_{n-j,\lambda^{(I)},s} \cdot \epsilon_j \mathbf{z}_j^I \quad (3.2.33)$$

represents a basis of W_r . Since μ_r maps a basis to a basis, it is an isomorphism of graded S_{n-j} -modules, hence $V_r \cong W_r$ as graded S_{n-j} -modules. This completes the proof. \square

Proof of Theorem 3.2.2. The identity (2.0.28) implies the first equality in Theorem 3.2.2. By Lemma 3.2.12, we have the equality of formal power series

$$\text{Frob}_q(\epsilon_j R_{n,\lambda,s}) = \sum_{I \in \mathcal{I}_s^j} q^{\Sigma(I)} \text{Frob}_q(R_{n-j,\lambda^{(I)},s}), \quad (3.2.34)$$

which is the second equality in Theorem 3.2.2. \square

3.2.2 Exact sequences

In this subsection, we show that the rings $R_{n,\lambda,s}$ fit into certain exact sequences of S_n -modules. We use these exact sequences to obtain identities for the graded Frobenius characteristic of $R_{n,\lambda,s}$.

Lemma 3.2.13. *Let $k < n$ be distinct positive integers, and let $\lambda \in \text{Par}(k, s)$ with $\ell(\lambda) < s$. Then there is an exact sequence of S_n -modules*

$$0 \rightarrow R_{n,\lambda,s-1} \rightarrow R_{n,\lambda,s} \rightarrow R_{n,\lambda \bullet (1),s} \rightarrow 0 \quad (3.2.35)$$

such that the first map shifts degree by $n - k$ and the second map is degree-preserving. Equivalently, we have the identity

$$\text{Frob}_q(R_{n,\lambda,s}) = \text{Frob}_q(R_{n,\lambda\bullet(1),s}) + q^{n-k}\text{Frob}_q(R_{n,\lambda,s-1}). \quad (3.2.36)$$

Proof. It can be checked that $I_{n,\lambda\bullet(1),s} = I_{n,\lambda,s} + \langle e_{n-k}(\mathbf{x}_n) \rangle$. Hence, we have the quotient map $\pi : R_{n,\lambda,s} \rightarrow R_{n,\lambda\bullet(1),s}$ with $\ker \pi = \langle e_{n-k}(\mathbf{x}_n) \rangle$. To construct a map $R_{n,\lambda,s-1} \rightarrow R_{n,\lambda,s}$, first define the map

$$\tilde{\phi} : \mathbb{Q}[\mathbf{x}_n] \rightarrow R_{n,\lambda,s} \quad (3.2.37)$$

of $\mathbb{Q}[\mathbf{x}_n]$ -modules given by multiplication by $e_{n-k}(\mathbf{x}_n)$.

We claim that $I_{n,\lambda,s-1} \subseteq \ker(\tilde{\phi})$. Since $I_{n,\lambda,s}$ and $I_{n,\lambda,s-1}$ have the same elementary symmetric polynomial generators, it suffices to show that $\tilde{\phi}(x_i^{s-1}) = 0$ for all $i \leq n$, or equivalently that $x_i^{s-1}e_{n-k}(\mathbf{x}_n) \in I_{n,\lambda,s}$. By symmetry, it suffices to show that $x_n^{s-1}e_{n-k}(\mathbf{x}_n) \in I_{n,\lambda,s}$. Observe that $x_n^{s-1}e_{n-k}(\mathbf{x}_n) \equiv x_n^{s-1}e_{n-k}(\mathbf{x}_{n-1}) \pmod{I_{n,\lambda,s}}$. Furthermore, since $e_{n-k}(\mathbf{x}_{n-1})$ is a generator of $I_{n-1,\lambda,s}$, then by Lemma 3.1.13(d) applied to $S = \mathbf{x}_{n-1}$ and $i = s$, we have $x_n^{s-1}e_{n-k}(\mathbf{x}_{n-1}) \in x_n^s\mathbb{Q}[\mathbf{x}_n] + I_{n,\lambda,s} = I_{n,\lambda,s}$. Hence, $x_n^{s-1}e_{n-k}(\mathbf{x}_n) \in I_{n,\lambda,s}$, so $I_{n,\lambda,s-1} \subseteq \ker(\tilde{\phi})$ as claimed.

Therefore, $\tilde{\phi}$ descends to a map

$$\phi : R_{n,\lambda,s-1} \rightarrow R_{n,\lambda,s} \quad (3.2.38)$$

whose image is exactly $\langle e_{n-k}(\mathbf{x}_n) \rangle = \ker \pi$. Therefore, the sequence

$$R_{n,\lambda,s-1} \xrightarrow{\phi} R_{n,\lambda,s} \xrightarrow{\pi} R_{n,\lambda\bullet(1),s} \rightarrow 0 \quad (3.2.39)$$

is exact, where ϕ shifts degree by $n - k$. Furthermore, we have

$$|\mathcal{OP}_{n,\lambda,s}| = |\mathcal{OP}_{n,\lambda\bullet(1),s}| + |\mathcal{OP}_{n,\lambda,s-1}|, \quad (3.2.40)$$

which can be seen by partitioning $\mathcal{OP}_{n,\lambda,s}$ into a disjoint union of two subsets by considering those elements $(B_1 | \cdots | B_s)$ such that $B_{\ell(\lambda)+1} \neq \emptyset$ and those such that $B_{\ell(\lambda)+1} = \emptyset$. By Theorem 3.1.20, then (3.2.40) implies

$$\dim_{\mathbb{Q}}(R_{n,\lambda,s}) = \dim_{\mathbb{Q}}(R_{n,\lambda\bullet(1),s}) + \dim_{\mathbb{Q}}(R_{n,\lambda,s-1}). \quad (3.2.41)$$

Hence, ϕ is injective, and we have an exact sequence

$$0 \rightarrow R_{n,\lambda,s-1} \xrightarrow{\phi} R_{n,\lambda,s} \xrightarrow{\pi} R_{n,\lambda\bullet(1),s} \rightarrow 0. \quad (3.2.42)$$

To complete the proof, observe that ϕ and π are S_n -module homomorphisms. \square

Theorem 3.2.14. *Let $k \leq n$ be positive integers and let $\lambda \in \text{Par}(k, s)$ such that $\ell(\lambda) < s$.*

We have

$$\text{Frob}_q(R_{n,\lambda,s}) = \sum_{m \geq 0} q^{(s-\ell(\lambda)-m)(n-k-m)} \begin{bmatrix} s - \ell(\lambda) \\ m \end{bmatrix}_q \text{Frob}_q(R_{n,\lambda\bullet(1^m),\ell(\lambda)+m}), \quad (3.2.43)$$

where $\begin{bmatrix} s - \ell(\lambda) \\ m \end{bmatrix}_q = 0$ for $m > s - \ell(\lambda)$ and $\text{Frob}_q(R_{n,\lambda\bullet(1^m),\ell(\lambda)+m}) = 0$ for $m > n - k$.

Proof. Proceed by induction on $s - \ell(\lambda)$. The base case where $s - \ell(\lambda) = 1$ holds by Lemma 3.2.13. Fix $a > 1$, and assume by way of induction that (3.2.43) holds for s and λ such that $0 < s - \ell(\lambda) < a$. Let $k \leq n$ be positive integers, and let $\lambda \in \text{Par}(k, s)$ such that $s - \ell(\lambda) = a$. By (3.2.36), we have

$$\text{Frob}_q(R_{n,\lambda,s}) = \text{Frob}_q(R_{n,\lambda\bullet(1),s}) + q^{n-k} \text{Frob}_q(R_{n,\lambda,s-1}). \quad (3.2.44)$$

Applying our inductive hypothesis, we have

$$\begin{aligned} \text{Frob}_q(R_{n,\lambda,s}) &= \sum_{m \geq 0} q^{(s-\ell(\lambda)-1-m)(n-k-1-m)} \begin{bmatrix} s-\ell(\lambda)-1 \\ m \end{bmatrix}_q \text{Frob}_q(R_{n,\lambda \bullet (1^{m+1}), \ell(\lambda)+m+1}) \\ &\quad + q^{n-k} \left(\sum_{p \geq 0} q^{(s-1-\ell(\lambda)-p)(n-k-p)} \begin{bmatrix} s-\ell(\lambda)-1 \\ p \end{bmatrix}_q \text{Frob}_q(R_{n,\lambda \bullet (1^p), \ell(\lambda)+p}) \right), \end{aligned} \quad (3.2.45)$$

which we may rewrite as

$$\begin{aligned} \text{Frob}_q(R_{n,\lambda,s}) &= q^{(s-\ell(\lambda))(n-k)} \text{Frob}_q(R_{n,\lambda, \ell(\lambda)}) \\ &\quad + \sum_{m \geq 1} \left(q^{(s-\ell(\lambda)-m)(n-k-m)} \begin{bmatrix} s-\ell(\lambda)-1 \\ m-1 \end{bmatrix}_q \right. \\ &\quad \left. + q^{(s-1-\ell(\lambda)-m)(n-k-m)+(n-k)} \begin{bmatrix} s-\ell(\lambda)-1 \\ m \end{bmatrix}_q \right) \text{Frob}_q(R_{n,\lambda \bullet (1^m), \ell(\lambda)+m}). \end{aligned} \quad (3.2.46)$$

The identity $\begin{bmatrix} a \\ b \end{bmatrix}_q = \begin{bmatrix} a-1 \\ b-1 \end{bmatrix}_q + q^b \begin{bmatrix} a-1 \\ b \end{bmatrix}_q$ with $a = s - \ell(\lambda)$ and $b = m$ shows that the coefficient of $\text{Frob}_q(R_{n,\lambda \bullet (1^m), \ell(\lambda)+m})$ on the right-hand side of (3.2.46) is equal to

$$q^{(s-\ell(\lambda)-m)(n-k-m)} \begin{bmatrix} s-\ell(\lambda) \\ m \end{bmatrix}_q. \quad (3.2.47)$$

Hence, the right-hand side of (3.2.46) is equal to the right-hand side of (3.2.43), which completes the induction. \square

3.3 Inversions and diagonal inversions

In this section, we define inversion and diagonal inversion statistics on labeled objects which are in bijection with $\mathcal{OP}_{n,\lambda,s}$ and use them to give formulas for the Hilbert series and graded Frobenius characteristic of $R_{n,\lambda,s}$. In Subsection 3.3.1, we define extended column-increasing fillings and the statistics inv and dinv . In Subsection 3.3.2, we prove that the statistics inv and dinv are equidistributed and their monomial generating functions are equal. In Subsection 3.3.3, we then prove our Hilbert series and graded Frobenius characteristic formulas for

1		
3	2	
5	9	
7	10	4
8		6

Figure 3.2: The standard extended column-increasing filling associated to $(1, 3, 5, 7, 8 | 2, 9, 10 | 4, 6) \in \mathcal{OP}_{10,(3,2),3}$, where the basement cells are bolded.

$R_{n,\lambda,s}$ in terms of these statistics.

3.3.1 The statistics inv and dinv

Given $(B_1 | \cdots | B_s) \in \mathcal{OP}_{n,\lambda,s}$, define a *standard extended column-increasing filling* of λ' as follows. First, define a column-increasing filling of the Young diagram of λ' by labeling the i th column with the λ_i smallest elements of B_i . Then, for each i , place the rest of the elements of B_i in their own cells vertically below the i th column of the diagram in increasing order from top to bottom. We call the labels below the diagram the *basement labels* and the cells containing them the *basement cells*.

Let $\text{SECI}_{n,\lambda,s}$ be the set of standard extended column-increasing fillings of λ' . See Figure 3.2 for the standard extended column-increasing filling associated to the ordered set partition $(1, 3, 5, 7, 8 | 2, 9, 10 | 4, 6) \in \mathcal{OP}_{10,(3,2),3}$, where 4, 6, 7, 8, and 10 are basement labels.

Given a composition $\alpha \in \text{Comp}(k, s)$, we define an *extended column-increasing filling* φ of $\text{dg}'(\alpha)$ to consist of

- A diagram $D(\varphi) = \text{dg}'(\alpha) \cup B(\varphi)$, where $B(\varphi)$ is a possibly empty collection of *basement cells* in columns $1 \leq i \leq s$ and rows $j \leq 0$, such that in each column i the basement cells are top justified so that the top basement cell is at coordinates $(i, 0)$,
- A labeling of the cells of $D(\varphi)$ which weakly increases down each column.

We denote by $\sigma(\varphi)$ the column-increasing filling obtained by restricting φ to $\text{dg}'(\alpha)$. Given

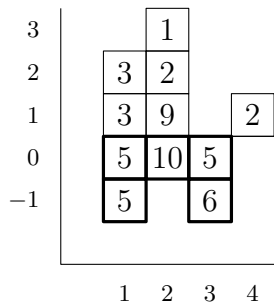


Figure 3.3: An extended column-increasing filling in $\text{ECI}_{11,(2,3,0,1),4}$. We have labeled the rows and columns for the reader’s aid.

a cell $(i, j) \in D(\varphi)$, we denote by $\varphi_{i,j}$ the label of φ in the cell (i, j) . Let $\text{ECI}_{n,\alpha,s}$ be the set of extended column-increasing fillings φ of $\text{dg}'(\alpha)$ with n cells. Let $\text{SECI}_{n,\alpha,s}$ be the subset of $\text{ECI}_{n,\alpha,s}$ consisting of *standard* extended column-increasing fillings which use the letters in $[n]$ without repetition. See Figure 3.3 for an example of an extended column-increasing filling in $\text{ECI}_{11,(2,3,0,1),4}$.

The *reading word* of φ is the concatenation $\text{rw}(\varphi) := \text{rw}(\sigma(\varphi))v$, where v is the word obtained by reading the labels of $B(\varphi)$ left to right across each row, from top to bottom. The *inversion reading word* of φ is the concatenation $\text{irw}(\varphi) := \text{rw}(\sigma(\varphi))w$, where w is the word obtained by reading the labels of $B(\varphi)$ down each column, starting with column 1 and ending with column s . The ordering of the cells of φ corresponding to $\text{rw}(\varphi)$ and $\text{irw}(\varphi)$ are the *reading order* and the *inversion reading order* of φ , respectively. For the standard extended column-increasing filling φ in Figure 3.2, we have $\text{rw}(\varphi) = 13259710486$ and $\text{irw}(\varphi) = 13259781046$.

Definition 3.3.1. Given $\varphi \in \text{ECI}_{n,\alpha,s}$, an *inversion* of φ is one of the following,

- (I1) A diagonal inversion of $\sigma(\varphi)$, as defined in Subsection 2.0.5,
- (I2) A pair $((i, 1), (i', j'))$ where $(i, 1) \in \text{dg}'(\alpha)$ and $(i', j') \in B(\varphi)$, and such that $i > i'$ and $\varphi_{i,1} > \varphi_{i',j'}$,

(I3) A pair $(i, (i', j'))$, where $(i', j') \in B(\varphi)$ and i is an integer such that $1 \leq i < i'$.

Let $\text{inv}(\varphi)$ be the number of inversions of φ .

For φ in Figure 3.3, we have the following inversions,

Type (I1): $((1, 2), (2, 2))$, $((1, 1), (4, 1))$, and $((2, 1), (4, 1))$

Type (I2): $((2, 1), (1, 0))$ and $((2, 1), (1, -1))$

Type (I3): $(1, (2, 0))$, $(1, (3, 0))$, $(2, (3, 0))$, $(1, (3, -1))$, and $(2, (3, -1))$.

In total, we have $\text{inv}(\varphi) = 10$.

We also introduce a diagonal inversion statistic on extended column-increasing fillings. Let an α -attacking pair be a pair of coordinates $((i, j), (i', j'))$ such that $1 \leq i \leq s$, $1 \leq i' \leq s$, $j \leq \alpha_i$, and $j' \leq \alpha_{i'}$, and such that one of the following holds,

- We have $j = j'$ and $i < i'$,
- We have $j = j' + 1$ and $i > i'$.

Definition 3.3.2. A *diagonal inversion* of φ is an α -attacking pair $((i, j), (i', j'))$ such that one of the following holds,

(D1) We have $(i, j), (i', j') \in D(\varphi)$ such that $\varphi_{i,j} > \varphi_{i',j'}$,

(D2) We have $(i, j) \notin D(\varphi)$ and $(i', j') \in D(\varphi)$.

Let $\text{dinv}(\varphi)$ be the number of diagonal inversions of φ .

For φ in Figure 3.3, we have the following diagonal inversions,

Type (D1): $((1, 2), (2, 2))$, $((1, 1), (4, 1))$, $((2, 1), (4, 1))$, and $((2, 1), (1, 0))$,
 $((2, 0), (3, 0))$ and $((2, 0), (1, -1))$

Type (D2): $((4, 0), (1, -1))$, $((4, 0), (3, -1))$, and $((2, -1), (3, -1))$.

In total, we have $\text{dinv}(\varphi) = 9$.

Remark 3.3.3. The notion of an extended column-increasing filling is a variation of the fillings introduced in [71]. To translate between our conventions and theirs, simply flip our labelings across the horizontal axis and convert each basement label into a floating number. Under this identification between $\mathcal{OP}_{n,\lambda,s}$ and $\text{SECI}_{n,\lambda,s}$, the inversion statistic in Definition 3.3.1 is similar to, but not the same as, the coinv statistic in [71]. In particular, coinv uses a different definition of attacking pair, where $((i, j), (i', j'))$ forms an attacking pair in λ' if $j = j'$ and $i < i'$, or $j' = j + 1$ and $i > i'$. The condition (I1) in Definition 3.3.1 is then replaced with the condition that this alternate type of attacking pair contributes to the number of coinversions if $\varphi_{i,j} < \varphi_{i',j'}$.

3.3.2 Equidistribution of inv and dinv

We need the following equidistribution theorem for the statistics inv and dinv . Before proving the theorem, we prove several lemmata.

Theorem 3.3.4. *For $\lambda \in \text{Par}(k, s)$, we have the identity of multivariate generating functions,*

$$\sum_{\varphi \in \text{ECI}_{n,\lambda,s}} q^{\text{inv}(\varphi)} \mathbf{x}^\varphi = \sum_{\varphi \in \text{ECI}_{n,\lambda,s}} q^{\text{dinv}(\varphi)} \mathbf{x}^\varphi, \quad (3.3.1)$$

where $\mathbf{x}^\varphi := \prod_{i \geq 1} x_i^{\#\text{ } i \text{'s in } \varphi}$.

Our strategy for proving Theorem 3.3.4 is inspired by generalizations of the Carlitz bijection constructed by Gillespie [61], Rhoades-Wilson [70], Rhoades-Yu-Zhao [71], and Wilson [97]. Since [71] uses our theorems to prove their results, we give independent proofs in order to avoid creating a cycle in the logical flow of the proofs.

Let $\gamma = (\gamma_1, \dots, \gamma_m)$ be a composition of n with $\gamma_i > 0$ for all i . A sequence $c = (c_1, \dots, c_n)$ of nonnegative integers is γ -weakly decreasing if every subsequence

$$c_{\gamma_1 + \dots + \gamma_i}, c_{\gamma_1 + \dots + \gamma_{i+1}}, \dots, c_{\gamma_1 + \dots + \gamma_i + \gamma_{i+1} - 1}$$

is weakly decreasing. For example, if $\gamma = (3, 2, 1, 1)$, then the sequence $c = (5, 5, 2, 6, 3, 1, 2)$ is γ -weakly decreasing. Let

$$\mathcal{C}_{n,\lambda,s}(\gamma) := \{c \in \mathcal{C}_{n,\lambda,s} : c \text{ is } \gamma\text{-weakly decreasing}\}. \quad (3.3.2)$$

Observe that when $\gamma = (1^n)$, then $\mathcal{C}_{n,\lambda,s}(1^n) = \mathcal{C}_{n,\lambda,s}$. For $0 \leq i \leq s-1$ and any λ , let

$$\mathcal{C}_{n-1,\lambda,s}^{(i)}(\gamma) := \{(c_1, \dots, c_{n-1}) \in \mathcal{C}_{n-1,\lambda,s}(\gamma) : c_{n-1} \geq i\}. \quad (3.3.3)$$

Lemma 3.3.5. *Let $\gamma = (\gamma_1, \dots, \gamma_m)$ be a composition of n with $\gamma_i > 0$ for all i . If $\gamma_m > 1$, we have*

$$\mathcal{C}_{n,\lambda,s}(\gamma) = \bigcup_{i=0}^{\ell(\lambda)-1} \mathcal{C}_{n-1,\lambda^{(i)},s}^{(i)}(\gamma_1, \dots, \gamma_m - 1) \bullet (i) \cup \bigcup_{i=\ell(\lambda)}^{s-1} \mathcal{C}_{n-1,\lambda,s}^{(i)}(\gamma_1, \dots, \gamma_m - 1) \bullet (i). \quad (3.3.4)$$

If $\gamma_m = 1$, we have

$$\mathcal{C}_{n,\lambda,s}(\gamma) = \bigcup_{i=0}^{\ell(\lambda)-1} \mathcal{C}_{n-1,\lambda^{(i)},s}(\gamma_1, \dots, \gamma_m - 1) \bullet (i) \cup \bigcup_{i=\ell(\lambda)}^{s-1} \mathcal{C}_{n-1,\lambda,s}(\gamma_1, \dots, \gamma_m - 1) \bullet (i). \quad (3.3.5)$$

Proof. In the case where $\gamma_m > 1$, then for $i < \ell(\lambda)$, the set $\mathcal{C}_{n-1,\lambda^{(i)},s}^{(i)}(\gamma_1, \dots, \gamma_m - 1) \bullet (i)$ is the subset of $\mathcal{C}_{n-1,\lambda^{(i)},s} \bullet (i)$ of γ -weakly decreasing sequences, by definition. Similarly, for $i \geq \ell(\lambda)$, the set $\mathcal{C}_{n-1,\lambda,s}^{(i)}(\gamma_1, \dots, \gamma_m - 1) \bullet (i)$ is the subset of $\mathcal{C}_{n-1,\lambda,s} \bullet (i)$ of γ -weakly decreasing sequences, by definition. Therefore, the partition (3.3.4) follows immediately from Lemma 3.1.8 by restricting both sides to γ -weakly decreasing sequences. The case where $\gamma_m = 1$ follows from Lemma 3.1.8 by similar reasoning. \square

Given a word w on positive integers, we say that it has *content* γ if i appears as a letter in w exactly γ_i many times for $1 \leq i \leq m$. Let $\text{ECI}_{n,\alpha,s}(\gamma)$ be the set of $\varphi \in \text{ECI}_{n,\alpha,s}$ such that $\text{rw}(\varphi)$ has content γ .

To each $\varphi \in \text{ECI}_{n,\alpha,s}(\gamma)$, we assign an *inversion code* and a *diagonal inversion code*,

defined next. These codes are inspired by the generalized Carlitz codes in [61] and the coinversion codes in [70, 71].

Given $\varphi \in \text{ECI}_{n,\alpha,s}(\gamma)$, let $(i_1, j_1), (i_2, j_2), \dots, (i_n, j_n)$ be the cells of $D(\varphi)$, listed so that $\varphi_{i_1, j_1} \leq \varphi_{i_2, j_2} \leq \dots \leq \varphi_{i_n, j_n}$, and breaking ties in **inversion** reading order. For example, for φ in Figure 3.3, the cells of $D(\varphi)$ are listed in the order

$$(2, 3), (2, 2), (4, 1), (1, 2), (1, 1), (1, 0), (1, -1), (3, 0), (3, -1), (2, 1), (2, 0). \quad (3.3.6)$$

Let c_p be the total number of inversions of φ of the following types,

- Type (I1) inversions of the form $((i, j), (i_p, j_p))$ for some i and j ,
- Type (I2) inversions of the form $((i, 1), (i_p, j_p))$ for some i ,
- Type (I3) inversions of the form $(i, (i_p, j_p))$ for some i .

Define the *inversion code* of φ to be $\text{invcod}^\alpha(\varphi) := (c_n, c_{n-1}, \dots, c_1)$. Observe that the sum of the entries of $\text{invcod}^\alpha(\varphi)$ is equal to $\text{inv}(\varphi)$. For φ in Figure 3.3, we have

$$\text{invcod}^\alpha(\varphi) = (0, 1, 2, 0, 0, 1, 1, 2, 2, 0, 1), \quad (3.3.7)$$

where $\alpha = (2, 3, 0, 1)$.

Given $\varphi \in \text{ECI}_{n,\alpha,s}(\gamma)$, let $(i_1, j_1), (i_2, j_2), \dots, (i_n, j_n)$ be the cells of $D(\varphi)$, listed so that $\varphi_{i_1, j_1} \leq \varphi_{i_2, j_2} \leq \dots \leq \varphi_{i_n, j_n}$, and breaking ties in reading order. For example, for φ in Figure 3.3, the cells of $D(\varphi)$ are listed in the same order as in (3.3.6), except with the three cells labeled by 5 listed in the order $(1, 0), (3, 0), (1, -1)$. Let d_p be the total number of diagonal inversions of type (D1) and (D2) of the form $((i, j), (i_p, j_p))$ for some i and j . Define the *diagonal inversion code* to be $\text{dinvcod}^\alpha(\varphi) := (d_n, d_{n-1}, \dots, d_1)$. For φ in Figure 3.3, we have

$$\text{dinvcod}^\alpha(\varphi) = (0, 1, 2, 0, 0, 1, 1, 2, 2, 0, 0), \quad (3.3.8)$$

where $\alpha = (2, 3, 0, 1)$.

Lemma 3.3.6. *Sending $\varphi \in \text{ECI}_{n,\alpha,s}(\gamma)$ to its inversion code gives a map*

$$\text{invcode}^\alpha : \text{ECI}_{n,\alpha,s}(\gamma) \rightarrow \mathcal{C}_{n,\text{sort}(\alpha),s}(\text{rev}(\gamma)). \quad (3.3.9)$$

Proof. Let $\lambda = \text{sort}(\alpha)$ for convenience. It suffices to show that if $\varphi \in \text{ECI}_{n,\alpha,s}(\gamma)$, then $(c_n, c_{n-1}, \dots, c_1) \in \mathcal{C}_{n,\lambda,s}(\text{rev}(\gamma))$. We first show that $(c_n, c_{n-1}, \dots, c_1) \in \mathcal{C}_{n,\lambda,s}$.

Let $(i_1, j_1), (i_2, j_2), \dots, (i_n, j_n)$ be the cells of φ , listed so that $\varphi_{i_1, j_1} \leq \dots \leq \varphi_{i_n, j_n}$ and then breaking ties in inversion reading order. Suppose the λ'_m cells in the m th row of $\text{dg}'(\alpha)$ are the p_1 th, p_2 th, \dots , $p_{\lambda'_m}$ th cells in the list, with $p_1 < p_2 < \dots < p_{\lambda'_m}$. Then we have $c_{p_t} \leq \lambda'_m - t$ for $1 \leq t \leq \lambda'_m$. Therefore, $(c_{p_1}, \dots, c_{p_{\lambda'_m}}) \subseteq \beta^m(\lambda) = (0, \dots, \lambda'_m - 1)$. Furthermore, for each p such that $(i_p, j_p) \in \text{B}(\varphi)$, we have $c_p \leq s - 1$. Therefore, $(c_n, c_{n-1}, \dots, c_1)$ is contained in a (n, λ, s) -shuffle, hence it is in $\mathcal{C}_{n,\lambda,s}$.

To complete the proof, it suffices to show that $(c_n, c_{n-1}, \dots, c_1)$ is $\text{rev}(\gamma)$ -weakly decreasing. Equivalently, we show that for $1 \leq p \leq n - 1$ such that $\varphi_{i_p, j_p} = \varphi_{i_{p+1}, j_{p+1}}$, then we have $c_p \leq c_{p+1}$. Given p such that $\varphi_{i_p, j_p} = \varphi_{i_{p+1}, j_{p+1}}$, define an injection from the set of inversions counted toward c_p to the set of inversions counted toward c_{p+1} as follows.

First, suppose that both (i_p, j_p) and (i_{p+1}, j_{p+1}) are cells of $\text{dg}'(\alpha)$. In this case, all inversions counting toward c_p and c_{p+1} are among labels in $\sigma(\varphi)$. Let $((a, b), (i_p, j_p))$ be an inversion of $\sigma(\varphi)$, so $\varphi_{a,b} > \varphi_{i_p, j_p} = \varphi_{i_{p+1}, j_{p+1}}$. Since (a, b) appears before (i_{p+1}, j_{p+1}) in reading order, then we have $b \geq j_{p+1}$. Therefore, we have $a \neq i_{p+1}$ since φ is column-increasing.

If $a < i_{p+1}$, then map $((a, b), (i_p, j_p))$ to $((a, j_{p+1}), (i_{p+1}, j_{p+1}))$, which is an inversion since $\varphi_{a, j_{p+1}} \geq \varphi_{a,b} > \varphi_{i_{p+1}, j_{p+1}}$ by the fact that φ is column-increasing. Otherwise, if $a > i_{p+1}$, then $b \geq j_{p+1} + 1$. Map $((a, b), (i_p, j_p))$ to $((a, j_{p+1} + 1), (i_{p+1}, j_{p+1}))$, which is an inversion since $\varphi_{a, j_{p+1} + 1} \geq \varphi_{a,b} > \varphi_{i_{p+1}, j_{p+1}}$. It is clear that this map is an injection, since each such inversion $((a, b), (i_p, j_p))$ has a unique a value. Hence, $c_p \leq c_{p+1}$.

Second, suppose that $(i_p, j_p) \in \text{dg}'(\alpha)$ and $(i_{p+1}, j_{p+1}) \in \text{B}(\varphi)$. Let $((a, b), (i_p, j_p))$ be

an inversion of φ . We have that $a \neq i_{p+1}$ by the fact that φ is column-increasing. If $a > i_{p+1}$, then map $((a, b), (i_p, j_p))$ to the inversion $((a, 1), (i_{p+1}, j_{p+1}))$ of type (I2). This is indeed an inversion since $\varphi_{a,1} \geq \varphi_{a,b} > \varphi_{i_p, j_p} = \varphi_{i_{p+1}, j_{p+1}}$. Otherwise, if $a < i_{p+1}$, then map $((a, b), (i_p, j_p))$ to the inversion $(a, (i_{p+1}, j_{p+1}))$ of type (I3). By the same reasoning as the first case, this map is an injection, hence $c_p \leq c_{p+1}$.

Finally, suppose that $(i_p, j_p), (i_{p+1}, j_{p+1}) \in B(\varphi)$. Since (i_p, j_p) precedes (i_{p+1}, j_{p+1}) in inversion reading order, we have $i_p \leq i_{p+1}$. Let $((a, 1), (i_p, j_p))$ be an inversion of type (I2), so that $\varphi_{a,1} > \varphi_{i_p, j_p} = \varphi_{i_{p+1}, j_{p+1}}$. By the fact that φ is column-increasing, we have that $a \neq i_{p+1}$. If $a > i_{p+1}$, map $((a, 1), (i_p, j_p))$ to the inversion $((a, 1), (i_{p+1}, j_{p+1}))$ of type (I2). Otherwise, if $a < i_{p+1}$, then map $((a, 1), (i_p, j_p))$ to the inversion $(a, (i_{p+1}, j_{p+1}))$ of type (I3). If $(a, (i_p, j_p))$ is an inversion of type (I3) with $a < i_p$, then map it to the inversion $(a, (i_{p+1}, j_{p+1}))$ of type (I3), which is an inversion since $a < i_p \leq i_{p+1}$. By the same reasoning as the previous two cases, this map is an injection. Hence, we have $c_p \leq c_{p+1}$ in all cases. \square

Lemma 3.3.7. *Sending $\varphi \in \text{ECI}_{n,\alpha,s}(\gamma)$ to its diagonal inversion code gives a map*

$$\text{dinvcode}^\alpha : \text{ECI}_{n,\alpha,s}(\gamma) \rightarrow \mathcal{C}_{n,\text{sort}(\alpha),s}(\text{rev}(\gamma)). \quad (3.3.10)$$

Proof. It suffices to show that if $\varphi \in \text{ECI}_{n,\alpha,s}(\gamma)$, then $(d_n, d_{n-1}, \dots, d_1) \in \mathcal{C}_{n,\text{sort}(\alpha),s}(\text{rev}(\gamma))$. The fact that $(d_n, \dots, d_1) \in \mathcal{C}_{n,\text{sort}(\alpha),s}$ follows by the same reasoning as in the proof of Lemma 3.3.6, so it suffices to prove that $(d_n, d_{n-1}, \dots, d_1)$ is $\text{rev}(\gamma)$ -weakly decreasing. Let $(i_1, j_1), (i_2, j_2), \dots, (i_n, j_n)$ be the cells of φ , listed so that $\varphi_{i_1, j_1} \leq \dots \leq \varphi_{i_n, j_n}$ and then breaking ties in reading order. It suffices to prove that for $1 \leq p \leq n-1$ such that $\varphi_{i_p, j_p} = \varphi_{i_{p+1}, j_{p+1}}$, then we have $d_p \leq d_{p+1}$. Given p such that $\varphi_{i_p, j_p} = \varphi_{i_{p+1}, j_{p+1}}$, define an injection from the set of diagonal inversions counted toward d_p to the set of diagonal inversions counted toward d_{p+1} as follows.

First, suppose that $(i_p, j_p), (i_{p+1}, j_{p+1}) \in \text{dg}'(\alpha)$. Then $d_p = c_p$ and $d_{p+1} = c_{p+1}$, where $(c_n, \dots, c_1) = \text{invcode}^\alpha(\varphi)$, since all diagonal inversions counting toward d_p and d_{p+1} are among entries in $\sigma(\varphi)$, hence we have $d_p \leq d_{p+1}$ by Lemma 3.3.6.

Second, suppose $(i_p, j_p) \in \text{dg}'(\alpha)$ and $(i_{p+1}, j_{p+1}) \in \text{B}(\varphi)$. Let $((a, b), (i_p, j_p))$ be a diagonal inversion of φ . Just as in the first case of the proof of Lemma 3.3.6, we have $b \geq j_p \geq j_{p+1}$ and $a \neq i_{p+1}$. If $a < i_{p+1}$, map $((a, b), (i_p, j_p))$ to the diagonal inversion $((a, j_{p+1}), (i_{p+1}, j_{p+1}))$, which is either a diagonal inversion of type (D1) or of type (D2), depending on whether (a, j_{p+1}) is in $\text{D}(\varphi)$ or not, respectively. If $a > i_{p+1}$, map $((a, b), (i_p, j_p))$ to $((a, j_{p+1} + 1), (i_{p+1}, j_{p+1}))$, which is similarly either a diagonal inversion of type (D1) or (D2) depending on whether $(a, j_{p+1} + 1)$ is in $\text{D}(\varphi)$ or not, respectively. The details of the fact that this is a well-defined injection are similar to the second case of the proof of Lemma 3.3.6, hence we omit them. We conclude that $d_p \leq d_{p+1}$.

Finally, suppose that $(i_p, j_p), (i_{p+1}, j_{p+1}) \in \text{D}(\varphi)$. The map on diagonal inversions $((a, b), (i_p, j_p))$ of type (D1) is the same as in the second case above, and we omit the details. Let $((a, b), (i_p, j_p))$ be a diagonal inversion of type (D2), so that $(a, b) \notin \text{D}(\varphi)$. Since $(i_{p+1}, j_{p+1}) \in \text{D}(\varphi)$ and $b \geq j_p \geq j_{p+1}$, we have $a \neq i_{p+1}$. Observe that since $(a, b) \notin \text{D}(\varphi)$, then $(a, j_{p+1}) \notin \text{D}(\varphi)$. If $a < i_{p+1}$, map $((a, b), (i_p, j_p))$ to the diagonal inversion $((a, j_{p+1}), (i_{p+1}, j_{p+1}))$ of type (D2). Otherwise, if $a > i_{p+1}$, then $(a, j_{p+1} + 1) \notin \text{D}(\varphi)$, and we map $((a, b), (i_p, j_p))$ to the diagonal inversion $((a, j_{p+1} + 1), (i_{p+1}, j_{p+1}))$ of type (D2). Hence, we have $d_p \leq d_{p+1}$ in all cases. \square

We define a map inverse to invcode^α via an insertion algorithm based on [70, 71]. Given $(c_n, \dots, c_1) \in \mathcal{C}_{n, \text{sort}(\alpha), s}(\text{rev}(\gamma))$, construct an element $\iota^\alpha(c_n, \dots, c_1) \in \text{ECI}_{n, \alpha, s}(\gamma)$ by the following procedure. At each step in the algorithm, label the columns of the partial labeling of $\text{dg}'(\alpha)$ with $0, 1, 2, \dots, s - 1$ inductively as follows. Suppose we have already used the column labels $0, 1, \dots, j - 1$. We say that a column $1 \leq i \leq s$ is *unfilled* if there is a cell of $\text{dg}'(\alpha)$ in column i which is unfilled. If there is an unfilled column of $\text{dg}'(\alpha)$ which does not have a label, scan through the diagram in reading order until an unfilled cell of the diagram is reached whose column is unlabeled, and label that column with j . Otherwise, label the leftmost unlabeled column with j .

At the 0th step in the algorithm, start with the unfilled diagram $\text{dg}'(\alpha)$, and let $a_1 a_2 \cdots a_n$

be the unique word with content γ such that $a_1 \leq a_2 \leq \dots \leq a_n$. At the i th step for $1 \leq i \leq n$, assume we have already inserted a_1, \dots, a_{i-1} into the filling. Label the columns of the partially filled diagram according to the procedure above. Let column j be the unique column labeled c_i . If column j is unfilled, label the highest unfilled cell in that column with a_i . Otherwise, add a basement cell to column j and label it with a_i . Let $\iota^\alpha(c_n, \dots, c_1)$ be the filling obtained after step n of the algorithm. Although $\iota^\alpha(c_n, \dots, c_1)$ also depends on γ , we suppress γ from the notation for convenience.

See Figure 3.4 for an example of the insertion algorithm when $n = 9$, $\alpha = (3, 2, 0)$, $s = 3$, $(c_n, \dots, c_1) = (0, 0, 2, 1, 1, 0, 0, 1, 0)$, and $\gamma = (2, 2, 1, 2, 1, 1)$. Observe that the final output is a filling with inversion code (c_n, \dots, c_1) .

We also define a map inverse to dinvcode^α via an insertion algorithm. The insertion algorithm is the same, except that at each step in the algorithm, we label the columns by a different procedure.

Let $(d_n, \dots, d_1) \in \mathcal{C}_{n, \text{sort}(\alpha), s}(\text{rev}(\gamma))$. At step $1 \leq i \leq n$ in the algorithm, suppose we have a partial filling of $\text{dg}'(\alpha)$. Label the columns of the diagram with $0, 1, 2, \dots, s - 1$ inductively as follows. Suppose we have already used the labels $0, 1, \dots, j - 1$. Scan through the coordinates (a, b) with $1 \leq a \leq s$ and $b \leq \alpha_a$ left to right across each row, starting with the top row. When an unfilled coordinate (a, b) is reached in a column which is unlabeled, label that column with j .

