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An Inverse Source Problem in Radiative Transfer

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Abstract

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We consider the inverse source problem for the radiative transfer equation, under various assumptions on the scattering and absorption parameters, as well as on the accessible data. In such a setup, we measure the outgoing radiation intensity of the solution to the equation when an unknown source is present in the interior of the domain. The goal is to reconstruct the source from such measurements and obtain some form of stability estimate of the reconstruction on the data if possible. First, we extend the result of [50] to the case of partial data, where the absorption and scattering coefficients may lie in a certain dense open subset of $C^2(\bar{\Omega} \times \mathbb{S}^{n-1}) \times C^2(\bar{\Omega}, \mathbb{S}_{\theta'}^{n-1}; C^{n+1}(\mathbb{S}_{\theta}^{n-1}))$. Here it is shown one can recover sources supported in a particular subset of the domain, which we call the *visible set*. We next show that for an open dense set of C^∞ absorption and scattering coefficients, one can recover the part of the wave front set of the source that is supported in the *microlocally visible set*, modulo a function in the Sobolev space H^k for k arbitrarily large. Following, we consider the case where the scattering kernel k is small in suitable norm, and in this case we can reduce the smoothness requirements on k from $C^2(\bar{\Omega} \times \mathbb{S}_{\theta'}^{n-1}; C^{n+1}(\mathbb{S}_{\theta}^{n-1}))$ to $C^2(\bar{\Omega} \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$. Finally, we demonstrate a numerical scheme based on the method in [30], which we use to solve the inverse source problem in specific cases.

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DEDICATION

to my parents, Graham and Deb, and to my sisters, Grace and Kathryn, who have always
been a light in my life

Chapter 1

INVERSE SOURCE PROBLEM

1.1 Motivations

I suppose that like many young graduate students in mathematics, upon starting graduate school I didn't have a very clear idea of an area to which I wanted to devote my research. In retrospect however, I couldn't imagine having taken any different path than delving further into the field of inverse problems. It is a field rich in both application and theory, incorporating interesting physical phenomena into mathematical models that can be studied quite abstractly in very meaningful ways. Perhaps it is also my own undergraduate background in both math and physics which led me to walk a fine line between the theoretical and the applied. Indeed, one of the most popular applications for which mathematics has helped to propel forward a great deal is that of medical imaging modalities. The problem of determining an unknown parameter function in the interior of a domain while only being able to access measurements on the domain's boundary is one which has garnered much interest for decades.

Throughout this work, we consider one particular medical imaging modality, that which is called Optical Tomography. In particular, we study the propagation of photons in a medium where both scattering and absorption effects can occur. Such a system is typically modeled by the so-called Radiative Transfer Equation or Linear Boltzmann Equation. In Chapters 2 and 3 I present two main results in the case of recovering an unknown optical source inside a domain given only partial data of outgoing photon intensities on the boundary. This work has been published in *Inverse Problems* [24].

In early 2012, I had the opportunity to make a short visit to the University of Helsinki where there is a vibrant inverse problems group, as well as to visit the University of Jyväskylä

for an extended period under the advisement of Mikko Salo. During that time I investigated further possible results for the inverse source problem. In Chapter 4 we consider the case where the scattering kernel is suitably small in norm and show that slightly less smoothness is required to obtain unique and stable recovery.

Inspired by the work of Francois Monard in developing a novel solver of the forward radiative transfer equation using the idea of rotating grids [30], I developed a solver for the inverse source problem mostly as a means to provide visual confirmation of the aforementioned theoretical partial data results obtained. In Chapter 5, I provide documentation of the schemes used to compute the solution operator to the inverse source problem as well as the normal operator, which provides a representation of the source modulo a smooth error.

1.2 Background and Physical Setup

Radiative transfer equations (also sometimes referred to as linear transport or linear boltzmann equations) are often used to model the propagation of particles that exhibit absorption and scattering in various contexts, including behavior of photons within biological tissues or neutrons in a reactor. In what follows, we consider a problem relevant to optical molecular imaging (OMI), which is a fast-growing research area. In this application, biochemical markers can be used to detect the presence of specific molecules or genes, and suitably designed markers could potentially identify diseases before phenotypical symptoms even appear. The markers are typically light-emitting molecules, such as fluorophores or luminophores. In contrast to Single Positron Emission Computed Tomography (SPECT), Positron Emission Tomography (PET), or Magnetic Resonance Imaging (MRI), optical markers emit low-energy near-infrared photons that are relatively harmless to human tissue. However, because of their low-energy level, the photons scatter before they are measured. Further specifics can be found in the bioengineering literature such as [9, 13, 25, 55, 56, 34].

The inverse problem we consider consists of reconstructing the spatial distribution of a radiation source from measurements of photon intensities at the boundary of the medium in specific outgoing directions. In many applications, the propagation of photons emitted can be modeled as inverse source problems of steady-state radiative transfer equations. Once we know the optical properties of the underlying medium, the problem of determining the

source is feasible. It is shown in [50] that under mild assumptions on the scattering and absorption parameters of the medium this is possible. However, in the partial data case, which will be made more clear shortly, one can only hope to recover information about the singularities of the source. In particular, we seek to recover information about the wavefront set of the source function. We now describe more precisely the mathematical problem.

Suppose that Ω is a bounded domain in \mathbb{R}^n with smooth boundary and outer unit normal vector $\nu(x)$. Consider the radiative transfer equation

$$\begin{aligned} \theta \cdot \nabla_x u(x, \theta) + \sigma(x, \theta)u(x, \theta) - \int_{\mathbb{S}^{n-1}} k(x, \theta, \theta')u(x, \theta') d\theta' &= f(x), \\ u|_{\partial_- S\Omega} &= 0, \end{aligned} \tag{1.2.1}$$

where the absorption σ and the collision kernel k are functions with regularity specified later, the solution $u(x, \theta)$ gives the intensity of photons at x moving in the direction θ , and $\partial_{\pm} S\Omega$ is the set of points $(x, \theta) \in \partial\Omega \times \mathbb{S}^{n-1}$ such that $\pm\nu(x) \cdot \theta > 0$. That is, $\partial_{\pm} S\Omega$ is the set of points $(x, \theta) \in \partial\Omega \times \mathbb{S}^{n-1}$ such that θ is pointing outward or inward, respectively. The source term f will be assumed to depend on x only for our purposes. We also remark that Equation (1.2.1) is only applicable at a single frequency, as the parameters σ and k typically depend highly on frequency. In particular, for high energy photons there is a coupling between energy and angle, whereas for photons with low energy scattering is not typically accompanied by an energy change.

In the case of full data, we have boundary measurements

$$Xf(x, \theta) = u|_{\partial_+ S\Omega}. \tag{1.2.2}$$

In [50], it is shown that for an open, dense set of absorption and scattering coefficients $(\sigma, k) \in C^2(\overline{\Omega} \times \mathbb{S}^{n-1}) \times C^2(\overline{\Omega} \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$, one can recover $f \in L^2(\Omega)$ uniquely from boundary measurements Xf on all of $\partial_+ S\Omega$. To set up the case of partial data, first let $V \subset \partial_+ S\Omega$ be open and let $\tilde{V} \Subset V$. Let $\chi_V \in C_0^\infty(\partial_+ S\Omega)$ be a smooth cutoff function such that $\chi_V(x, \theta) \equiv 1$ for $(x, \theta) \in \tilde{V}$ and $\chi_V(x, \theta) \equiv 0$ for $(x, \theta) \notin V$. The boundary measurements are then given by

$$X_V f(x, \theta) = \chi_V(x, \theta)u|_{\partial_+ S\Omega}. \tag{1.2.3}$$

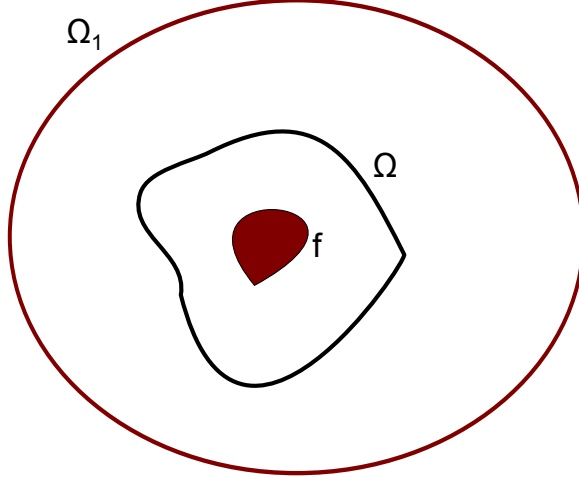


Figure 1.1: Extending Ω to a convex smooth domain Ω_1 .

To make notation a bit simpler, if $V = \partial_+ S\Omega$ (complete data) we will just write X , since in this case $X_V = X$.

When dealing with the inverse problem, which we will describe in detail in Section 2.4, we need to take a larger domain (strictly convex for convenience) that compactly contains Ω . That is, fix a strictly convex open set Ω_1 with smooth boundary such that $\Omega_1 \ni \Omega$. The strict convexity of Ω_1 ensures that the functions $\tau_{\pm}(x, \theta)$ are smooth, where $\tau_{\pm}(x, \theta)$ is the travel time from $x \in \Omega_1$ to $\partial\Omega_1$ in the direction $\pm\theta$. In other words

$$(x + \tau_{\pm}(x, \theta)\theta, \theta) \in \partial_{\pm} S\Omega_1. \quad (1.2.4)$$

We will extend σ and k to functions on Ω_1 with the same regularity. We choose and fix this extension as a continuous operator in those spaces. Now define $X_1 : L^2(\Omega_1) \rightarrow L^2(\partial_+ S\Omega_1)$ in the same way as for X . From this we can look at the restriction of X_1 applied to functions f supported in $\bar{\Omega}$ by first extending such f as zero on $\Omega_1 \setminus \Omega$. Essentially, we are moving the observation surface outward a bit and taking measurements on $\partial\Omega_1$. When dealing with the inverse problem, we will usually abuse notation and write X instead of X_1 , with the understanding that we've already extended the domain Ω to Ω_1 .

1.3 Overview of Previous Results

The direct problem of determining uniqueness and existence of solutions u to (1.2.1) and its variations thereof has been investigated quite extensively (e.g. see [14, 50, 39]). In particular, assuming (σ, k) to be an *admissible* pair as defined in [12] is more than sufficient to guarantee existence and unique of solutions. However, many results are based on the RTE not with an internal source, but instead with some boundary illumination supported on $\partial_- S\Omega$. For the time dependent linear Boltzmann equation with null source,

$$\frac{\partial}{\partial t}u(x, \theta, t) = -\theta \cdot \nabla_x u(x, \theta, t) - \sigma(x, \theta)u(x, \theta, t) + \int k(x, \theta', \theta)u(x, \theta', t) d\theta', \quad (1.3.1)$$

there is a wealth of literature dealing with the associated scattering theory ([54, 44, 39, 15]), and this has a nice connection with the so-called *albedo operator* ([10, 45]) used to recover σ and k (also see [3, 12, 11]).

Consider, for the moment, the time-independent case for (1.3.1) with null internal source where one is free to prescribe a boundary condition of the form $u|_{\partial_- S\Omega} = g$, usually with $g \in L^1(\partial_- S\Omega, d\xi)$. The albedo operator is then given by

$$\mathcal{A} : g \mapsto u|_{\partial_+ S\Omega},$$

where u solves (1.3.1) with $u|_{\partial_- S\Omega} = g$. The work of [12, 10] then aims to recover σ and k under modest assumptions by decomposing the Schwarz kernel of \mathcal{A} into singular components of different orders. It is then possible to recover σ and some information about k by choosing g to lie in different carefully chosen one-parameter families of boundary illuminations.

As for the inverse source problem, the primary, recent, existing results to the authors knowledge are from [7, 50]. The work of Bal and Tamaskan in [7] shows the recovery of a compactly supported L^2 source given that $\sigma \in C_0^2(\Omega)$ (isotropic) and $k(x, \theta', \theta) = k(x, \theta \cdot \theta')$, as well as an additional smallness condition on the Fourier transform of the Fourier coefficients of k with respect to $\theta - \theta'$. Note that such a condition on k in phase space indirectly implies its level of smoothness. A more general result is proven in (Theorem 2, [50]), which establishes that the aforementioned operator X is injective for an open,

dense set of coefficients (σ, k) as given in (Theorem 1, [50]) with $f \in L^2(\Omega)$. Moreover, a stability result is obtained for the normal operator $X^*X : L^2(\Omega) \rightarrow L^2(\Omega)$. Here the adjoint $X^* : L^2(\partial_+ S\Omega, d\Sigma) \rightarrow L^2(\Omega \times \mathbb{S}^{n-1})$ is defined with respect to the measure $d\Sigma$, which we return to in more detail later. More specifically, for an open and dense set of pairs $(\sigma, k) \in C^2(\bar{\Omega} \times \mathbb{S}^{n-1}) \times C^2(\bar{\Omega}_x \times \mathbb{S}_{\theta'}^{n-1}; C^{n+1}(\mathbb{S}_{\theta}^{n-1}))$, including a neighborhood of $(0, 0)$, we have that the conclusions of (Theorem 1, [50]) hold in Ω_1 , that X_1 is injective on $L^2(\Omega)$, and the stability estimate $\|f\|_{L^2(\Omega)} \leq C\|X_1^*X_1 f\|_{H^1(\Omega_1)}$ for a constant $C > 0$ locally uniform in (σ, k) .

We remark that the methods in [50] and [7] rely on treating the operator X (which restricts the solution to (1.2.1) to $\partial_+ S\Omega$) as a perturbation of the attenuated ray transform (see [5, 8, 16, 17, 31, 32, 33, 37] for some background on that problem). Additionally, the work of [50], and in the sequel, uses some microlocal results related to the geodesic ray transform with generic weights (see [18]). Earlier works on the inverse source problem based on different methods can be found in [2, 26, 35, 42, 43].

Chapter 2

THE CASE OF PARTIAL DATA

2.1 Introduction

In this chapter we consider the inverse source problem for the stationary radiative transfer equation where the data $u|_{\partial_+ S\Omega}$ is only known on some open subset $V \subset \partial_+ S\Omega$. What follows is a brief outline for the chapter. In Section 2.3 we will review the direct problem and some relevant results of use in the partial data case. We also establish some results about singular integrals that will be needed to prove the main theorem. In Section 2.4 we consider the inverse problem with partial data, which consists of determining the source term f from measuring $X_V f$. We also compute the normal operator $X_V^* X_V$ when the scattering coefficient $k = 0$. Note that when $\sigma = k = 0$, the operators X and X_V are the standard X-ray transforms with full and limited data, respectively. When $k \neq 0$, then X_V is more generally a weighted X-ray transform.

Following, in Section 2.5 we establish an injectivity result, Theorem 2.5.1, for $f \in L^2(\Omega)$ supported in the *visible set* assuming analytic σ . This result is based on the microlocal approach used in [18] as well as the original proof for full data in [50]. We then apply it in order to prove Theorem 2.2.1. Results needed pertaining to singular integral operators are located in appendices A and B.

2.2 Statement of the Main Result

To understand the main theorem of this chapter, we first need to define the set of points such that X_V is injective when restricted to sources supported there. This set will clearly depend on V . We also denote by $l_{x,\theta}(t)$ as the line starting at x with direction θ .

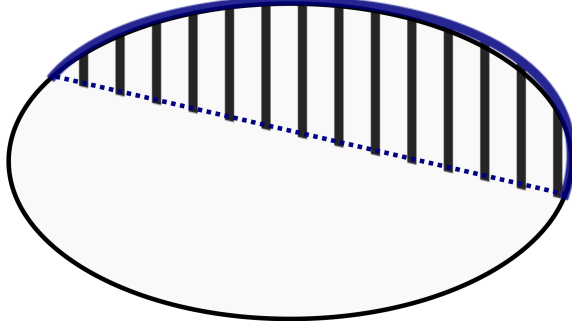


Figure 2.1: The visible set is denoted here with hashes in the case where V consists of a connected open subset of the boundary together with all outgoing directions.

Definition 1. We define the *visible set* $\mathcal{M} \subset \Omega$ by

$$\begin{aligned} \mathcal{M} = \{x \in \Omega \mid \forall \theta \in \mathbb{S}^{n-1} \exists (z, \theta^\perp) \in V \text{ with } \theta^\perp \cdot \theta = 0 \\ \text{such that } l_{z, \theta^\perp} \text{ intersects } x\}. \end{aligned} \quad (2.2.1)$$

It is relatively straightforward to show since V is open, \mathcal{M} is open as well. The proof is left to the reader.

Now we can state the main result, which is an injectivity condition adapted from results in [18].

Theorem 2.2.1. *Let $V \in \partial_+ S\Omega_1$ be an open set and let \mathcal{M} be as defined above. Let $W \Subset \mathcal{M}$. Then there exists an open and dense set of pairs*

$$(\sigma, k) \in C^2(\overline{\Omega} \times \mathbb{S}^{n-1}) \times C^2(\overline{\Omega}_x \times \mathbb{S}_{\theta'}^{n-1}; C^{n+1}(\mathbb{S}_\theta^{n-1})), \quad (2.2.2)$$

including a neighborhood of $(0, 0)$, such that for each (σ, k) in that set, the direct problem (1.2.1) has a unique solution $u \in L^2(\Omega_1 \times \mathbb{S}^{n-1})$ for any $f \in L^2(\Omega \times \mathbb{S}^{n-1})$, X_V extends to a bounded operator from $L^2(\Omega_1 \times \mathbb{S}^{n-1})$ to $L^2(\partial_+ S\Omega_1, d\Sigma)$, and

1. the map X_V is injective on $L^2(W)$,
2. the following stability estimate holds:

$$\|f\|_{L^2(\Omega)} \leq C \|X_V^* X_V f\|_{H^1(\Omega_1)}, \quad \forall f \in L^2(W), \quad (2.2.3)$$

with a constant $C > 0$ locally uniform in (σ, k) .

Remark 1. The proof of uniqueness and stability for the direct problem (1.2.1) as stated in Theorem 2.2.1 is essentially the same as the one contained in [50], so we will focus on the subtle differences. Furthermore, the proof that $X_V = \chi_V X$ extends to a bounded operator from $L^2(\Omega_1)$ to $L^2(\partial_+ S\Omega_1, d\Sigma)$ follows from the proof in [50] that X is bounded and the fact that multiplication by χ_V is bounded on $L^2(\partial_+ S\Omega_1, d\Sigma)$.

2.3 The Direct Problem

For notational convenience and to be consistent with convention, we set

$$T_0 = \theta \cdot \nabla_x, \quad T_1 = T_0 + \sigma, \quad T = T_0 + \sigma - K, \quad (2.3.1)$$

where σ denotes the operation of multiplication by $\sigma(x, \theta)$, and K is defined by

$$Kf(x, \theta) = \int_{\mathbb{S}^{n-1}} k(x, \theta, \theta') f(x, \theta') d\theta'. \quad (2.3.2)$$

If $k = 0$, we have that

$$Xf(x, \theta) = I_\sigma f(x, \theta) := \int_{\tau_-(x, \theta)}^0 E(x + t\theta, \theta) f(x + t\theta) dt, \quad (x, \theta) \in \partial_+ S\Omega, \quad (2.3.3)$$

where $\tau_\pm(x, \theta)$ is the arrival time defined by $(x + \tau_\pm(x, \theta)\theta, \theta) \in \partial_\pm S\Omega$ for $(x, \theta) \in \Omega \times \mathbb{S}^{n-1}$.

Here E is defined by

$$E(x, \theta) = \exp\left(-\int_0^\infty \sigma(x + s\theta, \theta) ds\right). \quad (2.3.4)$$

Note that if $\sigma > 0$ depends on x only, then I_σ is just the attenuated X-ray transform along the line through x in the direction θ . Moreover, in this case it is injective and [32] gives an explicit inversion formula.

In the general case with $k \neq 0$, it is shown in (Theorem 1, [50]) that the direct problem (1.2.1) is well-posed even for f depending on x and θ . That is, for an open and dense set of pairs

$$(\sigma, k) \in C^2(\overline{\Omega} \times \mathbb{S}^{n-1}) \times C^2(\overline{\Omega}_x \times \mathbb{S}_{\theta'}^{n-1}; C^{n+1}(\mathbb{S}_\theta^{n-1})),$$

including a neighborhood of $(0, 0)$, the direct problem $Tu = f$ with $u|_{\partial_- S\Omega} = 0$ has a unique solution $u \in L^2(\Omega \times \mathbb{S}^{n-1})$ for any $f \in L^2(\Omega \times \mathbb{S}^{n-1})$ depending on both x and θ .

Furthermore, the complete data operator X , which is only a priori bounded when restricted to sufficiently smooth f , extends to a bounded operator

$$X : L^2(\Omega \times \mathbb{S}^{n-1}) \rightarrow L^2(\partial_+ S\Omega, d\Sigma).$$

The proof of this relies on the fact that

$$[T_1^{-1}f](x, \theta) = \int_{-\infty}^0 \exp\left(-\int_s^0 \sigma(x + \tau\theta, \theta) d\tau\right) f(x + s\theta, \theta) ds, \quad (2.3.5)$$

as well as Fredholm Theory applied to the resolvent $(\text{Id} - (\lambda T_1^{-1}K)^2)^{-1}$ where λ is a complex parameter.

In order to solve $Tu = f$, we observe that $Tu = T_1u - Ku = f$, and so applying T_1^{-1} to both sides yields

$$u = T_1^{-1}(Ku + f). \quad (2.3.6)$$

This is equivalent to the integral equation

$$(\text{Id} - T_1^{-1}K)u = T_1^{-1}f. \quad (2.3.7)$$

Thus, if $\text{Id} - T_1^{-1}K$ is invertible, we can solve the forward problem uniquely for

$$u = T^{-1}f = (\text{Id} - T_1^{-1}K)^{-1}T_1^{-1}f. \quad (2.3.8)$$

To find k such that T^{-1} exists, we note that $(\text{Id} - T_1^{-1}K)^{-1}T_1^{-1} = T_1^{-1}(\text{Id} - KT_1^{-1})^{-1}$ and look at the operator

$$A(\lambda) = (\text{Id} - (\lambda KT_1^{-1})^2)^{-1} \quad (2.3.9)$$

It is shown in [50] that the operator $(KT_1^{-1})^2$ is compact, and for $\lambda = 0$ the resolvent (2.3.9) exists. By the analytic Fredholm theorem (Theorem VI.14, [38]), we have that $A(\lambda)$ is a meromorphic family of bounded operators with poles contained in a discrete set. It can be shown that

$$(\text{Id} - \lambda KT_1^{-1})^{-1} = (\text{Id} + \lambda KT_1^{-1}) A(\lambda). \quad (2.3.10)$$

In particular, the r.h.s above is easily seen to be a right inverse. To show that it is a left inverse as well, we can expand $A(\lambda)$ as a Neumann series for $\|KT_1^{-1}\| \ll 1$ and then use analytic continuation to show that it remains true for all λ that are not poles of $A(\lambda)$. These ideas will be useful later when proving Theorem 2.2.1.

2.4 The Inverse Source Problem with Partial Data

Let $V \subset \partial_+ S\Omega$ be some open subset. Then the boundary measurements for the problem (1.2.1) with partial data are modelled by

$$X_V f(x, \theta) := \chi_V(x, \theta) u|_{\partial_+ S\Omega}, \quad (x, \theta) \in \partial_+ S\Omega \quad (2.4.1)$$

where $u(x, \theta)$ is a solution of (1.2.1), and $\chi_V : \partial_+ S\Omega \rightarrow [0, 1]$ is a smooth function equal to 0 for $(x, \theta) \notin V$ and $\chi_V(x, \theta) = 1$ for $(x, \theta) \in \widetilde{V} \Subset V$ for some open \widetilde{V} . We also define the operator $J : L^2(\Omega) \rightarrow L^2(\Omega \times \mathbb{S}^{n-1})$ by

$$Jf(x, \theta) = f(x).$$

If $k = 0$, we have that

$$X_V f(x, \theta) = I_{\sigma, V} f(x, \theta) := \chi_V(x, \theta) I_\sigma f(x, \theta). \quad (2.4.2)$$

We will proceed as in [50] by looking at X_V as a perturbation of $I_{\sigma, V}$. Wishful thinking suggests that $X_V^* X_V$ is a relatively compact perturbation of $I_{\sigma, V}^* I_{\sigma, V}$, the normal operator corresponding to $k = 0$. Here X_V^* is the adjoint of X_V with respect to the measure $d\Sigma$ on $\partial_+ S\Omega$ given by

$$d\Sigma = |\theta \cdot \nu(x)| dS_x dS_\theta, \quad (2.4.3)$$

where as stated earlier, $\nu(x)$ is the outward unit normal to the boundary $\partial\Omega$.

Again, consider the case when $k = 0$ and compute $I_{\sigma, V}^*$. Note that $I_{\sigma, V} : L^2(\Omega \times \mathbb{S}^{n-1}) \rightarrow L^2(\partial_+ S\Omega, d\Sigma)$, and hence $I_{\sigma, V}^* : L^2(\partial_+ S\Omega, d\Sigma) \rightarrow L^2(\Omega \times \mathbb{S}^{n-1})$. For now we will restrict ourselves to applying $I_{\sigma, V}$ to functions f that depend on x only. Given $h(x, \theta) \in L^2(\partial_+ S\Omega, d\Sigma)$ and $f(x) \in L^2(\Omega)$, one can show that

$$\langle I_{\sigma, V}^* h(x), f(x) \rangle_{L^2(\Omega \times \mathbb{S}^{n-1})} = \int_{\Omega} \int_{\mathbb{S}^{n-1}} h^\#(y, \theta) \chi_V^\#(y, \theta) \overline{E}(y, \theta) \overline{f}(y) d\theta dy,$$

where $g^\#(x, \theta)$ is the extension of $g : \partial_+ S\Omega \rightarrow \mathbb{R}$ to $\Omega \times \mathbb{S}^{n-1}$ defined by $g^\#(x, \theta) = g(x + \tau_+(x, \theta)\theta, \theta)$. We also made use of the diffeomorphism $\phi : \mathcal{O} \rightarrow \Omega \times \mathbb{S}^{n-1}$ where $\mathcal{O} = \{(x, \theta, t) \mid t \in (\tau_-(x, \theta), 0) \text{ and } (x, \theta) \in \partial_+ S\Omega\}$. This map is defined by $\phi(x, \theta, t) = (x + t\theta, \theta)$. The Jacobian determinant of ϕ is $|\nu(x) \cdot \theta|$; see (Lemma 2.1, [12]). Note that

$\phi^{-1} : \Omega \times \mathbb{S}^{n-1} \rightarrow \mathcal{O}$ is given by $\phi^{-1}(x, \theta) = (x + \tau_+(x, \theta)\theta, \theta, -\tau_+(x, \theta))$. Hence the adjoint in the no-scattering case has the equation

$$I_{\sigma, V}^* h(x, \theta) = \int_{\mathbb{S}^{n-1}} \overline{E}(x, \theta) h^\#(x, \theta) \chi_V^\#(x, \theta) d\theta. \quad (2.4.4)$$

2.4.1 The Normal Operator $I_{\sigma, V}^* I_{\sigma, V}$

Similar to the way in which we derived the adjoint operator $I_{\sigma, V}^*$, we may compute the normal operator $I_{\sigma, V}^* I_{\sigma, V} : L^2(\Omega) \rightarrow L^2(\Omega)$ as

$$\begin{aligned} & \langle I_{\sigma, V}^* I_{\sigma, V} f(x), g(x) \rangle_{L^2(\Omega)} \\ &= \int_{\Omega} \left[\int_{\mathbb{S}^{n-1}} \overline{E}(x, \theta) \chi_V^\#(x, \theta) (I_{\sigma, V} f(x, \theta))^\# d\theta \right] \overline{g}(x) dx \\ &= \int_{\Omega} \left[\int_{\mathbb{S}^{n-1}} \overline{E}(x, \theta) \chi_V^\#(x, \theta) \left(\chi_V(x, \theta) \int_{\mathbb{R}} E(x + t\theta, \theta) f(x + t\theta) dt \right)^\# \right] \overline{g}(x) d\theta dx \quad (2.4.5) \\ &= \int_{\Omega} \int_{\Omega} \frac{\overline{E}(x, \frac{y-x}{|y-x|}) \left| \chi_V^\#(x, \frac{y-x}{|y-x|}) \right|^2 E(y, \frac{y-x}{|y-x|}) f(y)}{|y-x|^{n-1}} \overline{g}(x) dy dx. \end{aligned}$$

In the last line we used the substitution $y = x + t\theta$ to convert the integral over $\mathbb{S}^{n-1} \times \mathbb{R}$ in (θ, t) to an integral over Ω in y . Thus

$$I_{\sigma, V}^* I_{\sigma, V} f(x) = \int_{\Omega} \frac{E(y, \frac{y-x}{|y-x|}) \overline{E}(x, \frac{y-x}{|y-x|}) \left| \chi_V^\#(x, \frac{y-x}{|y-x|}) \right|^2 f(y)}{|y-x|^{n-1}} dy. \quad (2.4.6)$$

In the case that σ is C^∞ , we would like to know where $I_{\sigma, V}^* I_{\sigma, V}$ is elliptic. One can show using (Theorem 3.4, [20]) that

$$I_{\sigma, V}^* I_{\sigma, V} f(x) = (2\pi)^{-n} \int e^{i(x-y) \cdot \xi} b(x, \xi) f(y) dy d\xi \quad (2.4.7)$$

where

$$b(x, \xi) = (2\pi)^{-n} \int e^{-i(x-y) \cdot \xi} \frac{E(y, \frac{y-x}{|y-x|}) \overline{E}(x, \frac{y-x}{|y-x|}) \left| \chi_V^\#(x, \frac{y-x}{|y-x|}) \right|^2}{|y-x|^{n-1}} dy. \quad (2.4.8)$$

Proposition 1 of [50] then gives that $I_{\sigma, V}^* I_{\sigma, V} : L^2(\Omega) \rightarrow H^1(\Omega)$. See also [51].

