

Manipulating Elastic Waves in Topological Mechanical Metamaterials

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Abstract

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Due to the recent discovery of topological insulators in condensed matter physics, a new notion of topology has emerged in association with the intrinsic dispersion behavior of a structure. In this work, we have taken this notion of topology and have shown some novel wave manipulation capabilities in one- and two-dimensional (1D and 2D) elastic systems. In particular, we have taken four different type of elastic systems. First, a 1D elastic system made of a highly tunable chain of granular particles, in which we analytically, numerically, and experimentally demonstrate an in situ topological band transition and existence of a topologically-protected vibration mode. Second, we take a 1D time-dependent system made of a granular chain, in which nonreciprocal wave dynamics is invoked. Using numerical simulations, we show that this nonreciprocity can be quantified by a topological invariant of 2D, i.e., Chern number. Third, we consider a 2D discrete lattice consisting of units inspired by the Stewart platform. By harnessing the bistability of the unit, we show the capability of in situ manipulation of wave path in this tunable 2D topological structure. We also demonstrate the uniqueness of topological waveguides in comparison to conventional (non-topological) waveguides in this platform. Lastly, we propose a continuum structure, i.e., a thin plate, with systematically arranged local resonators on its top. We show numerically that by judiciously arranging the local resonators, we can create a subwavelength topological waveguide in the

system, which can robustly guide flexural waves along a path with sharp bends. In principle, this is analogous to the quantum spin Hall effect discovered in condensed matter physics. We employ a ubiquitous design of a bolted plate to experimentally demonstrate waveguiding capability of this plate. These results show that the new notion of topology—emerging from topological insulators in electronic and optical systems—can provide a fresh outlook towards manipulating elastic waves in mechanical structures, such as directional propagation, filtering, and localization of wave modes. These novel effects investigated in this study can be exploited for engineering applications such as impact mitigation, vibration isolation, smart sensing, and energy harvesting.

DEDICATION

to my parents and teachers

“One who finds inaction in action, and action in inaction, he (she) is the wise one among humans; he (she) is engaged in yoga and is a performer of all actions!”

– Bhagavad Gita 4.18

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Chapter 1

INTRODUCTION

1.1 *Mechanical metamaterials*

The study of elastic wave propagation in solids spans across many engineering and science disciplines. Starting from large-wavelength earth waves in geophysics to ultra-small-wavelength surface acoustic waves in modern sensing devices, the relevance of elastic waves has been evident at a wide range of length scales. Elastic waves, in the form of ultrasound, are also an essential part of well-established imaging techniques in medical diagnostics and structural health monitoring. Therefore, any fundamental studies related to elastic waves in solids, especially to acquire the ability to manipulate them at will, have potential to influence a wide range of engineering applications.

In this regard, the advent of mechanical metamaterials in recent years offers several new possibilities [1–4]. In a very generic sense, a mechanical metamaterial is a material that is artificially designed to show unprecedented mechanical properties (the Greek root “meta” means “beyond”). The design philosophy of mechanical metamaterial gives more emphasis to the geometric arrangement of material rather than the type of material itself; and the enhanced mechanical properties are usually dynamic and, are related to wave propagation in the medium.

This geometry-based design philosophy thus leverages on numerous degrees of freedom in a structure, to create basic structural ingredients that invoke versatile wave physics. Periodicity [5], nonlinearity [6], and structural resonances [7] are usually these structural ingredients of the metamaterial recipe. See some typical architectures in Fig. 1.1. A stricter definition of mechanical metamaterials, however, necessitates the presence of structural resonances, so that even large wavelength waves can be manipulated using in a subwavelength medium [8,9].

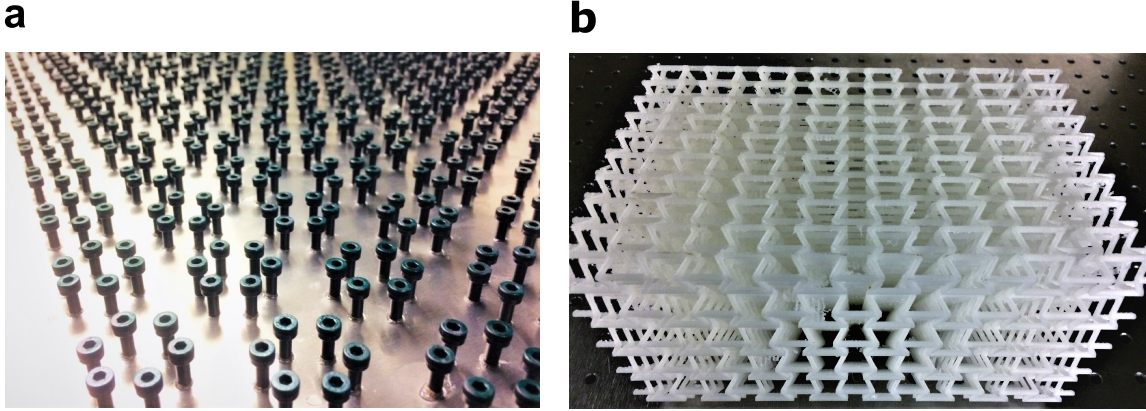


Figure 1.1: Mechanical metamaterials. (a) A 2D mechanical metamaterial to control the flow of flexural waves in thin plates. The bolts act as local resonator and invoke unique physics (cf. Chapter 6). (b) A 3D-printed 3D mechanical metamaterial.

Clearly, the design approach of mechanical metamaterials opens new doors for physics-based structural design paradigm that can not only provide more efficient solutions to existing applications, but also can pave the way for building novel multifunctional devices. For example, cloaking [10], vibration isolation [11], negative refraction for superlensing [12, 13], mechanical energy harvesting [14], and novel impact attenuating systems [15, 16] are some of the areas where the use of mechanical metamaterials have been suggested. Moreover, this design approach has been more relevant today than ever because of the rapid advancement of fabrication techniques, such as additive manufacturing, which can enable the realization of complex geometrical structures with remarkable precision [17].

In this thesis, we present a new structural ingredient in the mechanical metamaterial recipe. This compliments the existing ingredients in some sense and invokes nontrivial wave dynamics, potentially leading to novel engineering devices. The ingredient is topology.

1.2 Topology

Topology in mathematics is the study of objects under continuous deformations. To give a simple example, topology of a sphere remains intact even when it is squeezed, provided we do not allow any cut or hole. It is natural to ask what makes the topology of a sphere same as that of its homeomorphic counterpart, e.g., an ellipsoid. The answer lies in the topological invariant that remains the same while a body is smoothly deformed. It also means that if we take another body, e.g., a doughnut, its topological invariant would be different from that of a sphere, simply because a doughnut and a sphere are not homeomorphic, i.e., they cannot be transformed to each other by a smooth deformation. Mathematically, one can calculate the topological invariant of these objects performing a surface integral as per the Gauss-Bonnet theorem [18]. It states that the Euler characteristic

$$\chi = \frac{1}{2\pi} \int_S K dA, \quad (1.1)$$

where K is the Gaussian curvature. In addition, the genus $g = 1 - \chi/2$ represent the number of holes in these objects. Therefore it should, of course, remain intact under a smooth deformation.

Though the field of topology in mathematics is huge in its own right, it would be enough at this point to grasp the aforementioned basic definition. However, the key would be to take this simple analogy and apply in the realm of physics. In particular, we would import this understanding of topology not to characterize the shape of an object, rather we will use it in a subtler sense, i.e., to characterize the dispersion behavior of a system. It is well known that a rich dispersion can be obtained by several methods, for example, periodicity in a structure [5]. We can engineer a structure to support band gaps that do not allow propagation of waves with frequencies inside them [19]. However, an interesting question in this context is: Can two band gaps obtained from two distinct structures have a qualitative difference on topological ground? The answer is yes. We will therefore use topology not to characterize the real space [as is the case in Eq. (1.1)], but the wavevector space (“k-space”) of lattice structures. Hence, it would directly relate to the flow of elastic energy in solids, and

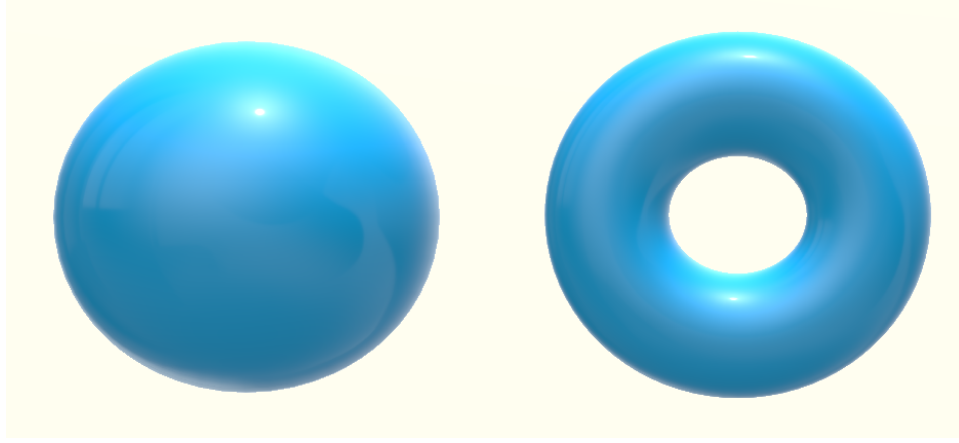


Figure 1.2: Topologically distinct objects. A sphere has genus $g = 0$, and a doughnut has $g = 1$.

such materials would be called topological mechanical metamaterials. But first, let us briefly go over the evolution of this notion of topology, which has its origin not in metamaterial engineering but in condensed matter physics of electronic systems.

1.3 *Topology in condensed matter physics*

A new notion of topology gained popularity in the context of the quantum Hall effect. It was the Nobel-prize-winning discovery when, in 1980, Klaus von Klitzing *et al.* [20] reported the exact quantization of Hall resistance in 2D electron gas under strong external magnetic field. In 1981, Laughlin's [21] and Halperin's [22] arguments indicated that a quantum mechanical phenomenon is at work and the quantized Hall conductance was the manifestation of robust *chiral* edge currents, propagating on the edges of the sample in one direction while the Fermi energy was between separated Landau levels. However, it was in 1982, when Thouless, Kohmoto, Nightingale, and den Nijs (TKNN) explicitly showed that this quantization is a “bulk property”, i.e., related to integers obtained from the periodic potential of the sample [23]. These integers were the topological invariants of the system, which related the topological nature of the bulk to the robustness of the edge currents in the quantum Hall

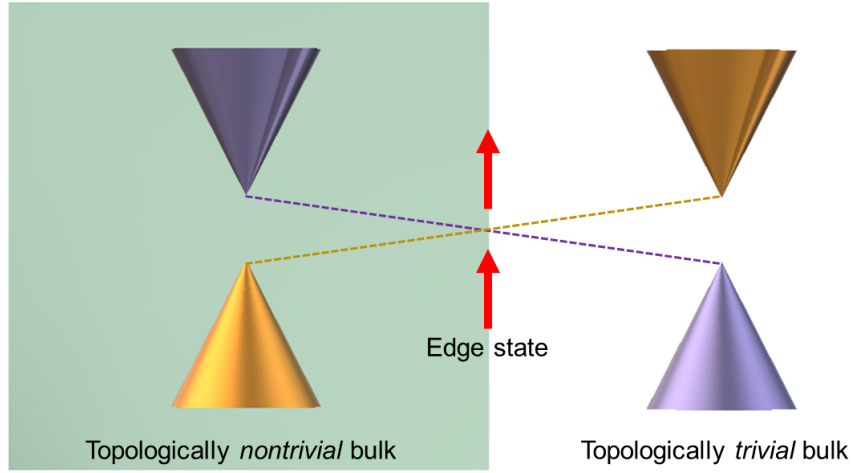


Figure 1.3: Band inversion between inside and outside of a sample indicates the closing of band gap on the edges of the sample. This manifests as an edge state.

effect. It was shortly pointed out by Simon [24] that these topological invariants, also known as the Chern numbers in mathematics, had a direct connection with the concept of geometric phase, first discussed by Pancharatnam [25] in 1956 and popularized by Berry [26] in 1984. In 1988, Haldane proposed a “2D-graphite” (graphene) model for the quantum Hall effect [27]. As opposed to the previous efforts, this semimetal did not rely on *net* external magnetic field and associated Landau levels. Rather, it had two bands meeting at high-symmetry points in Brillouin zone, and Haldane imagined creating a band gap (lifting the degeneracy) by breaking the time reversal symmetry in this lattice using *magnetic ordering*. Inside this band gap, the system showed chiral edge conductance that was quantized. This quantization again relates to the Chern number.

One can think of the Chern number as the k -space counterpart of the topological invariant mentioned in Eq. (1.1). For instance, for a 2-dimensional lattice, its k -space would be of the shape of a torus due to the periodicity of two wave vector components, and therefore, the integral over surface in *real space* in Eq. (1.1) gets substituted by an integral over the surface of a torus in k -space. Also, the Gaussian curvature is replaced by the Berry curvature

$\mathcal{F} = \nabla \times \mathbf{A}$, where the Berry connection $\mathbf{A}$ is obtained by the eigenvectors $|u_n(\mathbf{k})\rangle$ of n th band, such that $\mathbf{A} = -i\langle u_n(\mathbf{k})|\nabla_{\mathbf{k}}|u_n(\mathbf{k})\rangle$. Therefore, the Chern number of n th band has the following form [28]:

$$C_n = \frac{1}{2\pi} \int_S \mathcal{F} d^2\mathbf{k}. \quad (1.2)$$

Once the Chern numbers of bands are calculated, one can estimate the topological nature of a band gap by summing up the Chern numbers of all the bands below it [23]. Its nonzero value indicates that the band gap is topologically nontrivial. Physically, if the Fermi energy lies inside this topologically nontrivial band gap, one observes topological chiral currents at boundaries of the sample. This is also known as the principle of “bulk-boundary correspondence” [29–31], as the bulk topological properties directly dictate what happens at the boundaries of a sample.

More intuitive way to understand this process is by so-called “band inversion” [32], which is usually observed in topological physics. In Fig. 1.3, we show a topologically nontrivial bulk (e.g., the quantum Hall state) has an inverted band gap compared to the surrounding topologically trivial bulk (e.g., vacuum). Because both the bulks have a band gap, electrons at Fermi energy inside the band gap would flow neither inside nor outside of the sample. However, for the band inversion to occur, band gap will have to close, and that can only happen on the edges, leading to energy localization. This analogy fits very well when one compares the two topologically distinct spaces to two topologically distinct objects, for example, a sphere and a doughnut. If one wants to make a transition between these two objects, one must make a hole. Similarly, if one wants to go from one bulk to a topologically distinct bulk, one must have a band-closing on the edge. In this way, a new notion of topology emerges in relation to k -space characterization in band theory and predicts topological edge states. This therefore explained the extreme robustness of the Hall conductivity in the quantum Hall effect in the presence of disorder, where the disorder could be thought of as a smooth deformation of the Hamiltonian and the robustness was the manifestation of the topological invariance [29].

The role of topology in band theory, however, received a significant boost in recent years

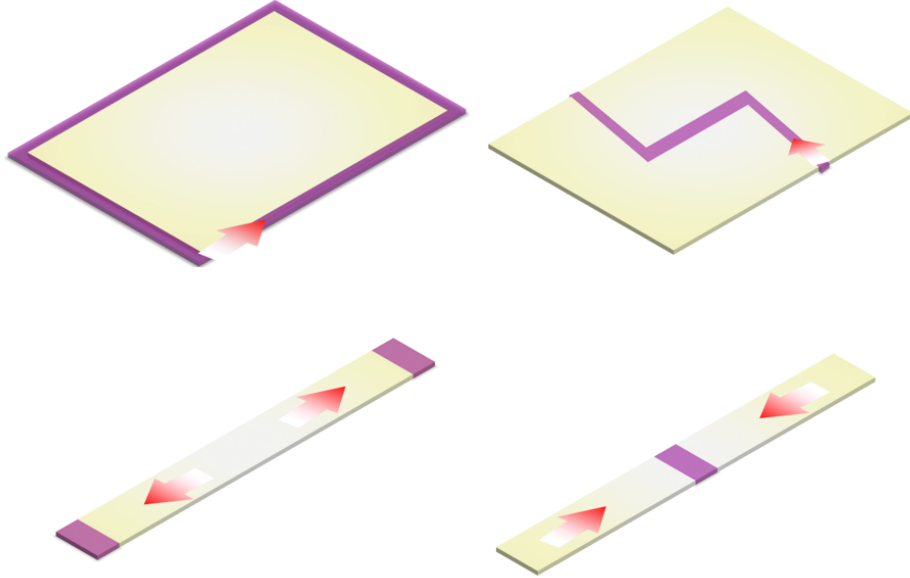


Figure 1.4: Some schematics of robust energy localization/guiding (purple) in 1D and 2D mechanical structures on boundaries.

with the proposal of the quantum spin Hall effect shown by topological insulators [33–35] . As opposed to the requirement of the breakage of the time reversal symmetry or an external magnetic field in the quantum Hall effect, the quantum spin Hall effect *did not* require any external magnetic field and preserved the time reversal symmetry, but still showed robust edge currents. It was analogous to a superimposition of two quantum Hall systems, each designed for two opposite electron spins. Consequently, two opposite spins propagated in opposite directions robustly on the edges of the sample while the bulk remained an insulator. Clearly, the net charge flow on the edges was 0, and that also implies the Chern number to be 0. Therefore, a new topological invariant, i.e., the spin Chern number or $\mathbb{Z}_2$ index was used to prove the topological robustness of an individual spin.

Since then, there have been a plethora of research to invoke nontrivial physics of electrons in 1D, 2D, and 3D systems. In fact, a systematic table for topological characterization, i.e., the periodic table, encompasses a variety of systems depending on their spatial dimension

and symmetry [36]. However, the next question is whether such a topological framework, originally developed for electronic (fermionic) systems, can also be used to characterize elastic (bosonic) systems? If yes, such systems would show robust energy confinement at the boundaries between two topologically distinct structures (Fig. 1.4). This is especially interesting in two aspects. First, they could provide a new way to study and control elastic energy in solids. Second, given the design freedom mechanical structures provide, topological physics can be studied in a controllable way in lab settings. Therefore, it is relevant more than ever to study this new notion of topology in mechanical metamaterials [37].

1.4 Topology in mechanical metamaterials

One of the earliest examples of topological band theory in mechanical systems were found in relation to the modeling of microtubules [38] and microfilaments [39], where the existence of robust edge phonon modes was associated with their topological nature. Since then, manipulation of finite-frequency (nonzero frequency) have been demonstrated in several acoustic metamaterials [40–46]. Similarly, for elastic vibrations, there are systems that require external bias or field to invoke a robust one-directional edge wave, analogous to the quantum Hall effect [47–49]. Also, there are systems that, in fact, do not require any external bias, and therefore are passive, but still possess topological wave physics [50–52]. Owing to the fundamental difference between a fermionic and a bosonic system, which lies in a bosonic system not having intrinsic spins as electrons do in a fermionic system, one creates “psuedospins” in these passive systems by judicious geometric designs. In this way, spin-dependent one-way propagation of edge modes have been shown, analogous to the quantum spin Hall effect.

Apart from manipulating finite-frequency wave modes, there have also been several studies on the topological nature of zero-frequency modes in mechanical structures. In 2013, Kane and Lubensky [53] proposed a mechanical counterpart of topological 1D electronic chain, in which they mapped the zero-energy edge state in electronics to zero-frequency edge modes in mechanics. This has therefore led to a number of studies in 1D and higher dimensions [54–56].

However, in this work, we only focus on manipulating finite-frequency modes in elastic structures, as those are directly relevant to applications in energy harvesting, vibration isolation, and structural health monitoring [37]. We list some key limitations and challenges of the research on topological mechanical metamaterials to date:

- Most of the efforts have been limited to theoretical studies due to the complexity in fabrication and testing of mechanical structures.
- Most of the work involve discrete spring-mass systems. This makes them hard to practically implement in real-world applications. There have been recent numerical studies on elastic plates [51, 57–59], which indicate the possibility of using continuum structures as topological mechanical metamaterials.
- Uniqueness of elastic systems in terms of achieving tunability from geometric manipulations is usually not harnessed to demonstrate a topological transition within a same structure. For example, the role of structural effects such as bistability [60], has not been explored to tune topological mechanical metamaterials *in situ*.
- Elastic systems do possess several types of wave modes as opposed to the wave phenomena in acoustics. This makes them far challenging in terms of manipulating wave modes on topological grounds. Researchers have therefore focused on only one type of wave mode and ignored others for topological manipulations [57].

Thus, these points motivate our research, and we have taken a variety of 1D and 2D elastic systems to address these challenges.

1.5 Significance and organization of this thesis

This work is a synthesis of several standalone studies on topological elastic systems (already published or under review in peer-reviewed journals), and those are arranged as different chapters in the order of their complexity, i.e, 1D to 2D. Each chapter has its own introduction,

which includes the motivation and relevant literature. A section on authors' contributions have been added at the end of each chapter.

In chapter 2, we take a 1D elastic system that is made of highly tunable granular particles. To the best of authors' knowledge, this is the first experimental demonstration of in situ topological transition in a 1D elastic system. Also, using laser Doppler vibrometer, we are able to precisely measure the shape of topologically protected mode in the system, and that matches well with numerical mode shape. We have also derived, for the first time, closed-form solutions for the frequencies of topologically protected modes in a 1D elastic system.

In chapter 3, we add a synthetic dimension of *time* to the aforementioned 1D granular chain, and make it a 1D time-dependent chain. We introduce the time dependency by using a secondary pump wave. We show that such a system supports a nonreciprocal wave behavior. This nonreciprocity is the manifestation of asymmetric dispersion curves in the system. In quasistatic limit, this time-dependent 1D system can be understood as a 2D system with 1 space and 1 time dimensions. Therefore, we show that the asymmetry in band dispersion curves can be explained by calculating topological indices in 2D, i.e., the Chern number. To the best of authors' knowledge, this is the first instance when such a connection is made in acoustic and elastic systems.

In chapter 4, we study a discrete 2D lattice. For the first time, we propose the unique idea of harnessing bistability in structural constituents to manipulate topological configurations in situ. We show the ability to guide mechanical wave in the lattice, and also show that this waveguide can be tailored in situ. This tunability allows us to closely compare the performance of topological waveguide and conventional (non-topological) waveguide.

In chapter 5, we propose a continuum plate structure with local resonators mounted on top, and numerically demonstrate the ability to guide flexural waves crisply around the bends. For the first time, we implement a symmetry-protected approach in locally resonant plates to invoke an effect analogous to the quantum spin Hall effect shown by topological insulators. We show the existence of two counter-rotating pseudospins and their propagation in opposite directions at the boundary between two topologically distinct lattices. We

devise a strategy to selectively excite these pseudospin, and demonstrate a spin-dependent robust one-way wave propagation, which is particularly novel in elastic systems. Also, the system is a subwavelength metamaterial and the frequency of the local resonators limits the operating frequencies on the plate. Therefore, low frequency modes, generally requiring large structures, can also be guided in limited-sized plates.

In chapter 6, we include the experimental study of topological waveguiding in elastic plate with local resonators. We choose a ubiquitous design of a bolted plate for this purpose, where bolts act as local resonators. Due to the complex interplay between the bolts and the plate, and local resonance of the bolts, we observe several unique effects, such as coexistence of two double Dirac cones, and a complete band gap that forbids *all* the plate modes. Consequently, we experimentally demonstrate the ability to guide flexural wave crisply along a path with minimal losses into the bulk of the structure. This suggests a new design strategy to manipulate waves in elastic systems on topological grounds, while efficiently managing other unwanted elastic modes.

Chapter 2

ONE-DIMENSIONAL TOPOLOGICAL METAMATERIAL

We numerically investigate and experimentally demonstrate an *in situ* topological band transition in a highly tunable mechanical system made of cylindrical granular particles. This system allows us to tune its interparticle stiffness in a controllable way, simply by changing the contact angles between the cylinders. The spatial variation of particles' stiffness results in an *in situ* transition of the system's topology. This manifests as the emergence of a boundary mode in the finite system, which we observe experimentally via laser Doppler vibrometry. When two topologically different systems are placed adjacently, we analytically predict and computationally and experimentally demonstrate the existence of a finite-frequency topologically protected mode at their interface.

2.1 Introduction

Topological insulators (TIs) constitute an intense area of recent interest within condensed matter physics. TIs support directional electron transport on their surface, and this transport is immune to defects [29]. The existence of such a surface state has a remarkable correspondence to the nontrivial topological invariants of the bulk of TIs [30]. Therefore, by knowing the topology of the bulk, one can predict the response on the surface or edge of the material. This response is robust against defects on the surface as long as the topology of the bulk is preserved. Such topological characterization has naturally emerged as a tool to design novel mechanical structures with unconventional vibration properties on their surfaces [47, 49–52, 57]. The study of these mechanical structures not only realizes the topological phenomenon in easily controllable and observable macroscale systems, but also has potential to shape a new design paradigm for structural applications, such as vibration isola-

tion and energy harvesting [37]. At the same time, the paradigm of TIs has had a significant impact in other areas, e.g., linear and nonlinear optics [61–65].

A dimer system such as the 1D Su-Schrieffer-Heeger (SSH) model provides the basic framework to understand band topology [66–69]. The direct mechanical analogue of such a dimer has been recently suggested [53]. This system has a phononic band gap near zero frequency and, in turn, can support *zero-frequency* topological modes. Deviating from this model, one can also arrive at a mechanical dimer that results in a non-zero-frequency band gap, potentially leading to *finite-frequency* topological modes [70]. This could be achieved by tailoring the on-site potentials in the standard SSH model. Such a system would be especially interesting as it would enable us to understand how the topological framework can be used to predict and tailor energy localization on the edges or interfaces of a structure. To study this phenomenon in detail, one would ideally need a system that allows a *smooth* control over its topological characteristics, in order to closely observe its effects on the vibration response of the structure. Achieving such tunability is extremely challenging in experimental settings, since it requires well-controlled, *in situ* manipulation of either the mass or stiffness. Changing the masses of the dimer lattice demands manual reassembly of the systems’ constituents [71, 72]. Altering the stiffness values via external couplings, e.g., magnetoelastic [73], electroelastic [74], and photoelastic [75] interactions, could make the system cumbersome for structural uses. The need of precise actuation and measurement adds up to this challenge. This explains why an *in situ* topological band transition for finite-frequency elastic vibrations has not been demonstrated experimentally. A design that can address this challenge would not only contribute to our general understanding of topological mechanical systems; it also holds promise towards catalyzing the implementation of standalone structures with tunable energy localization characteristics.

For this purpose, we use a granular system made of cylindrical particles interacting through the Hertz contact law [76]. This system is highly tunable in that the interparticle stiffness can be changed simply by altering the contact angles between cylinders [77]. Such a versatile structure has been recently exploited for manipulating stress waves in linear [78],

linear time-dependent [79], and nonlinear media [16]. To overcome the experimental challenges with regard to controlling the contact angles and conducting precise measurements, we devise an experimental setup of a tunable, stand-alone cylindrical particle system. This involves 3D-printed enclosures intended to support and tune the granular chain and to facilitate particles' velocity measurements through a laser Doppler vibrometer (LDV). Using this tabletop setup, we validate the *in situ* topological band transition in the system by detecting the emergence of an experimentally measured boundary mode. We further demonstrate the existence of a topologically protected mode at the interface of two topologically distinct granular chains. Last, we theoretically calculate the frequency of the topologically protected mode using symmetries in its shape and show that it has an excellent agreement with numerics and experiments.

2.2 Experiment and numerical setup

The experimental setup consists of a chain of short cylinders placed inside 3D-printed enclosures stacked vertically (Fig. 2.1). Each enclosure along with its cylinder can be independently rotated about its central axis to change the stacking angles of the particles. We maintain periodically varying two contact angles α_0 and α , such that the system resembles a dimer configuration. To demonstrate the topological transition, we fix α_0 to 20° and vary only α from 5° to 40° . The chain is composed of 27 cylinders, and all cylinders are made of fused quartz (Young's modulus $Y = 72$ GPa, Poisson's ratio $\mu = 0.17$, and density $\rho = 2187$ kg/m³) with identical diameter and a length of 18 mm. A piezoactuator excites the bottom of the chain to send a frequency sweep signal from 3 to 30 kHz. A free weight (25 N) is placed on the top of the chain to provide initial static compression to restrict the system dynamics to the linear regime. We track the velocity of each cylinder using an LDV mounted on a guide rail. Note that we have judiciously designed the enclosure to facilitate the passage of the laser beam emanating from the LDV in various angles (top inset in Fig. 2.1).

We use a discrete element method to model the system dynamics [80]. We represent the cylinders by lumped masses and the contacts by springs following the Hertz contact law. The

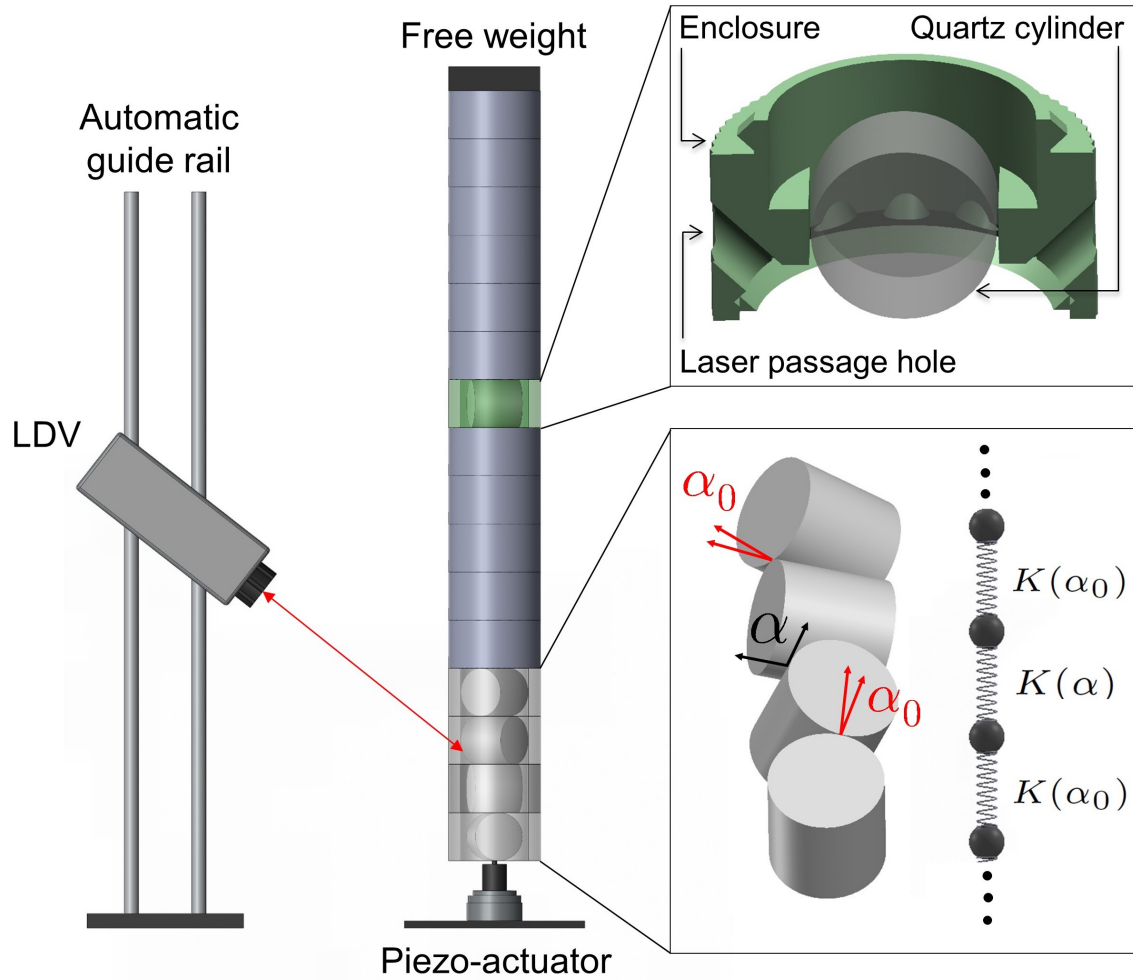


Figure 2.1: Schematic of the experimental setup. The top inset illustrates a cut section of the 3D-printed enclosure. The bottom inset shows the contact angles in the dimer chain and the representative spring-mass system.

force between the i th and $(i + 1)$ th cylinders can be written as $F_i = \beta(\alpha_i)[\delta_i + u_i - u_{i+1}]^{3/2}$, where $\beta(\alpha_i)$ is the stiffness coefficient for the contact angle of α_i ; δ_i is the initial static compression due to the free weight; u_i and u_{i+1} denote the dynamic displacements of the i th and $(i + 1)$ th cylinders in the longitudinal direction, respectively (see Appendix A). If $|u_i - u_{i+1}| \ll \delta_i$, as is the case here, we can linearize the contact law. Hence, the contact between the i th and $(i + 1)$ th cylinders can be assigned to a linear stiffness coefficient, $K(\alpha_i) = \frac{3}{2}\beta(\alpha_i)\delta_i^{1/2}$. This means that a dimer configuration with alternating α_0 and α angles can be represented by a lumped mass model with linear stiffness coefficients, $K(\alpha_0)$ and $K(\alpha)$, varying along the chain (bottom inset in Fig. 2.1).

For an infinitely long dimer chain, it is straightforward to establish a linear dispersion relation and calculate the edges of Bloch bands [81]. For a finite lattice, however, we expect to observe boundary effects. To this end, we perform the relevant eigenvalue analysis. For an N particle chain, we use $u = [u_1, u_2, u_3, \dots, u_N] = U \exp(j\omega t)$, where U and ω represent amplitude of displacement vector and angular frequency respectively, t is time, and j is an imaginary unit. Thus, by neglecting dissipation in the system, we obtain $\mathbf{\Lambda}U = \omega^2 mU$, where m is the particle mass and $\mathbf{\Lambda}$ is a $N \times N$ tridiagonal matrix consisting of stiffness coefficients $K(\alpha_0)$ and $K(\alpha)$. This finite system also accounts for the boundary condition of the finite system. Specifically, we fix the boundaries by choosing stiffness values $K_a = 2.78 \times 10^7$ N/m and $K_w = 1.62 \times 10^7$ N/m at the beginning (actuator side) and the end of the chain, respectively, to match the experimental data. Using this finite setup, we evaluate eigenfrequencies and eigenmodes of the system in comparison with analytical and experimental data.

2.3 Results and discussions

2.3.1 Topological band transition in infinite lattices

We first investigate the topological characteristics of the infinite dimer lattices. Figure 2.2 shows three dimer configurations that represent a topological band transition within our

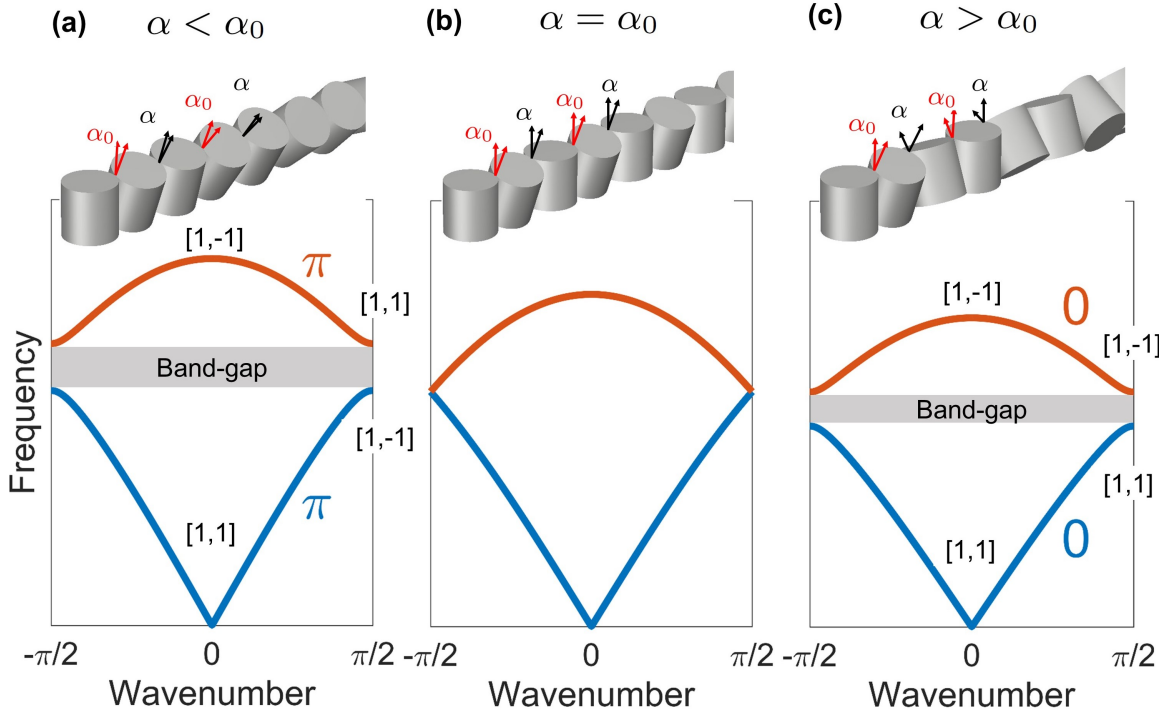


Figure 2.2: Representative configurations of the infinite dimer chain to show its topological band transition as a function of α . (a),(b), and (c) include the Bloch dispersion curves for $\alpha < \alpha_0$, $\alpha = \alpha_0$, and $\alpha > \alpha_0$, respectively. Bands are marked with the corresponding topological indices (0 and π) and the Bloch eigenmodes on their edges ($[1, 1]$ and $[1, -1]$ for symmetric and antisymmetric oscillations in the dimer unit cell, respectively).

system. Theoretically obtained Bloch dispersion curves are plotted below, showing acoustic (lower) and optical (upper) branches. A frequency band gap spans from $(1/2\pi)\sqrt{2K(\alpha_0)/m}$ to $(1/2\pi)\sqrt{2K(\alpha)/m}$, and the upper band edge of the optical band is at $(1/2\pi)\sqrt{2[K(\alpha_0) + K(\alpha)]/m}$. We notice that if α increases, the frequency band gap first closes at $\alpha = \alpha_0$ and then opens again. In this process, Bloch eigenmodes on the edges of the band gap are also flipped [see the change between $[1, 1]$ and $[1, -1]$ in Figs. 2.2(a) and (c)]. This indicates a typical topological band transition in our system as a function of the angle α . The system shifts between distinct dimer configurations that cannot be transformed to each other without closing the band gap. The mathematical quantification of this notion can be made by calculating the so-called Zak phase for each band; see Appendix A for detailed calculations.