After all columns are labeled, let column j be the unique column labeled d_i . If column j is unfilled, label the highest unfilled cell in column j with a_i . Otherwise, add a basement cell to column j and label it with a_i . Let $\iota_d^\alpha(d_n, \dots, d_1)$ be the filling obtained after step n of the algorithm.

See Figure 3.5 for an example of the insertion algorithm for ι_d^α with $n = 9$, $\alpha = (3, 2, 0)$, $s = 3$, $(d_n, \dots, d_1) = (0, 0, 2, 1, 1, 0, 0, 1, 0)$, and $\gamma = (2, 2, 1, 2, 1, 1)$. Observe that the final output is a filling with diagonal inversion code (d_n, \dots, d_1) . Further observe that the two insertion algorithms for ι^α and ι_d^α output different fillings for the same sequence.

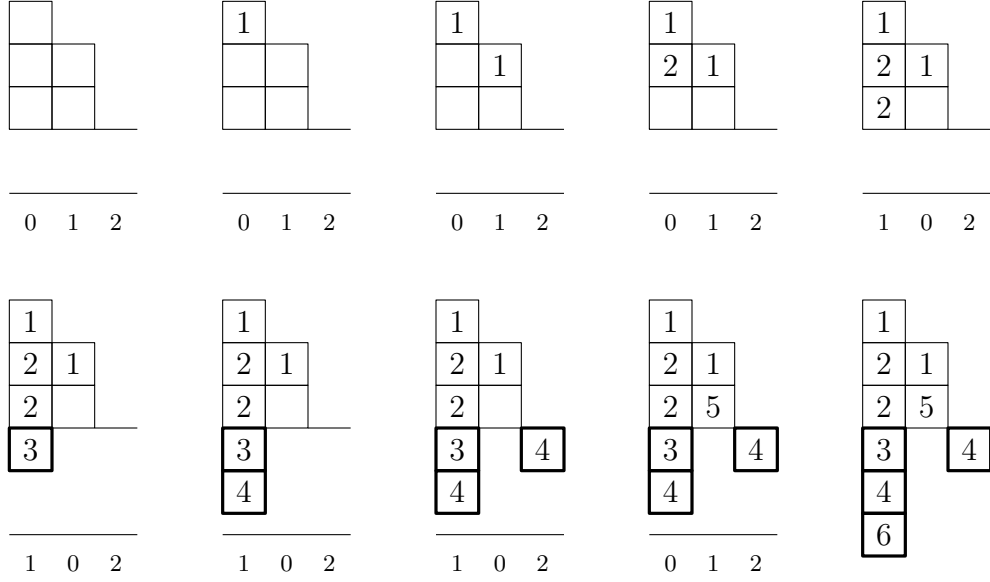


Figure 3.4: An example of the insertion algorithm for $\iota^\alpha(c_n, \dots, c_1)$ for $n = 9$, $\alpha = (3, 2, 0)$, $s = 3$, $(c_n, \dots, c_1) = (0, 0, 2, 1, 1, 0, 0, 1, 0)$, and $\gamma = (2, 2, 1, 2, 1, 1)$.

Lemma 3.3.8. *Sending (c_n, \dots, c_1) to $\iota^\alpha(c_n, \dots, c_1)$ gives a map*

$$\iota^\alpha : \mathcal{C}_{n, \text{sort}(\alpha), s}(\text{rev}(\gamma)) \rightarrow \text{ECI}_{n, \alpha, s}(\gamma). \quad (3.3.11)$$

Sending (d_n, \dots, d_1) to $\iota_d^\alpha(d_n, \dots, d_1)$ gives a map

$$\iota_d^\alpha : \mathcal{C}_{n, \text{sort}(\alpha), s}(\text{rev}(\gamma)) \rightarrow \text{ECI}_{n, \alpha, s}(\gamma). \quad (3.3.12)$$

Proof. Given $(c_n, \dots, c_1) \in \mathcal{C}_{n, \text{sort}(\alpha), s}(\text{rev}(\gamma))$, it is immediate that the reading word of $\iota^\alpha(c_n, \dots, c_1)$ has content γ and that the resulting filling is column-increasing, since $a_1 \leq \dots \leq a_n$ are inserted in increasing order. Therefore, in order to prove that $\iota^\alpha(c_n, \dots, c_1) \in \text{ECI}_{n, \alpha, s}(\gamma)$, it suffices to prove that all cells of $\text{dg}'(\alpha)$ are filled at the end of the algorithm. We proceed by induction on n . In the base case $n = 1$, if $\text{dg}'(\alpha)$ is empty then the conclusion is immediate. Otherwise, $\text{dg}'(\alpha)$ consists of one cell, whose column is labeled 0 at the beginning of the algorithm. The only $(n, \text{sort}(\alpha), s)$ -staircase is (0) , hence the unique cell of

$\text{dg}'(\alpha)$ is filled after step 1.

In the inductive step, let $n > 1$ and assume that ι^α gives a map $\iota^\alpha : \mathcal{C}_{m, \text{sort}(\alpha), s}(\text{rev}(\gamma)) \rightarrow \text{ECI}_{m, \alpha, s}(\gamma)$ for all $m < n$ and all α, s , and γ . Fix α, s , and γ , and let $\lambda = \text{sort}(\alpha)$. We claim that ι^α gives a map $\iota^\alpha : \mathcal{C}_{n, \lambda, s}(\text{rev}(\gamma)) \rightarrow \text{ECI}_{n, \alpha, s}(\gamma)$. Let $(c_n, \dots, c_1) \in \mathcal{C}_{n, \lambda, s}(\text{rev}(\gamma))$, and initialize the unfilled diagram $\text{dg}'(\alpha)$.

At the first step of the algorithm, the label $a_1 = 1$ is inserted into the column labeled c_1 . Suppose this column is the j th column from the left. If $\alpha_j > 0$, then the remaining unfilled cells of $\text{dg}'(\alpha)$ form the conjugate diagram of the composition $\alpha^{(c_1)} := (\alpha_1, \dots, \alpha_j - 1, \dots, \alpha_s)$. By the way we have labeled the columns of $\text{dg}'(\alpha)$, we have $\text{sort}(\alpha^{(c_1)}) = \lambda^{(c_1)}$.

If $\gamma_1 > 1$, then by Lemma 3.3.5, we have $(c_n, \dots, c_2) \in \mathcal{C}_{n-1, \lambda^{(c_1)}, s}^{(c_1)}(\gamma_m, \dots, \gamma_2, \gamma_1 - 1)$. By our inductive hypothesis, we have the map

$$\iota^{\alpha^{(c_1)}} : \mathcal{C}_{n-1, \lambda^{(c_1)}, s}(\gamma_m, \dots, \gamma_2, \gamma_1 - 1) \rightarrow \text{ECI}_{n-1, \alpha^{(c_1)}, s}(\gamma_1 - 1, \gamma_2, \dots, \gamma_m), \quad (3.3.13)$$

so there are no unfilled cells of $\text{dg}'(\alpha^{(c_1)})$ in $\iota^{\alpha^{(c_1)}}(c_n, \dots, c_2)$. Observe that by the construction of the insertion algorithm, $\iota^{\alpha^{(c_1)}}(c_n, \dots, c_2)$ is obtained from $\iota^\alpha(c_n, \dots, c_1)$ by deleting the cell labeled a_1 . Hence, there are no unfilled cells of $\text{dg}'(\alpha)$ in $\iota^\alpha(c_n, \dots, c_1)$, so $\iota^\alpha(c_n, \dots, c_1) \in \text{ECI}_{n, \alpha, s}(\gamma)$. The case when $\alpha_i > 0$ and $\gamma_1 = 1$, and the cases when $\alpha_i = 0$ and $\gamma_1 = 1$ or $\gamma_1 > 1$, follow by similar applications of the inductive hypothesis and Lemma 3.3.5. Hence, ι^α gives a map to $\text{ECI}_{n, \alpha, s}(\gamma)$ in all cases, and the induction is complete.

The fact that ι_d^α is well-defined follows from the fact that ι^α is well-defined. Indeed, it is immediate from the construction of ι_d^α that the reading word of $\iota_d^\alpha(c_n, \dots, c_1)$ has content γ and that the columns are weakly increasing. Furthermore, since unfilled columns of $\text{dg}'(\alpha)$ are labeled in the same way in both insertion algorithms, then the fillings $\iota^\alpha(c_n, \dots, c_1)$ and $\iota_d^\alpha(c_n, \dots, c_1)$ are the same when restricted to $\text{dg}'(\alpha)$. Hence, $\iota_d^\alpha(c_n, \dots, c_1)$ is also an element of $\text{ECI}_{n, \alpha, s}(\gamma)$, so ι_d^α is a map to $\text{ECI}_{n, \alpha, s}(\gamma)$. \square

Lemma 3.3.9. *The map invcode^α is a bijection with inverse ι^α , and the map dinvcodes^α is a bijection with inverse ι_d^α .*

Proof. First, we prove that $\text{invcode}^\alpha(\iota^\alpha(c_n, \dots, c_1)) = (c_n, \dots, c_1)$ for all $(c_n, \dots, c_1) \in \mathcal{C}_{n, \text{sort}(\alpha), s}(\text{rev}(\gamma))$. At step i in the algorithm constructing $\iota^\alpha(c_n, \dots, c_1)$, let (a, b) be the coordinates at which the label a_i is inserted.

First, suppose the column labeled c_i is unfilled, so that $b \geq 1$. Since (c_n, \dots, c_1) is $\text{rev}(\gamma)$ -weakly decreasing, then all other labels a_j with $j > i$ and $a_j = a_i$ will be inserted in a position which is after (a, b) in inversion reading order. Therefore, for every $q < a$ such that column q has label less than c_i then the cell (q, b) is unfilled at step i . Therefore, the cell (q, b) is filled with a label strictly larger than a_i by the end of the algorithm. Furthermore, for every $q > a$ such that the column q has label less than c_i , the cell $(q, b + 1)$ is unfilled at step i , hence $(q, b + 1)$ must be filled with a label strictly larger than a_i by the end of the algorithm. Since there are c_i many labels less than c_i , then the number of inversions of type (I1) of the form $((i, j), (a, b))$ in $\iota^\alpha(c_n, \dots, c_1)$ is c_i . Hence, the $(n - i + 1)$ th entry of $\text{invcode}^\alpha(\iota^\alpha(c_n, \dots, c_1))$ is c_i , as desired.

Second, suppose column a is filled, so that a_i is inserted into a basement cell in column a . By construction of the algorithm, for each column $q > a$ which is unfilled at step i , the entry $(q, 1)$ will be filled with a number strictly greater than a_i . Furthermore, each column $q < a$ has a label which is smaller than c_i . Hence, the number of inversions in $\iota^\alpha(c_n, \dots, c_1)$ of type (I2) of the form $((q, 1), (a, b))$ plus the number of type (I3) of the form $(q, (a, b))$ is c_i , so the $(n - i + 1)$ th entry of $\text{invcode}^\alpha(\iota^\alpha(c_n, \dots, c_1))$ is c_i . We conclude that $\text{invcode}^\alpha(\iota^\alpha(c_n, \dots, c_1)) = (c_n, \dots, c_1)$, so $\text{invcode}^\alpha \circ \iota^\alpha = \text{Id}$.

We claim that $\iota^\alpha \circ \text{invcode}^\alpha = \text{Id}$. Let $\varphi \in \text{ECI}_{n, \alpha, s}(\gamma)$ with $(c_n, \dots, c_1) = \text{invcode}^\alpha(\varphi)$. Suppose column a is the unique column labeled c_1 at step 1 of the insertion algorithm for ι^α . By the definition of c_1 , the first 1 in φ in inversion reading order is in the top-most cell of column a . Therefore, the location of the first 1 in φ is the same as the location of the label a_1 in $\iota^\alpha(c_n, \dots, c_1)$. Since $\iota^{\alpha(c_1)}(c_n, \dots, c_2)$ is obtained from $\iota^\alpha(c_n, \dots, c_1)$ by deleting the first cell labeled 1 (and shifting labels in the case that $\gamma_1 = 1$), then a straightforward induction on n shows that $\iota^\alpha(c_n, \dots, c_1) = \varphi$, so the claim follows.

We conclude that ι^α and invcode^α are mutually inverse, and hence they are bijections.

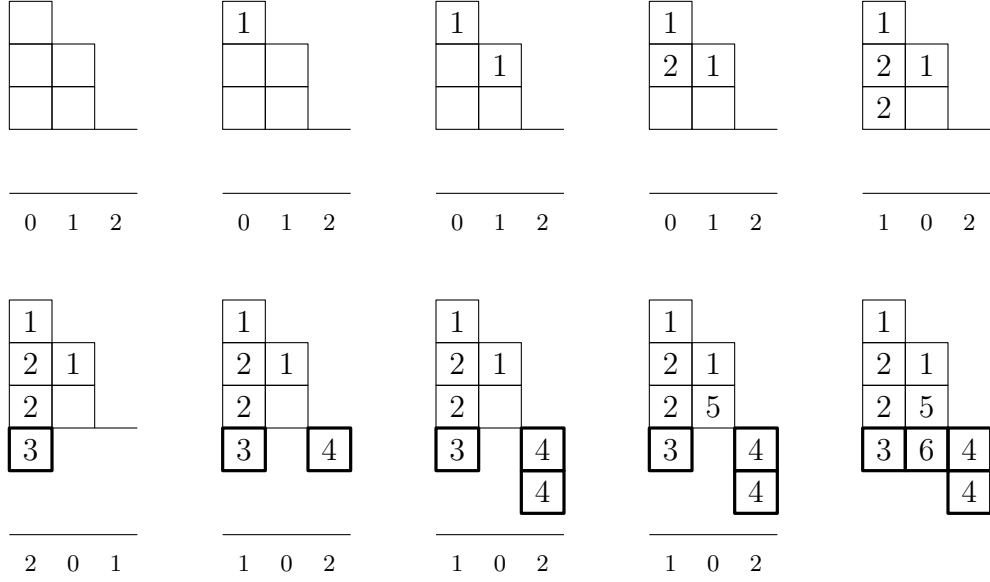


Figure 3.5: An example of the insertion algorithm for $\iota_d^\alpha(c_n, \dots, c_1)$ when $n = 9$, $\alpha = (3, 2, 0)$, $s = 3$, and $(d_n, \dots, d_1) = (0, 0, 2, 1, 1, 0, 0, 1, 0)$.

The fact that ι_d^α and dinvcode^α are bijections which are mutually inverse follows by a similar argument using reading order in place of inversion reading order. \square

Proof of Theorem 3.3.4. We have the bijection

$$\iota^\lambda \circ \text{dinvcode}^\lambda : \text{ECI}_{n,\lambda,s}(\gamma) \rightarrow \text{ECI}_{n,\lambda,s}(\gamma), \tag{3.3.14}$$

which maps $\varphi \in \text{ECI}_{n,\lambda,s}(\gamma)$ to $\varphi' \in \text{ECI}_{n,\lambda,s}(\gamma)$ with $\text{dinvcode}^\lambda(\varphi) = \text{invcode}^\lambda(\varphi')$. Therefore, we have $\text{dinv}(\varphi) = \text{inv}(\varphi')$. Hence, we have

$$\sum_{\varphi \in \text{ECI}_{n,\lambda,s}(\gamma)} q^{\text{inv}(\varphi)} = \sum_{\varphi \in \text{ECI}_{n,\lambda,s}(\gamma)} q^{\text{dinv}(\varphi)}, \tag{3.3.15}$$

for all $\gamma = (\gamma_1, \dots, \gamma_m)$ a composition of n into positive parts. Let $\text{set}(\gamma) := \{\gamma_1, \gamma_1 + \gamma_2, \dots, \gamma_1 + \dots + \gamma_{m-1}\}$. Since both sides of (3.3.1) are quasisymmetric, we have the following

expansions into monomial quasisymmetric functions,

$$\sum_{\varphi \in \text{ECI}_{n,\lambda,s}} q^{\text{inv}(\varphi)} \mathbf{x}^\varphi = \sum_{\gamma} \left(\sum_{\varphi \in \text{ECI}_{n,\lambda,s}(\gamma)} q^{\text{inv}(\varphi)} \right) M_{n,\text{set}(\gamma)}(\mathbf{x}), \quad (3.3.16)$$

$$\sum_{\varphi \in \text{ECI}_{n,\lambda,s}} q^{\text{dinv}(\varphi)} \mathbf{x}^\varphi = \sum_{\gamma} \left(\sum_{\varphi \in \text{ECI}_{n,\lambda,s}(\gamma)} q^{\text{dinv}(\varphi)} \right) M_{n,\text{set}(\gamma)}(\mathbf{x}), \quad (3.3.17)$$

where the sums are over compositions γ of n such that $\gamma_i > 0$ for all i . By (3.3.15), the right-hand sides of (3.3.16) and (3.3.17) are equal. This completes the proof. \square

Definition 3.3.10. Let $D_{n,\lambda,s}$ be the multivariate generating function in Theorem 3.3.4,

$$D_{n,\lambda,s} := \sum_{\varphi \in \text{ECI}_{n,\lambda,s}} q^{\text{inv}(\varphi)} \mathbf{x}^\varphi = \sum_{\varphi \in \text{ECI}_{n,\lambda,s}} q^{\text{dinv}(\varphi)} \mathbf{x}^\varphi. \quad (3.3.18)$$

Corollary 3.3.11. *We have*

$$D_{n,\lambda,s} = \sum_{\varphi \in \text{SECI}_{n,\lambda,s}} q^{\text{inv}(\varphi)} F_{n,\text{iDes}(\text{rw}(\varphi))}(\mathbf{x}) = \sum_{\varphi \in \text{SECI}_{n,\lambda,s}} q^{\text{dinv}(\varphi)} F_{n,\text{iDes}(\text{rw}(\varphi))}(\mathbf{x}), \quad (3.3.19)$$

where for any permutation $\pi \in S_n$, $\text{iDes}(\pi) := \text{Des}(\pi^{-1})$.

Proof. Recall from (2.0.25) that

$$F_{n,D}(\mathbf{x}) = \sum_{\substack{w \in \mathbb{N}^n, \\ \text{std}(w) = \pi}} \mathbf{x}^w, \quad (3.3.20)$$

where $\pi \in S_n$ is a fixed permutation such that $\text{iDes}(\pi) = D$. Since the statistic inv is only dependent on λ , the positions of the labels in φ , and $\text{rw}(\varphi)$, then the result follows immediately by Theorem 3.3.4 and Definition 3.3.10. \square

Theorem 3.3.12. *The generating function $D_{n,\lambda,s}$ is a symmetric function which expands as a positive sum of LLT symmetric functions, each shifted by some power of q .*

Proof. The right-most side of (3.3.18) is the sum over all $\beta \in \text{Comp}(n - k, s)$ of

$$\sum_{\varphi} q^{\text{dinv}(\varphi)} \mathbf{x}^{\varphi}, \quad (3.3.21)$$

where the sum ranges over column-increasing fillings $\varphi \in \text{ECI}_{n,\lambda,s}$ which have β_i many basement cells in column i for $1 \leq i \leq s$. Fix β , and let m be the number of diagonal inversions of type (D2) in a filling $\varphi \in \text{ECI}_{n,\lambda,s}$ which has β_i many basement cells in column i for $1 \leq i \leq s$. Observe that m is only dependent on β and s .

Recall the LLT polynomial $G_{\nu}(\mathbf{x}; q)$ defined in Subsection 2.0.5. For $1 \leq i \leq \ell(\lambda)$, let $\nu^{(i)}$ be the single row of size $\beta_i + \lambda_i$, shifted so that the cells have contents

$$\lambda_i, \lambda_i - 1, \dots, 1, 0, -1, \dots, -\beta_i + 1 \quad (3.3.22)$$

from left to right. For $\ell(\lambda) < i \leq s$, let $\nu^{(i)}$ be the single row of size β_i , shifted so that the cells have contents $0, -1, \dots, -\beta_i + 1$. Let $\nu = (\nu^{(1)}, \dots, \nu^{(s)})$.

We claim that

$$\sum_{\varphi} q^{\text{dinv}(\varphi)} \mathbf{x}^{\varphi} = q^m G_{\nu}(\mathbf{x}; q), \quad (3.3.23)$$

where the sum is over column-increasing fillings $\varphi \in \text{ECI}_{n,\lambda,s}$ which have β_i many basement cells in column i for $1 \leq i \leq s$. Indeed, to each $T = (T^{(1)}, \dots, T^{(s)}) \in \text{SSYT}(\nu)$ associate the column-increasing filling $\varphi \in \text{ECI}_{n,\lambda,s}$ whose i th column is filled with the same multiset of labels as $T^{(i)}$. It can then be checked that inversions in T correspond to diagonal inversions of type (D1) in φ , which proves our claim. Since $D_{n,\lambda,s}$ is a sum over shifted LLT polynomials of the form (3.3.23), and each LLT polynomial is symmetric in \mathbf{x} by Theorem 2.0.5, then $D_{n,\lambda,s}$ is symmetric in \mathbf{x} . \square

In [72], Roberts gives expansions of certain families of LLT polynomials in terms of Schur functions. It might be interesting to apply Roberts's formula to our LLT expansions to find

an explicit Schur expansion for $D_{n,\lambda,s}$.

3.3.3 Monomial and quasisymmetric function formulas for $\text{Frob}_q(R_{n,\lambda,s})$

In this subsection, we prove an expansion of $\text{Frob}_q(R_{n,\lambda,s})$ into Gessel's fundamental quasisymmetric functions in terms of the coinversion statistic. Our main tool to prove this result is the skewing operator $e_j(\mathbf{x})^\perp$. We obtain both the monomial expansion of $\text{Frob}_q(R_{n,\lambda,s})$ stated in the introduction as (3.0.29) and the formula for the Hilbert series of $R_{n,\lambda,s}$ stated as (3.0.25).

We first state the main theorem in this section, Theorem 3.3.13, and two corollaries. The remainder of this subsection is dedicated to proving Theorem 3.3.13.

Theorem 3.3.13. *We have*

$$\text{Frob}_q(R_{n,\lambda,s}) = D_{n,\lambda,s} = \sum_{\varphi \in \text{ECI}_{n,\lambda,s}} q^{\text{inv}(\varphi)} \mathbf{x}^\varphi = \sum_{\varphi \in \text{ECI}_{n,\lambda,s}} q^{\text{dinv}(\varphi)} \mathbf{x}^\varphi, \quad (3.3.24)$$

Corollary 3.3.14. *We have*

$$\text{Hilb}_q(R_{n,\lambda,s}) = \sum_{\varphi \in \text{SECI}_{n,\lambda,s}} q^{\text{inv}(\varphi)} = \sum_{\varphi \in \text{SECI}_{n,\lambda,s}} q^{\text{dinv}(\varphi)}. \quad (3.3.25)$$

Proof. Apply (2.0.23) to $V = R_{n,\lambda,s}$ and the monomial expansions in (3.3.24). \square

Corollary 3.3.15. *Consider $S_k \subseteq S_n$ as the subgroup permuting the first k elements. If $s > \ell(\lambda)$, then the top degree component of $R_{n,\lambda,s}$ is isomorphic to*

$$\text{Ind} \uparrow_{S_k}^{S_n} S^\lambda. \quad (3.3.26)$$

Proof. Recall that the top nonzero component of $R_{n,\lambda,s}$ is in degree $n(\lambda) + (s-1)(n-k)$. In the case when $s > \ell(\lambda)$, then the only terms in (3.3.24) with $\text{inv}(\varphi) = n(\lambda) + (s-1)(n-k)$ are those such that $\sigma(\varphi)$ is a column-increasing filling such that each row is strictly decreasing from left to right and all of the basement cells of φ are in the s th column. It can be checked

that the contribution of the $\sigma(\varphi)$ gives the Schur function $s_\lambda(\mathbf{x})$, and the contribution of the labeling of the basement cells in the s th column gives the complete homogeneous symmetric function $h_{n-k}(\mathbf{x})$. Hence, we have

$$[q^{n(\lambda)+(s-1)(n-k)}] \text{Frob}_q(R_{n,\lambda,s}) = s_\lambda(\mathbf{x})h_{n-k}(\mathbf{x}). \quad (3.3.27)$$

The corollary follows from the fact that

$$\text{Frob} \left(\text{Ind} \uparrow_{S_k}^{S_n} S^\lambda \right) = s_\lambda(\mathbf{x})h_{n-k}(\mathbf{x}). \quad (3.3.28)$$

□

Lemma 3.3.16. *Given $\alpha \in \text{Comp}(k, s)$, we have*

$$\sum_{\varphi \in \text{ECI}_{n,\alpha,s}} q^{\text{inv}(\varphi)} \mathbf{x}^\varphi = D_{n,\text{sort}(\alpha),s}. \quad (3.3.29)$$

Proof. Let $\lambda = \text{sort}(\alpha)$ for convenience. By Lemma 3.3.9, we have the bijection

$$\iota^\lambda \circ \text{invcode}^\alpha : \text{ECI}_{n,\alpha,s}(\gamma) \rightarrow \text{ECI}_{n,\lambda,s}(\gamma) \quad (3.3.30)$$

which maps $\varphi \in \text{ECI}_{n,\alpha,s}(\gamma)$ to $\varphi' \in \text{ECI}_{n,\lambda,s}(\gamma)$ with $\text{invcode}^\alpha(\varphi) = \text{invcode}^\lambda(\varphi')$. Hence, we have $\text{inv}(\varphi) = \text{inv}(\varphi')$. Therefore,

$$\sum_{\varphi \in \text{ECI}_{n,\alpha,s}(\gamma)} q^{\text{inv}(\varphi)} = \sum_{\varphi \in \text{ECI}_{n,\lambda,s}(\gamma)} q^{\text{inv}(\varphi)}. \quad (3.3.31)$$

Since both sides of (3.3.29) are quasisymmetric, the proof follows by the same reasoning as in the proof of Theorem 3.3.4. □

Observe that $D_{n,\lambda,s}$ and $\text{Frob}_q(R_{n,\lambda,s})$ are homogeneous in \mathbf{x} of positive degree, so their constant terms are both equal to 0. Further observe that $\text{Frob}_q(R_{1,\lambda,s}) = D_{1,\lambda,s}$ for all λ

and s . Recall that $D_{n,\lambda,s}$ is a symmetric function by Theorem 3.3.12. Furthermore, by Lemma 2.0.2 any symmetric function with zero constant term is uniquely determined by its images under the e_j^\perp skewing operators. By Theorem 3.2.2 and induction on n , in order to prove Theorem 3.3.13 it suffices to show that $D_{n,\lambda,s}$ satisfies the same identity under the skewing operators as $\text{Frob}_q(R_{n,\lambda,s})$, namely that

$$e_j^\perp D_{n,\lambda,s} = \sum_{I \in \mathcal{I}_\ell^j} q^{\Sigma(I)} D_{n-j,\lambda^{(I)},s} \quad (3.3.32)$$

for all $j \geq 1$.

Fix a composition $\beta = (\beta_1, \dots, \beta_p)$ of n into positive parts. A β -shuffle is a shuffle of the decreasing sequences

$$(\beta_1, \dots, 2, 1), (\beta_1 + \beta_2, \dots, \beta_1 + 1), \dots, (n, n-1, \dots, n - \beta_p + 1),$$

of lengths $\beta_1, \beta_2, \dots, \beta_p$, respectively. Let $\text{SECI}_{n,\lambda,s}^\beta$ be the set of $\varphi \in \text{SECI}_{n,\lambda,s}$ such that $\text{rw}(\varphi)$ is a β -shuffle. For convenience of notation, let $j := \beta_1$ so that $\beta = (j, \beta_2, \dots, \beta_p)$. Observe that if $\varphi \in \text{SECI}_{n,\lambda,s}^\beta$, then the cells labeled with $1, 2, \dots, j$ are in distinct columns, and hence each of the labels $1, 2, \dots, j$ is either in the top-most cell in its column or a basement label.

Proof of Theorem 3.3.13. Following the strategy in [45, pp. 878], $D_{n,\lambda,s}$ is the unique symmetric function such that for any composition $\beta = (j, \beta_2, \beta_3, \dots, \beta_p)$ of n into positive parts,

$$\langle D_{n,\lambda,s}, e_j(\mathbf{x})e_{\beta_2}(\mathbf{x})e_{\beta_3}(\mathbf{x}) \cdots e_{\beta_p}(\mathbf{x}) \rangle = \sum_{\varphi \in \text{SECI}_{n,\lambda,s}^\beta} q^{\text{inv}(\varphi)}. \quad (3.3.33)$$

By the definition of e_j^\perp , (3.3.33) is equivalent to

$$\langle e_j^\perp D_{n,\lambda,s}, e_{\beta_2}(\mathbf{x}) \cdots e_{\beta_p}(\mathbf{x}) \rangle = \sum_{\varphi \in \text{SECI}_{n,\lambda,s}^\beta} q^{\text{inv}(\varphi)}. \quad (3.3.34)$$

Since elementary symmetric functions form a basis of symmetric functions, then (3.3.32) is equivalent to the identity

$$\langle e_j^\perp D_{n,\lambda,s}, e_{\beta_2}(\mathbf{x}_n) \cdots e_{\beta_p}(\mathbf{x}_n) \rangle = \sum_{I \in \mathcal{I}_s^j} q^{\Sigma(I)} \langle D_{n-j,\lambda(I),s}, e_{\beta_2}(\mathbf{x}_n) \cdots e_{\beta_p}(\mathbf{x}_n) \rangle, \quad (3.3.35)$$

which can be rewritten as the identity

$$\sum_{\varphi \in \text{SECI}_{n,\lambda,s}^\beta} q^{\text{inv}(\varphi)} = \sum_{I \in \mathcal{I}_s^j} \sum_{\varphi \in \text{SECI}_{n-j,\lambda(I),s}^{\beta'}} q^{\Sigma(I) + \text{inv}(\varphi)}, \quad (3.3.36)$$

where $\beta' := (\beta_2, \dots, \beta_p)$.

The identity (3.3.36) has a simple bijective proof, as follows. Let $\varphi \in \text{SECI}_{n,\lambda,s}^\beta$, and let $i_1 + 1, \dots, i_j + 1$ be the columns of φ containing $j, j-1, \dots, 1$, respectively, and let $I = (i_1, \dots, i_j) \in \mathcal{I}_s^j$. Let α^I be the composition defined by

$$\alpha_i^I = \begin{cases} \lambda_i - 1 & \text{if } i + 1 \in I \\ \lambda_i & \text{if } i + 1 \notin I \end{cases}. \quad (3.3.37)$$

Let $\varphi' \in \text{SECI}_{n-j,\alpha^I,s}^{\beta'}$ be the extended column-increasing filling obtained from φ by removing cells labeled $1, \dots, j$ and then standardizing the remaining labels to the set $[n-j]$. The map sending φ to (I, φ') is a bijection between $\text{SECI}_{n,\lambda,s}^\beta$ and $\{(I, \varphi') : I \in \mathcal{I}_s^j, \varphi' \in \text{SECI}_{n-j,\alpha^I,s}^{\beta'}\}$.

Recall that $1, \dots, j$ appear in $\text{rw}(\varphi)$ in decreasing order, so each label $h \leq j$ is in column $i_{j-h+1} + 1$. Each $h \leq j$ which is a label in $\sigma(\varphi)$ forms a diagonal inversion with each of the other cells in its row to its left, hence it contributes i_{j-h+1} to $\text{inv}(\varphi)$. For each label $h \leq j$ which is the label of a basement cell (a, b) of φ , h does not form any inversions of type (I2). Therefore, the contribution of h to $\text{inv}(\varphi)$ is the number of inversions of type (I3) of the form $(i, (a, b))$, which is i_{j-h+1} . Hence, we have $\text{inv}(\varphi) = \Sigma(I) + \text{inv}(\varphi')$.

Using the bijection above, we have

$$\sum_{\varphi \in \text{SECI}_{n,\lambda,s}^\beta} q^{\text{inv}(\varphi)} = \sum_{I \in \mathcal{I}_s^j} \sum_{\varphi \in \text{SECI}_{n-j,\alpha^I,s}^{\beta'}} q^{\Sigma(I) + \text{inv}(\varphi)}. \quad (3.3.38)$$

For each $I \in \mathcal{I}_s^j$, we have $\text{sort}(\alpha^I) = \lambda^{(I)}$ by the definitions of α^I and $\lambda^{(I)}$. By Lemma 3.3.16,

$$\sum_{\varphi \in \text{SECI}_{n-j,\alpha^I,s}} q^{\text{inv}(\varphi)} \mathbf{x}^\varphi = D_{n-j,\lambda^{(I)},s}. \quad (3.3.39)$$

Taking the inner product of both sides of (3.3.39) with $e_{\beta_2}(\mathbf{x}) \cdots e_{\beta_p}(\mathbf{x})$, we have

$$\sum_{\varphi \in \text{SECI}_{n-j,\alpha^I,s}^{\beta'}} q^{\text{inv}(\varphi)} = \sum_{\varphi \in \text{SECI}_{n-j,\lambda^{(I)},s}^{\beta'}} q^{\text{inv}(\varphi)}. \quad (3.3.40)$$

Hence, (3.3.36) follows by combining (3.3.38) with (3.3.40), which completes the proof. \square

3.4 Applications to rank varieties

In this section, we apply our results on the rings $R_{n,\lambda,s}$ to the geometry of rank varieties. In particular, we show that the ring $R_{n,\lambda}$ defined in the Introduction is the coordinate ring of the scheme-theoretic intersection of a rank variety with diagonal matrices. We then find a monomial basis for $R_{n,\lambda}$. Furthermore, we compute the Hilbert series and Frobenius characteristic of $R_{n,\lambda}$ in terms of the inversion statistic on extended column-increasing fillings. We will show that each of these formulas for $R_{n,\lambda}$ is a ‘‘limit’’ as $s \rightarrow \infty$ of the corresponding formula for $R_{n,\lambda,s}$.

Let \mathfrak{gl}_n be the space of $n \times n$ matrices over \mathbb{Q} . Let $x_{i,j}$ for $1 \leq i, j \leq n$ be the coordinate functions corresponding to the entries of an $n \times n$ matrix. Then the coordinate ring of \mathfrak{gl}_n is $\mathbb{Q}[\mathfrak{gl}_n] = \mathbb{Q}[x_{i,j}]$.

For $\lambda \vdash n$, let $O_\lambda \subseteq \mathfrak{gl}_n$ be the conjugacy class of nilpotent $n \times n$ matrices over \mathbb{Q} whose Jordan canonical form has block sizes recorded by λ . Let \overline{O}_λ be the closure of O_λ in \mathfrak{gl}_n in

the Zariski topology. The set of diagonal matrices \mathfrak{t} is the variety defined by the ideal

$$I(\mathfrak{t}) = \langle x_{i,j} : i \neq j \rangle. \quad (3.4.1)$$

The *scheme-theoretic intersection* of the varieties \overline{O}_λ and \mathfrak{t} is the affine scheme whose coordinate ring is defined by the sum of the defining ideals of \overline{O}_λ and \mathfrak{t} ,

$$\mathbb{Q}[\overline{O}_\lambda \cap \mathfrak{t}] := \frac{\mathbb{Q}[x_{i,j}]}{I(\overline{O}_\lambda) + I(\mathfrak{t})}. \quad (3.4.2)$$

The symmetric group S_n of permutation matrices acts by conjugation on \mathfrak{t} , which descends to an action of S_n on $\mathbb{Q}[\overline{O}_\lambda \cap \mathfrak{t}]$. Observe that the variables $x_{i,i}$ generate this coordinate ring. Reindexing the generators $x_{i,i}$ of $\mathbb{Q}[\overline{O}_\lambda \cap \mathfrak{t}]$ by x_i , then S_n acts by permuting the x_i variables.

Motivated by work of Kostant [53] on the coinvariant algebra, Kraft [54] conjectured that the coordinate ring (3.4.2) is isomorphic to the cohomology ring of a Springer fiber. De Concini and Procesi [17] proved Kraft's conjecture. Tanisaki [90] then simplified the arguments of De Concini and Procesi and further proved that these rings have the explicit presentation as the quotient ring $R_\lambda = R_{n,\lambda,\ell(\lambda)}$.

Let $\text{Fl}(n)$ be the *complete flag variety* of flags $V_\bullet = (V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n)$, where V_i is an i -dimensional complex vector subspace of \mathbb{C}^n for each i . Given a matrix $X \in O_\lambda$, the *Springer fiber* of X is

$$\mathcal{B}^X := \{V_\bullet \in \text{Fl}(n) : XV_i \subseteq V_i, 1 \leq i \leq n\}. \quad (3.4.3)$$

The Springer fiber gets its name from the fact that it is the fiber over X of the Springer resolution of the nilpotent cone, see for example [13]. If $X, X' \in O_\lambda$, then we have an isomorphism of varieties $\mathcal{B}^X \cong \mathcal{B}^{X'}$. We denote by \mathcal{B}^λ the Springer fiber of any $X \in O_\lambda$.

In [81], Springer proved that there is an action of S_n on the cohomology ring $H^*(\mathcal{B}^\lambda; \mathbb{Q})$, even though S_n does not act directly on the space \mathcal{B}^λ . Furthermore, Springer proved that

these graded representations have the remarkable property that the top nonvanishing cohomology group $H^{2n(\lambda)}(\mathcal{B}^\lambda; \mathbb{Q})$ is isomorphic to an irreducible representation of S_n , and that all irreducible representations of S_n appear in this manner. It is well known that \mathcal{B}^λ has no nontrivial odd cohomology groups [51], so we consider $H^*(\mathcal{B}^\lambda; \mathbb{Q})$ as a graded ring by declaring that the i th graded piece is $H^{2i}(\mathcal{B}^\lambda; \mathbb{Q})$. Therefore, $H^*(\mathcal{B}^\lambda; \mathbb{Q})$ has the structure of a graded S_n -module. Since Springer's construction of this symmetric group module action, much work has been done to understand the combinatorics and geometry of Springer fibers and their generalizations, the Hessenberg varieties [26, 27, 79, 92, 93, 94].

Theorem 3.4.1 ([17, 90]). *We have isomorphisms of graded rings and S_n -modules*

$$R_\lambda \cong \mathbb{Q}[\overline{O}_\lambda \cap \mathfrak{t}] \cong H^*(\mathcal{B}^\lambda; \mathbb{Q}), \quad (3.4.4)$$

where S_n acts on $H^*(\mathcal{B}^\lambda; \mathbb{Q})$ via Springer's representation tensored with the sign representation.

In [23], Eisenbud and Saltman study varieties generalizing the varieties \overline{O}_λ . These varieties are the main focus of this section.