Now, unfortunately equation (2.4.8) isn't particularly useful when trying to analyze the singularities of $I_{\sigma, V}^* I_{\sigma, V} f$, even if σ is C^∞ . But recall (2.4.5), which shows that

$$I_{\sigma, V}^* I_{\sigma, V} f(x) = \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} A(x, t, \theta) f(x + t\theta) dt d\theta, \quad (2.4.9)$$

for a particular function A . By Lemma 4.2 of [18] we have that if $A \in C^\infty(\Omega_1 \times \mathbb{R} \times \mathbb{S}^{n-1})$ (which occurs if $\sigma \in C^\infty(\Omega_1 \times \mathbb{S}^{n-1})$), then $I_{\sigma,V}^* I_{\sigma,V}$ is a classical Ψ DO of order -1 with full symbol

$$b(x, \xi) \sim \sum_{m=0}^{\infty} b_m(x, \xi), \quad (2.4.10)$$

where

$$b_m(x, \xi) = 2\pi \frac{i^m}{m!} \int_{\mathbb{S}^{n-1}} \partial_t^m A(x, 0, \theta) \delta^{(m)}(\theta \cdot \xi) d\theta. \quad (2.4.11)$$

To check for ellipticity of b , we need only look at the principal symbol corresponding to when $m = 0$. This is just

$$b_0(x, \xi) = 2\pi \int_{\theta \cdot \xi = 0} |E(x, \theta)|^2 |\chi_V^\#(x, \theta)|^2 dS(\theta). \quad (2.4.12)$$

Since E is nonvanishing, we immediately have by (2.4.12) that $b(x, \xi)$ is elliptic on the set \mathcal{M}' .

2.5 Injectivity of X_V Restricted to the Visible Set

2.5.1 Injectivity of $I_{\sigma,V}$ and $I_{\sigma,V}^* I_{\sigma,V}$

Since we are only able to access some open subset V of $\partial_+ S\Omega_1$, we cannot expect for the operator $I_{\sigma,V}$ or the normal operator $I_{\sigma,V}^* I_{\sigma,V}$ to be injective. However, from [18] we can obtain injectivity for sources f supported in a particular subset of Ω . But first we must introduce the notion of a regular family of curves. We will use the notation $l_{x,\theta}$ to denote the line segment through $x \in \Omega$ in the direction $\theta \in \mathbb{S}^{n-1}$ with endpoints on $\partial\Omega_1$. We can also assume that $l_{x,\theta}(0) = x$ and $l'_{x,\theta}(0) = \theta$. It is also clear that the lines $l_{x,\theta}$ depend smoothly on $(x, \theta) \in T\Omega$ in the sense that the function $l(x, \theta, t) = l_{x,\theta}(t)$ depends smoothly on x, θ and t separately. In fact, we have $l(x, \xi, t) = x + t\xi$ where $t \in (a(x, \xi), b(x, \xi))$, an interval containing 0, and $l(x, \xi, a(x, \xi)), l(x, \xi, b(x, \xi)) \in \partial\Omega_1$.

Definition 2. Let Γ be an open family of smooth (oriented) curves on Ω , with a fixed parametrization on each one of them, with endpoints on $\partial\Omega$, such that for each $(x, \xi) \in T\Omega \setminus 0$, there is at most one curve $\gamma_{x,\xi} \in \Gamma$ through x in the direction ξ , and the dependence on (x, ξ) is smooth. We say that Γ is a *regular* family of curves, if for any $(x, \xi) \in T^*\Omega$, there exists $\gamma \in \Gamma$ through x normal to ξ without conjugate points.

Remark 2. In our specific case, all curves taken are straight lines and have no conjugate points. Moreover, if we let $\Gamma_{\mathcal{M}}$ be the set of lines in Ω_1 which intersect \mathcal{M} , then it turns out (as is shown in the proof of Theorem 2.5.1) that $\Gamma_{\mathcal{M}}$ is a regular family when restricted to \mathcal{M} . This is the motivation behind how \mathcal{M} was defined in the first place.

Theorem 2.5.1. *Let σ be analytic on Ω . If $I_{\sigma,V}f = 0$ for $f \in \mathcal{D}'(\Omega_1)$ supported in $W \Subset \mathcal{M}^{\text{int}}$, then $f = 0$. In particular, $I_{\sigma,V}$ is injective on $L^1(W)$.*

Proof. It is clear that the collection Γ of lines in Ω_1 is an analytic regular family of curves (see [18]). Let $\Gamma_{\mathcal{M}}$ be only those lines which pass through \mathcal{M} . We claim that $\Gamma_{\mathcal{M}}$ is a regular family of curves when restricted to \mathcal{M} . To see this, let $x \in \mathcal{M}$ and $\theta \in \mathbb{S}^{n-1}$. Then by definition of \mathcal{M} there exists $z \in V$ and an angle θ^\perp normal to θ such that the line l_{z,θ^\perp} passes through x and $(z, \theta^\perp) \in \partial_+ S\Omega$. We also have by definition that $l_{z,\theta^\perp} \in \Gamma_{\mathcal{M}}$, which proves the claim. Now, by Theorem 1 of [18] we have that f is analytic on \mathcal{M} with support properly contained in \mathcal{M}^{int} . In particular, $f = 0$ on an open subset of each component of \mathcal{M} . Therefore $f = 0$. \square

Although the definition of \mathcal{M} is a bit cryptic and difficult to visualize, it is possible to easily visualize an important subset of \mathcal{M} when V has a certain form, as shown by Lemma 2.5.2. Here we use the notation $\text{ch}A$ to denote the closed convex hull of a set $A \subset \mathbb{R}^n$.

Lemma 2.5.2. *Suppose that $V = \pi^{-1}(W)$ where W consists of a countable collection of disjoint connected open subsets of $\partial\Omega_1$, and $\pi : \partial_+ S\Omega_1 \rightarrow \partial\Omega_1$ is the natural projection. Then $\bigcup_j (\text{ch}W_j)^{\text{int}} \subset \mathcal{M}$ where W_j is a given component of W .*

Proof. Suppose $W = \bigcup_\alpha W_\alpha$ where $W_\alpha \subset \partial\Omega$ are disjoint connected open sets. Let $x \in \bigcup_\alpha (\text{ch}W_\alpha)^{\text{int}}$. Let $\theta \in \mathbb{S}^{n-1}$ and let θ^\perp be any vector perpendicular to θ . If we consider that

$$\text{ch}W_\alpha = \overline{\{\text{hyperplanes } P \subset \mathbb{R}^n \mid P \cap W_\alpha = \emptyset\}^c},$$

then l_{x,θ^\perp} must intersect W_α at some point z . Changing the direction of θ^\perp if necessary and using the strict convexity of Ω_1 , we have that $(z, \theta^\perp) \in \partial_+ S\Omega_1$. This proves that $(\text{ch}W_\alpha)^{\text{int}} \subset \mathcal{M}$ for all α . \square

2.5.2 Computing X_V as a perturbation of $I_{\sigma,V}$ for $k \neq 0$

In order to approach the case that $k \neq 0$, we will compute explicitly how X_V differs from $I_{\sigma,V}$. Note that

$$Xf = \chi_V R_+ T^{-1} f = \chi_V R_+ (\text{Id} - T_1^{-1} K)^{-1} T_1^{-1} f, \quad (2.5.1)$$

where

$$R_+ h = h|_{\partial_+ S\Omega}.$$

If f depends on x only (the case we are primarily interested in), then

$$X_V f = \chi_V R_+ T^{-1} J f = \chi_V R_+ (\text{Id} - T_1^{-1} K)^{-1} T_1^{-1} J f. \quad (2.5.2)$$

Now consider the identity

$$(\text{Id} - T_1^{-1} K)^{-1} T_1^{-1} = T_1^{-1} (\text{Id} - K T_1^{-1})^{-1}, \quad (2.5.3)$$

which implies that

$$X_V f = \chi_V R_+ T_1^{-1} (\text{Id} - K T_1^{-1})^{-1} J f. \quad (2.5.4)$$

Writing $X_V = I_{\sigma,V} + L_V$ and noting that

$$I_{\sigma,V} f = \chi_V R_+ T_1^{-1} J f,$$

we have that

$$X_V = I_{\sigma,V} + \chi_V R_+ (-\text{Id} + (\text{Id} - T_1^{-1} K)^{-1}) T_1^{-1} J. \quad (2.5.5)$$

and so we have

$$L_V := \chi_V R_+ (-\text{Id} + (\text{Id} - T_1^{-1} K)^{-1}) T_1^{-1} J. \quad (2.5.6)$$

Furthermore,

$$X_V^* X_V = I_{\sigma,V}^* I_{\sigma,V} + \mathcal{L}_V, \quad \mathcal{L}_V := I_{\sigma,V}^* L_V + L_V^* I_{\sigma,V} + L_V^* L_V. \quad (2.5.7)$$

Lemma 2.5.3. *The operators*

$$\partial_x I_{\sigma,V}^* L_V, \quad \partial_x L_V^* I_{\sigma,V}, \quad \partial_x L_V^* L_V$$

are compact as operators mapping $L^2(\Omega_1)$ into $L^2(\Omega_1)$.

Proof. Following the steps of the proof of Lemma 3 in [50], first note that

$$(-\text{Id} + (\text{Id} - T_1^{-1}K)^{-1})T_1^{-1} = T_1^{-1}KT_1^{-1}(\text{Id} - KT_1^{-1})^{-1}. \quad (2.5.8)$$

To prove it, we note that

$$T_1^{-1}KT_1^{-1} = (-\text{Id} + (\text{Id} - T_1^{-1}K)^{-1})T_1^{-1}(\text{Id} - KT_1^{-1}).$$

Thus L_V can be written as

$$L_V = \chi_V R_+ T_1^{-1} K T_1^{-1} (\text{Id} - K T_1^{-1})^{-1} J. \quad (2.5.9)$$

We note that multiplication by χ_V to obtain L_V from L is bounded and hence preserves compactness. First we need to analyze $I_{\sigma,V}^* L_V = I_{\sigma,V}^* \chi_V R_+ T_1^{-1} K T_1^{-1} h$, where $h = h(x, \theta)$. Recall that

$$[I_{\sigma,V}^* h](x) = \int_{\mathbb{S}^{n-1}} \overline{E}(x, \theta) h^\#(x, \theta) \chi_V^\#(x, \theta) d\theta.$$

Again, as in [50] we notice that $\chi_V R_+ T_1^{-1} g$ looks like $I_{\sigma,V}$, except that now the source depends on θ and x . Thus

$$\begin{aligned} [I_{\sigma,V}^* \chi_V R_+ T_1^{-1} g](x) &= \int_{\mathbb{S}^{n-1}} \overline{E}(x, \theta) \left[\chi_V(x, \theta) \int_{-\infty}^0 E(x + t\theta, \theta) g(x + t\theta, \theta) dt \right]^\# d\theta \\ &= 2 \int_{\Omega_1} \frac{\left[\overline{E}\left(x, \frac{y-x}{|y-x|}\right) \chi_V^\#\left(y, \frac{y-x}{|y-x|}\right) E\left(y, \frac{y-x}{|y-x|}\right) g\left(y, \frac{y-x}{|y-x|}\right) \right]_{\text{even}}}{|x-y|^{n-1}} dy, \end{aligned} \quad (2.5.10)$$

where $F_{\text{even}}(x, \theta)$ is the even part of F with respect to θ (i.e. $F_{\text{even}}(x, \theta) = \frac{1}{2}(F(x, \theta) + F(x, -\theta))$). To get back to $I_{\sigma,V}^* L_V$, we can let $g = K T_1^{-1} h$.

To proceed, we will now make a slightly weaker assumption on k than stated in (2.2.2) (see [50]). We will assume that k can be written as the infinite sum

$$k(x, \theta, \theta') = \sum_{j=1}^{\infty} \Theta_j(\theta) \kappa_j(x, \theta') \quad (2.5.11)$$

where Θ_j and κ_j are functions such that

$$\sum_{j=1}^{\infty} \|\Theta_j\|_{H^1(\mathbb{S}^{n-1})} \|\kappa_j\|_{L^\infty(\Omega_1 \times \mathbb{S}^{n-1})} < \infty \quad (2.5.12)$$

In particular, we could take Θ_j to be the spherical harmonics Y_j , and then κ_j would be the corresponding Fourier coefficients in such a basis. As discussed in [50], uniform convergence of (2.5.11) is guaranteed if $k \in L^\infty(\Omega_1 \times \mathbb{S}_{\theta'}^{n-1}; C_\theta^{m+1}(\mathbb{S}^{n-1}))$, which is indeed a weaker assumption.

Now let K_j be the integral operator with kernel $\Theta_j \kappa_j$ and $B_j = \kappa_j T_1^{-1}$, where we regard κ_j as integration in θ' against the kernel κ_j . Thus,

$$[K_j T_1^{-1} h](x, \theta) = \Theta_j(\theta) [B_j h](x), \quad (2.5.13)$$

$$B_j h(x) = \int_{\Omega_1} \frac{\Sigma\left(x, |x-y|, \frac{x-y}{|x-y|}\right) \kappa_j\left(x, \frac{x-y}{|x-y|}\right)}{|x-y|^{n-1}} h\left(y, \frac{x-y}{|x-y|}\right) dy. \quad (2.5.14)$$

By the proof of Lemma 1 in [50], we have that $B_j(\text{Id} - K T_1^{-1})^{-1} J : L^2(\Omega_1) \rightarrow L^2(\Omega_1)$ is compact. Now observe that

$$\begin{aligned} \partial_x I_{\sigma, V}^* L_V &= \partial_x I_{\sigma, V}^* \chi_V R_+ T_1^{-1} K T_1^{-1} (\text{Id} - K T_1^{-1})^{-1} J \\ &= \sum_{j=1}^{\infty} [\partial_x I_{\sigma, V}^* \chi_V R_+ T_1^{-1} \Theta_j J] [B_j (\text{Id} - K T_1^{-1})^{-1} J] \end{aligned} \quad (2.5.15)$$

By (2.5.10) and Proposition 1(b) of [50], we have that $\partial_x I_{\sigma, V}^* \chi_V R_+ T_1^{-1} \Theta_j J : L^2(\Omega_1) \rightarrow L^2(\Omega_1)$ is bounded with a norm bounded above by $C \|\sigma\|_{C^2(\bar{\Omega} \times \mathbb{S}^{n-1})} \|\Theta_j\|_{H^1(\mathbb{S}^{n-1})}$. Thus each summand of (2.5.15) is a compact operator with norm bounded above by $C \|\Theta_j\|_{H^1} \|\kappa_j\|_{L^\infty}$, with C depending on σ . By the condition (2.5.12), we have that $\partial_x I_{\sigma, V}^* L_V$ is compact.

Now, the proof for $\partial_x L_V^* L_V$ is similar. In light of the fact that $B_j(\text{Id} - K T_1^{-1})^{-1} J$ is compact, it suffices to show that $\partial_x L_V^* \chi_V R_+ T_1^{-1} J$ is bounded. Note that $K T_1^{-1}$ commutes with $(\text{Id} - K T_1^{-1})^{-1}$, and hence

$$\begin{aligned} L_V^* \chi_V R_+ T_1^{-1} \Theta_j J &= (\chi_V R_+ T_1^{-1} K T_1^{-1} (\text{Id} - K T_1^{-1})^{-1} J)^* \chi_V R_+ T_1^{-1} \Theta_j J \\ &= (K T_1^{-1} J)^* (\chi_V R_+ T_1^{-1} (\text{Id} - K T_1^{-1})^{-1})^* \chi_V R_+ T_1^{-1} J. \end{aligned} \quad (2.5.16)$$

As proven in [50], by the boundedness of $\chi_V R_+ T_1^{-1}$, the compactness of $\partial_x L_V^* \chi_V R_+ T_1^{-1} J$ relies on $\partial_x (K T_1^{-1} J)^*$, and indeed it is.

Finally, to show that $\partial L_V^* I_{\sigma, V}$ is compact, we can proceed similarly to the case of $\partial L_V^* L_V$. Observe that $\partial_x L_V^* I_{\sigma, V} = L_V^* \chi_V R_+ T_1^{-1} J$, which is equivalent to (2.5.16) with $\Theta_j = 1$. \square

Now we are ready to prove Theorem 2.2.1 regarding the injectivity of X_V when restricted to sources f supported compactly in the visible set \mathcal{M} .

Proof of Theorem 2.2.1. Our proof mostly parallels the proof of Theorem 2 in [50]. By Lemma 2.5.3, we have that $X_V^* X_V$ is equal to $I_{\sigma,V}^* I_{\sigma,V}$ plus a relative compact operator \mathcal{L}_V . First assume that σ and k are C^∞ . In this case, $I_{\sigma,V}^* I_{\sigma,V}$ is elliptic on \mathcal{M} , and thus there is a parametrix Q of order 1 which we view as an operator $Q : H^1(\Omega_1) \rightarrow L^2(\Omega)$ (We've restricted the image to Ω , though \mathcal{M} would do based on our assumption on the support of f). Thus, for f supported in $W \Subset \mathcal{M}$, we have

$$Q I_{\sigma,V}^* I_{\sigma,V} f = f + K_1 f, \quad (2.5.17)$$

where K_1 is of order -1 near \mathcal{M} . Now apply Q to $X_V^* X_V$ to get

$$Q X_V^* X_V f = f + K_1 f + Q \mathcal{L}_V f =: f + K_2 f. \quad (2.5.18)$$

By Lemma 2.5.3, we have that $Q \mathcal{L}_V$ is compact. Furthermore, $K_1 : L^2(\mathcal{M}) \rightarrow L^2(\mathcal{M})$ is compact by Rellich's lemma since it is smoothing near \mathcal{M} . This reduces the problem of inverting $X_V^* X_V$ to a Fredholm equation. By Theorem 2.5.1, we have that for σ real analytic on $\overline{\Omega} \times (\mathbb{S}^{n-1})$, $I_{\sigma,V}$ is injective when restricted to f supported in $W \Subset \mathcal{M}$. Therefore, $I_{\sigma,V}^* I_{\sigma,V}$ is as well by an integration by parts. That is, if $\langle I_{\sigma,V}^* I_{\sigma,V} f, g \rangle_{L^2(\Omega)} = 0$ for all $g \in L^2(\Omega)$, then in particular it holds for $g = f$, and hence $\|I_{\sigma,V} f\|_{L^2(\Omega)}^2 = 0$. In turn, we obtain that $f = 0$. In fact, by the remark after (Theorem 2, [18]) injectivity holds for small enough C^1 perturbations.

Now fix σ real analytic in $\overline{\mathcal{M}}$. Also fix k so that (σ, k) belongs to the generic set in (Theorem 1, [50]), related to Ω_1 , and the regularity assumption (2.2.2) is satisfied. This can be done for an open dense set of k 's by (Theorem 1, [50]). Now consider X_V related to $(\sigma, \lambda k)$ with λ belonging to some complex neighborhood \mathcal{C} of $[0, 1]$. The operator K_2 in (2.5.18) depends meromorphically on λ . This is because K_1 is related to $(\sigma, 0)$ (i.e. the unperturbed case) and is therefore independent of λ . Furthermore, the parametrix Q is also independent of λ since it is also related to the unperturbed operator $I_{\sigma,V}$. Finally, the remainder term \mathcal{L}_V is a meromorphic function of λ because L_V has that property, see (2.3.10) and (2.5.6). For $\lambda = 0$, we have $\mathcal{L}_V = 0$ and hence $K_2 = K_1$ in this case.

From the proof of Theorem 1.5 in [49], we can add a finite rank operator to Q to ensure that $\text{Id} + K_1$ is injective, see (2.5.17). We include the details here for clarity. Since K_1 is compact on $L^2(W)$, we have that $\text{Id} + K_1$ is a Fredholm operator, and hence it has a finite dimensional kernel \mathcal{F} . Let $\{f_1, \dots, f_l\}$ be a basis in it. Since $I_{\sigma,V}^* I_{\sigma,V}$ is injective on $L^2(W)$, we can choose this basis such that $\{I_{\sigma,V}^* I_{\sigma,V} f_1, \dots, I_{\sigma,V}^* I_{\sigma,V} f_l\}$ is an orthonormal basis for $I_{\sigma,V}^* I_{\sigma,V} \mathcal{F} \subset L^2(\Omega_1)$. Now we define the finite rank operator $Q_0 : L^2(\Omega_1) \rightarrow L^2(\Omega)$ by

$$Q_0 h = \sum_{j=1}^l \langle h, I_{\sigma,V}^* I_{\sigma,V} f_j \rangle_{L^2(\Omega_1)} f_j. \quad (2.5.19)$$

Then set $\tilde{Q} = Q + Q_0$ so that

$$\tilde{Q} I_{\sigma,V}^* I_{\sigma,V} f = (\text{Id} + K_1^\#) f \quad (2.5.20)$$

where $K_1^\# = K_1 + Q_0 I_{\sigma,V}^* I_{\sigma,V}$ is compact. We claim that $\text{Id} + K_1^\#$ is injective on $L^2(W)$. So suppose that $(\text{Id} + K_1) f + Q_0 I_{\sigma,V}^* I_{\sigma,V} f = 0$. Note that K_1 is self-adjoint since $I_{\sigma,V}^* I_{\sigma,V}$ and hence Q are. Thus the term $(\text{Id} + K_1) f$ is orthogonal to \mathcal{F} , because for any $g \in \mathcal{F}$ we have $\langle (\text{Id} + K_1) f, g \rangle_{L^2(\Omega_1)} = \langle f, (\text{Id} + K_1) g \rangle_{L^2(\Omega_1)} = 0$. Moreover, the term $Q_0 I_{\sigma,V}^* I_{\sigma,V} f \in \mathcal{F}$ by definition of Q_0 . Therefore, $f = 0$. Relabeling \tilde{Q} as Q , we can now assume that $\text{Id} + K_1$ is injective.

Let $\tilde{\mathcal{C}}$ be \mathcal{C} with the poles of $(\text{Id} - \lambda K T_1^{-1})^{-1}$ removed, and recall that the poles form a discrete set. Thus $\tilde{\mathcal{C}}$ is a connected set containing $\lambda = 0$ and $\lambda = 1$. Applying the analytic Fredholm theorem again on $\tilde{\mathcal{C}}$ implies that $Q X_V^* X_V$ is invertible for all λ in $\tilde{\mathcal{C}}$ with the possible exception of a discrete set. In particular, there are λ 's as close to 1 as we like with that property. For these values of λ , the invertibility of $Q X_V^* X_V$ implies that $X_V^* X_V$ and X_V are both injective. This shows that there is a dense set of pairs (σ, k) in the space (2.2.2) such that X_V is injective, which we will denote by \mathcal{U} . Since we need an open dense set of pairs (σ, k) , it remains to show that X_V is still injective for (σ, k) in some neighborhood of \mathcal{U} .

Let $(\sigma, k) \in \mathcal{U}$ so that $X_V : L^2(W) \rightarrow L^2(\partial\Omega_1, d\Sigma)$ is injective. An integration by parts can be used to show that $X_V^* X_V : L^2(W) \rightarrow H^1(\Omega_1)$ is injective as well.

To proceed, we will need the following lemma from [49], and for convenience the proof is repeated.

Lemma 2.5.4 (Lemma 2, [49]). *Let X , Y , and Z be Banach spaces, let $A : X \rightarrow Y$ be a closed linear operator with domain $\mathcal{D}(A)$, and let $K : X \rightarrow Z$ be a compact linear operator. Suppose that*

$$\|f\|_X \leq C(\|Af\|_Y + \|Kf\|_Z), \quad \forall f \in \mathcal{D}(A). \quad (2.5.21)$$

If A is injective, then

$$\|f\|_X \leq C'\|Af\|_Y, \quad \forall f \in \mathcal{D}(A).$$

Proof. We show first that one can assume that A is bounded. Indeed, let $\|\cdot\|_{\mathcal{D}(A)}$ denote the graph norm given by

$$\|f\|_{\mathcal{D}(A)} = \sqrt{\|f\|_X^2 + \|Af\|_Y^2}.$$

Note that $A : X \rightarrow Y$ with the norm $\|\cdot\|_{\mathcal{D}(A)}$ on X is bounded since $\|Af\|_Y \leq \sqrt{\|f\|_X^2 + \|Af\|_Y^2} = \|f\|_{\mathcal{D}(A)}$. Furthermore, (2.5.21) implies that

$$\|f\|_{\mathcal{D}(A)} \leq C(\|Af\|_Y + \|Kf\|_Z), \quad \forall f \in \mathcal{D}(A).$$

Assuming the lemma is true for bounded operators, we then have $\|f\|_{\mathcal{D}(A)} \leq C\|Af\|_Y$. Since $\|f\|_X \leq \|f\|_{\mathcal{D}(A)}$ the result follows.

For bounded A , assume on the contrary that such an estimate does not hold. Then there exists a sequence f_n in X with $\|f_n\|_X = 1$ and $Af_n \rightarrow 0$ in Y . Since $K : X \rightarrow Z$ is compact, there exists a subsequence, which we still denote by f_n , such that Kf_n converges in Z , and is therefore a Cauchy sequence in Z . Applying (2.5.21) to $f_n - f_m$, we have that $\|f_n - f_m\|_X \rightarrow 0$ as $n, m \rightarrow \infty$. That is, f_n is a Cauchy sequence in X . Therefore, there exists $f \in X$ such that $f_n \rightarrow f$, which implies that $\|f\|_X = 1$. Since A is closed, we have $Af_n \rightarrow Af = 0$, which contradicts the injectivity of A . \square

Now observe that

$$\begin{aligned} \|QX_V^*X_V f\|_{L^2(\Omega_1)} &= \|f + K_2 f\|_{L^2(\Omega_1)} \\ &\geq \|f\|_{L^2(\Omega_1)} - \|K_2 f\|_{L^2(\Omega_1)} \\ &= \|f\|_{L^2(W)} - \|K_2 f\|_{L^2(\Omega_1)}, \end{aligned} \quad (2.5.22)$$

and hence by elliptic regularity of Q

$$\begin{aligned} \|f\|_{L^2(W)} &\leq \|QX_V^*X_V\|_{L^2(\Omega_1)} + \|K_2f\|_{L^2(\Omega_1)} \\ &\leq C(\|X_V^*X_Vf\|_{H^1(\Omega_1)} + \|K_2f\|_{L^2(\Omega_1)}). \end{aligned} \quad (2.5.23)$$

Since $X_V^*X_V$ is closed, linear and injective on $L^2(W)$ and K_2 is compact, by (Lemma 2, [49]) we have that

$$\|f\|_{L^2(W)} \leq C'\|X_V^*X_Vf\|_{H^1(\Omega_1)}. \quad (2.5.24)$$

So (2.2.3) holds for $(\sigma, k) \in \mathcal{U}$. Furthermore, this estimate implies that the norm $\|X_V^*X_V\|_{L^2(W) \rightarrow H^1(\Omega_1)}$ depends continuously on (σ, k) as in (2.2.2), and certainly for C^∞ σ and k . Therefore, we can perturb (σ, k) and (2.2.3) will still hold true, since the r.h.s will be absorbed by the l.h.s. Thus $X_V^*X_V$ is injective for (σ, k) in an open dense subset of the generic set of pairs for which the direct problem is uniquely solvable (see Theorem 1 of [50]), and for such (σ, k) the estimate (2.2.3) holds. On the other hand, $X_V^*X_V$ being injective implies that X_V is injective. Finally, for a given $(\sigma_0, k_0) \in \mathcal{U}$, we can take C' as the supremum over all constants such that (2.5.24) holds for (σ, k) in a neighborhood of (σ_0, k_0) . That is, C' is locally uniform. This completes the proof of Theorem 2.2.1. From this point, the proof follows the same as that for Theorem 2 of [50], and so we conclude. \square

Chapter 3

MICROLOCAL RECOVERY OF A SOURCE

3.1 Introduction

Although injectivity of the data operator X_V is too much to ask for when V is some proper open subset of $\partial_+ S\Omega$, it is still possible to analyze how singularities are propagated under the normal operator $I_{\sigma,V}^* I_{\sigma,V}$, see (2.4.5). This draws from the theory of pseudodifferential operators and how singularities are propagated (see [20, 53]). Writing $X_V^* X_V = I_{\sigma,V}^* I_{\sigma,V} + \mathcal{L}_V$ and assuming suitable smoothness conditions on k , we will prove that one can partially recover the wavefront set of f from $X_V^* X_V$. This is somewhat analogous to Proposition 1 in [18], which allows one to partially recover the analytic wave front set of f when $k = 0$. Of course, much care must be taken to make sense of and analyze \mathcal{L}_V .

3.2 Statement of Main Result

For sources f with more general supports, we hope to be able to recover certain covectors in the wavefront set of f . Those covectors $(x, \xi) \in T^*\Omega$ that can be detected will depend on V in the following way:

Definition 3. The *microlocally visible set* corresponding to partial measurements on $\partial_+ S\Omega_1$ is given by

$$\mathcal{M}' := \{(x, \xi) \in T^*\Omega \mid \exists \theta \in \mathbb{S}^{n-1} \text{ such that } \theta \cdot \xi = 0 \text{ and } \chi_V^\#(x, \theta) \neq 0\}. \quad (3.2.1)$$

Recall $\chi_V^\#(x, \theta)$ is the extension of $\chi_V : \partial_+ S\Omega_1 \rightarrow \mathbb{R}$ to $\Omega_1 \times \mathbb{S}^{n-1}$ defined by $\chi_V^\#(x, \theta) = \chi_V(x + \tau_+(x, \theta)\theta, \theta)$.

