INVESTIGATION OF STUDENT UNDERSTANDING OF THE WAVE-LIKE PROPERTIES OF LIGHT AND MATTER

by

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Abstract

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The research described in this dissertation is part of an ongoing investigation of student learning of the wave-like properties of light and matter. The investigation was conducted in the context of a broad range of physics courses, including courses for introductory students, advanced undergraduate majors, and graduate students. The initial emphasis was to strengthen the current research base on student understanding of physical optics. The scope of the investigation was later expanded to probe the ability of students to interpret the interference and diffraction of matter in terms of a wave model. The findings led us to extend the investigation further to include student understanding of the de Broglie wavelength and some very basic concepts in quantum mechanics. Additional difficulties in interpreting formal representations were found to cause confusion about the nature of light as an electromagnetic wave and inhibit the ability of students to predict the outcome of simple quantum mechanical measurements. The results from the investigation were used to guide the design of supplementary instructional materials that have been shown to address specific difficulties identified in the research.

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My most heartfelt thanks are forever due to my very best friend, Alice, to my parents, Al and Jan Ambrose, and to Alice's parents, Dill and Anita Bass, for their constant love and support. Any success of my own is theirs also to share.

To Alice

CHAPTER 1: INTRODUCTION TO THE DISSERTATION

A. MOTIVATION

The Physics Education Group at the University of Washington has for many years been engaged in an ongoing investigation of student understanding in geometrical and physical optics.[1] The main emphasis in the part of the investigation described in this dissertation is on the wave-like properties of light and matter. The findings from this study indicated that students often do not develop a functional understanding of geometrical optics nor recognize the conditions under which the application of a ray model is appropriate.[2] We therefore sought to identify specific difficulties that students encounter in developing a wave model to account for interference and diffraction effects. This research base provided a guide for the design of instructional strategies for the enhancement of student learning of the wave properties of light.

Results from our research in the context of light suggested that students might have considerable difficulty in applying a wave model to matter. We therefore extended our investigation to student understanding of the de Broglie wavelength and some very basic concepts in quantum mechanics. We were particularly interested in determining how well students beyond the introductory level are able to recognize the conditions under which the concepts of classical physics do not apply.

Past experience has shown that conceptual difficulties are often interwoven with those in interpreting formal representations.[3] Specific difficulties identified in the context of diffraction and polarization revealed confusion about the nature of light as a transverse electromagnetic (EM) wave. We therefore expanded the investigation still further to probe the ability of students to interpret common representations of EM waves and to understand the connections between these representations and physical phenomena. In addition, we started to explore the ability of students to predict the outcome of simple quantum mechanical measurements. Some of the errors that we identified suggested an overall failure to recognize which observable quantities were well-defined for a particle in a particular quantum mechanical state.

B. ORGANIZATION AND OVERVIEW OF THE DISSERTATION

The main body of the dissertation is divided into three parts. Part One, which consists of the first three chapters, reports on the ability of students to interpret interference and diffraction effects of light and matter in terms of a wave model. In Part Two, which consists of the next three chapters, we describe our investigation of the ability of students beyond the introductory level to choose between a classical or quantum mechanical model for matter and to apply it appropriately. The last two chapters, which comprise Part Three, deal with how students relate formal representations of light to physical phenomena and how they interpret quantum mechanical observables.

Part One: Investigation of student understanding of interference and diffraction effects of light and matter in terms of a wave model. The dissertation begins with the description of a study for which the primary objective was to identify conceptual and reasoning difficulties that students have with the wave nature of light.[4] These findings formed the basis for the development of a series of tutorials on introductory physical optics that have proved to be effective.[5] When we expanded the scope of the investigation to student understanding of interference and diffraction of matter, we discovered additional complications that need to be explicitly addressed. Results from research guided the development of tutorial materials that were tested in a variety of courses in which this material is covered.

Part Two: Investigation of student ability to choose between a classical or quantum mechanical model of matter and to apply the model appropriately. Results presented in Part One indicated the need to probe the ability of students in advanced courses to interpret reflection, transmission, and bound states of matter in terms of a wave model. We found that many students did not understand some basic quantum mechanical ideas nor recognize the conditions under which a classical or quantum mechanical interpretation is appropriate. For meaningful learning to take place, it seems that most students require considerably more detailed guidance than is ordinarily given in a standard course. Accordingly, we have begun to develop tutorials for this purpose. Research on the effectiveness of the tutorials is discussed.

Part Three: Investigation of student understanding of the connections between physical phenomena and common formal representations that describe light and matter. In addition to looking at the conceptual understanding of students, we investigated their ability to make connections between actual physical phenomena and some common formal representations of light and matter. At the introductory level, we identified and addressed some specific difficulties with the algebraic and graphical formalism commonly used to represent light as an electromagnetic wave.[6] In an exploratory study that we conducted among students in more advanced courses, we found that many are unable to relate the formalism of quantum mechanics to the outcome of simple measurements involving basic quantum mechanical observables.

C. REVIEW OF PREVIOUS RESEARCH

Much research has been conducted to explore student understanding of geometrical optics.[7] Only recently has the scope of the research been extended to the context of physical optics. Although many articles have been published on teaching modern physics, quantum mechanics, and physical chemistry,[8] the instructional strategies described have not been based on systematic investigations of student understanding, nor has the effect on student learning been documented in the literature.

1. RELATED RESEARCH ON STUDENT UNDERSTANDING OF WAVES AND OPTICS

Substantial research on student understanding of waves and optics has been done by the Physics Education Group in the context of the introductory calculus-based course. The work of Karen Wosilait and other members of the group has led to the identification of a number of common difficulties in understanding geometrical optics and physical optics.[9] We found that many students had serious difficulty applying such concepts as superposition, wavelength, path length, path length difference, and phase difference to describe the behavior of light passing through one or more slits. Many of the difficulties persisted after standard instruction of physical optics. Results from the research motivated the initial design of a set of tutorial materials intended to address student difficulties in these subjects. Further assessment of the materials has led to revisions and has shown that student understanding has been significantly enhanced as a result of implementing the materials in the introductory calculus-based course.

2. DESCRIPTIVE STUDIES ON STUDENT UNDERSTANDING OF QUANTUM PHYSICS

Research in student understanding of topics from quantum mechanics has been considerably more limited. Descriptive studies have been conducted in the context of the wave-particle duality, atomic physics, and quantum states.[10-14] However, few specific student difficulties have been identified and analyzed in a systematic manner.

Wave-particle duality and complementarity. One study has been reported in the context of university students during and after standard instruction on quantum mechanics. Ian Johnston, *et. al.*, at the University of Sydney have recently reported results from surveys given to third-year physics students who have completed courses in modern physics and quantum mechanics.[10] The original version of the survey included two broad questions: "What is a particle?" and "What is a wave?" The questions were expected to elicit student conceptions about particles (*e.g.*, that particles are localized, travel along well-defined trajectories, and respond to forces) and waves (*e.g.*, that waves are non-localized and have such properties as superposition, reflection, and diffraction). The results from the original survey did not provide substantial insight into student thinking. The survey questions were subsequently modified. However, the questions did not require students to consider specific experiments or physical situations. Analysis of the responses to the modified survey did not reveal particular difficulties but only suggested that student understanding of quantum mechanics tends to be fragmented and unstructured.

Atomic orbitals. Descriptive studies have been carried out in Italy to try to characterize how secondary students begin to learn concepts from quantum mechanics and physical chemistry. Emphasis has been on the behavior of electrons in atoms. One prevalent difficulty that has been identified is the tendency to describe the motion of atomic electrons in terms of classical trajectories around the nucleus. R. Cervellati and D. Perugini report on the results from a questionnaire given to several hundred first-year students at the University of Bologna.[11] The students were asked questions on the basis of what they learned in high school chemistry, including the question, "What is an atomic orbital?" The intent was to gauge the ability of students who, after taking high school chemistry, could describe atomic electrons in terms of a (wave) function that gives the probability of finding the electron at a given distance from the nucleus (or at a given energy level). Very few students gave such explanations. Most of the incorrect responses were based on the belief that electrons are localized, occupy well-defined positions at every instant, and move along specified trajectories around the nucleus. It was expected that, by the time they began their studies at the university, students would have recognized that ideas from classical physics were inadequate to describe atomic electrons.

Stability of atoms. Results similar to those obtained by Cervellati and Perugini were obtained through studies conducted by Helmut Fischler and Michael Lichtfeldt at the Free University of Berlin[12] and by Hans Niedderer, *et. al.*, at the University of Bremen.[13] Fischler and Lichtfeldt report on results from interviews and questionnaires given to students from the upper secondary level of the German Gymnasium. Questionnaires were administered before instruction in quantum physics and again after five weeks of instruction. The students were asked general questions such as "Why are atoms stable?" Correct answers required specific reference to concepts in modern physics but not in classical physics. For example, it was expected that students would be able to account for the stability of atoms by applying ideas such as the delocalization of the electron, the Heisenberg uncertainty principle, and the spatial confinement of the electron around the nucleus. However, many students tended to think of the electrons as circling the nucleus in well-defined paths, as in the Bohr model. Niedderer and his colleagues at the University of Bremen have also conducted descriptive studies with upper secondary school students. Through the use of questionnaires and videotaped lessons they have identified several "states of learning" through which quantum physics students seem to evolve. Although the questions used in their studies typically did not ask about specific physical situations, the language used by students suggested a slow but gradual shift from classical ideas (such as forces and trajectories) to a probabilistic interpretation of wave functions.

Formalism of quantum states. Finally, an article written by Daniel Styer describes several interrelated misconceptions about quantum states and the formalism of quantum mechanics.[14] Students were found to have difficulty interpreting the meanings of "expectation value" and "eigenvalue." For example, some students held the incorrect belief that energy eigenstates were the only allowed states of a potential well. These difficulties and others were identified by informal methods as a result of his teaching experience at Oberlin College.

Substantial research has been conducted on student understanding of geometrical and physical optics. However, few specific conceptual difficulties have been identified and analyzed in sufficient detail to be used as a guide for curriculum development. The research described in this dissertation is intended to strengthen the current research base in physical optics and to extend it to include topics that involve the wave properties of matter, including interference and diffraction, reflection and transmission, bound states, and the interpretation of the wave function as a probability amplitude.

D. INSTRUCTIONAL CONTEXT

The research was conducted in the context of several different physics courses at the University of Washington at a wide range of instructional levels, ranging from the introductory to the graduate level. In this section we describe the overall structure and format of each course.

1. INTRODUCTORY CALCULUS-BASED COURSE ON WAVES AND OPTICS (PHYSICS 123)

The introductory calculus-based course at the University of Washington is taken mostly by science and engineering majors. Less than 5% of the students are physics majors.

Prerequisites for the waves and optics portion of the course include the calculus-based courses in mechanics and in electricity and magnetism. Topics that are typically covered in the waves and optics course include mechanical waves (such as waves on strings and sound waves), Maxwell's equations, electromagnetic waves, geometrical optics, and physical optics.

The course consists of three inter-related components: lecture, tutorial, and laboratory. Each week there are three 50-minute lectures, a 50-minute tutorial section, and a 3-hour laboratory. Each course examination explicitly tests on material from each component. Each of the three course components is described below.

a) Lecture

Lectures are taught by instructors who are assigned to teach the course for as many as three consecutive years. Students are usually expected to complete a lecture homework assignment each week consisting of questions and problems from the textbook (*Physics,* 4th ed., by Resnick, Halliday, and Krane).[15]

b) Tutorial system

The objective of the tutorials is to strengthen the reasoning skills of the students and to help them develop a functional understanding of the material in the course. This is done by trying to engage students intellectually at a sufficiently high level for them to confront and resolve conceptual and reasoning difficulties on their own. The tutorial system consists of several components, each with a specific purpose. Below we discuss each component individually.[16]

(1) Pretest

A ten-minute written, ungraded "pretest" is administered at the start of the first lecture each week. The pretest serves to indicate to the students the topic of the tutorial for that week as well as their current understanding of the subject matter. In addition, the student responses on the pretest provide information to the lecture and tutorial instructors about the knowledge state of the students. Often the pretest is posed after lecture instruction on the relevant topic, in which case the analysis of the student responses can give insight into the nature of student difficulties that persist after lecture instruction.

(2) Tutorial

In the tutorial sections students work collaboratively in groups of three or four through carefully structured worksheets.[17] The questions on the worksheets are intended to help them build a solid conceptual understanding of the material.

The tutorials use a variety of instructional strategies. For example, many follow an *elicit-confront-resolve* strategy.[18] The identification of specific student difficulties is a necessary first step in implementing this strategy. The tutorial worksheet includes questions that are designed to *elicit* the known difficulties. Students are then guided to *confront* the fact that the incorrect ideas lead to a mistaken prediction about a physical situation or to a logical inconsistency. The students are then led through the reasoning necessary to *resolve* the inconsistency and to develop a stronger understanding of the relevant concepts.

The tutorial instructors teach by questioning rather than by telling. The instructors manage small-group discussions and engage the students by asking them questions that the students may not have recognized the need to ask. In this way the tutorial instructors guide students through the reasoning necessary for them to arrive at a solid understanding of the material on their own. This type of instruction is very difficult to master and in turn requires explicit TA preparation. A graduate teaching seminar, discussed later in this chapter, serves as the context for the TA preparation.

(3) Tutorial homework

A tutorial homework is assigned each week. The homework questions are designed to help students reflect upon what they learned in tutorial and to apply their results to situations that they had not encountered previously.[19]

(4) Examination

Each examination includes at least one problem that tests on material directly from tutorials. The problem, which comprises one-fourth of the examination, serves as a post-test to assess the effectiveness of the tutorials.

c) Laboratory

Students are required to attend one 3-hour laboratory each week. The topics for each laboratory are more or less synchronized with those covered in lecture and in tutorial. One problem on each examination is written by the laboratory professor and tests directly on material covered in the lab.

Starting in Spring 1995 the Physics Education Group developed some pre-lab tutorials for the waves and optics (Physics 133) laboratories. Each pre-lab tutorial, which is done in the first hour of each lab session, is designed to strengthen conceptual understanding about the material covered in the corresponding laboratory. The standard laboratory experiments were modified so that they could be completed in the remaining two hours.

2. *INTRODUCTORY ALGEBRA-BASED COURSE ON WAVES AND OPTICS (PHYSICS 116)*

The introductory algebra-based course is a large lecture-based course in which almost all of the students enrolled plan to be life science majors. Typically there are four 50-minute lectures each week, with no tutorial or recitation sections. An associated laboratory course is offered but enrollment in this course is not required for taking the lecture course.

Prerequisites for the algebra-based waves and optics course include the corresponding courses in mechanics and in electricity and magnetism (or their equivalents). Topics covered in the course include mechanical waves, electromagnetic waves, geometrical and physical optics, and selected topics from modern physics, such as the photon nature of light, the de Broglie wavelength, electron diffraction, and radioactivity.

In Spring 1997 one lecture section of the algebra-based waves and optics course was restructured at the request of the instructor so that one of the four lectures was replaced with a tutorial section. Part of the research described in Chapters 3, 4, and 8, was conducted in this class. The tutorial system that was adopted is identical to that used in the introductory calculus-based course, described above.

3. *SOPHOMORE-LEVEL MODERN PHYSICS COURSE (PHYSICS 225)*

The second-year modern physics course represents the fifth course in the five-quarter introductory calculus-based sequence offered by the Physics Department at the University of Washington. In each class involved in this study, about half of the students were physics majors. Most, but not all, of the students took introductory physics with tutorials at the University of Washington. The size of the class is usually between 20 and 40 students. There are three 50-

minute lectures each week with no associated lab, recitation, or tutorial. The textbook that has been typically used for this course is *Modern Physics*, by P.A. Tipler.[20]

The calculus-based waves and optics course (Physics 123) or equivalent is a prerequisite. The topics covered in the modern physics course vary somewhat from instructor to instructor. Typically the course includes special relativity, the quantization of charge, the quantization of energy, the photon model for light, the de Broglie wavelength, electron diffraction, the Heisenberg uncertainty principle, the Bohr model for the hydrogen atom, and the wave function.

Occasionally, tutorials are conducted in the modern physics course. These are done by invitation of the instructor and are held during normal class time in the lecture room. A pretest is administered during the first ten minutes of the class, and the remaining time is spent working through the tutorial worksheet.

4. *JUNIOR-LEVEL QUANTUM MECHANICS COURSES (PHYSICS 324 AND 325)*

The quantum mechanics courses are taught in lecture format, with three 50-minute lectures each week. Almost all of the students who enroll in these courses are physics majors. In each class that was involved in this study, a majority of the students took introductory physics with tutorials at the University of Washington. As with the modern physics course, the quantum mechanics courses have no associated laboratory, recitation, or tutorial. Completion of the modern physics course or its equivalent is a prerequisite. Physics majors at the University of Washington are required to complete the first quarter of quantum mechanics (Physics 324) and are encouraged to take the second quarter (Physics 325).

The first quarter in quantum mechanics, which is taught in the Autumn quarter, typically begins with a review of some of the material covered in the modern physics course, including electron diffraction, de Broglie wavelength, the Heisenberg uncertainty principle, and the Bohr model for the hydrogen atom. Students then are taught how to solve the time-independent Schrödinger equation in one dimension. They consider various examples of bound states (*e.g.*, the infinite square well, the finite square well, the harmonic oscillator) and scattering states (*e.g.*, tunneling through a potential barrier). Topics that follow include an introduction to the formalism of quantum mechanics, angular momentum, the hydrogen atom, spin, and many-particle systems. The second quarter of the quantum mechanics sequence, taught the following

Winter quarter, is typically devoted to applications of quantum mechanics, including perturbation theory and the WKB semi-classical approximation.

In Autumn 1996-Winter 1997 and Autumn 1997-Winter 1998, a set of tutorials were incorporated into the syllabus and used during regularly scheduled class sessions by invitation of the instructor. The tutorial system, consisting of pretest, tutorial, tutorial homework, and post-test, emulated that of the calculus-based course. When a tutorial was conducted in class, the first ten minutes were spent administering the pretest and the remaining forty minutes were devoted to working through the tutorial worksheet.

Either two or three tutorials were used each quarter, for a total of five or six for the entire quantum mechanics sequence. In this dissertation, we report results obtained from testing three of the six tutorials that were used in the quantum mechanics classes.

5. GRADUATE LEVEL TEACHING SEMINAR (PHYSICS 503)

In conjunction with the tutorial system for the introductory calculus-based course, a weekly graduate teaching seminar is required for all first-time teaching assistants (graduate students) and other instructors who are assigned to teach in the tutorial sections. The seminar meets every Monday afternoon for the purpose of preparing the instructors to teach the tutorial designated for that week.

The activities in the seminar closely parallel those done by the students enrolled in the introductory course. The teaching assistants first take the same pretest that the students took earlier that day. They then look over student pretests to identify common errors and to characterize student difficulties. The teaching assistants then work through the tutorial in small groups in the same way as their students will later that week. Experienced graduate students demonstrate by example how to address specific difficulties that arise during the tutorial.

The responses given by the teaching assistants on written pretests are analyzed. The results are used to assess the conceptual understanding and reasoning skills of the teaching assistants before working through the tutorial and thus to identify difficulties that may persist after advanced instruction in physics. The results are also used to determine whether or not a tutorial has been effective in addressing difficulties with introductory students. A tutorial (or set of tutorials) is considered reasonably successful if the level of understanding demonstrated by introductory students on tutorial post-tests matches (or surpasses) that of the teaching assistants on tutorial pretests.

6. *FIRST-YEAR GRADUATE LEVEL QUANTUM MECHANICS COURSE (PHYSICS 517)*

In this dissertation we also present results from ungraded "pretests" that were written and administered by the instructor of the first-year graduate level quantum mechanics course in Autumn 1995. Although the pretests were not followed by tutorials, the instructor reviewed the pretest responses and modified subsequent lectures on the basis of the results.

E. RESEARCH METHODS

The research methods that have been used in the study consist of both formal and informal methods. These methods are described below.

1. *FORMAL METHODS*

The formal methods used in the study are the same as those used in other projects by the Physics Education Group. These methods include individual interviews, with and without demonstrations, and specially designed written questions posed during the course of instruction.

a) Individual student interviews

The students who participate in individual interviews volunteer to do so. Each interview lasts for about one hour. One or more investigators meet with the student and pose a series of tasks about a particular physical situation, which is often demonstrated or illustrated using props or a computer simulation. In most cases, the student is asked to predict how the outcome of an experiment would be affected by changing specific parameters. The students are asked to articulate the reasoning they are using in making their predictions. The tasks that are posed to the student are open-ended in order to provide the opportunity to ask follow-up questions and probe student reasoning in detail. The interviews are videotaped for later transcription and analysis. It has been our experience that students who participate in the interviews are in the top half of the class.

b) Written questions

We have written and administered specially designed questions for use on tutorial pretests, homework assignments, and examinations. Questions used on pretests are usually used to determine the prevalence and persistence of conceptual and reasoning difficulties that have been identified previously in the research, usually through interviews or other written questions. The questions are primarily qualitative in nature. When particular difficulties are already known, the questions are designed so that the difficulties would lead to specific incorrect responses. Examinations that are posed after tutorial instruction are used in order to assess the effectiveness of a tutorial (or set of tutorials) intended to address particular difficulties.

2. *INFORMAL METHODS*

We also have the benefit of interacting with students extensively throughout the course of instruction. As a result, we have the opportunity to observe and interact with students in lecture, tutorial, and laboratory. Informal observations and conversations with students often provide insight into the nature of certain difficulties. These are then investigated more systematically by the formal methods described above.

PART ONE:

INVESTIGATION OF STUDENT UNDERSTANDING OF INTERFERENCE AND DIFFRACTION EFFECTS OF LIGHT AND MATTER IN TERMS OF A WAVE MODEL

CHAPTER 2: AN INVESTIGATION OF STUDENT UNDERSTANDING OF SINGLE-SLIT DIFFRACTION AND DOUBLE-SLIT INTERFERENCE

This chapter has been published in the *American Journal of Physics.* The reference for this article is: Bradley S. Ambrose, Peter S. Shaffer, Richard N. Steinberg, Lillian C. McDermott, "An investigation of student understanding of single-slit diffraction and double-slit interference," Am. J. Phys. **67**, 146-155 (1999).

AN INVESTIGATION OF STUDENT UNDERSTANDING OF SINGLE-SLIT DIFFRACTION AND DOUBLE-SLIT INTERFERENCE

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Abstract

Results from an investigation of student understanding of physical optics indicate that university students who have studied this topic at the introductory level and beyond often cannot account for the pattern produced on a screen when light is incident on a single or double slit. Many do not know whether to apply geometrical or physical optics to a given situation and may inappropriately combine elements of both. Some specific difficulties that were identified for single and double slits proved to be sufficiently serious to preclude students from acquiring even a qualitative understanding of the wave model for light. In addition, we found that students in advanced courses often had mistaken beliefs about photons, which they incorporated into their interpretation of the wave model for matter. A major objective of this investigation was to build a research base for the design of curriculum to help students develop a functional understanding of introductory optics.

A. INTRODUCTION

In this paper, we report on the ability of introductory and more advanced students to select and apply an appropriate model to account for the pattern produced on a screen when light is incident on a single or double slit. The motivation for this research was our observation that many students who had completed the introductory calculus-based course lacked a consistent conceptual framework for the nature and behavior of light. Even the strongest students, who could readily solve standard textbook problems, were frequently unable to relate the results to the models for geometrical and physical optics that they had been taught. There was evidence of serious difficulties with the wave model beyond the introductory course. In addition, we found that students in advanced courses often had mistaken beliefs about photons, which they incorporated into their interpretation of the wave model for matter.

The work described is part of a long-term investigation in which the Physics Education Group at the University of Washington has been examining student understanding of geometrical and physical optics.[22-25] A major objective has been to establish a research base that can be used as a resource by physics instructors. Examples of the application of this research can be found in curriculum developed by our group.[26,27]

B. OVERVIEW OF THE INVESTIGATION

Most of the investigation was conducted at the University of Washington among students enrolled in one of two courses: the introductory calculus-based course or a subsequent modern physics course. The study also included students in the algebra-based course, physics majors in a junior-level quantum mechanics course, and graduate teaching assistants. Additional data were obtained from an introductory calculus-based physics course at the University of Maryland.

Although all prospective physics majors at the University of Washington enroll in the first-year calculus-based course, they constitute less than 5% the class. Most students intend to major in engineering, in mathematics, or in other sciences. In contrast, about half of the modern physics students are physics majors at the sophomore level or above.

1. INSTRUCTIONAL CONTEXT

During their first year of physics, students are introduced to two different ways of thinking about the behavior of light. It is generally assumed at the beginning of the course that the class is already familiar with the idea that light from an object travels outward in all directions in straight lines. The students study reflection and refraction, draw ray diagrams, and solve numerical problems for a variety of simple optical systems. Later, in physical optics, the students learn that light is a transverse electromagnetic wave that propagates through space. They are taught the concepts and formal representations that are used to predict and explain diffraction, interference, and polarization.

In a typical modern physics course, an understanding of the wave nature of light is assumed at the outset. When students learn about the photoelectric experiment, their attention is called to the inconsistency between the results and the predictions of physical optics. The discrepancy between theory and experiment is resolved by the introduction of the photon. The instructor explains that light appears to behave like a wave in certain situations and like a particle in others. Discussion of the wave-particle duality leads to study of the wave-like properties of matter and the foundations of quantum mechanics.

2. *RESEARCH METHODS*

The same research methods were used in this investigation as in others by our group.[28] The data consist of interview transcripts and student responses to written questions.

a) Initial interviews

An initial set of tasks provided the basis for a series of individual demonstration interviews that were conducted with 46 students from the introductory and modern physics courses. The 16 introductory and 30 modern physics students who participated were volunteers from several lecture sections in both courses. The interviews with the introductory students took place during the last week of the quarter, after all instruction had taken place. In the modern physics course, the students had reviewed interference and diffraction of light and had covered similar topics in the context of matter waves. All earned grades at or above the mean in their respective courses. The interviews, which lasted from 45 minutes to an hour, were videotaped and analyzed in detail.

(1) Interview tasks

At the beginning of each interview, the student was shown a small unlit bulb, a white paper screen, and a cardboard mask containing a slit 1 cm wide and 3 cm tall. (See Figure 2-1.) The student was told to assume that the room was darkened and that the bulb was lit. The main part of the interview was based on the following three questions:

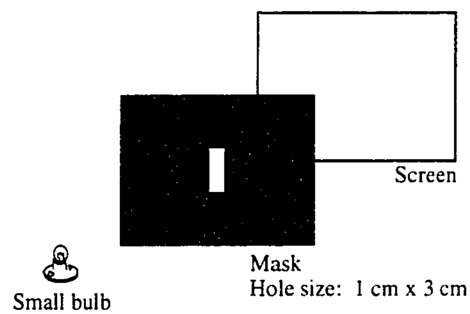


Figure 2-1: Apparatus from individual demonstration interviews.

1. What would you see on the screen if the bulb, mask, and screen were arranged as shown?

2. How would your prediction change if the bulb were moved farther and farther from the mask?

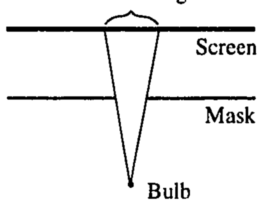
3. Assuming the bulb to be very far from the mask (and very bright), how would your prediction change if the slit were made narrower and narrower?

In addition to these three main questions, the investigator often posed others. Many of the students were asked about light incident on two slits. In some of the interviews, students expressed their ideas about photons, either spontaneously or at the suggestion of the investigator. During the interviews with the modern physics students and subsequent interviews with the quantum mechanics students, questions were posed about electron diffraction and interference. In all cases, students were asked to explain their reasoning.

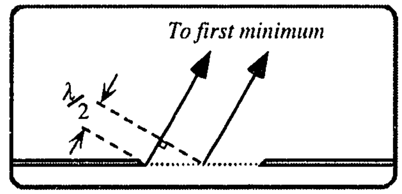
The investigator tried to ensure, either tacitly or overtly, that certain simplifying assumptions would be made. If students believed that the bulb would not be bright enough to produce a pattern on the screen, they were asked to suppose that the bulb was extremely bright. If they seemed to think of the bulb as an extended source, they were told to treat it as a point source. If they recognized that the light would be composed of many colors, they were told to imagine a red bulb. The students would have been told to consider only Fraunhofer diffraction had any attempted to treat the narrow slit otherwise.

(2) Correct responses to interview tasks

Correct responses to the three main interview questions could be given on the basis of material covered in the waves and optics portion of the introductory course. A student can answer the first two questions, which are based on a wide slit, by drawing straight lines from the bulb through the slit to the screen. [See Figure 2-2(a).] The size and shape of the illuminated region can be determined by considering similar triangles or by using trigonometry. As the bulb is moved farther and farther from the mask, the student needed to recognize that the rays through the slit become more nearly parallel. Eventually, the *geometric image* approaches the same size as the slit.[29]



Wide slit
(a)



Narrow slit
(b)

Figure 2-2: Diagrams that illustrate (a) the width of the illuminated region on the screen for the case of light incident on a wide slit and (b) the location of the first diffraction minimum produced by light incident on a narrow slit.

In the third question, as the slit is made narrower, the illuminated region becomes narrower. Eventually, geometrical optics is no longer valid and a single-slit diffraction pattern with a bright central maximum and smaller subsidiary maxima appears on the screen. In an introductory course, Huygens' Principle is used to account for the pattern.[30] Light from the slit is treated as emanating from many coherent, closely-spaced, secondary point sources. At the angle θ corresponding to the first minimum, light from one edge of the slit is 180° out of phase with light from the center. [See Figure 2-2(b).] The remaining secondary point sources also cancel in pairs. From trigonometry, the condition for the first diffraction minimum is $a \sin \theta = \lambda$ (where a is the slit width).

If the slit width is decreased further, the first minimum moves away from the center of the pattern and the central maximum becomes wider. Diffraction minima occur only when the slit width is larger than the wavelength. If the slit width is smaller, no minima appear since the waves from all the secondary point sources along the width of the slit cannot all cancel at any point on the screen.

b) Written questions and additional interview tasks

The results from the interviews guided the design of written questions that were administered to large numbers of students. Analysis of the interview transcripts and the written responses yielded information on the nature of specific difficulties, provided a rough measure of their prevalence, and suggested other questions and interview tasks. The written questions and additional interview tasks are discussed later in the paper.

C. IDENTIFICATION AND ANALYSIS OF STUDENT DIFFICULTIES

We have organized the student difficulties that we identified into three broad, overlapping categories: misapplication of geometrical and physical optics, lack of a qualitative understanding of the wave model, and difficulties with modern physics concepts. (Difficulties specific to geometrical optics are discussed in other papers.[22,23]) This classification provides a convenient structure for discussing a wide variety of student errors and for drawing inferences about the nature of the underlying conceptual and reasoning difficulties.

The errors that are used as illustrations are not intended to be an exhaustive list. Rather, they are symptoms of the lack of a coherent conceptual framework for optics. In order to address this more general (and far more serious) difficulty, it is necessary to be aware of common mistaken beliefs that can hinder the progress of students.

The particular interview and written questions presented in this paper are representative of a broad range of tasks that we have used to probe student understanding. The process of identifying specific student difficulties includes a long sequence of interviews, pre- and post-testing, and extended discussions with students. Questions are developed, modified, and tested over an extended period on the basis of extensive research. In analyzing student responses, we try to ascertain what students are thinking, rather than interpreting responses to isolated questions. In this paper, we report only difficulties that are widespread, persistent, and independent of the instructor. The quotes have been selected because they are typical and not idiosyncratic.

1. MISAPPLICATION OF GEOMETRICAL AND PHYSICAL OPTICS

During the interviews many students failed to consider whether geometrical or physical optics was valid in a given situation. They tried to apply one model when the other was appropriate or to combine ideas from both in a "hybrid" model.

a) Use of ideas from physical optics for light through a wide slit

Students often tried to apply ideas from physical optics to predict the appearance of the screen with a 1-cm wide slit. There were a variety of errors. Some students thought that diffraction fringes or a fuzzy boundary around the geometric image would be readily visible. These students did not recognize that the slit was sufficiently wide that, to a very good approximation, geometrical optics would be valid. Other students had serious difficulties that indicated a profound lack of understanding of either geometrical or physical optics. As illustrated by the student quotes in this paper, it was evident that none of the difficulties arose due to subtleties of the apparatus (*e.g.*, edge diffraction or the type of light source).[31] The example below is taken from the transcript of a modern physics student.

Mistaken belief that all slits, regardless of width or shape, can be treated like a single secondary point source of light. The student drew a set of curved wavefronts emanating from the bulb and a second set from the slit. (See Figure 2-3.) She described the light as "spreading out" from the slit, and said this "overcomes the problem of [light] passing through a little hole." In effect, she treated the entire slit as a single, secondary point source of light.

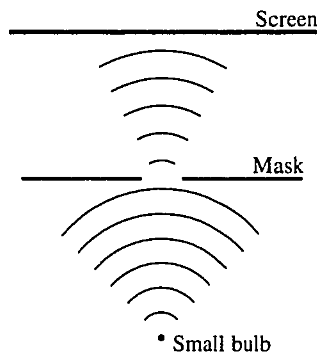


Figure 2-3: Diagram drawn by a student who incorrectly treated the wide slit ($\approx$1 cm) as a single secondary source of light waves. On the basis of her diagram, she concluded that the illuminated portion of the screen would be larger than the geometric image.

This student also predicted that the illuminated region on the screen would be approximately circular in shape, even if the slit were rectangular: "I'm trying to think if it will happen the same no matter what the shape of the opening is.... It shouldn't matter what shape it is." When asked how her prediction would change as the slit was made narrower or wider, she claimed that there would always be a broad, continuous, bright region. She never predicted the appearance of maxima and minima.

b) Use of geometrical optics for light through a narrow slit

Students also made the converse error to the one discussed above. Many attempted to apply ideas from geometrical optics to account for the pattern produced by light incident on a narrow slit. One example is provided by responses to a question given to about 410 introductory students from several classes with different instructors. The students had completed all standard lecture and laboratory instruction on single-slit diffraction. They had also taken an examination that tested on this material. They were shown a single-slit diffraction pattern with several maxima and minima and asked to predict how covering the right half of the slit (*i.e.,* narrowing the slit) would affect the distance between the first minima. The results are given in part 1 of Table 2-1 below.

Table 2-1: Results from a two-part written question on single-slit diffraction given to introductory students and to physics graduate students. The students had completed all standard lecture and laboratory instruction on single-slit diffraction and had been tested on this material. [Percentages have been rounded to the nearest 5%.]

Part 1: Prediction for the effect of covering half of the slit (*i.e.,* decreasing the slit width) on the location of the minima.	Introductory calculus-based course ($N \approx 410$)	Graduate teaching seminar ($N \approx 95$)
Correct: minima move farther from center	**70%**	**95%**
with correct reasoning	20%	60%
Incorrect: minima move closer to center	**20%**	**5%**
Incorrect: minima remain at same locations	**10%**	**<5%**
Part 2: Comparison of the slit width, *a*, to the wavelength, λ, of the light.		
Correct: $a > \lambda$	**40%**	**80%**
with correct reasoning	10%	55%
Incorrect: $a < \lambda$	**40%**	**15%**
Incorrect: $a = \lambda$	**10%**	**5%**

Mistaken belief that narrowing a slit produces a narrower central diffraction maximum. Despite having seen lecture demonstrations that showed the effect of narrowing a slit, about 20% of the students predicted a decrease in the distance to the first minimum. They seemed to be applying ideas from geometrical optics, in which narrowing a wide slit results in a narrower geometric image. Analysis of the final course grades revealed that the 20% did not consist only of students at the bottom of the class. An equal number of students with grades above and below the mean had this difficulty.

c) Use of a hybrid model with elements of geometrical and physical optics

Many of the errors made in the interviews and on responses to written questions suggested that students were not only confused about whether geometrical or physical optics applied but also that they could not separate one model from the other. They did not seem to be aware of critical differences and applied ideas from both indiscriminately.

(1) Inappropriate, simultaneous application of ideas from both geometrical and physical optics to account for diffraction

Students often used geometrical optics for light through the center of a slit and physical optics for light at the edges. They drew straight lines to show the path of light through the center of the slit and attributed diffraction to some sort of interaction between the incident light and the edges. Instead of thinking of many point sources along the entire width of the slit, they often drew circular waves that emanated only from the edges.

Mistaken belief that the central maximum in a diffraction pattern is the geometric image. A firm belief that the central maximum is due to light through the center of the slit was expressed by several students during the interviews. One introductory student stated that for a sufficiently narrow slit, "some of the light will go through without being diffracted and then the light that strikes the edges will be diffracted off." As was the case with this student, the incorrect use of geometrical optics for light through the center of the slit was almost always accompanied by a misapplication of physical optics.

Mistaken belief that a diffraction pattern is produced only by incident light that strikes the slit edges. At least 25% of the students who participated in the interviews seemed to think that only the light incident on the slit edges contributed to diffraction effects. Many of these students claimed that light "bounced" off the edges. One modern physics student referred to Huygens' Principle but drew semicircular wavefronts emanating from each slit edge. (See Figure 2-4.) He attempted to account for a diffraction pattern by attributing the fringes only to interference of light from the edges of the slit.

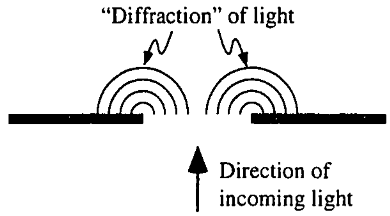


Figure 2-4: Sketch by a student who attributed the presence of "diffraction" maxima and minima to the interference of light waves emanating only from the edges of the slit.

Another modern physics student took this idea one step further. He said that, since a single slit of width d has two edges that act as point sources, the diffraction pattern should look identical to that of two point sources a distance d apart. He wrote the double-slit equation $d \sin\theta = m\lambda$ and decided that the *maxima* (not the minima) of a diffraction pattern occur at angles θ. From his explanation, it was clear that the student was not simply confusing the formulas for diffraction and interference.

(2) Failure to recognize the difference between superposition in geometrical and in physical optics

It was not only in single-slit diffraction that we identified a tendency among students to use a hybrid model. Failure to distinguish between geometrical and physical optics also became obvious when students tried to account for the interference pattern for light incident on a double slit. In particular, many students did not seem to understand that the superposition of light must be treated differently in the two cases. This difficulty became apparent during the analysis of student responses to the following written question that was given to more than 200 introductory students after lecture instruction on interference.

The students were shown a photograph of the central portion of a double-slit interference pattern. (The fringes were of approximately equal intensity.) The students were asked to predict how the pattern would change if the left slit were covered. Only 40% of the students gave the correct answer that the screen would be almost uniformly illuminated with no diffraction minima visible. Almost half of the students made errors based on the ideas described below.

Mistaken belief that each slit alone produces the same pattern (only dimmer) as that produced by the pair of slits. About 25% of the students thought that all the fringes would remain when one slit was closed. Most of these students stated that covering one slit would only make the fringes become dimmer. They seemed to think that each slit alone must produce the same pattern as the pair of slits. To account for the double-slit pattern, they were effectively adding intensities from the individual slits without regard to the phase difference of the light from the two slits. They were trying to apply ideas from geometrical optics to an interference phenomenon.

Mistaken belief that each slit alone produces half the pattern produced by the pair of slits. Another 20% of the students treated the double-slit interference pattern as the result of a juxtaposition of two patterns, each formed by one of the slits. These students seemed to believe that each slit was responsible for half of the interference pattern. Some claimed that all the bright fringes on the left half of the pattern would disappear when the left slit was covered. However, only under the conditions of geometrical optics is it possible to associate a particular bright region with a particular slit.

2. *LACK OF A QUALITATIVE UNDERSTANDING OF THE WAVE MODEL*

The results from the interviews and written questions discussed above indicated that many students did not know whether to use ideas from geometrical or physical optics in a particular situation. Even when they recognized that the wave model was appropriate, however, students often did not understand how to apply it in accounting for diffraction or interference. Some common errors are described below.

a) Misuse of comparisons between slit size and wavelength (or amplitude) to account for diffraction

Students usually recognized that the width of the slit affects the diffraction pattern. However, in many instances they made inappropriate comparisons between the physical dimensions of the slit and the wavelength or amplitude of the light. (Students usually referred to the wavelength, even though they seemed to know that it is measured along the direction of propagation, which was perpendicular to the slit.) Many attributed diffraction to the failure of light to "fit" through the slit.

Mistaken belief that no light passes through a slit if $a < \lambda$ and therefore the screen is totally dark. Some students who knew the appropriate equation for single-slit diffraction believed that light may or may not "fit" through the slit. In the interviews, some students predicted that for the case in which the slit width is less than the wavelength no light would get through the slit at all.

This claim was made not only by introductory students. The quote below was taken from an interview with a modern physics student.

S: [Pointing to the equation, "$a \sin \theta = \lambda$," that he had previously written down.] If we wanted a to be shorter than this wavelength...then this $\sin \theta$ would have to be greater than one, which it can't be.

I: And what do you conclude from that?

S: That no light can pass through a slit that's smaller than the wavelength of light.

Mistaken belief that diffraction occurs only if $a < \lambda$ and that there are diffraction fringes in that case. In addition to students who thought that no light would pass through a slit with width less than the wavelength, there were others who seemed to believe that diffraction occurs only if the slit width is less than the wavelength (or amplitude).

One example is provided by an interview with a modern physics student who was asked to consider the situation in which a bright red bulb was placed very far from a single slit. She correctly used ideas from geometrical optics for the case of the 1-cm wide slit but continued to use these same ideas as she considered narrower and narrower slits. Even when the investigator asked what would be seen on the screen if the wavelength were just 1 nm larger than the slit width, the student used geometrical optics to predict that the geometric image of the slit would appear on the screen. When asked, however, what she would see on the screen for a slit width 1 nm smaller than the wavelength, she stated that the light "has to bend in order to fit through" the slit. She predicted that a diffraction pattern would appear because "it's hard to put something that's big through something that's smaller than it is."

This type of reasoning was common among introductory students as was shown by a written question administered to about 410 students. (These are the same students whose performance on another question was discussed earlier. They had completed traditional lecture and laboratory instruction on single-slit diffraction and had been tested on this material.) The students were shown a diffraction pattern with several minima. They were asked to compare the wavelength to the slit width.

As can be seen in Part 2 of Table 2-1 (shown previously in section 1.b), only 40% correctly stated that the slit width would be larger than the wavelength. About 40% said that it would be smaller than the wavelength. An additional 10% claimed that the slit width was equal to the wavelength. Incorrect explanations included "slit width is less than λ or else there would not be so much diffraction" and "less than λ because the light didn't travel in a straight path."

The above question has also been posed to about 95 physics graduate students who were teaching assistants in the introductory course. Approximately 15% incorrectly stated that the slit width had to be less than the wavelength of the light. Many of these students gave explanations similar to those given by the introductory students.

Mistaken belief that diffraction and polarization are directly related. In several of the interviews, we included an additional task in which we asked students to predict how placing a polarizing filter between the small bulb and the slit would affect what they would see on the screen. We found that after polarization had been introduced in lecture, students often treated the slit as having the characteristics of a polarizer. In order to determine the prevalence of this difficulty, we posed the following written question to about 400 students in three introductory classes after polarization had been covered in class. Most of the students had also completed the relevant laboratory experiment.

The students were shown a diagram containing a light source, a single narrow vertical slit, and the resulting diffraction pattern. The pattern had several minima. The students were told the light is unpolarized. They were asked how, if at all, the appearance of the screen would change if a polarizer were placed in front of the slit with its transmission axis oriented (a) vertically and (b) horizontally. Explanations were required.

The data were similar in all three classes. About 25% of the students correctly predicted that for both orientations of the polarizer the intensity would decrease, but the pattern would not otherwise change.[32] Analysis of the other responses indicated that many students were treating the slit as if it were a polarizer. They predicted that the pattern remains unchanged when the transmission axis is vertical (parallel to the slit) and becomes significantly dimmer or completely dark when the transmission axis is horizontal. This error was made by about 40% of the students and was the most common incorrect answer. The following quote illustrates this type of student response.

> "There would be no pattern because [when the transmission axis is horizontal, the]...polarizer would block one plane of light and the slit would block the other."

The second most common error seemed to be based on the belief that the relative orientation of the slit and transmission axis of the polarizer determines whether physical or geometrical optics applies. About 10% of the students said that when the transmission axis is parallel to the slit, there is no diffraction. Instead, the geometric image of the slit appears on the screen. Typical student explanations were based on the incorrect idea that "the vertical waves passing through the slit do not get diffracted." One student, who seemed to attribute diffraction effects to only the edges of the slit, predicted that the light would not "spread out" because "there would be no interaction with the sides of the slit."

b) Failure to recognize the critical role of the path length (or phase) difference

During the interviews and on responses to the written questions, many students did not seem to recognize the critical role of the difference in path length (or phase) in determining the locations of the maxima and minima. Analysis of student responses to two written questions provides evidence for this generalization. The first, which was administered to introductory and graduate students, asks about the interference pattern produced by two small objects vibrating up and down in unison in a large water tank. The second question, which was administered to students in the modern physics course, asks about the effect on a double-slit interference pattern of a change in the slit separation. In this case, the pattern was produced by electrons. For both written questions, however, the reasoning required is the same as for light waves. The first question is described immediately below and the second later in this paper.

On the first of the two questions, the students were shown a top view of a water tank in which two sources of circular waves are separated by 2.5λ. Three points *(A, B,* and *C)* are labeled. (See Figure 2-5.) For each point, students are asked to determine whether there is maximum constructive interference, complete destructive interference, or neither. They are also asked to determine the phase difference at each point.

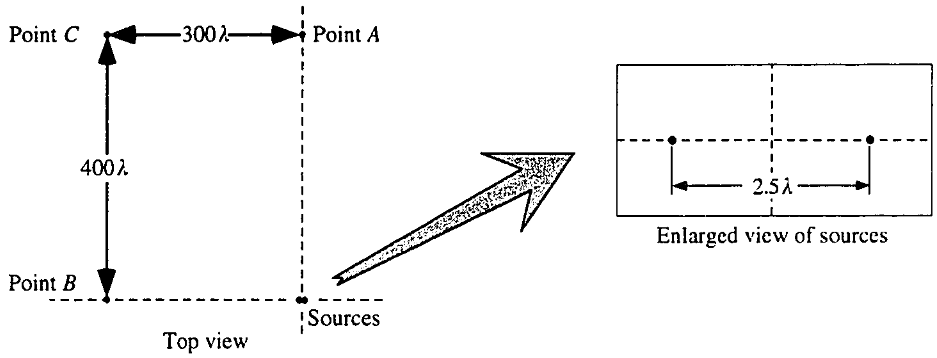


Figure 2-5: Top-view diagram of a water tank with two sources of circular waves. For each of the three labeled points *(A, B,* and *C),* students are asked to determine whether there is maximum constructive interference, complete destructive interference, or neither. They are also asked to determine the phase difference at each point.

Students needed to recognize that at point *A* the path length difference is zero, and at point *B* it is equal to the source separation (2.5λ). Thus, there is maximum constructive interference at point *A* and complete destructive interference at point *B*. (Small differences in amplitude at points far from the sources are neglected.) For point *C*, students could use the Pythagorean theorem to calculate the path length from each source and then find the difference. Alternatively, they could use the approximation that the path length difference, ΔD, is approximately equal to $d \sin\theta$, where d is the source separation. The angle, θ, for point *C* is given by $\sin\theta \approx 300\lambda / 500\lambda = 0.6$. Thus there would be destructive interference at point *C*, since ΔD is equal to an odd half-integer multiple of λ: $\Delta D \approx (2.5\lambda)(0.6) = 1.5\lambda$.

This written question has been given to about 1200 introductory students. The results do not seem to depend on whether or not the students have had lectures on interference. About 35% have given correct answers for points *A* and *B*. Only about 10% have attempted to find the difference in path length from the sources to point *C*. The question has also been given to about 95 physics graduate students in their first year as teaching assistants. Although more than 80% gave correct answers for points *A* and *B*, their success rate for point *C* was only 55%. Most incorrect responses revealed that, like the introductory students, they did not try to find the path length difference for point *C*.

Mistaken belief that the interference of waves at a point is determined by the path length or the direction from each source. Students sometimes used path length (instead of difference in path length) to decide whether the waves from the two sources would interfere constructively or destructively. For example, some reasoned that since point B is 300λ from the sources (a whole number of wavelengths), the waves would interfere constructively. Other students based their reasoning on the relative directions of propagation of the waves from the two sources. Some stated that at point B the waves are moving in the same direction and therefore interfere constructively. Similarly, at points A and C, students claimed that some destructive interference occurs since the waves are not moving in exactly the same direction. They did not recognize that the superposition of waves involves phase difference and amplitude, not direction of propagation.

Mistaken belief that the path length difference becomes negligible at large distances. Some students seemed to believe that the difference in path length (or phase) is negligible at all points far from the sources of the waves. For example, one student stated that point C would experience constructive interference because the distance "from source to C is much greater than [the distance] between the sources." Another student gave the same answer and explained, "I suppose that 2.5λ [the slit separation] is small compared to 400λ and 300λ, so the sources here act like a single source."

c) Memorization of algebraic formulas without an understsanding of the derivations

In addition to the investigation that took place at the University of Washington, we had the opportunity to collect some data from students in the introductory calculus-based course at the University of Maryland. This is one of the test sites for *Tutorials in Introductory Physics*, the curriculum that our group is developing to supplement the lectures and textbook of the standard introductory course. The structure of the waves and optics portion of the calculus-based physics course is similar at the two universities. At both, students attend three lectures each week, one small-group 50-minute session, and an associated laboratory course. At the University of Washington, all the small-group sessions are tutorials that focus on the development of concepts and qualitative reasoning. Very little time is spent on numerical problems. For some classes at the University of Maryland, the small group sessions are tutorials. For others, they are more traditional recitations, in which the emphasis is on problem solving.

It has been possible to give identical examination questions in the tutorial and recitation sessions at the University of Maryland. On qualitative questions similar to those that have been described, the students at the University of Maryland who were enrolled in classes with tutorials did significantly better than those in classes with traditional lecture sections. We have also been able to compare the performance of students in the two groups on a quantitative examination problem based on material typically covered in the lectures, homework, and recitation of an introductory course.

On the examination question, the students are told that light of wavelength $\lambda = 500$ nm is incident on a mask containing two slits separated by a distance $d = 30\ \mu$m. The distance between the center of the screen and the first dark fringe is $x = 1.5$ cm. The students are asked to determine the distance L between the screen and the mask.

To determine the answer, students must recognize that the first minimum corresponds to a difference in path length from the two slits, ΔD, that is equal to $\lambda/2$. If the screen is far from the mask, the path length difference is approximately $d \sin\theta$ or $d \tan\theta$, where θ is the angle measured from the center of the slits. From geometry, $\tan\theta$ is equal to x/L. The correct answer, $L = 1.8$ m, can be found by solving the equation $\lambda/2 = d \tan\theta = dx/L$.

This problem was administered to 165 students in the traditional lecture-recitation-laboratory course. Only 15% gave the correct response. The most common incorrect answer, $L = 0.9$ m, was given by approximately 40% of the students. Most of these students used the formula $(d \sin\theta = m\lambda)$ that they had memorized for the location of an interference *maximum* and simply substituted numerical values. In contrast, 60% of the approximately 115 tutorial students gave correct responses. Only 10% of these students gave the incorrect answer that $L = 0.9$ m. The tutorial students were much more likely than the recitation students to consider difference in path length and to use superposition.

3. DIFFICULTIES WITH MODERN PHYSICS CONCEPTS

The primary purpose of the initial interviews with the introductory and modern physics students was to elucidate the ideas that they had about the wave nature of light. However, as they attempted to justify their predictions for light incident on a single or double slit, 16 of the 30 modern physics students referred to photons. We used the opportunity to explore how these students related their ideas about photons to the behavior of light. In 11 of the interviews, we also had the opportunity to probe the ideas that the students had about the wave properties of electrons.

Later in the investigation, we conducted a set of 14 interviews with physics majors from a junior-level quantum mechanics course. Like the modern physics students in the initial interviews, almost all of these students earned final grades above the mean for their class. In addition to tasks that were similar to those posed during the initial interviews, the quantum mechanics students were asked questions to elicit their beliefs about the de Broglie wavelength of electrons. The results guided the design of written questions that were posed on examinations in the modern physics and quantum mechanics courses.

a) Light as photons

Twelve of the 16 modern physics students and 6 of the 14 quantum mechanics students who referred to photons in their explanations articulated ideas that were unanticipated from students who had progressed beyond introductory physics. In most instances, the naive beliefs that they expressed about the properties of photons could not possibly account for diffraction and interference effects. Below are a few examples.

Mistaken belief that photons move along straight paths that "bend" near slit edges. Many students tried to account for the behavior of light by treating photons as point particles that travel along paths that "bend" near the edges of a slit. For example, one student stated that the paths of the photons "would bend around an obstruction so that the light rays ... can spread out." She applied this idea to light passing through the 1-cm slit. She drew a top-view diagram like the one shown in Figure 2-6 and concluded that the illuminated region on the screen would be considerably wider than would be predicted by geometrical optics. Moreover, she drew on the same belief for the case of a narrow slit and never predicted that the pattern on the screen would contain bright and dark fringes. Thus, the idea that the path of a photon "bends" at the slit edges led this student to incorrect predictions for situations in both geometrical and physical optics.

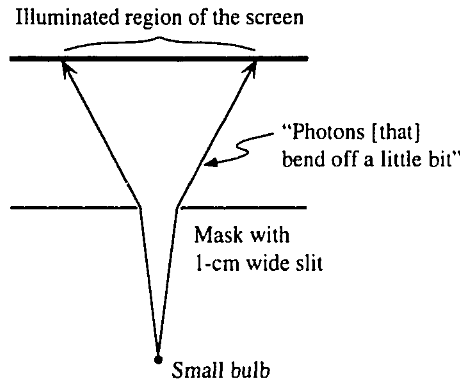


Figure 2-6: Diagram drawn by a modern physics student who believed that photons travel on straight paths that "bend" near the edges of a slit. She used the same diagram to predict the appearance of the screen for both a wide and a narrow slit.

Mistaken belief that photons move along sinusoidal paths. Almost half of the modern physics students who referred to photons (7 out of 16) tried to account for wave phenomena by thinking of photons as point particles that travel along sinusoidal curves.[33] For example, one student drew a diagram like the one shown in Figure 2-7 to show light passing through a narrow slit. He drew a sinusoidal curve with several dots along the curve, stating that "each point on the wave is a little particle," which he later called a "photon." He stated that "some of these particles will crash into this wall and the other ones will travel through." Thus, "part of the amplitude is cut off." He initially thought that his diagram illustrated diffraction but later realized that he could not use it to account for the bright and dark fringes.

SIDE-VIEW DIAGRAM

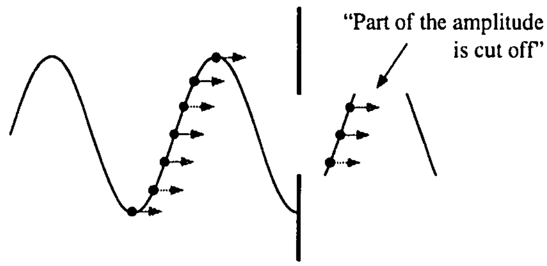


Figure 2-7: Diagram drawn by a modern physics student while trying to account for diffraction. The student believed that photons (represented by the dots) travel along sinusoidal curves.

Another modern physics student articulated a similar idea when he stated that a sinusoidal curve represents "the path that a particle [of light] would take." To illustrate, he drew a top-view diagram of a narrow vertical slit with a sinusoidal curve incident on the mask. [See Figure 2-8.] He said that "this particle would run into the wall..., so it wouldn't go through." He then drew a curve with a smaller amplitude to represent a wave that passes through the slit. He summarized his thoughts by saying "light that's oriented more vertically is going to pass through," and that "when your slit kind of gets too narrow...it's going to block off the photons from passing through like a polarizer would."

TOP-VIEW DIAGRAM

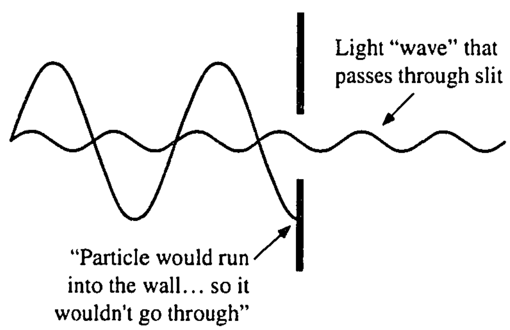


Figure 2-8: Diagram drawn by a modern physics student who believed that photons travel along sinusoidal paths. The student drew the above diagram to try to account for polarization.

Mistaken belief that two or more photons are required for diffraction or interference minima to occur. Those modern physics students who predicted that a sufficiently narrow slit would produce a pattern with maxima and minima were also asked how (if at all) the pattern would differ if the intensity of the incident light were decreased. Several students predicted that no interference would occur if the light were sufficiently dim. When asked what would appear on a photographic plate in a single-slit diffraction experiment with extremely low intensity light, one student responded as follows:

> S: If [the intensity] was really really low, ... you would just see random distribution of light coming in, just like random spots.... They're not going to be interfering with each other anymore, because there's so few of them....
>
> I: What if we let that experiment go on for a month? What would you think you'd see on the film?
>
> S: Pretty much lots and lots of dots.... I guess the whole plate would be lit.

Many of the students were also asked essentially the same question for the case of a double slit. Similar difficulties emerged. For example, while pointing to a minimum in a photograph of a double-slit interference pattern, a quantum mechanics student explained that if the photograph were replaced by a photographic plate:

> "Every time a photon would hit the plate, it would make a little spot.... Photons would still hit here, but since only one could pass through, there wouldn't be another one that could destructively interfere with it. This part [the minimum] would also appear to be light."

The difficulties illustrated above only arose when students were asked about diffraction or interference experiments with low intensity light. Many of these students failed to recognize that each photon can be treated as a wave incident on the slit, and thus the predictions of physical optics remain valid for extremely low intensities.[34,35]

b) Electrons as waves

The incorrect beliefs described above were not restricted to photons. Many of the errors made by the modern physics and quantum mechanics students revealed that they had the same mistaken beliefs about electrons as they did about photons. They also failed to recognize some key differences between the wavelength of light and the de Broglie wavelengths of particles that have mass, such as the electron.

(1) Extension of incorrect ideas about photons to electrons

During the initial interviews, 5 of the 11 modern physics students indicated that they thought that electrons move along straight paths that "bend" at the slit edges, that electrons move along sinusoidal paths, or that two or more electrons are necessary to produce interference effects. We also posed similar questions to 11 quantum mechanics students in the interviews, and 3 of those 11 students made similar errors.

(2) Failure to recognize that the de Broglie wavelength is not an inherent property of an electron (or other particle) but varies with momentum

In many of the interviews with quantum mechanics students, there was sufficient time to probe their understanding of the de Broglie wavelength of electrons and other particles. The results prompted us to design written questions that have been given on final course examinations to about 80 modern physics students and about 40 quantum mechanics students. The first of these two questions is similar to that given in the interviews. Students were asked to predict how a diffraction (or interference) pattern produced by electrons would change if the velocity (or kinetic energy) of the electrons were increased. In the second question, students were asked what would happen to the pattern if the experiment involved particles with greater mass (*e.g.*, neutrons) but the same kinetic energy as the electrons.

To arrive at a correct answer, students could use the inverse relationship between the de Broglie wavelength and the momentum of the particle. In both cases, the wavelength would decrease. Therefore, the interference fringes would move closer together. For each of the questions, about 75% of the students in each class failed to give a correct answer with a correct explanation.

On both the examination questions and during the interviews, there were a few students who expressed uncertainty about the relationship between the de Broglie wavelength and the velocity of an electron. One student articulated his confusion during an interview in the following way: "I can't think if there is a correlation between velocity and the actual wavelength of the electron, or if its wavelength is a property of the electron itself, regardless of velocity."

Analysis of the remaining written responses revealed two common types of difficulties that were similar to ideas expressed by students during the interviews. Most of the students either failed to recognize that the de Broglie wavelength depends on the electron velocity or applied certain equations for the wavelength of light in a way that was invalid for electrons. The students who made both of these errors were evenly distributed above and below the final mean grades for their classes.

Mistaken belief that diffraction and interference effects are independent of velocity. On the first written question, about 40% of both the modern physics and quantum mechanics students predicted that the pattern would not change. Many stated that a change in electron speed would affect only the overall intensity of the pattern and not the fringe locations. One student justified his incorrect answer by stating that the "electrons are just moving faster and λ is the same, so screen would not change."

Mistaken belief that all equations that apply to the wavelength of light apply to the de Broglie wavelength of electrons. About 20% of the students in each class wrote down formulas that are valid for photons but not for electrons. Some modern physics students incorrectly concluded that the de Broglie wavelength is directly proportional to the electron velocity by misinterpreting the equation $v = \lambda \nu$. Others, in both the modern physics and quantum mechanics classes, used the expression for photon energy, $E = hc/\lambda$, to claim that particles with different mass but with the same kinetic energy have the same de Broglie wavelength. Lacking a functional understanding of the wave properties of matter, the students tended to focus on memorized formulas and not on the physics.

D. CONCLUSION

In a standard introductory course, students are presented with at least two different conceptual models for the nature and behavior of light. We found that many students in introductory and more advanced courses do not understand the basic features of these models nor do they recognize the conditions under which each should be applied. Even the strongest students in the introductory course had serious conceptual and reasoning difficulties. Some of these were prevalent in a sophomore level modern physics course and persisted even among students taking a junior-level course in quantum mechanics.

Some of the difficulties that we identified were so severe that they had precluded students from attaining the type of qualitative understanding that should be a minimum outcome of an introductory course. It is impossible to construct a coherent model for the treatment of light as a wave without being able both to distinguish and to relate certain basic ideas (*e.g.*, wavelength, path length, path length difference, and phase difference). Moreover, since many of the critical features of the wave model are based on inferences and not on direct observations, students need to be able to interpret various abstract representations. Some difficulties with the formal representations of light as an electromagnetic wave are described in another paper, which also includes the discussion of a tutorial expressly designed to address these difficulties.[36]

The results of this study have clear implications not only for the teaching of optics in the introductory course but also for reform efforts directed toward introducing topics from modern physics into that course. We found that after traditional instruction, many students could not apply the wave model to account for the diffraction or interference of light incident on a single or double slit. The ability to do so, however, would seem to be a necessary condition for understanding the wave nature of matter.

In this and other instances, we have found that advanced study in physics does not necessarily overcome serious difficulties with basic material.[37] Unless these are explicitly addressed in the introductory course, they are likely to persist and preclude the learning of more advanced topics. There is a need for instruction that addresses the conceptual and reasoning difficulties that students have with the wave model for light. There is evidence that the tutorials that our group is developing help to fulfill this need.[24]

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CHAPTER 3: IDENTIFYING STUDENT DIFFICULTIES IN APPLYING A WAVE MODEL TO INTERFERENCE AND DIFFRACTION OF MATTER

A. INTRODUCTION

The idea that electrons and other particles of matter can exhibit wave-like properties is typically introduced by extending the treatment of phenomena from physical optics. The concept of de Broglie wavelength is presented to explain and account for double-slit electron interference, the Davisson-Germer experiment, and other experiments in which the wave-like properties of matter are relevant.[38] In addition, many instructors emphasize the de Broglie wavelength as a tool that can be used to account for the qualitative shape of wave functions, particularly those of stationary bound states.[39]

In this chapter we describe research to identify specific difficulties in applying a wave model in the context of the interference and diffraction of matter waves. Many of the difficulties that were found to be prevalent among students from a wide variety of instructional settings, from the introductory level through junior-level quantum mechanics. The difficulties that were identified can be grouped into two broad categories:

- difficulty in recognizing the relevance of the de Broglie wavelength in situations in which the wave-like properties of matter are important, and
- difficulty in relating the de Broglie wavelength to the velocity, momentum, or kinetic energy of a particle.

B. DESCRIPTION OF RESEARCH TASKS AND ANALYSIS PROCEDURE

Qualitative written problems on electron diffraction have been given in two sections of the algebra-based course (Physics 116), one section of the introductory calculus-based course (Physics 123), three sections of the modern physics course (Physics 225), and three sections of a junior-level quantum mechanics course (Physics 324). We have also designed an interview task that has been posed to students in the quantum mechanics course. All were administered after standard instruction on matter waves had taken place. The interview task and written questions are discussed below. We then describe the procedure we used to analyze the student responses.

1. *INTERVIEW TASK*

Preliminary research on student understanding of the de Broglie wavelength took place during a series of interviews on physical optics with 14 students from the junior-level quantum mechanics course. (The tasks chosen for these interviews were adapted from those used in interviews with students from the introductory calculus-based course and the sophomore-level modern physics course.[40]) We found that many of the students had difficulty with concepts that were more basic than the de Broglie wavelength. Only 7 of the 14 student volunteers did well enough to reach the part of the interview on de Broglie wavelength, which is described below.

Description of interview task. In each interview, the student was shown a photograph of a double-slit pattern, like the one shown in Figure 3-1. The students recognized that electrons could be used in a double-slit experiment to produce a similar pattern. For such an experiment, the students were asked to predict how the locations of the interference maxima would would change if the velocity of the incident electrons were decreased.

Figure 3-1: Photograph of double-slit pattern used in interview.

Correct response to interview task. Decreasing the speed of the incident electrons corresponds to a decrease in the momentum of the electrons. As a result, the de Broglie wavelength $\lambda = h/p$ of the electrons would increase. The first interference maximum, which corresponds to a path-length difference of exactly one wavelength, would move farther away from the center of the screen. Similarly, all the interference maxima would move farther apart if the velocity of the electrons were decreased. One can argue this geometrically or use the equation for interference maxima, $d \sin\theta = n\lambda$ ($n = 0, 1, 2, \ldots$), to show that as λ increases, the angle θ to each maximum also increases, and the interference pattern broadens as a result.

2. *WRITTEN QUESTIONS*

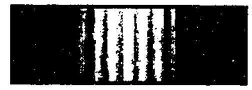
Written problems given to the various classes in the study were similar in format to the interview task described above. Some classes were posed questions about double-slit interference of electrons, others were asked about single-slit diffraction of electrons. In addition, pretests given in the junior-level quantum mechanics course were in the context of the Davisson-Germer experiment. We found, however, that the different versions of the problems did not seem to significantly affect student performance on questions testing on the de Broglie wavelength.

Format of pretest and examination problems. The problems on single-slit diffraction or double-slit interference of electrons begin by showing the students the photograph of the pattern produced on a distant screen when electrons are incident on a mask containing a single-slit or a double-slit. (Pretest problems on the Davisson-Germer experiment included a diagram and the condition for Bragg scattering.) Students were told either that the incident electrons are monoenergetic or that they were accelerated from rest through a potential difference V_o. The students were asked to predict how a single change to the original apparatus would affect the locations of the interference (or diffraction) maxima (or minima) and to explain their reasoning in each case. Although each class was given a different version of the problem, each version included at least two of the three cases listed below:

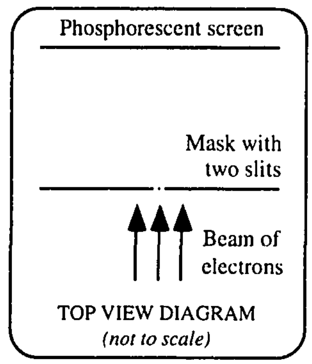
- the slit separation (or slit width, or lattice spacing) is increased or decreased (all versions)
- the accelerating voltage V_o (or speed, or kinetic energy of the incident electrons) is increased or decreased
- the electrons are replaced with particles of larger mass, such that (a) the new particles each have the same kinetic energy as each of the original electrons, or (b) the resulting pattern is identical to the pattern produced by the original electrons

Sample problems are presented below in Figure 3-2 and Figure 3-3.

A beam of monoenergetic electrons is incident on a mask that contains two very narrow slits (see top view diagram at right). The photograph shows the pattern seen on a phosphorescent screen placed far from the slits. The bright regions indicate the concentrations of electrons hitting the screen.



Front view of screen



Suppose this experiment were repeated with a ***single*** change made to the original setup. For each possible change described below, predict whether the bright regions would get *closer together,* move *farther apart,* or *stay in the same location.* Explain your reasoning in each case.

i. The slit separation is decreased.

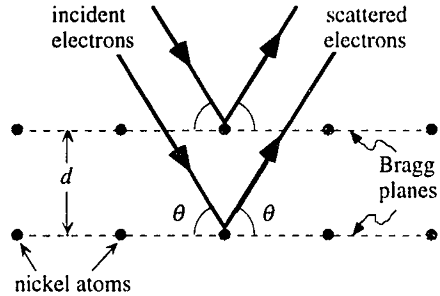
ii. The speed of the electrons is increased.

iii. The electrons are replaced with muons, with each muon having the *same kinetic energy* as each of the original electrons. (Recall that $m_\mu \approx 200\, m_{e^-}$)

Figure 3-2: Typical written problem on double-slit interference of electrons. Problems like this one and similar problems on single-slit diffraction of electrons were used on pretests and examinations after standard instruction on the de Broglie wavelength.

(Davisson-Germer experiment.) Monoenergetic electrons are incident on a nickel crystal. It is observed that intense scattering occurs at angles θ according to the Bragg condition, $2d \sin\theta = n\lambda$. (See diagram at right.)

Suppose that this experiment were repeated, each time with a single change made to the apparatus. For each change below, determine whether each of the angles θ at which intense scattering occurs would become larger, smaller, or stay the same. Explain your reasoning in each case.



i. The nickel target is replaced with another crystal that has the same lattice structure but a smaller lattice spacing. (Consider the Bragg planes that are analogous to those indicated in the diagram above.)

ii. The speed of the incident electrons is decreased.

ii. The electrons are replaced with neutrons, with each neutron having the same kinetic energy as each of the original electrons.

Figure 3-3: Written pretest problem on the Davisson-Germer experiment given to students in the junior-level quantum mechanics course. Unlike the other pretests and the examination problems on de Broglie wavelength, this problem included an equation relating λ to the angles θ of the interference maxima.

Each written problem began with a question in which the students were asked to predict the effect on the interference or diffraction pattern of varying the slit separation, slit width, or lattice spacing. Such questions could be answered solely on the basis of concepts covered in introductory physical optics (*e.g.*, superposition, path length difference, and phase difference) and thus served to determine a baseline understanding of interference and diffraction. In this chapter we refer to questions that pertain to changing the slit separation or lattice spacing as "Δd" questions, for short. Those that involve varying the slit width are denoted as "Δa" questions.

All pretest and written problems also contained one or more parts that tested student understanding of de Broglie wavelength. Those questions that asked students to predict the effect of changing the speed or kinetic energy of the incident electrons are denoted as "Δv" and "ΔE_K" questions, respectively. The questions that instead asked students to consider replacing the electrons with particles of larger mass are called "Δm" questions. In all cases, the term "de Broglie wavelength" was never used in the problem statement. The students were expected to recognize how the concept of de Broglie wavelength was relevant and how to apply it. Table 3-1 below summarizes the questions used in each of the classes discussed in this chapter.

Table 3-1 Outline of various pretest and examination problems on the de Broglie wavelength given in several different classes after all traditional instruction.

	Algebra-based		**Calc-based**	**Modern physics**			**Quantum mechanics**	
	Exam	Pretest	Exam	Exam	Pretest	Pretest	Exam	Pretest
	Au 97 $N=102$	Sp 97 $N=103$	Wi 98 $N=105$	Sp 95 $N=45$	Au 95 $N=37$	Wi 97 $N=18$	Au 95 $N=43$	Au 96 Au 97 $N=42$
# of slits (1 or 2) or Davisson-Germer (DG)	2	2	1	2	2	2	1	DG
Varying a			✓				✓	
Varying d	✓	✓		✓	✓	✓		✓
Varying v	✓	✓	✓			✓		✓
Varying E_K				✓	✓			
Varying m, same v or E_K		✓		✓	✓	✓	✓	✓
Varying m, same pattern	✓		✓					

Correct method to predict the effect of changing the slit separation, slit width, or lattice spacing (Δa and Δd questions). If the slit separation or slit width is decreased, the interference or diffraction maxima move farther apart. Conversely, an increase in slit separation or slit width results in the maxima moving closer together. To see this for the case of double-slit interference, consider the location of the first interference maximum on the screen. The path length difference at this location is equal to the wavelength λ. If the slits were brought closer together, the path-length difference at this location on the screen would become less than λ. The first maximum would therefore change location and be found farther from the center of the pattern. For similar reasons the other maxima would do the same. Equivalently, one can obtain this result by using the equation for interference maxima, $d \sin\theta = n\lambda$ ($n = 0, 1, 2, \ldots$). As d decreases, the angle θ to each maximum increases, and thus the interference pattern broadens.