Definition 3.4.2. Let $k \leq n$, and let $\lambda \vdash k$. The *Eisenbud-Saltman rank variety* is the variety

$$\overline{O}_{n,\lambda} := \{X \in \mathfrak{gl}_n : \dim \ker X^d \geq \lambda'_1 + \cdots + \lambda'_d, d = 1, 2, \dots, n\} \quad (3.4.5)$$

$$= \{X \in \mathfrak{gl}_n : \text{rk}(X^d) \leq (n - k) + p_{n-d}^n(\lambda), d = 1, 2, \dots, n\}. \quad (3.4.6)$$

The variety $\overline{O}_{n,\lambda}$ is the same as X_r defined in [23], where r is the rank function $r(d) = (n - k) + p_{n-d}^n(\lambda)$. Eisenbud and Saltman proved that rank varieties are Gorenstein and normal with rational singularities [23, Theorem 1]. When $n = k$, we have $\overline{O}_{n,\lambda} = \overline{O}_\lambda$. When $n > k$, then $\overline{O}_{n,\lambda}$ contains matrices which are not nilpotent. In particular, the variety $\overline{O}_{n,\lambda}$

contains all block diagonal matrices of the form

$$X_\lambda \oplus A_{n-k} = \left[\begin{array}{c|c} X_\lambda & 0 \\ \hline 0 & A_{n-k} \end{array} \right] \quad (3.4.7)$$

where $X_\lambda \in \overline{O}_\lambda \subseteq \mathfrak{gl}_k$ and $A_{n-k} \in \mathfrak{gl}_{n-k}$, as well as any matrix which is conjugate to a matrix in this form.

One can compute the defining ideal of $\overline{O}_{n,\lambda}$ as the radical of the ideal generated by the $((n-k) + p_{n-d}^n(\lambda) + 1)$ -minors of X^d for $d \geq 1$. A more explicit description of the defining ideal was conjectured by Eisenbud-Saltman [23] and proven by Weyman [96], which we state next. For an integer m , let $\bigwedge^m(tI - X)$ be the exterior power of the matrix $tI - X$, where I is the identity matrix. Recall that the entries of the exterior power are the $m \times m$ minors of $tI - X$, each of which is a polynomial in t and the variables $x_{i,j}$. Let f_d^m be the matrix of coefficients of t^{m-d} in $\bigwedge^m(tI - X)$, and let $I(f_d^m)$ be the ideal generated by the entries of f_d^m . Note that f_d^m is the same as λ_d^m in [23], which is the same as $V_{n-m,d}$ in [96].

Theorem 3.4.3 (Weyman [96]). *The ideal $I(\overline{O}_{n,\lambda'})$ is the sum of the ideals $I(f_d^m)$ for d and m which satisfy $d > m - p_m^n(\lambda)$.*

Recall the ideals $I_{n,\lambda}$ and rings $R_{n,\lambda}$, defined by

$$I_{n,\lambda} := \langle e_d(S) : S \subseteq \mathbf{x}_n, d > |S| - p_{|S|}^n(\lambda) \rangle \quad (3.4.8)$$

$$R_{n,\lambda} := \mathbb{Q}[\mathbf{x}_n]/I_{n,\lambda}. \quad (3.4.9)$$

We have the following corollary of Theorem 3.4.3

Corollary 3.4.4. *We have an isomorphism of graded rings,*

$$R_{n,\lambda} \cong \mathbb{Q}[\overline{O}_{n,\lambda'} \cap \mathfrak{t}], \quad (3.4.10)$$

where the right-hand side is the coordinate ring of the scheme-theoretic intersection, and the

isomorphism is given by mapping x_i to $x_{i,i}$.

Proof. By Theorem 3.4.3,

$$\mathbb{Q}[\overline{\mathcal{O}}_{n,\lambda} \cap \mathfrak{t}] = \frac{\mathbb{Q}[x_{i,j}]}{I(f_d^m : d > m - p_m^n(\lambda)) + I(\mathfrak{t})}. \quad (3.4.11)$$

Recall that f_d^m is the matrix of coefficients of t^{m-d} in $\bigwedge^m(tI - X)$. Observe that any $m \times m$ minor of $tI - X$ which is not a principal minor is contained in $I(\mathfrak{t})$. Furthermore, given $A \subseteq [n]$ of size m , the principal minor of $tI - X$ with row and column set A is equivalent modulo $I(\mathfrak{t})$ to

$$\sum_{d=0}^m (-1)^d e_d(\{x_{i,i} : i \in A\}) t^{m-d}. \quad (3.4.12)$$

Hence, we have

$$\mathbb{Q}[\overline{\mathcal{O}}_{n,\lambda} \cap \mathfrak{t}] \cong \frac{\mathbb{Q}[x_{i,i}]}{\langle e_d(\{x_{i,i} : i \in A\}) : A \subseteq [n], d > |A| - p_{|A|}^n(\lambda) \rangle}. \quad (3.4.13)$$

Identifying $x_{i,i}$ with x_i completes the proof. \square

Recall that $\beta^j(\lambda) = (0, 1, \dots, \lambda_j - 1)$ for $1 \leq j \leq \lambda_1$. Define an (n, λ) -staircase to be a shuffle of $\beta^1(\lambda), \beta^2(\lambda), \dots, \beta^{\lambda_1}(\lambda)$, and (∞^{n-k}) , which is the sequence consisting of ∞ repeated $n-k$ many times. Given α a weak composition of length n and β an (n, λ) -staircase, we say α is contained in β if $\alpha_i \leq \beta_i$ for all i . Let $\mathcal{C}_{n,\lambda}$ be the set of weak compositions with finite integer parts

$$\mathcal{C}_{n,\lambda} := \{\alpha = (\alpha_1, \dots, \alpha_n) : \alpha \subseteq \beta \text{ for some } (n, \lambda)\text{-staircase } \beta\}. \quad (3.4.14)$$

For example if $n = 3$ and $\lambda = (1, 1)$, then $\mathcal{C}_{3,(1,1)}$ is the set of compositions of the form

$$(0, 0, a), (0, a, 0), (a, 0, 0), (0, 1, a), (0, a, 1), (a, 0, 1), \quad (3.4.15)$$

ranging over all integers $0 \leq a < \infty$. Let

$$\mathcal{A}_{n,\lambda} := \{\mathbf{x}_n^\alpha : \alpha \in \mathcal{C}_{n,\lambda}\}. \quad (3.4.16)$$

Theorem 3.4.5. *The set $\mathcal{A}_{n,\lambda}$ represents a basis of $R_{n,\lambda} \cong \mathbb{Q}[\overline{O}_{n,\lambda} \cap \mathfrak{t}]$.*

Proof. Observe that for $d \geq 0$, the degree d components of $R_{n,\lambda,d+1}$ and $R_{n,\lambda}$ coincide, since the x_i^{d+1} generators of $I_{n,\lambda,d+1}$ are in degree $d+1$. By Theorem 3.1.17, the set

$$\{\mathbf{x}_n^\alpha : \alpha \in \mathcal{C}_{n,\lambda,d+1}, \alpha_1 + \cdots + \alpha_n = d\} \quad (3.4.17)$$

represents a basis of the degree d component of $R_{n,\lambda,d+1}$, and hence also for the degree d component of $R_{n,\lambda}$. Observe that the set (3.4.17) is equal to

$$\{\mathbf{x}_n^\alpha : \alpha \in \mathcal{C}_{n,\lambda}, \alpha_1 + \cdots + \alpha_n = d\}, \quad (3.4.18)$$

which is the subset of $\mathcal{A}_{n,\lambda}$ consisting of degree d monomials. Hence, we have that $\mathcal{A}_{n,\lambda}$ represents a basis for $R_{n,\lambda}$. \square

Let

$$\text{ECI}_{n,\lambda} := \bigcup_{s \geq \ell(\lambda)} \text{ECI}_{n,\lambda,s}, \quad (3.4.19)$$

$$\text{SECI}_{n,\lambda} := \bigcup_{s \geq \ell(\lambda)} \text{SECI}_{n,\lambda,s}, \quad (3.4.20)$$

where we identify $\varphi \in \text{ECI}_{n,\lambda,s}$ with the extended column-increasing filling in $\text{ECI}_{n,\lambda,s+1}$ obtained by appending an empty $(s+1)$ th column to φ . We similarly identify each element of $\text{SECI}_{n,\lambda,s}$ with its counterpart in $\text{SECI}_{n,\lambda,s+1}$. Observe that for each $\varphi \in \text{ECI}_{n,\lambda,s}$, the statistic $\text{inv}(\varphi)$ does not depend on the parameter s . Hence, we may consider inv to be a statistic on elements of $\text{ECI}_{n,\lambda}$. Note that this is not true for the statistic dinv , which is why our formulas below are only stated in terms of inv .

Theorem 3.4.6. *For any $k \leq n$ and $\lambda \vdash k$,*

$$\text{Frob}_q(R_{n,\lambda}) = \text{Frob}_q(\mathbb{Q}[\overline{O}_{n,\lambda'} \cap \mathfrak{t}]) = \sum_{\varphi \in \text{ECI}_{n,\lambda}} q^{\text{inv}(\varphi)} \mathbf{x}^\varphi. \quad (3.4.21)$$

Proof. The first equality in (3.4.21) follows by Corollary 3.4.4, so it suffices to show the left-most side and right-most side of (3.4.21) are equal. Recall that for $d \geq 0$, the degree d components of $R_{n,\lambda,d+1}$ and $R_{n,\lambda}$ coincide. Combining this with Theorem 3.3.13, we have

$$[q^d] \text{Frob}_q(R_{n,\lambda}) = [q^d] \text{Frob}_q(R_{n,\lambda,d+1}) = \sum_{\substack{\varphi \in \text{ECI}_{n,\lambda,d+1}, \\ \text{inv}(\varphi)=d}} \mathbf{x}^\varphi. \quad (3.4.22)$$

Observe that for $s > d + 1$, each element $\varphi \in \text{ECI}_{n,\lambda,s} \subseteq \text{ECI}_{n,\lambda}$ which is not identified with an element of $\text{ECI}_{n,\lambda,d+1}$ must have a basement cell in some column to the right of column $d + 1$. Hence, each of these fillings φ has $\text{inv}(\varphi) > d$. Therefore,

$$\sum_{\substack{\varphi \in \text{ECI}_{n,\lambda,d+1}, \\ \text{inv}(\varphi)=d}} \mathbf{x}^\varphi = \sum_{\substack{\varphi \in \text{ECI}_{n,\lambda}, \\ \text{inv}(\varphi)=d}} \mathbf{x}^\varphi = [q^d] \sum_{\varphi \in \text{ECI}_{n,\lambda}} q^{\text{inv}(\varphi)} \mathbf{x}^\varphi. \quad (3.4.23)$$

Combining (3.4.22) and (3.4.23), the q^d coefficients of the left-most side and the right-most side of (3.4.21) are equal for all $d \geq 0$, hence (3.4.21) follows. \square

We have the following two corollaries of Theorem 3.4.6. The first follows from Theorem 3.4.6 by the same argument as in the proof of Corollary 3.3.11, and the second follows by applying (2.0.23) to $V = R_{n,\lambda}$.

Corollary 3.4.7. *We have*

$$\text{Frob}_q(R_{n,\lambda}) = \text{Frob}_q(\mathbb{Q}[\overline{O}_{n,\lambda'} \cap \mathfrak{t}]) = \sum_{\varphi \in \text{SECI}_{n,\lambda}} q^{\text{inv}(\varphi)} F_{n, \text{iDes}(\text{rw}(\varphi))}(\mathbf{x}). \quad (3.4.24)$$

Corollary 3.4.8. *We have*

$$\mathrm{Hilb}_q(R_{n,\lambda}) = \mathrm{Hilb}_q(\mathbb{Q}[\overline{O}_{n,\lambda'} \cap \mathfrak{t}]) = \sum_{\varphi \in \mathrm{SECI}_{n,\lambda}} q^{\mathrm{inv}(\varphi)}. \quad (3.4.25)$$

Chapter 4

**LABELED BINARY TREES AND HYPERPLANE
ARRANGEMENTS**

The study of permutation statistics is a classical theme in algebraic combinatorics with its genesis in work by MacMahon [58]. An important statistic introduced by MacMahon is the descent statistic on permutations. The generating function for the distribution of this statistic gives rise to the well-known Eulerian polynomials, which show up in many areas in mathematics. The reader is referred to [66] for a detailed survey. Since the work of MacMahon, the descent statistic on permutations has been studied in depth, and yet it continues to inspire new research [75, 74].

In this chapter, which is based on joint work with Gessel and Tewari [35, 36], we study a multivariate generating function G first defined by Gessel encoding ascent-descent statistics on labeled plane binary trees. We prove that this generating function is a Schur-positive symmetric function and find an expansion for G in terms of ribbon Schur functions. We then use our results to study symmetric group actions on hyperplane arrangements.

For brevity's sake, by a tree, we will always mean a plane binary tree. Whether the tree is labeled or not will be clear from context. We remark that the notion of a descent has been studied with regards to other combinatorial objects before, such as in the case of standard Young tableaux (SYTs). However, viewing SYTs as P -partitions reveals that their descents are in fact descents of permutations in disguise. In contrast, the ascent-descent statistics that we study here are indeed different, as they depend on the embedding of the labeled trees in the plane and take into account the orientation of the edges.

More specifically, the ascent and descent statistics on labeled trees each come in two flavors depending on whether one compares the label of the parent node to the label of its

right child or its left child. Note that in our trees, we are allowing nodes to have just a left child or just a right child, see Subsection 4.1.1. We always draw our trees embedded in the plane with the root on top. Given a positive integer n , let \mathcal{T}_n^ℓ (respectively \mathcal{T}_n) denote the set of labeled (respectively unlabeled) plane binary trees on n nodes. The labels on the nodes are drawn from the set of positive integers $\mathbb{Z}_{>0}$, allowing repeats. A *standard labeling* of a tree $T \in \mathcal{T}_n$ is a labeling of its nodes with distinct labels drawn from $[n] := \{1, \dots, n\}$, and we call a tree with a standard labeling a *standard labeled tree*. For a labeled node v in T , denote by v^ℓ the label of v . If v is the left child of w , we say the edge between them is a *left ascent* if $v^\ell \leq w^\ell$. Otherwise, we say the edge is a *left descent*. Similarly, if a node v has a right child w , we say the edge between them is a *right ascent* if $v^\ell \leq w^\ell$. Otherwise, we say the edge is a *right descent*. One can think of these four statistics by listing the labels of the edge from left to right and then considering whether this pair is an ascent or descent. For any labeled tree $T \in \mathcal{T}_n^\ell$, let $\text{lasc}(T)$, $\text{ldes}(T)$, $\text{rasc}(T)$ and $\text{rdes}(T)$ denote the number of left ascents, left descents, right ascents, and right descents in T , respectively. See Figure 4.3 for two examples of labeled trees with 3 left ascents, 1 left descent, 3 right ascents, and 1 right descent.

We recover the case of ascents and descents of permutations by considering labeled trees in which no node has a left child, or alternatively, by considering labeled trees in which no node has a right child. Thus, the study of these statistics on labeled binary trees is a natural generalization of the study of ascents and descents on permutations.

Gessel, in the 1990s, initiated the study of these statistics and considered the following generating function tracking their distribution over the set of standard labeled trees.

$$B := B(x; \bar{\lambda}, \lambda, \bar{\rho}, \rho) = \sum_{n \geq 1} \sum_{\substack{T \in \mathcal{T}_n^\ell \\ T \text{ standard}}} \bar{\lambda}^{\text{lasc}(T)} \lambda^{\text{ldes}(T)} \bar{\rho}^{\text{rasc}(T)} \rho^{\text{rdes}(T)} \frac{x^n}{n!}. \quad (4.0.1)$$

In unpublished work, Gessel showed that B satisfies the functional equation

$$\frac{(1 + \bar{\lambda}B)(1 + \bar{\rho}B)}{(1 + \lambda B)(1 + \rho B)} = e^{[(\bar{\lambda}\bar{\rho} - \lambda\rho)B + \bar{\lambda} + \bar{\rho} - \lambda - \rho]x}. \quad (4.0.2)$$

Subsequently, different proofs of Equation (4.0.2) were given by Kalikow [52] and Drake [20]. From the definition of B , we observe that $B(x; \bar{\lambda}, \lambda, \bar{\rho}, \rho) = B(x; \rho, \bar{\rho}, \lambda, \bar{\lambda})$ and $B(x; \bar{\lambda}, \lambda, \bar{\rho}, \rho) = B(x; \lambda, \bar{\lambda}, \rho, \bar{\rho})$. The former is explained by reflecting a standard labeled tree across a vertical line passing through its root, while the latter follows from changing the label of a node from i to $n - i + 1$ in a standard labeled tree in \mathcal{T}_n^ℓ . Equation (4.0.2) brings to light another pair of symmetries, that

$$B(x; \bar{\lambda}, \lambda, \bar{\rho}, \rho) = B(x; \bar{\rho}, \lambda, \bar{\lambda}, \rho) = B(x; \bar{\lambda}, \rho, \bar{\rho}, \lambda). \quad (4.0.3)$$

These equalities are not obvious from the definition and a simple bijective proof for them remains elusive, although a complicated bijection can be derived from the work of Kalikow [52].

One impetus to return to the study of B has been fueled by connections with enumerative aspects of the theory of hyperplane arrangements. Let $B_n := B_n(\bar{\lambda}, \lambda, \bar{\rho}, \rho)$ denote the coefficient of $x^n/n!$ in B for $n \geq 1$. Gessel observed that certain evaluations of B_n coincide with the number of regions in various well-known deformations of Coxeter arrangements [33]. This viewpoint has been pursued in [15, 24, 91], and a complete explanation has been offered by Bernardi [8]. Given a subset A of $\{-1, 0, 1\}$, we can consider the arrangement in \mathbb{R}^n consisting of all hyperplanes $x_i - x_j = a$ where $i < j$ and $a \in A$. For $A = \{0\}$, $A = \{-1, 0, 1\}$, $A = \{-1, 1\}$, $A = \{0, 1\}$ and $A = \{1\}$, the corresponding hyperplane arrangements in \mathbb{R}^n are the *braid arrangement* \mathcal{B}_n , the *Catalan arrangement* \mathcal{C}_n , the *semiorder arrangement* \mathcal{I}_n , the *Shi arrangement* \mathcal{S}_n , and the *Linial arrangement* \mathcal{L}_n respectively. These arrangements are very well studied [7, 49, 68, 76, 77, 85] and are instances of deformations of Coxeter arrangements called *truncated affine arrangements* [68]. Various aspects of truncated affine

arrangements have been studied in great detail in [5, 4, 3, 68] and we refer the reader to them for further information. Remarkably, we have the following equalities in which the left-hand side is an evaluation of B_n and the right-hand side is the number of regions in a Coxeter arrangement deformation,

$$B_n(1, 1, 1, 1) = \text{number of regions in } \mathcal{C}_n = \frac{n!}{n+1} \binom{2n}{n}, \quad (4.0.4)$$

$$B_n(1, 0, 1, 1) = \text{number of regions in } \mathcal{S}_n = (n+1)^{n-1}, \quad (4.0.5)$$

$$B_n(1, 1, 0, 0) = \text{number of regions in } \mathcal{B}_n = n!, \quad (4.0.6)$$

$$B_n(1, 0, 1, 0) = \text{number of regions in } \mathcal{L}_n = \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k} (k+1)^{n-1}, \quad (4.0.7)$$

$$B_n(1, \zeta_6^{-1}, 1, \zeta_6) = \text{number of regions in } \mathcal{I}_n. \quad (4.0.8)$$

In (4.0.8), ζ_6 denotes a primitive sixth root of unity. Section 4.4 of this chapter is devoted to a representation-theoretic understanding of these equalities.

Our primary object of study is a multivariate generalization of B introduced by Gessel. Let $\mathbf{x} = \{x_1, x_2, \dots\}$ be a set of commuting indeterminates. With every $T \in \mathcal{T}_n^\ell$, we associate a monomial \mathbf{x}^T as follows. For a node $v \in T$ labeled i , let x_v be x_i . Then

$$\mathbf{x}^T := \prod_{v \in T} x_v. \quad (4.0.9)$$

Now consider the formal power series in \mathbf{x} with coefficients in $\mathbb{Q}[\bar{\lambda}, \lambda, \bar{\rho}, \rho]$,

$$G := G(\mathbf{x}; \bar{\lambda}, \lambda, \bar{\rho}, \rho) = \sum_{n \geq 1} \sum_{T \in \mathcal{T}_n^\ell} \bar{\lambda}^{\text{lasc}(T)} \lambda^{\text{des}(T)} \bar{\rho}^{\text{rasc}(T)} \rho^{\text{rdes}(T)} \mathbf{x}^T. \quad (4.0.10)$$

It transpires that G is a symmetric function in \mathbf{x} with coefficients in $\mathbb{Q}[\bar{\lambda}, \lambda, \bar{\rho}, \rho]$. This non-obvious fact follows from the following functional equation satisfied by G .

Theorem 4.0.1. *Let $H(z) = \sum_{n \geq 0} h_n z^n$ where h_n denotes the n th complete homogeneous*

symmetric function. We have

$$\frac{(1 + \bar{\lambda}G)(1 + \bar{\rho}G)}{(1 + \lambda G)(1 + \rho G)} = H((\bar{\lambda}\bar{\rho} - \lambda\rho)G + \bar{\lambda} + \bar{\rho} - \lambda - \rho). \quad (4.0.11)$$

Observe that the functional equation for B in (4.0.2) can be obtained from (4.0.11) by applying the homomorphism sending h_n to $x^n/n!$. This homomorphism has the crucial feature of sending the coefficient of $x_1x_2 \cdots x_n$ in any symmetric function to the coefficient of $x^n/n!$ in its image.

Given that G is a symmetric function, it is natural to ask for its expansion in the basis of Schur functions. This brings us to our first new result, which was originally conjectured by Gessel [31] in 1995.

Theorem 4.0.2. *G is Schur positive.*

Here we mean that G may be expressed as a sum of Schur functions s_λ with coefficients in the semiring $\mathbb{N}[\bar{\lambda}, \lambda, \bar{\rho}, \rho]$. Theorem 4.0.2 follows from another recursive functional equation satisfied by G , which is also one of our main results. We have the following expansion of G in terms of ribbon Schur functions r_α , defined in Chapter 2

Theorem 4.0.3. *We have*

$$G = \sum_{n \geq 1} \sum_{\alpha \models n} (\bar{\lambda}\bar{\rho}G + \bar{\lambda} + \bar{\rho})^{n-\ell(\alpha)} (\lambda\rho G + \lambda + \rho)^{\ell(\alpha)-1} r_\alpha, \quad (4.0.12)$$

where recall that r_α denotes the ribbon Schur function indexed by the strong composition α and $\ell(\alpha)$ denotes the length of α . All

In fact, Theorem 4.0.3 implies the much stronger fact that G may be expressed as a sum of ribbon Schur functions with coefficients in the semiring $\mathbb{N}[\bar{\lambda}\bar{\rho}, \lambda\rho, \bar{\lambda} + \bar{\rho}, \lambda + \rho]$. For $n \geq 1$, let $G_n := G_n(\mathbf{x}; \bar{\lambda}, \lambda, \bar{\rho}, \rho)$ denote the sum of the terms in G of total degree n in \mathbf{x} . We use a certain class of decorated noncrossing partitions called *marked interlacing partitions* to give an expansion of G_n in terms of ribbon Schur functions, stated next. Here, $\text{wt}(\pi)$ is a product

of expressions of the form $\bar{\lambda}\bar{\rho}$, $\lambda\rho$, $\bar{\lambda} + \bar{\rho}$, and $\lambda + \rho$ depending on the marked interlacing partition π . See Section 4.3 for the relevant definitions and notation, and see Section 5.2 for the proof of Theorem 4.0.4.

Theorem 4.0.4. *The formal power series G_n has the following expansion in terms of ribbon Schur functions.*

$$G_n = \sum_{\substack{\pi \in \text{mNC}(n) \\ \pi = B_1/\dots/B_k}} \text{wt}(\pi) r_{c(B_1)} r_{c(B_2)} \cdots r_{c(B_k)}. \quad (4.0.13)$$

A fact worth noting is that after expanding out the products of ribbon Schur functions on the right hand side of (4.0.13), the coefficient of r_α for every $\alpha \models n$ evaluates to the *Catalan number* $\text{Cat}_n := \frac{1}{n+1} \binom{2n}{n}$ upon setting $\bar{\lambda} = \lambda = \bar{\rho} = \rho = 1$. Additionally, the coefficients of both $r_{(1,\dots,1)}$ and $r_{(n)}$ upon setting $\bar{\lambda} = \lambda = t$, and $\bar{\rho} = \rho = q$ respectively are the *homogenized Narayana polynomials*

$$\text{Nar}_n(q, t) := \sum_{k=0}^{n-1} \frac{1}{n} \binom{n}{k} \binom{n}{k+1} q^k t^{n-1-k}.$$

We provide two proofs of Theorem 4.0.3, each with its own merits. In Section 4.2, we give an algebraic proof that follows from Theorem 4.0.1 combined with a result of MacMahon [58, Vol. 1, p. 186]. We postpone our second proof until Chapter 5 on tree enumeration bijections. We use our weight-preserving bijection to prove Theorem 4.0.4 in Section 5.2. The weight-preserving bijection will then allow us to establish a further refinement of Theorem 4.0.2, which was conjectured by Gessel [31].

Theorem 4.0.5. *Fix a positive integer n . Let ν be a word of length $n - 1$ in the alphabet $\{U, D\}$, and let $\mathcal{T}_{n,\nu}^\ell$ denote the set of labeled trees on n nodes with canopy ν . We have that the generating function*

$$G_{n,\nu} := G_{n,\nu}(\mathbf{x}; \bar{\lambda}, \lambda, \bar{\rho}, \rho) = \sum_{T \in \mathcal{T}_{n,\nu}^\ell} \bar{\lambda}^{\text{lasc}(T)} \lambda^{\text{ldes}(T)} \bar{\rho}^{\text{rasc}(T)} \rho^{\text{rdes}(T)} \mathbf{x}^T \quad (4.0.14)$$

is Schur positive.

Details on the terminology used in Theorem 4.0.5 can be found in Section 4.1.

In fact, using our weight-preserving bijection, we obtain an expansion of $G_{n,\nu}$ in terms of ribbon Schur functions using a class of decorated noncrossing partitions called *augmented interlacing partitions*. See Section 4.3 for the relevant definitions and notation, and see Section 5.2 for the proof of Theorem 4.0.6.

Theorem 4.0.6. *For $n \geq 1$ and $\nu \in \{U, D\}^{n-1}$, we have*

$$G_{n,\nu} = \sum_{\substack{\pi^* \in \text{mNC}^*(n,\nu) \\ \pi^* = B_1/\dots/B_k}} \text{wt}(\pi^*) r_{c(B_1)} r_{c(B_2)} \dots r_{c(B_k)}. \quad (4.0.15)$$

In Section 4.4, we connect specializations of G to deformations of Coxeter arrangements, focusing in particular on semiorder and Linial arrangements. Our main results in this setting are the following.

Theorem 4.0.7. *The Frobenius characteristic of the natural S_n -action on the set of regions of the semiorder arrangement \mathcal{I}_n is $G_n(\mathbf{x}; 1, \zeta_6^{-1}, 1, \zeta_6)$.*

Theorem 4.0.8. *There exists an S_n -action on the set of regions of the Linial arrangement \mathcal{L}_n whose graded Frobenius characteristic is given by $G_n(\mathbf{x}; \bar{\lambda}, 0, \bar{\rho}, 0)$.*

The proof of Theorem 4.0.7 utilizes a cycle indicator computation relying on a result of Postnikov-Stanley [68], whereas the proof of Theorem 4.0.8 utilizes crucially a recent bijection of Bernardi [8] relating regions of \mathcal{L}_n to a certain class of labeled trees that we call Bernardi trees.

It is worth mentioning that the set of regions of the braid arrangement \mathcal{B}_n and that of the Catalan arrangement \mathcal{C}_n also carry a natural S_n -action stemming from the fact that the set of hyperplanes defining both these arrangements is itself S_n -stable. In the case of \mathcal{B}_n , the S_n -action gives rise to the regular representation of S_n , whereas in the case of \mathcal{C}_n we obtain a direct sum of Cat_n many copies of the regular representation. It can be seen that the Frobenius characteristics of the S_n -actions on the regions of \mathcal{B}_n and \mathcal{C}_n are $G_n(\mathbf{x}; 1, 1, 0, 0)$ and $G_n(\mathbf{x}; 1, 1, 1, 1)$, respectively.

More interesting and equally well known is the fact that the regions of the Shi arrangement \mathcal{S}_n carry a S_n -action as well. This can be realized, for instance, by identifying the regions of \mathcal{S}_n with parking functions via the Pak-Stanley labeling [86]. The Frobenius characteristic of the resulting S_n -action on the regions of the Shi arrangement equals PF_n , the Frobenius characteristic of the character of the well-known parking function representation [46]. It is straightforward to see that the functional equation in Theorem 4.0.1 reduces to the functional equation satisfied by PF_n when we set $\bar{\lambda} = \lambda = \bar{\rho} = 1$ and $\rho = 0$. Thus, we obtain the following theorem.

Theorem 4.0.9. *The Frobenius characteristic of the S_n -action on the set of regions of the Shi arrangement \mathcal{S}_n coming from their identification with parking functions is given by $G_n(\mathbf{x}; 1, 1, 1, 0)$.*

In Section 4.1, we introduce the main combinatorial objects considered in this chapter and develop notation. Section 4.2 provides a generating function proof of Theorem 4.0.1. We then prove Theorem 4.0.3 using a generating function identity of MacMahon. Theorem 4.0.2 will then follow as a corollary. In Section 4.3, we define all definitions and notation used in Theorem 4.0.4 and Theorem 4.0.6.

In Section 4.4, we discuss applications of our results to studying actions of the symmetric group on Coxeter deformations focusing in particular on semiorder and Linial arrangements. In Section 4.5, we prove γ -nonnegativity for the distribution of right edges over local binary search trees. I

4.1 Binary trees

In this section, we introduce the main combinatorial objects of this chapter, unlabeled and labeled binary trees. For further details on binary trees, we refer the reader to [89].

4.1.1 Unlabeled trees and associated notions

A *plane binary tree* is a rooted tree in which every node has at most two children, of which at most one is called a *left child* and at most one is called a *right child*. We denote the set of all plane binary trees by \mathcal{T} and the set of plane binary trees on n nodes for $n \geq 1$ by \mathcal{T}_n . Recall that we use the term *tree* to mean a plane binary tree. Elements of \mathcal{T} will be considered *unlabeled trees*. We denote the set of nodes of T by $\text{Nodes}(T)$, the set of edges of T by $\text{Edges}(T)$, and the root of T by $\text{root}(T)$. We abuse notation on occasion and write $v \in T$ when we mean $v \in \text{Nodes}(T)$. The nodes of a tree can be categorized as *terminal nodes*, which are nodes with no children, and *internal nodes*, which are nodes with at least one child.

Given a binary tree $T \in \mathcal{T}_n$, let \bar{T} be the binary tree obtained by appending two children to every terminal node and one child to every node which is the parent of a single child. We call these appended nodes *leaves*, and we call \bar{T} the *completion* of T . In total, the completion \bar{T} has $n + 1$ leaves.

Given a binary tree $T \in \mathcal{T}$, we can define a partial order \leq_T on $\text{Nodes}(T)$ by drawing the tree with its root on top and leaves below and declaring this to be the Hasse diagram of the partial order. Precisely, we define a relation such that for $v, w \in \text{Nodes}(T)$ with v a child of w , we have $v <_T w$. Then \leq_T is defined as the transitive closure of this relation. If $v <_T w$, then we say that v is a *descendant* of w and that w is an *ancestor* of v .

Additionally, we work with two different types of total orderings on the nodes of a binary tree T derived from a traversal of its nodes. The *preorder traversal* is defined recursively, where we first visit the root, then traverse the right subtree of T in preorder, and finally the left subtree of T in preorder. It should be noted that this is slightly different from the usual convention for preorder traversal, where one traverses the left subtree before the right subtree. The *inorder traversal* is also defined recursively, where we first traverse the left subtree of T in inorder, then visit the root of T , and finally traverse the right subtree of T in inorder. If we order the nodes of a given tree T according to when they are visited in

the preorder (respectively inorder) traversal, we obtain a total order on $\text{Nodes}(T)$ which we call *preorder* (respectively *inorder*) and denote by \preceq_p (respectively \preceq_i). Figure 4.1 gives an example each of preorder and inorder, respectively.

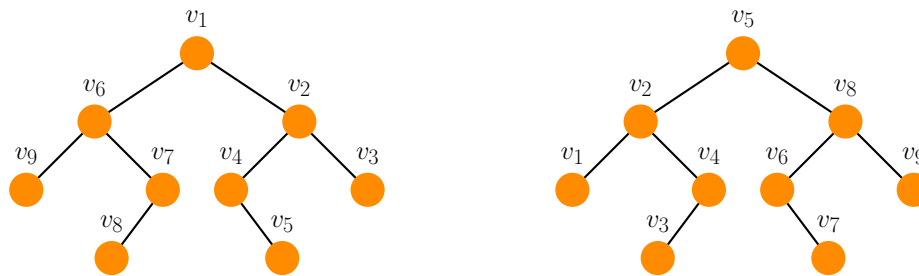


Figure 4.1: The nodes are labeled according to preorder on the left and according to inorder on the right.

The preorder \preceq_p on $T \in \mathcal{T}_n$ allows us to associate a composition of size n with it, which we call the *composition type* of T and denote by $\mathbf{c}(T)$. To compute $\mathbf{c}(T)$, let $v_1 <_p \cdots <_p v_n$ be the nodes of T in preorder. Assume further that $v_{i_1} <_p \cdots <_p v_{i_k}$ are all the terminal nodes in T . Note that $1 \leq i_1 < \cdots < i_k = n$. We now define $\mathbf{c}(T) := \text{comp}\{i_1, \dots, i_{k-1}\} = (i_1, i_2 - i_1, i_3 - i_2, \dots, i_k - i_{k-1})$. Clearly $\mathbf{c}(T) \models n$. The reader can verify that for the tree T in Figure 4.1, we have $\mathbf{c}(T) = (3, 2, 3, 1)$.

We work under the convention that the edge vw refers to the edge joining v and w in the tree, where v comes before w in inorder. If v is the left child of w , we say that vw is the *left edge* of w . If w is the right child of v , we say that vw is the *right edge* of v .

Next, we give a definition of the canopy of a binary tree as a word on $\{U, D\}$. To translate between the canopy defined in [69] and the one defined here, simply flip the tree vertically and replace $U \leftrightarrow \bar{a}$ and $D \leftrightarrow b$ in the word $v(T)$ defined in their paper. See [69, Proposition 2.2] for other equivalent definitions of canopy.

Definition 4.1.1. Given a binary tree $T \in \mathcal{T}_n$, label each node v except for the last node of T in inorder with either a D if v has a right child or a U if v does not have a right child. Traverse the tree in inorder and read off the labels. The resulting word of length $n - 1$ on

$\{U, D\}$ is the *canopy* of T , denoted by $\text{can}(T)$.

Figure 4.2 shows a tree T with its nodes labeled U and D according to Definition 4.1.1. Reading off the labels in inorder yields $\text{can}(T) = UDUUDDUD$.

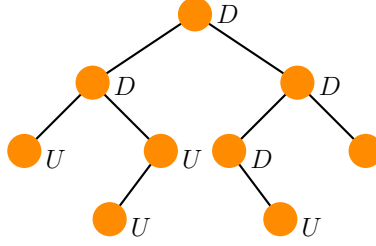


Figure 4.2: A tree T with canopy $\text{can}(T) = UDUUDDUD$.

4.1.2 Labeled trees

A *labeled plane binary tree* (or simply a *labeled tree*) is a tree whose nodes have labels drawn from the set of positive integers $\mathbb{Z}_{>0}$. We denote the set of labeled trees by \mathcal{T}^ℓ and the set of labeled trees on n nodes for $n \geq 1$ by \mathcal{T}_n^ℓ . Given $T \in \mathcal{T}_n^\ell$, we denote by $\text{sh}(T)$ the unlabeled tree obtained by removing the labels on the nodes of T . Given a node u in T , we refer to the label on u as u^ℓ . We associate two reading words with T : the *preorder reading word* denoted by $\text{pre}(T)$, and the *inorder reading word* denoted by $\text{in}(T)$. If v_1, \dots, v_n are the nodes of T in preorder, then $\text{pre}(T) := v_1^\ell \cdots v_n^\ell$. On the other hand, if v_1, \dots, v_n are the nodes of T in inorder, then $\text{in}(T) := v_1^\ell \cdots v_n^\ell$.

Let us recall from the Introduction that for a labeled tree, we have a refined classification for its edges given by left and right ascents and descents. First, suppose that v is the left child of w . If $v^\ell \leq w^\ell$, then vw is a *left ascent*. Otherwise it is a *left descent*. Second, suppose that v has a right child w . If $v^\ell \leq w^\ell$, then vw is a *right ascent*. Otherwise it is a *right descent*. Using this classification, we associate a weight $\text{wt}(T)$ to a labeled tree T as follows,

$$\text{wt}(T) := \bar{\lambda}^{\text{lasc}(T)} \lambda^{\text{ldes}(T)} \bar{\rho}^{\text{rasc}(T)} \rho^{\text{rdes}(T)}. \quad (4.1.1)$$

Recall that $\text{lasc}(T)$ (respectively, $\text{lides}(T)$, $\text{rasc}(T)$, and $\text{rdes}(T)$) is the number of left ascents (respectively, left descents, right ascents and right descents) in T . For the labeled tree T in Figure 4.3, each edge is labeled with $\bar{\lambda}$, λ , $\bar{\rho}$, or ρ , corresponding to the edge's orientation and whether the labels on the edge form an ascent or descent. It may help the reader to remember that λ corresponds to left edges, ρ corresponds to right edges, unbarred parameters correspond to strict inequalities, and barred parameters correspond to weak inequalities.

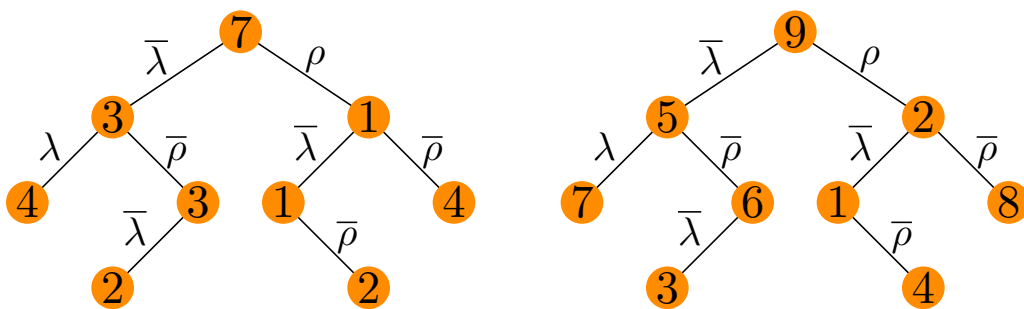


Figure 4.3: On the left, a tree with weight $\text{wt}(T) = \bar{\lambda}^3 \lambda^1 \bar{\rho}^3 \rho^1$. On the right, the inorder standardization $\text{sort}(T)$.