Theorem 3.2.1. *Let l be a positive integer. There exists an open dense set \mathcal{O}_l of pairs $(\sigma, k) \in C^\infty(\overline{\Omega} \times \mathbb{S}^{n-1}) \times C^\infty(\overline{\Omega}_x \times \mathbb{S}_{\theta'}^{n-1} \times \mathbb{S}_\theta^{n-1})$ depending on l such that given $(\sigma, k) \in \mathcal{O}_l$, if $(z, \xi) \in \mathcal{M}'$, then there exists a function $v \in H^l(\Omega)$ such that*

$$(z, \xi) \notin WF(X_V^* X_V f) \implies (z, \xi) \notin WF(f + v). \quad (3.2.2)$$

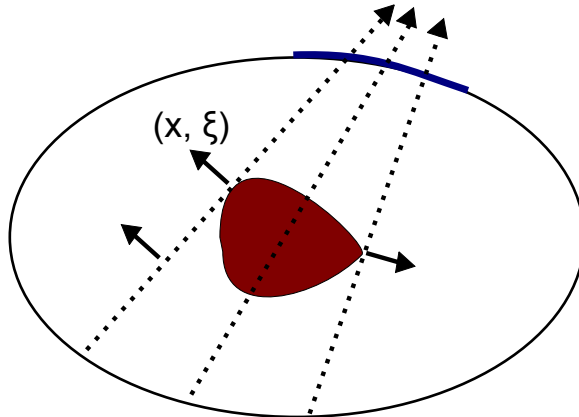


Figure 3.1: An example where (x, ξ) is in the microlocally visible set \mathcal{M}' , given that the source f is the characteristic function of the shaded set.

Corollary 3.2.2. *Suppose that $\|KT_1^{-1}\| < 1$. Then in Theorem 3.2.1 we have $v = 0$.*

3.3 A Microlocal Result

To get more of an intuition for how to proceed, consider the case when $\|T_1^{-1}K\| < 1$. Since $X_V = \chi_V R_+ T_1^{-1} (\text{Id} - KT_1^{-1})^{-1} J$, we may use a Neumann series expansion to get

$$X_V = \chi_V R_+ T_1^{-1} \sum_{j=0}^{\infty} (KT_1^{-1})^j J. \quad (3.3.1)$$

The term corresponding to $j = 0$ is exactly $I_{\sigma, V}$, which does not account for scattering. Subsequent terms in the expansion incorporate scattering of higher and higher orders. So if we can show that KT_1^{-1} is smoothing in some sense, then the most singular part of the data will be captured in the ballistic term.

Before proceeding, we need to establish a bit of notation. We define the space $\mathcal{H}_l(\Omega \times \mathbb{S}^{n-1})$ as the completion of $C^\infty(\Omega \times \mathbb{S}^{n-1})$ with respect to the norm $\|\cdot\|_{\mathcal{H}_l(\Omega \times \mathbb{S}^{n-1})}$ given by

$$\|g(x, \theta)\|_{\mathcal{H}_l(\Omega \times \mathbb{S}^{n-1})} := \sum_{m=0}^{\infty} \sum_{k=1}^{k_{m,n}} \|a_m^{(k)}\|_{H^l(\Omega \times \mathbb{S}^{n-1})} \|Y_{m,n}^{(k)}\|_{H^1(\mathbb{S}^{n-1})}, \quad (3.3.2)$$

where $g(x, \theta) = \sum_{m=0}^{\infty} \sum_{k=1}^{k_{m,n}} a_m^{(k)}(x) Y_{m,n}^{(k)}(\theta)$ is the series representation of g with respect to the spherical harmonics $Y_{m,n}^{(k)}(\theta)$ (see Appendix A). Similarly, we define the space $\mathcal{C}_l(\Omega \times$

\mathbb{S}^{n-1}) as the completion of $C^\infty(\Omega \times \mathbb{S}^{n-1})$ with respect to the norm $\|\cdot\|_{C_l(\Omega \times \mathbb{S}^{n-1})}$ given by

$$\|g(x, \theta)\|_{C_l(\Omega \times \mathbb{S}^{n-1})} := \sum_{m=0}^{\infty} \sum_{k=1}^{k_{m,n}} \|a_m^{(k)}\|_{C^l(\Omega)} \|Y_{m,n}^{(k)}\|_{H^1(\mathbb{S}^{n-1})}. \quad (3.3.3)$$

The following lemma establishes the regularizing properties of the operator KT_1^{-1} .

Lemma 3.3.1. *Let $f \in \mathcal{H}_l(\Omega_1 \times \mathbb{S}^{n-1})$ with $\text{supp}(f) \subseteq \Omega \times \mathbb{S}^{n-1}$ and $l \geq 0$, and suppose that $\sigma \in C^\infty(\overline{\Omega} \times \mathbb{S}^{n-1})$ and $k \in C^\infty(\overline{\Omega}_x \times \mathbb{S}_{\theta'}^{n-1} \times \mathbb{S}_\theta^{n-1})$. Then $KT_1^{-1}f \in \mathcal{H}_{l+1}(\Omega_1 \times \mathbb{S}^{n-1})$.*

Proof. First let us recall from the proof of Lemma 1 in [50] that

$$[KT_1^{-1}f](x, \theta) = \int_{\Omega} \frac{\Sigma\left(x, |x-y|, \frac{x-y}{|x-y|}\right) k\left(x, \theta, \frac{x-y}{|x-y|}\right)}{|x-y|^{n-1}} f\left(y, \frac{x-y}{|x-y|}\right) dy, \quad (3.3.4)$$

where $\Sigma(x, s, \theta') = \exp\left(-\int_{-s}^0 \sigma(x + \tau\theta', \theta') d\tau\right)$. The characteristic $\Sigma(x, |x-y|, \theta')k(x, \theta, \frac{x-y}{|x-y|})$ satisfies the hypotheses of Proposition A.1.1, and the result follows. \square

Corollary 3.3.2. *Suppose that $\sigma \in C^\infty(\overline{\Omega} \times \mathbb{S}^{n-1})$ and $k \in C^\infty(\overline{\Omega}_x \times \mathbb{S}_{\theta'}^{n-1} \times \mathbb{S}_\theta^{n-1})$. Then $(KT_1^{-1})^j Jf : L^2(\Omega) \rightarrow \mathcal{H}_j(\Omega_1 \times \mathbb{S}^{n-1})$ for all $j \geq 0$.*

From this result, we see that in the case that $\|KT_1^{-1}\| < 1$, $X_V f$ is equal to $I_{\sigma, V} f$ plus a remainder consisting of a series of terms with successively higher regularity, corresponding to higher order scattering.

Lemma 3.3.3. *Suppose that $\sigma \in C^\infty(\overline{\Omega} \times \mathbb{S}^{n-1})$ and $k \in C^\infty(\overline{\Omega} \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$. Then $KT_1^{-1}K : H^l(\Omega_1 \times \mathbb{S}^{n-1}) \rightarrow H^l(\Omega_1 \times \mathbb{S}^{n-1})$ is compact for all $l \geq 0$.*

Proof. Recall from the proof of Lemma 2 in [50] that

$$[KT_1^{-1}Kf](x, \theta) = \int \int_{\Omega_1 \times \mathbb{S}^{n-1}} \frac{\alpha\left(x, y, |x-y|, \frac{x-y}{|x-y|}, \theta, \theta'\right)}{|x-y|^{n-1}} f(y, \theta') dy d\theta' \quad (3.3.5)$$

with some C^∞ α compactly supported in x and y . The integral in y is a weakly singular integral of the form in Proposition A.1.1, and so by part (a) we gain a derivative in x for each fixed θ' . Moreover, the smoothness in θ of $KT_1^{-1}Kf(x, \theta)$ is dependent only on the smoothness of α . Therefore, $KT_1^{-1}K : H^l(\Omega_1 \times \mathbb{S}^{n-1}) \rightarrow H^{l+1}(\Omega_1 \times \mathbb{S}^{n-1})$. By Rellich's Lemma, the inclusion $H^{l+1}(\Omega_1 \times \mathbb{S}^{n-1}) \hookrightarrow H^l(\Omega_1 \times \mathbb{S}^{n-1})$ is compact, which completes the proof. \square

Again suppose that σ and k are C^∞ . Since $L_V = \chi_V R_+ T_1^{-1} K T_1^{-1} (\text{Id} - K T_1^{-1})^{-1} J$, we have by Lemma 3.3.1 that $L_V : H^l(\Omega) \rightarrow \mathcal{H}_{l+1}(\Omega_1 \times \mathbb{S}^{n-1})$. Since $X_V^* X_V = I_{\sigma,V}^* I_{\sigma,V} + \mathcal{L}_V$ and for smooth σ , $I_{\sigma,V}^* I_{\sigma,V}$ is a pseudodifferential operator of order -1 , we would like to show that \mathcal{L}_V maps $H^l(\Omega)$ into $H^{l+2}(\Omega_1)$. We have the following proposition:

Proposition 3.3.1. *Let $l \geq 0$ be a nonnegative integer. There exists an open dense set \mathcal{O}_l of pairs $(\sigma, k) \in C^\infty(\overline{\Omega} \times \mathbb{S}^{n-1}) \times C^\infty(\overline{\Omega}_x \times \mathbb{S}_{\theta'}^{n-1} \times \mathbb{S}_\theta^{n-1})$ depending on l such that for all $0 \leq l' \leq \frac{l}{2}$, the operator*

$$\mathcal{L}_V = I_{\sigma,V}^* L_V + L_V^* I_{\sigma,V} + L_V^* L_V = I_{\sigma,V}^* L_V + L_V^* X_V \quad (3.3.6)$$

maps $H^{l'}(\Omega)$ into $H^{l'+2}(\Omega_1)$. Moreover, we can write $\mathcal{L}_V = F + R$ where F is a pseudodifferential operator of order -2 , and $R : L^2(\Omega) \rightarrow H^l(\Omega)$.

Proof. First we write

$$\begin{aligned} \mathcal{L}_V f &= I_{\sigma,V}^* L_V f + L_V^* X_V f \\ &= (\chi_V R_+ T_1^{-1} J)^* \chi_V R_+ T_1^{-1} K T_1^{-1} (\text{Id} - K T_1^{-1})^{-1} J f \\ &\quad + (\chi_V R_+ T_1^{-1} K T_1^{-1} (\text{Id} - K T_1^{-1})^{-1} J)^* (\chi_V R_+ T_1^{-1} (\text{Id} - K T_1^{-1})^{-1} J) f \\ &=: I_1 f + I_2 f. \end{aligned} \quad (3.3.7)$$

Given equation (2.5.10), by Proposition A.1.1 we have that $(\chi_V R_+ T_1^{-1} J)^* \chi_V R_+ T_1^{-1}$ maps $\mathcal{H}_{l'+1}(\Omega_1 \times \mathbb{S}^{n-1})$ into $H^{l'+2}(\Omega_1)$ for all $l' \geq 0$.

We claim that for an open dense set of $(\sigma, k) \in C^\infty \times C^\infty$, $(\text{Id} - K T_1^{-1})^{-1} J$ maps $H^{l'}(\Omega_1)$ to $\mathcal{H}_{l'}(\Omega_1 \times \mathbb{S}^{n-1})$ for all $0 \leq l' \leq l+1$. First note that by Lemma 3.3.3 $\lambda(K T_1^{-1})^2 : H^{l'}(\Omega_1 \times \mathbb{S}^{n-1}) \rightarrow H^{l'}(\Omega_1 \times \mathbb{S}^{n-1})$ is compact for all $l' \geq 0$. Using the analytic Fredholm theorem on the resolvent $A(\lambda)$ with (2.3.9) and (2.3.10), we conclude that $(\text{Id} - \lambda K T_1^{-1})^{-1}$ exists and is bounded on $H^{l'}(\Omega_1 \times \mathbb{S}^{n-1})$ for all λ in some complex neighborhood of $[0, 1]$ except for possibly a discrete set, which depends on l' . Taking the complement of the union of all such discrete sets for $0 \leq l' \leq l+1$, we obtain that $(\text{Id} - K T_1^{-1})^{-1}$ is bounded on each $H^{l'}(\Omega_1 \times \mathbb{S}^{n-1})$ for all $0 \leq l' \leq l+1$ and for all but a discrete set of λ . So the set of pairs $(\sigma, k) \in C^\infty \times C^\infty$ for which (1.2.1) has a unique solution and $(\text{Id} - K T_1^{-1})^{-1} : H^{l'}(\Omega_1 \times \mathbb{S}^{n-1}) \rightarrow H^{l'}(\Omega_1 \times \mathbb{S}^{n-1})$ is bounded for $0 \leq l' \leq l+1$, is open and dense. Now, we

just apply Lemma 3.3.1 to the KT_1^{-1} factor in I_1 , which shows that I_1 maps $H^{l'}(\Omega)$ into $H^{l'+2}(\Omega_1)$ for $0 \leq l' \leq l$.

To analyze I_2 , we will use the series expansion of $k(x, \theta, \theta')$ previously defined in (2.5.11).

Observe that

$$\begin{aligned} I_2 f &= \left(\sum_{j=1}^{\infty} \chi_V R_+ T_1^{-1} K_j T_1^{-1} (\text{Id} - KT_1^{-1})^{-1} J \right)^* (\chi_V R_+ T_1^{-1} (\text{Id} - KT_1^{-1})^{-1} J) \\ &= \left(\sum_{j=1}^{\infty} [\chi_V R_+ T_1^{-1} \Theta_j J] [B_j (\text{Id} - KT_1^{-1})^{-1} J] \right)^* (\chi_V R_+ T_1^{-1} (\text{Id} - KT_1^{-1})^{-1} J) \\ &= \sum_{j=1}^{\infty} (B_j (\text{Id} - KT_1^{-1})^{-1} J)^* (\chi_V R_+ T_1^{-1} \Theta_j J)^* (\chi_V R_+ T_1^{-1} (\text{Id} - KT_1^{-1})^{-1} J). \end{aligned} \quad (3.3.8)$$

Similar to (2.5.10), we can compute

$$\begin{aligned} & (\chi_V R_+ T_1^{-1} \Theta_j J)^* \chi_V R_+ T_1^{-1} g(x) \\ &= 2 \int_{\Omega_1} \frac{\left[\overline{E} \left(x, \frac{y-x}{|y-x|} \right) \chi_V^\# \left(y, \frac{y-x}{|y-x|} \right) \Theta_j \left(\frac{y-x}{|y-x|} \right) E \left(y, \frac{y-x}{|y-x|} \right) g \left(y, \frac{y-x}{|y-x|} \right) \right]_{\text{even}}}{|x-y|^{n-1}} dy. \end{aligned} \quad (3.3.9)$$

By Proposition A.1.1, it is then evident that $(\chi_V R_+ T_1^{-1} \Theta_j J)^* \chi_V R_+ T_1^{-1}$ maps $\mathcal{H}_{l'}(\Omega \times \mathbb{S}^{n-1})$ into $H^{l'+1}(\Omega)$ for any $l' \geq 0$. Applying Proposition A.1.1 to (2.5.14) gives that $B_j (\text{Id} - KT_1^{-1})^{-1} J$ maps $H^{l'+1}(\Omega_1)$ to $H^{l'+2}(\Omega_1)$ for $0 \leq l' \leq l$. Therefore, so does its adjoint. Altogether, we have that $I_2 f \in H^{l'+2}(\Omega_1)$ for $f \in H^{l'}(\Omega)$ where $0 \leq l' \leq l$.

For the next part of the proposition, note that for $m \geq 1$

$$(\text{Id} - KT_1^{-1}) \sum_{j=0}^m (KT_1^{-1})^j J = J - (KT_1^{-1})^{m+1} J.$$

By Lemma 3.3.1 each summand $(KT_1^{-1})^j J$ is a pseudodifferential operator of order $-j$ with symbol depending smoothly on the parameter θ . Therefore, we may construct a symbol

$$\rho(x, \xi, \theta) \sim \sum_{j=0}^{\infty} \sigma_L((KT_1^{-1})^j J)(x, \xi, \theta), \quad (3.3.10)$$

where $\sigma_L : L_{1,0}^{-j}(\Omega) \rightarrow S_{1,0}^{-j}(\Omega \times \mathbb{R}^n)$ is the full left symbol map. Here we are using the notation $L_{\delta,\rho}^m(\Omega)$ to refer to pseudodifferential operators with symbols in the class $S_{\delta,\rho}^m(\Omega \times \mathbb{R}^n)$ (see [20]). The symbol ρ corresponds to a pseudodifferential operator \widetilde{F} of order 0 with smooth

parameter θ , and we have

$$\begin{aligned} (\text{Id} - KT_1^{-1}) \circ \widetilde{F} &= J + \widetilde{R}_0, \quad \widetilde{R}_0 \in (L_{1,0}^{-\infty}(\Omega); C^\infty(\mathbb{S}^{n-1})) \\ \widetilde{F} - \widetilde{R} &= (\text{Id} - KT_1^{-1})^{-1} J \\ \widetilde{R} &= (\text{Id} - KT_1^{-1})^{-1} \widetilde{R}_0 : L^2(\Omega) \rightarrow H^l(\Omega; C^\infty(\mathbb{S}^{n-1})). \end{aligned} \quad (3.3.11)$$

Substituting (3.3.11) into the expression (3.3.7) for \mathcal{L}_V gives that

$$\begin{aligned} \mathcal{L}_V &= \left[(\chi_V R_+ T_1^{-1} J)^* \chi_V R_+ T_1^{-1} K T_1^{-1} \widetilde{F} + (\chi_V R_+ T_1^{-1} K T_1^{-1} \widetilde{F})^* (\chi_V R_+ T_1^{-1} \widetilde{F}) \right] + R \\ &=: F + R, \end{aligned} \quad (3.3.12)$$

where R involves all the terms with \widetilde{R} . Specifically,

$$\begin{aligned} R &= (\chi_V R_+ T_1^{-1} J)^* \chi_V R_+ T_1^{-1} K T_1^{-1} \widetilde{R} + (\chi_V R_+ T_1^{-1} K T_1^{-1} \widetilde{R})^* (\chi_V R_+ T_1^{-1} \widetilde{F}) \\ &\quad + (\chi_V R_+ T_1^{-1} K T_1^{-1} \widetilde{F})^* (\chi_V R_+ T_1^{-1} \widetilde{R}) + (\chi_V R_+ T_1^{-1} K T_1^{-1} \widetilde{R})^* (\chi_V R_+ T_1^{-1} \widetilde{R}) \\ &=: A_1 + A_2 + A_3 + A_4. \end{aligned}$$

It is clear from the fact that $\widetilde{R} : L^2(\Omega) \rightarrow H^l(\Omega_1 \times \mathbb{S}^{n-1})$ that $A_1, A_3 : L^2(\Omega) \rightarrow H^l(\Omega_1)$. A bit more care is needed to analyze A_2 and A_4 . To this end, we consider the adjoint of $(\text{Id} - KT_1^{-1})^{-1}$. Recall that

$$\begin{aligned} (KT_1^{-1})^2 f(x, \theta) &= KT_1^{-1} K [T_1^{-1} f](x, \theta) \\ &= \int_{\Omega} \int_{\mathbb{S}^{n-1}} \frac{\Sigma(x, |x-y|, \widehat{x-y}) k(x, \theta, \widehat{x-y}) k(y, \widehat{x-y}, \theta')}{|x-y|^{n-1}} T_1^{-1} f(y, \theta') dy d\theta'. \end{aligned}$$

It can be shown that

$$\begin{aligned} [(KT_1^{-1})^2]^* f(x, \theta) &= \int_{\Omega} \int_{\mathbb{S}^{n-1}} \int_0^\infty \frac{\alpha(y, x+s\theta, |y-x-s\theta|, \widehat{y-x-s\theta}, \theta', \theta)}{|y-x-s\theta|^{n-1}} \\ &\quad \cdot \exp\left(-\int_0^s \sigma(x+\tau\theta, \theta) d\tau\right) f(y, \theta') dy d\theta' ds, \end{aligned} \quad (3.3.13)$$

where

$$\alpha(x, y, |x-y|, \widehat{x-y}, \theta, \theta') = \Sigma(x, |x-y|, \widehat{x-y}) k(x, \theta, \widehat{x-y}) k(y, \widehat{x-y}, \theta').$$

In particular, $[(KT_1^{-1})^2]^*$ is compact by the same argument as in Lemma 2 of [50]. Thus, one can argue that the resolvent $(\text{Id} - \lambda[(KT_1^{-1})^2]^*)^{-1}$ exists for all λ in a simply connected

complex neighborhood of $[0, 1]$ except possibly for a discrete subset. By the same reasoning as for $(I - KT_1^{-1})^{-1}$, we have that $[(I - KT_1^{-1})^{-1}]^*$ exists and is bounded on $H^l(\Omega \times \mathbb{S}^{n-1})$ for such a dense subset of σ, k . From this point, we may as well assume that this property holds for the dense subset corresponding to $(I - KT_1^{-1})^{-1}$.

Next, consider that since $R_0 \in L_{1,0}^{-\infty}(\Omega; C^\infty(\mathbb{S}^{n-1}))$, we have by definition of the adjoining of a pseudodifferential operator (see Theorem 3.5 of [20]) that

$$R_0^* f(x, \theta) = (2\pi)^{-n} \int \int e^{i(x-y) \cdot \xi} a(x, \xi, \theta) f(y) dy d\xi,$$

where $a(x, \xi, \theta)$ belongs to the symbol class $S_{1,0}^{-\infty}(\Omega \times \mathbb{R}^n)$ with the smooth parameter θ . Thus $\tilde{R}^* = R_0^*[(I - KT_1^{-1})^{-1}]^*$ is smoothing on L^2 , which takes care of A_2 and A_4 . It then follows that R maps $L^2(\Omega) \rightarrow H^l(\Omega_1)$, which completes the proof. \square

For reference, given an operator $A \in L_{1,0}^m(\Omega)$, we define $\text{WF}(A)$ as the smallest closed cone $\mathcal{C} \subset T^*\Omega \setminus 0$ such that $\sigma_A|_{\mathcal{C}^c} \in S^{-\infty}(\mathcal{C}^c)$, where σ_A is the symbol of A . We are now ready to prove the main theorem:

Proof of Theorem 3.2.1. Assume first that $(\sigma, k) \in C^\infty \times C^\infty$ is in the same open, dense set as in Proposition 3.3.1. Since σ is $C^\infty(\overline{\Omega} \times \mathbb{S}^{n-1})$, we have that $I_{\sigma,V}^* I_{\sigma,V}$ is a pseudodifferential operator of order -1 . Furthermore, it is elliptic on $N^*l(x_0, \theta_0)$ by (3.2.1). Let $(z, \xi) \in N^*l(x_0, \theta_0)$. Then there exists a microlocal parametrix $Q \in L_{1,0}^1(\Omega_1)$ elliptic at (z, ξ) and $S_1 \in L_{1,0}^0(\Omega)$ such that

$$Q I_{\sigma,V}^* I_{\sigma,V} = \text{Id} + S_1,$$

and $(z, \xi) \notin \text{WF}(S_1)$. We will also restrict the image of Q so that $Q : H^1(\Omega_1) \rightarrow L^2(\Omega)$. Since $\text{WF}(S_1 f) \subset \text{WF}(S_1) \cap \text{WF}(f)$ (e.g. by Lemma 7.2 of [20]), we have that $S_1 f$ is microlocally smooth near (z, ξ) , i.e. $(z, \xi) \notin \text{WF}(S_1 f)$.

Now we apply Q to the normal operator $X_V^* X_V = I_{\sigma,V}^* I_{\sigma,V} + \mathcal{L}_V$ to get

$$Q X_V^* X_V = \text{Id} + S_1 + Q \mathcal{L}_V.$$

By Proposition 3.3.1, we have that $\mathcal{L}_V \in H^k(\Omega) \rightarrow H^{k+2}(\Omega_1)$ for $0 \leq k \leq l$, and hence $Q \mathcal{L}_V : H^k(\Omega) \rightarrow H^{k+1}(\Omega)$. Moreover, from (3.3.12) we have that

$$Q X_V^* X_V = \text{Id} + QF + S_1 + QR.$$

We can then construct $(\text{Id} + QF)^{-1}$ modulo a smoothing operator on Ω_1 , since $\text{Id} + QF$ has principal symbol 1, and we can ignore the smooth error term for our purposes. Thus

$$(\text{Id} + QF)^{-1}QX_V^*X_V = \text{Id} + (\text{Id} + QF)^{-1}S_1 + (\text{Id} + QF)^{-1}QR.$$

We let

$$v = (\text{Id} + QF)^{-1}QRf \tag{3.3.14}$$

and note that $v \in H^l(\Omega)$. The other term $(\text{Id} + QF)^{-1}S_1f$ is microlocally smooth near (z, ξ) . Thus, $(z, \xi) \notin \text{WF}((\text{Id} + QF)^{-1}QX_V^*X_Vf) \implies (z, \xi) \notin \text{WF}(f + v)$. Since

$$\text{WF}((\text{Id} + QF)^{-1}QX_V^*X_Vf) \subset \text{WF}((\text{Id} + QF)^{-1}Q) \cap \text{WF}(X_V^*X_Vf),$$

the result follows. □

Remark 3. It is easy to see that the open dense sets of pairs (σ, k) that are dependent on l form a nested sequence. Moreover, one can eliminate the remainder function v by taking the intersection of all such open dense subsets. By the Baire category theorem, this limiting set of pairs will still be dense, but it is not clear if it remains open.

Proof of Corollary 3.2.2. In this case the series $\sum_{j=0}^{\infty}(KT_1^{-1})^jJ$ converges to the identity plus a weakly singular integral operator F , which altogether is a pseudodifferential operator of order 0. In light of (3.3.11) this implies that $\widetilde{R}_0 = 0$, and hence $R = 0$. By (3.3.12) and (3.3.14) we have $v = 0$. □

Chapter 4

RECOVERY OF A SOURCE FOR SMALL K IN C^2

4.1 Introduction

Consider the inverse source problem for the stationary RTE (1.2.1) with complete data when the collision kernel k is a function in $C^2(\overline{\Omega} \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$. The goal of this chapter is to make a first step in removing the rather technical assumption that $k \in C^2(\Omega \times \mathbb{S}_{\theta'}^{n-1}; C^{n+1}(\mathbb{S}_{\theta}^{n-1}))$ as introduced in [50]. For now, we limit ourselves to the case that k have small C^2 norm. Note that the smallness condition on the Fourier coefficients of k introduced in [7] seems to suggest a weaker smoothness condition. However, in that result they also need to assume more about the anisotropic structure of k .

Remark 4. In the 2-dimensional case, if we also assume that the absorption σ is a function of the form $\sigma(x, \theta) = \sigma_0(x) + \theta \cdot \mathbf{b}(x)$, where \mathbf{b} a smooth vector field, then from [36] we have injectivity for the attenuated ray transform $[I_{\sigma}f](x, \theta) = \int \exp\left(-\int_s^0 \sigma(x + \tau\theta, \theta) d\tau\right) f(x + s\theta) ds$ with $f \in C^{\infty}(\Omega)$ and $(x, \theta) \in \partial_+ S\Omega$.

More specifically, the authors consider the following operator, which in the Euclidean case takes the form

$$\theta \cdot \nabla_x u + Au = -f \quad \text{in } S\Omega, \quad u|_{\partial_- S\Omega} = 0, \quad (4.1.1)$$

where $A(x, \theta) : T\Omega \rightarrow \mathbb{C}^{2 \times 2}$ is a smooth map (called a connection) which is linear in θ for each x . In other words, A is a complex matrix whose entries are 1-forms. In this case, $u : S\Omega \rightarrow \mathbb{C}^2$ is a complex vector-valued function on $S\Omega$ and f is a smooth C^2 valued function on either $S\Omega$ or Ω , depending on the assumptions. The attenuated ray transform is then familiarly defined as $I_A f = u^f|_{\partial_+ S\Omega}$, where u^f solves (4.1.1). [36] shows that I_A is injective on the space of smooth functions $f \in C^{\infty}(\Omega)$. More generally, they show injectivity of I_A if $f : S\Omega \rightarrow \mathbb{C}^2$ is smooth and has the form $f(x, \theta) = F(x) + \alpha_j(x)\theta^j$, where $F : \Omega \rightarrow \mathbb{C}^2$ is a smooth function and α is a \mathbb{C}^2 valued 1-form.

However, this result is quite more general than we need here, as it treats the case of an arbitrary 2-dimensional compact, simple Riemannian manifold. So we leave it as merely an aside remark.

In this chapter, we seek to establish injectivity and stability of the operator X in (1.2.2) for such σ and k with slightly weaker smoothness by adapting the proof of Theorem 2.2.1. This presents a slight improvement over the conclusions of [50], since in that result we have injectivity and stability of X for (σ, k) in a C^2 neighborhood of $(0, 0)$ with the additional C^{n+1} regularity of k in θ and dependence of k on the size of σ . In the future, one problem of interest is to be able to remove the same C^{n+1} regularity assumption for k with no smallness condition.

Remark 5. As in Chapter 2 we assume the domain Ω has been extended to Ω_1 , which is convex for convenience. As such, the source function f is always compactly supported inside of Ω_1 , and k and σ have the same smoothness on Ω_1 as on Ω . However, for simplicity of notation, we will replace Ω_1 by Ω with the understanding that the domain has already been extended.

4.2 Statement of Main Result

Recall the relevant integro-differential equation:

$$\begin{aligned} \theta \cdot \nabla_x u(x, \theta) + \sigma(x, \theta)u(x, \theta) - \int_{\mathbb{S}^{n-1}} k(x, \theta, \theta')u(x, \theta')d\theta' &= f(x) \\ u|_{\partial_- \Omega} &= 0 \end{aligned}$$

The measurements are then given by $Xf = u|_{\partial_+ \Omega}$ where u solves the above equation.

