Matroid-Theoretic Approaches to Compactified Jacobians and Vector Bundles

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Abstract

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This thesis is composed of three distinct projects lying at the interface of combinatorics and algebraic geometry. Though not directly related to one another, these projects are united by the underlying themes of viewing graphs and matroids as discrete analogs of objects arising in algebraic geometry, and of viewing these combinatorial objects as tools for encoding discrete structure in algebraic-geometric contexts. The first project included in this thesis studies the combinatorics of certain subdivisions of Euclidean spaces into highly-structured polyhedra known as zonotopes. By taking periodic subdivisions of this type and passing to the quotient under the lattice of periodicity, one obtains a class of cell decompositions of real tori. We prove an enumerative result on the f-vectors of these decompositions. These subdivisions arise in the context of compactified Jacobians of nodal curves, where the cell structure encodes the inclusions of orbit closures under a torus action. We then use our f-vector result to give a formula for the classes of these compactifications in the Grothendieck ring of varieties. The second project is of a more tropical geometric nature, and represents

a context in which the categorical and K-theoretic understanding of algebraic geometry carries over well into the tropical context. We establish in particular that many categories of matroids arising in modern matroid theory naturally possess the structure of proto-abelian categories, forming non-additive analogs of the abelian categories appearing in many areas of algebra. We extend this result to show that the category of vector bundles in tropical geometry possesses a similar categorical structure, and use this to give an account of tropical Harder-Narasimhan filtrations mirroring those appearing in the moduli theory of algebraic vector bundles. Finally, the third project describes a purely combinatorial result motivated by the Riemann-Roch theory of graphs. We study two group actions of the critical group of a graph on the set of its spanning trees. Prior to our work, it was known that these group actions coincide if this graph is planar, and we resolve a conjecture that the converse also holds. For each non-planar graph, these two actions are necessarily distinct. By viewing the acting group as a subgroup of the tropical Jacobian of the graph, this affords a connection to the first project mentioned above.

The background for these projects is considerable and possesses little overlap. For the sake of readability, the thesis contains multiple sections and appendices dedicated to exposition of this background. The second and third projects mentioned above appear independently as papers, and the text of this thesis contains overlap with these documents.

Dedicated to my wife, Valerie.

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Chapter 1

Introduction

This thesis synthesizes a collection of projects regarding several interactions between combinatorics and algebraic geometry, in particular by exploring several facets of matroid theory and the analysis and use of matroidal structure in algebraic contexts. The present document is structured in such a way as to place matroids in the center of our attention, beginning with an exposition of matroid theory as we conceive of it and subsequently considering several contexts in which the analysis of matroids or analysis of algebraic-geometric objects through a matroidal lens yields a deepened understanding.

We now survey the contributions of the present document to the literature. Some of the work mentioned below is currently either in publication or has been submitted for publication, while other results appear here for the first time. When another reference for the included work is available, we mention it explicitly.

1.1. Enumerating Faces of Toric Arrangements

The first contributions of this paper appear in Chapter 4. We use methods of geometric combinatorics, in particular matroid theory, Möbius functions of posets, and some rudimentary topology in order to enumerate the number of cells of a fixed dimension in certain highly-structured arrangements of subtori in a topological torus $T = \mathbb{R}^n/\mathbb{Z}^n$. The necessary background for all material in Chapter 4 is surveyed in Sections 2.1 and 2.3 of Chapter 2 and Section 3.2 of Chapter 3.

Geometric combinatorics has a long history of studying arrangements of subspaces in ambient spaces of interest. Perhaps the most notable examples of such systems are hyperplane arrangements and subspace arrangements. In this context, a classical question asks, given a finite collection of k hyperplanes in a vector space $\mathbb{R}^n$, how many connected components there are in the complement of a hyperplane arrangement. This question was answered for arbitrary such arrangements by the thesis of Zaslavsky ([55]). The solution uses the Möbius function (an enumerative invariant of posets, largely introduced in combinatorics by Crapo in [17]) associated with a natural poset associated with the arrangement in order to recursively compute the number of regions of restrictions of the original arrangement. One then uses the Euler characteristic of the ambient spaces of these restrictions (each of which is a Euclidean space) in order to obtain a closed-form solution to the problem.

In our context, we generalize the approach of Zaslavsky in order to answer an analogous question for certain arrangements of topological subtori in a topological torus T. The arrangements we analyze are those arising from regular matroids. Given a regular matroid M, fix a realization $A = [a_1 \cdots a_n]$ of M in $\mathbb{R}^g$ and consider the zonotope $Z(M)$ defined as the Minkowski sum of the symmetric segments $\mathrm{conv}\{-a_i, a_i\}$. The free abelian group $\mathbb{Z}\langle a_1, \dots, a_n\rangle$ is a lattice in $\mathbb{R}^g$, and identifying faces of Z which are lattice translates of one another yields a real torus of dimension g. Through the realization A of M, we specify the structure of an oriented matroid on M, and a (realizable) single-element lifting of this oriented matroid can be associated with a hyperplane arrangement in $\mathbb{R}^{g+1}$ which then induces an arrangement of topological subtori of codimension 1 in T. We call such a toric arrangement *matroidal*. This matroidal toric arrangement induces a cell decomposition of the torus T, and the first main result of this thesis, presented in Chapter 4, provides a formula for the number of cells of dimension k in this cell decomposition.

Theorem A (Theorem 4.2.1)**.** *Let $\mathcal{T}$ be a matroidal toric arrangement in $\mathbb{R}^N$ defined by a* *single-element lifting $\widehat{M}$ of a regular matroid M, and let μ be the Möbius function of the lattice of flats of $\widehat{M}$. Denote by $\overline{\mathcal{Z}}$ the cell decomposition dual to $\mathcal{T}$. Then we have*

$$f_k(\overline{\mathcal{Z}}) = (-1)^{N-k} \sum_{\substack{R,S\in\mathcal{L}(\mathcal{T}) \\ \dim S=0 \\ \dim F=g-k}} \mu(F,G).$$

1.2. Matroids and Compactified Jacobians

The Jacobian of an algebraic curve X defined over an algebraically closed field $\mathbf{k}$ is a moduli space of degree 0 line bundles X. When X is smooth of genus g, the Jacobian J_X is an abelian variety of dimension g, and a wealth of information about X can be gleaned from the study of J_X. In fact, due to a theorem of Torelli, knowledge of the Jacobian J_X determines X up to isomorphism. If the curve X has mild (i.e. nodal) singularities, then the Jacobian J_X fails to be an abelian variety, and is in fact an extension of an abelian variety by an algebraic torus. Explicitly, we have a short exact sequence of algebraic groups as follows:

$$0 \longrightarrow \mathbb{G}_m^g \longrightarrow J_X \longrightarrow J_{\tilde{X}} \longrightarrow 0.$$

Here, $\tilde{X}$ is the (smooth) normalization of X, so that $J_{\tilde{X}}$ is an abelian variety. It is a classical problem to provide compactifications $\overline{J}_X$ of the Jacobian J_X which parameterize geometric objects associated with X. In [44], Oda and Seshadri utilize toric geometry in order to provide such compactifications $\overline{J}_X(\psi)$. These compactifications depend on a choice of parameter ψ drawn from the cut space of the dual graph G_X of X, and their orbit structure under $\mathbb{G}_m^g$ is determined by a certain periodic polyhedral decomposition $\mathrm{Del}_\psi(G_X)$ of the cycle space of G_X. The subdivisions $\mathrm{Del}_\psi(G_X)$ are referred to as *Namikawa subdivisions*. These Namikawa subdivisions encode the structure of $\overline{J}_X(\psi)$ and determine it uniquely. Moreover, the number of translation equivalence classes of k-dimensional cells of $\mathrm{Del}_\psi(G_X)$ is equal to the number of codimension-k $\mathbb{G}_m^g$-orbits in $\overline{J}_X(\psi)$. See Section 5.3 for details of this construction and the associated correspondence.

In Chapter 5, we prove a structural result about Namikawa subdivisions, establishing that for *orientable* choices of ψ (cf. Section 5.4) the Namikawa subdivisions are periodic continuations of coherent subdivisions of a zonotope naturally associated with G_X.

Theorem B (Theorem 5.5.1)**.** *Suppose that ψ is a choice of orientable polarization for G_X. Then the restriction of the Namikawa decomposition $Del_\psi(G_X)$ to a distinguished cell $Z_\psi \subseteq H^1(G_X;\mathbb{R})$ is a ρ-coherent subdivision with respect to the projection $\rho : C_\psi \to Z_\psi$ of a full-dimensional cube onto the zonotope Z_ψ. Moreover, every ρ-coherent subdivision arises this way.*

This result allows us to take advantage of the theory of coherent subdivisions and their relationship to oriented matroid theory (cf. Section 2.3) in order to enumerate translation equivalence classes of cells of $\mathrm{Del}_\psi(G_X)$. Indeed, the restricted Namikawa subdivision is dual to a matroidal toric arrangement of the sort described in Theorem 4.2.1, and so we can apply this result in our context to enumerate cells. In view of the correspondence between these equivalence classes and the torus orbits in $\overline{J}_X(\psi)$, we therefore obtain a count for the number of torus orbits of each dimension. Since algebraic group actions on a variety yield a stratification of the associated variety, these $\mathbb{G}_m^g$-orbits yield a stratification of $\overline{J}_X(\psi)$. By analyzing the geometry of the orbits themselves and applying Theorem 4.2.1, we obtain a formula for the class of $\overline{J}_X(\psi)$ in the Grothendieck ring of varieties $K_0(\mathrm{Var}_{\mathbf{k}})$.

Theorem C (Corollary 5.6.5)**.** *Let $Del_\psi(G_X)$ be a Namikawa subdivision associated with an orientable polarization ψ and let $\overline{N}_1, \dots, \overline{N}_k$ be the $H_1(G_X;\mathbb{Z})$-orbits of the cells of $Del_\psi(G_X)$. Then the class of the compactified Jacobian $\overline{J}_X(\psi)$ in the Grothendieck ring $K_0(Var_{\mathbf{k}})$ is given by the expression*

$$[\overline{J}_X(\psi)] = [J_{\tilde{X}}] \cdot \sum_{k=0}^{g} (-1)^{g-k} \sum_{\substack{R,S \in \mathcal{L}(\mathcal{M}_\varphi) \\ \dim S = 0 \\ \dim R = g-k}} \mu_\psi(R,S)(\mathbb{L}-1)^{g_{N_k}}.$$

Here, $\tilde{X}$ is the normalization of X, $\mathcal{M}_\psi$ is the oriented matroid lifting associated to ψ, μ_ψ is the Möbius function of $\mathcal{M}_\psi$, and g_{N_k} is the first Betti number of the subgraph of G_X obtained by deleting edges associated to Minkowski summands of the associated cell in $Del_\varphi(G_X)$.

1.3. Matroids as a Context for K-Theory

A large body of recent work in the area of matroid theory has revealed that an abstract approach to matroids with coefficients provides an avenue for simultaneously generalizing the theory of (classical) matroids, oriented matroids, valuated matroids, and linear subspaces of vector spaces. This theory, introduced by Baker and Bowler in [6], introduces the notion of a *matroid over an idyll F*. An *idyll* (also known as a *tract* or an emphidyllic tract by some authors) is a generalization of a field that includes fields, partial fields, and hyperfields as special cases (see Appendix C for details). The study of matroids over a idylls has proven to be an appropriate level of abstraction in which to phrase many important results in matroid theory, and among other things has afforded deeper study of the established bridge between (valuated) matroid theory and tropical geometry.

Among other things, the perspective of matroids over idylls underscores a strong analogy between matroid theory and linear algebra, and indeed one can ask whether there is an appropriate notion of morphisms for matroids over idylls which yields a category sharing structure with that of vector spaces over a field. Our next main result, drawn from joint work with Jaiung Jun and Alex Sistko ([32]), establishes that for a (perfect, see Section 2.4) idyll F, the resulting category F-$\mathbf{Mat}_\bullet$ shares a wealth of structure with the categories $\mathbf{Vect}_\mathbf{k}$ of vector spaces over a field $\mathbf{k}$.

Theorem D (Theorem 6.1.11, Theorem 6.1.22)**.** *For each perfect idyll F, the category of pointed F-matroids is proto-abelian (hence proto-exact) with exact direct sum and exact duality. The category of simple pointed F-matroids is also proto-abelian with exact direct sum, but does not admit exact duality.*

In Chapter 6 we also establish a variety of propositions regarding the category F-$\mathbf{Mat}_\bullet$ and morphisms of F-matroids which, to our knowledge, are first studied in [32].

1.4. Tropical Vector Bundles

Following the theme described in the previous section, we utilize our results on F-$\mathbf{Mat}_\bullet$ for the idyll $F = \mathbb{T}$ of tropical numbers to study tropical vector bundles in the sense of [38]. In particular, we introduce the notion of morphisms of tropical toric reflexive sheaves (cf. Section 6.4) and establish that the resulting category $\mathbf{TRS}_\bullet$ is proto-abelian.

Theorem E (Theorem 6.4.16, Corollary 6.4.17, Corollary 6.4.18)**.** *The category* $\boldsymbol{TRS}_\bullet$*, with morphisms as defined in Section 6.4, is proto-abelian, as is the subcategory* $\boldsymbol{MTRS}_\bullet$ *of modular tropical toric reflexive sheaves.*

Finally, in Section 6.4.1 we utilize this structure, togehter with some results of categorical stability theory, to illustrate that the Harder-Narasimhan filtration of tropical toric vector bundles as introduced in [38] is a categorical slope filtration as studied by [4] and [41].

Theorem F (Corollary 6.4.25)**.** *The Harder-Narasimhan filtrations of tropical reflexive sheaves are an instance of categorical slope filtrations on* $\boldsymbol{MTRS}_\bullet$.

In particular, these results yield a path toward studying vector bundles in tropical geometry through the modern lens of moduli theory. This affords implications for the study of moduli spaces of sheaves on tropical varieties in a spirit similar to that of [46].

1.5. Group Actions on Spanning Trees

The final main chapter of this thesis is Chapter 7, which includes the main text of the paper [52]. This work deals with the action of the critical group of a finite graph on the set of spanning trees of that graph. For any finite graph G there exists a finite group $\mathrm{Pic}^0(G)$ of divisors on that graph, known alternately as the *Picard group*, *critical group*, or *Jacobian* of the graph. One observes, via the Kirchhoff matrix-tree theorem, that the cardinality of this group is equal to the number of spanning trees of this graph. Further, it is known that there exist several combinatorial bijections between the group itself and the set $\mathcal{T}(G)$ of spanning trees of G. Each such bijection defines an identification between the elements of Pic0(G) and spanning trees of G, and therefore defines an action on $\mathcal{T}(G)$. Any such action is necessarily simply transitive and endows the set $\mathcal{T}(G)$ with the structure of a *torsor* over $\mathrm{Pic}^0(G)$.

Two such actions which have been considered in the literature are the *rotor-routing* torsor and the *Bernardi* torsor, each of which depends on the choice of a base vertex q of G ([28], [9], [14]). Despite the fact that these combinatorial actions differ in their construction and flavor, the associated action maps

$$\begin{aligned}\rho_q &: \mathrm{Pic}^0(G) \times \mathcal{T}(G) \longrightarrow \mathcal{T}(G)\\ \beta_q &: \mathrm{Pic}^0(G) \times \mathcal{T}(G) \longrightarrow \mathcal{T}.\end{aligned}$$

are known to coincide when G is planar. However, when G is nonplanar it was conjectured in [9] that there should always exist some vertex q of G for which $\rho_q \neq \beta_q$. The main result of [52] confirms this conjecture by analyzing the combinatorics of the actions β_q and ρ_q and non-crossing embeddings of nonplanar G in surfaces of positive genus. This result was independently proven by Ding in [18] in a separate work using distinct methods.

The chapter itself reads as a standalone paper, and contains significant overlap with the published version of [52], which was published in 2023.

This work, though apparently of a distinct flavor from the other chapters included in this paper, is related in particular to the results of Chapter 5. Indeed, Chapter 5 features a combinatorial and convex-geometric analysis of certain polyhedral subdivisions of the tropical Jacobian $J(G)$ of a finite graph G. The tropical Jacobian admits the structure of a tropical abelian variety, hence is given by a real torus. However, $J(G)$ contains the Picard group

$\mathrm{Pic}^0(G)$ as a finite subgroup. In the notation of Chapter 5, the group $\mathrm{Pic}^0(G)$ appears as the quotient group $H^1(G;\mathbb{Z})/\rho H_1(G;\mathbb{Z})$ inside of the Jacobian $J(G) = H^1(G;\mathbb{R})/\rho H_1(G;\mathbb{Z})$. Moreover, when G arises as the dual graph of some algebraic curve X, the actions ρ_q and β_q can be seen as combinatorial shadows of the action of the Picard group $\mathrm{Pic}^0(X)$ on the set of (algebraic-geometric) divisors on the curve X.

Chapter 2

Matroids

Matroids are combinatorial objects which capture the notion of independence and which have become increasingly prevalent in modern mathematics. For example, in addition to their study in purely combinatorial contexts, matroids arise naturally in tropical geometry and so have many subtle connections to various flavors of algebraic geometry. Matroids are perhaps most famous, however, for their variety of definitions; there are many ways to define matroids that are not obviously equivalent to one another. This variety of definitions is a lot to get used to for those new to matroid theory, but as we shall see there is much to be gained by having a degree of fluency with at least a handful of different definitions. In this chapter, we survey several flavors of matroid theory, including in particular classical matroids, oriented matroids, and the more general class of matroids over tracts.

2.1. (Classical) Matroids

Let E be a finite set. We begin by giving several definitions of a matroid on E twice, including axiomatizations for bases, circuits, and rank functions (together with a description of the lattice of flats of a matroid). The first formulation of matroids is particularly amenable to analogies with graph theory, while the second affords a direct bridge to the language of hyperplane arrangements. Both of these perspectives will be of particular use to us throughout this document, and so the reader will benefit from having both treated directly. For any of the other (plentiful) axiomatizations of matroids, the reader is encouraged to either wait until Section 2.4 or to refer to the standard text [45]. The first definition of matroids we will consider is among the most popular.

Definition 2.1.1. A **matroid** of rank r on E is a pair $M = (E, \mathcal{B}(M))$, where $B \subseteq \binom{E}{r}$ is a collection of r-element subsets of E called **bases** which must satisfy the following axioms:

(B1) $\mathcal{B}(M) \neq 0$

(B2) For all distinct bases $B_1, B_2 \in \mathcal{B}(M)$ there are elements $e_1 \in B_1 - B_2$ and $e_2 \in B_2 - B_1$ for which $(B_1 - e_1) \cup e_2$ and $(B_2 - e_2) \cup e_1$ are again bases.

The **base polytope** or **matroid polytope** of M is a polytope which is constructed from the bases of M. This polytope is denoted P_M and is defined as the convex hull of indicator vectors of bases in $\mathbb{R}^E$. Two vertices of P_M are adjacent precisely when they differ by an exchange as in (B2). ♦

Another typical axiomatization of matroids uses *independent sets*, which in the above formulation are subsets of bases. In this formulation, bases are the maximal independent sets, and any set which is not contained in a basis is referred to as *dependent*. This leads naturally to the second formulation of matroids which we describe, which is given in terms of a family of certain dependent sets.

Definition 2.1.2. A **matroid** on E is a pair $M = (E, \mathcal{C}(M))$, where $\mathcal{C}(M)$ is a collection of subsets of E called **circuits** which satisfy the following axioms:

(C1) $\varnothing \notin \mathcal{C}$

(C2) If $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$

(C3) If $C_1, C_2 \in \mathcal{C}$ are distinct and $e \in C_1 \cap C_2$, then there is some $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - e$.

♦

In the language of independent sets, a circuit is a dependent set which is inclusion-wise minimal. This means in particular that for any matroid $M = (E, \mathcal{B})$ given in terms of its bases, we can readily construct many circuits of M as follows.

Definition 2.1.3. Given a matroid $M = (E, \mathcal{B})$, any basis $B \in \mathcal{B}$, and any element $e \in E - B$, the set $B \cup e$ is necessarily dependent, and so contains some minimal dependent subset $C = C(B, e)$, referred to as the **fundamental circuit of B with respect to e**. ♦

The collection of all fundamental circuits of all bases of $M = (E, \mathcal{B})$ form a family $\mathcal{C}$ satisfying the axioms of Definition 2.1.2.

Circuits provide particularly useful descriptions of matroid arising from graphs, as we shall see later in Subsection 2.2.2. Given a matroid $M = (E, \mathcal{C})$, defined in terms of its circuits, one can easily extract the independent sets of M by declaring a set to be independent precisely when it contains no circuit.

Given a matroid in either of the above forms (i.e. either as a set with a collection of bases or a collection of circuits), one naturally obtains a rank function on the Boolean lattice of subsets of E by declaring the rank of a set $S \subseteq E$ to be the cardinality of the largest independent set contained in S. Doing so yields yet another formulation of matroids.

Definition 2.1.4. A **matroid** on E is an ordered pair $M = (E, r_M)$, where $r_M : \mathcal{P}(E) \to \mathbb{N}$ is a function, the **rank function** of M, satisfying the following axioms:

(R1) If $X \subseteq E$, then $r_M(X) \leq |X|$

(R2) If $X \subseteq Y \subseteq E$, then $r_M(X) \leq r_M(Y)$

(R3) For all $X, Y \subseteq E$, we have the *submodular inequality*

$$r_M(X) + r_M(Y) \geq r_M(X \cup Y) + r_M(X \cap Y).$$

A maximal set of a given rank is referred to as a **flat** of M, and the collection L of flats of M forms a geometric lattice, referred to as the **lattice of flats** of M. ♦

Here, a *geometric lattice* is a finite atomic lattice (poset) L which admits a submodular rank function. That is, L is a ranked lattice in which every element can be written as a join of atoms and whose rank function $r : L \to \mathbb{N}$ satisfies the submodular inequality above.

Remark 2.1.5. Note that each of the above definitions of a matroid is phrased in terms of subsets of the ground set E. In particular, each definition above affords a description of a matroid M either as a collection of vectors in $\{0, 1\}^E$. This perspective is important to keep in mind in order to understand the relationship between matroids and oriented matroids (see Section 2.3) and in order to better view each of these as matroids over tracts (See Section 2.4). ♦

Given a matroid $M = (E, \mathcal{B}(M))$, say that an element $e \in E$ is a **loop** if it is not contained in any bases. Equivalently, a loop is an element e for which $\{e\} \in \mathcal{C}(M)$ or for which $r_M(e) = 0$. A **coloop** of $M = (E, \mathcal{B}(M))$ is an element $f \in E$ for which $f \in \cap\mathcal{B}(M)$, i.e. an element which is in every basis. For a matroid $M = (E, r_M)$, we say that two elements $e, e' \in E$ are **parallel** if $r_M(\{e, f\}) = 1$. Say that a matroid M is **simple** if it contains no loops or pairs of parallel elements.

2.1.1 New Matroids from Old

One can also create new matroids from others via constructions what mirror the notions of subobjects and quotients. In fact, as we shall see in Chapter 6 these will be shown to define morphisms whose directly analogous to subobjects and quotients in abelian categories.

Definition 2.1.6. Given a matroid $M = (E, \mathcal{B})$ and an element $e \in E$, the **deletion** of e from M, also known as the the **restriction** of M to $E - e$, is the matroid $M \setminus e = (E - e, \mathcal{B} - e)$. When e is not a coloop of M, the bases of $M - e$ are those $B \in \mathcal{B}$ with $e \notin B$. When e is a coloop of M, the bases of $M - e$ are the sets $B - e$, where $B \in \mathcal{B}$. The matroid $M - e$ is also sometimes denoted by $M|(E - e)$ in accordance with the terminology of restriction.

Deleting all elements of some subset $A \subseteq E$ yields a matroid denoted by $M \setminus A$. Correspondingly, we denote such a matroid by $M|(E - A)$ and refer to it as the restriction of M to $E - A$. ♦

Definition 2.1.7. If a matroid M is of the form $\widetilde{M} - e$ for some other matroid $\widetilde{M}$ and an element e of $\widetilde{M}$, we say that $\widetilde{M}$ is a **single-element extensions** of M by e. When no confusion is possible, we refer to such a matroid simply as an **extension** of M. If e is neither a loop not a coloop of $\widetilde{M}$, say that the extension $\widetilde{M}$ is **non-trivial**. ♦

Remark 2.1.8. For a flat $F \in L$, the lattice of flats of the restriction $M|F$ is isomorphic to the sublattice $L_{\leq F}$ of L. On the other hand, the lattice of flats of the contraction M/F is isomorphic to the sublattice $L_{\geq F}$. ♦

Definition 2.1.9. Given $M = (E, \mathcal{B})$ and an element $e \in E$, the **contraction** of M by e is the matroid $M/e = (E - e, \mathcal{B}/e)$. When e is not a loop of M, bases of M/e are sets of the form $B - e$, where $B \in \mathcal{B}$ is such that $e \in B$. If e is a loop of M, M/e is identified with the deletion $M - e$.

Contracting all elements of some subset $A \subseteq E$ yields a matroid denoted by M/A. ♦

Definition 2.1.10. If M is of the form $\widehat{M}/e$ for some other matroid $\widehat{M}$ and an element e of $\widehat{M}$, say that $\widehat{M}$ is a **single-element lifting** of M by e. We often refer to $\widehat{M}$ simply as a **lifting** of M. If e is neither a loop not a coloop of $\widehat{M}$, say that the lifting $\widehat{M}$ is **non-trivial**. ♦

Matroids also possess a notion of duality which is central to many developments in matroid theory and which provides a simultaneous generalization of planar duals of graphs and of orthogonal complements of vector subspaces.

Definition 2.1.11. Given a matroid $M = (E, \mathcal{B}(M))$, the **dual matroid** to M is the matroid $M^* = (E, \mathcal{B}(M^*))$, where $B \in \mathcal{B}(M^*)$ if and only if $E - M \in \mathcal{B}(M)$. ♦

One immediately sees that the dualizing twice yields the original matroid. The circuits of M^* are also referred to as the **cocircuits** of M, and we write $\mathcal{C}^*(M) = \mathcal{C}(M^*)$ for the set of cocircuits of M. Loops of M are coloops of M^* and vice versa. For a simple matroid M, the dual M^* may not be simple, see Example **??**.

Remark 2.1.12. Deletion and contraction of matroids also plays nicely with dualization. In particular, for any matroid M and any $e \in E$, we have $(M-e)^* = M^*/e$ and $(M/e)^* = M^*-e$. This will be shown explicitly in Proposition 6.1.26 in the setting of matroids over (idyllic) tracts. ♦

2.2. Examples of Matroids

Matroids themselves are certain set systems which are subject to axioms, but much of the intuition needed to work from matroids comes from considering examples of matroids which arise in mathematics. Indeed, matroids were first introduced by Whitney in [CITE WHITNEY] for the purpose of generalizing certain results in graph theory, and accordingly graphs yield an important motivating class of matroids, which we describe shortly. First, however, we describe a broader class of matroids which contain all those arising from graphs.