Letting v_1, v_2, \dots, v_n be the nodes of T listed in inorder, we can obtain a standard labeled tree by relabeling node v_i with the i th letter of the permutation $\text{sort}(\text{in}(T))$. We call the resulting standard labeled tree the *inorder standardization* of T , denoted by $\text{sort}(T)$. Given an edge $v_i v_j$ in T , it forms a (left or right) descent if and only if $(i, j) \in \text{Inv}(\text{in}(T))$. Since standardization preserves the inversion set of a word, we have $\text{wt}(T) = \text{wt}(\text{sort}(T))$.

4.2 Two functional equations for G

We begin by proving Theorem 4.0.1 which establishes a functional equation for G . We then use this identity to give our first proof of Theorem 4.0.3.

A labeled tree is *increasing* if it has no left ascents or right descents. The term *increasing* is motivated by the fact that if T is an increasing tree, then the labels increase —weakly toward the right and strictly toward the left— along the path from the root of T to any

terminal node. The next lemma shows that these trees are in bijection with words on positive integers.

Lemma 4.2.1. *Fix n and $1 \leq j \leq n$. Then the map $\varphi_n(T) = \text{in}(T)$ from the set of increasing binary trees T on n nodes to $\mathbb{Z}_{>0}^n$ is a bijection. Furthermore, φ_n restricts to a bijection between the subset of trees whose completion has j right leaves and the subset of words with $j - 1$ descents.*

Proof. First, let T be an increasing binary tree on n nodes whose completion has j right leaves, and let $w = \text{in}(T)$. It can be checked that the i th node of T in inorder has no right child if and only if $i \in \text{Des}(w)$ or $i = n$. Recalling the definition of the completion \bar{T} from Subsection 4.1.1, a node of T has a right leaf in \bar{T} if and only if it has no right child. Therefore, we see that $j = \#\text{Des}(w) + 1$, so w has $j - 1$ descents.

Next, we recursively define a map ψ_n from $\mathbb{Z}_{>0}^n$ to the set of increasing binary trees on n nodes which maps a word with $j - 1$ descents to a tree whose completion has j right leaves. Let ψ_1 be the map which sends a word w_1 with length one to the tree on one node labeled with w_1 . Clearly, ψ_1 is the inverse of ϕ_1 . For $n > 0$, suppose we have constructed ψ_i for $i < n$ such that $\phi_i \circ \psi_i(w) = \text{in}(\psi_i(w)) = w$. Fixing $w \in \mathbb{Z}_{>0}^n$ with $j - 1$ descents, let us define $\psi_n(w)$ as follows. Let i be the smallest index of the smallest letter in w . Let $T_1 = \psi_{i-1}(w_1 \dots w_{i-1})$, and let $T_2 = \psi_{n-i}(w_{i+1} \dots w_n)$. Then define $\psi_n(w) = T$ to be the tree whose root is labeled w_i and whose left and right subtrees are T_1 and T_2 , respectively.

Finally, we show that ψ_n is the inverse of ϕ_n . By our inductive assumptions on the maps ψ_{i-1} and ψ_{n-i} , we have that $\text{in}(T_1) = w_1 \dots w_{i-1}$ and $\text{in}(T_2) = w_{i+1} \dots w_n$. Therefore, we have $\phi_n \circ \psi_n(w) = \text{in}(T) = \text{in}(T_1)w_i\text{in}(T_2) = w$ by the recursive definition of the inorder traversal. Furthermore, it follows from our choice of i that T is increasing, hence ψ_n is the right inverse of ϕ_n . We leave it to the reader to check that ψ_n is also the left inverse of ϕ_n , hence it is the inverse of ϕ_n . This completes our recursive construction of ψ_n , and hence ϕ_n is a bijection with inverse ψ_n . \square

Proof of Theorem 4.0.1. Recall that Theorem 4.0.1 claims that

$$\frac{(1 + \bar{\lambda}G)(1 + \bar{\rho}G)}{(1 + \lambda G)(1 + \rho G)} = H((\bar{\lambda}\bar{\rho} - \lambda\rho)G + \bar{\lambda} + \bar{\rho} - \lambda - \rho). \quad (4.2.1)$$

Let $\text{ll}(T)$ and $\text{rl}(T)$ denote the number of left leaves and right leaves in the completion \bar{T} , respectively. Let

$$D(\mathbf{x}; \bar{\lambda}, \lambda, \lambda_{\circ}, \bar{\rho}, \rho, \rho_{\circ}) = \sum_{n \geq 1} \sum_{T \in \mathcal{T}_n^{\ell}} \bar{\lambda}^{\text{lasc}(T)} \lambda^{\text{lides}(T)} \lambda_{\circ}^{\text{ll}(T)} \bar{\rho}^{\text{rasc}(T)} \rho^{\text{rdes}(T)} \rho_{\circ}^{\text{rl}(T)} \mathbf{x}^T. \quad (4.2.2)$$

In \bar{T} , since every node has a left child that is either a node or a leaf, we have

$$n = \text{lasc}(T) + \text{lides}(T) + \text{ll}(T) \quad (4.2.3)$$

and similarly,

$$n = \text{rasc}(T) + \text{rdes}(T) + \text{rl}(T). \quad (4.2.4)$$

Thus,

$$D(\mathbf{x}; \bar{\lambda}, \lambda, \lambda_{\circ}, \bar{\rho}, \rho, \rho_{\circ}) = G(\lambda_{\circ}\rho_{\circ}\mathbf{x}; \bar{\lambda}/\lambda_{\circ}, \lambda/\lambda_{\circ}, \bar{\rho}/\rho_{\circ}, \rho/\rho_{\circ}), \quad (4.2.5)$$

where $\lambda_{\circ}\rho_{\circ}\mathbf{x}$ means that each variable x_i is replaced with $\lambda_{\circ}\rho_{\circ}x_i$.

Let T be a labeled tree. If pq is a left ascent of T , such that p is the left child of q , we call p a *left ascent-child*. If pq is a right descent of T , such that q is the right child of p , we call q a *right descent-child*. We obtain a *marked tree* from T by “marking” some of the nodes of T . We require that every left ascent-child and right descent-child must be marked, the root must not be marked, and other nodes may be either marked or unmarked. See Figure 4.4 for an example of a marked tree. Let $\text{lm}(M)$ be the number of marked left children in M , and let $\text{rm}(M)$ be the number of marked right children in M .

Let

$$C(\mathbf{x}; \lambda_{\circ}, \lambda_m, \rho_{\circ}, \rho_m) = \sum_M \lambda_{\circ}^{\text{ll}(M)} \lambda_m^{\text{lm}(M)} \rho_{\circ}^{\text{rl}(M)} \rho_m^{\text{rm}(M)} \mathbf{x}^M \quad (4.2.6)$$

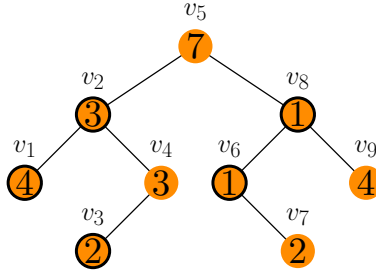


Figure 4.4: An example of a marked tree on 9 nodes with nodes $v_1, v_2, v_3, v_6,$ and v_8 in inorder marked.

where the sum is over all marked labeled trees M and where \mathbf{x}^M , $\text{ll}(M)$ and $\text{rl}(M)$ are equal to \mathbf{x}^T , $\text{ll}(T)$, and $\text{rl}(T)$ for T the underlying tree of M , respectively. Then by the definition of marked trees we have

$$C(\mathbf{x}; \lambda_\circ, \lambda_m, \rho_\circ, \rho_m) = D(\mathbf{x}; \lambda_m, 1 + \lambda_m, \lambda_\circ, 1 + \rho_m, \rho_m, \rho_\circ), \quad (4.2.7)$$

and it follows from (4.2.5) that

$$C(\mathbf{x}; \lambda_\circ, \lambda_m, \rho_\circ, \rho_m) = G\left(\lambda_\circ \rho_\circ \mathbf{x}; \frac{\lambda_m}{\lambda_\circ}, \frac{1 + \lambda_m}{\lambda_\circ}, \frac{1 + \rho_m}{\rho_\circ}, \frac{\rho_m}{\rho_\circ}\right). \quad (4.2.8)$$

Substituting $x_i \mapsto x_i/(\lambda_\circ \rho_\circ)$, $\lambda_\circ \mapsto 1/(\lambda - \bar{\lambda})$, $\lambda_m \mapsto \bar{\lambda}/(\lambda - \bar{\lambda})$, $\rho_\circ \mapsto 1/(\bar{\rho} - \rho)$ and $\rho_m \mapsto \rho/(\bar{\rho} - \rho)$ into (4.2.8), we find that

$$G(\mathbf{x}; \bar{\lambda}, \lambda, \bar{\rho}, \rho) = C\left((\lambda - \bar{\lambda})(\bar{\rho} - \rho)\mathbf{x}; \frac{1}{\lambda - \bar{\lambda}}, \frac{\bar{\lambda}}{\lambda - \bar{\lambda}}, \frac{1}{\bar{\rho} - \rho}, \frac{\rho}{\bar{\rho} - \rho}\right). \quad (4.2.9)$$

We shall find a functional equation for $C = C(\mathbf{x}; \lambda_\circ, \lambda_m, \rho_\circ, \rho_m)$ which through (4.2.9) will yield (4.2.1).

Next we count increasing labeled trees by left and right leaves. Let

$$K(\mathbf{x}; \lambda_\circ, \rho_\circ) = \sum_T \lambda_\circ^{\text{ll}(T)} \rho_\circ^{\text{rl}(T)} \mathbf{x}^T, \quad (4.2.10)$$

where the sum is over all increasing trees T . Recall the 2-parameter weighted power series analogue of the Eulerian polynomial $A(\mathbf{x}; s, t)$ defined in (2.0.14) and the identity 2.0.16, reproduced here for the reader's convenience,

$$A(\mathbf{x}; s, t) = \frac{\sum_{n \geq 1} (s-t)^{n-1} h_n}{1 - t \sum_{n \geq 1} (s-t)^{n-1} h_n} = \frac{H(s-t) - 1}{s - tH(s-t)}. \quad (4.2.11)$$

Therefore, we have

$$K(\mathbf{x}; \lambda_\circ, \rho_\circ) = \lambda_\circ \rho_\circ A(\mathbf{x}; \lambda_\circ, \rho_\circ) = \lambda_\circ \rho_\circ \frac{H(\lambda_\circ - \rho_\circ) - 1}{\lambda_\circ - \rho_\circ H(\lambda_\circ - \rho_\circ)}, \quad (4.2.12)$$

where the first equality follows from Lemma 4.2.1, and the second equality follows from (2.0.16).

We claim that C satisfies the functional equation

$$C = K(\mathbf{x}; \lambda_\circ + \lambda_m C, \rho_\circ + \rho_m C). \quad (4.2.13)$$

To see this, given a marked labeled tree M on $[n]$, let P be the set of unmarked nodes p of M with the property that no ancestor of p is marked. It is clear that the induced subtree of M on P is an increasing tree I . Then M can be recovered from I by attaching marked trees to the leaves of I and marking the roots of the attached trees. Equation (4.2.13) follows from this decomposition.

Using (4.2.12), we may expand (4.2.13) as

$$C = (\lambda_\circ + \lambda_m C)(\rho_\circ + \rho_m C) \frac{H(\lambda_\circ + \lambda_m C - \rho_\circ - \rho_m C) - 1}{(\lambda_\circ + \lambda_m C) - (\rho_\circ + \rho_m C)H(\lambda_\circ + \lambda_m C - \rho_\circ - \rho_m C)}. \quad (4.2.14)$$

Solving for H in (4.2.14) gives

$$\frac{(\rho_\circ + (\rho_m + 1)C)(\lambda_\circ + \lambda_m C)}{(\lambda_\circ + (\lambda_m + 1)C)(\rho_\circ + \rho_m C)} = H(\lambda_\circ - \rho_\circ + (\lambda_m - \rho_m)C). \quad (4.2.15)$$

Finally, set $\lambda_o = 1/(\lambda - \bar{\lambda})$, $\lambda_m = \bar{\lambda}/(\lambda - \bar{\lambda})$, $\rho_o = 1/(\bar{\rho} - \rho)$ and $\rho_m = \rho/(\bar{\rho} - \rho)$ in (4.2.15), and replace \mathbf{x} with $(\lambda - \bar{\lambda})(\bar{\rho} - \rho)\mathbf{x}$. Applying (4.2.9) to the resulting identity gives (4.2.1). \square

Remark 4.2.2. Our proof above of Theorem 4.0.1 is based on a proof given in [32] of a result equivalent to the case $\bar{\lambda} = 0$ of (4.0.2). The paper [32] counts forests of labeled rooted trees by descents and leaves instead of counting binary trees, but one can convert between forests of rooted trees and binary trees by applying a simple bijection.

As a corollary to Theorem 4.0.1, we see that G has some surprising symmetries. We state these next in Corollary 4.2.3.

Corollary 4.2.3. *The following identities hold,*

$$G(\mathbf{x}; \bar{\lambda}, \lambda, \bar{\rho}, \rho) = G(\mathbf{x}; \bar{\rho}, \lambda, \bar{\lambda}, \rho) = G(\mathbf{x}; \bar{\lambda}, \rho, \bar{\rho}, \lambda), \quad (4.2.16)$$

$$\omega(G(\mathbf{x}; \bar{\lambda}, \lambda, \bar{\rho}, \rho)) = G(\mathbf{x}; \lambda, \bar{\lambda}, \rho, \bar{\rho}). \quad (4.2.17)$$

Proof. We can rewrite the functional equation (4.2.1) as

$$G = (1 + \lambda G)(1 + \rho G) \sum_{n \geq 1} ((\bar{\lambda}\bar{\rho} - \lambda\rho)G + \bar{\lambda} + \bar{\rho} - \lambda - \rho)^{n-1} h_n. \quad (4.2.18)$$

This gives a recursive method for computing G_n in terms of G_m for $m < n$. Therefore, G is uniquely determined by the functional equation. Identity (4.2.16) then follows from the fact that the functional equation is invariant under swapping $\bar{\lambda}$ and $\bar{\rho}$ and under swapping λ and ρ . To see identity (4.2.17), apply ω to both sides of the functional equation (4.2.1), and then use the fact that $H(-z)E(z) = 1$. \square

Letting $A := A(\mathbf{x}; s, t)$, we may rewrite (2.0.16) as

$$H(s - t) = \frac{1 + sA}{1 + tA}. \quad (4.2.19)$$

At this point, we have all the tools we need to establish Theorem 4.0.3, which also implies Theorem 4.0.2.

Proof 1 of Theorem 4.0.3. Substituting $s = \bar{\lambda}\bar{\rho}G + \bar{\lambda} + \bar{\rho}$ and $t = \lambda\rho G + \lambda + \rho$ into (4.2.19), then $H(s - t)$ becomes the right-hand side of (4.2.1). Thus, we can rewrite (4.2.1) as

$$\frac{(1 + \bar{\lambda}G)(1 + \bar{\rho}G)}{(1 + \lambda G)(1 + \rho G)} = \frac{1 + sA}{1 + tA}. \quad (4.2.20)$$

Since $(1 + \bar{\lambda}G)(1 + \bar{\rho}G) = 1 + sG$ and $(1 + \lambda G)(1 + \rho G) = 1 + tG$, the identity (4.2.20) is equivalent to

$$\frac{1 + sG}{1 + tG} = \frac{1 + sA}{1 + tA}, \quad (4.2.21)$$

which implies that

$$G = A = A(\mathbf{x}; \bar{\lambda}\bar{\rho}G + \bar{\lambda} + \bar{\rho}, \lambda\rho G + \lambda + \rho), \quad (4.2.22)$$

and Theorem 4.0.3 follows by (2.0.15). \square

Remark 4.2.4. Theorem 4.0.3 together with (2.0.10) gives a way to recursively expand each G_n positively in terms of ribbon Schur functions. Therefore, the Schur positivity of G follows immediately from Theorem 4.0.3 by using Proposition 2.0.1. However, it is not clear how to prove the Schur positivity of the more refined generating functions $G_{n,\nu}$ using these same techniques. Instead, we prove Theorem 4.0.6 in Section 5.2 after developing our weight-preserving bijection in Section 5.1. Theorem 4.0.5 then follows from Theorem 4.0.6 by Proposition 2.0.1.

4.3 Marked and augmented interlacing partitions

In this section, we present the definitions and notation used in the statements of Theorems 4.0.4 and 4.0.6. We delay the proofs of these two theorems until Section 5.2 after developing all of the key concepts in Section 5.1.

4.3.1 Noncrossing partitions

We need some notions concerning the lattice of noncrossing partitions, so we recall the relevant definitions briefly. A *partition* π of $[n]$ is a collection of pairwise disjoint nonempty

subsets B_1, \dots, B_k whose union is $[n]$. We write this as $\pi := B_1/\dots/B_k$ where the B_i are ordered in increasing order of their minimal elements. The subsets B_1, \dots, B_k are the *blocks* of π , and the number of blocks is denoted by $\text{bk}(\pi)$. The set of partitions of $[n]$, denoted by Π_n , can be endowed with the structure of a *graded lattice* by defining a partial order as follows. Given partitions σ and τ , we say that $\sigma \leq_{\Pi_n} \tau$ if each block in σ is contained in a block in τ . In particular, a partition τ covers a partition σ in Π_n if τ is obtained by merging two distinct blocks in σ . If $\sigma \leq_{\Pi_n} \tau$, we say that σ is *finer* than τ or equivalently that τ is *coarser* than σ . The rank of $\pi \in \Pi_n$ is given by $n - \text{bk}(\pi)$. The partition of $[n]$ into singleton sets gives the unique minimal element in Π_n , and the partition of $[n]$ consisting of a single block gives the unique maximal element.

We identify a partition of $[n]$ with its *arc diagram*, which is defined as follows. Consider n nodes v_1, \dots, v_n representing the integers 1 through n from left to right, and connect two nodes v_i and v_j with $i < j$ by an undirected arc if i and j belong to the same block and if there is no k in that block such that $i < k < j$. If i and j belong to the same block in π , then we denote this equivalence by $i \sim_\pi j$.

A partition π of $[n]$ is said to be *noncrossing* if there do not exist $1 \leq a < b < c < d \leq n$ such that $a \sim_\pi c$, $b \sim_\pi d$ and $a \not\sim_\pi b$. The set of noncrossing partitions of n , denoted by $\text{NC}(n)$, inherits a graded lattice structure from that on Π_n . For the many interesting properties of $\text{NC}(n)$, the reader is referred to the beautiful survey by Simion [78] and references therein. For a more recent survey on the relevance of $\text{NC}(n)$ in various areas of mathematics, the reader is referred to McCammond [59].

4.3.2 Interlacing partitions

We now introduce a special type of noncrossing partition, called an interlacing partition, and decorated generalizations of interlacing partitions.

Let $\pi = B_1/\dots/B_k \in \text{NC}(n)$. If $i \sim_\pi i + 1$, then we say the node v_i is a *stepper* and the arc connecting v_i and v_{i+1} is a *short arc*. If $i \sim_\pi j$ with $j \geq i + 2$ and there is an arc connecting v_i and v_j in the arc diagram, then we say the node v_i is a *jumper*

and the arc between v_i and v_j is a *long arc*. Let $\max(\pi) := \{\max(B_1), \dots, \max(B_k)\}$ and $\min(\pi) := \{\min(B_1), \dots, \min(B_k)\}$. We say that π is *interlacing* if $i \in \max(\pi)$ implies that $i + 1 \notin \min(\pi)$ for all i . Finally, an interlacing partition $\pi \in \text{NC}(n)$ is said to be *marked* if a subset of nodes in $\{v_1, \dots, v_n\} \setminus \{v_i : i \in \max(\pi)\}$ is marked. Let $\text{mNC}(n)$ denote the set of all marked interlacing partitions whose underlying interlacing partition is in $\text{NC}(n)$. For the marked interlacing partition π in Figure 4.5, the nodes v_2 , v_4 , and v_8 are marked. Furthermore, the nodes v_1 , v_2 , v_7 , and v_9 are *steppers*, and the nodes v_3 , v_4 , and v_8 are *jumpers*.

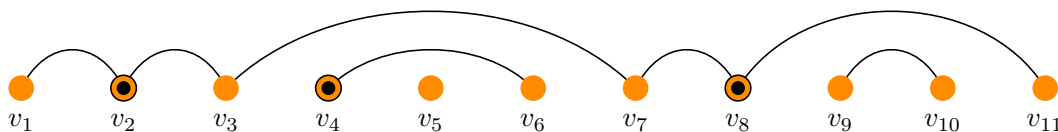


Figure 4.5: The arc diagram of the interlacing partition $\pi = 1\ 2\ 3\ 7\ 8\ 11 / 4\ 6 / 5 / 9\ 10$, with nodes v_2 , v_4 , and v_8 marked.

Remark 4.3.1. The cardinality of the set of interlacing partitions for $n \geq 1$ is given by the sequence of Motzkin numbers [63, A001006], which are well known to enumerate Motzkin paths, which are lattice paths from $(0, 0)$ to $(n, 0)$ where the steps allowed are up, down and level. The cardinality of the set of marked interlacing partitions for $n \geq 1$ is given by [63, A071356], which counts Motzkin paths where the up and level steps are bicolored. In theory we could have phrased our results in the language of Motzkin paths but it is (marked) interlacing partitions that arise naturally in our context.

To each $\pi \in \text{mNC}(n)$, we associate a sequence of compositions as follows. Partition each block B of π into disjoint subsets by breaking B after each marked node. Define $c(B)$ to be the composition obtained by recording the sizes of these subsets. For $\pi = B_1/B_2/B_3/B_4$ in

Figure 4.5, we have

$$\begin{aligned}
 B_1 &= \{1, 2, 3, 7, 8, 11\} & c(B_1) &= (2, 3, 1), \\
 B_2 &= \{4, 6\} & c(B_2) &= (1, 1), \\
 B_3 &= \{5\} & c(B_3) &= (1), \\
 B_4 &= \{9, 10\} & c(B_4) &= (2).
 \end{aligned}$$

Consider the following four statistics associated to a marked interlacing partition π . Let

$$\text{sa}(\pi) = \# \text{ Unmarked steppers in } \pi \quad (4.3.1)$$

$$\text{sd}(\pi) = \# \text{ Marked steppers in } \pi \quad (4.3.2)$$

$$\text{ja}(\pi) = \# \text{ Unmarked jumpers in } \pi \quad (4.3.3)$$

$$\text{jd}(\pi) = \# \text{ Marked jumpers in } \pi. \quad (4.3.4)$$

Let the weight of a marked interlacing partition be

$$\text{wt}(\pi) = (\bar{\lambda} + \bar{\rho})^{\text{sa}(\pi)} (\lambda + \rho)^{\text{sd}(\pi)} (\bar{\lambda}\bar{\rho})^{\text{ja}(\pi)} (\lambda\rho)^{\text{jd}(\pi)}. \quad (4.3.5)$$

Remark 4.3.2. The letters a and d in the names of the statistics in (4.3.1)–(4.3.4) correspond to *ascent* and *descent*. We will see in Section 5.2 how each marked interlacing partition π corresponds to a set of labeled binary trees, where each marked node in π represents a descent in the labeling, and each unmarked node represents an ascent in the labeling.

We define an *augmented interlacing partition* π^* to be a marked interlacing partition such that each short arc in its arc diagram is labeled either U or D . To each augmented

interlacing partition π^* , we associate the following six statistics. Let

$$\begin{aligned}
\text{saU}(\pi^*) &= \# \text{ Unmarked steppers in } \pi^* \text{ whose corresponding short arc is labeled } U, \\
\text{sdU}(\pi^*) &= \# \text{ Marked steppers in } \pi^* \text{ whose corresponding short arc is labeled } U, \\
\text{saD}(\pi^*) &= \# \text{ Unmarked steppers in } \pi^* \text{ whose corresponding short arc is labeled } D, \\
\text{sdD}(\pi^*) &= \# \text{ Marked steppers in } \pi^* \text{ whose corresponding short arc is labeled } D, \\
\text{ja}(\pi^*) &= \# \text{ Unmarked jumpers in } \pi^*, \\
\text{jd}(\pi^*) &= \# \text{ Marked jumpers in } \pi^*.
\end{aligned}$$

Combining all six statistics, we define

$$\text{wt}(\pi^*) = \bar{\lambda}^{\text{saU}(\pi^*)} \lambda^{\text{sdU}(\pi^*)} \bar{\rho}^{\text{saD}(\pi^*)} \rho^{\text{sdD}(\pi^*)} (\bar{\lambda}\bar{\rho})^{\text{ja}(\pi^*)} (\lambda\rho)^{\text{jd}(\pi^*)}. \quad (4.3.6)$$

For π^* in Figure 4.6, we have $\text{wt}(\pi^*) = \bar{\rho}\lambda(\bar{\lambda}\bar{\rho})(\lambda\rho)\bar{\rho}(\lambda\rho)\bar{\lambda} = \bar{\lambda}^2\lambda^3\bar{\rho}^3\rho^2$.

To each augmented interlacing partition π^* , associate words $w(\pi^*)$ and $\hat{w}(\pi^*)$ defined recursively as follows. If π^* contains a single block, define $w(\pi^*)$ to be the word obtained by recording the labels on the short arcs from left to right, and define $\hat{w}(\pi^*)$ to be $Dw(\pi^*)U$. If π^* contains more than 1 block, let B_1 be the block of π^* which contains v_1 and v_n . Partition B_1 into blocks C_1, \dots, C_p that are maximal under connectedness by short arcs. For $1 \leq i \leq p-1$, let π_i^* denote the augmented interlacing partition induced by π^* on the nodes v_j for $j \in (\max(C_i), \min(C_{i+1}))$. Define

$$w(\pi^*) := w(C_1)\hat{w}(\pi_1^*) \cdots w(C_{p-1})\hat{w}(\pi_{p-1}^*)w(C_p) \in \{U, D\}^{n-1}, \quad (4.3.7)$$

where the dots signify that the concatenation continues and n is the number of nodes of π . Given $\nu \in \{U, D\}^{n-1}$, let $\text{mNC}^*(n, \nu)$ be the set of augmented interlacing partitions π^* such that $w(\pi^*) = \nu$. For the augmented interlacing partition π^* in Figure 4.6, we have that $\pi^* \in \text{mNC}^*(11, DUDDUDDUU)$.

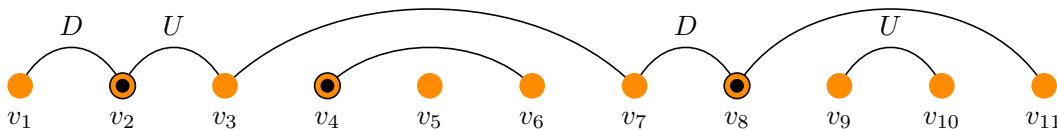


Figure 4.6: An augmented interlacing partition π^* with $w(\pi^*) = DUDDUDDUU$.

4.4 S_n -modules from deformations of Coxeter arrangements

In this section, we provide a representation-theoretic meaning to some of the equalities in the introduction relating specializations of $B_n(\mathbf{x}; \bar{\lambda}, \lambda, \bar{\rho}, \rho)$ to the number of regions in hyperplane arrangements related to the Coxeter arrangement. In particular, we focus on the semiorder and Linal arrangements. While there is an obvious symmetric group action on regions of the semiorder arrangement \mathcal{I}_n by permutation of the coordinates, the question of finding one on regions of the Linal arrangement \mathcal{L}_n is a bit more subtle. Throughout this section, we assume the ribbon Schur expansions in Theorem 4.0.4 and Theorem 4.0.6, which are proven in Section 5.2.

We briefly introduce our notation pertaining to hyperplane arrangements. For a detailed introduction, we refer the reader to [64, 82]. A *hyperplane arrangement* is a finite collection of affine hyperplanes in a vector space. Let \mathcal{A} be a hyperplane arrangement in a finite-dimensional vector space V over \mathbb{R} . A *region* of \mathcal{A} is a connected component of $V - \bigcup_{H \in \mathcal{A}} H$. We denote the set of regions of \mathcal{A} by $\text{Regions}(\mathcal{A})$ and the number of regions of \mathcal{A} by $r(\mathcal{A}) = |\text{Regions}(\mathcal{A})|$.

4.4.1 Semiorder arrangements

Recall that the semiorder arrangement \mathcal{I}_n is the hyperplane arrangement in \mathbb{R}^n given by the hyperplanes $H_{ij} : x_i - x_j = 1$ for $1 \leq i \neq j \leq n$. Note that this set of hyperplanes is stable under the natural action of the symmetric group S_n . This implies that $\text{Regions}(\mathcal{I}_n)$ inherits an action of S_n . We first expand a certain specialization of G_n in terms of the Motzkin numbers and the Frobenius characteristics of Foulkes characters. We then use this formula

to show in Theorem 4.4.4 that the Frobenius characteristic of the action on $\text{Regions}(\mathcal{I}_n)$ is a specialization of G_n .

Figure 4.7 demonstrates the action of S_3 on $\text{Regions}(\mathcal{I}_3)$. In this case, we can compute the Frobenius characteristic of this action to be $h_3 + 2h_{21} + 2h_{111}$. In terms of ribbon Schur functions, this may be written as $5r_3 + 3(r_{21} + r_{12}) + 2r_{111}$. Note that since ribbon Schur functions do not form a basis for the ring of symmetric functions, this expansion in terms of ribbon Schur functions is not unique.

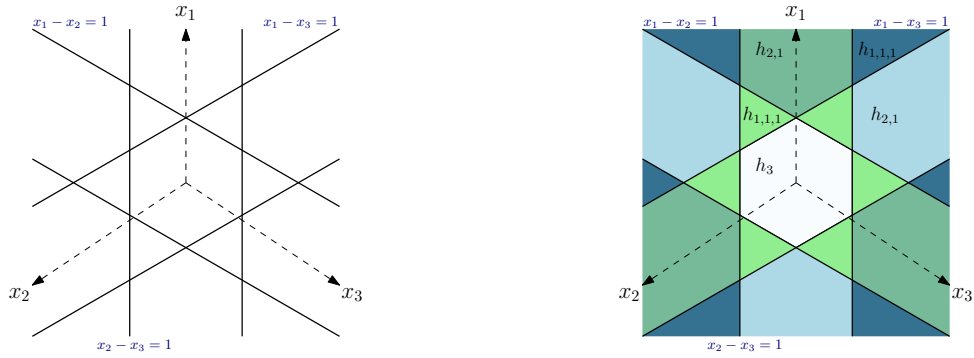


Figure 4.7: On the left is the arrangement \mathcal{I}_3 (projected onto $x_1 + x_2 + x_3 = 0$). On the right is a depiction of the S_3 -action on $\text{Regions}(\mathcal{I}_3)$ with regions of the same color belonging to the same orbit (see the electronic version).

By Theorem 4.0.4, we know G_n expands in terms of ribbon Schur functions with coefficients which are polynomials in $\bar{\lambda}\bar{\rho}$, $\bar{\lambda} + \bar{\rho}$, $\lambda\rho$, and $\lambda + \rho$. Consider the case $\bar{\lambda} = \bar{\rho} = 1$, $\lambda = \zeta_6^{-1}$, and $\rho = \zeta_6$, where ζ_6 is a primitive sixth root of unity. Then we have that $\bar{\lambda}\bar{\rho} = \lambda + \rho = \lambda\rho = 1$, whereas $\bar{\lambda} + \bar{\rho} = 2$. Thus, for this specialization we have $\text{wt}(\pi) = 2^{\text{sa}(\pi)}$ for $\pi \in \text{mNC}(n)$. From Theorem 4.0.4 and (2.0.11), we obtain

$$G_n(\mathbf{x}; 1, \zeta_6^{-1}, 1, \zeta_6) = \sum_{\pi \in \text{mNC}(n)} 2^{\text{sa}(\pi)} \left(\sum_{\delta \in [\check{\mathbf{c}}(\pi), \hat{\mathbf{c}}(\pi)]} r_\delta \right), \tag{4.4.1}$$

where if $\pi = B_1 / \dots / B_k$, then $\check{\mathbf{c}}(\pi) = c(B_1) \bullet \dots \bullet c(B_k)$ and $\hat{\mathbf{c}}(\pi) = c(B_1) \odot \dots \odot c(B_k)$.

For $0 \leq k \leq n - 1$, let

$$F_{n,k} := \sum_{\substack{\alpha \vdash n \\ \ell(\alpha) = k+1}} r_\alpha. \quad (4.4.2)$$

It can be shown that the dimension of the S_n -module corresponding to $F_{n,k}$ is the Eulerian number $A_{n,k}$ enumerating the number of permutations in S_n with k descents [25]. The symmetric group character corresponding to $F_{n,k}$ under the Frobenius characteristic map is known as the *Foulkes character*, introduced by Foulkes in his study of descents in permutations. They show up in various areas such as counting permutations by descents and cycle types [37], enumerating alternating permutations according to cycle type [88], and the analysis of the carrying process [18, 50, 62]. Foulkes characters have been generalized to other reflection groups in [60].

Throughout this section, we encode most of our formulas in terms of marked and augmented interlacing partitions. However, it should be noted that all of these formulas could just as easily be encoded in terms of binary trees via the following bijection between \mathcal{T}_n and $\text{NC}(n)$, which is a special case of a bijection due to Edelman. Figure 4.8 shows a binary tree T and its corresponding noncrossing partition.

Theorem 4.4.1 (Edelman [21]). *Given $T \in \mathcal{T}_n$, let v_1, \dots, v_n be its vertices listed in preorder. Define $\text{nc}(T)$ to be the finest partition of $[n]$ with the property that distinct positive integers $1 \leq i < j \leq n$ are in the same block if v_j is the left child of v_i . The map $T \mapsto \text{nc}(T)$ defines a bijection between \mathcal{T}_n and $\text{NC}(n)$.*

Given a binary tree $T \in \mathcal{T}_n$, we say that it is *left-leaning* if for every node v of T , whenever v has a right child, then it also has a left child. We denote the set of left-leaning trees on n nodes by \mathcal{LT}_n . From the definition of $\text{nc}(T)$, we see that if $T \in \mathcal{LT}_n$, then $\text{nc}(T)$ is an interlacing partition of $[n]$. In fact, Edelman's bijection restricts to a bijection between \mathcal{LT}_n and the set of interlacing partitions of $[n]$.

For $n \geq 1$, let Mot_n denote the n th *Motzkin number* [63, A001006] which is defined to be

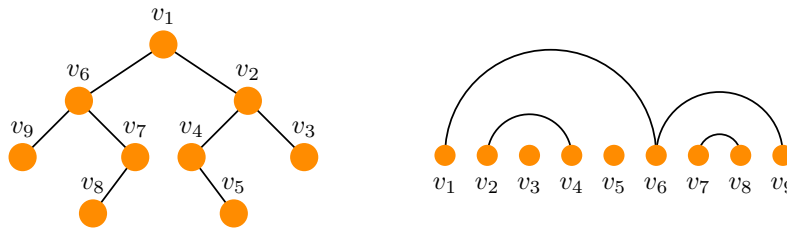


Figure 4.8: On the left is a binary tree T , and on the right is its corresponding noncrossing partition $\text{nc}(T)$.

the number of interlacing partitions on $[n]$, or equivalently the number of left-leaning binary trees on n nodes. Let $M(x)$ denote the generating function for the Motzkin numbers,

$$M(x) := \sum_{n \geq 1} \text{Mot}_n x^n = x + x^2 + 2x^3 + 4x^4 + \dots . \quad (4.4.3)$$

Let $\text{Cat}_n = |\mathcal{T}_n|$ be the n th *Catalan number* [63, A000108]. Let $C(x)$ denote the generating function for the Catalan numbers,

$$C(x) := \sum_{n \geq 1} \text{Cat}_n x^n = x + 2x^2 + 5x^3 + 14x^4 + \dots . \quad (4.4.4)$$

Then it can be checked that $M(x)$ and $C(x)$ are related by the following identity,

$$M\left(\frac{x}{1-x}\right) = C(x). \quad (4.4.5)$$

Equation (4.4.5) implies the following identity that will come in handy later,

$$M(e^x - 1) = C(1 - e^{-x}). \quad (4.4.6)$$

Decomposing a tree into its root with two subtrees, each of which is potentially empty, yields the identity $C = x(1 + C)^2$. By using (4.4.5) again, we see that $M(x)$ satisfies the functional

equation

$$M = x(1 + M + M^2). \quad (4.4.7)$$

Compare this to [19, Equation 1], for example, where our indexing differs from theirs by 1.

Theorem 4.4.2. *For $n \geq 1$, we have the following expansion of $G_n(\mathbf{x}; 1, \zeta_6^{-1}, 1, \zeta_6)$ in terms of the Frobenius characteristics of Foulkes characters,*

$$G_n(\mathbf{x}; 1, \zeta_6^{-1}, 1, \zeta_6) = \sum_{\alpha \models n} \text{Mot}_{\ell(\alpha)} h_\alpha = \sum_{j=1}^n \left(\sum_{k=0}^{n-j} \binom{n-j}{k} \text{Mot}_{n-k} \right) F_{n,j-1}.$$

Proof. Just for this proof, we denote $G(\mathbf{x}; 1, \zeta_6^{-1}, 1, \zeta_6)$ by G . Similarly, let $H = H(1) = \sum_{i \geq 0} h_i$. To obtain the h -expansion for $G_n(\mathbf{x}; 1, \zeta_6^{-1}, 1, \zeta_6)$, we use the functional equation for G from Theorem 4.0.1. When $\bar{\lambda} = \bar{\rho} = 1$, $\lambda = \zeta_6^{-1}$, and $\rho = \zeta_6$, we get that

$$G = (1 + G + G^2)(H - 1). \quad (4.4.8)$$

A comparison of the functional equation in (4.4.8) with the functional equation satisfied by $M(x)$ in (4.4.7) reveals that

$$G = \sum_{m \geq 1} \text{Mot}_m (H - 1)^m \quad (4.4.9)$$

$$= \sum_{m \geq 1} \sum_{\ell(\alpha)=m} \text{Mot}_m h_\alpha, \quad (4.4.10)$$

where the above sum is over all compositions α of any size with length m . Focusing on terms indexed by compositions of size n in (4.4.10) yields the first equality in the statement of the theorem.