The first question to address before proceeding any further is the well-posedness of the forward problem under such assumptions on σ and k . Recall that the solution to (1.2.1) is given by

$$u = (\text{Id} - T_1^{-1}K)^{-1}T_1^{-1}Jf = T_1^{-1}(\text{Id} - KT_1^{-1})^{-1}Jf.$$

It was previously proven in [50] that for a generic open dense set of pairs (σ, k) in $C^2(\overline{\Omega} \times \mathbb{S}^{n-1}) \times C^2(\overline{\Omega} \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$, the problem is well-posed. The set of pairs is obtained by using the analytic Fredholm theory in order to determine where $\text{Id} - KT_1^{-1}$ has a bounded

inverse. But assuming that k is small, we can use a Neumann series for $(\text{Id} - KT_1^{-1})^{-1}$. In this way, one can solve the forward problem uniquely for any C^2 σ and k , so long as $\|k\|_{C^2}$ is small enough.

Recall one of the main steps in showing injectivity of X lies in writing $X = I_\sigma + L$ where I_σ involves no scattering and

$$L = R_+ T_1^{-1} K T_1^{-1} (\text{Id} - K T_1^{-1})^{-1} J = R_+ T_1^{-1} K T_1^{-1} \sum_{m=0}^{\infty} (K T_1^{-1})^m J.$$

Then we consider the normal operator

$$\begin{aligned} X^* X &= I_\sigma^* I_\sigma + I_\sigma^* L + L^* I_\sigma + L^* L^* \\ &= I_\sigma^* I_\sigma + \mathcal{L}. \end{aligned}$$

If σ is C^∞ , then it has been shown that $I_\sigma^* I_\sigma$ is an elliptic pseudodifferential operator of order -1 , and so has a parametrix Q of order 1. Applying Q to $X^* X$, we get $\text{Id} + K_1 + Q\mathcal{L}$, where K_1 is a smoothing operator and hence compact. By adding a finite rank operator to Q , we can also arrange that $\text{Id} + K_1$ is injective.

In order to proceed further, we want to be able to show that $\text{Id} + K_1 + Q\mathcal{L}$ is injective assuming that $\text{Id} + K_1$ is injective. One way is to show that $Q\mathcal{L}$ is compact and then to use the analytic Fredholm theorem, but this is a more difficult task than simply showing $Q\mathcal{L}$ has small $L^2 \rightarrow L^2$ norm, which in turn implies that the associated Neumann series converges absolutely. Furthermore, this simpler approach gives a way to estimate how large an open set around 0 to which k can belong. We now state the main theorem of this chapter:

Theorem 4.2.1. There exists a generic dense, open set of $\sigma \in C^2(\overline{\Omega} \times \mathbb{S}^{n-1})$, such that for each σ in that set, there exists a number $\epsilon > 0$ depending on Ω, n, σ , so that for any $k \in C^2(\overline{\Omega} \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$ with $\|k\|_{C^2} < \epsilon$, the result of Theorem 1 in [50] holds. That is, the direct problem (1.2.1) has a unique solution $u \in L^2(\Omega_1 \times \mathbb{S}^{n-1})$ for any $f \in L^2(\Omega \times \mathbb{S}^{n-1})$, X extends to a bounded operator from $L^2(\Omega_1 \times \mathbb{S}^{n-1})$ to $L^2(\partial_+ S\Omega_1, d\Sigma)$, and

1. the map X is injective on $L^2(\Omega)$,

2. the following stability estimate holds:

$$\|f\|_{L^2(\Omega)} \leq C \|X^* X f\|_{H^1(\Omega_1)}, \quad \forall f \in L^2(\Omega),$$

with a constant $C > 0$ locally uniform in (σ, k) .

4.3 The Simpler Case $k \in C^2(\overline{\Omega} \times \mathbb{S}_{\theta'}^{n-1}; C^{n+1}(\mathbb{S}_{\theta}^{n-1}))$

Let us first consider the original simpler case, in order to get some idea of the size of $\|k\|_{C^2}$ for which the inverse problem has guaranteed uniqueness and stability. We thus make the additional assumption (as in [50]) that $k \in C^2(\overline{\Omega} \times \mathbb{S}_{\theta'}^{n-1}; C^{n+1}(\mathbb{S}^{n-1}))$. This ensures that we can write k as a harmonic series expansion in θ , given by

$$k(x, \theta, \theta') = \sum_{j=1}^{\infty} \kappa_j(x, \theta') \Theta(\theta).$$

4.3.1 Bounding $I_{\sigma}^* L$

We then have the decomposition

$$\begin{aligned} \partial_x I_{\sigma}^* L &= \partial_x I_{\sigma}^* R_+ T_1^{-1} K T_1^{-1} (\text{Id} - K T_1^{-1})^{-1} J \\ &= \sum_{j=1}^{\infty} [\partial_x I_{\sigma}^* R_+ T_1^{-1} \Theta_j J] [B_j (\text{Id} - K T_1^{-1})^{-1} J], \end{aligned}$$

where

$$B_j g(x) = \int \frac{\Sigma \left(x, |x-y|, \frac{x-y}{|x-y|} \right) \kappa_j \left(x, \frac{x-y}{|x-y|} \right)}{|x-y|^{n-1}} g \left(y, \frac{x-y}{|x-y|} \right) dy.$$

Recall from [50]

$$[I_{\sigma}^* R_+ T_1^{-1} \Theta_j J g](x) = 2 \int_{\Omega_1} \frac{\left[E \left(x, \frac{y-x}{|y-x|} \right) E \left(y, \frac{y-x}{|y-x|} \right) \Theta_j \left(\frac{y-x}{|y-x|} \right) \right]_{\text{even}}}{|y-x|^{n-1}} g(y) dy.$$

From Proposition 1 of [50] we have that the $L^2 \rightarrow H^1$ norm of $I_{\sigma}^* R_+ T_1^{-1} \Theta_j J$ is bounded by

$$C_n \|\alpha\|_{C^2(\Omega \times \Omega \times \mathbb{S}^{n-1})} \|\Theta_j\|_{H^1(\mathbb{S}^{n-1})}$$

where $\alpha(x, y, \theta) = E(x, \theta) E(y, \theta)$ and C_n is a constant depending only on n .

It remains to analyze $B_j (\text{Id} - K T_1^{-1})^{-1} J$, and we need only derive an $L^2 \rightarrow L^2$ bound for it. We start with $(\text{Id} - K T_1^{-1})^{-1}$. Recall that

$$K T_1^{-1} f(x, \theta) = \int \frac{\exp \left(- \int_{-|x-y|}^0 \sigma \left(x + \tau \frac{x-y}{|x-y|}, \frac{x-y}{|x-y|} \right) d\tau \right) k \left(x, \theta, \frac{x-y}{|x-y|} \right)}{|x-y|^{n-1}} f \left(y, \frac{x-y}{|x-y|} \right) dy.$$

We then estimate

$$\begin{aligned}
& \|KT_1^{-1}f\|_{L^2(\Omega \times \mathbb{S}^{n-1})}^2 \\
&= \int_{\mathbb{R}^n \times \mathbb{S}^{n-1}} \left| \int_{\mathbb{R}^n} \frac{\Sigma(x, |x-y|, \frac{x-y}{|x-y|}) k(x, \theta, \frac{x-y}{|x-y|})}{|x-y|^{n-1}} f\left(y, \frac{x-y}{|x-y|}\right) dy \right|^2 dx d\theta \\
&= \int_{\mathbb{R}^n \times \mathbb{S}^{n-1}} \left| \int_{\mathbb{R}^n} \frac{\Sigma(x, |z|, \frac{z}{|z|}) k(x, \theta, \frac{z}{|z|})}{|z|^{n-1}} f\left(x-z, \frac{z}{|z|}\right) dz \right|^2 dx d\theta \\
&\leq \int_{\mathbb{R}^n \times \mathbb{S}^{n-1}} \left(\int_{\mathbb{R}^n} \frac{\Sigma(x, |z|, \frac{z}{|z|}) k(x, \theta, \frac{z}{|z|})}{|z|^{n-1}} \left| f\left(x-z, \frac{z}{|z|}\right) \right| dz \right)^2 dx d\theta \\
&\leq \int_{\mathbb{R}^n \times \mathbb{S}^{n-1}} \left(\int_{\mathbb{R}^n} \frac{k(x, \theta, \frac{z}{|z|})}{|z|^{n-1}} \left| f\left(x-z, \frac{z}{|z|}\right) \right| dz \right)^2 dx d\theta \\
&\leq \int_{\mathbb{S}^{n-1}} \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{k(x, \theta, \frac{z}{|z|})}{|z|^{n-1}} \left| f\left(x-z, \frac{z}{|z|}\right) \right| dz dx \right)^2 d\theta \\
&\leq \int_{\mathbb{S}^{n-1}} \left(\sup_{x, \theta'} k(\cdot, \theta, \cdot) \right)^2 \left(\int_{\mathbb{R}_+ \times \mathbb{S}^{n-1}} \int_{\mathbb{R}^n} |f(x-r\eta, \eta)| dx dr d\eta \right)^2 d\theta \\
&= \int_{\mathbb{S}^{n-1}} \left(\sup_{x, \theta'} k(\cdot, \theta, \cdot) \right)^2 \left(\int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}_+} \int_{\mathbb{R}^n} \chi(x) \chi(x-r\eta) |f(x-r\eta, \eta)| dx dr d\eta \right)^2 d\theta \\
&\leq \int_{\mathbb{S}^{n-1}} \left(\sup_{x, \theta'} k(\cdot, \theta, \cdot) \right)^2 \left(\int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}_+} \|\chi(\cdot) \chi(\cdot - r\eta)\|_{L^2(\mathbb{R}^n)} \|f(\cdot, \eta)\|_{L^2(\mathbb{R}^n)} dr d\eta \right)^2 d\theta \\
&= \int_{\mathbb{S}^{n-1}} \left(\sup_{x, \theta'} k(\cdot, \theta, \cdot) \right)^2 \left(\int_{\mathbb{S}^{n-1}} \left[\int_{\mathbb{R}_+} \|\chi(\cdot) \chi(\cdot - r\eta)\|_{L^2(\mathbb{R}^n)} dr \right] \|f(\cdot, \eta)\|_{L^2(\mathbb{R}^n)} d\eta \right)^2 d\theta \\
&\leq \int_{\mathbb{S}^{n-1}} \left(\sup_{x, \theta'} k(\cdot, \theta, \cdot) \right)^2 \int_{\mathbb{S}^{n-1}} \left[\int_{\mathbb{R}_+} \|\chi(\cdot) \chi(\cdot - r\eta)\|_{L^2(\mathbb{R}^n)} dr \right]^2 d\eta \int_{\mathbb{S}^{n-1}} \|f(\cdot, \eta)\|_{L^2(\mathbb{R}^n)}^2 d\eta d\theta \\
&= C(\Omega)^2 \|f\|_{L^2(\Omega \times \mathbb{S}^{n-1})}^2 \int_{\mathbb{S}^{n-1}} \left(\sup_{x, \theta'} k(\cdot, \theta, \cdot) \right)^2 d\theta,
\end{aligned}$$

where

$$\begin{aligned}
C(\Omega)^2 &:= \int_{\mathbb{S}^{n-1}} \left[\int_{\mathbb{R}_+} \|\chi(\cdot)\chi(\cdot - r\eta)\|_{L^2(\mathbb{R}^n)} dr \right]^2 d\eta \\
&= \int_{\mathbb{S}^{n-1}} \left(\int_{\mathbb{R}_+} \left(\int_{\mathbb{R}^n} \chi(z)\chi(z - r\eta) dz \right)^{1/2} dr \right)^2 d\eta \\
&= \int_{\mathbb{S}^{n-1}} \left(\int_{\mathbb{R}_+} \left(\int_{\Omega} \chi(z - r\eta) dz \right)^{1/2} dr \right)^2 d\eta \\
&\leq \int_{\mathbb{S}^{n-1}} \left(\int_0^{\text{diam}(\Omega)} \left(\int_{\Omega} dz \right)^{1/2} dr \right)^2 d\eta \\
&\leq \int_{\mathbb{S}^{n-1}} (\text{diam}(\Omega)\text{Vol}(\Omega)^{1/2})^2 d\eta \\
&= \omega_n \text{diam}(\Omega)^2 \text{Vol}(\Omega).
\end{aligned}$$

Thus

$$\begin{aligned}
\|KT_1^{-1}\|_{L^2 \rightarrow L^2} &\leq \omega_n^{1/2} \text{diam}(\Omega)\text{Vol}(\Omega)^{1/2} \left\| \sup_{x, \theta'} k(\cdot, \theta, \cdot) \right\|_{L^2(\mathbb{S}_\theta^{n-1})} \\
&\leq \omega_n \text{diam}(\Omega)\text{Vol}(\Omega)^{1/2} \|k\|_{L^\infty},
\end{aligned} \tag{4.3.1}$$

and so

$$\|(\text{Id} - KT_1^{-1})^{-1}\| \leq \frac{1}{1 - \omega_n \text{diam}(\Omega)\text{Vol}(\Omega)^{1/2} \|k\|_{L^\infty}}. \tag{4.3.2}$$

Similarly, we need to compute the $L^2(\Omega \times \mathbb{S}^{n-1}) \rightarrow L^2(\Omega)$ norm of B_j . Identical computations show that

$$\|B_j\|_{L^2(\Omega \times \mathbb{S}^{n-1}) \rightarrow L^2(\Omega)} \leq \omega_n \text{diam}(\Omega)\text{Vol}(\Omega)^{1/2} \|\kappa_j\|_{L^\infty} \tag{4.3.3}$$

Finally, we estimate the $L^2(\Omega) \rightarrow L^2(\Omega)$ norm of $I_\sigma^* R_+ T_1^{-1} \Theta_j J$. We do this by means of Schur's Lemma. Denoting by $K(x, y)$ the Schwarz kernel of the integral operator $I_\sigma^* R_+ T_1^{-1} \Theta_j J$, we note $K(x, y)$ is symmetric in x and y , so we need only compute $\sup_y \int |K(x, y)| dx$. Ob-

serve that

$$\begin{aligned}
& \sup_y 2 \int_{\Omega} \frac{\left[E\left(x, \frac{y-x}{|y-x|}\right) E\left(y, \frac{y-x}{|y-x|}\right) \left| \Theta_j\left(\frac{y-x}{|y-x|}\right) \right| \right]_{\text{even}}}{|y-x|^{n-1}} dx \\
& \leq \sup_y 2 \int_{\Omega} \left[\frac{|\Theta_j(\frac{y-x}{|y-x|})|}{|y-x|^{n-1}} \right]_{\text{even}} dx \\
& = \sup_y 2 \int_{\mathbb{R}_+} \int_{\mathbb{S}^{n-1}} \chi(y-r\eta) [\Theta_j(\eta)]_{\text{even}} d\eta dr \\
& \leq \sup_y \int_0^{\text{diam}(\Omega)} \|\chi(y-r\eta)\|_{L^2(\mathbb{S}_\eta^{n-1})} \|\Theta_j\|_{L^2(\mathbb{S}_\eta^{n-1})} dr \\
& \leq \omega_n^{1/2} \text{diam}(\Omega) \|\Theta_j\|_{L^2(\mathbb{S}^{n-1})}.
\end{aligned}$$

So

$$\begin{aligned}
& \| [I_{\sigma}^* R_+ T_1^{-1} \Theta_j J] [B_j (\text{Id} - K T_1^{-1})^{-1} J f] \|_{H^1} \\
& \leq \| I_{\sigma}^* R_+ T_1^{-1} \Theta_j J \|_{L^2 \rightarrow H^1} \| B_j \|_{L^2(\Omega \times \mathbb{S}^{n-1}) \rightarrow L^2(\Omega)} \| (\text{Id} - K T_1^{-1})^{-1} \|_{L^2 \rightarrow L^2} \| J \|_{L^2(\Omega) \rightarrow L^2(\Omega \times \mathbb{S}^{n-1})} \| f \|_{L^2} \\
& \leq C_n \| E(x, \theta) E(y, \theta) \|_{C^2(\Omega \times \Omega \times \mathbb{S}^{n-1})} \| \Theta_j \|_{H^1(\mathbb{S}^{n-1})} \omega_n \text{diam}(\Omega) \text{Vol}(\Omega)^{1/2} \\
& \quad \cdot \| \kappa_j \|_{L^\infty} \left(\frac{1}{1 - \| K T_1^{-1} \|} \right) \omega_n^{1/2} \| f \|_{L^2} \\
& = \| E(x, \theta) E(y, \theta) \|_{C^2(\Omega \times \Omega \times \mathbb{S}^{n-1})} \| \Theta_j \|_{H^1(\mathbb{S}^{n-1})} \\
& \quad \cdot \| \kappa_j \|_{L^\infty} \left(\frac{C_n \omega_n^{3/2} \text{diam}(\Omega) \text{Vol}(\Omega)^{1/2}}{1 - \omega_n \text{diam}(\Omega) \text{Vol}(\Omega)^{1/2} \| k \|_{L^\infty}} \right) \| f \|_{L^2} \\
& = \| E(x, \theta) E(y, \theta) \|_{C^2(\Omega \times \Omega \times \mathbb{S}^{n-1})} \| \Theta_j \|_{H^1(\mathbb{S}^{n-1})} \| \kappa_j \|_{L^\infty} \left(\frac{C_n \omega_n^{1/2}}{\omega_n^{-1} \text{diam}(\Omega)^{-1} \text{Vol}(\Omega)^{-1/2} - \| k \|_{L^\infty}} \right) \| f \|_{L^2},
\end{aligned}$$

and thus

$$\begin{aligned}
& \| I_{\sigma}^* L \|_{L^2 \rightarrow H^1} \\
& \leq \sum_{j=1}^{\infty} \| [I_{\sigma,0}^* R_+ T_1^{-1} \Theta_j J] [B_j (\text{Id} - K T_1^{-1})^{-1} J] \|_{L^2 \rightarrow H^1} \\
& \leq \| E(x, \theta) E(y, \theta) \|_{C^2(\Omega \times \Omega \times \mathbb{S}^{n-1})} \left(\frac{C_n \omega_n^{1/2}}{\omega_n^{-1} \text{diam}(\Omega)^{-1} \text{Vol}(\Omega)^{-1/2} - \| k \|_{L^\infty}} \right) \sum_{j=1}^{\infty} \| \Theta_j \|_{H^1(\mathbb{S}^{n-1})} \| \kappa_j \|_{L^\infty} \\
& = \| E(x, \theta) E(y, \theta) \|_{C^2(\Omega \times \Omega \times \mathbb{S}^{n-1})} \left(\frac{C_n \omega_n^{1/2}}{\omega_n^{-1} \text{diam}(\Omega)^{-1} \text{Vol}(\Omega)^{-1/2} - \| k \|_{L^\infty}} \right) \| k \|_{\mathcal{H}_\infty}. \quad (4.3.4)
\end{aligned}$$

4.3.2 L^*L

From the series expansion for k , we have that

$$L^*L = \sum_{j=1}^{\infty} (KT_1^{-1}J)^* (R_+T_1^{-1}(\text{Id} - KT_1^{-1})^{-1})^* R_+T_1^{-1}\Theta_j JB_j (\text{Id} - KT_1^{-1})^{-1}J$$

We use the fact that given an operator $A : X \rightarrow Y$ for Hilbert spaces X and Y , we have $\|A\| = \|A^*\|$. Also, it is shown in [50] that $\|R_+T_1^{-1}\|_{L^2(\Omega \times \mathbb{S}^{n-1}) \rightarrow L^2(\partial_+ S\Omega, d\Sigma)} \leq \text{diam}(\Omega)^{1/2}$. So

$$\begin{aligned} & \|L^*L\|_{L^2 \rightarrow H^1} \\ & \leq \sum_{j=1}^{\infty} \|KT_1^{-1}J\|_{L^2(\Omega \times \mathbb{S}^{n-1}) \rightarrow H^1(\Omega)} \|R_+T_1^{-1}\|_{L^2 \rightarrow L^2} \left(\frac{1}{1 - \|KT_1^{-1}\|} \right) \\ & \quad \cdot \|R_+T_1^{-1}\|_{L^2 \rightarrow L^2} \|\Theta_j\|_{L^2(\mathbb{S}^{n-1})} \|B_j\|_{L^2 \rightarrow L^2} \left(\frac{1}{1 - \|KT_1^{-1}\|} \right) \omega_n^{1/2} \\ & \leq \sum_{j=1}^{\infty} C_n \omega_n^{1/2} \|\Sigma(y, x, \theta')k(y, \theta, \theta')\|_{C^2} \text{diam}(\Omega) \left(\frac{1}{1 - \omega_n \text{diam}(\Omega) \text{Vol}(\Omega)^{1/2} \|k\|_{L^\infty}} \right)^2 \\ & \quad \cdot \|\Theta_j\|_{L^2} \omega_n \text{diam}(\Omega) \text{Vol}(\Omega)^{1/2} \|\kappa_j\|_{L^\infty} \omega_n^{1/2} \\ & \leq C_n \omega_n^2 \text{diam}(\Omega)^2 \text{Vol}(\Omega)^{1/2} \left(\frac{1}{1 - \omega_n \text{diam}(\Omega) \text{Vol}(\Omega)^{1/2} \|k\|_{L^\infty}} \right)^2 \|k \cdot \Sigma\|_{C^2} \sum_{j=1}^{\infty} \|\kappa_j\|_{L^\infty} \|\Theta_j\|_{L^2}. \end{aligned} \tag{4.3.5}$$

In the second third line above we applied the result of Proposition 1 of [50] to bound $\|KT_1^{-1}J\|_{L^2(\Omega \times \mathbb{S}^{n-1}) \rightarrow H^1(\Omega)}$ by a constant times $\|\Sigma(y, x, \theta')k(y, \theta, \theta')\|_{C^2}$.

Similarly, for $L^*I_\sigma = (KT_1^{-1}J)^* (R_+T_1^{-1}(\text{Id} - KT_1^{-1})^{-1})^* R_+T_1^{-1}J$ we have

$$\begin{aligned} \|L^*I_\sigma\|_{L^2 \rightarrow H^1} & \leq \|KT_1^{-1}J\|_{L^2(\Omega \times \mathbb{S}^{n-1}) \rightarrow H^1(\Omega)} \|R_+T_1^{-1}\|_{L^2 \rightarrow L^2} \left(\frac{1}{1 - \|KT_1^{-1}\|} \right) \|RT_1^{-1}\|_{L^2 \rightarrow L^2} \omega_n^{1/2} \\ & \leq \frac{C_n \omega_n \text{diam}(\Omega) \|k(y, \theta, \theta')\Sigma(y, x, \theta')\|_{C^2}}{1 - \omega_n \text{diam}(\Omega) \text{Vol}(\Omega)^{1/2} \|k\|_{L^\infty}}. \end{aligned} \tag{4.3.6}$$

Altogether, (4.3.4), (4.3.5) and (4.3.6) imply that

$$\begin{aligned}
& \|X^*X\|_{L^2 \rightarrow H^1} \\
& \leq \|E(x, \theta)E(y, \theta)\|_{C^2} \left(\frac{C_n \omega_n^{1/2}}{\omega_n^{-1} \text{diam}(\Omega)^{-1} \text{Vol}(\Omega)^{-1/2} - \|k\|_{L^\infty}} \right) \|k\|_{\mathcal{H}_\infty} \\
& \quad + C_n \omega_n^2 \text{diam}(\Omega)^2 \text{Vol}(\Omega)^{1/2} \left(\frac{1}{1 - \omega_n \text{diam}(\Omega) \text{Vol}(\Omega)^{1/2} \|k\|_{L^\infty}} \right)^2 \|k(y, \theta, \theta') \Sigma(y, x, \theta')\|_{C^2} \|k\|_{\mathcal{H}_\infty} \\
& \quad + \frac{C_n \omega_n \text{diam}(\Omega) \|k(y, \theta, \theta') \Sigma(y, x, \theta')\|_{C^2}}{1 - \omega_n \text{diam}(\Omega) \text{Vol}(\Omega)^{1/2} \|k\|_{L^\infty}} \\
& = \left(\frac{C_n \omega_n \text{diam}(\Omega)}{1 - \omega_n \text{diam}(\Omega) \text{Vol}(\Omega)^{1/2} \|k\|_{L^\infty}} \right) \left[\omega_n^{1/2} \text{Vol}(\Omega)^{1/2} \|E(x, \theta)E(y, \theta)\|_{C^2} \|k\|_{\mathcal{H}_\infty} \right. \\
& \quad \left. + \frac{\omega_n \text{diam}(\Omega) \text{Vol}(\Omega)^{1/2} \|k(y, \theta, \theta') \Sigma(y, x, \theta')\|_{C^2}}{1 - \omega_n \text{diam}(\Omega) \text{Vol}(\Omega)^{1/2} \|k\|_{L^\infty}} \|k\|_{\mathcal{H}_\infty} + \|k(y, \theta, \theta') \Sigma(y, x, \theta')\|_{C^2} \right].
\end{aligned} \tag{4.3.7}$$

So the radius about $k = 0$ is $\frac{1}{\omega_n \text{diam}(\Omega) \text{Vol}(\Omega)^{1/2}}$.

4.4 The Case $k \in C^2$

Returning to the original problem assumptions, we start by analyzing \mathcal{L} more closely. L involves a series of terms of the form $(KT_1^{-1})^m J$ where $m \geq 1$. In order to deal with such less restrictive assumptions on k , we will need to analyze the explicit form of the integral operator $(KT_1^{-1})^m J$. The case $m = 1$ is already computed explicitly in [50], which takes the form of a weakly singular integral operator with C^2 numerator. Such operators map $L^2(\Omega)$ to $H^1(\Omega)$ with the additional C^2 parameter θ . In the case $m \geq 2$, we apply the results of Appendix B. Define

$$\alpha(x, y, r, \eta, \theta) = \Sigma(x, r, \eta)k(x, \theta, \eta),$$

where

$$\Sigma(x, s, \theta') = \exp \left(- \int_{-s}^0 \sigma(x + \tau \theta', \theta') d\tau \right).$$

Then

$$[KT_1^{-1} J f](x, \theta) = \int_{\Omega} \frac{\alpha \left(x, y, |x - y|, \frac{x - y}{|x - y|}, \theta \right)}{|x - y|^{n-1}} f(y) dy$$

More generally, for f depending on x and θ we have

$$[KT_1^{-1} f](x, \theta) = \int_{\Omega} \frac{\alpha \left(x, y, |x - y|, \frac{x - y}{|x - y|}, \theta \right)}{|x - y|^{n-1}} f \left(y, \frac{x - y}{|x - y|} \right) dy.$$

By the results of Appendix B, we have for $m \geq 2$ that

$$[(KT_1^{-1})^m Jf](x, \theta) = \int_{\Omega} K_m(x, y, \theta) f(y) dy,$$

where

$$K_m(x, y, \theta) = \int \cdots \int \frac{\alpha_m(x, \mathbf{y}_m, |x - y_1|, r_m, \widehat{x - y_1}, \widehat{\mathbf{r}_m}, \theta)}{|x - y_1|^{n-1} |y_1 - y_2|^{n-1} \cdots |y_{m-1} - y|^{n-1}} dy_1 \cdots dy_{m-1}$$

and α_m is defined inductively from α according to (B.0.1). The key point to realize is that α_m is a C^2 function of the variables $x, y_1, \dots, y_{m-1}, y, r_1, \dots, r_m, \eta_1, \dots, \eta_{m-1}, \theta$ and is, roughly speaking, a product of m copies of $k \cdot \Sigma$.