For the case of single-slit diffraction, one finds that decreasing the slit width causes the pattern to broaden. This result can be obtained by using a "pairing" method (similar to that described in Chapter 2) or by applying the equation for diffraction minima: $a \sin\theta = n\lambda$, where a represents the slit width and n is any positive integer.

The pretests on the Davisson-Germer experiment, which were given in the junior-level quantum mechanics course, included the condition for Bragg scattering ($2d \sin\theta = n\lambda$) in the problem statement. On the Δd question, students were asked to predict the effect of using a crystal with a smaller lattice spacing than before (while considering the same Bragg planes as before). Students were expected to use the equation for Bragg scattering to determine that a decrease in the lattice spacing d (while keeping λ unchanged) would result in increasing the angle θ to each maximum.

Correct method to predict the effect of changing the accelerating voltage, velocity, or kinetic energy of the electrons (Δv and ΔE_K questions). This part of the problem is similar to the interview task described earlier. If the accelerating voltage, velocity, or kinetic energy is increased, then the momentum of the electrons would also increase and thus the de Broglie wavelength $\lambda = h/p$ would decrease. From the equations given above for double-slit interference maxima or single-slit diffraction minima, it can be seen that a decrease in the de Broglie wavelength would cause the interference (or diffraction) maxima and minima to move closer together. Similarly, decreasing the velocity or kinetic energy would cause an increase in de Broglie wavelength and thus would cause the maxima and minima to move farther apart.[41]

Correct method to predict the effect of replacing the electrons with different particles having the same kinetic energy (Δm question). In most versions of the Δm question, students were asked to consider how the pattern would change if the electrons were replaced with neutrons having the same kinetic energy as the electrons. A correct answer is to predict that the interference maxima get closer together. To arrive at this answer, one must recognize that the momentum of a (massive) particle is related to its mass and kinetic energy by $p = (2mE_K)^{1/2}$. The mass of the neutron is larger than that of the electron, so a neutron with the same kinetic energy as an electron would therefore have a smaller de Broglie wavelength $\lambda = h/p$. Thus the interference (or diffraction) maxima and minima would move closer to the center.[42]

Correct method to compare the speed (or kinetic energy) of electrons and different particles that produce the same pattern. An alternative version of the "Δm" question was designed for use in the Autumn 1997 section of the algebra-based course (Physics 116) and the Winter 1998 section of the calculus-based course (Physics 123). For this question, students were told that particles of mass $m > m_e$ (*e.g.*, protons) were used to produce an interference or diffraction pattern that was identical to that of the original electrons. They were asked to compare the speed or kinetic energy of the new particles to that of the original electrons. To answer, students needed to recognize that the de Broglie wavelengths of the new particles and the electrons had to be equal. By the de Broglie relation, $\lambda = h/p$, the momenta of the new particles and the electrons also had to be equal. Because the mass of the new particles is larger than that of the electrons, the speed $v = p/m$ of the new particles must be less than that of the electrons. Thus, their kinetic energy $E_K = p^2/2m$ is also less than that of the electrons.

3. ANALYSIS PROCEDURE

A correct response to any of the questions described above requires careful multi-step reasoning. For example, students must first determine how the de Broglie wavelength would change in a particular situation. They then must relate the change in wavelength to the resulting change in locations of the maxima and minima of the interference (or diffraction) pattern. An advantage in using this type of question was that it could be used to determine which students recognized the relevance of the de Broglie wavelength and which of those students understood how to apply it. Incorrect responses could arise not only due to difficulties with the de Broglie wavelength but with other concepts that underlie interference and diffraction. We devised a procedure by which we could isolate only the errors related to the de Broglie wavelength. The procedure consisted of two steps, described below.

We first determined which students gave a correct response to the Δd or Δa question, in which they were asked to predict the effect of changing the slit separation, slit width, or lattice spacing. This allowed us to determine which students seemed to understand basic concepts of interference and diffraction. We counted as correct all responses that were justified with correct reasoning, incomplete reasoning, or reasoning that would *not* have led to errors on the subsequent questions about the de Broglie wavelength. (For example, a student who uses the equation "$a \sin \theta = n\lambda$" to justify an answer about the *maxima* [not minima] in a diffraction pattern would arrive at a correct answer on the Δa question. If the student then correctly determined how the de Broglie wavelength λ would change in a given situation, the incorrect formula would still lead to a correct answer about the resulting qualitative change to the interference or diffraction pattern.)

After determining which students gave correct responses on the Δd or Δa question, we then analyzed only the responses given *by these students* to the questions that pertain specifically to de Broglie wavelength (*i.e.*, the Δv, ΔE_K, and Δm questions). In support of this procedure, virtually all students who determined correctly how the de Broglie wavelength would change had also determined correctly the corresponding change in the interference or diffraction pattern. In addition, practically none of the students who answered the Δd or Δa questions *incorrectly* went on to give correct responses on the subsequent questions about the de Broglie wavelength.

In summary, for each question that tested on the de Broglie wavelength (*i.e.*, the Δv, ΔE_K, or Δm questions), we used two criteria. We counted as correct any student response that included (1) a correct response for the (preceding) Δd or Δa question *and* (2) a correct response with essentially correct reasoning about the de Broglie wavelength.

C. OVERVIEW OF RESULTS FROM WRITTEN PROBLEMS AFTER STANDARD INSTRUCTION

In this section we present the statistics from written problems given to several populations of students after standard instruction on electron interference and diffraction. The identification and analysis of specific student difficulties is discussed in the following section.

1. RESULTS FROM QUESTIONS THAT INVOLVE VARYING THE SLIT SEPARATION, SLIT WIDTH, OR LATTICE SPACING

In this section we present results from the questions that pertain to changing the slit separation, slit width, or lattice spacing (*i.e.*, the Δd and Δa questions). We first examine results from the written problems given to students in the introductory and sophomore-level modern physics courses. These results are compared to those from a related question given to introductory students in the context of single-slit diffraction of light. We then discuss results from written questions given to students in the junior-level quantum mechanics class. As discussed earlier, the pretest given to these students, which was in the context of the Davisson-Germer experiment, included a formula the condition for Bragg scattering.

a) Results from introductory and modern physics courses

Results from the pretest and examination problems on double-slit interference and single-slit diffraction of electrons are shown in Table 3-2. The statistics from examinations in two modern physics classes (Spring 1995 and Autumn 1995) were similar, therefore these results have been combined in the table.

Table 3-2: Results from questions given in first-year and second-year courses in which students were asked to predict the effect of changing the slit separation or slit width (*i.e.*, the Δd and Δa questions). Results from examinations in the sophomore-level modern physics courses were similar and thus have been combined. [Percentages rounded to the nearest 5%.]

	Algebra-based		**Calculus-based**	**Modern physics**	
	Exam Au 1997 $N = 102$	Pretest Sp 1997 $N = 103$	Exam* Wi 1998 $N = 105$	Exams Sp 1995 Au 1995 $N = 82$	Pretest Wi 1997 $N = 18$
Correct responses	**70% (69)**	**35% (36)**	**95% (98)**	**55% (47)**	**70% (13)**
With correct reasoning	15% (13)	5% (5)	45% (48)	25% (19)	20% (4)
Incorrect responses	**25% (28)**	**65% (67)**	**5% (7)**	**35% (30)**	**20% (4)**

* Examination given after tutorial instruction on single-slit diffraction of light.

In most classes listed in Table 3-2, the percentage of correct answers that were justified with correct reasoning was between 15% and 25%. The pretest in the Spring 1997 algebra-based course and the examination in the Winter 1998 calculus-based course are two exceptions. (These results are discussed separately below.) The percentage of correct answers from the remaining pretests and examinations varied by 15% (55% through 70%), and when correct answers with correct reasoning are taken into account the variation is only about 10% (15% through 25%).

In all classes, those students who gave correct responses to the Δd or Δa question without correct reasoning usually based their answers on memorized responses instead of explanations (*e.g.*, saying that a smaller slit width would cause "more diffraction"). Others used formulas that are relevant to interference or diffraction but interpreted them incorrectly (*e.g.*, using "$a \sin \theta = m\lambda$" to refer to diffraction *maxima* instead of diffraction minima). Such errors reflect a lack of a qualitative understanding of a wave model for light.[43] However, these errors would not necessarily affect the ability of students to answer correctly the subsequent parts of the exam related to the de Broglie wavelength.

Commentary on the results from the algebra-based courses. Although the results from the two algebra-based courses in Table 3-2 differ considerably, there were factors that could account for the differences. The question in the Autumn 1997 class was given on an examination, and students were allowed to use a sheet of self-written notes during the examination. During the Spring 1997 pretest the students did not have notes from which to work. These differences between the two classes support our interpretation of the data in Table 3-2: the Autumn 1997 students seemed to be significantly better than the Spring 1997 students in obtaining a correct answer to the Δd question but not significantly better in interpreting the equations that they were using. In both cases, however, the students had completed all standard instruction of the material. The observation that essentially the same percentage of students in both groups obtained the correct answer with correct reasoning (5% *vs* 15%) suggest that the two groups can be treated as equivalent.

Commentary on the results from the calculus-based course. The highest percentage of correct answers (95%) and highest percentage of correct answers with correct reasoning (45%) were obtained in the calculus-based course. We can account for these results by noting that the calculus-based course is the only course in which a series of tutorials on physical optics were used.[44] It is interesting to compare the results of the calculus-based course on this question to those obtained on similar questions given to other classes in the context of light. In these questions, students are shown a photograph of a single-slit diffraction pattern and are asked to predict how the distance between the first diffraction minima would change if the right half of the slit were covered (*i.e.*, if the width of the slit were halved). Table 3-3 below summarizes student performance on this question when it was given (1) after lecture, laboratory, and an examination on diffraction but *before tutorial instruction*, and (2) after modified instruction (with tutorial).[45] (The students in the Winter 1998 class, who took an examination on electron diffraction, are *not* included in either of the two groups.)

Table 3-3: Results from questions given in the introductory calculus-based course after standard instruction and after tutorial instruction on single-slit diffraction of light. The question required students to predict the effect on the diffraction pattern of covering the right half of the slit (*i.e.*, decreasing the width of the slit). The statistics in the left column are from tutorial pretests that were given after lecture, laboratory, and examination on diffraction but before tutorial instruction. (See Chapter 2.)

	Calculus-based course	
	Pretest after standard instruction, but before tutorial $N \approx 410$	Post-test after tutorial instruction $N \approx 455$
Correct responses	**70%**	**95%**
With correct reasoning	20%	65%
Incorrect responses	**30%**	**5%**

As shown in Table 3-3, the percentage of correct responses with correct reasoning is only 20% after standard instruction on diffraction. This result is consistent with those shown in Table 3-2 from the Δd and Δa questions given after standard instruction in the algebra-based course and modern physics course. After the tutorial, however, about 65% of the students give correct answers supported with correct reasoning. This increase is expected because the tutorials in the calculus-based course were developed to address student difficulties in applying a wave model to the interference and diffraction of light.

The results from the electron diffraction problem given to the Winter 1998 calculus-based course (shown previously) can be compared to those above in Table 3-3. From Table 3-2, about 45% of the students in the Winter 1998 course gave correct answers with correct reasoning on the Δa question. The corresponding percentages from questions given in the context of light after standard instruction and after tutorial instruction are 20% and 65%, respectively. (See Table 3-3 above.) These results suggest that many of the students in the Winter 1998 were able to transfer what they learned in tutorial to the more difficult context of electron diffraction. Furthermore, the fact that these students outperformed the other classes represented in Table 3-2 indicates that any attempt to address difficulties in understanding interference and diffraction of matter should include a review of basic concepts from physical optics. (The development and assessment of such an instructional strategy is discussed in the following chapter.)

b) Results from the quantum mechanics course

The results from Δd and Δa questions given to students in the junior-level quantum mechanics course are summarized in Table 3-4. The Autumn 1995 examination problem, which was in the context of single-slit diffraction of electrons, yielded results that were very similar to those from the modern physics courses.[46] (Compare the first column of Table 3-4 below to the fourth and fifth columns of Table 3-2.) The pretests in Autumn 1996 and Autumn 1997 were in the context of the Davisson-Germer experiment and included the equation for Bragg scattering. (See Figure 3-3.)

Table 3-4: Results from questions given in the junior-level quantum mechanics course in which students were asked to predict the effect of changing the slit width or lattice spacing (*i.e.*, the Δa and Δd questions).

	Quantum mechanics course		
	Exam Au 1995 $N = 43$	Pretest* Au 1996 $N = 27$	Pretest* Au 1997 $N = 25$
Correct responses	**60% (26)**	**65% (18)**	**70% (17)**
With correct reasoning	20% (8)	35% (9)†	60% (15)†
Incorrect responses	**35% (14)**	**30% (8)**	**30% (7)**

* The pretest questions were given in the context of the Davisson-Germer experiment and included the equation for Bragg scattering ($2d \sin\theta = n\lambda$).

† A total of 16 students on the Autumn 1996 pretest and 17 students on the Autumn 1997 pretest gave correct answers justified by reasoning that was either completely correct or incomplete (but not incorrect).

Commentary on the results from the pretests in the quantum mechanics course. The fact that the pretests included an equation for the diffraction maxima (while the examinations did not) seemed to affect the percentage of correct answers with correct reasoning, but not the total percentage of correct answers. Although the percentages of correct answers with correct reasoning are considerably different from the two pretests (35% and 60%), almost all of the students who gave correct responses supported their answers with reasoning that was either completely correct or incomplete, but not incorrect. It is possible that, because the equation for Bragg scattering was given, students did not feel it was necessary to refer to the equation in their response. Furthermore, we found that approximately equal percentages of students gave correct answers with either correct or incomplete reasoning: 16 of 27 (60%) in Autumn 1996, and 17 of 25 (70%) in Autumn 1997. These results suggest that the two classes can be treated equivalently.

Despite the use of different questions on the examination and pretests, approximately the same percentage of students in all classes (30%–35%) obtained incorrect answers for the Δa and Δd questions. On the pretests, many did not use the given equation for Bragg scattering and argued on the basis of other ideas. Some stated, for example, that a smaller spacing between the Bragg planes would make adjacent beams of scattered electrons closer to each other (which is correct) but that this change would not affect the angles at which the most intense scattering would occur (which is incorrect). Such errors reflected an overall failure to recognize that a smaller lattice spacing would change the path length difference between beams reflected from adjacent Bragg planes, which would therefore affect the angles at which constructive interference would occur.[47]

2. *RESULTS FROM QUESTIONS THAT INVOLVE VARYING THE ACCELERATING VOLTAGE, VELOCITY, OR KINETIC ENERGY OF THE ELECTRONS*

In this section we present an overview of the analysis of the written questions that required students to determine the effect on the de Broglie wavelength of varying the accelerating voltage, velocity, or kinetic energy of the electrons. Below we present the results obtained from the analysis of questions from several classes.

Table 3-5 summarizes the responses given by students on the Δv and ΔE_K questions given to students from the introductory courses, the sophomore-level modern physics course, and the junior-level quantum mechanics course. The table shows the results from only those students who gave a correct response to the Δd or Δa question.

Table 3-5: Results from questions that ask students to predict the effect of changing the speed or kinetic energy of the electrons. The table shows the results from only the N' students who correctly answered the question about varying the slit separation, slit width, or lattice spacing. The results from all questions given in each course were similar and thus have been combined.

	Algebra-based	**Calculus-based***	**Modern physics**	**Quantum mechanics**
	Sp 1997 Au 1997	Wi 1998	Sp 1995 Au 1995 Wi 1997	Au 1996 Au 1997
Predicting effect of changing v or E_K	$N' = 105$	$N' = 98$	$N' = 60$	$N' = 33$
Correct responses (correct reasoning used to relate λ to v or E_K)	**15% (17)**	**50% (48)**	**35% (22)**	**50% (17)**
Incorrect responses	**75% (78)**	**50% (49)**	**60% (37)**	**45% (15)**
Failure to recognize relevance of λ (*e.g.*, maxima change only in intensity)	40% (40)	20% (19)	35% (21)	10% (3)
Mistaken beliefs about λ (*e.g.*, does not depend on v; $v = \lambda f$)	35% (38)	30% (30)	25% (16)	35% (12)

* Written question given after tutorial instruction on single-slit diffraction of light.

Students in all four classes had serious difficulty recognizing that change in speed or kinetic energy of the electrons would affect the interference or diffraction pattern, including students who answered the Δd or Δa question correctly *with correct reasoning.* Many failed to recognize that varying the velocity or kinetic energy of the electrons would affect the locations of the interference (or diffraction) maxima and minima. Other students seemed to recognize that the wavelength was the critical issue in answering the question but used incorrect reasoning about the relationship between velocity (or kinetic energy) and de Broglie wavelength. Examples of student explanations are discussed later in section D.

3. *RESULTS FROM QUESTIONS THAT INVOLVE REPLACING THE ELECTRONS WITH DIFFERENT PARTICLES*

In Table 3-6 below we present overall results from each version of the Δm questions, in which the students considered replacing the electrons with particles of different mass. Although the two versions posed slightly different questions, both versions were challenging for students.

Table 3-6: Results from questions that ask students to predict the effect of changing the mass of the particles. The table shows the results from only the N' students who correctly answered the question about varying the slit separation, slit width, or lattice spacing. The results from all questions given in each course were similar and thus have been combined.

	Algebra-based	**Calculus-based***	**Modern physics**	**Quantum mechanics**
	Sp 1997 Au 1997	Wi 1998	Sp 1995 Au 1995 Wi 1997	Au 1995 Au 1996 Au 1997
Predicting effect of changing *m*	$N' = 105$	$N' = 98$	$N' = 60$	$N' = 59$
Correct responses (correct reasoning used to relate λ to *m*)	**20% (22)**	**50% (48)**	**25% (16)**	**30% (19)**
Incorrect responses	**65% (69)**	**45% (42)**	**60% (37)**	**50% (29)**
Failure to recognize relevance of λ (*e.g.*, maxima change only in intensity)	35% (37)	15% (16)	35% (22)	25% (15)
Mistaken beliefs about λ (*e.g.*, does not depend on *m*; $E = hc/\lambda$)	30% (32)	25% (26)	25% (15)	25% (14)

* Written question given after tutorial instruction on single-slit diffraction of light.

In general the results were similar to those from the Δv and ΔE_K questions posed on the examinations. Only 50% of the students in the calculus-based course who answered the Δd or Δa question correctly could answer the Δm question correctly as well. The corresponding percentages from the other courses ranged between 20% and 30%. As before, many students did not recognize that the Δm question was a question involving the wavelength of the particles.

These students based their explanations on concepts other than de Broglie wavelength. Others recognized the need to apply the concept of de Broglie wavelength but did so incorrectly.

D. IDENTIFICATION AND ANALYSIS OF SPECIFIC DIFFICULTIES WITH THE DE BROGLIE WAVELENGTH

In this section we discuss specific difficulties that arose from questions relating directly to the de Broglie wavelength. (Difficulties that were elicited by the Δd and Δa questions from each pretest and examination problem were similar to those from physical optics, which are discussed in Chapter 2.) The analysis yielded numerous conceptual and reasoning difficulties that we have grouped into three broad, overlapping categories:

- failure to recognize the relevance of the de Broglie wavelength,
- mistaken belief that the de Broglie wavelength is independent of velocity or kinetic energy, and
- mistaken belief that all equations that pertain to the wavelength of light also apply to the de Broglie wavelength of electrons and other particles.

Examples of each are discussed in the following sections.

1. FAILURE TO RECOGNIZE THE RELEVANCE OF THE DE BROGLIE WAVELENGTH

Many students, even after answering the Δd or Δa questions correctly, neglected to take into account ideas about de Broglie wavelength. The most common type of incorrect response from this group of students was that a change in the mass, velocity, or kinetic energy of the incident particles would affect the intensity of the interference or diffraction pattern but not the locations of the maxima and minima. These students did not mention de Broglie wavelength at all in their answers. Examples of student responses are given below.

Student quotes from questions that involve varying the accelerating voltage, speed, or kinetic energy of the electrons. When predicting the effect of varying the velocity of the electrons, many students in the introductory and modern physics courses explicitly stated that the locations of the maxima and minima would not change. Listed below are examples of student responses that illustrate this difficulty. (Boldface added by the author.)

> "[Increasing the speed] would **cause the pattern to be the same** but the bright regions of the screen would become brighter." [Algebra-based course]
>
> "[If the speed is decreased, then the minima] would **stay in the same location.** The location of the minima is not dependent on the speed of the electrons, if anything the screen would be dimmer overall." [Calculus-based course]
>
> "[If the speed is increased, then the] spots get brighter, **but don't move (still in or out of phase in the same spots)**" [Modern physics course]

Student quotes from questions that involve replacing the electrons with different particles. The failure to relate the pattern to de Broglie wavelength also arose on the Δm question in which students were asked to predict the effect of replacing the electrons with heavier particles of the same kinetic energy. Although many students correctly recognized that the heavier particles would have a speed that was less than that of the original electrons, they incorrectly believed that this change would not cause the maxima and minima to change location. Students from all levels used this type of incorrect reasoning. (Boldface added by the author.)

> "If the *KE* is the same for the neutron as each of the original electrons it means that the velocity of the neutrons [is] much slower than the electrons so the screen will not be hit as fast with as many neutron particles as electron particles. The intensity graph will decrease but **the shape will remain the same.**" [Algebra-based course]
>
> "So if neutron replaces electron, the $v \downarrow$. Thus the bright region will become dimmer **while showing the same pattern.**" [Algebra-based course]
>
> "Since the neutrons' masses are so much greater than those of the electrons, the neutrons would have to be moving much slower than the electrons were moving initially. Since the neutrons would be moving to the order of 10^3 times slower, again there would fewer particles striking the screen each second.... **The pattern would remain in the same form as before** (picture shown above, except a lot dimmer)." [Modern physics course]
>
> "[The only difference] is rate that muons go. $\therefore$ No change in intensity, just takes longer to make pattern." [Quantum mechanics]

All of the quotes above are from students who obtained a correct answer on the Δd or Δa question of the problem, in which they were asked to predict how changing the slit separation or slit width would affect the interference or diffraction pattern. Although they knew how to relate wavelength and path length difference to slit separation or slit width, they failed to recognize that questions referring to the mass, velocity, or kinetic energy of particles were questions about the de Broglie wavelength of the particles.

2. *MISTAKEN BELIEF THAT THE DE BROGLIE WAVELENGTH OF A PARTICLE IS INDEPENDENT OF THE VELOCITY OR KINETIC ENERGY OF THE PARTICLE*

Many students explicitly stated that varying the velocity or kinetic energy would not affect the de Broglie wavelength of the electrons. The responses suggested a lack of intuition that as the momentum decreases, the de Broglie wavelength increases. Such an intuition is necessary for students to recognize that quantum mechanical effects become more important as energy and momentum decrease. These difficulties are discussed below.

Student quotes from written questions that involve varying the accelerating voltage, speed, or kinetic energy of the electrons. When asked to predict how an interference or diffraction pattern would be affected by varying the speed (or accelerating voltage) of the electrons, some students explicitly stated that such changes would not affect the wavelength of the electrons. These students consequently predicted that the interference or diffraction maxima would remain in the same locations as before. Example student responses illustrating this difficulty are given below.

> "It doesn't matter what the speed of the electrons is, the wavelength stays the same and the same pattern is produced — no change." [Algebra-based course]

> "[The diffraction minima] stay at the same location. Even at lower speeds, electrons still exhibit wave like motion. As long as its wavelength stays the same, the pattern should stay the same." [Calculus-based course]

> "[The diffraction minima] stay at the same locations: I don't think that slowing the electrons down would change anything. We've been comparing electrons to light. We never took velocity into account when dealing with light, so the same should hold here." [Calculus-based course]

> "[The diffraction minima] stay [at] the same locations. Changing the voltage will not effect [sic] the λ of the electrons, only the acceleration of the electrons." [Modern physics]

Students often used equations such as $d \sin\theta = m\lambda$ or $a \sin\theta = m\lambda$ in order to determine how a change in the slit separation d or slit width a would affect the locations of the interference or diffraction fringes. However, in addition to the errors described above, many later failed to interpret "λ" as a quantity that depended upon the mass or velocity of the particles in question.

> [After writing and using the equation $D \sin\theta = m\lambda$ for minima from a single slit of width D.] "Speed is not a influential factor in this event. If it were, a v would be included in the single slit equations." [Algebra-based course]

> "Nothing because [the pattern] is independent from the speed. It will stay the same. $a \sin\theta = m\lambda$." [Calculus-based course]

Student quotes from interviews. Seven (7) of the 14 quantum mechanics (Physics 324) students who were interviewed in Autumn 1995 were given an interview task on the de Broglie wavelength. (The task is described earlier in part 1 of section B of this chapter.) They were asked how, if at all, decreasing the velocity of electrons used in a double-slit interference experiment would affect the locations of the interference maxima.

The results were consistent with those obtained from written questions. Three of the seven students incorrectly predicted that no change would occur, while another student gave a correct response but did not support his response with correct reasoning. All four had serious difficulty relating the de Broglie wavelength to the velocity of the electrons.

For example, when asked how the pattern would change if the speed of the electrons were decreased, one student responded:

> S_1: "I don't see a reason why [the pattern] would change.... I can't think if there is a correlation between velocity and the actual wavelength of the electron, or if its wavelength is a property of the electron itself, regardless of velocity."

Eventually, this student incorrectly predicted that no change would occur to the interference pattern as a result of decreasing the electron speed.

Another of the quantum mechanics students seemed to recognize that the electrons have a wavelength that is associated with their mass. Early in the interview when the topic of electron interference first came up, the student stated:

> S_2: "The wavelength is inversely proportional to mass. Only small masses have wavelengths we can detect."

However, later in the interview, the interviewer asked:

> I: "What if we...make the electrons go slower as they hit the mask with two slits. Would that change the pattern?"

> S_2: "No effect. Same pattern.... Because I said that the interference pattern depends on wavelength, not on how fast [the electron] goes through."

Although this student correctly said that the wavelength is inversely proportional to *mass*, he failed to recognize also that the wavelength is inversely proportional to the *speed* of the electrons. This error led the student to the incorrect prediction that the pattern would not change.

3. *MISTAKEN BELIEF THAT ALL EQUATIONS THAT APPLY TO THE WAVELENGTH OF LIGHT ALSO APPLY TO THE DE BROGLIE WAVELENGTH OF MASSIVE PARTICLES*

Many students in each class incorrectly related the wavelength of the electrons (or other particles) to velocity or kinetic energy. In particular, many applied certain relationships that are valid for light but not for particles of matter. Two common examples of this error are described below.

a) ***Incorrect belief that the relationship, $v = \lambda f$, for light waves also relates the velocity of a particle to its de Broglie wavelength***

When attempting to relate the speed and de Broglie wavelength of an electron, many students used the equation $v = \lambda f$ that relates phase velocity, wavelength, and frequency of light.[48] Rather than recognize that the de Broglie wavelength is inversely proportional to momentum (and thus inversely proportional to velocity), they incorrectly predict that an increase (or decrease) in the velocity of electrons would result in a proportional increase (or decrease) of the wavelength of the electrons.

Student quotes from questions that involve varying the accelerating voltage, speed, or kinetic energy of the electrons. About 30% of the students in the algebra-based course (Physics 116) and 25% of those in the calculus-based course (Physics 123) incorrectly used the equation $v = \lambda f$ to predict that increasing the velocity of the electrons would cause an increase in the wavelength of the electrons. A few modern physics students also made this error. Examples of incorrect responses that illustrate this difficulty are given below:

> "$\lambda = v/f$ ... If the frequency remains the same, then increasing the speed of the electrons will also increase their wavelength. An increase in wavelength would lead to an increase in $\sin\theta$ ($D\sin\theta = m\lambda$), so the pattern would get bigger, more spread out." [Algebra-based course]

> "$v = \lambda f$, frequency constant. Decreasing the speed will cause the wavelength property to also decrease. Decreasing λ will decrease the angle to the first minima, thus bringing the minima closer together." [Calculus-based course]

> "$v = \lambda f$ ... By increasing the acceleration, the velocity of the electron will also increase. To compensate for this, the wavelength would also have to increase. Increasing the wavelength has the same effect as narrowing the slit and therefore, the maxima will appear farther apart." [Modern physics course]

Student quotes from questions that involve replacing the electrons with different particles. Many students used the equation $v = \lambda f$ when considering particles of different mass. On the Δm questions in which they consider replacing electrons with heavier particles having the same kinetic energy, students correctly stated that the heavier particles would have a smaller velocity than the original electrons. However, they concluded incorrectly that the reduced velocity would necessarily imply a smaller wavelength.

On the Δm questions in which students were asked to compare the velocity of neutrons that made the same pattern as electrons, some ignored the difference in the masses of the particles. These students stated that the velocities of the electrons and neutrons would have to be equal in that case. For example, one student gave the following response:

> "Neutrons are more massive than electrons. But since we are looking at the wave nature of the electrons and neutrons we are concerned with $v = \lambda f$. Since we have same pattern and same λ, f & velocity must be the same as well." [Algebra-based course]

Commentary. Hardly any of the students who used the equation $v = \lambda f$ articulated what they believed the frequency "f" represented with regard to the electrons. A few students seemed to interpret the "frequency" as the number of electrons that reach the screen per unit time. Such an interpretation would lead students to predict that a change in the flux of electrons incident on the mask would affect the "wavelength," $\lambda = v/f$. Further research is required to gauge the prevalence of this idea.

b) Incorrect belief that the relationship, $E = hc/\lambda$, for the photon energy relates the kinetic energy of a particle to its de Broglie wavelength

Many students referred to the equation for photon energy, $E = hc/\lambda$, in their responses to the problem on electron diffraction. Some derived this relationship using the equation, $c = \lambda f$, for light and the Einstein relation, $E = hf$. Such errors led students to apply an equation that would be valid for relating the photon energy to the wavelength of light but not for relating the kinetic energy of an electron to its de Broglie wavelength.

Student quotes from questions that involve varying the accelerating voltage or kinetic energy of the electrons. On the ΔE_K questions, some students correctly stated that an increase in the accelerating voltage or kinetic energy would decrease the de Broglie wavelength. However, many justified their answer on the basis of the incorrect reasoning described above. Examples of their responses are given below:

> "Increase of V_o would decrease λ (wavelength), so the interference maxima get closer together... $E = hc/\lambda$, $E \uparrow$ [so] $\lambda \downarrow$." [Modern physics course]

> "When you increase V_o the λ decreases, $\lambda = hc/eV_o$, so the bright spots would be closer together." [Modern physics course]

Student quotes from questions that involve replacing the electrons with different particles. Many students, particularly in the modern physics course, applied the reasoning described above to the Δm question in which they tried to predict the effect of using heavier particles with the same kinetic energy as the original electrons. These students applied the equation $E = hc/\lambda$ to predict incorrectly that the de Broglie wavelength would remain unchanged. Examples of student responses are given below.

> "Neutrons have a higher mass than the e^- but because $\lambda = hc/E$, they have the same wavelength and therefore the pattern doesn't change." [Modern physics course]

> "$E = eV_o = hc/\lambda$. No change." [Modern physics course]

Other students applied similar reasoning on the other version of the Δm question, in which the they are asked to compare the kinetic energy of electrons and protons that produce exactly the same diffraction pattern (using the same value of slit width). Some students incorrectly answered that the kinetic energy of the protons had to be the same as that of the electrons:.

> "$\lambda \propto 1/E$ of light waves, so $\sin\theta \propto m/aE$, ... For $\sin\theta_e = \sin\theta_{protons}$, ... the KE_e equals $KE_{protons}$." [Calculus-based course]

The student who gave this response correctly recognized that the locations of the diffraction minima are given by the equation $a \sin\theta = m\lambda$ but incorrectly believed that the wavelength of each particle is inversely proportional to the kinetic energy. The reference to "light" in his explanation suggests the incorrect application of the photon energy equation $E = hc/\lambda$.

E. SUMMARY

In this chapter we have presented evidence from individual student interviews and from written questions that many students lack a functional understanding of the interference and diffraction of matter waves after standard instruction. The difficulties persist beyond the first course in which the material is taught.

Many students fail to recognize that changing the momentum of a particle affects its de Broglie wavelength. Some explicitly state that varying the velocity or kinetic energy of a particle does not affect the de Broglie wavelength. Others fail to recognize that particles with different mass can have different de Broglie wavelengths. Still others did not seem to even incorporate the de Broglie wavelength in their explanations. These students typically stated that a change in velocity would only change the overall the intensity of the interference or diffraction pattern.

Even students who recognized that the de Broglie wavelength depends upon mass and velocity had difficulties. Many incorrectly used formulas that are valid for light to predict changes in the de Broglie wavelength for particles with mass. Both introductory and modern physics students seem to believe that equations such as $v = \lambda f$ and $E = hc/\lambda$ are equally valid for matter as for light.

The difficulties that were identified seemed to be serious and to persist beyond the first or second exposure to this material. The results presented in this chapter support the need for a review of basic concepts relating to interference and diffraction. We therefore decided to attempt to address these difficulties with tutorial materials that would be appropriate for use in a variety of courses. In the next chapter we describe how the results from research were used in the design, development, and assessment of these materials.

CHAPTER 4: ADDRESSING STUDENT DIFFICULTIES IN APPLYING A WAVE MODEL TO INTERFERENCE AND DIFFRACTION OF MATTER

A. INTRODUCTION

In this chapter we discuss research to address student difficulties in developing and applying a wave model to the interference and diffraction of matter. The difficulties, described in detail in the preceding chapter, were prevalent among students from a wide variety of instructional levels. Many students had difficulty applying the underlying concepts of diffraction and interference. Other students who seemed to understand these concepts had other conceptual and reasoning difficulties when considering the wave properties of matter. In particular, many failed to recognize how changing the mass, speed, or kinetic energy of the particles would affect the diffraction or interference pattern.

In this chapter we describe a tutorial, entitled *Wave properties of matter,* that was developed on the basis of the results of research. The tutorial was used with several different undergraduate student populations. To assess the effectiveness of the tutorial at addressing specific difficulties, results from written questions given after standard instruction are compared to those questions posed after modified instruction that included the tutorial. We also investigate the extent to which the tutorial helped students of different academic ability.

B. DESIGN OF THE TUTORIAL *WAVE PROPERTIES OF MATTER*

The primary objective of the tutorial *Wave properties of matter* is to help students correctly apply the concept of de Broglie wavelength to physical situations in which it is relevant. Possibly the simplest such situation is double-slit interference. However, previous research, described in Chapter 2, has shown that even situations involving two-source interference of water waves and double-slit interference of light pose serious difficulties for students at many levels.[49] Thus the tutorial begins with a series of questions that help students review the important relevant concepts of superposition and path-length difference. After an introductory section on double-slit interference for light, students make an analogy between this situation and double-slit interference of electrons. Finally, students are led to recognize the relationship between the momentum of the electrons and the de Broglie wavelength of the electrons. The version of the tutorial described below is the current version, which was developed on the basis of the research discussed in the preceding chapter.

1. REVIEW OF DOUBLE-SLIT INTERFERENCE OF LIGHT

At the beginning of the tutorial, students are shown a photograph of the pattern produced near the center of the screen in a typical double-slit experiment for light. (See Figure 4-1.) Students notice that they cannot apply concepts from geometrical optics to account for the pattern (*e.g.*, that the light is not passing in straight lines from the light source through the slits to the screen). Students recognize that the light emanating from each slit illuminates the entire portion of screen shown and that the bright and dark fringes arise from constructive and destructive interference. They are then guided through the reasoning to derive the equations, $d\theta \approx m\lambda$ and $d\theta \approx (m + 1/2)\lambda$, that describe the angles to the maxima and minima, respectively.[50] (The small-angle approximation is used to simply the expression for the path length difference $d \sin\theta. \approx d\theta$.)

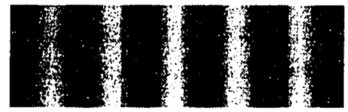


Figure 4-1: Photograph of double-slit pattern for light.

Students conclude the first part of the tutorial by applying their results in the following qualitative problem. They are shown two interference patterns that are different in that the maxima and minima are farther apart in the second pattern than in the first. (See Figure 4-2.) Students are told to suppose that only one change was made to the experimental setup to cause the maxima to move farther apart. They are first asked whether a change in slit separation could have been responsible for the change in the pattern, and if so, to determine whether the slit separation would have been increased or decreased in that case. By using qualitative arguments about path-length difference or the equations that they derived earlier, they

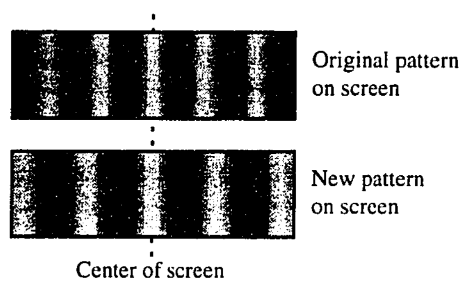


Figure 4-2: Double-slit patterns in which the maxima in one pattern are farther apart than those in the other.

recognize that the slit separation could have decreased in this case. Students are then asked whether a change in the wavelength of the light could have been responsible for the change in the pattern, and if so, to determine whether the wavelength would have been increased or decreased. Using similar arguments, they recognize that the wavelength must have increased in order to obtain the second pattern from the first.

2. *DOUBLE-SLIT INTERFERENCE OF ELECTRONS*

Figure 4-3: Photograph of a double-slit pattern using electrons.

The second part of the tutorial deals with electrons that are accelerated from rest through a known potential difference V_o and are incident on two very narrow slits. The students are shown a photograph of the pattern seen on a phosphorescent pattern placed far from the slits.[51] (See Figure 4-3.) As they had done previously for the case of double-slit interference of light, students reason that the electrons are not passing in straight lines through the slits to produce the pattern. Instead, the students recognize that the presence of maxima and minima on the screen suggests that the electrons are exhibiting wave-like properties in this experiment.

The structure of the remainder of the tutorial follows an *elicit-confront-resolve* strategy.[52] First, students are asked to predict how, if at all, the locations of the interference maxima would change if the accelerating voltage V_o (and thus the kinetic energy) were halved. The question is intended to *elicit* incorrect responses from students who have difficulty relating the de Broglie wavelength of an electron to its speed, momentum, or kinetic energy. One common incorrect response to this question is that the maxima would stay in the same place but become dimmer. Some students recognize that the speed of the electrons would decrease but incorrectly apply the equation $v = \lambda f$ to predict that the wavelength of the electrons would also decrease. Other students correctly state that the maxima would get farther apart but predict that the angles to the maxima would increase by a factor of two. The latter response is common among students who inappropriately use equations such as $E = hc/\lambda$ to relate the kinetic energy to the de Broglie wavelength. (These errors are discussed in detail in Chapter 3.)

Next, students are led to *confront* any errors they may have made in their predictions. After making their predictions, students are given photographs that illustrate the interference patterns made by electrons before and after the accelerating voltage is halved (see Figure 4-4). They are then asked a series of questions in which they utilize the results that they derived in the preceding review of double-slit interference of light. Students observe that halving the accelerating voltage causes the angles to the interference maxima to increase by a factor that is less than two. Thus those students who made any of the incorrect predictions described in the preceding paragraph are led to *confront* their errors.

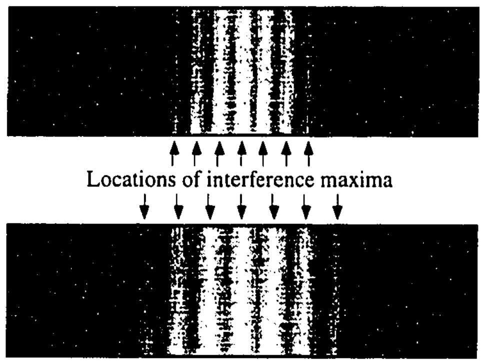


Figure 4-4: Photographs that illustrate the effect of halving the accelerating voltage in a double-slit experiment with electrons.

Finally, students are given the opportunity to *resolve* any errors that they made in their predictions. On the basis of the patterns shown in the photographs and the equation $d\,\theta \approx m\,\lambda$ for the interference maxima (assuming small θ), students infer that the de Broglie wavelength increased by a factor less than two. They then determine that halving the accelerating voltage would cause the kinetic energy to decrease by a factor of two and the momentum to decrease by a factor less than two. Students confirm that the increase in de Broglie wavelength and decrease in momentum are consistent with the definition $\lambda = h/p$ discussed previously in lecture. Finally, students are led to review their work and reflect on how their reasoning had evolved during the tutorial.

C. ASSESSMENT OF THE TUTORIAL *WAVE PROPERTIES OF MATTER*

In order to assess the effectiveness of the tutorial, we developed several post-tests that were used with the target populations. In almost all cases, the post-test was posed in a context different from double-slit interference, which was covered explicitly in the tutorial. The results from the qualitative questions administered after standard instruction were compared to those from questions given after modified instruction that included the tutorial *Wave properties of matter.*

The methods by which we analyzed the results from the written questions are described below in section 1. These methods were used for each population in the study. The results that were obtained from using the tutorial in the introductory algebra-based course, the sophomore-level modern physics course, and the junior-level quantum mechanics course are discussed in sections 2, 3, and 4, respectively. (These courses are described in general terms in Chapter 1.)

1. DESCRIPTION OF ASSESSMENT TECHNIQUES

The organization of the sections in which we present pretest and post-test results follows the general outline below. In part a of each section we present a general description of the questions that were used in each class that used the tutorial. In parts b and c we describe in detail how the student responses were analyzed.

a) General description of pretest and post-test questions

In part a of each section, we describe the written questions that were given before and after tutorial instruction in the class under discussion. Each pretest involved double-slit interference or Bragg scattering of electrons or other particles. All versions of the post-test, with one exception, were on single-slit diffraction of electrons or other particles. (The lone exception, which was given to about half of the students in an algebra-based course that used the tutorial, was in the context of double-slit interference.) The terms "de Broglie wavelength" and "wavelength" were never used in the phrasing of the questions. In this way, students needed to decide on their own that it was appropriate to apply their knowledge of de Broglie wavelength to the situation.

We believed that post-test questions on single-slit diffraction of electrons (or other particles) would be adequate in assessing their understanding of de Broglie wavelength after instruction in the context of double-slit interference. On the basis of prior research of student understanding of physical optics, students do not always treat diffraction as an interference phenomenon.[53] Thus we believed that it would sufficiently challenging to the students to be forced to extend their results from the context of double-slit interference to the context of single-slit diffraction.

Each pretest and post-test problem consisted of two or more questions. All problems contained one question on some of the basic ideas that underlie interference or diffraction. One or more additional questions specifically probed student understanding of the de Broglie wavelength. Each post-test question was designed to correspond to a pretest question given to the same class after standard instruction. Below we describe the types of questions that were included, however the specific pretest and post-test problems used in each class are presented when the results from that particular class are discussed.

Varying the slit width, slit separation, or lattice spacing. The first question on each pretest or post-test required students to predict the effect of changing the slit width, slit separation, or lattice spacing, depending upon the specific situation. In this way all of the students were tested on basic ideas that underlie interference or diffraction but not specifically on the de Broglie wavelength. The questions that pertain to changing the slit separation (in a double-slit interference problem) or lattice spacing (in a Davisson-Germer problem) are denoted in this chapter as "Δd" questions, for short. Those that pertain to changing the slit width (in a single-slit diffraction problem) are denoted as "Δa" questions. (We continue to refer to each question by the shorthand notation that was introduced in the preceding chapter.)

Varying the speed or kinetic energy. Some questions tested understanding of de Broglie wavelength by requiring students to predict the effect on the interference or diffraction pattern of changing the speed or kinetic energy of the particles. These questions are denoted as "Δv" or "ΔE_K" questions, respectively.

Using particles of different mass. On other questions students had to consider the effect on the interference or diffraction pattern of using particles of different mass. In some versions, students had to predict the effect of using different particles with the same kinetic energy or speed as the original electrons (while keeping the slit spacing, slit width, or lattice spacing constant). In other versions, the students were told to suppose that the new particles produced the same interference or diffraction pattern as did the original electrons. They then had to infer how the speed or kinetic energy of the new particles compared to that of the original electrons. Collectively, the questions that pertain to using particles of different mass are denoted as "Δm" questions.

b) Comparison of student responses after standard instruction and after modified instruction

In part b of each section we compare the overall results from questions given after standard instruction and after modified instruction. In those classes in which the tutorial was used, we considered only those students who took both the pretest and post-test and who worked through the tutorial.

A primary goal of the tutorial *Wave properties of matter* was to help students extend their understanding of the concepts underlying interference and diffraction to situations that involve the wave properties of matter. In order to gauge effectiveness of the tutorial in meeting this goal, we determined which students gave correct responses on those questions that pertain to changing the slit width, slit separation, or lattice spacing (*i.e.*, the Δa or Δd questions). We compared the percentage of correct responses that were accompanied with correct reasoning from post-standard and post-modified instruction. This comparison allowed us to determine whether the first part of the tutorial helped students acquire a deeper understanding of basic concepts in interference and diffraction.

From the subset of N' students who gave correct responses on each Δa and Δd question, we then determined the percentage *of this subset* who answered the Δv, ΔE_K, and Δm questions correctly with correct reasoning about de Broglie wavelength. We compared the percentages from questions given after standard instruction and after modified instruction in order to measure the effectiveness of the second part of the tutorial, which was intended to help address specific difficulties with the de Broglie wavelength.

c) Comparison of student responses according to overall academic performance in the class

In addition to the overall results from the written questions, we analyzed the student responses according to the academic ability of the students in the class. These results are presented in part c of each subsequent section.

Each class is subdivided into three groups, the highest quartile, and lowest quartile, and middle half of the class, according to performance on homework and examination questions that tested on material not specifically covered in the tutorial *Wave properties of matter.* (In all cases, the performance of the second and third quartiles were similar, so these groups were combined.) For each question, we tabulated the percentage of correct responses (as described in part b above) for each of the three separate groups. We then compared the results from questions given after standard instruction and after modified instruction for each group.

d) *Comparison of pretest results according to prior tutorial instruction in the modern physics course*

Eight students in the Autumn 1997 quantum mechanics class had had the tutorial *Wave properties of matter* previously in their modern physics course (Winter 1997). Therefore, the Autumn 1997 pretest served as a long-term assessment of the Winter 1997 tutorial for those students. We compared the performance of those eight students to the performance of the other 25 students who had not yet worked through *Wave properties of matter.*

2. *RESULTS FROM THE INTRODUCTORY ALGEBRA-BASED COURSE*

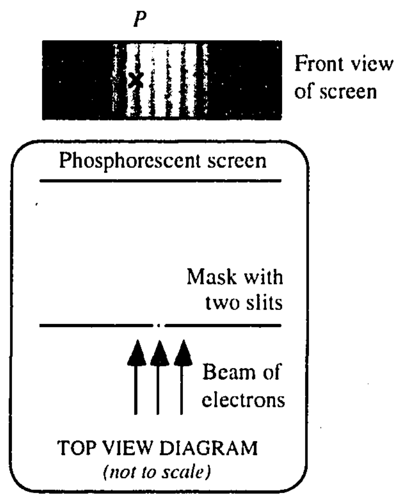
In this section we present results from the algebra-based waves and optics course (Physics 116). We focus primarily on one class (Autumn 1997) in which only standard instruction occurred and another class (Spring 1997) in which the students had had several tutorials in waves and optics besides *Wave properties of matter.*

In Autumn 1997 a written problem was included on the final examination that tested on double-slit interference of electrons. (This problem is described in the preceding chapter and so is not discussed here.) In Spring 1997, a similar question was administered as a pretest after all standard instruction in double-slit interference and the de Broglie wavelength, after tutorial instruction on two-source interference, but before the tutorial *Wave properties of matter.* Two slightly different versions of a post-test question were used in the same class to assess the effect of the tutorial. Each was taken by approximately half of the class on a course examination, given after all instruction.

a) *Description of pretest and post-test questions given in the algebra-based class that used the tutorial*

Description of pretest. The pretest used in Spring 1997 is shown in Figure 4-5. All parts of the pretest proved challenging to the students. We discuss in particular the results from the questions given in part B, which included questions that tested on the de Broglie wavelength. (Part A, which tests on basic concepts from physical optics, was difficult for many students. However, responses to part A were not taken into account in the analysis of the pretest in order to be consistent with the analysis of other problems on electron interference and diffraction, which did not include a similar question.)

A beam of monoenergetic electrons is incident on a mask that contains two very narrow slits (see top view diagram at right). The photograph shows the pattern seen on a phosphorescent screen placed far from the slits. The bright regions indicate the concentrations of electrons hitting the screen.



A. Consider a point P, a point of minimum intensity. If the left slit were covered, would the intensity at point P increase, decrease, or stay the same? Explain your reasoning.

B. Suppose this experiment were repeated with a ***single*** change made to the original setup. For each possible change described below, predict whether the bright regions would get *closer together*, move *farther apart*, or *stay in the same location*. Explain your reasoning in each case.

   i. The slit separation is decreased.

   ii. The speed of the electrons is increased.

   iii. The electrons are replaced with neutrons, with each neutron having the *same speed* as each of the original electrons. (Recall that $m_n > m_e$.)

Figure 4-5: Pretest administered in the Spring 1997 algebra-based course after standard instruction on double-slit interference and the wave properties of matter. In this chapter we discuss results from part B of the pretest.

Correct answers to part B of the pretest. In the case in which the slit separation is decreased (part i), one can reason qualitatively about changes in the path length difference or, alternatively, use the relationship $d \sin\theta = n\lambda$ (n = 0, 1, 2,...) to describe the angle θ to each interference maximum. If d were to decrease, the angle θ to any particular maximum would increase, so the maxima would move farther apart. If the speed of the electrons were increased (part ii), the de Broglie wavelength $\lambda = h/p$ of the electrons would decrease. Therefore, according to the equation for the angles to the maxima, the maxima would move closer together. Finally, if neutrons were used having the same speed as the original electrons (in part iii), the de Broglie wavelength would decrease, and the maxima would move closer together.[54]

Description of double-slit version of post-test. One version of the post-test (given to half the class) was in the context of double-slit interference. (See Figure 4-6.) Part A probed student understanding of the underlying ideas behind double-slit interference. Part B required students to infer from a comparison of electron and neutron interference patterns how the speed of the neutrons compared to that of the electrons. This task was different from any that the students had done on the pretest or in tutorial.

A beam of monoenergetic electrons is incident on a mask containing two very narrow slits. The photograph at right shows the resulting pattern on a distant screen.


front view of screen

A. If the distance between the slits were increased, would the bright regions on the screen get farther apart, closer together, or stay in the same locations? Explain your reasoning.

B. Suppose instead that neutrons were used (instead of electrons) with the same mask to produce **exactly the same pattern as before.** Would the speed of the neutrons in this case be greater than, less than, or equal to the speed of the electrons? Explain your reasoning.

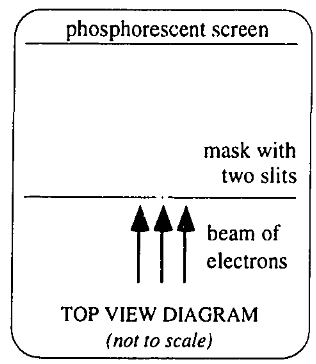


Figure 4-6: Double-slit version of post-test, posed on a midterm examination in the Spring 1997 algebra-based course.

Description of single-slit version of post-test. The other version of the post-test (given to the remainder of the class) presented questions about single-slit diffraction. (See Figure 4-7.) Part A, which pertains to changing the slit width, probed student ideas about diffraction without requiring knowledge of the factors affecting the de Broglie wavelength. Part B is similar to part B of the double-slit version of the post-test.

A beam of monoenergetic electrons is incident on a mask containing a single narrow slit. The photograph at right shows the resulting pattern on a distant screen.

A. If the width of the slit were decreased, would the bright regions on the screen get farther apart, closer together, or stay in the same locations? Explain your reasoning.

B. Suppose instead that neutrons were used (instead of electrons) with the same mask to produce **exactly the same pattern as before.** Would the speed of the neutrons in this case be greater than, less than, or equal to the speed of the electrons? Explain your reasoning.

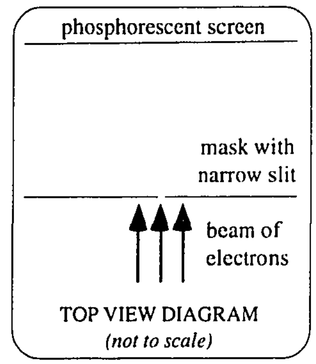


Figure 4-7: Single-slit version of post-test, posed on a midterm examination in the algebra-based course.

Correct answers to part A of each post-test. In part A of both post-tests, as on the pretest, one can either reason qualitatively in terms of path length difference or use the formulas for interference or diffraction maxima. On the double-slit post-test one can apply the condition $d \sin\theta = n\lambda$ ($n = 0, 1, 2, \ldots$) that describes the angle θ to each interference maximum. Increasing d would cause the angle θ to decrease, and the maxima would move closer together. In part A of the single-slit post-test, the equation $a \sin\theta = n\lambda$ ($n = 1, 2, \ldots$) can be used to determine the angle θ to each diffraction minimum. Decreasing the width of the slit a would cause the minima to occur at larger values of θ than before. The diffraction maxima move in a similar fashion, and thus decreasing the slit width would cause the bright regions to move farther apart.

Correct answers to part B of each post-test. In part B of each post-test, students needed to recognize that if neutrons produced the same interference or diffraction pattern as electrons (using the same slit), then the neutrons and electrons would have equal de Broglie wavelengths $\lambda = h/p$. Thus the electrons and neutrons would have equal momenta. Since neutrons are more massive than electrons, the speed of the neutrons must be less than that of the electrons.

b) Comparison of student responses after standard instruction and after modified instruction

In this section and the following section we compare the performance of students in the Spring 1997 and Autumn 1997 classes on pretest and post-test questions described above. In Spring 1997, in which the tutorial was used, we considered only those students who took the pretest before the tutorial, worked through the tutorial, and took the post-test.

We first compare student performance on questions that pertained to changing the slit separation or slit width. These results are summarized below in Table 4-1. Results from the two versions of the Spring 1997 post-test were similar and thus are combined.

Table 4-1: Comparison of performance of students in algebra-based courses on questions that involve varying the slit separation or slit width (*i.e.*, the Δd and Δa questions). Only those students from the Spring 1997 class who took the pretest and post-test and worked through the tutorial are included. [Percentages have been rounded to the nearest 5%.]

	Algebra-based course		
	After standard instruction		After modified instruction
	Au 1997 $N = 102$	Sp 1997 $N = 103$	Sp 1997 $N = 103$
Correct responses	**70% (69)**	**35% (36)**	**60% (62)**
Correct with correct reasoning	15% (13)	5% (5)	40% (42)

The results above seem to show a discrepancy between the performance of students on the Δd and Δa questions on the Spring 1997 pretest and Autumn 1997 post-test, when the questions were given after standard instruction. These two problems were given under different conditions; one was used as an in-class pretest, the other was included on an examination in which the students were allowed to use a sheet of notes. (These differences are discussed in detail in section C of the preceding chapter.) The observation that approximately equal percentages of students in both groups obtained the correct answer with correct reasoning (5% and 15%) suggest that the two groups can be treated as equivalent.

On Δd and Δa questions that were included on post-tests, the percentages of total correct responses were approximately the same in the two classes (70% in Autumn 1997, 60% in Spring 1997). However, 40% of the students in Spring 1997 gave correct responses supported with correct reasoning, as compared to only 15% in Autumn 1997. This result suggests that the review on the first part of the tutorial *Wave properties of matter* was helpful in enhancing student understanding of the basic concepts of diffraction and interference.

After determining the subset of students in each class who gave correct answers (ignoring reasoning) on the Δd and Δa questions, we determined the percentage of the N' students in each subset who correctly answered the questions that pertain to using particles of different mass (*i.e.*, the Δm questions). These results are shown below in Table 4-2. The results from the Autumn 1997 examinations and Spring 1997 pretests (both given after standard instruction) were similar and are therefore combined in the table.

Table 4-2: Comparison of performance of students in algebra-based courses on questions that involve using particles of different mass (*i.e.*, the Δm questions). Results are shown only for the subset of N' students who correctly determined the effect of varying the slit separation or slit width, shown in Table 4-1. Results from the Δm questions given after standard instruction were similar and therefore have been combined.

	Algebra-based course	
	After standard instruction	After modified instruction
	Au 1997 Sp 1997 $N' = 105$	Sp 1997 $N' = 62$
Correct responses (correct reasoning used to relate λ to m)	20% (22)	70% (42)
Incorrect responses	65% (69)	30% (19)
Blank or incomplete responses	15% (14)	< 5% (1)

The results from the Δm questions suggest improvement in relating de Broglie wavelength to the mass of the particles. Of those students who gave correct responses to the Δd question after standard instruction, only 20% (22 of 105) used correct reasoning about de Broglie wavelength when considering particles of different mass. On the Spring 1997 post-tests after the tutorial, however, this percentage increases to 70% (42 of 62).

c) Comparison of student responses according to overall academic performance in the class

The pretest and post-test results presented above were analyzed with respect to the overall academic standing of the students in the class. The measure of ability used was their performance on homework and examination questions that did not relate directly to the tutorial *Wave properties of matter.*

In Figure 4-8 below we show the results from the Δd and Δa questions from post-tests given in the Autumn 1997 class (after standard instruction) and the Spring 1997 class (after modified instruction that included the tutorial). (The pretests from the Spring 1997 class, which were given under different conditions than the examinations, are not shown.) Each row of pie charts in the figure represents the performance of a particular group of students, with each individual chart corresponding to either the lowest quartile, middle half, or highest quartile.

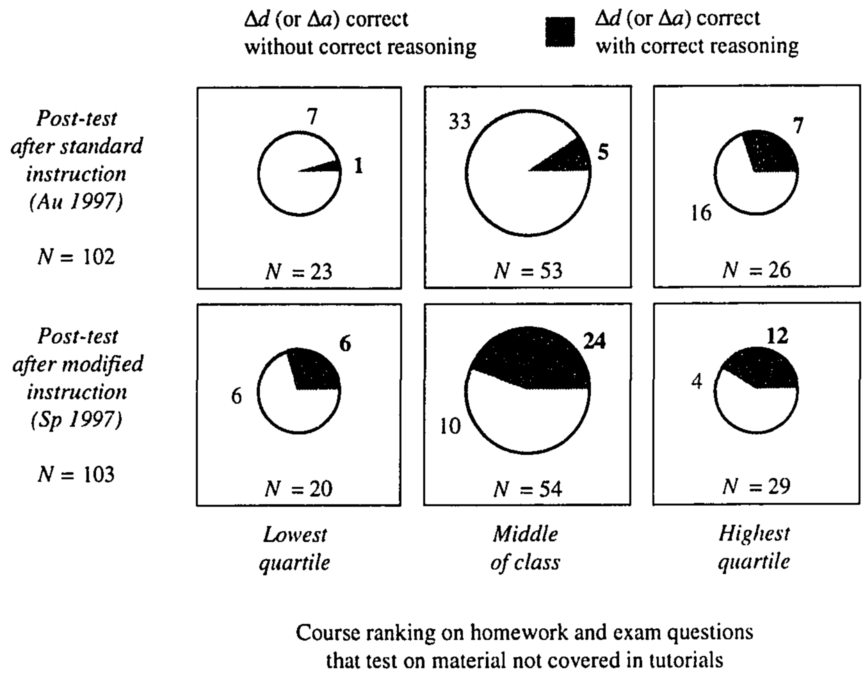


Figure 4-8: Comparison by course ranking of performance of students in algebra-based courses in which the tutorial was and was not used. Results are from questions that involve varying the slit width or slit separation. (Only those students from the Spring 1997 class who took the pretest and post-test and worked through the tutorial are included.)

In each pie chart shown above, correct responses with correct reasoning on the Δd or Δa questions are represented by regions in dark gray; correct responses without correct reasoning, in light gray. Taking into account all correct answers regardless of reasoning, the results from the Autumn 1997 and Spring 1997 are approximately the same. However, a larger percentage of students gave correct answers with correct reasoning after modified instruction than after standard instruction. This trend was most notable in the middle half of the class. About 45% of the students in the middle half of the Spring 1997 class (24 of 54) gave correct answers with correct reasoning, compared to less than 10% in Autumn 1997 (5 of 53).

Figure 4-9 below summarizes the results from the Δm questions by those students who gave a correct response on the preceding Δd or Δa question. The black portion of each pie chart indicates the fraction of students in a particular group who correctly answered both the Δd (or Δa) question (ignoring reasoning) and the Δm question with correct reasoning. The results from these subsets of students on the Autumn 1997 post-test and Spring 1997 pretest were similar (see Table 4-2), so the results from these groups are combined in the figure.

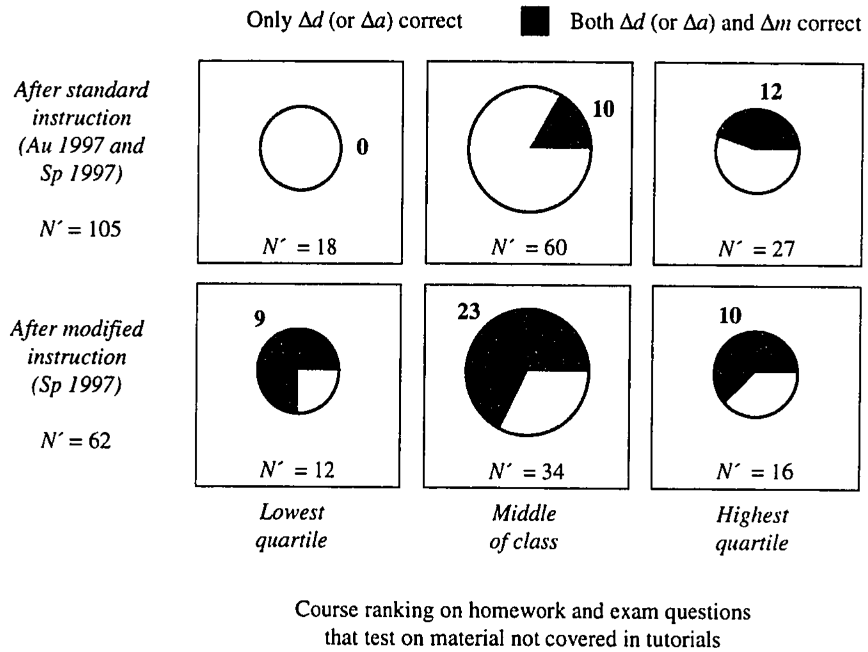


Figure 4-9: Comparison by course ranking of performance of students in algebra-based courses in which the tutorial was and was not used. Results are from questions that involve using particles of different mass (*i.e.*, the Δm questions). Results are shown only for the subset of N' students who correctly determined the effect of varying the slit separation or slit width, shown in Table 4-1. Results from the Δm questions given after standard instruction were similar and therefore have been combined.

As indicated by the pie charts in the top row of the figure, very few students gave correct answers on *both* the Δd (or Δa) *and* Δm questions after standard instruction. The performance of the students in the top quartile was the best of all three groups, as might be expected, with less than 50% (12 of 27) answering both questions correctly.

In contrast, in each of the three groups (highest quartile, middle half, and lowest quartile), more than half of the Spring 1997 students who correctly answered the Δd or Δa post-test questions also answered the Δm question correctly. This result suggests that the tutorial helped students, particularly in the middle half and lowest quartile of the class, in understanding the relationship between the de Broglie wavelength and the mass of the particles.

d) Summary of results from the algebra-based course

The results presented in this section indicate that the tutorial *Wave properties of matter* helped address specific difficulties that persist after standard instruction in the algebra-based physics course. The analysis of responses to questions that pertained to varying the slit width or slit separation suggest that the first part of the tutorial helped students develop a deeper understanding of the basic concepts of interference and diffraction. Similar findings were obtained from questions that tested student understanding of the de Broglie wavelength. Improvement was observed at all levels of academic ability, most notably in the middle half and bottom quartile of the class.

3. RESULTS FROM THE SECOND-YEAR MODERN PHYSICS COURSE

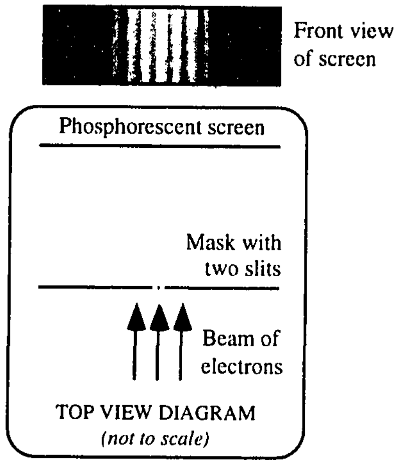
In this section we present pretest and post-tests results from the sophomore-level modern physics course (Physics 225). We discuss results from two different classes (Spring 1995 and Autumn 1995) in which standard instruction occurred and from one class (Winter 1997) that used the tutorial *Wave properties of matter.* All three classes had different instructors.

In Spring 1995 and Autumn 1995, a problem was included on the final examination. (This problem is described in the preceding chapter and so is not discussed here.) In Winter 1997, the tutorial *Wave properties of matter* was used after all standard instruction on double-slit interference and de Broglie wavelength. A pretest was administered in the first 10 minutes of the class period. The pretest was similar to the examinations posed after standard instruction in the modern physics course. After the pretest, the remaining 40 minutes of the class period was spent working through the tutorial worksheet in the regular lecture room. A post-test was included on the final examination.

a) Description of pretest and post-test questions given in the modern physics class that used the tutorial

Description of pretest. The pretest questions used in the Winter 1997 modern physics course are identical to those in part B of the pretest given in the Spring 1997 algebra-based course (Figure 4-5). The Winter 1997 pretest had one question that pertained to changing the slit separation and two other questions that tested on de Broglie wavelength. The pretest is shown in its entirety in Figure 4-10.

A beam of monoenergetic electrons is incident on a mask that contains two very narrow slits (see top view diagram at right). The photograph shows the pattern seen on a phosphorescent screen placed far from the slits. The bright regions indicate the concentrations of electrons hitting the screen.



Suppose this experiment were repeated with a ***single*** change made to the original setup. For each possible change described below, predict whether the bright regions would get *closer together*, move *farther apart*, or *stay in the same location*. Explain your reasoning in each case.

i. The slit separation is decreased.

ii. The speed of the electrons is increased.

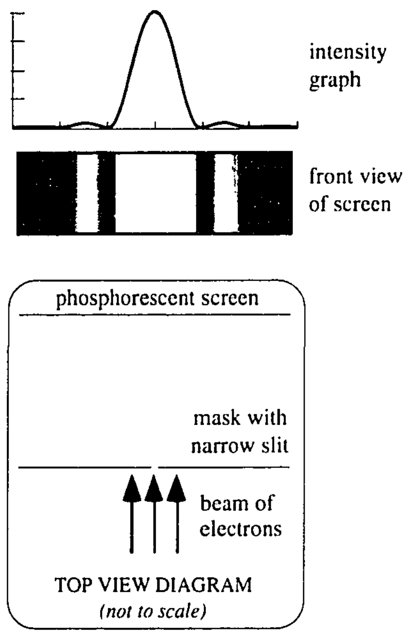
iii. The electrons are replaced with muons, with each muon having the *same kinetic energy* as each of the original electrons. (Recall that $m_\mu \approx 200\, m_e$)

Figure 4-10: Pretest administered in the Winter 1997 modern physics course after all relevant instruction on the wave properties of matter.

Correct answers to pretest. The answers to the pretest are the same as those to the questions included on part B of the pretest from the Spring 1997 algebra-based course. (See part a of section 2.) In part i, the interference maxima would move farther apart, while in parts ii and iii they would move closer together.

Description of post-test. Figure 4-11 shows the post-test used on the final examination in Winter 1997. Part A of the problem included questions about changing the slit width and the kinetic energy of the electrons. Part B is almost identical to part B of the single-slit version of the post-test used Spring 1997 in the algebra-based course. (See part a of section 2.)

A beam of monoenergetic electrons is incident on a mask containing a single narrow slit. The photograph at right shows the pattern that you would see on a distant phosphorescent screen.



A. Suppose this experiment were repeated with a *single* change made to the original setup. For each change described below, would that change cause the minima on the screen to get closer together, move farther apart, or stay at the same locations? Explain your reasoning.

   i. The slit width is halved.

   ii. The kinetic energy of the electrons is halved.

B. Suppose instead that the electrons were replaced with neutrons. In order for neutrons to produce **exactly the same pattern** as the electrons in the original experiment (shown at the top of the page), would the kinetic energy of the neutrons be greater than, less than, or equal to the kinetic energy of the original electrons? Explain your reasoning.

Figure 4-11: Post-test question included on the final examination given in the Winter 1997 modern physics course.

Correct answers to post-test. For both questions in part A, one can use the equation $a \sin\theta = n\lambda$ for the angles to the diffraction minima. In part i, halving the slit width a would cause the angle θ to each minima to increase, so the minima would move farther apart. In part ii, halving the kinetic energy of the electrons results in a decrease in the momentum p of each electron and thus an increase in de Broglie wavelength $\lambda = h/p$. In this case the minima would move farther apart.

On part B, students needed to infer from the given information that the de Broglie wavelength of the neutrons would be equal to that of the original electrons. Each neutron would therefore have the same momentum as each of the original electrons. Because $m_n > m_e$, the kinetic energy $E_K = p^2/2m$ of each neutron would be less than that of each of the original electrons.

b) Comparison of student responses after standard instruction and after modified instruction

In the following sections we compare student performance on corresponding pairs of questions given after standard instruction and after modified instruction. In the Winter 1997 class, which used the tutorial, we counted only the responses by the 18 students (out of 25 in the class) who took the pretest, worked through the tutorial, and took the post-test.

We first discuss the results from the first question, in which students were required to predict the effect of varying the slit separation or slit width. These results are shown below in Table 4-3. Results from the post-tests given after standard instruction (Spring 1995 and Autumn 1995) were similar and therefore have been combined.

Table 4-3: Comparison of performance of students in modern physics courses on questions that involve varying the slit separation or slit width (*i.e.*, the Δd and Δa questions). Only those students from the Winter 1997 class who took the pretest and post-test and worked through the tutorial are included.

	Modern physics course		
	After standard instruction		After modified instruction
	Sp 1995 Au 1995 $N = 82$	Wi 1997 $N = 18$	Wi 1997 $N = 18$
Correct responses	**55% (47)**	**70% (13)**	**85% (15)**
Correct with correct reasoning	20% (19)	20% (4)	65% (10)

The number of students who answered the Δd and Δa questions on the pretest and post-test increased slightly, from 70% (13 to 18) to 85% (15 of 18). However, the percentage of correct responses accompanied by correct reasoning increased from 20% (4 of 18) to 65% (10 of 18). This result suggests that the first part of the tutorial, in which students review basic concepts of superposition and interference, was helpful for the students.

Table 4-4 below summarizes the performance on the Δv and ΔE_K questions by those students who gave correct responses to the Δd or Δa questions. The results from all questions given standard instruction were similar and so are combined in the table.

Table 4-4: Comparison of performance of students in modern physics courses on questions that involve varying the speed or kinetic energy of the electrons (*i.e.*, the Δv and ΔE_K questions). Results are shown only for the subset of N' students who correctly determined the effect of varying the slit separation or slit width, shown in Table 4-3. Results from the Δv and ΔE_K questions given after standard instruction were similar and therefore have been combined.

	Modern physics course	
	After standard instruction	After modified instruction
	Sp 1995 Au 1995 Wi 1997 $N' = 60$	Wi 1997 $N' = 15$
Correct responses (correct reasoning used to relate λ to v or E_K)	35% (22)	75% (11)
Incorrect responses	60% (37)	25% (4)
Blank or incomplete responses	< 5% (1)	0% (0)

After standard instruction, only 35% (22 of 60) of the students who correctly answered the Δd question also gave correct reasoning in relating the de Broglie wavelength to speed or kinetic energy. On the Winter 1997 post-test, however, this percentage increased to 75% (11 of 15).

Improvement is also evident from the analysis of responses to the Δm questions. These results are summarized shown below in Table 4-5.

Table 4-5: Comparison of performance of students in modern physics courses on questions that involve using particles of different mass (*i.e.*, the Δm questions). Results are shown only for the subset of N' students who correctly determined the effect of varying the slit separation or slit width, shown in Table 4-3. Results from the Δm questions given after standard instruction were similar and therefore have been combined.

	Modern physics course	
	After standard instruction	After modified instruction
	Sp 1995 Au 1995 Wi 1997 $N' = 60$	Wi 1997 $N' = 15$
Correct responses (correct reasoning used to relate λ to m)	25% (16)	75% (11)
Incorrect responses	60% (37)	25% (4)
Blank or incomplete responses	15% (7)	0% (0)

Of those students who correctly answered the Δd question after standard instruction, only 25% (16 of 60) gave correct responses on the Δm question. The corresponding percentage from the Winter 1997 post-test (after the tutorial) was 75% (11 of 15).