One notices that the Zak phase of a band directly relates to the symmetry types of Bloch eigenmodes at its lower and upper band edges. The Zak phase of 0 indicates the same symmetry type [see Fig. 2.2(c) where the Bloch eigenmodes remain the same within the branch]. On the other hand, the Zak phase of π implies the formation of the opposite symmetry type within a band [Fig. 2.2(a) where the Bloch eigenmodes are flipped]. We evidently see the topological characteristics of a band gap are linked to the sum of the Zak phases of all the bands below the gap [40]. Therefore, the change in the Zak phase of the acoustic band confirms the *in situ* topological transition of our infinite lattice system.

2.3.2 Emergence of boundary mode in finite lattices

The topological nature of bulk band gaps observed for an *infinite* lattice manifests itself through the emergence of boundary mode(s) inside the band gaps for a *finite* lattice (i.e., the principle of bulk-boundary correspondence) [29]. Therefore, we expect the existence of a boundary (local or edge) mode in the current topologically nontrivial dimer configurations. We extract the cutoff frequencies and the local mode information from experimental data with α varying from 5° to 40° in steps of 5° (see Appendix A for details). Figure 2.3(a) summarizes the modal frequencies of the system as we vary α . Shaded areas denote modes corresponding to acoustic and optical branches of dispersion curves, constructed theoretically.

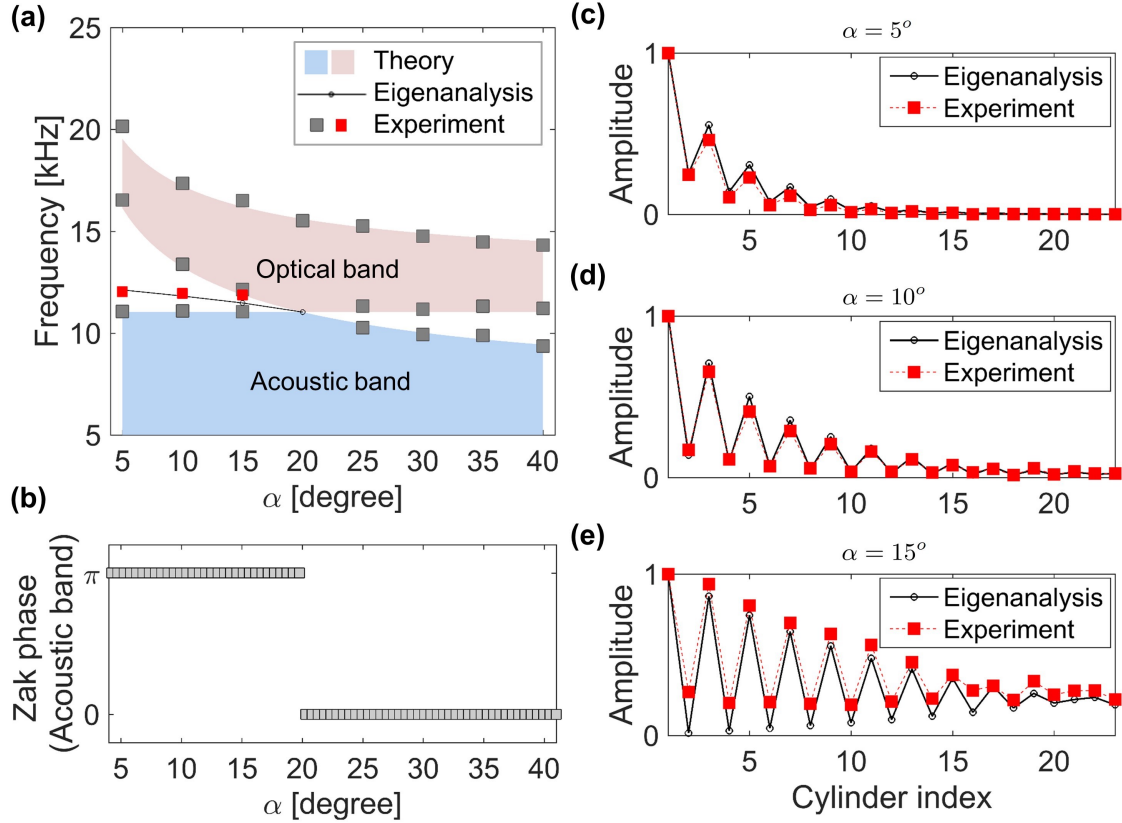


Figure 2.3: Topological transition and emergence of a boundary mode in the finite system. (a) Frequency spectrum evolution as a function of α . Extracted experimental data (gray markers) match the edges of theoretical bulk bands (shaded area). Also, experiments (red markers) show the emergence of a boundary mode inside the band gap, which follows the trend of numerical simulations (black curve). (b) Zak phase of the band gap and its transition at $\alpha = \alpha_0$. (c)-(e) Numerical (black markers) and experimental (red markers) boundary mode profiles in terms of normalized velocity amplitude at $\alpha = 5^\circ$, 10° , and 15° .

We observe an excellent match of the experimental data (gray square markers) with the edges of the theoretical bands. This successfully demonstrates the band gap closing and opening mechanism in our tunable system.

Because of the finite size of the chain, we experimentally observe a local mode residing inside the band gap (red square markers for $\alpha < \alpha_0$). Black curve data, obtained through the numerical eigenvalue analysis, follow the same trend. These are boundary modes localized in the front of the chain, which are not witnessed when $\alpha > \alpha_0$. The emergence of these modes from all configurations that satisfy $\alpha < \alpha_0$ complies with the change in the band topology at $\alpha = \alpha_0$. In Fig. 2.3(b), we plot the Zak phase of the acoustic band as a function of α to support the aforementioned argument.

In Figs. 2.3(c)-2.3(e), we show the normalized velocity profiles of the boundary modes extracted from the experiments and the eigenanalysis. These exponentially decaying profiles match closely between experiments and numerics. The localization length, however, increases as we move from $\alpha = 5^\circ$ to $\alpha = 15^\circ$. This can be explained by extending the intuitive arguments in Ref. [82], according to which the localization length depends on the stiffness ratio as $\xi \propto 1/\ln[K(\alpha)/K(\alpha_0)]$. Therefore, the localization length becomes large as we increase α for $\alpha < \alpha_0$, and it is natural to expect that the mode becomes extended and is *lost* inside the band in the limit of $\alpha \rightarrow \alpha_0$.

We note that the study of these boundary modes, i.e., the *gap modes* in classical lattices, can be traced back to the pioneering work of Wallis [83]. He extensively explored the effect of finite boundaries and defects on the spectrum of ordered lattices in terms of the generation of gap modes. However, the novelty of the first part of this work is that we have kept the boundary conditions and the length of the system the same and experimentally demonstrated how the change in bulk properties reflects in the emergence of a finite-frequency boundary mode as per the topological band theory for 1D dimers.

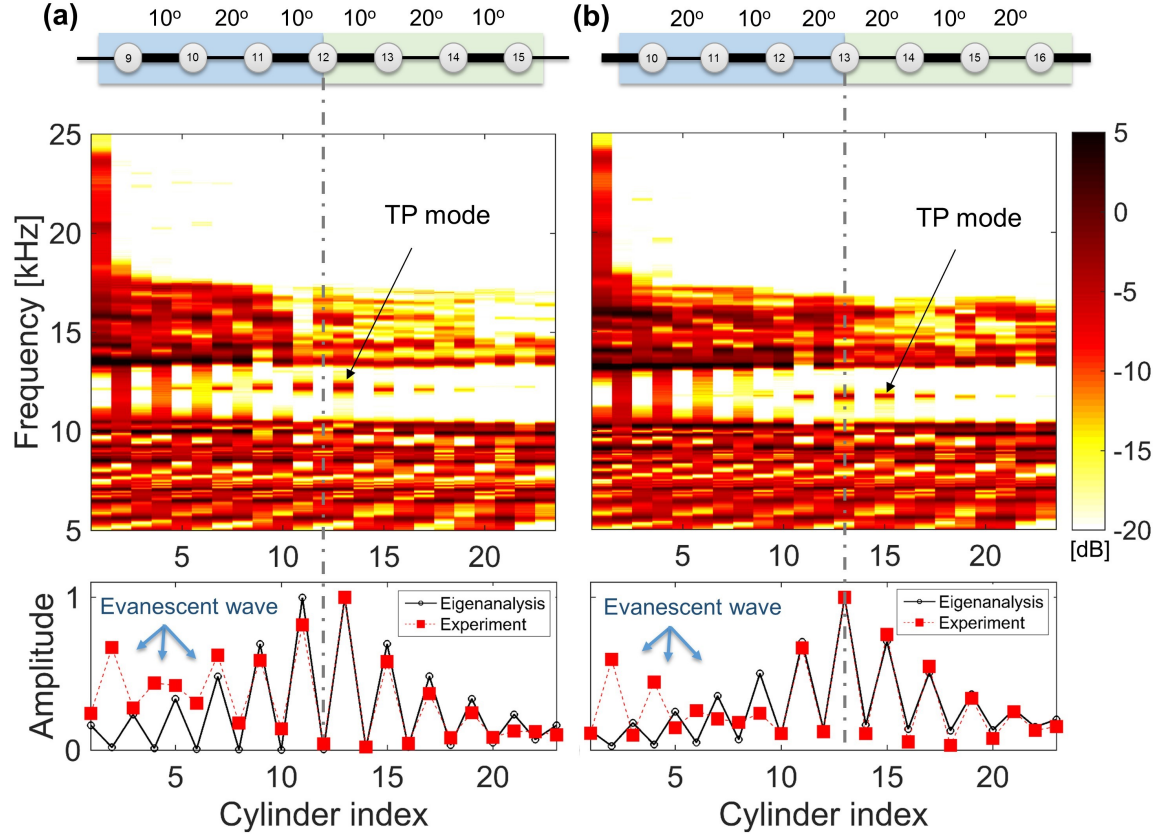


Figure 2.4: Topologically protected modes arising from dimer configurations with (a) *hard-hard* and (b) *soft-soft* interfaces. Below are PSD plots obtained from the particles' velocities measured experimentally. At the bottom are the TP mode shapes extracted from experiments (red markers) and eigenanalysis (black markers).

2.3.3 Topological defect and protected modes

We now study the ramifications of having two topologically different dimer configurations connected to each other. We assemble two dimer chains both with $\alpha_0 = 10^\circ$ (hard) and $\alpha = 20^\circ$ (soft), which connect with each other in mirror symmetry, such that we introduce a topological defect at their interface (see the top panels in Fig. 2.4 and also Refs. [57, 84] for similar settings). The connecting interface can be of two types: 10° - 10° or 20° - 20° , representing *hard-hard* and *soft-soft* interfaces, respectively. To understand why this defect is topological and how it is different from a trivial defect, we present the following argument. Suppose we can modify any interparticle contact stiffness along the chain. Now, in a dimer system, a topologically *trivial* defect can be introduced by going to the desired location in space and changing the stiffness locally. Similarly, the argument can be made that if there is a trivial defect, we can go to the defect site and perturb the stiffness locally to restore the original defect-free dimer configuration. However, the case of a topologically *nontrivial* defect is special in that it cannot be straightforwardly reverted to such a defect-free scenario through a *compact* (i.e., *local*) perturbation. In other words, if one tries to remove the topological defect by changing the local stiffness, one can see that it is not possible unless *all* stiffness values on one side of the chain are modified (i.e., a *noncompact* perturbation). In essence, a topological defect can be removed only by changing the topology of one side of the configurations adjacently placed. Therefore, the vibration mode caused by this defect is topologically protected and robust against local perturbations around the interface location (see Appendix A for details).

To demonstrate the topologically nontrivial modes, we follow the same procedure as mentioned in the earlier section for the experimental study. A piezoactuator is used for sending a frequency sweep signal from one end. It is understood that if a vibration mode caused due to a defect is localized in the middle of the chain, it is not easy to excite it using the input signal sent from the end of the chain. However, our current system is short enough that the topologically protected (TP) mode—localized in the middle of our short chain—can still be

excited by coupling it to the evanescent waves inside the band gap. In this way, we detect the existence of TP modes for hard-hard and soft-soft configurations (Fig. 2.4). Power spectral density (PSD) plots are obtained experimentally by performing Fourier transformation on the temporal velocity profiles of all particles in each chain. As indicated by the arrows, we can evidently observe the existence of the TP modes in both cases. In the bottom panel, we show the extracted normalized mode shape from the experimental data, which agree with the corresponding computed eigenmodes. The deviation in the initial part of the chain is due to the evanescent wave in experiments, which the numerical eigenanalysis does not incorporate.

Last, we derive analytical expressions for the frequencies of the TP modes by utilizing the symmetries of their eigenmodes, which are evident through their spatial waveform (bottom panel in Fig. 2.4). Let K_h and K_s denote the linear stiffness corresponding to hard (10°) and soft (20°) contacts. For the hard-hard interface, we observe from the mode shape that alternate particles do not move, while the rest of the particles oscillate around their equilibrium positions. Thus, considering that their net stiffness is $K_h + K_s$, the frequency of the TP mode in this hard-hard case would be $f_h = \frac{1}{2\pi} \sqrt{\frac{K_h + K_s}{m}}$. Similarly, for the soft-soft interface, one can derive that the frequency is $f_s = \frac{1}{2\pi} \sqrt{\frac{2K_s(1+\gamma^{-1})}{m}}$, where $\gamma = \frac{3r-1+\sqrt{9r^2-14r+9}}{2(r-1)}$ with $r = \frac{K_h}{K_s}$ (see Appendix A for further details). We compare these analytical frequencies with the experimental values extracted directly from the spectrum plots in Fig. 2.4 and the numerical ones obtained from the eigenanalysis. For the current configuration, we find the frequency of the hard-hard TP mode is 12.18 (theory), 12.18 (numerics), and 12.15 ± 0.19 kHz (experiment). Similarly, the frequency of the soft-soft TP mode is 11.80 (theory), 11.80 (numerics), and 11.70 ± 0.08 kHz (experiment). Here, the standard deviations in experiments are based on the frequencies measured from all cylinder locations.

Judging from the agreement of analytical frequencies with computational and experimental results, these mathematical expressions can be used for distinguishing the TP modes from trivial defect modes. We see that for any values of K_h and K_s (complying with $K_h > K_s$), the interface modes exist and the corresponding frequencies reside *inside* the band gap without coalescing with the bulk bands. Hence, the modes are protected as long as we have a

topological defect (hard-hard or soft-soft) created at the intersection of two topologically distinct dimer configurations. Again, these TP modes are robust against perturbations near the interface in contrast to trivial defect modes, and we verify such a nature of the TP modes via numerics in Appendix A (see also other related works, such as Refs. [85–87], on 1D topological interface modes).

2.4 Conclusions

In this work, we proposed a highly tunable mechanical system made of cylindrical granular particles, which can demonstrate an *in situ* topological band transition in a controllable manner. Using noncontact laser vibrometry, we precisely captured the smooth topological transition and showed how it leads to the emergence of a boundary mode in the system. We demonstrated the existence of topologically protected modes at the interface of two topologically distinct dimer configurations. The experimental observations of the resulting modes are supported by the theory and numerics. We also confirmed that these topologically protected modes are robust under perturbations, unlike trivial defect modes observed in granular chains. Though the current study is limited to linear dynamics, the proposed system can be tuned to incorporate nonlinear effects. In that light, this framework can provide a promising test bed for future studies involving the interplay of nonlinearity and topologically protected modes.

2.5 Author contributions

This chapter is based on the following manuscript: R. Chaunsali, E. Kim, A. Thakkar, P. G. Kevrekidis, J. Yang, “Demonstrating an *In Situ* Topological Band Transition in Cylindrical Granular Chains,” *Physical Review Letters* 119, 024301, 2017 [88]. R. Chaunsali and E. Kim conceived the idea of the project. E. Kim designed the experimental setup. R. Chaunsali carried out numerical studies and experiments. A. Thakkar contributed in the experiments. R. Chaunsali wrote the manuscript. E. Kim, P. G. Kevrekidis, and J. Yang supervised the project.

Chapter 3

ONE-DIMENSIONAL TIME-DEPENDENT TOPOLOGICAL METAMATERIAL

We present an active, purely mechanical stress wave isolator that consists of short cylindrical particles arranged in helical manner. This phononic architecture allows us to change interparticle stiffness dynamically by controlling the contact angles of the cylinders. We use *torsional* traveling wave to control the contact angles, thereby imposing a desired spatio-temporal stiffness variation to the phononic crystal along the *longitudinal* direction. We note that such torsional excitation is a form of parametric pumping in the system, which results in the breakage of the time-reversal symmetry. We report that, in quasistatic sense, the system shows topologically nontrivial band gaps. However, in a dynamic regime where the pumping effects are significant, these band gaps become asymmetric with respect to the frequency and wavenumber domains in the dispersion relationship. Using numerical simulations, we show that such asymmetry has a direct correspondence to the topological invariant, i.e., Chern number, of the system. We propose that this asymmetry, accompanied by selective interband transition, can be utilized for directional isolation of the stress wave propagating along the phononic crystal.

3.1 Introduction

A wide variety of researchers have been striving to attain the ability to control the flow of energy in a medium for centuries. In particular, one-way transmission of electrical energy by semiconductor diode and transistor has revolutionized the modern era. On the similar lines, one-way propagation of light, heat, and sound have captured the attention of the scientific community, leading to the invention of novel devices, namely photonic [89], thermal [90],

and acoustic diodes [91], respectively. With regard to controlling the mechanical waves in structures, however, researchers constantly face challenges mainly due to the complexity stemming from different modes of stress waves that can simultaneously propagate in the medium [92]. Recently, the advent of phononic crystals and mechanical metamaterials has opened the possibility of controlling such versatile wave propagation in solids [4, 93].

One of the most common ways to realize one way transmission in a mechanical system is to break the time-reversal symmetry [94]. Such a nonreciprocal system can also be termed as an isolator in one-dimensional settings. The nonreciprocity can be achieved, for example, by applying an external bias/field [95] or by introducing nonlinearity [96]. In the scenarios when *in situ* tunability of wave transmission direction is desired, a linear medium would take the precedence over a nonlinear one. This is because the allowed direction of wave propagation can be reversed simply by flipping the external bias, which is not the case with *passive* settings of nonlinearity-based isolators. Moreover, the isolation characteristics could be tuned by adjusting the magnitude of the external bias directly, thereby making such systems *active*. One could use different types of biases in these active systems. For example, in an acoustic domain, external moving fluid can serve as a bias [97, 98]. However, nonfluidic structural settings require an external inertial, optical, magnetic, electric or thermal field [99–102]. These external fields can then couple with the elastic properties of the host structure in order to introduce time modulation of elastic parameters. For the structures that cannot afford aforementioned couplings or are required to be placed in the settings where the application of external nonmechanical fields is not a viable option, it is desired that a purely mechanical system is devised in order to isolate the stress waves.

In this study, we propose a one-dimensional phononic crystal that consists of a chain of cylinders, to achieve the stress wave isolation in a purely mechanical manner. The array of cylinders has shown extraordinary tunability in terms of tailoring the stress wave propagation through them [78, 103, 104]. This tunability stems from the Hertz contact law [76] between adjacent cylinders, where their contact stiffness can be easily varied by changing the contact angle of the cylinders [77]. With this fundamental ability, we explore versatile,

one-directional stress wave propagation in the helicoidal phononic crystal. The structure allows both longitudinal and torsional modes of wave propagation in the medium. Remarkably, we use the complex interaction of these two different wave modes to our advantage, and dynamically tune the system's stiffness for the *longitudinal* wave by having a direct control over the *torsional* modes. In particular, we use a torsional *traveling* wave to induce a specific spatio-temporal modulation of the system's stiffness, which resembles the sliding lattice potential of the topological Thouless pump in quantum mechanics [105]. This, in turn, makes the system behave as a topological mechanical pump, and thus breaks the time-reversal symmetry. In comparison with such symmetry breaking in topological acoustic [41, 106, 107] and mechanical [47, 49] metamaterials in two dimensions, which results in a robust one-way propagation of edge mode, our time-dependent one-dimensional system results in asymmetric *bulk* wave propagation along the phononic crystal. Moreover, the topological invariants quantify the amount of asymmetry in our system, in contrast to a two-dimensional system, in which, those are related to the number of edge modes.

Therefore, inspired by the mechanics of Hertzian contact, the architecture of helix, and the physics of Thouless pump, this study claims threefold novelty as following. First, in terms of design, we propose a purely mechanical system utilizing an interesting coupling of two mechanical wave modes to break the time-reversal symmetry. Second, in terms of the fundamental physics of a time-dependent, one-dimensional mechanical system, we show that this system is topologically nontrivial and can be characterized by the topological invariant in two dimensions, i.e., Chern number [28]. Its nonzero value remarkably acts as a bridge connecting quasistatic and dynamic scenarios, and quantifies the asymmetry observed in dispersion curves. The third novelty is in terms of the mechanism that facilitates the controllable, directional wave propagation, as a consequence of the asymmetry in dispersion curves. To the best of the authors' knowledge, such type of mechanism—reminiscent of interband transition in electronics [108], and photonics [109]—has not been observed previously in a purely mechanical system.

3.2 Setup and theoretical background

3.2.1 Phononic crystal architecture

The phononic crystal to be investigated in this study consists of short cylindrical particles stacked helically as a one-dimensional chain (Fig. 3.1a). Each cylinder is oriented in a way that the contact angle with its neighboring particle is uniform throughout the chain at equilibrium. Under the application of compression to the chain, the contact stiffness among the particles in the longitudinal direction is governed by the Hertzian contact law [76] (see Fig. 3.1b for the contact between two neighboring cylinders). When there are no dynamic perturbations of the torsional angles, all cylinders will maintain constant contact stiffness over time (Fig. 3.1c). However, if a torsional traveling wave is sent from one end of the chain, we would expect it to perturb the contact angles from the equilibrium. The perturbation in the contact angle will, in turn, change the contact stiffness following the relationship in Fig. 3.1b. This will eventually cause a spatio-temporal modulation of the contact stiffness along the longitudinal direction of the medium (Fig. 3.1d).

Based on the previous work by Li *et al.* [104], we establish a framework to define and calculate instantaneous contact stiffness, which governs the wave propagation in the system. For a chain of cylinders, we write the absolute orientation angle of i th cylinder as

$$\alpha_i(t) = i\alpha_0 + A_\alpha \cos(k_\alpha i \mp 2\pi f_\alpha t), \quad (3.1)$$

where α_0 is the contact angle between any two cylinders at equilibrium; A_α is the dynamic perturbation in the contact angle caused by a traveling torsional wave; $k_\alpha = 2\pi/N$ represents a wavenumber with N being the spatial period of the contact angle variation; f_α denotes the frequency of the traveling torsional wave; and $\mp$ signs indicate *forward* and *backward* direction of the traveling torsional wave, respectively.

We can define the contact angle, i.e., relative angle between two cylinders, as $\Delta\alpha_i(t) = \alpha_i(t) - \alpha_{i-1}(t)$. Using the Hertz's contact law, instantaneous contact force between i th and $(i+1)$ th cylinders would be $\beta_i(t)\delta^{3/2}$, with δ as the relative longitudinal displacement between

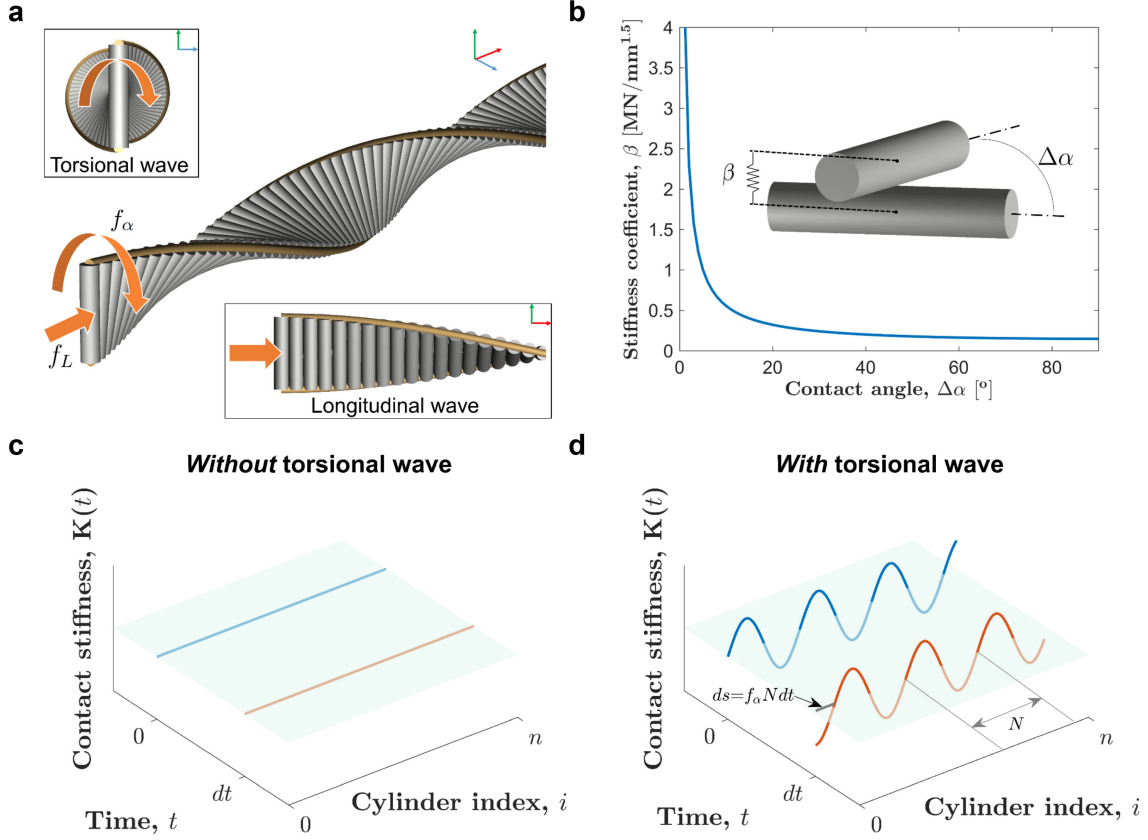


Figure 3.1: Phononic crystal model. (a) Cylinders stacked helically to form a one-dimensional chain. Insets show the two types of wave modes the structure supports. f_L and f_α are the frequencies of longitudinal and torsional traveling waves, respectively. (b) Contact stiffness dependence on the angle between two cylinders ($\Delta\alpha$), which is perturbed by the torsional wave. (c) Uniform contact stiffness for the longitudinal wave propagation in the system *without* torsional excitation. (d) Spatio-temporal variation of the same stiffness *with* torsional excitation. The spatial distribution of stiffness moves by ds distance in time dt .

two cylinders. The time-dependent stiffness coefficient $\beta_i(t)$ is of the following form [76, 77]:

$$\beta_i(t) = \frac{2Y}{3(1-\nu^2)} \sqrt{\frac{R}{\sin \Delta\alpha_i(t)}} \left[\frac{2K_{e(t)}}{\pi} \right]^{-3/2} \left\{ \frac{4}{\pi e(t)^2} \sqrt{\left[\left(\frac{a}{b} \right)^2 E_{e(t)} - K_{e(t)} \right] [K_{e(t)} - E_{e(t)}]} \right\}^{1/2} \quad (3.2)$$

Here, Y , μ and R denote Young's modulus, Poisson's ratio, and the radius of a cylinder. $K_{e(t)}$ and $E_{e(t)}$ represent the complete elliptical integrals of the first and second kinds, respectively. The elliptical contact area has eccentricity $e(t) = \sqrt{1 - (b/a)^2}$, with a and b as its semi-major and semi-minor axes, respectively. b/a is equal to $[(1 + \cos \Delta\alpha_i(t))/(1 - \cos \Delta\alpha_i(t))]^{-2/3}$, approximately [76].

3.2.2 Equations of motion

Once we have formulated the instantaneous contact stiffness between constituents elements, we can now write the equations of motion for the system consisting of n cylinders of equal masses m . We fix the longitudinal motion and keep the torsional motion free at both the ends of the chain. Let $x_i(t)$ represent instantaneous longitudinal displacement of the i th cylinder, and δ_i be the initial deformation between i th and $(i+1)$ th cylinders under static precompression. For a generic scenario, under the excitation forces $p_1(t)$ and $p_n(t)$ applied to the first and last cylinders of the chain, respectively, we write

$$\begin{aligned} m\ddot{x}_1(t) &= p_1(t) + \beta_0(t)[\delta_0 - x_1(t)]^{3/2} - \beta_1(t)[\delta_1 + x_1(t) - x_2(t)]^{3/2}, \\ m\ddot{x}_i(t) &= \beta_{i-1}(t)[\delta_{i-1} + x_{i-1}(t) - x_i(t)]^{3/2} - \beta_i(t)[\delta_i + x_i(t) - x_{i+1}(t)]^{3/2} \quad \forall \quad 2 \leq i \leq (n-1) \\ m\ddot{x}_n(t) &= \beta_{n-1}(t)[\delta_{n-1} + x_{n-1}(t) - x_n(t)]^{3/2} - \beta_n(t)[\delta_n + x_n(t)]^{3/2} - p_n(t). \end{aligned} \quad (3.3)$$

For small instantaneous displacements $x_i(t) \ll \delta_i$, we linearize Eq. (3.3) and define linear stiffness $K_i(t) = \frac{3}{2}\beta_i(t)\delta_i^{1/2}$ to obtain the following matrix form

$$\mathbf{M}\ddot{\mathbf{X}}(t) = \mathbf{K}(t)\mathbf{X}(t) + \mathbf{F}(t), \quad (3.4)$$

where displacement vector $\mathbf{X}(t)$, stiffness matrix $\mathbf{K}(t)$, mass matrix $\mathbf{M}$, and forcing vector

$F(t)$ are expressed as

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \dots \\ x_n(t) \end{bmatrix}, \mathbf{K}(t) = \begin{bmatrix} K_0(t) + K_1(t) & -K_1(t) & \dots & 0 \\ -K_1(t) & K_1(t) + K_2(t) & -K_2(t) & 0 \\ \dots & \dots & \dots & -K_{n-1}(t) \\ 0 & 0 & -K_{n-1}(t) & K_{n-1}(t) + K_n(t) \end{bmatrix},$$

$$\mathbf{M} = \begin{bmatrix} m & 0 & \dots & 0 \\ 0 & m & \dots & 0 \\ \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & m \end{bmatrix}, F(t) = \begin{bmatrix} p_1(t) \\ 0 \\ \dots \\ -p_n(t) \end{bmatrix} + \begin{bmatrix} \beta_0(t)\delta_0^{3/2} - \beta_1(t)\delta_1^{3/2} \\ \beta_1(t)\delta_1^{3/2} - \beta_2(t)\delta_2^{3/2} \\ \dots \\ \beta_{n-1}(t)\delta_{n-1}^{3/2} - \beta_n(t)\delta_n^{3/2} \end{bmatrix}.$$

Let us now look at the Eq. (3.4)—which includes multiple time-dependent terms—in more detail. We have a time-dependent stiffness matrix $\mathbf{K}(t)$ and a forcing vector $F(t)$, which make the system dynamics fairly complex yet interesting. $\mathbf{K}(t)$ imbibes spatio-temporal modulation of contact stiffness in the system, whereas the forcing term $F(t)$ has an *external* forcing component as well as a component dependent on stiffness modulation. Under no external force $p(t)$ applied to the system, we can easily deduce that a traveling torsional wave as in Eq. (3.1) would enforce time periodicity on these terms. Therefore, we have $\mathbf{K}(t) = \mathbf{K}(t + T)$ and $F(t) = F(t + T)$, where $T = 1/f_\alpha$. This periodic evolution of $\mathbf{K}(t)$ represents *parametric pumping* in the system caused by torsional wave. The mechanism is analogous to the pumping action performed, for example, when one wants to change the amplitude of oscillation of a swing without exerting an external force. Instead, one changes the center-of-mass and thus vary the effective swing length with time [110]. $\mathbf{K}(t)$ acts as such a time varying parameter in this case, which is pumped with frequency f_α of the torsional wave.

Note that this parametric pumping has direct consequences on vibration modes the system supports. If the perturbation in contact angles were to depend solely on space but not time, i.e., $f_\alpha = 0$, we see that the second component of $F(t)$ vanishes, and this equation reduces to the familiar standard form: $\mathbf{M}\ddot{X}(t) = \mathbf{K}X(t)$ with time *independent* $\mathbf{K}$. We

can easily analyze it further to obtain eigenfrequencies and Bloch states. However, due to a nonzero f_α , simultaneous space-time stiffness modulation is enforced, and a *time-periodic* set of equations with a period T represents our system. This period is dictated by the *pumping* frequency f_α of the torsional wave. In this way, our system not only supports the Bloch states caused by spatial variation of parameters, but also allows us to tailor them by an additional degree of freedom, i.e., time, hence taking us to the realm of Floquet-Bloch engineering [111].

3.3 Results and discussion

3.3.1 Quasistatic analysis: nontrivial topological band gaps

To understand the system dynamics, we start with analyzing the normal modes of vibration in the system. We would take the homogenized form of Eq. (3.4) by suppressing the forcing term such that

$$\mathbf{M}\ddot{\mathbf{X}}(t) = \mathbf{K}(t)\mathbf{X}(t) \quad (3.5)$$

where $\mathbf{K}(t + T) = \mathbf{K}(t)$. This time-periodic system of equation resembles the well studied Mathieu's equation, and it can be solved using Floquet theory [110]. However, we take a different approach to connect our system to an *adiabatic* topological quantum pump [105]. We first analyze the system in *quasistatic* scenario, and then extend the key characteristics to explain *dynamic* scenario of pumping. With regard to quasistatic evolution over time, we assume that the variation in the stiffness matrix $\mathbf{K}(t)$ with respect to time is significantly small, such that the natural modes of vibration can be calculated directly from the matrix at any time instant. Natural frequencies corresponding to such quasistatic evolution of system are named as *static* frequencies [110]. It is worth emphasizing again that k_α in Eq. (3.1), which is the measure of spatial variation in stiffness, is responsible for frequency band gaps [104]. On the other hand, the term f_α , i.e., the pumping frequency, causes temporal variations in stiffness, and is the measure of how quickly static frequencies are varied over time. If we understand how static frequencies evolve over time, that would give us direct insights into the actual system dynamics for quasistatic pumping. It would also

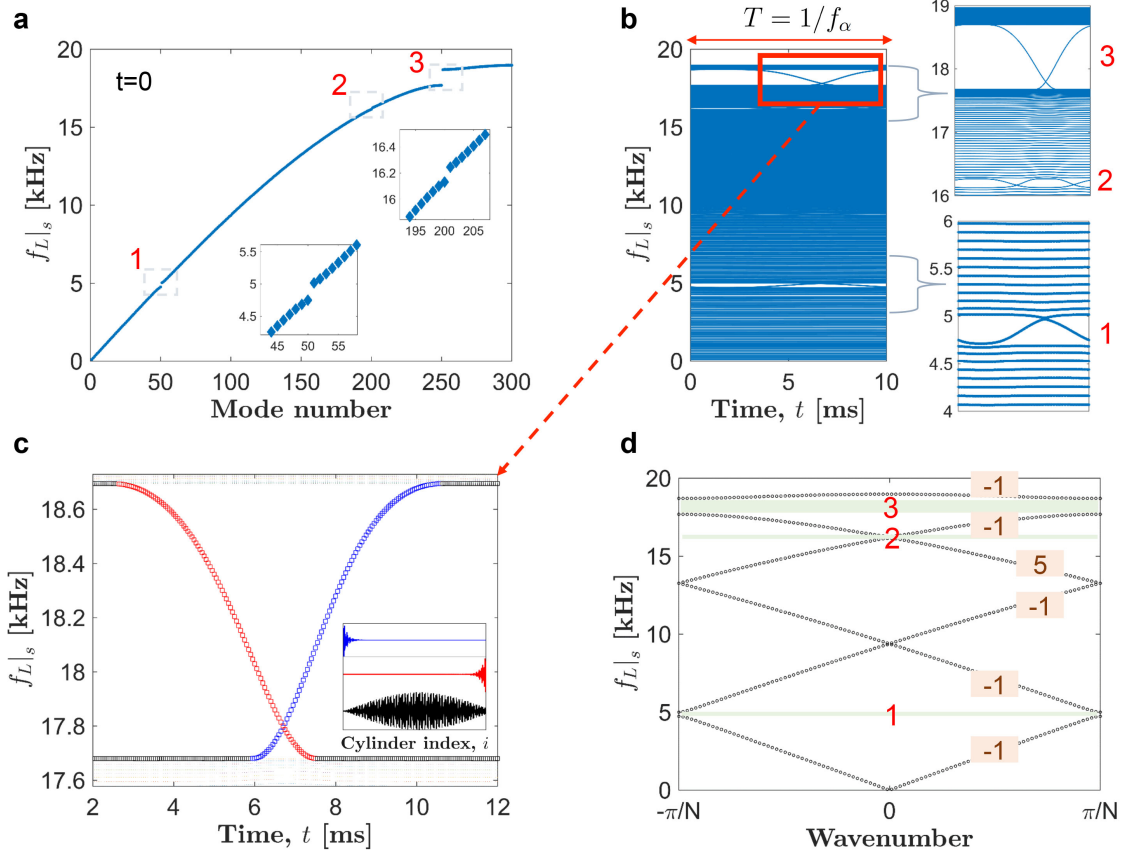


Figure 3.2: Nontrivial band topology due to *quasistatic* torsional wave pumping. (a) Spatio-temporal variation in stiffness, caused by torsional wave, forces the system's eigenfrequencies for longitudinal vibration mode to change over time. We plot the eigenfrequencies of a finite size system at one such time instant highlighting the prominent band gaps: 1, 2, and 3 (insets show zoomed-in view). (b) Evolution of eigenfrequencies over one period of pumping. Insets show close view of frequency evolution inside the band gaps: 1, 2, and 3. The points of degeneracy (crossing) are noticeable. (c) A closer view of the degenerate point inside band gap 3 showing that edge modes interchange their polarities across the degenerate point, a signature of the bulk-boundary correspondence in topological transition. Inset shows the corresponding eigenmodes, both edge and global, represented in different colors. (d) Bloch dispersion curve of an infinitely periodic lattice, with Chern numbers assigned to each bands. Nonzero values of gap Chern numbers (summation of Chern numbers of the bands below) confirm the topological nature of all three prominent band gaps.

be useful to decipher the complex dynamics in case of rapid pumping (with high torsional wave frequency) where secondary mode of vibration with frequency offsets, namely frequency cascades [112], can no more be neglected (to be discussed in the later sections).