4.4.1 Estimating $I_{\sigma}^* L$

It follows that

$$I_{\sigma}^* R_+ T_1^{-1} (KT_1^{-1})^m Jf(x) = 2 \int_{\Omega} \int_{\Omega} \frac{[\overline{E}\left(x, \frac{y-x}{|y-x|}\right) E\left(y, \frac{y-x}{|y-x|}\right) K_m\left(y, z, \frac{y-x}{|y-x|}\right)]_{\text{even}}}{|x-y|^{n-1}} f(z) dy dz.$$

From Appendix B, we have that

$$\begin{aligned} & I_{\sigma}^* R_+ T_1^{-1} (KT_1^{-1})^m Jf(x) \\ &= 2 \int_{\Omega} \int_{\Omega} \frac{[\overline{E}\left(x, \frac{y-x}{|y-x|}\right) E\left(y, \frac{y-x}{|y-x|}\right) \beta_m(y, z, |y-z|, \frac{y-z}{|y-z|}, \frac{y-x}{|y-x|})]_{\text{even}}}{|x-y|^{n-1} |y-z|^{n-m}} f(z) dy dz \end{aligned}$$

where $\beta_m(y, z, r, \eta, \theta)$ is C^2 . Furthermore,

$$\|\beta_m\|_{C^2} \leq C^m \|k\|_{C^2}^m$$

where C depends on σ, Ω and n . Using a similar approach as in Appendix B and applying Proposition 1 of [50], we then obtain that

$$\|I_{\sigma}^* R_+ T_1^{-1} (KT_1^{-1})^m J\|_{L^2 \rightarrow H^1} \leq (C')^m \|k\|_{C^2}^m. \quad (4.4.1)$$

Thus

$$\|I_{\sigma}^* L\|_{L^2 \rightarrow H^1} \leq \sum_{j=0}^{\infty} \|I_{\sigma}^* R_+ T_1^{-1} (KT_1^{-1})^m J\|_{L^2 \rightarrow H^1} \leq \frac{1}{1 - C' \|k\|_{C^2}^m}. \quad (4.4.2)$$

4.4.2 Estimating L^*L and L^*I_σ

First note that when computing L^*L , after expanding the Neumann series for $(\text{Id} - KT_1^{-1})^{-1}$, we will end up with a series of terms of the form

$$(R_+T_1^{-1}(KT_1^{-1})^{m_1}J)^* R_+T_1^{-1}(KT_1^{-1})^{m_2}J.$$

In particular, estimating L^*I_σ is the special case when $m_2 = 0$, so it suffices to provide norm estimates for the above operators. Let

$$P_m := R_+T_1^{-1}(KT_1^{-1})^m J.$$

Then

$$\begin{aligned} P_m f(x, \theta) &= \left[R_+T_1^{-1} \int_{\Omega} K_m(\cdot, y, \cdot) f(y) dy \right] (x, \theta) \\ &= \int_{-\infty}^0 \exp\left(-\int_s^0 \sigma(x + \tau\theta, \theta) d\tau\right) \int_{\Omega} K_m(x + s\theta, y, \theta) f(y) dy ds. \end{aligned}$$

Recall the diffeomorphism between $\Omega \times \mathbb{S}^{n-1}$ and $\mathcal{O} = \{(x, \theta, t) \mid (x, \theta) \in \partial_+ S\Omega \text{ and } t \in (\tau_-(x, \theta), 0)\}$ given by $\phi(x, \theta) = (x + \tau_+(x, \theta)\theta, \theta, -\tau_+(x, \theta))$ and $\phi^{-1}(x, \theta, t) = (x + t\theta, \theta)$. For functions g, h in L^2 , it is straightforward to show that

$$\begin{aligned} \langle P_m^* g, h \rangle &= \langle g, P_m h \rangle \\ &= \int_{\partial_+ S\Omega} g(x, \theta) \left[\int_{-\infty}^0 \exp\left(-\int_s^0 \sigma(x + \tau\theta, \theta) d\tau\right) \int_{\Omega} K_m(x + s\theta, y, \theta) h(y) dy ds \right] d\Sigma \\ &= \int_{\Omega} \left[\int_{\Omega \times \mathbb{S}^{n-1}} g(x + \tau_+(x, \theta)\theta, \theta) \exp\left(-\int_{-\tau_+(x, \theta)}^0 \sigma(x + (\tau_+(x, \theta) + \tau)\theta, \theta) d\tau\right) \right. \\ &\quad \left. \cdot K_m(x, y, \theta) dx d\theta \right] h(y) dy \\ &= \int_{\Omega} \left[\int_{\Omega \times \mathbb{S}^{n-1}} g^\#(x, \theta) \exp\left(-\int_0^{\tau_+(x, \theta)} \sigma(x + \tau\theta, \theta) d\tau\right) K_m(x, y, \theta) dx d\theta \right] h(y) dy. \end{aligned}$$

Thus

$$P_m^* g(x) = \int_{\Omega \times \mathbb{S}^{n-1}} g^\#(y, \theta) \exp\left(-\int_0^{\tau_+(y, \theta)} \sigma(y + \tau\theta, \theta) d\tau\right) K_m(y, x, \theta) dy d\theta. \quad (4.4.3)$$

From this we can compute

$$\begin{aligned}
& P_{m_1}^* P_{m_2} f(x) \\
&= \int_{\Omega \times \mathbb{S}^{n-1}} [P_{m_2} f]^\#(y, \theta) \exp\left(-\int_0^{\tau_+(y, \theta)} \sigma(y + \tau\theta, \theta) d\tau\right) K_{m_1}(y, x, \theta) dy d\theta \\
&= \int_{\Omega} \left[\int_{\Omega \times \mathbb{S}^{n-1}} \int_{\mathbb{R}} \exp\left(-\int_s^{\tau_+(y, \theta)} \sigma(y + \tau\theta, \theta) d\tau\right) \right. \\
&\quad \left. \cdot K_{m_2}(y + s\theta, z, \theta) K_{m_1}(y, x, \theta) ds dy d\theta \right] f(z) dz.
\end{aligned}$$

We then want to analyze the integral

$$\int_{\Omega \times \mathbb{S}^{n-1}} \int_{\mathbb{R}} \exp\left(-\int_s^{\tau_+(y, \theta)} \sigma(y + \tau\theta, \theta) d\tau\right) K_{m_2}(y + s\theta, z, \theta) K_{m_1}(y, x, \theta) ds dy d\theta.$$

Using the definition of K_m as well as the computations conducted in Appendix B, this integral has the form

$$\begin{aligned}
& \int_{\Omega \times \mathbb{S}^{n-1}} \int_{\mathbb{R}} \exp\left(-\int_s^{\tau_+(y, \theta)} \sigma(y + \tau\theta, \theta) d\tau\right) \frac{\beta_{m_2}(y + s\theta, z, |y + s\theta - z|, \widehat{y + s\theta - z}, \theta)}{|z - y - s\theta|^{n-1}} \\
&\quad \cdot \frac{\beta_{m_1}(y, x, |y - x|, \widehat{y - x}, \theta)}{|x - y|^{n-1}} ds dy d\theta,
\end{aligned}$$

where $\beta_{m_i}(x, y, r, \eta, \theta)$ are C^2 functions. Let $w = s\theta$ so that $dw = |w|^{n-1} ds d\theta$. Then using the method of cutoffs employed in Appendix B, we have

$$\begin{aligned}
& \int_{\Omega \times \mathbb{R}^n} \exp\left(-\int_s^{\tau_+(y, \widehat{w})} \sigma(y + \tau\widehat{w}, \widehat{w}) d\tau\right) \frac{\beta_{m_2}(y + w, z, |y + w - z|, \widehat{y + w - z}, \theta)}{|z - y - w|^{n-1} |w|^{n-1}} \\
&\quad \cdot \frac{\beta_{m_1}(y, x, |y - x|, \widehat{y - x}, \theta)}{|x - y|^{n-1}} dw dy \\
&= \frac{\beta(x, z, |x - z|, \widehat{x - z})}{|x - z|^{n-m_1-m_2}}
\end{aligned}$$

where $\beta(x, z, r, \theta) \in C^2$ and $\|\beta\|_{C^2} \leq (3C)^{m_1+m_2} \|k\|_{C^2}^{m_1+m_2}$. Thus

$$\|L^* L\|_{L^2 \rightarrow H^1} \leq \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} (C')^{m_1+m_2} \|k\|_{C^2}^{m_1+m_2} = \left(\frac{C' \|k\|_{C^2}}{1 - C' \|k\|_{C^2}} \right)^2. \quad (4.4.4)$$

4.4.3 Proof of Theorem

We are now ready to prove the main result, stated in Theorem 4.2.1. As mentioned earlier, the argument is mostly identical to that of Theorem 2.2.1, except that now we need not resort to the analytic Fredholm theorem.

Proof of Theorem 4.2.1. First assume that $\sigma \in C^\infty$ and is such that I_σ is injective. This is the case if σ is real analytic. Since injectivity of $X^*X = I_\sigma^*I_\sigma + \mathcal{L}$ implies injectivity of X , we may ensure that X is injective simply by bounding the norm of \mathcal{L} . We do this by making the restriction $\|k\|_{C^2} < \epsilon_1$ for a suitably small $\epsilon_1 > 0$, since the $L^2 \rightarrow H^1$ norm of \mathcal{L} depends continuously on $\|k\|_{C^2}$. Moreover, since \mathcal{L} is also continuously dependent $\|\sigma\|_{C^2}$, we may assume that ϵ_1 is chosen locally uniformly so that for any $\bar{\sigma} \in C^2$ with $\|\bar{\sigma} - \sigma\|_{C^2} < \epsilon_1$, we have the same bound on \mathcal{L} for any $\|k\|_{C^2} < \epsilon_1$.

From Equation (4.3.1) we have that $(\text{Id} - KT_1^{-1})^{-1}$ exists and is bounded if $\|k\|_{C^2} < [\omega_n \text{diam}(\Omega) \text{Vol}(\Omega)^{1/2}]^{-1} =: \epsilon_2$. Recall that $I_\sigma^*I_\sigma$ is elliptic and thus there is a parametrix Q of order 1 which we view as an operator $Q : H^1(\Omega_1) \rightarrow L^2(\Omega)$. So for $f \in L^2(\Omega)$, we have

$$QI_\sigma^*I_\sigma f = f + K_1 f,$$

where K_1 is a pseudodifferential operator of order -1 . Now apply Q to X^*X to get

$$QX^*Xf = f + K_1 f + Q\mathcal{L}f. \quad (4.4.5)$$

As in the proof of Theorem 2.2.1, we may add a finite rank operator to Q to ensure that $\text{Id} + K_1$ is injective. Then using the estimates (4.4.4), (4.4.2) on $\|L^*L\|_{L^2 \rightarrow H^1}$, $\|L^*I_\sigma\|_{L^2 \rightarrow H^1}$, and $\|I_\sigma^*L\|_{L^2 \rightarrow H^1}$, and choosing $\epsilon_3 > 0$ small enough, we may ensure that $\|Q\mathcal{L}\|_{L^2 \rightarrow L^2} < 1$ whenever $\|k\|_{C^2} < \epsilon_3$. Hence $\text{Id} + Q\mathcal{L}$ is invertible. We take a moment to remark that ϵ_3 is also locally uniform in k and σ in the sense that σ and k can be slightly perturbed in C^2 and the estimate $\|Q\mathcal{L}\|_{L^2 \rightarrow L^2} < 1$ still holds. Applying the inverse to both sides, we have

$$(\text{Id} + Q\mathcal{L})^{-1}QX^*Xf = f + (\text{Id} + Q\mathcal{L})^{-1}K_1 f, \quad (4.4.6)$$

where the operator involving K_1 is still compact. We then have the estimate

$$\begin{aligned} \|f\|_{L^2(\Omega)} &\leq \|(\text{Id} + Q\mathcal{L})^{-1}QX^*Xf\|_{L^2(\Omega)} + \|(\text{Id} + Q\mathcal{L})^{-1}K_1 f\|_{L^2(\Omega)} \\ &\leq C \left(\|X^*Xf\|_{H^1(\Omega)} + \|(\text{Id} + Q\mathcal{L})^{-1}K_1 f\|_{L^2(\Omega)} \right). \end{aligned}$$

X^*X is injective by the fact that $\|k\|_{C^2} < \epsilon_1$, and so the estimate $\|f\|_{L^2(\Omega)} \leq C\|X^*Xf\|_{H^1(\Omega)}$ then follows from Lemma 2.5.4. Finally, by perturbing σ slightly in C^2 we get the injectivity of X for a generic dense open set, and for each such σ we can take any $k \in C^2$ with $\|k\|_{C^2} < \epsilon =: \min\{\epsilon_1, \epsilon_2, \epsilon_3\}$. \square

Chapter 5

NUMERICAL SCHEME FOR SOURCE RECOVERY

5.1 Introduction and Assumptions

We now consider the complementary problem of numerical computation of solutions to (1.2.1) and using the accessible part of the synthesized data to visualize the results of Chapters 2 and 3. To this end, we employ a technique from [30] that uses rotations applied to the parameters in the spectral domain to help eliminate the ray effect, which is a byproduct of the discrete ordinates method. It should be noted that in principle such a method could be applied in any dimension, but for the actual computations we will restrict ourselves to the two dimensional case. We also refer the reader to [19] for another approach to solving the direct transport equation using finite element methods, which we do not use here.

The main goal of this chapter is to approximate the operators X^*Xf and $X_V^*X_Vf$ in the two-dimensional case. Ignoring the technical details for the moment, the idea is to utilize a Neumann series to approximate $(\text{Id} - KT_1^{-1})^{-1}$ when computing the forward solution operator X and adjoint X^* . As such, one must truncate the series to a finite number of terms when computing X^*X , and the difference of the approximation from the true function X^*Xf will have some Sobolev regularity. The only theoretical result of this chapter is Lemma 5.4.1, which essentially asserts that, if m_1 terms are computed in the approximation of X and m_2 terms are computed in the approximation of X^* , then $[X^*X]_{\text{approx}}f - X^*Xf \in H^{l+m+1}(\Omega)$, where $m = \min\{m_1, m_2\}$ and $f \in H^l(\Omega)$.

As previously assumed, let Ω be a convex bounded domain in \mathbb{R}^2 and again recall the transport equation

$$\begin{aligned} \theta \cdot \nabla_x u + \sigma(x, \theta)u - \int_{\mathbb{S}^1} k(x, \theta', \theta)u(x, \theta') d\theta' &= f(x), \quad \text{in } \Omega \times \mathbb{S}^1 \\ u(x, \theta) &= 0 \quad \text{on } \partial_- S\Omega. \end{aligned}$$

We need to generate the solution to the radiative transport equation, given by

$$\begin{aligned} u &= T_1^{-1}(\text{Id} - KT_1^{-1})^{-1}Jf \\ &= T_1^{-1} \sum_{m=0}^{\infty} (KT_1^{-1})^m Jf \end{aligned}$$

If we set $u_0(x, \theta) := [T_1^{-1}f](x, \theta)$ and set $u_l := T_1^{-1}Ku_{l-1}$ for $l \geq 1$, then $u = \sum_{l \geq 0} u_l$. In practice, it is quite simple to compute the scattering term Ku_{l-1} at each iteration. In order to compute T_1^{-1} we will use a simple first order Euler method to solve the associated differential equation to which T_1^{-1} is the solution operator:

$$\begin{aligned} \theta \cdot \nabla_x u(x, \theta) + \sigma(x, \theta)u(x, \theta) &= g(x, \theta), \quad (x, \theta) \in \Omega \times \mathbb{S}^{n-1} \\ u|_{\partial_- S\Omega} &\equiv 0. \end{aligned}$$

5.2 Numerical Method for the Direct Problem

We now present a detailed summary of the method used to solve (1.2.1) for $n = 2$. Note that at this stage it is not important that the source f be isotropic, but such an assumption will be used when computing the normal operator later on. As in [30] we use the source iteration method, which requires one to solve problems of the form

$$\begin{aligned} \theta \cdot \nabla_x u + \sigma(x, \theta)u &= g(x, \theta) \\ u|_{\partial_- S\Omega} &= 0. \end{aligned} \tag{5.2.1}$$

Without loss of generality, we may take $\Omega = \mathbb{D} \subset \mathbb{R}^2$ where \mathbb{D} is the unit disk, and assume that σ, k and f are all supported compactly in $\mathbb{D} \times \mathbb{S}^1$ or \mathbb{D} as appropriate. Such an assumption can be justified by finding a ball $B(0, R)$ large enough and rescaling the coordinates accordingly. The main advantage here from a numerical standpoint is that Ω remains invariant under rotations. Now, for actually computing T_1^{-1} , it will be easier to have boundary conditions on a cartesian domain. To that end, for each $\eta \in (0, 2\pi)$ denote $\theta = \theta(\eta) = (\cos(\eta), \sin(\eta))$ and let $\theta^\perp = \theta(\eta)^\perp$ be the counterclockwise rotation of θ by $\pi/2$. That is $\theta(\eta)^\perp = (-\sin \eta, \cos \eta)$. For each $\eta \in (0, 2\pi)$ define the η -dependent square

$$C_\eta = \{x \in \mathbb{R}^2 \text{ such that } |x \cdot \theta| < 1 \text{ and } |x \cdot \theta^\perp| < 1\}.$$

In short, C_η is the square of side length 2 rotated by an angle η . The corresponding incoming and outgoing sets (analogous to $\partial_\pm S\Omega$) are given by

$$\Gamma_{\pm,\eta} = \{x \in \partial C_\eta \text{ such that } x \cdot \theta = \pm 1 \text{ and } |x \cdot \theta^\perp| < 1\}.$$

At the heart of the rotational method, we perform a rotational change coordinates so that the derivative in the direction θ becomes ∂_x . Fix an angle $\eta \in [0, 2\pi)$ and for $x = (\mathbf{x}, \mathbf{y}) \in [-1, 1]^2$ define

$$u_\eta(\mathbf{x}, \mathbf{y}) = [u]_\eta(\mathbf{x}, \mathbf{y}) := u(\mathbf{x} \cos \eta - \mathbf{y} \sin \eta, \mathbf{x} \sin \eta + \mathbf{y} \cos \eta, \eta) \quad (5.2.2)$$

$$v_\eta(\mathbf{x}, \mathbf{y}) = [v]_\eta(\mathbf{x}, \mathbf{y}) := v(\mathbf{x} \cos \eta - \mathbf{y} \sin \eta, \mathbf{x} \sin \eta + \mathbf{y} \cos \eta, \eta) \quad (5.2.3)$$

where u, v are functions on \mathbb{R}^2 and $\mathbb{R}^2 \times \mathbb{S}^1$, respectively. The bracket notation will only be used for functions which already have a subscript. With this change of variables, we can rewrite (5.2.1) as

$$\frac{\partial}{\partial x_1} u_\eta(\mathbf{x}, \mathbf{y}) + \sigma_\eta(\mathbf{x}, \mathbf{y}) u_\eta(\mathbf{x}, \mathbf{y}) = g_\eta(\mathbf{x}, \mathbf{y}), \quad (5.2.4)$$

$$u_\eta(-1, \mathbf{y}) = 0 \quad \text{on } \Gamma_{-,\eta}$$

We remark that for a general (possibly non-zero) function $h \in L^1(\partial_- S\mathbb{D})$, we would have the boundary condition \tilde{h}_η on $\Gamma_{-,\eta}$, obtained by projecting h onto $\Gamma_{-,\eta}$ via the relation $\tilde{h}_\eta(x_1, x_2) = h(P_-^{-1}(x_1, x_2, \eta))$, where

$$P_\pm : \partial_\pm S\mathbb{D} \ni (x, \eta) \mapsto P_\pm(x, \eta) = \pm\theta(\eta) - \det(x, \theta(\eta))\theta(\eta)^\perp \in \Gamma_{\pm,\eta}.$$

Since the measurements are necessarily discrete, we introduce the fixed parameters N_x , N_d and N_{scat} to represent the number of grid points in each spatial dimension, the number of directions measured, and the number of scattering terms computed in the series $T_1^{-1} \sum_{m=0}^{\infty} (KT_1^{-1})^m J$, respectively. Typically, we will take each such parameter to be a power of 2, since FFT algorithms are used to rotate the grids. Then the basic idea for computing T_1^{-1} is to solve equations of the type (5.2.4) by first computing g_η and σ_η by rotating the images of σ and $g(\cdot, \eta)$ clockwise by the angle η . Then we solve (5.2.4) for u_η , which can be done using a standard ODE solver along each column of the cartesian grid. Specifically, denote $s_x := \frac{2}{N_x}$, set $u_\eta(x_1, :) = 0$ and consider the cartesian grid

$\{\mathbf{x}_1, \dots, \mathbf{x}_{N_x}\} \otimes \{\mathbf{y}_1, \dots, \mathbf{y}_{N_x}\}$ where $\mathbf{x}_i, \mathbf{y}_i = -1 + (i - \frac{1}{2})s_x$ for $1 \leq i \leq N_x$. Using a first order Euler method, we have the iterates

$$u_\eta(x_j, \mathbf{y}) = u_\eta(x_{j-1}, \mathbf{y}) + s_x (g_\eta(x_{j-1}, \mathbf{y}) - \sigma_\eta(x_{j-1}, \mathbf{y})u_\eta(x_{j-1}, \mathbf{y})), \quad 2 \leq j \leq N_x.$$

In order to compute K , we define the angular step size $\delta := \frac{2\pi}{N_d}$ and sum over the set of angles $\{\eta_1, \dots, \eta_{N_d}\}$ where $\eta_i = (1 - \frac{1}{2})\delta$. We can then approximate K by the discrete operator K_δ which has the formula

$$K_\delta g(x, \eta_i) = \delta \sum_{j=1}^{N_d} g(x, \eta_j) k(x, \eta_j, \eta_i), \quad x \in \Omega, 1 \leq i \leq N_d. \quad (5.2.5)$$

5.2.1 Computing X_V

We can now describe the iterative scheme to numerically solve (1.2.1) and simulate the partial data $X_V f$ with respect to some subset $V \subset \partial_+ S\Omega$. We summarize the steps in Algorithm 1.

What we have not yet described in much detail is the method used to compute the rotations of each function, which we discuss briefly as follows. For more details however, we refer the reader to [30]. The general idea is to write the rotation map $r_\eta(\mathbf{x}, \mathbf{y}) = (\mathbf{x} \cos \eta + \mathbf{y} \cos \eta, -\mathbf{x} \sin \eta + \mathbf{y} \cos \eta)$ as a composition of dilations and shearing/slant operations in each variable separately. In particular, we can write

$$\begin{aligned} r_\eta &= d_{\mathbf{x}, \frac{1}{\cos \eta}} \circ s_{\mathbf{x}, -\sin \eta} \circ d_{\mathbf{y}, \cos \eta} \circ s_{\mathbf{y}, \tan \eta}, \\ s_{\mathbf{y}, \alpha}(\mathbf{x}, \mathbf{y}) &= (\mathbf{x}, \mathbf{y} - \alpha \mathbf{x}), \quad s_{\mathbf{x}, \beta}(\mathbf{x}, \mathbf{y}) = (\mathbf{x} - \beta \mathbf{y}, \mathbf{y}), \quad \alpha, \beta \in \mathbb{R} \\ d_{\mathbf{x}, t}(\mathbf{x}, \mathbf{y}) &= (t\mathbf{x}, \mathbf{y}), \quad d_{\mathbf{y}, t}(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, t\mathbf{y}), \quad t \in \mathbb{R}. \end{aligned}$$

The shearing/slant operations $s_{\mathbf{x}, \beta}$ and $s_{\mathbf{y}, \alpha}$ are implemented in phase space using a periodic interpolation function together with some identities utilizing the discrete forward and inverse Fourier transform. Specifically, we first embed the $N_x \times N_x$ image into a $2N_x \times N_x$ image, padding the top and bottom of the image with $\frac{N_x}{2} \times N_x$ zeros. We then perform the vertical shearing operation $s_{\mathbf{y}, \alpha}$ by independently shifting each column of the image by an amount that depends linearly on the column index with factor $-\alpha$.

Algorithm 1 Computing $X_V f$

- 1: Let u_0 solve $\theta \cdot \nabla u + \sigma u = f$
 - 2: **for** $i = 1$ to N_d **do**
 - 3: **compute** σ_{η_i} and f_{η_i}
 - 4: **solve** $\partial_{x_1}[u_0]_{\eta_i} + \sigma_{\eta_i}[u_0]_{\eta_i} = f_{\eta_i}$ for $[u_0]_{\eta_i}$
 - 5: $u_0(\mathbf{x}, \mathbf{y}, \eta_i) \leftarrow [[u_0]_{\eta_i}]_{-\eta_i}(\mathbf{x}, \mathbf{y})$
 - 6: **end for**
 - 7: **for** $j = 1$ to N_{scat} **do**
 - 8: $f_j \leftarrow K_\delta u_{j-1}$
 - 9: **for** $i = 1$ to N_d **do**
 - 10: **compute** $[f_j]_{\eta_i}$
 - 11: **solve** $\partial_{x_1}[u_j]_{\eta_i} + \sigma_{\eta_i}[u_j]_{\eta_i} = [f_j]_{\eta_i}$ for $[u_j]_{\eta_i}$
 - 12: $u_j(\mathbf{x}, \mathbf{y}, \eta_i) \leftarrow [[u_j]_{\eta_i}]_{-\eta_i}(\mathbf{x}, \mathbf{y})$
 - 13: **end for**
 - 14: **end for**
 - 15: $u \leftarrow \sum_{j=0}^{N_{\text{scat}}} u_j$
 - 16: **compute** $\chi_V u \Big|_{\partial_+ S\mathbb{D}}$
-

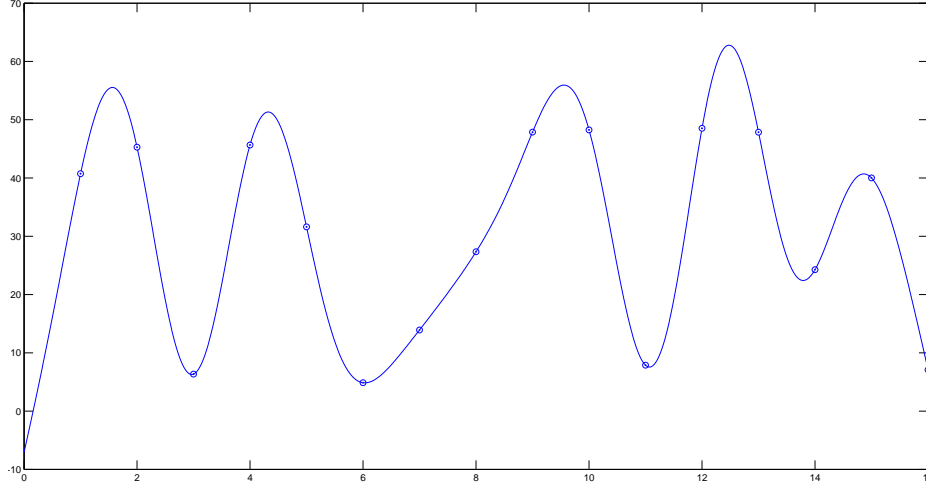


Figure 5.1: The values of a random 16-vector x are plotted while overlaid with the spectral interpolant $\tilde{x}(y) = \sum_{l=1}^m x_l D_m(y-l)$ for $y \in [0, 16]$.

The operation of shifting a vector $x = [x_1, \dots, x_m]^T$ by an amount s is done in phase space. First we define the $2m$ -periodic function

$$D_m(y) = \frac{\sin(\pi y)}{m \sin\left(\frac{\pi y}{m}\right)}, \quad y \in \mathbb{R}. \quad (5.2.6)$$

We then define the spectral interpolant

$$\tilde{x}(y) := \sum_{l=1}^m x_l D_m(y-l), \quad (5.2.7)$$

which coincides with x_j when $y = j$ and interpolates between those values sinusoidally (see Figure 5.1). The spectrally shifted vector x_s is given by

$$x_s = [\tilde{x}(1+s), \tilde{x}(2+s), \dots, \tilde{x}(m+s)]^T.$$

Note that x_s is not a priori defined for $s \in \mathbb{Z}$, but it can be extended continuously to such points due to the structure of the singularities of D_m .

Now, in practice we have first padded the image above and below with zeros, so we have $m = 2N_x$ which is even. It is then straightforward to verify that

$$D_{2N_x}(t) = \frac{\sin(\pi t)}{2N_x \sin\left(\frac{\pi t}{2N_x}\right)} = \frac{1}{2N_x} \sum_{l=-N_x}^{N_x-1} e^{i\frac{\pi}{N_x}(l+\frac{1}{2})t}, \quad t \in [0, 2N_x].$$

Recall the N -point discrete Fourier and inverse Fourier transforms, given by

$$\begin{aligned} X(k) &= F_{j \rightarrow k}^N[x(j)] = \sum_{j=1}^N x(j) e^{-\frac{2\pi i}{N}(j-1)(k-1)}, \quad k = 1, \dots, N, \\ x(j) &= F_{k \rightarrow j}^{-1,N}[X(k)] = \frac{1}{N} \sum_{k=1}^N X(k) e^{2\pi i N(j-1)(k-1)}, \quad j = 1, \dots, N. \end{aligned}$$

It is then possible to write $\tilde{x}(l-s)$ for $l = 1, \dots, 2N_x$ as a composition of discrete Fourier transforms and inverse transforms and multiplication by complex exponentials. In particular,

$$\tilde{x}(l-s) = e^{i\frac{\pi}{N_x}(-N_x+\frac{1}{2})(l-1)} F_{k \rightarrow l}^{-1,2N_x} \left[e^{-i\frac{\pi}{N_x}(k-1-N_x+\frac{1}{2})s} F_{j \rightarrow k}^{2N_x} \left[x(j) e^{-i\frac{\pi}{N_x}(-N_x+\frac{1}{2})(j-1)} \right] \right]. \quad (5.2.8)$$

The dilation operations $d_{\mathbf{x},t}$ and $d_{\mathbf{y},t}$ are computed via a resampling done in the Fourier domain. In particular, we must resample a vector x of size $2N_x$ down to a vector \bar{x} of size N_x but with a different step size. We can do this by using the N -point fractional Fourier transform with coefficient α , defined by

$$X(l) = G_{k \rightarrow l}^{N,\alpha}[x] = \sum_{k=1}^N x(k) e^{-2\pi i \alpha (k-1)(l-1)}. \quad (5.2.9)$$

For example, if we start with a vector x sampled at the gridpoints $\{j-1\}_{j=1}^{2m}$ and we want to shift x by s (i.e. sample at $\{j-1+s\}_{j=1}^{2m}$) and then resample back to a vector \bar{x} taking values at the m points $y_l = s + h(l-1)$ for $l = 1, \dots, m$, then

$$\bar{x}(y_l) = \frac{1}{2m} e^{i\frac{\pi}{m}(-m+\frac{1}{2})h(l-1)} G_{k \rightarrow l}^{2m, \frac{-h}{2m}} \left[e^{i\frac{\pi}{m}(k-m-1+\frac{1}{2})s} F_{j \rightarrow k}^{2m} \left[x(j) e^{-i\frac{\pi}{m}(-m+\frac{1}{2})(j-1)} \right] \right]. \quad (5.2.10)$$

This corresponds to a vertical dilation composed with a vertical shift (see [4, 30]).

5.3 Approximating the Normal Operator

After approximating the solution to the forward problem and restricting to $\partial_+ S\Omega$ to obtain the simulated data Xf , we can then proceed with computing X^*Xf , and similarly, $X_V^*X_V f$.