It is also informative to combine the results from the two tables above. To answer *both* the Δv (or ΔE_K) *and* Δm questions correctly with correct reasoning demonstrates a more complete understanding of the de Broglie wavelength than is not necessarily shown by correctly answering just one question. The combined results from these questions are presented in Table 4-6 below.

Table 4-6: Comparison of overall performance of students in modern physics courses on questions that involve varying parameters that affect de Broglie wavelength. Results are shown only for the subset of N' students who correctly determined the effect of varying the slit separation or slit width, shown in Table 4-3.

	Modern physics course	
	After standard instruction	After modified instruction
	Sp 1995 Au 1995 Wi 1997 $N' = 60$	Wi 1997 $N' = 15$
Correct reasoning on both Δv (or ΔE_K) and Δm questions	25% (14)	55% (8)
Correct reasoning on one but not both	15% (9)	40% (6)
Incorrect reasoning on both	60% (37)	5% (1)

About 55% of the Winter 1997 students (8 of 15) who answered the Δa post-test question correctly also gave correct responses with correct reasoning on *both* the ΔE_K and Δm questions. The corresponding percentage from questions after standard instruction was only 25% (14 of 60). Moreover, after standard instruction about 40% of the students (23 of 60) correctly answered at least one of the two questions pertaining to the de Broglie wavelength. After modified instruction, the corresponding percentage was about 95% (14 of 15).

c) Comparison of student responses according to overall academic performance in the class

The results presented in the previous section indicate that the tutorial enhanced student understanding. It is also useful to analyze the results according to academic ability in the class. To accomplish this, we organized the results according to student performance on homework and examinations not directly related to the tutorial.

In Figure 4-12 below we compare the performance of students on the Δd and Δa questions given after standard instruction and after modified instruction. The results are shown separately for the highest quartile, middle half, and lowest quartile of the classes. (For the Winter 1997 class, only the 18 students who took the pretest, did the tutorial, and took the post-test are included.)

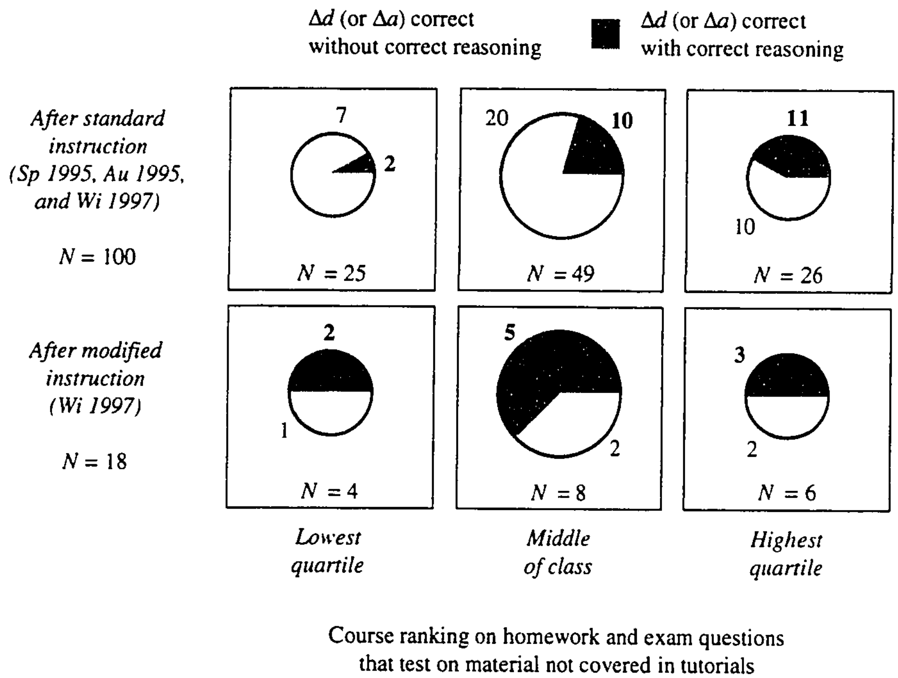


Figure 4-12: Comparison by course ranking of performance of students in modern physics courses in which the tutorial was and was not used. Results are from post-test questions that involve varying the slit width or slit separation. (Only those students from the Winter 1997 class who took the pretest and post-test and worked through the tutorial are included.)

The light gray and dark gray regions of the pie charts shown in Figure 4-12 indicate that more students correctly answered the Δa question after modified instruction than did the corresponding Δd questions after standard instruction. The most noticeable improvement occurred in the middle half of the class, in which about 20% (10 of 49) gave correct answers with correct reasoning before (or without) the tutorial, while more than half (5 of 8) did after the tutorial.

Figure 4-13 below summarizes the responses given on the Δv and ΔE_K questions by the subset of N' students who correctly answered the Δd or Δa question. The dark regions indicate those students who gave correct answers on the Δv and ΔE_K questions with correct reasoning about de Broglie wavelength.

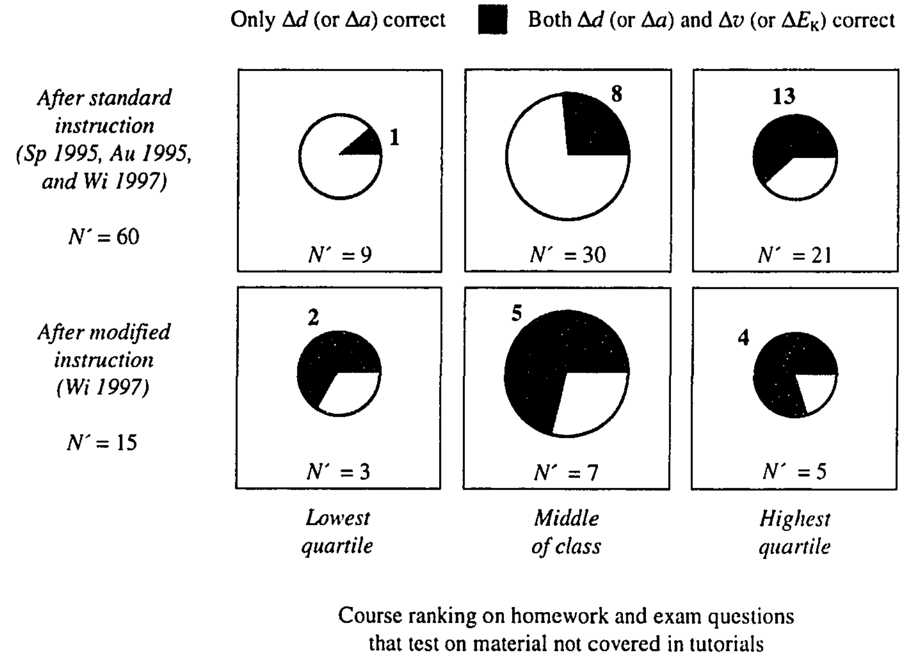


Figure 4-13: Comparison by course ranking of performance of students in modern physics courses in which the tutorial was and was not used. Results are from questions that pertain to varying the kinetic energy of the electrons (*i.e.*, the ΔE_K questions). Results are shown only for the subset of N' students who correctly determined the effect of varying the slit separation or slit width, shown in Table 4-3. Results from ΔE_K questions given after standard instruction were similar and therefore have been combined.

After the tutorial, improvement seemed to occur in all three groups, but most significantly in the middle half of the class. In this group, a majority of the students (5 of 7) who answered the Δa question correctly also answered the ΔE_K question correctly. The percentage from the corresponding group after standard instruction was only about 25% (8 of 30).

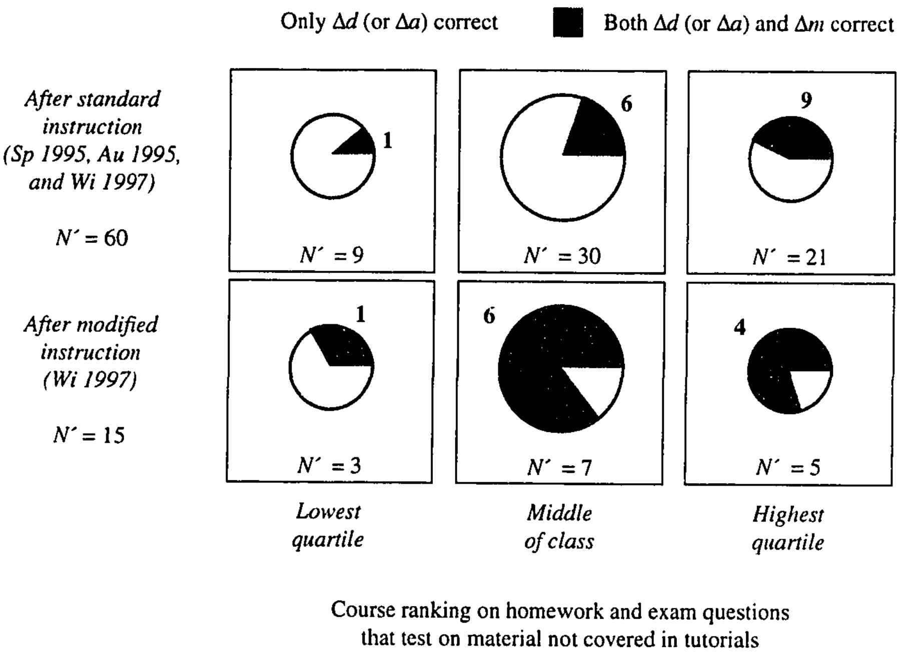


Figure 4-14: Comparison by course ranking of performance of students in modern physics courses in which the tutorial was and was not used. Results are from questions that involve using particles of different mass (*i.e.*, the Δm questions). Results are shown only for the subset of N' students who correctly determined the effect of varying the slit separation or slit width, shown in Table 4-3. Results from the Δm questions given after standard instruction were similar and therefore have been combined.

The pie charts shown above in Figure 4-14 illustrate the results from the Δm questions. Of those students in the middle half of the class who correctly answered the Δa question, only about 20% after standard instruction (6 of 30) correctly determined the effect on the de Broglie wavelength of changing the mass of the particles. After the tutorial, most students in the middle half (6 of 7) answered the Δm question correctly with correct reasoning. Similarly, in the highest quartile of the class, the percentage of correct answers with correct reasoning increased from about 45% (9 of 21) after standard instruction to well more than half (4 of 5) after modified instruction. Thus the results above suggest that improvement occurred among the highest quartile and middle half of the class as a result of using the tutorial.

d) Summary of results from the modern physics course

Comparison of results from questions given after standard instruction and after modified instruction suggest that the tutorial *Wave properties of matter* helped elicit and address difficulties among students in the sophomore-level modern physics course. Student performance on questions that pertain to changing the slit separation or slit width increased as a result of using the tutorial. This observation suggests that the first part of the tutorial, which reviews basic concepts from interference and diffraction, is helpful to the students.

The analysis of student responses on the written questions also reflect an increase in student performance on the Δv (or ΔE_K) questions and the Δm questions. In addition, more students could correctly answer *both* of these questions after standard instruction than after modified instruction. Improvement was observed at all levels of ability, especially in the middle half of the class. All of these results suggest that the tutorial helped students develop a functional understanding of de Broglie wavelength.

4. *RESULTS FROM THE THIRD-YEAR QUANTUM MECHANICS COURSE*

The tutorial *Wave properties of matter* was incorporated twice in the junior-level quantum mechanics course in two different years, Autumn 1996 and Autumn 1997. Each time, the pretest and tutorial were implemented in a single 50-minute class period. The pretest was given in the first 10 minutes, and the remaining 40 minutes were spent working through the tutorial.

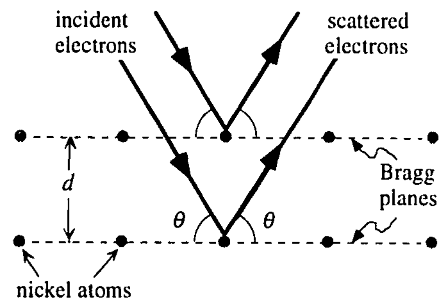
In order to assess the effectiveness of the tutorial, post-tests were included on course examinations. Each post-test was in the context of single-slit diffraction. Due to length constraints for each post-test, each post-test included one question that pertained to changing the width of the slit and one other question that tested on de Broglie wavelength. One post-test included a Δv question; the other, a Δm question.

a) Description of pretest and post-test questions given in the quantum mechanics classes that used the tutorial

Description of pretest. The pretest administered in both the Autumn 1996 and Autumn 1997 quantum mechanics courses is shown in Figure 4-15. Two features made this pretest unlike the post-test used in the quantum mechanics course in Autumn 1995 after standard instruction. First, the pretest was posed in the context of the Davisson-Germer experiment rather than double-slit interference of electrons. (Both topics are usually covered in the modern physics course, a prerequisite to quantum mechanics.) Second, the problem statement on the pretest included the equation for Bragg scattering, $2d \sin\theta = n\lambda$, that related λ to the angles θ at which constructive interference occurred. This change was intended to help elicit more strongly the particular difficulties with de Broglie wavelength that we expected to find.

(Davisson-Germer experiment.) Monoenergetic electrons are incident on a nickel crystal. It is observed that intense scattering occurs at angles θ according to the Bragg condition, $2d \sin\theta = n\lambda$. (See diagram at right.)

Suppose that this experiment were repeated, each time with a single change made to the apparatus. For each change below, determine whether each of the angles θ at which intense scattering occurs would become larger, smaller, or stay the same. Explain your reasoning in each case.



i. The nickel target is replaced with another crystal that has the same lattice structure but a smaller lattice spacing. (Consider the Bragg planes that are analogous to those indicated in the diagram above.)

ii. The speed of the incident electrons is decreased.

ii. The electrons are replaced with neutrons, with each neutron having the same kinetic energy as each of the original electrons.

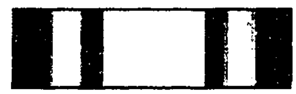
Figure 4-15: Written pretest administered in the Autumn 1996 and Autumn 1997 quantum mechanics classes.

Correct answers to pretest. On all parts of the pretest, the students were expected to use the equation for Bragg scattering given in the problem statement. In part i, decreasing the lattice spacing d would cause the angle θ for each diffraction maximum to increase. In part ii, a decrease in the speed of the electrons would mean that the de Broglie wavelength would increase, resulting in an increase for each angle θ. In part iii, replacing the electrons with neutrons of equal kinetic energy would correspond to a larger momentum $p = (2mE_K)^{1/2}$ than before, and thus a smaller de Broglie wavelength than before. The angle θ to each diffraction maximum would decrease as a result of the change.

Description of post-test #1. The first post-test, shown in Figure 4-16, was included on a midterm examination in the Autumn 1997 class. Two photographs (A and B) of single-slit diffraction patterns are given, with the pattern in the photograph B being broader than that in photograph A. Students are asked how to account for the change illustrated in the photographs assuming that (1) the width of the slit were changed, and (2) the speed of the electrons were changed (while keeping the slit width unchanged).

Monoenergetic electrons are incident on a single narrow slit of width w. The pattern shown in Photograph A is seen on a screen placed far from the slit.

Photograph A



Photograph B

Suppose that a single change were made to the original apparatus, resulting in the new pattern shown in photograph B. For each quantity below, state whether the new pattern could be produced by increasing that quantity, decreasing that quantity, or neither. Explain your reasoning in each case.

- the width of the slit
- the speed of the incident electrons

Figure 4-16: Post-test #1 given on a midterm examination in the Autumn 1997 quantum mechanics class.

Correct answers to post-test #1. The diffraction minima are farther apart after the change compared to before, meaning that the angle θ to each minimum increased. In answering the first part of the post-test question, one could use the condition for diffraction minima ($a \sin\theta = n\lambda$) to infer that the slit width a must have become smaller in order to produce the new diffraction pattern in photograph B. On the second part, a Δv question, the increase in θ can be explained by inferring that the de Broglie wavelength $\lambda = h/p$ must have increased. In order for this to occur, the speed of the incident electrons must have decreased to produce the new pattern.

Description of post-test #2. The second post-test, given on a final examination in the Autumn 1996 class, is shown in Figure 4-17. (A similar problem was included on the final examination in the Autumn 1995 quantum mechanics class and served as a post-test after all standard instruction.) As was the case with post-test #1, one question (part i) probed student understanding of ideas that underlie single-slit diffraction. Part ii of the post-test is a Δm question that requires students to relate de Broglie wavelength to mass and kinetic energy.

A beam of monoenergetic electrons is incident on a mask containing a single narrow slit of width *a*. The photograph at right shows the pattern that you would see on a distant phosphorescent screen. The bright regions indicate high concentrations of electrons hitting the screen.

Suppose this experiment were repeated with a *single* change made to the original setup. For each change described below, how would that change affect what you see on the screen? (For example, if a similar pattern is produced, explain how it would be different.)

i. The slit width is doubled.

ii. The beam of electrons is replaced with a beam of neutrons, with each neutron having the *same kinetic energy* as each of the original electrons. (Recall that $m_n \approx 1{,}800\ m_e$.)

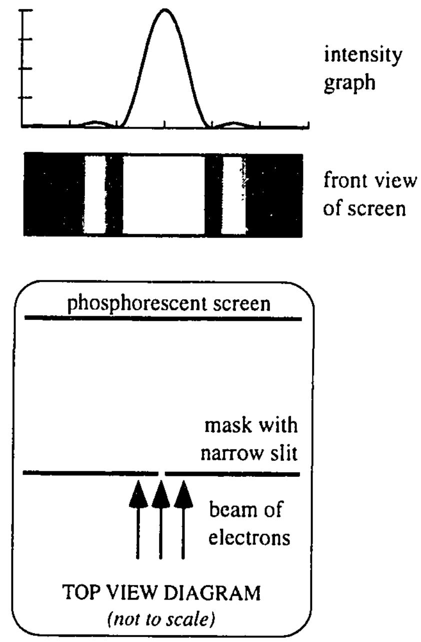


Figure 4-17: Post-test #2 given on a final examination in the Autumn 1996 quantum mechanics class.

Correct answers to post-test #2. In part i, doubling the slit width would cause the angle θ to each diffraction minimum decrease, according to $a \sin \theta = n\lambda$. Thus in the case the new pattern would have diffraction minima (and maxima) that are closer together than before. In part ii, replacing the electrons with neutrons of equal kinetic energy would cause the de Broglie wavelength to decrease. Thus the minima would get closer together as a result of the change.

Note about background of students in Autumn 1996 and Autumn 1997. The number of students who took the pretest, did the tutorial *Wave properties of matter,* and took the post-test was 27 in Autumn 1996 and 25 in Autumn 1997. An additional eight students in the Autumn 1997 section who took the pretest had done the tutorial previously during the course of instruction in modern physics (in Winter 1997). In parts b and c of this section we refer only to the 25 students in Autumn 1997 for whom the tutorial was new. In the last part of this section (part d) we discuss separately the eight students who had already worked through the tutorial before taking the Autumn 1997 pretest.

b) Comparison of student responses after standard instruction and after modified instruction

Table 4-7 below summarizes the results from questions that pertain to varying the lattice spacing or slit width (*i.e.*, the Δd and Δa questions). The results from post-tests given after modified instruction were similar and thus have been combined in the table.

Table 4-7: Comparison of performance of students in quantum mechanics courses on questions that involve varying the lattice spacing or slit width (*i.e.*, the Δd and Δa questions). Only those students from the Autumn 1996 and Autumn 1997 classes who took the pretest and post-test and worked through the tutorial are included.

	Quantum mechanics course			
	After standard instruction			After modified instruction
	Au 1995 $N = 43$	Au 1996* $N = 27$	Au 1997* $N = 25$	Au 1996 Au 1997 $N = 52$
Correct responses	**60% (26)**	**65% (18)**	**70% (17)**	**75% (40)**
With correct reasoning	20% (8)	35% (9)†	60% (15)†	45% (24)

* Problem statement on Autumn 1996 and Autumn 1997 pretests included the Bragg condition, $2d \sin\theta = m\lambda$. No other problems included equations for interference or diffraction maxima or minima.

† A total of 16 students on the Autumn 1996 pretest and 17 students on the Autumn 1997 pretest gave correct answers justified by reasoning that was either correct or incomplete (but not incorrect).

Commentary on results from the Δd and Δa questions given after standard instruction. As shown in the first three columns of Table 4-7, the percentage of correct answers with correct reasoning was larger in Autumn 1996 and Autumn 1997 (35% and 60%, respectively) than in Autumn 1995 (20%). These results suggest that including an equation for the locations of the interference or diffraction fringes, as was done on the Autumn 1996 and Autumn 1997 pretests, affected the percentage of correct responses that were justified by correct reasoning. Although the percentages were different in Autumn 1996 and Autumn 1997, it is possible that students in these classes treated the equation for the Bragg condition as sufficient for correct reasoning on the Δd question. We therefore determined the percentage of students who gave correct answers accompanied by reasoning that was either correct or incomplete but *not incorrect.* These percentages were similar in the two classes: 60% (16 of 27) in Autumn 1996, 70% (17 of 25) in Autumn 1997. (See Chapter 3.) Furthermore, all Δd and Δa questions posed after standard instruction, whether or not they included an equation for the diffraction or interference fringes, yielded similar percentages of correct answers ignoring reasoning (between 60% and 70%). These results suggest that the responses from the Δd and Δa questions given after standard instruction can be treated as coming from approximately equivalent groups of students.

Commentary on results from Δa questions given after standard instruction and after modified instruction. The first and fourth columns of Table 4-7 show the results from post-tests on single-slit diffraction of electrons. The Δa questions included no equations for the diffraction minima. When given after standard instruction, 60% of the students (26 of 43) gave correct answers to the Δa question, with 20% (8 of 43) doing so with correct reasoning. In contrast, 75% of the students (40 of 52) gave correct answers after modified instruction that included the tutorial, with 45% (24 of 52) supporting their answers with correct reasoning. The improvement indicates that the first part of the tutorial, which was intended as a review of the basic concepts of diffraction and interference, was helpful to students.

For each of the pretests and post-tests shown in Table 4-7, we determined the subset of N' students who correctly answered the Δd and Δa questions (ignoring reasoning). We then found the percentage of students in each subset who gave correct answers with correct reasoning on the remaining questions on the de Broglie wavelength. Table 4-8 below summarizes the results from the questions that pertained to changing the speed of the electrons (*i.e.*, the Δv questions) given after standard instruction (in Autumn 1996 and Autumn 1997) and after modified instruction (in Autumn 1997). The results from questions given after standard instruction were similar and have been combined in the table.

Table 4-8: Comparison of performance of students in quantum mechanics courses on questions that involve varying the speed of the electrons (*i.e.*, the Δv questions). Results are shown only for the subset of N' students who correctly determined the effect of varying the lattice spacing or slit width, shown in Table 4-7. Results from the Δv questions given after standard instruction were similar and therefore have been combined.

	Quantum mechanics course	
	After standard instruction	After modified instruction
	Au 1996 Au 1997 $N' = 35$	Au 1997 $N' = 18$
Correct responses (correct reasoning used to relate λ to v)	50% (18)	80% (14)
Incorrect responses	45% (16)	15% (3)
Blank or incomplete responses	< 5% (1)	5% (1)

Of the students who gave correct responses on the Δd or Δa question, 50% (18 of 35) answered the Δv question correctly after standard instruction. In contrast, 80% of those (14 of 18) on the Autumn 1997 post-test who correctly answered the Δa question also answered the Δv question correctly with good reasoning. The difference in the percentage of correct responses to the Δv question suggest improvement in understanding the de Broglie wavelength as a result of using the tutorial.

The results shown in Table 4-9 summarize the results from Δn questions given after standard instruction and after modified instruction. The results are shown only for the subset of students who correctly predicted the effect of changing the slit width or lattice spacing (ignoring reasoning) on an earlier part of the pretest or post-test.

Table 4-9: Comparison of performance of students in quantum mechanics courses on questions that involve using particles of different mass (*i.e.,* the Δm questions). Results are shown only for the subset of N' students who correctly determined the effect of varying the lattice spacing or slit width, shown in Table 4-7. Results from the Δm questions given before the tutorial were similar and therefore have been combined.

	Quantum mechanics course	
	After standard instruction	After modified instruction
	Au 1995 Au 1996 Au 1997 $N' = 61$	Au 1996 $N' = 22$
Correct responses (correct reasoning to relate λ to m)	30% (19)	50% (11)
Incorrect responses	50% (31)	50% (11)
Blank or incomplete responses	20% (11)	0% (0)

After standard instruction, 30% of the students (19 of 61) who correctly answered the Δd and Δa questions also gave correct responses with correct reasoning on the Δa question. This percentage increases to 50% (11 of 22) on the Autumn 1996 post-test, given after tutorial instruction. These results indicate a trend in the desired direction, although they also suggest that the students in the quantum mechanics classes found the Δm questions to be more difficult than the Δv questions.

c) Comparison of student responses according to overall academic performance in the class

The pretest and post-test results discussed above were analyzed with respect to the academic performance of the students in each of the classes. In this way we attempted to identify the group of students—those in the highest quartile, middle half, or lowest quartile—in which the tutorial had a positive effect.

We first present results from the Δa questions given after standard instruction and after modified instruction that included the tutorial. The first set of pie charts in Figure 4-18 below shows the results from each subdivision of the class after standard instruction; the second set of charts, after modified instruction. (The Δd questions from the pretests in Autumn 1996 and Autumn 1997 included the equation for the Bragg condition in the problem statement. As discussed previously, this feature of the questions seemed to affect the percentage of correct answers with correct reasoning. Therefore, the results from these questions are not shown.) The findings from the Autumn 1996 and Autumn 1997 post-tests were similar, so these results are combined in the second set of charts.

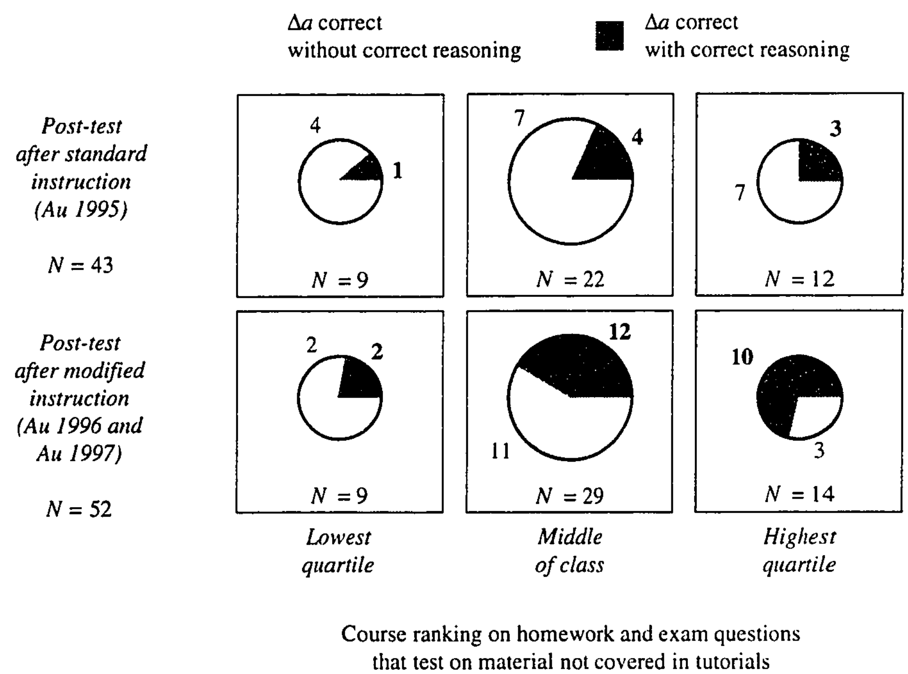


Figure 4-18: Comparison by course ranking of performance of students in quantum mechanics courses in which the tutorial was and was not used. Results are from questions that involve varying the slit width. (Only those students from the Autumn 1996 and Autumn 1997 classes who took the pretest and post-test and worked through the tutorial are included.)

As shown by the dark gray regions of each pie chart above, a greater fraction of each group of students answered the Δa question correctly with correct reasoning after modified instruction than after standard instruction. The improvement is especially notable in the middle half and upper quartile of the class. In the middle half, the percentage of correct answers with correct reasoning increased from 20% (4 of 22) after standard instruction to about 40% (12 of 29) after modified instruction. In the upper quartile, the corresponding percentages were 25% (3 of 12) and 70% (10 of 14). These results suggest that the first part of the tutorial was helpful to students in the middle half and upper quartile of the class in reviewing the basic concepts of diffraction and interference.

In the following two figures we summarize the results from the questions on the de Broglie wavelength (*i.e.*, the Δv and Δm questions) by those students who correctly determined the effect of varying the lattice spacing or slit width. Figure 4-19 below shows the responses to the Δv questions on the Autumn 1996 and Autumn 1997 pretests (after standard instruction) and the Autumn 1997 post-test (after modified instruction). The dark regions of each pie chart indicate correct responses on the Δv question with correct reasoning about de Broglie wavelength.

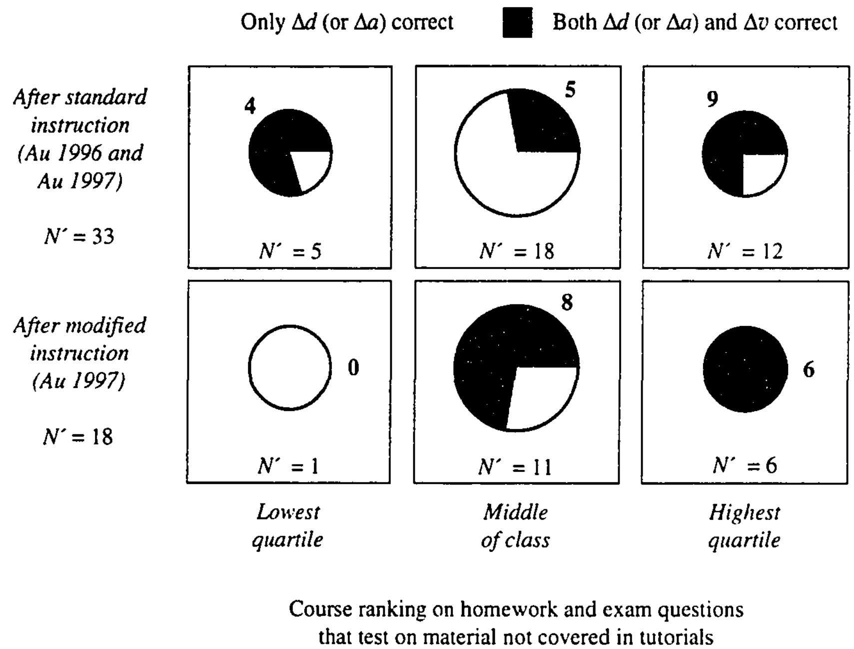


Figure 4-19: Comparison by course ranking of performance of students in quantum mechanics courses in which the tutorial was and was not used. Results are from questions that involve varying the speed of the electrons (*i.e.*, the Δv questions). Results are shown only for the subset of N' students who correctly determined the effect of varying the lattice spacing or slit width, shown in Table 4-7. Results from the Δv questions given after standard instruction were similar and therefore have been combined.

Of those students who correctly answered the Δd and Δa questions, the greatest improvement on the Δv questions seemed to occur in the middle half of the class. After standard instruction, about 30% (5 of 18) correctly answered the Δv question with correct reasoning. After modified instruction this percentage increased to about 75% (8 of 11). The only group that did not show improvement was the lowest quartile of the class, although on the Autumn 1997 post-test this group consisted of only one student.

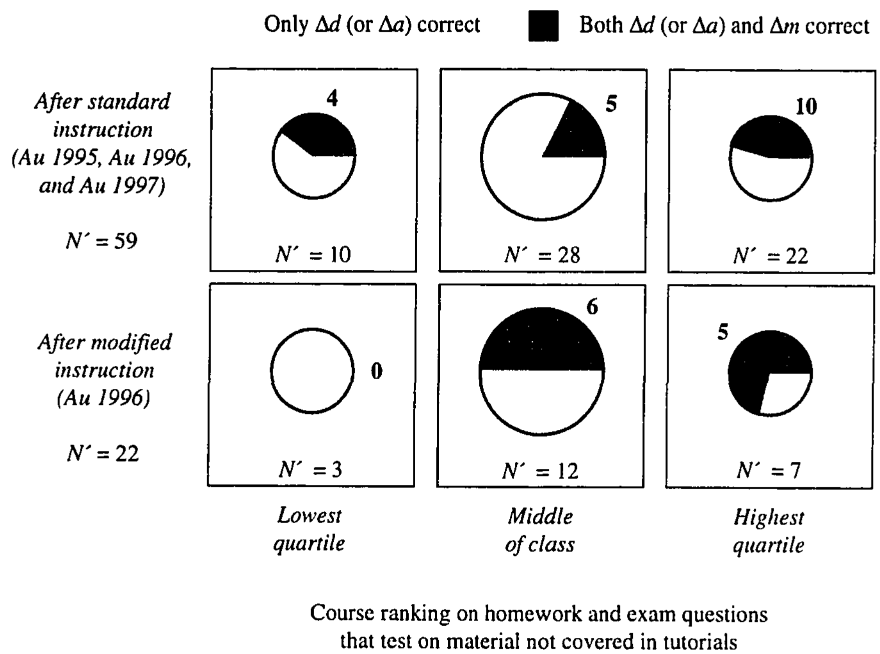


Figure 4-20: Comparison by course ranking of performance of students in quantum mechanics courses in which the tutorial was and was not used. Results are from questions that involve using particles of different mass (*i.e.*, the Δm questions). Results are shown only for the subset of N' students who correctly determined the effect of varying the lattice spacing or slit width, shown in Table 4-7. Results from the Δm questions given after standard instruction were similar and therefore have been combined.

The results from the Δm questions given after standard instruction and after modified instruction are illustrated in Figure 4-20 above. The pie charts that correspond to the middle half of the class reflect the most improvement. In this group, only 20% (5 of 28) of those who correctly answered the Δd and Δa questions after standard instruction also gave correct responses on the Δm question. After modified instruction, this percentage increases to 50% (6 of 12). The results presented above in Figure 4-19 and Figure 4-20 suggest that the tutorial *Wave properties of matter* helps students understand the de Broglie wavelength, particularly those in the middle half of the class.

d) Comparison of pretest and post-test results according to prior tutorial instruction in modern physics

The Autumn 1997 pretest (shown in Figure 4-15) was given to eight students who had already worked through the tutorial *Wave properties of matter* in their modern physics course (in Winter 1997, described previously in this chapter). For these students, the Δd and Δm questions on the pretest were not new; an optional tutorial homework that was used in Winter 1997 included these questions in the context of the Davisson-Germer experiment. However, the Δv question on the pretest had never been posed to them before.

In Table 4-10 we summarize the results from all 33 students in the Autumn 1997 quantum mechanics class. The first column shows the results from the 25 students for whom the tutorial was new, and the second column shows the corresponding results from the eight students who had had the tutorial previously.

Table 4-10: Comparison of pretest results from the Autumn 1997 quantum mechanics course ($N = 33$). Students are grouped according to whether or not they worked through the tutorial *Wave properties of matter* prior to the pretest.

	Autumn 1997 quantum mechanics course	
	Pretest *before* tutorial ($N = 25$)	Pretest *after* tutorial (Winter 1997) ($N = 8$)
Correct responses for Δd question (ignoring reasoning) and Δv question (with correct reasoning)	35% (9)	100% (8)
Correct responses for Δd question (ignoring reasoning) and Δm question (with correct reasoning)	25% (6)	75% (6)

Of those students for whom the tutorial was new, only 35% (9 of 25) gave correct answers to both the Δd question (ignoring reasoning) and the Δv question (with correct reasoning). Similarly, only 25% of these students (6 of 25) gave correct responses on both the Δd and Δm questions. In contrast, for the eight students who took the pretest after the tutorial, the corresponding percentages increased to 100% (8 of 8) for the Δd and Δv questions and 75% (6 of 8) for the Δd and Δm questions.

Although the two groups of students are small in size, the above results indicate a trend that the eight students who had already worked through *Wave properties of matter* had an advantage over the other 25 students for whom the tutorial was new. This claim is strengthened by the facts that (i) the eight students worked through the tutorial in Winter 1997, more than three quarters prior to the time they took the Autumn 1997 pretest, and (ii) a question that was included on the pretest (the Δv question) had never been posed to these eight students. Thus one cannot attribute the enhanced performance of these students to repeated exposure of the same question.

e) Summary of results from the quantum mechanics course

We have analyzed student responses to qualitative questions posed after standard instruction and after modified instruction on the interference and diffraction of matter. The results of the analysis indicated that the tutorial *Wave properties of matter* enhanced student understanding of the basic concepts of interference and diffraction. Improvement was also observed on questions that test on the relationship between the de Broglie wavelength and momentum (*i.e.*, the Δv and Δm questions), especially in the middle half of the class.

In addition, a subset of the quantum mechanics students in Autumn 1997 already worked through the tutorial *Wave properties of matter* in their Winter 1997 modern physics course. This subset of students significantly outperformed the other students on the Autumn 1997 pretest. These results provide evidence that the tutorial can have long-term effect on student understanding.

D. SUMMARY

The research described in this chapter and the preceding chapter illustrates how research in the teaching and learning of physics can be used to modify instruction with the objective of increasing student conceptual understanding. Knowledge of specific student difficulties helped guide our work in the design of a tutorial intended to address those difficulties. The tutorial was based in part on an *elicit-confront-resolve* strategy, which is designed to engage students at a sufficiently deep mental level that they can be guided to articulate and resolve conceptual difficulties on their own.

The results presented in this chapter demonstrate that the instructional strategy described above is an effective way to address difficulties that students have in developing and applying a wave model to the interference and diffraction of matter. In Part II of this dissertation we extend the results of our study to begin to identify and address specific difficulties in developing and applying a quantum mechanical model for matter.

PART TWO:

INVESTIGATION OF STUDENT ABILITY TO CHOOSE BETWEEN A CLASSICAL OR QUANTUM MECHANICAL MODEL OF MATTER AND TO APPLY THE MODEL APPROPRIATELY

CHAPTER 5: IDENTIFYING STUDENT DIFFICULTIES IN APPLYING A WAVE MODEL TO REFLECTION, TRANSMISSION, AND BOUND STATES OF MATTER

A. INTRODUCTION AND MOTIVATION

In this chapter we describe an investigation of student understanding of the behavior of bound states and scattering states in one dimension. Early in their study of quantum mechanics, students are taught that matter exhibits properties not only of particles but also of waves. They are introduced to the idea that matter propagates like a wave in classically allowed regions. Students are expected to recognize that matter undergoes reflection and transmission at a boundary between regions of different potential and that a particle in a stationary bound state can be treated as a standing wave. They are shown how the curvature of the wave function $\psi(x)$ for a particle depends upon the local kinetic energy or potential. In addition, they are taught that the wave function can exist even in classically forbidden regions (*e.g.*, like a wave within a gap between disjoint allowed regions, such as is the case during frustrated total internal reflection).

In Part I of this dissertation, we showed that after standard instruction many students have serious difficulty applying concepts of waves to situations that involve the interference and diffraction of matter. We suspected the presence of additional difficulties that would hinder the ability of students to predict or account for the behavior of scattering states and bound states in terms of wave phenomena. We have identified numerous, interrelated difficulties that indicate a lack of understanding not only of wave phenomena but also of probability density. In this chapter we focus on those difficulties that pertain to basic wave properties of matter. (Difficulties with probability density in both the classical and quantum mechanical regimes are discussed in the following chapter.) In this chapter we address the following questions that have formed part of the focus of our investigation: After standard instruction in quantum mechanics:

- Do students attribute properties of waves (*e.g.*, wavelength, reflection, and transmission) to matter?
- Do students apply knowledge of pulse width and de Broglie wavelength in the context of scattering states and bound states?

B. CONTEXT FOR RESEARCH

We designed and posed qualitative questions to students after standard instruction of basic topics in quantum mechanics. Below we specify the student populations in the study and give an overview of the research tasks used.

1. *STUDENT POPULATIONS*

We report on the analysis of student responses to a variety of specially-designed qualitative questions administered in quantum mechanics courses taught at the University of Washington. (See Chapter 1 for details about the structure of the courses.) The majority of the results presented here involve undergraduate students enrolled in three consecutive sequences of the junior-level quantum mechanics course: Autumn 1995–Winter 1996, Autumn 1996–Winter 1997, and Autumn 1997–Winter 1998. We also present results from questions that were posed in Autumn 1995 to physics graduate students enrolled in a first-year graduate level course in quantum mechanics. Unless otherwise specified, the students had had lecture instruction on the relevant material before attempting to answer the written questions.

2. *RESEARCH TASKS*

The research described in this chapter was conducted primarily through the analysis of student responses to specially designed qualitative questions given after standard lecture instruction. Some questions were posed on course examinations, others on ungraded quizzes given during regular class time.

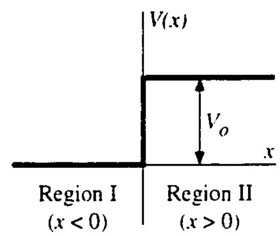
Below we provide a general description of the written questions discussed in this chapter and of the concepts that must be understood in order to answer the questions correctly. We first present those questions that pertain to scattering states, then those about bound states. (The questions, together with the intended correct responses, are described in detail in Appendix B.)

a) Questions about particles in scattering states

We have used two types of questions that pertain to scattering states in one dimension. One set of questions deals with particles scattering from a potential step, shown below in Figure 5-1. Another set of questions pertains to scattering from a potential barrier. Figure 5-2 illustrates a question about a beam of monoenergetic particles incident on a rectangular potential barrier.

Monoenergetic electrons travel through a region in which the potential energy $V(x)$ varies with x as follows:

$$V(x) = \begin{cases} 0, & x < 0 \\ V_o, & x > 0 \end{cases}$$

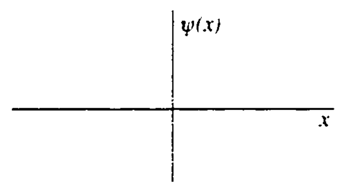


Let E represent the total energy of each electron, and let $\psi(x)$ represent the wave function associated with the electrons.

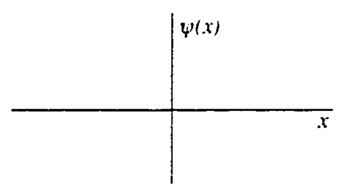
For each case described below:

- Describe in words the behavior of the electrons in Regions I and II. In particular: (i) How does the speed of the electrons in Region I compare to the speed of the electrons in Region II? (ii) Do all of the electrons that start in one Region end up in the other Region?
- On the set of axes provided, draw the shape of $\psi(x)$. Clearly indicate how, if at all, the wave function in $x < 0$ is qualitatively different from the wave function in $x > 0$.
- Briefly explain the reasoning you used in each case.

A. The electrons initially move *from left to right* (*i.e.*, in the $+x$ direction) with $E > V_o$.



B. The electrons initially move *from right to left* (*i.e.*, in the $-x$ direction) with $E > V_o$



C. The electrons initially move *from left to right* (*i.e.*, in the $+x$ direction) with $E < V_o$

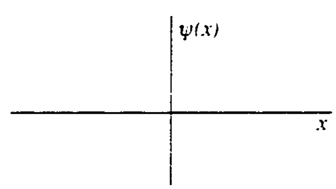


Figure 5-1: Potential step question.[55] This question is the only one in this chapter that was given to students before relevant instruction. Difficulties elicited by this question were very similar to those elicited by others (such as the potential barrier question) after standard instruction.

A beam of particles is incident from the left on the potential barrier shown at right. Consider the case for which the total energy E of each particle is ***less than*** V_o.

In the space provided below right, sketch a qualitatively correct graph of the real part of the wave function, Re{$\psi(x)$}, for this case. Explain the reasoning you used to draw your graph.

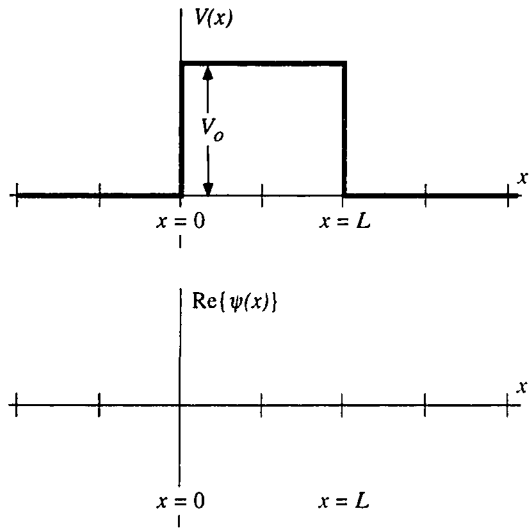
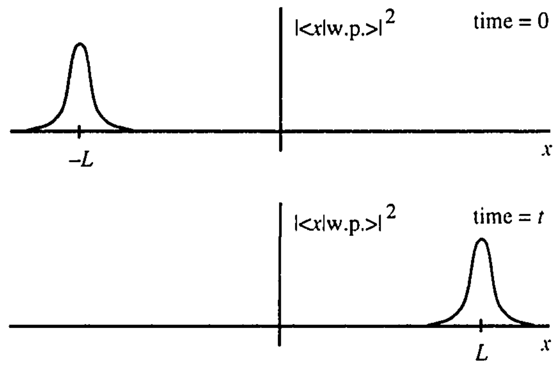


Figure 5-2: Version of the potential barrier question that pertains to a beam of monoenergetic particles incident on a rectangular barrier. Similar versions of this question included one or more other parts that asked for (1) the functional form of $\psi(x)$, (2) a comparison of the kinetic energy of the incident and transmitted particles, or (3) a comparison of the probability currents on either side of the barrier.

Both questions shown above required students to recognize that matter exhibits wave-like properties in situations that involve scattering from a potential step or barrier. For example, in regions where the potential is constant, a beam of monoenergetic particles can be treated as propagating like a wave with well-defined momentum and de Broglie wavelength. Particles that encounter an abrupt boundary between regions of different potential (*e.g.*, at $x = 0$ in each of the two questions above) undergo transmission and reflection at that boundary. In situations in which there are two or more classically allowed regions, the local de Broglie wavelength of the particles is smaller where the local kinetic energy is larger. In classically forbidden regions, on the other hand, particles do not have a well-defined de Broglie wavelength, even though the wave function is not equal to zero in those regions.

Another version of the potential barrier question, shown below in Figure 5-3, differs from the first in two ways. First, the physical situation involves a wave packet incident on a barrier, not a beam of monoenergetic particles. Second, the potential barrier is not rectangular but has arbitrary shape.

1. Consider a free particle moving in 1-dimension. At time $t = 0$, the particle is described by a wave packet having $\langle x \rangle = -L$ and $\langle p \rangle = p_o$. At a much later time t, $\langle x \rangle = L$. This situation is shown in the pictures below (note the spread of the wave packet is not noticeable). What is p_o?



2. Now suppose there is a potential barrier around $x = 0$, as sketched below. Assume $p_o^2/2m$ is less than the maximum height of the barrier. On the axes below, sketch the probability density of the wave packet at the later time t.

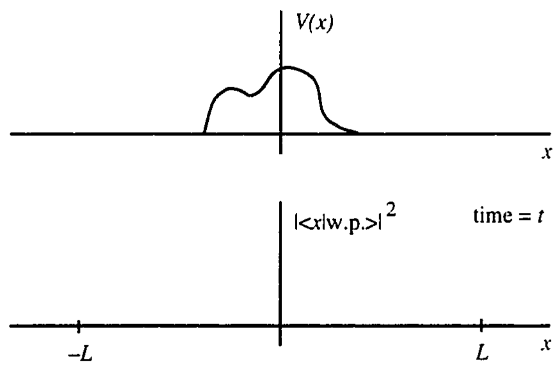


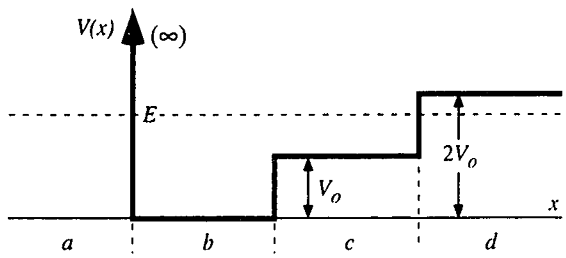
Figure 5-3: Version of the potential barrier question in which a particle, represented by a wave packet, is incident on a potential barrier. This version was given as an ungraded quiz to students in a graduate-level quantum mechanics course.

Despite the differences between the versions of the potential barrier question shown in Figure 5-2 and Figure 5-3, the concepts that the students are expected to apply in answering the wave-packet version of the question are very similar to those mentioned previously. Upon encountering the barrier, the wave packet would be partially reflected and partially transmitted. (Students were not expected to describe in detail the behavior of the wave packet inside the barrier.) A change in the average momentum or speed of a wave packet would cause a proportionate change to the width of the wave packet, as is the case for pulses on springs. Thus, if the potential is the same on both sides of the potential barrier, and if we ignore the spread of the wave packets, then the widths of the reflected and transmitted wave packets would be equal to each other and equal to that of the incident wave packet.

b) Questions about particles in bound states

We have given several different questions that require students to describe the behavior of particles in one-dimensional bound states. Results from the questions illustrated in Figure 5-4 and Figure 5-5 are discussed in this chapter.

An electron is present in a region in which the potential energy $V(x)$ varies with x as shown below. The wave function $\psi(x)$ for the electron is a stationary state corresponding to a total energy E such that $V_o < E < 2V_o$.



A. Is any value between V_o and $2V_o$ allowed for the energy E, or are only certain values of E allowed? Explain your reasoning. (You do not need to solve any equations.)

B. Rank the regions *a–d* according to the de Broglie wavelength of the electron in that region, from largest to smallest *(e.g.,* $\lambda_x > \lambda_y = \lambda_z$*...).* If λ is not well-defined in a particular region, state that explicitly and do not include that region in your ranking. Explain your reasoning.

C. For the regions in which the de Broglie wavelength is well-defined, rank those regions according to the maximum value of $\psi(x)$ in that region, from largest to smallest. Explain your reasoning.

D. In the space at right, carefully draw a qualitatively correct graph of $\psi(x)$ for the electron described above.

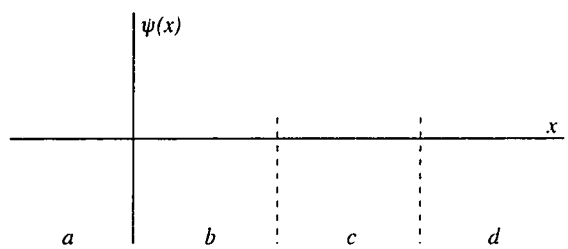


Figure 5-4: Double-tiered square well question. The results from parts B and C are discussed in detail in this chapter.

Consider a particle in a high energy eigenstate of the potential shown. (That is, suppose the energy E of the particle is an energy eigenvalue that is *much larger than* the ground state energy.)

$$V(x) = \begin{cases} \infty, & x < 0 \\ +\beta x, & x > 0 \end{cases}$$

A. On the graph of $V(x)$ *vs* x, indicate the classical turning point(s).

B. In the space at right, sketch a qualitatively correct graph of the wave function $\psi(x)$. Use the same horizontal scale used in the graph of $V(x)$ *vs* x. Explain the reasoning you used to draw your graph.

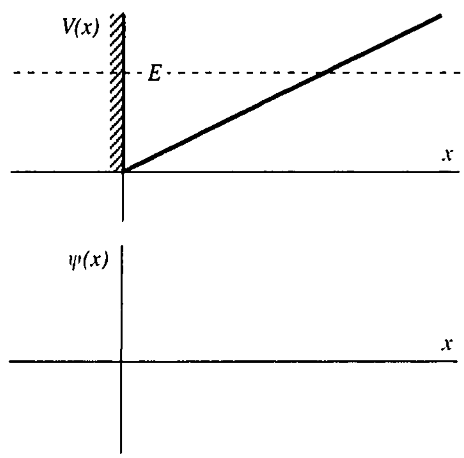


Figure 5-5: V-well question. The results from part B are discussed in detail in this chapter.

The questions shown above probe student understanding of concepts that are similar to those needed to answer the questions about scattering states. Students are expected to treat a particle in a stationary bound state of a potential well as a standing wave corresponding to a well-defined total energy. In the classically allowed region of the well, the wave function has a local de Broglie wavelength that depends upon the potential $V(x)$. In particular, as the potential increases, the local kinetic energy decreases, causing the local wavelength (and thus the distance between consecutive nodes of the wave function) to increase. In classically forbidden regions, however, the particle does not have a well-defined de Broglie wavelength, even when the wave function is not equal to zero.

C. IDENTIFICATION AND ANALYSIS OF SPECIFIC DIFFICULTIES

The analysis of student responses allowed us to identify several specific difficulties that seemed to hinder the ability of students to apply basic ideas about the wave-like behavior of matter. Although many student responses were incorrect on multiple levels, the discussion in this chapter focuses on difficulties from the following two broad categories:

- failure to attribute properties of waves (*e.g.*, wavelength, reflection, and transmission) to matter, and
- failure to apply the concepts of pulse width and de Broglie wavelength in the context of wave functions in one dimension

In the subsequent sections, we present in detail specific difficulties that were identified.

1. FAILURE TO ATTRIBUTE PROPERTIES OF WAVES TO MATTER

Analysis of student responses to the questions described above revealed that many students failed to attribute basic properties of waves to matter. Some students failed to recognize that the wave function of a free particle would be sinusoidal and thus have a well-defined de Broglie wavelength. Others failed to recognize that particles undergo reflection at a boundary between regions of different potential or transmission through a potential barrier of finite height.

a) Failure to recognize that a beam of monoenergetic particles corresponds to a well-defined wavelength

The written questions that dealt with scattering from a potential step or barrier were used to probe student understanding of the wave function for a beam of free particles. A beam of freely propagating particles with well-defined momentum p and total energy E can be represented by a complex wave function $\Psi(x, t) \sim e^{\pm ikx - i\omega t}$, where k and ω satisfy the de Broglie and Einstein relations, $p = \hbar k = h/\lambda$ and $E = \hbar\omega$. Both the real and imaginary parts of such a wave function vary sinusoidally with x and t. Thus students should recognize that the (real part of the) wave function has a well-defined de Broglie wavelength in regions where the potential is constant and less than E.

One common difficulty that arose on questions about scattering from a potential step or barrier was the failure to recognize that the wave function $\Psi(x, t)$ of a beam of free particles is a plane wave with a well-defined wavelength. Such an error was made by students both above and below the mean of the class.

Particles scattering from a potential step

In the question on scattering from a potential step (shown previously in Figure 5-1) students were asked to sketch the behavior of the wave function. For a case in which $E > V_o$, many students did not recognize that the wave function would be sinusoidal in shape in both regions. Specific examples of the errors they made are described below.

Several students drew graphs like the one shown in Figure 5-6. One student seemed to associate the value of the wave function with the velocity or kinetic energy of the electrons in that region. He explained his graph by stating that "electrons will slow down as they enter Region II because they enter a region of lower energy than they had originally been in.... Electrons will be slowed down through region II but will have a constant speed through [Region] I." The student also failed to recognize that the speed of the electrons would be constant for $x > 0$. The response by this student suggests a confusion between the wave function and the velocity or energy of the particle described by the wave function. His response also indicates the failure to recognize that the particles propagate with constant speed in the region $x > 0$, where the potential is constant.

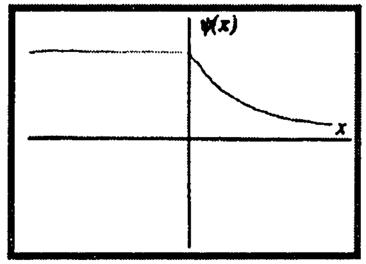


Figure 5-6: Incorrect student graph in which the wave function in a region of constant potential ($x < 0$) is constant rather than sinusoidal.

Another student indicated that $\psi(x)$ was sinusoidal in one region but not in the other. The graph drawn by this student is shown in Figure 5-7. The student drew $\psi(x)$ as sinusoidal only in the region containing the incident particles. He correctly explained that the incident wave experiences "interference from [the] reflected wave" propagating to the left from the boundary at $x = 0$. However, the student seems to believe that the wave function would be constant in the region $x < 0$ if there had not been such "interference." In Region II ($x > 0$), which would contain only particles that are transmitted past the boundary, the wave function is shown to have a constant value. The student made a similar error when considering electrons incident from Region II

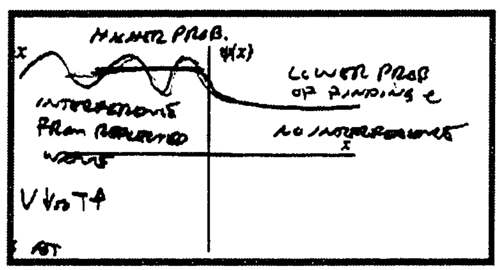


Figure 5-7: Incorrect student graph in which the wave function is sinusoidal only for $x < 0$ due to "interference from [the] reflected wave."

rather than from Region I; in that case his graph indicated a sinusoidally varying wave function in Region II but a constant wave function in Region I. It is also interesting that the student incorrectly showed $\psi(x)$ oscillating about a horizontal line above the $\psi = 0$ axis (x-axis).

Particles scattering from a potential barrier

Similar errors were made by students who attempted to sketch the real part of the wave function for particles incident on a finite potential barrier of width L (Figure 5-2). Many students failed to recognize that in the regions $x < 0$ and $x > L$ the wave function would be sinusoidal, despite having completed standard instruction on scattering in one dimension.

For example, one student indicated on her sketch that the real part of the wave function would be constant everywhere in the region $x < 0$. The student justified her sketch by explaining "to the left [of the barrier] ($x < 0$), there is equal probability of position, if the beam of particles keeps on coming." The student apparently understood that a stationary scattering state would correspond to a situation in which the beam of incident particles is always present. However, the student seemed to believe that the (real part of the) wave function would be constant to represent "an equal probability of position" rather than sinusoidal with a well-defined de Broglie wavelength.

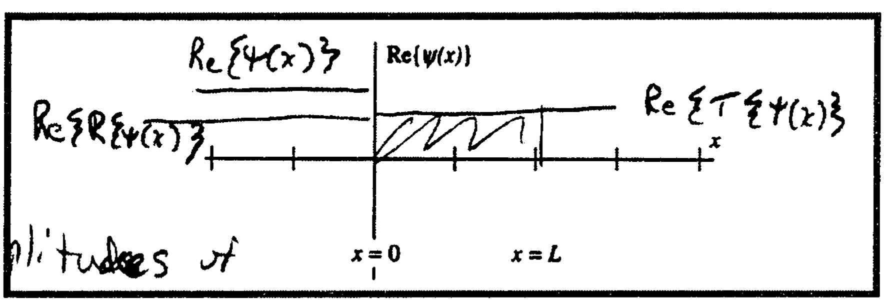


Figure 5-8: Incorrect student graph showing that the real part of $\psi(x)$ is constant on either side of the potential barrier.

Another student, who drew the graph shown in Figure 5-8, indicated that the real part of the wave function $\psi(x)$ was a certain constant value for all $x < 0$ and a smaller constant value for all $x > L$. Although the student seems to understand that the transmission probability through the barrier is less than unity, the student did not recognize that the real part of the wave function is sinusoidal as a function of x. It is also interesting to note that the student used the symbols "$\psi(x)$," "R{$\psi(x)$}," and "T{$\psi(x)$}" to denote *separately* the wave functions of the incident, reflected, and transmitted particles, respectively, rather than interpret "$\psi(x)$" as a single wave function representing the incident, reflected, and transmitted particles.

b) Failure to recognize that reflection occurs at the boundary between regions of different potential or wave speed

We have found that many students fail to recognize that matter can undergo reflection at a boundary between regions of different potential (or different wave speed). This difficulty persists not only after lecture instruction but after students had solved quantitative problems in which they determined the analytical form of wave functions for similar problems. Apparently, this type of mathematical modeling did not help students understand conceptually that matter reflects as well as transmits at a boundary. Below we discuss this particular difficulty as well as a related difficulty that has been identified in the context of pulses on springs.

Particles scattering from a potential step

On the potential step question (Figure 5-1), students were asked to consider monoenergetic electrons incident on a potential step. For both cases in which $E > V_o$ (whether the incident beam propagated toward the step from left to right or from right to left), nearly half of the students incorrectly stated that all of the electrons that started in one region would end up in the other. Most explained that each electron would have sufficient kinetic energy to reach the other region and did not take into account the fact that any electrons would be reflected where the potential changes (at $x = 0$).

For example, when considering the case in which electrons were incident from the region of lower potential (Region I), some students responded as follows:

> "All make it across [the boundary at $x = 0$] since the E of electron is greater than V_o."
>
> "[The electrons incident from Region I] will end up in region II because they will have KE there and [Region II] is an equipotential so there's nothing to reverse its momentum."
>
> "I would think they would all make the transition between regions. Why would they not go into R[egion] II?"

These errors were slightly more prevalent when students considered the case in which the electrons are incident from the region of higher potential (Region II):

> "All electrons get to Region I. (They all can lose *PE.*)"

> "I think [the electrons] would all get through—as before [when the electrons were incident from Region I], what's to stop them?"

About half of the students in the class based their answers only on the comparison between the total energy of the electrons and the value of the potential in each region. Although their reasoning would be correct for classical particles, it would not be correct for waves (or matter in the quantum mechanical regime). Similar reasoning was also used by students who correctly stated that some of the electrons would be reflected at $x = 0$ but who justified their answer by assuming that the electrons were not monoenergetic. These errors are discussed in greater detail in the subsequent chapter.

Particles scattering from a potential barrier

The potential barrier question, shown in Figure 5-2, also revealed a strong tendency for students to fail to take into account reflection at a boundary. On one version of the question that was posed after standard instruction, students were asked to compare the probability current on the left of the barrier (the region $x < 0$, which contains the incident particles) to that on the right of the barrier (the region $x > L$). They were expected to treat the situation as if it had existed for a long time and to recognize that the region $x < 0$ would contain both incident and reflected electrons. Because probability would not accumulate inside the barrier (or in any other region), the probability current must be the same in both regions.

About half of the students who attempted the question incorrectly stated that the probability current would be larger in the region containing the incident particles ($x < 0$) than in the region containing the transmitted particles ($x > L$). Most of these students did not seem to recognize that some of the incident particles would be reflected at $x = 0$. For example, one student approached the problem using a correct expression for the probability current, $j = \text{Re}\{\psi^* (p/m)\, \psi\}$, where p is the momentum operator $p = -i\hbar(d/dx)$. However, the student stated that applying the momentum operator to ψ, which would involve "taking the derivative of ψ," would "give a constant times ψ." She apparently failed to recognize that the *direction* of the momentum in the region $x < 0$ is not well-defined. Thus the student did not take into account that reflection would occur at $x = 0$ and that, as a result, the region $x < 0$ would contain both left-going and right-going electrons.

Other students based their answer purely on a comparison of the amplitude of the wave function in the two regions. For instance, one student simply said that "the magnitude [of the probability current] will be higher in the region of $x < 0$ than in the region of $x > L$ since ψ itself will be a lot smaller in [the region $x > L$]." Their responses indicated a failure to differentiate between a quantity and a rate of change of the quantity, namely probability density and probability current. This type of difficulty that has been identified in numerous contexts of introductory physics, including kinematics, and electricity and magnetism.[56]

Wave packet scattering from a potential barrier

The failure to recognize that reflection would occur at abrupt changes in potential was also found to be prevalent among graduate physics students in a graduate-level quantum mechanics course at the University of Washington. Evidence of this difficulty was obtained through the analysis of responses to the wave-packet version of the potential barrier question. (See Figure 5-3.)

The students were asked to draw a graph of the probability density for an instant after a wave packet reached a potential barrier of arbitrary shape. Almost half of the students who attempted the question did not include a reflected wave packet in their graphs. These students did not seem to recognize that reflection would be relevant in this situation.

One student who did not show a reflected wave packet gave the following explanation:

> "I guess the potential barrier acts like a blob of goop that a mass on a spring might travel through; it would slow it down ($\langle x(t) \rangle < L$)."

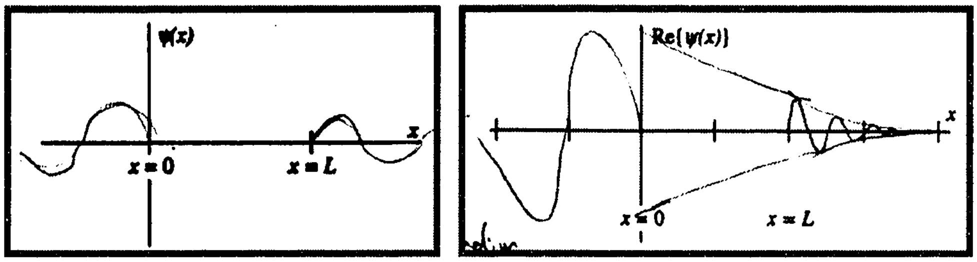
This student apparently thought that the only effect that the potential barrier had on the wave packet was to "slow it down," without producing a reflected wave packet. It is also interesting to note that the student attempted to make an analogy between the wave packet and a pulse on a spring, although the student referred to a "mass on a spring" rather than a pulse. Though the student attempted to make such an analogy, the student failed to recognize that such a pulse on a spring will undergo reflection as well as transmission.

Unfortunately, very few students wrote down the reasoning that they used in making their sketches. We were therefore unable to determine how similar their errors were to those made by the undergraduate students on the pretests about monoenergetic particles incident on a potential step. The responses by the graduate students, however, suggest a serious failure to attribute basic properties of waves, including reflection, to the wave packet, even after all undergraduate-level instruction in quantum mechanics.

Commentary. Similar errors have been identified through the analysis of student responses to a written question in the context of reflection and transmission of pulses on springs.[57] After standard instruction on this topic, as many as 20% of introductory students and 15% of participants in a graduate teaching seminar gave answers that suggested an overall failure to recognize that reflection would occur at a junction between springs of different wave speeds. (See Appendix A for details.) In comparison, the analogous error was made by about half of the quantum mechanics students and half of the graduate students after standard instruction in quantum mechanics. While standard instruction in the introductory physics seems to help students recognize that reflection occurs at a junction between two springs, difficulties in attributing the same property to matter do not seem to be addressed by standard instruction in the quantum mechanics course.

c) Failure to recognize that for transmission to occur through a finite barrier, the wave must be non-zero inside the barrier

After standard instruction, some students seem to understand that particles can tunnel through a potential barrier of finite height and width. However, many of these students fail to recognize that, in order for tunneling to occur, the wave function must be non-zero inside the barrier. Some students even fail to recognize that tunneling can occur at all. In order to illustrate these difficulties, we present below some examples of student responses to the potential barrier question (Figure 5-2).



(a) (b)

Figure 5-9: Examples of incorrect student graphs that show the wave function to be equal to zero inside the potential barrier.

Two examples of incorrect student graphs are shown in parts (a) and (b) of Figure 5-9. Both students correctly predicted that for the case in which electrons of energy $E < V_o$ (where V_o is equal to the height of the barrier) some electrons would tunnel through the potential barrier. However, they indicated in their sketches that the wave function would be equal to zero inside the barrier. These students gave the following explanations:

> "Electrons will end up in region $x > L$. Although a classical physical body would not make it into the region $0 < x < L$, the wave function will propagate through to the region $x > L$ and therefore there would be a probability of finding an electron there."

> "In the region of the potential barrier, the wave function exponentially decays, but has no real component. Past $x = L$ the real component appears again."

The first student seemed to believe that the wave function would be equal to zero inside the barrier because "a classical physical body would not make it into" that region. The second student explicitly states that the *real part* of the wave function is zero inside the barrier and non-zero outside the barrier. Students who gave responses like those above seemed to believe that the "real part" of the wave function would be non-zero in classically allowed regions, *i.e.*, where a "classical physical body" could exist, and zero in classically forbidden regions. (This particular error is discussed in detail in the subsequent chapter.)

Other students who appeared to have the same difficulty incorrectly predicted that no particles would tunnel through the barrier at all. For example, one such student drew the graph shown below in Figure 5-10.

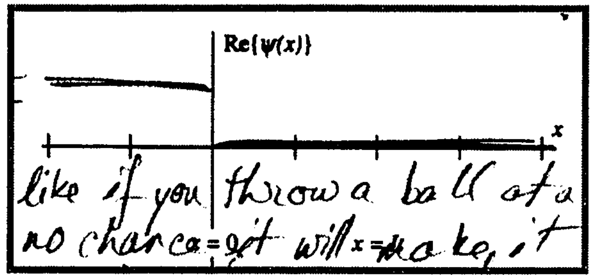


Figure 5-10: Incorrect student graph showing that Re{ $\psi(x)$ } is equal to zero everywhere in the region $x > 0$.

The graph has numerous errors, but in particular the student indicated that the real part of the wave function would be equal to zero inside and to the right of the barrier. The student explained as follows:

> "The 'real part' [of $\psi(x)$] means where a particle can physically exist, I think. If so, Re{ $\psi(x)$ } = 0 anywhere right of the barrier (like if you throw a ball at a brick wall, there's pretty much no chance it will make it past that wall)."

The student tried to relate the wave function $\psi(x)$ to the probability of finding a particle at various locations in the region $x < 0$. However, she seemed to associate the probability density with the "real" part of $\psi(x)$ instead of $|\psi(x)|^2$. This error led the student to predict incorrectly that particles "can physically exist" to the left of the barrier but not to the right of it.

d) *Summary of difficulties*

The results presented in this section suggest that students often fail to attribute properties of waves to matter in scattering states. Many do not recognize that free particles, propagating with well-defined momentum and therefore with well-defined de Broglie wavelength, can be represented by wave functions $\psi(x)$ that are sinusoidal. Many students fail to treat scattering from a potential step or barrier in the same way as the reflection and transmission of pulses at a boundary between two springs. They fail to recognize that reflection would occur as well as transmission at a boundary between regions of different potential. Others fail to recognize that transmission (or tunneling) can occur through a potential barrier of finite height. Many of these difficulties seem to persist after standard lecture instruction in quantum mechanics, even at the graduate level.

2. *FAILURE TO APPLY THE CONCEPTS OF PULSE WIDTH AND DE BROGLIE WAVELENGTH IN THE CONTEXT OF WAVE FUNCTIONS IN ONE DIMENSION*

In the preceding section we discussed several difficulties that students had in attributing wave properties to particles that are incident on a potential step or potential barrier. In this section we describe other difficulties that arose even among those students who attribute wave properties such as wavelength, reflection, and transmission to matter. In particular, we describe specific difficulties in applying the concepts of pulse width and de Broglie wavelength in the context of scattering states and bound states in one dimension.

Students are expected to recognize how the curvature of an energy eigenstate $\psi(x)$ varies with the potential $V(x)$. In classically allowed regions the local de Broglie wavelength obeys the relationship $\lambda = h/p$, where p represents the (magnitude of the) local momentum as a function of position (instead of an operator).[58] Similarly, the width of a wave packet propagating in one dimension changes in proportion to the average momentum of the wave packet. (This effect, separate and distinct from the usual "spreading" of a wave packet, is observed when pulses are transmitted from one spring to a connected spring.) In classically forbidden regions, however, local momentum is not a real quantity, and thus local de Broglie wavelength is not well-defined.

We have found that students have difficulty developing an understanding of the concepts described above. We describe several conceptual and reasoning difficulties that led to a number of student errors.

a) Failure to relate local de Broglie wavelength to local kinetic energy

The results of the analysis of responses to several written questions have revealed the presence of several difficulties that students have in relating the local de Broglie wavelength to the local kinetic energy in classically allowed regions. The responses reflected an overall failure to associate larger values of local kinetic energy to smaller values of local de Broglie wavelength. This difficulty did not seem to be addressed by standard instruction.

Particle in a bound state of an asymmetric square well

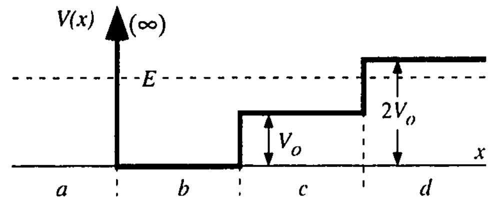


Figure 5-11: Potential energy diagram shown on the double-tiered square well question.

One of the questions that elicited the difficulty described above dealt with a particle in a double-tiered square well, shown at right in Figure 5-11. (See also Figure 5-4 for a description of the entire question.) In order to answer one part of the question, students were required to recognize that the local kinetic energy is larger in region *b* than in region *c*, and that the local de Broglie wavelength is therefore smaller in region *b* than in region *c*.