To obtain the expansion in terms of the Frobenius characteristics of Foulkes characters,

we utilize the fact that $h_\alpha = \sum_{\beta \succ \alpha} r_\beta$ and (4.4.10) to obtain

$$G = \sum_{m \geq 1} \sum_{\ell(\alpha)=m} \text{Mot}_m \sum_{\beta \succ \alpha} r_\beta \quad (4.4.11)$$

$$= \sum_{m \geq 1} \sum_{\beta \models m} r_\beta \left(\sum_{\beta \succ \alpha} \text{Mot}_{\ell(\alpha)} \right) \quad (4.4.12)$$

$$= \sum_{m \geq 1} \sum_{\beta \models m} r_\beta \left(\sum_{k=0}^{m-\ell(\beta)} \binom{m-\ell(\beta)}{k} \text{Mot}_{m-k} \right). \quad (4.4.13)$$

Note that the innermost sum is only dependent on $\ell(\beta)$. Collecting terms corresponding to compositions of a fixed size and grouping them according to their lengths gives us the second equality in the statement of the theorem. \square

For example, $G_3(\mathbf{x}; 1, \zeta_6^{-1}, 1, \zeta_6) = \text{Mot}_1 h_3 + \text{Mot}_2(h_{12} + h_{21}) + \text{Mot}_3 h_{111} = h_3 + 2h_{21} + 2h_{111}$, which is the Frobenius characteristic of the S_3 -action on \mathcal{I}_3 as shown in Figure 4.7. To compute the Frobenius characteristic in the general case we need the following result.

Lemma 4.4.3 ([68, Lemma 7.6]). *Let $\sigma \in S_n$ be a permutation with k cycles. Then the number of regions in \mathcal{I}_n fixed by σ is equal to the number of regions in \mathcal{I}_k .*

We remark here that the statement in Lemma 4.4.3 differs slightly from that in [68] as Postnikov-Stanley consider the hyperplane arrangement \mathcal{I}_n projected onto the hyperplane $x_1 + \cdots + x_n = 0$ as their definition of the semiorder arrangement. This induces a harmless shift in indices and does not affect the mathematical content.

Theorem 4.4.4. *The symmetric function $G_n(\mathbf{x}; 1, \zeta_6^{-1}, 1, \zeta_6)$ is the Frobenius characteristic of the action of S_n on $\text{Regions}(\mathcal{I}_n)$.*

Proof. Our proof uses the fact that the cycle indicator Z_n of the S_n -action on $\text{Regions}(\mathcal{I}_n)$, is also the Frobenius characteristic of the character of this action. See [87, Section 7.24] for more background on the cycle indicator. In view of Lemma 4.4.3, we find that the cycle

indicator is given by

$$Z_n = \sum_{\lambda \vdash n} r(\mathcal{I}_{\ell(\lambda)}) \frac{p_\lambda}{z_\lambda} \quad (4.4.14)$$

$$= \sum_{k=1}^n r(\mathcal{I}_k) \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=k}} \frac{p_\lambda}{z_\lambda}. \quad (4.4.15)$$

Let $Z := \sum_{n \geq 1} Z_n$. Then we have that

$$Z = \sum_{n \geq 1} \sum_{k=1}^n r(\mathcal{I}_k) \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=k}} \frac{p_\lambda}{z_\lambda} \quad (4.4.16)$$

$$= \sum_{k \geq 1} \frac{r(\mathcal{I}_k)}{k!} \left(\sum_{j \geq 1} \frac{p_j}{j} \right)^k. \quad (4.4.17)$$

Now let $H = \sum_{i \geq 0} h_i$ again. Then we have $\sum_{j \geq 1} p_j/j = \log H$. Using this in (4.4.17) gives

$$Z = \sum_{k \geq 1} r(\mathcal{I}_k) \frac{(\log H)^k}{k!}. \quad (4.4.18)$$

We have the expansion

$$\frac{(\log H)^k}{k!} = \sum_{m \geq 0} (-1)^{m-k} \text{Stir}(m, k) \frac{(H-1)^m}{m!} \quad (4.4.19)$$

where $(-1)^{m-k} \text{Stir}(m, k)$ is the *signed Stirling number of the first kind* [63, A008275] enumerating permutations in S_m having exactly k cycles in their cycle factorization. Using this equality, we can rephrase (4.4.18) as

$$Z = \sum_{k \geq 1} r(\mathcal{I}_k) \sum_{m \geq 0} (-1)^{m-k} \text{Stir}(m, k) \frac{(H-1)^m}{m!} \quad (4.4.20)$$

$$= \sum_{m \geq 1} \frac{(H-1)^m}{m!} \sum_{k=1}^m (-1)^{m-k} r(\mathcal{I}_k) \text{Stir}(m, k), \quad (4.4.21)$$

where, in changing the order of summation, we have used the fact that $\text{Stir}(0, k) = 0$ for $k \geq 1$. By [85, Theorem 2.3], the following equality holds

$$\sum_{m \geq 1} r(\mathcal{I}_m) \frac{x^m}{m!} = C(1 - e^{-x}). \tag{4.4.22}$$

Recall (4.4.6) states that $M(e^x - 1) = C(1 - e^{-x})$, and hence the left hand side of (4.4.22) also equals $M(e^x - 1)$. This implies that

$$m! \text{Mot}_m = \sum_{k=1}^m (-1)^{m-k} r(\mathcal{I}_k) \text{Stir}(m, k). \tag{4.4.23}$$

Substituting (4.4.23) into (4.4.21), we conclude that

$$Z = \sum_{m \geq 1} \text{Mot}_m (H - 1)^m = G(\mathbf{x}; 1, \zeta_6^{-1}, 1, \zeta_6). \tag{4.4.24}$$

The second equality in (4.4.24) follows from (4.4.10). Then (4.4.24) implies that the cycle indicator Z_n is equal to $G_n(\mathbf{x}; 1, \zeta_6^{-1}, 1, \zeta_6)$, which completes the proof. \square

We conclude this subsection with a generalization of Theorem 4.4.4. Given a positive integer p , define the p -semiorder arrangement $\mathcal{I}_{n,p}$ to be the hyperplane arrangement in \mathbb{R}^n defined by the hyperplanes $x_i - x_j = \pm 1, \pm 2, \dots, \pm p$ for $1 \leq i < j \leq n$. As before, the symmetric group S_n acts on $\text{Regions}(\mathcal{I}_{n,p})$ and one can ask about the corresponding Frobenius characteristic. We provide a brief description of its computation next, omitting details.

Let $C_p(x) = \sum_{n \geq 1} \text{Cat}_{n,p} x^n$ denote the generating function for the *Fuss-Catalan* numbers $\text{Cat}_{n,p} := \frac{1}{pn+1} \binom{(p+1)n}{n}$. Let \mathcal{T}_n^p be the set of rooted plane $(p + 1)$ -ary trees T such that each node of T has at most one i th child for each $1 \leq i \leq p + 1$. Then $\text{Cat}_{n,p}$ is the cardinality of the set \mathcal{T}_n^p . Let $\text{Mot}_{n,p}$ denote the cardinality of the set \mathcal{LT}_n^p consisting of rooted plane $(p + 1)$ -ary trees on n nodes such that every internal node which has a $(p + 1)$ th child also has at least one other child.

Let $M_p(x) := \sum \text{Mot}_{n,p} x^n$. As was the case earlier, we have that

$$M_p \left(\frac{x}{1-x} \right) = C_p(x), \quad (4.4.25)$$

which implies $M_p(e^x - 1) = C_p(1 - e^{-x})$.

Crucially for us, Lemma 4.4.3 continues to hold for $\mathcal{I}_{n,p}$ as well. Thus, we can repeat the cycle indicator computation of Theorem 4.4.4. We only need the relation analogous to (4.4.22). Indeed, by [85, Theorem 2.3], we have that

$$\sum_{m \geq 1} r(\mathcal{I}_{m,p}) \frac{x^m}{m!} = C_p(1 - e^{-x}). \quad (4.4.26)$$

In arriving at the above equality we have used the fact that the number of regions in the p -Catalan arrangement defined by the hyperplanes $x_i - x_j = 0, \pm 1, \dots, \pm p$ for $1 \leq i < j \leq n$ is given by $\text{Cat}_{n,p}$ (see [85, Section 2] or [2, Section 5]). Thus, we obtain the following theorem whose $p = 1$ case is Theorem 4.4.4.

Theorem 4.4.5. *For $p \geq 1$, the Frobenius characteristic of the action of S_n on $\text{Regions}(\mathcal{I}_{n,p})$ is given by $M_p(H-1)$, where $H = \sum_{i \geq 0} h_i$. Additionally, the Frobenius characteristic expands positively in terms of the $F_{n,k}$.*

4.4.2 Linial arrangements and local binary search trees

We turn our attention to studying the Linial arrangement \mathcal{L}_n and defining an S_n -action on its regions. Observe that, unlike in the case of the semiorder arrangement, the symmetric group S_n does not stabilize the set of hyperplanes defining \mathcal{L}_n . Hence it is not immediate how to construct an S_n -action on $\text{Regions}(\mathcal{L}_n)$. Another well-studied arrangement with the property that the set consisting of its defining hyperplanes is not stable under the S_n -action is the Shi arrangement \mathcal{S}_n . In spite of this limitation, one can define an S_n -action on $\text{Regions}(\mathcal{S}_n)$ by using one of the many ways to index its regions by parking functions of length n , and then using the natural S_n -action on them. Drawing inspiration from this, we use certain labeled

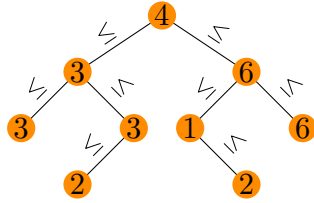


Figure 4.9: An example of a local binary search (LBS) tree on 9 nodes.

trees that we call Bernardi trees instead of parking functions to index regions of \mathcal{L}_n , and then construct a natural S_n -action on Bernardi trees to derive one on $\text{Regions}(\mathcal{L}_n)$. First, we need to understand how the symmetric function G_n relates to $\text{Regions}(\mathcal{L}_n)$.

The problem of enumerating $r(\mathcal{L}_n)$ was first considered by Postnikov [67], inspired by a question of Linial and Ravid. He showed that $r(\mathcal{L}_n)$ equals the number of intransitive trees and gave a bijection between intransitive trees and local binary search trees. It is the latter that carries a description in terms of ascents and descents in labeled trees.

Consider the case where $\lambda = \rho = 0$. In the setting of the introduction, this corresponds to considering labeled trees that only have left ascents and right ascents. We refer to such trees as *local binary search trees* (henceforth *LBS trees*). See Figure 4.9 for an example of a local binary search tree on 9 nodes. We construct another subset of \mathcal{T}_n^ℓ that is equinumerous with standard LBS trees on n nodes and use it to define an S_n -action on $\text{Regions}(\mathcal{L}_n)$. It is worth emphasizing that the same subset of trees has been considered by Bernardi [8] to solve the long-standing problem of finding a bijection between Linial regions and standard LBS trees.

By writing G_n as a sum of the $G_{n,\nu}$ where ν runs over all possible canopies of length $n - 1$ and using Theorem 4.0.6, we have

$$G_n(\mathbf{x}; \bar{\lambda}, 0, \bar{\rho}, 0) = \sum_{\pi^* = B_1/\dots/B_k} \bar{\lambda}^{\text{saU}(\pi^*)} \bar{\rho}^{\text{saD}(\pi^*)} (\bar{\lambda}\bar{\rho})^{\text{ja}(\pi^*)} r_{c(B_1)} \dots r_{c(B_k)} \tag{4.4.27}$$

$$= \sum_{\pi^* = B_1/\dots/B_k} \bar{\lambda}^{\text{saU}(\pi^*)} \bar{\rho}^{\text{saD}(\pi^*)} (\bar{\lambda}\bar{\rho})^{\text{ja}(\pi^*)} h_{(|B_1|, \dots, |B_k|)} \tag{4.4.28}$$

where the sum runs over all augmented interlacing partitions π^* on n nodes such that all nodes are unmarked. The second equality comes from the fact that, for a block B_i in π^* , since all steppers and jumpers are unmarked, then $r_{c(B_i)} = r_{(m_i)} = h_{m_i}$ where $m_i = |B_i|$.

Next, we use a bijection between the set of augmented interlacing partitions on n nodes such that all nodes are unmarked and the set \mathcal{T}_n in order to get an expansion in terms of binary trees. Given an augmented interlacing partition π^* on n nodes such that all nodes are unmarked, delete all short arcs in π^* which are labeled with a U , and then remove all U and D labels. This provides a bijection between these objects and $\text{NC}(n)$. Then compose this bijection with the inverse of Edelman’s bijection from $\text{NC}(n)$ to \mathcal{T}_n . Finally, apply the bijection from \mathcal{T}_n to itself which flips a tree across the vertical line passing through its root. See Figure 4.10 for an example of an augmented interlacing partition on 8 nodes with all nodes unmarked, together with its image under the composition of these bijections.

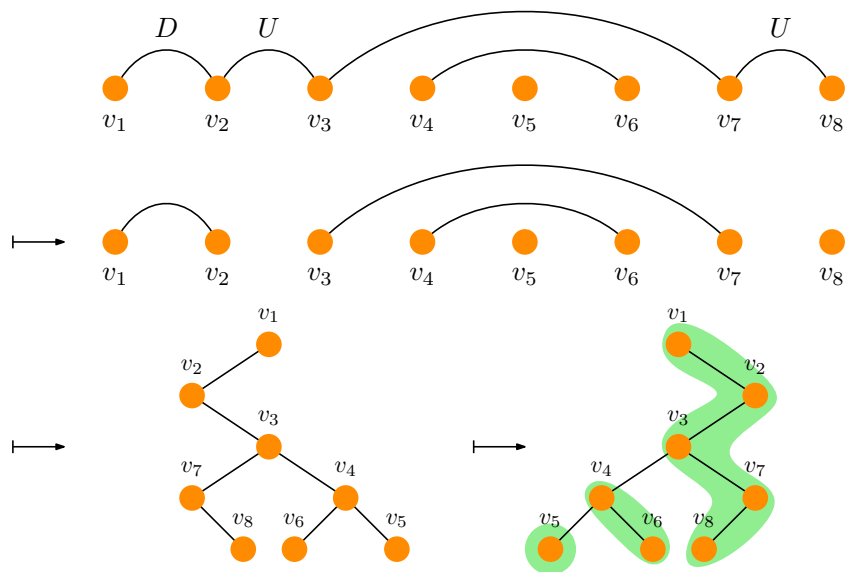


Figure 4.10: An example of an augmented interlacing partition on 8 nodes with all nodes unmarked, and its corresponding elements of $\text{NC}(8)$ and \mathcal{T}_8 .

We claim that $\text{sort}(\mathbf{c}(T)) = \text{sort}(|B_1|, \dots, |B_k|)$, where $\text{sort}(\alpha)$ is the partition underlying a composition α and $\mathbf{c}(T)$ denotes the composition type of T defined in Subsec-

tion 4.1.1. We make note of two pertinent aspects of this composition of three bijections. Let $\pi^* = B_1/\dots/B_k$ be an augmented interlacing partition on n nodes such that all nodes are unmarked, and let $T \in \mathcal{T}_n$ be the corresponding binary tree.

1. If v_i and v_{i+1} are connected by a short arc labeled U (respectively D) in π^* , then in T the node v_i has only a left child (respectively right child) which is v_{i+1} .
2. If nodes v_i and v_j where $i < j$ are connected by a long arc in π^* , then v_i has two children in T . In particular, the right child is v_j and the left child is v_{i+1} .

It follows that any two nodes in T connected by an edge which are visited in succession in the preorder traversal correspond to nodes in π^* connected by an arc. Hence, we have $\text{sort}(\mathbf{c}(T)) = \text{sort}(|B_1|, \dots, |B_k|)$. Additionally, we see that $\text{saD}(\pi^*) + \text{ja}(\pi^*) = \text{r}(T)$, $\text{saU}(\pi^*) + \text{ja}(\pi^*) = \ell(T)$, where $\ell(T)$ and $\text{r}(T)$ denote the number of left edges and right edges in T , respectively.

Hence, from (4.4.28) we obtain the following expansion,

$$G_n(\mathbf{x}; \bar{\lambda}, 0, \bar{\rho}, 0) = \sum_{T \in \mathcal{T}_n} \bar{\lambda}^{\ell(T)} \bar{\rho}^{\text{r}(T)} h_{\mathbf{c}(T)}. \quad (4.4.29)$$

Since $\ell(T) + \text{r}(T) = n - 1$ for any $T \in \mathcal{T}_n$, we set $\bar{\lambda} = 1$ and $\bar{\rho}$ equal to an indeterminate q . In order to give a representation-theoretic interpretation of $G_n(\mathbf{x}; \bar{\lambda}, 0, \bar{\rho}, 0)$, it suffices to find a S_n -module whose graded Frobenius characteristic is $G_n(\mathbf{x}; 1, 0, q, 0)$, which we give next.

A *Bernardi tree* is a standard labeled binary tree satisfying the condition that every internal node has a label that is greater than the label of its right child provided it exists, otherwise it is greater than the label of its left child. Let \mathcal{T}_n^B denote the set of Bernardi trees on n nodes. It can be checked that the image of the homomorphism ex applied to the right-hand side of (4.4.29) is a generating function enumerating Bernardi trees by the number of left and right edges. Since $G_n(\mathbf{x}; \bar{\lambda}, 0, \bar{\rho}, 0)$ is the generating function corresponding to LBS trees on n nodes, applying ex to the left-hand side of (4.4.29) establishes that the number

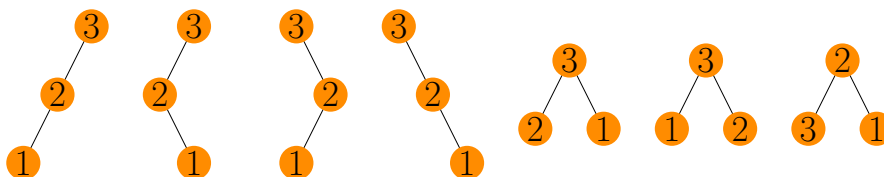


Figure 4.11: The Bernardi trees on 3 nodes.

of standard LBS trees on n nodes is equal to the cardinality of \mathcal{T}_n^B . Figure 4.11 shows all trees in \mathcal{T}_3^B .

Remark 4.4.6. For the reason behind the name Bernardi trees, we refer the reader to [8, Example 1.1] where Bernardi gives a bijection between \mathcal{T}_n^B and $\text{Regions}(\mathcal{L}_n)$. In the notation of [8], the bijection associates to $T \in \mathcal{T}_n^B$ the region of \mathcal{L}_n defined by the inequalities $x_i - x_j < 1$ where $1 \leq i < j \leq n$ and either $\text{drift}(i) \leq \text{drift}(j)$, or $\text{drift}(i) = \text{drift}(j) + 1$ and $i <_p j$. Here, $\text{drift}(v)$ is the number of ancestors of v (including v) that are right children. Figure 4.12 shows the regions of \mathcal{L}_3 indexed by the corresponding Bernardi trees according to this bijection.

An equivalent characterization of Bernardi trees is as follows. Let $v_1 <_p \dots <_p v_n$ be the nodes of a standard labeled binary tree T listed in preorder. If for all cover relations $v_i <_p v_{i+1}$ where v_i is the parent of v_{i+1} , we have that $v_i^\ell > v_{i+1}^\ell$, then T is a Bernardi tree.

For $T \in \mathcal{T}_n^B$, let $v_1 <_p \dots <_p v_n$ be the nodes of T in preorder. Among these nodes, let $v_{i_1} <_p \dots <_p v_{i_k}$ be all the terminal nodes and set $i_0 := 1$. Observe that the preorder reading word $\text{pre}(T) = v_1^\ell \dots v_n^\ell$ can be factorized as $W_1 \dots W_k$, where for $j \geq 1$ we have

$$W_j = v_{i_{j-1}+1}^\ell v_{i_{j-1}+2}^\ell \dots v_{i_j-1}^\ell v_{i_j}^\ell, \tag{4.4.30}$$

Given the definition of \mathcal{T}_n^B , we know that each W_j is strictly decreasing when read from left to right. For the tree $T \in \mathcal{T}_9^B$ in Figure 4.13 on the left, the shaded regions give us the words $W_1 = 731$, $W_2 = 86$, $W_3 = 952$ and $W_4 = 4$ and their concatenation gives us $\text{pre}(T) = 731\ 86\ 952\ 4$.

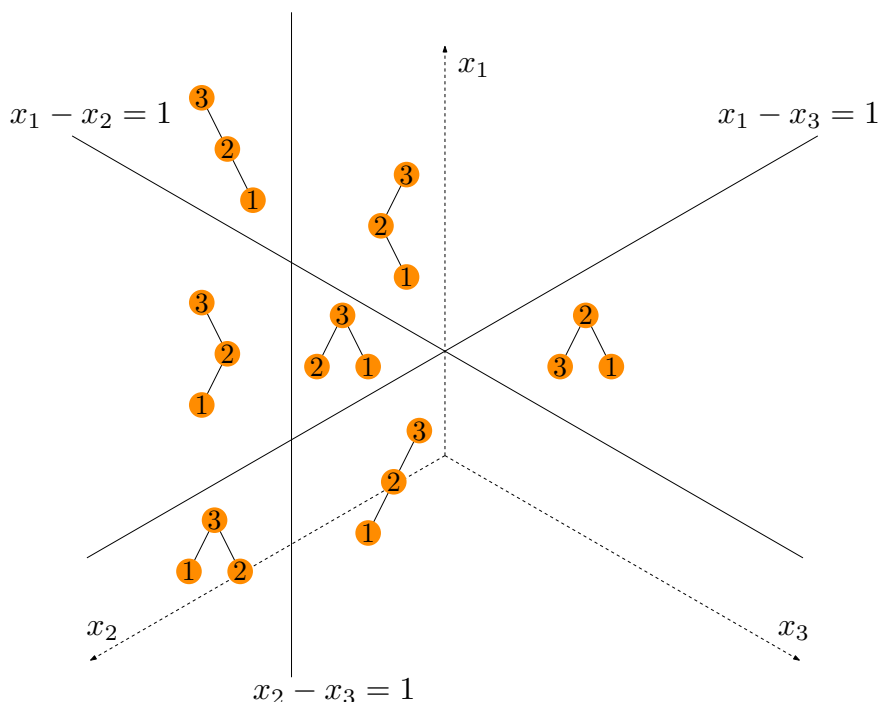


Figure 4.12: Regions of \mathcal{L}_3 indexed by trees in \mathcal{T}_3^B .

The factorization of $\text{pre}(T)$ as $W = W_1 \cdots W_k$ as described earlier allows us to define an obvious S_n -action on \mathcal{T}_n^B as follows. Given $\sigma \in S_n$, define $\tilde{\sigma}(W_i)$ to be the word obtained by replacing every letter in W_i by its image under σ , and then sorting the resulting word so that it is strictly decreasing when read from left to right. Now define $\tilde{\sigma}(W)$ to be $\tilde{\sigma}(W_1) \cdots \tilde{\sigma}(W_k)$, and let $\sigma(T)$ be the unique labeled tree such that $\text{sh}(\sigma(T)) = \text{sh}(T)$ and $\text{pre}(\sigma(T)) = \tilde{\sigma}(W)$. By the alternative characterization for Bernardi trees in Remark 4.4.6, these two conditions ensure that $\sigma(T) \in \mathcal{T}_n^B$.

This is a well-defined S_n -action on \mathcal{T}_n^B . One way to see this is to observe that Bernardi trees T with a fixed $\text{sh}(T)$ are in bijection with tabloids [73] of partition shape $\text{sort}(\mathbf{c}(T))$. Indeed, given the factorization $W_1 \cdots W_k$ of $\text{pre}(T)$ defined above, form k rows of boxes with $i_j - i_{j-1}$ many boxes in row $j \leq k$, where the boxes in row j are labeled with the letters of W_j in order from left to right. Then sort the rows to get a tabloid on the partition shape $\text{sort}(\mathbf{c}(T))$. This map defines the desired bijection and the action of S_n on Bernardi trees

with fixed $\text{sh}(T)$ corresponds to the usual action of S_n on tabloids under the bijection.

In Figure 4.13, for the tree T on the left, we have $\sigma(T)$ on the right where $\sigma = (38)$ in cycle notation. Note that for the instance under discussion, we have $\tilde{\sigma}(W) = 871\ 63\ 952\ 4$. Thus, $\sigma(T)$ is the unique tree whose preorder reading word is 871639524 and whose underlying shape is that of T . Had we chosen $\sigma = (37)$, then $\sigma(T)$ would be T itself.

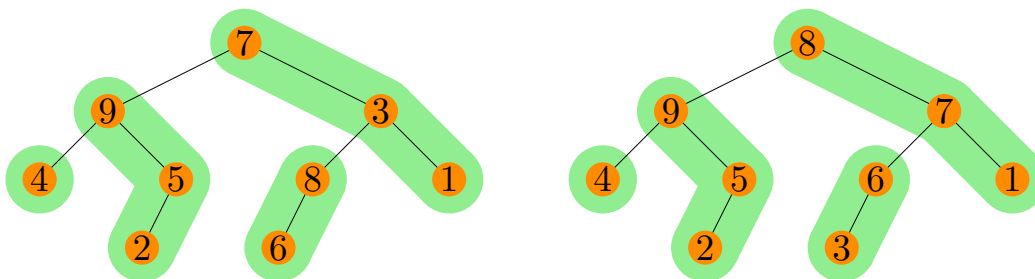


Figure 4.13: On the left, a tree $T \in \mathcal{T}_9^B$. On the right, $\sigma(T)$ where $\sigma = (38) \in S_9$.

Let $\mathbb{C}\mathcal{T}_n^B$ denote the S_n -module whose underlying vector space is generated by formal linear combinations of trees in \mathcal{T}_n^B . Denote by $\mathcal{T}_{n,k}^B$ the set of Bernardi trees with exactly k right edges. Let $\mathbb{C}\mathcal{T}_{n,k}^B \subseteq \mathbb{C}\mathcal{T}_n^B$ be the S_n -submodule spanned by Bernardi trees in $\mathcal{T}_{n,k}^B$. We have the following equality of S_n -modules

$$\mathbb{C}\mathcal{T}_n^B = \bigoplus_{k=0}^{n-1} \mathbb{C}\mathcal{T}_{n,k}^B. \tag{4.4.31}$$

Thus, we can think of $\mathbb{C}\mathcal{T}_n^B$ as being graded by the number of right edges.

Theorem 4.4.7. *The graded Frobenius characteristic of the S_n -module $\mathbb{C}\mathcal{T}_n^B$ is given by*

$$G_n(\mathbf{x}; 1, 0, q, 0) = \sum_{T \in \mathcal{T}_n} q^{\text{r}(T)} h_{\mathbf{c}(T)}. \tag{4.4.32}$$

Using Bernardi’s bijection between \mathcal{T}_n^B and $\text{Regions}(\mathcal{L}_n)$, our S_n -action lifts to an action on $\text{Regions}(\mathcal{L}_n)$ whose graded Frobenius characteristic is also $G_n(\mathbf{x}; 1, 0, q, 0)$.

Proof. Observe that the identity (4.4.32) follows from (4.4.29). Therefore, it suffices to prove that the Frobenius characteristic of $\mathbb{C}\mathcal{T}_{n,k}^B$ is the q^k coefficient of the right-hand side of (4.4.32). Given $\alpha \models n$ of length m , let $S_\alpha := S_{\alpha_1} \times \cdots \times S_{\alpha_m}$ be the corresponding Young subgroup of S_n . It is well known (see, e.g. [73]) that h_α is the Frobenius characteristic of $\mathbf{1} \uparrow_{S_\alpha}^{S_n}$, the trivial S_α -module induced up to S_n .

Given an unlabeled tree $T \in \mathcal{T}_n$, the Bernardi trees $T' \in \mathcal{T}_{n,k}^B$ such that $\text{sh}(T') = T$ span a S_n -submodule of $\mathbb{C}\mathcal{T}_{n,k}^B$ isomorphic to $\mathbf{1} \uparrow_{S_{\mathbf{c}(T)}}^{S_n}$. Hence, we have an isomorphism of S_n -modules,

$$\mathbb{C}\mathcal{T}_{n,k}^B \cong \bigoplus_{\substack{T \in \mathcal{T}_n \\ k \text{ right edges}}} \mathbf{1} \uparrow_{S_{\mathbf{c}(T)}}^{S_n} . \tag{4.4.33}$$

The proof is then completed by using the fact that the Frobenius characteristic of a direct sum of submodules is the sum of the Frobenius characteristics. □

Observe that by applying the homomorphism ex to $G_n(\mathbf{x}; 1, 0, 1, 0)$, we can compute the cardinality of \mathcal{T}_n^B . This gives another formula for $r(\mathcal{L}_n)$, which is not in any way more concise than the known formula and hence is omitted.

In fact, we can recover Postnikov’s formula [67] for $r(\mathcal{L}_n)$ by computing the character of our action on Bernardi trees. This requires us to write $G_n(\mathbf{x}; 1, 0, 1, 0)$ in terms of power sum symmetric functions. See [87] for definitions and background pertaining to power sum symmetric functions. We proceed by a generating function argument involving Lagrange inversion. It would be interesting to establish the same by a combinatorial argument.

Set $q = 1$ for the remainder of this subsection. By Theorem 4.0.1, we know that $G(\mathbf{x}; 1, 0, 1, 0)$ satisfies the functional equation

$$(1 + G(\mathbf{x}; 1, 0, 1, 0))^2 = H(2 + G(\mathbf{x}; 1, 0, 1, 0)). \tag{4.4.34}$$

In order to expand $G(\mathbf{x}; 1, 0, 1, 0)$ in terms of power sum symmetric functions, we need the following general result.

Theorem 4.4.8. *Let $Q(t)$ be a polynomial, let y be an indeterminate, and let F be the solution of*

$$F = H(Q(F))^y.$$

For every positive integer k , we have

$$F^k = \sum_{\lambda} \frac{p_{\lambda}}{z_{\lambda}} y^{\ell(\lambda)} \sum_{m=1}^{\infty} km^{\ell(\lambda)-1} [t^{m-k}] Q(t)^{|\lambda|}, \quad (4.4.35)$$

Proof. For $k > 0$, by Lagrange inversion (see, e.g., [34, equation (2.4.4)]) we have

$$F^k = \sum_{m=1}^{\infty} \frac{k}{m} [t^{m-k}] H(Q(t))^{my}. \quad (4.4.36)$$

We can write $H(Q(t))^{my}$ as

$$H(Q(t))^{my} = \exp\left(my \sum_{i=1}^{\infty} \frac{p_i}{i} Q(t)^i\right) \quad (4.4.37)$$

$$= \sum_{\lambda} \frac{p_{\lambda}}{z_{\lambda}} (my)^{\ell(\lambda)} Q(t)^{|\lambda|}, \quad (4.4.38)$$

where the sum in (4.4.38) is over all partitions λ with $|\lambda| \geq 0$. Substituting (4.4.38) into the right-hand side of (4.4.36) gives

$$F^k = \sum_{m=1}^{\infty} \frac{k}{m} [t^{m-k}] \sum_{\lambda} \frac{p_{\lambda}}{z_{\lambda}} (my)^{\ell(\lambda)} Q(t)^{|\lambda|} \quad (4.4.39)$$

$$= \sum_{\lambda} \frac{p_{\lambda}}{z_{\lambda}} y^{\ell(\lambda)} \sum_{m=1}^{\infty} km^{\ell(\lambda)-1} [t^{m-k}] Q(t)^{|\lambda|}, \quad (4.4.40)$$

and the theorem follows. □

Corollary 4.4.9. *Under the same assumptions as Theorem 4.4.8, if we expand F^k as $\sum_{\lambda} c_{\lambda} p_{\lambda}/z_{\lambda}$ then c_{λ} depends only on $\ell(\lambda)$, $|\lambda|$, y , k and Q .*

Using Theorem 4.4.8, we have the following result.

Theorem 4.4.10. *We have*

$$G_n(\mathbf{x}; 1, 0, 1, 0) = \sum_{\lambda \vdash n} c_\lambda p_\lambda / z_\lambda,$$

where

$$c_\lambda = \frac{1}{2^{\ell(\lambda)}} \sum_{m=1}^{n+1} m^{\ell(\lambda)-1} \binom{n}{m-1}. \quad (4.4.41)$$

Proof. Let $F := 1 + G(\mathbf{x}; 1, 0, 1, 0)$. From (4.4.34) we know that

$$F = (H(1 + F))^{1/2}. \quad (4.4.42)$$

Using $y = 1/2$, $k = 1$, and $Q(t) = 1 + t$ in Theorem 4.4.8, then (4.4.35) gives

$$F = \sum_{\lambda} \frac{p_\lambda}{2^{\ell(\lambda)} z_\lambda} \sum_{m=1}^{\infty} m^{\ell(\lambda)-1} [t^{m-1}] (1+t)^{|\lambda|}, \quad (4.4.43)$$

For each $\lambda \vdash n$ with $n \geq 1$, the coefficient c_λ of p_λ / z_λ in $G_n(\mathbf{x}; 1, 0, 1, 0)$ is equal to the coefficient of p_λ / z_λ in F . Therefore, we have

$$c_\lambda = \frac{1}{2^{\ell(\lambda)}} \sum_{m=1}^{n+1} m^{\ell(\lambda)-1} \binom{n}{m-1}, \quad (4.4.44)$$

which completes the proof. □

As a corollary of Theorem 4.4.10, we obtain the following generalization of Postnikov's formula for $r(\mathcal{L}_n)$ [67]. The reader is invited to compare it with the statement of Lemma 4.4.3.

Corollary 4.4.11. *Given a permutation $\sigma \in S_n$ with k cycles, the number of Bernardi trees fixed by σ equals*

$$\frac{1}{2^k} \sum_{m=1}^{n+1} m^{k-1} \binom{n}{m-1}.$$

Observe that Postnikov's formula is obtained by setting $k = n$ in Corollary 4.4.11.

Remark 4.4.12. Given the similarity of Lemma 4.4.3 and Corollary 4.4.11, the reader may wonder if $G_n(\mathbf{x}; 1, 0, 1, 0)$ expands as a positive integer linear combination of the Frobenius characteristics of Foulkes characters. This is not the case. Indeed, $G_3(\mathbf{x}; 1, 0, 1, 0)$ serves as a counterexample.

We conclude this subsection with a curious relation between $G_n(\mathbf{x}; 1, 0, 1, 0)$ and certain symmetric functions that arise as special cases of Jack symmetric functions when one sets the Jack parameter $y = 2$. Following Macdonald [56, Page 407, Equation 2.20], consider the symmetric function g_m defined by

$$g_m = \sum_{\lambda \vdash m} \frac{p_\lambda}{2^{\ell(\lambda)} z_\lambda}. \quad (4.4.45)$$

The key property of g_m for our purposes is expressed in the following relation

$$H(y)^{\frac{1}{2}} = \prod_{i \geq 1} (1 - x_i y)^{-\frac{1}{2}} = \sum_{m \geq 0} g_m y^m. \quad (4.4.46)$$

Instead of solving (4.4.34) using Lagrange inversion, one may alternatively proceed by ‘taking square roots’ on both sides and then rewrite the resulting functional equation in terms of the g_m . Thus, the functional equation (4.4.34) translates to

$$1 + G = \sum_{m \geq 0} g_m (2 + G)^m. \quad (4.4.47)$$

One can solve this functional equation in terms of Dyck paths, similar to the case of the h -expansion of the parking function representation. We keep our exposition brief.

For $\lambda = (\lambda_1, \dots, \lambda_k)$, set $g_\lambda := g_{\lambda_1} \cdots g_{\lambda_k}$. For $n \geq 1$, let \mathcal{D}_n be the set of Dyck paths, which are lattice paths that start at $(0, 0)$, end at (n, n) , take North and East steps, and stay weakly above the diagonal $y = x$. Given $D \in \mathcal{D}_n$, let $\lambda(D)$ be the partition of n obtained by sorting the lengths of the vertical runs of D in decreasing order. Recall that a vertical run in a Dyck path is any maximal contiguous sequence of North steps. Let $\text{peak}(D)$ denote the

number of peaks in D . Solving (4.4.47) for G_n yields the expansion

$$G_n = \sum_{D \in \mathcal{D}_n} 2^{n+1-\text{peak}(D)} g_{\lambda(D)}. \quad (4.4.48)$$

Stanley (see [84, Equation 10] and [84, Proposition 2.4]) describes the expansion of g_μ in terms of Jack symmetric functions J_λ at $y = 2$ (also called *zonal symmetric functions*). Note that Stanley uses scaled versions of our g_μ , which he denotes by \mathcal{J}_μ . Thus, we may expand G_n in terms of zonal symmetric functions. Since we already know by Theorem 4.4.7 that G_n is h -positive, we arrive indirectly upon a curious combination of zonal symmetric functions that is h -positive.

4.5 Local binary search trees and γ -nonnegativity

Theorem 4.0.4 implies that G_n can be written in terms of ribbon Schur functions with coefficients in $\mathbb{N}[\bar{\lambda} + \bar{\rho}, \bar{\lambda}\bar{\rho}, \lambda + \rho, \lambda\rho]$. Suppose that $G_n(\mathbf{x}; \bar{\lambda}, \lambda, \bar{\rho}, \rho) = \sum_{\alpha \models n} c_\alpha r_\alpha$ where the c_α belong to $\mathbb{N}[\bar{\lambda} + \bar{\rho}, \bar{\lambda}\bar{\rho}, \lambda + \rho, \lambda\rho]$. Applying the homomorphism ex yields

$$B_n(\bar{\lambda}, \lambda, \bar{\rho}, \rho) = \sum_{\alpha \models n} c_\alpha |\{\pi \in S_n : \text{Des}(\pi) = \text{set}(\alpha)\}| \frac{x^n}{n!}. \quad (4.5.1)$$

Note that the functional equation for B in (4.0.2) does not immediately imply an expansion of the form in (4.5.1). We use this expansion to turn our discussion to another notion of importance both in algebraic combinatorics and discrete geometry, that of γ -nonnegativity.

We say that a polynomial $P(t)$ of degree $n \geq 0$ is γ -nonnegative if it has an expansion of the form

$$P(t) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_{n,j} t^j (1+t)^{n-2j}, \quad (4.5.2)$$

where $\gamma_{n,j} \geq 0$. If such an expansion exists, then $P(t)$ is also palindromic and unimodal. We refer the reader to [66, Chapter 4] for a book exposition and [6] for a detailed exhaustive

survey on γ -nonnegativity. For another recent survey on the relevance and prevalence of γ -nonnegativity and real-rootedness of polynomials arising naturally in combinatorics, the reader is referred to [11]. Our focus here is the connection between intransitive trees of Postnikov [67] and regions of Linial arrangements.