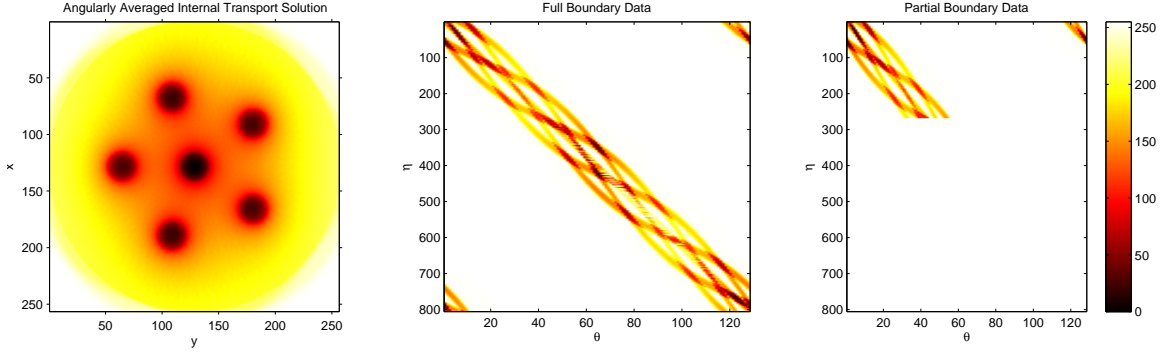


Figure 5.2: The angularly averaged solution to (1.2.1) corresponding to the internal source given in Figure 5.3 (left). The other two images show the complete and partial data Xf and $X_V f$, respectively, where V is the outgoing conic set above the arc of $\partial\mathbb{D}$ defined by $\eta \in [0, \pi/3]$. The axes are labeled such that θ is the transport direction and η is the position along the boundary $\partial\mathbb{D}$, with $\eta = 0$ corresponding to the point $(1, 0)$.

Recall that for any $N \in \mathbb{N}$

$$\begin{aligned} X_V f &= \chi_V R_+ T_1^{-1} (\text{Id} - K T_1^{-1})^{-1} J f \\ &= \chi_V R_+ T_1^{-1} \sum_{m=0}^N (K T_1^{-1})^m J f + \chi_V R_+ T_1^{-1} (K T_1^{-1})^{N+1} (\text{Id} - K T_1^{-1})^{-1} J f \end{aligned}$$

so that

$$X_V^* = J^* \left(\sum_{m=0}^{\infty} ([T_1^{-1}]^* K^*)^m \right) [\chi_V R_+ T_1^{-1}]^* \quad (5.3.1)$$

To numerically compute X_V^* , we separately compute the three types of terms appearing in (5.3.1).

Since $J : L^2(\Omega) \rightarrow L^2(\Omega \times \mathbb{S}^{n-1})$, let us compute

$$\begin{aligned} \langle [J^* g](x), h(x) \rangle_{L^2(\Omega)} &= \langle g(x, \theta), [Jh](x, \theta) \rangle_{L^2(\Omega \times \mathbb{S}^{n-1})} \\ &= \int_{\Omega \times \mathbb{S}^{n-1}} h(x) g(x, \theta) d\theta dx \\ &= \int_{\Omega} h(x) \int_{\mathbb{S}^{n-1}} g(x, \theta) d\theta dx. \end{aligned}$$

Thus

$$J^* g(x) = \int_{\mathbb{S}^{n-1}} g(x, \theta) d\theta. \quad (5.3.2)$$

Next, recall that $K : L^2(\Omega \times \mathbb{S}^{n-1}) \rightarrow L^2(\Omega \times \mathbb{S}^{n-1})$ and we compute

$$\begin{aligned} \langle K^*g(x, \theta), h(x, \theta) \rangle_{L^2(\Omega \times \mathbb{S}^{n-1})} &= \langle g(x, \theta), Kh(x, \theta) \rangle_{L^2(\Omega \times \mathbb{S}^{n-1})} \\ &= \int_{\Omega \times \mathbb{S}^{n-1}} g(x, \theta) \int_{\mathbb{S}^{n-1}} k(x, \theta, \theta') h(x, \theta') d\theta' dx d\theta \\ &= \int_{\Omega \times \mathbb{S}^{n-1}} h(x, \theta) \int_{\mathbb{S}^{n-1}} k(x, \theta', \theta) g(x, \theta') d\theta' dx d\theta. \end{aligned}$$

Thus

$$K^*g(x, \theta) = \int_{\mathbb{S}^{n-1}} k(x, \theta', \theta) g(x, \theta') d\theta'. \quad (5.3.3)$$

In the isotropic scattering case, this gives

$$K^*g(x, \theta) = k(x) \int_{\mathbb{S}^{n-1}} g(x, \theta') d\theta'. \quad (5.3.4)$$

We will need the discretized version of (5.3.3) in the same vain as (5.2.5), which is given by

$$K_\delta^*g(x, \eta_i) = \delta \sum_{j=1}^{N_d} k(x, \eta_j, \eta_i) g(x, \eta_j), \quad 1 \leq i \leq N_d \quad (5.3.5)$$

It is also a straightforward computation to verify that $T_1^{-1} : L^2(\Omega \times \mathbb{S}^{n-1}) \rightarrow L^2(\Omega \times \mathbb{S}^{n-1})$ has adjoint

$$[T_1^{-1}]^*g(x, \theta) = \int_0^\infty \exp\left(-\int_0^s \sigma(x + \tau\theta, \theta) d\tau\right) g(x + s\theta, \theta) ds. \quad (5.3.6)$$

However, in order to apply $[T_1^{-1}]^*$ numerically, it is easier to use the associated first order differential equation for which it is the solution operator. We already know that T_1^{-1} is the solution operator for the ODE $\theta \cdot \nabla + \sigma$. So naturally $-\theta \cdot \nabla + \sigma$ is the associated ODE for $[T_1^{-1}]^*$. To see this, given $f, \phi \in C^\infty(\Omega \times \mathbb{S}^{n-1})$ we have

$$\begin{aligned} &\langle (-\theta \cdot \nabla + \sigma(x, \theta))[T_1^{-1}]^*f(x, \theta), \phi(x, \theta) \rangle_{L^2(\Omega \times \mathbb{S}^{n-1})} \\ &= \langle [T_1^{-1}]^*f(x, \theta), \theta \cdot \nabla \phi(x, \theta) + \sigma(x, \theta)\phi(x, \theta) \rangle_{L^2(\Omega \times \mathbb{S}^{n-1})} \\ &= \langle f(x, \theta), T_1^{-1}(\theta \cdot \nabla \phi(x, \theta) + \sigma(x, \theta)\phi(x, \theta)) \rangle_{L^2(\Omega \times \mathbb{S}^{n-1})} \\ &= \langle f(x, \theta), \phi(x, \theta) \rangle_{L^2(\Omega \times \mathbb{S}^{n-1})}. \end{aligned}$$

Thus we can again use a first order Euler method to compute $[T_1^{-1}]^*$ just as with T_1^{-1} .

Finally, we must compute $[\chi_V R_+ T_1^{-1}]^* : L^2(\partial_+ S\Omega, d\Sigma) \rightarrow L^2(\Omega \times \mathbb{S}^{n-1})$. Observe that for $g \in L^2(\partial_+ S\Omega, d\Sigma)$ and $h \in L^2(\Omega \times \mathbb{S}^{n-1})$

$$\begin{aligned}
& \langle [\chi_V R_+ T_1^{-1}]^* g(x, \theta), h(x, \theta) \rangle_{L^2(\Omega \times \mathbb{S}^{n-1})} \\
&= \langle g(x, \theta), \chi_V R_+ T_1^{-1} h(x, \theta) \rangle_{L^2(\partial_+ S\Omega, d\Sigma)} \\
&= \int_{\partial_+ S\Omega} g(x, \theta) \chi_V(x, \theta) \int_{-\infty}^0 \exp\left(-\int_s^0 \sigma(x + \tau\theta, \theta) d\tau\right) h(x + s\theta, \theta) ds d\Sigma \\
&= \int_{\Omega \times \mathbb{S}^{n-1}} g(x + \tau_+(x, \theta)\theta, \theta) \chi_V(x + \tau_+(x, \theta)\theta, \theta) \\
&\quad \cdot \exp\left(-\int_{-\tau_+(x, \theta)}^0 \sigma(x + (\tau_+(x, \theta) + \tau)\theta, \theta) d\tau\right) h(x, \theta) dx d\theta \\
&= \int_{\Omega \times \mathbb{S}^{n-1}} g^\#(x, \theta) \chi_V^\#(x, \theta) \exp\left(-\int_0^{\tau_+(x, \theta)} \sigma(x + \tau\theta, \theta) d\tau\right) h(x, \theta) dx d\theta \\
&= \int_{\Omega \times \mathbb{S}^{n-1}} g^\#(x, \theta) \chi_V^\#(x, \theta) E(x, \theta) h(x, \theta) dx d\theta
\end{aligned}$$

Thus

$$[\chi_V R_+ T_1^{-1}]^* g(x, \theta) = g^\#(x, \theta) \chi_V^\#(x, \theta) E(x, \theta). \quad (5.3.7)$$

Recall the truncation parameter $N_{\text{scat}} \in \mathbb{N}$ corresponding to how many terms in the Neumann series $\sum_{m=0}^{\infty} ([T_1^{-1}]^* K^*)^m$ to use. We proceed as in Algorithm 2.

Algorithm 2 Computing $X_V^* X_V f$

- 1: $v_0(x, \eta) \leftarrow [\chi_V R_+ T_1^{-1}]^* X_V f(x, \eta) = (X_V f)^\#(x, \eta) E(x, \eta)$
 - 2: **for** $j = 1$ to N_{scat} **do**
 - 3: $v_j \leftarrow K_\delta^* v_{j-1}$ ▷ Apply K^*
 - 4: **for** $i = 1$ to N_d **do**
 - 5: $w_{\eta_i}(\mathbf{x}_{N_x}, \mathbf{y}) = 0$ ▷ Apply $[T^{-1}]^*$
 - 6: **solve** $-\partial_{x_1} w_{\eta_i} + \sigma_{\eta_i} w_{\eta_i} = [v_j]_{\eta_i}$ for w_{η_i} .
 - 7: $v_j(\mathbf{x}, \mathbf{y}, \eta_i) \leftarrow w = [w_{\eta_i}]_{-\eta_i}(\mathbf{x}, \mathbf{y})$
 - 8: **end for**
 - 9: **end for**
 - 10: $v(\mathbf{x}, \mathbf{y}) \leftarrow \delta \sum_{i=1}^{N_d} \sum_{j=1}^{N_{\text{scat}}} v_j(\mathbf{x}, \mathbf{y}, \eta_i)$ ▷ Apply J^*
 - 11: ▷ v is an approximation to $X_V^* X_V f$
-

5.4 Smoothness Analysis

When trying to recover microlocally the most singular part of the source via $X_V^* X_V$ as analyzed in Chapter 3, it turns out that in theory only the first term of the Neumann series is needed. We summarize this in the following Lemma.

Lemma 5.4.1. Suppose $f \in H^l(\Omega)$ and $\sigma, k \in C^\infty$. Let $m_1, m_2 \geq 0$. Then

$$X_V^* X_V f - \sum_{j=0}^{m_2} \sum_{i=0}^{m_1} [\chi_V R_+ T_1^{-1} (K T_1^{-1})^j J]^* \chi_V R_+ T_1^{-1} (K T_1^{-1})^i J f \in H^{l+m+1}(\Omega_1)$$

where $m = \min\{m_1, m_2\}$.

Proof. Observe that

$$\begin{aligned} X_V^* &= [\chi_V R_+ T_1^{-1} (I - K T_1^{-1})^{-1} J]^* \\ &= \sum_{j=0}^{m_2} [\chi_V R_+ T_1^{-1} (K T_1^{-1})^j J]^* + [\chi_V R_+ T_1^{-1} (K T_1^{-1})^{m_2+1} (I - K T_1^{-1})^{-1} J]^* \\ X_V &= \chi_V R_+ T_1^{-1} (I - K T_1^{-1})^{-1} J \\ &= \sum_{i=0}^{m_1} \chi_V R_+ T_1^{-1} (K T_1^{-1})^i J + \chi_V R_+ T_1^{-1} (K T_1^{-1})^{m_1+1} (I - K T_1^{-1})^{-1} J. \end{aligned}$$

Thus

$$\begin{aligned} X_V^* X_V f &- \sum_{j=0}^{m_2} \sum_{i=0}^{m_1} [\chi_V R_+ T_1^{-1} (K T_1^{-1})^j J]^* \chi_V R_+ T_1^{-1} (K T_1^{-1})^i J f \\ &= \sum_{i=0}^{m_1} [\chi_V R_+ T_1^{-1} (I - K T_1^{-1})^{-1} (K T_1^{-1})^{m_2+1} J]^* \chi_V R_+ T_1^{-1} (K T_1^{-1})^i J f \\ &+ \sum_{j=0}^{m_2} [\chi_V R_+ T_1^{-1} (K T_1^{-1})^j J]^* \chi_V R_+ T_1^{-1} (K T_1^{-1})^{m_1+1} (I - K T_1^{-1})^{-1} J f \\ &+ [\chi_V R_+ T_1^{-1} (I - K T_1^{-1})^{-1} (K T_1^{-1})^{m_2+1} J]^* \chi_V R_+ T_1^{-1} (K T_1^{-1})^{m_1+1} (I - K T_1^{-1})^{-1} J f \\ &=: A_1 + A_2 + A_3. \end{aligned}$$

First note that

$$[\chi_V R_+ T_1^{-1}]^* \chi_V R_+ T_1^{-1} f(x, \theta) = \chi_V(x, \theta)^2 \int \exp\left(-\int_s^\infty \sigma(x + \tau\theta, \theta) d\tau\right) f(x + s\theta, \theta) ds.$$

In particular, $[\chi_V R_+ T_1^{-1}]^* \chi_V R_+ T_1^{-1}$ is bounded on $H^j(\Omega \times \mathbb{S}^{n-1})$ for all j . Moreover, from the proof of Proposition 3.3.1 we have that $[(\text{Id} - K T_1^{-1})^{-1}]^*$ can be taken to be

bounded on $H^j(\Omega \times \mathbb{S}^{n-1})$ whenever $(\text{Id} - KT_1^{-1})^{-1}$ is, which we can assume is the case for $j \leq l + \max\{m_1, m_2\} + 1$. From the discussion in Appendix B, we have that

$$[(KT_1^{-1})^{m_2+1}J]^*g(x) = \int \int \frac{\beta_{m_2+1}(y, x, |y-x|, \widehat{y-x}, \theta)}{|y-x|^{n-m}} g(y, \theta) dy d\theta.$$

Using a similar argument as in Lemma 2 of [50] and applying Proposition B.2.1, we have that $[(KT_1^{-1})^{m_2+1}J]^*$ is bounded from $H^l(\Omega \times \mathbb{S}^{n-1}) \rightarrow H^{l+m_2+1}(\Omega \times \mathbb{S}^{n-1})$.

Now, to analyze A_1 we write

$$\begin{aligned} A_1 &= \sum_{i=0}^{m_1} [\chi_V R_+ T_1^{-1} (\text{Id} - KT_1^{-1})^{-1} (KT_1^{-1})^{m_2+1} J]^* \chi_V R_+ T_1^{-1} (KT_1^{-1})^i J \\ &= [(KT_1^{-1})^{m_2+1} J]^* [(\text{Id} - KT_1^{-1})^{-1}]^* [\chi_V R_+ T_1^{-1}]^* \chi_V R_+ T_1^{-1} \sum_{i=0}^{m_1} (KT_1^{-1})^i J, \end{aligned}$$

and from this it is clear that A_1 maps $H^l(\Omega)$ to $H^{l+m_2+1}(\Omega_1)$. Similarly,

$$\begin{aligned} A_2 &= \sum_{j=0}^{m_2} [\chi_V R_+ T_1^{-1} (KT_1^{-1})^j J]^* \chi_V R_+ T_1^{-1} (\text{Id} - KT_1^{-1})^{-1} (KT_1^{-1})^{m_1+1} J \\ &= \sum_{j=0}^{m_2} [(KT_1^{-1})^j J]^* [\chi_V R_+ T_1^{-1}]^* \chi_V R_+ T_1^{-1} (\text{Id} - KT_1^{-1})^{-1} (KT_1^{-1})^{m_1+1} J \end{aligned}$$

and so A_2 maps $H^l(\Omega)$ to $H^{l+m_1+1}(\Omega_1)$. A similar argument shows that A_3 maps $H^l(\Omega)$ to $H^{l+m_1+m_2+2}(\Omega_1)$. \square

5.5 Numerical Computations

For our numerical computations, the goal is to provide visual verification of Theorem 3.2.1, and more specifically, of the related result of Lemma 5.4.1. In all examples, we use a cartesian grid of 256 by 256 with 128 directions θ . We'll use the notation (x, y) to denote a point in \mathbb{R}^2 . In order to incorporate anisotropy in k , we use the Henyey-Greenstein (H-G) phase function [22], which is very commonly used in optical imaging:

$$p(\theta, \theta') = \begin{cases} \frac{1}{2\pi} \frac{1-g^2}{1+g^2-2g\theta \cdot \theta'} & n = 2, \\ \frac{1}{4\pi} \frac{1-g^2}{(1+g^2-2g\theta \cdot \theta')^{3/2}} & n = 3. \end{cases} \quad (5.5.1)$$

To set $g = 0$ corresponds to the isotropic case, while $g = 1$ and $g = -1$ correspond to forward scattering and backscattering, respectively. In typical applications g is around 0.80 to 0.95 ([19]), which is characteristic of highly forward-peaked scattering.

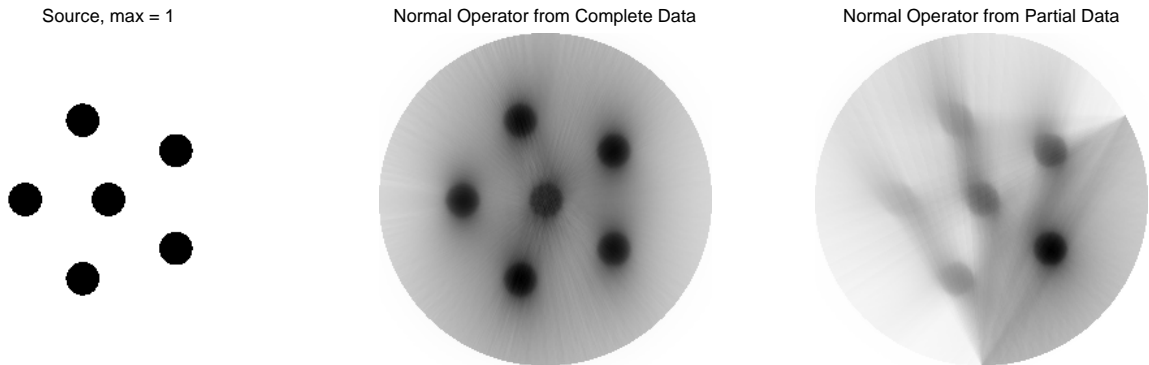


Figure 5.3: A source consisting of circular bump functions with height 1. The partial data is measured without noise on the set $V = \{(\eta, \theta) \in \partial_+ S\Omega = \partial\mathbb{D} \times \mathbb{S}^1 \mid \eta \in [0, \pi/3] \text{ and } \eta \cdot \theta > 0\}$.

We also incorporate noise into the boundary data in the following way. The noiseless data Xf is an $m \times 128$ matrix, with the rows corresponding to the position along the unit circle and the columns corresponding to the direction considered. Given a parameter $\mu > 0$, for the j th column v_j of Xf (i.e. $[Xf]_j$) we compute $\|v_j\|_2 = \sqrt{v_j^T v_j}$ and generate a vector w of length m with values chosen randomly according to the standard normal distribution (variance 1 and mean 0). We then define the j th column of the noisy data by

$$[Xf_{\text{noise}}]_j := [Xf]_j + \mu \sqrt{[Xf]_j^T [Xf]_j} \frac{w}{\sqrt{w^T w}} = v_j + \mu \|v_j\|_2 \frac{w}{\|w\|_2}. \quad (5.5.2)$$

Thus μ represents the fraction of $\|v_j\|_2$ to which we would like to rescale the variance of the noise.

With these details in mind, for all computed examples we have taken

$$k(x, y, \theta, \theta') = \frac{1}{2\pi} \chi_{\{x^2+y^2 < 1\}}(x, y) (0.05 + \sin^2 xy) \frac{1 - 0.85^2}{1 + 0.85^2 - 2 \cdot 0.85 \cdot \theta \cdot \theta'}$$

and

$$\sigma(x, y, \theta) = 0.5 \chi_{\{x^2+y^2 < 1\}}(x, y) [0.05 + \cos^2 xy] \sin^2 \theta.$$

Moreover, using the notation of Lemma 5.4.1, we take $m_1 = 8$ and $m_2 = 2$. This corresponds to computing the first 9 terms of the Neumann series expansion for Xf and only 3 terms in the series representation of X^* .

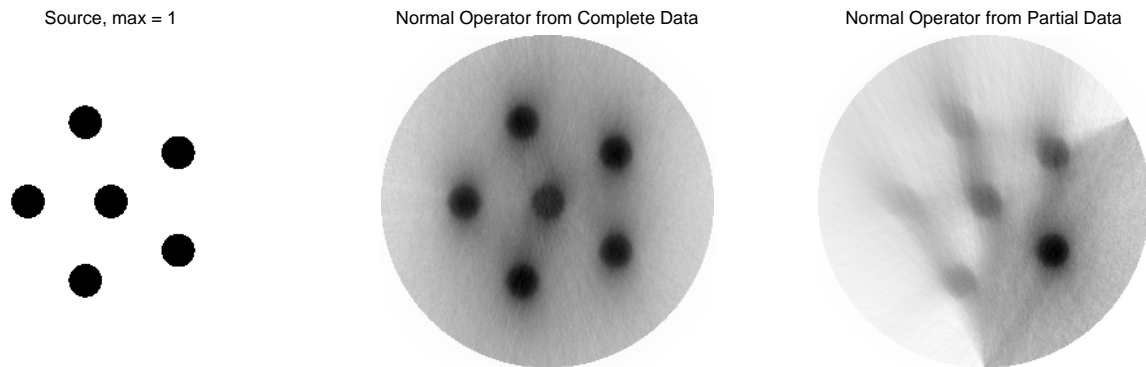


Figure 5.4: Same parameters as Figure 5.3 except with a noise coefficient $\mu = 0.50$.

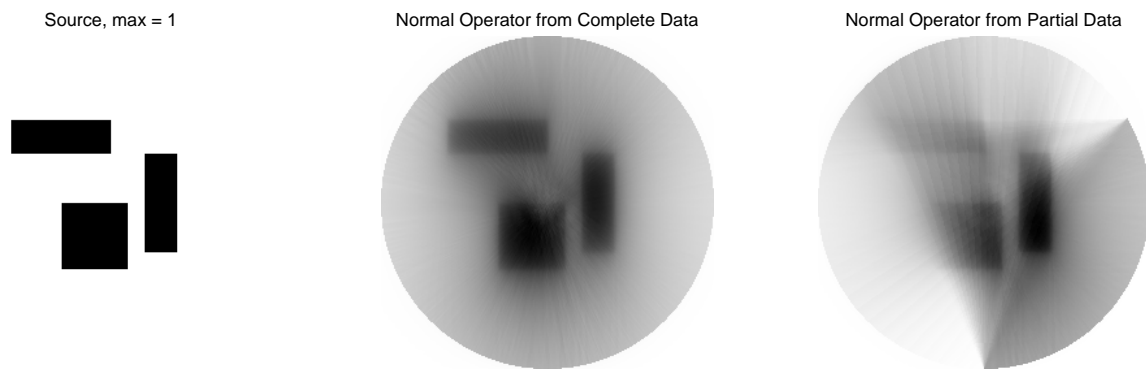


Figure 5.5: A source consisting of rectangular bump functions with height 1. The partial data is measured without added noise on the same set V as given in Figure 5.3.

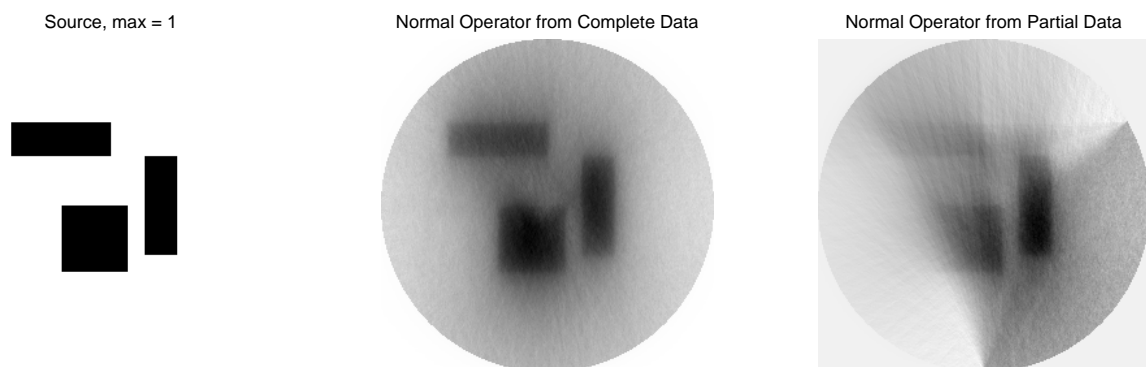


Figure 5.6: Same parameters as Figure 5.5 except with a noise coefficient $\mu = 0.5$.

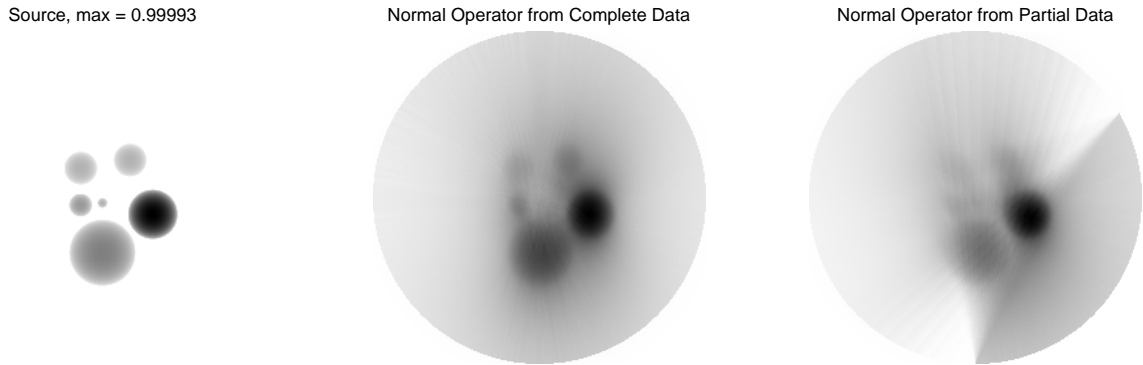


Figure 5.7: A spiral pattern of continuous circular bump functions of the form $g(x, y) = A\sqrt{1 - \frac{1}{r^2}(x - x_0)^2 - \frac{1}{r_0^2}(y - y_0)^2}$, where A is the maximum height and r is the radius. In this example, starting from the largest bump and moving counterclockwise, we have the set of heights and radii $A = \{0.5, 1, 0.3, 0.3, 0.4, 0.3\}$ and $r = \{0.2, 0.15, 0.1, 0.1, 0.07, 0.03\}$, respectively. The partial data is measured with no added noise on the set V given in Figure 5.3.

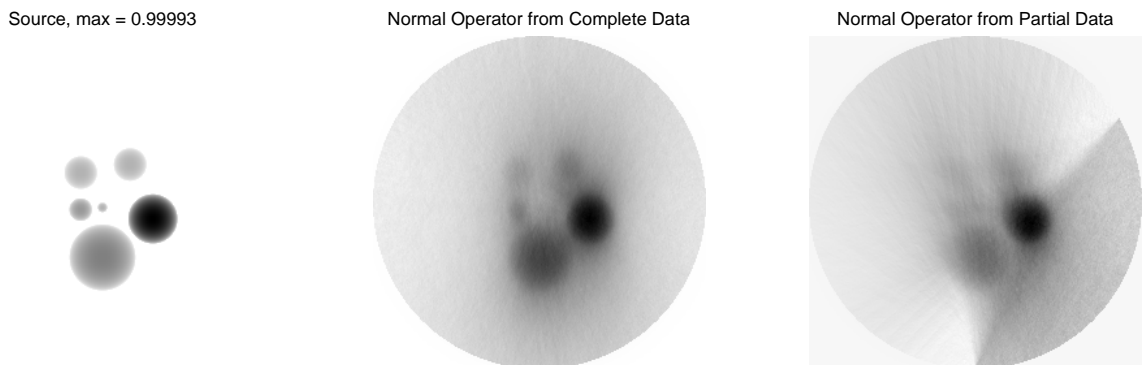


Figure 5.8: Same parameters as Figure 5.7 but with an added noise coefficient $\mu = 0.50$.

5.6 Conclusions

We have shown that from partial knowledge of the solution to (1.2.1) on $\partial_+ S\Omega$, one can recover an unknown isotropic radiative source $f(x)$ which is supported in the *visible set*. This gives a slight generalization of the result in [50]. More generally, we have shown that even for isotropic f supported on any part of Ω , one can still recover information about the singularities of f supported in the *microlocally visible set*. And although the result presented here relies on the classical theory of pseudodifferential operators, thus requiring that σ and k be C^∞ functions, in light of the theory of paradifferential operators and H^s wavefront sets, it is very likely such results can be extended to at least C^2 coefficients with analogous statements. Note that we have assumed that σ and k are known. Just as in X-ray tomography with incomplete data, in order to “see” a singularity, one must have knowledge of the ray transform along a line that intersects the associated wavefront covector normally. Thus, it is somewhat surprising that the presence of scattering does not alter this relationship.

In a first effort to reduce the regularity assumption on the scattering coefficient k , we have shown in Chapter 4 that for a generic open, dense set of $\sigma \in C^2$ it is enough that k lie in $C^2(\overline{\Omega} \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$ with suitably small C^2 norm. In light of the weakly-singular type integrals used to compute the normal operator X^*X , it is the author’s own intuition that no improvements in regularity can be made beyond $C^{1,\alpha}$. In the future, hopefully one can improve the smallness condition on k to simply requiring that it lie in a generic dense set.

Finally, we have presented a numerical method to solve the direct problem (1.2.1) based on the work of [30], which utilizes the discrete Fourier transform and fractional discrete Fourier transform to implement a rotation-based method for computing the necessary line integrals. Moreover, we have computed the normal operator X^*X in a similar manner. Ultimately, this has given us some nice visual examples in the presence of anisotropic (σ, k) with or without added noise, where the anisotropic parts of the parameters have physically meaningful structure as given in [19, 22]. Such examples visually demonstrate the ability to detect the singularities of an unknown source with only partial data of the transport solution u at the boundary.