Although some students recognized that de Broglie wavelength was related to kinetic energy, their responses suggested a confusion between kinetic energy and potential energy, or between kinetic energy and total energy. These students incorrectly stated that $\lambda_b > \lambda_c$ or that $\lambda_b = \lambda_c$. Example student responses are given below.

> "$\lambda = h/\sqrt{2mE}$. λ_a is not defined.... $\lambda_b > \lambda_c > \lambda_d$ because $E_b < E_c < E_d$"

> "Because we have a stationary state, E is constant. λ_a is not defined for $x < 0$, since the electron is never there. For $x > 0$, $\lambda_b = \lambda_c = \lambda_d$, since E is constant and $\lambda = h/\sqrt{2mE}$."

The first student began by writing an expression for the de Broglie wavelength that would be correct if the student had intended the symbol "E" to denote kinetic energy. However, from the student's ranking of "$E_b < E_c < E_d$" it seems that the student intended "E" to represent *potential energy,* which made the final comparison of de Broglie wavelengths incorrect. A similar error was made by the second student who incorrectly answered that the de Broglie wavelengths would be the same in regions *b* and *c*. This student used the expression "$\lambda = h/\sqrt{2mE}$" but interpreted "E" as the *total energy* of the particle, not the kinetic energy.

It is interesting to note that the students who gave the responses discussed above not only failed to relate the local de Broglie wavelength to local kinetic energy but also incorrectly believed that the wavelength would be well-defined in region *d*, a classically forbidden region.

The mistaken belief that the wave function $\psi(x)$ can have nodes in forbidden regions is discussed later in this chapter.

Particle in a bound state of a V-shaped potential well

Students also had serious difficulty relating local de Broglie wavelength to local kinetic energy in the classically allowed region of a V-shaped well, shown previously in Figure 5-5. When students were asked to draw a qualitatively correct graph of the wave function for an energy eigenstate of the well, more than half did not recognize that the nodes of the wave function would become gradually farther apart going from left to right.

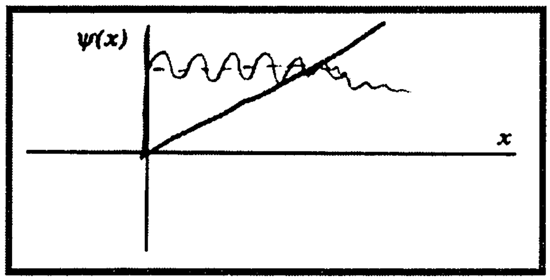


Figure 5-12: Incorrect student graph of $\psi(x)$ for the V-well question. The nodes are shown to be equally spaced in the classically allowed region.

The most common error was to draw a graph like that shown in Figure 5-12. The graph indicates that the nodes in the classically allowed region are all approximately equally spaced. This result would imply that the Broglie wavelength remains *constant* from $x = 0$ to the classical turning point on the right-hand side of the graph. Such a result would be correct for a region in which the potential does not change (*e.g.*, an infinite square well). It is also interesting to note that the student who drew the graph shown in Figure 5-12 superimposed the graph of the potential $V(x)$ with his graph of the wave function. Many students did this and drew the wave function as oscillating about a horizontal line *above* (not about) the $\psi = 0$ axis (x-axis).

Another prevalent error was to show that that the nodes of the wave function $\psi(x)$ gradually become closer together from left to right. An example of this type of response is reproduced in Figure 5-13. The student who drew this sketch wrote on the far left of the graph of the wave function, "large λ, high E," while nearer the center he wrote, "λ falls off, as does E." If we interpret the student's remarks about "E" to refer to the kinetic energy, which seems the most reasonable interpretation, then we would also conclude that he incorrectly related larger values of kinetic energy to larger values of the de Broglie wavelength. (It is worth noting that, in both examples above, the amplitude of the wave function was also qualitatively incorrect.)

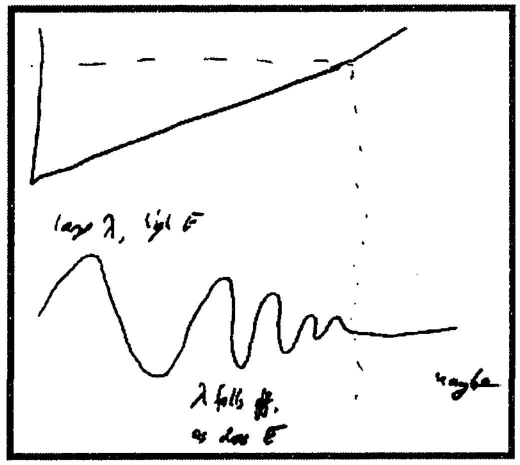


Figure 5-13: Example of incorrect student graph showing consecutive nodes becoming closer and closer together as local kinetic energy decreases.

Particles scattering from a potential barrier

In addition to difficulties that arose in the context of bound states, students often had serious difficulty relating the local de Broglie wavelength of particles transmitted through a potential barrier to the local kinetic energy. Evidence is drawn from the analysis of student responses to the potential barrier question, in which students consider a beam of monoenergetic particles incident on a rectangular barrier (Figure 5-2). In this problem, the potential was the same on both sides of the barrier ($x < 0$ and $x > L$). Many students who drew a graph of the real part of the wave function did not recognize that the local de Broglie wavelength would therefore be the same in the two regions. Instead, they indicated that the de Broglie wavelength of the transmitted particles (in the region $x > L$) would be noticeably smaller than that of the incident particles (in the region $x < 0$). Most of the students who made this error were above the mean of the class.

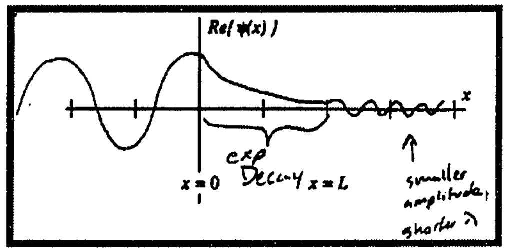
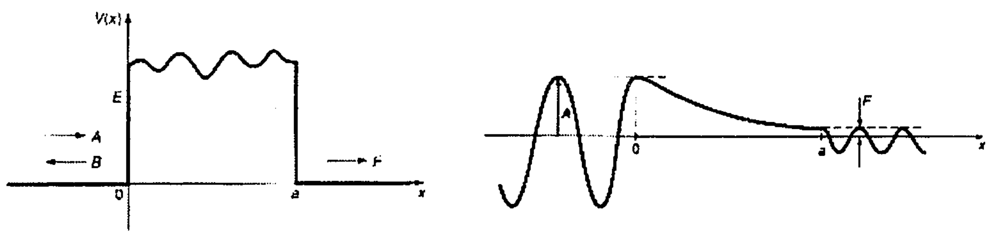


Figure 5-14: Reproduction of student response that shows a smaller wavelength on the right side of barrier than on the left side.

An example of a common incorrect student graph is shown in Figure 5-14. In this example, the student recognized that the Re{ $\psi(x)$ } had sinusoidal behavior in both classically allowed regions and approximately exponentially decaying behavior in the region containing the barrier. However, to the right of the barrier the student explicitly indicates that the wave function has both a "smaller amplitude" *and* a "shorter λ" than on the left hand side of the barrier. This student, as did others, seemed to think that a smaller amplitude must also correspond to a shorter local wavelength.



(a) (b)

Figure 5-15: (a) Graph of potential barrier shown in the textbook used in some of the quantum mechanics classes. (b) Sketch of corresponding wave function shown in the text. Like many student-drawn graphs, the wavelength of the transmitted particles is considerably less than that of the incident particles.

Correspondence with the author of a quantum mechanics textbook. In some of the quantum mechanics classes in the study, the textbook contained an error in one of the figures used in a discussion about "tunneling" through a potential barrier. The situation involved a beam of monoenergetic particles incident from the left on the tall, broad potential barrier shown in Figure 5-15(a). A sketch of the corresponding wave function, reproduced in Figure 5-15(b), indicated that the de Broglie wavelength of the transmitted particles was less than that of the incident particles.

We had the opportunity to ask the author for an explanation about the way the possible source of the error. He replied that the figure was drawn in such a way that "the 'aspect ratio' (amplitude-to-wavelength) should be roughly preserved" and acknowledged the error. This line of reasoning about the "aspect ratio" between the amplitude and the wavelength is consistent with the explanations given by those students who made similar errors on the potential barrier question.

Some students who mistakenly drew a shorter de Broglie wavelength for the transmitted particles than for the incident particles used the textbook with the errant figure described above. One could interpret this fact to suggest that the mistake in the textbook was the source of the errors made by the students on the final exams and on the pretests from these classes. However, the same error also arose among students who used a different textbook. Therefore, it seems unlikely that the errors were caused solely by the mistake in the textbook.

Commentary. After all standard instruction on the Schrödinger equation in one dimension, students have difficulty recognizing how the potential $V(x)$ in a particular region affects the local de Broglie wavelength of a particle (or the relative distance between consecutive nodes of the wave function) within that region. Even those who do recognize the existence of a relationship between local de Broglie wavelength and local kinetic energy (or potential) fail to apply the definition of de Broglie wavelength correctly. These types of errors are very similar to those that arise in the context of interference and diffraction of matter (see Chapter 3).

b) Failure to relate the width of a transmitted wave packet to the (average) momentum of the wave packet

We have found that many physics graduate students have difficulty relating the width of a wave packet to the average momentum. The presence of this difficulty was evident from the analysis of responses to the version of the potential barrier question shown in Figure 5-3, in which students consider a wave packet incident on a barrier. In particular many students drew sketches of transmitted wave packets as if they believed that an "amplitude-to-width ratio" must be preserved for the wave packet.

Almost all students who attempted the question correctly sketched a transmitted wave packet in the region to the right of the barrier. However, many indicated in their sketches that the width of the transmitted wave packet would be noticeably less than that of the incident wave packet, rather than the same width as the incident packet. Student graphs like the one shown in Figure 5-16 indicate the tendency to preserve the "amplitude-to-width aspect ratio" of the wave packet.

Figure 5-16: Reproduction of student response showing a transmitted wave packet much narrower than the reflected wave packet.

Commentary. Similar errors have been documented on a pretest question in the context of reflection and transmission of pulses on springs. This pretest required students to sketch the shape of two connected springs for an instant after a given incident pulse has passed by the junction point. Nearly half of the introductory students and physics graduate students who were given the pretest failed to recognize that the difference in wave speeds of the connected springs would affect the width of the transmitted pulse. (These results are described in detail in Appendix A.) The results suggest that standard instruction does little to address this difficulty, whether in the introductory course (in the context of springs) or in advanced courses (in the context of wave packets).

c) ***Mistaken belief that particles lose kinetic energy as a result of tunneling through a potential barrier***

One version of the potential barrier question included a part that required students to explicitly compare the de Broglie wavelength of particles incident on a potential barrier to that of particles transmitted through it (Figure 5-2). Many of the responses indicated an incorrect belief that particles lose kinetic energy as they tunnel through a barrier. This idea led several students to state incorrectly that the de Broglie wavelength of the transmitted particles would be *greater than* that of the incident particles. Example student responses are given below (boldface added by the author):

> "**The particle must lose *KE* tunneling through a potential.** $KE = p^2/2m$, $p^2 = h^2/\lambda^2$, $p \propto 1/\lambda$, p goes down, $\lambda \uparrow$."

> "The particles **expend kinetic energy 'breaking [through]' the barrier**....
> In the region $x > L$, the particle must have less kinetic energy, so $p_{x>L} < p_{x<0}$,
> so $\lambda_{x>L} < \lambda_{x<0}$ (the incident wavelength)."

Explanations like those above suggest the belief that "work" is done on or by the particles while inside the potential barrier. This belief may in fact be a symptom of a deeper conceptual difficulty that students may have in relating the potential energy $V(x)$ over a given region to the force (in the classical sense) exerted on particles located at various positions x in that region, $F_x = -dV(x)/dx$. Part of the difficulty seems to be a failure to interpret the potential energy graph. It may also be related to failure in distinguishing between a quantity and its rate of change,[59] namely $V(x)$ and $dV(x)/dx$.

d) *Incorrect belief that the wave function has oscillatory behavior in classically forbidden regions*

Several of the difficulties discussed thus far in this chapter suggest an overall failure to relate local de Broglie wavelength to local kinetic energy in classically allowed regions. Many students also had difficulty understanding the qualitative behavior of the wave function in classically forbidden regions. In particular, many seemed to believe that the wave function would have oscillatory behavior, and thus a well-defined de Broglie wavelength, in the forbidden regions of a potential well or potential barrier. Specific examples of student errors are described below.

Particle in a bound state of an asymmetric square well

Students who were given the double-tiered square well question (Figure 5-4) were asked to identify in which regions *(a–d)* the de Broglie wavelength was well-defined. A majority of the students incorrectly believed that the de Broglie wavelength would be well-defined in either (or both) of the classically forbidden regions, *a* and *d*. Examples of student responses are as follows:

> "The greater V the smaller p. p_a = undefined, $p_b > p_c > p_d$. λ_a = undefined,
> $\lambda_b < \lambda_c < \lambda_d$."

> "The momentum of the particles is ordered, $p_b > p_c > p_d$ due to the increasing potential. p_d is almost zero because of the high potential, so λ_d is very large and decreases with x in an exponential fashion."

> "λ_a not defined! Particle not allowed in that region. $\lambda_d > \lambda_c > \lambda_b$. Greater E means smaller λ, so order is inverse of E order."

Students who gave explanations like those above seemed to overgeneralize the rule that the de Broglie wavelength of a particle increases as the potential energy increases. Although this result is true for classically allowed regions, it is incorrect to extend this line of reasoning to the behavior of the wave function in classically forbidden regions.

Particle in a bound state of a V-shaped potential well

Related errors arose when students attempted to sketch qualitatively correct graphs of the wave function for a particle in a V-shaped potential well (Figure 5-5). Many students incorrectly indicated that the wave function would have oscillatory behavior in one or both of the forbidden regions of the well.

An example of an incorrect student graph is shown in Figure 5-17. The student has shown correctly that both the amplitude of the wave function and the local de Broglie wavelength gradually increase with distance to the right of $x = 0$. However, the behavior of the wave function to the right of the classical turning point (labeled by the student as $x = L$) is incorrect as shown. Rather than show $\psi(x)$ steadily decreasing to the right of the turning point, the student sketched an "envelope" of steadily decreasing amplitude in this region and drew a wave function that oscillated about the x-axis inside the envelope. While the student seemed to recognize that the qualitative shape of $\psi(x)$ would not be the same in the regions $0 < x < L$ and $x > L$, the student failed to realize that in the classically forbidden region $x > L$ the wave function cannot have a local wavelength and thus cannot have nodes.

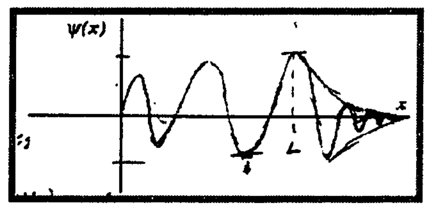


Figure 5-17: Incorrect student graph showing the wave function oscillating in the classically forbidden region.

Particles scattering from a potential barrier

The mistaken belief to attribute oscillatory behavior to the wave function in classically forbidden regions was elicited on questions about simple scattering states in one dimension. Earlier in this chapter we discussed the failure of students to recognize that the wave function must be non-zero inside a potential barrier in order for tunneling to occur. Here we present evidence of a related difficulty, namely the belief that the wave function would have oscillatory behavior inside the barrier for tunneling to occur.

In response to the potential barrier question shown in Figure 5-2, some students drew graphs like that shown in Figure 5-18. The student who made this sketch apparently believed that, inside the barrier, the wave function oscillates within an envelope whose amplitude is exponentially decreasing with distance to the right of $x = 0$. Although the students who made this error seemed to recognize correctly that there is a non-zero probability of a particle tunneling through the barrier, their graphs suggested the incorrect belief that the wave function must propagate in a *classical* sense through the barrier in order for tunneling to occur. (It is also interesting to note that the student who drew this graph indicated that the wavelength in the region $x > L$ would be much smaller than that in the region $x < 0$. This error has been discussed previously.)

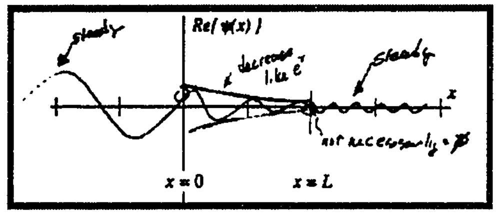


Figure 5-18: Graph drawn by a student showing that the wave function oscillates inside the barrier.

e) ***Summary of difficulties***

The results presented in this section suggest that after standard instruction in quantum mechanics students have difficulty relating local de Broglie wavelength or pulse width to the potential *V(x)*. Many errors that arose were similar to those in the context of interference and diffraction of matter. (See Chapter 3.) Students often fail to recognize that there is a relationship between the spacing of consecutive peaks and the kinetic energy. On questions about bound states, many students associated larger de Broglie wavelengths with larger kinetic energies. Others incorrectly associated de Broglie wavelength with the potential energy or total energy of the particle. On questions about scattering states, many students seemed to believe that the amplitude-to-wavelength ratio (for stationary scattering states) or the amplitude-to-width ratio (for wave packets scattering from potential barriers) would be the same for the transmitted particles as for the incident particles. In addition, students often fail to recognize that in regions where the potential is larger than E but still finite, the wave function does not have a well-defined local de Broglie wavelength and thus does not have nodes.

D. SUMMARY

In order for students to develop a coherent wave model for matter, it is necessary that they recognize that matter can exhibit behavior of classical waves, such as superposition, reflection, and transmission. They must also recognize under what conditions particles can be described with a well-defined wavelength and how the value of the wavelength is related to the mass and velocity of the particles.

We have found that after standard instruction students often fail to attribute properties of waves to particles incident on a potential step or potential barrier. Many fail to recognize that the (real part of the) wave function for a free particle is a plane wave and that the wave function would have a well-defined wavelength. In addition, students often fail to recognize that particles can undergo reflection at a boundary between regions of different potential or transmission through a potential barrier of finite height. Related difficulties that arise in the context of reflection and transmission of pulses on springs appear to be addressed somewhat by lecture instruction, but analogous difficulties in the quantum mechanical regime persist after lecture instruction in quantum mechanics.

We also have found that students have serious difficulty developing a functional understanding of the concept of de Broglie wavelength. Many students fail to recognize the relevance of local de Broglie wavelength in the context of wave functions in one dimension, and even those that do have difficulty trying to apply the concept. These difficulties are held approximately equally by academically strong and poor students and seem to persist even after standard instruction of more advanced topics in quantum mechanics.

CHAPTER 6: IDENTIFYING STUDENT DIFFICULTIES WITH PROBABILITY DENSITY IN THE CLASSICAL AND QUANTUM MECHANICAL REGIMES AND IN THE TRANSITION BETWEEN THEM

A. INTRODUCTION AND MOTIVATION

In this chapter we present results from an investigation of student understanding of probability density in both the classical and quantum mechanical regimes. As students begin to study quantum mechanics, they are expected to become familiar with the idea that matter exhibits wave-like properties, such as superposition, wavelength, reflection, and transmission.[60] They are then introduced to the wave function and to the interpretation of the wave function as a probability amplitude. As students develop a quantum mechanical model for matter, they need to recognize how to incorporate relevant concepts from the mechanics of both classical particles and waves. Furthermore, they are expected to recognize that quantum mechanics should yield classical results in the appropriate limit. Whether or not it is explicitly covered in class, students are expected to recognize for which physical situations it is valid to apply concepts from classical mechanics and for which situations concepts from quantum mechanics are more appropriate.[61]

In the preceding chapter, we identified specific difficulties that students have in applying a wave model for matter in the context of bound states and scattering states. Few students, for example, could correctly relate the curvature of the wave function to the potential. We suspected that additional complications would arise in relating the amplitude of the wave function to the potential. Because students are expected to interpret the wave function as a probability amplitude, such difficulties would hinder the ability of students to grasp more abstract ideas in quantum mechanics. In addition, in Part I of this dissertation we have shown that students have serious difficulty recognizing under what conditions it is more appropriate to use geometrical optics or physical optics to describe the passage of light through one or more slits. We anticipated the presence of similar student difficulties in deciding whether to use ideas from classical or quantum mechanics to describe a particle in a simple physical situation.

In this chapter we present evidence of numerous conceptual and reasoning difficulties that students have in understanding probability density. In particular, we attempt to answer the following research questions that have formed the focus of this part of the study: After standard instruction in quantum mechanics:

- Are students able to relate probability density (or the amplitude of the wave function $\psi(x)$) to potential $V(x)$?
- Are students able to recognize when it is more appropriate to use concepts from classical or quantum mechanics to explain a phenomenon and to apply correctly the relevant concepts?

B. CONTEXT FOR RESEARCH

In order to begin to answer the research questions above, we developed qualitative questions that were designed to probe for the presence of specific conceptual and reasoning difficulties. These questions were used in individual demonstration interviews protocols and on written tests at various stages of instruction. Below we describe the student populations included in the study and give an overview of the questions.

1. STUDENT POPULATIONS

The research described in this chapter has included several different student populations at the University of Washington and elsewhere. We focus primarily on students enrolled in the junior-level sequence of quantum mechanics courses taught at the University of Washington. We also discuss results from tasks that were posed in interviews to students in a sophomore-level modern physics course, a junior-level quantum mechanics course, and a graduate-level quantum mechanics course. (See Chapter 1 for details about the structure of the courses.) Additional results are presented from students in a traditional sophomore-level modern physics course taught at another large research university.

2. RESEARCH TASKS

Below we describe the questions used to identify the difficulties discussed in this chapter. Several are on particles in bound states and others are on scattering. Unless otherwise specified, all questions were posed after standard instruction of the relevant material. (All of the questions, together with the intended correct responses, are described in detail in Appendix B.)

a) Questions about particles in bound states

We have developed a variety of questions that deal with a particle in a bound state of a one-dimensional potential well. In each of the questions the students are shown a potential energy diagram illustrating the well and are asked to describe or sketch the wave function for a possible energy eigenstate. Figure 6-1 below shows examples of potential wells that were used. The infinite square well potential (in part (a) of the figure) served as the context for a series of individual student interviews. The other potential wells were used on written questions. (These written questions were discussed in the preceding chapter.)

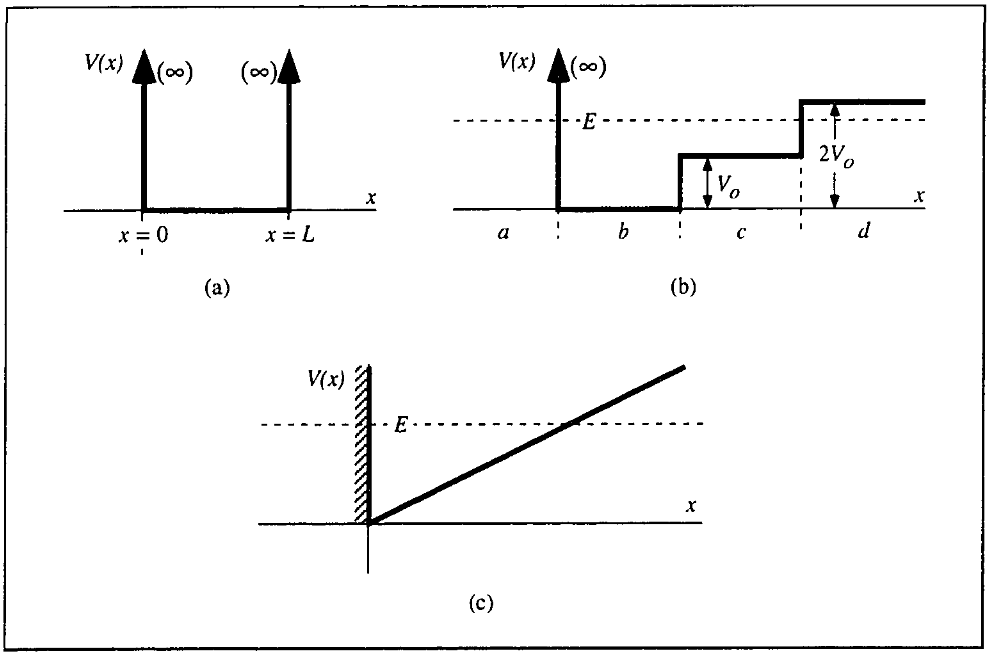


Figure 6-1: Potential energy diagrams from questions that pertain to a particle in a stationary bound state of: (a) an infinite square well, (b) a double-tiered square well, and (c) a V-shaped well.

To answer the interview tasks and written questions correctly, students must understand how the amplitude of $\psi(x)$ for a highly-excited energy state varies qualitatively with the potential. For sufficiently large values of the energy, *i.e.*, for a wave function $\psi(x)$ containing several nodes, the amplitude of $\psi(x)$ follows the same qualitative behavior as probability density $P_{cl}(x)$ in the classical regime. In this regime, the probability $P_{cl}(x)dx$ of finding the particle in a small segment of width dx is proportional to the time interval required for the particle to traverse that segment.

Thus probability density (and, quantum mechanically, the amplitude of $\psi(x)$) is relatively large in regions where the local kinetic energy is relatively small, and relatively small where the local kinetic energy is large. For example, in an infinite square well (case (a) above), the amplitude of $\psi(x)$ is constant everywhere within the classically allowed region of the well because the potential is constant in this region. In the double-tiered square well, the wave function would have a greater amplitude in region *c* than in region *b*.[62] In the V-shaped well, the amplitude gradually increases as local kinetic energy decreases from left to right within the classically allowed region. Figure 6-2 illustrates qualitatively correct graphs of $\psi(x)$ for each of the potential wells shown in Figure 6-1.

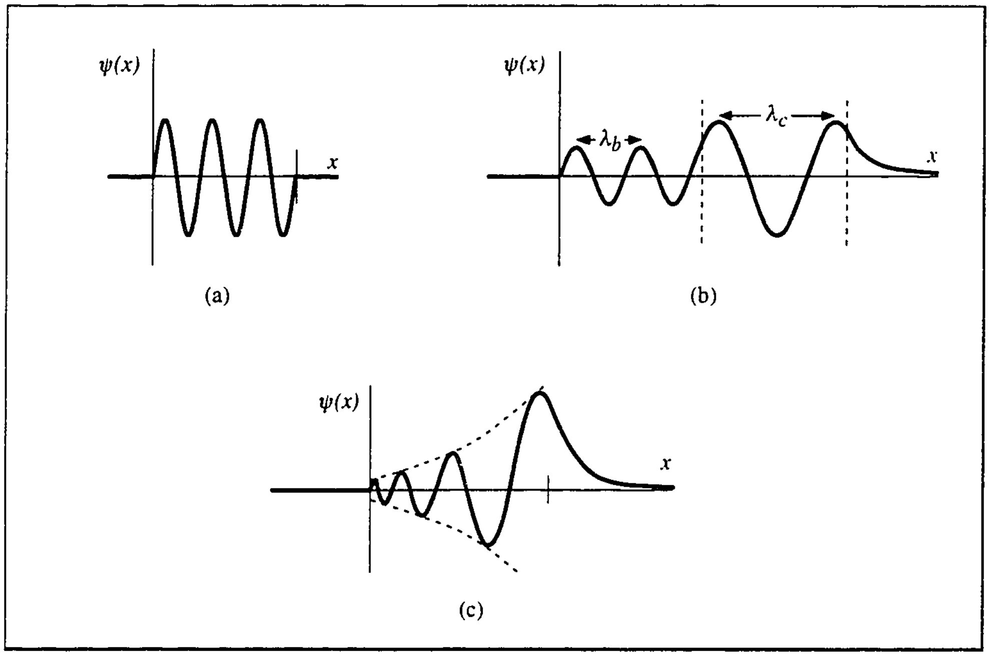


Figure 6-2: Qualitatively correct graphs of wave functions for the (a) infinite square well, (b) double-tiered square well, and (c) V-shaped well shown in Figure 6-1.

Another question, shown below in Figure 6-3, was adapted from an interview protocol that was also used in the study. (The similarities and differences between the written version of the question and the interview protocol are described later in this chapter.) In each part of the question the student is asked to consider a physical situation in which a particle is confined to a specified region. For each situation, the student must choose which of the three graphs of probability density best corresponds to that situation.

A. A small pebble of mass 1 g is placed inside a rigid box of length 20 cm. It is given an initial horizontal speed of 1.8 cm/s in a direction parallel to the length of the box. Ignore the friction and the rotational motion of the pebble, and assume that the pebble collides elastically with each end of the box. Which graph (i, ii, or iii) would best describe the probability density *P(x)* of the pebble as a function of its position *x* along the length of the box?

B. An electron ($m = 9 \times 10^{-28}$ g) is now placed inside a rigid box of length 20 cm. It is given an initial horizontal speed of 1.8 cm/s in a direction parallel to the length of the box. Assume that the electron collides elastically with each end of the box. Which graph (i, ii, or iii) would best describe the probability density *P(x)* of the electron?

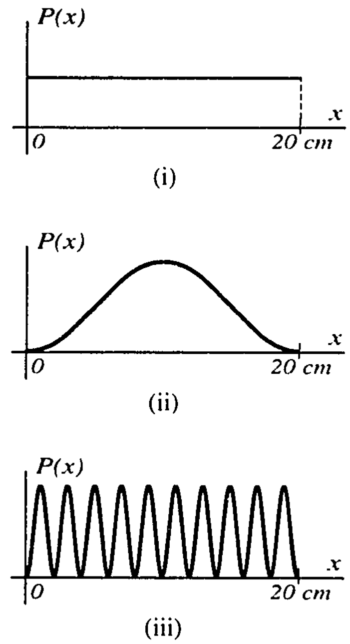


Figure 6-3: Multiple-choice written version of questions from an interview protocol about the probability density for a pebble in a shoebox. This question was given to students on a final examination after all standard instruction in a sophomore-level modern physics course taught at a large research university.

For the first situation (part A), in which a 1 g pebble undergoes (classical) periodic motion within the box, the pebble would have equal probability of being located at any location inside the box. (This result is true even near the ends of the box, which are stated as being rigid.) Graph (i) would therefore be the correct answer to part A. On the second part (part B), in which an electron undergoes periodic motion with the same speed as the 1 g pebble, students should recognize that classical reasoning would not be valid and that graph (i) would be incorrect. From the given information, the students must determine that the de Broglie wavelength of the electron has a value equal to one-fifth the length of the box. The wave function for such an electron would be sinusoidal and have ten equally-spaced anti-nodes, so squaring the wave function would yield a probability density graph that is identical to graph (iii).

b) Questions about particles in scattering states

Some questions that we have used pertain to particles scattering from a potential step or potential barrier. Potential energy diagrams of the step and barrier are shown in Figure 6-4. (The questions were also discussed in the preceding chapter.)

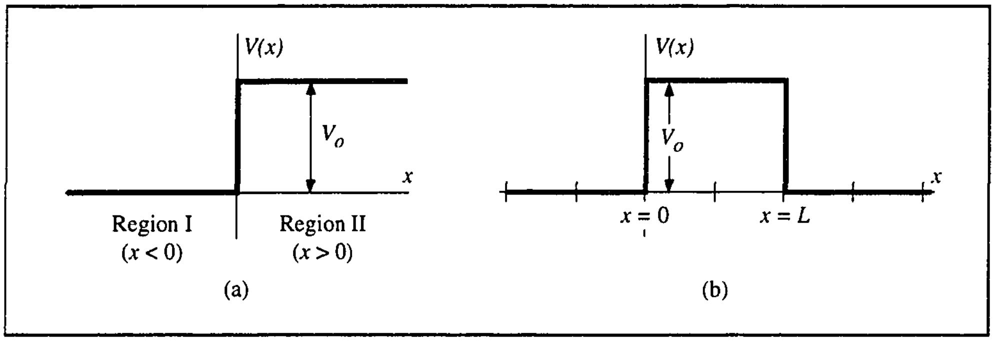


Figure 6-4: Potential energy diagrams from questions that pertain to a beam of monoenergetic particles incident from the left on: (a) a potential step, and (b) a potential barrier.

One question asked to students to consider a beam of monoenergetic particles each with energy $E > V_o$ that is incident on the potential step shown above in Figure 6-4(a). For this case, students were asked to predict whether all of the incident particles would be transmitted across the boundary ($x = 0$) between the two regions of the step. The question, given before standard instruction on scattering, was intended to probe whether or not students would treat the incident particles as a wave that would be partially reflected and transmitted at the boundary. On a later part of the same question, students were asked to sketch the behavior of the wave function for the case in which particles were incident from the left on the potential step with energy $E < V_o$. We were interested in gauging the ability of students to recognize that the real part of the wave function would be non-zero in the region $x > 0$, even though this region would be classically forbidden.

The question about scattering from a finite potential barrier was posed after standard instruction. Students were asked to draw the real part of the wave function for the case in which monoenergetic particles of energy $E < V_o$ were incident on the barrier from the left. Like the question on scattering from a potential step, the potential barrier question was designed to probe the ability fo students to recognize that the real part of the wave function is non-zero inside the barrier, even though it would be a classically forbidden region.

C. IDENTIFICATION AND ANALYSIS OF SPECIFIC DIFFICULTIES

Most students found the written questions and interview tasks difficult. Analysis of the student responses revealed the presence of several common conceptual and reasoning difficulties. Although many responses were incorrect for multiple reasons, we focus the discussion in this section on difficulties with probability density. These difficulties were organized into the three broad, overlapping categories below:

- lack of understanding of probability density for a particle undergoing classical motion,
- tendency to explain quantum mechanical behavior using concepts from the classical mechanics of particles, and
- failure to identify the small-momentum (large-de Broglie wavelength) limit in which classical mechanics should be abandoned for quantum mechanics

For the remainder of this chapter, we discuss the specific difficulties that were identified and present illustrative examples of student responses.

1. LACK OF UNDERSTANDING OF PROBABILITY DENSITY FOR A PARTICLE UNDERGOING CLASSICAL MOTION

Many instructors who teach quantum mechanics give analogies between classical physics and quantum mechanics. Some may expect their students to understand on a qualitative level how the probability density of a particle undergoing periodic motion in one dimension varies with the potential or kinetic energy of the particle. Such knowledge can be useful in relating the classical mechanics of particles to quantum mechanical results for excited states. However, we found that many students have difficulty relating the probability density of a particle, the total energy of the particle, and the potential even in the classical regime. These difficulties were prevalent even after standard instruction in quantum mechanics.

a) Mistaken belief that the probability density is larger in regions of lower potential

Many incorrect responses indicated a mistaken belief that the probability density for a particle is larger in regions of lower potential than regions of higher potential. Rather than relate potential or local kinetic energy in a particular region to the amount of time that a particle would spend in that region, students tended to think that it is "easiest" for a particle to be found in regions where the potential is lowest.

Particle in a bound state of an asymmetric square well

After standard instruction, students had serious difficulty comparing the amplitude of the wave function in regions b and c of the double-tiered potential well shown in Figure 6-1 (and reproduced at right in Figure 6-5). Very few students correctly stated that the maximum value of $\psi(x)$ would be larger in region c than in region b, and no one justified a correct answer with correct reasoning. A majority of the students answered instead that $\psi(x)$ would have a larger value in region b than in region c. Examples of incorrect explanations are given below:

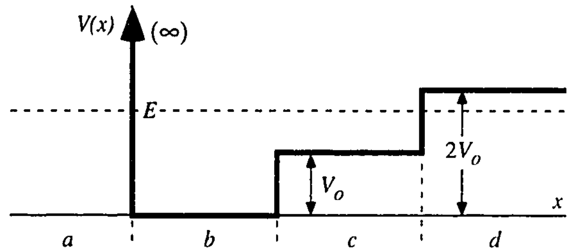


Figure 6-5: Graph of $V(x)$ *vs.* x shown on the double-tiered square well question.

> "Because b has a lower potential, a particle is more likely to be found in region b. This implies $|\psi_b|^2 > |\psi_c|^2$ and $\psi_{b\,max} > \psi_{c\,max}$."
>
> "If the potential is higher, the particle has a harder time being in that spot."
>
> "ψ is more likely to be in b than c because b is a 'deeper hole.'"

The question about a particle in the double-tiered square well shown above included a part in which students were asked to draw a qualitatively correct sketch of the wave function. (See Appendix B.) All of the students who expressed the type of reasoning illustrated above drew sketches that were consistent with their written responses.

Particle in a bound state of a V-shaped well

Students made similar errors on a question involving a particle in a V-shaped potential well. (See Figure 6-1.) Many students drew qualitatively incorrect sketches of a possible stationary state wave function for the well. Although the probability density (and thus the amplitude of the wave function) is smallest near $x = 0$, where the local kinetic energy is greatest, many students drew graphs in which the amplitude of the wave function was largest near $x = 0$.

The graph shown at right in Figure 6-6 is an example of an incorrect response by a student. The student gave the following explanation to justify his graph: "The particle is most likely to be in the lowest potential, so [the wave function] has the greatest amplitude there." For this reason, the student drew a graph of $\psi(x)$ in which the amplitude is largest on the left side of the well, where the kinetic energy is largest. This type of reasoning would be incorrect not only in the quantum mechanical regime but in the classical regime as well.

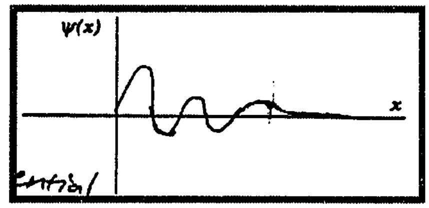


Figure 6-6: Incorrect graph of $\psi(x)$ with *larger* amplitude in region of higher kinetic energy.

b) Failure to recognize that probability density (or the amplitude of the wave function) is constant only in regions of constant potential

Whether in the classical or quantum mechanical regime, the infinite square well potential is a relatively simple situation in which to determine probability density. Both potential energy and kinetic energy are constant, so probability density (classically) or the amplitude of the wave function (quantum mechanically) do not vary with location inside the well. We have found that students often generalize this result to potential wells of any shape.

Particle in a bound state of a double-tiered square well

As mentioned previously, very few students correctly compared the amplitude of the wave function in the two classically allowed regions (regions *b* and *c*) of the double-tiered square well shown in Figure 6-1. (See also Figure 6-5.) Some students stated incorrectly that the wave function would have the same amplitude in both regions, even though the potential was larger in region *c* than in region *b*. For example, the explanation by one student included a reference to the energy eigenstates of an infinite square well, *i.e.*, $\psi_n(x) = (2/L)^{1/2}\sin(n\pi x/L)$. The student seemed to believe that this expression was valid for the wave function in both regions *b* and *c*, stating that the amplitude of the wave function in region *b* would be equal to that in region *c*. Another student said that the amplitude of $\psi(x)$ "should be about the same in *b* and *c* [because] ψ_{max} goes like $L/2$ [and the width] L is the same in both regions." Both the reference to the factor of $L/2$ and the fact that regions *b* and *c* have the same width suggest that the student incorrectly applied the expression for the eigenstates $\psi_n(x)$ of an infinite square well to regions *b* and *c* of the double-tiered square well.

Particle in a bound state of a V-shaped well

Many students seemed to believe that the amplitude of the wave function would be constant within the classically allowed region of the V-shaped well. (See Figure 6-1.) About half of the students who attempted the V-well question after standard instruction made this error. An example of an incorrect student graph is shown in Figure 6-7. Notice that this graph shows qualitatively incorrect features for both the amplitude of $\psi(x)$ and local de Broglie wavelength. (Difficulties in relating de Broglie wavelength to potential are described in detail in the preceding chapter.)

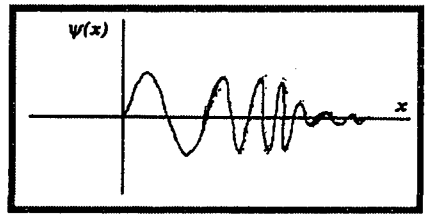


Figure 6-7: Incorrect student graph in which $\psi(x)$ has *constant* amplitude in classically allowed region.

Pebble in a shoebox

In both potential well questions discussed above, students failed to recognize that the amplitude of the wave function is not constant in regions of varying potential. We have also probed the ability of students to recognize specifically that the probability density for a particle undergoing classical motion in a region of constant potential is constant throughout that region.

The written question shown in Figure 6-3 was given on the final examination to 440 students in a sophomore-level modern physics course at a large research university other than the University of Washington. In part A of the question, students are told to consider a 1 g pebble that is given an initial speed of 1.8 cm/s inside a rigid, frictionless box with perfectly elastic ends. They are expected to recognize that the subsequent motion of the pebble can be described classically and that the probability density of the pebble does not vary with respect to location in the box. Of the three graphs shown in the problem, the graph of uniform probability density (graph (i)) is the correct choice.

Only about 60% of the students selected the correct graph. The most common incorrect choice, given by 30% of the class, was graph (ii), which would be correct only for a particle in the *ground state* of an infinite square well having the same length as the box. This response suggests the presence of a mistaken belief that the 1 g pebble would spend more time in the center of the box than near the walls. Further evidence of this belief was found in the analysis of student interviews on a particle in an infinite square well, which are discussed in the following section.

c) ***Summary of difficulties***

Analysis of student responses to written questions has revealed the presence of several common conceptual difficulties in relating the probability density of a particle in the bound state of a potential well to the potential *V(x)*. For example, rather than recognize that a particle is more likely to be found in regions of relatively large potential (and thus relatively small local kinetic energy), students often believe that it is "easier" for the particle to be in regions of lower or "deeper" potential. Other students seem to believe that probability density (or the amplitude of the wave function) would vary in regions of constant potential. These difficulties arose on questions pertaining to quantum mechanical situations and on others about classical situations.

2. *TENDENCY TO EXPLAIN QUANTUM MECHANICAL BEHAVIOR USING CONCEPTS FROM THE CLASSICAL MECHANICS OF PARTICLES*

Many student errors seemed to be caused by the application of ideas that would be correct in the regime of classical mechanics but that were inappropriate for quantum mechanical situations. These ideas often led to incorrect predictions about the behavior of particles incident on a potential step or potential barrier. Similar difficulties arose in the context of bound states in an infinite square well potential.

a) ***Mistaken belief that reasoning from classical mechanics can be used to account for the ground state wave function of a square well***

The tendency to apply reasoning from the classical mechanics of particles in an inherently quantum mechanical situation was observed during individual student interviews in which the infinite square well task was used. A total of nine students enrolled in a junior-level quantum mechanics course participated. They had finished all standard instruction on wave functions in one dimension. All were at or above the mean of the course.

The interview volunteers were asked to draw qualitatively correct sketches of the wave functions for several energy eigenstates of an infinite square well potential and to explain their reasoning. Although almost all students sketched qualitatively correct wave functions, some applied reasoning about the classical motion of a particle in such a potential in order to account for the shape of the ground state wave function.

One student drew the graph shown in Figure 6-8 for the ground state wave function. Although the wave function does not meet the appropriate boundary conditions at the endpoints of the well, the student correctly indicated that the wave function would attain a maximum value in the center of the well. She justified this feature of her graph, however, by thinking "classically" about a ball being passed "back and forth" by people sitting next to each other side by side:

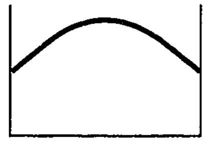


Figure 6-8: Illustration of a student sketch of the ground state wave function for an infinite square well.

> S_1: "Let's say you have five people sitting down on a couch, right? And they're passing this ball...back and forth.... The guy in the middle is going to touch the ball the most, more than the two on the end."

This statement is correct in that the person in the middle would touch the ball more than the two people at the ends of the couch. However, the student extended this result to the probability density for the particle in the infinite square well, saying that the particle would more likely be found in the center of the square well than at any other location within the well.

Another student who was interviewed used similar reasoning. The student drew qualitatively correct wave functions, but the reasoning that she used to justify the shape of the ground state wave function was similar to that used by the first student.

> S_2: "You can think of [the electron] as trying to get out and constantly running into a wall, but it can't go through, so it bounces off the wall and comes back.... Since it's constantly just going back and forth, it spends more time in the center because it crosses the center twice for every one time that it hits a wall."

The interviewer probed the student's reasoning more deeply by pointing to the sketch that the student made for the first excited state wave function, which correctly included a node at the center of the well. The interviewer asked:

> I: "What about the next one [the first excited state]? Would the 'bouncing' be any different?"

S_2: [Pause.] "That's an unfair question...because I can't answer it! Well, see, it's not as though the electron ends up skipping the middle."

The student apparently recognized that the node at the center of well meant that, for this case, the probability of finding the electron near the center of the well would be extremely low. She also seemed to recognize the inconsistency between the presence of the node in the first excited state wave function and her claim that the electron bounces "back and forth" in the well. The student could resolve this inconsistency only by saying:

S_2: "With an electron I can cover the first case [the ground state], but after that, it's the math that covers beyond that."

Commentary. We have shown that some of the students in the interviews inappropriately applied their intuition about classical physics to try to account for the shape of the ground state wave function. In particular, these students seemed to believe that the probability density for a particle undergoing classical motion in an infinite square well would attain a maximum value at the center of the well. This mistaken belief is consistent with the incorrect responses to the (multiple-choice) written question about a 1 g pebble in a shoebox. As discussed in section 1, about one-third of the students in a standard sophomore-level modern physics course believed that the probability density of the 1 g pebble would resemble that for the ground state wave function. This result suggests that the errors that arose on the written question resulted from the same underlying difficulty that was identified from the interviews on the infinite square well.

b) Mistaken belief that reflection and transmission of a beam of particles is due to a range of energies of the particles in the beam

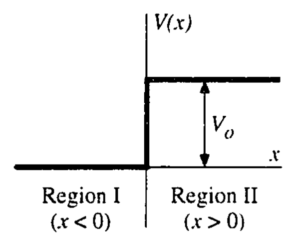


Figure 6-9: Potential energy diagram from the potential step question.

The indiscriminate use of ideas from the classical mechanics of particles was also evident in the analysis of student responses to questions about scattering states. For example, in the situation described on the potential step question, monoenergetic particles are incident on a potential step of height V_o such that $E > V_o$. (See the potential energy diagram shown at right in Figure 6-9 and previously in Figure 6-4.) Students are taught that reflection and transmission will occur at the abrupt change in potential. They are expected to be able to account for this result by treating the particles as having the wave properties of reflection and transmission, even though classical mechanics would predict that all of the particles would be transmitted.

As discussed in the preceding chapter, about half of the students incorrectly believed that all of the monoenergetic electrons with energy $E > V_o$ incident from Region I ($x < 0$) would end up in Region II ($x > 0$). These students failed to take into account the reflection at the boundary. Many seemed to base their answers only on a comparison between the total energy of the electrons and the value of the potential in that region. For example, one student incorrectly predicted that "all [electrons] make it across [the boundary at $x = 0$] since the E of electron is greater than V_o."

In this section we focus on responses given by students who correctly recognized that electrons would be reflected and transmitted at the boundary ($x = 0$) but who justified their answer with incorrect reasoning. Rather than attribute such phenomena to the wave properties of the electrons, these students based their answer on the belief that some electrons did not have sufficient kinetic energy to transmit into the region $x > 0$. This inference was in contradiction to the problem statement, which describes the incident electrons as monoenergetic.

An example response by a student who correctly predicted that some of the incident electrons would be reflected at the boundary ($x = 0$) between Regions I and II ($x > 0$) is given below:

> S: Region two has a potential, kind of like running through jello. No, not all [electrons] make it [to Region II]. Assuming some Boltman [sic] spread, some electrons do not have enough energy to make it.

We focus on the student's reference to a "Boltman spread," or, as he mentioned later on the pretest, to a "Boltman energy dist[ribution]." We interpret this reference, apparently to a "Boltzmann distribution," to mean that the student was thinking of the incident electrons as having a distribution of energy values rather than a single value. Although this idea is contrary to the problem statement, it served as the basis of the student's explanation that "some electrons do not have enough energy to make it" into Region II.

Another student engaged in a conversation with the author (via electronic mail) about the physical situation on the potential step question. The student was confused about how to account for the result that electrons would be reflected at the boundary ($x = 0$):

> S: I don't quite understand why some of the electrons would not cross over [the boundary at $x = 0$] if they had enough energy to do so. Is it because the energy is really just the expected value of the energy and so some electrons have higher energy while others have lower?

The question that the student asked suggests a line of reasoning similar to that used by the first student above. To probe more deeply this student's reasoning, the author responded by asking what the student meant by saying that "the energy is really just the expected value of the energy." The student replied as follows (boldface added by the author):

> S: What I meant by "the energy ...", was that each electron has slightly different energies. **Therefore the energy of an electron could be represented by an energy distribution similar to –or identically– psi.**... The proportion that did make it [across the boundary] would be equal to the probability of an electron to have that high of energy.

On the basis of the above response, it seems that the student interpreted the "wave-function psi" as a probability distribution of energy values. To further explain what he was thinking, the student referred to a discussion that had occurred in class. The discussion centered around the problem of interpreting a quantum state that is an equally-weighted linear combination of energy eigenstates. The student recognized that the expectation value of the energy would not be equal to either of the energy eigenvalues and that a measurement of the energy would have a 50% probability of yielding either of the two eigenvalues. However, the student seemed to interpret the incident electrons in the potential step pretest in a similar way, even though the electrons were stated as being monoenergetic. It made more sense to the student to associate "E" with the expectation ("expected") value of the energy rather than the energy of each electron, because then "the proportion [of electrons] that did make it [across the boundary] would be equal to the probability of an electron to have that high of energy."

We have documented other evidence of difficulties that students have in interpreting wave functions that are "mixed" states, which are described in Chapter 9 of this dissertation. This result is consistent with informal findings about the difficulties students have in interpreting the meaning of expectation values and eigenvalues.[63]

c) *Mistaken belief that the real part of the wave function is zero in the forbidden region of a finite potential step or barrier*

Many students applied their intuition about the classical motion of the particles to make incorrect sketches of the real part of the wave function in classically forbidden regions. In particular, students often believed that wave function is always equal to zero in any region where the total energy of the particle is less than the potential in that region. Although this argument is valid for the probability density of particles undergoing classical motion, it is not correct quantum mechanically unless the potential step or potential barrier is infinite.

Particles scattering from a potential step

In one part of the potential step question, students were asked to consider a beam of monoenergetic electrons incident on a potential step whose height was greater than the kinetic energy of the particles. (See also Figure 6-4.) Almost all students recognized that the electrons would all be reflected at the potential step in this case. However, when asked to sketch the behavior of the wave function for this case, many indicated that the wave function would be zero in the region containing the potential step (Region II, or $x > 0$). Two students who made this error in their sketches gave the following explanations:

> "Since the $E < V_o$, none of the electrons make it to region II. You can't have negative *KE*, and since $E < V_o$, the electrons can never be at V_o.... Wave [function] is [non-zero] only in region I."
>
> "This is just like square well potential, the electrons bounce back."

Although the argument given by the first student would be correct for probability density in the classical regime, it is not correct for the quantum mechanical wave function of the electrons. It is interesting to note the reference made to a "square well potential" by the second student. The student seemed to treat the potential step in the same way as a perfectly rigid wall of an infinite square well, saying that "the electrons bounce back." Thus the student failed to discriminate between a finite potential and an infinite potential.

Particles scattering from a potential barrier

Similar errors arose on the potential barrier question, in which students considered a beam of monoenergetic electrons incident on the potential barrier shown in Figure 6-10. (See also Figure 6-4.) The students were asked to draw qualitatively correct graphs of the real part of the wave function for the case in which the total energy of each electron is less than V_o. After standard instruction, many students incorrectly believed that the real part of the wave function would be equal to zero inside the potential barrier and seemed to use reasoning from the classical mechanics of particles to arrive at their answer.

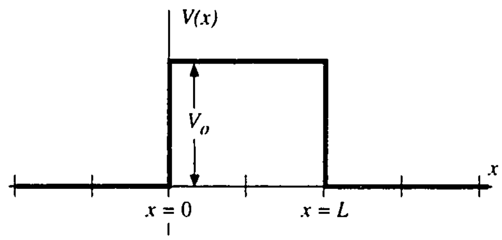
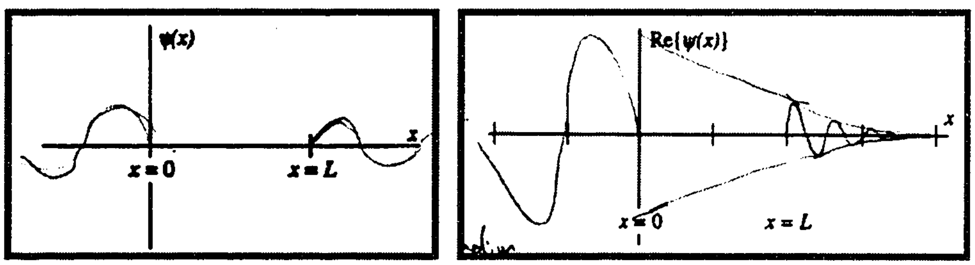


Figure 6-10: Potential energy graph shown on the potential barrier question.



(a) (b)

Figure 6-11: Examples of incorrect student graphs that show the wave function to be equal to zero inside the potential barrier.

Two examples of incorrect student graphs are shown in parts (a) and (b) of Figure 6-11 above. (These examples are also discussed in the preceding chapter.) Both students correctly predicted that some electrons would tunnel through the potential barrier but thought that the wave function would be equal to zero inside the barrier. These students gave the following explanations:

> "Electrons will end up in region $x > L$. Although a classical physical body would not make it into the region $0 < x < L$, the wave function will propagate through to the region $x > L$ and therefore there would be a probability of finding an electron there."

> "In the region of the potential barrier, the wave function exponentially decays, but has no real component. Past $x = L$ the real component appears again."

The first student seemed to believe that the wave function would be equal to zero inside the barrier because "a classical physical body would not make it into" that region. The second student explicitly stated that the *real part* of the wave function is zero inside the barrier and non-zero outside the barrier. Yet another student expressed similar reasoning when she said, "The 'real part' [of $\psi(x)$] means where a particle can physically exist, I think." Explanations such as these indicate a strong belief that the "real part" of the wave function would be non-zero in classically allowed regions (where a "classical physical body" could exist) and zero in classically forbidden regions.

d) Summary of difficulties

The results presented above indicate that quantum mechanics students frequently apply ideas from classical mechanics in order to predict or account for the behavior of bound states and scattering states. When approaching a problem that involves scattering from a potential step or barrier of finite height, many students believe that the wave function (or the real part) is always zero inside the classically forbidden region. In situations that involve stationary bound states, many students try to account for the behavior of low-energy eigenstates, including the ground state wave function, by considering the classical motion of a particle in the potential well. All of these difficulties seem to inhibit student understanding of the behavior of matter in the quantum mechanical regime.

3. FAILURE TO IDENTIFY THE SMALL-MOMENTUM (LARGE-DE BROGLIE WAVELENGTH) LIMIT IN WHICH CLASSICAL MECHANICS SHOULD BE ABANDONED FOR QUANTUM MECHANICS

The difficulties discussed in the preceding section involve the indiscriminate use of ideas from classical mechanics to predict or account for quantum mechanical behavior. The presence of such difficulties suggests a more general failure to recognize that there are separate regimes in which either reasoning from classical physics or quantum mechanics would apply.

We designed an interview protocol in order to probe in depth the ability of students to decide under what conditions it is appropriate to treat a simple physical situation as belonging in one regime or the other. In this section we describe the students who participated in the interviews, an overview of the interview protocol, and specific difficulties that were identified. (The entire protocol is described in detail in Appendix B.)

Description of interview population. A total of nine students from a variety of instructional levels volunteered for these interviews. Three students had completed the sophomore-level course in modern physics. These students had been introduced to the concept of the wave function as a probability amplitude. Three other students had completed the junior-level courses in quantum mechanics, and the remaining three students were physics graduate students who had completed at least two quarters of quantum mechanics at the graduate level. All of the students were at or above the mean of their respective classes.

Overview of interview protocol. Each interview began by asking the student to consider the translational motion of a 1 g pebble that is given an initial speed of 4.2 cm/s in a rigid, frictionless box of length 20 cm. The student was told to assume that the pebble collides elastically with each end of the box. First, the student was asked to describe in words the resultant motion of the pebble. The student was then asked to imagine conducting the experiment in a darkened room and that a light bulb flashed on and off with random time intervals between flashes. At this point the student was given the task of predicting at which locations in the box, if any, the pebble would more likely be located at a time when the bulb flashed. The student was then asked to draw a graph of the probability density of the pebble. (This task is essentially identical to the first part of the written multiple-choice question shown earlier in Figure 6-3.) Next, the student was told to imagine cutting the pebble in half, setting one of the pieces in motion with the same initial speed as before, and then describing the probability density of the new pebble. Finally, the student imagined repeating this process, cutting the pebble in half each time the experiment is performed, and was asked whether the probability density would ever be different from what it was for the original 1 g pebble.[64]

For the case involving the original 1 g pebble, the students were expected to recognize that the motion of the pebble can be described classically and that the pebble would have equal probability of being found anywhere in the box. The probability density of the 1 g pebble would therefore be uniform. For subsequent experiments in which pebbles of smaller and smaller mass are used, the students are expected to recognize that at some point reasoning from classical mechanics would no longer apply to the probability density of the pebble.

Almost all of the students who were interviewed drew a correct graph of probability density for the 1 g pebble. (An explanation for this result, which is different from those from the written version of the same question, is addressed later in this section.) However, it was much more difficult for students to recognize that the classical result would not always be valid. In particular, many had difficulty recognizing that reasoning from quantum mechanics would be appropriate if the de Broglie wavelength of the pebble were comparable to the length of the box. Below we describe some of the specific difficulties that were identified.

a) Failure to recognize that the classical result for probability density is not valid for smaller and smaller values of momentum

The students in the sophomore-level modern physics course incorrectly said that the probability density for the pebble would remain uniform for all values of the mass. These students based their prediction on the given information that the speed of the pebble never changed. Although this reasoning is correct for the 1 g pebble, the student continued to use this reasoning even when considering the hypothetical situation of a "pebble" with the same mass as an electron. For example, one student explained: "If [the mass is] just really really really small, I still think it's going to be equal distribution over the length of the box." When asked to consider replacing the pebble with an actual electron (and when told to ignore the effect of the charge of the electron), the student gave the same incorrect prediction.

One of the undergraduate quantum mechanics students recognized that classical mechanics would eventually break down but failed to recognize that the probability density would not remain uniform. For the hypothetical case of a pebble of mass m_e, the student correctly stated that for such a pebble to remain inside the box (having a minimum value of uncertainty in position Δx) it must have a minimum momentum given by the Heisenberg uncertainty principle $\Delta x \, \Delta p \geq \hbar/2$. However, when the student was asked to assume that the momentum of the pebble was equal to its minimum value, he believed that the probability density would be uniform rather than the same as that for the ground state wave function: "Δx would be as large as it can be.... You would have no preferred position for it."

b) Failure to express correctly the limit in which classical mechanics should be abandoned for quantum mechanics

All six students who had taken undergraduate-level quantum mechanics before the interview correctly recognized that classical mechanics would eventually break down for sufficiently small values for the mass of the pebble. Many also recognized the existence of a limit in which results from quantum mechanics would be consistent with those from classical mechanics. However, only one of the six students could correctly express this limit in terms of the length of the box and the de Broglie wavelength of the pebble.

One student who recognized the importance of de Broglie wavelength incorrectly believed in this case that it was necessary to compare the de Broglie wavelength to the size of the *particle*, not the length of the box. The student explained as follows:

> "When the mass gets really really small, the de Broglie wavelength is going to approach the size of the actual object.... And then we're going to start to see quantum mechanical effects."

Many other students failed to recognize that the de Broglie wavelength of the pebble was an important parameter in determining the importance of quantum mechanical effects. Some seemed to believe that reasoning from classical mechanics would be valid except only for fundamental particles, a term that some students interpreted as "quantum mechanical particles." Other students recognized the existence of a limit in which results from quantum mechanics correspond to those from classical mechanics, but they could not correctly express the limit in terms of the de Broglie wavelength or any other relevant parameter.

c) Summary of difficulties

The interview protocol described above allowed us to probe the ability of students to express and apply the large-de Broglie wavelength (low-energy) limit in which classical mechanics should be abandoned for quantum mechanics. Most students correctly stated that classical mechanics would break down for sufficiently small values for the mass of the pebble and recognized the need to use ideas from quantum mechanics. However, many students failed to recognize that classical mechanics would not be valid if the de Broglie wavelength of the pebble were the same order of magnitude as the dimensions of the box. As a result, these students were unable to express the limit in which quantum mechanical effects would become important.

Commentary. The students in the pebble-and-shoebox interviews were asked to describe the motion and probability density for a 1 g pebble in the rigid, frictionless shoebox. All answered correctly that the pebble would have no preferred location inside the shoebox at a given instant. In contrast, on a written, multiple-choice version of essentially the same task, only 60% of the students from a large sophomore-level modern physics course chose the correct probability density graph. (See section 1 of this chapter.)

We account for the apparent difference in the above results to the fact that the students in the interviews were asked additional questions that seemed to help them understand the probability density of the 1 g pebble. They were first asked to describe in words the motion of the pebble. They were then asked to imagine that the pebble-and-shoebox experiment were being conducted in a darkened room and that a light bulb flashed on and off with random time intervals between flashes. Students recognized that the pebble would always move at a constant speed (assuming elastic collisions at the walls) and that, at each instant the bulb flashes, the pebble would have equal probability of being at any location in the box. This result suggests that similar tasks could be incorporated in an instructional strategy that would address difficulties in understanding probability density in the classical regime. In the following chapter we discuss some promising results from quantum mechanics classes in which this instructional strategy was tested.

D. SUMMARY

In this chapter we have described the results of interviews and written questions designed to probe student understanding of probability density in both the classical and quantum mechanical regimes. We have documented a number of specific conceptual and reasoning difficulties in describing the probability density of a particle in one dimension. Many of these difficulties seem to hinder the ability of students to predict and account for quantum mechanical phenomena.

For wave functions of highly-excited bound states in a potential well, many students make errors that reflect difficulties in understanding probability density for even the classical motion of a particle. For example, many believe that a particle would more likely be found in regions of lower potential rather than higher potential. Many students recognize that probability density is uniform in a confined region of constant potential, but some inappropriately extend this result to potential wells of any shape. Such errors indicate a critical failure to relate local kinetic energy or potential to probability density.

Several other difficulties seem to stem from the tendency to use ideas from classical mechanics to predict and account for the behavior of bound states or scattering states. For situations that involve scattering from a potential step or barrier, many students treat the wave function (or its real part) as being non-zero only in classically allowed regions. Some fail to recognize that particles can tunnel through a potential barrier. Others tend to use reasoning about the classical motion of particles in a potential well in order to determine the behavior of stationary bound states, including the ground state. Thus, for both bound states and scattering states, students often indiscriminately used ideas from classical mechanics in order to account for quantum mechanical phenomena.

Other errors reveal an overall failure to recognize whether reasoning from classical mechanics or quantum mechanics would be more appropriate for a given physical situation. These difficulties are analogous to many that were identified in the context of interference and diffraction. (These results are described in detail in Chapter 2.) Such difficulties reflect a general lack of understanding in which regime a particular set of ideas is most useful or appropriate.

The research presented in this chapter is not intended to elucidate all possible difficulties that students have in developing an understanding of the wave function. Many difficulties, in fact, are intertwined with others in attributing basic wave properties to matter, as described in the preceding chapter. The difficulties that we identified, though, seem sufficiently serious as to inhibit the ability of many students to grasp more abstract properties of the wave function. The results have been used to guide the initial development of instructional materials to address several specific difficulties. In the following chapter, we describe preliminary research undertaken to design and assess the effect of the materials on student understanding.

CHAPTER 7: ADDRESSING STUDENT DIFFICULTIES IN UNDERSTANDING DE BROGLIE WAVELENGTH AND PROBABILITY DENSITY IN THE CONTEXT OF WAVE FUNCTIONS IN ONE DIMENSION

A. INTRODUCTION

In this chapter we describe how the results from research on student understanding of local de Broglie wavelength and probability density (Chapters 5 and 6) guided the preliminary development of two tutorials. The context of the tutorials is simple scattering states and bound states in one dimension. The two tutorials—*Reflection and transmission* and *Relating classical mechanics to quantum mechanics*—each target specific difficulties students have in relating the potential $V(x)$ to both the local de Broglie wavelength and the amplitude of the wave function $\psi(x)$. The materials include pretests, tutorial worksheets, and associated tutorial homework assignments. These materials have been tested with students in junior-level quantum mechanics courses.

This chapter begins with a description of the tutorials and the instructional objectives of each. We then discuss the specific classes in which the tutorials were used and tested. Finally, we compare results from written pretest and post-test questions designed to help us assess the effect of the tutorials on student learning.

B. DEVELOPMENT OF RESEARCH-BASED TUTORIALS

In this section we describe the tutorials *Reflection and transmission* and *Relating classical mechanics to quantum mechanics.* The general instructional strategy that was used in designing both was to engage students intellectually, to pose carefully structured sequences of questions that focus on crucial concepts, and to help the students themselves confront and resolve specific difficulties. Both of the tutorials were designed to address conceptual and reasoning difficulties students have in relating local de Broglie wavelength to local kinetic energy. In addition, *Relating classical mechanics to quantum mechanics* focused on probability density in both the classical and quantum mechanical regimes.

1. *DESCRIPTION OF THE TUTORIAL* REFLECTION AND TRANSMISSION

The tutorial *Reflection and transmission* begins with a series of questions about the classical motion of particles in a region of space where the potential is not uniform. The students are then guided to recognize how the behavior of the particles differs from that of quantum mechanical particles in the same potential.

a) Classical motion of particles incident on a potential "ramp"

The students first consider a beam of monoenergetic electrons incident on an extended potential step. As shown in Figure 7-1(a) below, the potential in the region $x < 0$ is equal to zero and over the interval $0 < x < x_o$ increases linearly from zero to a constant value V_o. The students are asked to compare the total energy E of the electron to the kinetic energy of the electron in the regions $x < 0$ and $x > x_o$. They are then asked to describe the motion of electrons having energy E as they move from left to right through the region shown. They are told to treat the electrons as particles undergoing classical motion.

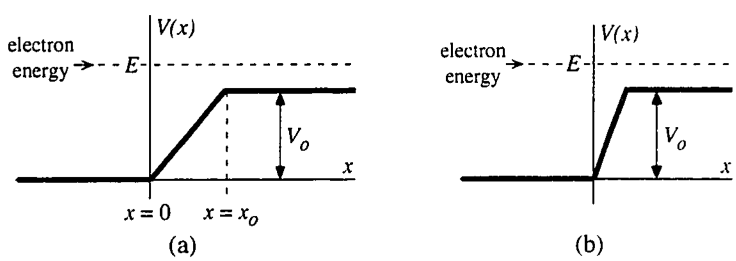


Figure 7-1: Potential energy diagrams that show the potential linearly increasing from zero to V_o (a) over a distance from $x = 0$ to $x = x_o$, and (b) over a distance shorter than x_o.

Students interpret the potential energy diagram to determine the magnitude and direction of the net force on an electron at all locations shown in the graph. From the relationship $F_x = -dV/dx$, they recognize that in the region $0 < x < x_o$ the net force is in the $-x$ direction and has magnitude $|F_x| = (V_o - 0)/(x_o - 0) = V_o/x_o$. Students recognize that their description of the net force is consistent with their knowledge of the motion of the electrons.

The students continue by considering the case in which the potential energy increases from zero to V_o over a smaller interval than before, as shown in Figure 7-1(b). Students come to recognize that in the region in which the potential is varying the net force on the particle is larger in magnitude than what it was in the original situation. They determine that the electrons experience the same change in momentum as in the original case. Students conclude that the net force in the new case must be exerted over a shorter time interval than in the original case.

Finally, students consider the limiting case in which the potential changes abruptly from zero to V_o at $x = 0$, as shown in Figure 7-2. In this case students are guided to treat the magnitude of the net force on the electrons as being extremely large and the duration of the net force as being extremely small. They are asked whether any of the electrons that are incident from Region I ($x < 0$) would reach Region II ($x > 0$). Using strictly classical arguments they determine that all of the electrons would cross from Region I to Region II, and that their kinetic energy in Region II would be less than that in Region I.

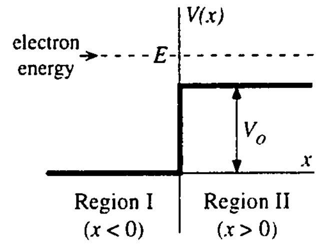


Figure 7-2: Limiting case of a potential ramp in which the potential is equal to zero in the region $x < 0$ and V_o in the region $x > 0$.

b) Quantum mechanical behavior of electrons incident on a potential step

In the next part of the tutorial, students are posed a series of questions about the actual (wave-like) behavior of electrons incident on the potential step shown in Figure 7-2. They again consider the case in which electrons are incident from the left (*i.e.*, from Region I) with each electron having a total energy E that is greater than V_o. First, students compare the de Broglie wavelength of the electrons that reach Region II ($x > 0$) to the de Broglie wavelength of the original electrons in Region I ($x < 0$). Through discussion with one another, they come to recognize that since the kinetic energy in Region II is less than that in Region I, the de Broglie wavelength must be larger in Region II than in Region I. Second, students use their comparison of the de Broglie wavelengths in Regions I and II to sketch a qualitatively correct graph of the real part of the wave function. Third, students predict whether all of the incident electrons would reach Region II. Many students fail to recognize that some of the electrons are reflected at the potential step; this error is addressed through the following sequence of activities.

On the basis of the wave-like nature of electrons students are led to make an analogy between the electrons incident on the potential step shown in Figure 7-2 and a pulse on a spring that is incident on a junction with a different spring. A demonstration is shown using two different springs. First, the springs are stretched to (approximately) the same tension and pulses are created along each. Students observe that under these conditions the wave speeds of the springs are different.

The springs are then joined end-to-end, and the students predict what will happen when a pulse is incident on the junction from each side. Many incorrect ideas are elicited at this point, including the belief that reflection will not occur. (This difficulty is discussed in detail in Chapter 5.) Students then observe the demonstration and recognize that reflection and transmission occurs at the junction in all cases. On the basis of the analogy they made between pulses on springs and electrons incident on the potential step, students recognize that electrons could be reflected and transmitted at $x = 0$ when the total energy E of the electrons is larger than the potential on each side of the step.[65]

c) *Algebraic expressions for the wave function of electrons incident on a potential step*

The tutorial worksheet concludes with questions designed to guide students through the reasoning necessary to determine algebraic expressions for the wave function $\psi(x)$ in the two regions (I and II) of the potential step, shown in Figure 7-2. Students consider two different cases: electrons incident from Region I on the potential step (i) with energy $E > V_o$, and (ii) with energy $E < V_o$. The intent was to help students relate what they had done in parts a and b to the mathematical formalism. The tutorial leads students to apply the appropriate boundary conditions at $x = 0$ for both $\psi(x)$ and the slope of $\psi(x)$ and then to determine expressions for $\psi(x)$ in Regions I and II.[66]

2. *DESCRIPTION OF THE TUTORIAL* RELATING CLASSICAL MECHANICS TO QUANTUM MECHANICS

Like *Reflection and transmission,* the tutorial *Relating classical mechanics to quantum mechanics* begins by having students consider the classical motion of a particle. The initial focus, however, is on probability density. The students then extend their results for the probability density in the classical case to an analogous quantum mechanical situation. They determine qualitatively how the amplitude of $\psi(x)$ varies for a particle in a highly-excited state of the same potential. The students are also guided to recognize how the local de Broglie wavelength is related to the potential. As a result, they go through the reasoning necessary to determine qualitatively how both the amplitude and curvature of a bound state wave function vary with potential.

a) Probability density for the classical motion of a bead moving on a frictionless wire

Students start by considering a bead that is free to slide on a straight wire tied to a wall, as shown in Figure 7-3(a). The students are told to assume that there is negligible friction between the bead and wire, and that the bead always collides elastically with the wall. They are asked to describe the motion of the bead in words and then to compare the amount of time that the bead spends on two equal-length segments of the wire, as shown in Figure 7-3(b). Most students recognize that, because the bead moves with greater speed through segment 1 than through segment 2, it spends less time in segment 1 than in segment 2.

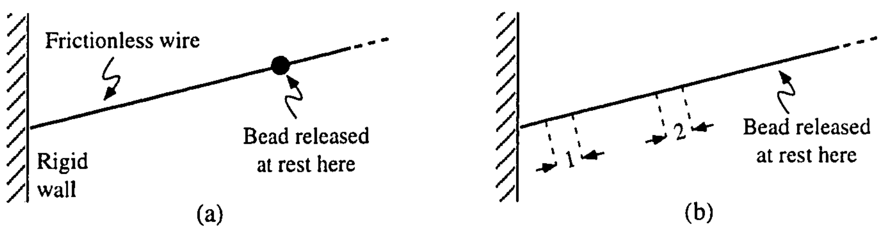


Figure 7-3: Two side view diagrams that illustrate a bead released from rest on a frictionless wire tied to a rigid wall.