Following Postnikov [67], an *intransitive tree* on n nodes is a tree whose nodes are labeled with distinct positive integers from $[n]$ such that the label of a node is either greater than labels of its neighbors, in which case we call it a *right vertex*, or is less than the labels of its neighbors, in which case we call it a *left vertex*. Note that the trees considered by Postnikov are neither plane nor rooted, and they do not have to be binary. We refer the reader to [67] for further details on the terminology. Let $f_n(t) := \sum_{k \geq 1} f_{nk} t^k$ where f_{nk} is the number of intransitive trees on $[n+1]$ with k right vertices. Consider the generating function

$$F(t, x) = \sum_{n \geq 0} f_n(t) \frac{x^n}{n!}. \quad (4.5.3)$$

By [67, Theorem 3], we have that $F := F(t, x)$ satisfies the functional equation

$$F(F + t - 1) = t e^{x(F+t)}. \quad (4.5.4)$$

We note that in Postnikov's statement of the above functional equation, the roles of x and t are switched. Consider the functional equation satisfied by $\tilde{B} := 1 + \bar{\rho} B(x; 1, 0, \bar{\rho}, 0)$. From (4.0.2), it can be seen that

$$\tilde{B}(\tilde{B} + \bar{\rho} - 1) = \bar{\rho} e^{x(\tilde{B} + \bar{\rho})}. \quad (4.5.5)$$

By comparing (4.5.4) and (4.5.5), we obtain the following proposition.

Proposition 4.5.1. *For $n \geq 1$, the number of intransitive trees on $[n+1]$ with k right vertices equals the number of standard LBS trees on $[n]$ with $k-1$ right edges.*

Setting $\bar{\lambda} = 1$, $\lambda = 0$, $\bar{\rho} = t$, and $\rho = 0$ in Theorem 4.0.4, we have

$$G_n(\mathbf{x}; 1, 0, t, 0) = \sum_{\text{LBS } T \in \mathcal{T}_n^\ell} t^{r(T)} \mathbf{x}^T = \sum_{\pi = B_1/\dots/B_k} t^{ja(\pi)} (t + 1)^{sa(\pi)} h_{(|B_1|, \dots, |B_k|)}, \tag{4.5.6}$$

where the last sum is over $\pi \in \text{mNC}(n)$ such that all nodes are unmarked. Hence, it is a sum over all interlacing noncrossing partitions on n nodes. Applying the homomorphism ex to the second and third expressions in (4.5.6), we obtain

$$\sum_{\substack{T \in \mathcal{T}_n^\ell \\ \text{standard LBS}}} t^{r(T)} = \sum_{\substack{\pi = B_1/\dots/B_k \\ \text{interlacing on } [n]}} t^{ja(\pi)} (1 + t)^{sa(\pi)} \binom{n}{|B_1|, \dots, |B_k|}, \tag{4.5.7}$$

where $\binom{n}{m_1, \dots, m_k} = \frac{n!}{m_1! \dots m_k!}$ for a composition $(m_1, \dots, m_k) \models n$. Thus, we have established that the distribution of right edges over standard LBS trees is γ -nonnegative.

In fact, we can obtain an explicit combinatorial description for the coefficients in the γ -nonnegative expansion. For $0 \leq j \leq \frac{n-1}{2}$, let $\gamma_{n,j}$ denote the number of left-leaning Bernardi trees on n nodes such that exactly j nodes have two children. Using Edelman’s bijection between $NC(n)$ and \mathcal{T}_n , one can show that (4.5.7) implies

$$\sum_{\substack{T \in \mathcal{T}_n^\ell \\ \text{standard LBS}}} t^{r(T)} = \sum_{0 \leq j \leq \frac{n-1}{2}} \gamma_{n,j} t^j (1 + t)^{n-1-2j}. \tag{4.5.8}$$

Combining this with Proposition 4.5.1, we have the following theorem.

Theorem 4.5.2. *For $n \geq 1$, the distribution of right edges over the set of standard LBS trees on n nodes is γ -nonnegative. Equivalently, the polynomials $f_n(t)$ in (4.5.3) considered by Postnikov are γ -nonnegative. As a corollary, we have that the sequence of coefficients of $f_n(t)$ is unimodal.*

In the spirit of the theme in [11], we offer the following stronger conjecture.

Conjecture 4.5.3. *The polynomials $f_n(t)$ are real-rooted with all roots negative for all $n \geq 1$. In particular, the coefficients of $f_n(t)$ form a log-concave sequence.*

Chapter 5

A BIJECTIVE APPROACH TO LABELED TREE ENUMERATION

In this chapter, after setting up the necessary notation, we present a bijective proof of Theorem 4.0.3 using a weight-preserving bijection involving path-tree sequences. Our inverse bijection is inspired by the Push-Glide algorithm of Préville-Ratelle and Viennot [69]. Corollary 5.1.16 gives a natural noncommutative analogue of Theorem 4.0.3. In Section 5.2, we use the weight-preserving bijection defined in Section 5.1 to prove the ribbon Schur expansions stated in Theorems 4.0.4 and 4.0.6.

5.1 A bijective proof of Theorem 4.0.3

We begin by restating the identity that we seek to establish for the convenience of the reader,

$$G = \sum_{n \geq 1} \sum_{\alpha \neq n} (\bar{\lambda} \bar{\rho} G + \bar{\lambda} + \bar{\rho})^{n-\ell(\alpha)} (\lambda \rho G + \lambda + \rho)^{\ell(\alpha)-1} r_{\alpha}. \quad (5.1.1)$$

Since our proof is intricate, we present a broad outline of this section.

1. We first interpret the right-hand side of (5.1.1) as the multivariate generating function of the set of alternating sequences of labeled trees and lattice paths. This is accomplished in Lemma 5.1.1.
2. We then discuss in detail the case of labeled trees that contribute to $G_{3,\nu}$ for all possible canopies $\nu \in \{UU, UD, DD, DU\}$. To each standard labeled tree on 3 nodes we associate an alternating sequences of labeled trees and lattice paths.
3. The insight gained from understanding the $n = 3$ case leads us to the crucial notions

of distinguished triples, life-sustaining nodes, and prunable nodes in Subsection 5.1.1.

4. The prunable nodes in a labeled tree determine a particular partition of the nodes of the tree, which in turn determines an alternating sequence of labeled trees and lattice paths. In Subsection 5.1.3, we show that this correspondence is in fact a weight-preserving bijection between the set of labeled binary trees and the set of alternating sequences of labeled trees and lattice paths.
5. Finally, in Section 5.2, we use this weight-preserving bijection to prove Theorem 4.0.5. The notions of (augmented and marked) interlacing partitions defined in Section 4.3 can be seen to arise naturally from our act of dismantling a tree into an alternating sequence of labeled trees and lattice paths.

A *lattice path* is a sequence $\nu = (\nu_1, \dots, \nu_k)$ of points in the plane starting at $\nu_1 = (0, 0)$ such that ν_{i+1} is either $\nu_i + (1, 1)$ or $\nu_i + (1, -1)$. We identify the lattice path ν with its corresponding path graph where points are nodes and for each $i < k$, we have an edge joining ν_i and ν_{i+1} . Let $\text{start}(\nu) := \nu_1$ and $\text{end}(\nu) := \nu_k$. If $\nu_{i+1} = \nu_i + (1, 1)$, we say that the edge between ν_i and ν_{i+1} is an *up step* in the path, and denote it by U . Otherwise, if $\nu_{i+1} = \nu_i + (1, -1)$, we say that the edge between the nodes is a *down step* in the path and denote it by D . A lattice path consisting of k nodes is said to be of *length* $k - 1$. Let w_ν be the length $k - 1$ word on the alphabet $\{U, D\}$ recording the up and down steps of ν from left to right.

Given a positive integer k , a *labeled lattice path* of *length* $k - 1$ is the data of a lattice path ν of length $k - 1$ and a labeling of the nodes of ν with positive integers. If ν_1, \dots, ν_k are the nodes of ν , we let ν_i^ℓ denote the label of ν_i . If ν is a labeled lattice path, its *inorder reading word* $\text{in}(\nu)$ is defined to be $\nu_1^\ell \nu_2^\ell \dots \nu_k^\ell$.

To the edge joining the labeled nodes ν_i and ν_{i+1} in ν , we associate an *edge weight* based on its orientation as indicated in Figure 5.1. Precisely, if $\nu_i \nu_{i+1}$ form a U step, then the edge is assigned the weight $\bar{\lambda}$ if $\nu_i^\ell \leq \nu_{i+1}^\ell$, or the weight λ if $\nu_i^\ell > \nu_{i+1}^\ell$. On the other hand, if

$\nu_i \nu_{i+1}$ form a D step, then the edge is assigned the weight $\bar{\rho}$ if $\nu_i^\ell \leq \nu_{i+1}^\ell$, or the weight ρ if $\nu_i^\ell > \nu_{i+1}^\ell$. Note that the barred parameters $\bar{\lambda}$ and $\bar{\rho}$ correspond to weak inequalities, while the unbarred parameters λ and ρ correspond to strict inequalities. We define the *weight* of

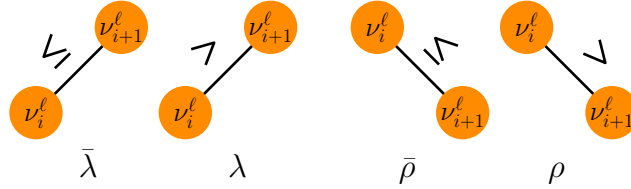


Figure 5.1: Determining the weight of a labeled edge.

the labeled lattice path ν to be the product of the weights on its edges, and denote it by $\text{wt}(\nu)$. Observe that the weights in Figure 5.1 are consistent with the weights assigned to the edges of a labeled tree based on their orientation. Suppose we draw a labeled (plane binary) tree with the root on top, left edges going Southwest to Northeast, and right edges going Northwest to Southeast. If we defined the weight of a labeled (plane binary) tree to be the product of the weights along its edges according to the rules in Figure 5.1, then this new weight in fact coincides with the one in Subsection 4.1.2.

We define a *path-tree sequence* to be an alternating sequence

$$S = (\nu^{(0)}, T_1, \nu^{(1)}, \dots, \nu^{(m-1)}, T_m, \nu^{(m)}) \tag{5.1.2}$$

of labeled lattice paths $\nu^{(h)}$ and labeled trees T_h , with $m \geq 0$, such that the sequence starts and ends with a lattice path. We define the *inorder reading word* of S , denoted by $\text{in}(S)$, to be the concatenation of the inorder reading words of the labeled lattice paths and labeled trees in the order in which they appear in S from left to right,

$$\text{in}(S) := \text{in}(\nu^{(0)})\text{in}(T_1)\text{in}(\nu^{(1)}) \dots \text{in}(\nu^{(m-1)})\text{in}(T_m)\text{in}(\nu^{(m)}), \tag{5.1.3}$$

where the ellipsis here denotes that the concatenation continues. Define a monomial \mathbf{x}^S

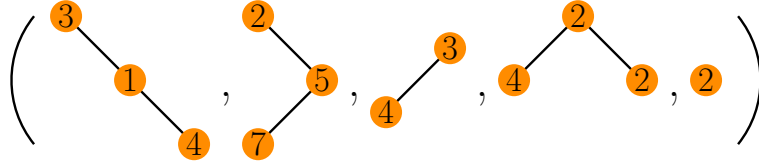


Figure 5.2: A path-tree sequence S .

associated to S by

$$\mathbf{x}^S := \mathbf{x}^{\text{in}(S)} = \mathbf{x}^{\text{in}(\nu^{(0)})} \prod_{h=1}^m \mathbf{x}^{T_h} \mathbf{x}^{\text{in}(\nu^{(h)})}. \tag{5.1.4}$$

Let the *weight* of S be

$$\text{wt}(S) := \text{wt}(\nu^{(0)}) \prod_{h=1}^m \gamma_h \text{wt}(T_h) \text{wt}(\nu^{(h)}), \tag{5.1.5}$$

where for each $1 \leq h \leq m$,

$$\gamma_h := \begin{cases} \bar{\lambda}\bar{\rho} & \text{if } \text{end}(\nu^{(h-1)})^\ell \leq \text{start}(\nu^{(h)})^\ell, \\ \lambda\rho & \text{if } \text{end}(\nu^{(h-1)})^\ell > \text{start}(\nu^{(h)})^\ell. \end{cases} \tag{5.1.6}$$

We define the *canopy* of S to be the word on $\{U, D\}$ given by the concatenation

$$\text{can}(S) = w_{\nu^{(0)}} D \text{can}(T_1) U w_{\nu^{(1)}} \cdots w_{\nu^{(m-1)}} D \text{can}(T_m) U w_{\nu^{(m)}}. \tag{5.1.7}$$

Given ν a word on $\{U, D\}$ of length $n - 1$, let $\mathcal{PT}_{n,\nu}^\ell$ be the set of all path-tree sequences with canopy ν . See Figure 5.2 for a path-tree sequence S with $\text{in}(S) = 314275434222$, $\mathbf{x}^S = x_1x_2^4x_3^2x_4^3x_5x_7$, $\text{wt}(S) = (\rho\bar{\rho})(\bar{\lambda}\bar{\rho})(\lambda\bar{\rho})(\lambda)(\lambda\rho)(\lambda\bar{\rho})(1) = \bar{\lambda}\lambda^4\bar{\rho}^4\rho^2$ and $\text{can}(S) = DDDDUUUDUDU$.

We denote the set of all path-tree sequences of any finite length by \mathcal{PT}^ℓ . Recall that by our convention, the ℓ in this notation refers to the fact that all path-tree sequences have

labeled nodes. The weighted generating function of \mathcal{PT}^ℓ is equal to the right-hand side of the equation in Theorem 4.0.3 by the following lemma.

Lemma 5.1.1. *We have that*

$$\sum_{S \in \mathcal{PT}^\ell} \text{wt}(S) \mathbf{x}^S = \sum_{n \geq 1} \sum_{\alpha \models n} (\bar{\lambda} \bar{\rho} G + \bar{\lambda} + \bar{\rho})^{n-\ell(\alpha)} (\lambda \rho G + \lambda + \rho)^{\ell(\alpha)-1} r_\alpha. \quad (5.1.8)$$

Proof. By the definition of r_α in (2.0.9), we can rewrite the right-hand side of (5.1.8) as

$$\sum_{n \geq 1} \sum_{\alpha \models n} (\bar{\lambda} \bar{\rho} G + \bar{\lambda} + \bar{\rho})^{n-\ell(\alpha)} (\lambda \rho G + \lambda + \rho)^{\ell(\alpha)-1} r_\alpha \quad (5.1.9)$$

$$= \sum_{n \geq 1} \sum_{w \in \mathbb{Z}_{>0}^n} (\bar{\lambda} \bar{\rho} G + \bar{\lambda} + \bar{\rho})^{\text{asc}(w)} (\lambda \rho G + \lambda + \rho)^{\text{des}(w)} \mathbf{x}^w. \quad (5.1.10)$$

For $w \in \mathbb{Z}_{>0}^n$, consider the summand

$$(\bar{\lambda} \bar{\rho} G + \bar{\lambda} + \bar{\rho})^{\text{asc}(w)} (\lambda \rho G + \lambda + \rho)^{\text{des}(w)} \mathbf{x}^w, \quad (5.1.11)$$

which can be rewritten as

$$\left[\prod_{i \in \text{Asc}(w)} (\bar{\lambda} \bar{\rho} G + \bar{\lambda} + \bar{\rho}) \prod_{j \in \text{Des}(w)} (\lambda \rho G + \lambda + \rho) \right] \mathbf{x}^w = \sum_{(A_1, \dots, A_{n-1})} A_1 \dots A_{n-1} \mathbf{x}^w, \quad (5.1.12)$$

where the sum in the right-hand side of (5.1.12) is over all tuples (A_1, \dots, A_{n-1}) such that

$$A_i \in \begin{cases} \{\bar{\lambda} \bar{\rho} \text{wt}(T) \mathbf{x}^T \mid T \in \mathcal{T}^\ell\} \cup \{\bar{\lambda}, \bar{\rho}\} & \text{if } i \in \text{Asc}(w), \\ \{\lambda \rho \text{wt}(T) \mathbf{x}^T \mid T \in \mathcal{T}^\ell\} \cup \{\lambda, \rho\} & \text{if } i \in \text{Des}(w). \end{cases} \quad (5.1.13)$$

For each $w \in \mathbb{Z}_{>0}^n$, define a function \mathfrak{f}_w from the collection of tuples (A_1, \dots, A_{n-1}) satisfying (5.1.13) to \mathcal{PT}^ℓ as follows. First, define a sequence on $\{U, D\} \cup \mathcal{T}^\ell$ by replacing each $\bar{\lambda} \bar{\rho} \text{wt}(T) \mathbf{x}^T$ and $\lambda \rho \text{wt}(T) \mathbf{x}^T$ with T in (A_1, \dots, A_{n-1}) and then replacing each remaining $\bar{\lambda}$ and λ with a U and each remaining $\bar{\rho}$ and ρ with a D . Then replace each maximum

consecutive subsequence of U s and D s with its corresponding lattice path. If two trees appear consecutively in the sequence, then we insert a lattice path consisting of a single node between the two trees in the sequence. If the sequence begins with a tree, we insert a lattice path consisting of a single node at the beginning of the sequence. Likewise, if the sequence ends with a tree, we insert a lattice path consisting of a single node at the end of the sequence. Observe that the total number of nodes in the disjoint union $\nu^{(0)} \sqcup \nu^{(1)} \sqcup \dots \sqcup \nu^{(m)}$ is n . Finally, label the nodes of the lattice paths we have just constructed from left to right with the letters in w to obtain a path-tree sequence

$$S = (\nu^{(0)}, T_1, \nu^{(1)}, \dots, \nu^{(m-1)}, T_m, \nu^{(m)}) \in \mathcal{PT}^\ell \tag{5.1.14}$$

with

$$\text{in}(\nu^{(0)})\text{in}(\nu^{(1)}) \dots \text{in}(\nu^{(m)}) = w. \tag{5.1.15}$$

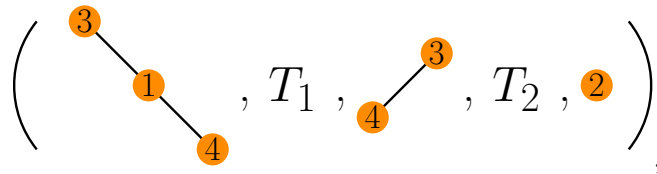
Observe that

$$\text{wt}(S) \mathbf{x}^S = A_1 \dots A_{n-1} \mathbf{x}^w. \tag{5.1.16}$$

For example, in the case when $w = 314432$, one possible tuple is

$$(A_1, A_2, A_3, A_4, A_5) = (\rho, \bar{\rho}, \bar{\lambda}\bar{\rho} \text{wt}(T_1) \mathbf{x}^{T_1}, \lambda, \lambda\rho \text{wt}(T_2) \mathbf{x}^{T_2}), \tag{5.1.17}$$

for some labeled trees T_1 and T_2 . Then $\mathbf{f}_w(A_1, \dots, A_5) =$



which is an element of \mathcal{PT}^ℓ .

The map f_w is a bijection onto the subset of \mathcal{PT}^ℓ of alternating sequences $S = (\nu^{(0)}, T_1, \nu^{(1)}, \dots, \nu^{(m-1)}, T_m, \nu^{(m)})$ which satisfy (5.1.15). Indeed, given a sequence $S = (\nu^{(0)}, T_1, \nu^{(1)}, \dots, \nu^{(m-1)}, T_m, \nu^{(m)}) \in \mathcal{PT}^\ell$, we can recover w by reading off the labels of the lattice paths in S from left to right. Furthermore, the tuple (A_1, \dots, A_{n-1}) can be recovered by recording the weights of the edges of the lattice paths together with $\gamma_h \text{wt}(T_h) \mathbf{x}^{T_h}$ in the order in which the edges and trees appear in S . Therefore,

$$\sum_{S \in \mathcal{PT}^\ell} \text{wt}(S) \mathbf{x}^S = \sum_{n \geq 1} \sum_{w \in \mathbb{Z}_{>0}^n} \sum_{(A_1, \dots, A_{n-1})} A_1 \dots A_{n-1} \mathbf{x}^w \quad (5.1.18)$$

$$= \sum_{n \geq 1} \sum_{\alpha = n} (\bar{\lambda} \bar{\rho} G + \bar{\lambda} + \bar{\rho})^{n-\ell(\alpha)} (\lambda \rho G + \lambda + \rho)^{\ell(\alpha)-1} r_\alpha, \quad (5.1.19)$$

where the inner sum in the right side of (5.1.18) is the right-hand side of (5.1.12). \square

By Lemma 5.1.1, in order to prove Theorem 4.0.3 it suffices to show that

$$G = \sum_{S \in \mathcal{PT}^\ell} \text{wt}(S) \mathbf{x}^S. \quad (5.1.20)$$

Before we proceed to prove this identity, we first verify that G_3 is equal to the degree 3 homogeneous component of the right-hand side of (5.1.20). This example will give us insight into how to prove the identity and will motivate many of the ideas in the rest of this section.

Example 5.1.2. Recall from (4.0.14) that for ν a word on $\{U, D\}$ of length $n - 1$, we have $G_{n, \nu}$ is the weighted generating function summing over trees with canopy ν . We have that

$$G_3 = G_{3, UU} + G_{3, DD} + G_{3, UD} + G_{3, DU}.$$

Recall from (5.1.7) the definition of the canopy of a path-tree sequence. We show that for

each canopy ν , we have that

$$G_{3,\nu} = \sum_{S \in \mathcal{PT}_{3,\nu}^\ell} \text{wt}(S) \mathbf{x}^S. \quad (5.1.21)$$

In the case $\nu = UU$, it is not hard to verify that

$$G_{3,UU} = \bar{\lambda}^2 r_{(3)} + \bar{\lambda} \lambda r_{(2,1)} + \lambda \bar{\lambda} r_{(1,2)} + \lambda^2 r_{(1,1,1)} \quad (5.1.22)$$

$$= \sum_{S \in \mathcal{PT}_{3,UU}^\ell} \text{wt}(S) \mathbf{x}^S. \quad (5.1.23)$$

Indeed, the path-tree sequences S with $\text{can}(S) = UU$ are those consisting of a single lattice path whose steps are given by UU . On the other hand, the only binary tree T with $\text{can}(T) = UU$ is a path graph starting from the root and going down and to the left. Therefore, we can define a bijection $\Phi_{3,UU} : \mathcal{T}_{3,UU}^\ell \rightarrow \mathcal{PT}_{3,UU}^\ell$ which simply converts the two left edges of T into U steps in a lattice path. Furthermore, the map $\Phi_{3,UU}$ preserves weight, meaning that $\text{wt}(T) = \text{wt}(\Phi_{3,UU}(T))$. Similar statements hold with UU replaced by UD and DD .

In the case $\nu = DU$, there is no obvious way to convert a labeled tree T with $\text{can}(T) = DU$ into a labeled path with steps DU while preserving weight. Recall from Subsection 4.1.2 that the weight of a labeled tree T is the same as the weight of its inorder standardization. Let us partition the set of labeled trees T with canopy DU according to their inorder standardization $\text{sort}(T)$. The 12 possibilities for $\text{sort}(T)$ are listed in Figure 5.3. One can verify that if we restrict G_3 to summing over labelings of the trees in the first row of Figure 5.3, the resulting multivariate formal power series is *quasisymmetric* but not symmetric. Therefore, if we wish to expand G_3 in terms of ribbon Schur functions, we must group terms corresponding to labeled trees in the first row of Figure 5.3 with terms corresponding to labeled trees in the second row of Figure 5.3.

It remains to show that (5.1.21) holds for $\nu = DU$. Observe that if S has canopy DU , then either $S = (\nu^{(0)})$ where $w_{\nu^{(0)}} = DU$, or $S = (\nu^{(0)}, T_1, \nu^{(1)})$ where $\nu^{(0)}$, T_1 and $\nu^{(1)}$

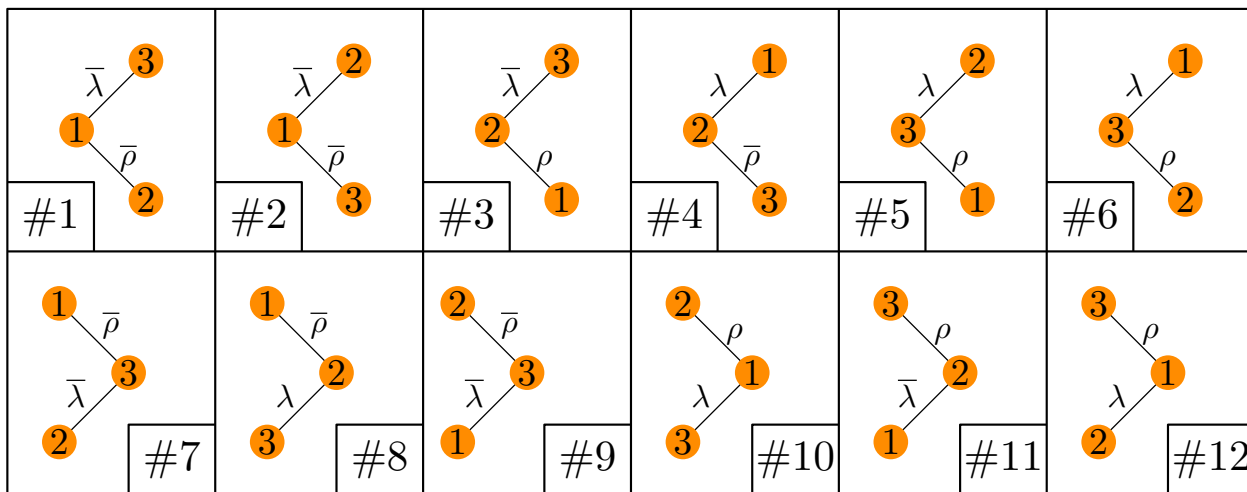


Figure 5.3: The 12 possibilities for $\text{sort}(T)$ for T with $\text{can}(T) = DU$.

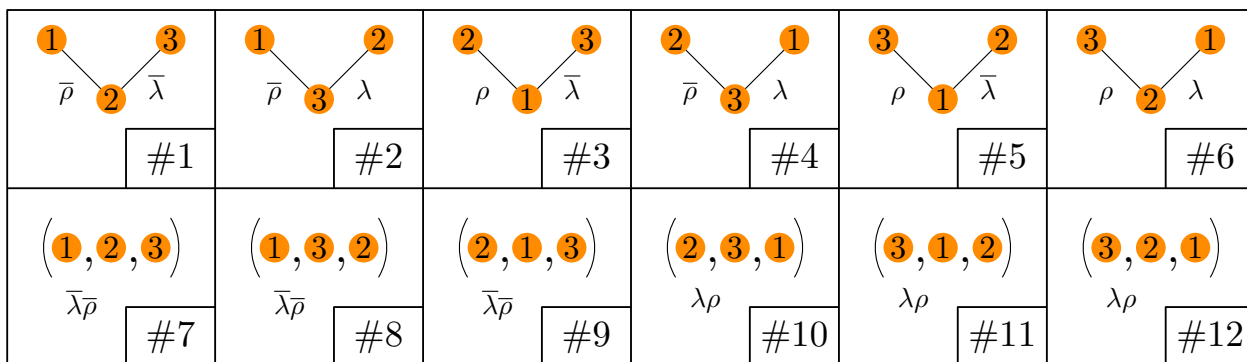


Figure 5.4: The 12 possible standardizations of the path-tree sequences with canopy DU .

each consist of a single node. Define the *inorder standardization* of a path-tree sequence S , denoted by $\text{sort}(S)$ to be the unique path-tree sequence whose underlying unlabeled paths and trees are the same as those of S , and whose inorder reading word is $\text{sort}(\text{in}(S))$. Figure 5.4 lists the possible standardizations of the path-tree sequences with canopy DU . Our goal is to find a bijection $\Phi_{3,DU}$ between the 12 labeled trees T whose inorder standardization is listed in Figure 5.3 and the 12 path-tree sequences S whose inorder standardization is listed in Figure 5.4. We would also like to construct the map so that $\Phi_{3,DU}$ preserves weight and inorder reading word. See Figure 5.5 for a summary of the map $\Phi_{3,DU}$ we defined on standard

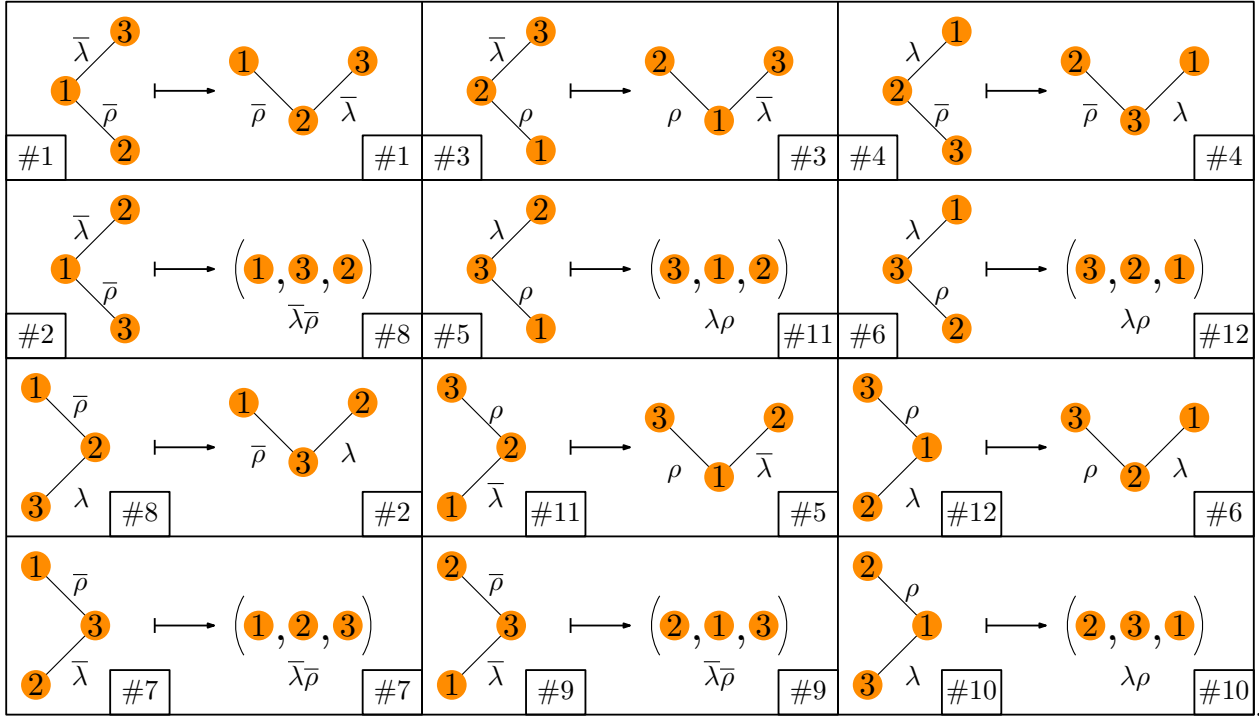


Figure 5.5: A summary of the map $\Phi_{3,DU}$ defined on standard trees.

trees with canopy DU . The reader is encouraged to verify that this map preserves weight and inorder reading word using Figure 5.5.

Observe that for some trees the image is uniquely determined by the weight and inorder reading word requirements, while for other trees, there was some choice for the image. For example, looking at tree #2 in Figure 5.3, we see that its inorder reading word is 132 and its weight is $\bar{\lambda}\bar{\rho}$. In order to preserve both its inorder reading word and weight, $\Phi_{3,DU}$ must map it to path-tree sequence #8 in Figure 5.4. On the other hand, both trees #1 and #7 have inorder reading word 123 and weight $\bar{\lambda}\bar{\rho}$. Similarly, both path-tree sequences #1 and #7 have inorder reading word 123 and weight $\bar{\lambda}\bar{\rho}$. We make the arbitrary choice to map tree #1 to path-tree sequence #1 and tree #7 to path-tree sequence #7.

We extend the definition of $\Phi_{3,DU}$ to a map $\Phi_{3,DU} : \mathcal{T}_{3,DU}^\ell \rightarrow \mathcal{PT}_{3,DU}^\ell$ as follows. Given $T \in \mathcal{T}_{3,DU}^\ell$, suppose that $\text{sort}(T)$ maps to the path-tree sequence S in Figure 5.5. Then define

$\Phi_{3,DU}(T)$ to be the unique path-tree sequence whose underlying unlabeled paths and trees are the same as those in S , such that the inorder reading word is $\text{in}(T)$. Therefore, $\Phi_{3,DU}$ is a bijection which preserves weight, inorder reading word, and canopy. Then we have

$$G_{3,DU} = \sum_{T \in \mathcal{T}_{3,DU}^\ell} \text{wt}(T) \mathbf{x}^T \quad (5.1.24)$$

$$= \sum_{T \in \mathcal{T}_{3,DU}^\ell} \text{wt}(\Phi_{3,DU}(T)) \mathbf{x}^{\Phi_{3,DU}(T)} \quad (5.1.25)$$

$$= \sum_{S \in \mathcal{PT}_{3,DU}^\ell} \text{wt}(S) \mathbf{x}^S, \quad (5.1.26)$$

so (5.1.21) holds for $\nu = DU$.

We make one more observation from this example. Let T be a tree whose inorder standardization is a tree listed in either the second or fourth row of Figure 5.5. Let v_2 be the second node of T in inorder, which is the lowest node drawn in each of the trees in these rows. We can think of the map $\Phi_{3,DU}$ as “pruning” off v_2 from T and mapping T to (v_1, v_2, v_3) , where we think of v_2 as a single node tree sandwiched between v_1 and v_3 .

The outline of the proof of Theorem 4.0.3 is organized as follows. In Subsection 5.1.1, we define the notion of a “prunable node” in a tree and show that we can identify the weights of a corresponding set of edges in the tree. In Subsection 5.1.3, we use the antichain of maximal prunable nodes in a labeled tree to partition the node set of a labeled tree. We then use this partition to define a map $\Phi_{n,\nu} : \mathcal{T}_{n,\nu}^\ell \rightarrow \mathcal{PT}_{n,\nu}^\ell$ which generalizes the map $\Phi_{3,DU}$ in Example 5.1.2. Finally, we use $\Phi_{n,\nu}$ to prove the identity

$$G = \sum_{S \in \mathcal{PT}^\ell} \text{wt}(S) \mathbf{x}^S$$

from (5.1.20), completing the proof of Theorem 4.0.3.

5.1.1 Distinguished triples and prunable nodes

Throughout this section, let us fix a labeled tree $T \in \mathcal{T}_n^\ell$. Let the nodes of T be v_1, v_2, \dots, v_n listed in inorder, as in Figure 4.1. After setting up some notation, we identify a special subset of nodes of T as “prunable”. This set of nodes will allow us to give a simple description of the weights of a corresponding set of edges in T .

Let the *roof* of T be the set of nodes of T which can be reached from the root by either traversing only left edges or traversing only right edges. Let the set of nodes which are not in the roof be the *attic*.

Definition 5.1.3. Let $v_j \in \text{Nodes}(T)$ be a node in the attic of T .

- If v_j is the left child of v_k for some $k > j$, let i be the greatest index smaller than j such that v_i is an ancestor of v_j . Define the *distinguished triple* corresponding to v_j to be (v_i, v_j, v_k) , and define the *distinguished ancestor* of v_j to be $v_i = \text{da}(v_j)$.
- If v_j is the right child of v_i for some $i < j$, let k be the smallest index greater than j such that v_k is an ancestor of v_j . Define the *distinguished triple* of v_j to be (v_i, v_j, v_k) , and define the *distinguished ancestor* of v_j to be $v_k = \text{da}(v_j)$.

Observe that if $v_j \in \text{Nodes}(T)$ is in the attic of T , then its distinguished ancestor always exists. Indeed, assuming v_j is the left child of v_k for some $k > j$, if there exists no $i < j$ such that v_i is an ancestor of v_j , then v_j must be in the roof of T , a contradiction, hence the distinguished ancestor of v_j exists. Similar reasoning holds when v_j is the right child of v_i for some $i < j$. See Figure 5.6 for a schematic diagram of the two cases in Definition 5.1.3, where the blue triangles indicate arbitrary subtrees of T .

We call the unique path from v_j to $\text{da}(v_j)$, including both v_j and $\text{da}(v_j)$, the *distinguished path* of the node v_j . Figure 5.7 shows a tree T with its roof and attic highlighted, with a list of the distinguished triples for each v_j in the attic of T .

Observe that in each case of Definition 5.1.3, we have $i < j < k$. Furthermore, v_i has a

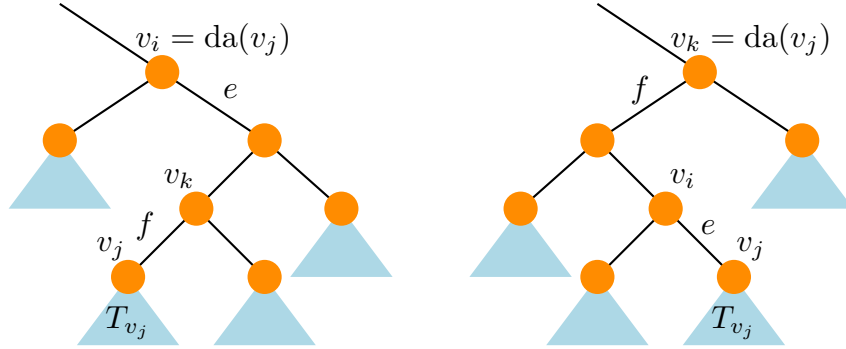


Figure 5.6: A schematic diagram for distinguished triples (v_i, v_j, v_k) .

right edge e , and v_k has a left edge f . Further observe that

$$\text{Nodes}(T_{v_j}) = \{v_h : i + 1 \leq h \leq k - 1\}, \quad (5.1.27)$$

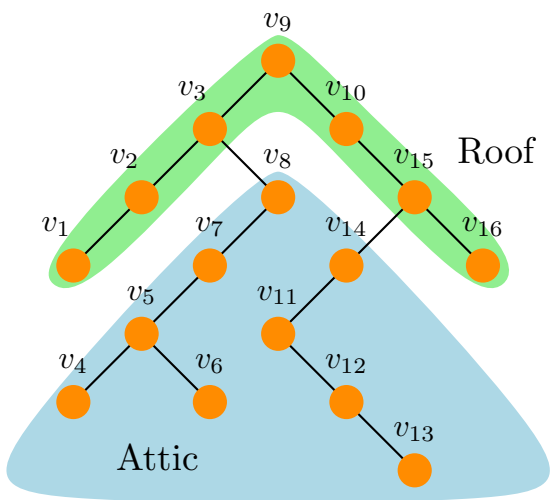
where T_{v_j} is the subtree of T consisting of v_j and all of its descendants. Indeed, it can be seen from Figure 5.6 that i is maximum such that $i < j$ and v_i is not in T_{v_j} , and k is minimum such that $k > j$ and v_k is not in T_{v_j} .

Next we prove a lemma relating the canopy of T to distinguished ancestors. Recall from Definition 4.1.1 that $\text{can}(T)$ is the word on $\{U, D\}$ whose j th letter is a D if and only if v_j has a right child.