2.2.1 Linear Matroids and Realizations

Fix a field $\mathbf{k}$. Consider the vector space $\mathbf{k}^r$ and suppose we are given a finite configuration $E = \{e_1, \ldots, e_n\}$ of vectors in $\mathbf{k}^r$ for which span $E = \mathbf{k}^r$. Assemble the vectors e_i into an $r \times n$ matrix $A = [e_1 \cdots e_n]$. Then we obtain a matroid $M(A)$, the **linear matroid** of A, by declaring a collection $\{e_{i_1}, \ldots, e_{i_r}\}$ to be a basis of $M(A)$ precisely when the maximal $r \times r$ minor of A formed by taking columns $i_1, \ldots, i_r$ has nonvanishing determinant. In other words, the bases of $M(A)$ are the subsets of columns which form bases of the column space of A. The circuits of $M(A)$ are the minimal subsets of columns of A which support a linear dependence relation, and so can be though of as the minimal supports of vectors in the

Ker(A).

Definition 2.2.1. Say that a matroid M is **realizable over k** if there is some matrix A over **k** for which $M = M(A)$. Any such matrix A is referred to as a **realization** of M. Say that M is **regular** if it is realizable over any field **k**. ♦

The question of whether a given matroid M is realizable over a field **k** is a classical question in matroid theory, with a history going back to the inception of the field. The reason for the interest in such questions can largely be traced to the fact that for realizable matroids much can be deduced using linear algebraic study of the matrix A and the associated vector configuration.

2.2.2 (Co)Graphic Matroids

As demonstrated in the previous Subsection, it is clear that much is to be gained from considering matroids as generalizations of objects from linear algebra. However, matroids were initially introduced by Whitney as a generalization of graphs. In this section we provide some of the important connections between graph theory and matroid theory.

Definition 2.2.2 (Graphic and Cographic Matroids)**.** Let $G = (V, E)$ be a finite graph. Declare a subset $B \subset E$ to be a *basis* if it is a spanning forest of G, i.e. if it is a maximal acyclic subset of edges. Then the set $\mathcal{B}$ of bases of G defines a matroid $M(G) = (E, \mathcal{B})$, known as the **graphic matroid** of G.

It immediately follows that a collection of edges is independent in $M(G)$ precisely when it is acyclic. The circuits of $M(G)$ are therefore the inclusion-wise minimal acyclic sets of edges in G.

In case G is planar, the graphic matroid of its planar dual G^* is related to $M(G)$ by the equality $M(G^*) = M(G)^*$. That is, the (matroidal) dual of a graphic matroid is the graphic matroid of the (planar) dual of G. Note that even for nonplanar graphs G, it still makes sense to construct the matroid $M^*(G) := M(G)^*$. This matroid is known as the **cographic matroid** of G. The circuits of the cographic matroid $M^*(G)$ are readily seen to be the *fundamental cuts* or *bonds* of the graph G. ♦

2.2.3 Failure of Realizability

Despite the fact that linear algebra is a powerful tool in matroid theory, this tool also has its limitations. Indeed, there are myriad matroids which are known not to be realizable over certain fields, and the study of obstructions to realizability is also storied, with many approaches centering on certain forbidden minor characterizations. Many of these approaches are classical, but there also exist modern approaches to this question, some of which will be mentioned below in Section 2.4. For now, we consider a famous example in matroid theory which is known not to be realizable over certain fields.

Example 2.2.3 (Uniform Matroids)**.** Let $E = [n] = \{1, \ldots, n\}$ and fix some $r \leq n$. Then there is a matroid $U(r, n)$ known as the **uniform matroid** of rank r on n elements defined as follows. The bases of $U(r, n)$ are all r-element subsets of E, and so the independent subsets are all subsets of E of size at most r. The circuits are all subsets of size $r + 1$, and for a subset $A \subseteq E$ we have $r_{U(r,n)}(A) = \min\{\#A, r\}$. The dual of $U(r, n)$ is the uniform matroid $U(n - r, n)$.

For example, the matroid $U(2, 4)$ has bases $\mathcal{B} = \{12, 13, 14, 23, 24, 34\}$ and the base polytope is the octahedron.

The matroid $U(2, n)$ cannot be realized over any finite field with fewer than $n-1$ elements, since in this case the projective line over the field would have fewer than n points. ◊

2.2.4 Single-Element Extensions of Matroids

In this section we survey some results characterizing single-element extensions of matroids. The problem of classifying all such extensions was a major problem in matroid theory for some time, and a variant of the problem for oriented matroids still persists today (see Section

2.3). Fortunately, an early classification result was proved by Crapo in [16]. In this section we survey this result and a related result for extensions of graphic matroids, where more concrete descriptions can be given.

Remark 2.2.4. For this section, we use the Roman letter C to denote the set of (co)circuits of a classical matroid. This is mainly to contrast with the notation of Section 2.3, where we also describe single-element extensions of oriented matroids using the calligraphic $\mathcal{C}$. ♦

Proposition 2.2.5 ([12], Proposition 7.1.2)**.** *Suppose that M is a matroid on a finite set E and let $C^* = C^*(M) \subseteq \{0,1\}^E$ the set of cocircuits of M. Let $\widetilde{M}$ be a single-element extension of M on the ground set $E \sqcup p$ and denote by $\widetilde{C}^* = C^*(\widetilde{M})$ its set of cocircuits.*

Then for all $Y \in C^*$ there is a unique $Z \in \widetilde{C}^*$ such that $Z \subseteq Y \sqcup p$. That is, there is a unique assignment $\sigma : C^* \to \{0,1\}$ such that

$$\{(Y, \sigma(Y)) \ : \ Y \in C^*\} \subseteq \widetilde{C}^*.$$

Moreover, $\widetilde{M}$ is uniquely determined by σ and we have

$$\begin{aligned}\widetilde{C}^* = &\{(Y, \sigma(Y)) \ : \ Y \in C^*\} \sqcup \\ &\{(Y_1 \circ Y_2, 0) \ : \ (Y_1, Y_2) \text{ a modular pair with } \sigma(Y) = 1 \text{ for all } Y \subseteq Y_1 \circ Y_2\}\end{aligned}$$

Here, we write $- \circ -$ to denote the union of two vectors $X, Y \in \{0,1\}^E$. Explicitly, this is given by declaring that $(X \circ Y)_e$ is 1 if either $X_e = 1$ or $Y_e = 1$ and 0 otherwise. Viewing X and Y as subsets of E, $X \circ Y$ is just the union $X \cup Y$. The notation $- \circ -$ is chosen primarily to emphasize the relation with oriented matroid (cf. Subsection 2.3.2).

Recently, work by Slilaty and Zaslavsky has provided a similar characterization of single-element extensions of graphic matroids by using graph-theoretic means. We briefly describe this characterization for the sake of using it in the sequel.

Definition 2.2.6. Let $G = (V, E)$ be a graph and $M = M(G)$ its graphic matroid. Recall that the set of cocircuits $C^*(M)$ is exactly the collection of bonds in G. A **tribond** in G is a union of three bonds of G. Note that the removal of a tribond from G increases the number of connected components by 2. A **dibond** of G is a subset of edges which is not a tribond but whose removal increases the number of connected components by 2.

A **linear class of bonds** is a collection $\mathcal{L} \subseteq C^*(M)$ of bonds such that each tribond of G contains (as a subset) 0, 1, or 3 elements of $\mathcal{L}$. ♦

Theorem 2.2.7 ([53], Theorem 3.1)**.** *Let G be a graph and let $\mathcal{L}$ be a linear class of bonds for $M(G)$. Then there is a matroid $J_0(G, \mathcal{L})$ on $E \sqcup e_0$ such that e_0 is neither a loop nor a coloop and whose circuits consist of*

(i) $\mathcal{L} \subseteq C^*(M)$

(ii) sets of the form $B \cup e_0$, where $B \in C^*(M) - \mathcal{L}$

(iii) dibonds and tribonds of G not containing an element of $\mathcal{L}$.

Note that $J_0(G, \mathcal{L})$ is a single-element extension of M. Each single-element extension of M for which e_0 is neither a loop nor a coloop is of the form $J_0(G, \mathcal{L})$ for some linear class of bonds $\mathcal{L}$.

2.3. Oriented Matroids

The previous section portrays matroid theory as a field of combinatorics which deals with set systems and which affords connections to graph theory and linear algebra. We now turn our attention to a related field which maintains the strong connections with combinatorics, but which is of a more geometric flavor. This field is the theory of oriented matroids, which has a long history going back to the work of Bland and Las Vergnas, which introduced oriented matroids as a means of abstracting certain phenomena in linear programming and convex geometry.

Like their non-oriented relatives, oriented matroids can be axiomatized in several ways, although again we do not give all such descriptions. The axiomatizations most relevant to us are phrased in terms of sign vectors, which we now introduce. As before, let E be a finite set.

Definition 2.3.1. A **sign vector** over E is an element $X \in \{0, +, -\}^E$. Denote the eth coordinate of X by X_e. Each sign vector X has a negative $-X$, obtained by exchanging $+$ and $-$ in every coordinate of X. The collection of all sign vectors over E is a partially ordered set, with partial order given by the coordinate-wise extension of the partial order on the set $\{0, +, -\}$ given by $0 \leq +$ and $0 \leq -$.

Given two sign vectors X and Y, the **composition** $X \circ Y$ is another sign vector given by

$$(X \circ Y)_e = \begin{cases} X_e & \text{if } X_e \neq 0 \\ Y_e & \text{else} \end{cases}.$$

note that composition gives a non-commutative operation on sign vectors over E. The **separation set** of two sign vectors X and Y is the set $S(X, Y) = \{e \in E \ : \ X_e = (-Y)_e\}$. The **support** of a sign vector is $|X| = \{e \in E \ : \ X_e \neq 0\}$, also sometimes denoted by X^0 depending on the context. ♦

The definitions of oriented matroids with which we will be most concerned are those given in terns of sign vectors. We give two such descriptions, and provide a third below in Section 2.4.

Definition 2.3.2. An **oriented matroid** on E is a pair $\mathcal{M} = (E, \mathcal{V}^*)$, where $\mathcal{V}^*$ is a subposet of sign vectors on E called **covectors**, subject to the following axioms.

(OM0) $(0, ..., 0) \in \mathcal{V}^*$

(OM1) For all $X, Y \in \mathcal{V}^*$, we have that $X \circ (-Y) \in \mathcal{V}^*$.

(OM2) For each pair $X, Y \in \mathcal{V}^*$ with $e \in S(X, Y)$, there is some $Z \in \mathcal{V}^*$ for which $Z_e = 0$ and $Z_f = (X \circ Y)_f$ for all $f \in E - e$.

Note that (OM1) ensures that $-\mathcal{V}^* = \mathcal{V}^*$ and that $\mathcal{V}^*$ is closed under composition. ♦

As it happens, the collection $\mathcal{V}^*(\mathcal{M})$ of covectors of an oriented matroidhas the structure of a lattice. Moreover, this lattice admits a strong relation with lattices of flats of (classical) matroids.

For a fixed finite set E, let $Z : \{+,-,0\}^E \to \{0,1\}^E$ be the map defined by

$$Z(X)_e = \begin{cases} 0 & X_e \in \{+,-\} \\ 1 & X_e = 0 \end{cases}.$$

That is, for each sign vector X, Z records the set of elements $e \in E$ for which $X_e = 0$.

Proposition 2.3.3 ([12]). *Given an oriented matroid $\mathcal{M}$ on E, let $\mathcal{V}^*(\mathcal{M})$ be its lattice of covectors. Then the lattice $Z(\mathcal{V}^*(\mathcal{M}))$ of subsets of E is a geometric lattice and therefore corresponds to a (classical) matroid M on E.*

Definition 2.3.4. For an oriented matroid $\mathcal{M}$, the matroid associated with the geometric lattice $Z(\mathcal{V}^*(\mathcal{M}))$ is referred to as the **underlying matroid** of $\mathcal{M}$, and is denoted by $\underline{\mathcal{M}}$.

The **rank** of an oriented matroid $\mathcal{M}$ is the rank of its underlying matroid $\underline{\mathcal{M}}$. ♦

Each regular matroid M possesses a unique (up to reorientation) oriented matroid structure.

2.3.1 Topological Representation of Oriented Matroids

In many respects, oriented matroids can be considered as a more geometric variant of ordinary matroids, with strong relations to real and convex geometry by virtue of the fact that the field $\mathbb{R}$ is ordered (hence signs have meaning). A concrete manifestation of this geometric nature is provided by the celebrated Folkman-Lawrence Topological Representation Theorem, which establishes that all oriented matroids admit a realization by a cell decomposition of a sphere coming from an arrangement of *pseudospheres*.

Definition 2.3.5. Let T be a topological space homeomorphic to the d-sphere S^d. Say that a subspace $S \subseteq T$ is a **pseudosphere** if there is a homeomorphism $H : T \to S^d$ for which $H(S) = \{x \in S^d : x_{d+1} = 0\}$. We also define T to itself be a pseudosphere, namely the **degenerate pseudosphere**.

A **signed pseudosphere** is a triple (S, S^+, S^-), where $S \subseteq T$ is a pseudosphere in T and S^+, S^- are the **sides** of S. In particular, if S is degenerate we have $S^+ = S^- \varnothing$, while if S is nondegenerate then S^+ and S^- are the (topological) open d-balls in T bounded by S.

A **pseudosphere arrangement** in T is a multiset $\mathcal{S} = \{S_e : e \in E\}$ of (finitely many) signed pseudospheres for which

(SA1) For all $A \subseteq E$, the intersection $S_A := \bigcap_{e \in A} S_e$ is a topological sphere

(SA2) For all $e \in E$ and all $A \subseteq E - e$, we have one of the following:

- $S_A \subseteq S_e$
- $S_A \cap S_e$ is a pseudosphere in S_A with sides $S_e^+ \cap S_A$ and $S_e^- \cap S_A$

(SA3) Each intersection of closed sides is either a topological sphere or a topological ball.

♦

Remark 2.3.6. The axiom (SA3) above actually follows from (SA1) and (SA2) by a result of Mandel (see [3]). ♦

Definition 2.3.7. Say that a pseudosphere arrangement $\mathcal{S} = \{S_e : e \in E\}$ is **centrally symmetric** if there exists a homeomorphism $H : T \to S^d$ for which $H(S_e) = -H(S_e)$ for all $e \in E$. It is **essential** if $\bigcap_{e \in E} S_e = \varnothing$.

Given a pseudosphere arrangement $\mathcal{S} = \{S_e : e \in E\}$ in S^d and a point $x \in S^d$, we define a sign vector $\mathrm{sgn}_{\mathcal{S}}(x) \in \{+, -, 0\}^E$ by

$$\mathrm{sgn}_{\mathcal{S}}(x)_e = \begin{cases} + & x \in S_e^+ \\ 0 & x \in S_e \\ - & x \in S_e^- \end{cases}.$$

We write $\mathcal{V}^*(\mathcal{S}) = \{\mathrm{sgn}_{\mathcal{S}}(x) : x \in S^d\}$. ♦

Theorem 2.3.8 (Folkman-Lawrence Topological Representation Theorem, see [3])**.** *For any oriented matroid $\mathcal{M}$ of rank $d+1$, there exists an centrally symmetric essential arrangement $\mathcal{S}$ of pseudospheres in S^d for which*

$$\mathcal{V}^*(\mathcal{S}) = \mathcal{V}^*(\mathcal{M}),$$

and moreover every centrally symmetric essential arrangement $\mathcal{S}$ in S^d arises this way.

Remark 2.3.9. This theorem establishes the perspective that an oriented matroid can be viewed as a purely topological object, rather than an abstract combinatorial object. ♦

2.3.2 Single-Element Extensions of Oriented Matroids

This section includes two results regarding the description of single-element lifts of oriented matroids. The first is a result of Las Vergnas characterizing single-element extensions of oriented matroids in a similar capacity to the characterization in Proposition 2.2.5. This result is included primarily to address the compatibility of the construction of two extensions which arise in the sequel (cf. Proposition 5.5.2).

The second result is referred to as the Bohne-Dress Theorem; it establishes a bijection (in fact a poset isommorphism) between single-element lifts of a realizable oriented matroid $\mathcal{M}$ and zonotopal subdivisions of any zonotope Z realizing $\mathcal{M}$. Together with the Topological

Representation Theorem above, this provides further basis for the claim that oriented matroids are inherently of geometric/topological nature.

Proposition 2.3.10 ([12], Proposition 7.1.4)**.** *Let $\mathcal{M}$ be an oriented matroid on a finite set E, and let $\mathcal{C}^* := \mathcal{C}^*(\mathcal{M}) \subseteq \{+,-,0\}^E$ be its set of oriented cocircuits. Suppose that $\widetilde{\mathcal{M}}$ is a single-element extension of $\mathcal{M}$ on ground set $E \sqcup p$ and denote by $\widetilde{\mathcal{C}}^* := \mathcal{C}^*(\widetilde{\mathcal{M}})$ its set of cocircuits.*

Then there exists a unique assignment $\sigma : \mathcal{C}^* \to \{+,-,0\}$ with $\sigma(-Y) = -\sigma(Y)$ such that

$$\{(Y,\sigma(Y)) \ : \ Y \in \mathcal{C}^*\} \subseteq \widetilde{\mathcal{C}}^*.$$

Moreover, $\widetilde{\mathcal{M}}$ is uniquely determined by σ, with

$$\begin{aligned}\widetilde{\mathcal{C}}^* = &\{(Y,\sigma(Y)) \ : \ Y \in \mathcal{C}^*\} \sqcup \\ &\{(Y_1 \circ Y_2, 0) \ : \ Y_1, Y_2 \in \mathcal{C}^*,\ \sigma(Y_1) = -\sigma(Y_2) \neq 0,\ S(Y_1,Y_2) = \varnothing,\ \text{and } r_{\underline{\mathcal{M}}}(Y_1 \circ Y_2) = 2\},\end{aligned}$$

where $r_{\mathcal{M}}$ is the rank function on the lattice $\mathcal{V}^*(\mathcal{M})$.

Theorem 2.3.11 (Bohne-Dress Theorem; [50], Theorem 1.8; [30], Theorem 4.3)**.** *Let $\mathcal{M}$ be a realizable oriented matroid on a set E, and let Z be a zonotope realizing $\mathcal{M}$. Then there is a canonical bijection (poset isomorphism) between the poset of zonotopal subdivisions $\mathcal{Z}$ of Z (under refinement) and the set of single-element lifts $\widehat{\mathcal{M}}$ of $\mathcal{M}$ on the ground set $E \sqcup e_0$ (under weak maps).*

In addition to the bijection above, we have that the subposet $\mathcal{V}^*_{\geq 0}(\widehat{\mathcal{M}})$ is anti-isomorphic to the face poset of the zonotopal subdivision $\mathcal{M}$, where

$$\mathcal{V}^*_{\geq 0}(\widehat{\mathcal{M}}) = \{X \in \mathcal{V}^*(\widehat{\mathcal{M}}) : X_{e_0} \in \{+,0\}\}.$$

Here, the faces of Z correspond to sign vectors with $X_{e_0} = 0$ and the cells $Z' \in \mathcal{Z}$ meeting $\mathrm{relint}(Z)$ *correspond to sign vectors with $X_{e_0} = +$.*

We address the Bohne-Dress Theorem further in Chapter 3, where we re-examine arrangements and subdivisions.

2.4. Matroids over Idylls

In the previous sections we have considered some elementary aspects (mostly definitions) of both matroids and oriented matroids. These two theories, though closely related, largely developed separately from one another. Still, several results and constructions in matroid theory were appropriated to the context of oriented matroids and vice versa. However, in view of the development of *matroids with coefficients* by Dress and Wenzel [19], it became clear that there should be some abstract notion of a matroid which specializes to matroids, oriented matroids, and the *valuated matroids* later introduced also by Dress and Wenzel [19].

The modern formulation of the common generalization was provided by Baker and Bowler in [6] via the theory of *matroids over idylls*. Briefly, a tract is an algebraic object which mimics those parts of fields which allow for reasoning about linear independence. Seeing as matroids themselves are generalizations of linear independence, this turns out to be an appropriate context in which to do matroid theory. A brief introduction to the theory of idylls and related structures is given in C. Sometimes an idyll is referred to as a *tract*. Idylls and Tracts are similar objects which were introduced with separate definitions, but recently in the literature there has been a conflation of terminology. In the present document, whenever we say a tract we implicitly mean an idyll.

We define matroids over idylls as equivalence classes of Grassmann-Plücker functions, following notation and definitions of [6] and [31] . For the duration of this section F will always denote an idyll.

Definition 2.4.1. A **Grassmann-Plücker function** of rank r on a finite ground set E over F is a function $\mu : E^r \to F$ satisfying the following properties:

(GP0) If $r > 0$ then μ is not identically zero.

(GP1) μ is alternating in the sense that $\mu(\mathbf{x}) = \operatorname{sign}(\sigma)\mu(\mathbf{x}^\sigma)$ for each permutation $\sigma \in S_r$ and $\mu(\mathbf{x}) = 0$ whenever there are $i \neq j$ with $x_i = x_j$.

(GP2) For every $\mathbf{x} = (x_0, \ldots, x_r) \in E^{r+1}$ and every $\mathbf{y} = (y_2, \ldots, y_r) \in E^{r-1}$, we have that

$$\sum_{k=0}^{r} (-1)^k \mu(\mathbf{x}_{\hat{k}}) \cdot \mu(x_k, \mathbf{y}) \in N_F.$$

♦

By convention, if $r = 0$ and E is a pointed set with basepoint $*$, we set $E^0 := \{*\}$. We say that two Grassmann-Plücker functions μ and μ' are *equivalent* if there is some $\lambda \in F^\times$ such that $\mu' = \lambda \cdot \mu$.

Definition 2.4.2. An F-**matroid** over F of rank r is an equivalence class $M = [\mu]$ of a Grassmann-Plücker function μ of rank r. ♦

In the case that $F = \mathbb{K}$ is the Krasner hyperfield, an F-matroid of rank r on E is a matroid of rank r on the set E in the classical sense. Likewise, a $\mathbb{T}$-matroid is a *valuated matroid* and an $\mathbb{S}$-matroid is an *oriented matroid*. For a field $\mathbf{k}$, a $\mathbf{k}$-matroid of rank r on E is an r-dimensional linear subspace of $\mathbf{k}^E$. See [6] for more on these characterizations.

Definition 2.4.3. Let $M = [\mu]$ be an F-matroid of rank r on the set E_M and fix a total order on E_M. Let $n = \#|E_M|$. The **dual** of M is the F-matroid $M^* = [\mu^*]$, where $\mu^* : E_M^{n-r} \to F$ is the Grassmann-Plücker function defined by

$$\mu^*(\mathbf{x}) = \begin{cases} 0 & \exists i \neq j \text{ with } x_i = x_j, \\ \operatorname{sign}(\mathbf{x}, \mathbf{x}') \cdot \mu(\mathbf{x}') & \text{else,} \end{cases}$$

where $\mathbf{x}' \in E_M^r$ is such that the support of $\mathbf{x}$ and $\mathbf{x}'$ form a partition of E_M and the function $\operatorname{sign}(\cdot)$ assigns the sign of the permutation of E_M defined by the string $(\mathbf{x}, \mathbf{x}')$. The dual M^* is independent of the choice of total order on E_M. If M is a pointed matroid then we define M^* to be the matroid obtained by deleting 0_M, taking the dual, and appending a loop 0_{M^*}. ♦

Remark 2.4.4. Some idylls F come equipped with a natural involution which interact directly with the structure of matroids over F. For example, the hyperfield of signs $\mathbb{S}$ is equipped with the involution which exchanges $+$ and $-$. For F-matroids over idylls with involution $\overline{(-)}$, the dual matroid is instead defined by

$$\mu^*(\mathbf{x}) = \begin{cases} 0 & \exists i \neq j \text{ with } x_i = x_j, \\ \operatorname{sign}(\mathbf{x}, \mathbf{x}') \cdot \overline{\mu(\mathbf{x}')} & \text{else.} \end{cases}$$

See [31] Subsection 2.7 for more details. ♦

Circuits and Vectors of F-matroids

Following [6] and [2], to each F-matroid M of rank r is associated collections $\mathcal{C}_M, \mathcal{V}_M \subseteq F^r$ of *circuits* and *vectors*, respectively. In the case that $F = \mathbb{K}$ (resp. $F = \mathbb{S}$), these are the circuit and vector sets of the associated (resp. oriented) matroid. We will use these notions only in the next subsection in order to define morphisms of matroids, and mention them primarily to emphasize that this theory is a direct generalization of classical (oriented) matroid theory. As such, we refer the reader to [6], [2], and [31] for more details.

For set E, we let F^E be the set of vectors indexed by E, i.e., $F^E = \{X = (X_e)_{e \in E} \mid X_e \in F\}$. For $X \in F^E$, the *support* of X is the set $\underline{X} := \{e \in E \mid X_e \neq 0_F\}$ and for a subset $A \subseteq F^E$, the support of A is $\underline{A} := \{\underline{X} \mid X \in A\}$. For $X, Y \in F^E$, we say that X and Y are *orthogonal*, denoted by $X \perp Y$, if

$$X \cdot Y := \sum_{e \in E} X_e \cdot Y_e \in N_F,$$

where $X_e \cdot Y_e$ is the multiplication in F and N_F is the null set of F. For $A \subseteq F^E$, we let $A^\perp$ be the set of vectors $X \in F^E$ which are orthogonal to all vectors in A. For $A, B \subseteq F^E$, we write $A \perp B$ if $X \perp Y$ for any $X \in A$ and $Y \in B$.

Suppose that $M = [\mu]$ is an F-matroid of rank r and define $\Xi_M \subseteq E_M^{r+1}$ to be the set of $\mathbf{y} = (y_0, \ldots, y_r) \in E_M^{r+1}$ for which $\#|\mathbf{y}| = r + 1$ and such that there is some $i \in \{0, 1, \ldots, r\}$

for which $\mu(\mathbf{y}_{\hat{i}}) \neq 0$. For any $\mathbf{y} \in \Xi_M$ define the *fundamental circuit* of $\mathbf{y}$ with respect to μ to be the vector $X_{\mathbf{y}} \in F^{E_M}$ given by

$$X_{\mathbf{y}}(x) = \begin{cases} (-1)^k \cdot \mu(\mathbf{y}_{\hat{k}}) & x = y_k, \\ 0 & \text{else}. \end{cases}$$

Then the **set of circuits** of M is $\mathcal{C}_M = \{a \cdot X_{\mathbf{y}} \mid a \in F^\times,\ \mathbf{y} \in \Xi_M\}$ and the **set of cocircuits** of M is defined to be $\mathcal{C}_M^* := \mathcal{C}_{M^*}$. Correspondingly, define the **set of vectors** of M to be $\mathcal{V}_M = (\mathcal{C}_M^*)^\perp$ and the **set of covectors** to be $\mathcal{V}_M^* = \mathcal{C}_M^\perp$. An idyll F is said to be **perfect** when $\mathcal{V}_M \perp \mathcal{V}_M^*$ for all F-matroids M.

Given an F-matroid $M = [\mu]$, a morphism $\Phi : F \to G$ of idylls yields a G-matroid $\Phi_* M = [\Phi \circ \mu]$ on the same ground set, referred to as the push-forward of M along Φ. The push-forward of any F-matroid M along the unique map $F \to \mathbb{K}$ is referred to as the **underlying matroid** of M and is denoted by $\underline{M}$. Note that $\underline{M}$ is a classical matroid.