After answering the above questions, students are asked a series of questions that is very similar to that used in individual student interviews on the pebble and the shoebox (discussed previously in Chapter 6). They imagine that the bead is set in motion in a darkened room and that a flash bulb in the room flashes on and off with random time intervals between the flashes. The students determine that, at an instant when the bulb flashes, they would have a greater probability of finding the bead in segment 2 than in segment 1. (See Figure 7-3(b).)

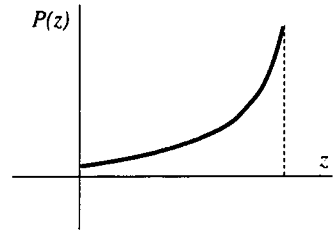


Figure 7-4: Qualitatively correct graph of *P(z)* *vs. z* for a bead on the frictionless wire shown in Figure 7-3.

Students quantify their results and come to recognize that the probability *P(z)dz* of finding the bead along a segment of wire of length *dz* is proportional to the time interval required for the bead to traverse that segment. They find that the probability density *P(z)* for the bead is inversely proportional to the speed of the bead *v(z)* as a function of location along the wire. Students use this result in order to draw a qualitatively correct graph of probability density *P(z)* as a function of position *z* along the wire. An example of such a graph is shown in Figure 7-4.

b) Quantum mechanical analogue to the bead on the frictionless wire

In the second part of the tutorial, students consider a quantum mechanical analogue to the situation involving the bead on the frictionless wire. They first describe the physical situation in terms of the potential energy function *V(z)* for the bead. They are then asked to imagine a particle in a potential well of the same shape. The students are specifically told to consider the case in which the particle is in a quantum mechanical eigenstate with an energy much higher than the ground state energy. They are then led through the instructional sequence described below by which they determine a qualitatively correct wave function for the particle.

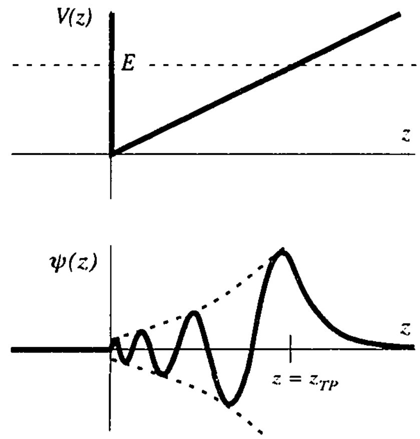


Figure 7-5: Graphs of the potential $V(z)$ and wave function $\psi(z)$ for the quantum mechanical analogue of the bead.

First, students draw a graph of the potential $V(z)$ for the bead, shown at right in Figure 7-5. (The correct graph is identical to that shown on the V-well question, which appears on the pretest corresponding to this tutorial. See Appendix B for a description of this question.) The students verify that their graph is consistent with their knowledge about the motion of the bead. Next, for the quantum mechanical case, students are asked to relate the curvature of $\psi(z)$ to the potential $V(z)$. Through discussions with each other and with the tutorial instructors, they come to recognize that the local de Broglie wavelength increases as the kinetic energy decreases. This result implies that the nodes in the wave function would gradually become farther apart as the kinetic energy decreases from left to right inside the well. Finally, students combine their results from the first part of the tutorial, in which they sketched the probability density for the classical motion of the bead, together with their considerations about de Broglie wavelength. Using both sets of results, students determine a qualitatively correct wave function $\psi(z)$ for a highly-excited energy state of the potential well. Figure 7-5 illustrates an example of a sketch that would be accepted as correct.

Students conclude the tutorial by critiquing a student statement that is designed to elicit a common difficulty about the amplitude of the wave function:

> "A particle in a potential 'well' prefers to be where the potential energy is lowest, so that is where the amplitude of the wave function is largest."

Students reflect upon their results for the probability density for the bead on the wire, and recognize the flaw in the reasoning implicit in the above statement. They are then guided to check that their results from the tutorial are consistent with mathematical expressions for the wave function derived on the basis of the WKB semi-classical approximation.

c) Tunneling through a finite potential barrier

Tutorial homework exercises were designed to accompany the tutorial *Relating classical mechanics to quantum mechanics.* The homework used in the Winter class of Sequence 1 (Winter 1997) was slightly different than that used in the Autumn class of Sequence 2 (Autumn 1997). The homework was optional in each case. The exercises used on each version were designed to address difficulties in relating local de Broglie wavelength to the potential in the context of scattering from a finite rectangular barrier. Each version included questions to help students reinforce and extend their results from tutorial.

In both Sequences 1 and 2, not all of the students who worked through the tutorial completed the optional homework. Our discussion will involve those students in each class who took the tutorial pretest, completed the tutorial worksheet (but not necessarily the homework), and took the post-test.

C. CONTEXT FOR ASSESSMENT OF THE TUTORIALS

In this section we discuss the courses in which we conducted the research described in this chapter. For each class, we specify which tutorials were used.

1. Description of student populations

The research described in this chapter involved the two-quarter sequence of junior-level quantum mechanics courses at the University of Washington. (The general format of these courses is described in Chapter 1.) Research was conducted during three consecutive sequences: Autumn 1995–Winter 1996 (Sequence 0), Autumn 1996–Winter 1997 (Sequence 1) and Autumn 1997–Winter 1998 (Sequence 2).

a) Sequence 0 (Winter 1996)

In Sequence 0 only standard lecture instruction took place. Neither *Reflection and transmission* nor *Relating classical mechanics to quantum mechanics* were used.

b) Sequence 1 (Autumn 1996–Winter 1997)

In Sequence 1, both tutorials were conducted in the lecture room during regularly scheduled class time. For each class meeting time in which a tutorial was used, the class began with a 10-

minutes pretest. For the remainder of the class time the students worked collaboratively through the worksheets. The tutorials were facilitated by the course instructor and by members of the Physics Education Group.

The tutorial *Reflection and transmission* was tested only once in the Autumn class of Sequence 1 (Autumn 1996). It was used after the instructor had covered applications of the Schrödinger equation in one dimension. The tutorial *Wave properties of matter,* designed to address student difficulties in the context of interference and diffraction of matter (Chapter 4), had also been used before *Reflection and transmission.*

Most of the students from the Autumn class of Sequence 1 later enrolled in the Winter class of the same sequence (in Winter 1997) and worked through the tutorial *Relating classical mechanics to quantum mechanics.* This second tutorial was used after standard instruction on the WKB semi-classical approximation and other advanced topics.

c) Sequence 2 (Autumn 1997)

In Sequence 2, the overall implementation of the tutorials was the same as that done in Sequence 1. However, the course syllabus was changed slightly from Sequence 1 to Sequence 2. The most substantial change was that the WKB semi-classical approximation was covered in the autumn quarter instead of the winter quarter. This decision was made by the instructor for the course, who had also taught during Sequence 1. The intent was to provide students the opportunity to develop a stronger qualitative understanding of wave functions before covering situations in which the solution to the Schrödinger equation can be determined analytically (*e.g.*, the square well, harmonic oscillator, and hydrogen atom). In order to adapt to the new syllabus, it was decided to use the tutorial *Relating classical mechanics to quantum mechanics* in the Autumn class of Sequence 2. Due to time constraints, this tutorial followed *Wave properties of matter* without using *Reflection and transmission.*

2. *COMPARISON OF OVERALL ACADEMIC PERFORMANCE OF STUDENTS IN THE STUDY*

The fact that different classes used various combinations of the tutorials at different stages of instruction complicates the analysis of the results. We decided it was important to be able to assess the overall academic ability of the students in each class in the study. In order to accomplish this, we analyzed student responses to similar examination questions that were written by the course instructor in the Autumn classes of Sequences 1 and 2 (Autumn 1996 and

Autumn 1997). We also determined the total score earned by each student on each examination in answering questions that were written by the lecturer on topics other than those addressed in the tutorials. We compared the distribution of these total scores for the Autumn and Winter classes of Sequences 1 and 2. (These results are described in detail in Appendix C.)

Our results indicated that the overall performance of students in the Autumn class of Sequence 1 (Autumn 1996) was very similar to that of the students in the Autumn class of Sequence 2 (Autumn 1997). We therefore treat both populations of students as approximately equal in overall ability. In Autumn 1995 (Sequence 0) there were no examination questions that were similar to those used in the other Autumn classes, so we did not have the information necessary to make a comparison between the Autumn 1995 students and those in the other Autumn classes. However, we feel it is reasonable to assume that the student populations from all three Autumn classes (including those from Sequence 0, in Autumn 1995) were approximately equal in overall ability.

Likewise, there were similarities among the groups of students from the Autumn classes of Sequence 0 and 1 who went on to finish their respective sequences. In each of these Autumn classes, about 65%-70% of the students from the highest quartile, 60% of those from the middle half, and 15% of those from the lowest quartile went on to take the Winter class of the same sequence. We therefore regard the students from the Winter classes of Sequences 0 and 1 as having slightly better overall academic ability than those from the Autumn classes of the same sequences. In addition, the assumption that the Autumn classes of Sequences 0 and 1 were approximately equal in overall academic ability (as stated in the preceding paragraph) suggests that the students from the corresponding Winter classes were also approximately equal in ability. We made this assumption in interpreting the results presented in the following section.

D. RESEARCH TO ASSESS THE EFFECTIVENESS OF THE TUTORIALS

In this section we describe the results from research to try to assess the effect of the tutorials *Reflection and transmission* and *Relating classical mechanics to quantum mechanics* on student understanding of local de Broglie wavelength and the local amplitude of the wave function. We were interested in probing for the persistence of difficulties in the context of scattering states and in the context of bound states. However, specific difficulties in one context were not all the same as those in the other. For example, difficulties with local amplitude were prevalent in the context of bound states but not with scattering states. Other errors were specific to the context of scattering, such as the tendency to preserve the "amplitude-to-wavelength" ratio for particles transmitted through a potential barrier. We therefore designed and administered written questions about both types of situations. The analysis of student responses allowed us to compare student performance at various stages of instruction.

Below we describe the written questions used in this part of the study, the instructional preparation of the student populations who were given the questions, and the results from the analysis of student responses. Results from questions on scattering states are presented in part 1; those on bound states, in part 2. In both parts, results that pertain to the tutorial *Reflection and transmission* are indicated by the abbreviation *"R&T;"* those that pertain to *Relating classical mechanics to quantum mechanics,* by the abbreviation *"CM-QM."*

1. EFFECT ON STUDENT UNDERSTANDING OF LOCAL DE BROGLIE WAVELENGTH IN THE CONTEXT OF SCATTERING STATE WAVE FUNCTIONS

Both of the tutorials *Reflection and transmission* and *Relating classical mechanics to quantum mechanics* included activities that were developed to help students relate local de Broglie wavelength to local kinetic energy. Below we present results that helped us to assess the effect of these tutorials on student understanding of the de Broglie wavelength in the context of scattering states. We describe pretest and post-test questions that were used in different junior-level quantum mechanics courses. We then present results from the analysis of student responses in an attempt to determine whether either or both of the tutorials seemed to address specific student difficulties.

a) Description of questions given after standard instruction and after tutorial instruction

Two variations of a question were used as a pretest (after standard instruction) and a post-test (after tutorial instruction). In each case, the problem statement describes a situation in which a beam of monoenergetic particles is incident on a potential barrier of finite height and width. On one variation of the question, referred to as the "potential barrier pretest," the potential on either side of the barrier was the same. On the other, referred to as the "tunneling post-test," the transmitted particles are located in a region in which the potential is higher than in the region containing the incident particles. The potential energy diagrams presented on both questions are shown in Figure 7-6 below.

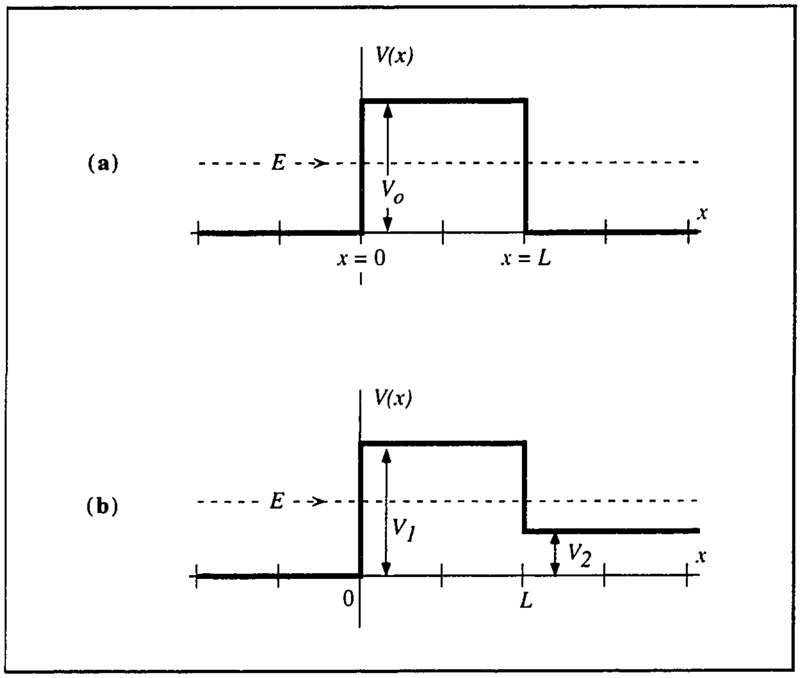


Figure 7-6: Potential energy diagrams from (a) the potential barrier pretest and (b) the tunneling post-test. On each question the students were required to sketch a qualitatively correct graph of Re{ $\psi(x)$ }.

On both the potential barrier pretest and tunneling post-test, students were required to sketch a qualitatively correct graph of the real part of the wave function. Although many criteria could be required for their graphs to be counted as qualitatively correct, the students needed to recognize that the particles in each of the classically allowed regions can be treated as propagating with a well-defined de Broglie wavelength. They were also expected to understand that the de Broglie wavelength of the transmitted particles is (a) equal to that of the incident particles on the potential barrier pretest and (b) larger than that of the incident particles on the tunneling post-test.

Note about the design of the tunneling post-test. On homework exercises associated with the tutorial *Relating classical mechanics to quantum mechanics,* students consider tunneling through a barrier like that shown in Figure 7-6(a), with the potential equal on either side. The tunneling post-test, which was given only after tutorial instruction on scattering state wave functions, required students to extend their results from tutorial to a situation in which the de Broglie wavelength for the transmitted particles was different from that of the incident particles. Both of the post-test questions allowed us to probe the prevalence of a particular student difficulty that was found to persist after standard instruction, namely a tendency to preserve the "amplitude-to-wavelength" aspect ratio of the wave function. (This difficulty is discussed in detail in Chapter 5.) The tunneling post-test was designed so that a correct answer would require students to overcome the tendency of preserving the "amplitude-to-wavelength" ratio and instead recognize that the de Broglie wavelength for the transmitted particles would be *longer* than that of the incident particles.

Commentary. The problem statement on each question did not include any reference to local de Broglie wavelength, so the students were required to recognize on their own that this feature was necessary to consider in their response. If students included written explanations in which they made an explicit comparison, then their explanation was used to determine whether their answer was correct or incorrect, and any inconsistency between their explanation and their sketch of Re{ $\psi(x)$} *vs.* x was ignored. If no written explanation was made by a given student about the de Broglie wavelengths, then a comparison of the de Broglie wavelengths was inferred from the sketch of Re{ $\psi(x)$} *vs.* x made by that student.

b) Instructional context for questions

The potential barrier pretest was used both on ungraded quizzes and course examinations. The tunneling post-test was included only on course examinations that follow tutorial instruction. Below we describe the specific contexts in which the potential barrier pretest and tunneling post-test have been posed to students. See also Table 7-1, which lists the specific conditions under which each pretest and post-test was administered.

Table 7-1: Pretest and post-test questions on scattering state wave functions posed during Sequences 0, 1 and 2 of junior-level quantum mechanics. (See Figure 7-6 for potential energy diagrams of the barriers used in the questions.)

Sequence/Qtr	Written question(s)	Instructional context
Sequence 0 Winter 1996	Potential barrier pretest	Final examination, after all standard instruction (no tutorials)
Sequence 1 Winter 1997	Potential barrier pretest	Pretest before *CM-QM* but after *R&T* and standard instruction
	Tunneling post-test	Final examination, after all instruction, including *R&T* and *CM-QM*
Sequence 2 Autumn 1997	Potential barrier pretest	Pretest prior to *CM-QM*, after standard instruction
	Tunneling post-test	Final examination, after all instruction, including *CM-QM*

(1) Sequence 0 (Winter 1996)

In Winter 1996 a question that was essentially identical to the potential barrier pretest was posed on the final examination. A total of 25 students took the examination.

(2) Sequence 1 (Winter 1997)

In the Winter 1997 class, the students were posed the potential barrier pretest immediately before the tutorial *Relating classical mechanics to quantum mechanics*. The question was posed after standard instruction on scattering and after the tutorial *Reflection and transmission* (used in the preceding Autumn quarter). The same group of students was given the tunneling post-test on the final examination, and thus after both tutorials. A total of 18 students took the pretest, completed both tutorials, and then took the final examination.

(3) Sequence 2 (Autumn 1997)

Students in the Autumn 1997 class were given the potential barrier pretest after standard instruction on the Schrödinger equation in one dimension, scattering in one dimension, and the WKB semi-classical approximation. The pretest immediately preceded the tutorial *Relating classical mechanics to quantum mechanics*. The final examination in the Autumn class included the tunneling post-test. A total of 33 students took the pretest, worked through *Relating classical mechanics to quantum mechanics,* and took the final examination.

c) Comparison of student responses

The results from the potential barrier pretests and tunneling post-tests are shown in Table 7-2 below. In Sequences 0 and 2, in which the potential barrier pretest was posed after all standard instruction and without any tutorials, student performance was very similar. The results from these classes are therefore combined in the first column of the table.

Table 7-2: Results from written questions given after standard instruction and after tutorial instruction to students in the junior-level quantum mechanics course. Results indicate the ability of students to relate local de Broglie wavelength to the potential on both sides of a rectangular barrier. When the potential barrier pretest was given after standard instruction in the Winter 1996 and Autumn 1997 classes, the student performance was similar, so these results are combined in the table.

	Potential barrier pretest ($\lambda_{left} = \lambda_{right}$)		Tunneling post-test ($\lambda_{left} < \lambda_{right}$)	
	Sequence 0 (Wi 1996) Sequence 2 (Au 1997) $N = 58$	Sequence 1 (Wi 1997) $N = 18$	Sequence 2 (Au 1997) $N = 33$	Sequence 1 (Wi 1997) $N = 18$
	After standard instruction	*After one tutorial that included a focus on wavelength*		*After both tutorials*
	No tutorials	*R&T* only	*CM-QM* only	*R&T* and *CM-QM*
Correct responses	**35% (20)**	**40% (7)**	**35% (11)**	**85% (15)**
Incorrect responses	**65% (38)**	**60% (11)**	**65% (22)**	**15% (3)**
Constant "amplitude-to-wavelength" ratio	25% (14)	55% (10)	5% (2)	0% (0)
Re{ $\psi(x)$ } not sinusoidal	15% (10)	0% (0)	10% (4)	0% (0)
Other responses	25% (14)	5% (1)	50% (16)[†]	15% (3)[†]

[†] The most common incorrect response in this category was $\lambda_{left} = \lambda_{right}$.

The first column of Table 7-2 summarizes the results from classes in which the potential barrier pretest was given after standard instruction on wave functions (Winter 1996 and Autumn 1997). Only about one third of students in each class indicated in their sketches of the wave function or in their explanations that the de Broglie wavelength would be the same on both sides of the potential barrier. The most common error was to draw a much smaller de Broglie wavelength for the transmitted particles, indicating a tendency for students to preserve the "amplitude-to-wavelength" aspect ratio of the wave function. (This error is discussed in Chapter 5.) Another common error was the failure to recognize that the real part of the wave function would have oscillatory behavior in the classically allowed regions. Both errors were approximately equally prominent in both the Winter 1996 and Autumn 1997 classes, even though the Winter 1996 class had had more instruction than the Autumn 1997 class when the potential barrier pretest was administered.

The second and third columns of Table 7-2 show the results from the potential barrier pretest and the tunneling post-test when they were administered to students who had completed either *Reflection and transmission* (in Winter 1997) or *Relating classical mechanics to quantum mechanics* (in Autumn 1997). The percentages of correct responses from these two classes were essentially the same as that from the classes that had had no tutorials. At first glance these cumulative results might suggest that neither tutorial had an effect on student understanding. It is, however, important to note that the distribution of errors was different in the two classes.

In the Winter 1997 class (Sequence 1) almost all of the errors reflected the tendency to keep the "amplitude-to-wavelength" ratio the same. (See the second column of the table.) These students had already worked through the tutorial *Reflection and transmission,* which had not included any exercises to address this difficulty. However, this issue is explicitly addressed in *Relating classical mechanics to quantum mechanics,* and only 5% of the students who had worked through this tutorial in Autumn 1997 (Sequence 2) made that particular error. (See the third column of the table.)

In the Autumn 1997 class many students failed to recognize that the real part of the wave function for a beam of free monoenergetic particles would vary sinusoidally with respect to position. This class had worked through *Relating classical mechanics to quantum mechanics* but not *Reflection and transmission,* which includes exercises that address this error. None of the students who had worked through *Reflection and transmission* (in Sequence 1) made that error in answering the pretest or post-test. (See the second and fourth columns of the table.) Thus, in the classes that had worked through one tutorial or the other (but not both), the results suggest that each tutorial helps address a different difficulty but that neither by itself addresses all difficulties.

Finally, the fourth column of Table 7-2 summarizes the results from the Winter 1997 students, who were given the tunneling post-test after working through both *Reflection and transmission* and *Relating classical mechanics to quantum mechanics.* In this class, 85% of the students (15 of 18) drew qualitatively correct graphs of the real part of the wave function, with most including correct explanations to support their sketches. Furthermore, the two most common errors that arose after standard instruction (failure to recognize that Re{ $\psi(x)$ } is sinusoidal in the allowed regions and the tendency to preserve the "amplitude-to-wavelength" ratio) were made by none of the students. Thus specific difficulties in understanding local de Broglie wavelength in the context of scattering states seem to be addressed most effectively when both tutorials were incorporated into instruction rather than just one.

Commentary. The students in the Winter 1997 (Sequence 1) class demonstrated a gain of about 45% from the potential barrier pretest (40%) to the tunneling post-test (85%). (See the second and fourth columns of the table.) It is important to ask whether this improvement could be attributed solely to the fact that these students were attempting a similar task for a second time. The results from the Autumn 1997 class, who also were given both the potential barrier pretest and the tunneling post-test, suggest otherwise. In this class the percentage of correct answers was essentially the same on both questions (about 35%). (See the first and third columns of the table.)

We have also noted that the overall academic ability of the students in the Winter classes has tended to be stronger than that of students in the Autumn classes. (See part 2 of section C in this chapter.) However, we cannot use this observation to account for the improved post-test performance of the Winter 1997 class. The success rate from another Winter class (Winter 1996, Sequence 0) on the potential barrier pretest was essentially the same as those from the Autumn classes (about 35% or 40%). (See the first two columns of the table.) We therefore account for the enhanced performance of the Winter 1997 students on the tunneling post-test to the fact that the students in this class worked through both *Reflection and transmission* and *Relating classical mechanics to quantum mechanics,* whereas the Autumn 1997 students had done only the latter tutorial.

2. *EFFECT ON STUDENT UNDERSTANDING OF LOCAL DE BROGLIE WAVELENGTH AND LOCAL AMPLITUDE IN THE CONTEXT OF BOUND STATE WAVE FUNCTIONS*

A primary objective of the pair of tutorials is to help students understand how to relate the qualitative shape of a stationary state wave function to the total energy E of the particle and the potential $V(x)$. *Reflection and transmission* was designed to address particular difficulties in relating the local de Broglie wavelength to the potential. *Relating classical mechanics to* *quantum mechanics* was developed to address difficulties not only with local de Broglie wavelength but also with probability density and the amplitude of the wave function.

In order to test the effectiveness of the tutorials, we have posed numerous examination questions that test student understanding of the qualitative shape of bound state wave functions in one dimension. Below we discuss the questions that were used and the results from the analysis of student responses.

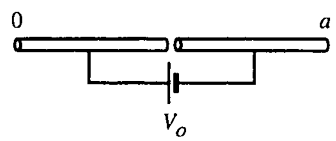
a) Description of questions given after standard instruction and after tutorial instruction

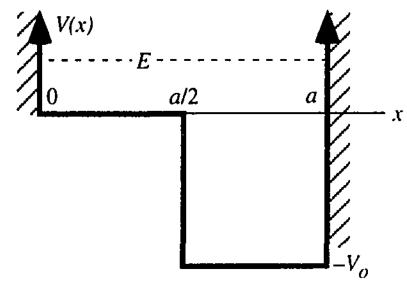
Various written questions on bound state wave functions have been posed after standard instruction and after modified instruction. Two general types of questions have been used, which are referred to as "asymmetric square well questions" and "unknown potential well questions." (See Appendix B for detailed descriptions of the questions.)

(1) "Asymmetric square well" questions

Each of the asymmetric square well questions involves a square well containing two classically allowed regions, with each region having a different, constant value of potential. On each post-test, the students were asked to specifically relate or compare (i) the local de Broglie wavelengths and (ii) the amplitudes of the wave function in the two regions. These questions are shown in Figure 7-7 and Figure 7-8 below.

Behavior of the wave function $\psi(x, t)$. Consider an electron of energy E confined to a broken wire, with potential $V = 0$ on the left side $0 \le x \le a/2$, and $V = -V_o$ on the right $a/2 < x \le a$; you may assume that V approaches infinity outside the wire.





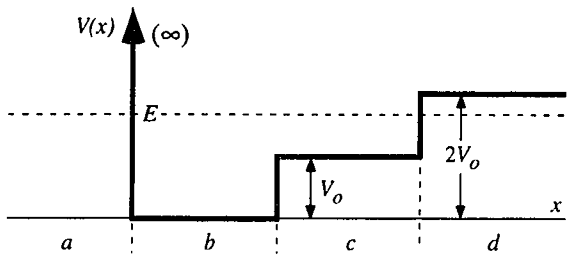
a. Write down the time dependent Schrodinger equation for this problem and a complete solution $\psi(x, t)$ that satisfies an appropriate boundary condition on the left hand side.

b. Assume that the kinetic energy of the electron on the right side of the wire is 4 times that on the left. Given the wave properties on the left side of wavelength λ_L, angular frequency ω_L, and amplitude A_L, determine the corresponding properties on the right side λ_R, ω_R, and A_R.
Hint: Make use of the WKB approximation.

c. Show that the current density $J(x, t)$ is a constant J independent of x for an eigenstate of the Schrodinger equation, *i.e.*, $\psi_n(x, t) = \psi_n(x) \exp(iE_n t/\hbar)$ and determine the constant.
Hint: Determine $J(x, t)$ on the left side using the result of part a.

Figure 7-7: Version #1 of an asymmetric square well question (broken wire question). In this chapter we discuss student responses to part b about the wavelength λ_R and amplitude A_R on the right side of the wire.

An electron is present in a region in which the potential energy $V(x)$ varies with x as shown below. The wave function $\psi(x)$ for the electron is a stationary state corresponding to a total energy E such that $V_o < E < 2V_o$.



A. Is any value between V_o and $2V_o$ allowed for the energy E, or are only certain values of E allowed? Explain your reasoning. (You do not need to solve any equations.)

B. Rank the regions *a–d* according to the de Broglie wavelength of the electron in that region, from largest to smallest *(e.g.,* $\lambda_x > \lambda_y = \lambda_z$*...)*. If λ is not well-defined in a particular region, state that explicitly and do not include that region in your ranking. Explain your reasoning.

C. For the regions in which the de Broglie wavelength is well-defined, rank those regions according to the maximum value of $\psi(x)$ in that region, from largest to smallest. Explain your reasoning.

D. In the space at right, carefully draw a qualitatively correct graph of $\psi(x)$ for the electron described above.

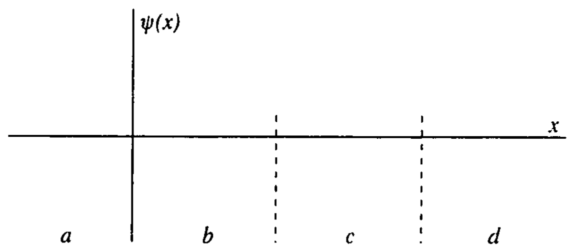


Figure 7-8: Version #2 of an asymmetric square well question (double-tiered square well question). In this chapter we discuss student responses to parts B and C about the de Broglie wavelengths and amplitudes of the wave function in the classically allowed regions *(b* and *c)*.

The questions shown above were designed to probe student understanding of the qualitative behavior of the wave function in the classically allowed regions of a potential well. In each case, students needed to recognize that for a larger potential $V(x)$, the local kinetic energy is smaller and thus the local de Broglie wavelength (and thus the distance between consecutive nodes of the wave function) is larger. Therefore, on version #1 of the asymmetric square well question (the broken-wire question, shown in Figure 7-7), the wavelength λ_R in the right part of the wire is less than the wavelength λ_L on the left side (by a factor of two). Similarly, on version #2 (the double-tiered square well question (shown in Figure 7-8), the local de Broglie wavelength is larger in region *c* than in region *b*.

Each of the asymmetric square well questions also required students specifically to compare or relate the amplitude of the wave function in the allowed regions of the well. The students needed to understand that a particle would have a greater probability of being found in the region of lower local kinetic energy than in the region of higher local kinetic energy. Probability density is equal to the absolute square of the wave function, so on version #1 (the broken wire question) the amplitude of the wave function is larger in the left half of the wire than in the right half (by a factor of $\sqrt{2}$). Similarly, on version #2 (the double-tiered square well question) the maximum value of $\psi(x)$ is larger in region *c* than in region *b*. On both questions, the reasoning described above would be valid for sufficiently high-energy states of the potential well; analysis of the responses indicated that none of the students seemed to be confused by thinking of low-energy states. (See Appendix B.)

(2) "Unknown potential" well questions

Another set of post-tests were included on course examinations. Like the asymmetric square well questions described above, these questions required students to relate specified quantities at various locations within a potential well. The questions, each about a particle in an "unknown" potential well, are described below.

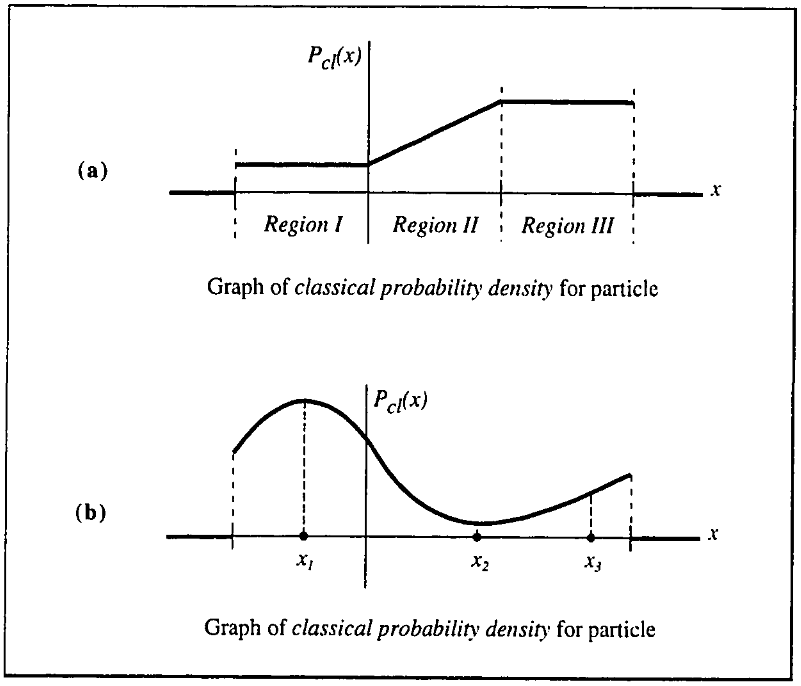


Figure 7-9: Probability density graphs from versions #1 and #2 of the unknown potential well questions, shown respectively in parts (a) and (b). The students were essentially asked to rank the labeled regions or points according to (i) the potential and (ii) the local de Broglie wavelength.

In each of the unknown potential well questions, the students were given a probability density graph for the classical motion of a particle in an unknown potential well. (Figure 7-9 above shows the graphs provided on the two versions of the question.) Students were first asked to rank the labeled points or regions according to the potential $V(x)$. They were then asked to consider the quantum mechanical analogue for a particle in the same potential well and rank the labeled points or regions according to the number of nodes (or relative spacing between nodes) of the wave function $\psi(x)$.

First, in regions of relatively high probability density the local kinetic energy of the particle is relatively low because the particle requires a relatively large interval of time to traverse that region. From this result the students conclude that at locations of relatively high probability density, the potential energy is relatively low. Therefore, the correct rankings according to the potential are $V_{III} > V_{II} > V_I$ for version #1 and $V(x_1) > V(x_3) > V(x_2)$ for version #2. (In version #1, students were not required to explain that the potential in Region II was not constant, only that it was intermediate between the potential in Regions I and III.)

Second, in regions of relatively high local kinetic energy, the local de Broglie wavelength will be relatively small and thus the nodes of the wave function will be relatively close together. On version #1, therefore, in which the students were asked to rank the regions according to the number of nodes n in that region, the correct ranking is $n_{I} > n_{II} > n_{III}$. On version #2, in which the students were instead asked to rank the labeled points according to the spacing between consecutive nodes near that point, the correct answer is that the nodes are closest together at x_2, followed by x_3, then x_1.

Note about the design of the unknown potential well questions. The unknown potential well questions were specially designed so that students were required to extend their results from tutorial to situations that they had not covered before. For instance, the initial information provided to the students was different from that given in the tutorial. Rather than begin with a description of an actual physical situation or a graph of the potential $V(x)$, students were instead given a graph of probability density for a classical situation. Furthermore, a correct answer on parts of each question required multiple-step reasoning. Students had to relate probability density to local kinetic energy, and from that point infer information about both the potential and local de Broglie wavelength. Thus students could not arrive at a correct answer simply by memorizing a previous result.

b) Instructional context for questions

Below we describe the context in which the asymmetric square well questions and unknown potential well questions were administered. In one case a question was given after standard instruction. Other questions were given after the students had worked through either *Reflection and transmission* or *Relating classical mechanics to quantum mechanics.* Yet another question was administered after both tutorials. In all cases, the questions were included on course examinations. Table 7-3 summarizes the instructional context for each question.

Table 7-3: Examination questions on bound state wave functions posed during Sequences 1 and 2 of junior-level quantum mechanics.

Sequence/Qtr	Written question(s)	Instructional context
Sequence 1 Autumn 1996	Asymmetric square well question #1 (see Figure 7-8)	Midterm examination, after standard instruction and after *R&T*
Sequence 1 Winter 1997	Unknown potential well question #1 (see Figure 7-9)	Final examination, after all instruction, including *R&T* and *CM-QM*
Sequence 2 Autumn 1997	Asymmetric square well question #2 (see Figure 7-8)	Midterm examination, after standard instruction (including the WKB approximation)
	Unknown potential well question #2 (see Figure 7-9)	Final examination, after all instruction, including *CM-QM*

(1) Sequence 1 (Autumn 1996 – Winter 1997)

Version #1 of the asymmetric square well question was given on the Autumn midterm examination in Sequence 1 (Autumn 1996). At the time of the examination, the students had had standard instruction on the Schrödinger equation in one dimension. They had also worked through the tutorial *Reflection and transmission.* A total of 25 students in the class worked through the tutorial and took examination.

In the Winter quarter of the same sequence (Winter 1997), students worked through the tutorial *Relating classical mechanics to quantum mechanics.* A total of 18 students in Sequence 1 completed the tutorial and took version #1 of the unknown potential well question, which was included on the final examination. Thus these students had worked through both *Reflection and transmission* (from the preceding quarter) and *Relating classical mechanics to quantum mechanics.*

(2) Sequence 2 (Autumn 1997)

Version #2 of the asymmetric square well question was included on the Autumn midterm examination in Sequence 2 (Autumn 1997). At the time of the examination, the students had had standard instruction on the Schrödinger equation in one dimension and the WKB semi-classical approximation. Later that same quarter, students worked through the tutorial *Relating classical* *mechanics to quantum mechanics.* On the final examination in Sequence 2 the students were given version #2 of the unknown potential well question. A total of 33 students in the class took the midterm examination (after standard instruction), completed the tutorial *Relating classical mechanics to quantum mechanics,* and then took the final examination (after the tutorial).

c) Comparison of student responses

Results from the analysis of student responses on the post-tests are presented in Table 7-4 below. Results from questions that test on the relationship between the potential and the amplitude of the wave function (or probability density) are summarized in part (a) of the table. Results from questions on the local de Broglie wavelength are shown in part (b). The combined results from each pair of questions are given in part (c).

Table 7-4: Results from post-tests on examinations given after standard instruction and after tutorial instruction in the junior-level quantum mechanics course. Part (a) of the table shows results from questions on the amplitude of the wave function (or probability density), part (b) shows results from questions on local de Broglie wavelength, and part (c) shows the combined results from both questions.

	Asymmetric square well post-tests		Unknown-potential well post-tests	
	Version #2 Sequence 2 (Au 1997) $N = 33$	Version #1 Sequence 1 (Au 1996) $N = 25$	Version #2 Sequence 2 (Au 1997) $N = 33$	Version #1 Sequence 1 (Wi 1997) $N = 18$
	No tutorials	*R&T* only	*CM-QM* only	*R&T* and *CM-QM*

Part (a): Relating amplitude of $\psi(x)$ to $V(x)$	*After standard instruction*		*After relevant tutorial (CM-QM)*	
Correct responses	**10% (3)**	**5% (2)**	**70% (23)**	**95% (17)**
With correct reasoning	10% (3)	0% (0)	50% (16)	85% (15)

Part (b): Relating local wavelength to $V(x)$	*After standard instruction*	*After one tutorial* (*R&T*)	(*CM-QM*)	*After both tutorials*
Correct responses	**20% (6)**	**80% (20)**	**60% (19)**	**95% (17)**
With correct reasoning	20% (6)	70% (18)	35% (12)†	85% (15)

Part (c): Combined results for both parts	*After standard instruction*	*After one tutorial (R&T)*	*After one tutorial (CM-QM)*	*After both tutorials*
Both parts correct	**10% (3)**	**5% (2)**	**45% (15)**	**90% (16)**
With correct reasoning	10% (3)	0% (0)	30% (10)	70% (13)

† Two of these 12 students incorrectly related potential to probability density but used correct and consistent reasoning to rank the points according to the spacing between nodes.

(1) Results from questions pertaining to the local amplitude of $\psi(x)$ (or probability density)

Part (a) of Table 7-4 can be used to assess the effectiveness of the tutorial *Relating classical mechanics to quantum mechanics* in helping students relate the potential $V(x)$ to the amplitude of the wave function $\psi(x)$ (or the probability density). The percentages in the first two columns of Table 7-4(a) show the results from questions that were given after standard instruction but before the tutorial. (*Note: Reflection and transmission* does not contain exercises to address difficulties with the amplitude of the wave function.) On each of these questions, no more than 10% of the students correctly related the amplitude in the two classically allowed regions of the well. In contrast, on the unknown potential well post-tests given after the tutorial *Relating classical mechanics to quantum mechanics,* most students correctly ranked the labeled regions or points on the graph of (classical) probability density according to potential. At least half of the students in each class gave correct rankings that were supported with correct reasoning.

It is noteworthy that the students who attempted version #1 of the asymmetric square well question in Autumn 1996 (Sequence 1) had not yet studied the WKB approximation, and those who tried version #2 of the same question in Autumn 1997 (Sequence 2) had covered this topic. Despite the additional instruction in the Autumn 1997 class (and the hint included on version #2 referring explicitly to the WKB approximation), there is little difference in the percentage of correct responses in the Autumn 1996 and Autumn 1997 classes. The larger percentages of correct answers on the unknown-potential well questions suggests that the tutorial *Relating classical mechanics to quantum mechanics* helped students understand on a qualitative level how probability density varies with potential.

(2) Results from questions pertaining to the local de Broglie wavelength

Both of the tutorials *Reflection and transmission* and *Relating classical mechanics to quantum mechanics* were designed to address specific student difficulties in relating local de Broglie wavelength to local kinetic energy. Part (b) of Table 7-4 summarizes the results from questions that pertain to local de Broglie wavelength. Different groups of students were tested after purely standard instruction, after one of the tutorials, and after both of the tutorials.

According to the results shown in the first column of Table 7-4(b), only 20% of the students in the Autumn 1997 class (without either tutorial) correctly related the local de Broglie wavelength in the two classically allowed regions of the asymmetric square well. As shown in the other three columns, on questions that were given to students who had worked through one or both of the tutorials, the percentage of correct responses is between 60% and 95%. In all cases, most of the students who gave correct answers supported them with correct reasoning. These results suggest that each of the tutorials *Reflection and transmission* and *Relating classical mechanics to quantum mechanics* seems to help students develop an understanding of the relationship between local de Broglie wavelength and potential.

(3) Combined results from pairs of questions on the local amplitude of $\psi(x)$ and local de Broglie wavelength

It is also insightful to evaluate the percentage of students who gave completely correct answers for both the amplitude of the wave function and local de Broglie wavelength. In the first two columns of Table 7-4(c) we show the percentage of students who gave correct responses on the asymmetric square well questions for both the amplitude of $\psi(x)$ and the local de Broglie wavelength. Similarly, in the last two columns we show the percentage of students who gave correct rankings on the unknown potential well questions for both the potential $V(x)$ and the nodes in the wave function $\psi(x)$.

The asymmetric square well questions were both given before students worked through the tutorial *Relating classical mechanics to quantum mechanics,* which addresses difficulties with probability density. The tutorial *Reflection and transmission* did not address such difficulties, so it is not surprising that the Autumn 1996 students who had done this tutorial performed at about the same level as those in Autumn 1997 who had had only standard instruction (10% or fewer completely correct). In Autumn 1997, the overall performance appeared to improve after working through the tutorial *Relating classical mechanics to quantum mechanics,* which is intended to address difficulties both with the amplitude of the wave function and local de Broglie wavelength. Nearly half of the students in this class gave correct responses to both questions. In Winter 1997, the students who had worked through both tutorials outperformed those in Autumn 1997, with a majority of the class giving correct answers with correct reasoning on both questions.

Commentary. As was the case in interpreting the results from the tunneling post-tests (see part 2 of section C in this chapter), it is necessary to consider whether the better performance in the Winter 1997 (Sequence 1) class is due to the fact that (i) this was the only group of students that had worked through both tutorials or that (ii) this class, being a Winter class, was academically stronger than any of the Autumn classes. In addition to the results presented earlier, the evidence described below suggests that the Winter 1997 students would not have performed as well as they did on the unknown potential well question if they had not had the tutorial *Relating classical mechanics to quantum mechanics.*

The evidence comes from the analysis of student responses to a written question in which the students were asked to sketch a possible wave function for a highly-excited state of a V-shaped well, shown in Figure 7-10. (The question is presented in its entirety in Appendix B.) This question was posed to students in the Winter 1997 class as a pretest prior to the tutorial *Relating classical mechanics to quantum mechanics.* Only about 10% of the students (2 of 18) gave responses that reflected an understanding of *both* the local de Broglie wavelength *and* the local amplitude of the wave function. Such a low percentage of correct answers suggests that serious difficulties had persisted among the students after standard instruction on the WKB semi-classical approximation, despite the greater overall academic ability of the class in comparison to those of the Autumn classes. We therefore attribute the high level of student performance on the unknown potential well post-tests in Winter 1997 to the fact that they had worked through the activities in the tutorials.

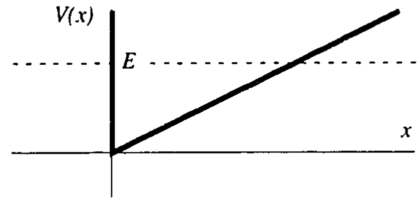


Figure 7-10: Graph of *V(x) vs. x* on the V-well question.

E. SUMMARY

In this chapter we presented results from the assessment of preliminary versions of research-based tutorials that have been designed to address student difficulties in understanding basic concepts from quantum mechanics. The research discussed in Chapters 5 and 6 revealed the presence of difficulties that persisted after standard instruction in undergraduate quantum mechanics, and in some cases even after graduate level instruction in physics. Although students in these courses had developed sophisticated skills in solving quantitative problems, they had serious difficulty answering questions that require a solid qualitative understanding of local de Broglie wavelength and probability density.

The tutorials *Reflection and transmission* and *Relating classical mechanics and quantum mechanics* were developed to help students understand how local de Broglie wavelength and the local amplitude of the wave function vary with potential. The underlying strategy of the tutorials was to engage students intellectually, to pose carefully constructed sequences of questions that direct their attention to crucial concepts, and to help them confront and resolve their errors by themselves. The results from qualitative questions posed at various stages of instruction suggest that each tutorial is helpful to students and that the level of improvement is enhanced when both tutorials are used together. Although the tutorials are still in development, the results presented in this chapter are quite promising. These results suggest that a similar approach to research-based curriculum development can be taken to help address other conceptual and reasoning difficulties that have been identified.

PART THREE:

INVESTIGATION OF STUDENT UNDERSTANDING OF THE CONNECTIONS BETWEEN PHYSICAL PHENOMENA AND COMMON FORMAL REPRESENTATIONS THAT DESCRIBE LIGHT AND MATTER

CHAPTER 8: STUDENT UNDERSTANDING OF LIGHT AS AN ELECTROMAGNETIC WAVE: RELATING THE FORMALISM TO PHYSICAL PHENOMENA

This chapter will be published in the *American Journal of Physics.* The reference is: Bradley S. Ambrose, Paula R.L. Heron, Stamatis Vokos, and Lillian C. McDermott, "Student understanding of light as an electromagnetic wave: Relating the formalism to physical phenomena."

STUDENT UNDERSTANDING OF LIGHT AS AN ELECTROMAGNETIC WAVE: RELATING THE FORMALISM TO PHYSICAL PHENOMENA

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Abstract

During an investigation of student understanding of physical optics, we found that some serious difficulties that students have with this topic may be due, at least in part, to a lack of understanding of the nature of light as an electromagnetic wave. We therefore decided to look carefully at how students interpret the diagrammatic and mathematical formalism commonly used to represent a plane EM wave. The results of this research have guided the development and modification of tutorials that address some of the difficulties that we identified. These instructional materials are an example of how, within a relatively short time allotment, a curriculum developed on the basis of research can help students relate the concepts and formal representations associated with EM waves to physical phenomena.

A. INTRODUCTION

During a long-term investigation of student understanding of physical optics, we found that many students do not develop a basic wave model that they can apply to predict and explain diffraction and interference effects.[67,68] Moreover, they have serious difficulties with some very basic concepts, such as wavelength, path length difference, and phase difference.[69] We have described elsewhere how we have used results from this research to develop a curriculum that addresses specific difficulties and that helps students develop a scalar wave model for light.[70] Such a model is inadequate, however, for some optical phenomena. For example, to be able to account for polarization, it is necessary to understand the nature of light as a transverse electromagnetic (EM) wave. In order to extend the wave model to include oscillating vector fields, students at the introductory level must be able to interpret the diagrammatic and mathematical formalism that describes a plane EM wave.

The concept of an infinite plane in which two orthogonal vector quantities each have the same magnitude and direction everywhere at an instant is highly abstract. Experienced instructors know that the diagrammatic representation of a plane EM wave commonly used in introductory textbooks is often incomprehensible to students. (See Figure 8-1.) Students also have difficulties with the mathematical expressions for the oscillating electric and magnetic fields and with the equations that relate each field to changes in the other. We therefore decided to look carefully at how students interpret the formalism and how they relate the representations to physical phenomena.[71,72] This paper describes how this study has provided a research base and guided the development of curriculum to improve student learning of this topic.

As implemented by the Physics Education Group, research, curriculum development, and instruction take place in an iterative cycle. The instructional context for most of the work described here is the tutorial system associated with the introductory calculus-based course at the University of Washington. The core of the system is provided by a set of tutorials designed to supplement the lectures and textbook of a traditional introductory calculus-based or algebra-based course.[73] The emphasis in the tutorials is on constructing concepts, developing reasoning skills, and relating the formalism of physics to the real world, not on transmitting information and solving end-of-chapter problems.

Every tutorial is preceded by a pretest that serves to *elicit* conceptual and reasoning difficulties that have been identified by research or teaching experience. During the subsequent 50-minute tutorial sessions, students collaborate in groups of 3 or 4 on worksheets designed to help them *confront* and *resolve* specific difficulties. Tutorial homework assignments help them reinforce and extend what they have learned during the tutorial session. The material covered is post-tested on course examinations by questions that require qualitative reasoning and written explanations.

It has been our experience that the study of advanced material does not necessarily deepen conceptual understanding of introductory physics.[74] Therefore, preparation of the TA's and other tutorial instructors is essential and takes place weekly in a required teaching seminar. The participants take the same pretests and work through the tutorials in the same way as the introductory students. The seminar provides an opportunity to assess the understanding of more advanced students and to set a standard by which to judge the effectiveness of the instructional materials produced. We consider that a tutorial is reasonably successful when the achievement of the introductory students on post-tests matches (or surpasses) that of the graduate students on corresponding pretests.

B. INVESTIGATION OF STUDENT DIFFICULTIES

The student difficulties that we identified in this investigation have been organized into three overlapping categories: (a) failure to interpret formal representations of a plane EM wave; (b) failure to apply the Lorentz force law to interactions involving EM waves; and (c) failure to recognize that the electric and magnetic fields in an EM wave are interdependent. Initial identification of these difficulties took place in interviews with individual students.[75] The findings were confirmed and extended through widely administered pretests and course examinations, which also served to give an estimate of the prevalence of specific difficulties.

During exploratory interviews, students who had completed the study of optics in the calculus-based course were shown a small unlit bulb, a mask with a small rectangular slit (~1 cm wide), and a screen. The students were asked (1) what they would see on the screen if the bulb were lit and placed far from the mask and (2) how their prediction would change for smaller values of the slit width.[3] For the first question, the students were expected to recognize they would see the geometric image of the slit on the screen. For narrower slit widths, we expected some reference to the appearance of diffraction effects. Some students were also asked what would happen if a polarizing filter were placed in front of the mask. They were expected to recognize that the pattern would become dimmer because the filter would transmit only some of the light.[76]

1. *IDENTIFYING FAILURE TO INTERPRET FORMAL REPRESENTATIONS OF A PLANE EM WAVE*

Responses to the first two questions indicated that many students had not developed a functional understanding of some basic ideas in geometrical optics. We have previously described how we have tried to address this problem by developing a set of tutorials to help students apply a ray model to account for some simple optical phenomena.[77,78]

The second interview question elicited elements of a rudimentary wave model from many students.[75] However, there were often serious flaws in their reasoning. There was a tendency to attribute a spatial extent to the amplitude of the wave. Some students drew diagrams of sinusoidal curves incident on a slit and based their reasoning on whether or not the light would "fit" through the slit. Some claimed that diffraction occurs only when the wavelength or amplitude is larger than the slit width because the light "has to bend in order to fit through" the narrow slit. These errors are consistent with a misinterpretation of a common representation of a linearly polarized plane EM wave. (See Figure 8-1.)

Recognizing the inherent complexity of this diagram, some instructors attribute the difficulties that students have with EM waves to the representation. During the interviews, however, it became clear that some difficulties transcend the representation and are unlikely to vanish by removing the diagram from the course. We decided to try (to the degree possible) to disentangle difficulties with concepts from difficulties with representations. We wanted to determine whether helping students interpret the representations could serve to deepen their understanding of light as an EM wave.

Pretest Question #1

Pretest Question #1 is based on the diagram in Figure 8-1, which shows a plane wave that propagates through empty space. Mathematical expressions for the fields are given. The students are asked to rank the magnitudes of the electric and magnetic fields at four designated points: *P, Q, R,* and *S.* In some versions of the pretest, students have been asked to explain their reasoning. To give a correct ranking, students must interpret either the diagram or the mathematical expressions for the fields. They must recognize that all the points are located in a plane perpendicular to the direction of propagation. Thus, the values of the fields must be the same at all four points; the correct rankings are $E_P = E_Q = E_R = E_S$ and $B_P = B_Q = B_R = B_S$.

$$\vec{E}(x,y,z,t) = E_o \sin(kx + \omega t)\hat{z} \qquad \vec{B}(x,y,z,t) = B_o \sin(kx + \omega t)\hat{y}$$

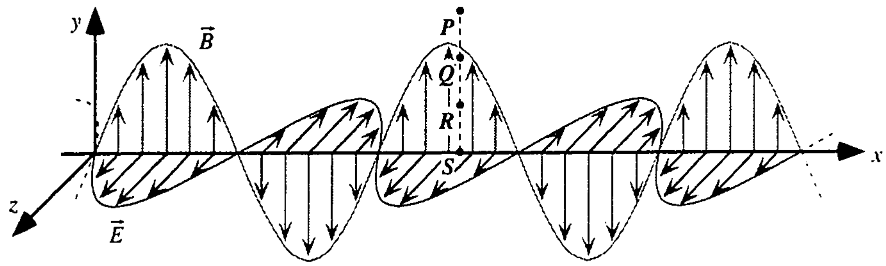

Figure 8-1: Diagrammatic and algebraic representations of an EM plane wave used in Pretest Question #1. Students are asked to rank the points *P, Q, R,* and *S* according to the magnitudes of the electric and magnetic fields at those points.

Variations of this question have been given to about 1275 students in the calculus-based course and about 130 students in the algebra-based course. Performance has been about the same, whether the question has been administered before or after traditional instruction in lecture and laboratory. About 10% of the students in the calculus-based course and in the algebra-based course have given the correct ranking. The results from all classes have been combined in Table 8-1.

Table 8-1: Results from Pretest Question #1 and Post-tests #1(a) and #1(b). The questions probe whether students understand that, at any instant, the $\vec{E}$- and $\vec{B}$-fields of a plane EM wave do not vary in a plane perpendicular to the direction of propagation. [Percentages on all tables have been rounded to the nearest 5%.]

	Introductory algebra and calculus-based courses[†]			Graduate teaching seminar
	Pretest Question #1 before tutorial* $N \approx 1400$	Post-test #1(a) after tutorial $N \approx 800$	Post-test #1(b) after tutorial $N = 355$	Pretest Question #1 before tutorial $N \approx 70$
Correct responses	**10%**	**85%**	**90%**	**70%**
Correct reasoning	NA**	55%***	70%	NA**
Incorrect responses	**80%**	**15%**	**10%**	**25%**
$E = 0$ or $B = 0$ outside sinusoidal curves	60%	10%	5%	20%
Blank or incomplete	**5%**	**< 5%**	**< 5%**	**< 5%**

† Most students were enrolled in the calculus-based physics course. About 130 students were in an algebra-based course that used the tutorials. These students took Pretest Question #1 and Post-test #1(a). Since the results in the calculus-based and algebra-based courses were similar, they have been combined.

* This column includes classes in which the material had and had not been covered in lecture. Since the results were similar, they have been combined.

** Students were not asked to explain their reasoning.

*** The version of Post-test #1(a) given in the algebra-based course did not require students to explain their reasoning.

The most common error has been to ascribe to the plane EM wave a finite spatial extent in the plane perpendicular to the direction of propagation. Many responses reflect the mistaken belief that the electric and magnetic fields are confined to the region within the sinusoidal curves. About 60% of the students have said that the electric or magnetic field is zero at point P, which lies outside both sinusoidal curves. Some have made explicit statements, such as "$P = 0$ because it is outside the boundary of the 'reach' of the *B* field," or that "$S = R = Q$, $P = 0$ [because] *P* lies off the wave where there is no field."

Students who treat the plane wave as having finite spatial extent often attribute special significance to the *x*-axis (the axis of propagation) shown in the diagram. About 35% have stated that the electric or magnetic fields are zero along this axis. For example, one student wrote: "At point *S*, *y* & *z* equal zero, so both $\vec{B}$ and $\vec{E}$ will be zero there (from the above formula)." Often these students think the magnitudes of the electric and magnetic fields increase with distance from the *x*-axis. About 40% have attributed a greater magnitude to the $\vec{B}$-field at point *Q* than at point *R*. Some have argued on the basis of the algebraic expressions for the $\vec{E}$- and $\vec{B}$-fields.[79] Many of these errors seem to be related to a failure to distinguish between the *y* and *z* coordinates and the corresponding unit vectors. For example, one student responded: "$P > Q > R > S$, since *y* [referring to $\hat{y}$] corresponds to the strength of the magnetic field, and *P* is higher than *Q*, *etc.*" Other students have treated the magnitude as decreasing with distance from the x-axis. Some even seem to believe the fields are "emitted" from the *x*-axis: "*S* is closest to the source of the wave, so it experiences the most magnetic field. The field drops in value as you get farther from the source."

Some students who have correctly stated that points *Q*, *R*, and *S* have fields of equal magnitude have confused the vector and field-line representations.[80] For example, one student wrote "$Q = R = S$ because lines have the same spacing (the field is uniform below the curve). $P = 0$ because [there are] no field lines above the curve."

Of the approximately 70 participants in the graduate teaching seminar who have taken this pretest, 70% have answered correctly. (See Table 8-1.) The pretest has also been given to more than 25 physics faculty at a national workshop, with a success rate of about 60%.

2. *IDENTIFYING FAILURE TO APPLY THE LORENTZ FORCE LAW TO INTERACTIONS INVOLVING EM WAVES*

It is often difficult to distinguish difficulties with concepts from difficulties with representations. The two are intertwined. During the interviews, many students stated that light consists of oscillating electric and magnetic fields. However, they often failed to recognize that the $\vec{E}$- and $\vec{B}$-fields in an EM wave exert forces on electric charges. To probe student understanding of the Lorentz force law in this context, we asked a pretest question about the interaction of an EM wave with electric charges in a straight antenna.

Pretest Question #2

On Pretest Question #2, the students are told that the EM wave in Figure 8-1 represents a radio wave. The electric field oscillates in a direction parallel to the z-axis, the magnetic field along the y-axis. In addition to the vector representation, algebraic expressions for the functional dependence of the fields are given. The students are asked in which direction they would orient the antenna of a portable radio for best reception.

In order to give a correct answer, it is necessary to recognize that the antenna should be parallel to the electric field (*i.e.*, parallel to the z-axis) and thus perpendicular to the direction of propagation of the wave. The electric force on the free charges in the antenna would then be parallel to the length of the antenna. In this case, the electric field would generate a maximum current in the antenna and therefore maximize the reception of the EM wave by the portable radio.[81] The force due to the magnetic field is perpendicular to the antenna and its effect on the charges can be neglected.

Table 8-2: Results from Pretest Question #2 and Post-test #2, which probe the ability of students to apply the Lorentz force law to the interaction between an EM wave and the free charges in an antenna.

	Introductory calculus-based physics course			Graduate teaching seminar
	Pretest Question #2 before tutorial and before lecture $N \approx 670$	Pretest Question #2 before tutorial and after lecture* $N \approx 620$	Post-test #2 (parts i and ii combined) after tutorial $N \approx 255$	Pretest Question #2 before tutorial $N \approx 70$
Correct responses (with correct reasoning)	**5%**	**15%**	**50%****	**60%**
Correct responses (ignoring reasoning)	15%	35%	50%	75%

* Pretest Question #2 has also been given to about 130 students in the algebra-based course after lecture instruction. The results were similar.

** Correct reasoning on one or both parts.

This question has been given to approximately 620 students in the introductory calculus-based course after lecture instruction. As can be seen in Table 8-2, about 5% of the students answered correctly with correct reasoning before any instruction and about 15% after standard lectures. It did not seem to matter whether or not the lectures had included a demonstration in which a long, straight antenna is used to detect a radio wave.

The most common incorrect response has been to propose aligning the antenna parallel to the direction of propagation. As one student explained, "I would orient the antenna along the x-axis. This is because that's the direction of the wave, and it gets a maximum electrical and magnetic field (strong signal)." This student failed to apply the idea that an EM wave is a transverse wave. Another common error was made by the student who gave the following response: "It seems that either the y- or z-axes would be good because [the antenna] would be perpendicular to the direction of propagation." His failure to identify the $\vec{E}$-field as the relevant field suggests that he also could not distinguish between the effects of electric and magnetic interactions.

About 75% of the participants in the graduate teaching seminar and the national faculty workshop have answered the pretest question correctly. About 60% of each group have justified their responses with correct reasoning. (See Table 8-2.)

3. *IDENTIFYING FAILURE TO RECOGNIZE THAT THE $\vec{E}$ AND $\vec{B}$ FIELDS IN AN EM WAVE ARE INTERDEPENDENT*

As mentioned earlier, in some of the interviews, we asked what would happen to the pattern on a screen if a polarizing filter were placed in front of a narrow slit. The students frequently failed to recognize that the $\vec{E}$- and $\vec{B}$-fields are interdependent and that a traveling electromagnetic wave cannot consist of either field alone.

Several students treated the oscillating electric and magnetic fields in a light wave as independent entities. For example, a student correctly predicted that a polarizing filter placed in front of a single slit would decrease the intensity at the screen. He supported his answer, however, by saying that the polarizer consists of long molecular chains that form "very little [parallel] grooves.... [The] only waves of light that are allowed to go through are the ones that are moving along that line, and the ones that are moving ... perpendicular to that line will be canceled out." When asked to consider the case in which the electric field of the incident light is parallel to the "little grooves," he stated that all of the electric field would be transmitted but *none* of the magnetic field. He referred to the spaces between molecular chains as slits that "block" part of the light. Like other students in the interviews, he had an incorrect model to explain polarization. Of much greater concern, however, was his lack of recognition that the $\vec{E}$- and $\vec{B}$-fields are not independent but must be transmitted or absorbed together.

C. DEVELOPMENT OF TUTORIAL ON EM WAVES

In order to address the difficulties described above, we developed a tutorial on electromagnetic waves.[73] The tutorial assumes that students are already familiar with some basic properties of waves, such as frequency, wavelength, and amplitude.[82] These concepts are developed earlier in a series of tutorials on transverse mechanical waves.[73]

The tutorial on EM waves is preceded by the two pretest questions discussed above. In this and other instances, we have found that questions that have proved effective at eliciting student difficulties often provide a good first step for addressing them. The tutorial directs the attention of students to the pretest questions and to consideration of the interdependence of the $\vec{E}$- and $\vec{B}$-fields.

In the first part of the tutorial, the students are shown the representation of a plane EM wave propagating in the negative x-direction and are given algebraic expressions for the electric and magnetic fields. (See Figure 8-2 below.) They are asked (1) to determine the direction of propagation of the wave and (2) to rank the four labeled points *(1-4)* according to the magnitudes of the $\vec{E}$- and $\vec{B}$-fields. The first question concerns the time dependence of the fields, while the second involves their spatial dependence.

$$\vec{E}(x,y,z,t) = E_o \sin(kx + \omega t)\hat{z} \qquad \vec{B}(x,y,z,t) = B_o \sin(kx + \omega t)\hat{y}$$

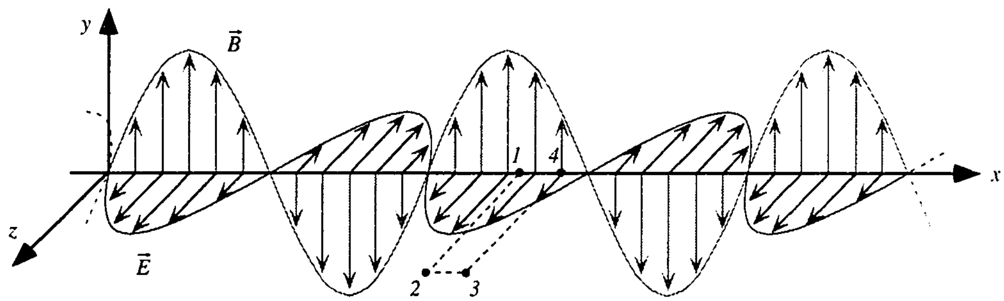

Figure 8-2: Diagrammatic and algebraic representations of an EM plane wave used in tutorial. After the students have ranked the four designated points according to the magnitudes of the electric and magnetic fields, they are guided through the reasoning necessary to apply Faraday's law to the time-varying fields at the loop *1→2→3→4→1*.

The students can infer that the wave propagates in the negative x-direction from the algebraic expressions for the electric and magnetic fields or by using the diagram to determine the direction of $\vec{E} \times \vec{B}$. To rank the magnitudes, they can use the diagram, the equations, or both. Whichever they choose, the students are asked to show that the result is also consistent with that obtained using the other representation. Thus, the students strengthen their understanding of both representations. After completing the ranking task, they are asked to sketch electric field vectors at four different points within a plane perpendicular to the direction of propagation. Most students now recognize that the vectors must have the same direction and magnitude. In this way, the students articulate for themselves a correct interpretation of the term "plane wave."

The second part of the tutorial helps students connect the graphical and mathematical formalism to the real world. They are asked to think about how an antenna can detect the presence of an EM wave. They first consider a long, thin conducting wire in the presence of a plane EM wave. The wire lies along the z-axis, as does the oscillating $\vec{E}$-field. The students are asked to determine what effect, if any, the electric and magnetic fields will have on the free charges in the wire. They then consider the same wire (in the same orientation) with a light bulb inserted in the middle. (See Figure 8-3 below.) They are asked how changing the orientation of the wire would affect the brightness of the bulb.

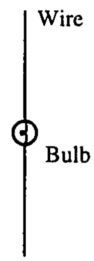

Figure 8-3: Diagram of a device consisting of a bulb connected to two long pieces of thin wire. In the tutorial, students predict how the orientation of the device relative to the electric and magnetic fields of the plane wave would affect the brightness of the bulb.

The purpose of the last part of the tutorial is to help students recognize that the electric and magnetic fields are interdependent. The focus is on the integral form of Faraday's law. On a diagrammatic representation of an EM wave, a closed rectangular loop is drawn in the x-z plane. (See Figure 8-2.) The students are asked to determine the sign of the quantity $\oint \vec{E} \cdot d\vec{l}$. They check their results for consistency with the changing magnetic flux through the surface bounded by the loop. The tutorial homework provides additional opportunities for students to extend their understanding of plane EM waves. For example, they are led to recognize the interdependence of the fields by being asked to show that the integral form of Ampère's law cannot be satisfied if the $\vec{E}$-field is identically zero, even if the $\vec{B}$-field has the usual sinusoidal spatial and time dependence.

D. ASSESSMENT OF THE TUTORIAL ON EM WAVES

In order to assess the effectiveness of the tutorial, we designed several post-test questions that were administered on midterm and final examinations. In addition to working through the *EM Waves* tutorial, the students had all had traditional lecture and laboratory instruction on this topic. Below are examples of three of the post-tests that we have used to probe student difficulties in the three categories described above.

1. ADDRESSING FAILURE TO INTERPRET FORMAL REPRESENTATIONS OF A PLANE EM WAVE

In trying to gauge how successful the tutorial on EM waves has been in helping students interpret the formalism used to represent light as an EM wave, we have given two types of post-tests. Post-test #1(a) is similar to Pretest Question #1, in that it asks for a direct interpretation of a diagrammatic representation. Post-test #1(b) requires a greater degree of transfer since students must make a prediction about a physical situation.

Post-test #1(a)

In Post-test #1(a), students are shown a diagram that represents an EM plane wave that propagates through empty space, similar to the one used on the pretest and in the tutorial. The students are asked to rank several points according to the magnitude of the electric field or, in some versions, the magnetic field. The points are different from those used in the pretest and the tutorial. An example is shown in Figure 8-4 below.

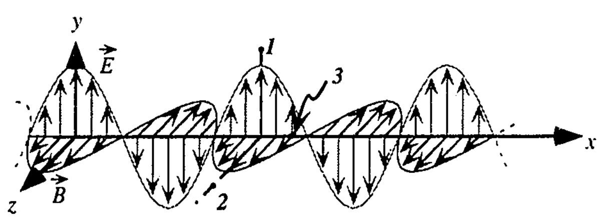

Figure 8-4: Diagram representing a "snapshot" of an electromagnetic plane wave used on several versions of Post-test #1(a).

Variations of this question have been given to about 675 students in the introductory calculus-based course and about 130 students in a section of the algebra-based course in which the tutorial was used. (The version of the tutorial in this course did not include Faraday's law.) Since the results were similar, they have been combined in Table 8-1.

After working through the tutorial, about 85% of the students gave a correct ranking, an increase from 10% on Pretest Question #1. About 55% of the students supported their answers with complete and correct reasoning. Responses that suggest a spatial interpretation of the amplitude of the wave dropped from 60% on the pretest to about 10% on the post-test. The post-test performance of the students in both courses was better than the pretest performance of the tutorial instructors.[83]

Post-test #1(b)

In Post-test #1(b), the students are shown a diagram of an EM plane wave that propagates past a device consisting of a light bulb and two long, conducting wires. (See Figure 8-5 below.) They are told that the device lies in a plane perpendicular to the direction of propagation of the wave and that the bulb glows. They are asked to predict whether shifting the device upward a specified distance would cause the brightness of the bulb to increase, decrease, or stay the same. A correct answer involves recognizing that, since the EM wave is a plane wave, shifting the device upward in a plane perpendicular to the direction of propagation would not affect the magnitude of the electric field. Therefore the current induced along the wire, and hence the bulb brightness, would not change.

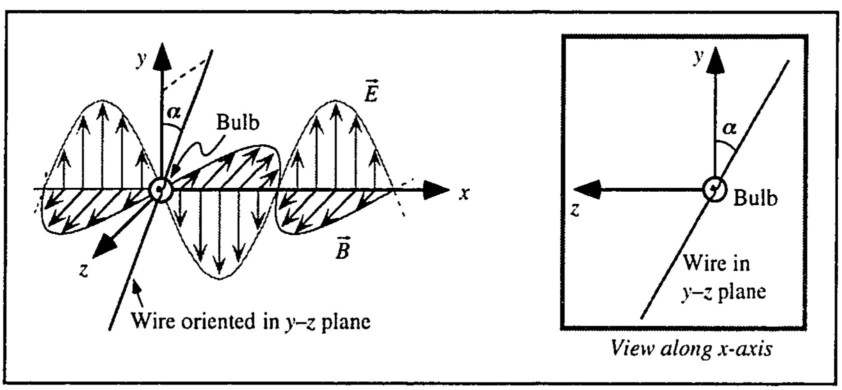

Figure 8-5: Diagram that illustrates the physical situation described in Post-test #1(b). The students are asked to predict whether shifting the wire-and-bulb device upward (*i.e.*, in the y-direction) would affect the brightness of the bulb.

Post-test #1(b) was given on an examination to 355 students in the introductory calculus-based course. The results are shown in Table 8-1. About 90% of the students gave correct responses, with 70% giving explanations that showed a correct interpretation of the concept of a plane wave. In contrast, about 10% of the students answered Pretest Question #1 correctly. Only 5% gave answers on Post-test #1(b) that suggest a spatial interpretation of the EM wave, as compared with 60% on the pretest.

2. *ADDRESSING FAILURE TO APPLY THE LORENTZ FORCE LAW TO INTERACTIONS INVOLVING EM WAVES*

We also designed a post-test question to determine whether students had improved in their ability to relate the Lorentz force law to the interaction of an EM wave with electric charge. To do so, they need to know that EM waves are transverse and to recognize how the orientation of the electric field is related to the motion of charge in the wire.

Post-test #2

Post-test #2 is based on a diagram in which there are three identical antennas, each consisting of long wires attached to a bulb. (See Figure 8-6 below.) The antennas have different orientations in the plane of the page. The students are told that a plane EM wave of radio frequency is propagating in the direction out of the page and that bulb A is not glowing.

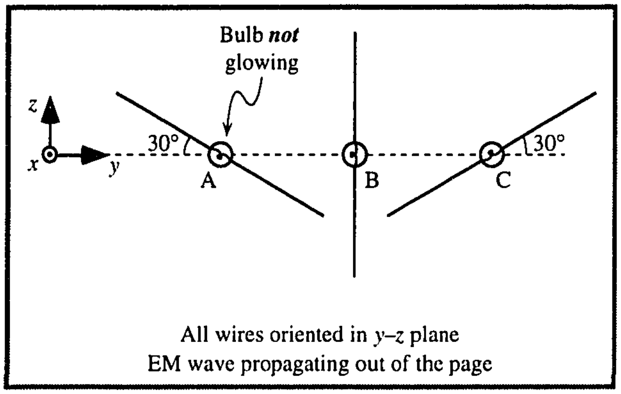

Figure 8-6: Diagram that illustrates the physical situation described in Post-test #2. The students are told that a plane EM wave is propagating in a direction out of the page and that bulb A does not glow. In part (i) of the post-test, the students are asked to compare the brightness of bulbs B and C. In part (ii), they are asked whether bulb A would glow if the device to which it is attached were oriented parallel to the x-axis.