Lemma 5.1.4. *Consider a node v_j in T for $j \leq n - 1$. Then the following hold.*

- (a) *Suppose the j th letter of $\text{can}(T)$ is a U . Then either v_j is the left child of v_{j+1} , or v_j is in the attic and $v_{j+1} = \text{da}(v_j)$.*
- (b) *Suppose the j th letter of $\text{can}(T)$ is a D . Then either v_{j+1} is the right child of v_j , or v_{j+1} is in the attic and $v_j = \text{da}(v_{j+1})$.*

Proof. If the j th letter of $\text{can}(T)$ is a U , then v_j does not have a right child. Therefore, v_{j+1} must be an ancestor of v_j . In this case, v_j cannot be the root, so v_j must have a parent. If v_j



v_i	v_j	v_k
v_3^*	v_4	v_5
v_3^*	v_5	v_7
v_5	v_6	v_7^*
v_3^*	v_7	v_8
v_3	v_8	v_9^*
v_{10}^*	v_{11}	v_{14}
v_{11}	v_{12}	v_{14}^*
v_{12}	v_{13}	v_{14}^*
v_{10}^*	v_{14}	v_{15}

Figure 5.7: On the left, a tree $T \in \mathcal{T}_{16}$ with its roof and attic highlighted. On the right, a table listing the distinguished triples (v_i, v_j, v_k) , where in each row the distinguished ancestor of the node v_j is starred.

is the left child of a node, then v_{j+1} must be its parent by the definition of inorder and the fact that v_j has no right child. In this case, the condition in (a) is satisfied, so we can assume that v_j is the right child of some node. By the definition of inorder and the fact that v_{j+1} is its ancestor, v_j must be in the attic. By Definition 5.1.3, $da(v_j) = v_k$ for the minimal $k > j$ such that v_k is an ancestor of v_j . Since v_{j+1} is an ancestor of v_j , then we have $j + 1 = k$. Hence, $v_{j+1} = da(v_j)$. This concludes the proof of part (a). The proof of part (b) follows similarly. □

The reader may verify the claim in Lemma 5.1.4 in the case of the tree in Figure 5.7 whose canopy equals $UUDUDUUUDDDDUUD$.

Definition 5.1.5. Given v_j in the attic of T , let (v_i, v_j, v_k) be its distinguished triple. We call the node v_j *life-sustaining* if and only if one of the following criteria holds:

(S1) The node v_j is a left child, and

$$\text{sort}(v_i^\ell v_j^\ell v_k^\ell) \in \{132, 312, 321\}.$$

(S2) The node v_j is a right child, and

$$\text{sort}(v_i^\ell v_j^\ell v_k^\ell) \in \{123, 213, 231\}.$$

If v_j is in the attic of T and it is not life-sustaining, we call it *prunable*.

See Figure 5.8 for a summary of the six cases where v_j is life-sustaining and Figure 5.9 for a summary of the six cases where v_j is prunable. Observe that if T is a labeled tree on 3 nodes with $\text{can}(T) = DU$, then v_2 is life-sustaining if and only if the standardization of T appears in either rows 1 or 3 of Figure 5.5. These are exactly the cases where $\Phi_{3,DU}(T)$ consists of a single lattice path in Example 5.1.2. On the other hand, v_2 is prunable if and only if the standardization of T appears in either rows 2 or 4 of Figure 5.5. These are exactly the cases where $\Phi_{3,DU}(T)$ is a path-tree sequence of length 3 in Example 5.1.2.

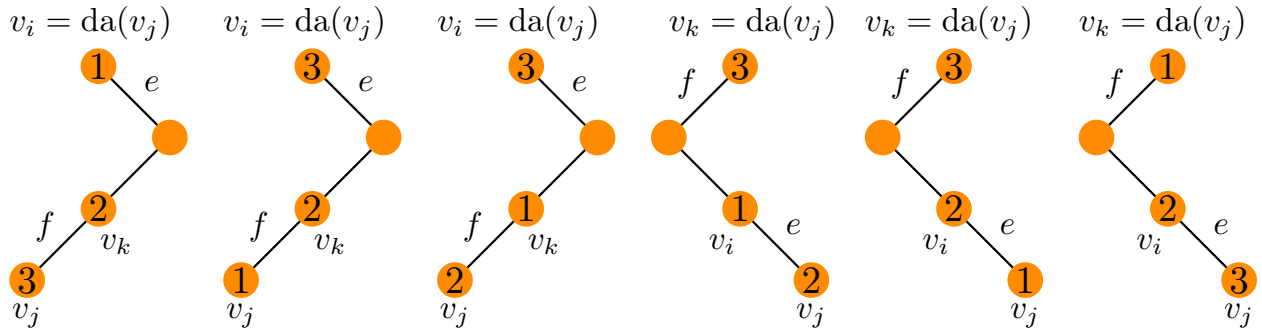
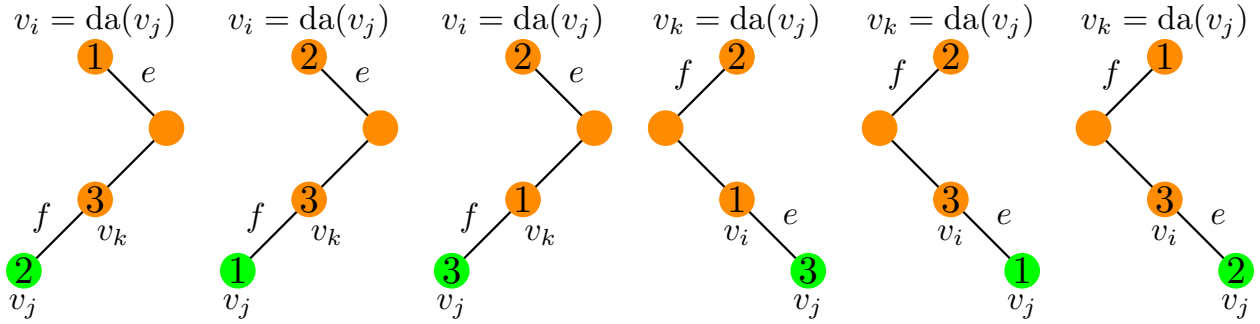


Figure 5.8: The six cases where v_j is life-sustaining.

Next we prove a lemma which identifies certain edge weights in the tree associated to life-sustaining nodes. We then use it to prove Lemma 5.1.7, which identifies products of edge weight of the form $\bar{\lambda}\bar{\rho}$ and $\lambda\rho$. Lemma 5.1.7 will help explain the appearance of the γ_h weights in the definition of the weight of a path-tree sequence (5.1.5).

Lemma 5.1.6. *Let v_j be a node in the attic of T such that all of the nodes in the distinguished path of v_j are life-sustaining nodes of T . Let (v_i, v_j, v_k) be the distinguished triple of v_j , let e be the right edge of v_i , and let f be the left edge of v_k . Then the following hold.*

Figure 5.9: The six cases where v_j is prunable.

(a) If v_j is the left child of v_k , then

$$\text{wt}(e) = \begin{cases} \bar{\rho} & \text{if } v_i^\ell \leq v_j^\ell, \\ \rho & \text{if } v_i^\ell > v_j^\ell. \end{cases}$$

(b) If v_j is the right child of v_i , then

$$\text{wt}(f) = \begin{cases} \bar{\lambda} & \text{if } v_j^\ell \leq v_k^\ell, \\ \lambda & \text{if } v_j^\ell > v_k^\ell. \end{cases}$$

Proof. (a) The proof is by case analysis. First, suppose that the distinguished path of v_j consists of only nodes v_i , v_j and v_k . Looking at Figure 5.8, we see that if $\text{sort}(v_i^\ell v_j^\ell v_k^\ell) = 132$, then v_i^ℓ is weakly smaller than both v_j^ℓ and v_k^ℓ . Therefore, we are in the case where $v_i^\ell \leq v_j^\ell$, and $\text{wt}(e) = \text{wt}(v_i v_k) = \bar{\rho}$. On the other hand, if $\text{sort}(v_i^\ell v_j^\ell v_k^\ell) \in \{312, 321\}$, then v_i^ℓ is strictly greater than both v_j^ℓ and v_k^ℓ . Therefore, we are in the case where $v_i^\ell > v_j^\ell$, and $\text{wt}(e) = \text{wt}(v_i v_k) = \rho$.

Now, suppose that the distinguished path of v_j consists of more than three nodes. Since all of the nodes along the distinguished path are life-sustaining, there are only two cases. Either v_i^ℓ is weakly smaller than all of the other labels along the distinguished path, or

v_i^ℓ is strictly greater than all of the other labels along the distinguished path. In the first case, $v_i^\ell \leq v_j^\ell$ and $\text{wt}(e) = \bar{\rho}$. In the second case, $v_i^\ell > v_j^\ell$ and $\text{wt}(e) = \rho$.

- (b) A similar case analysis shows that either v_k^ℓ is weakly greater than all of the other labels along the distinguished path of v_j , or v_k^ℓ is strictly smaller than all of the other labels along the distinguished path. In the first case, $v_j^\ell \leq v_k^\ell$ and $\text{wt}(f) = \bar{\lambda}$. In the second case, $v_j^\ell > v_k^\ell$ and $\text{wt}(f) = \lambda$. \square

Lemma 5.1.7. *Let v_j be a node in the attic of T . Suppose that v_j is prunable and that all other nodes in the distinguished path of v_j are life-sustaining. Let (v_i, v_j, v_k) be the distinguished triple of v_j , let e be the right edge of v_i , and let f be the left edge of v_k . Then*

$$\text{wt}(e)\text{wt}(f) = \begin{cases} \bar{\lambda}\bar{\rho} & \text{if } v_i^\ell \leq v_k^\ell, \\ \lambda\rho & \text{if } v_i^\ell > v_k^\ell. \end{cases} \quad (5.1.28)$$

Proof. It suffices to show that

$$\text{wt}(e) = \begin{cases} \bar{\rho} & \text{if } v_i^\ell \leq v_k^\ell, \\ \rho & \text{if } v_i^\ell > v_k^\ell, \end{cases} \quad (5.1.29)$$

$$\text{wt}(f) = \begin{cases} \bar{\lambda} & \text{if } v_i^\ell \leq v_k^\ell, \\ \lambda & \text{if } v_i^\ell > v_k^\ell. \end{cases} \quad (5.1.30)$$

First, suppose v_j is the left child of v_k , and let $v_i = \text{da}(v_j)$. If the distinguished path of v_j consists of only the nodes v_i , v_j and v_k , then v_k is the right child of v_i . Therefore, $e = v_i v_k$, so $\text{wt}(e) = \text{wt}(v_i v_k)$, and (5.1.29) holds. Otherwise, v_k is in the attic of T , and $v_i = \text{da}(v_k)$. By hypothesis, we know that every node on the distinguished path of v_k is life-sustaining. Therefore, applying Lemma 5.1.6(a) to the distinguished path of v_k , we have that (5.1.29) continues to hold.

Since v_j is the left child of v_k , then (5.1.30) follows from a case analysis. Indeed, looking

at Figure 5.9, then $v_i^\ell \leq v_k^\ell$ exactly when $\text{sort}(v_i^\ell v_j^\ell v_k^\ell) \in \{213, 123\}$. In both of these cases we have that $v_j^\ell \leq v_k^\ell$, which means exactly that $\text{wt}(f) = \text{wt}(v_j v_k) = \bar{\lambda}$, so (5.1.30) holds. On the other hand, $v_i^\ell > v_k^\ell$ exactly when $\text{sort}(v_i^\ell v_j^\ell v_k^\ell) = 231$. In this case, we have $v_j^\ell > v_k^\ell$, so (5.1.30) continues to hold.

This concludes our proof of the lemma in the case when v_j is the left child of v_k . The case when v_j is the right child of v_i follows by a similar application of Lemma 5.1.6(b) and a case analysis. \square

Lemma 5.1.8. *Consider a node v_j in T for $j \leq n - 1$. Suppose v_j and all its ancestors of v_j are life-sustaining. Then the following hold.*

- (a) *If the j th letter of $\text{can}(T)$ is a U , then v_{j+1} has a left edge which has the same weight and orientation as a U step in a lattice path labeled $v_j^\ell v_{j+1}^\ell$.*
- (b) *If the j th letter of $\text{can}(T)$ is a D , then v_j has a right edge which has the same weight and orientation as a D step in a lattice path labeled $v_j^\ell v_{j+1}^\ell$.*

Proof. If the j th letter of $\text{can}(T)$ is a U , then by Lemma 5.1.4(a), either v_j is the left child of v_{j+1} , or v_j is in the attic and $v_{j+1} = \text{da}(v_j)$. In the first case, simply observe that the left edge $v_j v_{j+1}$ has the same weight and orientation as a U step labeled $v_j^\ell v_{j+1}^\ell$. In the second case, apply Lemma 5.1.6(b) where $k = j + 1$ and f is the left edge of v_{j+1} . Then we have

$$\text{wt}(f) = \begin{cases} \bar{\lambda} & \text{if } v_j^\ell \leq v_{j+1}^\ell, \\ \lambda & \text{if } v_j^\ell > v_{j+1}^\ell, \end{cases} \quad (5.1.31)$$

which is exactly the weight of a U step labeled with $v_j^\ell v_{j+1}^\ell$. This concludes the proof of part (a). Part (b) follows from a similar application of Lemma 5.1.4(b) and Lemma 5.1.6(a) using the edge labeled e . \square

5.1.2 Partition of a tree

Throughout this subsection, let us continue to fix a labeled tree $T \in \mathcal{T}_n^\ell$. Let the nodes of T be v_1, v_2, \dots, v_n listed in inorder. Let us define some notation which we use throughout the rest of Section 5.1.

Definition 5.1.9. Let $T \in \mathcal{T}_n$, and let $\mathcal{A} = \{v_{j_1}, v_{j_2}, \dots, v_{j_m}\}$ be an antichain under the partial ordering \leq_T of nodes in the attic such that $j_1 < j_2 < \dots < j_m$. For each $1 \leq h \leq m$, define the following notation.

- Let T_h be the subtree of T consisting of v_{j_h} and all of its descendants.
- Let $(v_{i_h}, v_{j_h}, v_{k_h})$ be the distinguished triple of the node v_{j_h} .
- Let e_h be the right edge of v_{i_h} , and let f_h be the left edge of v_{k_h} .
- Let $k_0 := 1$ and $i_{m+1} := n$.
- Let $w^{(h)}$ be the word on $\{U, D\}$ given by traversing the nodes in the interval $[v_{k_h}, v_{i_{h+1}})$ of T in inorder and recording a D if the node has a right child, and a U otherwise.

Observe that $i_h < j_h < k_h \leq i_{h+1}$ for all h . Indeed, let v_m be the lowest common ancestor of v_{j_h} and $v_{j_{h+1}}$. Then we have $j_h < m < j_{h+1}$ by the definition of inorder. Hence, by the definition of k_h and i_{h+1} , we have $k_h \leq m \leq i_{h+1}$. Therefore, we have partitioned the set $[n]$ into a disjoint union of intervals,

$$[1, i_1] \sqcup [i_1 + 1, k_1 - 1] \sqcup [k_1, i_2] \sqcup \dots \sqcup [k_{m-1}, i_m] \sqcup [i_m + 1, k_m - 1] \sqcup [k_m, n]. \quad (5.1.32)$$

Further observe that $[i_h + 1, k_h - 1] = \{p : v_p \in \text{Nodes}(T_h)\}$ by (5.1.27).

Lemma 5.1.10. Let $T \in \mathcal{T}_n$, and let $\mathcal{A} = \{v_{j_1}, v_{j_2}, \dots, v_{j_m}\}$ be an antichain in the attic such that $j_1 < j_2 < \dots < j_m$. Using the notation in Definition 5.1.9, the canopy of T is equal to

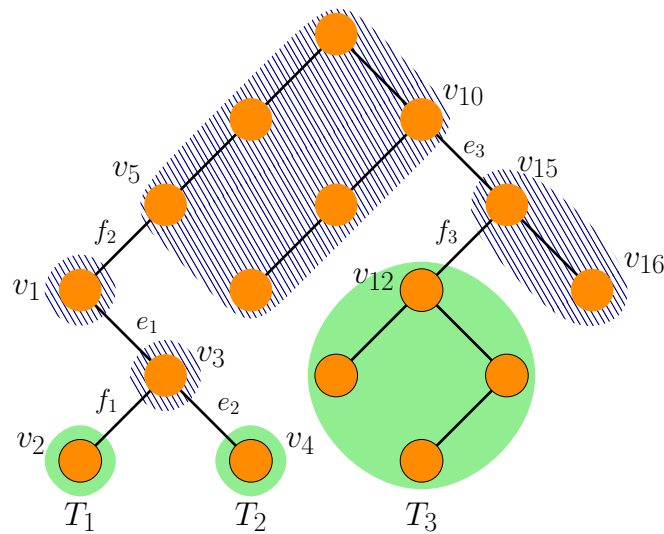


Figure 5.10: A tree $T \in \mathcal{T}_{16}$ with canopy $DUDUUUDUUDUUDUUD$, partitioned using the antichain $\{v_2, v_4, v_{12}\}$.

the following concatenation

$$\text{can}(T) = w^{(0)} (D \text{ can}(T_1) U) w^{(1)} \dots w^{(m-1)} (D \text{ can}(T_m) U) w^{(m)}. \quad (5.1.33)$$

Proof. Recalling the definition of the canopy of a tree, Definition 4.1.1, we label each node in $\text{Nodes}(T) \setminus \{v_n\}$ with a D if it has a right child, or with a U if it does not have a right child. Recall the partitioning (5.1.32) of the set $[n]$, which corresponds to a partition of the nodes of T , where the interval $[i_h + 1, k_h - 1]$ corresponds to the set of nodes of T_h . By Definition 5.1.9, we have that $w^{(h)}$ is the word read from the labels of the nodes in $[v_{k_h}, v_{i_{h+1}})$ in inorder. For $h < m$, the node v_{i_h} has a right edge, so it is labeled with a D . Let t_h be the final node of T_h in inorder. Then the word read from the labels of the nodes in $\text{Nodes}(T_h) \setminus \{t_h\}$ in inorder is exactly $\text{can}(T_h)$. Since t_h is the last node of T_h in inorder, it does not have a right child, so it is labeled with a U . By definition, $\text{can}(T)$ is the word read from the labels of the nodes in $\text{Nodes}(T) \setminus \{v_n\}$ in inorder, which is exactly the right-hand side of (5.1.33). \square

Figure 5.10 shows a binary tree T , together with a choice of antichain $\{v_{j_1}, v_{j_2}, v_{j_3}\}$. Each

blue striped region corresponds to an interval of nodes $[v_{k_h}, v_{i_{h+1}}]$, and each solid green region corresponds to a subtree T_h . In this case, we have

$$\begin{aligned} \text{can}(T) &= ()(DU)()(DU)(UUDUU)(DUDUU)(D) \\ &= DUDUUUDUUDUUDUUD, \end{aligned}$$

where $\text{can}(T_1) = \text{can}(T_2)$ are the empty word and $\text{can}(T_3) = UDU$. The reader can check that Lemma 5.1.10(b) holds in this case.

Remark 5.1.11. Note that even though the subgraph formed by each interval $[v_{k_h}, v_{i_{h+1}}]$ is connected in Figure 5.10, these subgraphs may not be connected in general.

5.1.3 Construction of the map $\Phi_{n,\nu}$

In this subsection, we define a map $\Phi_{n,\nu} : \mathcal{T}_{n,\nu}^\ell \rightarrow \mathcal{PT}_{n,\nu}^\ell$ which extends the map $\Phi_{3,\nu}$ defined in Example 5.1.2. We then prove that it is a weight-preserving bijection and use it to prove Theorem 4.0.3.

Definition 5.1.12. We define the map $\Phi_{n,\nu} : \mathcal{T}_{n,\nu}^\ell \rightarrow \mathcal{PT}_{n,\nu}^\ell$ as follows. Given $T \in \mathcal{T}_{n,\nu}^\ell$, let $\mathcal{A} = \{v_{j_1}, v_{j_2}, \dots, v_{j_m}\}$, ordered so that $j_1 < j_2 < \dots < j_m$, be the subset of prunable nodes of T which are maximal among prunable nodes under the partial order \leq_T . By definition, \mathcal{A} is an antichain, and all of its elements are in the attic of T . Recall the notation defined in Definition 5.1.9.

Let $\nu^{(h)}$ be the unique labeled lattice path whose steps are $w_{\nu^{(h)}} = w^{(h)}$ and whose in-order reading word is $\text{in}(\nu^{(h)}) = v_{k_h}^\ell v_{k_h+1}^\ell \cdots v_{i_{h+1}}^\ell$. Define

$$\Phi_{n,\nu}(T) := (\nu^{(0)}, T_1, \nu^{(1)}, \dots, \nu^{(m-1)}, T_m, \nu^{(m)}),$$

which lies in $\mathcal{PT}_{n,\nu}^\ell$ by Lemma 5.1.10.

Figure 5.11 shows a tree T with its prunable nodes highlighted in green. In this labeled tree, we have $\mathcal{A} = \{v_2, v_4, v_{12}\}$. Figure 5.12 shows the image of the tree T under the map

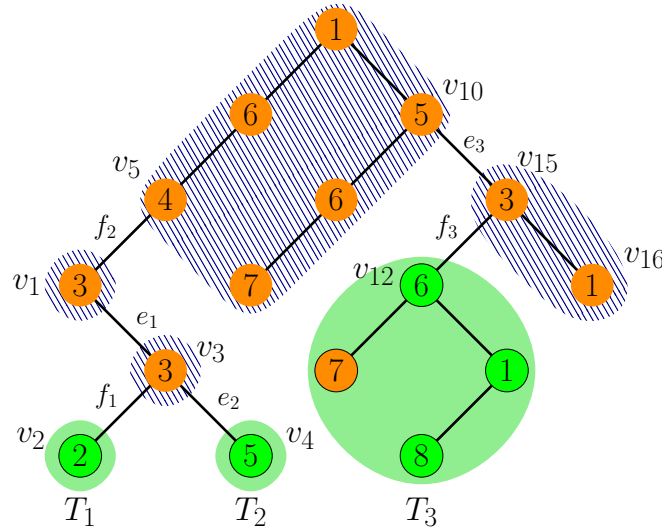


Figure 5.11: A tree $T \in \mathcal{T}_{16}^\ell$ whose prunable nodes are highlighted in green with $\text{can}(T) = DUDUUUDUUDUDUUD$.

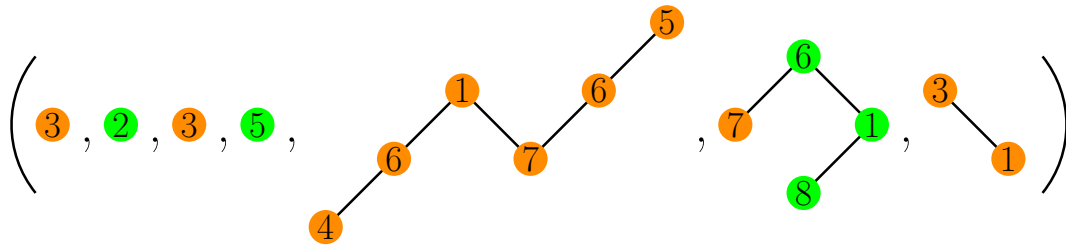


Figure 5.12: The image under $\Phi_{n,\nu}$ of the tree in Figure 5.11.

$\Phi_{n,\nu}$. The reader can check that for T in Figure 5.11 and $\Phi_{n,\nu}(T)$ in Figure 5.12,

$$\begin{aligned} \text{wt}(T) &= \bar{\lambda}^3 \lambda^6 \bar{\rho}^3 \rho^3 = \bar{\lambda} \lambda^3 \bar{\rho} \rho (\bar{\lambda} \bar{\rho} \text{wt}(T_1)) (\bar{\lambda} \bar{\rho} \text{wt}(T_2)) (\lambda \rho \text{wt}(T_3)) = \text{wt}(\Phi_{n,\nu}(T)), \\ \text{in}(T) &= 3235461765768131 = \text{in}(\Phi_{n,\nu}(T)). \end{aligned}$$

Lemma 5.1.13. *For all $T \in \mathcal{T}_{n,\nu}^\ell$, we have that $\text{wt}(T) = \text{wt}(\Phi_{n,\nu}(T))$ and $\text{in}(T) = \text{in}(\Phi_{n,\nu}(T))$.*

Proof. Definition 5.1.12 states that $\text{in}(\nu^{(h)}) = v_{k_h}^\ell v_{k_h+1}^\ell \cdots v_{i_{h+1}}^\ell$. Therefore, the fact that

$\text{in}(T) = \text{in}(\Phi_{n,\nu}(T))$ follows from the partition of $[n]$ in (5.1.32).

Recall from (5.1.5) that

$$\text{wt}(\Phi_{n,\nu}(T)) = \text{wt}(\nu^{(0)}) \prod_{h=1}^m \gamma_h \text{wt}(T_h) \text{wt}(\nu^{(h)}), \tag{5.1.34}$$

where

$$\gamma_h = \begin{cases} \bar{\lambda}\bar{\rho} & \text{if } \text{end}(\nu^{(h-1)})^\ell \leq \text{start}(\nu^{(h)})^\ell, \\ \lambda\rho & \text{if } \text{end}(\nu^{(h-1)})^\ell > \text{start}(\nu^{(h)})^\ell. \end{cases} \tag{5.1.35}$$

Observe that $v_{i_h}^\ell = \text{end}(\nu^{(h-1)})^\ell$ and $v_{k_h}^\ell = \text{start}(\nu^{(h)})^\ell$. Since v_{j_h} is maximal among prunable nodes in T , then all ancestors of v_{j_h} are life-sustaining. Applying Lemma 5.1.7 with $i = i_h$, $j = j_h$, $k = k_h$, $e = e_h$ and $f = f_h$, we have

$$\text{wt}(e_h)\text{wt}(f_h) = \begin{cases} \bar{\lambda}\bar{\rho} & \text{if } v_{i_h}^\ell \leq v_{k_h}^\ell, \\ \lambda\rho & \text{if } v_{i_h}^\ell > v_{k_h}^\ell, \end{cases} \tag{5.1.36}$$

$$= \begin{cases} \bar{\lambda}\bar{\rho} & \text{if } \text{end}(\nu^{(h-1)})^\ell \leq \text{start}(\nu^{(h)})^\ell, \\ \lambda\rho & \text{if } \text{end}(\nu^{(h-1)})^\ell > \text{start}(\nu^{(h)})^\ell, \end{cases} \tag{5.1.37}$$

$$= \gamma_h. \tag{5.1.38}$$

Let us define a map on the disjoint union of the sets of edges of the paths $\nu^{(h)}$,

$$\Lambda : \bigsqcup_{h=0}^m \text{Edges}(\nu^{(h)}) \rightarrow \text{Edges}(T) \setminus \left(\bigcup_{h=1}^m \{e_h, f_h\} \cup \text{Edges}(T_h) \right) \tag{5.1.39}$$

with the property that $\text{wt}(e) = \text{wt}(\Lambda(e))$. Given $e \in \text{Edges}(\nu^{(h)})$, let its endpoints be the j th and $(j + 1)$ th nodes of S in inorder. First, suppose that e is a U step. Since $\nu^{(h)}$ is defined so that $w_{\nu^{(h)}} = w^{(h)}$, then the j th letter of $\text{can}(T)$ is a U . It follows from Lemma 5.1.4(a) that v_{j+1} has a left edge, which we then define to be $\Lambda(e)$. Similarly, if e is a D step, then

the j th letter of $\text{can}(T)$ must be a D . It follows from Lemma 5.1.4(b) that v_j must have a right edge, which we then define to be $\Lambda(e)$. By construction, we have that Λ is a bijection.

In the case that e is a U step, then by Lemma 5.1.8(a) we have $\text{wt}(e) = \text{wt}(\Lambda(e))$. In the case that e is a D step, then by Lemma 5.1.8(b) we have $\text{wt}(e) = \text{wt}(\Lambda(e))$. Therefore, Λ is a weight-preserving bijection. Hence,

$$\text{wt}(\Phi_{n,\nu}(T)) = \prod_{h=0}^m \text{wt}(\nu^{(h)}) \prod_{h=1}^m \gamma_h \text{wt}(T_h) \tag{5.1.40}$$

$$= \prod_{h=0}^m \text{wt}(\nu^{(h)}) \prod_{h=1}^m \text{wt}(e_h) \text{wt}(f_h) \text{wt}(T_h) \tag{5.1.41}$$

$$= \text{wt}(T). \square \tag{5.1.42}$$

Next, we prove that the map $\Phi_{n,\nu}$ is a bijection by constructing an inverse map. Our inverse map to $\Phi_{n,\nu}$ involves an algorithm which is inspired by the Push-Gliding algorithm of Prévaille-Ratelle and Viennot [69], which gives a bijection between binary trees and certain pairs of lattice paths.

Define $\Psi_{n,\nu} : \mathcal{PT}_{n,\nu}^\ell \rightarrow \mathcal{T}_{n,\nu}^\ell$ as follows. Let

$$S = (\nu^{(0)}, T_1, \nu^{(1)}, \dots, \nu^{(m-1)}, T_m, \nu^{(m)})$$

be an element of $\mathcal{PT}_{n,\nu}^\ell$. Define $\hat{\nu}$ to be the unique labeled lattice path such that

$$w_{\hat{\nu}} = w_{\nu^{(0)}} DU w_{\nu^{(1)}} DU \cdots DU w_{\nu^{(m)}} \tag{5.1.43}$$

and

$$\text{in}(\hat{\nu}) = \text{in}(\nu^{(0)}) \text{root}(T_1)^\ell \text{in}(\nu^{(1)}) \cdots \text{in}(\nu^{(m-1)}) \text{root}(T_m)^\ell \text{in}(\nu^{(m)}). \tag{5.1.44}$$

For easy comparison with [69], we switch momentarily to drawing our lattice paths with North and East steps instead of U and D steps, respectively. Similarly, we draw our trees so that right edges are horizontal and left edges are vertical.

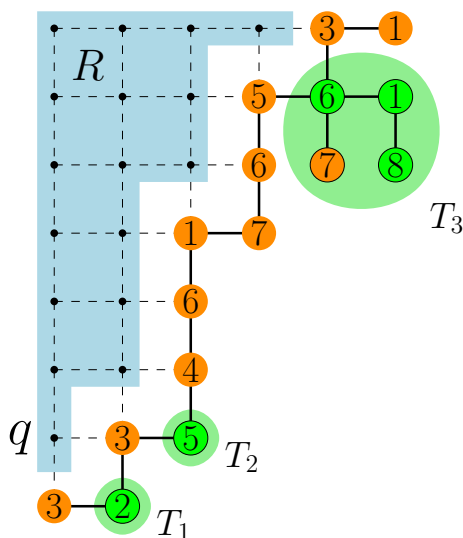


Figure 5.13: The Ω -tree corresponding to the path-tree sequence in Figure 5.12.

Recall from Prévaille-Ratelle and Viennot [69] that an Ω -tree T is defined to be a lattice path with some binary trees rooted at some nodes of the path. Define the *inorder traversal* of T by traversing the lattice path Southwest to Northeast, except that whenever a node is reached which is the root of a binary tree, traverse the binary tree in inorder and continue on with the rest of the path. If the highest node of T is at height $y = h$, let R denote the subset of points (x, y) on the integer lattice strictly above the lattice path with $x \geq 0$ and $y \leq h$.

Iteratively construct a tree from $\hat{\nu}$ and the trees T_h as follows. Initialize the Ω -tree T to be the path $\hat{\nu}$ together with each tree T_h rooted at the node of ν labeled $v_{j_h}^\ell$. Identify the j th node of T in inorder with the j th node of S in inorder. Both nodes will simultaneously be denoted by v_j . Draw T in the plane with the first node inorder situated at the origin $(0, 0)$ in Cartesian coordinates. For example, if S is the element of \mathcal{PT}^ℓ in Figure 5.12, then the initial Ω -tree T and set of lattice points R are shown in Figure 5.13. At any step, we keep track of an Ω -tree T , all of whose nodes are labeled. Suppose $\text{root}(T_h) = v_{j_h}$ for each h .

If R is nonempty, let q be the Eastern-most point of the bottom row of R . Let v_i, v_j

and v_k be the nodes of the path which are immediately South, Southeast, and East of q , respectively. We say the node v_j of T is *prunable* if $v_j = \text{root}(T_h)$ for some T_h and v_j is *life-sustaining* otherwise.

- (C1) If v_j is prunable and $\text{sort}(v_i^\ell v_j^\ell v_k^\ell) \in \{132, 312, 321\}$, or if v_j is life-sustaining and $\text{sort}(v_i^\ell v_j^\ell v_k^\ell) \in \{123, 213, 231\}$, then delete the edge $v_j v_k$, shift the part of T which is weakly Northeast of the node v_k 1 unit West, and draw a new edge between v_i and the new location of v_k . Update the Ω -tree T accordingly.
- (C2) If v_j is prunable and $\text{sort}(v_i^\ell v_j^\ell v_k^\ell) \in \{123, 213, 231\}$, or if v_j is life-sustaining and $\text{sort}(v_i^\ell v_j^\ell v_k^\ell) \in \{132, 312, 321\}$, then delete the edge $v_i v_j$, shift the part of the Ω -tree which is weakly Northeast of the node v_j 1 unit South, and create a new edge between v_i and the new location of v_k . Update the Ω -tree T accordingly.

Repeat this procedure until the set R is empty. At the end of this algorithm, we obtain a labeled plane binary tree T rooted at the Northwestern-most node.

Remark 5.1.14. This procedure of converting S into $\Psi_{n,\nu}(S)$ is essentially a modified version of the Push-Gliding algorithm of Préville-Ratelle and Viennot [69], where translating part of the path West by 1 unit is a Glide operation and translating part of the path South by 1 unit is part of a Push operation.

Define the canopy of an Ω -tree to be the word on $\{U, D\}$ obtained by recording the U and D steps of $\hat{\nu}$ from left to right, except that whenever a node is reached which is the root of a binary tree, we record the canopy of that binary tree and continue on with the rest of the path. By (5.1.43), the canopy of the initial Ω -tree is $\text{can}(S)$. From [69], Push and Glide operations preserve the canopy of the Ω -tree T . Define $\Psi_{n,\nu}(S) := T$, which is in $\mathcal{PT}_{n,\nu}^\ell$.

Figure 5.14 shows the construction of $\Psi_{n,\nu}(S)$ where $S = \Phi_{n,\nu}(T)$ shown in Figure 5.12 and T is the labeled tree in Figure 5.11. Note that we do indeed reconstruct the original tree in this example.

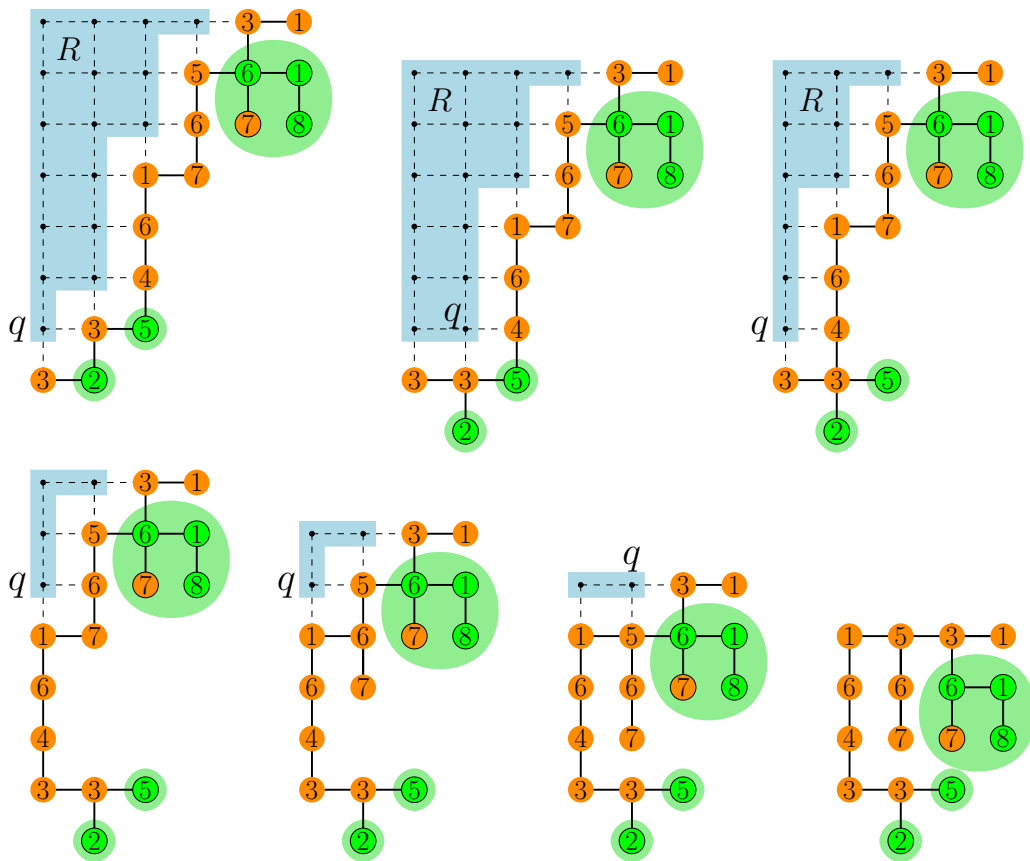


Figure 5.14: The inverse bijection $\Psi_{n,\nu}$ applied to the path-tree sequence S from Figure 5.12.

Note that throughout this procedure, we need to avoid creating cycles when drawing T . Therefore, some of the edge lengths may have to be drawn lengthened in order to avoid collisions in the grid (see the footnote at the bottom of [69, page 5230]).

Lemma 5.1.15. *The map $\Phi_{n,\nu}$ is a bijection.*

Proof. The choice of either applying (C1) or (C2) at each step guarantees that the node v_{j_h} of the tree $\Psi_{n,\nu}(S)$ is prunable and that it is maximal among prunable nodes in $\Psi_{n,\nu}(S)$. Therefore, the subtrees in the path-tree sequence $\Phi_{n,\nu}(\Psi_{n,\nu}(S))$ will be exactly T_1, \dots, T_m .

From [69], Push and Glide operations preserve the canopy and inorder reading word of an Ω -tree. Therefore, the sequences of U and D steps in the $m + 1$ lattice paths of $\Phi_{n,\nu}(\Psi_{n,\nu}(S))$

must be $w_{\nu^{(0)}}, w_{\nu^{(1)}}, \dots, w_{\nu^{(m)}}$, respectively. It is easy to see that their inorder reading words are $\text{in}(\nu^{(0)}), \text{in}(\nu^{(1)}), \dots, \text{in}(\nu^{(m)})$, respectively, so the lattice paths of $\Phi_{n,\nu}(\Psi_{n,\nu}(S))$ must be $\nu^{(0)}, \dots, \nu^{(m)}$. Therefore, $\Phi_{n,\nu}(\Psi_{n,\nu}(S)) = S$ so $\Phi_{n,\nu}$ is surjective.