Direct sum and pointed F-matroids

For any idyll F there are precisely two F-matroids defined on a one-element set. The first is $U(0,1)$, for which any associated Grassmann-Plücker function takes only the value 0_F. The second is $U(1,1)$, for which an associated Grassmann-Plücker function can take any value in $F^\times$. Since F is an idyll, the group $F^\times$ consists of all nonzero elements of F and so the Grassmann-Plücker function of $U(1,1)$ is unique up to nonzero scaling, hence $U(1,1)$ is unique.

Given two F-matroids $M = [\mu]$ and $N = [\nu]$ of ranks r_M and r_N on ground sets E_M and E_N, the **direct sum** of M and N is defined to be F-matroid given as the equivalence class $M \oplus N := [\mu \oplus \nu]$, where $\mu \oplus \nu : (E_M \sqcup E_N)^{r_M + r_N} \to F$ is a function defined as follows. For $\mathbf{x} \in (E_M \sqcup E_N)^{r_M + r_N}$, let $\mathbf{x}_M$ be the sub-tuple consisting of those entries of $\mathbf{x}$ which are in E_M, and define $\mathbf{x}_N$ similarly. Then define $\mu \oplus \nu$ by

$$(\mu \oplus \nu)(\mathbf{x}) := \begin{cases} \mu(\mathbf{x}_M) \cdot \nu(\mathbf{x}_N) & \#|\mathbf{x}_M| = r_M \text{ and } \#|\mathbf{x}_N| = r_N \\ 0 & \text{else} \end{cases}.$$

A **pointed matroid** of rank r over F is a matroid $M = [\mu]$ in the usual sense which is defined on a pointed ground set $E_M = \tilde{E}_M \sqcup \{*_M\}$ for which $\mu(\mathbf{x}) \in N_F$ whenever $\mathbf{x} \in E_M^r$ is such that $\mathbf{x}_i = *_M$ for some i. For the case that $F = \mathbb{K}$, this means that $*_M$ is a loop of the underlying matroid. In other words, a pointed matroid over an idyll F is a matroid with the choice of a distinguished loop in the ground set. We refer to $\tilde{E}_M$ as the set of *nonzero elements* of E_M.

Note that from each non-pointed F-matroid M we can obtain a pointed F-matroid by taking the direct sum with the matroid U_1^0, and in fact every pointed matroid arises this way.

Given two pointed matroids M and N, we define the pointed direct sum $M \oplus N$ to be an F-matroid on the ground set $\tilde{E}_M \sqcup \tilde{E}_N \sqcup \{*_{M\oplus N}\}$ and proceed to define the associated Grassmann-Plücker function in complete analogy to the above. Alternatively, one can form the pointed direct sum by taking the usual direct sum of pointed F-matroids and subsequently identifying the two distinguished loops of the summands.

The definition of matroids as equivalence classes of Grassmann-Plücker functions is not standard in many classic matroid texts, and so it may seem unintuitive to some familiar with the classical theory. The following result illustrates a concrete relation to the classical theory of matroids.

Proposition 2.4.5 ([6]). *Let $M = [\mu]$ be an F-matroid of rank r on E_M and let $\underline{M}$ be its underlying matroid. Then for any r-tuple $\mathbf{e} = (e_1, \ldots, e_r) \in E_M^r$, we have that $\mu(\mathbf{e}) \neq 0_F$ if and only if $\{e_1, \ldots, e_r\}$ is a basis of $\underline{M}$.*

Proof. This follows from Theorem 3.17 of [6] and the discussion following. □

Restriction and contraction of F-matroids

F-matroids also possess notions of restrictions and contractions which generalize those of classical matroid theory. Let $M = [\mu]$ be an F-matroid of rank r on E_M and $A \subseteq E_M$ an arbitrary subset. We define the *deletion*, *contraction*, and *restriction* of M with respect to A as follows. For each of the following definitions, suppose that $A \subseteq E_M$ is a set with $r_{\underline{M}}(A) = r'$ and $r_{\underline{M}}(E_M \setminus A) = r''$.

(1) (Contraction) Let $\mathbf{a} \in A^{r'}$ be a tuple of elements of A for which $|\mathbf{a}|$ is a basis of the classical matroid $\underline{M}|A$. Then define the function $\mu/A : (E_M \setminus A)^{r-r'} \to F$ by

$$(\mu/A)(\mathbf{x}) := \mu(\mathbf{x}, \mathbf{a}).$$

(2) (Deletion) Let $\mathbf{b} \in A^{r-r''}$ be a tuple of elements of A for which $|\mathbf{b}|$ is a basis of the classical matroid $\underline{M}/(E_M \setminus A)$. Then define the function $\mu \setminus A : (E_M \setminus A)^{r''} \to F$ by

$$(\mu \setminus A)(\mathbf{x}) := \mu(\mathbf{x}, \mathbf{b}).$$

(3) (Restriction) With all notations as above, define $\mu|A := \mu \setminus (E_M \setminus A) : A^{r'} \to F$. More precisely, let $\mathbf{c} \in (E_M \setminus A)^{r-r'}$ be a tuple for which $|\mathbf{c}|$ is a basis for the classical matroid $\underline{M}/A$. Then $\mu|A$ is given by

$$(\mu|A)(\mathbf{x}) = \mu(\mathbf{x}, \mathbf{c}).$$

The **contraction** of M by A is the F-matroid $M/A := [\mu/A]$, the **deletion** of M by A is $M \setminus A := [\mu \setminus A]$, and the **restriction** of M to A is $M|A := [\mu|A]$. In each case, the resulting function is again a Grassmann-Plücker function and the matroids they define independent of any of the choices of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ above. With these notions, by [6, Theorem 3.29], we have the usual identities $(M/A)^* = M^* \setminus A$ and $(M \setminus A)^* = M^*/A$. In the sequel, for any $A \subseteq E_M$

we denote by A^c the complement $E_M \setminus A$.

Remark 2.4.6. Some care should be taken when performing restriction and contraction on pointed matroids. When restricting to a subset S not containing the distinguished loop, we tacitly append a loop $*_{M|S}$ to the result. Likewise, when contracting a subset T which contains the loop, we contract the subset $T \setminus *_M$. These adjustments do not alter the constructions or the resulting matroids, but free us from worrying about the loop. ♦

As in the theory of classical matroids, the operations of contraction and deletion commute when applied to disjoint subsets of the ground set.

Proposition 2.4.7. *Let $M = [\mu]$ be a pointed F-matroid of rank m on a set E and let $S \subseteq T \subseteq E$ be subsets of the ground set. Then $(M|T)/S = (M/S)|T$.*

Proof. We prove in terms of deletion and contraction, establishing instead that for any disjoint $A, B \subseteq E$, we have $(M \setminus A)/B = (M/B) \setminus A$. It is easily verified that this is equivalent to the statement as written.

The proof proceeds by direct calculation of the associated Grassmann-Plücker functions. Let $A, B \subseteq E$ be disjoint sets with $\ell = \mathrm{rk}_{\underline{M}}(A)$ and $k = \mathrm{rk}_{\underline{M/A}}(E \setminus B)$.

We first consider $(M/A) \setminus B$. Let $\mathbf{a} \in A^\ell$ be a basis of $\underline{M}|A$ and $\mathbf{b} \in B^{m-\ell-k}$ be a basis of $(\underline{M/A})/(E \setminus B)$. Then $(M/A) \setminus B$ is given by the function $(\mu/A) \setminus B : (E \setminus (A \cup B))^k \to F$ which sends $\mathbf{y} \in (E \setminus (A \cup B))^k$ to

$$((\mu/A) \setminus B)(\mathbf{y}) = (\mu/A)(\mathbf{y}, \mathbf{b}) = \mu(\mathbf{y}, \mathbf{b}, \mathbf{a})$$

Note that the argument of μ above has $k + (m - \ell - k) + \ell = m$ entries.

To approach $(M \setminus B)/A$, note first that the disjointness of A and B ensures that $r_{\underline{M}}(A) = r_{\underline{M \setminus B}}(A)$. Thus $\mathbf{a}$ in the previous paragraph serves as a basis for $(\underline{M \setminus B})|A$. Likewise, $\mathbf{b}$ serves as a basis for $\underline{M}/(E \setminus B)$ since it was assumed to be a basis of $(\underline{M/A})/(E \setminus B)$. We now see that $(M \setminus B)/A$ is given by $(\mu \setminus B)/A : (E \setminus (A \cup B))^k \to F$ which sends

$\mathbf{y} \in (E \setminus (A \cup B))^k$ to

$$((\mu \setminus B)/A)(\mathbf{y}) = (\mu \setminus B)(\mathbf{y}, \mathbf{a}) = \mu(\mathbf{y}, \mathbf{a}, \mathbf{b})$$

Hence we have that $((\mu/A)\setminus B) = \pm((\mu\setminus B)/A)$. This implies in particular that the associated Grassmann-Plücker functions are scalar multiples of one another and so the matroids they define are identical. $\square$

Lattices of Flats and Modularity

A fundamental tool in the study of classical matroids is the lattice of flats $\mathcal{L}(M)$ of a matroid M (recall Section 2.1). Flats are certain subsets of the ground set of a matroid which generalize the linear spans of sets of vectors. Given a subset A of the ground set of M, denote by $\langle A \rangle$ the smallest flat of M containing A as a subset. The lattice of flats of M is a poset under the subset relation, and is a lattice with meet and join given by

$$F \wedge G = F \cap G,$$
$$F \vee G = \langle F \cup G \rangle.$$

It is well-known that the lattice of flats of any matroid is a geometric lattice. Moreover, each geometric lattice corresponds uniquely to a simple matroid. This observation will be of use to us in Section 6.2 where we introduce and study the category of simple F-matroids. For an F-matroid M, we write $\mathcal{L}(M)$ to denote the lattice of flats of the underlying (classical) matroid.

In the sequel, we use the notion of *modularity* of flats $F \in \mathcal{L}(M)$ in an essential way. For any classical matroid M, say that two flats $F, G \in \mathcal{L}(M)$ form a **modular pair** precisely when we have an equality in the submodular inequality for F and and G:

$$r_M(F) + r_M(G) = r_M(F \vee G) + r_M(F \wedge G).$$

Say that $F \in \mathcal{L}(M)$ is **modular** if (F, G) is a modular pair for every $G \in \mathcal{L}(M)$. The maximum element E_M and minimum element $\langle\varnothing\rangle$ of $\mathcal{L}(M)$ are always modular flats, as are all atoms of M. We have the following characterization of modular flats.

Proposition 2.4.8 ([38], Proposition 2.4)**.** *For $F \in \mathcal{L}(M)$, the following are equivalent.*

(1) F is a modular flat

(2) For all pairs of flats $G \leq H$ we have $(F \wedge H) \vee G = (F \vee G) \wedge H$.

(3) For all $G \in \mathcal{L}(M)$ and all $H \leq F$, we have $(F \wedge G) \vee H = F \wedge (G \vee H)$.

Say that the matroid M is **modular** if every flat $F \in \mathcal{L}(M)$ is modular. For an arbitrary F-matroid M, say that M is modular precisely when the underlying matroid $\underline{M}$ is. Modular matroids are closely related to projective geometry, as witnessed by the following characterization:

Proposition 2.4.9 ([45], Proposition 6.9.1)**.** *A (classical) matroid M is modular if and only if the simplification of each of its connected components is either a free matroid or the matroid of some finite projective geometry.*

This proposition will not be used explicitly, but instead is included primarily to emphasize that the class of modular matroids is much smaller than that of all matroids.

The class of modular matroids is closed under the operation of restriction and contraction of flats. Indeed, given a modular matroid M and a flat $F \in \mathcal{L}_M$, the restriction $M|F$ and contraction M/F are again modular matroids. This follows from the well-known fact that for any $F \in \mathcal{L}(M)$ we have

$$\mathcal{L}(M|F) \simeq [\langle\varnothing\rangle, F]$$

$$\mathcal{L}(M/F) \simeq [F, E_M]$$

where the right-hand side of the above expressions are intervals in the lattice $\mathcal{L}(M)$. These isomorphisms are canonical, with $\varphi : [\langle\varnothing\rangle, F] \to \mathcal{L}(M|F)$ given by the identity and $\psi : [F, E_M] \to \mathcal{L}(M/F)$ given by $G \mapsto G \setminus F$.

Simple F-matroids

For an F-matroid M, say that an element $e \in E_M$ is a **loop** when it is contained in no basis of $\underline{M}$. Similarly, $e, e' \in E_M$ are **parallel** precisely when $\{e, e'\}$ is dependent in $\underline{M}$ but neither are loops. In accordance with these notions, say that M is **simple** precisely when $\underline{M}$ is. That is, M is simple if and only if $\underline{M}$ contains no loops or parallel elements. When working with pointed matroids, we do not consider the distinguished loop for the purpose of simplicity. That is, a pointed matroid M is simple if and only if the matroid obtained by deleting the distinguished point is simple.

To each matroid, and so to each F-matroid M, we can associate a simple matroid known as the **simplification** of M, and denoted by $\mathrm{si}(M)$ or $\mathrm{si}_\bullet(M)$ in the pointed case. The matroid $\mathrm{si}(M)$ (resp. $\mathrm{si}_\bullet(M)$) is obtained from M by deleting all (non-distinguished) loops and all but one element from each class of parallel elements. This is well-defined up to isomorphism.

It is a standard fact of matroid theory that simple (classical) matroids are uniquely determined by their lattice of flats, and that each geometric lattice is isomorphic to the lattice of flats of a unique simple matroid.

Chapter 3

Some Geometric Combinatorics

In the sequel, we will crucially use several notions from convex geometry and from geometric combinatorics, including in particular the basics of the theory of polyhedra, hyperplane arrangements and zonotopes, and lattices. This section provides an overview of the background in each of these areas which we will require in later chapters. Several of the sections here afford close relationships with matroid theory, and as such we make explicit mention of connection to Chapter 2.

3.1. Convex Geometry

In this section we largely fix notation and terminology for use in the sequel. In particular, we survey those aspects of polyhedral geometry which will be of crucial use to us later on. Topics include faces and normal cones of polyhedra, zonotopes, polyhedral complexes and subdivisions (including fans), and some more specialized tools such as the theory of fiber polytopes. We make frequent reference to the connections with matroid theory, in particular as a means of illustrating connections with Chapter 2 which will be important in later chapters. Throughout this section we work in the ambient Euclidean space $\mathbb{R}^n$.

3.1.1 Polyhedra, Complexes, and Subdivisions

Definition 3.1.1. A **polyhedron** is a subset $P \subseteq \mathbb{R}^n$ which is described by finitely many linear inequalities. That is, a set for which there is some matrix $A \in \mathbb{R}^{m\times n}$ and some vector $b \in \mathbb{R}^m$ for which $P = \{x \in \mathbb{R}^n : Ax \leq b\}$, where the inequality is understood coordinate-wise. A polyhedron is a **polytope** if it is bounded.

The **dimension** of P is the dimension of its affine hull. A polyhedron P is **rational** if A can be taken to have integer entries and b can be taken to be rational. It is a **cone** if we may take $b = 0$. The **relative interior** of a polyhedron P is its interior, taken within in its affine hull.

Given two polyhedra P and Q in the same ambient space, their **Minkowski sum** is the polyhedron given by $P + Q = \{p + q \ : \ p \in P \text{ and } q \in Q\}$. ♦

Definition 3.1.2. A polytope Z is said to be a **zonotope** if it is a Minkowski sum of line segments or, equivalently, if it is (a translate of) a linear image of the standard cube $C^n = [-1, 1]^n$ in $\mathbb{R}^n$. Thus a zonotope Z can always be expressed as

$$Z = b(Z) + \sum_{i=1}^{n} [-a_i, a_i],$$

where $b(Z) \in \mathbb{R}^d$ is the **barycenter** of Z and we have denoted by $[-a_i, a_i]$ the segment $\operatorname{conv}\{-a_i, a_i\} \subseteq \mathbb{R}^d$. In this case, the matrix $A = [a_1 \cdots a_n]$ maps the cube C^n onto the zonotope $Z - b(Z)$. ♦

A polyhedron which is a proper subset of $\mathbb{R}^n$ always admits smaller polyhedra as its faces.

Definition 3.1.3. A **face** of a polyhedron P is the locus F of points in P which are maximized by some linear functional $\psi \in (\mathbb{R}^n)^*$. For convenience, we also declare the empty set to be a face of P. That is, nonempty faces satisfy $F = \{x \in P : \psi(x) = \max_{y\in P} \psi(y)\}$. In this case, we write $F = P^{\psi}$ and say that F is **exposed** by ψ. A face of dimension zero is a **vertex** and a face of dimension 1 is an **edge**. A face of codimension 1 is a **facet**.

The collection of faces of P form a lattice $\mathcal{F}(P)$ under inclusion, with maximal element $P = P^0$ and minimal element $\varnothing$. This lattice is ranked by dimension, where we declare that $\varnothing$ has dimension -1. We denote the face relation by $F \leq P$.

The collection of all ψ exposing a given face $F \leq P$ forms a cone in the dual space known as the **normal cone of P at F**, denoted by $N_P(F) = \{\psi \in (\mathbb{R}^n)^* \ : \ F = P^\psi\}$. ♦

Note that all nonempty faces of cones are again cones, and all nonempty faces of zonotopes are again zonotopes. We are particularly interested in those regions of space which can be decomposed into polyhedra in a way that respects their face structure.

Definition 3.1.4. Say that a collection $\mathcal{P} = \{P_1, P_2, \ldots\}$ of polyhedra in $\mathbb{R}^n$ forms a **polyhedral complex** if it is closed under taking faces and if $P_i \cap P_j$ is either empty or a face of both P_i and P_j for all i, j. We say that the polyhedra P_i are the **cells** of $\mathcal{P}$. Denote by $\mathrm{Sk}_\ell \mathcal{P}$ the subcomplex of $\mathcal{P}$ comprising all cells of $\mathcal{P}$ of dimension at most ℓ.

The cells of a polyhedral complex $\mathcal{P}$ form a poset under inclusion, and two complexes are said to be **combinatorially equivalent** if their posets of cells are isomorphic. The union $|\mathcal{P}| = \cup\mathcal{P} = \bigcup_i P_i$ is the **support** of $\mathcal{P}$. ♦

Definition 3.1.5. If $|\mathcal{P}| = P$ for some polyhedron P, we say that $\mathcal{P}$ is a **subdivision** of P. A subdivision, all of whose cells are simplices, is a **triangulation**. Given two subdivisions $\mathcal{P} = \{P_1, P_2, \ldots\}$ and $\mathcal{P}' = \{P'_1, P'_2, \ldots\}$ of P, say that $\mathcal{P}'$ **refines** $\mathcal{P}$ if each P'_j is contained within some P_i. In this case, write $\mathcal{P}' \prec \mathcal{P}$. The collection of subdivisions of P forms a poset under refinement. For any two subdivisions $\mathcal{P}$ and $\mathcal{P}'$ of P, their **common refinement** is the subdivision $\mathcal{P} \wedge \mathcal{P}'$ of P having cells $P_i \cap P'_j$. ♦

We include also some terminology on polyhedral complexes of cones, which will be of special interest to us in particular throughout Section 3.2.

Definition 3.1.6. A polyhedral complex $\mathcal{C} = \{C_1, C_2, \ldots\}$ whose cells are all cones is referred to as a **fan**. A fan whose support is $\mathbb{R}^n$ is said to be **complete**. For a polyhedron P, the collection $\mathcal{N}(P)$ of all normal cones of P at its faces is referred to as the **normal fan** of

P. Explicitly, $\mathcal{N}(P) = \{N_P(F) : F \in \mathcal{F}(P)\}$. The normal fan of a polytope is always a complete fan. ♦

3.1.2 Regular and Coherent Subdivisions

Subdivisions of polyhedra are central objects of study for us in the present document, and so we include some additional definitions which will be central to our analysis. In particular, we survey in this section two further kinds of subdivisions of polyhedra which have close relations with convex geometric constructions.

For each convex affine-linear function $f : \mathbb{R}^n \to \mathbb{R}$, there exists a countable collection of vectors $m_i \in \mathbb{R}^d$ and scalars $\alpha_i \in \mathbb{R}$ such that $f(x) = \max_{i \in I}\langle m_i, x\rangle + \alpha_i$. Given such a function, we denote its graph by $\mathcal{G}_f = \{(x, f(x)) : x \in \mathbb{R}^n\} \subseteq \mathbb{R}^{n+1}$. The graph of any such function is necessarily an n-dimensional polyhedral complex in $\mathbb{R}^{d+1}$, and moreover comes equipped with a natural projection map $\pi_{n+1} : \mathcal{G}_f \to \mathbb{R}^n$ given by forgetting the last coordinate. The image $\pi_{n+1}(\mathcal{G}_f)$ is surjective and therefore induces a polyhedral subdivision $\mathcal{S}_f$ of $\mathbb{R}^n$.

We can also obtain a similar subdivision of any polytope in a similar fashion. Given a subdivision $\mathcal{P}$ of a polytope P, suppose we are given a height function $h : \mathrm{Sk}_0\mathcal{P} \to \mathbb{R}$ which assigns to each vertex v of $\mathcal{P}$ a height $h(v) \in \mathbb{R}$. Then the convex hull $P_h = \mathrm{conv}\{(v, h(v)) : v \in \mathrm{Sk}_0\mathcal{P}\}$ is a polytope in $\mathbb{R}^{n+1}$. Define the *epigraph* of h to be the polyhedron $E_h \subseteq \mathbb{R}^{n+1}$ given by

$$E_h = \{(x, y) \in \mathbb{R}^{n+1} : x \in P,\ y \geq \alpha \text{ for some } (x, \alpha) \in P_h\}.$$

one then obtains a subdivision $\mathcal{P}_h$ of P consisting of the images $\pi_{n+1}(F)$ for all faces $F \leq E_h$.

Definition 3.1.7. A subdivision of a polytope P is **regular** if it is of the form $\mathcal{P}_h$ for some height function h on P. A subdivision of $\mathbb{R}^n$ is **regular** if it is of the form $\mathcal{S}_f$ for some convex affine linear function f on $\mathbb{R}^n$. ♦

Remark 3.1.8. A more colloquial definition of regular subdivisions defines them as the images under π_{n+1} of the *lower faces* of the graph $\mathcal{G}_f$ or of the epigraph E_h. Here, a face is called "lower" if it is visible from a point $(0, -\alpha)$ for some $\alpha \in \mathbb{R}_{\geq 0}$ sufficiently large. This definition is not precise as written, but better reflects the intuition of regularity. ♦

Regular subdivisions of polytopes are special cases of a more general construction which can be defined from any projection sending one polytope to another. We now introduce this more class of subdivisions.

Suppose that $N > n$ and let $\pi : \mathbb{R}^N \to \mathbb{R}^n$ be a linear projection. Suppose moreover that there is are polytopes $P \subseteq \mathbb{R}^N$ and $Q \subseteq \mathbb{R}^n$ for which $\pi(Q) = P$. In this case, we call π a projection of polytopes and write $\pi : P \to Q$. For any point $x \in Q$, write $P_x = P \cap \pi^{-1}(x) \subseteq P$ for the preimage of x in P, which is a polytope contained in P.

Definition 3.1.9. Given polytopes $P \subseteq \mathbb{R}^N$ and $Q \subseteq \mathbb{R}^n$ and a projection $\pi : P \to Q$, a subdivision $\mathcal{Q} = \{Q_1, \ldots, Q_k\}$ of a polytope Q is said to be **π-induced** if each $Q_i \in \mathcal{Q}$ satisfies $Q_i = \pi(F_i)$ for some face $F_i \leq P$.

Now let $\varphi \in (\mathbb{R}^N)^*$ be a linear functional. The subdivision $\mathcal{Q}$ is moreover said to be **coherent** with respect to π and the functional φ if it is π-induced and for each $x \in Q$ with $x \in Q_i = \pi(F_i)$ we have

$$P_x^{\varphi} = P_x \cap F_i.$$

♦

Note that this recovers the definition of a regular subdivision of Q by taking $P = Q_h$, $\pi = \pi_{n+1}$, and $\varphi = -e_{n+1}^*$. Indeed, in this case the lower faces of E_h agree with those of Q_h and they all necessarily satisfy the defining property of Definition 3.1.9. In this way, we see that all regular subdivisions are coherent. On the other hand, from each coherent subdivision is necessarily also regular.

Indeed, given a subdivision $\mathcal{Q}$ of Q which is coherent with respect to a projection $\pi : P \to Q$ with functional φ, we define a height function h on the vertices of $\mathcal{Q}$ as follows. Since a π-coherent subdivision is π-induced, for any such vertex $v \in Q$, there is some vertex

$w \in P$ with $\pi(w) = v$. Moreover, by the coherence of $\mathcal{Q}$ we can say that $P_v^{\varphi} = P_v \cap F$, where $F \leq P$ and $\pi(F) = \{v\}$. Define the height function h by declaring $h(v) = -\varphi(p)$ for $v \in \mathrm{Sk}_0\mathcal{Q}$. By construction, we now have that the faces of the epigraph E_h project via π to the cells of $\mathcal{Q}$, so that $\mathcal{Q}$ is regular.

From this perspective, it becomes clear that regular and coherent subdivisions are the same as subdivisions of a polytope Q, but coherence of a subdivision makes explicit reference to the extrinsic data of another polytope P and a projection $\pi : P \to Q$. Another notable fact, closely related to the regularity of coherent subdivisions, is that coherent subdivisions necessarily factor through a polyhedron of one dimension higher than the base. This is formalized in the following result.

Proposition 3.1.10. *Suppose that $\pi : P \to Q$ is a projection of polytopes and that $\mathcal{Q}$ is a π-coherent subdivision given by some $\varphi \in (\mathbb{R}^n)^*$. Then there is some polytope $Q' \subseteq \mathbb{R}^{d+1}$ and linear projections $\pi'' : P \to Q'$ and $\pi' : Q' \to Q$ such that $\pi = \pi' \circ \pi''$ and such that $\mathcal{Q}$ is coherent with respect to π'. Moreover, the π'-coherent subdivision $\mathcal{Q}$ is defined by the linear functional e_{d+1}^* on $\mathbb{R}^{d+1}$.*

Proof. Given π,$\mathcal{Q}$, and φ as in the statement, we construct a polytope $Q' \subseteq \mathbb{R}^{d+1}$ by

$$Q' = \mathrm{conv}\{(x, m_x), (x, M_x) : x \in Q\},$$

where $m_x := \min\{\varphi(y) : y \in P_x\}$ and $M_x := \max\{\varphi(y) : y \in P_x\}$. Define also projections $\pi'' : \mathbb{R}^n \to \mathbb{R}^{d+1}$ given by $\pi''(y) = (\pi(y), \varphi(y))$ and $\pi' : \mathbb{R}^{d+1} \to \mathbb{R}^d$ given by forgetting the last coordinate. Then by construction we have that $\pi''(P) = Q'$ and $\pi'(Q') = Q$. It thus remains only to show that the subdivision $\mathcal{Q}$ is π'-coherent with respect to the functional e_{d+1}^*, which we establish by constructing such a subdivision $\mathcal{S}$ from π'' and subsequently showing that $\mathcal{S} = \mathcal{Q}$.