In part (i) of the post-test, the students are asked to compare the brightness of the other two bulbs (B and C). To obtain a correct answer, they must use the observation that bulb A is not glowing to infer that the electric field of the radio wave is oriented perpendicular to bulb A's antenna. It can then be seen that the electric field makes a 30° angle with each of the other antennas. Since the parallel component of the electric field is the same for these two antennas, bulbs B and C are equally bright.

In part (ii) of the post-test, the students are asked whether or not bulb A would glow if the antenna to which it is attached were placed parallel to the direction of propagation of the wave. They need to recognize that the antenna would still be perpendicular to the electric field. The free charges in the wires would experience no force along the direction of the antenna due to the electric field. Therefore, bulb A would not glow. Although this question is in a similar context as the tutorial, the post-test questions require a longer chain of reasoning that precludes simple memorization of a result.

Post-test #2 was given on an examination to about 255 students in the introductory calculus-based course. The students had worked through the tutorial on EM waves. In part (i) of the post-test, about 65% of the students determined the direction of the electric field with correct reasoning. Most of these students arrived at a correct final comparison of bulbs B and C. The remainder made errors in geometry but used correct reasoning. In part (ii), 70% of the students correctly stated that bulb A would not glow.

In order to assess the effectiveness of the tutorial, it is useful to combine the results from parts (i) and (ii) of Post-test #2 to obtain an overall success rate. (See Table 8-2.) About 50% of the students answered both parts (i) and (ii) correctly, with correct reasoning on one or both parts. In contrast, correct answers with correct reasoning were given on Pretest Question #2 by only 15% of the students and only 60% of the participants in the graduate teaching seminar. After the tutorial, the introductory students performed nearly as well as the TA's had done before working through the tutorial.[83]

3. *ADDRESSING FAILURE TO RECOGNIZE THAT THE $\vec{E}$ AND $\vec{B}$ FIELDS IN AN EM WAVE ARE INTERDEPENDENT*

The question described below was asked on a final examination given to about 300 students in the introductory calculus-based course. The topics of EM waves and polarization had been covered in both lecture and laboratory and the students had also worked through the tutorial on EM waves. Thus, this question can serve as a post-test (Post-test #3) for the third part of the *EM Waves* tutorial, in which students show mathematically that the electric and magnetic fields in an EM wave are interdependent and cannot exist separately. The students had also worked through a tutorial on polarization that had been designed prior to the research described in this paper.[73]

Post-test #3

The students are told that linearly polarized light is incident normally on a polarizing filter and that the transmission axis of the filter is parallel to the magnetic field of the incident wave. They are asked to describe the electric and magnetic fields of the transmitted light in terms of the fields of the incident light and to explain their reasoning. Because the electric field is perpendicular to the transmission axis of the polarizer, no electric field is transmitted. Hence, no magnetic field is transmitted.

Table 8-3: Results from Post-test #3, which probes student understanding of the interdependence of the $\vec{E}$- and $\vec{B}$-fields. The table shows the percentage of students who answered correctly for both the electric and magnetic fields.

	Introductory calculus-based physics course	
	Post-test #3 after *EM Waves* tutorial and original *Polarization* tutorial** $N \approx 300$	Post-test #3 after *EM Waves* tutorial and modified *Polarization* tutorial** $N \approx 165$
Correct responses (with correct reasoning)*	**30%**	**50%**
Incorrect responses	**55%**	**40%**
Magnetic field completely transmitted	45%	25%
Blank or incomplete	**15%**	**10%**

* Essentially all students who gave the correct answer used correct reasoning.

** Traditional lecture and laboratory instruction had also been completed.

Only 30% of the students gave correct answers for both the transmitted electric and magnetic fields. The most common incorrect answer, given by 45% of the class, was that the magnetic field is completely transmitted. (See Table 8-3.) For example, one student who responded correctly for the electric but not the magnetic field stated: "100% of B field will be transmitted, $B_{trans} = B_o \cos 0°$. None of E field will be transmitted because it is $\perp$ to direction of polarizer, $E_{trans} = E_o \cos 90° = 0$." Such responses revealed the prevalent belief that the $\vec{E}$- and $\vec{B}$-fields of an EM wave are independent entities.

E. MODIFICATION OF TUTORIAL ON POLARIZATION

The results from the examination question indicated that many students could not apply their experience with the field equations in the context of polarization. The failure to recognize the interdependence of the $\vec{E}$- and $\vec{B}$-fields of an EM wave had been elicited by questions involving polarizers. Therefore, it seemed that polarization might be a useful context for addressing this difficulty. We modified the *Polarization* tutorial so that it reviews and reinforces ideas developed in the *EM Waves* tutorial.

The original version emphasizes the transmission of light through polarizing filters. The students find that the orientation of the polarizing filter affects the amount of light transmitted. They account for their observations by assuming that the polarizing filter has a transmission axis and that the component of the incident electric field that is parallel to this axis is transmitted. The students also develop a general expression relating the ratio of the amplitudes of the transmitted and incident electric fields to the angle between the incident electric field and the transmission axis of the polarizer.

In the modified tutorial, the students are also asked to consider the incident and transmitted *magnetic* fields. They find that the amplitudes of the $\vec{E}$- and $\vec{B}$-fields are reduced by the same factor. They can infer that neither field can be transmitted alone.

F. ASSESSMENT OF THE MODIFIED TUTORIAL ON POLARIZATION

Post-test #3 was administered as a post-test on the modified *Polarization* tutorial on a course examination to 165 students in the calculus-based course. These students had not been given this question before. As described earlier, however, this question had been used in other classes before the *Polarization* tutorial was modified. The results obtained helped establish a baseline for assessing the effectiveness of the modified *Polarization* tutorial and in this sense Post-test #3 has also served as a pretest.

As shown in Table 8-3, about 50% of the students recognized that no electric or magnetic field is transmitted (compared with 30% when the question was used as a pretest). The incorrect answer that the magnetic field is completely transmitted was given by 25% of the students (compared with 45% on the pretest). Thus, modifying the tutorial to include questions about transmission of both the $\vec{E}$- and $\vec{B}$-fields seems to have led to better understanding of their interdependence and inseparability.

G. RESULTS FROM PILOT SITES

For physics education research to be useful to the community of physics instructors, the findings must be reproducible beyond the institution in which they were obtained. The tutorials are being used at other universities and at two- and four-year colleges. Several of these institutions serve as pilot-sites at which we can assess the effectiveness of the materials in different instructional settings. Below we give an example from another large research university in which the tutorial on EM waves has been used.

Pretest Question #1 has been given in multiple-choice form to about 930 students in the calculus-based course. The distractors were based on our experience with open-ended questions. The success rate has been about 15% with the same distribution of difficulties as among our students. After working through the tutorial, about 400 students were given multiple-choice versions of Post-tests #1(a) and #1(b) on a final examination. The results were similar to those of our own students when reasoning is ignored: 85% on Post-test #1(a) and 75% on Post-test #1(b) as compared to 85% and 90%, respectively, at our university. Although it is unlikely that all students at the pilot site who gave correct answers used correct reasoning, the tutorial appears to have also been effective with those students, as has been the case with other tutorials at the pilot sites.[84]

H. CONCLUSION

Instruction in waves and optics relies heavily on the use of abstract representations of physical quantities. The diagrammatic, graphical, algebraic, and vector representations that are introduced are intended to help students to account for physical situations in terms of the concepts of physical optics. A new representation, however, is often accompanied by its own set of difficulties. One option is to avoid possible complications by eliminating a particular representation from the introductory course. However, it is difficult to design a diagram that effectively demonstrates the variation in space and time of two propagating, oscillating, orthogonal vector fields. The approach we have taken is to try to help students interpret the existing formalism in such a way that their learning of the material is enhanced.

Early in our investigation it became apparent that misinterpretations of Figure 8-1 were intertwined with other serious conceptual difficulties. It has been our experience that, if such issues are not explicitly addressed, it may be impossible for students to relate the formalism to physical phenomena. If a concerted effort to help students understand the diagram in Figure 8-1 is not made, we believe that it should not be used. We found that helping students understand the representations also helps them to understand the concepts and to relate both the concepts and the representations to the real world. The tutorials described are an example of how instructional materials developed on the basis of research can, within a relatively short time allotment, help students develop a deeper understanding of the wave model for light.

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CHAPTER 9: IDENTIFYING STUDENT DIFFICULTIES IN RELATING THE FORMALISM OF QUANTUM MECHANICS TO MEASUREMENTS OF BASIC QUANTUM MECHANICAL OBSERVABLES

A. INTRODUCTION

In this chapter we discuss preliminary research to investigate student understanding of measurements of basic quantum mechanical observables. Developing an understanding of this topic requires a solid understanding of the wave function as a probability amplitude. Also required is the ability to interpret the mathematical formalism of quantum mechanics, including algebraic and symbolic representations of operators and quantum states.

In a typical quantum mechanics course, students are taught the probabilistic interpretation of the wave function. They are also taught the postulate that only eigenvalues are observed in nature, that is, a measurement of a Hermitian operator A on an eigenstate ψ_n must yield the corresponding eigenvalue A_n. From this fundamental postulate come several important ramifications. For example, if a measurement on any state ψ yields a particular eigenvalue A_n, the system proceeds to time-evolve in the state ψ_n after the measurement. More colloquially, the system is said to "collapse" to the state ψ_n at the time of measurement. In this way, repeated measurements of observables that commute with the Hamiltonian (*e.g.*, total energy) give reproducible results. In addition, sequential measurements of incompatible observables, such as position and momentum, position and energy, and different components of angular momentum, can yield different results depending upon the order in which the measurements were made. All of these ideas are typically covered in undergraduate quantum mechanics. (Advanced concepts such as entangled states are introduced at the graduate level and are not discussed in this chapter.)

We have conducted research on student understanding of the above concepts in the context of the junior-level quantum mechanics courses at the University of Washington. In this chapter we describe specific difficulties that were found to persist after standard instruction. The results presented in this chapter are consistent with and expand upon those from informal studies published previously.[85]

B. MOTIVATION FOR RESEARCH

It has been our experience that after standard instruction practically all students correctly believe that a measurement on an eigenstate always yields the corresponding eigenvalue. However, many problems involve mixed states and therefore require a solid understanding of the wave function as a probability amplitude. Others require an understanding of the time-evolution of quantum states after a measurement is made. Examples include problems that involve repeated measurements of the same observable or measurements of incompatible observables.

In Chapter 6 of this dissertation we discussed specific conceptual and reasoning difficulties that students have in understanding probability density in the classical and quantum mechanical regimes. We have found that many students tend to use reasoning from the classical mechanics of particles to describe wave functions or probability densities for simple one-dimensional situations. We therefore suspected that students would also tend to use inappropriate reasoning from classical mechanics to predict the outcome of measurements of quantum mechanical observables. For example, in classical mechanics it is assumed that measurements do not affect the state of the system being measured. In addition, a particle undergoing classical motion can be treated as simultaneously having a well-defined position, momentum, and energy. In the quantum mechanical regime, however, these assumptions are not valid and could lead to incorrect predictions about a quantum mechanical system.

We have also found that some difficulties in understanding abstract physical concepts are closely related to those in interpreting the mathematical formalism. As discussed in the preceding chapter, mistaken beliefs about the nature of electromagnetic waves are often linked to misinterpretations of common algebraic and diagrammatic representations of such waves. We suspected that difficulties in understanding quantum mechanical observables would be similarly related to those in interpreting the formalism of quantum states.

C. CONTEXT FOR RESEARCH

The research was conducted in the context of the junior-level quantum mechanics courses taught in the Department of Physics at the University of Washington (Physics 324–325). (These courses are described in Chapter 1.) In part 1 below, we present the research tasks that were posed to students during the study. In part 2, we describe in detail the student populations involved in the study, including the instruction that had occurred prior to the administration of the research tasks.

1. RESEARCH TASKS

Qualitative questions were used to probe student understanding in individual student interviews, ungraded written quizzes, and course examinations. The results obtained from interviews and from the written forms of the questions were consistent.

a) Written questions

Each of the written questions pertained to a specific physical situation that is typically covered in a quantum mechanics course. One set of questions pertained to a particle in a one-dimensional infinite square well. Another set of questions was posed in the context of the Stern-Gerlach experiment. Each question was used in more than one quantum mechanics sequence, and no question was given more than once to the same student population.

(1) Infinite square well questions

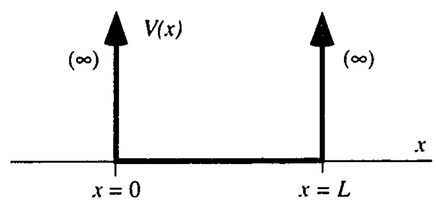
Three questions dealt with a particle in an infinite square well. Two questions, to which we refer as the "mixed-state" question and the "repeated-energy-measurement" question, are described together below. We then describe a third question, the "energy-position-energy" question.

Description of mixed-state question and repeated-energy-measurement question. This pair of questions is shown in Figure 9-1. In the "mixed-state" question (part A) the students are given an algebraic expression of an initial wave function for a particle in an infinite square well. They are asked to determine the possible results of a measurement of the energy of the particle. In the "repeated-energy-measurement" question (part B) students are asked to choose the statement that best describes the outcome of a second energy measurement on the particle. In both questions, the students are asked to explain their reasoning.

A particle in an infinite square well (shown at right) is prepared so that its wave function at $\psi(x, t)$ at time $t = 0$ is:

$$\psi(x, 0) = 0.6\, \psi_1(x, 0) + 0.8i\, \psi_2(x, 0),$$

where $\psi_1(x, t)$ and $\psi_2(x, t)$ are the two lowest energy states of the infinite square well, with $E_1 = \varepsilon$ and $E_2 = 4\varepsilon$.



A. Suppose you measured the energy of this particle at time $t_2 > 0$. What value or values would a measurement of the energy yield? Explain your reasoning.

B. Suppose a few minutes later, at time t_3, you measured the energy of the particle again. (Assume that, during the time interval $t_2 < t < t_3$ nothing happens to the particle while it is inside the square well.)

Which of the following statements would best describe the result of your second energy measurement? Circle one and explain your reasoning.

i. The second energy measurement would *definitely* give the same result as the first.
ii. The second energy measurement could *possibly* (but not necessarily) give the same result as the first.
iii. The second energy measurement wouid *definitely not* give the same result as the first.

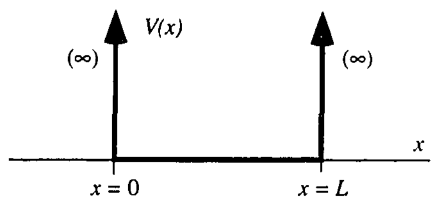
Figure 9-1: Two-part problem consisting of the "mixed-state" question (part A), and the "repeated-energy-measurement" question (part B).

Correct answers to mixed-state question and repeated-energy-measurement question. To answer the "mixed-state" question, students must recognize that the only possible results of measuring the energy are energy eigenvalues. In part A, therefore, the initial state of the particle would allow only two possible results: either ε, the eigenvalue corresponding to ψ_1, or 4ε, which corresponds to ψ_2. Thus the correct response to part A is that the energy measurement would yield either $\varepsilon\ (E_1)$ or $4\varepsilon\ (E_2)$.

Immediately after the first energy measurement is made the particle begins to time-evolve in the energy eigenstate corresponding to the measured eigenvalue. For example, if the first energy measurement yielded ε at $t = 0$, then for all later times t the quantum state of the particle would be described by the wave function $\psi_1(x, 0)\, e^{-i\varepsilon t/\hbar}$, *i.e.*, the particle is in a stationary state. Thus the second energy measurement would always yield the same value as did the first measurement, which means that the correct answer to part B is statement (i).

Description of energy-position-energy question. The third question, shown below in Figure 9-2, involves a particle that is known initially to be in the ground state of an infinite square well. The position of the particle has been measured, and its energy is about to be measured. The students are asked to choose the statement that best describes the result of the energy measurement and to explain their reasoning.

A particle in an infinite square well (shown at right) is prepared such that its wave function at time $t = 0$ is in the ground state: $\psi(x, 0) = \psi_1(x, 0)$.



At time $t_2 > 0$ you measure the position of the particle and find that at that time it is located at $x = 0.3L$.

Suppose a few minutes later, at a time t_3, you measured the energy of the particle.

Which of the following statements would best describe the result of your energy measurement? Circle one and explain your reasoning.

i. The value that you measure would *definitely* be the ground state energy.
ii. The value that you measure could *possibly* (but not necessarily) be the ground state energy.
iii. The value that you measure would *definitely not* be the ground state energy.

Figure 9-2: "Energy-position-energy" question. (When used in written form, this question followed the "mixed-state" question and "repeated-energy-measurement" question, shown in Figure 9-1.)

Correct answer to energy-position-energy question. To arrive at a correct answer, students must recognize that a state corresponding to a well-defined position (at $x = 0.3L$) is not a stationary state. In other words, after the position measurement, the particle time-evolves in a state that can be expressed as a superposition of energy eigenstates that includes the ground state. (Each eigenstate whose wave function $\psi_n(x)$ was non-zero at $x = 0.3L$ contributes to this superposition. It was not necessary, however, for a student to state this in order to be counted as correct.) Thus the second energy measurement could yield many possible energy eigenvalues, including the ground state energy; the correct statement is (ii).

Notes about energy-position-energy question. The Hamiltonian does not commute with position, *i.e.*, $[H, x] = i\,(\hbar/m)\,p \neq 0$. Students who gave this reason for choosing statement (ii) were counted as correct, however technically speaking this reasoning would be incomplete. The fact that $[H, x] \neq 0$ would justify eliminating only statement (i) but not necessarily statement (iii).

In order to rule out statement (iii), one can consider the total probability amplitude for the position measurement to yield $x = 0.3L$ and for the subsequent energy measurement to yield the energy eigenvalue E_n (corresponding to ψ_n). This total probability amplitude is equal to the product of the individual probability amplitudes, as shown below:

$$\langle \psi_n | x = 0.3L \rangle \langle x = 0.3L | \psi_1 \rangle = \psi_n^*(x = 0.3L)\ \psi_1(x = 0.3L)$$

The above expression is non-zero when $n = 1$, which means that the ground state energy is a possible outcome of the second energy measurement. Therefore, statement (iii) is incorrect. (Notice that if the value of n corresponds to an eigenstate for which the wave function $\psi_n(x)$ vanishes at $x = 0.3L$, such as $n = 10$, $n = 20$, $n = 30$, *etc.*, then the first term in the above expression is equal to zero.)

(2) Stern-Gerlach question

The question shown in Figure 9-3 requires students to apply the idea that eigenstates of the spin operator $S_z = (\hbar/2)\ \sigma_z$ are not the same as those of the operator $S_x = (\hbar/2)\ \sigma_x$. In the particular classes in which the question was given, spin operators were usually written as dimensionless operators, *i.e.*, with the overall factor of $\hbar$ suppressed. The wording of the examination problem is consistent with this choice of notation.

A beam of electrons prepared in the state $\chi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (that is, with $S_z = +1/2$) passes through a Stern-Gerlach (SG) apparatus with an inhomogeneous magnetic field along the x-direction. Two beams of electrons emerge from the other side of the SG.

A. Are the two beams from the first SG equally populated? Explain why or why not.

Note: The eigenstates of σ_x are $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

B. Suppose one of the two beams from the first SG (from part A) passes through a second SG that has its magnetic field along the z-direction. How many beams would emerge from the second SG? Explain.

Figure 9-3: Stern-Gerlach problem. Results from part B are discussed in this chapter.

In the next section we examine in detail only student responses to part B. For completeness, however, we briefly describe answers to both parts below.

Correct answers to Stern-Gerlach examination problem. For part A students must recognize that the spinor χ that describes the initial electron beam can be written in terms of an equally-weighted sum of the eigenstates of σ_x (denoted below as $\chi_{x,+}$ and $\chi_{x,-}$):

$$\chi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}(\chi_{x,+} + \chi_{x,-})$$

The probability of any electron populating either of the two outgoing beams is 50%, so the two beams from the first Stern-Gerlach apparatus would be equally populated.

For part B students should recognize that two beams (each with a well-defined value of S_z) would emerge from the second Stern-Gerlach apparatus. To arrive at a correct answer, students could explain that each eigenstate of σ_x can be written as an equally-weighted sum of the eigenstates of σ_z:

$$\chi_{x,\pm} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \pm \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}(\chi_{z,+} \pm \chi_{z,-})$$

Thus the electrons in either beam emerging from the first Stern-Gerlach apparatus have a well-defined value of S_x but not S_z. Alternatively, students can assert the fact that the operators S_x and

S_z do not commute, *i.e.*, $[S_x, S_z] = -i\, S_y \neq 0$. This result implies that S_x and S_z are incompatible observables. Either method described above would be accepted as correct with correct reasoning.

b) Interview tasks

The written questions described above were also posed in individual student interviews. A total of nine (9) student volunteers from the quantum mechanics sequence participated in the interviews. Each interview lasted about one hour. The interviews were videotaped for later analysis.

The format of the interview tasks differed in two minor ways from that of the written questions. First, although the content of the interview tasks and written questions was essentially the same, the interview tasks (particular those that pertain to the infinite square well) were not posed in the same order as they appeared on the written questions. The analysis of the interview transcripts and written responses suggest, however, that the order in which the questions were posed did not seem to affect student responses. Second, the open-endedness of the interviews allowed us to pose follow-up questions in order to probe for particular difficulties. We describe any variations between student responses on the written questions and interview tasks as the results become relevant to the discussion of specific difficulties.

2. STUDENT POPULATIONS

The students involved in this study come from the junior-level quantum mechanics sequence at the University of Washington. There are two courses in this sequence that are described in section B of Chapter 1.

The interview population consists of nine (9) student volunteers were from the Autumn 1995–Winter 1996 sequence, to which we shall refer as Sequence 0. The interviews were all conducted in Spring 1996 and thus after all (standard) instruction in the quantum mechanics sequence. The students were mostly at or above the mean of the class.

Written questions were posed to students during three consecutive quantum mechanics sequences. In addition to Sequence 0, described above, written questions were posed in the Autumn 1996–Winter 1997 and Autumn 1997–Winter 1998 sequences, to which we shall refer as Sequence 1 and Sequence 2, respectively. Each question or set of questions was posed after standard instruction of the relevant topics. In particular, the "mixed-state" question, the "repeated-energy-measurement" question, and the "energy-position-energy" question were all posed after instruction on the structure of quantum mechanics, including the interpretation of the wave function, operators, commutation relations, eigenvalues and eigenfunctions, expectation values. The Stern-Gerlach question was given after standard instruction on angular momentum, spin, and the Stern-Gerlach experiment.

D. DISCUSSION OF OVERALL RESULTS

In general, the students found the written questions and interview tasks quite difficult. The results from written questions were consistent in all of the classes in which they were posed. Therefore, the data from the questions have been combined. In this section we describe the overall results. In the following section we discuss specific difficulties that were identified.

1. OVERALL RESULTS FROM INFINITE SQUARE WELL QUESTIONS

We first present results from the "mixed-state" question, in which students were told that a particle in an infinite square well was initially in a mixed state consisting of a linear combination of the ground state and first excited state. Many students had difficulty predicting the possible values that an energy measurement would yield. The overall results from this question are given below in Table 9-1.

Table 9-1: Results from the "mixed-state" question given after standard instruction in the Autumn classes of Sequence 1 and Sequence 2 (N_{total} = 53). Students were asked to predict the possible values that an energy measurement would yield. [Percentages rounded to the nearest 5%.]

Correct (E_1 or E_2)	**30%**	**(17)**
With correct reasoning	30%	(16)
Incorrect responses	**40%**	**(21)**
Expectation value of energy	15%	(7)
$E_1 + E_2$	10%	(4)
E_1 only	10%	(5)
Other incorrect responses	10%	(5)
Blank or incomplete	**30%**	**(15)**

Only 30% of the students (17 of 53) correctly answered that the energy measurement would yield either the ground state energy or the first excited state energy, with another 30% (15 of 53) giving blank or incomplete responses. A variety of incorrect answers were given by the remaining 40% of the students (21 of 53). A larger percentage of students in the interviews gave a correct answer (5 of 7) to this question, although only 2 students did so with correct reasoning. However, the written question and interviews differed in that, on the written version of the "mixed-state" question, one of the coefficients in the algebraic expression for the mixed state was purely imaginary. In the following section, we discuss how this feature of the written version of the question seemed to make it more difficult than the version used in the interviews.

In the written version of the "repeated-energy-measurement" question, students were asked to imagine making a second measurement of the energy of the particle described in the "mixed-state" question. Most students failed to recognize that the result of the second measurement would be identical to that from the first. The results from this question are summarized below in Table 9-2.

Table 9-2: Results from the "repeated-energy-measurement" question given in the Autumn classes of Sequence 1 and Sequence 2 ($N_{total} = 53$). Students were asked whether the second energy measurement would *definitely, possibly,* or *definitely not* yield the same value obtained from the first energy measurement.

Correct (second energy measurement would *definitely* yield same value as the first)	**30% (15)**
With correct reasoning	10% (6)
Incorrect responses	**60% (33)**
Could *possibly* yield same value	55% (30)
Would *definitely not* yield same value	5% (3)
Blank or incomplete	**10% (5)**

The written version of the "repeated-energy-measurement" question was designed as a multiple-choice question. The success rate (15 of 53, or 30%) was approximately the same as would be obtained by random guessing. Only 10% of the students (6 of 53) defended their answers with correct reasoning. In the interviews, 3 of the 6 students who were posed this task answered correctly. Errors that arose in the interviews were similar to those that were elicited by the written questions.

In the "energy-position-energy" question, students were told that a particle initially in the ground state of an infinite square well was measured to be at the location $x = 0.3L$. They were asked to predict the likelihood that a subsequent measurement of the energy of the particle would yield the ground state energy. The results from this question are shown in Table 9-3 below.

Table 9-3: Results from the "energy-position-energy" question given in the Autumn classes of Sequence 1 and Sequence 2 ($N_{total} = 53$). Assuming that a particle initially in the ground state of an infinite square well was found to be located at $x = 0.3L$, student were asked to predict whether a subsequent energy measurement would *definitely*, *possibly*, or *definitely not* yield the ground state energy.

Correct (second energy measurement *could possibly* yield ground state energy)	**40% (21)**
With correct reasoning	5% (2)
Incorrect responses	**35% (20)**
Would *definitely* yield ground state energy	25% (14)
Would *definitely not* yield ground state energy	10% (6)
Blank or incomplete	**25% (12)**

As shown in Table 9-3 above, only 40% (21 of 53) of the students gave correct answers, with about 5% (2 of 53) doing so with correct reasoning. Again, the success rate was approximately the same as one would expect from completely random guessing (ignoring reasoning). In interviews, 4 of the 9 students who were given this task answered correctly, but none used complete and correct reasoning.

2. *Overall results from Stern-Gerlach question*

On the Stern-Gerlach problem described previously, practically all of the students gave correct answers on part A, saying that an electron beam prepared with $S_z = +1/2$ would split equally after passing through a Stern-Gerlach apparatus with its magnetic field oriented along the x-axis. Most gave correct answers with correct reasoning.

However, many students had serious difficulty in part B of the problem, in which they tried to predict the outcome of passing one of the beams from the first Stern-Gerlach apparatus through another apparatus oriented in the z-direction. These students failed to apply correctly the idea that different components of spin angular momentum (*e.g.*, S_x and S_z) are incompatible observables. The results from part B of the Stern-Gerlach question are summarized below in Table 9-4.

Table 9-4: Results from part B of Stern-Gerlach examination problem: How many beams of electrons emerge from the final Stern-Gerlach apparatus? Shown are the combined results from examinations given in Sequences 0, 1, and 2 (N_{total} = 81).

Correct (two beams emerge)	**55% (46)**
With correct reasoning	25% (21)
Incorrect responses	**35% (30)**
Only one beam emerges	35% (29)
Other incorrect responses	< 5% (1)
Blank or incomplete	**5% (5)**

About half of the students (46 of 81) got part B correct. Only 25% (21 of 81) justified their answers with correct reasoning. The most common incorrect response was that only one beam would emerge.

E. IDENTIFICATION OF SPECIFIC DIFFICULTIES

The analysis of student responses to the questions described above revealed numerous difficulties in understanding quantum mechanical observables. The specific difficulties have been organized into four broad, overlapping categories:

- failure to recognize that a measurement of a quantity yields an eigenvalue of that quantity,
- failure to interpret the coefficients that describe a mixed state as probability amplitudes,
- failure to recognize that repeated energy measurements must yield identical results, and
- failure to recognize that measurements of incompatible observables depend on the order of the measurements

Below we discuss in detail the difficulties that were identified and present illustrative student responses.

1. *FAILURE TO RECOGNIZE THAT A MEASUREMENT OF A QUANTITY YIELDS AN EIGENVALUE OF THAT QUANTITY*

Of fundamental importance in quantum theory is the idea that a measurement of an observable quantity can only yield an eigenvalue of that quantity. On the "mixed-state" question, in which a particle in an infinite square well is assumed to be in an initial state $0.6\ \psi_1(x) + 0.8i\ \psi_2(x)$, many students failed to recognize that the eigenvalues corresponding to ψ_1 and ψ_2 (given as $E_1 = \varepsilon$ and $E_2 = 4\varepsilon$ in the problem) would be allowed.

a) *Mistaken belief that a measurement on a mixed state yields the expectation value*

About 15% of the students believed that a single measurement of the energy would yield the expectation value of the energy. For example, one student wrote:

> "$(.6)^2\varepsilon + (.8)^2(4\varepsilon)$... The measured energy will be a combination of the possible energies, in proportion to the probability of the wave appearing."

Students who gave answers like the one above failed to recognize that measurements of observable quantities can only yield eigenvalues. Furthermore, because these students gave a single possible result for the energy measurement, they apparently failed to recognize that a mixed energy state cannot be described by a single, well-defined value of energy.

b) *Mistaken belief that a measurement on a mixed state yields a value related to the sum of the possible eigenvalues*

Other students seemed to apply incorrectly the principle of superposition in answering the "mixed-state" question. About 10% believed that an energy measurement would yield a value greater than or equal to the sum of the allowed energies (ε and 4ε). Specific examples of student responses are given below.

> "$E \geq 5\varepsilon$, principle of superposition."

> "$E \geq 5\varepsilon$. Since 5ε is the lowest possible energy the particle can have in the square well & the well is infinite, the particle can have any energy above 5ε."

The students who gave the above responses may have recognized that the energy eigenvalues of an infinite square well potential have no upper bound. However, they treated the sum of the two eigenvalues as the lower bound of the possible values that energy measurement could yield. These students clearly did not recognize that the two eigenvalues are the only possible results of the energy measurement.

2. *FAILURE TO INTERPRET THE COEFFICIENTS THAT DESCRIBE A MIXED STATE AS PROBABILITY AMPLITUDES*

When a quantum state is represented as a normalized superposition of eigenstates, the coefficients for each term can be interpreted as the probability amplitude that a measurement would yield the corresponding eigenvalue. We found that many students had difficulty in applying this idea.

a) Mistaken belief that the coefficients represent probabilities rather than probability amplitudes

A common difficulty that was elicited during the interviews as well as from the written questions was the belief that the coefficients in an expression of a mixed state represent probabilities rather than probability amplitudes. For example, in one interview the investigator wrote the expression "$\psi = a\psi_o + b\psi_1$" on a sheet of paper to represent the state of a particle in an infinite square well. (Unlike the labeling convention on the written questions, "ψ_o" represented the ground state and "ψ_1" represented the first excited state.) The investigator then posed the following question:

> I: "Suppose that I had written this ["$\psi = a\psi_o + b\psi_1$"] in such a way that I could tell you ahead of time that the values of a and b make that wave function normalized.... What's the probability of measuring the ground state energy and what's the probability of measuring the first excited state energy?"
>
> S: "The probability of measuring the ground state energy is a, and the probability of measuring the first excited state energy is b."

In the written version of the "mixed-state" question (in the context of the infinite square well) the same difficulty arose. Many students who made this error also failed to recognize that a measurement of a quantity must yield an eigenvalue of that quantity. (This difficulty is described earlier.) An example response is given below.

> "At $t > 0$, I would get a value in between the ε and 4ε. The value I'd get would be close to $.6\varepsilon + .8(4\varepsilon) = \ldots = 3.8\varepsilon$."

The claim that the energy measurement would yield "a value in between ... ε and 4ε" is incorrect because the energy measurement must yield an energy eigenvalue. In addition, the student seems to believe that the measured value would be close to the expectation value of the energy, and in attempting to calculate the expectation value the student weighted each of the energy eigenvalues according to the absolute value (rather than the absolute *square*) of the coefficient. Responses such as the one above indirectly indicated a tendency to treat the coefficients that described the mixed state as probabilities rather than a probability amplitudes.

b) Mistaken belief that only real-valued coefficients correspond to allowed eigenvalues or eigenstates

On the written version of the "mixed-state" question, about 10% of the students incorrectly stated that the energy measurement would yield only the ground state energy ε. These students seemed to base their prediction on the fact that, in the expression for the mixed state, the ground state wave function was the only eigenstate with a real-valued coefficient. Example student responses are as follows:

> "~ ε. Does the imaginary normalization constant on ψ_2 mean it is unmeasurable? I would think so!"
>
> "The measured energy would be ε... only real components of the wave function can be measured."
>
> "I suppose that the pure imaginary amplitude of the ψ_2 part means that its contribution cannot be measured."

Although it is true that eigenvalues of Hermitian operators are always real, these students seem to have extended that idea inappropriately to say that only the real-valued coefficients would correspond to allowed eigenstates. The error is also reminiscent of other errors documented in Chapter 6, in which students said that the real part of the wave function must be zero in classically forbidden regions. These students seemed to believe that the "real" part of the wave function corresponds only to those regions in which the particle "can physically exist."

3. FAILURE TO RECOGNIZE THAT REPEATED ENERGY MEASUREMENTS MUST YIELD IDENTICAL RESULTS

A fundamental postulate in quantum mechanics is that measurements yield only eigenvalues of the quantity being measured. Many students, as described already, had serious difficulty applying this idea when answering the "mixed-state" question. Even those who correctly answered this question did not seem to understand an important related idea, namely that at the time of measurement the system "collapses" to the eigenstate that corresponds to the particular eigenvalue that was measured.

a) Mistaken belief that an energy measurement on a mixed energy state leaves the system in the original mixed state

On the written version of the repeated-energy measurement question, some students correctly recognized that either of the two energies E_1 or E_2 would be possible outcomes of the first energy measurement. However, many of these students incorrectly predicted that the *same* set of outcomes were possible for the second energy measurement. These students articulated the belief that the measurement would cause the state to "collapse" but thought that the system would then return to the original mixed state. Example responses from the written questions are given below.

> "The particle is described by a wave function with elements in both eigenstates. Although a measurement of energy collapses it to one, the possibility of the other still exists, so a second measurement could get the other [energy eigenvalue]."

> "As long as you give the system sufficient time to return to its original state before the first measurement, the probability distribution is the same: $0.36E_1$ and $0.64E_2$."

It is interesting to note that the second quote seems to reflect the belief that a certain amount of time must elapse in order for the particle "to return to its original state." In informal conversations with students, some proceed further with this incorrect reasoning to state that this "relaxation time" is given by the uncertainty principle $\Delta E\ \Delta t > \hbar/2$, where ΔE is the difference in the allowed energy eigenvalues. A more proper interpretation of this time interval Δt would be that it represents the minimum time interval between consecutive energy measurements to ensure that the results of such measurements would be identical.

In one of the interviews in which the repeated-energy measurement question was posed, the student had difficulty in deciding which state the particle would occupy after the energy measurement was made. The student correctly stated that because the initial wave function was an equally weighted superposition of the ground state (ψ_o) and the first excited state (ψ_1), the first energy measurement could yield the first excited state energy. However, he had difficulty deciding how to describe the state of the particle after this first measurement [boldface added by the author]:

> "It depends on whether or not measuring the energy modifies the wave function. I mean, if ... [ψ] is what you look at, and you measure [the energy of] it, and after you're done measuring it what you have in your hand is something that looks like [ψ_1], then, no, there's no probability of ever measuring ... an energy eigenvalue corresponding to [ψ_o]. If you measure this [ψ] and you get this value, ψ_1, but you still have [ψ] sitting there after

> you measure the energy, then you still have the same probability you had before of measuring ψ_o... **After the first measurement, I don't know if it's in that state, or if that's just the eigenvalue that was returned."**

The concluding statement made by the student (in boldface) eloquently summarized the confusion that he held. The student had difficulty deciding whether or not the "eigenvalue that was returned" was relevant to the state of the particle after the first measurement occurred. He did not recognize that all subsequent energy measurements would be identical to the first.

b) Mistaken belief that an energy measurement on a mixed energy state leaves the system in a different mixed state

One common type of response—which has been documented previously in the context of incompatible observables[86]—is the idea that any measurement made on a quantum mechanical system "disturbs" the system. This belief was elicited on the "repeated-energy-measurement" question, both on the written questions and during interviews. Even though the question does not raise the issue of compatible or incompatible observables, many believed that repeated energy measurements would "disturb" the particle. Such students failed to recognize that the second energy measurement would always yield the same result as the first. Two example student responses are given below:

> "[The second energy measurement could possibly (but not necessarily give the same result as the first.] By measuring the particle we have changed its wave function forever, right?"

> "[The second energy measurement could possibly (but not necessarily give the same result as the first.] By measuring it you screw it up...could it possibly go back?"

In some cases, students thought that repeated measurements had a systematic effect. In one interview, the student was asked to consider a particle in an infinite square well whose initial wave function was an equally-weighted superposition of the ground state and first excited state. Although the student recognized that the first energy measurement would yield either of these two energy values, he failed to recognize that all subsequent energy measurements would yield the same value as the first. He argued that one must "measure the system by disturbing the system," and that making a measurement would "take up energy."

In order to probe more deeply how the student was thinking, the interviewer asked the student to consider taking repeated energy measurements (one after another) of the particle. The student explained how he believed the probabilities of measuring the ground state energy (denoted by E_0) and first excited state energy (denoted by E_1) would change with each measurement:

> "You'll start off with a 50-50 probability of measuring either [E_0] or [E_1]. Then after your next measurement it's going to become more heavily weighted [toward the ground state], to the point that if you've made enough measurements you get it down to just the ground energy.... It is possible, because these are only probabilities, that you would get really lucky, and all the way through you could measure E_1 all the way down. Although very unlikely, it could happen."

The student seemed to believe that each energy measurement leaves the particle in a mixed state "more heavily weighted" toward the ground state, thus "taking up energy." This belief led the student to state incorrectly that the probability of measuring E_1 (the first excited state energy) each time would be *between* 0% and 100%, rather than equal to 0% (if E_0 were measured the first time) or equal to 100% (if E_1 were measured the first time).

4. *FAILURE TO RECOGNIZE THAT MEASUREMENTS OF INCOMPATIBLE OBSERVABLES DEPEND ON THE ORDER OF THE MEASUREMENTS*

The idea that the quantum mechanical state of a particle or system "collapses" to an eigenstate is important to understanding the possible outcomes of consecutive measurements of different observables. The results from interviews and written questions revealed several common difficulties in recognizing that certain basic quantities, such as position and momentum, or different components of angular momentum, were incompatible. These results are discussed below.

a) Mistaken belief that a particle can have both a well-defined position and a well-defined energy (or momentum)

In our experience we have found that most quantum mechanics students can recite the Heisenberg uncertainty principle as expressed by the inequality $\Delta x \, \Delta p \geq \hbar/2$. We have also found that after all instruction most students can recite and derive commutation relations for quantum mechanical operators (*e.g.*, those that are based on the fundamental relation $[x, p] = i\hbar$). However, when posed certain qualitative questions like the "energy-position-energy" question and the Stern-Gerlach question (which are described earlier in this chapter), students had serious difficulty in interpreting and applying these relationships.

(1) Mistaken belief that position measurements "conserve energy"

In the "energy-position-energy" question, students consider a particle that begins in the ground state of an infinite square well of width L and that is measured to be at the location $x = 0.3L$. They are then asked to predict the likelihood that a subsequent energy measurement would yield the ground state energy. Many students incorrectly stated that the ground state energy would be the only possible outcome. Some of their explanations were based on the principle of energy conservation:

> "There is only one eigenstate for this situation, so unless energy is added/subtracted (ψ changed) there remains only one possibility."
>
> "Without additional energy supplied to the system, the particle will remain in the ground state."
>
> "Since no energy was added or subtracted, there MUST be the same E due to the law of conservation of energy."

Thus many students did not seem to recognize that, after the position measurement, the energy of the particle would not be well-defined. Instead, they stated that energy would be "conserved," as if they believed that the energy were well-defined. These students resorted to an idea from classical mechanics for a physical situation in which it was inappropriate to do so.

Similar errors arose during the interviews. A task that was used in the interviews was almost identical to the written version of the "energy-position-energy" question. The student was asked to imagine that everyone in the city of Seattle began with a particle in the ground state of an infinite square well, measured the energy of the particle, measured the position of the particle, and then measured the energy of the particle again. When asked to predict how the aggregate results from everyone's second energy measurement would compare to those from the first, one student gave the following response:

> "Well, I think that [everyone in Seattle will] still get that same original value that they got for the energy, because if the particle's in that same state then it's going to have that same energy, although my feeling is that there's some trick to this question that I'm missing."

Four of the nine interview volunteers gave the same type of incorrect response. Their reasoning suggested the mistaken belief that the position measurement would have no effect on the wave function of the particle, even if the initial state of the particle were an energy eigenstate.

(2) *Mistaken belief that a quantum state representing a well-defined position also has a well-defined energy*

In the individual student interviews, several students recognized that making a position measurement could affect the wave function of the particle. However, some incorrectly believed that once the wave function "collapsed" to a well-defined position it would evolve back into an energy eigenstate.

For example, one student said that the position measurement would result in the wave function becoming a Dirac delta-function, which he referred to as a "spike." When asked what would happen to the wave function at later times, he replied:

> "I'm assuming that this spike will go ahead and evolve back into whatever the ground state was.... As time goes on, this [spike] ends up looking like the original wave function."

In another interview, a student was given the "energy-position-energy" question in the form described earlier. The student imagined that everyone in Seattle were given a particle in the ground state of an infinite square well and measured its energy, its position, and then its energy once again. The investigator probed how the student was thinking about the position measurements by asking the following question:

> I: "If I asked you to draw the wave function of the particle before they make the position measurement, and then [do the same for the wave function] after they make the position measurement, would those wave functions be identical or would they be different?"
>
> S: "I would say they would be identical.... You measure [the position of the particle], it collapses its wave function, you get this delta function, because the particle was here. And then after you're done measuring it, if you will, I haven't done anything necessarily to the energy of the particle, so it would go back to its [original] state."

The student recognized that the position measurement would cause the wave function (initially in the ground state) to "collapse" to a "delta function," but he incorrectly believed that the same state would evolve back to the original energy eigenstate. Such responses reflect a mistaken belief that it is possible for a quantum state to be both a position eigenstate and an energy eigenstate, that is to have a well-defined position and well-defined energy simultaneously. This error also reflects the failure to recognize the relevance of the principle of superposition, in particular that a position eigenstate can be thought of as a superposition of many energy eigenstates but not a single energy eigenstate.

(3) Mistaken belief that a position measurement can also yield the momentum or kinetic energy at the time of measurement

Follow-up questions that were used in individual student interviews helped reveal the presence of additional difficulties. One particular difficulty that arose was the belief that information about the (well-defined) position of a particle in any potential well can be used to infer a well-defined value for the momentum or the kinetic energy of the particle.

For example, after students described the allowed energy eigenstates of an infinite square well, the investigator often probed the student's reasoning more deeply by asking what the lowest possible value of energy could be for a particle in the well. One of the students answered incorrectly that a particle could be at rest in the well and thus have zero kinetic energy and (assuming that the potential is zero inside the well) zero total energy. The conversation between investigator and student proceeded as follows:

> I: "What's the lowest value of energy that [a particle in infinite square well] can have?"
>
> S: [After thinking aloud.] "Zero. If it wasn't moving at all, then it would have kinetic energy of zero."
>
> I: "So if the kinetic energy were zero, can you draw qualitatively what the state would look like?"
>
> S: "[Draws a "spike" to represent a Dirac delta function at a location inside the well.] The particle would just be defined at one point if it weren't moving and had potential of zero, so it would be here, it would be a delta-function. Its position, or its wave function would have collapsed at that *x*."

The student seemed to understand that after a position measurement the wave function would "collapse" into a "delta-function." However, he failed to recognize that such a wave function would not evolve as a superposition of energy eigenstates. He incorrectly believed that the wave function would remain a "delta-function" as time went on and interpreted this behavior to mean that the kinetic energy of the particle would be zero.

In another interview, the conversation extended to the case of a particle in a one-dimensional harmonic oscillator potential. The student was asked to imagine that everyone in Seattle were given identical particles, each in the ground state of identical harmonic oscillator potential wells, and measured the momentum of their particles. The student correctly stated that the measurements would not all yield the same value. However, the student did not justify his answer by recognizing that energy eigenstates for the harmonic oscillator potential are not also momentum eigenstates. Instead, he began by writing down the equation "$p = (2mE - V(x))^{1/2}$" for the momentum p in terms of E and $V(x)$. (The expression is correct from a classical standpoint, except that the quantity $E - V(x)$ should be enclosed by parentheses.) The conversation then proceeded as follows (boldface added by the author):

I: "What does this suggest about the momentum measurements that everyone tabulates?"

S: **"They [the momentum measurements] are going to be all different, depending on where the particle is in the box when they looked at it."**

I: "But they're not looking for [the position of the particle], they're just making a momentum measurement."

S: "Right, right, right. But nevertheless it is going to be all different, I mean the particle is going to be in different places in the boxes when they take measurements of it. So, they're going to get different momentum values."

I: "Is that something we can conclude, that, if they bring back different momentum measurements, [the particles] must all have been in different positions at the time they took their measurements?"

S: "Yeah, I think so."

The student correctly claimed that the momentum measurements would not all be the same. However, the student incorrectly believed that the momentum measurements would be different because the particles would not all have the same position in the potential well when the momentum measurements are made. He used his expression for the momentum p to explain that the initial value of the total energy E and the measured value of p could be used to infer the position x of the particle when the momentum measurement was made. Thus the student tried to interpret the result of a momentum measurement to yield information about both the momentum and the position of the particle.

The error made by both students described in this section reflect an overall tendency to use reasoning from the classical mechanics of particles to account for quantum mechanical phenomena. The above results suggest a particularly strong tendency to think that a particle in a quantum mechanical state can simultaneously have a well-defined position *and* momentum, as if following a classical trajectory over time.

b) Failure to apply the knowledge that different components of angular momentum are incompatible observables

The Stern-Gerlach experiment is commonly used in order to illustrate the quantum mechanical properties of spin angular momentum, including the idea that different components of angular momentum are incompatible observables. However, many students failed to apply the idea that different components of spin angular momentum *(e.g.,* S_x and S_z*)* are incompatible. In the Stern-Gerlach problem described earlier, students consider a beam of electrons prepared with $S_z = \pm 1/2$ that pass through a Stern-Gerlach apparatus with its magnetic field directed along the *x*-direction. They imagine that one of the emerging beams passes through another Stern-Gerlach apparatus oriented along the z-direction. Many students incorrectly predicted that only one beam of electrons would emerge from the last Stern-Gerlach apparatus. Their reasoning illustrated a variety of incorrect ideas.

For example, one student did not seem to recognize the need for a different set of spin states to describe the electrons that pass through the first Stern-Gerlach apparatus. The response given by this student is as follows:

> "Only one beam would emerge: the first apparatus separated the spin up and spin down electrons, so repeating the process (even in a different direction) would not split the beam further."

The student apparently did not recognize that states describing "spin up and spin down" along different coordinate axes (*e.g.*, the *x*- and *z*-axes) are not the same.

Some students recognized that a spin state corresponding to "spin up" along the *z*-axis would somehow be different than that for "spin up" along the *x*-axis. However, many incorrectly believed that the electrons that pass through the first Stern-Gerlach apparatus (oriented along the *x*-direction) would retain the property of having spin $S_z = + 1/2$. As a result, they believed that all of the electrons passing through the second apparatus (along the *z*-direction) would behave identically:

> "One beam would emerge. At the start, all of the electrons were prepared in the state with $S_z = 1/2$. After the first split ... all of the electrons will still have $S_z = 1/2$, so when they go through the *z*-oriented [magnetic field] they will all act identically."

> "The initial $S_z = 1/2$ nature of the electrons is unchanged by passing through the first SG. Thus, all the electrons entering the second SG have the same S_z (of + 1/2), and they all react the same way to an SG with magnetic field along z-direction. Thus only one beam of electrons emerges from the second SG."

> "Only one beam. The original beam was preset with $S_z = 1/2$, so all electrons have same z-direction angular momentum."

The same incorrect reasoning was elicited during an interview in which the student was asked to describe how he tried to answer the Stern-Gerlach examination problem (which he had taken several weeks prior to the interview). The student predicted that one beam would emerge from the last Stern-Gerlach apparatus. In order to probe the student's reasoning more deeply, the investigator asked the student to describe the probability that the electrons entering the last Stern-Gerlach apparatus have spin $S_z = + 1/2$, or spin "up." The student responded:

> "[The electrons] are all up.... They're all mostly up and mostly left.... And so again when we send them through another magnet, they're already mostly up, none will be mostly down, so we should get a single beam out."

Thus the student incorrectly stated the electrons would have spin "up," meaning $S_z = + 1/2$, *and* spin "left," meaning $S_x = + 1/2$. It is also interesting to notice that the student described the electron spin as being "mostly up and mostly left." The student seemed to believe that the electrons in a beam described as being "spin up" ($S_z = + 1/2$) would actually have a distribution of well-defined spins, with the distribution of spin vectors being centered about the z-axis.

The errors above suggest that many students fail to recognize that different components of spin angular momentum are incompatible observables. It is interesting to note, however, that the error is context-dependent. In Winter 1996 the examination containing the Stern-Gerlach problem also contained a question (written by the lecture instructor) in which the students were asked to explain why the angular momentum components L_x and L_y are not simultaneously observable. (The expected correct answer was to state that the commutator $[L_x, L_y]$ is not equal to zero.) Of the 11 students that quarter who answered the Stern-Gerlach problem incorrectly, 9 of those students answered the question about L_x and L_y question *correctly*. Therefore, it seems that many students presumably understood that different components of angular momentum are incompatible observables but did not recognize how to apply that knowledge in the context of the Stern-Gerlach experiment.

F. SUMMARY

We have presented in this chapter results from individual student interviews and written questions designed to probe student understanding of measurements of basic quantum mechanical observables. We identified specific conceptual difficulties that were closely related to difficulties with symbolic representations of quantum states. The difficulties did not seem to be addressed by standard instruction in quantum mechanics.

In many ways students lack a functional understanding of the postulate that only eigenvalues can be observed or measured. Many believe that an energy measurement on a system described by a mixed state would yield the expectation value of the energy rather than one of the possible energy eigenvalues. Such responses would indicate a possible confusion between expectation values and eigenvalues.[87] Even those students who seem to understand the possible outcomes of the first energy measurement have difficulty recognizing that a second energy measurement must yield the same value as the first. Even students who articulated that the wave function would "collapse" after the first measurement believed incorrectly that, after sufficient time had elapsed, the state of the system would evolve once again as a mixed state.

Numerous additional difficulties seem to be rooted in the inappropriate application of ideas from the classical mechanics of particles. Many difficulties appear to result from an underlying belief that the quantum mechanical state of a particle can simultaneously have *both* a well-defined position *and* a well-defined momentum (or energy). For example, students often articulate the notion that position measurements "conserve energy," or that measuring the momentum of a particle can be interpreted to yield position measurement as well. These types of errors seem to be rooted in the tendency to predict or explain quantum mechanical phenomena purely on the basis of ideas from classical mechanics, particularly the idea that position and momentum can simultaneously be well-defined.[88]

Other errors appear to be rooted in the incorrect belief that only the "real" part of the wave function can be measured. This notion appears to be closely linked to the mistaken belief that $\mathrm{Re}\{\psi(x)\}$ exists only in classically allowed regions and $\mathrm{Im}\{\psi(x)\}$ only in classically forbidden regions (see Chapter 6 for details). Students who articulate this idea seem to interpret the "real" part of $\psi(x)$ quite literally as the "classical" or "observable" part of the wave function.

The research discussed in this chapter is not meant to represent an exhaustive identification of student understanding of quantum mechanical observables. For instance, the research tasks presented here have involved simple *gedanken* experiments in which students are asked to predict the possible outcomes of measurements of particular quantities, such as the position of a particle. However, students were not always asked to describe the specific steps necessary to actually carry out such a measurement. Examining how students would approach the task of making such measurements could provide further insight into the nature of their difficulties and possible instructional strategies to address them.

CHAPTER 10: CONCLUSION

In this dissertation we have presented the results from an investigation of student understanding of the wave-like properties of light and matter. The work described is part of the ongoing effort of the Physics Education Group at the University of Washington to establish a robust research base for curriculum development. During this investigation, we identified numerous conceptual and reasoning difficulties among students enrolled in physics courses ranging from the introductory to the graduate level. We have applied the results from this research to guide the design of instructional materials to address some of the specific difficulties that we identified. We have done preliminary testing, modification, and assessment of the materials. Below we summarize the overall results and draw some generalizations.

Analysis of student responses to specially designed research tasks has revealed the presence, prevalence, and persistence of specific difficulties. We found that difficulties with advanced material often are rooted in difficulties with more basic concepts. For example, serious difficulties in developing and applying a wave model to matter arise because of persistent confusion with the basic concepts that underlie interference and diffraction. Other errors in the context of reflection, transmission, and bound states of matter are the result of a persistent difficulty in relating kinetic or potential energy to the de Broglie wavelength, a concept typically introduced prior to a course in quantum mechanics. Many students who demonstrate confusion about the qualitative behavior of matter in bound states have difficulty in relating probability density to kinetic or potential energy in the classical regime. Although it is reasonable to expect that quantum mechanics would be a challenging subject to learn, most of the difficulties identified in the study are related to ideas that are not necessarily new to students who are studying this subject.

Additional difficulties appeared to stem from an overall inability of students to recognize for which situations one model or set of ideas is more appropriate than another. For example, many first- and second-year students have difficulty in recognizing when to use ideas from geometrical optics or physical optics to account for the behavior of light passing through slits. They often have difficulty applying a wave model to light and are unable to interpret the formalism used to describe light as an electromagnetic wave. Many attempt to explain interference phenomena in terms of geometrical optics. Similarly, advanced students frequently misapply their knowledge of the classical mechanics of particles in trying to account for inherently quantum mechanical phenomena. Many fail to recognize that the probability density determined from quantum mechanics, when taken to the appropriate (small de Broglie wavelength) limit, should be the same as that obtained classically.

A particular generalization can be made with regard to the types of written questions and interview tasks that proved useful in our research. We believe that an effective way of determining whether students have developed a functional understanding of a specific topic is to ask them to make a prediction about a specified physical situation and to explain the reasoning used in making their predictions. These questions should be carefully constructed so that students who apply particular incorrect ideas will arrive at incorrect answers. Such questions are especially useful when probing student reasoning in subjects as abstract as physical optics or quantum mechanics. These questions are far more fruitful than such broad questions as "What is a wave?" or "What is a particle?" The analysis of responses and explanations to a carefully designed question or task can provide detailed insight into student thinking. After questions that pertain to a sufficiently broad variety of physical situations are administered to a sufficiently large number of students, the patterns observed in the responses can reveal the nature of whatever underlying difficulties are prevalent. For example, in the study we asked qualitative questions on electron diffraction and interference, scattering from a potential step, and tunneling through a potential barrier. Each question elicited a different array of incorrect answers. The results suggested that students often fail to attribute properties of classical waves (such as superposition, reflection, and transmission) to matter in situations where those properties are relevant and important.

In this and other investigations, many of the same tasks that were used to identify particular difficulties have often been incorporated into carefully structured worksheets and other instructional materials that have proved effective in addressing these difficulties. When qualitative questions are used to *elicit* specific difficulties during instruction, students can be given the opportunity not only to articulate their incorrect beliefs but also to *confront* their difficulties and to recognize the need to *resolve* them. For example, we believe that the success of the tutorial *Wave properties of matter* can be attributed, at least in part, to the *elicit–confront–resolve* strategy used in the tutorial.

It is often necessary to address specific difficulties with a particular concept several times. For example, all three tutorials that were developed for use in the quantum mechanics course address difficulties related to the de Broglie wavelength in different contexts. The tutorial *Wave properties of matter* deals with de Broglie wavelength in the context of interference and diffraction of matter waves. The tutorial *Reflection and transmission* focuses on the relationship between local de Broglie wavelength and local kinetic energy in the context of scattering from a potential step. *Relating classical mechanics to quantum mechanics* addresses persistent difficulties in the context of bound states. Results from qualitative questions posed at various stages of instruction indicate that each tutorial is helpful to students and that the level of improvement is enhanced when all three tutorials are used together.

Research on the ability of students to apply a wave model to light and to matter has raised other interesting issues that merit further study. For example, in individual student interviews on diffraction, approximately half of the students from the modern physics and quantum mechanics courses articulated beliefs about light exhibiting *simultaneous* wave-like and particle-like properties. Many stated that photons were point-sized particles moving along sinusoidal paths, while others believed that multiple photons must interact with each other in order to produce interference effects. The prevalence of such beliefs suggests the need for more systematic studies of student interpretations of the wave-particle duality. Additional research is needed in order to determine effective instructional strategies that help students recognize the appropriateness of one set of ideas over another (*e.g.,* geometrical optics *vs.* physical optics, classical mechanics *vs.* quantum mechanics) for a given physical situation. The context of research on student interpretation of the wave function can be extended from wave functions in one dimension (*i.e.,* of the form $\psi(x)$) to those in two or more dimensions and thus to the quantum mechanical formalism necessary to describe the hydrogen atom.

We have identified through research some serious conceptual and reasoning difficulties that students have in developing an understanding of the wave-like properties of light and matter. We have applied results from our investigation to design, test, modify, and assess instructional materials that effectively address specific difficulties. We have shown that specific difficulties with the de Broglie wavelength and with basic concepts in quantum mechanics can be addressed at a wide variety of instructional levels, from the introductory through the junior and senior level. We have also demonstrated that efforts to address such difficulties can be made with a minimal allotment of class time and with minimal impact on the overall structure of the course.

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[23] K. Wosilait, P.R.L. Heron, P.S. Shaffer, and L.C. McDermott, "Development and assessment of a research-based tutorial on light and shadow," Am. J. Phys. **66**, 906-913 (1998).

[24] K. Wosilait, "Research as a guide for the development of tutorials to improve student understanding of geometrical and physical optics," Ph.D. dissertation, Department of Physics, University of Washington, 1996 (unpublished).

[25] Other papers that report on our research and curriculum development in physical optics include: K. Wosilait, P.R.L. Heron, P.S. Shaffer, and L.C. McDermott, "Addressing student difficulties in applying a wave model to the interference and diffraction of light," accepted for publication in Physics Education Research, a Supplement to the *American Journal of Physics;* B.S. Ambrose, P.R.L. Heron, S. Vokos, and L.C. McDermott, "An investigation of student understanding of light as an electromagnetic wave: Relating common formal representations to physical phenomena," to be published in the *American Journal of Physics.*

[26] L.C. McDermott, P.S. Shaffer, and the Physics Education Group at the University of Washington, *Tutorials in Introductory Physics,* Preliminary Edition (Prentice Hall, Upper Saddle River, NJ, 1998). The research reported in this paper has guided the development and assessment of several tutorials on optics.

[27] Research by our group on geometrical optics has also guided the development of a laboratory-based curriculum intended for the preparation of precollege teachers, for underprepared

students, and for liberal arts majors. See L.C. McDermott and the Physics Education Group at the University of Washington, *Physics by Inquiry* (Wiley, New York, 1996) Vols. I and II.

[28] For descriptions of how the Physics Education Group conducts research on student understanding of physics, see Refs. 22-25, 33, and 37. Also, see D.E. Trowbridge and L.C. McDermott, "Investigation of student understanding of the concept of velocity in one dimension," Am. J. Phys. **48**, 1020-1028 (1980); D.E. Trowbridge and L.C. McDermott, "Investigation of student understanding of the concept of acceleration in one dimension," *ibid.* **49**, 242-253 (1981); L.C. McDermott and P.S. Shaffer, "Research as a guide to curriculum development: an example from introductory electricity, Part I: Investigation of student understanding, *ibid.* **60**, 994-1003 (1992); Erratum to Part I, *ibid.* **61**, 81(1993).

[29] The term *geometric image* refers to the bright region on a screen that would be produced by the rectilinear propagation of light from a source through an aperture to the screen. For a discussion of the differences between this type of image and the real image formed by a converging lens, see F. Goldberg, S. Bendall, and I. Galili, "Lenses, pinholes, screens, and the eye," Phys. Teach. **29**, 221-224 (1991).

[30] In an introductory course, interactions between the electromagnetic waves and the material of which the slit edges are made are not considered. A more rigorous approach can be found in S.G. Lipson, H. Lipson, and D.S. Tannhauser, *Optical Physics,* (Cambridge University Press, Cambridge, UK, 1995).

[31] Edge diffraction is not typically emphasized in the course, although it is mentioned briefly in some texts and by some instructors during lecture.

[32] The students were not expected to recognize that the axis of polarization of the light has an effect on the diffraction pattern. For a discussion of how the polarization of light can change the diffraction pattern, see T.J. Mayes and B.F. Melton, "Fraunhofer diffraction of visible light by a narrow slit," Am. J. Phys. **62**, 397-403 (1994); T.J. Racey, P. Rochon, and N. Gauthier, "Effect of light polarization on the diffraction pattern of small wires," *ibid.* **53**, 783-786 (1985).

[33] This difficulty and others related to student beliefs about photons are discussed in R.N. Steinberg, G.E. Oberem, and L.C. McDermott, "Development of a computer-based tutorial on the photoelectric effect," Am. J. Phys. **64**, 1370-1379 (1996).

[34] Another explanation, not usually presented in introductory courses, is based on Feynman's path integral approach. In this formulation, all paths between the emitter and a point on the screen is ascribed a phase. The sum of the contributions of all paths yields the standard intensity pattern. See R.P. Feynman, *QED, The Strange Theory of Light and Matter,* (Princeton University Press, Princeton, NJ, 1985.)

[35] For a description of a demonstration of a low-intensity, double-slit interference experiment, see S. Parker, "Single-photon double-slit interference—a demonstration," Am. J. Phys. **40**, 1003-1007 (1972). An explanation intended for a first course in modern physics can be found in P.A. Tipler, *Modern Physics,* (Worth Publishers, New York, 1978) p. 185.

[36] See the second paper in Ref. 25.

[37] In addition to Refs. 23, 24, and 33, see L.C. McDermott, P.S. Shaffer, and M.D. Somers, "Research as a guide for teaching introductory mechanics: An illustration in the context of the Atwood's machine," Am. J. Phys. **62**, 46-55 (1994) and T. O'Brien Pride, S. Vokos, and L.C. McDermott, "The challenge of matching learning assessments to teaching goals: An example from the work-energy and impulse-momentum theorems," *ibid.* **66**, 147- 157 (1998).

[38] C. J. Davisson and L. H. Germer, "Diffraction of electrons by a crystal of nickel," Phys. Rev. **30**, 705-740 (1927). Results from single-slit, double-slit, and multiple-slit experiments with electrons have been published over 30 years ago: G. Möllenstedt and C. Jönsson, "Elektronen-Mehrfachinterferenzen an regelmässig hergestellten Feinspalten," Z. Phys. **155**, 472-474

(1959); C. Jönsson, "Elektroneninterferenzen an mehreren künstilich hergestellten Feinspalten," Z. Phys. **161**, (1961); C. Jönsson, "Electron diffraction at multiple slits," Am. J. Phys. **42**, 4-11 (1974); O. Donati, G. F. Missiroli and G. Pozzi, "An experiment on electron interference," Am. J. Phys. **41**, 639-644 (1973); P. G. Merli, G. F. Missiroli and G. Pozzi, "On statistical aspects of electron interference phenomena," Am. J. Phys. **44**, 306-307 (1976); G. Matteucci and G. Pozzi, "Two further experiments on electron interference," Am. J. Phys. **46**, 619-623 (1978); and A. Tonomura, J. Endo, T. Matsuda and T. Kawasaki, "Demonstration of single-electron buildup of an interference pattern," Am. J. Phys. **57**, 117-120 (1989). For an example of single-slit and double-slit experiments with neutrons, see R. Gahler and A. Zeilinger, "Wave-optical experiments with very cold neutrons," Am. J. Phys. **59**, 316-324 (1991).

[39] For an example of a semi-qualitative approach to bound state wave functions in one dimension, see A.P. French and E.F. Taylor, "Qualitative plots of bound state wave functions," Am. J. Phys. **39**, 961-962 (1971), and A.P. French and E. F. Taylor, *An Introduction to Quantum Mechanics,* M.I.T. Introductory Physics Series (Norton, New York, 1978).

[40] Details from this research are described in Chapter 2 and in B.S. Ambrose, P.S. Shaffer, R.N. Steinberg, and L.C. McDermott, "An investigation of student understanding of single-slit diffraction and double-slit interference," Am. J. Phys. **67**, 146-155 (1999).

[41] In most of the classes the students were asked explicitly to determine whether the bright regions move farther apart, closer together, or stay in the same locations as a result of the change. In the Autumn 1995 quantum mechanics course the students were asked instead to predict how, if at all, the pattern would be different. An increase in the velocity or kinetic energy of the electrons (while keeping the electron flux incident on the slit(s) the same) would cause the overall intensity on the screen to increase. However, the students were not required to explain how the intensity would change in order to be counted as correct. We found that the difference in wording did not seem to affect student performance.

[42] An error was made in the wording of the "Δm" question. It would not be possible to use a phosphorescent screen in order to visually observe an interference or diffraction pattern produced by neutrons. However, none of the student responses directly or indirectly indicated confusion for this reason.

[43] See Chapter 2.

[44] L.C. McDermott, P.S. Shaffer and the Physics Education Group at the Washington, *Tutorials in Introductory Physics,* preliminary edition (Prentice Hall, Upper Saddle River, NJ, 1998). For additional details on research conducted to assess the effectiveness of the tutorials, see: K. Wosilait, "Research as a guide for the development of tutorials to improve student understanding of geometrical and physical optics," Ph.D. dissertation, Department of Physics, University of Washington, 1996 (unpublished); K. Wosilait, P.R.L. Heron, P.S. Shaffer, and L.C. McDermott, "Addressing student difficulties in applying a wave model to the interference and diffraction of light," accepted for publication in Physics Education Research, a Supplement to the *American Journal of Physics.*

[45] The information in this table is also presented in the third reference in Ref. 44.

[46] About half of the quantum mechanics students who used correct reasoning to answer the Δa question on the examination problem about electron diffraction justified their answers by using the Heisenberg Uncertainty Principle. (These students are included in Table 3-4 and Table 3-6.) On a subsequent question about changing the mass of the particles in the diffraction experiment, many attempted to use reasoning about de Broglie wavelength but did so incorrectly.

[47] Similar difficulties in the context of double-slit interference are described in Chapter 2.

[48] Students who apply the equation $v = \lambda f$ for electrons or other massive particles may be failing to recognize the difference between phase velocity and group velocity. In general, the physical velocity of a particle is the group velocity, $v_g \equiv \partial\omega/\partial k$, *i.e.*, the first derivative with respect to wave number k of the dispersion relation $\omega(k)$ for the particle. The phase velocity, however, is simply the quotient of $\omega(k)$ and k.

From the application of the de Broglie and Einstein relations for momentum and energy ($p = \hbar k$ and $E = \hbar\omega$, respectively), the dispersion relation for massive particles moving (nonrelativistically) in free space is proportional to k^2:

$$\omega(k) = E/\hbar = p^2/2\hbar m = \hbar k^2/2m$$

From this result we find that the group velocity for massive particles is not equal to the phase velocity:

$$v_g = \partial\omega(k)/\partial k = \hbar k/m$$

$$v_p = \omega(k)/k = \hbar k/2m \neq v_g$$

For photons and other massless particles, however, the dispersion relation is linear in p and thus linear in k:

$$\omega(k) = E/\hbar = pc/\hbar = \hbar ck/\hbar = ck$$

The group velocity $v_g = d\omega(k)/dk$ for photons is therefore equal to the phase velocity:

$$v_g = \partial(ck)/\partial k = c = \omega(k)/k = v_p$$

Thus, $v = \lambda f$ is equivalent to the relationship $v = \omega/k$ that describes both the phase velocity and group velocity for the special case of light (and other massless particles).

[49] In addition to the research described in Chapter 2 of this dissertation, see: B.S. Ambrose, P.S. Shaffer, R.N. Steinberg, and L.C. McDermott, "An investigation of student understanding of single-slit diffraction and double-slit interference," Am. J. Phys. **67**, 146-155 (1999); K. Wosilait, "Research as a guide for the development of tutorials to improve student understanding of geometrical and physical optics," Ph.D. dissertation, Department of Physics, University of Washington, 1996 (unpublished); K. Wosilait, P.R.L. Heron, P.S. Shaffer, and L.C. McDermott, "Addressing student difficulties in applying a wave model to the interference and diffraction of light," accepted for publication in Physics Education Research, a Supplement to the *American Journal of Physics.*

[50] The strategy used in this part of the tutorial is very similar to that on the homework for two-source interference in L. C. McDermott, P. S. Shaffer and the Physics Education Group at the Washington, *Tutorials in Introductory Physics,* preliminary edition (Prentice Hall, Upper Saddle River, NJ, 1998).

[51] P.A. Tipler, *Modern Physics,* (Worth Publishers, New York, 1978), p. 185.

[52] A description of the *elicit-confront-resolve* instructional strategy is described in Chapter 1. For more details, see also L. C. McDermott, "What we teach and what is learned—Closing the gap," Am. J. Phys. **59**, 301-315 (1991).

[53] See Ref. 49.

[54] An error was made in the wording of the Δm question that was included on the pretest given to the Spring 1997 algebra-based class. It would be not be possible to use a phosphorescent screen in order to visually observe the interference pattern produced by neutrons. However, none of the student responses, on this pretest or on any other questions presented in this chapter, directly or indirectly indicated confusion for this reason.

[55] The instruction in the second bullet of the potential step question should read "draw a graph of Re{ $\psi(x)$ }" rather than "draw the shape of $\psi(x)$." However, none of the students expressed confusion while attempting to answer the question. Analysis of the student responses indicated that the students seemed to interpret the instruction in the intended fashion. The error in wording was corrected on all subsequent questions.

[56] Many difficulties stem from the failure to distinguish between a quantity and the rate of change of that quantity. For examples from kinematics in one dimension, see D. E. Trowbridge and L. C. McDermott, "Investigation of student understanding of the concept of velocity in one dimension," Am. J. Phys. **48**, 1020-1028 (1980) and D. E. Trowbridge and L. C. McDermott, "Investigation of student understanding of the concept of acceleration in one dimension," Am. J. Phys. **49**, 242-253 (1981). For examples from electric circuits, see L. C. McDermott and P. S. Shaffer, "Research as a guide for curriculum development: An example from introductory electricity, Part I: Investigation of student understanding," Am. J. Phys. **60**, 994-1003 (1992).

[57] L. C. McDermott, P. S. Shaffer and the Physics Education Group at the University of Washington, *Tutorials in Introductory Physics*, preliminary edition (Prentice Hall, Upper Saddle River, NJ, 1998).

[58] For an example of instructional approaches that rely on the determination of the curvature of the wave function in terms of the local potential, see A. P. French, and Edwin F. Taylor, "Qualitative plots of bound state wave functions," Am. J. Phys. **39**, 961-962 (1971); A. P. French and E. F. Taylor, *An Introduction to Quantum Mechanics,* M.I.T. Introductory Physics Series (Norton, New York, 1978). Examples of simulation software that place similar emphasis on the qualitative shape of the wave function include: *The M.U.P.P.E.T. Utilities,* by E.F. Redish, J.M. Wilson, and I.D. Johnston (Physics Academic Software, North Carolina State University, 1993); *Quantum Mechanics Simulations: The Consortium for Upper-Level Physics Software,* by J.R. Hiller, I.D. Johnston, and D.F. Styler (Wiley, New York, 1995); and *Visual Quantum Mechanics* by D. Zollman and the Physics Education Research Group at Kansas State University.

[59] See Ref. 56.