Given an integer n , a word $w \in \mathbb{Z}_{>0}^n$, and a sequence $\nu \in \{U, D\}^{n-1}$, let $\mathcal{T}_{n,\nu,w}^\ell$ be the subset of $\mathcal{T}_{n,\nu}^\ell$ of trees with inorder reading word w , and let $\mathcal{PT}_{n,\nu,w}^\ell$ be the subset of $\mathcal{PT}_{n,\nu}^\ell$ with inorder reading word w . Observe that $\Phi_{n,\nu}$ maps $\mathcal{T}_{n,\nu,w}^\ell$ surjectively onto $\mathcal{PT}_{n,\nu,w}^\ell$ by the argument above, so $|\mathcal{T}_{n,\nu,w}^\ell| \geq |\mathcal{PT}_{n,\nu,w}^\ell|$. Furthermore, observe that $\mathcal{T}_{n,\nu,w}^\ell$ is in bijection with the set of unlabeled trees on n nodes with canopy ν . Similarly, the set $\mathcal{PT}_{n,\nu,w}^\ell$ is in bijection with unlabeled path-tree sequences on n nodes with canopy ν .

A simple generating function argument shows that the number of unlabeled path-tree sequences on n nodes is $|\mathcal{T}_n| = \text{Cat}_n$, the n th Catalan number. Indeed, the ordinary generating function for unlabeled path-tree sequences is given by

$$\sum_{i \geq 1} \left(\frac{x}{1-2x} \right)^{i+1} C(x)^i = \frac{x}{1-2x-xC(x)}. \quad (5.1.45)$$

Recall from Subsection 4.4.1 that $C(x) = \sum_{n \geq 1} \text{Cat}_n x^n$ and that $C := C(x)$ satisfies the relation $C = x(1+C)^2$. This relation implies in turn that the right hand side of (5.1.45) equals C . Therefore, for fixed n and w we have

$$\text{Cat}_n = |\mathcal{T}_n| = \sum_{\nu} |\mathcal{T}_{n,\nu,w}^\ell| \geq \sum_{\nu} |\mathcal{PT}_{n,\nu,w}^\ell| = \text{Cat}_n. \quad (5.1.46)$$

Hence, the inequality in (5.1.46) must be an equality, and in fact each inequality $|\mathcal{T}_{n,\nu,w}^\ell| \geq |\mathcal{PT}_{n,\nu,w}^\ell|$ must be an equality, so $|\mathcal{T}_{n,\nu,w}^\ell| = |\mathcal{PT}_{n,\nu,w}^\ell|$.

Since $\Phi_{n,\nu}$ maps $\mathcal{T}_{n,\nu,w}^\ell$ surjectively onto $\mathcal{PT}_{n,\nu,w}^\ell$, and these two sets have the same cardinality, then $\Phi_{n,\nu}$ must restrict to a bijection between $\mathcal{T}_{n,\nu,w}^\ell$ and $\mathcal{PT}_{n,\nu,w}^\ell$. As we run over w , the sets $\mathcal{T}_{n,\nu,w}^\ell$ and $\mathcal{PT}_{n,\nu,w}^\ell$ partition the domain and codomain, respectively, of the map $\Phi_{n,\nu}$. Therefore, $\Phi_{n,\nu}$ is a bijection. \square

Proof 2 of Theorem 4.0.3. By Lemmas 5.1.13 and 5.1.15, we have that $\Phi_{n,\nu} : \mathcal{T}_{n,\nu}^\ell \rightarrow \mathcal{PT}_{n,\nu}^\ell$

is a bijection such that $\text{wt}(T) = \text{wt}(\Phi_{n,\nu}(T))$ and $\text{in}(T) = \text{in}(\Phi_{n,\nu}(T))$. Therefore,

$$\sum_{T \in \mathcal{T}^\ell} \text{wt}(T) \mathbf{x}^T = \sum_{n \geq 1} \sum_{\nu \in \{U, D\}^{n-1}} \sum_{T \in \mathcal{T}_{n,\nu}^\ell} \text{wt}(\Phi_{n,\nu}(T)) \mathbf{x}^{\Phi_{n,\nu}(T)} \quad (5.1.47)$$

$$= \sum_{S \in \mathcal{P}^\ell} \text{wt}(S) \mathbf{x}^S \quad (5.1.48)$$

$$= \sum_{n \geq 1} \sum_{\alpha \models n} (\bar{\lambda} \bar{\rho} G + \bar{\lambda} + \bar{\rho})^{n-\ell(\alpha)} (\lambda \rho G + \lambda + \rho)^{\ell(\alpha)-1} r_\alpha, \quad (5.1.49)$$

where the last equality holds by Lemma 5.1.1. \square

In addition to proving Theorem 4.0.3, the same proof allows us to obtain a noncommutative version of Theorem 4.0.3.

5.1.4 Noncommutative version

Let $\mathbf{x} = \{\mathbf{x}_1, \mathbf{x}_2, \dots\}$ be a set of noncommuting variables. Given a word $w = w_1 \dots w_n \in \mathbb{Z}_{>0}^n$, define the noncommutative monomial \mathbf{x}_w to be $\mathbf{x}_{w_1} \dots \mathbf{x}_{w_n}$. Let

$$\mathbf{G} := \mathbf{G}(\mathbf{x}; \bar{\lambda}, \lambda, \bar{\rho}, \rho) = \sum_{n \geq 1} \sum_{T \in \mathcal{T}_n^\ell} \text{wt}(T) \mathbf{x}_{\text{in}T}.$$

Let \mathbf{s} and \mathbf{t} be parameters which do not commute with the \mathbf{x}_i . For $w \in \mathbb{Z}_{>0}^+$, let $\mathbf{x}_w(\mathbf{s}, \mathbf{t})$ be the *extended noncommutative monomial* given by inserting \mathbf{s} between \mathbf{x}_{w_i} and $\mathbf{x}_{w_{i+1}}$ in the monomial \mathbf{x}_w for each ascent $w_i \leq w_{i+1}$ and inserting \mathbf{t} for each descent $w_i > w_{i+1}$. For instance, if $w = 21131$, then $\mathbf{x}_w(\mathbf{s}, \mathbf{t}) = \mathbf{x}_2 \mathbf{t} \mathbf{x}_1 \mathbf{s} \mathbf{x}_1 \mathbf{s} \mathbf{x}_3 \mathbf{t} \mathbf{x}_1$.

As a corollary of Proof 2 of Theorem 4.0.3, we obtain the following noncommutative analogue of the functional equation in Theorem 4.0.3.

Corollary 5.1.16. *We have the following functional equation in terms of substitutions into the extended noncommutative monomials,*

$$\mathbf{G} = \sum_{w \in \mathbb{Z}_{>0}^+} \mathbf{x}_w (\bar{\lambda} \bar{\rho} \mathbf{G} + \bar{\lambda} + \bar{\rho}, \lambda \rho \mathbf{G} + \lambda + \rho).$$

5.2 Proofs of Theorem 4.0.4 and Theorem 4.0.6

In this section, we use the weight-preserving bijection defined in Definition 5.1.12 to prove Theorem 4.0.6. We then show how Theorem 4.0.4 follows from Theorem 4.0.6.

Our key tool in this section is a weight-preserving surjective map $\Theta_{n,\nu}$ from $\mathcal{PT}_{n,\nu}^\ell$ onto $\text{mNC}^*(n,\nu)$ with the following crucial property: The preimage of $\pi^* \in \text{mNC}^*(n,\nu)$ consists of certain $S \in \mathcal{PT}_{n,\nu}^\ell$ such that the sum over the monomials \mathbf{x}^S is a product of ribbon Schur functions. The map $\Theta_{n,\nu}$ is defined recursively. Informally put, all nodes that belong to paths in a path-tree sequence S contribute the block B_1 in our augmented interlacing partition π^* . Furthermore, individual paths in S correspond to sub-blocks in B_1 that are maximal under connectedness by short arcs. The labels of the short arcs record the sequences of up and down steps in individual paths. To determine the remaining blocks in π^* , we first apply $\Phi_{n',\nu'}$ for the appropriate n' and ν' to each tree in S , and subsequently apply $\Theta_{n',\nu'}$ to the resulting path-tree sequence. In this manner we recursively compute all blocks in π^* . Finally, in view of the definitions of weights of elements of $\mathcal{PT}_{n,\nu}^\ell$ and $\text{mNC}^*(n,\nu)$, the choice of which nodes get marked is essentially ‘forced’ by our demand that $\Theta_{n,\nu}$ be weight-preserving. With this informal description serving as a crutch, we give precise details.

Given $S = (\nu^{(0)}, T_1, \dots, T_m, \nu^{(m)}) \in \mathcal{PT}_{n,\nu}^\ell$, let v_1, \dots, v_n be the nodes of S listed in inorder. For $0 \leq h \leq m$, let $[k_h, i_{h+1}]$ be the set of indices of nodes in $\nu^{(h)}$ and $[i_h + 1, k_h - 1]$ be the set of indices of nodes in T_h .

Define a map $\Theta_{n,\nu} : \mathcal{PT}_{n,\nu}^\ell \rightarrow \text{mNC}^*(n,\nu)$ recursively as follows. If $n = 1$, define $\Theta_{1,\emptyset}$ to be the map which sends each sequence $(\nu^{(0)})$ consisting of a single labeled node to the unique augmented interlacing partition on a single node. Now, given $n > 1$ and $S \in \mathcal{PT}_{n,\nu}^\ell$, define the arc diagram of an augmented interlacing partition as follows.

1. Initialize a set of nodes u_1, \dots, u_n , and draw arcs so that all of the nodes u_p with $p \in [1, i_1] \sqcup [k_1, i_2] \sqcup \dots \sqcup [k_m, n]$ are in the single block B_1 .
2. For $0 \leq h \leq m$ and $p \in [k_h, i_{h+1})$, observe that $v_p v_{p+1}$ is an edge e of $\nu^{(h)}$. If e is an up

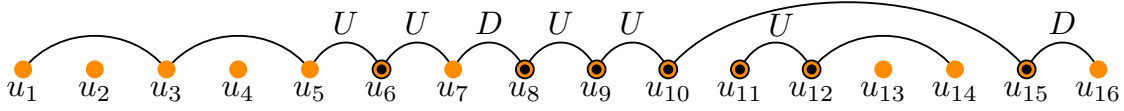


Figure 5.15: The augmented interlacing partition $\Theta_{n,\nu}(S)$ for S in Figure 5.12.

step, then label the short arc $u_p u_{p+1}$ drawn in step (1) with a U . Otherwise, label the short arc $u_p u_{p+1}$ with a D . In addition, if $v_p^\ell > v_{p+1}^\ell$, then mark the node u_p .

3. For $1 \leq h \leq m$, mark the node u_{i_h} if and only if $v_{i_h}^\ell > v_{k_h}^\ell$.
4. For $1 \leq h \leq m$, let $n_h = |\text{Nodes}(T_h)|$ and $\nu_h = \text{can}(T_h)$. Recursively construct the augmented interlacing partition $\pi_h^* := \Theta_{n_h, \nu_h}(\Phi_{n_h, \nu_h}(T_h))$ on the set of nodes $u_{i_h+1}, u_{i_h+2}, \dots, u_{k_h-1}$.
5. Define $\Theta_{n,\nu}(S) := B_1/\pi_1^*/\dots/\pi_m^*$, i.e. the augmented interlacing partition whose blocks are B_1 together with the blocks in π_1^*, \dots, π_m^* .

For the path-tree sequence S in Figure 5.12, its image $\Theta_{n,\nu}(S)$ is shown in Figure 5.15.

Lemma 5.2.1. *We have that $\Theta_{n,\nu}$ is a well-defined map to $\text{mNC}^*(n, \nu)$ which is weight-preserving and surjective.*

Proof. We prove that $\Theta_{n,\nu}$ is well-defined, weight-preserving, and surjective by induction on n . The $n = 1$ case of each part is trivial, because in this case the only element of $\mathcal{PT}_{n,\nu}^\ell$ is a path-tree sequence $(\nu^{(0)})$ consisting of a single node, which maps to the augmented interlacing partition consisting of a single node. Let us assume by way of induction that the lemma holds for all $n < N$ and all ν and S .

To show $\Theta_{N,\nu}$ is well-defined, we must show $w(\Theta_{N,\nu}(S)) = \text{can}(S)$ for all $S = (\nu^{(0)}, T_1, \dots, T_m, \nu^{(m)}) \in \mathcal{PT}_{N,\nu}^\ell$. Let the nodes of S be v_1, \dots, v_N listed in inorder, let $n_h = |\text{Nodes}(T_h)|$, and let $\nu_h = \text{can}(T_h)$. By our inductive hypothesis, we have

$$w(\Theta_{n_h, \nu_h}(\Phi_{n_h, \nu_h}(T_h))) = \text{can}(\Phi_{n_h, \nu_h}(T_h)) = \text{can}(T_h). \tag{5.2.1}$$

Letting B_1 be the block of $\Theta_{N,\nu}(S)$, partition B_1 into blocks C_0, C_1, \dots, C_p that are maximal under connected by short arcs. Then $w(C_i) = w_{\nu^{(i)}}$ for $0 \leq i \leq m$ by step (2) of the definition of $\Theta_{N,\nu}$. Therefore, by the definition of $w(\Theta_{N,\nu}(S))$, see (4.3.7), and the definition of $\text{can}(S)$, see (5.1.7), we have $w(\Theta_{N,\nu}(S)) = \text{can}(S)$.

To show $\Theta_{N,\nu}$ is weight-preserving, we must show $\text{wt}(\Theta_{N,\nu}(S)) = \text{wt}(S)$ for all $S \in \mathcal{PT}_{N,\nu}^\ell$. Using the same notation as above, by our inductive hypothesis we have

$$\text{wt}(\Theta_{n_h,\nu_h}(\Phi_{n_h,\nu_h}(T_h))) = \text{wt}(\Phi_{n_h,\nu_h}(T_h)) = \text{wt}(T_h). \tag{5.2.2}$$

Recall the definition of $\text{wt}(\Theta_{n,\nu}(S))$, see (4.3.6). Observe that by step (2) of the construction of $\Theta_{N,\nu}$, the number of unmarked steppers whose short arc is labeled U in C_i is equal to the number of up steps in $\nu^{(i)}$ which form a descent. Similar equalities hold for the three other types of steppers. Furthermore, the contribution of the marked and unmarked jumpers to $\text{wt}(\Theta_{n,\nu}(S))$ is equal to the contribution of the γ_i to $\text{wt}(S)$ by step (3). From these observations together with (5.2.2), we conclude $\text{wt}(\Theta_{N,\nu}(S)) = \text{wt}(S)$.

To show $\Theta_{N,\nu}$ is surjective, let $\pi^* \in \text{mNC}^*(N, \nu)$ an augmented interlacing partition on nodes u_1, \dots, u_N . Letting B_1 be the part of π^* containing u_1 and u_N , partition B_1 into blocks C_0, \dots, C_m that are maximal under connectedness by short arcs, and let π_h^* be the augmented interlacing partition induced by π^* on the nodes u_j for $j \in (\max(C_{h-1}), \min(C_h))$. By our inductive hypothesis, we have Θ_{n_h,ν_h} is surjective, where n_h is the number of nodes of π_h^* and $\nu_h = w(\pi_h^*)$. Hence, there exists S_h such that $\Theta_{n_h,\nu_h}(S_h) = \pi_h^*$. For $0 \leq i \leq m$, let $\nu^{(i)}$ be the unlabeled lattice path on $|C_i|$ many nodes such that $w_{\nu^{(i)}}$ is given by the labels of the short arcs in C_i . Furthermore, label the nodes of the $\nu^{(i)}$ so that if we define $S = (\nu^{(0)}, \Phi_{n_1,\nu_1}^{-1}(S_1), \dots, \Phi_{n_m,\nu_m}^{-1}(S_m), \nu^{(m)})$, then steps (2) and (3) would give us the marking of π^* back under applying $\Theta_{n,\nu}$. Finally, by step (4) we see that S has been chosen exactly so that we recover the π_h^* . Hence, we have $\Theta_{N,\nu}(S) = \pi^*$, so $\Theta_{N,\nu}$ is surjective, which completes the induction. □

The reader can verify that for the path-tree sequence S in Figure 5.12 and $\Theta_{n,\nu}(S)$ in

Figure 5.15, we have $\text{wt}(\Theta_{n,\nu}(S)) = \bar{\lambda}\lambda^4\bar{\rho}\rho(\bar{\lambda}\bar{\rho})^2(\lambda\rho)^2 = \text{wt}(S)$.

Proof of Theorem 4.0.6. By Lemma 5.2.1, we have

$$G_{n,\nu} = \sum_{T \in \mathcal{T}_{n,\nu}^\ell} \text{wt}(T) \mathbf{x}^T \tag{5.2.3}$$

$$= \sum_{S \in \mathcal{PT}_{n,\nu}^\ell} \text{wt}(S) \mathbf{x}^S \tag{5.2.4}$$

$$= \sum_{S \in \mathcal{PT}_{n,\nu}^\ell} \text{wt}(\Theta_{n,\nu}(S)) \mathbf{x}^S \tag{5.2.5}$$

$$= \sum_{\pi^* \in \text{mNC}^*(n,\nu)} \text{wt}(\pi^*) \sum_{S \in \Theta_{n,\nu}^{-1}(\pi^*)} \mathbf{x}^S, \tag{5.2.6}$$

where (5.2.4) follows from the fact that $\Phi_{n,\nu}$ is a weight-preserving bijection.

It remains to show that for an augmented interlacing partition $\pi^* = B_1/\dots/B_k \in \text{mNC}(n,\nu)$, we have

$$\sum_{S \in \Theta_{n,\nu}^{-1}(\pi^*)} \mathbf{x}^S = r_{c(B_1)}r_{c(B_2)} \dots r_{c(B_k)}. \tag{5.2.7}$$

We prove this by induction on n . The case of $n = 1$ is immediate since $G_{n,\emptyset} = r_1$. Let $n > 1$, and let $\pi^* \in \text{mNC}^*(n,\nu)$. Letting B_1 be the block of π^* containing 1 and n , decompose B_1 into blocks C_0, \dots, C_m maximal under connectedness by short arcs, as in (4.3.7). Let π_1^*, \dots, π_m^* be the induced augmented interlacing partitions on the remaining nodes. Let n_h be the number of nodes in π_h^* , and let $\nu_h = w(\pi_h^*)$. Then $S = (\nu^{(0)}, T_1, \dots, T_m, \nu^{(m)}) \in \Theta_{n,\nu}^{-1}(\pi^*)$ if and only if

- (a) The word obtained by concatenating the inorder reading words of $\nu^{(0)}, \dots, \nu^{(m)}$ has strict descents in positions corresponding to marked nodes of C_0, \dots, C_m and weak ascents in positions corresponding to unmarked nodes,
- (b) For $0 \leq h \leq m$, $w_{\nu^{(h)}}$ is equal to the word read from the labeling of the short arcs

connecting the nodes in C_h ,

(c) For $1 \leq h \leq m$, we have $\Phi_{n_h, \nu_h}(T_h) \in \Theta_{n_h, \nu_h}^{-1}(\pi_h^*)$.

For fixed π^* , the steps of $\nu^{(h)}$ are determined by (b). If T_1, \dots, T_m are fixed trees which satisfy (c), then by (a) and the definition of $c(B_1)$ in Section 4.3, we have that

$$\sum_S \mathbf{x}^{\text{in}(\nu^{(0)})} \mathbf{x}^{\text{in}(\nu^{(1)})} \dots \mathbf{x}^{\text{in}(\nu^{(m)})} = r_{c(B_1)}, \tag{5.2.8}$$

where the sum is over all $S = (\nu^{(0)}, T_1, \dots, T_m, \nu^{(m)}) \in \Theta_{n, \nu}^{-1}(\pi^*)$.

Finally, by (5.2.8) and our induction hypothesis applied to each π_h^* , we have

$$\sum_{S \in \Theta_{n, \nu}^{-1}(\pi^*)} \mathbf{x}^S = r_{c(B_1)} \prod_{h=1}^m \left(\sum_{T_h \in \Phi_{n_h, \nu_h}^{-1}(\Theta_{n_h, \nu_h}^{-1}(\pi_h^*))} \mathbf{x}^{T_h} \right) \tag{5.2.9}$$

$$= r_{c(B_1)} \prod_{h=1}^m \left(\sum_{S_h \in \Theta_{n_h, \nu_h}^{-1}(\pi_h^*)} \mathbf{x}^{S_h} \right) \tag{5.2.10}$$

$$= r_{c(B_1)} r_{c(B_2)} \dots r_{c(B_k)}, \tag{5.2.11}$$

which completes the proof. □

Proof of Theorem 4.0.4. We have $G_n = \sum_{\nu \in \{U, D\}^{n-1}} G_{n, \nu}$. Furthermore, for each $\pi \in \text{mNC}(n)$, summing over all augmented interlacing partitions π^* whose underlying element of $\text{mNC}(n)$ is π , we get

$$\sum_{\pi^*} \text{wt}(\pi^*) = \sum_{\pi^*} \bar{\lambda}^{\text{saU}(\pi^*)} \lambda^{\text{sdU}(\pi^*)} \bar{\rho}^{\text{saD}(\pi^*)} \rho^{\text{sdD}(\pi^*)} (\bar{\lambda} \bar{\rho})^{\text{ja}(\pi^*)} (\lambda \rho)^{\text{jd}(\pi^*)} \tag{5.2.12}$$

$$= (\bar{\lambda} + \bar{\rho})^{\text{sa}(\pi)} (\lambda + \rho)^{\text{sd}(\pi)} (\bar{\lambda} \bar{\rho})^{\text{ja}(\pi)} (\lambda \rho)^{\text{jd}(\pi)} \tag{5.2.13}$$

$$= \text{wt}(\pi). \tag{5.2.14}$$

The result follows by summing both sides of (4.0.15) over all $\nu \in \{U, D\}^{n-1}$. □

BIBLIOGRAPHY

- [1] E. ARTIN, *Galois Theory*, Notre Dame Mathematical Lectures, no. 2, University of Notre Dame, Notre Dame, Ind., second ed., 1944.
- [2] C. A. ATHANASIADIS, *Characteristic polynomials of subspace arrangements and finite fields*, Adv. Math., 122 (1996), pp. 193–233.
- [3] ———, *On free deformations of the braid arrangement*, European J. Combin., 19 (1998), pp. 7–18.
- [4] ———, *Extended Linnel hyperplane arrangements for root systems and a conjecture of Postnikov and Stanley*, J. Algebraic Combin., 10 (1999), pp. 207–225.
- [5] ———, *Deformations of Coxeter hyperplane arrangements and their characteristic polynomials*, in Arrangements—Tokyo 1998, vol. 27 of Adv. Stud. Pure Math., Kinokuniya, Tokyo, 2000, pp. 1–26.
- [6] ———, *Gamma-positivity in combinatorics and geometry*, Sémin. Lothar. Combin., 77 ([2016-2018]), pp. Art. B77i, 64.
- [7] C. A. ATHANASIADIS AND S. LINUSSON, *A simple bijection for the regions of the Shi arrangement of hyperplanes*, Discrete Math., 204 (1999), pp. 27–39.
- [8] O. BERNARDI, *Deformations of the braid arrangement and trees*, Adv. Math., 335 (2018), pp. 466–518.
- [9] S. BILLEY AND I. COSKUN, *Singularities of generalized Richardson varieties*, Comm. Algebra, 40 (2012), pp. 1466–1495.
- [10] A. BOREL, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math. (2), 57 (1953), pp. 115–207.
- [11] P. BRÄNDÉN, *Unimodality, log-concavity, real-rootedness and beyond*, in Handbook of enumerative combinatorics, Discrete Math. Appl. (Boca Raton), CRC Press, Boca Raton, FL, 2015, pp. 437–483.
- [12] C. CHEVALLEY, *Invariants of finite groups generated by reflections*, Amer. J. Math., 77 (1955), pp. 778–782.

- [13] N. CHRISS AND V. GINZBURG, *Representation theory and complex geometry*, Birkhäuser Boston, Inc., Boston, MA, 1997.
- [14] T. CHURCH, J. S. ELLENBERG, AND B. FARB, *FI-modules and stability for representations of symmetric groups*, *Duke Math. J.*, 164 (2015), pp. 1833–1910.
- [15] S. CORTEEL, D. FORGE, AND V. VENTOS, *Bijections between affine hyperplane arrangements and valued graphs*, *European J. Combin.*, 50 (2015), pp. 30–37.
- [16] D. COX, J. LITTLE, AND D. O’SHEA, *Ideals, varieties, and algorithms*, Undergraduate Texts in Mathematics, (2007).
- [17] C. DE CONCINI AND C. PROCESI, *Symmetric functions, conjugacy classes and the flag variety*, *Invent. Math.*, 64 (1981), pp. 203–219.
- [18] P. DIACONIS AND J. FULMAN, *Foulkes characters, Eulerian idempotents, and an amazing matrix*, *J. Algebraic Combin.*, 36 (2012), pp. 425–440.
- [19] R. DONAGHEY AND L. W. SHAPIRO, *Motzkin numbers*, *J. Combinatorial Theory Ser. A*, 23 (1977), pp. 291–301.
- [20] B. DRAKE, *An inversion theorem for labeled trees and some limits of areas under lattice paths*, ProQuest LLC, Ann Arbor, MI, 2008. Thesis (Ph.D.)—Brandeis University.
- [21] P. H. EDELMAN, *Multichains, noncrossing partitions and trees*, *Discrete Math.*, 40 (1982), pp. 171–179.
- [22] D. EISENBUD, *Commutative algebra*, vol. 150 of Graduate Texts in Mathematics, Springer-Verlag, New York, 1995.
- [23] D. EISENBUD AND D. SALTMAN, *Rank varieties of matrices*, *Commutative Algebra*, (1989), pp. 173–212.
- [24] D. FORGE, *Linial arrangements and local binary search trees*, preprint. arXiv:1411.7834, 2014.
- [25] H. O. FOULKES, *Eulerian numbers, Newcomb’s problem and representations of symmetric groups*, *Discrete Math.*, 30 (1980), pp. 3–49.
- [26] L. FRESSE AND A. MELNIKOV, *On the singularity of the irreducible components of a Springer fiber in \mathfrak{sl}_n* , *Selecta Math. (N.S.)*, 16 (2010), pp. 393–418.

- [27] F. Y. C. FUNG, *On the topology of components of some Springer fibers and their relation to Kazhdan-Lusztig theory*, Adv. Math., 178 (2003), pp. 244–276.
- [28] A. M. GARSIA AND M. HAIMAN, *A graded representation model for Macdonald’s polynomials*, Proc. Nat. Acad. Sci. U.S.A., 90 (1993), pp. 3607–3610.
- [29] A. M. GARSIA AND C. PROCESI, *On certain graded S_n -modules and the q -Kostka polynomials*, Adv. Math., 94 (1992), pp. 82–138.
- [30] I. M. GESSEL, *Multipartite P -partitions and inner products of skew Schur functions*, in Combinatorics and algebra (Boulder, Colo., 1983), vol. 34 of Contemp. Math., Amer. Math. Soc., Providence, RI, 1984, pp. 289–317.
- [31] —, *unpublished work*. 1995.
- [32] —, *Counting forests by descents and leaves*, vol. 3, 1996, pp. Research Paper 8, approx. 5. The Foata Festschrift.
- [33] —, *Oberwolfach reports (enumerative combinatorics)*. https://www.mfo.de/document/1410/OWR_2014_12.pdf, 2014.
- [34] —, *Lagrange inversion*, J. Combin. Theory Ser. A, 144 (2016), pp. 212–249.
- [35] I. M. GESSEL, S. T. GRIFFIN, AND V. TEWARI, *Schur positivity and labeled binary trees*, Sémin. Lothar. Combin., 78B (2017), pp. Art. 73, 12.
- [36] —, *Labeled binary trees, subarrangements of the Catalan arrangements, and Schur positivity*, Adv. Math., 356 (2019), p. 106814.
- [37] I. M. GESSEL AND C. REUTENAUER, *Counting permutations with given cycle structure and descent set*, J. Combin. Theory Ser. A, 64 (1993), pp. 189–215.
- [38] S. T. GRIFFIN, *Ordered set partitions, Garsia-Procesi modules, and rank varieties*. Trans. Amer. Math. Soc., in press, 2020.
- [39] —, *Ordered set partitions, Tanisaki ideals, and rank varieties*, Sémin. Lothar. Combin., (2020). Extended abstract.
- [40] J. HAGLUND, *The q, t -Catalan numbers and the space of diagonal harmonics*, vol. 41 of University Lecture Series, American Mathematical Society, Providence, RI, 2008.
- [41] J. HAGLUND, M. HAIMAN, AND N. LOEHR, *A combinatorial formula for Macdonald polynomials*, J. Amer. Math. Soc., 18 (2005), pp. 735–761.

- [42] J. HAGLUND, M. HAIMAN, N. LOEHR, J. B. REMMEL, AND A. ULYANOV, *A combinatorial formula for the character of the diagonal coinvariants*, Duke Math. J., 126 (2005), pp. 195–232.
- [43] J. HAGLUND, K. LUOTO, S. MASON, AND S. VAN WILLIGENBURG, *Quasisymmetric Schur functions*, J. Combin. Theory Ser. A, 118 (2011), pp. 463–490.
- [44] J. HAGLUND, J. B. REMMEL, AND A. T. WILSON, *The Delta Conjecture*, Trans. Amer. Math. Soc., 370 (2018), pp. 4029–4057.
- [45] J. HAGLUND, B. RHOADES, AND M. SHIMOZONO, *Ordered set partitions, generalized coinvariant algebras, and the Delta Conjecture*, Adv. Math., 329 (2018), pp. 851–915.
- [46] M. HAIMAN, *Conjectures on the quotient ring by diagonal invariants*, J. Algebraic Combin., 3 (1994), pp. 17–76.
- [47] ———, *Hilbert schemes, polygraphs and the Macdonald positivity conjecture*, J. Amer. Math. Soc., 14 (2001), pp. 941–1006.
- [48] ———, *Vanishing theorems and character formulas for the Hilbert scheme of points in the plane*, Invent. Math., 149 (2002), pp. 371–407.
- [49] P. HEADLEY, *On a family of hyperplane arrangements related to the affine Weyl groups*, J. Algebraic Combin., 6 (1997), pp. 331–338.
- [50] J. M. HOLTE, *Carries, combinatorics, and an amazing matrix*, Amer. Math. Monthly, 104 (1997), pp. 138–149.
- [51] R. HOTTA AND T. A. SPRINGER, *A specialization theorem for certain Weyl group representations and an application to the Green polynomials of unitary groups*, Invent. Math., 41 (1977), pp. 113–127.
- [52] L. H. KALIKOW, *Symmetries in trees and parking functions*, vol. 256, 2002, pp. 719–741. LaCIM 2000 Conference on Combinatorics, Computer Science and Applications (Montreal, QC).
- [53] B. KOSTANT, *Lie group representations on polynomial rings*, Amer. J. Math., 85 (1963), pp. 327–404.
- [54] H. KRAFT, *Conjugacy classes and Weyl group representations*, in Young tableaux and Schur functors in algebra and geometry (Toruń, 1980), vol. 87 of Astérisque, Soc. Math. France, Paris, 1981, pp. 191–205.

- [55] A. LASCoux, B. LECLERC, AND J.-Y. THIBON, *Ribbon tableaux, Hall-Littlewood functions, quantum affine algebras, and unipotent varieties*, J. Math. Phys., 38 (1997), pp. 1041–1068.
- [56] I. G. MACDONALD, *Symmetric functions and Hall polynomials*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, second ed., 1995.
- [57] P. A. MACMAHON, *Two Applications of General Theorems in Combinatory Analysis: (1) To the Theory of Inversions of Permutations; (2) To the Ascertainment of the Numbers of Terms in the Development of a Determinant which has Amongst its Elements an Arbitrary Number of Zeros*, Proc. London Math. Soc. (2), 15 (1916), pp. 314–321.
- [58] P. A. MACMAHON, *Combinatory analysis*, Two volumes (bound as one), Chelsea Publishing Co., New York, 1960.
- [59] J. MCCAMMOND, *Noncrossing partitions in surprising locations*, Amer. Math. Monthly, 113 (2006), pp. 598–610.
- [60] A. R. MILLER, *Foulkes characters for complex reflection groups*, Proc. Amer. Math. Soc., 143 (2015), pp. 3281–3293.
- [61] M. MONKS GILLESPIE, *A combinatorial approach to the q, t -symmetry relation in Macdonald polynomials*, Electron. J. Combin., 23 (2016), pp. P2.38, 64.
- [62] J.-C. NOVELLI AND J.-Y. THIBON, *Noncommutative symmetric functions and an amazing matrix*, Adv. in Appl. Math., 48 (2012), pp. 528–534.
- [63] OEIS FOUNDATION INC., *The online encyclopedia of integer sequences*. <http://oeis.org/>, 2020.
- [64] P. ORLIK AND H. TERAo, *Arrangements of hyperplanes*, vol. 300 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, 1992.
- [65] B. PAWLOWSKI AND B. RHOADES, *A flag variety for the Delta Conjecture*, Trans. Amer. Math. Soc., 372 (2019), pp. 8195–8248.
- [66] T. K. PETERSEN, *Eulerian numbers*, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser/Springer, New York, 2015.
- [67] A. POSTNIKOV, *Intransitive trees*, J. Combin. Theory Ser. A, 79 (1997), pp. 360–366.

- [68] A. POSTNIKOV AND R. P. STANLEY, *Deformations of Coxeter hyperplane arrangements*, vol. 91, 2000, pp. 544–597. In memory of Gian-Carlo Rota.
- [69] L.-F. PRÉVILLE-RATELLE AND X. VIENNOT, *The enumeration of generalized Tamari intervals*, *Trans. Amer. Math. Soc.*, 369 (2017), pp. 5219–5239.
- [70] B. RHOADES AND A. T. WILSON, *Line configurations and r -Stirling partitions*, *J. Comb.*, 10 (2019), pp. 411–431.
- [71] B. RHOADES, T. YU, AND Z. ZHAO, *Harmonic bases for generalized coinvariant algebras*. arXiv:2004.00767, 2020.
- [72] A. ROBERTS, *Dual equivalence graphs revisited and the explicit Schur expansion of a family of LLT polynomials*, *J. Algebraic Combin.*, 39 (2014), pp. 389–428.
- [73] B. E. SAGAN, *The symmetric group*, vol. 203 of Graduate Texts in Mathematics, Springer-Verlag, New York, second ed., 2001. Representations, combinatorial algorithms, and symmetric functions.
- [74] J. SHARESHIAN AND M. L. WACHS, *Eulerian quasisymmetric functions*, *Adv. Math.*, 225 (2010), pp. 2921–2966.
- [75] ———, *Chromatic quasisymmetric functions*, *Adv. Math.*, 295 (2016), pp. 497–551.
- [76] J. Y. SHI, *The Kazhdan-Lusztig cells in certain affine Weyl groups*, vol. 1179 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1986.
- [77] ———, *Sign types corresponding to an affine Weyl group*, *J. London Math. Soc. (2)*, 35 (1987), pp. 56–74.
- [78] R. SIMION, *Noncrossing partitions*, vol. 217, 2000, pp. 367–409. Formal power series and algebraic combinatorics (Vienna, 1997).
- [79] N. SPALTENSTEIN, *The fixed point set of a unipotent transformation on the flag manifold*, *Nederl. Akad. Wetensch. Proc. Ser. A*, 79 *Indag. Math.*, 38 (1976), pp. 452–456.
- [80] T. A. SPRINGER, *Trigonometric sums, Green functions of finite groups and representations of Weyl groups*, *Invent. Math.*, 36 (1976), pp. 173–207.
- [81] ———, *A construction of representations of Weyl groups*, *Invent. Math.*, 44 (1978), pp. 279–293.

- [82] R. P. STANLEY, *Hyperplane arrangements*. <http://www-math.mit.edu/~rstan/arrangements/arr.html>.
- [83] —, *Invariants of finite groups and their applications to combinatorics*, Bull. Amer. Math. Soc. (N.S.), 1 (1979), pp. 475–511.
- [84] —, *Some combinatorial properties of Jack symmetric functions*, Adv. Math., 77 (1989), pp. 76–115.
- [85] —, *Hyperplane arrangements, interval orders, and trees*, Proc. Nat. Acad. Sci. U.S.A., 93 (1996), pp. 2620–2625.
- [86] —, *Hyperplane arrangements, parking functions and tree inversions*, in Mathematical essays in honor of Gian-Carlo Rota (Cambridge, MA, 1996), vol. 161 of Progr. Math., Birkhäuser Boston, Boston, MA, 1998, pp. 359–375.
- [87] —, *Enumerative combinatorics. Vol. 2*, vol. 62 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.
- [88] —, *Alternating permutations and symmetric functions*, J. Combin. Theory Ser. A, 114 (2007), pp. 436–460.
- [89] —, *Enumerative combinatorics. Vol. 1*, vol. 49 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, second ed., 2012.
- [90] T. TANISAKI, *Defining ideals of the closures of the conjugacy classes and representations of the Weyl groups*, Tôhoku Math. J. (2), 34 (1982), pp. 575–585.
- [91] V. TEWARI, *Gessel polynomials, rooks, and extended Linial arrangements*, J. Combin. Theory Ser. A, 163 (2019), pp. 98–117.
- [92] J. TYMOCZKO, *The geometry and combinatorics of Springer fibers*, in Around Langlands correspondences, vol. 691 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2017, pp. 359–376.
- [93] J. S. TYMOCZKO, *Linear conditions imposed on flag varieties*, Amer. J. Math., 128 (2006), pp. 1587–1604.
- [94] J. A. VARGAS, *Fixed points under the action of unipotent elements of SL_n in the flag variety*, Bol. Soc. Mat. Mexicana (2), 24 (1979), pp. 1–14.

- [95] M. L. WACHS, *Poset topology: tools and applications*, in Geometric combinatorics, vol. 13 of IAS/Park City Math. Ser., Amer. Math. Soc., Providence, RI, 2007, pp. 497–615.
- [96] J. WEYMAN, *The equations of conjugacy classes of nilpotent matrices*, Invent. Math., 98 (1989), pp. 229–245.
- [97] A. T. WILSON, *An extension of MacMahon’s equidistribution theorem to ordered multiset partitions*, Electron. J. Combin., 23 (2016), pp. P1.5, 21.

VITA

Sean Griffin earned an A.B. from Dartmouth College in 2014 and is expected to earn a Ph.D. from the University of Washington in 2020. His research interests include algebraic combinatorics, representation theory of the symmetric group, and algebraic geometry. He has published papers in *Advances in Mathematics* and *Séminaire Lotharingien de Combinatoire*. He has also given research talks at Portland State University, Simon Fraser University, University of California at Davis, University of California at San Diego, University of Pennsylvania, University of Rome Tor Vergata, University of Washington, Washington University in St. Louis. He has also given contributed talks at the 32nd international conference on Formal Power Series and Algebraic Combinatorics (FPSAC), an AMS special session, and a JMM contributed paper session. In Fall 2020, he will be a visiting lecturer at the University of California at San Diego, and in Spring 2021 he will be a postdoctoral fellow at Brown University's Institute for Computational and Experimental Research in Mathematics (ICERM).