Let $\mathcal{G} = \{G \leq Q' :$ for all $(x, \alpha) \in G$ we have $\alpha = M_x\}$. Thus $\mathcal{G}$ consists of all of the upper faces of Q', namely those faces G whose normal cone $N_{Q'}(G)$ contains e_{d+1}^* in its interior. Now let $\mathcal{S} = \{\pi'(G) : G \in \mathcal{G}\}$. Thus $\mathcal{S}$ is π'-coherent and defined by the functional e^*_{d+1} by construction.

Let $\pi(F) \in \mathcal{Q}$ and $y \in F$. Then for all $x \in \mathrm{relint}(\pi(F))$, have $P^{\varphi}_x = P_x \cap F$ and F is the minimal face of P with this property. This implies that for all $y \in F$ we have that $\varphi(y) = M_{\pi(y)}$ and so $\pi''(F)$ is an upper face of Q', say $\pi''(F) = G \in \mathcal{G}$. But now $\pi(F) = \pi'(G) \in \mathcal{S}$. Thus $\mathcal{S}$ and $\mathcal{Q}$ are two subdivisions with support Q such that every cell of $\mathcal{Q}$ is a cell of $\mathcal{S}$. This implies that $\mathcal{Q} = \mathcal{S}$ and so we are done. □

The definitions of regular and coherent subdivisions presented here are in some sense rather complicated, and require several auxiliary definitions in order to state explicitly. As a result, one desires a more succinct criterion for a subdivision $\mathcal{Q}$ of Q to be coherent with respect to a projection $\pi : P \to Q$. In the case that π is an orthogonal projection, such a criterion can be obtained by considering normal cones.

Given an orthogonal projection $\pi : \mathbb{R}^N \to \mathbb{R}^n$ (with respect to the Euclidean inner product on $\mathbb{R}^N$), we identify the kernel of this map with $\mathbb{R}^{N-n}$, so that $\mathbb{R}^N$ has an orthogonal decomposition

$$\mathbb{R}^N \simeq \mathbb{R}^n \oplus \mathbb{R}^{N-n}.$$

Denote the corresponding inclusion of the second summand by $\iota_2 : \mathbb{R}^{N-n} \to \mathbb{R}^N$ and the projection by $\pi_2 : \mathbb{R}^N \to \mathbb{R}^{N-n}$. On dual spaces, we then have a projection $\iota_2^* : (\mathbb{R}^N)^* \to (\mathbb{R}^{N-n})^*$ and an inclusion $\pi_2^* : (\mathbb{R}^{N-n})^* \to (\mathbb{R}^N)^*$. With this notation, we have the following result.

Lemma 3.1.11. *For polytopes $P \subseteq \mathbb{R}^N$ and $Q \subseteq \mathbb{R}^n$, let $\pi : P \to Q$ be a projection of polytopes which is orthogonal on $\mathbb{R}^N$. For each $x \in Q$ and each $F \leq P$ satisfying $x \in \pi(F)$, we have that, as subsets of $(\mathbb{R}^N)^*$,*

$$\pi_2^* \iota_2^* (N_P(F)) \subseteq N_{P_x}(F_x).$$

In particular, the polyhedral fan in $(\mathbb{R}^N)^*$ having cones $\{\pi_2^* \iota_2^* (N_P(F)) : F \leq P,\ x \in \pi(F)\}$

is a refinement of the normal fan $\mathcal{N}(P_x)$.

Proof. Let $F \leq P$ be such that $x \in \pi(F)$ and let $\varphi \in N_P(F)$, so that for all $f \in F$ and $p \in P$ we have $\varphi(f) \geq \varphi(p)$. We would like to show that $\pi_2^* \iota_2^* \varphi \in N_{P_x}(F_x) \subseteq (\mathbb{R}^N)^*$. Suppose that $f \in F_x$ and $p \in P_x$. Then we may write $f = f' + x$ and $p = p' + x$ for some $f', p' \in \mathbb{R}^{N-n}$. Note that $\pi_2(f) = f'$ and $\pi_2(p) = p'$, and moreover that $\varphi(f) \geq \varphi(p)$ also guarantees that

$$\varphi(f') = \varphi(f) - \varphi(x) \geq \varphi(p) - \varphi(x) = \varphi(p').$$

We now see the following.

$$\pi_2^* \iota_2^* \varphi(f) = \iota_2^* \varphi(\pi_2(f)) = \varphi(f') \geq \varphi(p') = \pi_2^* \iota_2^* \varphi(p)$$

This completes the proof. □

Recasting the assumptions of Lemma 3.1.11 in terms of subdivisions, we obtain our desired criterion.

Lemma 3.1.12. *Let $P \subseteq \mathbb{R}^N$ and $Q \subseteq \mathbb{R}^n$ be polytopes and let $\pi : P \to Q$ be a projection of polytopes arising from an orthogonal projection $\pi : \mathbb{R}^N \to \mathbb{R}^n$. Suppose that $\mathcal{F}$ is a collection of faces F of P such that*

$$\mathcal{Q} = \{\pi(F) : F \in \mathcal{F}\},$$

is a π-induced subdivision of Q. If $\varphi \in (\mathbb{R}^N)^*$ satisfies $\varphi \in \pi_2^* \iota_2^*(N_P(F))$ for all $F \in \mathcal{F}$, then $\mathcal{Q}$ is a π-coherent subdivision defined by the functional φ.

Proof. Let $x \in P$. By the definition of a π-induced subdivision, there is a unique $F \in \mathcal{F}$ such that $x \in \operatorname{relint}(\pi(F))$. It suffices to show that for each such pair of x and F, we have $\varphi \in N_{P_x}(F_x)$. By Lemma 3.1.11 we have that $\pi_2^* \iota_2^*(N_P(F)) \subseteq N_{P_x}(F_x)$. This completes the proof. □

3.1.3 Fiber Polytopes

We now describe the construction of a polytope associated with a projection $\pi : P \to Q$ of polytopes. This polytope, the *fiber polytope* of the projection, was introduced by Billera and Sturmfels in [11] in order to illustrate additional convex geometric structure present in the theory of regular (and coherent) subdivisions. This construction produces from each projection π a **fiber polytope** $\Sigma(Q, \pi)$ whose face poset is isomorphic to the poset of π-coherent subdivisions of Q, ordered by refinement.

Theorem 3.1.13 ([11], Theorem 2.1)**.** *Given a projection of polytopes $\pi : P \to Q$, where $P \subseteq \mathbb{R}^N$ and $Q \subseteq \mathbb{R}^n$, there is a polytope $\Sigma(Q, \pi)$ of dimension $N - n$ in* $\ker \pi$ *such that the face lattice of $\Sigma(Q, \pi)$ is isomorphic to the poset of all π-coherent subdivisions of Q.*

We are most interested in this construction in the case where P is a cube and Z is a zonotope, in which case $\Sigma(Q, \pi)$ is itself a zonotope and we have an explicit description of its summands.

Let $C \subseteq \mathbb{R}^N$ be a full-dimensional cube $C = \sum_{i=1}^N [-e_i, e_i]$ equipped with a projection $\pi : \mathbb{R}^N \to \mathbb{R}^n$ sending C onto a full-dimensional zonotope $Z = \sum_{i=1}^N [-a_i, a_i]$, where $a_i = \pi(e_i)$. In this case, it is shown in [11] that the fiber polytope $\Sigma = \Sigma(Z, \pi)$ is again a zonotope.

Theorem 3.1.14 ([11], Theorem 4.1)**.** *For a projection $\pi : C \to Z$ of the N-cube onto a zonotope $Z \sum_{i=1}^N [-a_i, a_i]$, the fiber polytope of the projection is a zonotope given as the Minkowski sum $\Sigma(Z, \pi) = \sum_{S \in \binom{[n]}{d+1}} [-E_S, E_S]$, where $S = \{s_1 < \cdots < s_{d+1}\}$ is a $(d+1)$-subset of $[n]$ and*

$$E_S = \sum_{j=1}^{d+1} (-1)^j \det(a_{s_1}, \ldots, a_{s_{j-1}}, a_{s_{j+1}}, \ldots, a_{s_{d+1}}) \cdot e_{s_j}.$$

Note that since the face poset of Σ is anti-isomorphic to the face poset of the normal fan $\mathcal{N}(\Sigma)$ and isomorphic to the poset of coherent zonotopal subdivisions of Z, we have that the latter two posets are anti-isomorphic.

The linear projection π induces a direct sum decomposition $\mathbb{R}^n = \mathbb{R}^d \oplus \mathbb{R}^{n-d}$ with $\mathbb{R}^{n-d} = \ker \pi$. We consider the fiber zonotope Σ as a polytope in $\mathbb{R}^{n-d}$. Note that the inclusion $\iota : \mathbb{R}^{n-d} \to \mathbb{R}^n$ induces a dual map $\iota^* : (\mathbb{R}^n)^* \to (\mathbb{R}^{n-d})^*$. Now, any linear functional φ on $\mathbb{R}^n$ yields a linear functional $\iota^*\varphi$ on $\mathbb{R}^{n-d}$ and therefore distinguishes a cone N in the normal fan of Σ. At the same time, φ defines a π-coherent zonotopal subdivision $\mathcal{Z}$ of Z. It turns out that the projection π factors through another zonotope Z' in $\mathbb{R}^{d+1}$ such that $\mathcal{Z}$ is coherent with respect to the projection $Z' \to Z$.

Proposition 3.1.15. *Let $\pi : C \to Z$ be as above and let $\mathcal{Z}$ be a coherent zonotopal subdivision of Z with respect to π. Then there is a zonotope $Z' \subseteq \mathbb{R}^{d+1}$ and linear projections $\pi'' : C \to Z'$ and $\pi' : Z' \to Z$ such that $\pi = \pi'' \circ \pi'$ and such that $\mathcal{Z}$ is coherent with respect to π'. Moreover, the subdivision $\mathcal{Z}$ is defined by the linear functional e^*_{d+1} on $\mathbb{R}^{d+1}$.*

Proof. Appealing to Proposition 3.1.10, we have $C \to Z' \to Z$ and see that Z' is a zonotope. □

Because of this result, we are able to associate to each coherent zonotopal subdivision $\mathcal{Z}$ of Z a projection from a $(d+1)$-zonotope Z' with respect to which $\mathcal{Z}$ is coherent. This process greatly simplifies the study of the subdivision $\mathcal{Z}$, as the coherence of $\mathcal{Z}$ with respect to the projection π' and the functional e^*_{d+1} implies also that $\mathcal{Z}$ is combinatorially equivalent to the subcomplex Z'_+ of the boundary complex of Z' consisting of *upper* faces, i.e. those which lie strictly above the hyperplane $\{x_{d+1} = 0\}$.

Corollary 3.1.16. *The subdivision $\mathcal{Z}$ is combinatorially equivalent to the poset Z'_+ of upper faces of Z'.*

Proof. The projection π' is readily seen to induce a bijection between $\mathcal{Z}$ and Z'_+ which preserves the face relations. □

3.2. Arrangements

3.2.1 Hyperplane Arrangements

Definition 3.2.1. A **hyperplane arrangement** in $\mathbb{R}^d$ is a collection $\mathcal{A} = \{H_1, H_2, \ldots\}$ of hyperplanes in $\mathbb{R}^d$. In particular, each hyperplane H_i is of the form $H_i = \{x : \langle a_i, x\rangle = b_i\}$ for some $a_i \in \mathbb{R}^d$ and $b_i \in \mathbb{R}$. Say that $\mathcal{A}$ is **central** if the intersection of all hyperplanes in $\mathcal{A}$ is nonempty, and **affine** if it is not central. The **complement** of $\mathcal{A}$ is the set $M(\mathcal{A}) = \mathbb{R}^d \setminus \bigcup \mathcal{A}$, and a **chamber** of $\mathcal{A}$ is a connected component of $M(\mathcal{A})$. The number of chambers of $\mathcal{A}$ is denoted by $c(\mathcal{A})$. ♦

Definition 3.2.2. Given a hyperplane arrangement $\mathcal{A}$, the **intersection poset** of $\mathcal{A}$ is the poset $\mathcal{L}(\mathcal{A})$ of intersections of hyperplanes in $\mathcal{A}$, ordered by reverse inclusion. Explicitly, elements of $L(\mathcal{A})$ are of the form $F = \bigcap_{i\in I} H_i$, and $F \leq G$ if and only if $F \supseteq G$. The minimal element of $L(\mathcal{A})$ is therefore the empty intersection $\mathbb{R}^d$ and the atoms are the hyperplanes H_i. The intersection poset is, in fact a lattice, with join $F \vee G = F \cap G$ and meet being the smallest intersection of hyperplanes containing both F and G.

Another poset associated with $\mathcal{A}$ is the **face poset** $\mathcal{F}(\mathcal{A})$, which we define to be the face poset of the polyhedral complex induced by $\mathcal{A}$. In particular, the (topological) closures of chambers of $\mathcal{A}$ are polyhedra, and the collection of these polyhedra and all of their faces forms a complete fan in $\mathbb{R}^d$. $\mathcal{F}(\mathcal{A})$ is the face poset of this fan. ♦

Remark 3.2.3. For a matroid M realized by a vector arrangement $A = \{a_1, \ldots, a_n\}$ in $\mathbb{R}^n$, we obtain a central hyperplane arrangement $\mathcal{A}$ consisting of hyperplanes $H_i = \{x : \langle a_i, x\rangle = 0\}$. In this case, the intersection lattice $L(\mathcal{A})$ is naturally identified with the lattice of flats $L(M)$. Indeed, this situation provides the motivation for studying the lattice of flats of a matroid, although the definition in matroid theory applies also to matroids which do not admit a realization. ♦

Suppose now that Z is a zonotope in $\mathbb{R}^n$ given by $Z = b(Z) + \sum_{i=1}^{N} [-a_i, a_i]$. Then the normal fan $\mathcal{N}(Z)$ can be identified with the fan induced by the arrangement having hyperplanes $H_i = \{x : \langle a_i, x\rangle = 0\}$. In this way, each zonotope is dual to a central hyperplane arrangement.

3.2.2 The Möbius Function

For any poset P, an important invariant in the enumerative theory of P is the Möbius function of M, which is a function defined on intervals of P and which satisfies an inversion formula that makes it useful for probing the structure of P. As we shall see, this function is particularly important in the theory of hyperplane arrangements and contains sufficient information to recover the number of faces of each dimension in the polyhedral complex induced by the arrangement.

Definition 3.2.4. Given a poset P the **Möbius function** of P is a function μ_P which assigns a real number $\mathbb{R}$ to each pair (x, y) in P with $x \leq y$. This function is defined by setting $\mu_P(x, x) = 1$ and recursively defining

$$\mu_P(x, y) = -\sum_{x \leq w < y} \mu_P(x, w).$$

If P has unique minimal element $\hat{0}$, we write $\mu_P(x) := \mu_P(\hat{0}, x)$. ♦

3.2.3 Discriminantal Arrangements

For central arrangements $\mathcal{A}$, we can translate its hyperplanes along their normal vectors to obtain affine arrangements of different combinatorial types. The combinatorics of these arrangements depends on the choice of translates of the hyperplanes, and indeed one can parametrize the set of all such parallel translates by using another hyperplane arrangement in a different space. In this section, we review this construction and mention its relation to the fiber zonotopes mentioned in Subsection 3.1.3.

Fix a finite central hyperplane arrangement $\mathcal{A} = \{H_1, \ldots, H_N\}$ in $\mathbb{R}^n$ and suppose that the hyperplanes are given by $H_i = \{x : \langle x, a_i\rangle = 0\}$ with normal vectors $a_i \in \mathbb{R}^n$. Assume moreover that $\mathcal{A}$ is *essential*, i.e. that the set $\{a_1, \ldots, a_N\}$ of normal vectors spans $\mathbb{R}^N$.

Definition 3.2.5. Let $b = (b_1, \ldots, b_N) \in \mathbb{R}^N$. The **parallel translate** of $\mathcal{A}$ along b is the (affine) hyperplane arrangement $\mathcal{A}_b$ having hyperplanes $H_i' = \{x : \langle x, a_i \rangle = b_i\}$ for $i = 1, \ldots, N$. ♦

Definition 3.2.6. The **discriminantal arrangement** based on $\mathcal{A}$ is the central hyperplane arrangement $\mathcal{B}(\mathcal{A})$ whose hyperplanes have normal vectors of the form

$$E_S = \sum_{k=1}^{n+1} (-1)^k \det(a_{s_1}, \ldots, a_{s_{i-1}}, a_{s_{i+1}}, \ldots, a_{s_{n+1}}) \cdot e_{s_i},$$

where $S = \{s_1, < s_2 < \cdots < s_{n+1}\}$ ranges over all $(n+1)$-element subsets of $[n]$. ♦

Note that by definition the discriminantal arrangement $\mathcal{B}(\mathcal{A})$ is the arrangement dual to the fiber zonotope over the zonotope $Z = \sum_{k=1}^{N} [-a_k, a_k]$ dual to $\mathcal{A}$. As such, by Theorem 3.1.14 we see that the poset of cells of the fan induced by $\mathcal{B}(\mathcal{A})$ is isomorphic to the poset of (possibly trivial) coherent subdivisions of Z. Since the latter are identified with (possibly trivial) single-element lifts of the oriented matroid $\mathcal{M}(Z)$ associated with Z, we have the following.

Proof. For any oriented matroid $\mathcal{M}$ with a fixed realization as a hyperplane arrangement $\mathcal{A}$, there is a canonical bijection between (possibly trivial) realizable single-element lifts $\widehat{\mathcal{M}}$ of $\mathcal{M}$ and cones of the discriminantal arrangement $\mathcal{B}(\mathcal{A})$ over $\mathcal{M}$. □

3.3. (Euclidean) Lattices

In this section we review the basic definitions of Euclidean lattices and various convex geometric constructions which are naturally associated to them. We also include some basic results about these objects which will assist us in the sequel.

Definition 3.3.1. A **Euclidean lattice** of rank r is a free abelian group Λ together with a choice of inner product $\langle \cdot, \cdot \rangle$ on the vector space $\Lambda_{\mathbb{R}} = \lambda \otimes \mathbb{R}$.

The **volume** of a Euclidean lattice Λ is defined to be the (Lebesgue) volume of the quotient $\Lambda_{\mathbb{R}}/\Lambda$. A lattice Λ is **unimodular** if it has volume 1. ♦

The choice of inner product on $\Lambda_{\mathbb{R}}$ affords a metric perspective on lattices, which we use to define various subdivisions of the Euclidean space $E = \Lambda_{\mathbb{R}}$) of interest.

Definition 3.3.2. The **dual lattice** of Λ is the lattice $\Lambda^\vee = \{\lambda \in E : \langle \lambda, \xi \rangle \in \mathbb{Z} \text{ for all } \xi \in \Lambda\}$.

Note that if we identify $E \simeq \mathbb{R}^n$, we have that $\Lambda \leq \Lambda^\vee$ so long as the Gram matrix of $\langle \cdot, \cdot \rangle$ is diagonal with integer entries. In this case, we say that Λ is **integral** and define the **determinant group** of Λ to be the (finite) group $\Lambda^\vee/\Lambda$. ♦

Given a nontrivial orthogonal decomposition of E into subspaces which meet Λ, we have the following result which describes relationships between Λ and the lattices it induces within the summands.

Proposition 3.3.3 ([5], Lemma 1)**.** *Suppose that Λ is a unimodular integral lattice and that E admits an orthogonal decomposition $E = E_1 \oplus E_2$ and that $M := E_1 \cap \Lambda$ and $N := E_2 \cap \Lambda$ are both nonempty. Denote by $\pi_i : E \to E_i$ the associated orthogonal projections. Then we have the following.*

(1) M is a lattice in E_1 and N is a lattice in E_2,

(2) $M^\vee = \pi_1(\Lambda)$ and $N^\vee = \pi_2(\Lambda)$, where the duals are taken within the subspaces $E_1 = M \otimes \mathbb{R}$ and $E_2 = N \otimes \mathbb{R}$,

(3) M and N have isomorphic determinant groups.

We now turn our attention to convex geometric constructions in E arising from Λ. Below, we denote by $\|\cdot\|$ the norm on E defined by $\langle \cdot, \cdot \rangle$ in the usual way.

Definition 3.3.4. Given a lattice point $\xi \in \Lambda$, the **Voronoi cell of ξ** is the subset

$$V(\xi) = \{x \in E : \|x - \xi\| \leq \|x - \eta\| \text{ for all } \eta \in \Lambda\}.$$

Observe that Voronoi cells are centrally symmetric and that for any $\xi' \in \Lambda$ we have that $V(\xi + \xi') = V(\xi) + \xi'$. ♦

The property mentioned above immediately implies that the Voronoi cell of $0 \in \Lambda$ can be used to tile E by translations under Λ. Doing so results in a subdivision of E.

Definition 3.3.5. The **Voronoi subdivision** of E with respect to Λ is the subdivision $\mathrm{Vor}(E, \Lambda)$ consisting of all Voronoi cells of Λ together with all of their faces. Such a cell is of the form

$$V(\xi_1, \dots, \xi_k) = \{x \in E : \|x - \xi_1\| = \cdots = \|x - \xi_k\| \leq \|x - \eta\| \text{ for all } \eta \in \Lambda\}.$$

We refer to any cell of $\mathrm{Vor}(E, \Lambda)$ as a **Voronoi cell**. ♦

The Voronoi subdivision is dual to another subdivision of E arising from Λ, which we now define.

Definition 3.3.6. For any $x \in E$, we also define the **Delone cell** of x to be

$$D(x) = \mathrm{conv}\{\xi \in \Lambda : \|x - \xi\| = \min_{\eta \in \Lambda} \|x - \eta\|\}.$$

Note that if $x, x' \in \mathrm{relint}\, V$ for some Voronoi cell V then we necessarily have $D(x) = D(x') =: D$. I this case we write $V = D^*$ and $D = V^*$ and say that V and D are **dual** to one another. The **Delone subdivision** of E with respect to Λ is the collection $\mathrm{Del}(E, \Lambda)$ of all Delone cells $D(x)$ for $x \in E$. ♦

Remark 3.3.7. There are two notions of duality so far discussed in this section which are *distinct* from one another. In particular, the reader should note that the Delone subdivision

$\mathrm{Del}(E, \Lambda)$ is *not* the same as (or even isometric to) the Voronoi subdivision $\mathrm{Vor}(E, \Lambda^\vee)$. ♦

Remark 3.3.8. In the case where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product, we may identify $\Lambda \simeq \mathbb{Z}^n$ and $E \simeq \mathbb{R}^n$. In this case, we have that $\mathrm{Del}(E, \Lambda) = \frac{1}{2}\mathbf{1} + \mathrm{Vor}(E, \Lambda)$, where $\mathbf{1} \in E$ is the all-ones vector. This actually also holds so long as the Gram matrix of the inner product is diagonal. ♦

3.4. Tropical Toric Varieties

We include here a brief summary of the theory of tropical toric varieties, partially following the treatment in [34] and [42]. Let $K = \overline{K}$ be an algebraically closed field and let $\nu_K : K^\times \to \mathbb{R}$ be a nontrivial additive valuation. Note that necessarily we have $\nu_K(K^\times) \subseteq \mathbb{R}$ is dense. Let $M \cong \mathbb{Z}^n$ be a free abelian group of finite rank and let $N = \mathrm{Hom}(M, \mathbb{Z})$ be its dual. Then we have an algebraic torus $T = \mathrm{Hom}(M, K^\times) \cong (K^\times)^r$ having character group M. Here, the points of T are maps $p : M \to K^\times$, and each element $m \in M$ can be seen as a character of T as follows.

$$\begin{aligned} x^m : T &\longrightarrow K^\times \\ p = (t_1, \ldots, t_r) &\longmapsto t_1^{m_1} \cdots t_r^{m_r} =: x^m(p) \end{aligned}$$

Conversely, each $p \in T$ gives a map $x^m \mapsto x^m(p)$. Finally, p also gives a map $\nu_p \in N_\mathbb{R}$ via

$$\nu_p := \nu_K \circ p : M \longrightarrow K^\times,$$

and we write $\nu_T : T \to N_\mathbb{R}$ for the map $p \mapsto \nu_p$.

Write $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. We have a bijection $\mathbb{R}_{\geq 0} \to \overline{\mathbb{R}}$ given by sending $a \mapsto -\log a$. We use this map to endow $\overline{\mathbb{R}}$ with a topology. Now, extend ν_K to a map $\overline{\nu_K} : K \to \overline{\mathbb{R}}$ by declaring $\overline{\nu_K}(a) = \nu_K(a)$ for $a \neq 0$ and $\overline{\nu_K}(0) = \infty$.

Now let Σ be a fan in $N_{\mathbb{R}}$ and take a cone $\sigma \in \Sigma$. Then we define a topological space

$$U_\sigma^{\mathrm{tr}} := \mathrm{Hom}_{\mathbf{Mon}}(\sigma^\vee \cap M, \overline{\mathbb{R}}) \in \mathbf{Top},$$

equipped with the compact-open topology. Note that $\sigma \mapsto U_\sigma^{\mathrm{tr}}$ is a functor $\Sigma \to \mathbf{Top}$ since for all $\tau \leq \sigma$ we have a map

$$\begin{aligned} U_\tau^{\mathrm{tr}} &\longrightarrow U_\sigma^{\mathrm{tr}} \\ (\tau^\vee \cap M \to \overline{\mathbb{R}}) &\longmapsto (\sigma^\vee \cap M \hookrightarrow \tau^\vee \cap M \to \overline{\mathbb{R}}) \end{aligned}$$

For any such fan Σ, the **tropical toric variety** associated with Σ is the topological space

$$X_\Sigma^{\mathrm{tr}} := \mathrm{colim}_{\sigma \in \Sigma} U_\sigma^{\mathrm{tr}} \in \mathbf{Top}.$$

Moreover, if X_Σ is the (classical) toric scheme associated with Σ then we have a **tropicalization map** $X_\Sigma(K) \to X_\Sigma^{\mathrm{tr}}$ given by

$$\begin{aligned} U_\sigma(K) = \mathrm{Hom}(\sigma^\vee \cap M, K) &\longrightarrow U_\sigma^{\mathrm{tr}} = \mathrm{Hom}(\sigma^\vee \cap M, \overline{\mathbb{R}}) \\ p &\longmapsto \nu_K \circ p = \nu_p. \end{aligned}$$

Note that the construction above of a tropical toric variety is directly analogous to the scheme-theoretic description of a toric variety as a colimit of affines. Indeed, if M,N, and Σ are as above and R is any ring, we have a functor $\Sigma \to \mathbf{Sch}_R$ defined by mapping

$$\sigma \longrightarrow U_\sigma = \mathrm{Spec} R[\sigma^\vee \cap M].$$

One then defines

$$X_\Sigma(R) := \mathrm{colim}_{\sigma \in \Sigma} U_\sigma.$$

3.4.1 Tropical Toric Vector Bundles

The parallel works [38] and [37] established a notion of tropical vector bundles on tropical toric varieties. The paper [38] in particular also introduced a notion of tropical reflexive sheaves on such varieties and established that a large class of these objects exhibit a Harder-Narasimhan filtration as in the context of vector bundles on curves. In this subsection we review the basics of this theory, phrased in terms of pointed $\mathbb{T}$-matroids.