Appendix A

RELEVANT RESULTS ON SINGULAR INTEGRAL OPERATORS

Suppose we have an integral operator of the form

$$Kf(x) = a(x)f(x) + \int K(x, x-y)f(y) dy, \quad K(x, x-y) = r^{-n}\phi(x, \theta), \quad (\text{A.1.1})$$

where $f \in H^l(\Omega)$, $\theta = \frac{x-y}{|x-y|}$, and $r = |x-y|$. The function $\phi(x, \theta)$ is called the characteristic of the singular integral operator. We formally define the symbol $\Phi(x, \xi)$ of K by

$$\Phi(x, \xi) = \int e^{-iz \cdot \xi} K(x, z) dz. \quad (\text{A.1.2})$$

It is easy to see by a change of variables that under such assumptions on K , Φ is homogeneous of degree 0 in ξ . Letting $\omega = \frac{\xi}{|\xi|}$, we will write $\Phi(x, \omega)$ from now on. It can be shown that if $K(x, x-y) = \phi(x, \theta)r^{-n}$, then

$$\Phi(x, \omega) = \int_{\mathbb{S}^{n-1}} \phi(x, \theta) \left[\ln \left(\frac{1}{|\cos \gamma|} \right) + \frac{i\pi}{2} \text{sign}(\cos \gamma) \right] d\theta \quad (\text{A.1.3})$$

where γ is the angle between the vectors θ and ω .

Consider the singular operator with a variable symbol,

$$(Af)(x) := a(x)f(x) + \int_{\mathbb{R}^n} \frac{\phi(x, \theta)}{|x-y|^n} f(y) dy = \int_{\mathbb{R}^n} e^{ix \cdot \xi} \Phi_A(x, \omega) \widehat{f}(\xi) d\xi, \quad \omega = \frac{\xi}{|\xi|}. \quad (\text{A.1.4})$$

We introduce the class $\mathcal{B}_{l, \lambda}$ of those symbols that satisfy the condition

$$D_x^\alpha \Phi(x, \omega) \hat{\in} H^\lambda(\mathbb{S}^{n-1}), \quad \forall \alpha : |\alpha| \leq l. \quad (\text{A.1.5})$$

Here the relation $\beta(x, \omega) \hat{\in} H^l(\mathbb{S}^{n-1})$ means that

$$\int_{\mathbb{S}^{n-1}} |D_\omega^\alpha \beta(x, \omega)|^2 d\omega \leq C, \quad 0 \leq |\alpha| \leq l. \quad (\text{A.1.6})$$

In this case, we say that $\beta(x, \omega)$ belongs to $H^l(\mathbb{S}^{n-1})$ uniformly with respect to the parameter x . For symbols of singular integral operators that satisfy such a condition, we have the following useful theorem.

Theorem A.1.1 (Theorem XI.9.2, [29]). *If $\Phi_A(x, \omega) \in \mathcal{R}_{l, \lambda}$ where $\lambda > \frac{n-1}{2}$, then the operator (A.1.4) is bounded in $H^l(\mathbb{R}^n)$.*

For relating the characteristic $\phi(x, \theta)$ to its symbol $\Phi(x, \omega)$ we have the following theorem

Theorem A.1.2 (Theorem X.7.1, [29]). *The symbol of a singular integral operator satisfies the relation $\Phi(x, \omega) \hat{\in} H^\lambda(\mathbb{S}^{n-1})$ if and only if the characteristic of this integral satisfies the condition $\phi(x, \theta) \hat{\in} H^l(\mathbb{S}^{n-1})$ where $l = \lambda - \frac{n}{2}$.*

We also recall that the derivative of a weakly singular integral operator ([29], IX §7) is given by

$$\frac{\partial}{\partial x_k} \int_{\Omega} \frac{\phi(x, \theta)}{r^{n-1}} f(y) dy = \int_{\Omega} f(y) \frac{\partial}{\partial x_k} \left[\frac{\phi(x, \theta)}{r^{n-1}} \right] dy - f(x) \int_{\mathbb{S}^{n-1}} \phi(x, \theta) \theta_k dS(y). \quad (\text{A.1.7})$$

This formula holds for any $f \in L^2(\Omega)$ and for $\phi \in C^1(\Omega, \mathbb{S}^{n-1})$.

In dealing with weakly singular integral operators depending on a parameter θ and acting on functions f depending on x and θ' , it will be helpful to work with a particular type of space on which these operators work nicely. In particular, we will use expansions of functions in terms of spherical harmonics. Recall that any function $g(x, \theta) \in C^\infty(\mathbb{R}^n \times \mathbb{S}^{n-1})$ can be expanded as a series

$$g(x, \theta) = \sum_{m=0}^{\infty} \sum_{k=1}^{k_{m,n}} a_m^{(k)}(x) Y_{m,n}^{(k)}(\theta), \quad (\text{A.1.8})$$

where

$$k_{m,n} = \frac{(2m+n-2)(m+n-3)!}{(n-2)!m!}$$

denotes the number of linearly independent spherical functions of order m . Furthermore, if g has compact support we claim that

$$\sum_{m=0}^{\infty} \sum_{k=1}^{k_{m,n}} \|a_m^{(k)}\|_{H^l(\mathbb{R}^n)} \|Y_{m,n}^{(k)}\|_{H^1(\mathbb{S}^{n-1})} < \infty \quad \forall l \geq 0. \quad (\text{A.1.9})$$

In [50] it is stated that for $g \in L^\infty(\mathbb{R}^n; C^{n+1}(\mathbb{S}^{n+1}))$ with compact support, we have that

$$\sum_{m=0}^{\infty} \sum_{k=1}^{k_{m,n}} \|a_m^{(k)}\|_{L^\infty(\mathbb{R}^n)} \|Y_{m,n}^{(k)}\|_{H^1(\mathbb{S}^{n-1})} < \infty \quad (\text{A.1.10})$$

Under the assumption that g is compactly supported, we have that the L^∞ norm is comparable to the L^2 norm. Since all derivatives of g also satisfy (A.1.10) with $\|a_m^{(k)}\|_{L^\infty(\mathbb{R}^n)}$ replaced by $\|a_m^{(k)}\|_{L^2(\mathbb{R}^n)}$, we have that (A.1.9) holds.

Recall our definition of $\mathcal{H}_l(\Omega \times \mathbb{S}^{n-1})$ as the completion of $C^\infty(\Omega \times \mathbb{S}^{n-1})$ with respect to the norm $\|\cdot\|_{\mathcal{H}_l(\Omega \times \mathbb{S}^{n-1})}$, and $\mathcal{C}_l(\Omega \times \mathbb{S}^{n-1})$ as the completion of $C^\infty(\Omega \times \mathbb{S}^{n-1})$ with respect to the norm $\|\cdot\|_{\mathcal{C}_l(\Omega \times \mathbb{S}^{n-1})}$.

The following proposition related to weakly singular integral operators and its applications in context will prove useful.

Proposition A.1.1. Let A be the operator

$$[Af](x) = \int \frac{\alpha\left(x, y, |x-y|, \frac{x-y}{|x-y|}\right)}{|x-y|^{n-1}} f\left(y, \frac{x-y}{|x-y|}\right) dy$$

with $\alpha(x, y, r, \theta)$ compactly supported in x and y . Then for a constant $C > 0$ depending only on n and $l \in \mathbb{Z}_{\geq 0}$,

a) If $\alpha \in C^{2l+2}(\mathbb{R}_x^n \times \mathbb{R}_y^n \times \mathbb{R}_r \times \mathbb{S}_\theta^{n-1})$, then $A : H^l(\Omega) \rightarrow H^{l+1}(\mathbb{R}^n)$ is continuous with a norm not exceeding $C\|\alpha\|_{C^{2l+2}}$.

b) If $\alpha(x, y, r, \theta) = \alpha'(x, y, r, \theta)\phi(\theta)$ and also in C^{2l+2} , then

$$\|A\|_{H^l(\Omega) \rightarrow H^{l+1}(\mathbb{R}^n)} \leq C\|\alpha'\|_{C^{2l+2}}\|\phi\|_{H^1(\mathbb{S}^{n-1})}.$$

c) If $f \in \mathcal{H}_l(\Omega)$ and α is as in (a), then $A : \mathcal{H}_l(\Omega \times \mathbb{S}^{n-1}) \rightarrow H^{l+1}(\Omega)$ is continuous with $\|A\|_{\mathcal{H}_l(\Omega \times \mathbb{S}^{n-1}) \rightarrow H^{l+1}(\mathbb{R}^n)} \leq C\|\alpha\|_{C^{2l+2}}$.

d) If $\alpha = \alpha(x, y, r, \theta, \eta) \in C^\infty(\mathbb{R}_x^n \times \mathbb{R}_y^n \times \mathbb{R}_r \times \mathbb{S}_\theta^{n-1} \times \mathbb{S}_\eta^{n-1})$ is compactly supported in x and y , then $A : \mathcal{H}_l(\Omega \times \mathbb{S}^{n-1}) \rightarrow \mathcal{H}_{l+1}(\mathbb{R}^n \times \mathbb{S}^{n-1})$ is bounded with

$$\|A\|_{\mathcal{H}_l(\Omega \times \mathbb{S}^{n-1}) \rightarrow \mathcal{H}_{l+1}(\mathbb{R}^n \times \mathbb{S}^{n-1})} \leq C\|\alpha\|_{\mathcal{C}_{2l+2}(\mathbb{R}^n \times \mathbb{S}^{n-1})}.$$

Proof. First note that in all cases, since α is compactly supported in x and y , we can let $U \Subset \mathbb{R}^n$ be such that $\alpha(x, y, r, \theta, \eta) = 0$ for $x, y \notin U$. We can then replace α by $\alpha\psi$ without

affecting the integral, where $\psi \in C_c^\infty(\mathbb{R})$ satisfies $\psi(r) = 1$ for $r \in [0, \text{diam}(U)]$. Thus, we may as well assume that α is compactly supported in all variables.

Consider the case that f is independent of θ and α only depends on x and θ . From the Calderón-Zygmund theory of singular operators, we know that for an integral operator K with singular kernel $k(x, y) = \phi(x, \theta)r^{-n}$ where we recall $r = |x - y|$, if ϕ has mean value 0 as a function of θ for each x , then K is a well-defined operator on test functions, where the integration has to be understood in the principal value sense. Moreover, K extends to a bounded operator on L^2 satisfying $\|K\| \leq C \sup_x \|\phi(x, \cdot)\|_{L^2(\mathbb{S}^{n-1})}$ (Theorem XI.3.1, [29]). As a remark, the extension can be considered as a convolution in the sense of distributions, and then ϕ need not have mean value 0 in θ .

Let $(j_1, j_2, \dots, j_{l+1})$ be a multi index. To make notation a bit more consistent, let $\alpha_0 = \alpha$. Consider the derivative $\partial_{x_{j_1}} A$, which by (A.1.7) and ([50], Proposition 1) consists of a bounded term $a_1(x)f(x)$ plus the integral operator with kernel

$$\partial_{x_{j_1}} \frac{\alpha(x, \theta)}{r^{n-1}} = \frac{(1-n)\theta_{j_1}\alpha + \partial_{\theta_{j_1}}\alpha}{r^n} + \frac{\partial_{x_{j_1}}\alpha(x, \theta)}{r^{n-1}}. \quad (\text{A.1.11})$$

Letting $\phi_1(x, \theta) := (1-n)\theta_{j_1}\alpha(x, \theta) + \partial_{\theta_{j_1}}\alpha(x, \theta)$, since $\alpha \in C^{2l+2}$ and is compactly supported in x , we have that the symbol

$$\Phi_1(x, \omega) = \int e^{-iz \cdot \omega} \frac{\phi_1\left(x, \frac{z}{|z|}\right)}{|z|^n} dz$$

of ϕ_1 belongs to $C^{2l+1}(\mathbb{R}^n \times \mathbb{S}^{n-1})$. Since ϕ_1 is compactly supported in x , $\partial_x^\gamma \phi_1 \in H^{2l+1-|\gamma|}(\mathbb{S}^{n-1}) \subset H^{1+\lambda-\frac{n}{2}}(\mathbb{S}^{n-1})$ for all $0 \leq |\gamma| \leq l$ and for some fixed $\lambda > \frac{n-1}{2}$ (in particular we could take $\lambda = \frac{n}{2}$). By Theorem A.1.2, $\Phi_1(x, \omega) \in \mathcal{R}_{l, \lambda}$. By Theorem A.1.1 we have that the integral kernel $\phi_1(x, \theta)r^{-n}$ corresponds to a singular integral operator that is bounded on H^l . For the second term in (A.1.11), which is a weakly singular integral kernel, we have that $\alpha_1(x, \theta) := \partial_{x_{j_1}}\alpha(x, \theta) \in C^{2l+1}$. Similarly as before, we compute $\partial_{x_{j_2}}\left(\frac{\alpha_1(x, \theta)}{r^{n-1}}\right)$ which corresponds to an operator with a bounded multiplier $a_2(x)$, a singular integral operator, and a weakly singular integral operator. It can be shown analogously that the symbol $\Phi_2(x, \xi)$ corresponding to the characteristic $\phi_2(x, \theta)$ of the singular integral term belongs to $\mathcal{R}_{l-1, \lambda}$. Thus the integral operator with kernel $\phi_2(x, \theta)r^{-n}$ is bounded on H^{l-1} . We then focus our attention on the weakly singular integral operator that remains.

After repeating this process a total of $l+1$ times, which involves $l+1$ differentiations, the remaining weakly singular integral operator has a kernel $\alpha_{l+1}(x, \theta)r^{-n+1}$ with $\alpha_{l+1}(x, \theta) \in C^{l+1}$. We can then proceed as in the proof of ([50], Proposition 1) to obtain that this term is bounded on $L^2(\Omega)$. In particular, we use the criterion from Calderón Zygmund Theory which states that if K is an integral operator with integral kernel $k(x, y)$ satisfying

$$\sup_x \int |k(x, y)| dy \leq M, \quad \sup_y \int |k(x, y)| dx \leq M, \quad (\text{A.1.12})$$

then K is bounded in L^2 with a norm not exceeding M ([51], Prop. A.5.1).

Now we want to bound the operator norm $\|A\|_{H^l(\Omega) \rightarrow H^{l+1}(\mathbb{R}^n)}$ in this simpler case. Let ϕ_i be the characteristic of the i th singular integral operator obtained by the above process with symbol Φ_i and \tilde{A}_i the corresponding singular integral operator. Note that \tilde{A}_i is bounded on $H^{l-i+1}(\Omega)$. One can explicitly compute that

$$\alpha_i = \partial_{x_{j_1} x_{j_2} \dots x_{j_i}} \alpha, \quad (\text{A.1.13})$$

$$\phi_i = (1-n)\theta_{j_i} \alpha_{i-1} + \partial_{\theta_{j_i}} \alpha_{i-1}, \quad (\text{A.1.14})$$

$$\alpha_i(x) = \int \alpha_{i-1}(x, \theta) \theta_{j_i} dS(y), \quad \text{for } 1 \leq i \leq l+1. \quad (\text{A.1.15})$$

Also define

$$\tilde{R}_i f(x) := \int \frac{\alpha_i(x, \theta)}{r^{n-1}} f(y) dy. \quad (\text{A.1.16})$$

We have

$$\begin{aligned} \|\phi_i\|_{C^{l+1-i}} &\leq C\|\alpha\|_{C^{l+1}} \\ \|\alpha_i\|_{C^{l+1-i}} &\leq C\|\alpha\|_{C^{l+1}} \\ \|a_i\|_{C^{l+1-i}} &\leq C\|\alpha_{i-1}\|_{C^{l+1-i}} \leq C\|\alpha_{i-1}\|_{C^{l+1-(i-1)}} \leq C'\|\alpha\|_{C^{l+1}} \end{aligned} \quad (\text{A.1.17})$$

If $\|f\|_{H^l(\Omega)} = 1$, then ([29], Thm XI.3.2, Thm XI.9.2) imply that for $1 \leq i \leq l+1$,

$$\begin{aligned}
\|\tilde{A}_i f\|_{H^{l-i+1}(\mathbb{R}^n)} &= \sum_{|\beta| \leq l-i+1} \|D_x^\beta \tilde{A}_i f\|_{L^2(\mathbb{R}^n)} \\
&\leq \sum_{|\beta| \leq l-i+1} \sum_{\gamma \leq \beta} \frac{\beta!}{\gamma!(\beta-\gamma)!} \sup_x \|D_x^{\beta-\gamma} \Phi_i(x, \cdot)\|_{H^{\frac{\beta}{2}}(\mathbb{S}^{n-1})} \|f\|_{H^{|\gamma|}(\Omega)} \\
&\leq C \sum_{|\beta| \leq l-i+1} \sum_{\gamma \leq \beta} \sup_x \|D_x^{\beta-\gamma} \phi_i(x, \cdot)\|_{L^2(\mathbb{S}^{n-1})} \|f\|_{H^l(\Omega)} \\
&\leq C \|\phi_i\|_{C^{l-i+1}} \\
&\leq C \|\alpha\|_{C^{l+1}}. \tag{A.1.18}
\end{aligned}$$

Now we estimate $\|Af\|_{H^{l+1}}$ using (A.1.18) with the understanding that we sum over all indices j_1, j_2, \dots . Again, assume that $\|f\|_{H^l(\Omega)} = 1$.

$$\begin{aligned}
\|Af\|_{H^{l+1}} &= \|Af\|_{L^2} + \|\partial_{x_{j_1}} Af\|_{H^l} \\
&\lesssim \|\alpha\|_{C^0} + \|a_1 f\|_{H^l} + \|\tilde{A}_1 f\|_{H^l} + \|\tilde{R}_1 f\|_{H^l} \\
&\lesssim \|\alpha\|_{C^0} + \|a_1\|_{C^l} + \|\alpha\|_{C^{l+1}} + \|\tilde{R}_1 f\|_{L^2} + \|\partial_{x_{j_2}} \tilde{R}_1 f\|_{H^{l-1}} \\
&\lesssim \|\alpha\|_{C^{l+1}} + \|\alpha\|_{C^0} + \|\alpha_1\|_{C^0} + \|a_1\|_{C^l} + \|a_2 f\|_{H^{l-1}} + \|\tilde{A}_2 f\|_{H^{l-1}} + \|\tilde{R}_2 f\|_{H^{l-1}} \\
&\lesssim \|\alpha\|_{C^{l+1}} + \|\alpha\|_{C^0} + \|\alpha_1\|_{C^0} + \|a_1\|_{C^l} + \|a_2\|_{C^{l-1}} + \|\tilde{R}_2 f\|_{H^{l-1}} \\
&\vdots \\
&\lesssim \|\tilde{R}_{l+1}\|_{L^2} + \|\alpha\|_{C^{l+1}} + \sum_{i=1}^{l+1} \|\alpha_{i-1}\|_{C^0} + \|a_i\|_{C^{l+1-i}} \\
&\lesssim \|\alpha_{l+1}\|_{C^0} + \|\alpha\|_{C^{l+1}} + \sum_{i=1}^{l+1} \|\alpha\|_{C^{l+1}} \\
&\lesssim \|\alpha\|_{C^{l+1}}.
\end{aligned}$$

Thus in the simplified case where α only depends on x and θ and f is independent of θ , we have

$$\|A\|_{H^l(\Omega) \rightarrow H^{l+1}(\mathbb{R}^n)} \leq C \|\alpha\|_{C^{l+1}}. \tag{A.1.19}$$

To extend to $\alpha = \alpha_0$ depending also on y and r , we use a first order Taylor expansion

in y and r centered at $y = x$ and $r = 0$, similarly to in ([50], Prop. 1), to get

$$\begin{aligned}\alpha_0(x, y, r, \theta) &= \alpha_0(x, x, 0, \theta) + \sum_{|\beta|+|\gamma|=1} r^{|\beta|} (y-x)^\gamma \int_0^1 \partial_r^\beta \partial_y^\gamma \alpha_0(x, x+t(y-x), tr, \theta) dt \\ &= \alpha_0(x, x, 0, \theta) + \sum_{|\beta|+|\gamma|=1} r^{|\beta|} (-r\theta)^\gamma \int_0^1 \partial_r^\beta \partial_y^\gamma \alpha_0(x, x+t(y-x), tr, \theta) dt \\ &= \alpha_0(x, x, 0, \theta) + r \sum_{|\beta|+|\gamma|=1} (-1)^{|\gamma|} \theta^\gamma \int_0^1 \partial_r^\beta \partial_y^\gamma \alpha_0(x, x+t(y-x), tr, \theta) dt.\end{aligned}$$

We can then write

$$\alpha_0(x, y, r, \theta) = \alpha_0(x, x, 0, \theta) + r\gamma_1(x, y, r, \theta) \quad (\text{A.1.20})$$

where $\gamma_1 \in C^{2l+1}$. After dividing by r^{n-1} , the first term in (A.1.20) maps H^l to H^{l+1} by the previous argument. The second term corresponds to an integral operator with kernel $\gamma_1(x, y, r, \theta)r^{-n+2}$. If we differentiate this with respect to x , we get a weakly singular integral operator with kernel $\alpha_1(x, y, r, \theta)r^{-n+1}$ where $\alpha_1 \in C^{2l}$. Now repeat as before, writing

$$\alpha_1(x, y, r, \theta) = \alpha_1(x, x, 0, \theta) + r\gamma_2(x, y, r, \theta) \quad (\text{A.1.21})$$

where $\gamma_2 \in C^{2l-1}$. The first term $\alpha_1(x, x, 0, \theta)$ corresponds to a bounded operator $A_1 : H^{l-1}(\Omega) \rightarrow H^l(\mathbb{R}^n)$. Moreover, $\gamma_2(x, y, r, \theta)r^{-n+2}$ can be differentiated with respect to x to obtain a weakly singular integral operator with kernel $\alpha_2(x, y, r, \theta)r^{-n+1}$ where $\alpha_2 \in C^{2l-2}$.

After repeating this process a total of l times, we have a remainder term that is a weakly singular integral operator with kernel $\alpha_l(x, y, r, \theta)r^{-n+1}$ where $\alpha_l \in C^2$. Write

$$\alpha_l(x, y, r, \theta) = \alpha_l(x, x, 0, \theta) + r\gamma_{l+1}(x, y, r, \theta)$$

with $\gamma_{l+1} \in C^1$. Then γ_{l+1} corresponds to the operator $\gamma_{l+1}(x, y, r, \theta)r^{-n+2}$, which we can differentiate with respect to x to obtain a weakly singular operator that is bounded on L^2 with a bound not exceeding $\|\gamma_{l+1}\|_{C^1}$ by using the estimates in (A.1.12) and applying the Calderón Zygmund theorem. Since each weakly singular integral operator with kernel $\alpha_j(x, x, 0, \theta)r^{-n+1}$ is a bounded map from $H^{l-j}(\Omega) \rightarrow H^{l-j+1}(\mathbb{R}^n)$, we combine the remainder terms together to get that $A : H^l(\Omega) \rightarrow H^{l+1}(\mathbb{R}^n)$.

More explicitly, let A_i be the weakly singular integral operator with kernel $\alpha_i(x, x, 0, \theta)r^{-n+1}$ and R_i the integral operator with kernel $\gamma_{i+1}(x, y, r, \theta)r^{-n+2}$. In particular, $A_i = \partial_{x_j} R_{i-1}$.

We will also need the straightforward estimates

$$\|\alpha_i\|_{C^{l-i+1}} \lesssim \|\alpha\|_{C^{l+i+1}}, \quad \|\gamma_i\|_{C^m} \lesssim \|\alpha\|_{C^{m+2i-1}}. \quad (\text{A.1.22})$$

For $\|f\|_{H^l} = 1$ we have

$$\begin{aligned} \|Af\|_{H^{l+1}} &\leq \|A_0f\|_{H^{l+1}} + \|R_0f\|_{H^{l+1}} \\ &\leq \|\alpha_0\|_{C^{l+1}} + \|R_0f\|_{L^2} + \|\partial_{x_{j_1}}R_0f\|_{H^l} \\ &\lesssim \|\alpha_0\|_{C^{l+1}} + \|\gamma_1\|_{C^0} + \|A_1f\|_{H^l} + \|R_1f\|_{H^l} \\ &\lesssim \|\alpha_0\|_{C^{l+1}} + \|\gamma_1\|_{C^0} + \|\alpha_1\|_{C^l} + \|R_1f\|_{L^2} + \|\partial_{x_{j_2}}R_1f\|_{H^{l-1}} \\ &\lesssim \|\alpha_0\|_{C^{l+1}} + \|\alpha_1\|_{C^l} + \|\gamma_1\|_{C^0} + \|\gamma_2\|_{C^0} + \|A_2f\|_{H^{l-1}} + \|R_2f\|_{H^{l-1}} \\ &\vdots \\ &\lesssim \|A_lf\|_{H^1} + \|R_lf\|_{H^1} + \sum_{i=0}^{l-1} \|\alpha_i\|_{C^{l+1-i}} + \|\gamma_{i+1}\|_{C^0} \\ &\lesssim \|\alpha_l\|_{C^1} + \|R_lf\|_{L^2} + \|\partial_{x_{j_{l+1}}}R_lf\|_{L^2} + \sum_{i=0}^{l-1} \|\alpha_i\|_{C^{l+1-i}} + \|\gamma_{i+1}\|_{C^0} \\ &\lesssim \|\alpha\|_{C^{2l+1}} + \|\gamma_{l+1}\|_{C^0} + \|\gamma_{l+1}\|_{C^1} + \sum_{i=0}^{l-1} \|\alpha\|_{C^{l+1+i}} + \|\alpha\|_{C^{2i+1}} \\ &\lesssim \|\alpha\|_{C^{2l+1}} + \|\alpha\|_{C^{2l+2}} + \sum_{i=0}^{l-1} \|\alpha\|_{C^{l+1+i}} + \|\alpha\|_{C^{2i+1}} \\ &\lesssim \|\alpha\|_{C^{2l+2}}. \end{aligned}$$

This proves (a).

Now consider if $\alpha(x, y, r, \theta) = \alpha'(x, y, r, \theta)\phi(\theta)$. Then

$$(1-n)\theta_j\alpha + \partial_{\theta_j}\alpha = (1-n)\theta_j\alpha'\phi + \alpha'\partial_{\theta_j}\phi + \phi\partial_{\theta_j}\alpha'. \quad (\text{A.1.23})$$

In short, for each term in the decomposition of A by differentiation, ϕ is differentiated exactly once. Therefore

$$\|A\|_{H^l(\Omega) \rightarrow H^{l+1}(\mathbb{R}^n)} \leq C\|\alpha'\|_{C^{2l+2}}\|\phi\|_{H^1(\mathbb{S}^{n-1})}, \quad (\text{A.1.24})$$

which proves (b).

For $f \in \mathcal{H}_l(\Omega \times \mathbb{S}^{n-1})$ depending on θ as well, we expand f as a series

$$f(x, \theta) = \sum_{m=0}^{\infty} \sum_{k=1}^{k_{m,n}} a_m^{(k)}(x) Y_{m,n}^{(k)}(\theta),$$

Then

$$\begin{aligned} [Af](x) &= \int \frac{\alpha(x, y, r, \theta)}{r^{n-1}} f(y, \theta) dy \\ &= \int \frac{\alpha(x, y, r, \theta)}{r^{n-1}} \sum_{m=0}^{\infty} \sum_{k=1}^{k_{m,n}} a_m^{(k)}(y) Y_{m,n}^{(k)}(\theta) dy \\ &= \sum_{m=0}^{\infty} \sum_{k=1}^{k_{m,n}} \int \frac{\alpha(x, y, r, \theta) Y_{m,n}^{(k)}(\theta)}{r^{n-1}} a_m^{(k)}(y) dy. \end{aligned}$$

Hence

$$\begin{aligned} \|Af\|_{H^{l+1}(\mathbb{R}^n \times \mathbb{S}^{n-1})} &\leq \sum_{m=0}^{\infty} \sum_{k=1}^{k_{m,n}} \left\| \int \frac{\alpha(x, y, r, \theta) Y_{m,n}^{(k)}(\theta)}{r^{n-1}} a_m^{(k)}(y) dy \right\|_{H^{l+1}(\mathbb{R}^n \times \mathbb{S}^{n-1})} \\ &\leq \sum_{m=0}^{\infty} \sum_{k=1}^{k_{m,n}} \|A_{m,n}^{(k)}\|_{H^l \rightarrow H^{l+1}} \|a_m^{(k)}\|_{H^l(\Omega)} \\ &\leq C \|\alpha\|_{C^{2l+2}} \sum_{m=0}^{\infty} \sum_{k=1}^{k_{m,n}} \|Y_{m,n}^{(k)}\|_{H^1(\mathbb{S}^{n-1})} \|a_m^{(k)}\|_{H^l(\Omega)} \\ &\leq C \|\alpha\|_{C^{2l+2}} \|f\|_{\mathcal{H}_l(\Omega)}. \end{aligned}$$

Here the operator $A_{m,n}^{(k)}$ is given by $[A_{m,n}^{(k)}g](x) = \int \frac{\alpha(x, y, r, \theta) Y_{m,n}^{(k)}(\theta)}{r^{n-1}} g(y) dy$. This proves (c).