Under the quasistatic assumption, we can convert Eq. (3.5) to an eigenvalue problem by substituting time-periodic ansatz $X = \tilde{X} \exp\{j(2\pi f_L|_s)t\}$ and deduce

$$\mathbf{M}^{-1}\mathbf{K}(t)\tilde{X} = (2\pi f_L|_s)^2\tilde{X}. \quad (3.6)$$

Here $f_L|_s$ denotes the *static* frequency of the *longitudinal* waves, which should not be confused with pumping frequency f_α of the *torsional* waves. For the analysis, we choose $\alpha_0 = 10^\circ$, $A_\alpha = 2^\circ$, $N = 6$, and $n = 300$. We take geometrical dimensions and material properties of quartz cylinders from our previous experimental studies ($Y = 72$ GPa, $\nu = 0.17$, density $\rho = 2187$ kg/m³, height $h = 18$ mm and $R = 9$ mm) [78]. We apply a precompression of 40 N to the chain and solve Eq. (3.6) at each time step for a small pumping frequency $f_\alpha = 0.1$ kHz of the *forward* propagating torsional wave.

In Fig. 3.2a, we show eigenfrequencies for the longitudinal wave at one time instance, e.g., $t = 0$. In general, we expect five band gaps in frequency spectrum for the spatial period ($N = 6$) of stiffness variation. However, the perturbation A_α in contact angles is small, and therefore we notice only three prominent band gaps, labeled as 1, 2, and 3 (insets show zoomed-in view). We then calculate how these frequencies evolve over time due to the quasistatic pumping of torsional wave. Since the pumping is periodic, we expect evolution of the eigenfrequencies to be periodic as well. In Fig. 3.2b we show such time evolution for one pumping period T . It is important to note that only frequencies on the edges of band gaps evolve quite distinctly over time. Frequencies corresponding to global modes do not evolve as distinctly as the edge modes. This is intuitive if we rely on the following argument: spatio-temporal stiffness variation in our system is equivalent to a sliding potential, so at each time instant, stiffness variation is just spatially shifted from the one at last time instant. Quasistatically speaking, this only reflects as a change in the edge modes of a finite lattice, not the global modes. Hence, the modulation is only evident for the modes on the edges. It

is also important to note that all three band gaps evolve distinctly over time with a different number of degeneracies (crossings). We will now shed light on the same in more detail, and establish their nontrivial topological characteristics.

Figure 3.2c includes a zoomed-in view of the typical time evolution of modes inside the highest band gap 3, centered around 18.2 kHz. We observe that the pumping forces the band gap to open and close periodically over time. Interestingly, such transition is accompanied by a process of phase change for the edge modes across a point of degeneracy. For example, the highlighted mode at the bottom edge of the band gap transitions from being global to a left localized edge mode, then to a right localized, and global mode again. This implies that the system changes its characteristics with time, as we see edge modes appearing, interchanging their phases, and then disappearing. This time-dependent *bulk-boundary correspondence* indicates breaking of the time-reversal symmetry; and it is the classical analogue of the topological band gaps of the sliding lattice first proposed by Thouless in 1983 [105].

Such nontrivial topological nature of band gaps manifests the quantization of pumped charge in a time-dependent, one-dimensional quantum mechanical system. The average charge/photon flow in one period of pumping cycle can be quantified using a topological invariant in two dimensions, i.e., Chern number [28, 113, 114]. In the current case, we adopt a similar approach to characterize the band gaps in theoretically possible quasistatic limit. We choose wavevector and time as two independent variables and calculate the Chern number using the method detailed in Ref. [49] for a mechanical system. In Fig. 3.2d, we show the Bloch dispersion curve for an infinitely periodic system. We label each band with the corresponding Chern number. It is known that the topological characteristics of a band gap is dictated by the “gap Chern number”, which is the summation of the Chern numbers of all the bands below it. In this way, gap Chern numbers -1 , 2 , and 1 can be associated with three prominent band gaps: 1, 2, and 3, respectively. It is evident that these nonzero values indicate nontrivial band topology of the system under time evolution. It is also remarkable that these gap Chern numbers directly explain the number of degenerate points during one period of time evolution of band gaps as seen in Fig. 3.2b. Opposite signs of the numbers

associated with band gaps 1 and 3 simply denote the opposite phase reversal of the edge modes (interchanging the blue and red colors of Fig. 3.2c). In the later sections, we would discuss how these topological invariants relate to the asymmetry in the dispersion curve, and hence influence the dynamic response of our system.

3.3.2 One-way transmission of stress waves

We now verify the results obtained through quasistatic analysis, and demonstrate the implications of the bulk-boundary correspondence in the system by carrying out full numerical simulations at a *small* pumping frequency, $f_\alpha = 0.01$ kHz. In Fig. 3.3a, topological transitions are highlighted for the highest band gap 3, in which we notice that there are two time instances which are particularly important at a given input frequency of longitudinal wave. The first instance (indicated by Point 1 in Fig. 3.3a) is a point from which an edge mode—localized at the *right* end—evolves to a global mode. The other instance (Point 2 in Fig. 3.3a) is where an edge mode—localized at the *left* end—evolves to a global mode. To verify this wave evolution, we send a short pulse of longitudinal wave (specifically, 0.8 ms long Gaussian force excitation at $f_L = 18.2$ kHz) from both left and right ends of the chain separately and see how the pulse propagates in the medium. We send these pulses at the two different time points 1 and 2, and solve Eq. (3.3) using the Runge-Kutta scheme for all further numerical simulations. First, when we send pulses at the time point 1, as shown in Figs. 3.3b-c, we observe that the longitudinal waves can propagate only from the right to the left end. This is accompanied by a frequency shift shown in Figs. 3.3d-e, where the shifted frequency corresponds to the lower edge of the band gap. In the reverse direction, we do not see the transmission of the input pulse. In the second case when the pulse is sent at the time point 2, Figs. 3.3f-i show that the pulse can travel only from the left to the right end. In addition, frequency shifts to a higher value, which corresponds to the upper edge of the band gap. We also notice that wave speeds are not equal in these cases because of the inherent dispersion in the system, where higher frequency wave propagates slowly. Therefore, it is clear that the system responds selectively to excitation sent from left and right end of

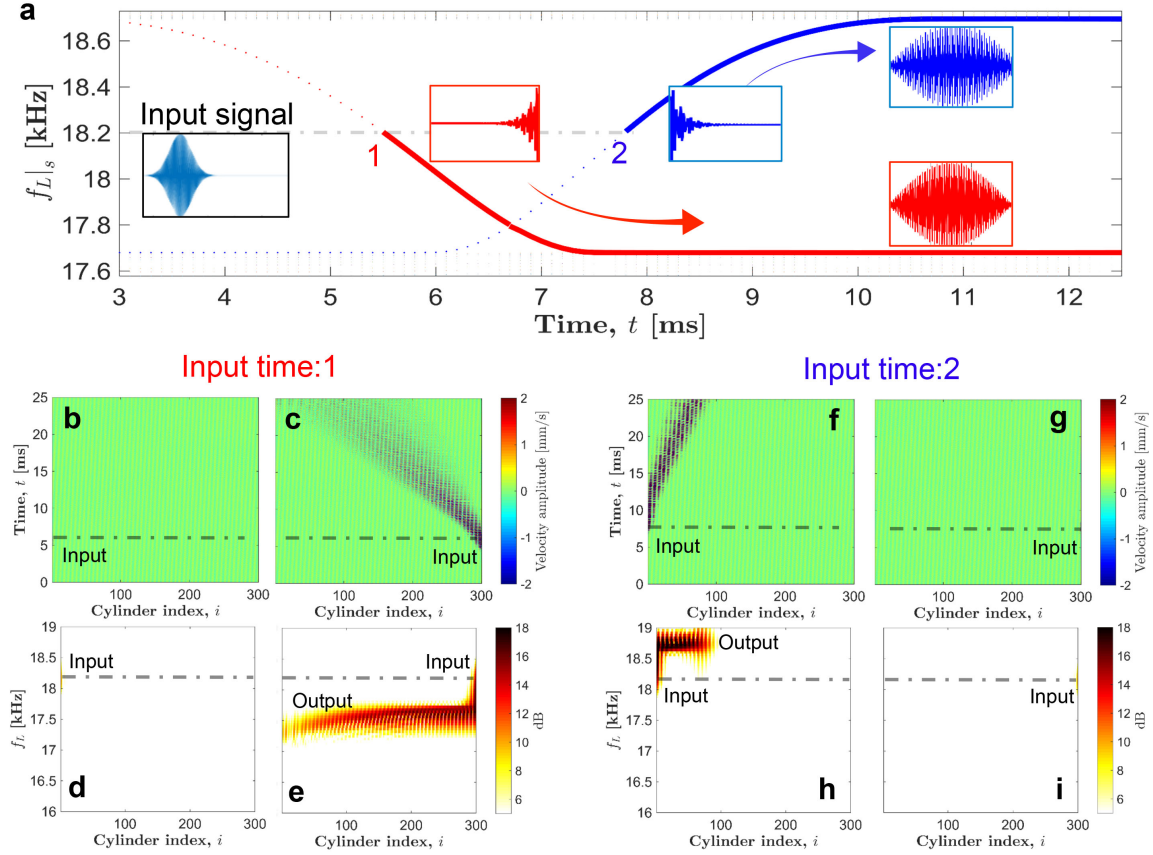


Figure 3.3: Implication of the bulk-boundary correspondence for *small* pumping frequency. (a) Typical topological transitions of the eigenmodes inside band gap 3 as also shown in Fig. 3.2c. System's ability to selectively convert a left (and right) localized edge mode to a global mode with higher (and lower) frequency is highlighted at two unique time points. A full numerical simulation is carried out with a Gaussian pulse centered at $f_L = 18.2$ kHz as an input to the system. (b)-(c) Spatio-temporal map of cylinder velocity when the input is from left and right end of the chain, respectively, at time point 1. (d)-(e) Propagating pulse frequency corresponding to the velocity maps. (f)-(i) The same plots obtained by sending the pulse at time point 2.

the chain, and introduces different frequency shifts in accordance with the bulk-boundary correspondence in quasistatic analysis.

It should be also noted that if we flip the sign of the pumping frequency [essentially of the phase factor $2\pi f_\alpha t$ in Eq. (3.1)], we can easily see that the above observations would be reversed. For example, the pulse sent from the left end would propagate with lower frequency, and the pulse sent from the right end would propagate with higher frequency. This would also reflect in the change of sign (positive to negative, and vice versa) of the Chern numbers associated with the bands. This implies that we break the time-reversal symmetry in the system for a non-zero pumping frequency. Therefore we show that our system offers non-reciprocity for a mechanical pulse, and the fundamental mechanism relies on the topological bulk-boundary correspondence inside the band gaps in a quasistatic limit. At this juncture, a natural question to ask would be: how is this nontrivial topology of band gaps important when we break the quasistatic assumption and start applying high-frequency torsional waves? We note that quasistatic evolution of eigenfrequencies (Fig. 3.2b) no longer would dictate the dynamics directly. We would have to take dynamic factors into our consideration once the torsional wave has a significantly high frequency. Would the nonreciprocity shown in the quasistatic limit still hold true? How do the topological invariants translate into the dynamic system? We would answer these questions in the following sections.

3.3.3 *Dynamic analysis: Asymmetry in dispersion curves*

To investigate the dynamic effect of the torsional waves on the longitudinal ones, we solve Eq. (3.3) computationally for two different torsional wave frequencies: 0 and 2 kHz, while other system parameters are kept the same. The first case represents a chain with only spatial variation of stiffness, whereas the later also incorporates the effect of simultaneous temporal modulation. We give an impulse excitation at one end of the chain to excite all possible vibration modes. By using the Fast Fourier Transformation (FFT) of velocity map, we obtain dispersion curves as per extended Brillouin zone scheme. For *zero* pumping frequency, the dispersion curves are perfectly symmetric with respect to the zero longitudinal wavenumber

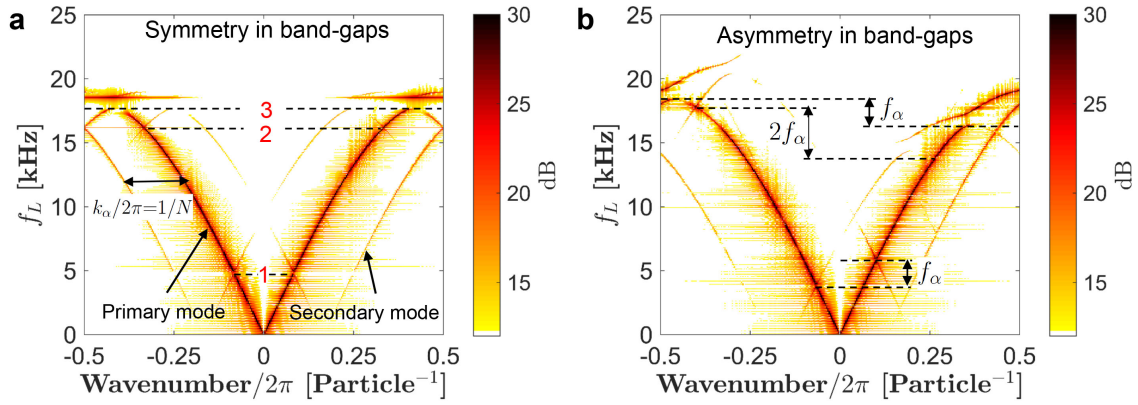


Figure 3.4: Dispersion curves obtained using full numerical simulations to show the effect of high pumping frequency. (a) $f_\alpha = 0$, i.e., a system with only spatial modulation in stiffness, has a symmetric dispersion curve. The three prominent band gaps: 1, 2, and 3 are marked. (b) Torsional wave (with $f_\alpha = 2$ kHz in this case) breaks the time-reversal symmetry, and thus results in asymmetric band gaps for rightward (*positive* wavenumber) and leftward (*negative* wavenumber) propagating longitudinal wave. The frequency offsets for the band gaps are indicated, and those are dictated by corresponding topological invariant, i.e., the summation of Chern numbers of bands below it, defined in quasistatic sense.

axis (Fig. 3.4a). In particular, we see symmetric Bloch band gaps on both wavenumber and frequency domains. Also noticeable are the symmetric secondary mode branches that are horizontally shifted by $\pm mk_\alpha/2\pi$ with m being an integer that represents the interference order as a result of spatial variation of stiffness in the system [104].

On the other hand, as we introduce a significant pumping frequency ($f_\alpha = 2$ kHz), the time-reversal symmetry is broken, and we notice asymmetry in the dispersion curves (Fig. 3.4b). Firstly, the secondary mode branches are asymmetric. This asymmetry can, in turn, be quantified as a vertical shift equivalent to $\pm lf_\alpha$, where l is an integer representing the mode index (primary mode being $l = 0$). However, the most remarkable observation is that each band gap, on the primary mode branch, shows a unique asymmetry with respect to *positive* and *negative* wavenumber axes. Those are not only distinct in terms of the magnitude of frequency and wavenumber offset but also in their directional shifts. For example, band gap 1 shifts *up* for positive wavenumbers, whereas band gap 3, even though showing the same frequency offset, shifts *down* for the positive wavenumbers. Band gap 2, however, shows entirely distinct offset with double the magnitude compared to those of band gap 1 and 3.

In order to explain this key observation, we refer to our earlier section on quasistatic analysis. We had shown that the system supports nonreciprocity for mechanical waves based on the bulk-boundary correspondence quantified by a topological invariant in two dimensions. In particular, the degeneracy (mode crossing)—which occurs during the time evolution of eigenfrequencies (Fig. 3.2b)—essentially influences the system response and results in nonreciprocity with distinct frequency shifts (Fig. 3.3). In this way, a degenerate point can be treated as the core element in the pumping due to the torsional wave. The *number* and the *type* of degenerate points would, thus, dictate the dynamics *in* and *around* the respective band gap. As stated earlier, the gap Chern numbers associated with the band gaps would give us this number and the type of degeneracy. We find that these numbers are -1 , 2 and 1 for band gaps 1, 2, and 3, respectively. Hence, for the pumping frequency of f_α , these band gaps would be individually pumped at the rate of $-f_\alpha$, $2f_\alpha$ and f_α , respectively. This is ex-

actly what is reflected as the distinct asymmetry of band gaps in the dispersion curve shown in Fig. 3.4b. Therefore, we find an interesting connection between the asymmetry of band gaps caused by dynamics pumping, and their topological characteristics in the quasistatic limit. We would like to mention that rapid pumping into the system might cause instability, so we limit ourselves to a finite system analyzed for a finite time. Detailed discussion on instability in such systems can be found in the earlier work by Li *et al.* [104].

3.3.4 Nonreciprocity and interband transition

We now explore the implication of these asymmetric band gaps, and their potential to tailor mechanical wave propagation in the medium. Referring to Fig. 3.5a, a zoomed-in view of the same asymmetric dispersion curve at pumping frequency $f_\alpha = 2$ kHz, we focus on the widest and highest band gap 3. We choose a longitudinal traveling pulse with frequency $f_L = 16.78$ kHz, such that it lies inside and outside of the band gap of the primary mode for the right and left branches of dispersion curve, respectively. We use 0.8 ms long Gaussian pulse (Fig. 3.5b) with the aforementioned central frequency f_L to give force excitation and send the pulse from both ends of the chain. Figures 3.5c-d show the spatio-temporal map of particles' velocities when the system is excited from left and right ends, respectively. Interestingly, we observe that the system allows the injected pulse to propagate in both directions. However, it is remarkable that wave frequency is direction dependent, which makes the wave speed direction dependent due to dispersion. We show the spectrum of the propagating wave in Figs. 3.5e-f. This is obtained by running the numerical simulation for 13 ms, and then performing FFT on the velocity map. It is evident that the wave traveling from left to right has a *higher* frequency compared to the injected one at 16.78 kHz, and the difference is equal to the pumping frequency f_α . Note that the stoppage of the highlighted parts in figures does not mean that waves stop propagating at that point. On the other hand, a pulse injected at the right end propagates with the same speed without any frequency shift. We observe even more interesting wave directional behavior in the presence of a defect in the medium (Appendix B).

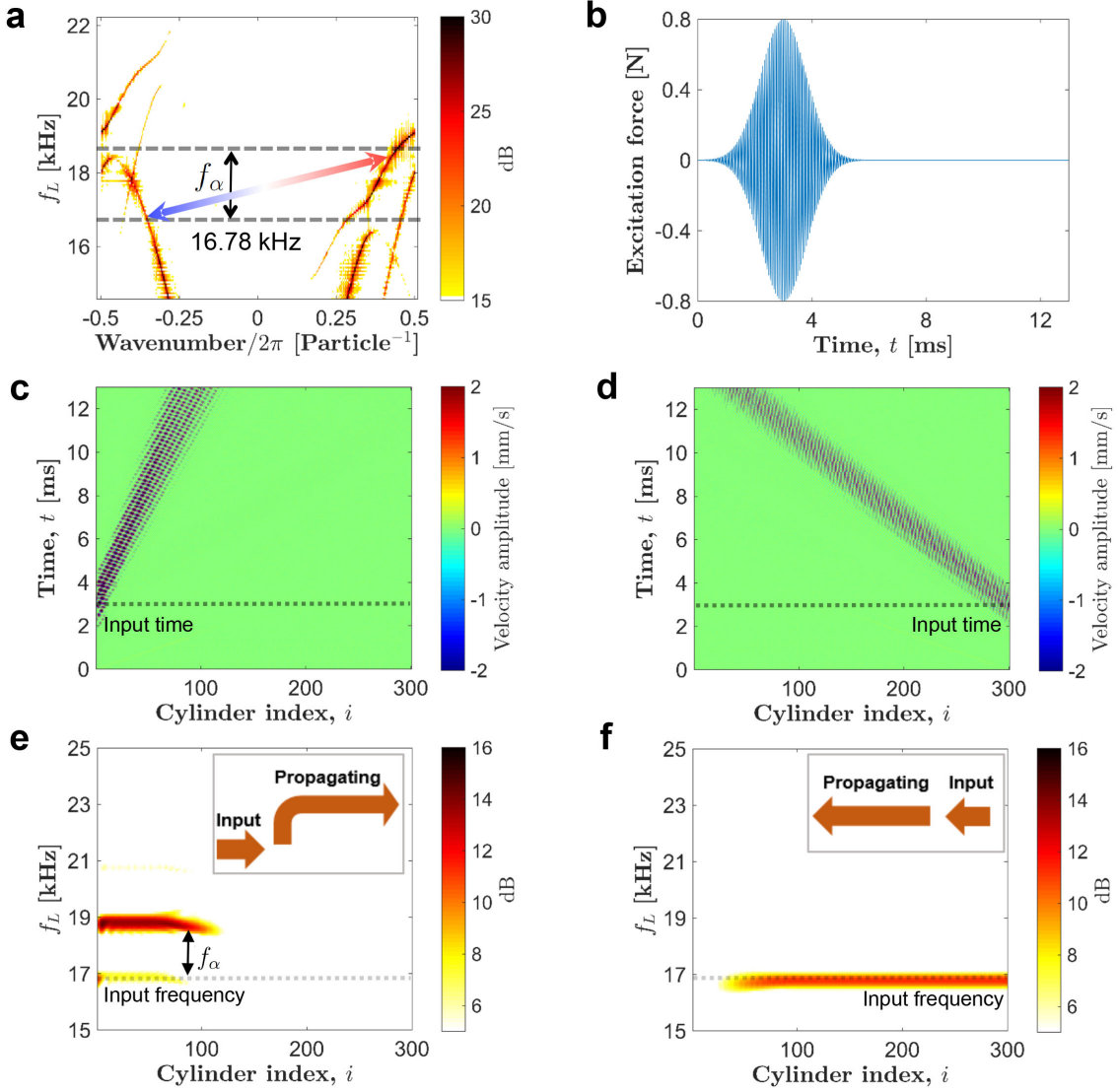


Figure 3.5: Implication of asymmetric band gaps. (a) A zoomed-in view of the highest band gap 3 in the dispersion curve at $f_\alpha = 2$ kHz. System's response is examined at input excitation frequency $f_\alpha = 16.78$ kHz. The arrow tips denote leftward and rightward propagating frequencies indicating interband transition. (b) Gaussian profile of the input force that is applied at the ends of the cylinder chain. (c)-(d) Spatio-temporal map of cylinder velocity when signal is sent from the left (right) end of the chain. (e)-(f) The corresponding frequency content of the propagating pulse. We observe that rightward traveling pulse propagates with a higher frequency (slower speed) compared to that of input pulse, however leftward propagating pulse does not see any such frequency shift (insets for summary).

This is indeed a peculiar type of wave dynamics we observe in our system. Though, an input wave can propagate in *both* the directions, there is nonreciprocity in terms of the frequency spectrum. Different frequencies propagate in opposite directions. This is in contrast to usual cascading process in time-dependent systems, in which we observe wave mixing effect due to the presence of *multiple* modes separated by the pumping frequency [112]. In the present case, however, we see only two types of modes propagating in the medium instead of multiple ones. We identify that the selection of these two modes is can be linked to the interband transition mechanism [108, 109]. Interband transition is facilitated in a way that a frequency *lower* than a band gap region propagates in one direction (left), and a frequency *higher* than the band gap region propagates in the opposite direction (right). These are indicated as the tips of the arrow marked in Fig. 3.5a. Clearly, those will have different wavenumbers as well. The difference in frequencies would be equal to the offset in band gaps. As we discussed previously, topological invariants (gap Chern number) dictate this offset, which is f_α for the current band gap. Hence, we see the modes traveling in opposite directions lie on different bands of the dispersion curve (one lower than gap, and one higher); and hence the term *interband* transition is justified.

3.4 Conclusions

In this study we have proposed a purely mechanical device based on a helical phononic crystal, which shows nonreciprocity in terms of stress wave propagation. We used the coupling of two mechanical waves—longitudinal and torsional modes—to achieve the breakage of the time-reversal symmetry. We related such symmetry breaking to the topological nature of band gaps induced by spatio-temporal variation of system parameters. In this phononic crystal, the topological characteristics represented by the Chern number directly dictate the level of asymmetry associated with each band gap in the dispersion curves. Notably, the asymmetry in the dispersion curve can be used to achieve the interband transition at a specific frequency range. Towards the practical significance, if the higher frequency mode in such asymmetric wave propagation can be filtered, e.g., by using the fact that it would attenuate more in the

presence of material dissipation or by taking advantage of its slower wave propagation speed, the current system would serve the purpose of stress wave isolator, allowing stress waves only in one direction. Moreover, the allowable direction as well as general characteristics of the longitudinal wave can be simply controlled by fine-tuning the torsional wave parameters, thereby making the proposed system active and tunable.

3.5 Author contributions

This chapter is based on the following manuscript: R. Chaunsali, F. Li, J. Yang, “Stress Wave Isolation by Purely Mechanical Topological Phononic Crystals,” *Scientific Reports* 6, 30662, 2016 [79]. All authors along with their collaborators conceived the idea of the project together. R. Chaunsali designed the study, performed the analysis, and wrote the manuscript. J. Yang supervised the project.

Chapter 4

TWO-DIMENSIONAL DISCRETE TOPOLOGICAL METAMATERIAL

Recently, there have been significant efforts to guide mechanical energy in structures by relying on a novel topological framework popularized by the discovery of topological insulators. Here, we propose a topological metamaterial system based on the design of the Stewart Platform, which can not only guide mechanical waves robustly in a desired path, but also can be tuned *in situ* to change this wave path at will. Without resorting to any active materials, the current system harnesses bistability in its unit cells, such that tuning can be performed simply by a dial-in action. Consequently, a topological transition mechanism inspired by the quantum valley Hall effect can be achieved. We show the possibility of tuning in a variety of topological and traditional waveguides in the same system, and numerically investigate key qualitative and quantitative differences between them. We observe that even though both types of waveguides can lead to significant wave transmission for a certain frequency range, topological waveguides are distinctive as they support robust, back scattering immune, one-way wave propagation.

4.1 Introduction

The discovery of topological insulators [29, 30], which are labeled as a new state of matter in the condensed matter physics, has galvanized research efforts in multiple fields. Topological framework enables us to understand fascinating phenomena such as quantum Hall effect (QHE) [105, 115], quantum spin Hall effect (QSHE) [33], and quantum valley Hall effect (QVHE) [116]. These are special because of the intriguing directional and robust edge states they support. It is recent that these fundamental concepts have been ex-

tended to the fields of photonics [117], acoustics [41–43, 106, 107, 118–120], and mechanics [37, 47, 49–52, 57, 79, 88, 121–123]. Such extension is of the fundamental interest to the research community as it has potential to set new design principles for *topological metamaterials* that aim to strategically tailor energy transport for waveguiding, isolating, switching, filtering, and related applications.

Acoustic and mechanical metamaterials that rely on QHE need active components (e.g., gyroscope and flow circulator) or applications of external fields (e.g., magnetic field) to break the time reversal symmetry [41, 47, 49, 79, 106, 107, 118, 121]. This adds complexity to the system, and thus, these metamaterials based on QHE are challenging to be realized in practical environments. QSHE-inspired metamaterials employ only passive components, but usually mandate intricate ways to achieve a double Dirac cone in their dispersion relation [43, 50–52, 119, 122]. Metamaterials based on QVHE [42, 57, 120, 123], however, rely on the breakage of the inversion symmetry to achieve topological properties, and these can be comparably easier to be designed and realized in practical settings. QVHE originates from the newly discovered valley degree of freedom (DOF) of electrons in the two-dimensional honeycomb lattice of graphene [116]. These valley DOFs are energetically degenerate but are largely separated in momentum space [42, 120]. Due to this large separation, inter-valley scattering can be avoided, and valley DOFs constitute two pseudospins, which have opposite-directional and robust properties on the topological interfaces. On this principle, backscattering immune and robust energy transport along topological waveguides with sharp bends have been proposed in acoustic [42, 120] and mechanical [57, 123] systems.

Though the aforementioned configurations have shown to be guiding energy in a specific path in the system, there remains a challenge; Can one change the path *in situ* and thus achieve a complete control over the waveguide? Once achieved, this will provide a fertile testbed for future experiments related to wave guiding capability of various types of topological interfaces, and at the same time those can be compared with traditional waveguides in the same system more closely than ever. However, adding such versatility into the system comes at a cost. Generally, *in situ* tunability requires complex components or mechanisms to

be present in the system, so that the wave path in the lattice structure can be reconfigured in a controllable and versatile manner. But such complexity in design could again make the system cumbersome for practical use, and it would defeat the purpose of building a simple QVHE-based system to some extent.

Here we show that *in situ* tunability in the QVHE-based mechanical metamaterial can be achieved by utilizing nonlinearity of the constituent elements in the system. More specifically, we use an assembly of the Stewart Platform (SP), in which translation and rotational degrees of freedoms of each SP are judiciously tailored to achieve a bistable response. Consequently, a simple dial-in action changes its configuration from one stable state to the other, and this feature can be used for *in situ* control of the wave path in the system. The SP already has a wide range of engineering applications such as vibration control, precise positioning, and flight simulation [124, 125]. Therefore, by integrating the elegant engineering of SP with the fascinating physics of QVHE, we propose a dial-in mechanical metamaterial for creating tunable topological waveguides. We use extensive numerical simulations and show that this metamaterial can be tuned *in situ* to design a variety of topological waveguides with robust wave propagation characteristics. The tunability allows us to also build traditional waveguides by suppressing topological variations in the same systems, making it possible to compare their performance with the topological counterpart to a remarkable detail. Such a comparison therefore plays a key role in extending our knowledge and appreciation towards the uniqueness of topological waveguides in the proposed system.

4.2 Design of the tunable topological metamaterial

The tunable system we propose is illustrated in Fig. 4.1. Each SP is bistable, i.e., it has two stable configurations: Y-state (yellow) and P-state (purple) as shown in Figs. 4.1a-b. This unit is made of two parallel disks connected with six linear springs. The conventional SP unit has six DOFs for the top disk, while the bottom disk is fixed [124]. However, by judiciously choosing the connecting springs, one can decouple some DOFs and reduce the total DOFs (see Appendix C.1). In this study, for the sake of simplicity, we assume that the

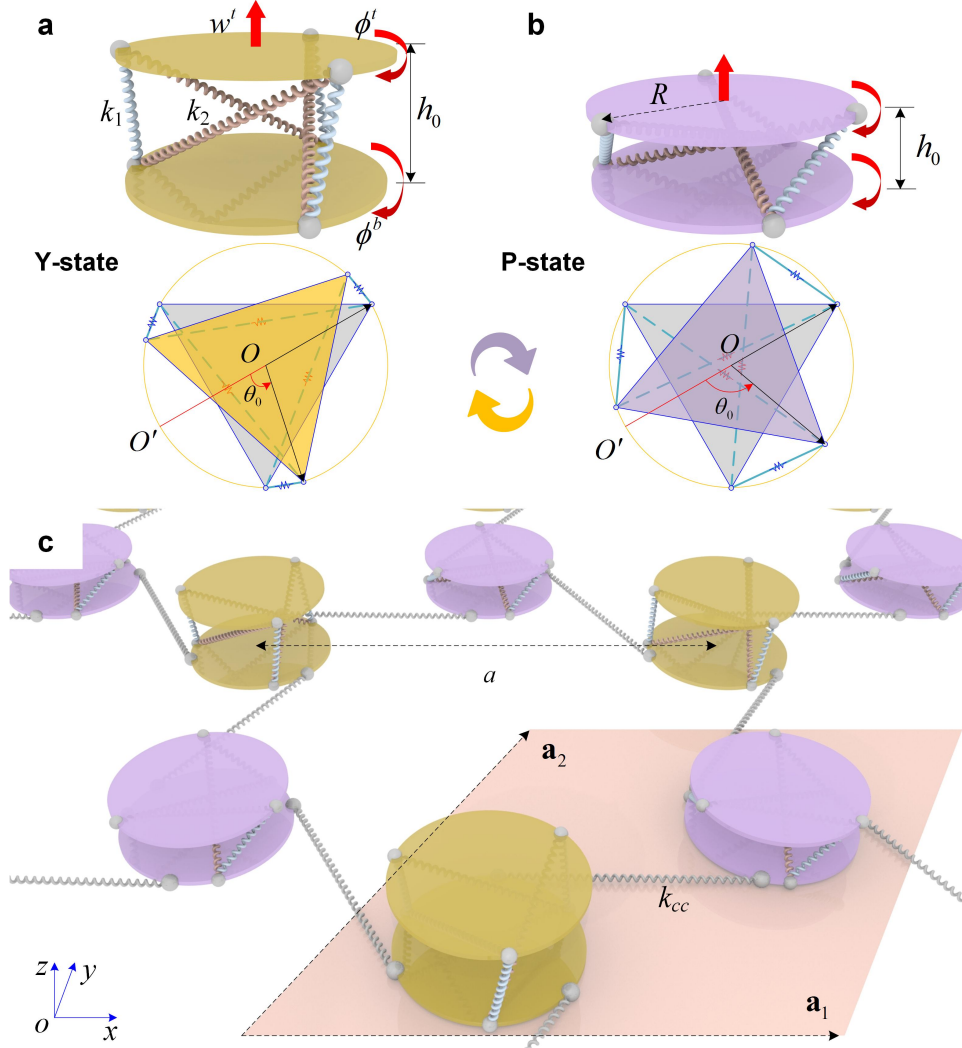


Figure 4.1: Design of the tunable topological metamaterial based on bistable SP. (a)-(b) The two stable states, Y- and P-states, and below are their top views. They differ in the height (h_0) and the relative rotation (θ_0) between the disks: Y-state ($h_0 = 0.04$ m, $\theta_0 = 70^\circ$) and P-state ($h_0 = 0.0215$ m, $\theta_0 = 110^\circ$). Disk radius $R = 0.0408$ m, connecting spring coefficients $k_1 = 10^2$ N/m and $k_2 = 10^5$ N/m. Three allowable degrees of freedom, ϕ^b , ϕ^t , and w^t , are marked with the red arrows. (c) Hexagonal arrangement by combining the two stable states. The lattice constant a of the hexagonal arrangement is 0.212 m, and the torsional spring coefficient k_{cc} of the reverse spring is 10^2 N·m. The periodic unit cell consisting of one Y-state and one P-state is highlighted along with its basis vectors $\mathbf{a}_1$ and $\mathbf{a}_2$.

bottom disk is pinned at its center, such that it can only rotate about the z direction. We denote this rotational DOF of the bottom disk by ϕ^b . The top disk can have only rotational (ϕ^t) and translational (w^t) motions along the z direction with respect to its equilibrium position. Note that these dynamic perturbation parameters, ϕ^b , ϕ^t , and w^t , should not be confused with θ_0 and h_0 , which denote the equilibrium parameters of the SP unit cell and vary depending on whether it is in the Y- or P-state (Fig. 4.1a). All three DOFs of these dynamic motions in terms of ϕ^b , ϕ^t , and w^t are governed by the six springs between the plates. The detailed mathematical relationships—including the derivations of the bistability in the SP unit cell—are described in Appendix C.2.

We design a hexagonal lattice by combining two stable states, such that the system breaks C_6 symmetry but retains C_3 symmetry (Fig. 4.1c). Only the bottom disk of each SP is connected with neighboring SPs with springs (indicated with a torsional spring coefficient k_{cc}). Note that this connection is a reverse spring, i.e., it induces opposite torque in the connected units (a similar setup can be found in a recent work [52]). Tunability comes from the fact that one can easily change the stable state of each SP—independently—to achieve a desired lattice configuration.

4.3 Results and discussion

4.3.1 Band inversion and topology

In this section, we evaluate dispersion characteristics of the system and observe a topological transition. For describing the dynamics of the (infinite) hexagonal lattice, we choose a periodic unit cell (highlighted in Fig. 4.1c), which consists of two SPs indexed as 1 and 2. Here, each SP can take either Y- or P-state by dial-in actions. Since a single SP has three DOFs, the unit cell would be represented by the following six parameters: $[\phi^{(1)b}, \phi^{(2)b}, \phi^{(1)t}, \phi^{(2)t}, w^{(1)t}, w^{(2)t}]$.

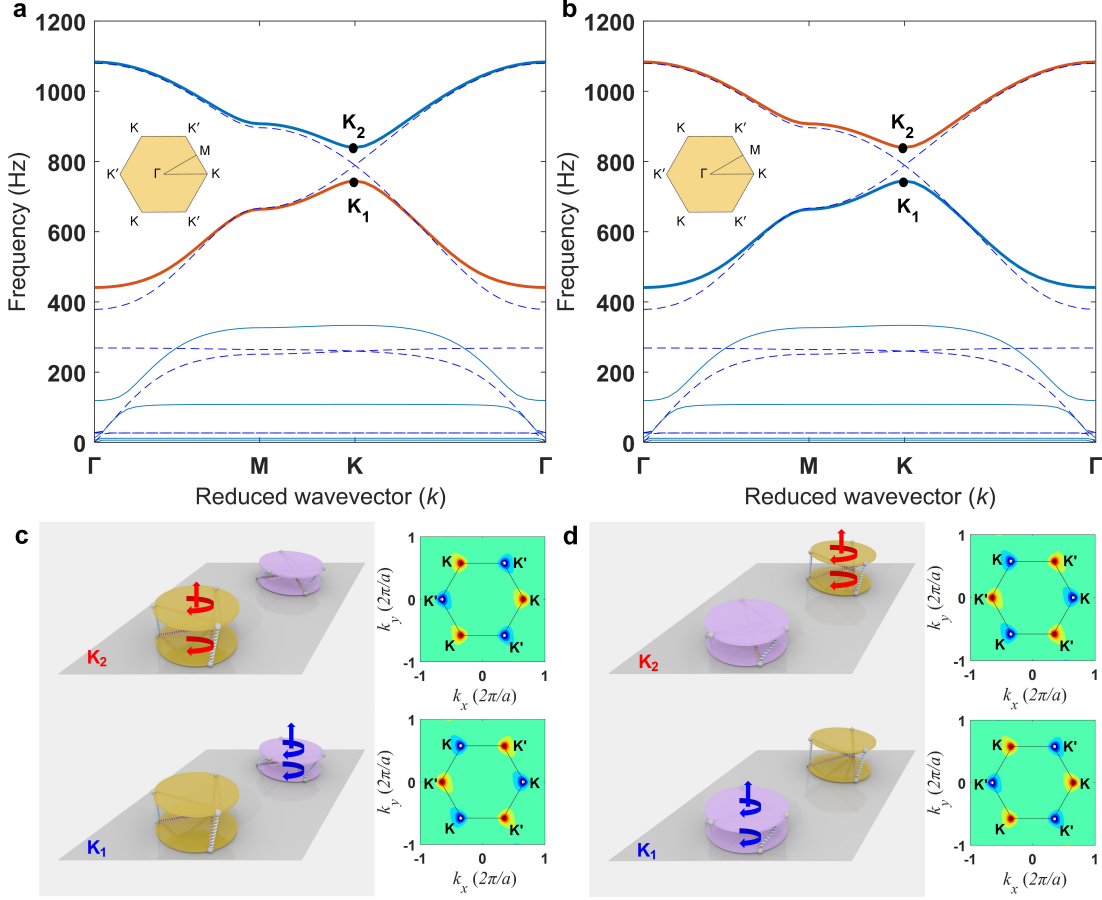


Figure 4.2: Band inversion when the unit cell is transformed from the Y-P to the P-Y configuration. (a) Unit cell dispersion relation for the Y-P configuration. A complete band gap emerges around 800 Hz due to broken inversion symmetry. The dashed lines with a Dirac cone at the $\mathbf{K}$ point represent the case with both SPs of the unit cell having the same effective stiffness. (b) The same for the P-Y state. Left panels in (c) and (d) represent the eigenmodes corresponding to $\mathbf{K}_1$ and $\mathbf{K}_2$ points in the dispersion relations of the Y-P and the P-Y configurations, respectively. Only one SP shows the motion, demonstrating a spatial inversion in the unit cell for the two configurations. In the right panels of (c) and (d), this reflects as the change in the sign of the Berry curvature at the $\mathbf{K}$ and $\mathbf{K}'$ points in the first Brillouin zone (solid line).