Let Σ be a simplicial fan in a lattice $L \cong \mathbb{Z}^n$ and $\Lambda = \mathrm{Hom}(L, \mathbb{Z})$ the dual lattice. Fix a field k and denote by X_Σ the toric k-variety with fan Σ. Let $\mathrm{trop}(X_\Sigma)$ be the tropical toric variety with fan Σ as in [34] or [47]. Denote by $\Sigma(1)$ be the set of rays ρ of Σ and, in case $\mathrm{trop}(X_\Sigma)$ is the tropicalization of a toric variety X_Σ, denote the corresponding torus-invariant divisors on X_Σ by D_ρ. For each ray ρ, we let v_ρ be the first lattice point on ρ. Fix a polarization $h = (h_\rho)_\rho \in \mathbb{R}^{\Sigma(1)}$.

Definition 3.4.1. [38] A **pointed tropical toric reflexive sheaf** of rank r on the tropical toric variety $\mathrm{trop}(X_\Sigma)$ is a tuple $\mathcal{F} = (N, \{F^\rho_\bullet\}_{\rho\in\Sigma(1)})$, where

1. N is a simple pointed $\mathbb{T}$-matroid of rank r on the ground set $E_N = \widetilde{E}_N \sqcup \{*_N\}$, and
2. for each ray $\rho \in \Sigma(1)$, $F^\rho_\bullet = \{F^\rho_j\}_{j\in\mathbb{Z}}$ is a chain of flats of the underlying matroid $\underline{N}$ such that $F^\rho_j \geq F^\rho_{j+1}$, $F^\rho_j = *_N$ for $j \gg 0$, and $F^\rho_j = E_N$ for $j \ll 0$.

Throughout, we write $\{F^\rho_\bullet\}_\rho$ to denote the collection of all flags $F^\rho_\bullet$ as ρ ranges in $\Sigma(1)$. ♦

Definition 3.4.2. [38] A **pointed tropical toric vector bundle** is a pointed tropical toric reflexive sheaf $\mathcal{F} = (N, \{F^\rho_\bullet\}_\rho)$ satisfying the following additional condition: for any maximal cone $\sigma \in \Sigma$ there is a multiset $\mathbf{u}(\sigma)$ in the dual lattice Λ, and a basis $B_\sigma = \{\mathbf{w_u} \mid \mathbf{u} \in \mathbf{u}(\sigma)\}$ of $\underline{N}$ such that for every ray $\rho \leq \sigma$ and $j \in \mathbb{Z}$, we have

$$F^\rho_j = \bigvee_{\mathbf{u}\in B_\sigma, \mathbf{u}\cdot\mathbf{v}_\rho \geq j} \mathbf{w_u}. \tag{3.1}$$

That is, F^ρ_j is the smallest flat containing all the $\mathbf{w_u}$ with $\mathbf{u}\cdot\mathbf{v}_\rho \geq j$. ♦

Remark 3.4.3. As noted in [38], for smooth toric varieties X_Σ, (3.1) is equivalent to the following: For each cone $\sigma \in \Sigma$, there is a basis B_σ of $\underline{M}$ such that whenever $\rho \leq \sigma$, each flat F_j^ρ is a join of elements of B_σ. ♦

In what follows, all tropical toric reflexive sheaves and tropical toric vector bundles are assumed to be pointed, and we will simply say tropical toric reflexive sheaves or tropical toric vector bundle if the context is clear. The **rank** of a tropical toric reflexive sheaf is defined to be $\mathrm{rk}(\mathcal{F}) := \mathrm{rk}(N)$.

A tropical toric reflexive sheaf of rank 1 is called a **tropical line bundle**. Since a tropical line bundle has an underlying matroid of rank 1, all tropical line bundles are indeed tropical vector bundles. For a tropical line bundle $\mathcal{L} = (M, \{L_\bullet^\rho\}_\rho)$, the only information contained in the chains $L_\bullet^\rho$ is the indices $a_\rho = \min\{j \ : \ L_j^\rho \neq *_M\}$. Thus a tropical line bundle can be specified by a vector $(a_\rho) \in \mathbb{Z}^{\Sigma(1)}$.

The paper [38] also defines a way to tropicalize a toric vector bundle on X_Σ to obtain a tropical toric vector bundle. In particular, the tropicalization of a line bundle $\mathcal{O}(\sum_\rho a_\rho \cdot D_\rho)$ is the tropical line bundle corresponding to the vector $(a_\rho) \in \mathbb{Z}^{\Sigma(1)}$ as above.

Definition 3.4.4 ([38], Definition 6.9)**.** Two tropical toric vector bundles $\mathcal{E}$ and $\mathcal{F}$ are **isomorphic** if the associated valuated matroids are isomorphic and if there is some $u \in \Lambda$ for which we have

$$F_j^\rho = G_{j+u\cdot v_\rho}^\rho$$

for every $\rho \in \Sigma(1)$ and every $j \in \mathbb{Z}$, where v_ρ is the first lattice point on ρ. ♦

The Harder-Narasimhan filtration for tropical toric vector bundles is defined using a slope function. The approach taken mirrors the classical setting for toric vector bundles as established in [40], expressing the degree of a tropical toric reflexive sheaf in terms of the filtration $F_\bullet^\rho$. Precisely, the **degree** of a tropical toric reflexive sheaf $\mathcal{E} = (M, \{F_\bullet^\rho\}_\rho)$ is defined to be

$$\deg(\mathcal{E}) = \sum_{\rho \in \Sigma(1)} h_\rho \sum_{j \in \mathbb{Z}} j \cdot (r_{\underline{M}}(F_j^\rho) - r_{\underline{M}}(F_{j+1}^\rho)). \tag{3.2}$$

For a tropical toric reflexive sheaf $\mathcal{E}$ as above and a tropical toric line bundle $\mathcal{L}$ corresponding to a vector $(a_\rho) \in \mathbb{Z}^{\Sigma(1)}$, define the **tensor product** of $\mathcal{E}$ and $\mathcal{L}$ to be the tropical toric reflexive sheaf $\mathcal{E} \otimes \mathcal{L} = (M, \{E^\rho_{\bullet - a_\rho}\}_\rho)$. Here, $E^\rho_{\bullet - a_\rho}$ is the chain of flats whose jth term is $E^\rho_{j-a_\rho}$ (cf. [38, Definition 6.7]).

Remark 3.4.5. Example 7.10 of [38] establishes the identity

$$\deg(\mathcal{E} \otimes \mathcal{L}) = \deg(\mathcal{E}) + r(\underline{M}) \cdot \deg(\mathcal{L}), \tag{3.3}$$

and Remark 7.6 of [38] establishes that the degree of a toric reflexive sheaf is preserved under tropicalization. This establishes that for a complete toric variety X_Σ with polarization $h = (c, c, \cdots, c) \in \mathbb{Z}^{\Sigma(1)}$, tensoring with the tropicalization of a principal divisor preserves the degree of a tropical reflexive sheaf. ♦

The existence and uniqueness of Harder-Narasimhan filtrations for tropical toric reflexive sheaves rely on certain modularity conditions on flats of the associated $\mathbb{T}$-matroid. Say that a tropical toric reflexive sheaf is **modular** when its underlying $\mathbb{T}$-matroid is.

In Section 6.4 we define a category $\mathbf{MTRS}_\bullet$ of modular tropical toric reflexive sheaves and, by using categorical techniques developed in [4] and [41], illustrate that the above rank and degree functions define a unique Harder-Narasimhan filtration.

Chapter 4

Face Numbers of Matroidal Toric Arrangements

In this section we describe a result which computes the number of cells of each dimension in a class of arrangements of subtori in a real torus. In the literature, several authors have considered various combinatorial questions associated with versions of toric arrangements, and the meaning of the term “toric arrangement" is not uniform across all of these works. The toric arrangements that we study here are those referenced in Subsection **??**. That is, for a fixed real torus $T = \mathbb{R}^N/\mathbb{Z}^N$, we study collections $\mathcal{T} = \{T_1, \dots, T_n\}$ of (piecewise-linearly) embedded topological tori $T_i \simeq \mathbb{R}^k/\mathbb{Z}^k$ inside of T.

We study a certain class of toric arrangements constructed explicitly from a regular matroid M. As metnioned in Section 2.3, regularity ensures that M possesses the structure of an oriented matroid $\mathcal{M}$ which is unique up to reorientation. If M is of rank r on n elements, we construct a toric arrangement $\mathcal{T}(\widehat{M})$ in $T = \mathbb{R}^r/\mathbb{Z}^r$ for each single-element lifting $\widehat{\mathcal{M}}$ of $\mathcal{M}$. Any toric arrangement $\mathcal{T}$ which is combinatorially equivalent to some $\mathcal{T}(\widehat{\mathcal{M}})$ is said to be *matroidal*, and it is these arrangements which we study here.

4.1. Construction of Matroidal Toric Arrangements

Suppose that M is a regular matroid of rank N on n elements and fix a realization $\{v_1, \ldots, v_n\}$ of M in $\mathbb{R}^r$. Then this also endows M with the structure of an oriented matroid $\mathcal{M}$. Define the zonotope associated with this representation by

$$Z = \sum_{i=1}^{n} [-v_i, v_i].$$

Chose some (possibly non-realizable) single-element lift $\widehat{\mathcal{M}}$ of the oriented matroid $\mathcal{M}$. Then by Theorem 2.3.11 we know that $\widehat{\mathcal{M}}$ is equivalent to choosing a zonotopal subdivision $\mathcal{Z}$ of Z. Write $\mathcal{Z}_i$ for the zone of $\mathcal{Z}$ associated with v_i (cf. Section 3.1).

By the regularity of M, we know that the (full-dimensional) zonotope Z tiles the ambient space $\mathbb{R}^N$ via translation by some lattice $\Lambda \leq \mathbb{R}^N$. This implies in particular that one can identify faces $Z' \leq Z$ with one another by translation via Λ, and that the quotient $T = Z/\Lambda$ is a real torus of dimension N. We take this section to construct a toric arrangement $\mathcal{T}(\widehat{\mathcal{M}})$ in T by using the information associated with the lifting $\widehat{\mathcal{M}}$ (or rather, with the zonotopal subdivision $\mathcal{Z}$).

For each zone $\mathcal{Z}_i \subseteq \mathcal{Z}$, let π_i denote the orthogonal projection of $\mathbb{R}^N$ onto the hyperplane $\{x \in \mathbb{R}^N : \langle x, v_i \rangle = 0\}$. We consider functions g_i, G_i on the image $\pi_i(Z)$, defined as follows.

$$g_i(x) := \min\{t : x + tv_i \in \bigcup \mathcal{Z}_i\}$$
$$G_i(x) := \max\{t : x + tv_i \in \bigcup \mathcal{Z}_i\}$$

From the proof of the Section Lemma (Lemma 3.2) of [50] that the functions g_i and G_i are well-defined, continuous, and piecewise-linear. We now use g_i and G_i to define a section γ_i of π_i in Z. For $x \in \pi_i(Z)$, we declare

$$\gamma_i(x) := x + \tfrac{1}{2}(g_i(x) + G_i(x))v_i.$$

Write $\widetilde{T_i} = \mathrm{im}(\gamma_i) \subseteq Z$. Note in particular that $\widetilde{T_i} \subseteq \bigcup \mathcal{Z}_i$. Since g_i and G_i are continuous and piecewise-linear, it follows that $\widetilde{T_i}$ is a piecewise-linear image of $\pi_i(Z)$ sitting within Z. Moreover, for all $Z' \in \mathcal{Z}_i$ we have that $b(Z') \in \widetilde{T_i}$ by the definition of the barycenter $b(Z')$. Indeed, we have that $\gamma_i(\pi_i(b(Z'))) = b(Z')$ since $b(Z')$ is the midpoint of the segment $\pi_i^{-1}(\pi_i(b(Z')))$. In fact, we have the following.

Proposition 4.1.1. *For any cell $Z' \in \mathcal{Z}_i$ and any $x \in \pi_i(Z')$, we have that $\gamma_i(x) = b(\pi_i^{-1}(x))$.*

Proof. Write $x = \sum_{\ell \neq i} \alpha_\ell \pi_i(v_\ell)$, where $\alpha_i \in [-1, 1]$. Then

$$\pi_i^{-1}(x) = \sum_{\ell \neq i} \alpha_\ell \pi_i(v_\ell) + [g_i(x)v_i, G_i(x)v_i].$$

In particular, this fiber is a segment with barycenter

$$\sum_{\ell \neq i} \alpha_\ell \pi_i(v_\ell) + \tfrac{1}{2}(g_i(x) + G_i(x))v_i = x + \tfrac{1}{2}(g_i(x) + G_i(x))v_i = \gamma_i(x).$$

□

Define $\widetilde{\mathcal{T}} = \{\widetilde{T_i} : i = 1, \ldots, n\}$. We now observe that $\widetilde{\mathcal{T}}$ descends to an arrangement of piecewise-linearly embedded tori in $T = Z/\Lambda$.

Lemma 4.1.2. *If $Z', Z'' \leq Z$ are faces of Z such that $Z' = Z'' + \lambda$ for some $\lambda \in \Lambda$, then for all i we have that*

$$\widetilde{T_i} \cap Z' = (\widetilde{T_i} \cap Z'') + \lambda.$$

Proof. for $x'' \in \pi_i(Z'')$, we have by definition that there is some $y'' \in Z''$ with $\pi_i(y'') = x''$ Then $y' := y'' + \pi(\lambda)$ lies in Z' and we immediately obtain $x' := \pi_i(y') = x'' + \pi_i(\lambda)$. Note that we have

$$\begin{aligned}\gamma_i(x') &= x' + \tfrac{1}{2}(g_i(x) + G_i(x))v_i \\ &= x'' + \pi_i(\lambda) + \tfrac{1}{2}(g_i(x'' + \pi_i(\lambda)) + G_i(x'' + \pi_i(\lambda)))v_i\end{aligned}$$

Write $\pi_i^\perp$ for the orthogonal projection of $\mathbb{R}^N$ onto $\operatorname{span}\{v_i\}$, so that for all $z \in \mathbb{R}^N$ we have $z = \pi_i(z) + \pi_i^\perp(z)$. Then, writing $\alpha_\lambda v_i := \pi_i^\perp(\lambda)$, we see

$$\begin{aligned}
g_i(x'' + \pi_i(\lambda)) &= \min\{t : x'' + \pi_i(\lambda) + tv_i \in \bigcup \mathcal{Z}_i\} \\
&= \min\{t : x'' + \pi_i(\lambda) + tv_i \in Z'\} \\
&= \min\{t : x'' + \pi_i(\lambda) + tv_i \in Z'' + \lambda\} \\
&= \min\{t : x'' + tv_i \in Z'' + \alpha_\lambda v_i\} \\
&= \min\{t : x'' + (t - \alpha_\lambda)v_i \in Z''\} \\
&= \min\{s + \alpha_\lambda : x'' + sv_i \in Z''\} \\
&= g_i(x'') + \alpha_\lambda
\end{aligned}$$

That is, we now have that $g_i(x')v_i = g_i(x'')v_i + \pi_i^\perp(\lambda)$ and, similarly, $G_i(x')v_i = g_i(x'')v_i + \pi_i^\perp(\lambda)$. Together, these then yield

$$\gamma_i(x') = x' + \tfrac{1}{2}(g_i(x') + G_i(x'))v_i = x'' + \pi_i(\lambda) + \tfrac{1}{2}g_i(x'') + G_i(x''))v_i + \pi_i^\perp(\lambda) = \gamma_i(x'') + \lambda.$$

In particular, this gives $\widetilde{T_i} \cap Z' = (\widetilde{T_i} \cap Z'') + \lambda$. □

This $\widetilde{T_i}$ descends to a subtorus T_i in T by identifying the intersection of $\widetilde{T_i}$ with faces of Z which differ only by lattice translation. That is, $\widetilde{\mathcal{T}}$ descends to a (piecewise-linear) toric arrangement $\mathcal{T} = \{T_i : i = 1, \dots, n\}$ in T.

Definition 4.1.3. Say that a toric arrangement $\mathcal{T}$ in T is **matroidal** if it arises in the above way from a single-element lifting $\widehat{\mathcal{M}}$ associated with a regular matroid M. ♦

We now prove a collection of results which together establish that the combinatorics of a matroidal toric attangement $\mathcal{T}(\widehat{\mathcal{M}})$ is captured by the matroid $\widehat{\mathcal{M}}$ itself. We do so in essentially three steps:

(1) Show that, in a ball $B_{Z'}$ about the barycenter $b(Z')$ of each cell $Z' \in \mathcal{Z}$, the (topological)

tori T_i intersect like a hyperplane arrangement $\mathcal{A}_{Z'}$.

(2) Show that the hyperplane arrangement $\mathcal{A}_{Z'}$ is dual to the zonotope Z'.

(3) Show that, in each cell $Z' \in \mathcal{Z}$, the tori T_i do not intersect in any unexpected ways. In particular, show that intersections $\widetilde{T}_{i_1} \cap \cdots \cap \widetilde{T}_{i_\ell}$ are piecewise-linear homeomorphic images polytopes of the expected dimension.

Together, these establish that the arrangement $\mathcal{T}(\widehat{\mathcal{M}})$ induces a cell complex which is dual to (the quotient by Λ of) the subdivision $\mathcal{Z}$ itself. We show each of the above in turn.

Step (1) is broken into two smaller results, namely Proposition 4.1.4 and Proposition 4.1.5.

Proposition 4.1.4. *For all cells $Z' \in \mathcal{Z}$ and all zones $\mathcal{Z}_i$ containing Z', we have that $\widetilde{T}_i \cap Z'$ is centrally symmetric about $b(Z')$.*

Proof. Without loss of generality, we assume that A' is centered at the origin. Then, up to reordering $v_1, \ldots, v_n$, we may write

$$Z' = \sum_{i=1}^{k} [-v_i, v_i]$$

. We show that $\widetilde{T}_k \cap Z'$ is centrally symmetric about 0. We have

$$\pi_k(Z') = \sum_{i=1}^{k-1} [-\pi_k(v_i), \pi_k(v_i)]$$

and $b(\pi_k(Z')) = b(Z') = 0$. Since $\pi_k(Z')$ is again a zonotope, it is centrally symmetric and therefore we have that $x \in \pi_k(Z')$ if and only if $-x \in \pi_k(Z')$. Fix $x \in \pi_k(Z')$ and consider the following.

$$\begin{aligned}
-g_k(x) &= \min\{t : -x + tv_k \in Z'\} \\
&= \min\{t : x - tv_k \in Z'\} \\
&= -\max\{-t : x - tv_k \in Z'\} \\
&= -\max\{s : x_s v_k \in Z'\} = -G_k(x)
\end{aligned}$$

Likewise, we similarly observe that $G_k(-x) = -g_k(x)$. Thus we see

$$\begin{aligned}
\gamma_k(-x) &= -x + \tfrac{1}{2}(g_k(-x) + G_k(-x))v_k \\
&= -x - \tfrac{1}{2}(g_k(x) + G_k(x))v_k = -\gamma_k(x)
\end{aligned}$$

This immediately gives that $\widetilde{T}_i \cap Z'$ is symmetric about $0 = b(Z')$, as desired. □

Proposition 4.1.5. *For all cells $Z' \in \mathcal{Z}$ and all zones $\mathcal{Z}_i$ containing Z', there is some open ball $B_{Z'} = B(b(Z'), \varepsilon)$ for which*

$$(\widetilde{T}_i \cap Z') \cap B_{Z'} = H_i' \cap B_{Z'}$$

for some affine hyperplane H_i passing through $b(Z')$.

Proof. This follows immediately from Proposition 4.1.4. Indeed, since the functions g_i and G_i are piecewise-linear and continuous for all i, it follows that $\widetilde{T}_i \cap Z'$ is a piecewise-linear homeomorphic image of $\pi_i(Z')$. Thus the central symmetry guaranteed by Proposition 4.1.4 ensures that $b(Z')$ is not in the corner locus of $\widetilde{T}_i \cap Z'$, so such a ball $B_{Z'}$ must exist. □

By translating the hyperplanes H_i' of Proposition 4.1.5 by $-b(Z')$, we obtain hyperplanes H_i passing through the origin. Collecting the H_i, we have then a central hyperplane arrangement

$$\mathcal{A}_{Z'} := \{H_i = H_i' - b(Z') : i = 1, \ldots, k\}.$$

With this arrangement in hand, we how observe that it is necessarily dual to the zonotope Z'.

Proposition 4.1.6. *For each $Z' \in \mathcal{Z}$, we have an isomorphism of fans $\mathcal{F}(\mathcal{A}_{Z'}) \simeq \mathcal{N}(Z')$.*

Proof. We continue to assume that $Z' = \sum_{i=1}^{k} [-v_i, v_i]$. Let n_i denote the normal of the hyperplane H_i. We seek to show that the matroid of the vector arrangement $\{n_1; \ldots, n_k\}$ is identical to that of $\{v_1, \ldots, v_k\}$. To do so, we show that a subset $\{v_{i_1}, \ldots, v_{i_\ell}\} \subseteq \{v_i, \ldots, v_k\}$ is linearly independent if and only if $\{n_{i_1}, \ldots, n_{i_\ell}\}$ is linearly independent.

Suppose that Z' is m-dimensional. Since we are currently concerned only with the cell Z', we fix an isomorphism between its affine hull and $\mathbb{R}^m$ and work there. Suppose that $\{v_{i_1}, \ldots, v_{i_k}\}$ is linearly independent. Then $m \geq \ell$ and $\{v_{i_1}, \ldots, v_{i_k}\}$ is contained in some basis $\{v_{i_1}, \ldots, v_{i_m}\}$ of $\mathbb{R}^m$. Since we are concerned only with the matroid and the normal fan associated with Z', we may rescale the vectors v_j for $j > k$ as we like. In particular, we may take a linear automorphism A of $\mathbb{R}^m$ sending $v_{i_1}, \ldots, v_{i_m}$ to the standard basis, denoted by $\tilde{v}_{i_1}, \ldots, \tilde{v}_{i_m}$, and all other v_i for $i \in [k] - \{i_1, \ldots, i_m\}$ to vectors $\tilde{v}_i$ with $\|\tilde{v}_i\| = \frac{\varepsilon}{2(k-m)} \ll 1$. Now

$$AZ' = \sum_{i=1}^{k} [-\tilde{v}_i, \tilde{v}_i] = \sum_{i=1}^{m} [-\tilde{v}_{i_j}, \tilde{v}_{i_j}] + \sum_{i \notin \{i_1, \ldots, i_m\}} [-\tilde{v}_i, \tilde{v}_i].$$

That is, AZ' is the Minkowski sum of a hypercube of side length 2 and $k - m$ additional segments of length $\frac{\varepsilon}{k-m}$.

For any $i = 1, \ldots, k$, let $\tilde{\pi}_i$ be the orthogonal projection onto $\tilde{v}_i^{\perp}$ and define $\tilde{g}_i, \tilde{G}_i, \tilde{\gamma}_i$, and $\tilde{H}_i$ for AZ' in analogy with those defined for Z and Z'. Then by the structure of AZ' as the Minkowski sum of a cube and of small segments, we may take $B_{AZ'} = B(2 - 2\varepsilon, 0)$ and observe that for $j = 1, \ldots, \ell$ we have that $\tilde{g}_{i_j}|_{\tilde{\pi}_{i_j}(B_{AZ'})}$ and $\tilde{G}_{i_j}|_{\tilde{\pi}_{i_j}(B_{AZ'})}$ are constant functions. That is, $\tilde{H}_{i_j}$ is parallel to the i_j coordinate hyperplane and thus n_{i_j} can be taken to be the standard basis vector $\tilde{v}_{i_j}$. In particular, $\{n_{i_1}, \ldots, n_{i_\ell}\}$ is linearly independent.

Conversely, if $\{n_{i_1}, \ldots, n_{i_\ell}\}$ is linearly independent, then we may perform an analogous process to the above, extending $\{n_{i_1}, \ldots, n_{i_\ell}\}$ to a basis $\{n_{i_1}, \ldots, n_{i_m}\}$ of $\mathbb{R}^m$. Note that here $\{n_1, \ldots, n_k\}$ is guaranteed to span $\mathbb{R}^m$ by the assumption that Z' is full-dimensional in $\mathbb{R}^m$

and the forward direction above.

This establishes that the matroids associated to the arrangements $\{v_1, \ldots, v_k\}$ and $\{n_1, \ldots, n_k\}$ give rise to isomorphic matroids, so that they are dual to combinatorially equivalent hyperplane arrangements. This establishes the desired isomorphism. $\square$

At this stage, we know that the arrangement $\mathcal{T}(\widehat{\mathcal{M}})$ arises from a collection $\widetilde{\mathcal{T}}$ of piecewise-linear sections $\widetilde{T}_i$, each of which satisfies that the restrictions to any cell Z' through which it passes should contain the barycenter of that cell. Moreover, each cell Z' contains a ball around its barycenter such that the restriction of the $\widetilde{T}_i$ to this ball form a hyperplane arrangement which is dual to the cell Z'.

Thus, if we establish that no two sections $\widetilde{T}_i$ and $\widetilde{T}_j$ having no-empty intersection in Z' intersect only at $b(Z')$, we can conclude that the cell decomposition of Z induced by $\widetilde{T}$ is dual to the subdivision $\mathcal{Z}$. We now prove the final step, establishing step (3) mentioned above.

Proposition 4.1.7. *For any $i_1, \ldots, i_\ell$ with $\{v_{i_1}, \ldots, v_{i_\ell}\}$ linearly independent, we have that $\widetilde{T}_{i_1} \cap \cdots \cap \widetilde{T}_{i_\ell}$ is a piecewise-linear homeomorphic image of the polytope $(\pi_{i_1} \circ \cdots \circ \pi_{i_\ell})(Z')$.*

Proof. We work with the case $\ell = 2$. The general case works in precisely the same way, but is notationally intensive and less illuminating. First, note that if $v_i = \alpha v_j$ for some $\alpha \in \mathbb{R}$, then we have $\widetilde{T}_i \cap \widetilde{T}_j$ empty. Thus if we have some $y \in \widetilde{T}_{i_1} \cap \widetilde{T}_{i_2}$ then we have $\{v_{i_1}, v_{i_2}\}$ linearly independent.

We again assume that Z' is of dimension m and centered at the origin in $\mathbb{R}^m$. As in the proof of Proposition 4.1.5, we extend $\{v_{i_1}, v_{i_2}\}$ to a basis $\{v_{i_1}, \ldots, v_{i_m}\}$ and apply an automorphism A of $\mathbb{R}^m$ to view $\{v_{i_1}, \ldots, v_{i_m}\}$ as the standard basis. In particular, we have $v_1 \perp v_2$ and $\pi_i \circ \pi_j = \pi_j \circ \pi_i$, which we denote by π.