[60] Other treatments of quantum mechanics, such as hidden-variables theories and pilot waves, are not usually treated at an undergraduate level.

[61] For a descriptive study of difficulties among German Gymnasium students that seem to stem from an overemphasis of the Bohr atomic model, see H. Fischler and M. Lichtfeldt, "Modern physics and students' conceptions," Int. J. Sci. Ed. **14**, 181-190 (1992).

[62] The reasoning described here to compare the amplitude of the wave function in different regions of the double-tiered square well would be correct only for a stationary state of sufficiently high energy, *i.e.,* a state for which the wave function would have many nodes in each of the regions *b* and *c.* As discussed in Appendix B, none of the students were confused by thinking of low-lying states.

[63] D. F. Styer, "Common misconceptions regarding quantum mechanics," Am. J. Phys. **64**, 31-34 (1996).

[64] The strategy employed in the third interview task was very similar to that used in preliminary interviews on diffraction (see Chapter 2). In those interviews, the students were asked to consider light incident on a slit 1.0 cm wide, a situation in which geometrical optics is valid. The students were then asked to consider slit widths of gradually smaller and smaller value, in order to probe their understanding of the relationship between wavelength and slit width when physical optics would be needed instead of geometrical optics. The strategy in the pebble-and-shoebox interviews is very similar. The students are asked to consider smaller and smaller values of the mass of the pebble, and thus smaller and smaller values of its momentum. The intent was to probe student understanding of the relationship between the de Broglie

wavelength of the pebble and the length of the shoebox when quantum mechanics would be needed instead of classical mechanics.

[65] An issue that the tutorial *Reflection and transmission* did not raise is the apparent inconsistency between the relationship between wave speed and wavelength for pulses on springs and for electrons incident on the potential step. While the analogy between the two situations is useful in recognizing that reflection and transmission both occur at a boundary, the wavelength of a wave on a spring is proportional to the wave speed, while the de Broglie wavelength of a particle is inversely proportional to group velocity. This apparent contradiction is resolved by considering the appropriate dispersion relation $\omega(k)$ for each type of wave. Subsequent versions of the worksheet will therefore include a series of questions intended to help students recognize the limitations of the analogy.

[66] When the tutorial *Reflection and transmission* was first tested, we found that, even after standard instruction on the Schrödinger equation in one dimension, many students had serious difficulties expressing the wave function for a beam of free particles as a plane wave (*i.e.*, $\Psi(x, t) \propto e^{\pm ikx - i\omega t}$), which had already been covered in class. The tutorial (and optional homework exercises accompanying the tutorial) had not been designed to address such a difficulty, so the exercises on the tutorial worksheet that dealt with the algebraic expressions for the wave function were covered in standard lecture format.

[67] K. Wosilait, "Research as a guide for the development of tutorials to improve student understanding of geometrical and physics," Ph.D. dissertation, Department of Physics, University of Washington, 1996 (unpublished).

[68] B.S. Ambrose, "Investigation of student understanding of the wave-like properties of light and matter," Ph.D. dissertation, Department of Physics, University of Washington, 1999 (unpublished).

[69] B.S. Ambrose, P.S. Shaffer, R.N. Steinberg, and L.C. McDermott, "An investigation of student understanding of two-source interference and single-slit diffraction," Am. J. Phys. **67**, 146-155 (1999).

[70] K. Wosilait, P.R.L. Heron, P.S. Shaffer, and L.C. McDermott, "Addressing student difficulties in applying a wave model to the interference and diffraction of light," accepted for publication in Physics Education Research, a Supplement to the *American Journal of Physics*.

[71] In an earlier study, we had examined the ability of students to relate the formal representations used in geometrical optics to physical phenomena. See F.M. Goldberg and L.C. McDermott, "Student difficulties in understanding image formation by a plane mirror," Phys. Teach. **24**, 472-480 (1986) and "An investigation of student understanding of the real image formed by a converging lens or concave mirror," Am. J. Phys. **55**, 108-119 (1987).

[72] For a report of another study in which the Physics Education Group examined the ability of students to make connections among concepts, their formal representations, and the real world, see M.L. Rosenquist and L.C. McDermott, "A conceptual approach to teaching kinematics," Am. J. Phys. **55**, 407-415 (1987) and L.C. McDermott, M.L. Rosenquist, and E. van Zee, "Student difficulties in connecting graphs and physics: Examples from kinematics," *ibid.* **55**, 503-513 (1987).

[73] L.C. McDermott, P.S. Shaffer, and the Physics Education Group at the University of Washington, *Tutorials in Introductory Physics*, Preliminary Edition (Prentice Hall, Upper Saddle River, NJ, 1998).

[74] In addition to Refs. 67–70, see L.C. McDermott, P.S. Shaffer, and M.D. Somers, "Research as a guide for teaching introductory mechanics: An illustration in the context of the Atwood's machine," Am. J. Phys. **62**, 46-55 (1994); T. O'Brien Pride, S. Vokos, and L.C. McDermott, "The challenge of matching learning assessments to teaching goals: An example from the work-energy and impulse-momentum theorems," *ibid.* **66**, 147-157 (1998).

[75] In addition to Refs. 67–69, see R.N. Steinberg, G.E. Oberem, and L.C. McDermott, "Development of a computer-based tutorial on the photoelectric effect," Am. J. Phys. **64**, 1370-1379 (1996).

[76] The students were not expected to recognize that the axis of polarization affects the diffraction pattern. For a discussion of this effect, see T.J. Mayes and B.F. Melton, "Fraunhofer diffraction of visible light by a narrow slit," Am. J. Phys. **62**, 397-403 (1994) and T.J. Racey, P. Rochon, and N. Gauthier, "Effect of light polarization on the diffraction pattern of small wires," *ibid.* **53**, 783-786 (1985).

[77] K. Wosilait, P.R.L. Heron, P.S. Shaffer, and L.C. McDermott, "Development and assessment of a research-based tutorial on light and shadow," Am. J. Phys. **66**, 906-913 (1998).

[78] P.R.L. Heron and L.C. McDermott, "Bridging the gap between teaching and learning in geometrical optics: The role of research," Opt. & Phot. News **9** (9), 30-36 1998.

[79] For a discussion of a related difficulty in the context of springs, see R.N. Steinberg, M.C. Wittmann, and E.F. Redish, "Sample class on mathematical tutorials in introductory physics" in Proceedings of the International Conference on Undergraduate Physics Education (ICUPE), College Park, MD, July 31-Aug. 3, 1996, edited by E.F. Redish and J.S. Rigden, AIP Conf. Proc. **399** (AIP, Woodbury, NY, 1997).

[80] Student difficulties with the field-line representation have been documented in electrostatics. See, for example, S. Törnkvist, K.A. Pettersson and G. Tranströmer, "Confusion by representation: On students' comprehension of the electric field concept," Am. J. Phys. **61**, 335-338 (1993) and R.R. Harrington, "An investigation of student understanding of electric concepts in the introductory university physics course," Ph.D. dissertation, Department of Physics, University of Washington, 1995 (unpublished).

[81] In most introductory courses, conductivity (σ) is treated as a scalar quantity. Thus the current and the electric field can be assumed to be collinear.

[82] Some student difficulties with mechanical waves are discussed in M.C. Wittmann, R.N. Steinberg, and E.F. Redish, "Making sense of how students make sense of mechanical waves," Phys. Teach. **37**, 15-21 (1999).

[83] It has been our experience that, after working through a tutorial, the success rate of the graduate TA's on post-test questions is close to 100%. See, for example, the last article in Ref. 74.

[84] For other examples, see Ref. 77 and the last paper in Ref. 74.

[85] D. F. Styer, "Common misconceptions regarding quantum mechanics," Am. J. Phys. **64**, 31-34 (1996).

[86] See Ref. 85.

[87] See Ref. 85.

[88] See I.D. Johnston, K. Crawford and P.R. Fletcher, """Student difficulties in learning quantum mechanics," Int. J. Sci. Ed. **20**, 427-446 (1998); H. Fischler and M. Lichtfeldt, "Modern physics and students' conceptions," Int. J. Sci. Ed. **14**, 181-190 (1992); and R. Cervellati and D. Perugini, "The understanding of the atomic orbital concept by Italian high school students," J. Chem. Ed. **58**, 568-569 (1981).

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APPENDIX A: DISCUSSION OF RESULTS FROM WRITTEN PRETEST QUESTIONS ON REFLECTION AND TRANSMISSION OF PULSES ON SPRINGS

In introductory waves and optics, a topic that is typically covered is the reflection and transmission of pulses on springs. Students are taught that when a pulse is incident on a junction between two different springs, both a reflected and a transmitted pulse are created at the junction. They also are shown how to relate qualitatively the widths of the incident, reflected, and transmitted pulses to the wave speeds of the springs.

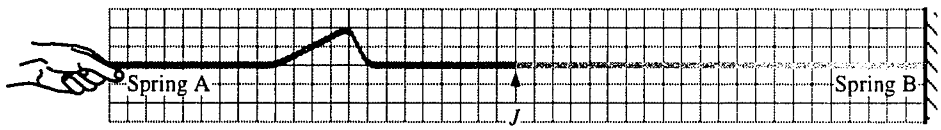
Results presented in Chapter 5 of this dissertation indicate, however, that quantum mechanics students at the undergraduate and graduate level have difficulty interpreting reflection and transmission of matter in terms of a wave model. The research presented in this Appendix was conducted in order to probe for the presence of similar difficulties among both introductory and advanced students in the simpler context of pulses on springs.

A. DESCRIPTION OF WRITTEN QUESTION

The problem shown in Figure A-1 has been administered on written pretests that precede a tutorial on reflection and transmission of pulses.[i] The question has been given to (i) introductory students before standard lecture instruction (and before tutorial instruction), (ii) introductory students after standard instruction (but before tutorial instruction), and (iii) participants in a graduate teaching seminar in which tutorial teaching assistants are trained to teach introductory tutorials. (The structure of the introductory course and teaching seminar are described in detail in Chapter 1.)

A student performs an experiment with two springs, A and B, attached end-to-end at point *J* as shown below. The student notes that pulses travel more quickly in spring A than in spring B.

The student generates an asymmetric pulse in spring A that moves to the right as shown below.



Sketch the shape of spring A and spring B a short time after the entire pulse has reached the junction point, *J*, between the two springs.

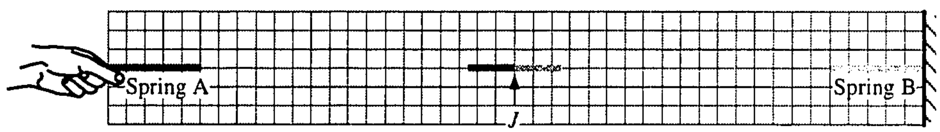


Figure A-1: Pretest question posed to introductory calculus-based waves and optics students on reflection and transmission of pulses on springs.

Qualitatively correct answer. To answer correctly, students needed to show both a reflected pulse and a transmitted pulse in their sketches. A response that would be accepted as qualitatively correct is shown in Figure A-2.

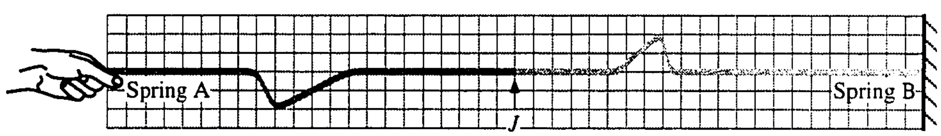


Figure A-2: Diagram showing the correct orientations, relative widths, and relative positions of the reflected and transmitted pulses. The amplitudes of the pulses are exaggerated for clarity.

It is given that the wave speed in spring B is less than that in spring A, which would imply that the linear mass density of spring A is less than that of spring B. The difference in wave speeds in the two springs would cause differences in the relative widths, relative locations, and orientations of the reflected and transmitted pulses.

First, the time interval required for a transmitted pulse to completely pass by a given point on spring B is equal to the corresponding time interval for the incident (or reflected) pulse to pass by a given point on spring A. This results implies that the ratio of the widths of the transmitted and incident pulses is equal to the ratio of the wave speeds of the two springs. Thus the width of the transmitted pulse would be smaller than that of the incident pulse, and the width of the reflected pulse would be equal to that of the incident pulse.

Second, the problem statement indicates that the wave speed in spring B is smaller than that in spring A. The distance between the transmitted pulse and the junction would therefore be smaller than the distance between the incident pulse and the junction.

Third, the linear mass density of spring B is greater than that of spring A because each spring has the same tension (otherwise the junction point *J* would accelerate horizontally) and spring B has the greater wave speed. This result means that the original pulse on spring A would be incident on another spring of greater mass density. The reflection at the junction would therefore be similar to the reflection at a fixed end of spring. Thus the reflected pulse would have the opposite orientation as the incident pulse.

Note: The difference in wave speed between the two springs would also affect the relative amplitudes of the reflected and transmitted pulses. However, students were not required to consider the relative amplitudes when making their sketches.

B. IDENTIFICATION OF SPECIFIC DIFFICULTIES

Below we present results from the analysis of student responses to the pretest question described above. In particular, we focus on two specific difficulties that arose in both the context of pulses on springs and in the more advanced context of scattering states of matter. We found that many students: (i) failed to recognize that reflection would occur at a junction between two regions of different wave speed and (ii) failed to relate the widths of the incident and transmitted pulses to the wave speeds.

1. FAILURE TO RECOGNIZE THAT REFLECTION ALWAYS OCCURS AT THE JUNCTION

Table A-1 summarizes part of the analysis of the student sketches of spring A, the spring containing the reflected pulse. In particular, we show only the percentages of students who did or did not include a reflected pulse in their sketches. (Other features of the reflected pulse, such as width, location, and orientation, were not taken into account.) The results from separate classes were similar and so are combined in the table.

Table A-1: Results from pretest question on reflection and transmission of pulses on springs. Shown are the percentages of students who did or did not recognize that reflection would occur at the junction point *J*.

	Introductory calculus-based physics students		**Participants in teaching seminar**
	Before standard instruction $N = 1180$	*After standard instruction* $N = 237$	*After standard instruction* $N = 70$
Reflected pulse shown *(correct)*	35%	70%	85%
No reflected pulse shown	50%	20%	15%

About 50% of the introductory students before lecture instruction included a transmitted pulse in their sketch but not a reflected pulse. This percentage decreases to 20% for introductory students after lecture instruction and to 15% for teaching assistants in the graduate seminar. At first glance, it would seem that standard instruction addresses this difficulty to some extent. However, it is remarkable that the difficulty was still prevalent among some of the physics graduate students.

Informal interactions with students have revealed some insight into the nature of this particular difficulty. After standard instruction, and even after observing that reflection occurs when a pulse is incident on a spring of *smaller* wave speed, students often say that a pulse approaching a different spring with *greater* wave speed would be completely transmitted. These students often articulate the idea that reflection only occurs when the incident pulse experiences "resistance" at the boundary. For many students, this "resistance" occurs only when the pulse is incident on a different spring with smaller wave speed. The pretest asks students to predict the outcome of one such experiment. Thus the failure to take into account reflection at the boundary may be more serious than the results in Table A-1 would indicate.

2. *FAILURE TO RELATE THE WIDTHS OF THE INCIDENT AND TRANSMITTED PULSES TO THE WAVE SPEEDS*

Table A-2 shows summarizes results from the analysis of student sketches for spring B, the spring containing the transmitted pulse. We restrict our attention to the width ascribed to the transmitted pulse by the students. As before, the results from separate classes were similar and so have been combined.

Table A-2: Results from pretest question on reflection and transmission of pulses on springs. Results indicate the ability of students to relate wave speed to pulse width.

	Introductory calculus-based physics students		**Participants in teaching seminar**
	Before standard instruction $N = 1180$	*After standard instruction* $N = 237$	*After standard instruction* $N = 70$
Correct (transmitted pulse narrower than incident pulse)	**30%**	**40%**	**50%**
Incorrect	**55%**	**45%**	**40%**
Transmitted pulse has *the same width* as does the incident pulse	40%	35%	30%
Transmitted pulse is *wider* than incident pulse	15%	10%	10%

The task of drawing the transmitted pulse proved to be difficult for all students, including graduate students. Only 30% of the introductory students before lecture instruction recognized that the width of the transmitted pulse in spring B would be smaller than that of the incident pulse in spring A. This percentage increases only slightly to 40% of the introductory students after lecture and to 50% of the teaching assistants in the graduate teaching seminar. The most common error in all three populations was to indicate that the transmitted pulse had the same width as the incident pulse.

C. SUMMARY

Students are expected to be familiar with basic wave phenomena by the time they begin to study quantum mechanics. When introduced to the idea that matter can exhibit wave properties, students are often led to make analogies between matter waves and other types of waves encountered previously during instruction, including mechanical waves and light. The results presented in this Appendix, however, suggest that specific difficulties in understanding reflection and transmission of pulses on springs are not always adequately addressed by standard instruction. Some of the difficulties are analogous to those that arose in the context of scattering from a potential step or barrier, which are described in Chapter 5 of this dissertation.

In introductory waves and optics, the mistaken belief that reflection would not occur at a junction between two springs decreases from 50% before standard instruction to 20% after standard instruction. (See Table A-1.) The corresponding percentage among physics graduate students is even smaller but is not zero. In the more abstract context of scattering from a potential step or potential barrier, however, the analogous difficulty recurs even after standard instruction in quantum mechanics. Thus the failure to take into account reflection at a boundary is a difficulty that appears context-dependent.

The ability to relate the relative wave speeds and widths of the transmitted and incident pulses is difficult after standard instruction at both the introductory and advanced levels. Only 40% of the introductory students and 50% of the graduate students seemed to recognize that the smaller speed for the transmitted pulse would imply a smaller width as well. This error is similar to those the errors that arose on tunneling questions given to students in a graduate-level quantum mechanics course. We do not have sufficient information to tell whether addressing this difficulties in the earlier context of pulses on springs (at the introductory level) would affect the prevalence of corresponding errors in the context of tunneling (at the advanced level). However, the prevalence of this difficulty among advanced students signals a need to address such errors at all levels of instruction.

NOTE FOR APPENDIX A

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[i] L. C. McDermott, P. S. Shaffer and the Physics Education Group at the Washington, *Tutorials in Introductory Physics,* Preliminary Edition (Prentice Hall, Upper Saddle River, NJ, 1998).

APPENDIX B: WRITTEN QUESTIONS AND INTERVIEW TASKS USED TO PROBE STUDENT UNDERSTANDING OF BASIC CONCEPTS IN QUANTUM MECHANICS

In this appendix we present specially designed qualitative questions that were used to probe student understanding of basic concepts in quantum mechanics. The questions were administered either in written form or in individual student interviews. Analysis of student responses indicated the prevalence of particular difficulties at various stages of instruction. (The results are discussed in detail in Part II of this dissertation.)

A. QUALITATIVE QUESTIONS ABOUT SCATTERING STATES

In this section we describe several questions that pertained to scattering states in one dimension. After a description of each question (and any variations of that question that were also used), we give a response that was accepted as correct with correct reasoning.

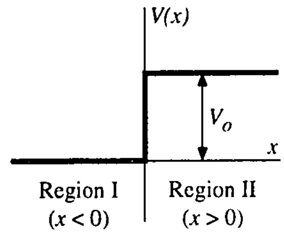
1. POTENTIAL STEP QUESTION

Figure B-1 below illustrates a question that involves a beam of monoenergetic particles incident on a potential barrier. Students consider three cases (A–C) that differ according to the preparation of the incident particles.

Monoenergetic electrons travel through a region in which the potential energy $V(x)$ varies with x as follows:

$$V(x) = \begin{cases} 0, & x < 0 \\ V_o, & x > 0 \end{cases}$$

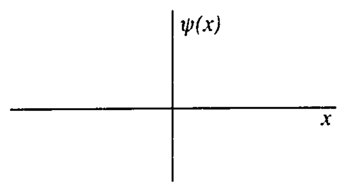
Let E represent the total energy of each electron, and let $\psi(x)$ represent the wave function associated with the electrons.



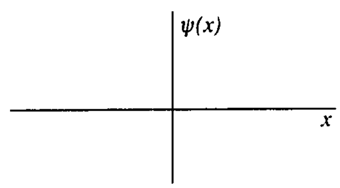
For each case described below:

- Describe in words the behavior of the electrons in Regions I and II. In particular: (i) How does the speed of the electrons in Region I compare to the speed of the electrons in Region II? (ii) Do all of the electrons that start in one Region end up in the other Region?
- On the set of axes provided, draw the shape of $\psi(x)$. Clearly indicate how, if at all, the wave function in $x < 0$ is qualitatively different from the wave function in $x > 0$.
- Briefly explain the reasoning you used in each case.

A. The electrons initially move *from left to right* (*i.e.*, in the $+x$ direction) with $E > V_o$.



B. The electrons initially move *from right to left* (*i.e.*, in the $-x$ direction) with $E > V_o$



C. The electrons initially move *from left to right* (*i.e.*, in the $+x$ direction) with $E < V_o$

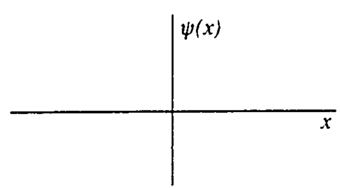


Figure B-1: Potential step question, given before standard instruction on scattering. Difficulties elicited by this question were very similar to those by other questions posed after standard instruction.

Note about wording of the question. The instruction in the second bullet should read "draw a graph of Re{ $\psi(x)$ }" rather than "draw the shape of $\psi(x)$." However, the students seemed to interpret the instruction in the intended fashion.

Predicting whether all electrons that start in one region would end up in the other region ($E > V_o$). The potential $V(x)$ undergoes a sudden, finite jump at $x = 0$, *i.e.,* the change in potential occurs over a spatial interval much smaller than the de Broglie wavelength of the incident electrons. Therefore, in cases A and B, in which the total energy of each electron is greater than the height of the potential step, the electrons would exhibit both reflection and transmission at the $x = 0$. Thus not all electrons that start in one region will end up in the other, regardless of the region from which the electrons are incident on the step. We counted as correct reasoning any explanation that included a reference to the reflection of electrons at $x = 0$ or to the wave-like nature of electrons.

Qualitatively correct sketch of the wave function: Cases A and B. A qualitatively correct sketch of (the real part of) the wave function for cases in which $E > V_o$ (*i.e.,* cases A and B) is shown in Figure B-2. To be counted as correct, the wave functions in the two regions had to be shown as varying sinusoidally with x about the horizontal axis with the correct relative de Broglie wavelengths in the two regions. Specifically, the kinetic energy of the particles in Region I ($x < 0$) is larger than that in Region II ($x > 0$), so the de Broglie wavelength, or the distance between consecutive peaks in the wave function, must be smaller in Region I than in Region II.

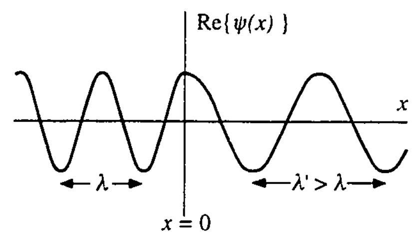


Figure B-2: Graph of Re{ $\psi(x)$ } showing qualitatively correct wavelengths for the case $E > V_0$ (parts A and B).

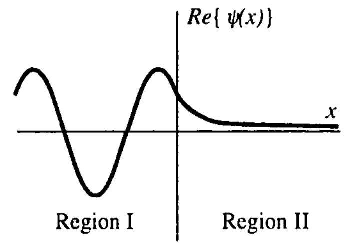


Figure B-3: Qualitatively correct graph of Re{ $\psi(x)$} for part C of the potential step question.

Qualitatively correct sketch of the wave function: Case C. A qualitatively correct sketch of (the real part of) the wave function for the case in which $E < V_o$ (*i.e.,* case C) is shown in Figure B-3. We counted as correct those graphs in which the wave function (i) varied sinusoidally as a function of x in Region I, (ii) was continuous at $x = 0$ (the boundary between the two regions), and (iii) asymptotically approached zero in Region II. Even though Region II is classically forbidden, the wave function is non-zero in this region.

There are several other criteria that one could require for a qualitatively correct answer but were not used in our analysis. For example, the slope of the wave function must be continuous at $x = 0$. The wave function must be non-zero at the boundary. In addition, for those cases in which $E > V_o$ (*i.e.,* cases A and B), one could apply the boundary conditions mentioned above to determine that Re{ $\psi(x)$} has the same amplitude in both regions.

2. *POTENTIAL BARRIER QUESTIONS*

We have also given a variety of written questions that deal with scattering from a potential barrier. Described below are three different categories of potential barrier questions that involve: (1) monoenergetic particles incident on a finite rectangular barrier, (2) a wave packet incident on an arbitrary potential barrier, and (3) the probability current on both sides of a rectangular barrier.

a) Scattering of monoenergetic particles from a potential barrier

The two versions of the potential barrier question shown below pertain to a beam of monoenergetic particles incident on a rectangular potential barrier. One version was used after standard instruction, the other was used after modified instruction.

A beam of particles is incident from the left on the potential barrier shown at right. Consider the case for which the total energy E of each particle is ***less than*** V_o.

In the space provided below right, sketch a qualitatively correct graph of the real part of the wave function, $\mathrm{Re}\{\psi(x)\}$, for this case. Explain the reasoning you used to draw your graph.

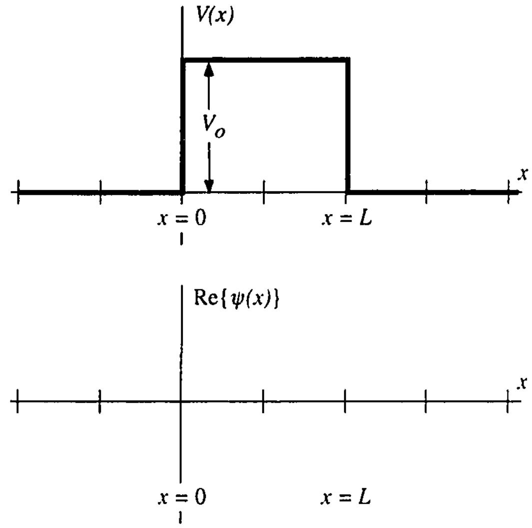


Figure B-4: Version of the potential barrier question given after standard instruction. Related versions of the question included one or more parts that asked for either (1) the functional form of $\psi(x)$ or (2) a comparison of the kinetic energy of the incident and transmitted particles.

Correct answer. The potential $V(x)$ is the same in both of the regions $x < 0$ and $x > L$. Therefore, the particles have the same kinetic energy and thus the same de Broglie wavelength in these regions. Inside the barrier, in the region $0 < x < L$, the total energy of the particles is less than the potential energy, so the wave function is purely real and decreases monotonically (approximately exponentially) in this region. Thus the wave function does not have a well-defined de Broglie wavelength inside the barrier. Figure B-5 illustrates a sketch of the wave function that would be accepted as qualitatively correct. (As was the case in the potential step question, we ignored subtler features of the wave function, such as the continuity of $d\psi(x)/dx$ as a function of x, when determining which responses were correct.)

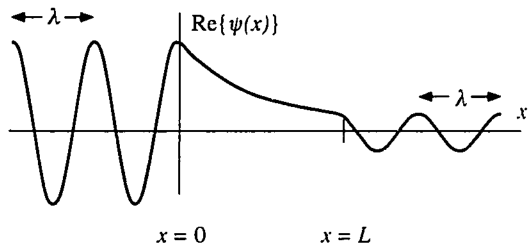


Figure B-5: Qualitatively correct graph of $\mathrm{Re}\{\psi(x)\}$ for the potential barrier question.

A second version of the potential barrier question, shown below, was used after modified instruction. In this version the potential was not equal to zero in the region containing the transmitted particles *(x > L)*.

Monoenergetic particles of energy E are incident from the left on the potential barrier of width L shown at right. (Assume that $V_1 > E > V_2$.)

In the space provided below right, carefully draw a qualitatively correct graph of Re{*ψ(x)*}, the real part of *ψ(x)*, for this case. Explain your reasoning.

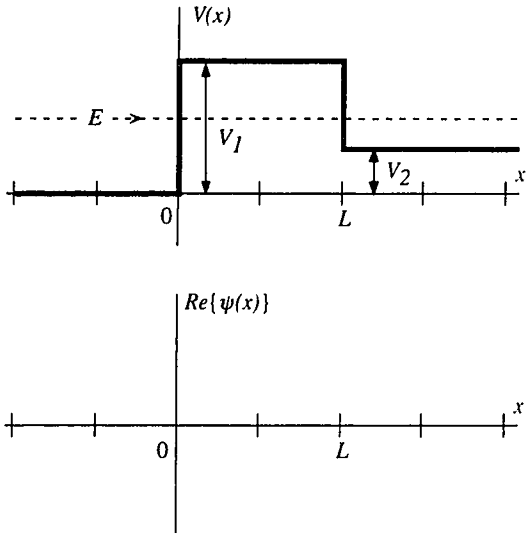


Figure B-6: Version of the potential barrier question in which the potential is not the same on both sides of the barrier.

Correct answer. A qualitatively correct graph is shown in Figure B-7. The graph is essentially the same as that for the preceding question (shown in Figure B-5) except that the de Broglie wavelength is larger in the region $x > L$ than in $x < 0$. This difference arises because the local kinetic energy of the transmitted particles is less than that of the incident particles.

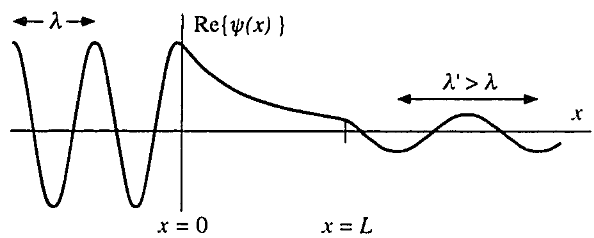
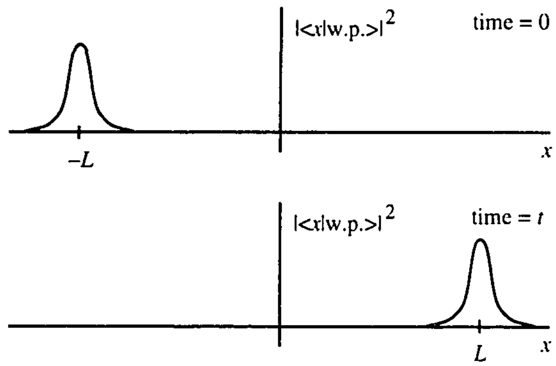


Figure B-7: Qualitatively correct graph of Re{ *ψ(x)*} *vs.* x for the tunneling post-test.

b) Scattering of a wave packet from a potential barrier

Another version of the potential barrier question pertained to a single particle, represented by a wave packet, that was incident on a potential barrier of arbitrary shape. This question is shown below.

1. Consider a free particle moving in 1-dimension. At time $t = 0$, the particle is described by a wave packet having $\langle x \rangle = -L$ and $\langle p \rangle = p_o$. At a much later time t, $\langle x \rangle = L$. This situation is shown in the pictures below (note the spread of the wave packet is not noticeable). What is p_o?



2. Now suppose there is a potential barrier around $x = 0$, as sketched below. Assume $p_o^2/2m$ is less than the maximum height of the barrier. On the axes below, sketch the probability density of the wave packet at the later time t.

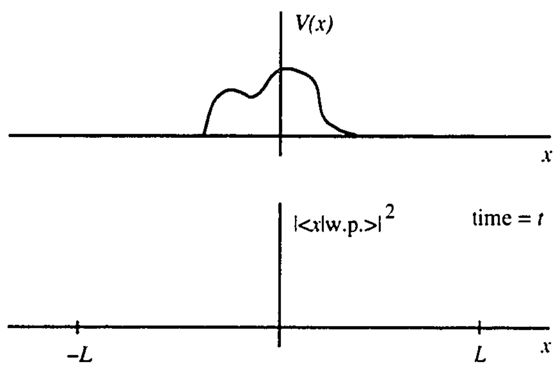


Figure B-8: Version of the potential barrier question in which a particle, represented by a wave packet, is incident on a potential barrier. Results from part 2 of the question are discussed in Chapter 5 of this dissertation.

Correct answers. In part 1, students must apply the idea that the expectation values of position and momentum obey the dynamical relationships from classical mechanics (Ehrenfest's theorem). The expectation value of the momentum (p_o) of the wave packet would therefore be equal to $m\ (2L/t)$, or $2mL/t$.

In part 2, a qualitative correct sketch of the probability density must include both a transmitted wave packet and a reflected wave packet.[i] If the spread of the wave packet is ignored, then a change in average momentum would be accompanied by a proportionate change in width, as is observed for pulses on springs. Thus the widths of the reflected and transmitted wave packets would be equal to that of the incident wave packet.

The presence of both reflected and transmitted wave packets and indicating (approximately) the correct relative widths of the packets were the only two criteria used to determine which responses would correct. For example, students were not expected to describe in detail the behavior of the wave packet inside the barrier. The incident packet would penetrate the barrier, so the reflected packet would not emerge immediately after the incident packet reaches the barrier. This result could be represented in the sketch at time t by showing that the peak of the reflected wave packet would not have reached $x = -L$ at that time. Similarly, the transmitted wave packet would require more time to propagate to $x = L$ than in the original situation. At time t the peak of the transmitted wave packet would therefore be located to the left of $x = L$. In our analysis, however, students did not have to consider these subtleties in order for their sketches to be counted as correct.

c) Probability current outside a potential barrier

A third type of question on scattering required students to consider probability current. The question pertained to a beam of particles incident on a finite rectangular potential barrier (see Figure B-4). Students were asked to compare the probability current to the left of the barrier ($x < 0$, which contains the incident particles) to that to the right of the barrier ($x > L$).

Correct answer. The probability current is the same in both regions. On the right side of the barrier, the current is due to only the transmitted particles. On the left side, the current arises from both the incident and reflected particles, which propagate in opposite directions. Because probability does not accumulate inside the barrier (or at any other location), the probability currents must be equal.

B. QUALITATIVE QUESTIONS ABOUT BOUND STATES

In this section we describe several questions that pertained to simple bound states in one dimension. As before, the description of each question is followed by a correct response.

1. INFINITE SQUARE WELL TASK

The students were told to consider a particle in an infinite square well potential. They were asked to draw the wave functions for several energy states, including the lowest state, and to explain their reasoning.

Correct answer. Shown below in Figure B-9 are graphs of the first several energy eigenstates of the infinite square well. Students could either derive these solutions to the Schrödinger equation or argue qualitatively that the wave function must be sinusoidal with respect to x (to represent a well-defined kinetic energy) and must obey the appropriate boundary conditions at $x = 0$ and $x = L$.

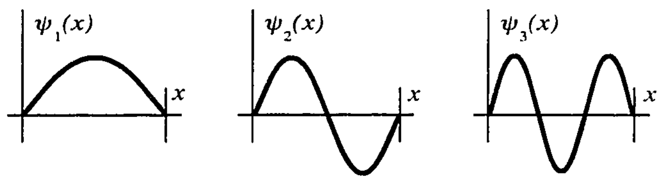


Figure B-9: The three lowest energy eigenstates for the infinite square well.

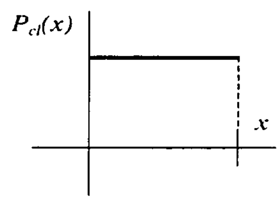


Figure B-10: Probability density for a particle undergoing classical motion in an infinite square well.

An important fact to emphasize here is that careful quantum mechanical reasoning can be taken to appropriate limits in which classical results are obtained, but the reverse process is not possible. One can consider larger and larger energy eigenvalues for the infinite square well and find that the graph of probability density ($P(x) = |\psi(x)|^2$) would contain more and more peaks. In the limit in which the energy of the particle becomes very large (compared to the ground state energy), or equivalently, in which the de Broglie wavelength of the particle becomes vanishingly small (compared to L), it becomes impossible to distinguish one peak from another. In this limit the probability density approaches the classical result, shown below in Figure B-10.

2. *PEBBLE-AND-SHOEBOX QUESTIONS*

Results from using the infinite square well task, described above, led us to develop a special task in order to probe student understanding of probability density, both in the classical and quantum mechanical regimes. It was also designed to probe the ability of students to recognize when it is valid to apply ideas from classical mechanics and when concepts from quantum mechanics are more appropriate. This task was given after standard instruction in individual demonstration interviews at the University of Washington as well as on a course examination in a sophomore-level modern physics course at another large research university.

a) Interview protocol

The interview protocol consisted of three related tasks. The interview opened by asking the student to consider a 1 g pebble that is placed inside a shoebox 20 cm long. The pebble has an initial speed of 42 mm/s so that it moves horizontally and directly toward one end of the box. The student was told that there is negligible friction between the pebble and the bottom surface of the box, the pebble collides elastically with the walls of the box, and that the box is fixed relative to the surface on which it is placed. The student was also told to consider only the translational (not the rotational) motion of the pebble. With this information given, the student was given the following tasks:

1. Describe the subsequent motion of the pebble after it has been set into motion. Explain your reasoning.

2. Suppose that this experiment were being conducted in a darkened room in which a flash bulb flashed on and off with random time intervals between flashes. Suppose also that you recorded the position of the pebble each time a flash occurred.

   Would you find that the pebble would more likely be found in certain places in the box than others? If so, where would it be more likely be found, and why? If not, why not? Explain.

3. Suppose that the above experiment were repeated many times, halving the mass of the pebble each time it is repeated. (Assume that the pebble is given the same initial speed before each measurement you make of its position.)

   Would your results ever be different than what they were for the case of the 1 g pebble? If so, how would they be different? If not, why not? Explain.

Correct answer to task 1: Motion of the 1 g pebble. Because the box is rigid, the bottom of the box is frictionless, and the pebble collides elastically with the side of the box, the pebble continually bounces back and forth between the ends of the box with a constant speed 42 mm/s. (For a non-rigid box, the velocity of the particle would reverse direction over a finite time interval. Students who encountered confusion with this idea were told explicitly to treat the box as perfectly rigid.) Most of the students recognized that the pebble would move back and forth with constant speed. However, it was very difficult for them to prove using the given information and their knowledge of classical mechanics.[i]

Correct answer to task 2: Flash bulb experiment using the 1 g pebble. Because the 1 g pebble moves with constant speed back and forth between the ends of the box, it takes the same amount of time to travel a unit distance anywhere in the box. This means that the probability density of the 1 g pebble is uniform across the length of the box. Thus the pebble would have no preferred location inside the box at a time when the flash bulb would flash. The graph of probability density shown earlier in Figure B-10 would be valid for the 1 g pebble.

Correct answer to task 3: Flash bulb experiments using pebbles of smaller and smaller mass. The first time the pebble's mass is halved there will be no difference in results; the pebble will still have no preferred location inside the box. However, when the mass of the pebble is reduced to a sufficiently small value such that the de Broglie wavelength of the particle becomes comparable to the length of the box, the probability density will no longer be uniform. In that regime the probability density will vary as $\sin^2 kx$, where $k = 2\pi/\lambda$. For example, when the mass of the pebble has been halved 90 times, the momentum of the pebble is such that the de Broglie wavelength of the pebble is (to within 3%) equal to 2.0 cm, or one-tenth the length of the shoebox. The wave function of the pebble is therefore an energy eigenstate that contains ten equally-spaced anti-nodes, at $x = 0.5$ cm, $x = 1.5$ cm, $x = 2.5$ cm, and so on until $x = 19.5$ cm. Students did not need to perform any calculations to be counted as correct.

Commentary. The strategy employed in the third interview task was very similar to that used in preliminary interviews on diffraction (see Chapter 2 of this dissertation). In those interviews, the students were asked to consider light incident on a slit 1.0 cm wide, a situation in which geometrical optics is valid. The students were then asked to consider slit widths of gradually smaller and smaller value, in order to probe their understanding of the relationship between wavelength and slit width in the regime of physical optics. The strategy in the pebble-and-shoebox interviews is very similar. The students are asked to consider smaller and smaller values of the mass of the pebble, and thus smaller and smaller values of its momentum. The intent was to probe student understanding of the relationship between the de Broglie wavelength of the pebble and the length of the shoebox in the quantum mechanical regime.

b) Written question

The above interview protocol was modified for use as a written question that was included on a final examination in a sophomore-level modern physics course at a large research university. The students were given the question after standard instruction on the behavior of wave functions in one dimension.

A. A small pebble of mass 1 g is placed inside a rigid box of length 20 cm. It is given an initial horizontal speed of 1.8 cm/s in a direction parallel to the length of the box. Ignore the friction and the rotational motion of the pebble, and assume that the pebble collides elastically with each end of the box. Which graph (i, ii, or iii) would best describe the probability density *P(x)* of the pebble as a function of its position *x* along the length of the box?

B. An electron ($m = 9 \times 10^{-28}$ g) is now placed inside a rigid box of length 20 cm. It is given an initial horizontal speed of 1.8 cm/s in a direction parallel to the length of the box. Assume that the electron collides elastically with each end of the box. Which graph (i, ii, or iii) would best describe the probability density *P(x)* of the electron?

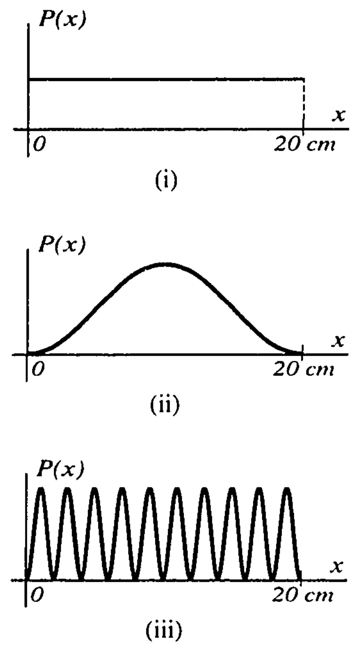


Figure B-11: Pebble-and-shoebox question, given to students after all standard instruction in a modern physics course taught at a large research university.

Correct answers. Part A is essentially identical to task 1 of the interview protocol described above. Ignoring the effects of friction and rotation of the pebble, and assuming that the ends of the box are rigid and elastic, the probability density for the 1 g pebble would be uniform, as shown in graph (i). In part B, in which an electron undergoes periodic motion with the same speed as the 1 g pebble, classical reasoning would no longer be valid. From the given information the de Broglie wavelength of the electron is found to be equal 4.0 cm, or one-fifth the length of the box. In this case the wave function of the electron would have 10 anti-nodes. The correct graph of probability density would therefore be graph (iii).

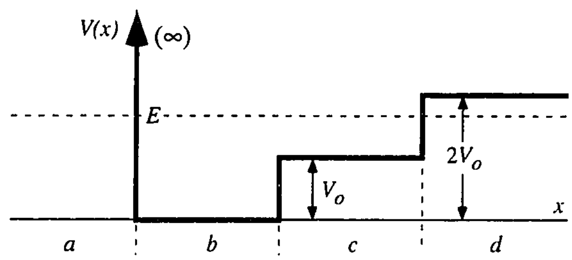
3. *ASYMMETRIC SQUARE WELL QUESTIONS*

A physical situation that has served as the basis for two informative questions is a particle in an asymmetric square well. Each written question requires students to compare or relate the local de Broglie wavelengths in different regions of the well. Each question also asks the students to describe how the amplitude of the wave function varies with position in the well.

a) Double-tiered square well question

One question, referred to as the "double-tiered square well question," is shown below in Figure B-12. The correct answers to parts B and C are discussed in detail.

An electron is present in a region in which the potential energy $V(x)$ varies with x as shown below. The wave function $\psi(x)$ for the electron is a stationary state corresponding to a total energy E such that $V_o < E < 2V_o$.



A. Is any value between V_o and $2V_o$ allowed for the energy E, or are only certain values of E allowed? Explain your reasoning. (You do not need to solve any equations.)

B. Rank the regions *a–d* according to the de Broglie wavelength of the electron in that region, from largest to smallest *(e.g.,* $\lambda_x > \lambda_y = \lambda_z$*...)*. If λ is not well-defined in a particular region, state that explicitly and do not include that region in your ranking. Explain your reasoning.

C. For the regions in which the de Broglie wavelength is well-defined, rank those regions according to the maximum value of $\psi(x)$ in that region, from largest to smallest. Explain your reasoning.

D. In the space at right, carefully draw a qualitatively correct graph of $\psi(x)$ for the electron described above.

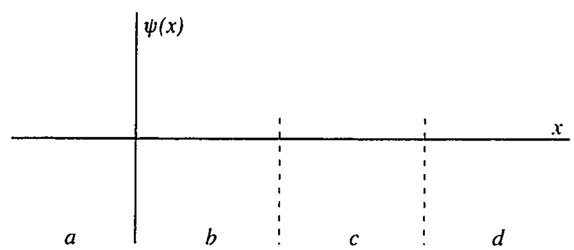


Figure B-12: Double-tiered square well question, given after modified instruction on reflection and transmission but after standard instruction on bound state wave functions in one dimension. The results from parts B and C are discussed in detail.

Correct answer to part B. In part B, a correct ranking would be that the de Broglie wavelength in region *c* is larger than that in region *b*, and in regions *a* and *d* the de Broglie wavelength is not well-defined. In region *a* the potential is infinite, so $\psi(x) = 0$ and $d^2\psi(x)/dx = 0$ in this region. In region *d* the quantity $E - V(x)$ is negative (and finite), so the wave function decreases exponentially with distance to the right of the boundary between regions *c* and *d*. Each of the remaining regions corresponds to a well-defined kinetic energy, with region *b* corresponding to a larger kinetic energy than region *c*. The same holds true for the (magnitude of the) momentum of the electron in these regions. Thus the local de Broglie wavelength $\lambda = h/p = h/[2m(E - V(x))]^{1/2}$ of the electron is smaller in region *b* than in region *c*.

Correct answer to part C. Part C of this question can be answered using semi-classical reasoning. The classically allowed region consists only of regions *b* and *c*, and the kinetic energy of the particle is larger in *b* than in *c*. A particle undergoing classical motion would therefore spend more time in region *c* than in *b*, since it would require less time to pass through *b* than through *c* during each cycle of its motion. This result would be consistent with the peak values of the quantum mechanical probability density $|\psi(x)|^2$ and thus would imply that $\psi(x)$ has a larger maximum value in region *c* than in region *b*. A qualitatively correct graph of the wave function for this case is shown in Figure B-13.

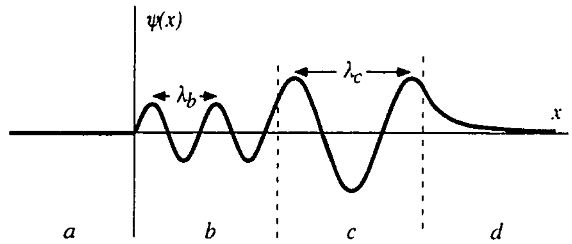


Figure B-13: Qualitatively correct graph of $\psi(x)$ for the double-tiered square well question.

Note about qualitatively correct graph of the wave function. The reasoning described above is valid for sufficiently highly-excited energy eigenstates. For a low-energy state, however, in which less than one (local) de Broglie wavelength fits within either region *b* or *c*, the task of determining the region in which the wave function

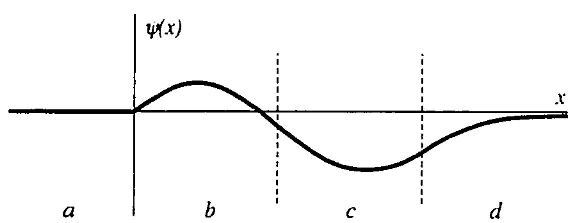


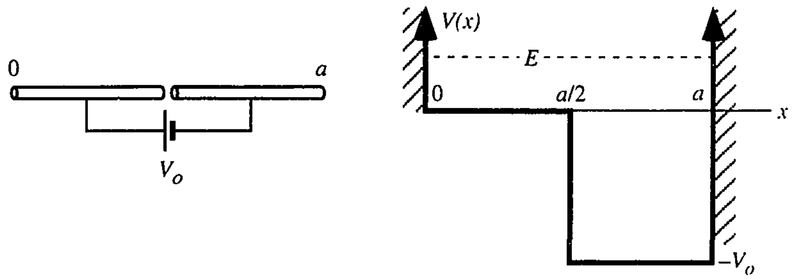
Figure B-14: Low-lying energy eigenstate that would be qualitatively correct.

has the largest maximum value becomes ill-defined. For example, consider the wave function shown in Figure B-14. The wave function shown is qualitatively correct with regard to the local de Broglie wavelength and with regard to overall amplitude, however the wave function would have a *larger maximum value* in region *b* than in region *c*. While correct, the above reasoning would have led to an answer contrary to the intended correct answer. However, none of the students who used correct reasoning were led to an incorrect answer by thinking of a low-lying energy state.

b) Broken wire question

Another question that pertains to an asymmetric square well, the "broken wire question," is shown below in Figure B-15. The square well is very similar to that used in the double-tiered square well question, except that in this case the potential is taken to be infinite in both classically forbidden regions. The correct answers to part b are presented in detail.

Behavior of the wave function $\psi(x, t)$. Consider an electron of energy E confined to a broken wire, with potential $V = 0$ on the left side $0 \leq x \leq a/2$, and $V = -V_o$ on the right $a/2 < x \leq a$; you may assume that V approaches infinity outside the wire.



a. Write down the time dependent Schrodinger equation for this problem and a complete solution $\psi(x, t)$ that satisfies an appropriate boundary condition on the left hand side.

b. Assume that the kinetic energy of the electron on the right side of the wire is 4 times that on the left. Given the wave properties on the left side of wavelength λ_L, angular frequency ω_L, and amplitude A_L, determine the corresponding properties on the right side λ_R, ω_R, and A_R.
Hint: Make use of the WKB approximation.

c. Show that the current density $J(x, t)$ is a constant J independent of x for an eigenstate of the Schrodinger equation, *i.e.*, $\psi_n(x, t) = \psi_n(x) \exp(iE_n t/\hbar)$ and determine the constant.
Hint: Determine $J(x, t)$ on the left side using the result of part a.

Figure B-15: The broken wire question. Results from part b are discussed in detail.

Correct answer to part b. Using the WKB approximation, we treat momentum as a function $p(x)$ rather than as an operator. The kinetic energy of the electron on the right side of the wire is four times that on the left side, as stated in the problem, so the absolute value of the momentum of the electron $p = (2mE_K)^{1/2}$ on the right side is twice that on the left. Thus the de Broglie wavelength on the right side is half that on the left: $\lambda_R = \lambda_L/2$. Also from the WKB approximation, the amplitude of the wave function is inversely proportional to the square root of the momentum. Because $p_R = 2p_L$, the amplitude A_R on the right is smaller than that on the left by a factor of $\sqrt{2}$: $A_R = A_L/\sqrt{2}$. Finally, the total energy $E = \hbar\omega$ is the same on either side of the wire, so $\omega_R = \omega_L$. In this dissertation we discuss in detail the student responses about the wavelength λ_R and amplitude A_R on the right side of the wire, not the angular frequency ω_R.

4. V-WELL QUESTION

The V-well question, shown in Figure B-16, was adapted from the double-tiered square well question. In the V-well question, however, the classically allowed region is not subdivided into parts in which the potential is constant in each part. Instead, the potential varies linearly along the entire length of the classically allowed region.

Consider a particle in a high energy eigenstate of the potential shown. (That is, suppose the energy E of the particle is an energy eigenvalue that is *much larger than* the ground state energy.)

$$V(x) = \begin{cases} \infty, & x < 0 \\ +\beta x, & x > 0 \end{cases}$$

A. On the graph of $V(x)$ *vs* x, indicate the classical turning point(s).

B. In the space at right, sketch a qualitatively correct graph of the wave function $\psi(x)$. Use the same horizontal scale used in the graph of $V(x)$ *vs* x. Explain the reasoning you used to draw your graph.

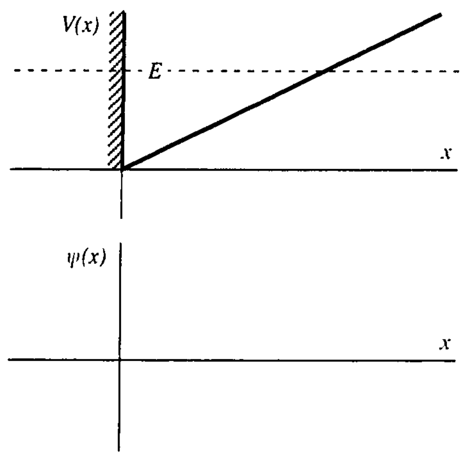


Figure B-16: The V-well question, given after standard instruction. Results from part B are discussed in detail in Chapters 5–7 of this dissertation.

Correct answer. A qualitatively correct graph of the wave function may be determined by carefully using both classical and quantum mechanical arguments. Because the particle is in a highly excited state, the wave function may be thought of as a standing wave oscillating within a gradually varying envelope that can be determined from classical arguments. The particular way in which $\psi(x)$ oscillates within this envelope must be obtained using quantum mechanical arguments, specifically, with regard to the de Broglie wavelength. A qualitatively correct envelope for this case is shown by the dashed curves in the graph in Figure B-17.

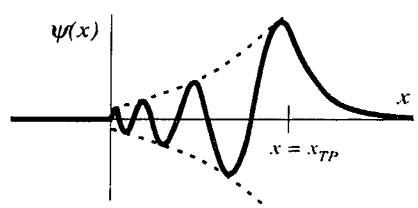


Figure B-17: Qualitatively correct graph of $\psi(x)$ for the V-well question.

A particle undergoing classical motion in the V-shaped well would oscillate between $x = 0$ and the classical turning point $x = x_{TP}$, where the total energy of the particle E would equal the potential energy. The envelope for the wave function within the classically allowed region follows the same qualitative behavior as does the classical probability density. The region where the particle has least kinetic energy would be where the particle spends the most time. Thus the envelope reaches its largest extent near $x = x_{TP}$, the turning point on the right-hand side, and then monotonically approaches $x = 0$ from the right.

Because the potential energy $V(x)$ in the classically allowed region is not constant, the kinetic energy varies. The local kinetic energy near $x = 0$ is larger than that near the classical turning point $x = x_{TP}$ on the right-hand side. Thus the local de Broglie wavelength starts out relatively small near $x = 0$ and gradually increases as one approaches $x = x_{TP}$. This result means that the nodes in the wave function are closest together near $x = 0$ and farthest apart near $x = x_{TP}$.

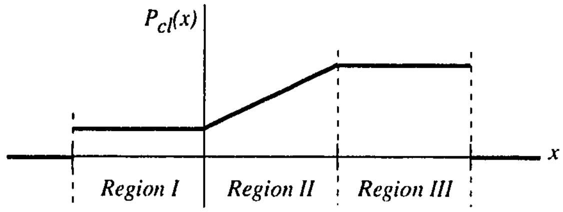
Finally, we consider the behavior of the wave function in the classically forbidden regions. In the region $x < 0$, where the potential energy is infinitely large, the wave function is everywhere equal to zero. In the region to the right of the classical turning point ($x > x_{TP}$) the potential is not only larger than the total energy of the particle but increases with distance. Therefore, the wave function will decrease at a rate more rapidly than exponential decay (*i.e.*, more dramatically than e^{-Kx}, where K is a real constant). For their responses to be considered correct, students simply needed to illustrate that the wave function decreased asymptotically toward zero with concave upward curvature (or increased asymptotically toward zero with concave downward curvature).

5. UNKNOWN POTENTIAL WELL QUESTIONS

Two similar versions of "unknown potential well questions" were developed to assess the effectiveness of two tutorials that were tested in two quantum mechanics classes. (The development and assessment of the tutorials are described in detail in Chapter 7.) In each version of the question presents students with a graph of probability density for a particle undergoing classical motion in a potential well. Students were required to relate specified quantities at various locations within the well. Both versions of the question are shown below.

For parts B and C, consider a particle in an *unknown* one-dimensional potential "well."

B. The *classical probability density, $P_{cl}(x)$*, associated with the potential is shown at right. Regions I, II and III all have the same width. Rank the three regions, I–III, according to the height of the potential, *V(x)*, in that region. Explain your reasoning.



Graph of ***classical probability density*** for particle

C. Now consider the *quantum mechanical* wave function, $\psi(x)$, for the case that the particle is in a highly-excited energy eigenstate. Rank the three regions (I–III) according to the number of nodes that $\psi(x)$ would have in that region, from greatest to smallest. Explain your reasoning.

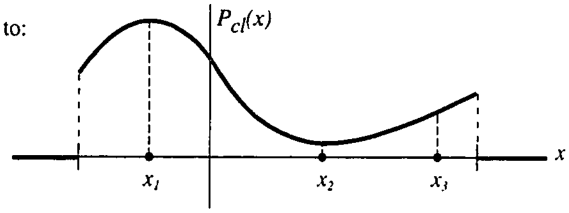
Figure B-18: Version #1 of the unknown potential well question. (When used on examinations, a different, unrelated question that was given as part A.)

A particle is in a bound state of an *unknown* one-dimensional potential. Consider the *classical probability density* $P_{cl}(x)$ for this potential (shown below) and the *quantum mechanical* wave function $\psi_n(x)$ for a high energy state of this same potential (not shown).

Rank the points x_1, x_2, and x_3, according to:

i. the potential *V(x)* at each point, and
ii. the spacing between consecutive nodes in the wave function $\psi_n(x)$ near each point.

Explain your reasoning.



Graph of ***classical probability density*** for particle

Figure B-19: Version #2 of the unknown potential well question.

Correct answers. In each version of the unknown potential well question, students were first asked to rank the labeled regions or points according to the potential *V(x)*. In terms of the classical motion of the particle, the particle would spend more time in regions of relatively low kinetic energy (or, relatively high potential energy). Thus regions of relatively high probability density correspond to relatively high potential. On version #1, the correct ranking according to potential is therefore: $V_{III} > V_{II} > V_I$. (Students did not have to state that the potential in Region II was not uniform.) Similarly, on version #2, the correct ranking is: $V(x_1) > V(x_3) > V(x_2)$.

As for the wave function $\psi(x)$ in each well, in regions of relatively high local kinetic energy the local de Broglie wavelength would be relatively short and thus the spacing between consecutive nodes of $\psi(x)$ would be relatively small. Conversely, regions of relatively low local kinetic energy (or, relatively high potential) correspond to relatively large local de Broglie wavelengths. Therefore, on version #1, in which the three labeled regions have the same width, the ranking according to the number of nodes *(n)* is: $n_{I} > n_{II} > n_{III}$. On version #2, the local de Broglie wavelength would be longest near x_1, intermediate near x_3, and shortest near x_2. Thus the ranking according to the spacing between consecutive nodes would follow the same way: greatest spacing near x_1, then x_3, then smallest spacing at x_2.

NOTES FOR APPENDIX B

[i] While there exist "reflectionless" barriers for specially prepared incident wave packets, the arbitrary potential barrier shown in Figure B-8 is not such a barrier. Discussion of reflectionless barriers may be found in L.D. Landau and E.M. Lifshitz, *Quantum Mechanics, Non-Relativistic Theory,* 2nd ed., transl. by Sykes, J.B. and Bell, J.S. (Addison-Wesley Publishing Co., Inc., Reading, MA, 1965).

[ii] T. O'Brien-Pride, "An investigation of student difficulties with two dimensions, two-body systems, and relativity in introductory mechanics," Ph.D. dissertation, Department of Physics, University of Washington, 1997 (unpublished). Because probing student understanding of classical mechanics was not the main objective in the pebble-and-shoebox interviews, those students who gave a correct answer to task 1 were allowed to proceed to the remaining two tasks, regardless of the reasoning they used on task 1.

APPENDIX C: DISCUSSION OF OVERALL ACADEMIC PERFORMANCE OF UNDERGRADUATE QUANTUM MECHANICS STUDENTS IN THE STUDY

In order to assess the effect of tutorials that were tested in the junior-level quantum mechanics course, it was necessary to compare the overall academic performance of the students in each class in the study. In this appendix we attempt to describe the academic ability of students in three consecutive sequences of the junior-level quantum mechanics courses. One sequence, in Autumn 1995–Winter 1996, had only standard lecture instruction and is referred to as Sequence 0. In the other two sequences, in Autumn 1996–Winter 1997 and Autumn 1997–Winter 1998, tutorials were used as part of the instruction. These sequences are referred to as Sequence 1 and Sequence 2, respectively.

At the University of Washington, approximately 35-45 students typically enroll in the first course (Physics 324) of the two-quarter sequence on quantum mechanics. In contrast, on the average 20-30 students take the second course (Physics 325) in the sequence. Although most physics majors take the entire sequence, only the first course is required for completion of the major in physics. With such small enrollments in the sequence and with fewer students in the Winter courses than in the Autumn courses, it is important to determine whether or not there are significant differences in aggregate student ability (1) between one sequence and another and (2) between courses within the same sequence.

A. COMPARISON OF STUDENT POPULATIONS IN THE AUTUMN SECTIONS OF DIFFERENT QUANTUM MECHANICS SEQUENCES

In this section we compare the overall student academic performance of students in the Autumn sections from two different quantum mechanics sequences. To accomplish this, we restrict our attention to the examination questions that were written by the instructor and that did not relate directly to the material covered in the tutorials from those classes. We compared the overall results from the midterm and final examinations in the two classes. We then compared the results of similar examination problems given on the midterm examinations in each of the two classes.

1. COMPARISON OF STATISTICS FROM MIDTERM AND FINAL EXAMINATION QUESTIONS WRITTEN BY THE LECTURE INSTRUCTOR OF THE COURSE

Shown in Table C-1 are the overall average and standard deviation of all examination questions taken by students in the Autumn classes of two consecutive quantum mechanics sequences, Sequence 1 (Autumn 1996) and Sequence 2 (Autumn 1997). The results pertain only to examination questions that were written by the lecture instructor and that tested on material not covered directly in the tutorials. The mean, standard error of the mean, and the median are given as a percentage of the total number of points allocated to the lecture questions on the examinations.

Table C-1: Overall results from examinations given in the Autumn classes of Sequences 1 and 2 on topics not directly related to those addressed in tutorials. Shown are the results for only those students who both took the midterm examination and worked through one tutorial on wave functions in one dimension (*Reflection and transmission* in Sequence 1, *Relating classical mechanics to quantum mechanics* in Sequence 2).

		Sequence 1 (Autumn 1996) $N = 25$	Sequence 2 (Autumn 1997) $N = 33$
Midterm exam	Mean score (standard error)	62% (4%)	61% (3%)
	Median score	66%	57%
Final exam	Mean score (standard error)	54% (4%)	60% (4%)
	Median score	53%	63%

As shown in each of the two rows in part b of the table, the average number of points earned on the midterm and final examinations differed by only 6%. The median scores for each examination are approximately the same as the averages. The standard deviation of the mean for each examination was approximately 3% or 4%. The difference in mean scores on the midterm examinations (1%) and the difference in mean scores on the final examinations (6%) are each less than the sum of the standard errors of the means (7% for the midterm examinations, 8% for the final examinations). In addition, the students in Sequence 1 performed slightly better than those in Sequence 2 on the midterm examinations, but the opposite was true for the final examinations. These results suggest that the level of academic ability among the students in Sequence 1 is similar to that of the students in Sequence 2.

2. *COMPARISON OF RESULTS FROM SIMILAR EXAMINATION PROBLEMS GIVEN ON MIDTERM EXAMINATIONS*

In Figure C-1 below are two similar examination problems that were included on the midterm examinations of the Autumn classes of Sequences 1 and 2. Each problem tested on Ehrenfest's theorem, a topic that is typically covered in the context of quantum mechanics in one dimension.

- Explain Ehrenfest's theorem (no calculation required) for the average velocity $d\langle x\rangle/dt$.
- An important theorem in Quantum Mechanics states that $d\langle x\rangle/dt = (i/\hbar)\langle[H, x]\rangle$. Evaluate $(i/\hbar)\,[H, x]$ where $H = p^2/2m + V(x)$ is the Hamiltonian operator and $[H, x]$ is the commutator of H and x.

(a)

- Evaluate the commutator $[H, p]$, where $H = p^2/2m + V(x)$ and hence determine the result for $d\langle p\rangle/dt$ [where $d\langle p\rangle/dt = (i/\hbar)\langle[H, p]\rangle$ is given on a previous part of the problem].
- You should have been able to guess the above result for $d\langle p\rangle/dt$ without any calculation from "Ehrenfest's theorem." Explain in words the meaning of Ehrenfest's theorem and how it applies to $d\langle p\rangle/dt$.

(b)

Figure C-1: Pairs of examination questions about Ehrenfest's theorem that were posed on midterm examinations in the Autumn classes of (a) Sequence 1 and (b) Sequence 2.

As shown in the figure, each problem contains a pair of questions. One question of each pair asks students to calculate a commutator of the Hamiltonian with another operator, while the other question of each pair requires asks for an interpretation of the meaning of Ehrenfest's theorem to the particular case. To answer both questions in each pair correctly requires an understanding of Ehrenfest's theorem on both a qualitative and a quantitative level.

Table C-2: Results from similar examination problems on Ehrenfest's theorem given in the Autumn classes of Sequences 1 and 2. Shown are the results for only those students who both took the midterm examination and worked through a tutorial that quarter on wave functions in one dimension (*Reflection and transmission* in Autumn 1996, *Relating classical mechanics to quantum mechanics* in Autumn 1997).

	Sequence 1 (Autumn 1996) $N = 25$	Sequence 2 (Autumn 1997) $N = 33$
Correct responses on *both* questions in each examination problem	12% (3)	15% (5)

In Table C-2 above, we present the percentage of students who correctly answered both questions in each examination problem on Ehrenfest's theorem. Between 10% and 15% of the students in each class answered both parts correctly. The results indicate that the performance by the students in the Autumn sections of quantum mechanics Sequences 1 and 2 are similar. The results provide further evidence that the overall academic ability of the students in the classes was very similar.

B. CONTRAST OF STUDENT POPULATIONS IN THE AUTUMN AND WINTER SECTIONS WITHIN THE SAME QUANTUM MECHANICS SEQUENCE

As mentioned earlier, enrollment in the Winter course (Physics 325) of the quantum mechanics sequence is typically smaller than that in the Autumn course (Physics 324). Some of the tutorial materials, including those associated with the tutorial *Relating classical mechanics to quantum mechanics,* were used in the Winter course in Sequence 1 and in the Autumn course in Sequence 2. It is therefore important to characterize the ability of the students in the Autumn course of a given sequence who continue in the Winter course of the same sequence.

The information shown in Figure C-2 allows us to describe the overall academic performance of students in two Winter quantum mechanics courses, from Sequence 0 (Winter 1996) and Sequence 1 (Winter 1997) relative to that of the students in Autumn courses that preceded them. Each set of three pie charts indicates the number of students who fell within the top quartile, bottom quartile, and middle half of the Autumn course in the given sequence. This ranking is determined on the basis of student performance on homework and examination questions that did not test directly on material covered in the tutorials. Thus the ranking is intended to provide a rough measure of student ability in ways other than those in which the tutorials were designed to have any direct effect.

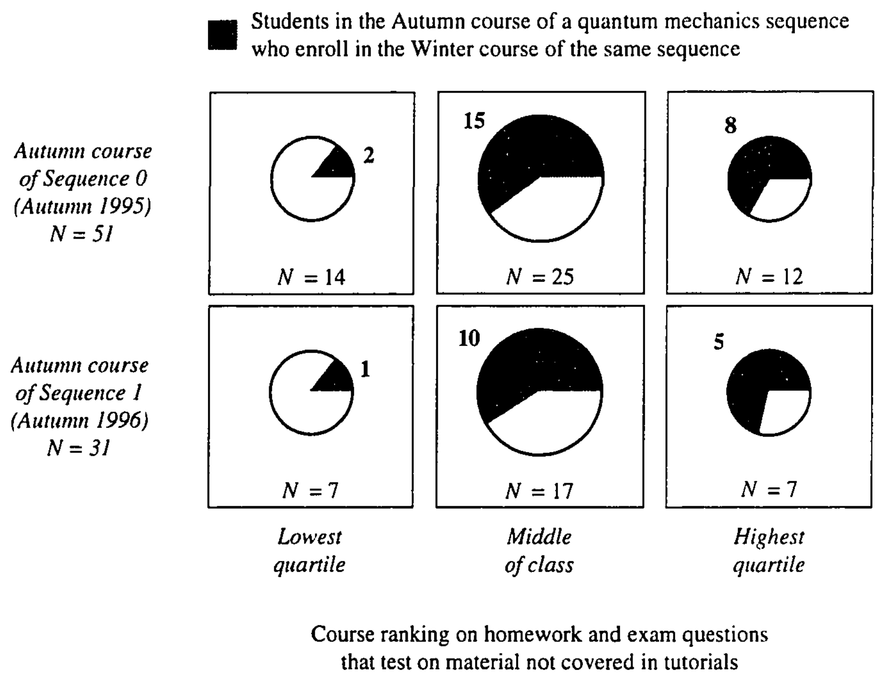


Figure C-2: Students in the Autumn courses of Sequences 0 and 1 grouped according to course ranking. The shaded regions represent students who went on to take the Winter course of the corresponding sequence.

Although the total enrollment at the beginning of Sequence 0 is less than that in Sequence 1—an observation that reflects the trend at the time of fewer students majoring in physics—the students in each Autumn course who continued on to take the following Winter course tended to come from the highest quarter and middle half of the class. In particular, about 65%–70% of those from the highest quarter of the Autumn course and 60% of those from the middle half go on to take the Winter course. Very few students in the lowest quarter of the Autumn course finished the sequence. These results suggest that the students in each Winter course were academically stronger than those in the preceding Autumn course.

C. DISCUSSION OF RESULTS

In section A above we have presented evidence that suggests the academic performance of students in the Autumn sections of Sequences 1 and 2 (*i.e.*, Autumn 1996 and Autumn 1997) are approximately equal. In Autumn 1995 (Sequence 0), however, there were no examination questions that were similar to those used in the other Autumn classes. We therefore did not have sufficient information to compare directly the academic performance of the Autumn 1995 class to the others. However, we believe it is reasonable to assume that the Autumn 1995 students had approximately the same aggregate academic ability as those in the Autumn sections of the other two sequences.

In section B above we have shown that the students in the Winter sections of Sequences 0 and 1 (*i.e.*, Winter 1996 and Winter 1997) are academically stronger than those who took the Autumn classes of the same sequences. Having assumed that the overall ability of the Autumn classes of these two sequences are approximately equal to each other, it is therefore reasonable to assume also that the overall ability of the Winter classes of the same sequences are also approximately equal to each other. In Chapter 7 of this dissertation we interpret the results of written questions given at various stages of instruction on the basis of the above assumptions.

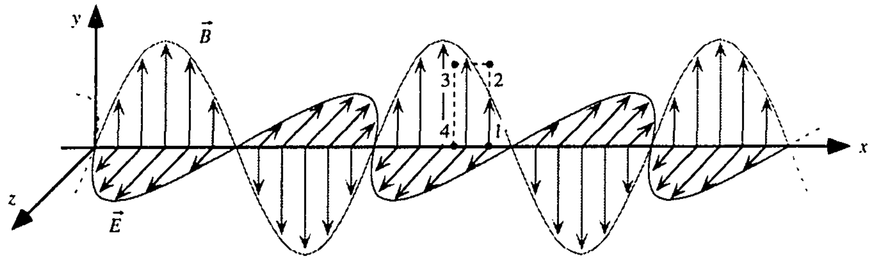
Appendix D: Pretest, tutorial worksheet, and homework for the tutorial *Electromagnetic waves*

- Pretest (original version)
- Pretest (current version)
- Tutorial worksheet
- Homework

Pretest: Electromagnetic waves (original version)

1. Shown below are mathematical and pictoral representations of an electromagnetic plane wave propagating through empty space. The electric field is parallel to the z-axis; the magnetic field, the y-axis.

$$\vec{E}(x, y, z, t) = E_o \sin(kx + \omega t)\,\hat{z} \qquad \vec{B}(x, y, z, t) = B_o \sin(kx + \omega t)\,\hat{y}$$



a. The points 1–4 in the diagram above lie in the x–y plane.

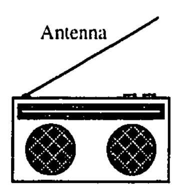
   i. For the instant shown, rank the magnitude of the *magnetic field* at these points from largest to smallest. If the magnetic field is zero at any of these points, state that explicitly.

   ii. For the instant shown, rank the magnitude of the *electric field* at these points from largest to smallest. If the electric field is zero at any of these points, state that explicitly.

b. Suppose that the above diagram represents a radio wave.

   i. In which direction is the radio wave propagating? Explain how you can tell.

   ii. In order to obtain the best reception with your portable radio (see figure at right), how would you orient the antenna relative to the radio wave? For example, would you orient it along the x-, y-, or z-axis? Explain your reasoning.