First, we write the equations of motion for the periodic unit cell (index as i, j) as:

$$I\ddot{\phi}_{i,j}^{(1)b} + k_{cc}(3\phi_{i,j}^{(1)b} + \phi_{i,j}^{(2)b} + \phi_{i,j-1}^{(2)b} + \phi_{i-1,j}^{(2)b}) - k_{\phi w}^{(1)}w_{i,j}^{(1)t} + k_{\phi\phi}^{(1)}(\phi_{i,j}^{(1)b} - \phi_{i,j}^{(1)t}) = 0 \quad (4.1a)$$

$$I\ddot{\phi}_{i,j}^{(1)t} + k_{\phi w}^{(1)}w_{i,j}^{(1)t} + k_{\phi\phi}^{(1)}(\phi_{i,j}^{(1)t} - \phi_{i,j}^{(1)b}) = 0 \quad (4.1b)$$

$$m\ddot{w}_{i,j}^{(1)t} + k_{ww}^{(1)}w_{i,j}^{(1)t} + k_{w\phi}^{(1)}(\phi_{i,j}^{(1)t} - \phi_{i,j}^{(1)b}) = 0 \quad (4.1c)$$

$$I\ddot{\phi}_{i,j}^{(2)b} + k_{cc}(3\phi_{i,j}^{(2)b} + \phi_{i,j}^{(1)b} + \phi_{i,j+1}^{(1)b} + \phi_{i+1,j}^{(1)b}) - k_{\phi w}^{(2)}w_{i,j}^{(2)t} + k_{\phi\phi}^{(2)}(\phi_{i,j}^{(2)b} - \phi_{i,j}^{(2)t}) = 0 \quad (4.2a)$$

$$I\ddot{\phi}_{i,j}^{(2)t} + k_{\phi w}^{(2)}w_{i,j}^{(2)t} + k_{\phi\phi}^{(2)}(\phi_{i,j}^{(2)t} - \phi_{i,j}^{(2)b}) = 0 \quad (4.2b)$$

$$m\ddot{w}_{i,j}^{(2)t} + k_{ww}^{(2)}w_{i,j}^{(2)t} + k_{w\phi}^{(2)}(\phi_{i,j}^{(2)t} - \phi_{i,j}^{(2)b}) = 0 \quad (4.2c)$$

where I and m are the rotational inertia and the mass of the disks, respectively. k_{ww} , $k_{\phi\phi}$, $k_{w\phi}$, and $k_{\phi w}$ are the stiffness coefficients for the relative translation and rotations, and their coupling. The detailed expressions of these coefficients as a function of k_1 , k_2 , and geometric parameters are described in Appendix C.1.

For a lattice length of a , we invoke the Bloch's theorem by using the periodicity in two directions: $\mathbf{a}_1 = [1, 0]a$ and $\mathbf{a}_2 = [1/2, \sqrt{3}/2]a$, and obtain the following eigenvalue problem:

$$\omega^2 \mathbf{M} \mathbf{U} = \mathbf{B} \mathbf{U} \quad (4.3)$$

where $\mathbf{U} = [\phi_{i,j}^{(1)b}, \phi_{i,j}^{(2)b}, \phi_{i,j}^{(1)t}, \phi_{i,j}^{(2)t}, w_{i,j}^{(1)t}, w_{i,j}^{(2)t}]$ is the generalized eigenvector, and ω is the angular frequency. $\mathbf{B}$ and $\mathbf{M}$ are the stiffness and mass matrices, respectively (see the detailed expressions in Appendix C.3).

There are two possible configurations of the C_3 symmetric hexagonal unit cell: Y-P configuration and P-Y configuration. Figure 4.2 displays the dispersion properties and associated band inversion when one state is transformed to the other. Due to the breakage of the inversion symmetry, a complete band gap emerges at the $\mathbf{K}$ point between the fifth and the sixth bands (see the highlighted red and blue bands in Figs. 4.2a-b). Note that these bands would have a degeneracy at the $\mathbf{K}$ point if the inversion symmetry is not broken (see the blue

dashed lines). Although the dispersion curves for these two Y-P and P-Y configurations look similar, there is a difference in terms of the topology. The highlighted bands are inverted in these configurations—so-called band inversion. We verify this by plotting in the left columns of Figs. 4.2c-d the mode shapes of the unit cell corresponding to the points $\mathbf{K}_1$ (744.5 Hz) and $\mathbf{K}_2$ (839.8 Hz) marked in the dispersion relations. For the Y-P configuration in Fig. 4.2c, we see that the low frequency vibration ($\mathbf{K}_1$ point) corresponds to the case when only the P-state is vibrating in the lattice. In Fig. 4.2d, the P-state vibrates in the P-Y configuration at the low frequency in the same way. This makes sense as the designed stiffness for the P-state is lower than that of the Y-state (see Appendix C.2). However, there is an inversion in terms of where in the unit cell the vibration is dominant due to the flipping of the Y- and P-states.

The aforementioned band inversion resembles the one seen in the valley Hall effect. We can calculate the valley Chern number in order to track the topological transition associated with this effect. This is achieved by integrating its Berry curvature over *half* of the first Brillouin zone. Mathematically, the valley Chern number $C = (1/2\pi) \iint \Omega(\mathbf{k}) d\mathbf{k}$, where the Berry curvature $\Omega(\mathbf{k}) = i \sum_{v=1, v \neq u}^6 [\langle \mathbf{u} | \partial \mathbf{B} / \partial k_x | \mathbf{v} \rangle \langle \mathbf{v} | \partial \mathbf{B} / \partial k_y | \mathbf{u} \rangle - c.c.] / (\omega_v^2 - \omega_u^2)^2$ [57]. Here, i is the imaginary unit, *c.c.* denotes complex conjugate, $\langle \cdot | \cdot \rangle$ represents the inner product, $\mathbf{k} = [k_x, k_y]$ is the wavevector, and ω_u and ω_v are the angular frequencies corresponding to the normalized eigenvectors $\mathbf{u}$ and $\mathbf{v}$, respectively. The calculated Berry curvature in the first Brillouin zone is shown in the right columns of Figs. 4.2c-d. One notices that it is localized at $\mathbf{K}$ and $\mathbf{K}'$ points in the Brillouin zone, and it changes its sign as we alter from the Y-P to the P-Y configuration—reflecting the band inversion process. Therefore, the calculated valley Chern numbers C^K (or $C^{K'}$) for the fifth and the sixth bands of the Y-P configuration (highlighted red and blue bands in Fig. 4.2a) are $-1/2$ and $1/2$ (or $1/2$ and $-1/2$), respectively. These are reversed for the P-Y configuration, and this confirms that the two configurations are topologically distinct. The quantized difference of the valley Chern numbers of the two configurations, i.e., $|C_{Y-P} - C_{P-Y}| = 1$, indicates the emergence of a topologically protected edge state at the interface if these configurations are placed

adjacently.

4.3.2 Topological defect and its manipulation

In this section, we will show how a topological defect can be created by placing topologically distinct lattices, P-Y (defined hereafter to be type I for the sake of simplicity) and Y-P (type II), adjacently. We will also show that this topological defect can easily be reconfigured to any other shapes—thanks to the extreme tunability of the system. First, we confirm the existence of topologically protected interface modes for a linear topological defect (see Fig. 4.3a). To this end, we take a supercell of size 1×20 with the top and the bottom boundaries fixed, and we apply the periodic boundary condition in the x direction.

The resulting dispersion of the supercell strip is plotted in Fig. 4.3b. We notice several bulk modes (curves in black color). These correspond to the bulk bands seen in the unit cell analysis done earlier (Figs. 4.2a-b). However, we also note some additional modes (dashed red and blue curves in Fig. 4.3b). In blue color are the two overlapping modes that appear at the top and the bottom of the supercell due to the identical boundaries on both sides. However, there is a distinguished mode (red color) *inside* the band gap. This corresponds to the interface mode emerging due to the distinct topological nature of the lattices I and II. Additionally, to the extreme top of the dispersion curve, we do observe another interface mode (red color) outside the cutoff frequency.

Now that we have shown the framework of creating a topological defect, it is also possible to achieve various complex shapes of the topological interfaces in a 2D lattice by strategic dial-in actions of the SP cells. In Figs. 4.3c-e, we assemble some of these shapes with various bends, showcasing the manipulation capability in the system. Below are the plots, obtained from the numerical experiments (see methods) for a harmonic excitation at 760 Hz applied to the 40×40 lattice. This frequency excitation, being inside the band gap, excites the topologically protected mode, and thus demonstrates a robust energy transport through various interfaces.

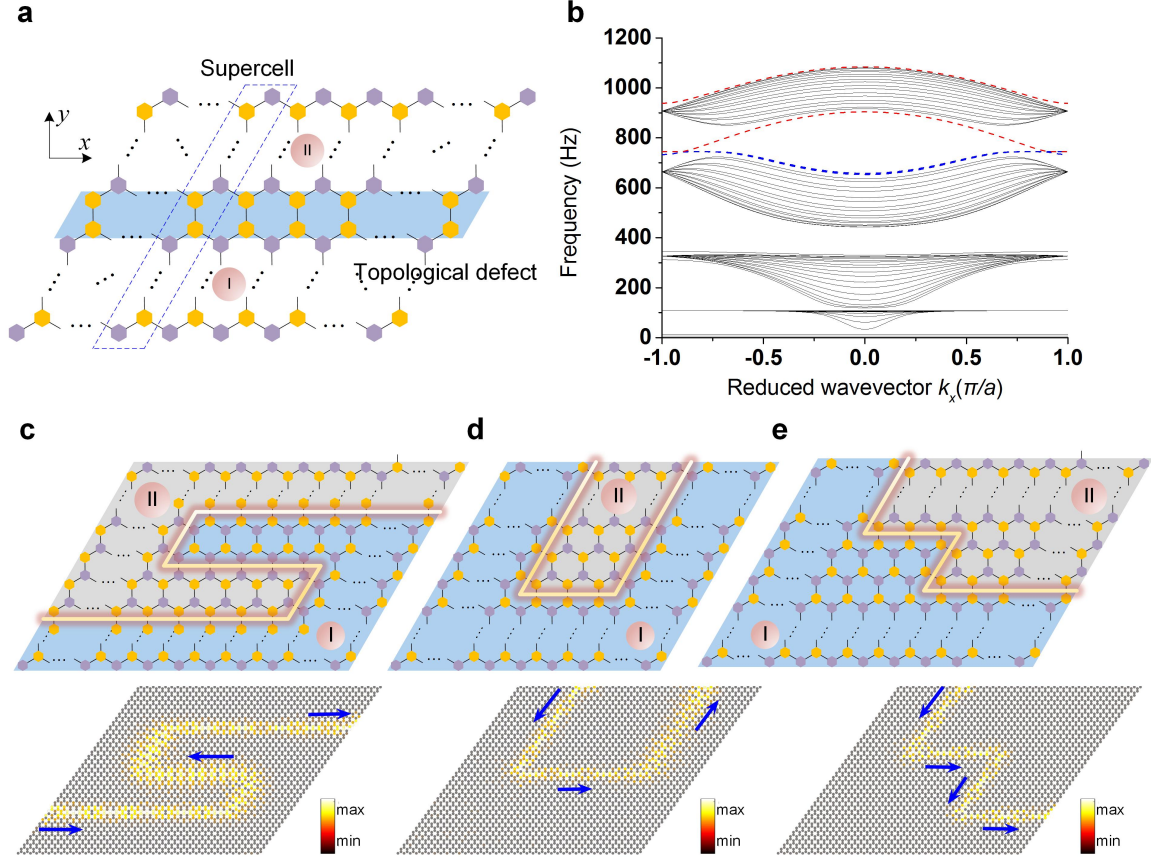


Figure 4.3: Topologically protected interface mode between two topologically distinct lattices, and waveguide path manipulation via dial-in actions. (a) A linear topological defect is highlighted by the blue background at the interface of the P-Y (I) and the Y-P (II) configurations. Dynamic properties are evaluated by taking a supercell enclosed by the dashed box. (b) Band structure for the supercell. The dashed red and blue curves represent the localized modes in the supercell. (c)-(e) A wide range of waveguide paths at the intersection of I and II can be achieved simply by dial-in actions in the lattice. Below are the numerical simulations to confirm the robust energy transport along these interfaces. Colors denote the rotation of the bottom disks.

4.3.3 Topological vs. traditional waveguides

It is now natural to ask—in what ways the *topological* waveguide different from a *traditional* waveguide? Here, by a traditional waveguide, we refer to a waveguide that is created without incurring topological disparities in the same system. One way to create such a waveguide is by introducing a *topologically trivial* defect along the desired wave path so as to utilize the localized defect modes for wave transmission [126]. Figures 4.4a-b show the exemplary cases of topological and traditional waveguides, respectively, realized in the same SP settings. Note that given the initially uniform C_3 symmetric hexagonal structure, we can achieve the creation of the traditional defect simply by transforming P-states into Y-states (i.e., *in situ* dial-in action) along the desired wave path (see the inset of Fig. 4.4b). However, the creation of the topological defects in the originally uniform lattice requires more actions, since it necessitates the border between the type I and the type II lattices. That is, we need to convert the one side of the SP cells (i.e., the upper side of the lattice with respect to the wave path in Fig. 4.4a) from the type I to the type II lattice, which demands the whole flipping (i.e., dial-in action) of the SP cells from the Y- to P-state and vice versa. The analogous operation in 1D lattice systems has been investigated by Chaunsali *et al.* [88]. The advantage of our system is that it can realize both types of traditional and topological waveguides in the same system by leveraging the bistable SP network and the *in situ* dial-in action on it. This provides an excellent opportunity to compare their transmission properties.

We perform an eigen analysis on a 40×40 structure (with 9600 DOF in total) for both types of waveguide structures. In Figs. 4.4c-d, we plot the eigenfrequencies against the modal order for the topological and traditional waveguides, respectively. The blue and yellow curves correspond to the bulk bands, highlighted in the unit cell dispersion in Figs. 4.2a-b. Sandwiched between these two branches are the modes localized at the waveguide interfaces (red color, as magnified in the insets). We observe that the topological waveguide ensures that the entire band gap is populated by the interface modes. This affirms our observations in the previous sections where a robust transmission was shown along topological interfaces.

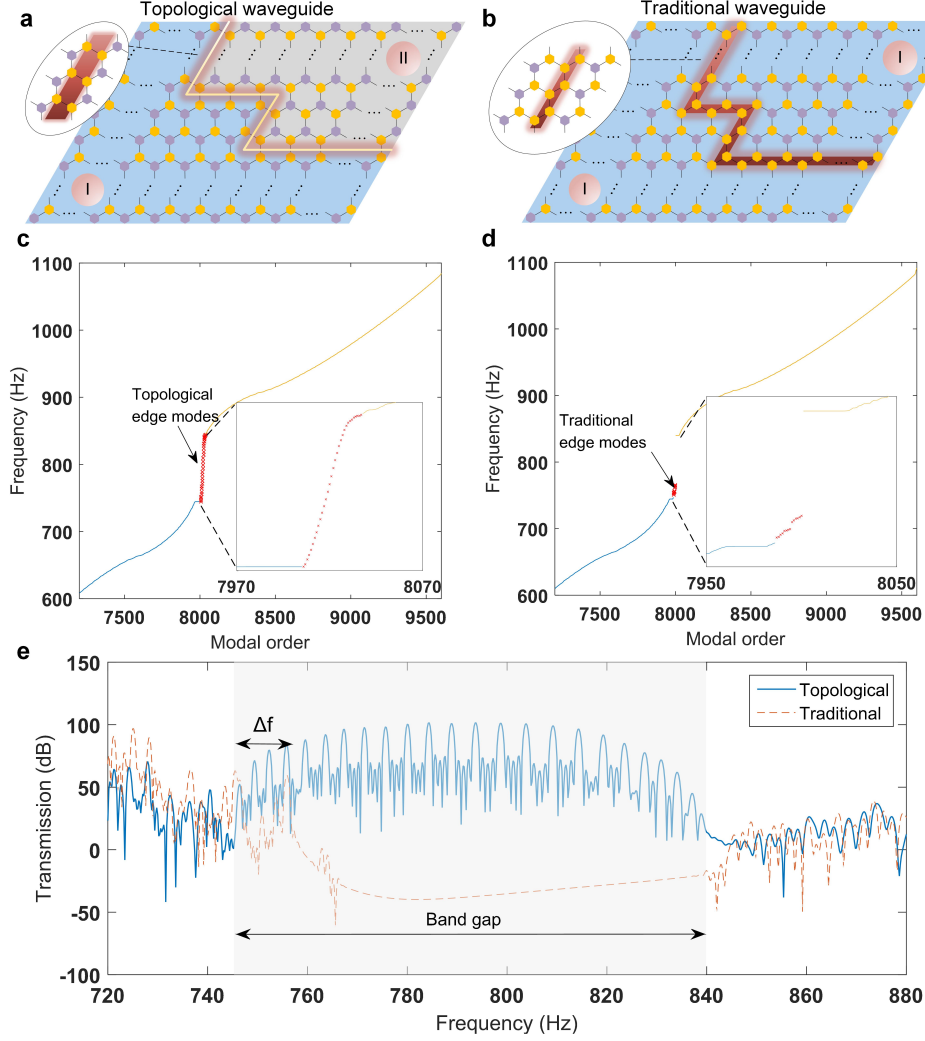


Figure 4.4: Comparison between topological and traditional waveguides. (a)-(b) A finite 40×40 lattice can either be dialed in to form a topological or a traditional waveguide. (c)-(d) Eigenfrequencies of these lattices arranged in the modal order. In (c), the entire band gap is populated with topologically protected edge modes at the interface (zoomed-in view in the lower inset). In (d), the traditional waveguide does not lead to the edge modes spanned over the entire band gap (zoomed-in view in the lower inset). (e) Transmission spectrum of both waveguides.

The traditional waveguide, however, does not lead to the modes spanned over the entire band gap (Fig. 4.4d)—indicating its limitations in terms of constructing a wide range of defect modes.

The aforementioned characteristics provide us with a deep insight into the differences of the topological and traditional waveguides, and these are now used to explain different transmission spectra along the waveguide channels. We perform numerical simulations and calculate the transmission spectra for a range of input frequencies (see methods). In Fig. 4.4e, we plot the obtained transmission. We can immediately notice a clear difference in the transmission inside the band gap (i.e., 744.5 to 839.8 Hz). The topological waveguide leads to a superior transmission all along the band gap, however, the traditional waveguide yields a significant transmission only in a small range of frequencies (Δf). This transmission limited to a small frequency window is due to the presence of the defect modes, previously shown in Fig. 4.4d.

Even though the topological waveguide shows a superior transmission overall for the frequencies inside the band gap, we investigate if it has any *qualitative* differences from the traditional waveguide for the small range of frequencies (Δf in Fig. 4.4e). To answer this question, we perform a numerical experiment on the waveguides by sending a 50 ms long Gaussian packet at a frequency (757 Hz) lying inside the range of interest Δf . In Fig. 4.5, we compare the transient responses for topological and traditional waveguides. We observe that the topological waveguide in Fig. 4.5a shows that the wave packet does not back scatter at the multiple sharp bends, and it smoothly travels along the path. However, the traditional waveguide in Fig. 4.5b indicates the presence of scattering around the bends. Note that the wave packet still propagates along the path and would lead to a significant transmission as shown in Fig. 4.4e. Nonetheless, it is qualitatively different from the topological waveguide in terms of guiding a wave packet by allowing back scattering around bends. The difference of wave transmission efficiency between the traditional and topological waveguides is further investigated systematically for bends of various angles in Appendix C.4.

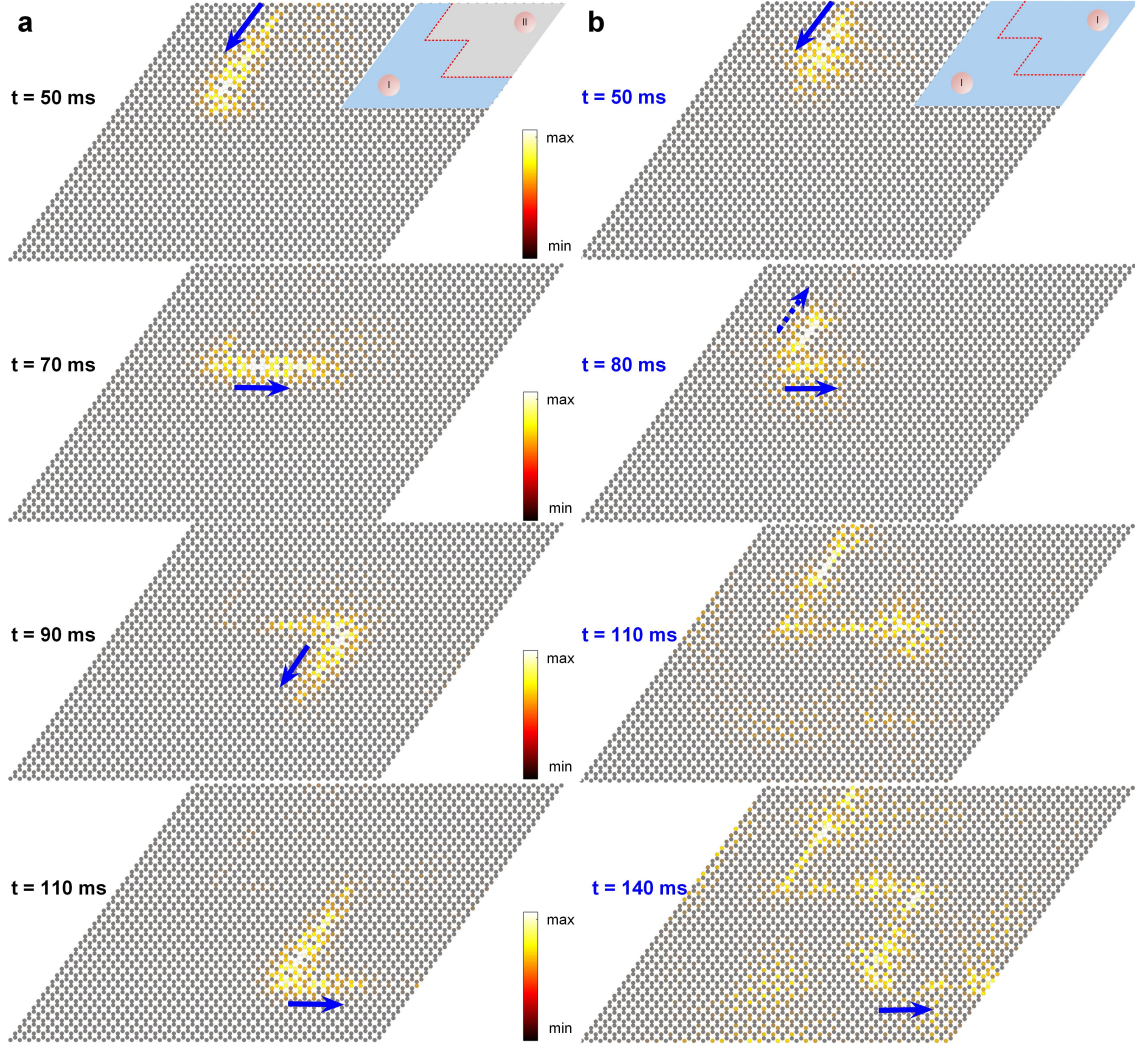


Figure 4.5: Wave packet scattering. (a) A topological waveguide showing back scattering suppressed wave propagation. (b) The same for a traditional waveguide. Though the wave packet reaches the exit, its propagation is not scattering free. Colors denote the rotation of the bottom disks.

4.3.4 One-way waveguide

Lastly, we demonstrate that a topological waveguide constructed through our system can support one-way wave propagation if a valley-selective excitation is given at the source. To this end, we first extract the amplitude and phase information of the topologically protected mode from the supercell analysis done earlier (Figs. 4.3a-b). In Figs. 4.6a-b, we calculate the amplitude ratio (i.e., A_1/A_2) and phase difference ($\varphi_1 - \varphi_2$) of the bottom disks' rotations between the Y-and P-states at the topological interface (see insets), and plot them as a function of reduced wavevector (k_x) in the band-gap range. We observe that the trend can be categorized into two *valleys*. One corresponds to the forward propagation (data points indicated with the subscript F in Figs. 4.6a-b), and the other to the backward propagation (subscript B). These can be verified by looking at the slope (i.e., group velocity) of the protected mode (red color) in Fig. 4.3b. In this way, appropriate amplitude ratio and phase difference can be applied (i.e., valley selection at either $\mathbf{K}$ or $\mathbf{K}'$) to the disks at the interface to excite either forward or backward propagating waves. This is confirmed by a transient analysis performed at 778 Hz in Fig. 4.6c. If the lattice is excited in the middle point (marked with a star) with the amplitude of $A_1/A_2 = 1.748$ and the phase difference of $\varphi_1 - \varphi_2 = 0.38\pi$, we observe a forward propagating wave (left panel in Fig. 4.6c). However, if the excitation with the same amplitude but opposite phase (i.e., $\varphi_1 - \varphi_2 = -0.38\pi$) is applied, we observe a backward propagating wave (right panel). These numerical results attest that the bistable SP-based metamaterial system proposed in this study allows not only the *in situ* manipulation of wave paths via dial-in actions, but also a selective one-way propagation of mechanical waves by strategic excitations.

4.4 Conclusions

Here, we propose a dial-in topological metamaterial system based on the bistable Stewart Platform (SP) and report robust one-way propagation of mechanical waves in it along tailorable wave paths. By arranging the bistable SP cells hexagonally in an alternating fashion,

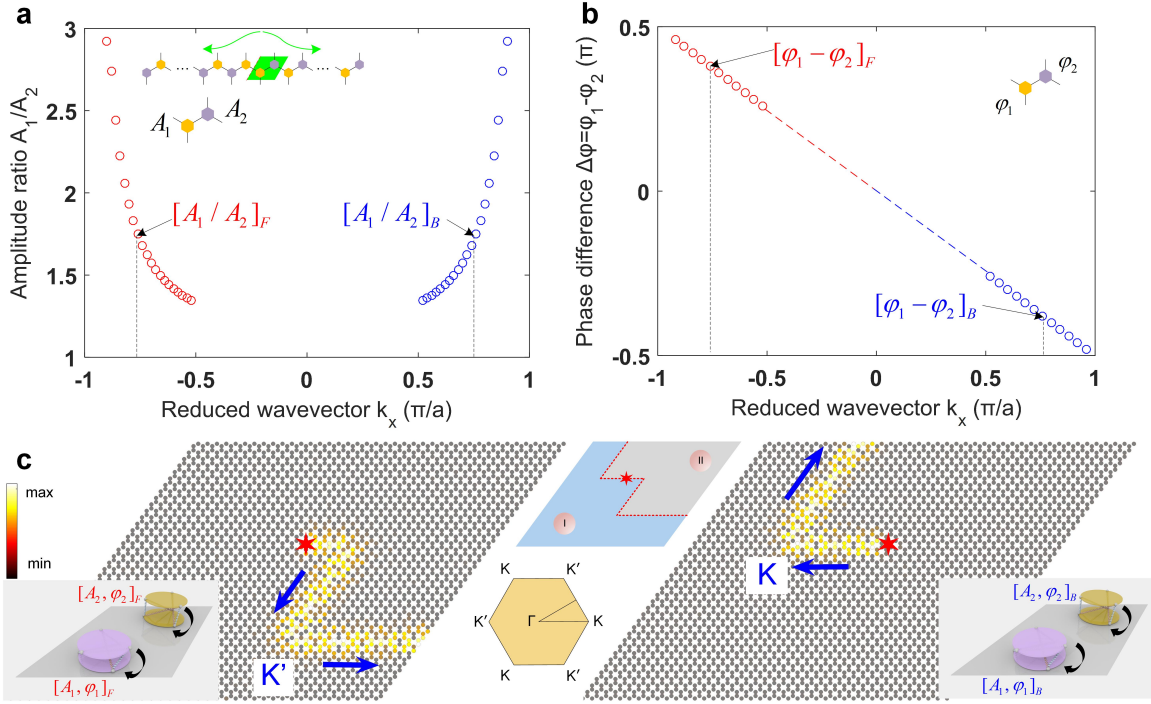


Figure 4.6: Valley-selective one-way wave propagation. (a) The extracted amplitude ratio (A_1/A_2) of Y- and P-states in the unit cell (green box in inset) from the supercell analysis. The data points on the left side (red dots) correspond to the solution with positive group velocity, i.e, forward propagation, while the right ones (blue dots) are for backward propagating wave. These are the solutions for two valleys. (b) Phase difference ($\varphi_1 - \varphi_2$) extracted for the same states. (c) Numerical simulation results that show one-way wave propagation along the topological waveguide. The red star indicates the place where the valley-selective excitation is given (see insets) to induce either forward- or backward-propagating waves. Colors denote the rotation of the bottom disks.

we can create two types of topologically distinctive lattices, which can be transformed to each other simply by a dial-in action. We prove that this transformation changes the topology of the system—quantified by the valley Chern numbers. When lattices with these two topologically distinct configurations are placed adjacently, we show the existence of a topologically protected mode at the interface. This idea is extended to tune the system *in situ* to create variety of waveguides, and we demonstrate a robust energy propagation along them using numerical simulations. We conduct eigenmode analysis and transient simulations of finite structures to highlight some key differences between topological and traditional waveguides in the same system. While the traditional waveguides also lead to significant wave transmission due to interfacial local modes inside the band gap, the edge modes generated in the topological waveguides are qualitatively different in that they support a wider range of frequencies and are immune to back scattering at sharp bends in the structure. We also show the strategy of giving valley-selective excitation to the system such that one-way wave propagation is achieved along the waveguide. Therefore, this tunable system opens up possibilities to realize various complex shapes of topological waveguides without resorting to external fields or adding/removing the masses from the system. Further studies including the experimental verification of the proposed tunable metamaterials would be reported in authors' future publications.

4.5 Methods

Numerical experiment: We employ the Runge-Kutta method (step size = 10^{-4} s) to get the response at any time instant for a variety of input signals: harmonic, Gaussian pulse, and sinusoidal frequency sweep.

Calculation of the transmission spectrum: We perform a numerical experiment and calculate the transmission spectra. A sweep frequency signal (20 to 1000 Hz in 5 s) is used as a rotational perturbation applied to the bottom disk at the input location. The transient rotation of the bottom disk is measured at the output location. Thus, the transmission $T(\omega) = 20 \log[\Phi_{output}^b(\omega)/\Phi_{input}^b(\omega)]$, where $\Phi(\omega)$ represents the power spectral density (PSD)

of transient rotation $\phi(t)$ of the disk.

4.6 Author contributions

This chapter is based on the following manuscript: Y. Wu, R. Chaunsali, H. Yasuda, K. Yu, J. Yang, “Dial-in topological metamaterial based on bistable Stewart platform,” *Scientific Reports* 8, 112, 2018 [127]. Y. Wu, J. Yang, and K. Yu conceived the original idea. Y. Wu performed the numerical simulations. H. Yasuda calculated the bistable characteristic of SP. Y. Wu and R. Chaunsali analyzed the results and wrote the manuscript. J. Yang and K. Yu supervised the project.

Chapter 5

TWO-DIMENSIONAL PLATE TOPOLOGICAL METAMATERIAL: NUMERICAL SIMULATIONS

Inspired by the quantum spin Hall effect shown by topological insulators, we propose a plate structure that can be used to demonstrate the pseudospin Hall effect for flexural waves. The system consists of a thin plate with periodically arranged resonators mounted on its top surface. We extend a technique based on the plane wave expansion method to identify a double Dirac cone emerging due to the zone-folding in frequency band structures. This particular design allows us to move the double Dirac cone to a lower frequency than the resonating frequency of local resonators. We then manipulate the pattern of local resonators to open subwavelength Bragg band gaps that are topologically distinct. Building on this method, we verify numerically that a waveguide at an interface between two topologically distinct resonating plate structures can be used for guiding low-frequency, spin-dependent one-way flexural waves along a desired path with bends.

5.1 Introduction

A topological insulator has emerged as a new state of matter in condensed matter physics. This is a special type of insulator that conducts electricity only on its boundary. Here topology is relevant because one can predict the boundary properties of these *finite* materials (i.e., finite-sized lattices) solely by knowing the bulk properties of *infinite* materials (i.e., infinitely large lattices). A topological framework provides an elegant way to categorize the bulk properties in terms of a topological invariant, and thus, one expects a *topological protection* and a degree of *robustness* for the boundary properties [29, 30].

Recently, that this whole framework dealing with the flow of electrons has evolved further

and influenced other areas, such as photonics [117] and acoustics [40–45, 119, 128]. It has also propelled a new design paradigm for artificial mechanical structures, so-called topological mechanical metamaterials, to achieve unconventional static (zero-frequency) [53–56, 129] and dynamic (nonzero-frequency) [47–52, 57, 58, 79, 85, 86, 88, 130–134] responses. In particular, tailoring nonzero-frequency responses, i.e., elastic waves in structures, on topological grounds shows tremendous potential to be used for energy-harvesting, sensing, and impact mitigation purposes [37].

One of the most unique topological effects is the quantum spin Hall effect, the underlying phenomenon shown by topological insulators [33, 35]. These systems are passive in the sense that they do not require any external field, but they still possess *directional* boundary states. This is due to the presence of Kramers partners, i.e., two opposite spins of an electron, which travel in the opposite directions on their boundaries, thereby keeping the time reversal symmetry intact. Although mechanical counterparts, being bosonic systems, do not possess these intrinsic spins, one can carefully design the system to have two pseudospins by imposing certain symmetries in the lattice, and thus realize the pseudospin Hall effect [50–52, 132, 133].

While previous studies have successfully reported the feasibility of the pseudospin Hall effect in mechanical settings, in this study, we focus on the feasibility of the same in less-explored continuum structures such as plates. One of the approaches that has recently been applied in plate structures is a so-called zone-folding technique [135], in which one considers a larger unit cell rather than an irreducible one in a hexagonal lattice arrangement, so that the frequency band structure folds onto itself, creating a *double* Dirac cone at the Γ point. Based on the same, Brendel *et al.* [132] and Yu *et al.* [133] showed that purely geometric manipulation of holes can invoke topological effects in plates. However, these topological effects have been restricted to high-frequency wave modes. Therefore, in this research, we ask the following question: How can one invoke the pseudospin Hall effect at *low frequencies* for a given plate dimension? It is important because of several reasons: (i) the low-frequency plate modes, such as flexural modes, carry a large amount of energy, and manipulating them could lead to relevant engineering applications; and (ii) these lower modes generally

require bigger lattice patterns of holes on conventional plates due to the Bragg condition, and thus, an improved way of controlling the low-frequency wave modes can relax the current stringent size limitations. Therefore, it would be a significant advancement to the current research trend if one could demonstrate a low-frequency pseudospin Hall effect in a continuum mechanical structure such as plates, which are ubiquitous in many engineering disciplines.

To address the aforementioned challenges, we propose a topological plate system that consists of a thin plate with periodically arranged local resonators mounted on its top surface. This locally resonant (LR) plate is reminiscent of sonic crystals [7]. Pal *et al.* [57] proposed such a structure for realizing the elastic analog of the quantum valley Hall effect. Building on a similar methodology, in this research, we employ a technique based on a combination of the classical plate theory and the plane wave expansion (PWE) method [136–138] that enables fast and efficient calculation of the wave dispersion relation. Furthermore, we integrate into this scheme the zone-folding technique to create a double Dirac cone for flexural wave modes. As a result, we report that the double Dirac cone can be formed in low-frequency regimes by tuning the resonating frequency of the resonators. In this way, we can demonstrate a subwavelength characteristic of the proposed plate system, i.e., the lattice size of the LR plate being smaller than the wavelengths in the bare plate at operating frequencies. We then show that a purely geometric manipulation of the local resonator pattern results in the opening of topologically trivial and non-trivial subwavelength Bragg band gaps around the double Dirac cone. Building on these findings, we verify numerically—by using the finite element method (FEM)—that a waveguide created at the interface of topologically distinct LR plates can guide low-frequency flexural waves along a designed path. Moreover, it shows a unique spin-dependent one-way propagation characteristic. Unlike the traditional plate-based waveguides studied in the past [74, 139–148], we show that this LR topological plate system has the potential to guide one-way flexural waves along a path with multiple bends—generally challenging in topologically trivial waveguides.

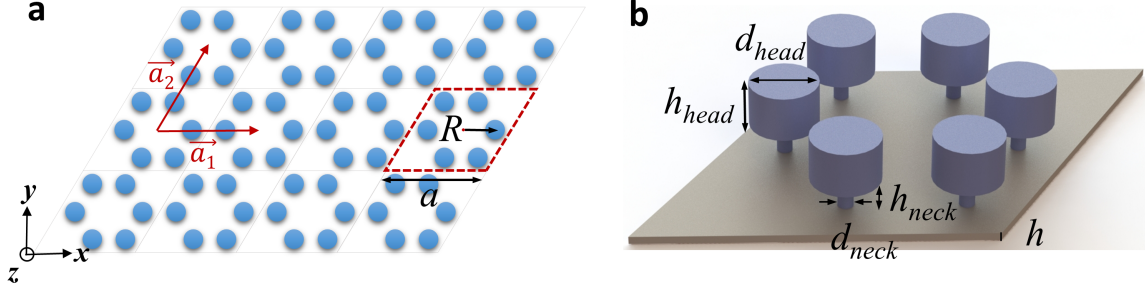


Figure 5.1: (a) Lattice arrangement of the resonators with a rhombus shaped unit cell in red. Dimensional parameters and lattice vectors are also shown. (b) A detailed view of the unit cell with six resonators mounted on top of a thin plate.