Take some $y \in \widetilde{T}_i \cap \widetilde{T}_j$, suppose that $y = \sum_{\ell=1}^{k} \beta_\ell v_\ell$ and write $x_i := \pi_i(y)$ and $x_j := \pi_j(y)$, so that

$$x_i = \sum_{\ell \neq i} \beta_\ell \pi_i(v_\ell) \ , \quad x_j = \sum_{\ell \neq j} \beta_\ell \pi_j(v_\ell).$$

Then from Proposition 4.1.1, we have that

$$y = b(\pi_i^{-1}(x_i)) = b(\pi_j^{-1}(x_j)).$$

We also write $x = \pi(y)$, so that

$$x = \sum_{\ell \neq i,j} \beta_\ell \pi(v_\ell),$$

and $x = \pi_i(x_j) = \pi_j(x_i)$. Note that by $x \perp v_i$ and $v_i \perp v_j$ we additionally have

$$\begin{aligned} g_i(x) &= \min\{t : x + t\pi_i(v_j) \in Z'\} \\ &= \min\{t : x + t\pi_i(v_j) \in \pi_i(Z')\} \\ &= \min\{t : x_i + tv_j \in Z'\} = g_i(x_i), \end{aligned}$$

and similar for G_i, g_j, and G_j. By applying Proposition 4.1.1 to the projections of zonotopes given by $\pi_i : \pi_j(Z') \to \pi(Z')$ and $\pi_i : \pi_j(Z') \to \pi(Z')$, we thus obtain that $x_i = b(\pi_j^{-1}(x))$ and $x_j = b(\pi_i^{-1}(x))$. Explicitly,

$$\begin{aligned} x_i &= \sum_{\ell \neq i,j} \beta_\ell \pi(v_\ell) + \tfrac{1}{2}(g_j(x) + G_j(x))\pi_i(v_j) \\ &= \sum_{\ell \neq i,j} \beta_\ell \pi(v_\ell) + \tfrac{1}{2}(g_j(x) + G_j(x))v_j \\ &= x + \tfrac{1}{2}(g_j(x) + G_j(x))v_j \\ x_j &= x + \tfrac{1}{2}(g_i(x) + G_i(x))v_i. \end{aligned}$$

Now, observe that we can express $\pi^{-1}(x)$ as follows.

$$
\begin{aligned}
\pi^{-1}(x) &= \pi_j^{-1}(\pi_i^{-1}(x)) \\
&= \pi_j^{-1}\left(\sum_{\ell \neq i,j} \beta_\ell \pi_i(\pi_j(v_\ell)) + [g_i(x)v_i, G_i(x)v_i]\right) \\
&= \sum_{\ell \neq i,j} \beta_\ell \pi(\pi_j(v_\ell)) + [g_i(x)v_i, G_i(x)v_i] + [g_j(x)v_j, G_j(x)v_j].
\end{aligned}
$$

We therefore also see that $\pi^{-1}(x)$ is a zonotope with barycenter

$$
b(\pi^{-1}(x)) = x + \tfrac{1}{2}(g_i(x) + G_i(x))v_i + \tfrac{1}{2}(g_j(x) + G_j(x))v_j.
$$

This implies that $y = b(\pi^{-1}(x))$. Indeed, combining $y = b(\pi_i^{-1}(x_i))$ with $x_i = b(\pi_j^{-1}(x))$ and the observation that the functions g_i and G_i behave well with respect to arguments x_i, x_j, and x as above, we have

$$
\begin{aligned}
y &= \sum_{\ell \neq i} \beta_\ell \pi_i(v_\ell) + \tfrac{1}{2}(g_i(x_i) + G_i(x_j))v_i \\
&= \sum_{\ell \neq i,j} \beta_\ell \pi(v_\ell) + \tfrac{1}{2}(g_j(x_j) + G_j(x_j))v_j + \tfrac{1}{2}(g_i(x_i) + G_i(x_i)) \\
&= \sum_{\ell \neq i,j} \beta_\ell \pi(v_\ell) + \tfrac{1}{2}(g_j(x) + G_j(x))v_j + \tfrac{1}{2}(g_i(x) + G_i(x))
\end{aligned}
$$

That is, $y = b(\pi^{-1}(x))$. In particular, this shows that each point in the intersection of a collection of sections $\widetilde{T}_{i_1}, \ldots, \widetilde{T}_{i_\ell}$ is the barycenter of the fiber over a point in the polytope $(\pi_{i_1} \circ \cdots \circ \pi_{i_\ell})(Z')$. Moreover, the locus $\widetilde{T}_{i_1} \cap \ldots \cap \widetilde{T}_{i_\ell}$ is an intersection of the piecewise-linear homeomorphic images $\widetilde{T}_{i_j}$ of $\pi_{i_j}(Z')$. That is, each such intersection is necessarily a piecewise-linear homeomorphic image of the polytope $(\pi_{i_1} \circ \cdots \circ \pi_{i_\ell})(Z')$. $\square$

We now see that the arrangement $\widetilde{\mathcal{T}}$ within Z consists of piecewise-linear images $\widetilde{T}_i$ of the centrally symmetric polytopes $\pi_i(Z)$ and that the restrictions $\widetilde{\mathcal{T}}^\circ_{Z'} := \{\widetilde{T}_i \cap \mathrm{relint}(Z') : Z' \in$

$\mathcal{Z}_i\}$ of $\widetilde{T}$ to the cells $Z' \in \mathcal{Z}$ induce cell decompositions of Z' which are combinatorially isomorphic to the normal fans $\mathcal{N}(Z')$. Further, Proposition 4.1.7 guarantees also that for any $\widetilde{T}_{i_1}, \ldots, \widetilde{T}_{i_\ell} \in \widetilde{\mathcal{T}}$ with nonempty intersection in $Z' \in \mathcal{Z}$ we have a homeomorphism

$$\widetilde{T}_{i_1} \cap \cdots \cap \widetilde{T}_{i_\ell} \cap Z' \simeq (\pi_{i_1} \circ \cdots \pi_{i_\ell})(Z').$$

That is, we now have isomorphisms of posets

$$\mathcal{F}(\widetilde{\mathcal{T}}^{\circ}_{Z'}) \simeq \mathcal{F}(\mathcal{A}_{Z'}) \simeq \mathcal{N}(Z').$$

Further, under face identifications of the full zonotope Z, any nonempty intersection of sections $\widetilde{T}_{i_1} \cap \cdots \cap \widetilde{T}_{i_\ell}$ descends to some subtorus $T_{i_1} \cap \cdots T_{i_\ell}$ in T. That is, $\mathcal{T}(\widehat{\mathcal{M}})$ is an arrangement of piecewise-linear tori in T whose intersections are also tori in $\mathcal{T}$. We can thus speak of the poset of layers $\mathcal{L}(\mathcal{T}(\widehat{\mathcal{M}}))$.

The construction of $\mathcal{T}$ by identifying faces of Z (that is, cells of $\mathcal{Z}$) also ensures that all intersections of tori in $\mathcal{T}(\widehat{\mathcal{M}})$ can already be seen in $\widetilde{\mathcal{T}}$. In particular, each face $F \in \mathcal{F}(\widetilde{\mathcal{T}})$ can be labeled with a sign vector corresponding to its relative position above, on, or below the section $\widetilde{T}_i$. Indeed, any point $y \in Z$ gives rise to a sign vector $X(y) \in \{+,-,0\}^[n]$, where $X(y)_i = +$ if y lies above the section $\widetilde{T}_i$, $X(y)_i = -$ if y lies below it, and $X(y)_i = 0$ when $y \in \widetilde{T}_i$. This sign vector is constant for all points in F, and so we declare F to correspond to the sign vector $X(F) = X(y)$ for any $y \in F$.

By sending each such sign vector X to the cell Z' with

$$Z' = \sum_{i:X_i=+} v_i - \sum_{j:X_j=-} v_j + \sum_{k:X_k=0} [-v_k, v_k],$$

the face poset $\mathcal{F}(\widetilde{\mathcal{T}})$ is now dual to the zonotopal subdivision $\mathcal{Z}$. Now, using the anti-isomorphism between the zonotopal subdivision $\mathcal{Z}$ of Z and the upper complex of the covector poset of the single-element lifting $\widehat{\mathcal{M}}$ (cf. Theorem 2.3.11), we deduce the following alternative characterization of the poset of layers $\mathcal{L}(\mathcal{T}(\widehat{\mathcal{M}}))$.

Proposition 4.1.8. *For a matroidal toric arrangement $\mathcal{T}(\widehat{\mathcal{M}})$ arising from a single-element lifting of $\mathcal{M}$, we have an isomorphism of posets:*

$$\mathcal{L}(\mathcal{T}(\widehat{\mathcal{M}})) \simeq \mathcal{L}(\underline{\widehat{\mathcal{M}}}),$$

where the poset on the left is the poset of layers of $\mathcal{T}(\widehat{\mathcal{M}})$ and the poset on the right is the lattice of flats of the underlying matroid $\underline{\widehat{\mathcal{M}}}$.

Proof. First, note that we have constructed $\widetilde{\mathcal{T}}$ such that $\mathcal{F}(\widetilde{\mathcal{T}})$ is anti-isomorphic to $\mathcal{Z}$. Since the latter is anti-isomorphic to $\mathcal{V}^*_{\geq 0}(\widehat{\mathcal{M}})$ by Bohne-Dress, we thus have an isomorphism

$$\mathcal{F}(\widetilde{\mathcal{T}}) \simeq \mathcal{V}^*_{\geq 0}(\widehat{\mathcal{M}}).$$

Observing that the map $\mathcal{F}(\widetilde{\mathcal{T}}) \to \mathcal{L}(\mathcal{T})$ given by sending a face F to the minimal intersection $T_{i_1} \cap \cdots \cap T_{i_\ell} \in \mathcal{T}$ containing F corresponds exactly to the map $z : \mathcal{V}^*_{\geq 0}(\widehat{\mathcal{M}}) \to \mathcal{L}(\underline{\widehat{\mathcal{M}}})$ induced by the zero map sending a covector X to the flat spanned by the elements e with $X_e = 0$. $\square$

We use this observation in the following section to show that we can actually enumerate the cells of fixed dimension in either $\overline{\mathcal{Z}}$ or of $\mathcal{F}(\mathcal{T}(\widehat{\mathcal{M}}))$.

4.2. Computing Face Numbers of Matroidal Arrangements

Suppose that M is a regular matroid of rank N equipped with a realization in $\mathbb{R}^N$ as a vector arrangement $\{v_1, \ldots, v_n\}$. Assume that this realization defines a zonotope $Z = \sum_{i=1}^n [-v_i, v_i]$ which tiles $\mathbb{R}^N$ via translation by some lattice Λ. This fixes an oriented matroid structure $\mathcal{M}$ on M (this structure is essentially unique, cf. Section 2.3). Then, in the way described in Section 4.1, any single-element lifting $\widehat{\mathcal{M}}$ of $\mathcal{M}$ associated with a zonotopal subdivision

$\mathcal{Z}$ determines a cell decomposition $\overline{\mathcal{Z}}$ of the associated torus $T = Z/\Lambda$ and a dual matroidal toric arrangement $\mathcal{T}(\widehat{\mathcal{M}})$.

In this section we prove that the f-vector of the cell decomposition $\overline{\mathcal{Z}}$ of T, and therefore the f-vector of the decomposition induced by the arrangement $\mathcal{T}(\widehat{\mathcal{M}})$, is expressible in terms of matroidal information about the lifting $\widehat{\mathcal{M}}$. In fact, it depends only on the lattice of flats of the underlying matroid $\widehat{M} = \underline{\widehat{\mathcal{M}}}$.

Theorem 4.2.1. *Consider the matroidal toric arrangement $\mathcal{T} = \mathcal{T}(\widehat{\mathcal{M}})$ associated with a single-element lift $\widehat{\mathcal{M}}$ of a realized regular oriented matroid $\mathcal{M} = \mathcal{M}(Z)$, and let $\mathcal{F}(\mathcal{T})$ denote the cell decomposition of the torus $T = Z/\Lambda$. Then the number of ℓ-dimensional cells of $\mathcal{F}(\mathcal{T})$ is given by*

$$f_\ell(\mathcal{F}(\mathcal{T})) = (-1)^\ell \sum_{\substack{S \leq R \in \mathcal{L}(\widehat{\mathcal{M}}) \\ r(S) = N - \ell \\ r(R) = N}} \mu(S, R).$$

Proof. First, observe that each ℓ-dimensional face of $\mathcal{F}(\mathcal{T})$ is contained in a unique ℓ-dimensional subtorus S in the poset of layers $\mathcal{L}(\mathcal{T})$. Thus we may write

$$f_\ell(\mathcal{F}(\mathcal{T})) = \sum_{S \in \mathcal{L}(\mathcal{T})} c(\mathcal{T}_S),$$

where $\mathcal{T}_S$ is the arrangement of subtori in S defined by intersections of S with the tori in $\mathcal{T}$ and $c(\mathcal{T}_S)$ is the number of *chambers*, i.e. full-dimensional cells in $\mathcal{F}(\mathcal{T}_S)$. Note moreover that this formula holds as well for all induced arrangements on subtori $S \in \mathcal{L}(\mathcal{T})$ of T, so that we similarly obtain a formula for each of the face numbers $f_\ell(\mathcal{F}(\mathcal{T}_S))$ in the decompositions induced by each such arrangement $\mathcal{T}_S$.

Now, since $\mathcal{F}(\mathcal{T}_S)$ is a cell decomposition of the piecewise-linear torus S, we may compute the topological Euler characteristic $\chi(S)$ as the alternating sum of the face numbers

$f_j(\mathcal{F}(\mathcal{T}_S))$.

$$\chi(S) = \sum_{j=0}^{\ell}(-1)^j f_j(\mathcal{F}(\mathcal{T}_S)) = \sum_{R\in\mathcal{L}(\mathcal{T}_S)}(-1)^{\dim(R)}c(\mathcal{T}_R) = \sum_{\substack{R\in\mathcal{L}(\mathcal{T})\\ R\geq S}}(-1)^{\dim(R)}c(\mathcal{T}_R).$$

In particular, we have now expressed the Euler characteristic $\chi(S)$ as an alternating sum of function values on the principal order ideal generated by S. Thus, applying the inversion formula for the Möbius function of $\mathcal{L}(\mathcal{T})$, we obtain a formula for the number of chambers of any such arrangement $\mathcal{T}_S$:

$$c(\mathcal{T}_S) = (-1)^{\dim(S)}\sum_{\substack{R\in\mathcal{L}(\mathcal{T})\\ R\geq S}}\chi(R)\mu(S,R).$$

By combining the formulas above for $f_\ell(\mathcal{T})$ and $c(\mathcal{T}_S)$ and recalling that the Euler characteristic of a torus is 0 for positive-dimensional tori and 1 for tori of dimension zero, we thus obtain the following:

$$\begin{aligned}
f_\ell(\mathcal{F}(\mathcal{T})) &= (-1)^\ell \sum_{\substack{S\in\mathcal{L}(\mathcal{T})\\ \dim(S)=\ell}}\sum_{\substack{R\in\mathcal{L}(\mathcal{T})\\ R\geq S}}\chi(R)\mu(S,R)\\
&= (-1)^\ell \sum_{\substack{S\in\mathcal{L}(\mathcal{T})\\ \dim(S)=\ell}}\sum_{\substack{R\in\mathcal{L}(\mathcal{T})\\ \dim(R)=0\\ R\geq S}}\mu(S,R)\\
&= (-1)^\ell \sum_{\substack{S\leq R\in\mathcal{L}(\mathcal{T})\\ \dim(S)=\ell\\ \dim(R)=0}}\mu(S,R).
\end{aligned}$$

Finally, by Proposition 4.1.8, we know that $\mathcal{L}(\mathcal{T}) \simeq \mathcal{L}(\widehat{\underline{\mathcal{M}}})$ and that the latter poset has rank function given by $r(\cdot) = N - \dim(\cdot)$. Thus we obtain the formula desired:

$$f_\ell(\mathcal{F}(\mathcal{T})) = (-1)^\ell \sum_{\substack{S\leq R\in\mathcal{L}(\widehat{\underline{\mathcal{M}}})\\ r(S)=N-\ell\\ r(R)=N}}\mu(S,R).$$

□

4.2.1 Alternative Proof for the case of Realizable Lifts

In this section we present an alternative proof of Theorem 4.2.1 in the case where the single-element lifting $\widehat{\mathcal{M}}$ is realizable over $\mathbb{R}$. This section uses methods more closely related to the polytope theory present in the Section 3.1 and avoids the heavier approach using posets and oriented matroids present in Theorem 4.2.1.

Suppose that $Z \subseteq \mathbb{R}^d$ is a zonotope whose translates by a full-rank lattice $\Lambda \subseteq \mathbb{R}^d$ produce a subdivision of the ambient space. Then we obtain a torus $T := \mathbb{R}^d/\Lambda$ by identifying faces of Z which differ only by a translation by an element of Λ. In this situation, we see that any polyhedral subdivision of Z induces a cell decomposition of the torus T. We shall be interested in this situation in the case that the polyhedral subdivision in question is a coherent zonotopal subdivision. More specifically, we investigate the number of faces of various dimensions which appear in the induced polyhedral subdivision of T.

As before, Let $\pi : C \to Z$ be a projection of a cube onto a zonotope and let $\mathcal{Z}$ be a coherent subdivision with respect to this projection. We define a cell complex $\overline{\mathcal{Z}}$ by identifying cells of $\mathcal{Z}$ which differ only by translation via Λ. Then in this case Proposition 3.1.15 yields a convenient means of enumerating the number of faces of each dimension in the polyhedral complex $\overline{\mathcal{Z}}$. By Corollary 3.1.16, identifying the cells of the boundary complex of Z'_+ according to the identifications defining $\overline{\mathcal{Z}}$ yields a combinatorially equivalent cell decomposition of the same torus. Thus in particular we see that the number of faces of dimension k in the complex $\overline{\mathcal{Z}}$ is equal to the number of faces of dimension k in the complex Z'_+ up to these identifications.

Since Z' is a zonotope, we see also that the boundary complex of Z' is anti-isomorphic to the poset of the fan $\mathcal{N}(Z')$, which is itself isomorphic to the face poset of the hyperplane arrangement $\mathcal{A}'$ dual to Z'. Considering the construction of Z' in the proof of Proposition 3.1.15, we have that $\mathcal{A}'$ contains a hyperplane $H_0 := \{x_{d+1} = 0\}$. Moreover, it is immediate that the face poset of the complex Z'_+ is anti-isomorphic to the poset $\mathcal{A}'_+$ of faces of $\mathcal{N}(Z')$ which lie on the positive side of H_0.

Without loss of generality, we may assume that Z' is centered at the origin and may embed $\mathcal{A}'$ in the same space as a central arrangement at the origin. Doing so ensures that the hyperplanes in $\mathcal{A}'$ intersect the faces of Z' and that these intersections define an arrangement $\mathcal{T}$ of subtori in the polyhedral subdivision of T defined by identifying faces of Z'_+. We may therefore enumerate the number of faces of dimension k in the cell complex defined by $\mathcal{T}$ in order to enumerate the faces of dimension $d-k$ in $\mathcal{Z}$. This is formalized in the following proposition.

Proposition 4.2.2. *Let C, Z, $\mathcal{Z}$, and Z' be as in Proposition 3.1.15. Suppose also that Z tiles $\mathbb{R}^d$ via translation by a lattice Λ and let $T := \mathbb{R}^d/\Lambda$ be the resulting torus and $\overline{\mathcal{Z}}$ the subdivision of T induced by $\mathcal{Z}$.*

There is an arrangement $\mathcal{T}$ of subtori in T such that the number $f_k(\mathcal{T})$ of faces of dimension k in the cell complex defined by T is equal to the number $f_{d-k}(\overline{\mathcal{Z}})$ of faces of dimension $d-k$ in $\overline{\mathcal{Z}}$:

$$f_k(\mathcal{T}) = f_{d-k}(\overline{\mathcal{Z}}).$$

The expression for these face numbers is given in terms of the Möbius function of the intersection poset $\mathcal{I}(\mathcal{T})$, as stated in the following theorem.

Theorem 4.2.3. *Suppose we are in the situation of Proposition 4.2.2. Then we have*

$$f_k(\overline{\mathcal{Z}}) = (-1)^{d-k} \sum_{\substack{R,S\in\mathcal{L}(\mathcal{T}) \\ \dim S=0 \\ \dim F=d-k}} \mu(F,G),$$

where $\mathcal{I}(\mathcal{T})$ is the intersection poset of the arrangement $\mathcal{T}$ and $\mu(\cdot,\cdot)$ is the associated Möbius function.

Proof. We note first that by Proposition 4.2.2, it is sufficient to compute the number $f_{d-k}(\mathcal{T})$. Denote by $\mathcal{L}(\mathcal{T})$ the intersection poset of $\mathcal{T}$. Observe that each ℓ-dimensional face in $\mathcal{I}(\mathcal{T})$

is incident to a unique ℓ-dimensional subtorus S in $\mathcal{L}(\mathcal{T})$. Thus we may write

$$f_\ell(\mathcal{T}) = \sum_{\substack{S\in\mathcal{L}(\mathcal{T})\\ \dim S=\ell}} c(\mathcal{T}_S),$$

where $\mathcal{T}_S$ is the arrangement of subtori in S defined by intersections of S with subtori in $\mathcal{T}$. Note moreover that this formula holds as well for all induced arrangements on subtori $S \in \mathcal{L}(\mathcal{T})$ of T, so that we similarly obtain a formula for each of the face numbers $f_\ell(\mathcal{T}_S)$ of each such arrangement $\mathcal{T}_S$.

Now, noting that the arrangement $\mathcal{T}_S$ induces a cell complex structure on the subtorus S, we may compute the topological Euler characteristic $\chi(S)$ as the alternating sum of face numbers $f_j(\mathcal{T}_S)$.

$$\chi(S) = \sum_{j=0}^{d}(-1)^j f_j(\mathcal{T}_S) = \sum_{R\in\mathcal{L}(\mathcal{T}_S)} (-1)^{\dim R} c(\mathcal{T}_R) = \sum_{\substack{R\in\mathcal{L}(\mathcal{T})\\ R\geq S}} (-1)^{\dim R} c(\mathcal{T}_R).$$

Now, applying the Möbius inversion formula to the equality of the leftmost and rightmost expressions above, we obtain a formula for the number of chambers of any such arrangement $\mathcal{T}_S$.

$$c(\mathcal{T}_S) = (-1)^{\dim S} \sum_{\substack{R\in\mathcal{L}(\mathcal{T})\\ R\geq S}} \chi(R)\mu(S,R).$$

By combining the formulas above for $f_\ell(\mathcal{T})$ and $c(\mathcal{T}_S)$ above and recalling that the Euler characteristic of a torus is 0 for positive-dimensional tori and 1 for tori of dimension zero, we thus obtain the following:

$$\begin{aligned} f_\ell(\mathcal{T}) &= (-1)^\ell \sum_{\substack{S\in\mathcal{L}(\mathcal{T})\\ \dim S=\ell}} \sum_{\substack{R\in\mathcal{L}(\mathcal{T})\\ R\geq S}} \chi(R)\mu(S,R) \\ &= (-1)^\ell \sum_{\substack{S\in\mathcal{L}(\mathcal{T})\\ \dim S=\ell}} \sum_{\substack{R\in\mathcal{L}(\mathcal{T})\\ \dim R=0\\ R\geq S}} \mu(S,R). \end{aligned}$$

Finally, appealing to Proposition 4.2.2 and setting $\ell = d - k$, we have

$$f_k(\overline{\mathcal{Z}}) = (-1)^{d-k} \sum_{\substack{R,S \in \mathcal{L}(\mathcal{T}) \\ \dim R = 0 \\ \dim S = d-k}} \mu(S, R).$$

□

Chapter 5

Compactified Jacobians

In this chapter we give an overview of the existing theory of compactified Jacobians, establishing the basic theory of algebraic curves, their Jacobians, and the combinatorial input required to define compactifications of Jacobians of nodal curves. We begin by reviewing the smooth case, before surveying the approach by Oda and Seshadri to the problem of constructing and describing compactifications.

Throughout this section, we work over an algebraically closed field $\mathbf{k}$, and will specify to the case $\mathbf{k} = \mathbb{C}$ when appropriate. We assume as background the basics of algebraic geometry, but review those definitions and constructions which will be most relevant to us.

Loosely, the Jacobian of a curve X is a space which parametrizes the collection of line bundles on X of degree 0. This theory arose in the context of complex geometry, but nowadays is most often approached through the lens of algebraic geometry. For smooth curves, the Jacobian J_X contains the same information as the curve X, and indeed one can recover X through analysis of J_X.

5.1. Jacobians of Curves

Let X be a smooth curve over $\mathbf{k}$, by which we mean a smooth, integral, reduced, and separated scheme over $\mathbf{k}$ of dimension 1. Note in particular that we do not assume that X is irreducible.

5.2. Nodal Curves and Dual Graphs

Let X be a nodal curve having irreducible components $X_1, \ldots, X_n$ and nodes $x_1, \ldots, x_m$. Since all singularities of X are of one type (i.e. nodal) and since these nodes are distinguished from one another primarily by the pair of irreducible components to which they are incident, the information of how to assemble X from the components X_i is fundamentally combinatorial. In particular, the incidence structure of the nodes and components of X can be cleanly encoded by a graph as follows.

Definition 5.2.1. For a nodal curve X having irreducible components $X_1, \ldots, X_n$ and nodes $x_1, \ldots, x_m$, the **dual graph** of X is the (combinatorial) graph G_X on vertices $v_1, \ldots, v_n$ with edges $e_1, \ldots, e_m$, constructed as follows. For each irreducible component X_i, G_X has one vertex v_i. We connect two vertices v_i and v_j by an edge e_k precisely when $x_k \in X_i \cap X_j$.

Note that G_X may have loops and parallel edges in general. ♦

Remark 5.2.2. The assignment of a dual graph to a nodal curve is far from injective, even failing to be injective on isomorphism classes. For example, note that the graph G_X does not encode any information about the genus of the irreducible components of X. In particular, all smooth curves have the same dual graph, namely a single vertex with no edges. ♦

Despite the fact that the dual graph G_X is a rather coarse invariant of a nodal curve, it nonetheless encodes a wealth of information about the curve X and is generally useful in understanding and classifying nodal curves. For example, the boundary of the Deligne-Mumford compactification of the moduli space of curves of genus g admits a stratification by dual graphs.

5.3. Oda-Seshadri Compactified Jacobians

The work [44] of Oda and Seshadri is among the first to approach the problem of compactifying Jacobians of nodal curves X, and takes an approach through Geometric Invariant

Theory which relies heavily on the dual graph $G = G_X$ and on the geometry of Euclidean spaces associated with G as described in Appendix A. Moreover, the constructions we survey below rely heavily on the notion of Voronoi and Delone subdivisions of Euclidean spaces with respect to lattices within them. As such, the reader may find it useful to refer back to Subsection 3.3 for these definitions. For the time being, we assume this material as given and present those definitions and results from [44] which are of interest to us in the present document.

5.3.1 Construction and Properties of Namikawa Subdivisions

Consider the space $C_1(G;\mathbb{R})$ of (simplicial) 1-chains associated with G. This space admits an orthogonal decomposition $C_1(G);\mathbb{R}) \simeq H_1(G;\mathbb{R}) \oplus \delta C_0(G;\mathbb{R})$ with summands the cycle space of G and the cut space of G, respectively. Fix an isomorphism $H_1(G;\mathbb{R}) \simeq H_1(G;\mathbb{R})^* =: H^1(G;\mathbb{R})$ and denote the composition of this isomorphism with the orthogonal projection from $C_1(G;\mathbb{R})$ by $\rho : C_1(G;\mathbb{R}) \to H^1(G;\mathbb{R})$.