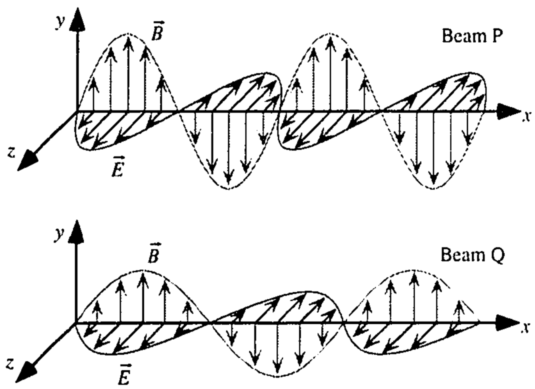


Portable radio

Pretest: Electromagnetic waves (original version)

2. The diagrams below right represent two beams of visible light that are projected onto a white screen. Both diagrams are drawn to the same scale.

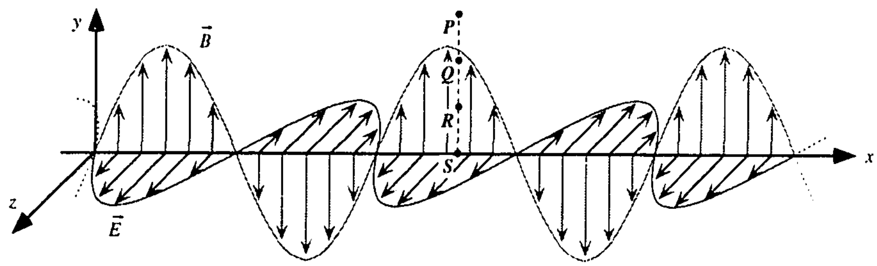
   How would the projections of the beams look different? Specify *two* ways in which they would look different. Explain how you can tell from the diagrams.



Pretest: Electromagnetic waves (current version)

1. Shown below are pictorial and mathematical representations of an electromagnetic plane wave propagating through empty space. The electric field is parallel to the z-axis; the magnetic field, the y-axis.

$$\vec{E}(x, y, z, t) = E_o \sin(kx + \omega t)\,\hat{z} \qquad \vec{B}(x, y, z, t) = B_o \sin(kx + \omega t)\,\hat{y}$$



a. The points P, Q, R, and S in the diagram above lie in the x–y plane.

   i. For the instant shown, rank the points P, Q, R, and S according to the magnitude of the *magnetic field* at these points, from largest to smallest. If the magnetic field is zero at any of these points, state that explicitly. Explain the reasoning you used to determine your ranking.

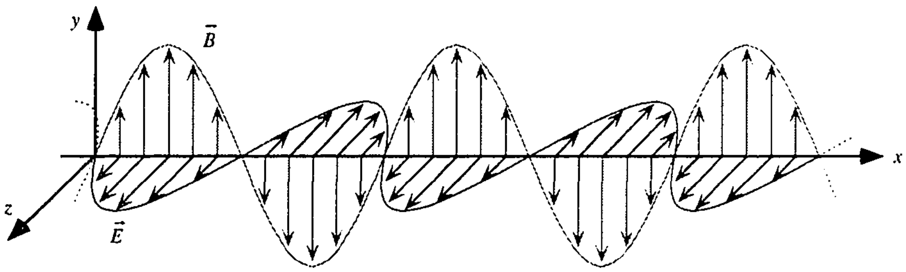
   ii. For the instant shown, rank the points P, Q, R, and S according to the magnitude of the *electric field* at these points, from largest to smallest. If the electric field is zero at any of these points, state that explicitly. Explain the reasoning you used to determine your ranking.

b. In which direction is the electromagnetic wave propagating? Explain how you can tell.

Pretest: Electromagnetic waves (current version)

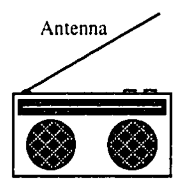
2. Reproduced below are the mathematical and pictorial representations shown in question 1 for an electromagnetic plane wave propagating through empty space.

$$\vec{E}(x, y, z, t) = E_o \sin(kx + \omega t)\,\hat{z} \qquad \vec{B}(x, y, z, t) = B_o \sin(kx + \omega t)\,\hat{y}$$



Suppose that the above diagram represents a radio wave.

In order to obtain the best reception with your portable radio (see figure at right), how would you orient the antenna relative to the radio wave? For example, would you orient it along the x-, y-, or z-axis? Explain your reasoning.



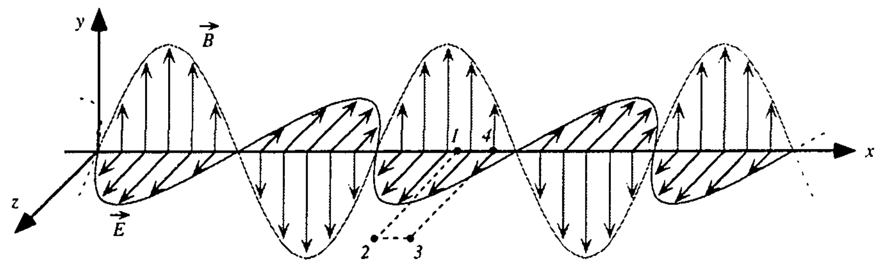
Portable radio

Tutorial worksheet: Electromagnetic waves

I. Representations of electromagnetic waves

A. Shown below are mathematical and pictorial representations of an electromagnetic plane wave propagating through empty space. The electric field is parallel to the z-axis; the magnetic field is parallel to the y-axis.

$$\vec{E}(x, y, z, t) = E_o \sin(kx + \omega t)\,\hat{z} \qquad \vec{B}(x, y, z, t) = B_o \sin(kx + \omega t)\,\hat{y}$$



1. In which direction is the wave propagating? Explain how you can tell from the expressions for the electric field and magnetic field.

   Is the wave transverse or longitudinal? Explain in terms of the quantity or quantities that are oscillating.

2. The points *1–4* in the diagram above lie in the x–z plane.

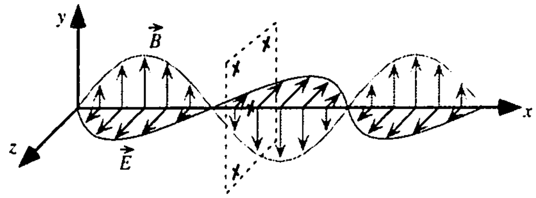
   For the instant shown, rank the magnitude of the *electric field* at these points from largest to smallest. If the electric field is zero at any point, state that explicitly.

   Is your ranking consistent with the mathematical expression for the electric field shown above? If not, resolve any inconsistencies. (For example, how, if at all, does changing the value of z affect the value of $\vec{E}(x, y, z, t)$?)

   For the instant shown, rank the magnitude of the *magnetic field* at points *1–4* from largest to smallest. Check that your ranking is consistent with the expression for the magnetic field, $\vec{B}(x, y, z, t)$, above.

Tutorial worksheet: Electromagnetic waves

3. In the diagram at right, the four points labeled "×" are all located in a plane parallel to the y–z plane. One of the labeled points is located on the x-axis.

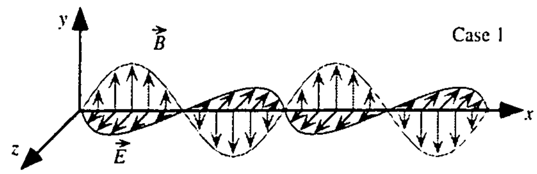
   

   On the diagram, sketch vectors to show the direction and relative magnitude of the electric field at the labeled points.

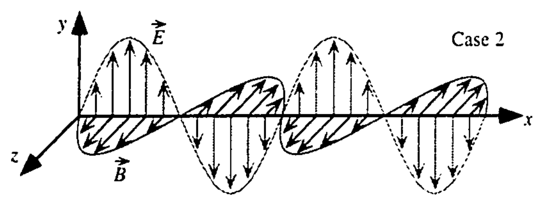
   Justify the use of the term *plane wave* for this electromagnetic wave.

➪ Check your answers to part A with a tutorial instructor.

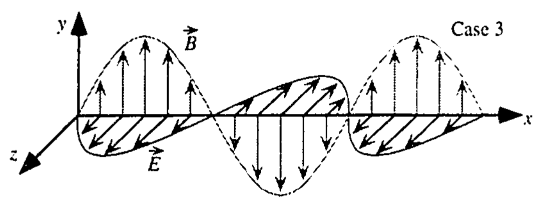
B. Three light waves are represented at right. The diagrams are drawn to the same scale.



1. How is the wave in case 1 different from the wave in case 2? Explain how you can tell from the diagrams.



2. If the wave in case 2 were green light, could the wave in case 3 be *red* light or *blue* light? Explain.



Tutorial worksheet: Electromagnetic waves

II. Detecting electromagnetic waves

A. Write an expression for the force exerted on a charge, q, by (1) an electric field, $\vec{E}$, and (2) a magnetic field, $\vec{B}$.

If an electric field and a magnetic field were both present, would a force be exerted on the charge even if the charge were initially not moving? Explain.

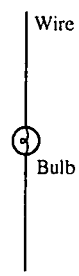
B. A long, thin conducting wire (see figure at right) is placed in the path of the radio wave represented in section I.

1. Suppose that the wire were oriented parallel to the z-axis.

   As the wave moves past the wire, would the *electric field* due to the radio wave cause the charges in the wire to move? If so, would the charges move in a direction along the length of the wire? Explain.

   As the wave moves past the wire, would the *magnetic field* due to the wave cause the charges in the wire to move in a direction along the length of the wire? Explain.

2. Imagine that the thin conducting wire is cut in half and that each half is connected to a different terminal of a light bulb. (See diagram at right.)

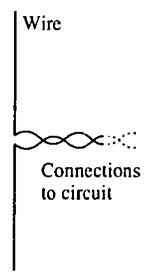
   If the wire were placed in the path of the radio wave and oriented parallel to the z-axis, would the bulb ever glow? Explain. (*Hint:* Under what conditions can a bulb glow even if it is not part of a closed circuit?)

   How, if at all, would your answer change if the wire were oriented:

   - parallel to the y-axis? Explain.

   - parallel to the x-axis? Explain.

3. Suppose that the bulb were disconnected and that each half of the wire were connected in a circuit, as shown. (A conducting wire or rod used in this way is an example of an *antenna.*)

   In order to best detect the oncoming radio wave (that is, to maximize the current through the circuit), how should the antenna be oriented relative to the wave? Explain.

Tutorial worksheet: Electromagnetic waves

III. Electromagnetic waves and Maxwell's equations

A. Recall Faraday's law, $\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$, from electricity and magnetism. We shall consider how each side of the equation for Faraday's law applies to the imaginary loop, $1 \to 2 \to 3 \to 4 \to 1$, in the figure for part A of section I.

1. For the instant shown in the figure, determine whether each quantity below is *positive, negative,* or *zero.* Explain your reasoning in each case.
   - the quantity $\int \vec{E} \cdot d\vec{l}$ evaluated over the path $1 \to 2$
   - the quantity $\int \vec{E} \cdot d\vec{l}$ evaluated over the path $2 \to 3$
   - the quantity $\oint \vec{E} \cdot d\vec{l}$ evaluated over the entire loop, $1 \to 2 \to 3 \to 4 \to 1$ (*Hint:* The answer is *not* zero!)

For an imaginary surface that is bounded by a closed loop, it is customary to use the right-hand rule to determine the direction of the area vector that is normal to that surface. For example, the vector that is normal to the flat, imaginary rectangular surface bounded by the loop $1 \to 2 \to 3 \to 4 \to 1$ points in the *positive* y-direction.

2. At the instant shown in the figure, is the magnetic flux through the loop $1 \to 2 \to 3 \to 4 \to 1$ *positive, negative,* or *zero?* Explain how you can tell from the figure.

   A short time later, will the magnetic flux through the loop be *larger, smaller,* or *the same?* Explain how you can tell from the figure.

3. According to your answers in part 2 above, is the quantity $\left(-\frac{d\Phi_B}{dt}\right)$, written on the right-hand side of the equation for Faraday's law, *positive, negative,* or *zero?* Explain.

   According to your results in part 1 above, is the quantity on the left-hand side of this equation *positive, negative,* or *zero?*

   Do you get the same answer for both sides of the equation for Faraday's law? If not, resolve the inconsistencies.

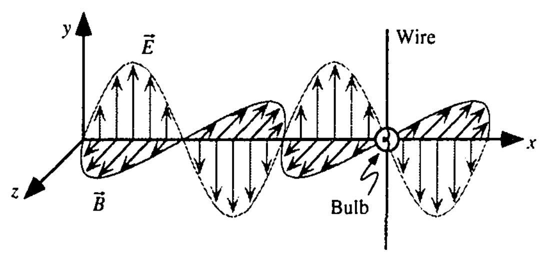
B. Suppose that the electric field in a light wave were $\vec{E}(x, y, z, t) = E_o \sin(kx + \omega t)\,\hat{z}$.

   Would it be possible to have a magnetic field that is *zero* for all x and t? Use Faraday's law to support your answer. *(Hint:* How, if at all, would your answers in part A above be different if the magnetic field were zero for all x and t?)

Homework: Electromagnetic waves

1. A long, thin steel wire is cut in half, and each half is connected to a different terminal of a light bulb. An electromagnetic (EM) plane wave ($\vec{E}(x, y, z, t) = E_o \sin(kx - \omega t)\,\hat{y}$, $\vec{B}(x, y, z, t) = B_o \sin(kx - \omega t)\,\hat{z}$) moves past the wire, as shown.

   a. In what direction is the wave propagating? Explain your reasoning.



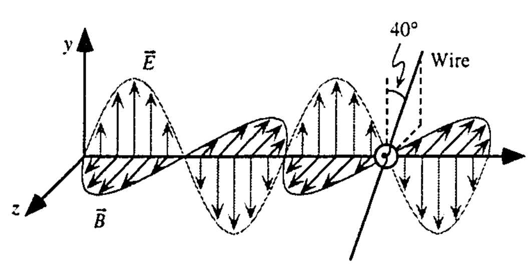
   b. Suppose the wire were oriented parallel to the y-axis, as shown above.

      Would the bulb glow in this case? Explain.

   c. Suppose instead that the wire were positioned as described below. Would the brightness of the bulb be *greater than, less than,* or *equal to* the brightness that it had in part b? Explain your reasoning in each case.

      i. The wire is parallel to the y-axis but with its bottom end located on the x-axis (*i.e.*, the wire is shifted upward a distance $L/2$).

      ii. The wire is tilted so that is makes an angle of 40° with respect to the y-axis but is still parallel to the y–z plane. (See diagram at right.)

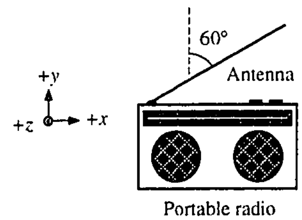


Homework: Electromagnetic waves

2. A radio-frequency EM plane wave (not shown) propagates in the $+z$ direction. A physics student finds that her portable radio obtains the best reception of the wave when the antenna is parallel to the x–y plane so that it makes an angle of 60° with respect to the y-axis. (See the diagram below.)

   a. Consider an instant t_o when the fields are non-zero at the location of the antenna.

      On the diagram at right, draw and labels arrows to indicate (1) the direction of the electric field and (2) the direction of the magnetic field at $t = t_o$. Explain your reasoning. (*Note:* More than one answer is possible.)



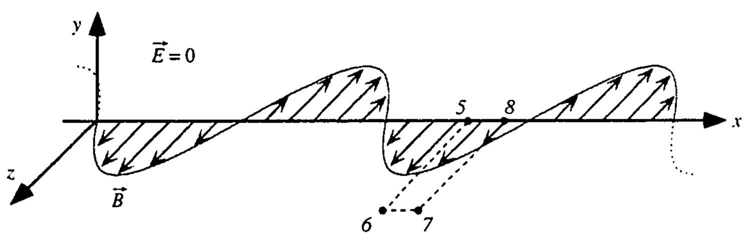
   b. How would your answers to part a be different if the wave were propagating in the $-z$ direction instead of the $+z$ direction? (*Note:* More than one answer is possible.) Explain.

3. In a region that does not contain current-carrying wires, the magnetic field is found to be $\vec{B}(x, y, z, t) = B_o \sin(kx - \omega t)\,\hat{z}$.

   Show that in such a region it is not be possible to have an electric field that is equal to *zero* for all x and t. (*Hint:* Set $i_{encl} = 0$ in Ampère's law:

$$\oint \vec{B} \cdot d\vec{l} = \mu_o i_{encl} + \mu_o \varepsilon_o \frac{d\Phi_E}{dt}$$

   and consider the quantity $\oint \vec{B} \cdot d\vec{l}$ evaluated around the loop $5 \rightarrow 6 \rightarrow 7 \rightarrow 8 \rightarrow 5$ in the x–z plane, shown below.)



Appendix E: Modified version of the pre-lab tutorial *Polarization*

Pre-lab tutorial: Polarization

I. Polarization of light

A. Look at the room lights through one of the polarizing filters provided.

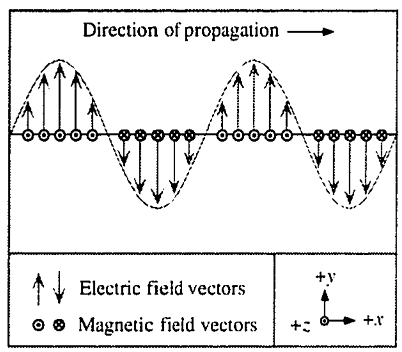
Describe how the filter affects what you see. Does rotating the filter have an effect?

B. Hold a second polarizing filter in front of the first, and look at the room lights again.

Describe how the filter affects the light that you see. How does rotating one of the filters affect what you see in this case?

On the basis of your observations so far, why is it appropriate to use the term *filter* to describe these pieces of apparatus?

How is the behavior of the polarizing filters *different* from the behavior of colored acetate filters?



You have learned that light may be thought of as a wave consisting of oscillating electric and magnetic fields. If the electric field in all parts of a light beam oscillates along a single axis, the light beam is said to be *linearly polarized,* or simply, *polarized.* For example, the diagram at right represents a polarized light wave moving in the x-direction in which the electric field oscillates only along the y-axis. By convention, the direction along which the electric field oscillates (in this case, the y-direction) is called the *direction of polarization* of a light beam. If the electric field oscillates in different, random directions within the same light beam, that beam is said to be *unpolarized.*

Pre-lab tutorial: Polarization

II. Polarizing filters

The light transmitted by a polarizing filter (or *polarizer*) depends upon the relative orientation of the polarizer and the electric field in the light wave. Every polarizer has a *direction of polarization*, which is often marked by a line drawn on it. The electric field of the transmitted wave is equal to the component of the electric field of the incident wave that is *parallel* to the direction of polarization of the polarizer.

A. Do the room lights produce polarized light? Explain how you can tell from your observations so far.

B. Suppose that you had two marked polarizers (*i.e.*, their directions of polarization are marked).

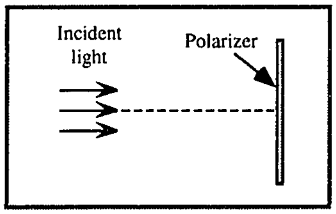
Predict how you should orient the polarizers with respect to one another so that the light transmitted through the polarizers would have (1) *maximum* intensity or (2) *minimum* intensity. Discuss your reasoning with your partners and then check your predictions.

When two polarizers are oriented with respect to each other such that the light transmitted through them has minimum intensity, the polarizers are said to be *crossed.*

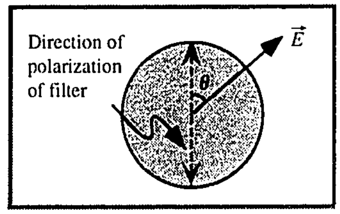
C. Suppose that you had a polarizer with its direction of polarization marked. How could you use this polarizer to determine the direction of polarization of another (unmarked) polarizer? Explain your reasoning.

Pre-lab tutorial: Polarization

D. A beam of light is incident on a polarizer, as shown in the side view diagram below. The direction of polarization of the light makes an angle θ with respect to the polarizer's direction of polarization. (See front view diagram.) The amplitude of the electric field of the incident light is E_o. The magnetic field (not shown) has an amplitude B_o.



Side view



Front view

The vector $\vec{E}$ represents the electric field of the incident light at the front surface of the polarizer at a particular time. Resolve $\vec{E}$ into two components: one that is *transmitted* by the polarizer and one that is *absorbed* by the polarizer.

What is the *direction* of the electric field of the transmitted light? How, if at all, is it different from the direction of the electric field of the incident light? Explain.

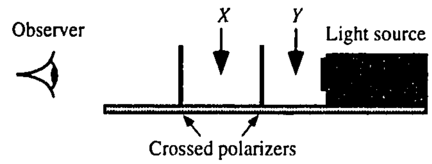
Write an expression for the *amplitude* of the electric field of the transmitted light, in terms of E_o and θ.

Write an expression for the *amplitude* of the magnetic field of the transmitted light, in terms of B_o and θ. Explain your reasoning.

Write an expression for the *intensity* of the transmitted light in terms of I_o, the intensity of the incident light, and θ. Show all work. (*Hint:* If the amplitude of the electric field were reduced by a factor of two, by what factor would the intensity be reduced?)

➪ Check your results from part D with a tutorial instructor.

E. An observer is looking at a light source through two polarizers as shown in the side view diagram at right. The polarizers are crossed, that is, they are oriented so that the light transmitted through them has *minimum* intensity.



1. Suppose that a third polarizer were inserted at the position marked *X*, shown above.

   Predict how, if at all, this change would affect the intensity of the light reaching the observer. Does your answer depend on the orientation of the third polarizer? Discuss your reasoning with your partners.

   Check your prediction experimentally. (Ask a tutorial instructor to show you the equipment that you need in order to do so.) If your prediction was incorrect, identify those parts of your prediction that were wrong.

   How can you apply your results from part D to help you account for your observations? Support your answer with one or more diagrams.

2. Suppose that instead a third polarizer were inserted at the position marked *Y*, shown above.

   Predict how, if at all, this change would affect the intensity of the light reaching the observer. Does your answer depend on the orientation of the third polarizer? Discuss your reasoning with your partners.

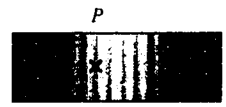
F. Consider a beam of *unpolarized* light that is incident on a polarizer. What is the intensity of the transmitted light in terms of I_o, the intensity of the incident light? (*Hint:* We can think of unpolarized light as equal amounts of light that are polarized *parallel* and *perpendicular* to the direction of polarization of the polarizer.)

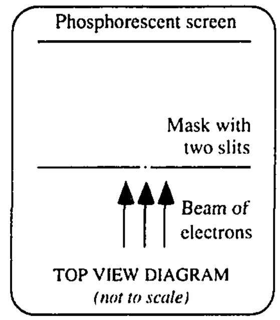
Appendix F: Pretest, tutorial worksheet, tutorial handout, and homework for the tutorial *Wave properties of matter*

- Pretest (algebra-based course)
- Pretest (modern physics course)
- Pretest (quantum mechanics course)
- Tutorial worksheet (includes tutorial handout)
- Homework (algebra-based course)
- Homework (modern physics course)
- Homework (Autumn 1996 quantum mechanics course)
- Homework (Autumn 1997 quantum mechanics course)

Pretest: Wave properties of matter (algebra-based course)

A beam of monoenergetic electrons is incident on a mask that contains two very narrow slits (see top view diagram at right). The photograph shows the pattern seen on a phosphorescent screen placed far from the slits. The bright regions indicate the concentrations of electrons hitting the screen.





A. Consider point P, a point of minimum intensity. If the left slit were covered, would the intensity at point P *increase, decrease,* or *stay the same?* Explain your reasoning.

B Suppose this experiment were repeated with a ***single*** change made to the original setup. For each possible change described below, **predict** whether the bright regions would get *closer together,* move *farther apart,* or *stay in the same location.*

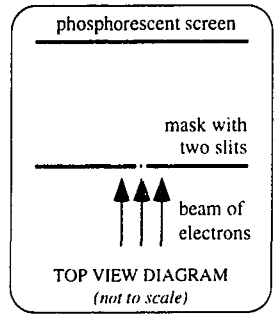
Explain your reasoning in each case.

i. The slit separation is *decreased.*

ii. The speed of the electrons is *increased.*

iii. The electrons are replaced with neutrons, with each neutron having the *same speed* as each of the original electrons. (Recall that $m_n > m_e$.)

Pretest: Wave properties of matter (modern physics course)



A beam of monoenergetic electrons is incident on a mask that contains two very narrow slits (see top view diagram at right). The photograph shows the pattern seen on a phosphorescent screen placed far from the slits. The bright regions indicate the concentrations of electrons hitting the screen.

Suppose this experiment were repeated with a ***single*** change made to the original setup. For each possible change described below, **predict** whether the bright regions would get *closer together,* move *farther apart,* or *stay in the same location.*

Explain your reasoning in each case.

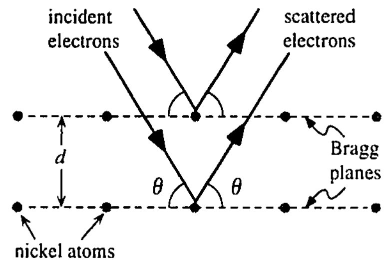
i. The slit separation is decreased.

ii. The speed of the electrons is increased.

iii. The electrons are replaced with muons, with each muon having the *same kinetic energy* as each of the original electrons. (Recall that $m_\mu \approx 200m_e$.)

Pretest: Wave properties of matter (quantum mechanics course)

(Davisson-Germer experiment.) Monoenergetic electrons are incident on a nickel crystal. It is observed that intense scattering occurs at angles θ according to the Bragg condition, $2\,d \sin \theta = n\lambda$. (See diagram at right.)



Suppose that this experiment were repeated, each time with a ***single*** change made to the apparatus. For each change below, determine whether each of the angles θ at which intense scattering occurs would become *larger, smaller,* or *stay the same.*

Explain your reasoning in each case.

a. The nickel target is replaced with another crystal that has the same lattice structure but a smaller lattice spacing. (Consider the Bragg planes that are analogous to those indicated in the diagram.)

b. The speed of the incident electrons is decreased.

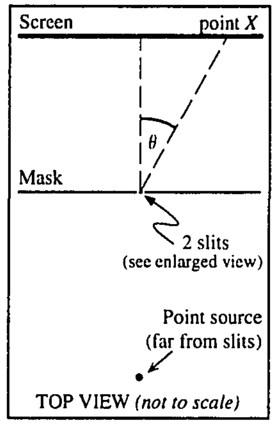
c. The electrons are replaced with neutrons such that each neutron has the *same kinetic energy* as each of the original electrons.

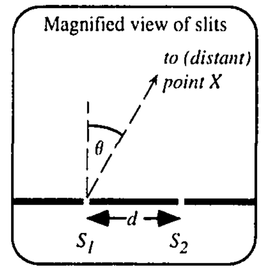
Tutorial worksheet: Wave properties of matter

I. Review of two-slit interference of light

Light of wavelength λ from a distant point source is incident on two very narrow slits, S_1 and S_2. (See diagrams at right.) The photograph above right shows the pattern seen on a distant screen.

Pattern on distant screen





A. How does this *differ* from what you would have predicted if you had used geometrical optics (*i.e.*, the idea that light travels in straight lines through slits)?

How can you use the principle of superposition to account for the presence of the bright and dark regions seen on the screen?

If one of the slits were completely covered, then the portion of the screen shown in the photograph above would become uniformly illuminated (although dimly).

Explain how you can use the principle of superposition to account for this observation.

B. In the magnified view of the slits, an arrow is drawn showing the direction from slit S_1 to an arbitrary point on the screen, point X. On the magnified view:

- Draw an arrow to indicate the approximate direction from slit S_2 to the *distant* point X.
- Identify and label the line segment that represents the difference in path lengths from the slits to point X. (This distance is referred to as the *path length difference.*)
- Show that the path length difference is approximately equal to $d \sin \theta$, where d is the slit separation and θ is the angle defined in the top view diagram above.

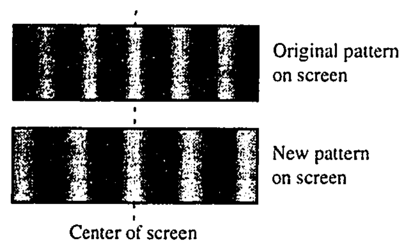
For small angles θ (θ in radians), what is $d \sin \theta$ approximately equal to?

C. For what particular values of the path length difference (written in terms of λ) will there be:

- maximum constructive interference *(i.e.,* a *maxima)?*

- complete destructive interference *(i.e.,* a *minima)?*

D. Suppose that a ***single*** change were made to the apparatus, resulting in the new pattern shown.



i. Are the angles to the interference maxima in the new pattern *greater than, less than,* or *equal to* those in the original pattern? Explain how you can tell from the photographs.

ii. If the wavelength of light *(λ)* was the ***only*** quantity changed, determine (a) whether λ was *increased* or *decreased,* and (b) whether it was changed by a factor that was *greater than, less than,* or *equal to* 2. Explain how you can use your results from parts B and C to justify your answer.

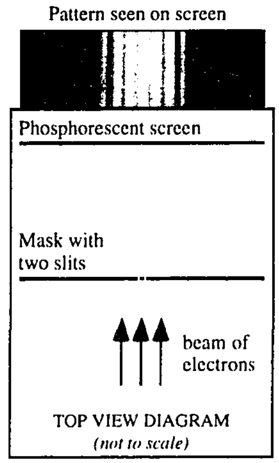
iii. If the slit separation *(d)* was the ***only*** quantity changed, determine (a) whether *d* was *increased* or *decreased,* and (b) whether it was changed by a factor that was *greater than, less than,* or *equal to* 2. Explain how you can use your results from parts B and C to justify your answer.

✔ ***Check your answers to part D with a staff member.***

Tutorial worksheet: Wave properties of matter

II. Two-slit interference of electrons

A beam of electrons is accelerated through a potential difference, *V*, and incident on two narrow slits. The photograph shows the pattern seen on a phosphorescent screen placed far from the slits. (When an electron hits a small portion of the screen, that portion of the screen glows.)



A. Which is a better model for how the electrons behave in this case: that they propagate in straight lines through the slits, or that they propagate like waves? Explain how you can tell.

B. Suppose that the above experiment were repeated but with the electrons accelerating through a potential difference equal to $0.5V$ instead of V.

i. **Predict** whether the bright regions on the screen would get closer together, farther apart, or stay at the same locations. Discuss your reasoning with your partners.

✔ ***Ask a staff member for a figure showing how the interference pattern would change if the accelerating voltage were halved so that you may check your prediction.***

ii. Considering *only* how the interference pattern changes (as shown in the figures provided), would you conclude that halving the accelerating voltage changes the wavelength of the electron wave?

If so: Does the wavelength increase or decrease?
Does the wavelength change by a factor that is *greater than, less than,* or *equal to* 2? Explain how you can tell from the figures.

If not: Explain how you can tell that the wavelength did not change.

Tutorial worksheet: Wave properties of matter

iii. How would halving the accelerating voltage affect each of the quantities listed below? In particular, determine (a) whether each quantity would *increase* or *decrease*, and (b) whether each quantity would change by a factor that is *greater than, less than,* or *equal to* 2. Explain your reasoning in each case.

- the kinetic energy of each electron that reaches the slits

- the momentum of each electron that reaches the slits

- the deBroglie wavelength of each electron that reaches the slits

iv. Are your answers to parts ii and iii regarding the de Broglie wavelength of the electron wave consistent? If not, resolve any inconsistencies.

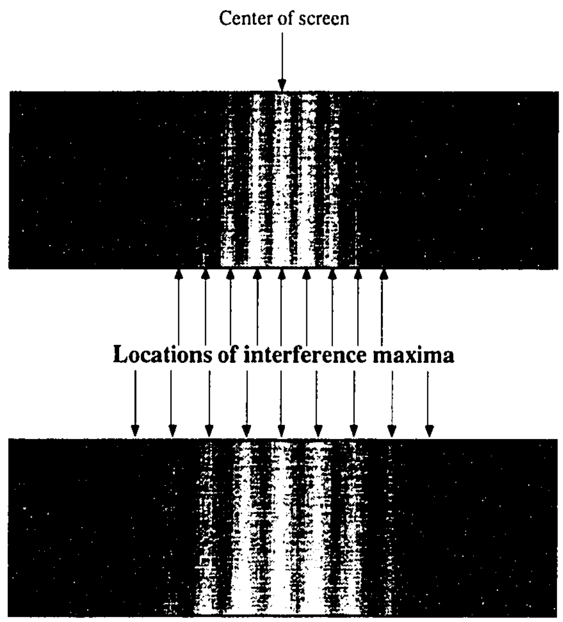
Now that you have worked through parts ii and iii, review your answer to part i. Do you still agree with your earlier reasoning? If not, how would you revise it?

C. Suppose you were to perform the electron interference experiment described in part A.

What measurements would you need to make and what steps would you need to follow in order to calculate the deBroglie wavelength of the electrons? Describe two independent methods that you could use in order to calculate the deBroglie wavelength.

Figures showing how halving the accelerating voltage would affect two-slit electron interference pattern:

Accelerating voltage equal to $V = V_o$



Accelerating voltage equal to $V = \mathbf{0.5}V_o$

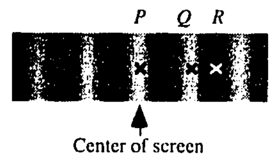
(Photograph from Tipler, *Modern Physics*, p. 185.)

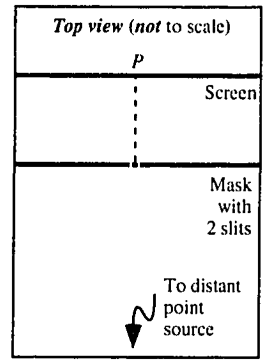
Handout for tutorial *Wave properties of matter*

Homework: Wave properties of matter (algebra-based course)

1. A distant point source of red light, a mask with two identical narrow slits, and a screen are arranged as shown in the top view diagram below right.

   The photograph at right shows the pattern that appears on a distant screen. Point *P*, the center of the pattern, and point *Q* are maxima. Point *R* marks a minimum to the right of point *Q*.





A. In the space above the diagram at right, clearly label each of the lettered points according to the path length difference from the slits to that point. Express your answers in terms of λ.

B. Suppose that the left slit were completely covered. Would the intensity at each of the following points *increase, decrease,* or *stay the same?* In each case, explain your reasoning.

- point *Q*

- point *R*

C. Suppose that the screen is 2.2 m from the slits, and that the distance from point *P* to point *R* is 1.6 mm.

   Determine the distance between the slits in terms of λ. Show all work.

Homework: Wave properties of matter (algebra-based course)

2. Consider the two-slit electron interference experiment as described in part II of the tutorial. (Shown at right is the pattern seen on a phosphorescent screen placed far from the slits.)

Pattern on phosphorescent screen

A. Consider the following statement below made by a student:

> "The wavelength of a wave is proportional to the speed of the wave. This means that if the speed of an electron decreases, then so does the wavelength of the electron wave."

Do you agree or disagree with this statement? Justify your answer on the basis of your results from part II of the tutorial.

B. Suppose that the two-slit experiment described above were repeated using muons that have the *same kinetic energy* as the original electrons. (Recall $m_\mu \approx 200\, m_e$.)

i. Would the *speed* of each muon be greater than, less than, or equal to the speed of each of the (original) electrons? Explain your reasoning.

ii. Would the *momentum* of each muon be greater than, less than, or equal to the momentum of each of the (original) electrons? Explain your reasoning.

iii. Would the bright regions on the screen be *closer together, farther apart,* or *stay at the same locations* as before? Explain your reasoning. (*Hint:* How does the de Broglie wavelength of the muons compare to that of the (original) electrons?)

Homework: Wave properties of matter (modern physics course)

1. Consider the two-slit electron interference experiment as described in part II.A of the tutorial. (Shown at right is the pattern seen on a phosphorescent screen placed far from the slits.)

Pattern on phosphorescent screen

A. Write down an expression for the kinetic energy of a (non-relativistic) electron in terms of its deBroglie wavelength, λ, and appropriate constants. (*Hint:* How can you express the kinetic energy of the electron in terms of its momentum?)

B. Suppose that the two-slit experiment described above were repeated using muons that have the *same kinetic energy* as the original electrons. (Recall $m_\mu \approx 200m_e$.)

Would the bright regions on the screen be closer together, farther apart, or stay at the same locations as before? Explain your reasoning.

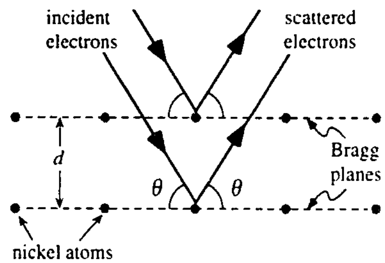
C. Consider the statement below made by a student:

"Muons have a higher mass than electrons. Because energy, E, is related to wavelength by $E = hc/\lambda$, muons that have the same kinetic energy as electrons will also have the same wavelength."

Do you agree or disagree with this statement? Explain your reasoning.

Homework: Wave properties of matter (modern physics course)

2. (Davisson-Germer experiment.) Monoenergetic electrons are incident on a nickel crystal. It is observed that intense scattering occurs at angles θ according to the Bragg condition, $2d\sin\theta = n\lambda$. (See diagram at right.)



A. Use trigonometry to show that the path length difference between the two scattered beams shown is equal to $2d\sin\theta$. Show all work.

B. Suppose that this experiment were repeated, each time with a ***single*** change made to the apparatus. For each change below, determine whether each of the angles θ at which intense scattering occurs would become *larger*, *smaller*, or *stay the same*. Explain your reasoning in each case.

   i. The electrons are replaced with neutrons, with each neutron having the same *momentum* as each of the original electrons.

   ii. The electrons are replaced with neutrons, with each neutron having the same *velocity* as each of the original electrons.

   iii. The electrons are replaced with neutrons, with each neutron having the same *kinetic energy* as each of the original electrons.

Homework: Wave properties of matter (Autumn 1996 quantum mechanics course)

1. Consider the two-slit electron interference experiment as described in part II.A of the tutorial. (Shown at right is the pattern seen on a phosphorescent screen placed far from the slits.)

Pattern on phosphorescent screen

A. Write down an expression for the kinetic energy of a (non-relativistic) electron in terms of its deBroglie wavelength, λ, and appropriate constants. (*Hint:* How can you express the kinetic energy of the electron in terms of its momentum?)

B. Suppose that the two-slit experiment described above were repeated using muons that have the *same kinetic energy* as the original electrons. (Recall $m_\mu \approx 200m_e$.)

Would the bright regions on the screen be closer together, farther apart, or stay at the same locations as before? Explain your reasoning.

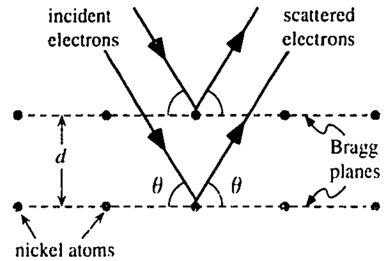
C. Consider the statement below made by a student:

"Muons have a higher mass than electrons, but because the energy, E, is related to the wavelength by $E = hc/\lambda$, muons that have the same kinetic energy as electrons will also have the same wavelength."

Do you agree or disagree with this statement? Explain your reasoning.

Homework: Wave properties of matter (Autumn 1996 quantum mechanics course)

2. (Davisson-Germer experiment.) Monoenergetic electrons are incident on a nickel crystal. It is observed that intense scattering occurs at angles θ according to the Bragg condition, $2d\sin\theta = n\lambda$. (See diagram at right.)



A. Use trigonometry to show that the path length difference between the two scattered beams shown is equal to $2d\sin\theta$. Show all work.

B. Suppose that this experiment were repeated, each time with a ***single*** change made to the apparatus. For each change below, determine whether each of the angles θ at which intense scattering occurs would become *larger*, *smaller*, or *stay the same*. Explain your reasoning in each case.

   i. The kinetic energy of the incident electrons is decreased.

   ii. The electrons are replaced with neutrons, with each neutron having the same velocity as each of the original electrons.

Homework: Wave properties of matter (Autumn 1997 quantum mechanics course)

1. Consider the two-slit electron interference experiment as described in the tutorial. (Shown at right is the pattern seen on a phosphorescent screen placed far from the slits.)

Pattern on phosphorescent screen

Suppose that this experiment were repeated using muons, with each muon having the *same kinetic energy* as each of the original electrons. (Recall $m_\mu \approx 200m_e$.)

a. Is the momentum of the each muon *greater than, less than,* or *equal to* the momentum of each of the original electrons? Explain your reasoning.

b. Is the de Broglie wavelength of the muons *greater than, less than,* or *equal to* the de Broglie wavelength of the original electrons? Explain your reasoning.

c. When the electrons are replaced with muons, would the bright regions on the screen be *closer together, farther apart,* or *stay at the same locations as before?* Explain your reasoning.

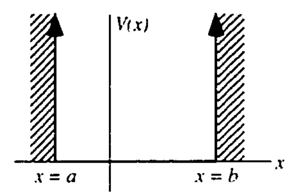
d. Consider the statement below made by a student:

"Muons have a higher mass than electrons, but because the energy, E, is related to the wavelength by $E = hc/\lambda$, muons that have the same kinetic energy as electrons will also have the same wavelength."

Do you agree or disagree with this statement? Explain your reasoning.

Homework: Wave properties of matter (Autumn 1997 quantum mechanics course)

2. Consider an infinite square well, shown at right. (In the case of an infinite square well, the potential, $V(x)$, is zero within some region, $a < x < b$, and is infinitely large outside this region.)



$$V(x) = \begin{cases} \infty, & x < a \\ 0, & a < x < b \\ \infty, & x > b \end{cases}$$

a. Would the kinetic energy of a particle in the ground state be *greater than, less than,* or *equal to* the kinetic energy of the same particle in an excited state?

b. Would the de Broglie wavelength of a particle in the ground state be *greater than, less than,* or *equal to* the de Broglie wavelength of the same particle in an excited state? Use your answer in part a to justify your answer here.

c. In the space below, sketch (as functions of x) the wave functions for the three lowest energy eigenstates: the ground state, the first excited state, and the second excited state.

For each wave function you have drawn, express the de Broglie wavelength in terms of the width, L, of the infinite square well (where $L = b - a$).

Are your expressions consistent with your results in part b? If not, resolve the inconsistencies.

Appendix G: Pretests, tutorial worksheet, and homework for the interactive tutorial lecture *Reflection and transmission*

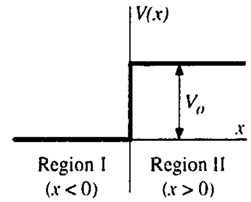
- Pretest (potential step question)
- Pretest (potential barrier question)
- Tutorial worksheet
- Homework

Pretest: Reflection and transmission (potential step question)

Monoenergetic electrons travel through a region in which the potential energy $V(x)$ varies with x as follows:

$$V(x) = \begin{cases} 0, & x < 0 \\ V_o, & x > 0 \end{cases}$$

(See figure at right.)

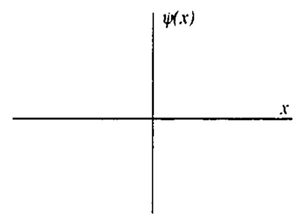


Let E represent the total energy of each electron, and $\psi(x)$, the wave function associated with the electrons.

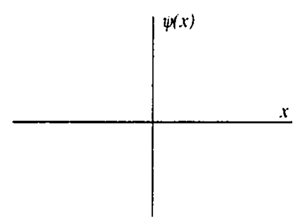
For each case described below:

- Describe in words the behavior of the electrons in Regions I and II. In particular: (i) How does the speed of the electrons in Region I compare to the speed of the electrons in Region II? (ii) Do all of the electrons that start in one Region end up in the other Region?
- On the set of axes provided, draw the shape of $\psi(x)$. Clearly indicate how, if at all, the wave function in $x < 0$ is qualitatively different than the wave function in $x > 0$.
- Briefly explain the reasoning you used in each case.

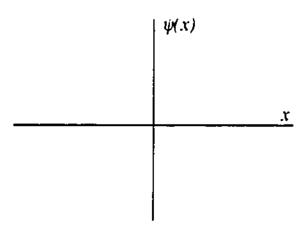
A. The electrons initially move ***from left to right*** (i.e., in the $+x$ direction) with $E > V_o$.



B. The electrons initially move ***from right to left*** (i.e., in the $-x$ direction) with $E > V_o$.



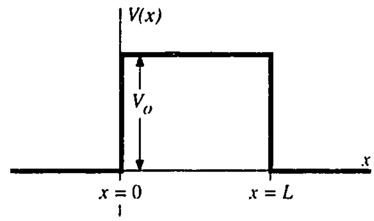
C. The electrons initially move ***from left to right*** (i.e., in the $+x$ direction) with $E < V_o$.



Pretest: Reflection and transmission (potential barrier question)

Monoenergetic electrons move from left to right into a region of space in which the potential energy $V(x)$ varies with x as follows:

$$V(x) = \begin{cases} 0, & x < 0 \\ V_o, & 0 < x < L \\ 0, & x > L \end{cases}$$

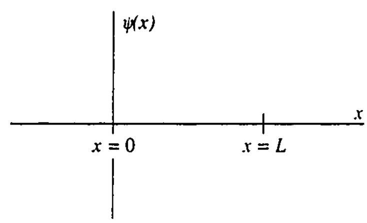


(A graph of $V(x)$ versus x is shown at right.)

Suppose that each electron has the same total energy, *E*, and that *E* is *less than* V_o.

A. Describe in words the behavior of the electrons in this case. In particular, do any of the electrons end up in the region $x > L$? Explain your reasoning.

B. On the set of axes provided below right, sketch the shape of the wave function, $\psi(x)$, for all values of x shown in the graph. If the wave function is equal to zero anywhere, show that explicitly. Explain the reasoning you used in drawing your graph.

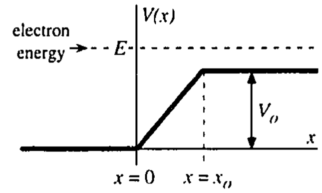


Tutorial worksheet: Reflection and transmission

In this interactive lecture, we will examine situations in which the wave functions are stationary states; *i.e.*, the full wave function, $\Psi(x, t)$, may be written as $\Psi(x, t) = \psi(x)\exp(-iEt/\hbar)$.

I. Behavior of classical particles in a region of varying potential energy

Monoenergetic electrons travel in the $+x$ direction and enter a region in which the potential energy, $V(x)$, increases linearly with x over the interval $0 < x < x_o$. (See figure at right.) Suppose that the energy, E, of each electron is larger than, V_o.



Answer the following questions by treating the electrons as ***classical*** particles (*i.e.*, by thinking of them as moving like billiard balls, not as having wave-like properties).

A. How, if at all, does the kinetic energy of an electron vary in the region $x < 0$? Explain.

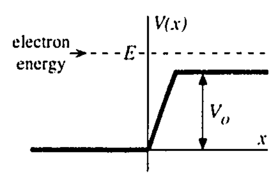
How does the *total energy* of an electron compare to its *kinetic energy* when the electron is located (i) in the region $x < 0$, (ii) in the region $x > x_o$? Explain.

Describe the motion of the electrons as they move through the region shown in the graph.

B. Use the graph of $V(x)$ versus x in order to determine the magnitude and direction of the net force on an electron at all locations shown in the graph. Explain how you used the graph to determine your answers.

Suppose that the potential energy increases from zero to V_o over a *smaller* interval than before, as shown at right.

How would this change affect (i) the magnitude of the net force on the electron and (ii) the duration of the net force? Explain your reasoning.

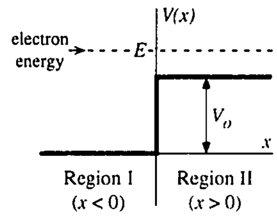


C. **Please STOP HERE.** After a brief class discussion, continue with the next page.

Tutorial worksheet: Reflection and transmission

II. Behavior of classical and quantum particles at a boundary

A. Suppose that the potential energy, $V(x)$, varied with x as shown in the graph at right. Let Region I be the region where $x < 0$, and Region II the region where $x > 0$. As before, the electron energy, E, is larger than V_o.



$$V(x) = \begin{cases} 0, & x < 0 \\ V_o, & x > 0 \end{cases}$$

i. In answering the following questions, treat the electrons as ***classical*** particles:

- Describe the magnitude, direction, and duration of the net force exerted on the electrons in this case.

- Would any electrons reach Region II? If so, in which region would the electrons have a larger kinetic energy? Explain.

ii. In answering the following questions, treat the electrons as ***quantum*** particles (*i.e.*, particles that have wave-like properties):

- In which region would the electrons have a larger de Broglie wavelength? Explain.

- Are there only *certain* allowed values for the total energy, E, of the incident electrons, or is *any* value of E allowed? Explain how you can tell.

- Would all of the incident electrons reach Region II? Explain why or why not.

- In the space below, sketch the behavior of the wave function associated with the electrons in Region I and in Region II. Be sure to take into account the relative sizes of the de Broglie wavelength in your sketch.

iii. **Please STOP HERE.** After a brief class discussion and demonstration, continue with the next page.

Tutorial worksheet: Reflection and transmission

iv. Write down the general form of the total wave function in Region I as a sum of two terms, with each term characterized by a definite direction of propagation. Label one of these terms, $\psi_{inc}(x)$, and the other, $\psi_{refl}(x)$. (Why are these names appropriate?)

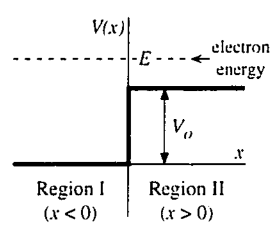
v. In this case, does the general form of the wave function in Region II contain terms that correspond to *one* direction of propagation or *many?* Explain.

Write down the general form of the wave function in Region II for this case. Label this wave function, $\psi_{tran}(x)$.

The wave function must satisfy certain conditions at the boundary between Regions I and II: (1) $\psi(x)$ must be continuous at $x = 0$, and (2) $d\psi(x)/dx$ is continuous at $x = 0$. (We will justify these conditions, called *boundary conditions,* in the homework.)

vi. Apply the above boundary conditions to your expressions for $\psi_{inc}(x)$, $\psi_{refl}(x)$, and $\psi_{tran}(x)$. If the amplitude of $\psi_{inc}(x)$ is treated as given information, do the boundary conditions provide enough information for you to solve the problem completely?

B. Now suppose the incident electrons arrived at $x = 0$ from the *right* instead of from the left.



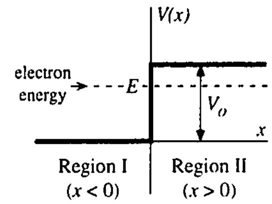
i. Would any of the incident electrons be reflected at the boundary in this case? Explain your reasoning.

ii. Write down the general form of the wave function in Regions I and II.

iii. **Please STOP HERE.** After a brief class discussion, continue with the next page.

Tutorial worksheet: Reflection and transmission

C. Now consider the case in which the incident electrons are incident from the left but their total energy, E, is ***smaller*** than V_o.



i. Write down the form of the total wave function in Region I (i.e., $\psi_{inc}(x) + \psi_{refl}(x)$). Explain your reasoning.

ii. Write down the general form of the wave function in Region II. If it is zero, state that explicitly and explain how you can tell that this is the case.

iii. **Please STOP HERE.** After a brief class discussion, continue with part iv.

iv. Does the transmitted wave function, $\psi_{tran}(x)$, correspond to a wave function for a traveling wave? Explain why or why not.

v. Using your expressions above for the wave function in Regions I and II, write down equations that illustrate the relevant boundary conditions. Solve the resulting system of equations in order to express the amplitude of $\psi_{refl}(x)$ in terms of that of $\psi_{inc}(x)$.

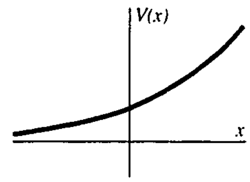
How does the absolute square of the amplitude of $\psi_{inc}(x)$ compare to that of $\psi_{refl}(x)$?

Interpret your result: How does the intensity of the reflected beam compare to that of the incident beam? (Is this result what you would expect *classically?*)

vi. **Please STOP HERE** for class discussion and wrap-up.

Homework: Reflection and transmission

1. Suppose that in a certain region of space the potential energy, $V(x)$, varies smoothly with x as shown at right.



In parts A and B below, you will justify one of the boundary conditions you used on p. 4 of the interactive lecture worksheet. In all parts of the problem, $x = -\varepsilon$ and $x = \varepsilon$ are locations just to the left and to the right of $x = 0$, respectively.

A. At a given instant in time, how does the probability density at $x = -\varepsilon$ compare to the probability density at $x = \varepsilon$? Explain.

What does your answer imply about the value of the wave function, $\psi(x)$, at $x = -\varepsilon$ and $x = \varepsilon$? In particular, consider the limit in which ε becomes vanishingly small. Explain.

B. Suppose that we integrated the (one-dimensional, time-independent) Schrödinger equation from $x = -\varepsilon$ and $x = \varepsilon$:

$$\int_{-\varepsilon}^{\varepsilon} -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x)dx + \int_{-\varepsilon}^{\varepsilon} V(x)\psi(x)dx = \int_{-\varepsilon}^{\varepsilon} E\psi(x)dx.$$

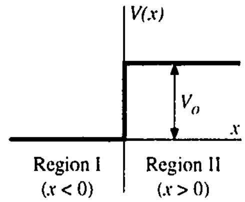
i. The first term on the left-hand side is an integral of the second-order derivative of $\psi(x)$. Simplify this term so that it can written in terms of *first-order* derivatives of $\psi(x)$.

ii. Now consider the resulting equation in the limit in which ε approaches ***zero***. Show that the only term that survives in this limit is the term that you simplified in part i. (*Hint:* You will need to apply the boundary condition from part A to do this.)

iii. Can you interpret the resulting equation from part ii to mean that $d\psi(x)/dx$ is continuous at $x = 0$? If not, check your work above and resolve any inconsistencies.

Homework: Reflection and transmission

C. Consider the potential energy curve shown at right (taken from the interactive lecture).



i. If electrons are incident from the left, how does the probability density at $x = -\varepsilon$ compare to that at $x = \varepsilon$? Explain.

(*Hint:* Although an electron detected in Region I could be an incident electron or a reflected one, which expression best describes the *total* probability density in Region I?)

- $|\psi_{inc}(x)|^2 + |\psi_{refl}(x)|^2$
- $|\psi_{inc}(x) + \psi_{refl}(x)|^2$

Is your answer consistent with the boundary condition that $\psi(x)$, is continuous at $x = 0$? If not, check your work above and resolve any inconsistencies.

How, if at all, would your answer above be different if the electrons were incident from the right (in Region II)? Explain.

ii. Apply the method you used in part B to show that $d\psi(x)/dx$ is continuous at $x = 0$ in this case.

D. Show that $d\psi(x)/dx$ is ***not*** continuous at $x = 0$ for each of the following cases:

i. $V(x) = -\lambda\delta(x)$, where λ is a constant and $\delta(x)$, the Dirac delta-function, is defined to have the property: $\lim_{\varepsilon\to 0} \int_{-\varepsilon}^{+\varepsilon} \delta(x)dx = 1$.

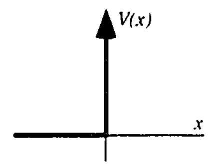
ii. $V(x) = \begin{cases} 0, & x < 0 \\ \infty, & x > 0 \end{cases}$

Homework: Reflection and transmission

2.

A. Monoenergetic electrons are incident from the left on a potential barrier of infinite height, as shown below right.

i. Write down the general form of the wave function in the region $x < 0$ (*i.e.*, $\psi_{inc}(x) + \psi_{refl}(x)$).

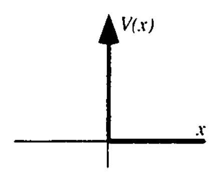


ii. What is the value of the wave function for $x > 0$? Explain.

$$V(x) = \begin{cases} 0, & x < 0 \\ \infty, & x > 0 \end{cases}$$

iii. Apply the appropriate boundary condition at $x = 0$ in order to solve for the amplitude of $\psi_{refl}(x)$ in terms of the amplitude of $\psi_{inc}(x)$. Show all work.

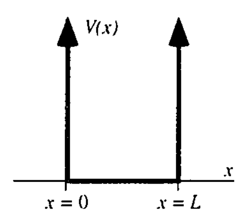
iv. How, if at all, would your result in part iii be different if the same electrons were incident from the *right* on a potential barrier of infinite height (see figure at right)? Explain.



$$V(x) = \begin{cases} \infty, & x < 0 \\ 0, & x > 0 \end{cases}$$

B. Now consider an electron in a region in which $V(x)$ varies with x as shown below right.

i. Write down the general form of the wave function in the region $0 < x < L$.



ii. Show that in order for the wave function to satisfy the relevant boundary conditions at $x = 0$ **and** at $x = L$, the total energy, E, takes on only ***certain*** values (i.e., not all values of E are allowed).

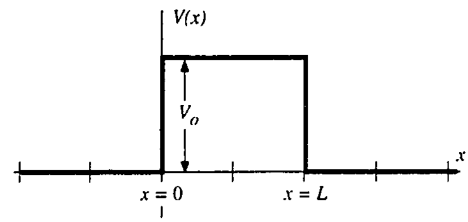
$$V(x) = \begin{cases} \infty, & x < 0 \\ 0, & 0 < x < L \\ \infty, & x > L \end{cases}$$

Appendix H: Pretest, tutorial worksheet, and homework for the tutorial *Relating classical mechanics to quantum mechanics*

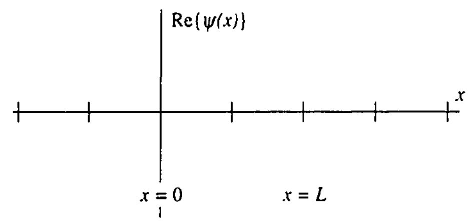
- Pretest
- Tutorial worksheet
- Homework (Winter 1997 quantum mechanics course)
- Homework (Autumn 1997 quantum mechanics course)

Pretest: Relating classical mechanics to quantum mechanics

1. A beam of particles is incident from the left on the potential barrier shown at right. Consider the case for which the total energy, E, of each particle is ***less than*** V_o.

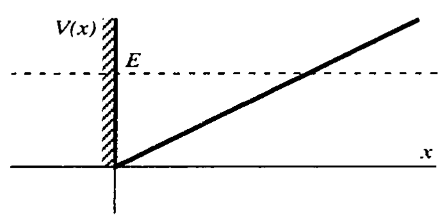


In the space provided below right, sketch a qualitatively correct graph of the real part of the wave function, Re{ $\psi(x)$ }, for this case. **Explain the reasoning you used to draw your graph.**



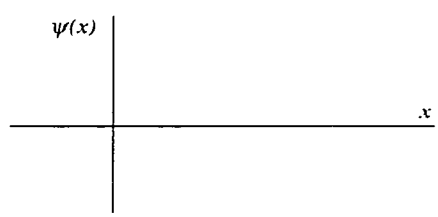
2. Consider a particle in a high energy eigenstate of the potential given below. (That is, suppose the energy, E_n, of the particle is an energy eigenvalue that is *much larger than* the ground state energy.)

$$V(x) = \begin{cases} \infty, & x < 0 \\ +\beta x, & x > 0 \end{cases}$$



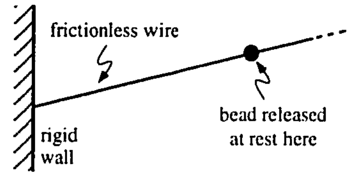
A. On the graph of $V(x)$ vs x, indicate the classical turning point(s).

B. In the space at right, sketch a qualitatively correct graph of the wave function, $\psi(x)$. Use the same horizontal scale used in the graph of $V(x)$ vs x. **Explain the reasoning you used to draw your graph.**



Tutorial worksheet: Relating classical mechanics to quantum mechanics

A. Consider a small bead that is free to slide along a taut, frictionless wire tied to a wall, shown at right. (Note that the wire is *not* horizontal.) The bead is initially at rest at a point along the wire and then released. Assume that the bead always collides elastically with the wall.

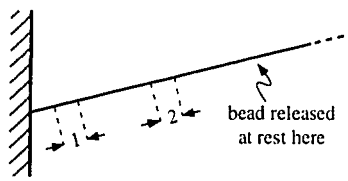


SIDE VIEW DIAGRAM

i. Describe the motion of the bead. In particular, how does the speed of the bead change as time goes on?

ii. Consider two small segments (1 and 2) of the wire shown at right. The segments are of equal length.

During one cycle of its motion, does the bead spend more time along segment 1 or 2? Explain.



Imagine the bead is set in motion as described above but in a darkened room. Also imagine that a light bulb in the room flashes on and off with random time intervals between flashes.

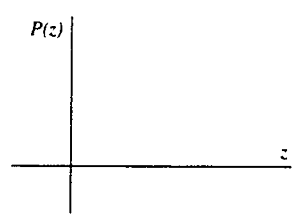
At an instant when the bulb flashes, how does the *probability* of finding the bead along segment 1 compare to the *probability* of finding it along segment 2? Explain.

iii. Let $P(z)dz$ represent the probability of finding the bead along the segment between z and $z+dz$ at a randomly selected time. (Let z refer to position *along the length of the wire.*)

Write an expression for $P(z)dz$ in terms of the time, dt, that the bead spends along this segment as it travels down the wire toward the wall (*i.e.*, during one-half cycle of its motion). You may ignore overall constants. Explain.

Rewrite your expression above by rewriting dt in terms of dz (the length of the segment) and other quantities that are related to the bead's motion.

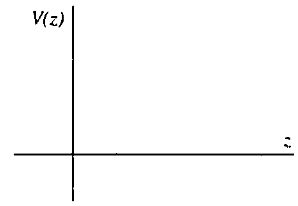
In the space at right, draw a qualitatively correct graph of $P(z)$ vs z. Explain how you can use your results above to justify the way you drew your graph.



Tutorial worksheet: Relating classical mechanics to quantum mechanics

B. Now consider the quantum-mechanical analogue to the physical situation described in part A.

i. In the space at right, draw a graph of the potential, $V(z)$ vs z, for this case. On the graph, indicate a possible value for the energy of the particle and the corresponding turning points.

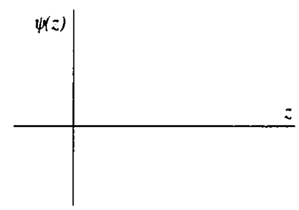


ii. Suppose that the particle occupied an energy eigenstate (call it $\psi(z)$) corresponding to an energy eigenvalue *much larger than* the ground state energy. (That is, consider an energy eigenstate for which the wave function has several nodes.)

How does the spacing between consecutive nodes vary with z? Explain. (*Hint:* Use your knowledge of the de Broglie wavelength.)

How does the amplitude of $\psi(z)$ vary with z? Explain. (*Hint:* How can you use your graph of $P(z)$ in part A to help you here?)

iii. In the space at right, draw a qualitatively correct graph of $\psi(z)$. Explain the reasoning you used to draw the graph.



iv. Consider the statement below made by a physics student:

"A particle in a potential well prefers to be where the potential energy is lowest. That means that the amplitude of the wave function is largest where the potential energy is lowest."

Do you agree or disagree with this statement? Explain your reasoning.

C. According to the WKB approximation, the wave function, $\psi(z)$, for the particle in part B may be expressed as a linear combination of the terms, $\psi_{\pm}(z) = \dfrac{C_{\pm}}{\sqrt{p(z)}} exp\left(\pm \dfrac{i}{\hbar}\int p(z)dz\right)$.

i. Are your results from part B regarding the ***amplitude*** of $\psi(z)$ and the ***spacing between consecutive nodes*** consistent with this result? If not, resolve any inconsistencies.

ii. How should the square of the wave function, $|\psi(z)|^2$ (using the WKB approximation), compare to your expression for $P(z)$ from part A.iii? If there are any inconsistencies, resolve them.

Tutorial worksheet: Relating classical mechanics to quantum mechanics

D. In your graph of the potential, $V(z)$, (in part B) identify all *classically forbidden regions* and all *classical turning points* of the particle.

For each statement below, state the conditions (if any) under which that statement is true. Explain. (*Hint:* None of the statements are true under *all* conditions!)

- the kinetic energy of the particle is greater than zero at all positions within each classically forbidden region

- $\psi(z)$ has nodes within each classically forbidden region

- $\psi(z)$ is equal to zero at all positions, z, in each classically forbidden region

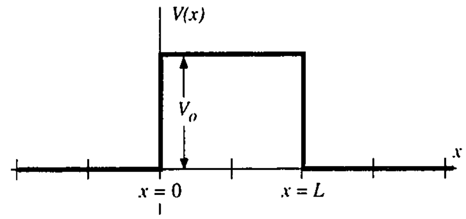
- $|\psi(z)|$ decreases as $e^{-|Kz|}$ (where K is a constant) in each classically forbidden region

Is your graph of the wave function, $\psi(z)$, (in part B) consistent with your results above? If not, resolve the inconsistencies.

Homework: Relating classical mechanics to quantum mechanics (Winter 1997)

1. ***Application of the WKB Approximation***

Consider a beam of monoenergetic particles incident from the left toward the region shown at right. The total energy, E, of each particle is less than V_o. (A region such as the one shown, in which the potential is larger than the total energy of each particle, is called a *potential barrier.*)



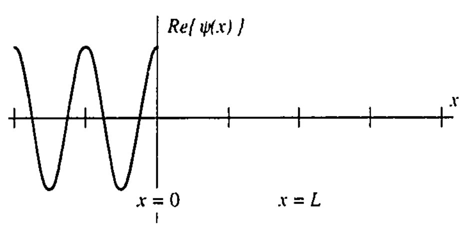
Some of the particles are observed to propagate away from the barrier in the region $x > L$.

A. Do *all* of the incident particles end up in the region $x > L$? On the basis of your answer, how does the amplitude of $\psi(x)$ in the region $x > L$ compare to that in the region $x < 0$? Explain.

B. How does the de Broglie wavelength of the particles in the region $x > L$ compare to the de Broglie wavelength of the incident particles? Explain.

C. The qualitative structure of the wave function (*e.g.*, the real part of $\psi(x)$) in the region $x < 0$ is shown at right. Complete the graph for $x > 0$.

Is your graph consistent with your results from parts A and B? If not, resolve any inconsistencies.

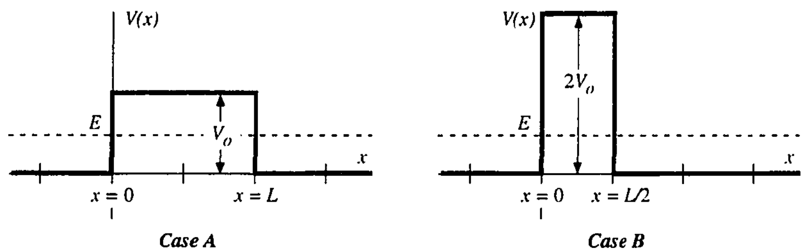


D. Recall that, in the WKB approximation, the transmission probability for a barrier of width L is given by: $T \approx \exp(-2\gamma)$, where $\gamma = \frac{1}{\hbar}\int_0^L |p(x)|dx$. This approximation holds as long as $\gamma \gg 1$.

i. For the barrier shown at the top of the page, how would increasing the energy, E, of the incident particles (while keeping E less than V_o) affect the transmission probability? Explain your reasoning.

Homework: Relating classical mechanics to quantum mechanics (Winter 1997)

ii. Consider the two situations shown below. In each case the energy, E, of the incident particles is the same, but the features of the potential barrier are different.



According to the WKB approximation, is the transmission probability for case A *greater than, less than,* or *equal to* that for case B? (*Hint:* How does the integral in the expression for transmission probability compare in each case?) Show all work.

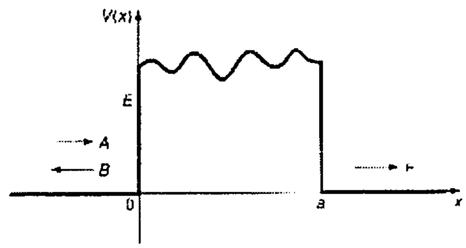
iii. Consider Case A above, for which we have assumed that the WKB approximation holds when $E = V_o/2$. Would the WKB approximation ***also*** hold for:

- all values of E that are larger than 0 but less than $V_o/2$? Explain.

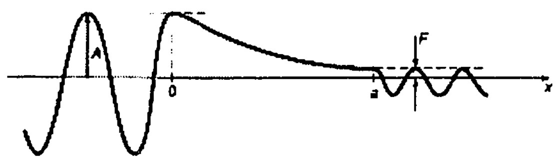
- all values of E that are larger than $V_o/2$ but less than V_o? Explain.

Homework: Relating classical mechanics to quantum mechanics (Winter 1997)

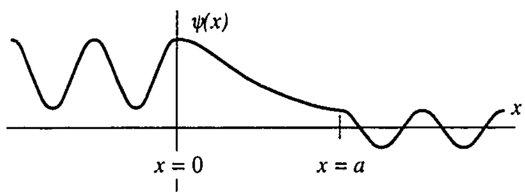
2. OOPS! Consider the potential barrier shown at right. Each figure below attempts to illustrate the (real part of the) wave function associated with the barrier, however each figure contains a serious error. Identify the error in each figure.



A.

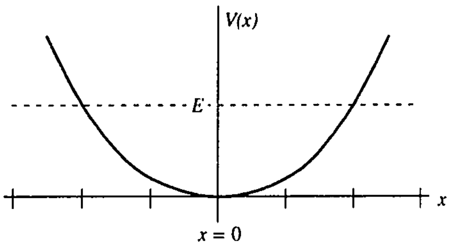


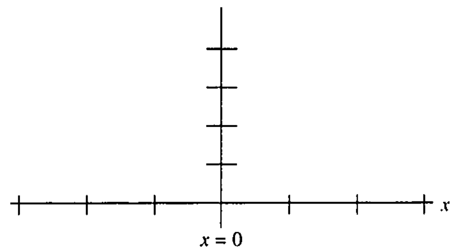
B.



Homework: Relating classical mechanics to quantum mechanics (Autumn 1997)

I. Consider a particle in a harmonic oscillator potential with total energy $E > 0$. (See graph below left.)





A. In the space above right, carefully draw and label qualitatively correct graphs of:

i. the probability density, $P_{cl}(x)$ *vs. x,* for a particle in a *classical* oscillator, and

ii. the probability density, $P_n(x)$ *vs. x,* for a particle in a *quantum mechanical* oscillator. In your answer, assume that E is equal to an energy eigenvalue, E_n, for which ***n* is large** (*i.e.*, that $n \gg 1$).

Explain the reasoning you used in drawing your graphs. In particular: How is your graph of $P_{cl}(x)$ consistent with your graph of $P_n(x)$? Should the nodes in the graph of $P_n(x)$ be equally spaced?

B. How, if at all, would your reasoning in part A have been different if you had been asked to draw a graph of $P_n(x)$ for the *ground state* ($n = 0$) rather than for a high ($n \gg 1$) energy state? Explain.

C. Consider the following *incorrect* statements made by two students:

Student 1: *"The ground state wave function for the harmonic oscillator has a maximum value at x = 0. That means I can think of the particle as spending more time in the center than at any other point."*

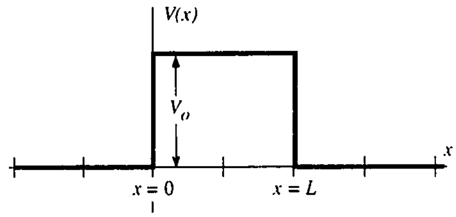
Student 2: *"This makes sense because the particle would pass through the center more often than it would pass any other point."*

For each statement, specify which parts are incorrect. Explain.

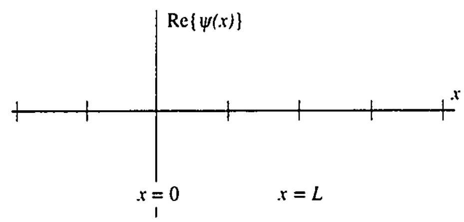
Homework: Relating classical mechanics to quantum mechanics (Autumn 1997)

2. A beam of monoenergetic particles is incident from the left toward the potential barrier shown below right. The total energy, E, of each particle is less than V_o. Some of the particles are observed to propagate away from the barrier in the region $x > L$.

A. Rank the regions (i) $x < 0$, (ii) $0 < x < L$, and (iii) $x > L$, according to the de Broglie wavelength, λ, of the particles in that region. If λ is not well-defined in a particular region, state that explicitly. Explain.



B. In the space at right, draw a qualitatively correct graph of the real part of $\psi(x)$. Make sure your graph is consistent with your results in part A.



C. Is the magnitude of the probability current in the region $x < 0$ greater than, less than, or equal to that in the region $x > L$? Explain your reasoning, or show all work.

VITA

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IMAGE EVALUATION TEST TARGET (QA-3)

150mm

6"

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Rochester, NY 14609 USA
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