5.2 Description of the locally resonant topological plate

Our system consists of a thin plate on which multiple local resonators are attached to form a lattice arrangement (Fig. 5.1). The rhombus-shaped unit cell is of length a and consists of six resonators in a hexagonal arrangement (Fig. 5.1a). Each resonator is at a distance R and rotationally symmetric from the center of the unit cell, showing the C_6 symmetry. $\vec{a}_1$ and $\vec{a}_2$ are the lattice vectors. We model the resonators as cylindrical heads attached to the plate with a thin neck (Fig. 5.1b). In order to invoke the topological effects in the system, we will only vary radius R , keeping the C_6 symmetry intact in this unit cell.

As a substrate material, we choose an aluminum plate ($E = 77.6$ GPa, $\rho = 2,730$ kg/m³, $\nu = 0.352$) of thickness $h = 1$ mm with the unit cell of size $a = 45$ mm. The resonator neck is made of acrylic plastic ($E_{neck} = 3.2$ GPa) with $h_{neck} = 5$ mm and $d_{neck} = 2$ mm, whereas the resonator head is made of tungsten ($\rho_{head} = 19,260$ kg/m³) with $h_{head} = 14$ mm and $d_{head} = 9$ mm. The aforementioned material properties are based on nominal values of standard materials.

5.3 Results and discussion

5.3.1 Calculation of the unit cell dispersion

We first evaluate the dispersion characteristics of the unit-cell design with six resonators for variable R . For fast calculations, we rely on the PWE method. To this end, we simplify the resonator design with a lumped mass ($m = \pi\rho_{head}d_{head}^2h_{head}/4$) connected to the plate with a linear spring ($\beta = \pi E_{neck}d_{neck}^2/4h_{neck}$). In this process, we neglect the bending motion of the resonators and only consider their motion in the out-of-plane z direction. The bending (and other modes) of resonators, though taken into account in full-scale models in later sections, does not affect the topological phenomenon in our system as those frequencies can be separated from the Dirac point we will be investigating, and they have minimal coupling with out-of-plane wave modes (see Appendix D.1). We have $h \ll a$, therefore the plate can be assumed to be thin [136], and the transverse motion of the plate can be calculated as per the classical plate theory (i.e., Kirchhoff-Love theory) [149]. Following the approaches taken by Pal *et al.* [57], Xiao *et al.* [138], and Torrent *et al.* [144], the governing equations for the time-harmonic vibration of the unit cell with angular frequency ω can be written as

$$D\nabla^4 w(\mathbf{r}) - \omega^2 \rho h w(\mathbf{r}) = -\beta \sum_{\alpha} [w(\mathbf{R}_{\alpha}) - \tilde{w}(\mathbf{R}_{\alpha})] \delta(\mathbf{r} - \mathbf{R}_{\alpha}), \quad (5.1a)$$

$$-\omega^2 m \tilde{w}(\mathbf{R}_{\alpha}) = \beta [w(\mathbf{R}_{\alpha}) - \tilde{w}(\mathbf{R}_{\alpha})], \quad (5.1b)$$

where $D = Eh^3/12(1 - \nu^2)$ represents the flexural rigidity of the plate, $\mathbf{r} = (x, y)$ denotes the generalized coordinate of the plate, $w(\mathbf{r})$ represents the transverse displacement of the plate, and $\tilde{w}(\mathbf{R}_{\alpha})$ represents the displacement of the resonating masses attached at points $\mathbf{R}_{\alpha}$. We have $\alpha = 1, 2, \dots, 6$ for six different resonating masses per unit cell, and $\delta(\mathbf{r} - \mathbf{R}_{\alpha})$ is a delta function in two dimensions.

We introduce the following non-dimensional angular frequency:

$$\Omega = \omega a^2 \sqrt{\frac{\rho h}{D}}.$$

Also, the mass of the resonator can be normalized as

$$\gamma = \frac{m}{\rho A_c h},$$

where $A_c = \sqrt{3} a^2/2$ is the area of a unit cell. We can write the normalized resonance frequency of a resonator as $\Omega_r = a^2 \sqrt{(\beta/m)\rho h/D}$.

Employing the PWE method, we write the displacement of the plate for a Bloch wave vector $\mathbf{K}$ as a superposition of multiple plane waves such that

$$w(\mathbf{r}) = \sum_{\mathbf{G}} W(\mathbf{G}) e^{-i(\mathbf{K}+\mathbf{G})\cdot\mathbf{r}}, \quad (5.2)$$

where $W(\mathbf{G})$ is a plane wave coefficient and $\mathbf{G}$ denotes the reciprocal lattice vector given by $\mathbf{G} = p\mathbf{b}_1 + q\mathbf{b}_2$, in which p and q are integers, and $\mathbf{b}_1$ and $\mathbf{b}_2$ are the basis vectors of the reciprocal lattice. We truncate the summation with respect to $\mathbf{G}$ by choosing both p and q as $-M, -(M-1), \dots, 0, \dots, (M-1), M$. Therefore, the reciprocal space is a $N \times N$ finite grid with $N = 2M + 1$.

The displacement of the plate at the locations where the resonators are attached can be simply deduced from Eq. (5.2) as

$$w(\mathbf{R}_\alpha) = \sum_{\mathbf{G}} W(\mathbf{G}) e^{-i(\mathbf{K}+\mathbf{G})\cdot\mathbf{R}_\alpha}. \quad (5.3)$$

Substituting Eq. (5.2) and Eq. (5.3) into Eq. (5.1a)

$$\begin{aligned} D \sum_{\mathbf{G}'} |\mathbf{K} + \mathbf{G}'|^4 W(\mathbf{G}') e^{-i(\mathbf{K}+\mathbf{G}')\cdot\mathbf{r}} &= \omega^2 \rho h \sum_{\mathbf{G}'} W(\mathbf{G}') e^{-i(\mathbf{K}+\mathbf{G}')\cdot\mathbf{r}} \\ &= \beta \sum_{\alpha} \left[\tilde{w}(\mathbf{R}_\alpha) - \sum_{\mathbf{G}'} W(\mathbf{G}') e^{-i(\mathbf{K}+\mathbf{G}')\cdot\mathbf{R}_\alpha} \right] \\ &\times \delta(\mathbf{r} - \mathbf{R}_\alpha). \end{aligned} \quad (5.4)$$

Multiplying both sides with $e^{i(\mathbf{K}+\mathbf{G})\cdot\mathbf{r}}$, we obtain

$$\begin{aligned} \sum_{\mathbf{G}'} \left[D |\mathbf{K} + \mathbf{G}'|^4 - \omega^2 \rho h \right] W(\mathbf{G}') e^{-i(\mathbf{G}'-\mathbf{G})\cdot\mathbf{r}} \\ = \beta \sum_{\alpha} e^{i(\mathbf{K}+\mathbf{G})\cdot\mathbf{r}} \left[\tilde{w}(\mathbf{R}_\alpha) - \sum_{\mathbf{G}'} W(\mathbf{G}') e^{-i(\mathbf{K}+\mathbf{G}')\cdot\mathbf{R}_\alpha} \right] \delta(\mathbf{r} - \mathbf{R}_\alpha). \end{aligned} \quad (5.5)$$

Taking the area integral over the entire unit cell of area A_c leads to

$$\begin{aligned}
\sum_{\mathbf{G}'} \left[D|\mathbf{K} + \mathbf{G}'|^4 - \omega^2 \rho h \right] W(\mathbf{G}') \iint_{A_c} e^{-i(\mathbf{G}' - \mathbf{G}) \cdot \mathbf{r}} d\mathbf{r}^2 \\
= \beta \sum_{\alpha} \left[\tilde{w}(\mathbf{R}_{\alpha}) - \sum_{\mathbf{G}'} W(\mathbf{G}') e^{-i(\mathbf{K} + \mathbf{G}') \cdot \mathbf{R}_{\alpha}} \right] \\
\times \iint_{A_c} e^{i(\mathbf{K} + \mathbf{G}) \cdot \mathbf{r}} \delta(\mathbf{r} - \mathbf{R}_{\alpha}) d\mathbf{r}^2.
\end{aligned} \tag{5.6}$$

We now use the following relations

$$\iint_{A_c} e^{-i(\mathbf{G}' - \mathbf{G}) \cdot \mathbf{r}} d\mathbf{r}^2 = \begin{cases} A_c, & \text{if } \mathbf{G} = \mathbf{G}' \\ 0, & \text{otherwise} \end{cases} \tag{5.7a}$$

$$\iint_{A_c} f(\mathbf{r}) \delta(\mathbf{r} - \mathbf{R}_{\alpha}) d\mathbf{r}^2 = f(\mathbf{R}_{\alpha}) \tag{5.7b}$$

to obtain

$$\begin{aligned}
A_c \left[D|\mathbf{K} + \mathbf{G}|^4 - \omega^2 \rho h \right] W(\mathbf{G}) &= \beta \sum_{\alpha} \left[\tilde{w}(\mathbf{R}_{\alpha}) - \sum_{\mathbf{G}'} W(\mathbf{G}') e^{-i(\mathbf{K} + \mathbf{G}') \cdot \mathbf{R}_{\alpha}} \right] \\
&\times e^{i(\mathbf{K} + \mathbf{G}) \cdot \mathbf{R}_{\alpha}}.
\end{aligned} \tag{5.8}$$

Using the Bloch's theorem for the resonators, we write $\tilde{w}(\mathbf{R}_{\alpha}) = \tilde{w}(\mathbf{0}_{\alpha}) e^{-i\mathbf{K} \cdot \mathbf{R}_{\alpha}}$, where $\tilde{w}(\mathbf{0}_{\alpha})$ represents the Bloch displacement of the resonator (indexed with α) at the reference unit cell. Thus, we deduce

$$\left[a^4 |\mathbf{K} + \mathbf{G}|^4 - \Omega^2 \right] W(\mathbf{G}) = \gamma \Omega_r^2 \sum_{\alpha} e^{i\mathbf{G} \cdot \mathbf{R}_{\alpha}} \left[\tilde{w}(\mathbf{0}_{\alpha}) - \sum_{\mathbf{G}'} W(\mathbf{G}') e^{-i\mathbf{G}' \cdot \mathbf{R}_{\alpha}} \right]. \tag{5.9}$$

Similarly, we simplify the second governing Eq. (5.1b) (for $\alpha = 1, 2, \dots, 6$) as

$$-\Omega^2 \tilde{w}(\mathbf{0}_{\alpha}) = \Omega_r^2 \left[\sum_{\mathbf{G}} W(\mathbf{G}) e^{-i\mathbf{G} \cdot \mathbf{R}_{\alpha}} - \tilde{w}(\mathbf{0}_{\alpha}) \right]. \tag{5.10}$$

Given the $N \times N$ size of the reciprocal space, we arrange Eq. (5.9) and Eq. (5.10) in the form of an eigenvalue problem to solve for Ω at a specific Bloch wave vector $\mathbf{K}$ and

obtain the dispersion relation. Note that we multiply Eq. (5.10) with γ to make the matrices Hermitian. Therefore, we have

$$\begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix} \begin{Bmatrix} W(\mathbf{G}) \\ \tilde{w}(\mathbf{0}_\alpha) \end{Bmatrix} = \Omega^2 \begin{bmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} \end{bmatrix} \begin{Bmatrix} W(\mathbf{G}) \\ \tilde{w}(\mathbf{0}_\alpha) \end{Bmatrix} \quad (5.11)$$

with

$$\begin{aligned} \mathbf{P}_{11} &= a^4 \begin{bmatrix} |\mathbf{K} + \mathbf{G}_1|^4 & 0 & \cdots & 0 \\ 0 & |\mathbf{K} + \mathbf{G}_2|^4 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & |\mathbf{K} + \mathbf{G}_{N^2}|^4 \end{bmatrix} \\ &+ \gamma \Omega_r^2 \exp \left\{ i \begin{bmatrix} \mathbf{G}_1 \\ \mathbf{G}_2 \\ \vdots \\ \mathbf{G}_{N^2} \end{bmatrix} \begin{bmatrix} \mathbf{R}_1 & \mathbf{R}_2 & \cdots & \mathbf{R}_6 \end{bmatrix} \right\} \exp \left\{ -i \begin{bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \vdots \\ \mathbf{R}_6 \end{bmatrix} \begin{bmatrix} \mathbf{G}_1 & \mathbf{G}_2 & \cdots & \mathbf{G}_{N^2} \end{bmatrix} \right\}, \\ \mathbf{P}_{12} &= \mathbf{P}_{21}^\dagger = -\gamma \Omega_r^2 \exp \left\{ i \begin{bmatrix} \mathbf{G}_1 \\ \mathbf{G}_2 \\ \vdots \\ \mathbf{G}_{N^2} \end{bmatrix} \begin{bmatrix} \mathbf{R}_1 & \mathbf{R}_2 & \cdots & \mathbf{R}_6 \end{bmatrix} \right\}, \quad \mathbf{P}_{22} = \gamma \Omega_r^2 \mathbf{I}_6, \\ \mathbf{Q}_{11} &= \mathbf{I}_{N^2}, \quad \mathbf{Q}_{12} = \mathbf{Q}_{21}^\dagger = \mathbf{0}_{\{N^2, 6\}}, \quad \mathbf{Q}_{22} = \gamma \mathbf{I}_6, \end{aligned}$$

where ‘exp’, $\dagger$, $\mathbf{I}$, and $\mathbf{0}$ represent the exponential function, the conjugate transformation, the identity matrix, and the null matrix, respectively. We choose $N = 7$ for further calculations.

5.3.2 Band folding and subwavelength unit design

By using the aforementioned technique, we calculate the dispersion relation for the hexagonal arrangement of resonators, i.e., $R = a/3$ (see Fig. 5.1a), and we plot it in Fig. 5.2a. Torrent *et al.* [144] showed the existence of a single Dirac cone in such a system. Building on this finding, in the current study, we create a *double* Dirac cone (two Dirac cone dispersion curves

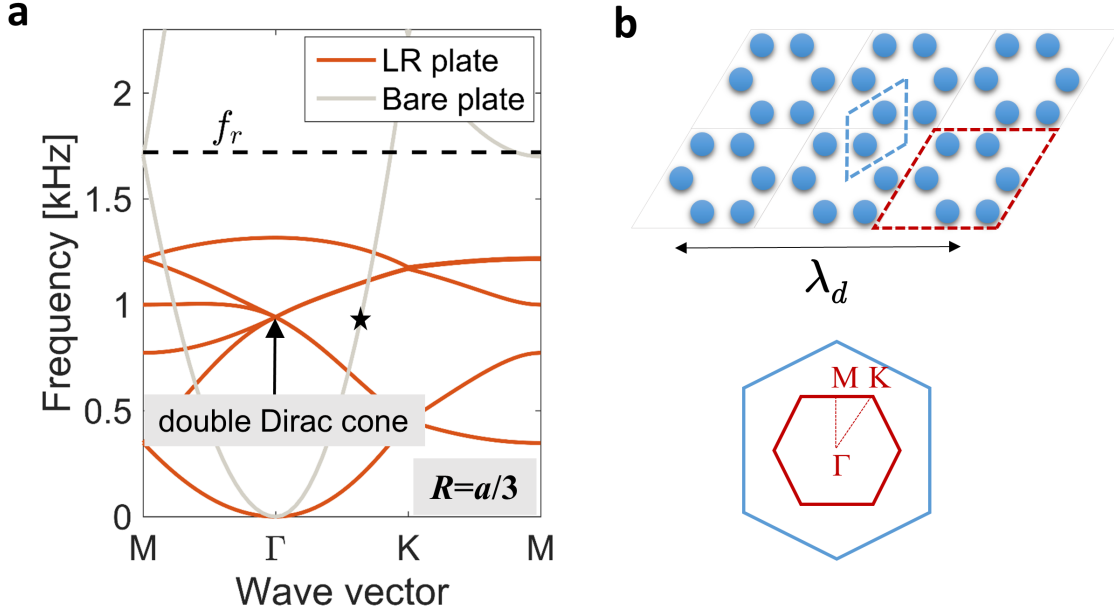


Figure 5.2: (a) Dispersion plot for the LR plate with a hexagonal arrangement, i.e., $R = a/3$ (red curves). It shows a double Dirac cone, which is lower than the resonating frequency f_r of the local resonators. Dispersion of the bare plate with no local resonators is shown in light gray for comparison. (b) Two unit-cell representations to enable the zone-folding (i.e., mathematical folding) of bands and the corresponding Brillouin zones below. The smaller cell (enclosed by blue dashed lines) represents the typical, irreducible unit-cell configuration for the hexagonal arrangement, while in this study, we consider the bigger unit cell (enclosed by red dashed lines) to create a double Dirac cone and for further topological manipulations. Their sizes are compared with the wavelength in the bare plate at the Dirac frequency (star mark), making it a subwavelength unit design.

superimposed) with the frequency $f_d = 0.94$ kHz at the Γ point. This is possible because we have chosen a bigger unit cell consisting of six resonators instead of two (compare the unit cells in Fig. 5.2b of different colors and corresponding Brillouin zones below). This results in the dispersion curves folded onto a smaller Brillouin zone [135]. Note that the physics is the same in both representations and it is simply a *mathematical* zone-folding of bands. However, achieving a double Dirac cone—which is a key ingredient of the spin Hall systems—guides us to realize topological effects by manipulating the geometrical configuration of the larger unit cell (to be further discussed in later sections).

In Fig. 5.2a, we also mark the resonating frequency f_r of the local resonator. It equals $f_r = (1/2\pi)\sqrt{\beta/m} = 1.72$ kHz. It is important to realize that $f_d \leq f_r$, as was thoroughly investigated by Torrent *et al.* [144]. Therefore, the resonator design can be used as a tuning knob to push the Dirac frequency further down in the dispersion relation. In the same figure, we also plot the dispersion relation for a bare plate (i.e., the identical plate to the substrate described in Section II, but without local resonators attached). This is to compare the wavelength (λ_d) of the flexural wave in the bare plate if excited at the Dirac frequency. This is indicated by the star marker on the dispersion curve. For the chosen set of design parameters, λ_d is approximately 2.3 times longer than the length of the large unit cell (i.e., a), and 4 times longer than the size of the small, irreducible unit cell ($a/\sqrt{3}$). Figure 5.2b shows the relative sizes of the unit cells compared to this wavelength, indicating subwavelength units of the LR plate. Therefore, as the topological effects will be seen around the Dirac frequency, this opens up new pathways to controlling large-wavelengths flexural waves by using a relatively small substrate. Again, by further reducing the resonant frequency, it is possible to shift the Dirac point to an even lower frequency regime, thereby making the plate design deep-subwavelength. However, practical challenges in designing such a system can limit the same, e.g., due to heavy resonating masses and soft neck structures.

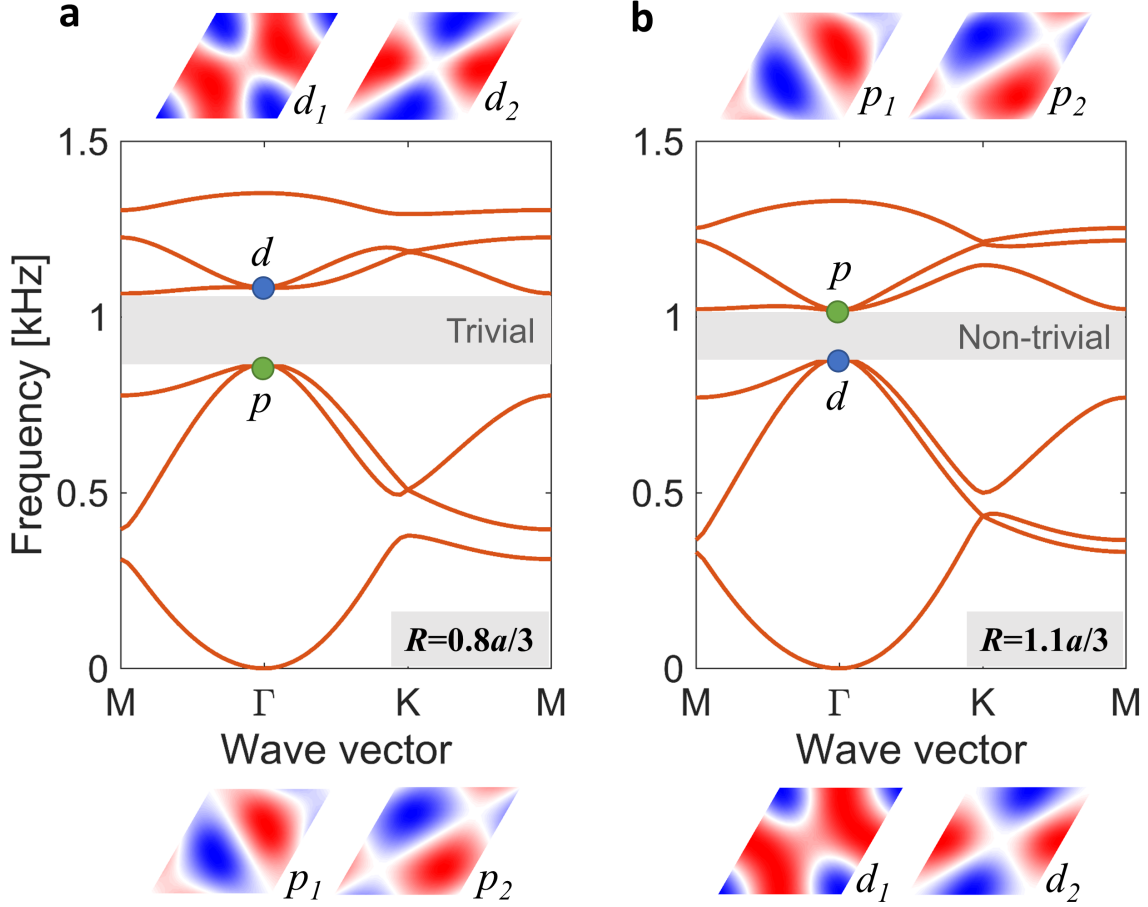


Figure 5.3: Emergence of subwavelength Bragg band gaps and their inversion when the unit cell of the LR plate is perturbed around the $R = a/3$ configuration. (a) A case with $R < a/3$ leading to a topologically trivial band gap with p -type modes having lesser frequency than d -type modes. See the insets for the corresponding mode shapes, with the colors indicating the out-of-plane displacements of the plate. (b) A case with $R > a/3$ leading to a topologically nontrivial band gap with p - and d -type modes inverted.

5.3.3 Band inversion and topology

We now vary the radius R and see its effects on the wave dispersion in the system. For $R < a/3$, as shown in Fig. 5.3a, there emerges a band gap near the Dirac frequency. We call it a subwavelength Bragg band gap because it lies in the subwavelength regimes as discussed above but emerges due to the change in translational periodicity of the resonators. By keeping the C_6 symmetry intact, two modes on each side (the lower or higher side) of the gap are degenerate at the Γ point. Seen in the insets are the corresponding degenerate mode shapes of the plate at the Γ point, which are obtained by the PWE method described in Section III. Here, the lower frequency modes are of p -type (p_1 and p_2 as shown in the bottom panel of Fig. 5.3a), and the higher frequency modes are d -type (d_1 and d_2 , upper panel in Fig. 5.3a) as per the analogy to electronic orbital shapes. For $R > a/3$, however, the band gap still exists, but its topological characteristic is different from the earlier case. As shown in Fig. 5.3b, the degenerate modes are flipped, i.e., d -type modes are at the lower frequency compared to p -type modes. This *band inversion* as we vary R around $R = a/3$ indicates a typical topological transition in the system. Again, the validity of this result based on the lumped mass model is verified and discussed in Appendix D.1 in comparison with the FEM (using COMSOL MULTIPHYSICS), which takes into account all geometrical features in the resonator design.

The presence of degenerate modes around the band gap has important implications in realizing the pseudospin Hall effect. One can take linear combinations of these modes and construct two alternate modes, i.e., pseudospin modes, without changing the physics of the system. Let $p_{\pm} = p_1 \pm ip_2$ and $d_{\pm} = d_1 \pm id_2$ represent such spin modes for these degenerate points. The sign in the middle determines if these are rotating clockwise or counterclockwise. We can interpret the dispersion near the Γ point in terms of the pair of spins by projecting the eigenstates onto the spin basis $\{p_{\pm}, d_{\pm}\}$. Therefore, the effective Hamiltonian of the system around the Γ point reduces to the one for the Cd/Te/HgTe/CdTe quantum well [35] and would resemble a mechanical pseudospin Hall system. One can show that the bands have

a nonzero spin Chern number for the case with $R > a/3$, hence, proving it to be topologically nontrivial [119, 128, 135].

5.3.4 Emergence of the topological interface state

Now that we verified the feasibility of the double Dirac cone formation and the band inversion in the unit-cell level, we move to the investigation of waveguiding characteristics in multi-cell configurations. To account for more complicated geometry and boundary conditions in such a multi-cell setting, we resort to the FEM henceforth. According to the bulk-boundary correspondence of topology [29], we expect distinct behaviors on the boundaries of topologically trivial and nontrivial lattices. One way to observe it clearly is to have topologically distinct lattices placed adjacently and investigate their connecting interface for a nontrivial local response. To this end, we take a supercell, which consists of both topologically trivial ($R = 0.8a/3$) and nontrivial ($R = 1.1a/3$) lattices, 10 units of each placed as one strip (Fig. 5.4a). The periodic boundary condition is introduced in the direction of another lattice vector (at 60° from the horizontal). In this way, such a system provides a quick way to calculate vibration responses at the interface and monitor their propagation along the periodic direction.

In Fig. 5.4b, we plot the eigenfrequencies of the supercell as a function of wave number in the periodic direction. The presence of two modes inside the band gap (shown in purple and yellow colors) is especially striking, since those have the following nontrivial properties. First, they represent two types of pseudospin modes localized at the interface: one rotates clockwise, while the other rotates counterclockwise. Second, both have opposite group velocities at a given frequency. Figure 5.4c shows the respective mode shapes corresponding to points S_1 and S_2 in Fig. 5.4b, which are excited at 0.91 kHz. Opposite spins and group velocities of these modes can be verified by looking at the harmonic evolution of these modes. We also plot in-plane time-averaged mechanical energy flux ($I_j = -\sigma_{ij}v_j$, where σ_{ij} and v_j are stress tensor and velocity vector, respectively) over a harmonic cycle as black arrows. This further confirms the spin nature of these flexural modes in the LR plate. There is a small

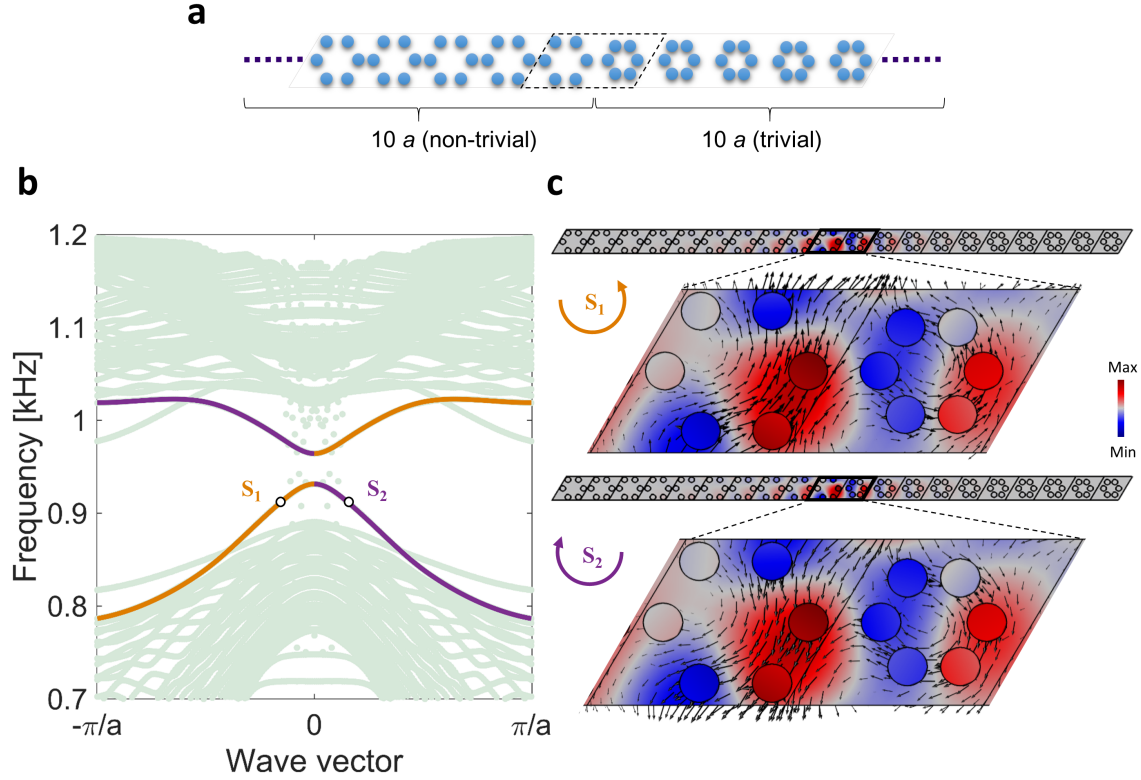


Figure 5.4: (a) A supercell made by placing topologically distinct lattices adjacently. (b) Eigenfrequencies of the supercell as a function of the wave number in the periodic direction by analyzing a full-scale design with the FEM. We highlight clockwise (purple) and counterclockwise (yellow) pseudospin modes localized at the interface between the trivial and nontrivial lattices. In the background are the other modes, i.e., bulk modes, and local modes that are at the extreme left and right end of the supercell. (c) pseudospin mode shapes corresponding to the points S_1 and S_2 in (b). The color intensity represents the out-of-plane displacement, and the arrows indicate the time-averaged mechanical energy flux, thereby confirming their spin nature.

frequency gap at the Γ point for these spin modes. The absence of topological interface modes indicates the absence of topological protection, and that suggests these pseudospin modes are not topologically protected in the full frequency band gap. This is because, in our system, the protection is guaranteed by the C_6 symmetry, which we break by introducing a sharp interface between topologically trivial and nontrivial lattices, and thereby resulting in an avoided crossing at the Γ point. Nevertheless, we will show in the next section that these modes can still be used to build robust and directional waveguides. The remedy to reduce the gap at the Γ point is to minimize the effect of the C_6 symmetry breaking at the interface. This can be done in several ways, including (1) by choosing the radii of trivial and nontrivial configurations as close as possible, or (2) by constructing a *graded* interface between two topologically distinct lattices (see Appendix D.2). This therefore leads to a “greater” degree of protection of the topological spin-modes.

5.3.5 Directional waveguides

In the previous section, we showed that the mechanical spin Hall effect enables us to have two pseudospins at one frequency but in opposite directions. In order to demonstrate how this property can be used to build unconventional waveguides on plates, we combine topologically trivial and nontrivial LR plates to form a 2D structure. Figure 5.5 shows the waveguide (three linear segments with two bends) along the interface of two types of lattices. We give a forced excitation in the z direction at the center of the plate (indicated by the star symbol) in such a way that we selectively excite spin modes. This could be done, for example, by choosing multiple points in the vicinity but with a phase difference in their forcing. Insets show the three excitation points in a nontrivial unit cell (i.e., with $R = 1.1a/3$) where the spin is predominantly d -type. We extract the phase information from the spin modes in the supercell analysis done earlier for 0.91 kHz (see the resonators in red and blue, representing out-of-phase oscillations in Fig. 5.4c). We apply this phased excitation as $F_1 = F \exp(i\omega t)$, $F_2 = F \exp[i(\omega t + 2\pi/3)]$, and $F_3 = F \exp[i(\omega t + \pi)]$. Note that the phase differences in the three excitation points are not equally spaced, but show $\pi/3$ and $2\pi/3$ differences between

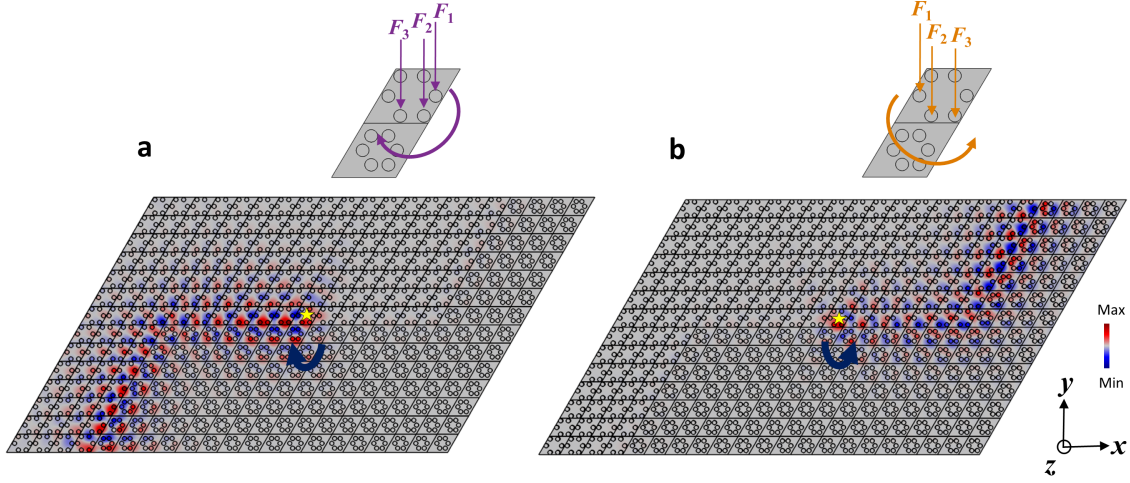


Figure 5.5: Pseudospin dependent one-way propagation of a flexural wave. The star indicates the zone of excitation. Multi-point phased excitation in force is used for exciting (a) clockwise and (b) counterclockwise spin modes. The color intensity represents the out-of-plane displacement. No backscattering around the bends is observed.

the neighboring ones (i.e., $\pi/3$ between F_2 and F_3 , and $2\pi/3$ between F_1 and F_2). This excitation tactic induces clockwise spin in Fig. 5.5a and counterclockwise spin in Fig. 5.5b. We enforce low-reflecting boundary conditions on the plate and perform harmonic analysis using the FEM.

We confirm the unique features of this topological waveguide. The clockwise spin mode propagates to the left (Fig. 5.5a) and the counterclockwise spin propagates only to the right (Fig. 5.5b). These spin waves propagate robustly along the waveguide interface in such a way that even though there are sharp bends, there is no backscattering and the spins remain intact. These simulation results imply that by using the pseudospin Hall effect induced in this LR plate structure, we can guide flexural waves in a selected path and direction without resorting to the breakage of time-reversal symmetry. That is, without using any active components, we can achieve directional control of low-frequency flexural waves simply by creating a topological boundary and exciting the host medium strategically via a phased

excitation. It is important to note that the robustness of these spin waves shown along a waveguide with sharp bends does not imply that these are also robust against any other types of “defects” along the waveguide, as shown in acoustics recently [150]. Though our elastic LR plate structure demands a thorough stand-alone study on this subject in future, we have explored exemplary cases here, in which certain defects along the waveguides can(not) back scatter these spin waves (Appendix D.3).

5.4 Conclusions

We have proposed a locally resonant plate structure to demonstrate the pseudospin hall effect for directional control of flexural waves. We show that the resonator design can be simplified with a lumped mass model and solved by employing the plane wave expansion method. This method enables us to efficiently investigate the key design parameters responsible for forming a double Dirac cone at a lower frequency than the resonating frequency of the local resonators. Keeping the C_6 symmetry intact, we perturb the unit cell and show an opening of subwavelength Bragg band gaps and the corresponding band-inversion process. This provides us with two topologically distinct lattice configurations. When these lattices are placed adjacently, we show the existence of two pseudospin modes traveling in the opposite directions along the interface. This unique feature is used to build topological waveguides with multiple bends and robustly guiding the spin-dependent flexural waves in a selected direction. The finding could be useful in designing compact and robust one-way channels for guiding low-frequency flexural waves in applications such as energy harvesting, sensing, and impact mitigation. Future studies include the optimization of the locally resonant plate configurations by using the proposed numerical techniques, as well as the experimental verification of the waveguiding effects, which will be reported by the authors’ future publications.

5.5 Author contributions

This chapter is based on the following manuscript: R. Chaunsali, C.-W. Chen, and J. Yang, “Subwavelength and directional control of flexural waves in zone-folding induced topological plates,” *Physical Review B* 97 (5), 054307, 2018 [122]. R. Chaunsali conceived the idea of project. C.-W. Chen performed the numerical simulations. R. Chaunsali calculated dispersion relations using PWE method. R. Chaunsali and C.-W. Chen analyzed the results and wrote the manuscript. J. Yang supervised the project.

Chapter 6

TWO-DIMENSIONAL PLATE TOPOLOGICAL METAMATERIAL: EXPERIMENTS

It is recent that the emergence of topological insulators in condensed matter physics has inspired analogous wave phenomena in mechanical systems. However, to date, the design of these mechanical systems has been limited mostly to discrete lattices or perforated structures. Here, we take a ubiquitous design of a bolted elastic plate and demonstrate that it can guide flexural waves crisply around sharp bends. We show that this continuum system eliminates unwanted in-plane plate modes and allows the manipulation of low-frequency flexural modes by exploiting the local resonance of the bolts. We report the existence of a pair of double Dirac cones near the resonant frequency of the bolts, one of which leads to the creation of a topological complete band gap that forbids all the plate modes. These finding open new possibilities of managing multiple wave modes in elastic solids for applications in energy harvesting, impact mitigation, and structural health monitoring.

6.1 Introduction

The discovery of topological insulators in condensed matter physics has prompted a new notion of topology in association with the intrinsic dispersion behavior of a material [29, 30]. By using this concept, one can characterize the dispersion behavior of an infinite (i.e., “bulk”) material, which consequently provides a tool to predict the response at the “boundaries” of a finite material. This “bulk-boundary correspondence” leads to a topologically protected boundary response of a non-trivial bulk, thereby offering a degree of robustness. At the physical level, the topological insulator has thus shown an exotic state, in which a robust and directional current flows along the material’s boundary, while it is forbidden in the bulk.