Fix some parameter $\psi \in \delta C_0(G;\mathbb{R})$ and consider the affine subspace $\psi + H_1(G;\mathbb{R})$ in $C_1(G;\mathbb{R})$. Since $\mathrm{Vor}(G) := \mathrm{Vor}(C_1(G;\mathbb{R}), C_1(G;\mathbb{Z}))$ is a subdivision of $C_1(G;\mathbb{R})$, this affine space meets the relative interiors of some cells of the Voronoi subdivision. We define this collection of cells by

$$K^\circ_\varphi(G) := \{D \in \mathrm{Vor}(G) : \varphi \in \mathrm{relint}\,\partial D\}.$$

Note here that the defining condition $\varphi \in \mathrm{relint}\,\partial D$ is equivalent to $\mathrm{relint}(D) \cap \psi + H_1(G;\mathbb{R})$ being nonempty, where $\partial\psi = \varphi$. This equivalence holds because ∂ induces an isomorphism $\delta C_0(G;\mathbb{R}) \simeq \partial C_1(G;\mathbb{R})$. Projecting the cells of $K^\circ_\varphi(G)$ to $H^1(G;\mathbb{R})$ via ρ then yields a polyhedral subdivison for each ψ.

Definition 5.3.1. We refer to each $\psi \in \delta C_0(G;\mathbb{R})$ as a **polarization**. In view of the isomorphism $\partial : \delta C_0(G;\mathbb{R}) \to \partial C_1(G;\mathbb{R})$, we also refer to $\varphi = \partial\psi$ as a polarization, since the information is equivalent.

Given a polarization $\psi \in \delta C_0(G;\mathbb{R})$, write $\varphi = \partial\psi$. The **Namikawa subdivision** associated with φ (equivalently, with ψ) is the collection of cells $\mathrm{Del}_\varphi(G) = \{\rho(C) : C^* \in K^\circ_\psi(G)\}$. ♦

We begin by listing some basic geometric properties of Namikawa subdivisions $\mathrm{Del}_\varphi(G)$, including the fact that it is actually a subdivision of $H^1(G;\mathbb{R})$.

Proposition 5.3.2 (Proposition 6.1, [44])**.** *Let $\varphi \in \partial C_1(G;\mathbb{R})$. The Namikawa subdivision $Del_\varphi(G)$ satisfies the following properties.*

(1) $Del_\varphi(G)$ is a subdivision of $H^1(G;\mathbb{R})$ into zonotopes.

(2) The vertices of $Del_\varphi(G)$ are contained in $\rho C_1(G;\mathbb{Z})$.

(3) $Del_\varphi(G)$ is invariant under translation by $\rho H_1(G;\mathbb{Z})$, and the quotient complex is finite.

Definition 5.3.3. In view of the finiteness guaranteed by part (3) of Proposition 5.3.2, we define the **Namikawa complex** associated to $\varphi \in \partial C_1(G;\mathbb{R})$ to be the cell complex given as the quotient $\overline{\mathrm{Del}}_\varphi(G) := \mathrm{Del}_\varphi(G)/\rho H_1(G;\mathbb{Z})$.

For a cell $\overline{N} \in \overline{Del}_\varphi(G)$, let $\dim(\overline{N})$, $\mathrm{supp}(\overline{N})$, and $\partial b(\overline{N})$ to be, respectively, $\dim(D)$, $\mathrm{supp}(D)$, and $\partial b(D)$ for any $D \in K^\circ_\varphi(G)$ with $\rho(D) \in \overline{N}$. ♦

We moreover have a clean description of the structure of those Namikawa subdivisions $\mathrm{Del}_\varphi(G)$ for those $\varphi \in \frac{1}{2}\mathbf{1} + \partial C_1(G;\mathbb{Z})$. They are simply translates (by a vector depending on φ) of a Voronoi subdivision. This is codified by the following proposition, whose proof we postpone to Section 5.3.3 (cf. Proposition 5.3.19).

Proposition 5.3.4 ([44], Corollary 6.2)**.** *Suppose that $\varphi \in \partial C_1(G;\mathbb{R})$ satisfies $\varphi \in \frac{1}{2}\partial\mathbf{1} + \partial C_1(G;\mathbb{Z})$. Then the associated Namikawa subdivision $Del_\varphi(G)$ is a translate of the Voronoi* *subdivision* $Vor(H^1(G;\mathbb{R}), \rho H_1(G;\mathbb{Z}))$. *In particular, for* $\xi \in C_1(G;\mathbb{Z})$ *and* $\varphi = \frac{1}{2}\partial\mathbf{1} + \partial\xi$, *we have*

$$Del_\varphi(G) = \rho(\tfrac{1}{2}\mathbf{1} + \xi) + Vor(H^1(G;\mathbb{R}), \rho H_1(G;\mathbb{Z})).$$

In this situation, we denote by $\mathrm{Vor}_\varphi(0)$ the translate by $\rho(\frac{1}{2}\mathbf{1}+\xi)$ of the Voronoi cell $V(0) \in \mathrm{Vor}(H^1(G;\mathbb{R}), \rho H_1(G;\mathbb{Z}))$. Note that, even though there may be several $\xi \in C_1(G;\mathbb{Z})$ with $\varphi = \frac{1}{2}\partial\mathbf{1} + \partial\xi$, the cell $\mathrm{Vor}_\varphi(0)$ is dependent only on φ and not on the choice of preimage ξ.

Remark 5.3.5. Since Voronoi subdivisions are regular (cf. Section 3.3), this immediately implies that for all $\varphi \in \frac{1}{2}\partial\mathbf{1}+\partial C_1(G;\mathbb{Z})$ we have that the Namikawa subdivision is regular. In fact, since the trivial subdivision of a polytope is coherent with respect to any projection onto that polytope, we have that the restriction of $\mathrm{Del}_\varphi(G)$ to $\mathrm{Vor}_\varphi(0)$ is coherent with respect to the projection $\rho : C \to \mathrm{Vor}_\varphi(0)$, where C is the Voronoi cell of 0 in $\mathrm{Vor}(C_1(G;\mathbb{R}), C_1(G;\mathbb{Z}))$.

In Theorem 5.5.1, we extend this coherence to those Namikawa subdivisions for which φ arises from an orientable polarization (cf. Definition 5.3.22). ♦

In [44], the authors construct compactifications $J_X(\varphi)$ of the Jacobian J_X whose structure can be described in terms of the combinatorics of the Namikawa subdivision $\mathrm{Del}_\varphi(G)$. Some preliminary properties of these compactifications and their relationship with $\mathrm{Del}_\varphi(G)$ and $\overline{\mathrm{Del}}_\varphi(G)$ are as follows.

Proposition 5.3.6 ([44], Proposition 12.17)**.** *The compactified Jacobian* $\overline{J}_X(\varphi)$ *is fully determined by the Namikawa complex* $\overline{Del}_\varphi(G)$. *Moreover, there is a bijection between the* $Pic^0(X)$*-orbits of the scheme* $\overline{J}_X(\varphi)$ *and the cells of the Namikawa complex* $\overline{Del}_\varphi(G)$.

For a given cell $\overline{N} \in \overline{Del}_\varphi(X)$, *let* $O(\overline{N})$ *denote the corresponding* $Pic^0(X)$ *orbit. Then* $dim\overline{N} = codim(O(\overline{N}))$. *Moreover, for two cells* $\overline{N}_1, \overline{N}_2 \in \overline{Del}_\varphi(X)$, *we have a face relation* $\overline{N}_1 \leq \overline{N}_2$ *if and only if* $O(\overline{N}_2)$ *is contained in the Zariski closure of* $O(\overline{N}_2)$ *in* $\overline{J}_X(\varphi)$. *Finally, we have a canonical isomorphism*

$$O(\overline{N}) \simeq Pic^n(X([m] \setminus supp(\overline{N}))),$$

where $n \in C_0(G;\mathbb{Z})$ is given by $n = \partial b(\overline{N}) - \frac{1}{2}d(supp(\overline{N}))$.

In particular, the above isomorphism implies that $dim(\overline{N}) = codim(O(\overline{N}))$.

Remark 5.3.7. The reader should be aware that the definition $\dim(\overline{N})$ of a cell in the Namikawa complex is distinct from its topological dimension. Instead, it is defined to be the dimension of its preimage $D \in K^{\circ}_{\varphi}(G)$. However, for *nondegenerate* polarizations φ (see Definition 5.3.14 and Proposition 5.3.15 below) we have that $\dim(\overline{N})$ is the same as the expected dimension $\dim(\rho(D))$. This observation will be of use to us in particular in the proof of Theorem 5.6.4 and Corollary 5.6.5. ♦

Remark 5.3.8. Recall from Section 5.1 that for a nodal curve Y we have that the Picard scheme Pic_Y admits connected components for each choice of multidegree $n \in C_0(G_Y;\mathbb{Z})$, and that moreover each such component Pic^n_Y is a torsor for the group scheme Pic^0_Y. This implies in particular that Pic^n_Y is (non-canonically) isomorphic to Pic^0_Y, so that their classes in the Grothendieck ring coincide. ♦

Even beyond this characterization of the orbit decomposition of the compactified Jacobians $\overline{J}_X(\varphi)$, Oda and Seshadri also give the following description of the Picard schemes of nodal curves and their subcurves.

Proposition 5.3.9 ([44], Proposition 12.3)**.** *Let X be a nodal curve, let $E' \subseteq E(G_X)$ be a subset of edges of the dual graph of X, and let $X(E')$ be the associated partial normalization of X. Then for any multidegree $n \in C_0(G;\mathbb{Z})$, $Pic^n_{X(E')}$ admits the structure of a $\mathbf{G}_m \otimes H^1(G(E');\mathbb{Z})$-bundle over $Pic^n_{\tilde{X}}$.*

5.3.2 Parameterizing Namikawa Subdivisions

As is clear from the construction and notation of Namikawa subdivisions, the structure of the subdivisions $\mathrm{Del}_{\varphi}(G)$ is sensitive to the choice of polarization $\psi = \partial^{-1}\varphi$. Indeed, varying ψ varies the collection of cells $K^{\circ}_{\psi}(G)$ and therefore affects both the Namikawa subdivision and the Namikawa complex, hence the compactification $J_X(\varphi)$. Fortunately, the authors of [44] give several tools with which to understand the dependence of $\mathrm{Del}_\varphi(G)$ on ψ. The main among these tools is a certain (infinite) hyperplane arrangement in $\partial C_1(G;\mathbb{R})$ whose induced subdivision serves as a parameter space for Namikawa subdivisions. In this subsection we describe this hyperplane arrangement and some tools for its analysis.

Recall (cf. Appendix A) that for each spanning forest T of G and edge $e \in T$ that there are distinguished elements $\omega_{T,e} \in \partial C_1(G;\mathbb{Z})$. Recall also that we have an inner product $\langle\cdot,\cdot\rangle$ on the ambient space $C_0(G;\mathbb{R})$.

Definition 5.3.10 ([44], pp. 16 and pp. 33)**.** Define $\mathrm{Arr}(G)$ to be the (infinite) hyperplane arrangement in $\partial C_1(G;\mathbb{R})$ having hyperplanes

$$H_{\omega,k} := \{\varphi \in \partial C_1(G;\mathbb{R}) : \langle\omega^+,\varphi\rangle = k\},$$

where ω is a fundamental cut of G and $k \in \mathbb{Z}$. Recall that the superscript $(\cdot)^+$ refers to the (linear combination in $C_0(G;\mathbb{Z})$ corresponding to) the collection of target vertices of the oriented cut ω.

For each spanning forest T of G, denote by $\mathrm{Arr}_T(G)$ the subarrangement of $\mathrm{Arr}(G)$ consisting of only those hyperplanes $H_{\omega,k}$ for which ω is a fundamental cut of T.

From here onward, we use $\mathcal{A}(G)$ and $\mathcal{A}_T(G)$ to denote the subdivisions of $\partial C_1(G;\mathbb{R})$ induced by $\mathrm{Arr}(G)$ and $\mathrm{Arr}_T(G)$, respectively. ♦

From this definition we immediately see that $\mathrm{Arr}(G)$ is given as the union of the arrangements $\mathrm{Arr}_T(G)$. The associated subdivisions satisfy

$$\mathcal{A}(G) = \bigwedge_{T\in\mathcal{T}(G)} \mathcal{A}_T(G).$$

These arrangements are particularly important from our perspective since, as mentioned above, they parameterize the Namikawa subdivisions $\mathrm{Del}_\varphi(G)$.

Remark 5.3.11. Note that for each $T \in \mathcal{T}(G)$ we have that $\partial C_1(G;\mathbb{Z}) \subseteq \mathrm{Sk}_0(\mathcal{A}_T(G))$. This immediately follows from the observation that for each fundamental cut ω is a $\mathbb{Z}$-linear combination of basis vectors $e \in C_1(G;\mathbb{Z})$. In particular, we have that

$$\langle \omega^+, \partial e \rangle = \langle \omega, e \rangle \in \mathbb{Z}.$$

♦

Proposition 5.3.12 (Theorem 7.1, Corollary 7.2, [44])**.** *With the notation above, we have the following.*

(1) For each cell $A \in \mathcal{A}(G)$, the set $K^\circ_\varphi(G)$ (and hence the Namikawa subdivision $Del_\varphi(G)$) is invariant as φ varies within $\frac{1}{2}\partial\mathbf{1} + \mathrm{relint}(A)$.

(2) For $\xi \in C_1(G;\mathbb{Z})$ and $\varphi \in \partial C_1(G;\mathbb{R})$, we have equalities

$$K^\circ_{\varphi+\partial\xi}(G) = K^\circ_\varphi(G) + \xi$$
$$Del_{\varphi+\partial\xi}(G) = Del_\varphi(G) + \rho(\xi)$$

(3) If $A' \leq A$ are cells of (G), $\varphi \in \frac{1}{2}\partial\mathbf{1} + \mathrm{relint}\, A$, and $\varphi' \in \frac{1}{2}\partial\mathbf{1} + \mathrm{relint}\, A'$, then the Namikawa subdivision Del_φ is a refinement of the subdivision $Del_{\varphi'}(G)$.

Observe that we trivially have $\partial\delta C_0(G;\mathbb{Z}) \subseteq \mathrm{Sk}_0\mathcal{A}(G)$, since for any $\psi \in \delta C_0(G;\mathbb{Z})$ we see that

$$\langle \omega, \psi \rangle = \langle \delta\omega^+, \psi \rangle = \langle \omega^+, \partial\psi \rangle \in \mathbb{Z},$$

so that any such $\partial\psi$ lies on some hyperplane from each family $\{H_{\omega,k} : k \in \mathbb{Z}\}$. Further, part (3) of Proposition 5.3.2 and part (2) of Proposition 5.3.12 together imply that translating $\varphi \in \partial C_1(G;\mathbb{R})$ by any $\partial\xi$ for $\xi \in C_1(G;\mathbb{Z})$ leaves the associated Namikawa subdivision $\mathrm{Del}_\varphi(G)$ unchanged. That is, the subdivision $\mathcal{A}(G)$ is invariant under translation by $\partial\delta C_0(G;\mathbb{Z})$ and translating any $\varphi \in \partial C_1(G;\mathbb{R})$ by $\partial\delta C_0(G;\mathbb{Z})$ results in the same Namikawa subdivision.

In particular, the quotient cell complex $\mathcal{A}(G)/\partial\delta C_0(G;\mathbb{R})$ serves as a parameter space for Namikawa subdivisions, in the sense that the Namikawa subdivision $\mathrm{Del}_\varphi(G)$ is dependent only on the cell of $\mathcal{A}(G)/\partial\delta C_0(G;\mathbb{R})$ containing the image of φ in its relative interior. Consider now the assignment

$$\mathcal{A}(G)/\partial\delta C_0(G;\mathbb{Z}) \ni \overline{A} \mapsto \mathrm{Del}_\varphi(G),$$

where $A \in \mathcal{A}(G)$ is such that $\varphi \in \frac{1}{2}\partial\mathbf{1}+\mathrm{relint}\, A$ and $\overline{A}$ is the class of A in the quotient. Part (3) of Proposition 5.3.12 further illustrates that this assignment is an order-reversing map from the poset of cells of $\mathrm{Arr}(G)/\partial\delta C_0(G;\mathbb{Z})$ (under inclusion) and the poset of subdivisions of $H^1(G;\mathbb{R})$ (under refinement). It immediately follows that this assignment is an anti-isomorphism from the former poset to the subposet of subdivisions consisting only of the Namikawa subdivisions. These observations establish the following result.

Proposition 5.3.13 ([44], Corollary 7.3)**.** *There are only a finite number of Namikawa decompositions $\mathrm{Del}_\varphi(G)$, the number being bounded by the cardinality of $\mathcal{A}(G)/\partial\delta C_0(G;\mathbb{Z})$.*

It is clear from the discussion above that the location of the parameter φ within the space $\partial C_1(G;\mathbb{R})$ is of critical importance in determining the structure of the associated Namikawa decomposition $\mathrm{Del}_\varphi(G)$. As a means of taking advantage of the additional insight provided by knowing the the precise location of φ, we introduce the following definition for use in the sequel.

Definition 5.3.14 ([44], Discussion preceding and including Proposition 7.6)**.** Let $\varphi \in \partial C_1(G;\mathbb{R})$. Say that φ is **nondegenerate** if there is some full-dimensional $A \in \mathcal{A}(G)$ with $\varphi \in \mathrm{relint}(A)$. In other words, φ is nondegenerate precisely when it lies on none of the hyperplanes in $\mathrm{Arr}(G)$. ♦

Oda and Seshadri provide the following equivalent condition to nondegeneracy of a polarization.

Proposition 5.3.15 ([44], Proposition 6.6)**.** *A polarization $\varphi \in \partial C - 1(G;\mathbb{R})$ is nondegenerate if and only if each $D \in K^\circ_\varphi(G)$ satisfies* $\dim D = \dim \rho D$.

Proof. On pp. 34 of [44], a polarization φ is defined to be nondegenerate precisely if each $D \in K^\circ_\varphi(G)$ is φ-stable (a definition not used in the present document). Earlier, Proposition 6.6 of [44] establishes that any $D \in K^\circ_\varphi(G)$ is φ-stable precisely when it satisfies $\dim D = \dim \rho D$. □

5.3.3 The Structure of Namikawa Subdivisions

In pursuit of a more explicit description of the cells of the Namikawa subdivisions $\mathrm{Del}_\varphi(G)$, we investigate more closely the structure of the subdivisions $\mathcal{A}(G)$ and $\mathcal{A}_T(G)$ of $\partial C_1(G;\mathbb{R})$ induced by $\mathrm{Arr}(G)$ and $\mathrm{Arr}_T(G)$.

Proposition 5.3.16. *For each spanning tree T of G, we have the following alternative description of the subdivision $\mathcal{A}_T(G)$.*

$$\mathcal{A}_T(G) = \{\partial D : D \in Del(C_1(G;\mathbb{R}), C_1(G;\mathbb{Z})),\ D^0 \subseteq T\}.$$

Proof. Let $D \in \mathrm{Del}(C_1(G;\mathbb{R}), C_1(G;\mathbb{Z}))$ be a Delone cell satisfying $D^0 \subseteq T$. Then we can write

$$D = \xi + \sum_{t\in D^0} [0, e_t]$$
$$\partial D = \partial\xi + \sum_{t\in D^0} [0, \partial e_t].$$

Then for $x \in \operatorname{relint} \partial D$, we have scalars $0 < \lambda_t < 1$ for which

$$x = \partial\xi + \sum_{t\in D^0} \lambda_t \partial e_t.$$

Let ω be a fundamental cut of T, say $\omega = \omega_{T,e}$. Then ω^+ defines a family of hyperplanes in

$\mathrm{Arr}_T(G)$. We have

$$\langle x, \omega^+\rangle = \langle \partial\xi, \omega^+\rangle + \sum_{t\in D^0} \lambda_t \langle \partial e_t, \omega^+\rangle.$$

Recalling that

$$\langle \partial e_t, \omega^+\rangle = \begin{cases} 0 & e_t \notin \omega \\ \pm 1 & e_t \in \omega \end{cases},$$

where the sign is dependent on whether the orientation of e agrees with that of ω. In particular, if $\langle \partial e_t, \omega^+\rangle \neq 0$ then we have $e_t \in D^0 \cap \omega$. However, since $D^0 \subseteq T$ and $\omega \cap T = \{e\}$, this means that $e_t = e$. Thus if $x \in \operatorname{relint} \partial D$, then we have $\langle x, \omega^+\rangle \notin \mathbb{Z}$ for all fundamental cuts ω of T with $D^0 \cap \omega$ nonempty and $\langle x, \omega^+\rangle \in \mathbb{Z}$ otherwise. In particular, ∂D is some cell of $\mathcal{A}_T(G)$. Moreover, by varying $D^0 \subseteq T$, we see that all cells of $\mathcal{A}_T(G)$ arise in this way. $\square$

Since $\mathcal{A}_T(G)$ is a subdivision of $\partial C_1(G;\mathbb{R})$, the relative interiors of cells of $\mathcal{A}_T(G)$ form a partition of $\partial C_1(G;\mathbb{R})$. In particular, this means that for each $\varphi \in \partial C_1(G;\mathbb{R})$ there is a unique cell $A_T(\varphi) \in \mathcal{A}_T(G)$ with $\varphi - \frac{1}{2}\partial\mathbf{1} \in \operatorname{relint} A_T(\varphi)$. We denote the collection of all such cells $A_T(\varphi)$ for $T \in \mathcal{T}(G)$ by

$$\mathcal{S}_\varphi = \{A_T(\varphi) : T \in \mathcal{T}(G)\}.$$

For any such φ, we also make a choice of preimage $D_T(\varphi) \in \mathrm{Del}(C_1(G;\mathbb{R}), C_1(G;\mathbb{Z}))$. In particular, for such φ we take $D_T(\varphi)$ to be a cell of minimal dimension satisfying $\partial D_T(\varphi) = A_T(\varphi)$. We write $D_T^*(\varphi)$ for the duals $(D_T(\varphi)^*)$ of these cells. For each $T \in \mathcal{T}(G)$, the choice of such the cell $D_T(\varphi)$ is well-defined up to translation by elements of $H_1(G;\mathbb{Z})$. Moreover, the next two results establish that the choice of these cells $D_T(\varphi)$ also specifies the collection of maximal cells in the associated Namikawa decomposition $\mathrm{Del}_\varphi(G)$.

Proposition 5.3.17. *For each $\varphi \in \partial C_1(G;\mathbb{R})$, the collection of cells*

$$\mathcal{D}(\varphi) = \{D_T(\varphi) \ : \ T \in \mathcal{T}(G)\}$$

and the collection of their translates under $H_1(G;\mathbb{Z})$ *comprise all minimal cells* $D \in$ *Del*$(C_1(G;\mathbb{R}), C_1(G;\mathbb{Z}))$ *satisfying* $\varphi \in \frac{1}{2}\partial\mathbf{1} + \operatorname{relint}(\partial D)$.

Proof. Suppose that φ and that we have minimal cells $D_T \in \operatorname{Del}(G)$ as above for each $T \in \mathcal{T}(G)$. Let D be a minimal cell of $\operatorname{Del}(C_1(G;\mathbb{R}), C_1(G;\mathbb{Z}))$ such that $\varphi \in \frac{1}{2}\partial\mathbf{1} + \operatorname{relint}(\partial D)$. If $D^0 \subseteq T'$ for some spanning forest T' of G, then $\partial D \in \mathcal{A}_{T'}(G)$. Observing that $\{\operatorname{relint}(P) : P \in \mathcal{A}_{T'}(G)\}$ is a partition of $\partial C_1(\mathbb{R})$ which is periodic with respect to translation by $\partial C_1(\mathbb{Z})$, it follows that the preimages in $\operatorname{Del}(C_1(G;\mathbb{R}), C_1(G;\mathbb{Z}))$ of the cells $P \in \mathcal{A}_{T'}(G)$ are periodic with respect to $C_1(\mathbb{Z})$. In particular, D is necessarily a $C_1(\mathbb{Z})$-translate of the cell $D_{T'}$. We thus assume that D^0 is not contained in any spanning forest T' of G.

We may write $D = \xi + \sum_{t \in D^0} [0, e_t]$ for some $\xi \in C_1(\mathbb{Z})$. We will construct a proper face $\tilde{D} < D$ and illustrate that $\varphi \in \frac{1}{2}\partial\mathbf{1} + \operatorname{relint}(\partial\tilde{D})$, contradicting the minimality of D. Let H be the graph with the same vertex set as G and with edge set D^0.

Since by assumption the set D^0 is not contained in any spanning forest of G, it must contain at least one cycle. Let $\gamma_1, \ldots, \gamma_r \subseteq D^0$ be the collection of fundamental cycles associated to some spanning forest S of H. We know by $\varphi \in \frac{1}{2}\partial\mathbf{1} + \operatorname{relint}(\partial D)$ that we may write

$$\varphi - \tfrac{1}{2}\partial\mathbf{1} = \partial\left(\xi + \sum_{t \text{ bridge}} \alpha_t e_t + \sum_{t' \text{ non-bridge}} \beta_{t'}^{(0)} e_{t'}\right).$$

where $\xi \in C_1(\mathbb{Z})$ and all coefficients appearing on edge summands are in the open interval $(0, 1)$. Here, the first sum ranges over edges t of H which are bridges (i.e. appear in no fundamental cycle of S) and the second ranges over all edges t' which appear in at least one fundamental cycle. Define $\beta_1 := \min\{\beta_{t'}^{(0)} : t' \in \gamma_1\}$. Then $\beta_1\gamma_1 \in \ker\partial$ and so we may write

$$
\begin{aligned}
\varphi - \tfrac{1}{2}\partial \mathbf{1} &= \partial\left(\xi + \sum_{t \text{ bridge}} \alpha_t e_t + \sum_{t' \text{ non-bridge}} \beta_{t'}^{(0)} e_{t'}\right) - \partial \beta_1 \gamma_1 \\
&= \partial\left(\xi + \sum_{t \text{ bridge}} \alpha_t e_t + \sum_{t' \text{ non-bridge}} \beta_{t'}^{(1)} e_{t'}\right)
\end{aligned}
$$

where the new coefficients in the second sum are given by

$$
\beta_{t'}^{(1)} = \begin{cases} \beta_{t'}^{(0)} & t' \notin \gamma_1 \\ \beta_{t'}^{(0)} - \beta_i & t' \in \gamma_1 \end{cases}.
$$

In particular, in the new expression for $\varphi = \frac{1}{2}\partial \mathbf{1}$ we have that one edge of γ_1 appears with coefficient zero, so that the collection of edges appearing with positive coefficients contains one fewer fundamental cycle. Performing this process iteratively, we define at the ith iteration $\beta_i := \min\{\beta_{t'}^{(i-1)} : t' \in \gamma_i\}$ and observe that subtracting $\partial \beta_i \gamma_i$ from the expression for $\varphi - \frac{1}{2}\partial \mathbf{1}$ yields a new expression

$$
\varphi - \tfrac{1}{2}\partial \mathbf{1} = \partial\left(\xi + \sum_{t \text{ bridge}} \alpha_t e_t + \sum_{t \text{ non-bridge}} \beta_{t'}^{(i)} e_{t'}\right),
$$

where the coefficients $\beta_{t'}^{i)}$ are defined analogously to in the first stage above. After performing this process r times, we may write

$$
\varphi - \tfrac{1}{2}\partial \mathbf{1} = \partial\left(\xi + \sum_{t \text{ bridge}} \alpha_t e_t + \sum_{t \text{ non-bridge}} \beta_{t'}^{(r)} e_{t'}\right).
$$

We now have by construction that the collection of edges appearing in the expression with positive coefficients is acyclic and therefore must be contained in some spanning forest of Δ. In particular, denoting by S' this subset, we have that S' is contained in some spanning forest T' of G. Defining $\tilde{D} := \xi + \sum_{t \in S'} [0, e_t]$, we therefore have that $\partial \tilde{D} \in \mathcal{A}_{T'}(G)$ and

$\varphi \in \frac{1}{2}\partial\mathbf{1} + \partial\tilde{D}$. Since $\tilde{D} < D$, this contradicts the minimality of D. $\square$

Proposition 5.3.18. *For each $\varphi \in \partial C_1(G;\mathbb{R})$, the collection of Delone cells $\{\frac{1}{2}\mathbf{1} + D_T^*(\varphi) : T \in \mathcal{T}(G)\}$ and their translates by elements of $H_1(G;\mathbb{Z})$ comprise the collection of maximal cells in $K_\varphi^\circ(G)$.*

In particular, the cells $\frac{1}{2}\rho\mathbf{1} + \rho D_T^*(\varphi)$ for $D_T(\varphi) \in \mathcal{D}(\varphi)$ and their translates by elements of $\rho H_1(G;\mathbb{Z})$ comprise all maximal cells of $Del_\varphi(G)$.