Finally, if $\alpha = \alpha(x, y, r, \theta, \eta)$ is C^∞ with compact support, we can expand it as a series

$$\alpha(x, y, r, \theta, \eta) = \sum_{m=0}^{\infty} \sum_{k=1}^{k_{m,n}} b_m^{(k)}(x, y, r, \theta) Y_{m,n}^{(k)}(\eta). \quad (\text{A.1.25})$$

Note that $\alpha \in \mathcal{C}_j(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$ for all $j \geq 0$. Then for $f \in \mathcal{H}_l(\Omega \times \mathbb{S}^{n-1})$, we

have

$$\begin{aligned}
& \|Af\|_{\mathcal{H}_{l+1}(\mathbb{R}^n \times \mathbb{S}^{n-1})} \\
&= \sum_{m_1=0}^{\infty} \sum_{k_1=1}^{k_{m_1,n}} \|Y_{m_1,n}^{(k_1)}\|_{H^1(\mathbb{S}^{n-1})} \left\| \int \frac{b_{m_1}^{(k_1)}(x,y,r,\theta)}{r^{n-1}} f(y,\theta) dy \right\|_{H^{l+1}(\mathbb{R}^n \times \mathbb{S}^{n-1})} \\
&= \sum_{m_1=0}^{\infty} \sum_{k_1=1}^{k_{m_1,n}} \|Y_{m_1,n}^{(k_1)}\|_{H^1(\mathbb{S}^{n-1})} \left\| \sum_{m_2=0}^{\infty} \sum_{k_2=1}^{k_{m_2,n}} \int \frac{b_{m_1}^{(k_1)}(x,y,r,\theta) Y_{m_2,n}^{(k_2)}(\theta)}{r^{n-1}} a_{m_2}^{k_2}(y) dy \right\|_{H^{l+1}(\mathbb{R}^n \times \mathbb{S}^{n-1})} \\
&\lesssim \sum_{m_1=0}^{\infty} \sum_{k_1=1}^{k_{m_1,n}} \sum_{m_2=0}^{\infty} \sum_{k_2=1}^{k_{m_2,n}} \|Y_{m_1,n}^{(k_1)}\|_{H^1(\mathbb{S}^{n-1})} \|b_{m_1}^{(k_1)}\|_{C^{2l+2}} \|Y_{m_2,n}^{(k_2)}\|_{H^1(\mathbb{S}^{n-1})} \|a_{m_2}^{k_2}\|_{H^l(\Omega)} \\
&= \|\alpha\|_{\mathcal{C}_{2l+2}} \|f\|_{\mathcal{H}_l(\Omega \times \mathbb{S}^{n-1})}.
\end{aligned}$$

□

Appendix B

SMOOTHING PROPERTIES OF COMPOSITIONS OF WEAKLY SINGULAR-TYPE INTEGRAL OPERATORS

Let's consider an operator of type

$$[Af](x, \theta) = \int \frac{\alpha(x, y, |x - y|, \widehat{x - y}, \theta)}{|x - y|^{n-1}} f(y, \widehat{x - y}) dy$$

where $\alpha(x, y, r, \eta, \theta) \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$. We would like to analyze compositions of the form $[A^m Jf](x, \theta)$ where $J : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n \times \mathbb{S}^{n-1})$ is the extension operator $Jf(x, \theta) = f(x)$ as used previously.

For $m = 2$, we compute

$$\begin{aligned} & [A^2 Jf](x, \theta) \\ &= \int \frac{\alpha(x, y_1, |x - y_1|, \widehat{x - y_1}, \theta)}{|x - y_1|^{n-1}} [AJf] \left(y_1, \frac{x - y_1}{|x - y_1|} \right) dy_1 \\ &= \int \frac{\alpha(x, y_1, |x - y_1|, \widehat{x - y_1}, \theta)}{|x - y_1|^{n-1}} \int \frac{\alpha(y_1, y_2, |y_1 - y_2|, \widehat{y_1 - y_2}, \widehat{x - y_1})}{|y_1 - y_2|^{n-1}} f(y_2) dy_2 dy_1 \\ &= \int \left[\int \frac{\alpha(x, y_1, |x - y_1|, \widehat{x - y_1}, \theta) \alpha(y_1, y_2, |y_1 - y_2|, \widehat{y_1 - y_2}, \widehat{x - y_1})}{|x - y_1|^{n-1} |y_1 - y_2|^{n-1}} dy_1 \right] f(y_2) dy_2. \end{aligned}$$

Similarly for $m = 3$,

$$\begin{aligned} & [A^3 Jf](x, \theta) \\ &= \int \left(\int \int \frac{\alpha(x, y_1, |x - y_1|, \widehat{x - y_1}, \theta) \alpha(y_1, y_2, |y_1 - y_2|, \widehat{y_1 - y_2}, \widehat{x - y_1})}{|x - y_1|^{n-1} |y_1 - y_2|^{n-1} |y_2 - y_3|^{n-1}} \right. \\ & \quad \left. \cdot \alpha(y_2, y_3, |y_2 - y_3|, \widehat{y_2 - y_3}, \widehat{y_1 - y_2}) dy_2 dy_1 \right) f(y_3) dy_3 \end{aligned}$$

In general, define

$$\begin{aligned}
\alpha_1 &:= \alpha, \\
\alpha_2(x, y_1, y_2, |x - y_1|, |y_1 - y_2|, \widehat{x - y_1}, \widehat{y_1 - y_2}, \theta) &:= \alpha_1(x, y_1, |x - y_1|, \widehat{x - y_1}, \theta) \\
&\quad \cdot \alpha_1(y_1, y_2, |y_1 - y_2|, \widehat{y_1 - y_2}, \widehat{x - y_1}), \\
\alpha_m(x, y_1, \dots, y_m, |x - y_1|, \dots, |y_{m-1} - y_m|, \widehat{x - y_1}, \dots, \widehat{y_{m-1} - y_m}, \theta) &:= \quad (B.0.1) \\
&\quad \alpha_{m-1}(y_1, y_2, \dots, y_m, |y_1 - y_2|, \dots, |y_{m-1} - y_m|, \widehat{y_1 - y_2}, \dots, \widehat{y_{m-1} - y_m}, \widehat{x - y_1}) \\
&\quad \cdot \alpha_1(x, y_1, |x - y_1|, \widehat{x - y_1}, \theta),
\end{aligned}$$

where $y_0 := x$. For simplicity of notation, define

$$\begin{aligned}
\mathbf{y}_m &:= (y_1, \dots, y_{m-1}, y) \\
r_m &:= (|y_1 - y_2|, \dots, |y_{m-1} - y|) \\
\widehat{\mathbf{r}}_m &:= (\widehat{y_1 - y_2}, \dots, \widehat{y_{m-1} - y}).
\end{aligned}$$

Then set for $m \geq 2$

$$a_m(x, y, \theta) := \int \dots \int \frac{\alpha_m(x, \mathbf{y}_m, |x - y_1|, r_m, \widehat{x - y_1}, \widehat{\mathbf{r}}_m, \theta)}{|x - y_1|^{n-1} |y_1 - y_2|^{n-1} \dots |y_{m-1} - y|^{n-1}} dy_1 \dots dy_{m-1}.$$

Also let

$$a_0 := J[\delta(\cdot - y)], \quad a_1(x, y, \theta) := \frac{\alpha_1(x, y, |x - y|, \widehat{x - y}, \theta)}{|x - y|^{n-1}},$$

so that $\int a_0(x, y, \theta) f(y) dy = \int \delta(x - y) f(y) dy = f(x)$. Then

$$[A^m Jf](x, \theta) = \int a_m(x, y, \theta) f(y) dy, \quad m \geq 0.$$

Now we would like to show that $A^m J$ has a kernel of the form

$$\frac{\beta_m(x, y, |x - y|, \widehat{x - y}, \theta)}{|x - y|^{n-m}} \quad (B.0.2)$$

where $\beta_m(x, y, r, \eta, \theta)$ is C^∞ . Ultimately, our goal is to rigorously show for $m \geq 2$ that $A^m J$ is a pseudodifferential operator of order $-m$, by adapting the proof of Lemma 2 in [18], which already directly applies to the case $m = 1$.

B.1 The Case $m = 2$

For $\eta \in \mathbb{S}^{n-1}$, define the set $\mathcal{D}_\eta := \mathbb{R}^n \setminus (B_0(1/4) \cup B_\eta(1/4))$. Also let A_η be the rotational matrix such that $A_\eta \eta = e_1$, the first unit basis vector of \mathbb{R}^n . Let $\psi(r)$ be a smooth even bump function such that $0 \leq \psi \leq 1$, $\psi \equiv 1$ for $|r| \leq 1/4$, and $\psi \equiv 0$ for $|r| \geq 1/2$. We will frequently use the notation $r = |x - y|$ and $\eta = \widehat{x - y}$. Note that differentiation of A_η with respect to x yields, from the fact that $\partial_{x_j} \widehat{x - y} = \frac{x_j - y_j}{|x - y|^2}$, that $|\partial_{x_j} A_\eta| \leq \frac{C}{|x - y|}$. Similarly, l th order derivatives of A_η with respect to x will have bounds of the form $\frac{C}{|x - y|^l}$. Of course, if we treat A_η as a matrix of functions of η only, and not implicitly dependent on x , then each such coordinate function is in $C^\infty(\mathbb{S}^{n-1})$.

We have

$$\alpha_2(x, y, \theta) = \int \frac{\alpha_2(x, y_1, y, |x - y_1|, |y_1 - y|, \widehat{x - y_1}, \widehat{y_1 - y}, \theta)}{|x - y_1|^{n-1} |y_1 - y|^{n-1}} dy_1.$$

We may then cut up the above integral, using ψ and suppressing some variables, to obtain

$$\begin{aligned} & \int \psi\left(\frac{|x - y_1|}{|x - y|}\right) \alpha_2 dy_1 + \int \psi\left(\frac{|y_1 - y|}{|x - y|}\right) \alpha_2 dy_1 + \int \left[1 - \psi\left(\frac{|x - y_1|}{|x - y|}\right) - \psi\left(\frac{|y_1 - y|}{|x - y|}\right)\right] \alpha_2 dy_1 \\ & =: I_1 + I_2 + I_3. \end{aligned}$$

Let's focus on the first term. Substitute $w = \frac{x - y_1}{|x - y|}$ so that $|x - y|^n dw = dy_1$, $y_1 = x - |x - y|w$ and $y_1 - y = |x - y|(\widehat{x - y} - w)$. We also note that the region of integration is $w \in B_0(\frac{1}{2})$.

We obtain

$$\begin{aligned} I_1 &= \int \frac{\psi\left(\frac{|x - y_1|}{|x - y|}\right) \alpha_2(x, y_1, y, |x - y_1|, |y_1 - y|, \widehat{x - y_1}, \widehat{y_1 - y}, \theta)}{|x - y_1|^{n-1} |y_1 - y|^{n-1}} dy_1 \\ &= \frac{1}{|x - y|^{n-2}} \int \frac{\psi(|w|) \alpha_2\left(x, x - |x - y|w, y, |x - y||w|, |x - y||\widehat{x - y} - w|, \widehat{w}, \widehat{\frac{x - y - w}{|x - y - w|}}, \theta\right)}{|w|^{n-1} |\widehat{x - y} - w|^{n-1}} dw \\ &= \frac{1}{r^{n-2}} \int_{B_0(1/2)} \frac{\psi(|w|) \alpha_2\left(x, x - |x - y|w, y, |x - y||w|, |x - y||\eta - w|, \widehat{w}, \widehat{\eta - w}, \theta\right)}{|w|^{n-1} |\eta - w|^{n-1}} dw \\ &= \frac{1}{r^{n-2}} \int_{B_0(1/2)} \frac{\psi(|w|) \alpha_2\left(x, x - rA_\eta^{-1}w, y, r|w|, r|e_1 - w|, \widehat{w}, A_\eta^{-1}\widehat{e_1 - w}, \theta\right)}{|w|^{n-1} |e_1 - w|^{n-1}} dw. \end{aligned}$$

Let's consider the other terms:

$$\begin{aligned}
I_2 &= \int \frac{\psi\left(\frac{|y_1-y|}{|x-y|}\right) \alpha_2(x, y_1, y, |x-y_1|, |y_1-y|, \widehat{x-y_1}, \widehat{y_1-y}, \theta)}{|x-y_1|^{n-1} |y_1-y|^{n-1}} dy_1 \\
&= \frac{1}{r^{n-2}} \int_{B_0(1/2)} \frac{\psi(|\widehat{x-y}-w|) \alpha_2(x, x-rA_\eta^{-1}w, y, r|w|, r|e_1-w|, \widehat{w}, A_\eta^{-1}\widehat{e_1-w}, \theta)}{|w|^{n-1} |e_1-w|^{n-1}} dw, \\
I_3 &= \int \frac{\left(1 - \psi\left(\frac{|x-y_1|}{|x-y|}\right) - \psi\left(\frac{|y_1-y|}{|x-y|}\right)\right) \alpha_2(x, y_1, y, |x-y_1|, |y_1-y|, \widehat{x-y_1}, \widehat{y_1-y}, \theta)}{|x-y_1|^{n-1} |y_1-y|^{n-1}} dy_1 \\
&= \frac{1}{r^{n-2}} \int_{\mathcal{D}_{e_1}} \frac{(1 - \psi(|w|) - \psi(|e_1-w|)) \alpha_2(x, x-rA_\eta^{-1}w, y, r|w|, r|e_1-w|, \widehat{w}, A_\eta^{-1}\widehat{e_1-w}, \theta)}{|w|^{n-1} |e_1-w|^{n-1}} dw.
\end{aligned}$$

Notice that after multiplying each term by r^{n-2} , the remaining integrals are smooth in the variables x, y, r, η and θ .

B.2 The General Case

We seek to record a general formula for the integral representation of $A^m Jf(x)$ which resembles a weakly singular integral with integral singularity $\frac{1}{|x-y|^{n-m}}$. Let's make some new definitions to simplify the notation. Set $y_0 = x$ and $y_m = y_{m+1} = y$, and given $1 \leq j \leq m-1$, define

$$w_j := \frac{x - y_j}{|x - y_{j+1}|}, \quad w'_j := \frac{y_j - y_{j+1}}{|x - y_{j+1}|}.$$

Note that $\eta = \widehat{x-y} = w_m$. Also define $w_0 = 0$ for convenience, and then set

$$\begin{aligned}
\psi_j^1(w_j, w'_j) &:= \psi(|w_j|), \\
\psi_j^2(w_j, w'_j) &:= \psi(|w'_j|), \\
\psi_j^3(w_j, w'_j) &:= 1 - \psi(|w_j|) - \psi(|w'_j|).
\end{aligned}$$

We notice a few useful formulas for $1 \leq j \leq m-1$

$$\begin{aligned}
w'_j &= \widehat{w_{j+1}} - w_j, \\
y_j &= x - |x-y| w_j \prod_{l=j+1}^{m-1} |w_l|, \\
y_j - y_{j+1} &= |x - y_{j+1}| (\widehat{w_{j+1}} - w_j).
\end{aligned}$$

Define

$$\mathbf{y} := [y_1, \dots, y_{m-1}, y] = \left[\left\{ x - |x - y| w_j \prod_{l=j+1}^{m-1} |w_l| \right\}_{j=1}^{m-1}, y \right] \in \mathbb{R}^m,$$

$$|\mathbf{w}| := \left[\prod_{l=j}^{m-1} |w_l| \right]_{j=1}^{m-1}.$$

We can partition the integrations involved in $\alpha_m(x, y, \theta)$ as

$$\int \sum_{\gamma \in \{1,2,3\}^m} \left[\prod_{j=1}^{m-1} \psi_j^{\gamma_j}(w_j, \widehat{w_{j+1}} - w_j) \right] \cdot \frac{\alpha_m(x, \mathbf{y}, r|\mathbf{w}|, \widehat{w_1}, \frac{\widehat{w_2} - w_1}{|\widehat{w_2} - w_1|}, \dots, \frac{\eta - w_{m-1}}{|\eta - w_{m-1}|}, \theta)}{r^{n-m} |\eta - w_{m-1}|^{n-1} \prod_{l=1}^{m-1} |w_l|^{n-l-1} \prod_{k=1}^{m-2} |\widehat{w_{k+1}} - w_k|^{n-1}} dw_1 \dots dw_{m-1}.$$

For $0 \leq j \leq m-1$ define

$$\mathcal{D}_j^1 := B_0(1/2)$$

$$\mathcal{D}_j^2 := B_{\widehat{w_{j+1}}}(1/2)$$

$$\mathcal{D}_j^3 := \mathbb{R}^n \setminus (B_0(1/4) \cup B_{\widehat{w_{j+1}}}(1/4)).$$

For each $\gamma \in \{1,2,3\}_{j=1}^{m-1}$, set $\mathcal{D}_\gamma := \prod_{j=1}^{m-1} \mathcal{D}_j^{\gamma_j}$. Then we have the integral

$$I_m(x, y, r, \eta, \theta) = \sum_{\gamma \in \{1,2,3\}_{j=1}^{m-1}} \int_{\mathcal{D}_\gamma} \left[\prod_{j=1}^{m-1} \psi_j^{\gamma_j}(w_j, \widehat{w_{j+1}} - w_j) \right] \cdot \frac{\alpha_m(x, \mathbf{y}, r|\mathbf{w}|, \widehat{w_1}, \frac{\widehat{w_2} - w_1}{|\widehat{w_2} - w_1|}, \dots, \frac{\eta - w_{m-1}}{|\eta - w_{m-1}|}, \theta)}{r^{n-m} |\eta - w_{m-1}|^{n-1} \prod_{l=1}^{m-1} |w_l|^{n-l-1} \prod_{k=1}^{m-2} |\widehat{w_{k+1}} - w_k|^{n-1}} dw_1 \dots dw_{m-1},$$
(B.2.1)

We then define

$$\beta_m(x, y, r, \eta, \theta) := r^{n-m} I_m(x, y, r, \eta, \theta),$$
(B.2.2)

so that

$$A^m Jf(x) = \int \frac{\beta_m(x, y, |x - y|, \widehat{x - y}, \theta)}{|x - y|^{n-m}} f(y) dy.$$

In order to justify the convergence of such an iterated integral (as we have changed the order around), we first note that x and y are restricted to Ω by assumption. Secondly, the

singularities in each variable w_j are all locally integrable since near such singularities we have local behavior like $\frac{1}{|w_j|^{n-j-1}}$ or $\frac{1}{|\widehat{w_{j+1}} - w_j|^{n-1}}$. It remains to show bounds on the terms in the sum which involve integrations over one or more of the domains $\mathcal{D}_j^3 := \mathbb{R}^n \setminus (B_0(1/4) \cup B_{\widehat{w_{j+1}}}(1/4))$. Let j be the first index where such an integration occurs in a given term. We may integrate out the variables $w_1 \dots w_{j-1}$ first since the domains of integration are bounded in those cases. Since $y_j \in \Omega$, we have $x - rw_j \prod_{l=j+1}^{m-1} |w_l| \in \Omega$. Thus

$$|w_j| \leq \frac{\text{diam}(\Omega)}{r \prod_{l=j+1}^{m-1} |w_l|} =: M.$$

So we may estimate the integral by

$$\begin{aligned} & \int_{\mathcal{D}'_\gamma} \frac{C}{r^{n-m} |\eta - w_{m-1}|^{n-1} |\widehat{w_{m-1}} - w_{m-2}|^{n-1} \dots |\widehat{w_{j+2}} - w_{j+1}|^{n-1}} \\ & \quad \cdot \frac{1}{\prod_{l=j+1}^{m-1} |w_l|^{n-l-1}} \left[\frac{1}{|u^{n-j-2}|} \right]_1^M dw_{j+1} \dots dw_{m-1} \\ & \leq \frac{\text{diam}(\Omega)^{2+j-n}}{r^{j+2-m}} \int_{\mathcal{D}'_\gamma} \frac{C}{|\eta - w_{m-1}|^{n-1} |\widehat{w_{m-1}} - w_{m-2}|^{n-1} \dots |\widehat{w_{j+2}} - w_{j+1}|^{n-1}} \\ & \quad \cdot \frac{1}{\prod_{l=j+1}^{m-1} |w_l|^{j+1-l}} dw_{j+1} \dots dw_{m-1}, \end{aligned}$$

where \mathcal{D}'_γ is the domain obtained from \mathcal{D}_γ by removing the domains of integration with respect to $w_1 \dots w_j$. Since $1 \leq j \leq m-1$, we have that $3-m \leq 2+j-m \leq 1$. In all cases, the behavior in r is an integrable singularity since $n \geq 2$. Further integrations over domains \mathcal{D}_j^3 will result in similar estimates, and in all cases the power on r remains integrable. It follows that, upon multiplying the whole integral by r^{n-1} , we obtain a function that is C^∞ in x, y, r and θ , since the multiplicative factor of r has exponent between $n-2$ and $n+m-3$, all of which are smooth at 0.

It remains to show smoothness in η . Notice that when $\gamma_{m-1} = 1$ or 3 (i.e. when $w_{m-1} \in B_0(1/2)$ or $w_{m-1} \in \mathbb{R}^n \setminus (B_0(1/4) \cup B_\eta(1/4))$), then $|\eta - w_{m-1}|$ is uniformly positive. Thus, we can differentiate under the integral sign with respect to η the term $\frac{1}{|\eta - w_{m-1}|^{n-1}}$ infinitely many times. On the other hand, to differentiate the terms with $\gamma_{m-1} = 2$ (i.e. $w_{m-1} \in B_\eta(1/2)$), we make the substitution $z_{m-1} = \eta - w_{m-1}$ and note that $z_{m-1} \in B_0(1/2)$. The term $\frac{1}{|w_{m-1}|^{n-m}} = \frac{1}{|\eta - z_{m-1}|^{n-m}}$ is uniformly bounded away from 0, so we may

differentiate it arbitrarily many times with respect to η . The only other potential problem will occur if in addition, $\gamma_{m-2} = 2$, or equivalently, $w_{m-2} \in \widehat{B_{w_{m-1}}}(1/2) = \widehat{B_{\eta-z_{m-1}}}(1/2)$. In this case, we make yet another change of variables $z_{m-2} = \widehat{w_{m-1}} - w_{m-2} = \widehat{\eta - z_{m-1}} - w_{m-2}$, so that $z_{m-2} \in B_0(1/2)$ and the term $\frac{1}{|\widehat{w_{m-1}} - w_{m-2}|^{n-1}}$ does not depend on η . Similarly, the term $\frac{1}{|w_{m-2}|^{n-m+1}} = \frac{1}{|\widehat{\eta - z_{m-1}} - z_{m-2}|^{n-m+1}}$ is uniformly bounded for $z_{m-2} \in B_0(1/2)$, and so we can differentiate it arbitrarily many times with respect to η . Continuing in this way, if at any point $\gamma_j = 2$, we can make the change of variables $z_j = \widehat{w_{j+1}} - w_j$ and obtain similar bounds. The main idea is to transfer the derivative to the numerator function α_m . It is a routine matter to check that under such changes of variables, differentiation of α_m with respect to η does not result in any unbounded factors involving η via the chain rule.

Remark 6. By the same reasoning, if $\alpha(x, y, r, \eta, \theta)$ is in $C_c^2(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$, then $\beta_m \in C_c^2$.

We now consider the following adaptation of Lemma 2 from [18]:

Lemma B.2.1. Let $m \geq 1$ and let $\mathcal{A} : C_0(\Omega) \rightarrow C(\Omega, \mathbb{S}^{n-1})$ be the operator

$$\mathcal{A}f(x, \theta) = \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} r^{m-1} \alpha(x, r, \omega, \theta) f(x + r\omega) dr d\omega,$$

where $\alpha \in C^\infty(\Omega \times \mathbb{R} \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$. Then \mathcal{A} is a classical Ψ DO of order $-m$ with full symbol

$$a(x, \xi) \sim \sum_{k=m-1}^{\infty} a_k(x, \xi),$$

where

$$a_k(x, \xi) = 2\pi \frac{i^k}{k!} \int_{\mathbb{S}^{n-1}} \partial_r^k A(x, 0, \omega, \theta) \delta^{(k)}(\omega \cdot \xi) d\omega.$$

Proof. The proof directly follows from the proof of the case $m = 1$ in [18]. Let $A'(x, r, \omega, \theta) = r^{m-1} A(x, r, \omega, \theta)$, and note that if A' is an odd function of (r, ω) , then $\mathcal{A}f = 0$. So we may replace $A' = r^{m-1} A$ by

$$\begin{aligned} A'_{\text{even}}(r, \omega) &= \frac{1}{2}(A'(r, \omega) + A'(-r, -\omega)) \\ &= \frac{1}{2}(r^{m-1} A(r, \omega) + (-r)^{m-1} A(-r, -\omega)) \\ &= \frac{r^{m-1}}{2}(A(r, \omega) + (-1)^{m-1} A(-r, -\omega)). \end{aligned}$$

We can then integrate over $r \geq 0$ only and double the result. Thus,

$$\mathcal{A}f(x) = 2 \int_{\mathbb{S}^{n-1}} \int_0^\infty A'_{\text{even}}(x, r, \omega, \theta) f(x + r\omega) dr d\omega.$$

Changing to polar coordinates via $z = r\omega$ and setting $y = x + r\omega$, we obtain

$$\mathcal{A}f(x) = 2 \int A'_{\text{even}} \left(x, |y - x|, \frac{y - x}{|y - x|}, \theta \right) \frac{f(y)}{|y - x|^{n-1}} dy.$$

We then use a finite Taylor expansion of $A_{\text{even}}(x, r, \omega, \theta)$ in r near $r = 0$ with $N > 0$ to get

$$A'_{\text{even}}(x, r, \omega, \theta) = \sum_{k=0}^{N-1} A'_{\text{even},k}(x, \omega, \theta) r^k + r^N R_N(x, r, \omega, \theta).$$

One can compute that $2A'_{\text{even},k}(x, \omega, \theta) = A'_k(x, \omega, \theta) + (-1)^k A'_k(x, -\omega, \theta)$, where $k!A'_k = \partial_r^k|_{r=0}A' = \partial_r^k|_{r=0}[r^{m-1}A]$. Clearly, $A'_k = 0$ for all $k < m - 1$. Therefore, following the proof of Lemma 2 in [18], we obtain that the terms $a_k(x, \xi, \theta) = 2\pi i^k \int_{\mathbb{S}^{n-1}} A'_k(x, \omega, \theta) \delta^{(k)}(\omega \cdot \xi) d\omega = 0$ for all $k < m - 1$. \square

Proposition B.2.1. Let $\alpha \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$ and consider the operator $A : L^2(\mathbb{R}^n \times \mathbb{S}^{n-1}) \rightarrow L^2(\mathbb{R}^n; C^\infty(\mathbb{S}^{n-1}))$ defined by

$$[Af](x, \theta) = \int \frac{\alpha(x, y, |x - y|, \widehat{x - y}, \theta)}{|x - y|^{n-1}} f(y, \widehat{x - y}) dy.$$

Then for $m \geq 1$, $A^m J$ is a classical pseudodifferential operator of order $-m$ with smooth parameter θ , where $J : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n \times \mathbb{S}^{n-1})$ is the extension operator $Jf(x, \theta) = f(x)$.

Proof. Recall that

$$\begin{aligned} [A^m Jf](x, \theta) &= \int I_m(x, y, |x - y|, \widehat{x - y}, \theta) f(y) dy \\ &= \int \frac{\beta_m(x, y, |x - y|, \widehat{x - y}, \theta)}{|x - y|^{n-m}} f(y) dy \\ &= \int_{\mathbb{S}^{n-1}} \int_0^\infty r^{m-1} \beta_m(x, x - r\omega, r, \omega, \theta) f(x - r\omega) dr d\omega \\ &= \int_{\mathbb{S}^{n-1}} \int_0^\infty r^{m-1} \beta_m(x, x + r\omega, r, -\omega, \theta) f(x + r\omega) dr d\omega. \end{aligned}$$

We can then apply Lemma B.2.1 to complete proof. \square

Let us also compute the adjoint of the operator $A^m J$. Given functions $f \in L^2(\mathbb{R}^n \times \mathbb{S}^{n-1})$ and $g \in L^2(\mathbb{R}^n)$, we have

$$\begin{aligned} \langle [A^m J]^* f, g \rangle_{L^2(\mathbb{R}^n)} &= \langle f, A^m J g \rangle_{L^2(\mathbb{R}^n \times \mathbb{S}^{n-1})} \\ &= \int \int \int \frac{\beta_m(x, y, |x-y|, \widehat{x-y}, \theta)}{|x-y|^{n-m}} g(y) f(x, \theta) dy dx d\theta \\ &= \int g(y) \left(\int_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} \frac{\beta_m(x, y, |x-y|, \widehat{x-y}, \theta)}{|x-y|^{n-m}} f(x, \theta) dx d\theta \right) dy. \end{aligned}$$

Thus

$$[A^m J]^* f(x) = \int_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} \frac{\beta_m(y, x, |y-x|, \widehat{y-x}, \theta)}{|x-y|^{n-m}} f(y, \theta) d\theta dy.$$

It is then clear that $[A^m J]^* : H^l(\Omega \times \mathbb{S}^{n-1}) \rightarrow H^{l+m}(\Omega)$ using a similar argument as in Lemma 2 of [50] together with Proposition B.2.1.

B.2.1 Estimates of $A^m J$ for C^2 α_m

Suppose that the characteristic α in the integral formula for A is in the space $C^2(\overline{\Omega} \times \overline{\Omega} \times \mathbb{R}_+ \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$. Then α_m , the kernel of $A^m J$ is also C^2 . From Proposition 1 of [50], we know that $A^m J$ is a bounded operator from $L^2(\Omega)$ to $H^1(\Omega; C^2(\mathbb{S}^{n-1}))$, and we have the estimate

$$\|A^m J\|_{L^2 \rightarrow H^1} \leq C \|\beta_m\|_{C^2}. \quad (\text{B.2.3})$$

Furthermore, by the definition of β_m and the inductive definition of α_m , see (B.0.1), we have that

$$\|A^m J\|_{L^2 \rightarrow H^1} \leq C \|\beta_m\|_{C^2} \leq C(3C')^m \|\alpha\|_{C^2}^m \leq (C'')^m \|\alpha\|_{C^2} \quad (\text{B.2.4})$$

where C'' only depends on n and m .

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