This tool of topology has paved a way for researchers to control the flow of energy in other areas, such as photonics [117] and acoustics [41, 43, 45, 151]. It has also given an impetus to a new way of designing elastic systems [47–50, 52, 79, 88, 127, 130, 131, 152, 153]. These topological structures—mostly in the setting of discrete lattices or perforated structures—aim at manipulating elastic vibrations and offer a tremendous degree of flexibility in controlling their dynamic responses. Therefore, these prototypical systems are excellent candidates for tabletop designs, in which topological physics can be systematically investigated [37]. However, one of the outstanding challenges in the topological manipulation of elastic systems is to have control over several types of wave modes that can exist in elastic solids, particularly in a low-frequency domain. This becomes even more relevant when one goes beyond the discrete lattice models and considers continuum structures such as plates. Though a judicious manipulation of plate modes has been shown in high frequencies [51, 58, 154], a simple design of low-frequency waveguiding remains yet challenging. Conventional designs thus far tend to focus selectively on certain plate modes, and therefore, are prone to energy leakage through other modes [57, 122, 144, 155].

Therefore, a natural question is whether there exists a way to manipulate flexural plate waves at low frequencies yet avoiding the wave leakage into other modes. This can be of significant importance, because low-frequency flexural modes typically carry high vibrational energy. Their efficient manipulation can be useful for several engineering applications, such as energy harvesting, vibration isolation, and structure health monitoring. However, lowering the operating frequencies generally demands large lattice sizes due to the Bragg condition. This can pose challenges especially under stringent size limitations of wave media. While recent numerical studies provide novel solutions to this issue [156, 157], the experimental realization of topological manipulation of low-frequency waves in elastic systems remains a formidable challenge to date.

Here we propose a ubiquitous design of bolted plates to tackle the aforementioned challenges. This strikingly simple design complements the topological physics in a continuum elastic plate with the mechanism of local resonance of bolts [7]. The design is rich in physics

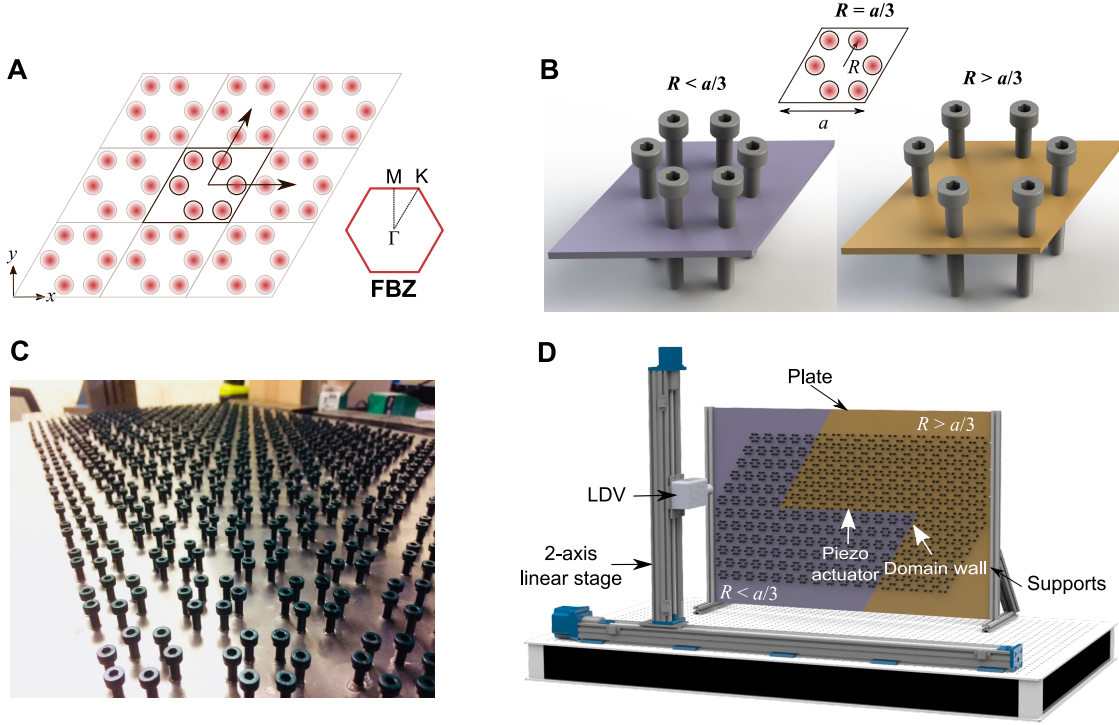


Figure 6.1: System configuration. (a) A hexagonal lattice arrangement with the unit cell containing six elements. The corresponding first Brillouin zone (FBZ) is indicated. (b) This unit cell is perturbed around $R = a/3$ to obtain two topologically distinct unit cells. Bolts act as local resonators mounted on a thin plate. (c) A snapshot of the actual system with multiple bolts mounted on the plate. (d) The experimental setup, in which a domain wall is created between two topologically distinct lattice patterns in yellow and violet. The plate is excited by a piezoelectric actuator positioned at the center, and out-of-plane velocity is measured point-by-point by an LDV mounted on a 2-axis linear stage.

as it allows not only the out-of-plane coupling between the mounted bolts and the plate, but also the interweaving of their bending with in-plane modes. This results in a simultaneous effect of guiding low-frequency flexural plate modes and removing unwanted in-plane modes. To achieve this, we employ the zone-folding technique [135] to invoke the C_6 symmetry-protected pseudo-spin Hall effect and manipulate flexural waves by strategically arranging the bolts on the plate. Remarkably, this bolted plate system forms a pair of *double* Dirac cones [158] in the low-frequency range of the bolt's bending resonance. We numerically investigate the opening of *multiple* topological band gaps by manipulating the bolt arrangement on the plate, one of which is notably a complete band gap that prevents the leakage of flexural waves into other modes. We experimentally verify efficient waveguiding capability of this bolted plate along a path with sharp bends, when excited at frequencies inside these topological band gaps.

6.2 The locally resonant topological plate

6.2.1 Design and fabrication

Our system consists of a thin plate and bolts that are tightly fastened to form a periodic pattern. As a starting point, we take a hexagonal lattice pattern shown in Fig. 6.1A. We focus on a large unit-cell representation, i.e., a rhombus-shaped unit cell (lattice constant $a = 45$ mm) that contains six bolts to facilitate zone-folding. We then perturb this unit cell by varying the circumferential radius R of the mounted bolts around $R = a/3$ (Fig. 6.1B). This is to obtain two topologically distinct unit-cell configurations of the bolted plate. Note that, in the process, we keep the C_6 symmetry intact in the unit cell, so that we obtain degenerate modes in the dispersion to be shown later. These two types of unit cells, specifically $R = 0.8a/3$ and $1.1a/3$ in this study, are then tessellated to form a periodic pattern in the plate (Fig. 6.1C). We construct a domain wall consisting of three linear segments (one horizontal and two inclined at 60°) by joining these two types of periodic patterns (Fig. 6.1D).

The substrate is an aluminum 6061-T6 plate (91.4×61.0 cm and thickness $t = 2$ mm).

We use CNC milling machine to drill 1320 holes, followed by manual tapping and fastening M4-0.7 mm black-oxide alloy steel bolts (91290A180, McMaster-Carr). To facilitate the firm contact between the bolts and the threads of the plate, we add a drop of glue (Henkel Loctite adhesive) in the process of mounting bolts using a screwdriver. The partial thread on the screws comes handy, as it allows tightening them firmly on the plate without using any additional nuts.

6.2.2 *Experimental measurements and processing*

To excite the system, we bond a piezoelectric ceramic disc (STEMiNC, diameter 10 mm, and thickness 1 mm) with silver epoxy adhesive on the plate as shown in Fig. 6.1D. We mount the Polytec OFV 534 laser Doppler vibrometer (LDV) on the 2-axis linear stage and control its motion using MATLAB to measure vibrations on the plate. The LDV takes point-by-point measurements by moving in a square grid of 7.5×7.5 mm. We measure 114 points horizontally and 60 points vertically, i.e., a total of 6840 points. All measurements are synchronized with respect to the onset of the input voltage signal from the function generator, which sends a signal to the piezoelectric actuator via an amplifier. To get the steady-state transmission profiles, we send a 100 ms wide frequency sweep signal from 2 to 40 kHz. Then, we employ fast Fourier transformation on the time-history obtained by each measurement and construct a 2D transmission profile for a frequency slice. To get the transient wavefields, we send a Gauss-modulated sine pulse with 80 cycles at a given center frequency. We apply a bandpass filter around the center frequencies to eliminate ground noise on the plate. Then, we use a cubic interpolation on the measured grid data for a better visualization of wavefield on the plate.

6.2.3 *Numerical modeling*

We use commercial finite element software (COMSOL Multiphysics) to perform numerical simulations. We take nominal material properties of aluminum plate (Young's modulus $E = 68.90$ GPa, density $\rho = 2,700$ kg/m³, Poisson ratio $\nu = 0.33$) and steel bolts (Young's

modulus $E = 210.60$ GPa, density $\rho = 7,800$ kg/m³, Poisson ratio $\nu = 0.30$). See Appendix E.1 for the geometric details of the bolt. The threaded contact between the bolt and plate is modeled using a simple approach, which captures the realistic contact stiffness by modeling an effecting contact area (see Appendix E.2). We mesh the model with tetrahedral elements. To obtain the dispersion diagrams, we take a unit-cell and apply Bloch-Floquet boundary conditions and solve for eigenfrequencies along the irreducible Brillouin zone.

6.3 Results and discussion

6.3.1 Unit-cell dispersion and band inversion

In Fig. 6.2, we numerically show dispersion diagrams for the unit-cell configurations with radii varying across $R = a/3$. The colormap indicates the polarization index defined as $P_z = \frac{\int_{V_u} |u_z|^2 dV}{\int_{V_u} (|u_x|^2 + |u_y|^2 + |u_z|^2) dV}$, where V_u is the plate volume in the unit cell, and u_x , u_y , and u_z are the displacement components in x , y , and z axis, respectively. P_z would be close to 1 for out-of-plane plate modes and near 0 for in-plane plate modes. For $R = 0.8a/3$, we observe two distinct band gaps in the range of 6.2 – 8.5 kHz in Fig. 6.2A. Clearly, the polarization index confirms that the band gaps are for out-of-plane modes (blue markers). The lower band gap is a partial band gap (PBG) because it supports nearly in-plane modes (S_0 and SH_0) shown with red and yellow markers. Notably, the upper one is a complete band gap (CBG). This is because the bending-dominated local resonance of the bolts occurs near 6.9 kHz (see the horizontal blue markers in Fig. 6.2A), and it is coupled with in-plane modes of the plate (further explanations to follow next). This results in a locally resonant band gap (see the two-sided arrow) for “undesired” in-plane modes (red and yellow markers) in the range of 6.9 – 9.4 kHz. Consequently, we observe a complete band gap between 6.9 kHz to 8.5 kHz for all plate modes.

This is an example of judicious engineering of overlapping two types of band gaps: locally resonant band gap for in-plane modes and Bragg band gap for out-of-plane modes in the same system. We call the later Bragg band gaps because those emerge solely when the

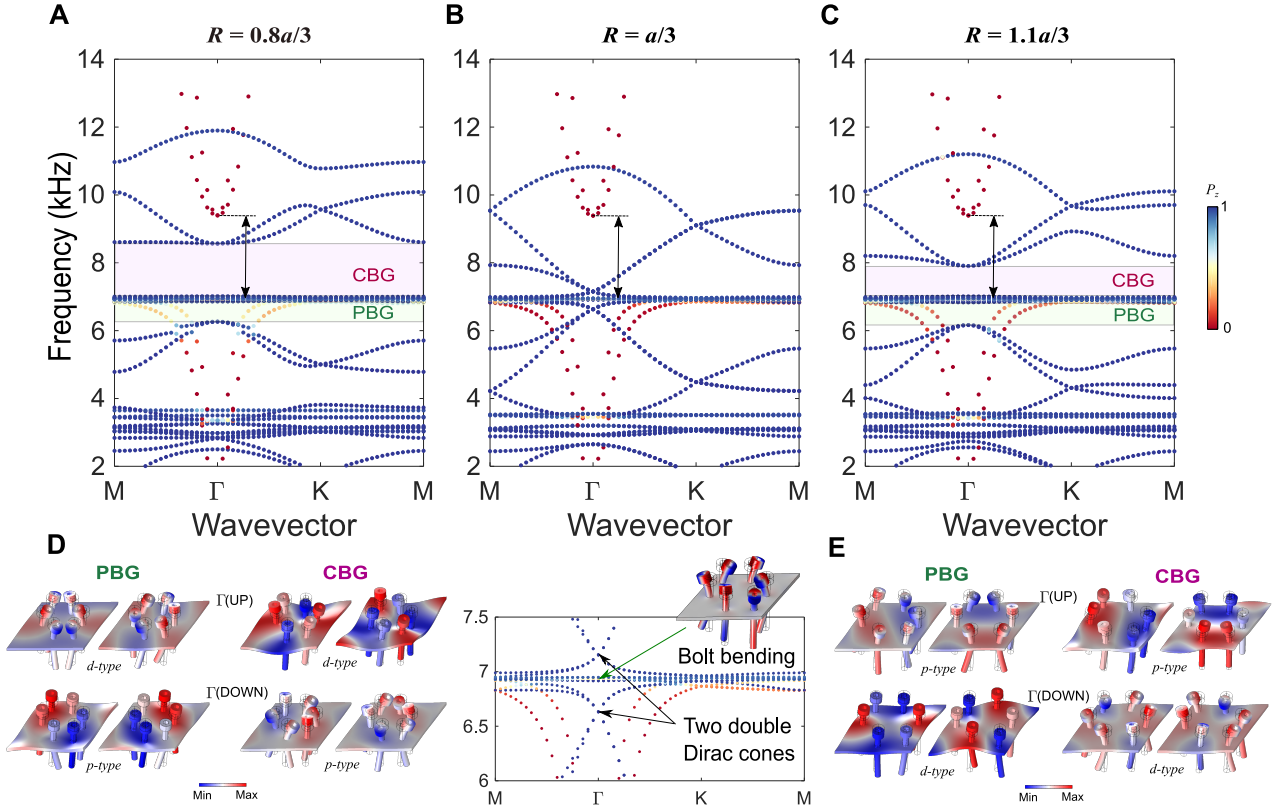


Figure 6.2: Presence of multiple band gaps and their inversion in the unit-cell dispersion. (a) A case with $R < a/3$ leading to the emergence of two band gaps. Color of the markers indicates the polarization of plate mode, i.e., it is nearly 1 for out-of-plane and nearly 0 for in-plane plate modes. The lower band gap is a partial band gap (PBG) due to the presence of other nearly in-plane plate modes. The upper band gap is a complete band gap (CBG) for all plate modes. The two-sided arrow indicates the locally resonant band gap for in-plane modes. (b) In the hexagonal lattice arrangement, i.e., $R = a/3$, these band gaps close and form two distinct double Dirac cones. Zoomed-in view is in the inset below to show the bolt bending mode between two double Dirac cones. (c) When $R > a/3$, the two double Dirac cones open again to form two distinct band gaps. (d) The mode shapes (out-of-plane) at the Γ point for the band gaps shown in (a). $\Gamma(\text{UP})$ and $\Gamma(\text{DOWN})$ indicate the upper and the lower edges of the band gap, respectively. Note that two degenerate p -type modes have smaller frequency than two degenerate d -type modes for both band gaps. (e) The mode shapes at the Γ point for the band gaps shown in (c). These mode shapes are inverted with respect to the ones shown in (d).

translational symmetry in the system is changed by varying the radius across $R = a/3$. This point can be further justified when we plot the dispersion diagram for the case of the perfectly hexagonal system, i.e., with $R = a/3$, in Fig. 6.2B. We observe that both band gaps (denoted by PBG and CBG above) close. However, the band gap for in-plane modes (red markers) remains intact, because it is a locally resonant band gap caused by the local bending of the bolts, which we do not change in the process (see the two-sided arrow). Remarkably, near the resonant frequency of the bolts, we observe the formation of a pair of distinct double Dirac cones (see the panel below Fig. 6.2B for the zoomed-in view of the dispersion curves along with the mode shape of the resonant bolts). To the best of the authors' knowledge, creation of such multiple double Dirac cones at different frequencies in an elastic system has not been reported in the literature.

In Fig. 6.2C, we plot the dispersion diagram for the case when we further increase the unit-cell radius to $R = 1.1a/3$. We choose this radius to avoid the bolts approaching too close to the unit cell boundary, and at the same time to ensure an overlap with the band gaps established in $R = 0.8a/3$ (compare the locations of band gaps between Figs. 6.2A and C). We observe that the two double Dirac cones open again and form two band gaps. However, these are different from the ones observed in Fig. 6.2A in terms of topology. For further investigation, we revisit Fig. 6.2A, examine the out-of-plane mode shapes, and plot them in Fig. 6.2D. For the lower band gap, i.e., PBG, we extract the mode shapes of its edges at the Γ point. We observe that two p -type modes are degenerate at the lower edge, i.e., $\Gamma(\text{DOWN})$, and two d -type modes are degenerate at the upper edge, i.e., $\Gamma(\text{UP})$. Naturally, the d -type modes show prominent bending of the bolts, approaching the vicinity of local resonance region around 6.9 kHz. A similar pattern is observed for the upper band gap (CBG) as well, wherein d -type modes are at the upper edge. Sandwiched between these two regions are the locally resonant bending modes of the bolts.

In Fig. 6.2E, we plot the mode shapes at the aforementioned band edges at the Γ point, but for the configuration with $R = 1.1a/3$ shown in Fig. 6.2C. A similar degeneracy of modes is observed. However, p -type modes are at higher frequencies than d -type modes

for both band gaps. This so-called band-inversion is a crucial ingredient of the topological phenomena at work and makes this configuration topologically non-trivial. This mechanism is reminiscent of the pseudo-spin Hall effect in photonics [135], where the degeneracy of p - and d -type modes is used for creating two counter-propagating pseudo-spin modes at the domain wall between two topologically distinct unit cells. Here we realize this effect for a flexural wave at a low frequency regime (i.e., in the vicinity of the bending mode of the bolts) in a continuum plate.

6.3.2 Emergence of topological interface state

Now moving to the multicell configuration, we first take a supercell consisting of eight unit cells of each type and join them to form a domain wall as shown in Fig. 6.3A. We apply a periodic boundary condition in parallel to the domain wall to investigate the wave dynamics of this tessellated system. We keep the other boundaries free. In Fig. 6.3B, we show the eigenfrequencies of the supercell as a function of wavevector $k_{//}$. The colormap indicates the localization index for out-of-plane wave modes. This is defined as $L_z = \frac{\int_{V_{s*}} |u_z|^2 dV}{\int_{V_s} |u_z|^2 dV}$, where V_s is the entire plate volume of the supercell and V_{s*} is the plate volume of two unit cells touching the domain wall (see the enclosed cells in Fig. 6.3A with a darker solid line). The high value of the localization index in the colormap clearly highlights the presence of localized modes at the domain wall (red markers) inside both band gaps (PBG and CBG). The modes in the upper region are desolate as expected in the complete band gap. However, the localized modes in the lower region are populated with other modes in this partial band gap.

To verify the polarization of these localized modes, we replot Fig. 6.3B in Fig. 6.3C but with a different colormap indicating the wave polarization, as was used in Fig. 6.2. We clearly observe that the localized modes at the domain wall are out-of-plane modes (blue markers). Again, the upper region shows clean out-of-plane modes. The additional branch in this band gap corresponds to the modes at the other ends of the supercell, and therefore is

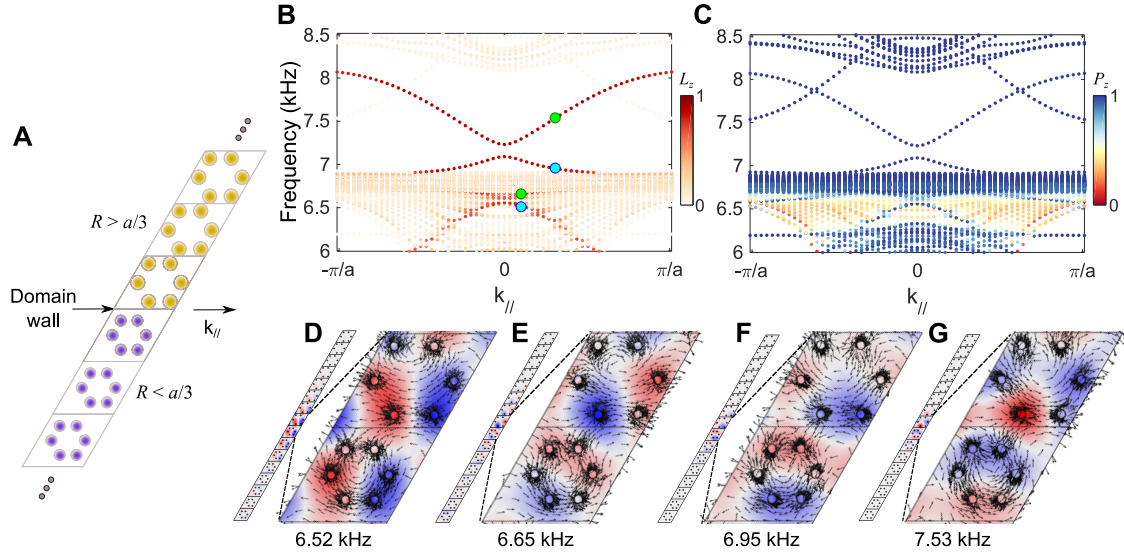


Figure 6.3: Emergence of topological modes at the domain wall. (a) A supercell made by placing two topologically distinct lattices adjacently at the domain wall. (b) Eigenfrequencies of the supercell vs. $k_{||}$, the wavevector parallel to the domain wall. The colormap indicates the localization index (L_z). Higher the value it has, the more localized are the out-of-plane modes at the domain wall. The upper region (i.e., CBG) shows clear localized modes. The lower region (PBG) also shows localized modes, but those are mixed with other extended modes. (c) The same eigenfrequencies diagram, but with a different colormap to show the polarization of plate modes. It denotes 1 (0) for out-of-plane (in-plane) plate modes. (d)-(g), Localized out-of-plane mode shapes at some selected frequencies inside PBG and CBG [marked by circles in (b)]. The arrows indicate the time-averaged mechanical energy flux to confirm their spin nature.

not of any important considerations here. The lower region, however, shows weaker out-of-plane localized modes, and those are intermixed with other in-plane modes (red and yellow markers). This fact leads us to understand the difference of topological wave localization between PBG and CBG. The advantage of having the CBG is evident as it facilitates a clean wave localization at the domain wall and eliminates the possibility of wave leakage into other modes. Note in passing that there is a small gap at the $k_{//} = 0$ point in both regions, where there is no localization at the domain wall. As highlighted in the previous studies [122,135], this is due the breakage of crystalline C_6 symmetry at the domain wall. By employing engineering tricks, such as a slowly-varying (i.e., gradient) domain wall, one can also reduce this small gap at the $k_{//} = 0$ point and impart greater protection to the topological modes.

We proceed to investigate the mode shapes of the aforementioned localized modes. We plot out-of-plane displacements at selected frequencies marked as circles in Fig. 6.3B. Figures 6.3D-E correspond to the frequencies inside PBG, and Figs. 6.3F-G correspond to the frequencies inside CBG, all arranged in the order of increasing frequencies. It is clear that these modes are localized at the domain wall. Insets show zoomed-in views of the mode shapes on which the superimposed arrows indicate in-plane time-averaged mechanical energy flux ($I_j = -\sigma_{ij}v_j$, where σ_{ij} and v_j are stress tensor and velocity vector, respectively) over a harmonic cycle. It is evident that these modes have spin characteristics. More specifically, in Figs. 6.3D,F that correspond to the cyan markers in Fig. 6.3B, we observe clockwise spins. In Figs. 6.3E,G (corresponding to the green markers in Fig. 6.3B), we witness counterclockwise spins (see the lattices with smaller R for clearer visualization). As the time-reversal symmetry is intact in our system, it is obvious that we will have exactly the opposite spins at these frequencies for a negative wavevector $k_{//}$. Therefore, judging from the group velocity, i.e., the slope of the dispersion branch at the green circles in Fig. 6.3B, we expect that the counterclockwise spin propagates along the domain wall towards positive wavevector, while the clockwise spin propagates in the opposite direction. We will look for these spins in the experimental results later.

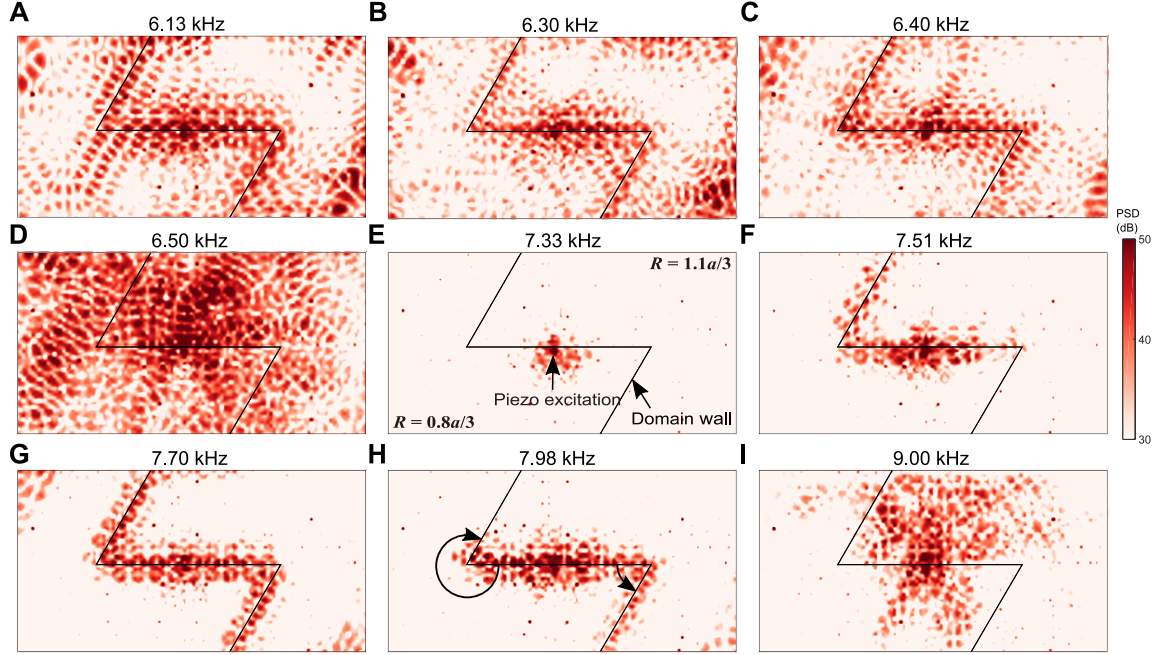


Figure 6.4: Experimentally obtained steady-state transmission maps at multiple excitation frequencies of the piezoelectric actuator located at the domain wall. Colors indicate the power spectral density (PSD) calculated from the measured out-of-plane velocity of the plate. (a)-(c) Wave localization at the domain wall for the frequencies inside PBG. (d)-(e) Disappearance of this localization indicates the dispersion region between PBG and CBG, where no localized topological mode exists at the domain wall. (f)-(h) Emergence of wave localization at the domain wall once again indicates the region of CBG. (i) Wave transmission at a frequency above CBG.

6.3.3 Steady-state transmission

Building from this numerical result, we experimentally verify the existence of topological modes at the domain wall for both PBG and CBG regions. In Fig. 6.4, we show the evolution of steady-state transmission plots on the plate, as we inject vibrational energy through a piezoelectric actuator attached at the center of the domain wall and increase its input frequency via a sweep signal. At low excitation frequencies (Figs. 6.4A-C), we observe that energy is primarily localized at the domain wall. This waveguiding pattern detected in this low frequency range corresponds to the PBG (see Figs. 6.3B-C). We observe some traces of elastic energy far away from the domain wall. This can be attributed to the nature of this PBG as it allows other wave modes to co-exist in the system. In addition, the smaller size of this band gap would generally mean higher localization length of the modes at the domain wall, and hence, the flexural wave would penetrate more into the bulk. Beyond this PBG, the wave localization is lost, marking the presence of global modes (extended in plate as shown in Fig. 6.3D). At 7.33 kHz (Fig. 6.3E), the wave is highly attenuated around the point of excitation.

Remarkably, for the frequencies above this point, we again observe wave localization along the domain wall (Figs. 6.4F-H). This corresponds to the localization region in the CBG as indicated in Figs. 6.3B-C. We clearly observe a persistent wave localization along the horizontal domain wall for a wide range of frequencies. Among these, the transmission at 7.70 kHz (Fig. 6.4G) is robust, in which the wave follows the entire domain wall without any significant back scattering around the sharp bends.

It is worth pointing out that at 7.51 and 7.98 kHz (Figs. 6.4F and H), the wave back scatters around the sharp bends. A similar behavior was also observed recently in experiments on a mechanical counterpart of quantum valley Hall effect [159], where guided topological waves are back scattered around a sharp bend for some frequencies inside the topological band gap. However, here we observe that the wave is able to pass through the left bend at a lower frequency (Fig. 6.4F) and through the right bend at a higher frequency (Fig. 6.4H).

This implies the preference of the topological wave propagation around one bend over the other, depending on the excitation frequency in the CBG. We explain this mechanism by the asymmetry of the two bends experienced by the spin waves. That is, as marked by the circular arrows in Fig. 6.4H, the bending angle experienced by the counterclockwise spin that propagates towards the right is different from that by the clockwise spin towards the left. Note that in this symmetry-protected topological system, we have intentionally broken C_6 symmetry at the domain wall, including the sharp bends. This makes the modes less “protected” (i.e., less robust), thereby resulting in the spin-dependent transmission efficiency around the bends.

6.3.4 Robust waveguiding

Now we demonstrate robust waveguiding capabilities of our system by capturing the transient wavefield when excited at the frequencies inside PBG and CBG. Navigating the steady-state profiles observed in the previous section, we choose two central frequencies of excitation, 6.3 kHz (in PBG) and 7.7 kHz (in CBG), and send Gaussian pulses from the piezo-actuator. In Figs. 6.5A-C, we show transient wave propagation on the plate at three different time steps inside PBG. We observe that the wave, emanating from the point of excitation at the middle of the domain wall, is effectively guided through the domain wall. We also verify (via plotting these figures as a movie) that a counterclockwise spin propagates towards the right and a clockwise spin propagates towards the left. Note Fig. 6.5C that there is some visible leakage into the bulk, complying well with the result from the steady-state response (Fig. 6.4B). In Figs. 6.5D-F, we show transient wave propagation inside CBG. It is evident that this wave is also being guided along the domain wall without any significant back scattering at the sharp bends. However, the key difference is that there is no evident leakage into the bulk due to the presence of CBG (Fig. 6.5F). This result is consistent with the supercell analysis (as shown in Fig. 6.3B) and the steady state response (Fig. 6.4G).

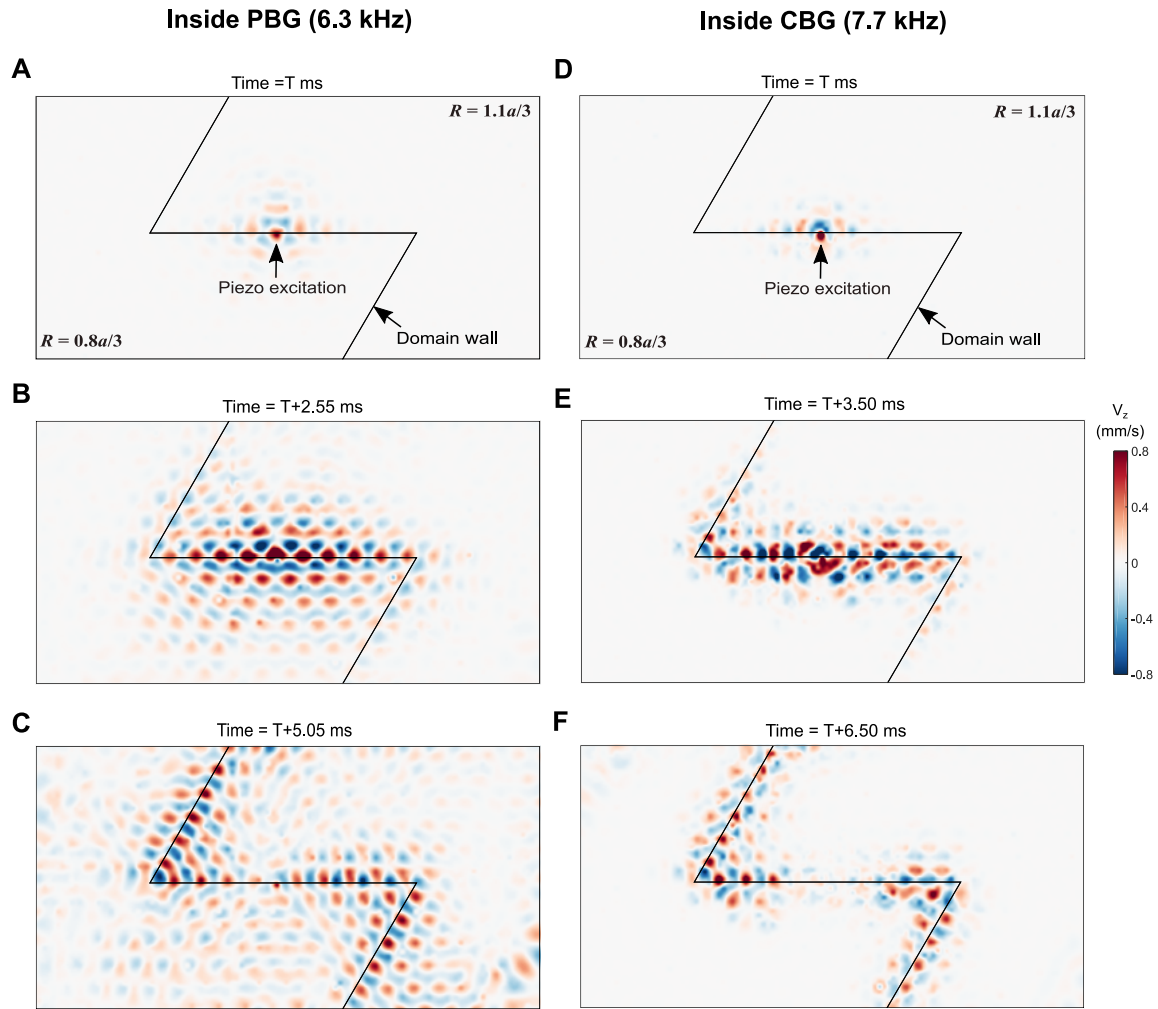


Figure 6.5: Experimentally obtained transient wavefields. (a)-(c) Velocity wavefields when a Gaussian input centered at 6.3 kHz is injected by the piezoelectric actuator in the middle of the domain wall. (d)-(f) Wavefields for the Gaussian input centered at 7.7 kHz.

6.4 *Conclusions*

In this study we have demonstrated numerically and experimentally the capability of topological waveguiding in a plate with local resonators. By incorporating the local resonance effects in the design, we observe some novel phenomena in this study. First, low-frequency bending mode of the bolt leads to the creation of a pair of double Dirac cones, thus invoking topological effects at low-frequencies. Second, it enables us to deal with undesired in-plane plate modes, which makes it possible to clearly guide flexural waves along a topological waveguide, eliminating their leakage to in-plane modes. This is highly desirable in designing efficient waveguides for low-frequency flexural waves. In addition, we have also observed spin-dependent transmission efficiency of our topological waveguide around sharp corners for certain frequencies inside topological band gaps. This is an interesting feature that can be exploited in applications, but at the same time, demands further exploration.

This simple and tunable design paves the way for a number of studies in future, especially with regard to studying the effect of disorder in such symmetry-protected topological systems. For example, the height of the bolts can be changed, or a nut can be attached to it, in order to introduce disorder in the system (see Appendix E.3 for a few examples). It is remarkable that even though a degree of disorder exists in the current setup in terms of bolt torques and plate threads, which in turn can change the effective coupling, we are still able to guide stress waves robustly. This is a hallmark of topological mechanical systems.

6.5 *Author contributions*

This chapter is based on the following preprint: R. Chaunsali, C.-W. Chen, and J. Yang, “Experimental demonstration of topological waveguiding in elastic plates with local resonators, arXiv:1806.00655” [160]. C.-W. Chen conceived the design and performed the numerical simulations. R. Chaunsali and C.-W. Chen fabricated the sample. R. Chaunsali performed experiments and interpreted results. R. Chaunsali and C.-W. Chen wrote the manuscript. J. Yang supervised the project.

Chapter 7

CONCLUSIONS AND OUTLOOK

Through a number of studies on 1D and 2D systems in this work, we showed how a new notion of topology comes into the design process of mechanical metamaterials. Topological characterization of dispersion in elastic systems allows us to treat structures in a different light. Using a combination of analytical, numerical, and experimental techniques we arrive at the following conclusions:

- We learned that structures having similar dispersion can still be qualitatively different on topological grounds. This has interesting repercussions when such lattices are joined together to create a boundary. We observed that such boundaries support localized vibration modes in 1D (chapter 2) and localized traveling wave in 2D (chapter 4, chapter 5, and chapter 6).
- Localization of vibration energy at the boundaries is robust against small perturbation (chapter 2). This is interesting because engineering uncertainties, which can generally alter vibration characteristics, would less likely influence topological modes, and hence, this fact might lead to better designs for containing elastic energy. Also, countering the traditional problems of backscattering at corners when a wave is bent, the topological waveguides in 2D (chapter 4, chapter 5, and chapter 6) do not show significant backscattering around the bends.
- Topological waveguides in 2D (chapter 4, chapter 5, and chapter 6) have unique spin characteristics such that counter-rotating spins propagate in opposite directions robustly. Therefore, this suggests new strategies to force the wave to flow in a specific directional line if the corresponding spin is excited selectively.

- We experimentally demonstrated how different elastic modes of a plate system can be managed to create efficient topological waveguides (chapter 6). This relies on creating a complete band gap, a combination of a Bragg type band gap and a locally resonant band gap, for all the elastic modes.
- We show that geometric manipulation of structures could be used to tune topological waveguiding capability. In particular, we have used bistable lattices to demonstrate that waves can be guided in different paths in the same structure by tuning it in situ (chapter 4).
- Apart from energy localization at boundaries of 1D and 2D structures, we have shown that topology can also dictate nontrivial bulk transport in 1D time-dependent nonreciprocal lattices (chapter 3). Here, the asymmetry in band dispersion could be explained by the topological indices in 2D, i.e., the Chern number.

These findings thus convince us that topological framework in mechanical structures could be further extended to exploring fundamental physics as well as designing novel application-oriented systems. For example, here we list the avenues that could be of interest to the research community:

- A natural extension of this research would be to explore 3D metamaterials with topological properties. One of the aims could be to design a structure that guides elastic waves only on its outer surfaces. If it is conceptually feasible, it would be challenging to fabricate and test these complex systems. However, using advanced additive manufacturing and high-precision measurement techniques, one can potentially demonstrate nontrivial wave dynamics.
- Most of the topological manipulations have been on rectilinear objects, e.g., plates in this study. As most of the thin wall structures in use are curved, it would be important

to study how any physical curvature of a plate affects the propagation of topological guided modes.

- Generally, topological vibration modes are robust to small perturbations in the system. However, an open question is to know how robust these are. Mechanical structures could be excellent platforms for this investigation. Their geometrical features can be systematically tailored to introduce perturbations (or disorder), which is not convenient in condensed matter electronic systems.
- Topological framework discussed here is limited to linear wave dynamics. A fundamental question is to know how this framework can be integrated with any nonlinearity in a system.
- It would be relevant to investigate if surface acoustic waves, such as Rayleigh waves, could also be robustly manipulated using the topological framework. If yes, it might open up new possibilities in practical scenarios ranging from manipulation of large-scale seismic waves to small-scale SAW devices.
- In chapter 4, chapter 5, and chapter 6, we showed that the presence of pseudospins results in nontrivial waveguiding in structures. Generally, one does not expect such pseudospins in conventional mechanical structures that do not allow any real spin degree of freedom. Therefore, it would be interesting to explore the possibility of using these beyond waveguiding applications discussed in this work.

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Appendix A

ONE-DIMENSIONAL TOPOLOGICAL METAMATERIAL

A.1 *Hertz contact law*

As defined in the main text, the contact force between two similar cylinders follows the Hertz contact law such that $F = \beta(\alpha)[\delta + \Delta u]^{3/2}$. The stiffness coefficient $\beta(\alpha)$ has the following dependence on α [76, 77]:

$$\beta(\alpha) = \frac{2Y}{3(1 - \nu^2)} \sqrt{\frac{R}{\sin \alpha}} \left[\frac{2K_e}{\pi} \right]^{-3/2} \left\{ \frac{4}{\pi e^2} \sqrt{\left[\left(\frac{a}{b} \right)^2 E_e - K_e \right] [K_e - E_e]} \right\}^{1/2}. \quad (\text{A.1})$$

Here, Young's modulus, Poisson's ratio and radius of each cylinder are denoted by symbols Y , μ and R , respectively. The contact area between the cylinders is assumed to be elliptical with eccentricity $e = \sqrt{1 - (b/a)^2}$, where a and b are semi-major and semi-minor axes, respectively. Correspondingly, K_e and E_e are the complete elliptical integrals of the first and second kind, respectively. The ratio b/a can be approximated to $[(1 + \cos \alpha)/(1 - \cos \alpha)]^{-2/3}$ [76].

A.2 *Effect of gravity*

In this study, we neglect the effect of gravity. The rationale behind this approximation is as follows. Given the total number of cylinders (27) and the weight of the quartz cylinders (10.0 g), the additional amount of the pre-compression experienced at the bottom of the chain is about 2.7 N. This corresponds to 11% of the static pre-compression used in this study ($F_0 = 25$ N). When the Hertzian contact law is linearized under the static pre-compression, we

see that contact stiffness $K \propto F_0^{1/3}$. This means that the contact stiffness would have a maximum deviation of approximately 3.6%. Now the band edge frequencies (f) are related to the stiffness as $f \propto K^{1/2}$, so frequencies would have 1.8% deviation. The average frequency near that band gap is about 12 kHz. Therefore, the self-weight will cause an error in the order of 0.21 kHz. This will be even smaller in the middle of the chain where topologically protected mode is centered. We see that an error of this order is indeed reflected in our experimental results for the topologically protected mode frequencies, which are based on measurements from all cylinder locations. While neglecting the effect of gravity is reasonable in the settings investigated in this study, it is necessary to account for this feature when the particle weight becomes comparable to the amount of the applied precompression.

A.3 Eigenvalue analysis

To get the modal frequencies of a finite chain, we solve the eigenvalue problem described in the main text. Let α_i denote the contact angle between i th and $(i+1)$ th cylinders. Therefore, for an N -particle granular system, with generalized contact angles $(\alpha_1, \alpha_2, \dots, \alpha_{N-1})$, we have the tridiagonal matrix $\mathbf{\Lambda}$ of the form:

$$\begin{pmatrix} K_a + K(\alpha_1) & -K(\alpha_1) & \dots & 0 \\ -K(\alpha_1) & K(\alpha_1) + K(\alpha_2) & \dots & 0 \\ \dots & \dots & \dots & -K(\alpha_{N-1}) \\ 0 & 0 & -K(\alpha_{N-1}) & K(\alpha_{N-1}) + K_w \end{pmatrix}$$

where K_a and K_w are the stiffness values at the front (actuator side) and the end of the chain. Here, cylinder indices start from the bottom of the chain to the top.

A.4 Differences with the SSH model

An infinitely long dimer chain can be represented by a unit cell consisting of two cylinders and having periodic boundary condition (Fig. A.1). This would be used to arrive at the governing eigenvalue problem, and we would compare the matrices involved with the standard

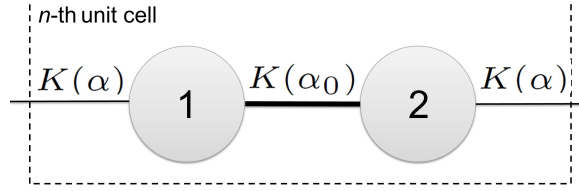


Figure A.1: A representative unit cell for a periodic dimer chain

Su-Schrieffer-Heeger (SSH) formulation. Let $K(\alpha_0)$ and $K(\alpha)$ be the two linear contact stiffnesses inside the n th unit cell made of cylinders of mass m ; and $\hat{u}_{r,s}$ represents the dynamic displacement of the r th cylinder in the s th unit cell. Therefore, the equations of motion can be written as

$$m\ddot{\hat{u}}_{1,n} = K(\alpha)[\hat{u}_{2,n-1} - \hat{u}_{1,n}] - K(\alpha_0)[\hat{u}_{1,n} - \hat{u}_{2,n}], \quad (\text{A.2})$$

$$m\ddot{\hat{u}}_{2,n} = K(\alpha_0)[\hat{u}_{1,n} - \hat{u}_{2,n}] - K(\alpha)[\hat{u}_{2,n} - \hat{u}_{1,n+1}]. \quad (\text{A.3})$$

Taking into account the periodicity, we now invoke the Bloch wave solutions of the following form:

$$\begin{bmatrix} \hat{u}_{1,n} \\ \hat{u}_{2,n} \end{bmatrix} = e^{j2\kappa} \begin{bmatrix} \hat{u}_{1,n-1} \\ \hat{u}_{2,n-1} \end{bmatrix} \quad (\text{A.4})$$

with κ being the wave-number defined in terms of the number of cylinders. The factor of 2 comes from the periodicity of dimer, which is of 2 cylinders. Considering the steady-state solution of the type $\hat{u} = [\hat{u}_{1,n}, \hat{u}_{2,n}] = \hat{U}e^{j\omega t}$, the equations of motion can be reduced to the following eigenvalue problem:

$$\mathbf{B}(\kappa)\hat{U}(\kappa) = \omega^2 m \hat{U}(\kappa). \quad (\text{A.5})$$

Here, ω is the angular frequency, $\hat{U}(\kappa)$ is the Bloch eigenmode, and the stiffness matrix reads

$$\mathbf{B}(\kappa) = \begin{bmatrix} K(\alpha_0) + K(\alpha) & -K(\alpha_0) - K(\alpha)e^{-j2\kappa} \\ -K(\alpha_0) - K(\alpha)e^{j2\kappa} & K(\alpha_0) + K(\alpha) \end{bmatrix}.$$

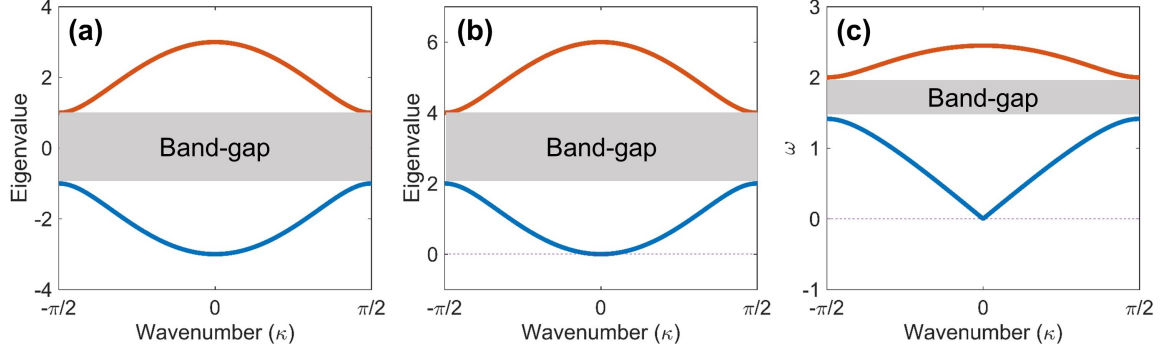


Figure A.2: Representative eigenvalues of the matrix $\mathbf{B}_g(\kappa)$ in Eq. (A.6). (a) SSH system with $t = 1$, $t' = 2$, and $\mu = 0$. (b) Granular chain system with $t = 1$, $t' = 2$, and $\mu = 3$; (c) Square-root of eigenvalues in (b).

This matrix is used for calculating the Bloch dispersion curve of the system, and therefore governs the nature of the band gap in the system. In order to compare this matrix form with the standard SSH Hamiltonian, we take the following generalized form

$$\mathbf{B}_g(\kappa) = \begin{bmatrix} \mu & -t - t'e^{-j2\kappa} \\ -t - t'e^{j2\kappa} & \mu \end{bmatrix}. \quad (\text{A.6})$$

In the SSH model, the on-site potential μ vanishes from the above matrix [66,161]. However, in the granular crystal system, μ is nonzero and equals the sum of coupling coefficients t and t' . This has an important implication on the nature of the resulting dispersion curves. As opposed to the dispersion curves that are symmetric with respect to the zero-eigenvalue in the SSH system [see Fig. A.2(a)], the cylindrical granular chain presented in this work exhibits the dispersion curves that have an offset [Fig. A.2(b)]. Consequently, all eigenvalues are positive, and the band gap is formed around a nonzero eigenvalue. By taking the square-root of these eigenvalues, we obtain the eigenfrequencies of the system. The results are shown in Fig. A.2(c), qualitatively describing the nature of dispersion curves in the current study.

A.5 Zak phase calculation

Continuing from the eigenvalue problem in the current system [Eq. (A.5)], we find the normalized eigenmodes $\hat{U}(\kappa)$ at each κ and then proceed towards calculating the Zak phase. The Zak phase for a band is the amount of phase the corresponding eigenmode $\hat{U}(\kappa)$ accumulates over the entire first Brillouin zone, i.e., $\kappa = [-\pi/2, \pi/2]$ in this case. It is given by the following relation [40, 67, 68]

$$Z = i \int_{-\pi/2}^{\pi/2} \hat{U}^\dagger(\kappa) \partial_\kappa \hat{U}(\kappa) d\kappa \quad (\text{A.7})$$

where $(\dagger)$ denotes the conjugate transpose. We then use numerical integration and calculate the Zak phases from the eigenmodes for acoustic and optical branches in the dispersion plot, separately. Note that the Zak phase in the current system is gauge dependent. This means that if we represent the same dimer chain by choosing a slightly different periodic unit cell [exchanging $K(\alpha_0)$ and $K(\alpha)$ in Fig. A.1], the Zak phases will change. Therefore, there is no absolute Zak phase for a dimer configuration. However, the *difference* in Zak phases of two different dimer configurations studied is, in fact, gauge invariant [68], and hence, the change of π explains the topological band transition at $\alpha = \alpha_0$.

A.6 Experimental results on topological band transition

We include the detailed experimental results here for the first part of the study on *in situ* topological transition. These have been taken for a dimer configuration with $\alpha_0 = 20^\circ$ and varying α from 5° to 40° in steps of 5° . By measuring the time history of velocity at each cylinder location, we generate the spectrum plots by performing the Fast Fourier Transformation (FFT). Figure A.3 shows the spectrum plots for all the configurations tested. One can see the boundary mode inside the band gap for all the configurations with $\alpha < \alpha_0$ (see the highlighted horizontal line around 12 kHz in the top three panels in Fig. A.3). When $\alpha = \alpha_0$ (i.e., the monomer case), the upper (optical) and the lower (acoustic) bands merge into a single band, and thus, the boundary mode vanishes (see the middle panel in Fig. A.3).

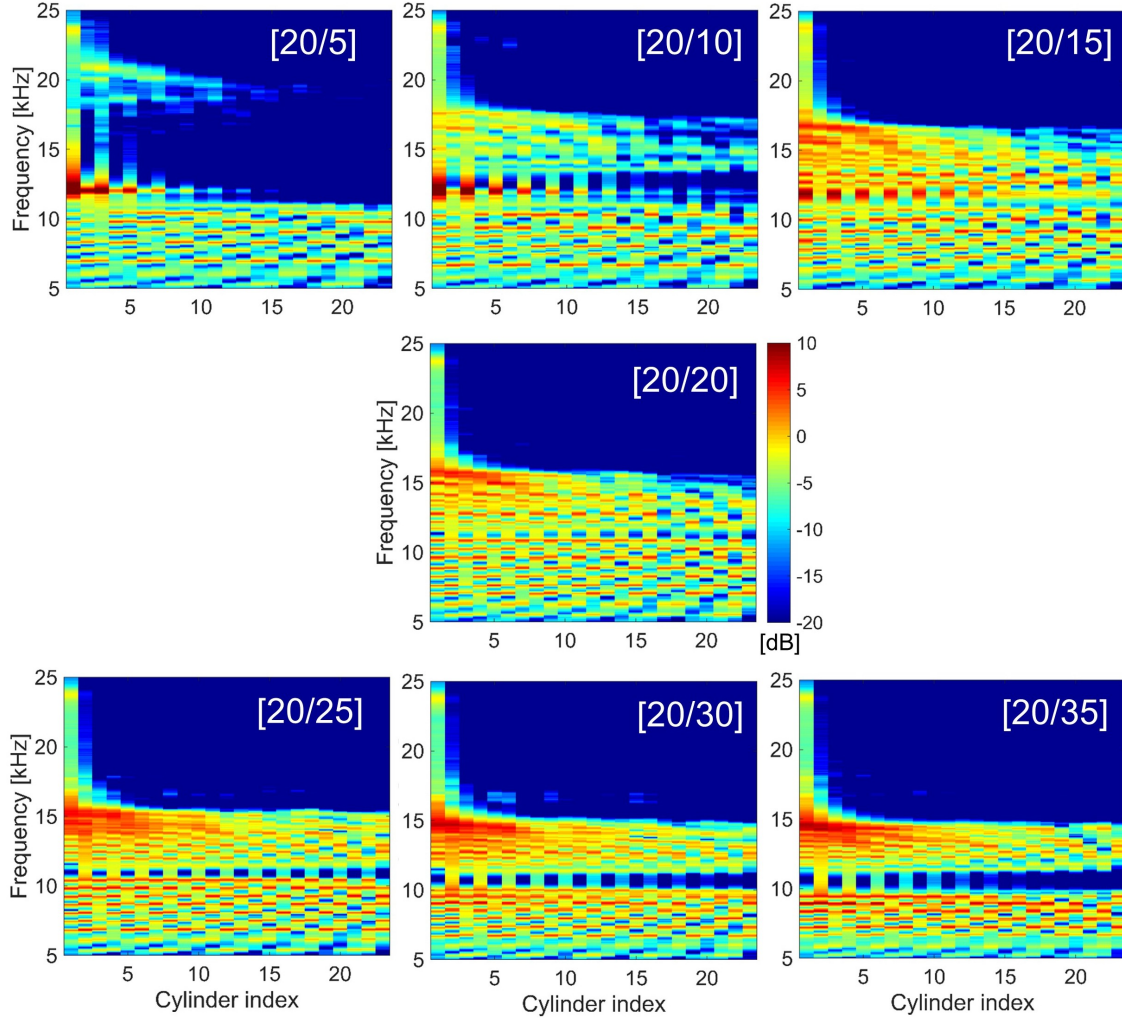


Figure A.3: Spectrum plots for different dimer configurations represented by the label $[\alpha_0/\alpha]$. The top three panels represent the cases that $\alpha < \alpha_0$. The middle panel shows the case when $\alpha = \alpha_0$. The bottom three panels depict the cases that $\alpha > \alpha_0$. It is evident that a boundary mode emerges inside the band gap for configurations with $\alpha < \alpha_0$.

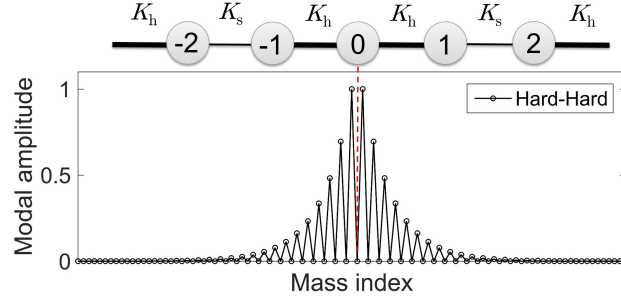


Figure A.4: Hard-hard topological defect, and corresponding eigenmode (absolute value)

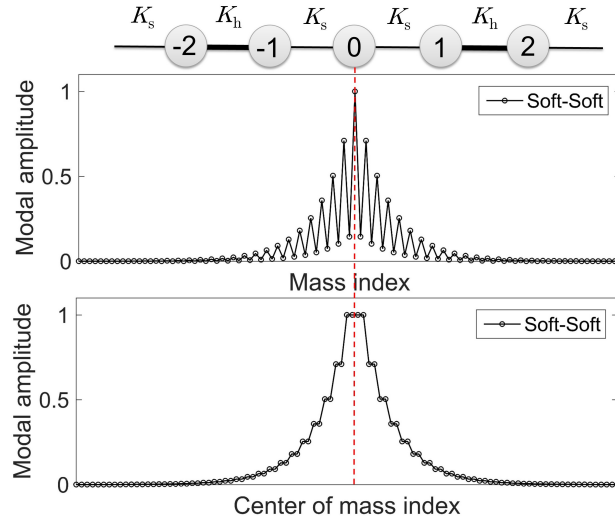


Figure A.5: Soft-soft topological defect, and corresponding eigenmode (absolute value) and center-of-mass motion (absolute value)

If we further increase α to the level of $\alpha > \alpha_0$ (bottom three panels in Fig. A.3), the band gap reopens. However, there exists no boundary modes in the band gap.

A.7 Frequency calculations of topologically protected modes

The frequency calculation of the topologically protected mode at the hard-hard interface of the dimer configuration is straightforward. Observing the corresponding mode shape

(absolute value) in Fig. A.4, it is clear that alternate mass particles do not move, and therefore the rest of the mass particles oscillate around their equilibrium positions through a spring of net stiffness $K_s + K_h$. This corresponds to a frequency $f_h = \frac{1}{2\pi} \sqrt{\frac{K_h + K_s}{m}}$.

On the other hand, for the topologically protected mode of the soft-soft interface of dimer, we observe in Fig. A.5 that it supports a special symmetry condition. In particular, it is symmetric about the topological interface, and hence the dynamic displacements of particles at any symmetric i th locations can be written as

$$u_{-i} = u_i \quad (\text{A.8})$$

Now, if the displacements of the center of mass of any two adjacent masses are calculated, we see that two consecutive centers of mass (COM) move equally on both sides of the interface (bottom panel of Fig. A.5). One can easily check that this movement of consecutive COMs is in the opposite direction, i.e., an out-of-phase motion (not shown in the figure). Therefore, we can enforce the following conditions on the displacements.

$$u_{3m} + 2u_{3m+1} + u_{3m+2} = 0; \quad m = \dots -1, 0, 1, \dots \quad (\text{A.9})$$

We know that at steady state, the ratio of (dynamic) displacements of two similar particles would be equal to the ratio of *net* dynamic forces (F) on them. Thus, for the particles at locations 0 and 1, we have

$$\begin{aligned} \frac{u_0}{u_1} &= \frac{F_0}{F_1} \\ &= \frac{K_s(u_{-1} - u_0) - K_s(u_0 - u_1)}{K_s(u_0 - u_1) - K_h(u_1 - u_2)}. \end{aligned} \quad (\text{A.10})$$

Using Eq. (A.8) and Eq. (A.9), we can rewrite Eq. (A.10) as

$$\frac{u_0}{u_1} = \frac{2K_s(u_1 - u_0)}{(K_s - K_h)u_0 - (K_s + 3K_h)u_1}. \quad (\text{A.11})$$

Solving this equation, we obtain:

$$\frac{u_0}{u_1} = \frac{-K_s + 3K_h \pm \sqrt{(K_s - 3K_h)^2 + 8K_s(K_s - K_h)}}{2(K_s - K_h)}.$$

For simplification, we let $r = K_h/K_s$. This leads to:

$$\frac{u_0}{u_1} = \frac{1 - 3r \pm \sqrt{9r^2 - 14r + 9}}{2(r - 1)} = -\gamma. \quad (\text{A.12})$$

As the mode in consideration is a decaying mode from the topological interface, it would satisfy $|u_0/u_1| > 1$. Hence, we exclude the irrelevant solution of γ by dropping (+) sign, and obtain the net force on particle 0 as

$$\begin{aligned} F_0 &= 2K_s(u_1 - u_0) \\ -m\omega_s^2 u_0 &= 2K_s(-\gamma^{-1} - 1)u_0 \\ \Rightarrow f_s &= \frac{\omega_s}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{2K_s(1 + \gamma^{-1})}{m}}, \end{aligned} \quad (\text{A.13})$$

where $\gamma = \frac{3r-1+\sqrt{9r^2-14r+9}}{2(r-1)}$.

A.8 Robustness of topologically protected modes

In this section, we discuss the robustness of the topologically protected mode against local perturbations near the interface. To this end, we first take two cases of trivial defects in the dimer chain, and plot their spectrum for different defect angles (α_d). Figure A.6 shows these two cases of (topologically) trivial defects and the corresponding eigen spectrum for a sufficiently long dimer chain. We can see that the gap mode seen (in red) due to the defect is highly dependent on α_d . It will coalesce with the bulk bands at a certain value of defect angle. Therefore, the local perturbation in the chain can dictate the presence (or absence) of a gap mode, thus rendering it not robust.

In Fig. A.7, however, we take the case of topological defect and check its robustness against local perturbations near the interface. We can clearly see that even though we change one of the interface angles, the resulting mode is always inside the band gap and does not coalesce with the bulk bands. Hence, it is robust, and it can not be removed simply by local perturbations near the interface.

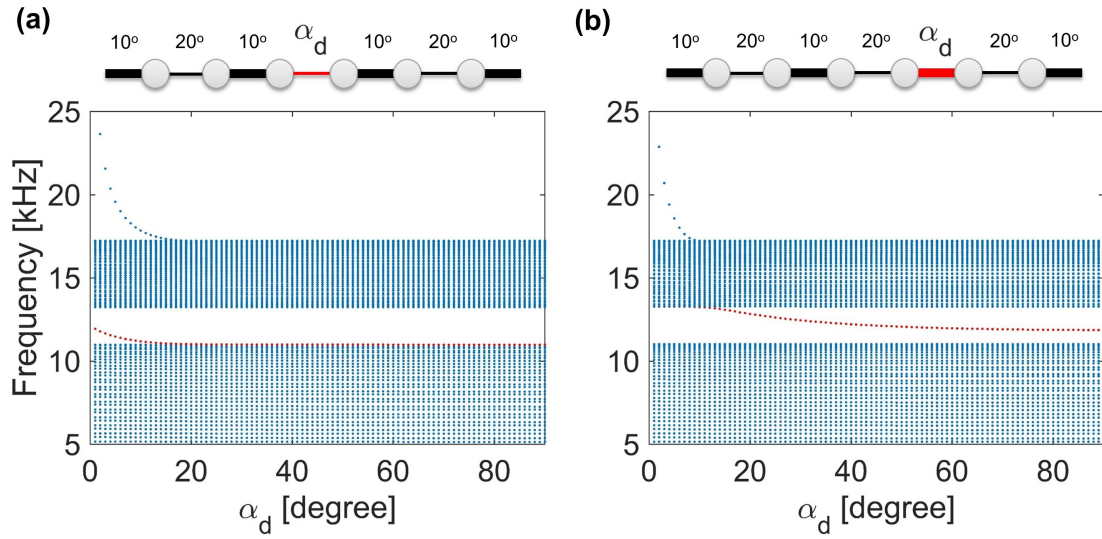


Figure A.6: Spectrum of the dimer chain with a (topologically) trivial defect and its perturbation: (a) Defect introduced at the 20° site. (b) Defect introduced at the 10° site. In both cases, the mode disappears within the band gap upon varying the defect angle α_d . In other words, we observe the absence of the boundary modes (red dots) when $\alpha_d > 20^\circ$ in the left panel. In the right panel, there exists no boundary modes in the band gap when $\alpha_d < 20^\circ$.

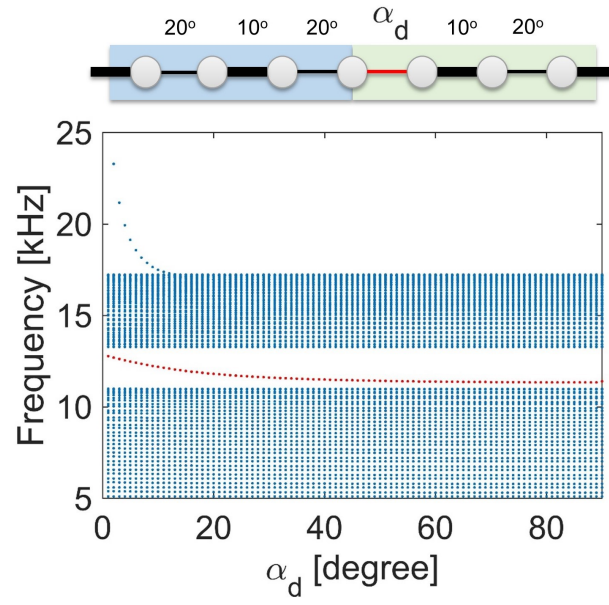


Figure A.7: Spectrum of dimer chain with a topological defect and its perturbation by varying the defect angle α_d . The topologically protected mode can be seen to be present always in the band gap.

Appendix B

ONE-DIMENSIONAL TIME-DEPENDENT TOPOLOGICAL METAMATERIAL

B.1 Scattering from large defects

In this section, we evaluate the effect of large defects on the nonreciprocal behavior of the current system. As a preliminary analysis, we consider a mass defect which is assigned by doubling the mass of one cylinder at the center of the chain. We keep rest of the parameters the same. We send a Gaussian pulse at 16.78 kHz from both ends of the chain and see how the propagating waves interact with the defect. We would expect that the defect would scatter a fraction of energy in the form of reflected waves; and the rest would be simply transmitted to the other side of the defect. In Fig. B.1a, we show the spectrum of propagating pulse injected from left end of the chain. As discussed in the main text, we first observe a frequency shift to a higher value as soon as the input excitation enters the system. The wave is then partially reflected and transmitted due to the defect. However, what is interesting is that reflected wave again is at the original input frequency, and that is because the system can allow the wave to propagate in backward direction only at the input frequency (see the main text). On the contrary, when the chain is excited from the right end, Fig. B.1b shows how the wave first travels in the same frequency. But after the scattering, the reflected wave attains a higher frequency. These interesting wave interband transfers in the presence of a defect have been summarized in the insets attached to each plot.

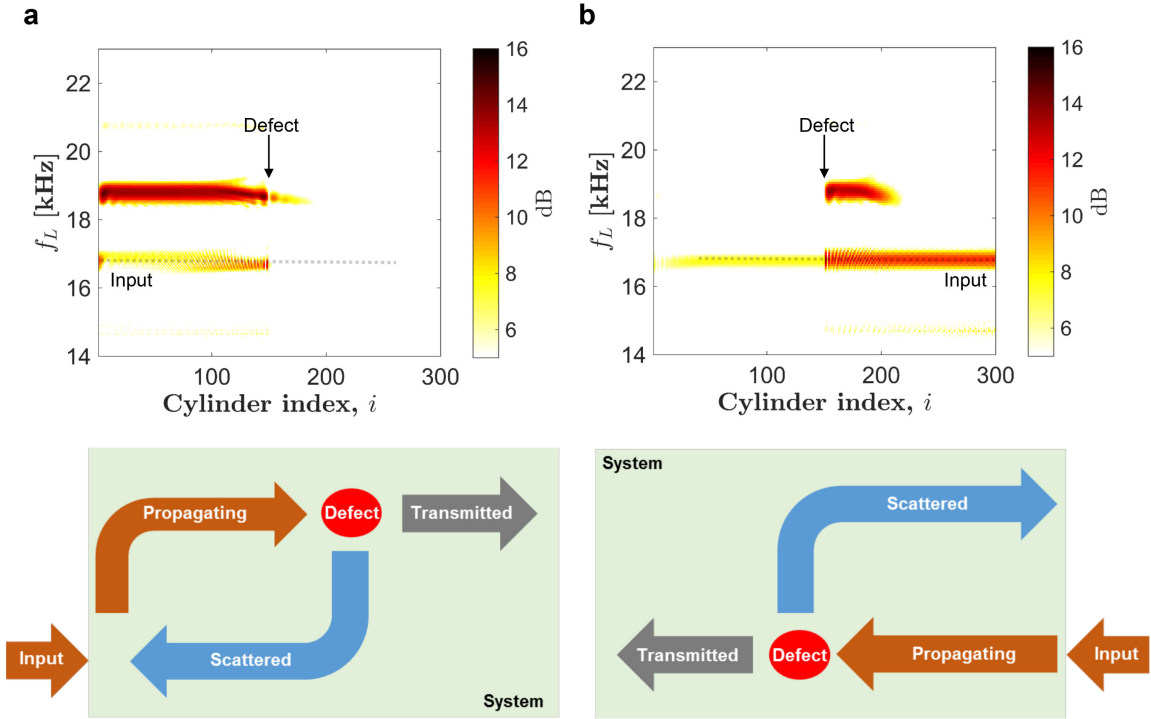


Figure B.1: Effect of large mass defect on nonreciprocity in the system: (a)-(b) Frequency content in the traveling pulse as input is given from left (right) end of the chain. Input, propagating, transmitted and scattered frequency components are highlighted using the schematics below.

Appendix C

TWO-DIMENSIONAL DISCRETE TOPOLOGICAL METAMATERIAL

C.1 Generalized model of SP

Here we show how the conventional SP that has six DOFs of the top plate and a fixed bottom plate can be reduced to a 3-DOF structure by maintaining symmetry in the connecting spring constant attached between two plates. Based on Kane's method [124], we formulate the full dynamic model of 6-DOF SP. These DOFs are labeled as $u, v, w, \psi, \vartheta, \phi$. Here, u, v , and w denote the displacements, and ψ, ϑ , and ϕ represent the rotations of the top plate along x, y , and z directions, respectively (see Fig. C.1a). Therefore, the full stiffness matrix of SP is

$$\mathbf{B}_{full}^{SP} = \mathbf{J} \mathbf{B}_{\Delta} \mathbf{J}^T$$

$$= \begin{bmatrix} k_{uu} & 0 & 0 & k_{u\psi} & k_{u\vartheta} & 0 \\ 0 & k_{vv} & 0 & k_{v\psi} & k_{v\vartheta} & 0 \\ 0 & 0 & k_{ww} & 0 & 0 & k_{w\phi} \\ k_{\psi u} & k_{\psi v} & 0 & k_{\psi\psi} & 0 & 0 \\ k_{\vartheta u} & k_{\vartheta v} & 0 & 0 & k_{\vartheta\vartheta} & 0 \\ 0 & 0 & k_{\phi w} & 0 & 0 & k_{\phi\phi} \end{bmatrix}, \quad (\text{C.1})$$

where, $\mathbf{B}_{\Delta} = \text{diag}[k_1 \ k_2 \ k_1 \ k_2 \ k_1 \ k_2]$ represents the symmetrical arrangement of spring constants connecting two disks, and $\mathbf{J} = \begin{bmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_6 \\ \tilde{\mathbf{p}}_1 \mathbf{u}_1 & \cdots & \tilde{\mathbf{p}}_6 \mathbf{u}_6 \end{bmatrix}$ is the Jacobian matrix, in which, $\mathbf{u}_i$ is the unit vector along the direction of i th spring connector, and $\tilde{\mathbf{p}}_i$ is the skew-symmetric matrix for the position vector $\mathbf{p}_i$ of the i th spring joint on the top disk (see Fig. C.1a). The entries marked in blue denote the displacement and rotation along the z direction, which are naturally coupled with each other but are independent with the rest

four DOFs. As a result, we are able to extract k_{ww} , $k_{w\phi}$, $k_{\phi w}$, and $k_{\phi\phi}$ from the stiffness matrix and add one rotation DOF to the bottom disk to eventually fabricate the stiffness matrix of our simplified 3-DOF SP as

$$\mathbf{B}_{reduced}^{SP} = \begin{bmatrix} k_{\phi\phi} & -k_{\phi\phi} & -k_{w\phi} \\ -k_{\phi\phi} & k_{\phi\phi} & k_{w\phi} \\ -k_{\phi w} & k_{\phi w} & k_{ww} \end{bmatrix}, \quad (\text{C.2})$$

with

$$\begin{aligned} k_{ww} &= 3k_1 h_0^2 / l_1^2 + 3k_2 h_0^2 / l_2^2 \\ k_{\phi\phi} &= 3k_1 R^4 [1 + \cos(2\theta_0 - 5\pi/3)] / l_1^2 \\ &\quad + 3k_2 R^4 \sin^2(\theta_0 - 2\pi/3) / l_2^2 \\ k_{w\phi} = k_{\phi w} &= 2R^2 k_1 h_0 \sin(\theta_0 - 2\pi/3) / l_1^2 \\ &\quad + 2\sqrt{3} R^2 k_1 h_0 \cos(\theta_0 - 2\pi/3) / l_1^2 \\ &\quad - 2R^2 k_2 h_0 \sin(\theta_0 - 2\pi/3) / l_2^2. \end{aligned}$$

Here, l_1 and l_2 represent the lengths of alternating springs with constants k_1 and k_2 as shown in Fig. C.1a.

C.2 Bistable behavior analysis of SP

To analyze the bistable behavior of the SP unit cell, we can further reduce 3-DOF SP structure to a 2-DOF structure. This is because only the relative height and angle between the two plates determine the stable configurations. So, for simplicity, we can restrict the rotation of the bottom disk and only analyze the rotation and translation of the top disk for this study. As shown in Fig. C.1a, let the initial height and rotation angle be h_0 and θ_0 , respectively (representing the Y-state). We examine the folding behavior of the SP structure by using the principle of minimum total potential energy. First, we consider the length of the spring elements l_1 and l_2 as shown in Fig. C.1a. Based on the geometry of the SP, we

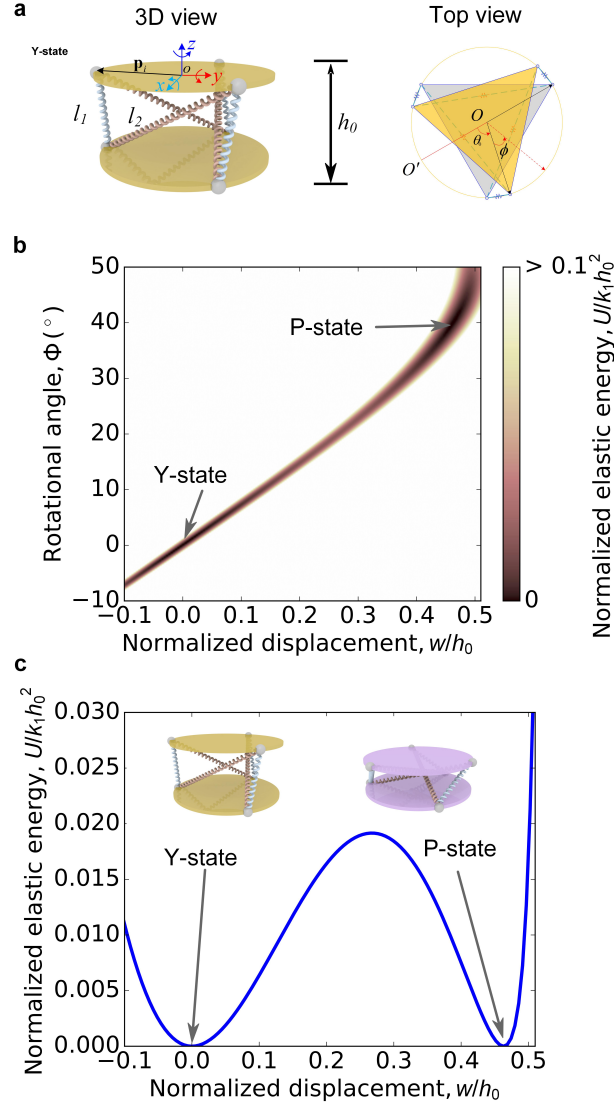


Figure C.1: Bistable behavior of the Stewart Platform (SP). (a) 3D (left) and top (right) views of the SP unit cell are shown. (b) The surface map of the elastic energy as a function of both w and ϕ shows the two local minimum states, i.e., Y- and P- states as denoted by arrows. The energy is normalized by $k_1 h_0^2$ and the displacement is normalized by h_0 . (c) The normalized energy as a function of w is plotted along the minimum potential energy trajectory which corresponds to the highlighted region in (b). The inset illustrations show the schematic configuration of the SP at each local minimum state.

can derive l_1 and l_2 as follows:

$$\begin{aligned} l_1 &= \sqrt{(h_0 - w)^2 + 4R^2 \sin^2 \left(\frac{\phi}{2} + \frac{\theta_0}{2} - \frac{\pi}{6} \right)}, \\ l_2 &= \sqrt{(h_0 - w)^2 + 4R^2 \sin^2 \left(\frac{\phi}{2} + \frac{\theta_0}{2} + \frac{\pi}{6} \right)}. \end{aligned} \quad (\text{C.3})$$

From Eq. (C.3), we can calculate the deformation in the spring elements and thus the total elastic energy as

$$U = \frac{3}{2}k_1(l_1 - L_1)^2 + \frac{3}{2}k_2(l_2 - L_2)^2, \quad (\text{C.4})$$

where, L_1 and L_2 are the initial lengths of the two springs.

Also, the work done on this system is

$$W = Fw + T\phi, \quad (\text{C.5})$$

where, F and T are the external force and torque, respectively.

Then, the total potential energy ($\Pi = U - W$) would simply reduce to the total elastic energy U under no external force/torque in our system, such that

$$\Pi(w, \phi) = U = \frac{3}{2}k_1(l_1 - L_1)^2 + \frac{3}{2}k_2(l_2 - L_2)^2. \quad (\text{C.6})$$

Therefore, applying the principle of minimum total potential energy (i.e., $\delta\Pi = 0$) to Eq. (C.6), we have

$$\partial\Pi/\partial w = 0 \quad \text{and} \quad \partial\Pi/\partial\phi = 0, \quad (\text{C.7})$$

which are used to obtain a trajectory to analyze the folding behavior of the SP unit cell.

We plot the total potential energy of SP unit cell during folding/unfolding motion by using Eq. (C.6) in Fig. C.1b. In the parameter space of w and ϕ , the highlighted region indicates low energy levels. The darkest spots indicate two local minima in the system, and hence, prove the existence of two stable equilibrium states: Y-state and P-state. By further focusing on this highlighted trajectory we deduce a double-well potential curve (Fig. C.1c) showing the energy level as a function of the normalized displacement. Note that this bistable

behavior can be tuned by the geometrical parameters such as the initial height (h_0) and angle (θ_0), as well as the materials properties such as spring constants k_1 and k_2 .

Therefore, for the initial stable configuration, i.e., Y-state ($h_0 = 0.04$ m, $\theta_0 = 70^\circ$), we can obtain the parameters for another stable P-state ($h_0 = 0.0215$ m, $\theta_0 = 110^\circ$). In addition, the stiffness matrices for these states would be

$$\mathbf{B}_{reduced}^{Y-state} = \begin{bmatrix} 69.13 & -69.13 & -2.167 \times 10^3 \\ -69.13 & 69.13 & 2.167 \times 10^3 \\ -2.167 \times 10^3 & 2.167 \times 10^3 & 6.813 \times 10^4 \end{bmatrix},$$

$$\mathbf{B}_{reduced}^{P-state} = \begin{bmatrix} 3.846 & -3.846 & -268.8 \\ -3.846 & 3.846 & 268.8 \\ -268.8 & 268.8 & 1.969 \times 10^4 \end{bmatrix}.$$

Judging from the magnitude of the entries, it is clear that the Y-state is stiffer than the P-state.

C.3 Eigenvalue problem for hexagonal unit cell

The unit cell of hexagonal lattice has two SP units, each with three DOFs. Hence, for a wave vector $\mathbf{k}$, 6×6 stiffness and mass matrices in Eq. 3 of the main text are given by

$$\mathbf{B} = \begin{bmatrix} 3k_{cc} + k_{\phi\phi}^{(1)} & -k_{\phi\phi}^{(1)} & -k_{\phi w}^{(1)} & k_{cc}(1 + e^{-i\mathbf{k}\cdot\mathbf{a}_1} + e^{-i\mathbf{k}\cdot\mathbf{a}_2}) & 0 & 0 \\ -k_{\phi\phi}^{(1)} & k_{\phi\phi}^{(1)} & k_{\phi w}^{(1)} & 0 & 0 & 0 \\ -k_{w\phi}^{(1)} & k_{\phi w}^{(1)} & k_{ww}^{(1)} & 0 & 0 & 0 \\ k_{cc}(1 + e^{i\mathbf{k}\cdot\mathbf{a}_1} + e^{i\mathbf{k}\cdot\mathbf{a}_2}) & 0 & 0 & 3k_{cc} + k_{\phi\phi}^{(2)} & -k_{\phi\phi}^{(2)} & -k_{\phi w}^{(2)} \\ 0 & 0 & 0 & -k_{\phi\phi}^{(2)} & k_{\phi\phi}^{(2)} & k_{\phi w}^{(2)} \\ 0 & 0 & 0 & -k_{\phi w}^{(2)} & k_{\phi w}^{(2)} & k_{ww}^{(2)} \end{bmatrix},$$

and

$$\mathbf{M} = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & m \end{bmatrix}.$$

To illustrate the analogy with systems exhibiting QVHE, we express the effective Hamiltonian of the system as [42, 57, 120]

$$\begin{aligned} \mathbf{H}(\mathbf{k}) &= \mathbf{M}^{-1/2} \mathbf{B} \mathbf{M}^{-1/2} \\ &= 3\sigma_0 \otimes \tau_0 + \sigma_0 \otimes \tau_1 + (c\sigma_1 - s\sigma_2) \otimes \tau_0 - \xi\sigma_3 \otimes \tau_1, \end{aligned} \tag{C.8}$$

where $\sigma_0, \sigma_1, \sigma_2$, and σ_3 are the Pauli matrices; $\tau_0 = \tau_{-1/2}\tau_e\tau_{-1/2}$, $\tau_1 = \tau_{-1/2}\tau_k\tau_{-1/2}$, $\tau_{-1/2} = \tau_m^{-1/2}$ are the parameter matrices, in which $\tau_e = \text{diag}(k_c, 0, 0)$, $\tau_{-1/2}$ = average of stiffness matrices of Y and P states, $\tau_m = \text{diag}(I, I, m)$, $c = 1 + \cos(\mathbf{k} \cdot \mathbf{a}_1) + \cos(\mathbf{k} \cdot \mathbf{a}_2)$, and $s = \sin(\mathbf{k} \cdot \mathbf{a}_1) + \sin(\mathbf{k} \cdot \mathbf{a}_2)$.

The first two terms in the Hamiltonian are the constant matrices that just translate the band dispersion curves without changing their topology. The third term, which is similar to the effective mass Hamiltonian of graphene, leads to the formation of a Dirac cone [33, 57]. The last term emerges due to the breakage of the inversion symmetry (thus keeping C_3 symmetry), and it will vanish when the two SPs inside a unit cell are at the same stable configuration (i.e., C_6 symmetry).

C.4 Angle-dependent waveguides

We have already shown in the main text that various complex shaped topological waveguides can be built in the system and a robust wave propagation can be observed. In this note, we highlight a subtle difference in transmission if one uses bends of different angles. In Fig. C.2, we show topological waveguides at two angles: 120° and 60°, and their traditional

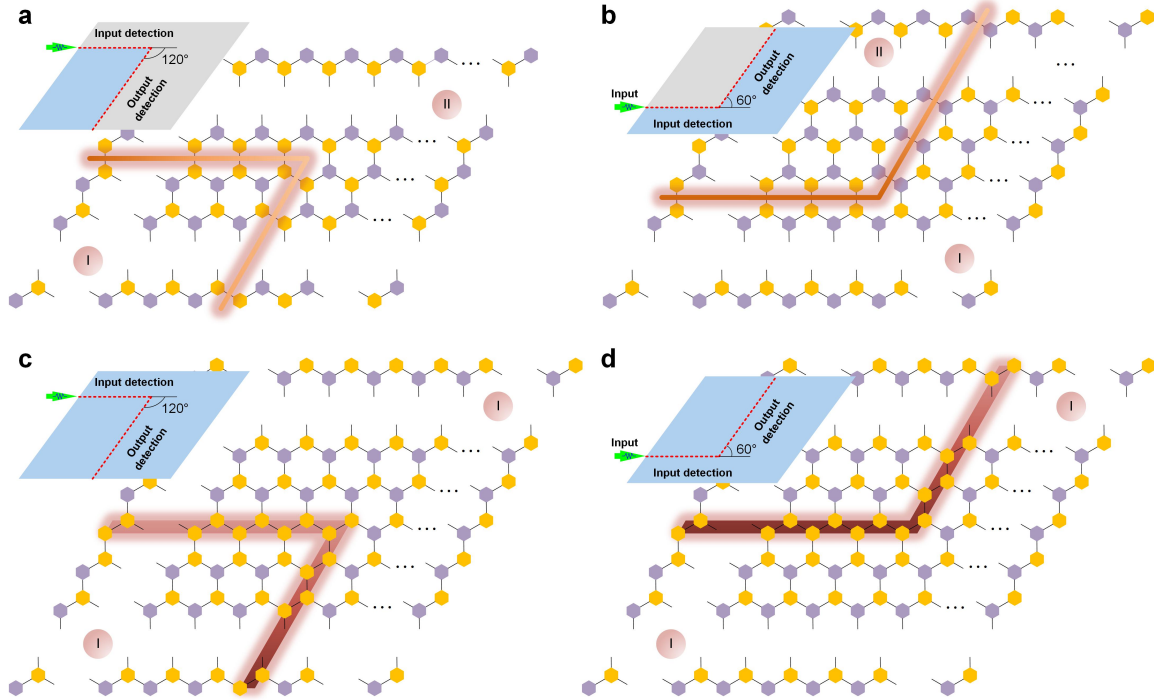


Figure C.2: Topological and traditional waveguides with two different bend angles. (a) and (b) are topological waveguides with 120° and 60° bends, respectively. Note that in the former case, Y-Y connection makes the entire waveguide channel. However, in the later, Y-Y connection is followed by a P-P connection. (c) and (d) Their traditional counterparts.

counterparts. Note that both waveguides in Fig. C.2a-b are topological. However, due to the difference in the bend angles, the interface connections are not identical along the waveguides. In the former, the interface all along the length of the waveguide is made of Y-Y connections. But, in the later, it is a mix of Y-Y and P-P connections. These distinct connections can be understood as the direct extension of *soft-soft* and *hard-hard* type topological interfaces in 1D systems [88]. We evaluate the transmission characteristics of these and compare them with their traditional counterparts shown in Fig. C.2c-d.

Shown in Fig. C.3 are the transmission profiles for a sinusoidal frequency sweep signal given at the input of the waveguides. One sees that the topological waveguides are clearly superior to traditional waveguides. However, there is not any discernible difference between the transmission of two distinct topological waveguides. This makes sense as this transmission profile is mainly dictated by the presence of local modes inside the band gap, and both the topological waveguides guarantee the presence of local modes all along the band gap.

In Fig. C.4, we compare these two topological waveguides in transient conditions by sending a 50 ms Gaussian wave packet at 760 Hz. We notice that the waveguide with 120° bend is absolutely back scattering free, however, 60° waveguide leads to a minute back scattering. This is the subtle difference in term of transmission properties of the topological waveguide at different angles, which arises due to the uneven interface along the length of the waveguide as mentioned above.

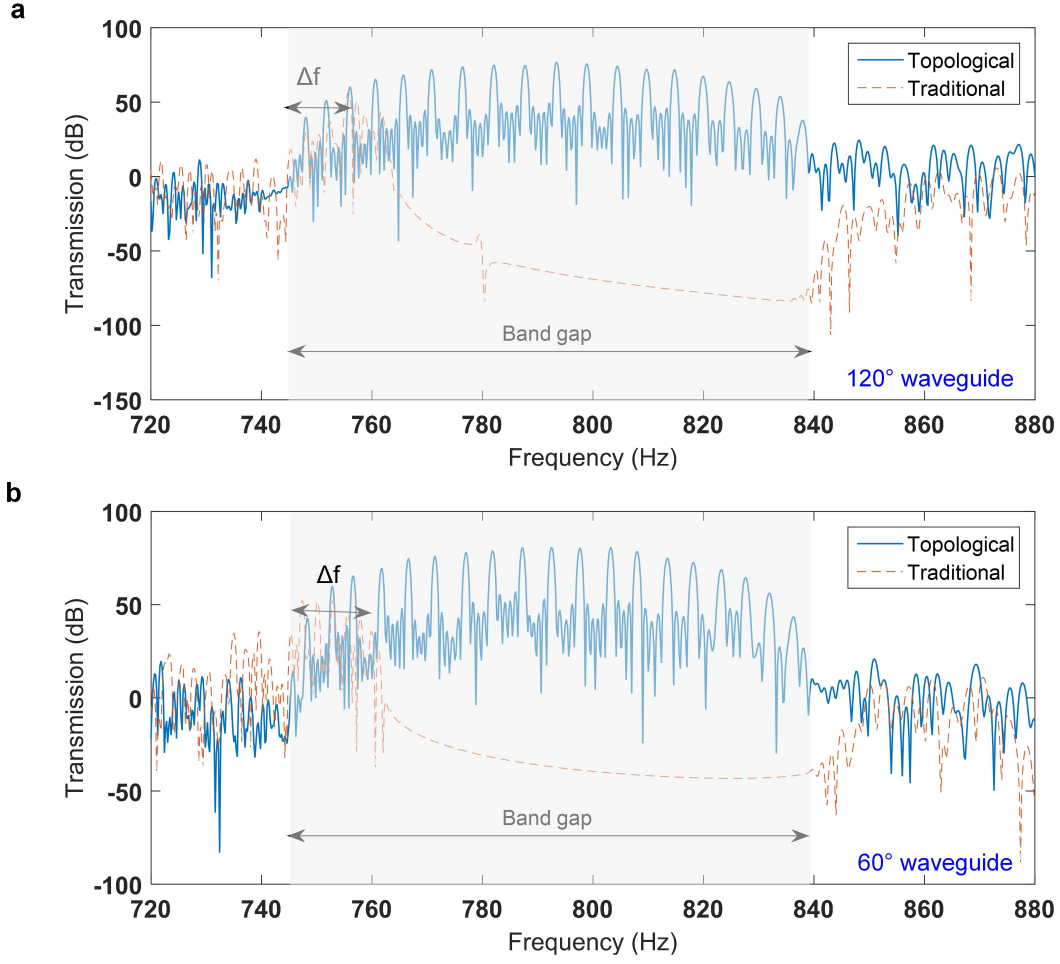


Figure C.3: Transmission of angle dependent waveguides. (a) Comparing transmissions of topological and traditional waveguides for 120° bend. (b) The same for 60° bend.

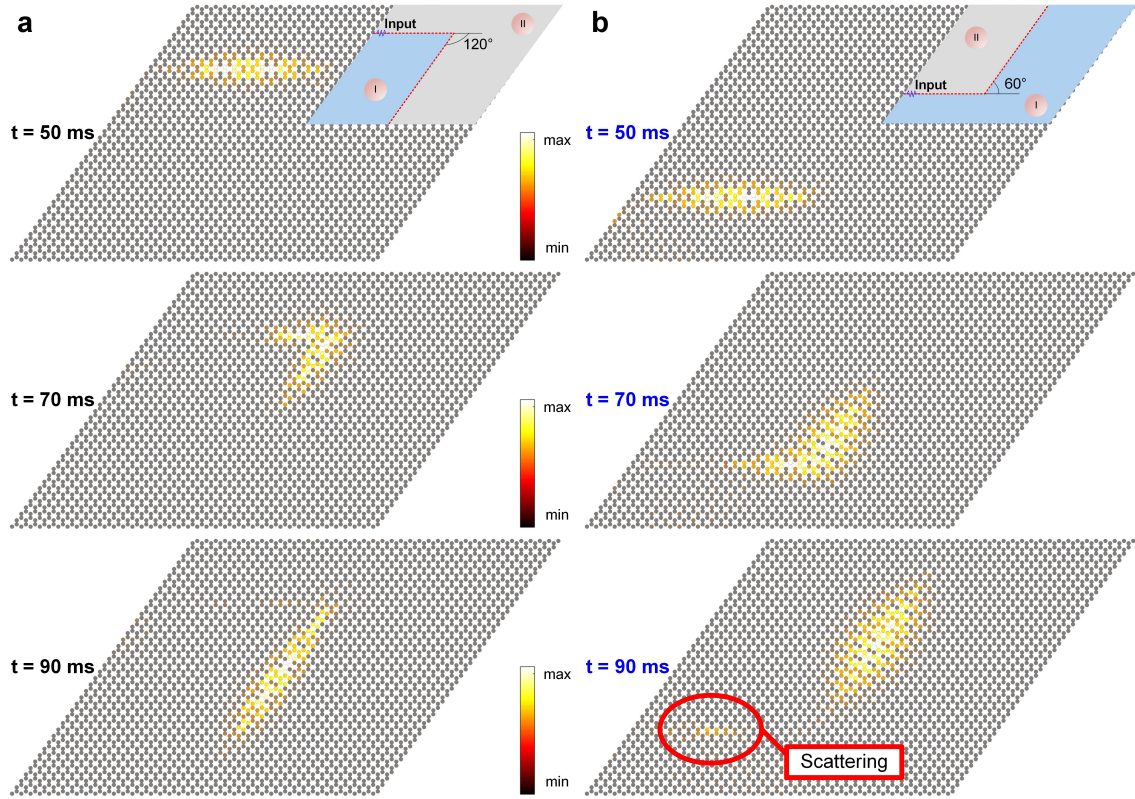


Figure C.4: Transient simulations for the topological waveguides. (a) 120° waveguide. (b) 60° waveguide, for which, a slight scattering is highlighted at 90 ms. Colors denote the rotation of the bottom disks.

Appendix D

TWO-DIMENSIONAL PLATE TOPOLOGICAL METAMATERIAL: NUMERICAL SIMULATIONS

D.1 Comparison between the lumped mass model (PWE) and the full-scale model (FEM)

We corroborate the results obtained earlier based on the lumped mass model now by using the FEM, in which we account for all geometrical features described in Fig. 5.1b. As shown in Fig. D.1, we see excellent agreement between the PWE method (red curves) and the FEM (green circles). It should also be noted that the p - and d -type degenerate mode shapes obtained through the lumped mass model comply with those obtained by the full-scale model (compare the inset images between Figs. 5.3 and D.1). In the dispersion relation obtained by the FEM, however, we observe that previously neglected shear-horizontal (SH_0) and shear (S_0) plate modes do appear for a thin plate. In our analysis, it is reasonable to ignore such modes and focus solely on anti-symmetric (A_0) flexural modes for the transverse source excitation at low frequencies, as was already demonstrated by full-scale simulations in Section VI and Section VII. We also see nearly flat dispersion curves due to the other modes of local resonators, which were not accounted for in the lumped mass model. Nevertheless, these modes have minimal coupling with the out-of-plane vibration of the plate, and these are away from the double Dirac cone. Therefore, the band-inversion process is not affected by them and it is reasonable to neglect them in the PWE method.

D.2 Graded interface between two topologically distinct lattices

Here we verify the scheme of reducing the gap observed at the Γ point in Fig. 5.4b. The gap emerges due to the breakage of the C_6 symmetry at the interface. Therefore, we minimize the

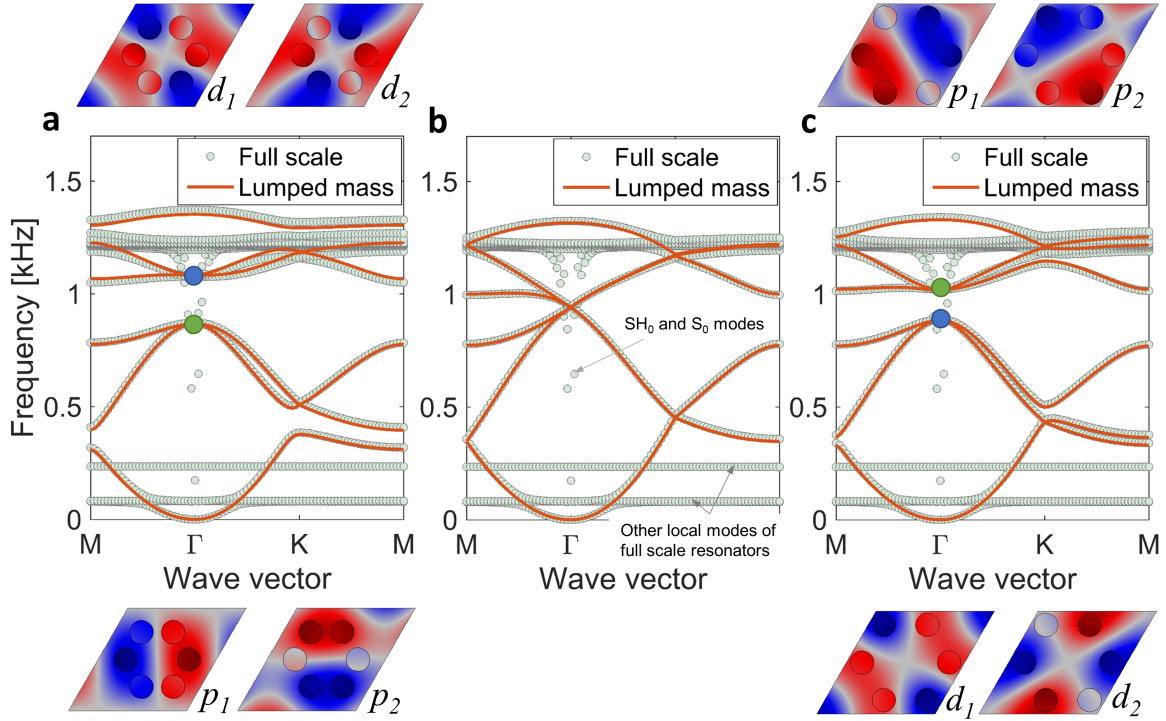


Figure D.1: Comparison of unit cell dispersion curves obtained from the lumped mass model (solid curves) and the full-scale FEM (dotted curves) for the cases: (a) $R = 0.8a/3$, (b) $R = a/3$, and (c) $R = 1.1a/3$. Excellent agreement is observed, and the band-inversion mechanism (insets with modes shapes obtained via the full-scale FEM) is confirmed for flexural modes (A_0). Additional modes emerging in the full-scale numerical simulation are SH_0 and S_0 guided plate modes, and other modes of local resonators.

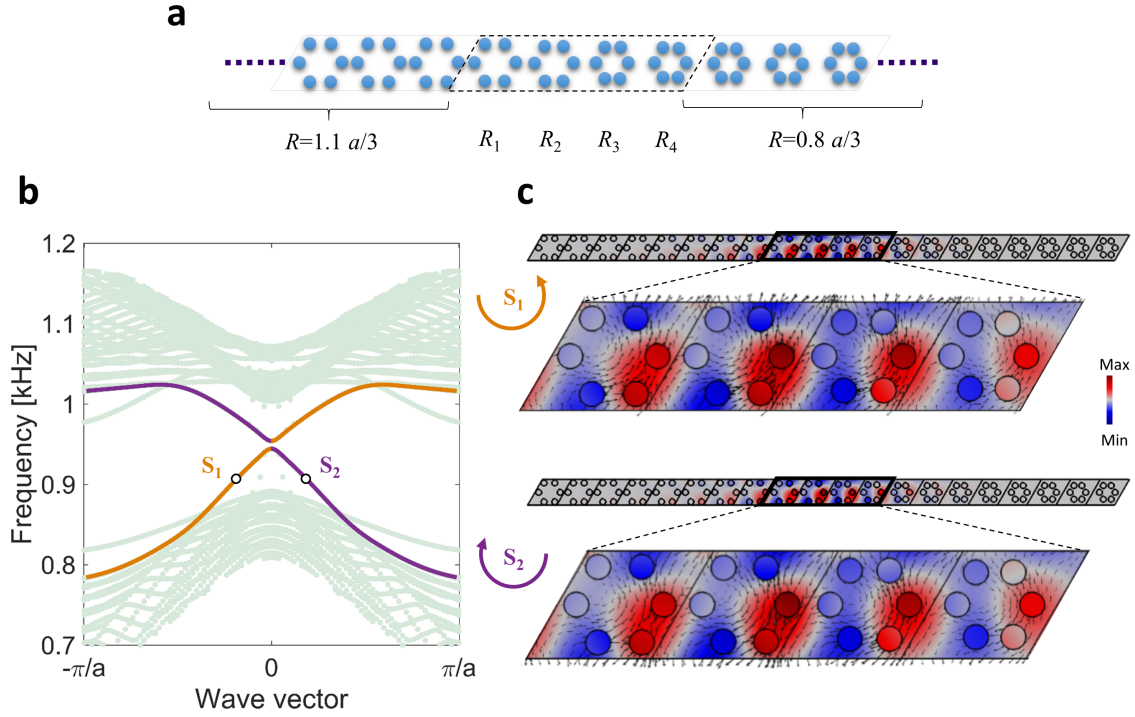


Figure D.2: The effect of a graded interface on the pseudospin modes at the interface. (a) A supercell with a smooth interface (radii $R_1 > R_2 > R_3 > R_4$) between topologically trivial and nontrivial lattices. (b) Eigenfrequencies of the modified supercell. The gap reduction at the Γ point can be observed, indicating a greater degree of topological protection. (c) Pseudospin modes corresponding to S_1 and S_2 points in the dispersion curve.

effect of symmetry breakage by modifying the interface. Figure D.2a shows a supercell, in which a topologically nontrivial lattice ($R = 1.1a/3$) smoothly transitions to a topologically trivial lattice ($R = 0.8a/3$) via four lattices with a gradient in their radii, i.e., $R_1 = 1.05a/3$, $R_2 = 1.02a/3$, $R_3 = 0.95a/3$, and $R_4 = 0.9a/3$. The gap reduction is confirmed in Fig. D.2b. It is about 3 times smaller than the one observed in Fig. 5.4b. The spin nature of these interface modes is also intact, as shown in Fig. D.2c.

D.3 The presence of defects along the waveguide

We further examine the robustness of spin-dependent one-way flexural waves against certain defects present at the topological interface, as shown in Fig. D.3. The first defect, i.e., Defect 1, is introduced by removing four local resonators from one of the topologically trivial unit cells along the interface (see Fig. D.3a). We then excite a point on the left most end of the waveguide (shown by a star marker) with low-reflecting boundary conditions, and we perform harmonic analysis for the same frequency as that in Fig. 5.5. We observe that the presence of this defect does not have any noticeable effect in terms of scattering the spin mode. The mode continues to propagate from the left to the right with a counterclockwise spin (the same as in Fig. 5.5b) and maintains a similar modal amplitude before and after the defect location. Therefore, this defect would qualify as a non spin-mixing defect [150].

We take another defect, i.e., Defect 2, in which we remove all six local resonators from the same unit cell (see Fig. D.3b). Thus this defect is ‘stronger’ than Defect 1. Under the same excitation and boundary conditions as those in the previous case, we observe that this defect affects the spin mode more drastically. In Fig. D.3b, we see that the modal amplitude is almost negligible on the right side of the defect because of strong back-scattering. The defect causes the spin modes to mix. Consequently, the rightward-propagating counterclockwise spin mode is converted to a leftward-propagating clockwise spin mode. Therefore, this represents a case of a spin-mixing defect [150].

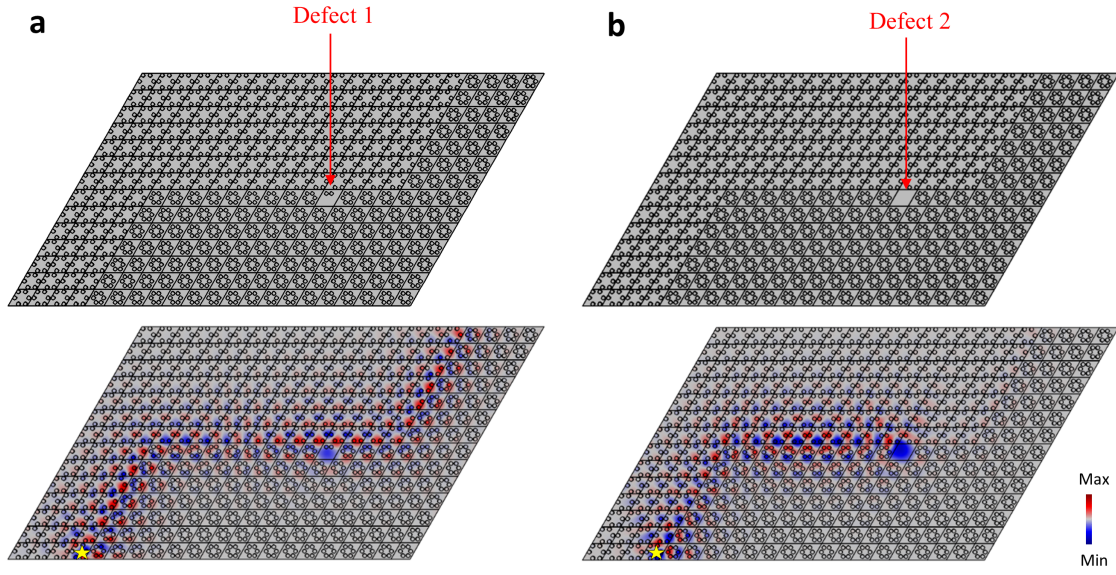


Figure D.3: The effect of defects on the spin-dependent one-way propagation of flexural waves. (a) Defect 1 is created by removing *four* resonators in one of the unit cells along the interface. The star indicates the point of excitation. No obvious backscattering is observed. (b) Defect 2 is created by removing *all six* resonators in the same unit cell. We observe backscattering of the spin mode from this defect.

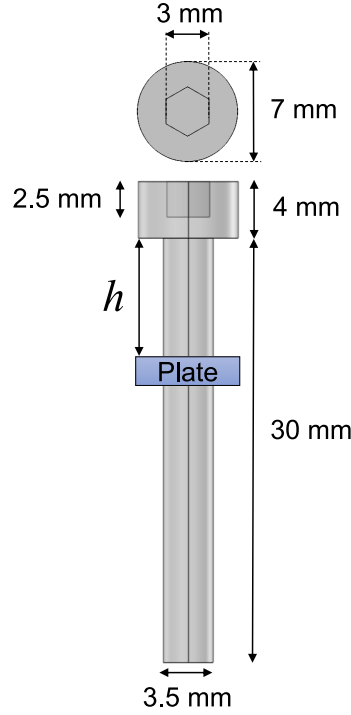


Figure E.1: Bolt geometry used for numerical simulations.

Appendix E

TWO-DIMENSIONAL PLATE TOPOLOGICAL METAMATERIAL: EXPERIMENTS

E.1 Geometrical details of the bolts

We model the bolts by assuming a uniform diameter along their shank and threaded areas as shown in Fig. E.1. We take the height h of the bolt above the plate as 8.5 mm (measured nominal value), which is dictated by the length of the thread and the applied torque to the bolt during the assembly process.

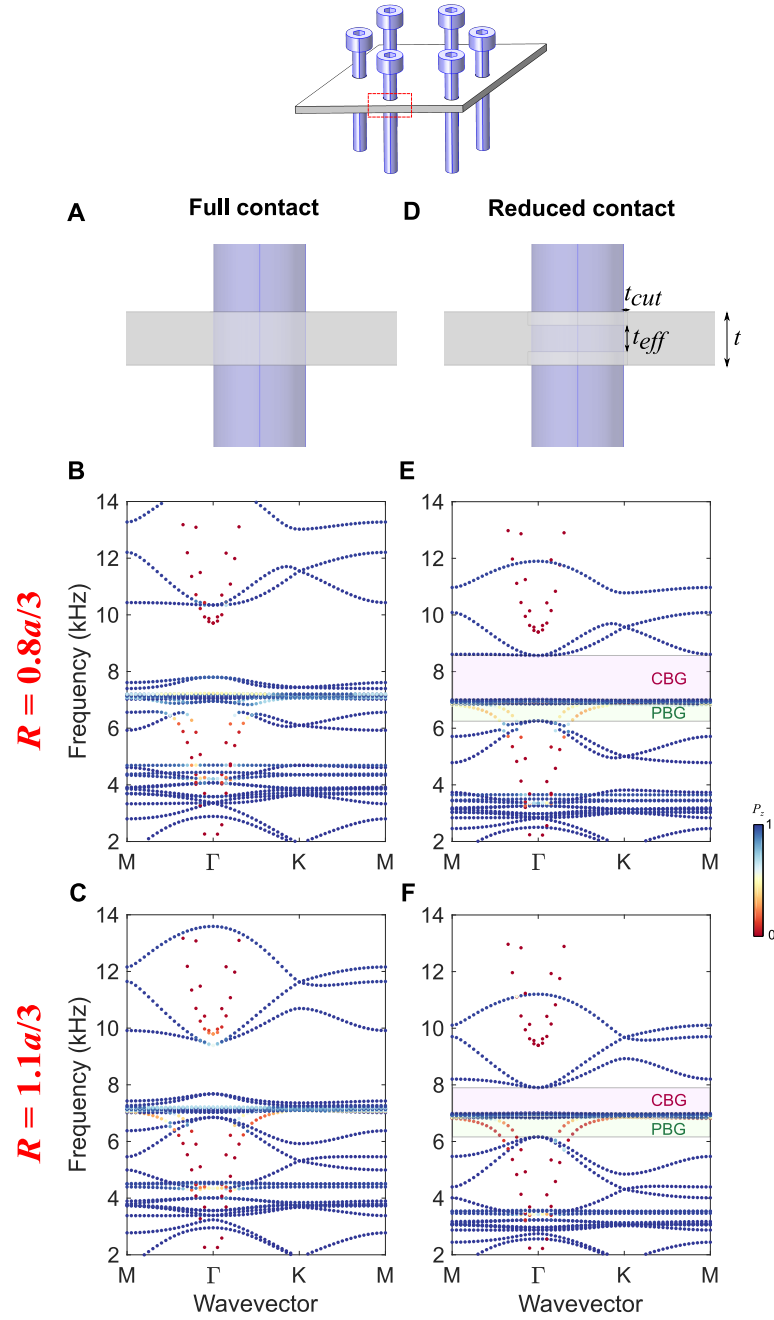


Figure E.2: Scaling of dispersion curves. (a) The “full contact” model, in which the plate and the bolt are rigidly connected throughout the entire contact area (whole thickness of the plate). (b)-(c) Dispersion curves for this model in configurations with $R = 0.8a/3$ and $R = 1.1a/3$, respectively. (d) The “reduced contact” model, in which the plate and the bolt are connected only through a limited area (a fraction of thickness of the plate). (e)-(f) The corresponding dispersion curves.

E.2 Modeling of the bolt-plate contact

We first model the bolt-plate contact using the simplest approach, i.e., assuming a rigid connection at the entire contact area between them (see Fig. E.2A). The resulting dispersion is plotted in Figs. E.2B,C for the cases with $R = 0.8a/3$ and $R = 1.1a/3$, respectively. The colormap indicates polarization of plate modes as discussed in the main text. Though we find the existence of topological band gap, the dispersion bands are generally up-shifted in comparison to the experimental observations (see steady-state transmission in Movie 2). This makes sense because by assuming this perfectly rigid connection, we overestimate the resulting out-of-plane stiffness that comes from the threaded contact between the bolt and the plate. Therefore it leads us to scale down the contact area as shown in Fig. E.2D. We introduce a tiny cut in the plate $t_{cut} = 0.15$ mm, from the top and the bottom as shown, to separate some area of plate from the bolt, and therefore maintain an effective thickness $t_{eff} = 0.5t$ at the contact. In this way, the resulting dispersion bands (see Figs. E.2D-F) explain the experimentally observed wave transmission to a reasonable degree. Note that a better match can be achieved if we model the thread connection between the bolt and plate with a greater detail. Nonetheless, our simple model with a reduced contact area captures the effective contact stiffness reasonably well, thus elucidating the nature of band gaps and topological localization in our bolted plate system.

E.3 Tuning dispersion by bolt height

In this section, we show numerically some exemplary cases to tune the plate dispersion by changing the height h of the bolts. In Fig. E.3, we show the dispersion curves for the case with $R = a/3$ for a number of bolt heights. We observe that the bolt height primarily influences the bending resonance of the bolts, and therefore affects the onset of the locally resonant band gap for in-plane plate modes (in red). However, by closely looking in the vicinity of the double Dirac cone for out-of-plane plate modes (in blue), we notice a slight change in these modes as well. Our current setup with the bolt height as 8.5 mm (Fig. E.3C)

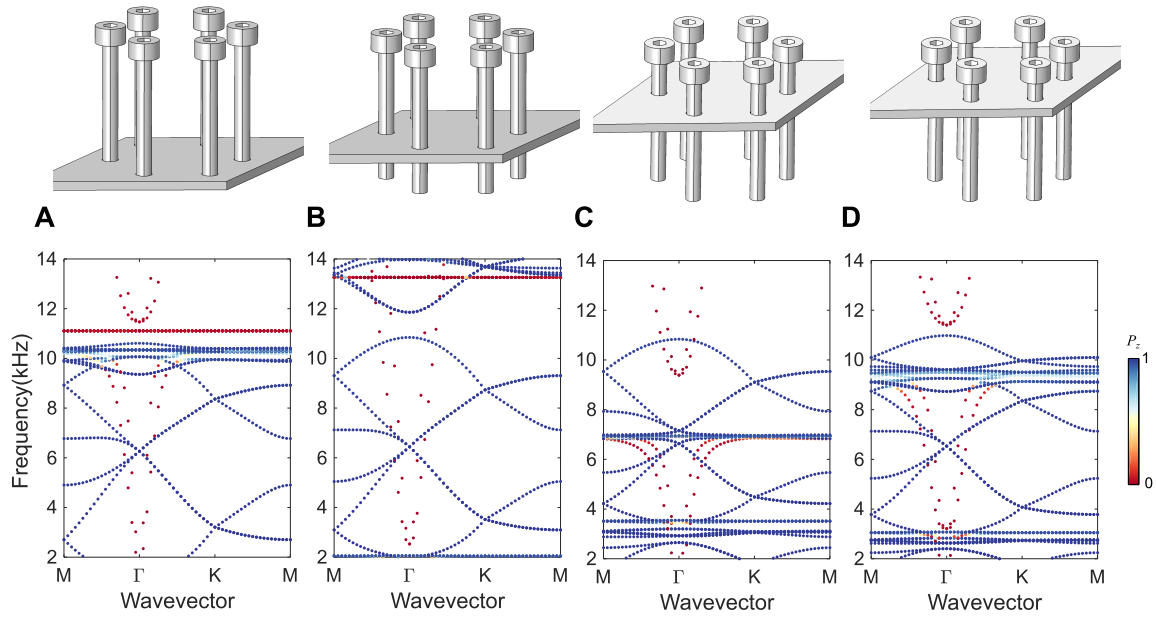


Figure E.3: Tuning dispersion by bolt height. (a)-(d) The configurations ($R = a/3$) with bolt height $h = 27.5, 19.5, 8.5$, and 5 mm, respectively.

is particularly unique as it shows the locally resonant band gap for in-plane modes and the double Dirac cones for out-of-plane plate modes in the similar frequency range. This facilitates the emergence of a complete band gap for all the plate modes as shown in the main text. Moreover, the emergence of two separate double Dirac cones in the vicinity of bolt bending resonances is also remarkable. Therefore, we see that such a dependence of dispersion modes on the bolt height can be further utilized to tune the system.

VITA

Rajesh Chaunsali was born in a town in the foothills of the Himalayas in Uttarakhand, India. He earned his Bachelors of Technology and Master of Technology in Mechanical Engineering from Indian Institute of Technology Madras, India in 2012. He then joined General Electric and designed composite fan blades for GE9X turbofan engine. In autumn 2014, he joined the Laboratory for Engineered Materials and Systems (LEMS), led by Professor Jinkyu Yang, at the University of Washington. In summer 2018, he earned a Ph.D. degree in Aeronautics & Astronautics.