Proof. This can immediately be seen from the previous proposition by taking duals of the minimal Delone cells $D_T(\varphi)$ and translating by $\frac{1}{2}\mathbf{1}$ to recover Delone cells $\frac{1}{2}\mathbf{1} + D_T^*(\varphi)$ satisfying the desired properties. Indeed, since $\varphi \in \frac{1}{2}\partial\mathbf{1} + \mathrm{relint}(\partial D_T(\varphi))$, we have that $\varphi - \frac{1}{2}\partial\mathbf{1} \in \mathrm{relint}(\partial D_T(\varphi))$ and so the dual Voronoi cells D_T^* satisfy $\varphi - \frac{1}{2}\partial\mathbf{1} \in \mathrm{relint}(\partial(D_T^*(\varphi))^*)$. Hence $\varphi \in \mathrm{relint}\left(\partial(\frac{1}{2}\mathbf{1} + D_T^*)^*\right)$ and so $\frac{1}{2}\mathbf{1} + D_T^*(\varphi)$ is a Delone cell whose dual contains φ. This establishes that $\frac{1}{2}\mathbf{1} + D_T^*(\varphi)$ is a Delone cell in $K_\varphi^\circ(G)$ and maximality of this cell follows from the minimality of $D_T(\varphi)$. $\square$

With this in hand, we can now explicitly describe the Namikawa subdivisions corresponding to those $\varphi \in \partial C_1(G;\mathbb{R})$ with $\varphi - \frac{1}{2}\partial\mathbf{1} \in \partial C_1(G;\mathbb{Z})$.

Proposition 5.3.19 ([44], Corollary 6.2)**.** *Suppose that $\varphi \in \partial C_1(G;\mathbb{R})$ satisfies $\varphi - \frac{1}{2}\partial\mathbf{1} \in \partial C_1(G;\mathbb{Z})$. Then the associated Namikawa subdivision $Del_\varphi(G)$ is a translate of the Voronoi subdivision $Vor(H^1(G;\mathbb{R}), \rho H_1(G;\mathbb{Z}))$. In particular, for $\xi \in C_1(G;\mathbb{Z})$ and $\varphi = \frac{1}{2}\partial\mathbf{1} + \partial\xi$, we have*

$$Del_\varphi(G) = \rho(\tfrac{1}{2}\mathbf{1} + \xi) + Vor(H^1(G;\mathbb{R}), \rho H_1(G;\mathbb{Z})).$$

Proof. Given such a $\varphi \in \partial C_1(G;\mathbb{R})$, suppose that $\xi \in C_1(G;\mathbb{Z})$ is such that

$$\varphi - \tfrac{1}{2}\partial\mathbf{1} = \partial\xi \in \partial C_1(G;\mathbb{Z}).$$

We consider the collections $\mathcal{S}_\varphi$ and $\mathcal{D}(\varphi)$. Recall from Proposition 5.3.16 that

$$\mathcal{A}_T(G) = \{\partial D : D \in \mathrm{Del}(C_1(G;\mathbb{R}), C_1(G;\mathbb{Z})),\ D^0 \subseteq T\}.$$

This implies in particular that $\partial C_1(G;\mathbb{Z}) \subseteq \mathrm{Sk}_0\mathcal{A}_T(G)$, since for each $\xi' \in C_1(G;\mathbb{Z})$ we have that $\xi' \in \mathrm{Del}(C_1(G;\mathbb{R}), C_1(G;\mathbb{Z}))$ and $(\xi')^0 = \varnothing$. We therefore see that $A_T(\varphi) = \partial\xi$ for all $T \in \mathcal{T}(G)$ and so we may take $D_T(\varphi) = \xi$ for all T.

From here, Proposition 5.3.17 gives the collection of all maximal cells in $\mathrm{Del}_\varphi(G)$. Indeed, we now see that

$$\tfrac{1}{2}\mathbf{1} + D_T^*(\varphi) = \xi + \sum_{i=1}^{m}[0, e_i] =: C_\varphi.$$

Thus the collection of maximal cells of $\mathrm{Del}_\varphi(G)$ is the collection of all $\rho H_1(G;\mathbb{Z})$-translates of the cell

$$\rho C_\varphi = \rho\xi + \sum_{i=1}^{m}[0, \rho e_i] = \tfrac{1}{2}\rho\mathbf{1} + \rho\xi + \rho V(0) =: Z_\varphi.$$

That is, by the description of the Voronoi subdivision of $H^1(G;\mathbb{R})$ given in Appendix A, we have

$$\mathrm{Del}_\varphi(G) = \tfrac{1}{2}\rho\mathbf{1} + \rho\xi + \mathrm{Vor}(H^1(G;\mathbb{R}), \rho\, H_1(G;\mathbb{Z})).$$

□

Remark 5.3.20. For any $\varphi \in \partial C_1(G;\mathbb{R})$ satisfying $\varphi - \frac{1}{2}\partial\mathbf{1} = \partial\xi$ for some $\xi \in C_1(G;\mathbb{Z})$, we have introduced the following notation:

$$C_\varphi := \xi + \sum_{i=1}^{m}[0, e_i]$$
$$Z_\varphi := \tfrac{1}{2}\rho\mathbf{1} + \rho\xi + \rho V(0) = \rho C_\varphi.$$

In particular, $\rho : C_\varphi \to Z_\varphi$ is a projection of a cube onto a zonotope. Viewing Z_φ as having the trivial subdivision $\mathrm{Del}_\varphi(G)|_{Z_\varphi}$, this illustrates that the Voronoi subdivision of $H^1(G;\mathbb{R})$ is a periodic continuation of a coherent zonotopal subdivision of Z_φ. Below, in Theorem 5.5.1, we use this notation and illustrate that for any *orientable* choice of φ (cf. Definition 5.3.22) ♦

In addition to providing additional insight into Namikawa subdivisions $\mathrm{Del}_\varphi(G)$ for $\varphi - \frac{1}{2}\partial\mathbf{1} \in \partial C_1(G;(Z))$, the collections $\mathcal{D}_\varphi$ of cells from Proposition 5.3.17 also encode the relation between the cells of $\mathcal{A}(G)$ containing different polarizations φ.

Proposition 5.3.21. *Suppose that $\varphi_0 = \frac{1}{2}\partial\mathbf{1} + \partial\xi$ for some $\xi \in C_1(G;\mathbb{Z})$ and let $\varphi \in \partial C_1(G;\mathbb{R})$ be such that the cell $A \in \mathcal{A}(G)$ containing $\varphi - \frac{1}{2}\partial\mathbf{1}$ in its relative interior admits $\partial\xi$ as a vertex. Then for each spanning tree T of G, the cell $D_T(\varphi_0)$ can be taken to be a face of $D_T(\varphi)$.*

Proof. Fix some spanning tree T of G. Then by the proof of Proposition 5.3.19 we know that we can choose $A_T(\varphi_0) = \partial\xi$ and $D_T(\varphi_0) = \xi$. In particular, $A_T(\varphi_0) = \partial\xi$ is a vertex of $\mathcal{A}_T(G)$.

By the observation that $\mathcal{A}(G)$ is the common refinement of the subdivisions $\mathcal{A}_T(G)$, we have that $A = \bigcap_{T'\in\mathcal{T}(G)} A_{T'}(\varphi)$. Because of this, $\partial\xi \leq A$ guarantees that $\partial\xi \in A_T(G)$. Thus, since $\partial\xi$ is a vertex of $\mathcal{A}_T(G)$ and $A_T(\varphi) \in \mathcal{A}_T(G)$, it must be that $\partial\xi \leq A_T(\varphi)$. This implies in particular that we can choose a preimage $D_T(\varphi)$ of $A_T(\varphi)$ with $\xi \leq D_T(\varphi)$. □

From the results and proofs of Proposition 5.3.19 and Proposition 5.3.21, it is clear that the property of some φ_0 lying in $\frac{1}{2}\partial\mathbf{1} + \partial C_1(G;\mathbb{Z})$ is particularly important for our analysis, as is the property of other φ lying in cells $A \in \mathcal{A}(G)$ satisfying $\varphi_0 \leq A$. This importance motivates the following definition, whose terminology will become clear in Section 5.5.

Definition 5.3.22. Say that a polarization $\varphi \in \partial C_1(G;\mathbb{R})$ is **orientable** if the cell $A \in \mathcal{A}(G)$ with $\varphi \in \mathrm{relint}(A)$ has some vertex φ_0, where $\varphi_0 = \frac{1}{2}\partial\mathbf{1} + \partial\xi$ for some $\xi \in C_1(G;\mathbb{Z})$. Given an orientable φ with a choice of such φ_0, say that φ is **orientable with respect to** φ_0. Note that a single φ may be orientable with respect to distinct vertices φ_0 and φ_0'.

Correspondingly, say that a Namikawa subdivision $\mathrm{Del}_\varphi(G)$, the Namikawa complex $\overline{\mathrm{Del}}_\varphi(G)$, and the associated compactified Jacobian $\overline{J}_X(\varphi)$ associated with an orientable polarization φ are themselves **orientable**. ♦

5.3.4 Relation to Discriminantal Arrangements and Fiber Polytopes

Observing the structure of the arrangement $\mathrm{Arr}(G)$, it is clear that the structure of the dual graph G is essential to the analysis of the carious Namikawa subdivisions $\mathrm{Del}_\varphi(G)$. In fact, for the purpose of understanding the parameterizing space $\mathrm{Arr}(G)/\partial\delta C_0(G;\mathbb{Z})$, there is much to be gained also by considering the relationship to the cographic matroid $M^*(G)$ and its corresponding oriented matroid structure as specified by the orientation on G.

Recall from Subsection 2.2.2 that the graphic matroid has as circuits the indecomposable cycles in G. This means that the cographic matroid $M^*(G)$ has cocircuits the indecomposable cycles in G and circuits the minimal cuts in G. Likewise, the cographic oriented matroid $\mathcal{M}^*(G)$ has oriented circuits the minimal cuts $\omega_{T,e}$ in G. As such, we identify the discriminantal arrangement over $\mathcal{M}^*(G)$ to be finite hyperplane arrangement $\partial\mathrm{Arr}_0(G)$, where $\mathrm{Arr}_0(G)$ is the subarrangement of $\mathrm{Arr}(G)$ given as

$$\mathrm{Arr}_0(G) = \{H_{\omega,0} : \omega \text{ is a fundamental cut of } G\}.$$

Thus, since the discriminantal arrangement over an oriented matroid is the normal fan of the fiber polytope associated with the matroid (cf. Subsections 3.1.3 and 3.2.3), we see that the cells of $\mathrm{Arr}_0(G)$ correspond to coherent zonotopal subdivisions of the zonotope of $\mathcal{M}^*(G)$.

These observations foreshadow the results of Section 5.5 below, where we identify these zonotopal subdivisions as restrictions of orientable Namikawa subdivisions (see Definition 5.3.22 below).

Note moreover that for each $\xi \in C_1(G;\mathbb{Z})$, the (affine central) hyperplane arrangement $\mathrm{Arr}_{\partial\xi}(G)$ consisting of those $H_{\omega,k}$ passing through $\partial\xi$ is combinatorially equivalent to

$\mathrm{Arr}_0(G)$. As we shall see below, each of these central arrangements can be seen to parameterize families of Namikawa subdivisions associated with various translates of the zonotope of $\mathcal{M}^*(G)$.

For now, we simply observe the combinatorial equivalence of the discriminantal arrangement over $\mathcal{M}^*(G)$ and various central subarrangements of $\mathrm{Arr}(G)$, keeping in mind that we will return to these observations later on in Section 5.5.

5.4. Stability Conditions and Degeneracy Sets

Definition 5.4.1. A **numerical polarization** on a nodal curve X is a map

$$\begin{aligned} \psi : \{\text{subcurves of } X\} &\longrightarrow \mathbb{R} \\ Y &\longmapsto \psi_Y \end{aligned}$$

which is additive. In particular, we need only define ψ_{X_i} for irreducible components X_i of X. Say that the polarization ψ is of **characteristic** χ if $|\psi| := \psi_X = \chi$. ♦

Note in particular that any polarization $\psi \in \delta C_0(G_X;\mathbb{R})$ defines a numerical polarization on the curve X, since a subcurve is uniquely identified by its set of irreducible components, which correspond to the vertices of G_X. Conversely, additivity ensures that any numerical polarization defines an element $\psi \in \delta C_0(G;\mathbb{R})$ by taking $\psi_i := \psi(X_i)$, where X_i is some irreducible component of X, and this correspondence is bijective. This perspective allows us to define the value of ψ on subsets of vertices of G_X. In particular, we may speak of values $\psi_W = \sum_{i\in W}\psi_i$ for biconnected subsets $W \in \mathrm{BCon}(G)$.

Definition 5.4.2. A V-stability condition of degree d on the graph G is a map

$$\begin{aligned} \mathfrak{n} : \mathrm{BCon}(G) &\longrightarrow \mathbb{Z} \\ W &\longmapsto \mathfrak{n}_W \end{aligned}$$

satisfying the following properties:

(VS1) For all $W \in \mathrm{BCon}(G)$ we have

$$\mathfrak{n}_W + \mathfrak{n}_{W^c} + \mathrm{val}(W) - d \in \{0, 1\}.$$

If this quantity is 0, we say that W is $\mathfrak{n}$-**degenerate**, and otherwise W is $\mathfrak{n}$-**nondegenerate**.

(VS2) For all pairwise disjoint triples $W_1, W_2, W_3 \in \mathrm{BCon}(G)$ for which $W_1 \sqcup W_2 \sqcup W_3 = V(G)$ we have that if two of W_1, W_2, W_3 are $\mathfrak{n}$-degenerate, the third is as well. Moreover, we require that

$$\sum_{i=1}^{3} \mathfrak{n}_{W_i} + \sum_{1 \le i < j \le 3} \mathrm{val}(W_i, W_j) - d \in \begin{cases} \{1, 2\} & W_i \text{ is } \mathfrak{n}\text{-nondegenerate} \\ \{1\} & \text{exactly one } W_i \text{ is } \mathfrak{n}\text{-degenerate} \\ \{0\} & W_i \text{ is } \mathfrak{n}\text{-degenerate for all } i \end{cases}$$

The collection $\mathcal{D}(\mathfrak{n})$ of $\mathfrak{n}$-degenerate biconnected subsets W is referred to as the **degeneracy set** of the condition $\mathfrak{n}$. ♦

Degeneracy sets of V-stability conditions are of great importance in the analysis in contemporary treatments of compactified Jacobians (see [23], [24], [25]). Moreover, they are particularly notable for our treatment in virtue of their connection to matroid extensions (cf. Proposition 5.4.6). Because of this, we provide a definition of abstract degeneracy subsets as follows.

Definition 5.4.3. A **degeneracy subset** for a graph G is a subset $\mathcal{D} \subseteq \mathrm{BCon}(G)$ satisfying the following.

(DS1) $W \in \mathcal{D}$ implies $W^c \in \mathcal{D}$.

(DS2) For all disjoint pairs $W_1, W_2 \in \mathrm{BCon}(G)$ for which $W_1 \cup W_2 \in \mathrm{BCon}(G)$, we have that $W_1, W_2 \in \mathcal{D}$ implies $W_1 \cup W_2 \in \mathcal{D}$.

$\blacklozenge$

Remark 5.4.4. Though we have seen a means of associating a degeneracy subset to a V-stability condition, not all degeneracy subsets satisfying the axioms above arise in this way. In particular, there are degeneracy subsets $\mathcal{D}$ which are not of the form $\mathcal{D}_{\mathfrak{n}}$ for any V-stability condition $\mathfrak{n}$. $\blacklozenge$

Remark 5.4.5. From each numerical polarization ψ on X we obtain a V-stability condition $\mathfrak{s}(\psi)$ given by declaring, for a subcurve Y of X,

$$\mathfrak{s}(\psi)_Y = \lceil \psi_Y \rceil.$$

Moreover, the associated degeneracy set is given by

$$\mathcal{D}(\psi) := \mathcal{D}(\mathfrak{s}(\psi)) = \{W \in \mathrm{BCon}(G) \ : \ \psi_W \in \mathbb{Z}\}.$$

In particular, given a numerical polarization $\psi \in \partial C_1(G;\mathbb{R})$, we may define its degeneracy subset as the collection of those $W \in \mathrm{BCon}(G)$ for which $\langle \delta W, \psi \rangle = \langle W, \partial\psi \rangle \in \mathbb{Z}$. By the characterization of the hyperplane arrangement $\mathrm{Arr}(G)$ from Section 5.3.2, we thus see that the degeneracy subset remembers precisely those cuts corresponding to hyperplanes containing ψ. $\blacklozenge$

5.4.1 Relation to Matroid Extensions and Liftings

Recently in [23], the authors established that for a nodal curve X, each *smoothable* compactified Jacobian arises from a V-stability condition, and that each V-stability condition determines a degeneracy subset $\mathcal{D} \subseteq \mathrm{BCon}(G)$. We now establish that choosing a degeneracy subset for G is equivalent to specifying a single-element extension of the graphic matroid $M(G)$, and so is equivalent to choosing a single-element lifting of the cographic matroid $M^*(G)$.

Given a degeneracy subset $\mathcal{D} \subseteq \mathrm{BCon}(G)$, define a subset of cocircuits of $M = M(G)$ by

$$\mathcal{L}_{\mathcal{D}} = \{\omega \in \mathcal{C}^*(M) : \omega^+ \in \mathcal{D}\}.$$

Conversely, for any linear class of bonds $\mathcal{L} \subseteq \mathcal{C}^*(M)$, define a subset of $\mathrm{BCon}(G)$ by

$$\mathcal{D}_{\mathcal{L}} := \{\omega^+, \omega^- \in \mathrm{BCon}(G) : \omega \in \mathcal{L}\}.$$

Proposition 5.4.6. *The poset of linear classes of bonds and the poset of degeneracy conditions are isomorphic via the assignments*

$$\{\textit{linear classes of bonds}\} \longleftrightarrow \{\textit{degeneracy subsets for } G\}$$

$$\mathcal{L} \longmapsto \mathcal{D}_{\mathcal{L}}$$

$$\mathcal{L}_{\mathcal{D}} \longleftarrow\!\shortmid \mathcal{D}$$

Here, both posets are ordered by inclusion.

Proof. Let $\mathcal{D} \subseteq \mathrm{BCon}(G)$ be a degeneracy subset and $\mathcal{L}_{\mathcal{D}}$ its image under the assignment above. Suppose that $\omega_1 \cup \omega_2 \cup \omega_3 \subseteq E$ is a tribond containing two distinct bonds $\omega, \omega' \in \mathcal{L}_{\mathcal{D}}$ (i.e. $\omega^+, \omega^-, (\omega')^+, (\omega')^- \in \mathcal{D}$). To see that $\mathcal{L}_{\mathcal{D}}$ is a linear class, we must show that there is some third bond $\omega'' \in \mathcal{L}_{\mathcal{D}}$ which is contained in $\omega_1 \cup \omega_2 \cup \omega_3$.

Suppose that $G - (\omega_1 \cup \omega_2 \cup \omega_3)$ has connected components $V_1, V_2, V_3 \subseteq V$. We may assume without loss of generality that $G - \omega$ has connected components V_1 and $V_2 \cup V_3$. Since $\omega \neq \omega'$, we may further assume that $G - \omega'$ has connected components $V_2, V_1 \cup V_3$. Now we see that $V_1, V_2 \in \mathcal{D}$ satisfy $V_1 \cap V_2 = \varnothing$ and $V_1 \cup V_2 = V_3^c \in \mathrm{BCon}(G)$, so that condition (DS2) of 5.4.3 ensures that $V_3, V_3^c \in \mathcal{D}$. In particular, there is some bond ω'' for which the connected components of $G - \omega''$ are V_3 and $V_1 \cup V_2 = V_3^c$. Thus $\omega'' \in \mathcal{L}_{\mathcal{D}}$ and $\omega'' \subseteq \omega_1 \cup \omega_2 \cup \omega_3$, so that $\mathcal{L}_{\mathcal{D}}$ is a linear class.

Conversely, let $\mathcal{L}$ be a linear class of bonds in G and let $\mathcal{D}_{\mathcal{L}}$ be as above. Note that for all

$V = \omega^+ \in \mathcal{D}_{\mathcal{L}}$, we have $V^c = \omega^- \in \mathcal{D}_{\mathcal{L}}$. Now, let $V_1, V_2 \in \mathcal{D}_{\mathcal{L}}$ be such that $V_1 \cup V_2 \in \mathrm{BCon}(G)$ and $V_1 \cap V_2$ is empty. To show that $\mathcal{D}_{\mathcal{L}}$ is a degeneracy subset, we need only establish that $V_1 \cup V_2 \in \mathcal{D}_{\mathcal{L}}$. Write $V_3 := (V_1 \cup V_2)^c$ and let $\omega_1, \omega_2, \omega_3 \in \mathcal{L}$ be defined by $V_i = \omega_i^+$. By $V_1, V_2, V_3 \in \mathrm{BCon}(G)$ we then have that there is some tribond C with $G - C$ having connected components V_1, V_2, V_3, so that $\omega_1, \omega_2 \subseteq C$. But since $\mathcal{L}$ is a linear class, there is some other $\omega \in \mathcal{L}$ distinct from ω_1, ω_2 for which $\omega \subseteq C$. Finally, we may now write without loss of generality that $\omega^+ = V_3^c$, hence $V_1 \cup V_2 = V_3^c \in \mathcal{D}_{\mathcal{L}}$ and $\mathcal{D}_{\mathcal{L}}$ is a degeneracy subset.

The assignments in the statement are readily seen to be mutually inverse injections and order-preserving, competing the proof. $\square$

Remark 5.4.7. We now see that for a given numerical polarization ψ for G we have assignments

$$\psi \longmapsto \mathcal{D}(\psi) \longmapsto \mathcal{L}_\psi = \mathcal{L}_{\mathcal{D}(\psi)} \longmapsto M_\psi = J_0(G, \mathcal{L}_\psi) \longmapsto M_\psi^*.$$

In particular, each numerical polarization defines a unique single-element lifting of the cographic matroid $M^*(G)$. It is important to note that the lifting M_ψ^* need not be orientable, so that a numerical polarization ψ may not correspond to a lifting of the cographic oriented matroid $\mathcal{M}^*(G)$. ♦

5.5. Coherence of Namikawa Subdivisions

In this section we establish that any Namikawa subdivision $\mathrm{Del}_\varphi(G)$ associated with an orientable polarization φ is a $\rho H_1(G;\mathbb{Z})$-periodic continuation of a ρ-coherent zonotopal subdivision. This result follows from the characterization in Proposition 5.3.19 of Namikawa subdivisions for φ satisfying $\varphi - \frac{1}{2}\mathbf{1} \in \partial C_1(G;\mathbb{Z})$ and from the criterion presented in Lemma 3.1.12 for coherence of a subdivision. Note that we use the notation and terminology of Remark 5.3.20 and Definition 5.3.22 in both the statements and the proofs of the results below.

Theorem 5.5.1. *Let φ be an orientable polarization with respect to some $\varphi_0 \in \frac{1}{2}\partial\mathbf{1} + \partial C_1(G;\mathbb{Z})$. Then the restriction of the Namikawa decomposition $Del_\varphi(G)$ to the cell $Z_{\varphi_0} = \frac{1}{2}\rho\mathbf{1}+\rho\xi+\rho V(0)$ is a ρ-coherent subdivision with respect to the projection $\rho : C_{\varphi_0} \to Z_{\varphi_0}$. In particular, $Del_\varphi(G)|_{Z_{\varphi_0}}$ is the ρ-coherent subdivision of Z_{φ_0} defined by the linear functional $\varphi - \varphi_0$.*

Proof. For such φ and φ_0, recall from the proof of Proposition 5.3.19 that we may take $D_T(\varphi_0) = \xi$ for all T and recall from Proposition 5.3.21 that we may moreover take $D_T(\varphi)$ to have $D_T(\varphi_0)$ as a vertex. By taking duals and translating, we therefore also see that

$$\tfrac{1}{2}\mathbf{1} + D_T^*(\varphi) \leq \tfrac{1}{2}\mathbf{1} + D_T^*(\varphi_0).$$

In the notation of our current situation, this says simply that $\frac{1}{2}\mathbf{1} + D_T(\varphi) \leq C_{\varphi_0}$ for all $T \in \mathcal{T}(G)$. Additionally, we know from Proposition 5.3.18 that the cells $\frac{1}{2}\rho\mathbf{1} + \rho D_T^*(\varphi)$ and their translates by $\rho H_1(G;\mathbb{Z})$ comprise all maximal cells of the Namikawa subdivision $\mathrm{D}_\varphi(G)$, while the cell $Z_\varphi = \rho C_\varphi$ and its $\rho H_1(G;\mathbb{Z})$-translates comprise all maximal cells of $\mathrm{Del}_{\varphi_0}(G)$. Further, by part (3) of Proposition 5.3.12, $\partial\xi \leq A$ guarantees also that $\mathrm{Del}_\varphi(G)$ is a refinement of $\mathrm{Del}_{\varphi_0}(G)$.

Together, we now see that the cells $\frac{1}{2}\rho\mathbf{1} + \rho D_T^*(\varphi)$ form the maximal cells of $\mathrm{Del}_\varphi(G)$ and form a refinement of the trivial subdivision of $Z_{\varphi_0} = \rho C_{\varphi_0}$. That is, the restriction $\mathrm{Del}_\varphi(G)|_{Z_{\varphi_0}}$ is composed of projections of faces of C_{φ_0} and so is a ρ-induced subdivsion of Z_{φ_0}. We now appeal to Lemma 3.1.12 to show that it is indeed a coherent subdivision.

Identify $C_1(G;\mathbb{R})$ with its dual vector space via the (Euclidean) inner product $\langle\cdot,\cdot\rangle$. This allows us to view elements $x \in C_1(G;\mathbb{R})$ as functionals on $C_1(G;\mathbb{R})$ and allows us to speak of normal cones of polyhedra as living in the same space as the polyhedron itself. Fix a spanning tree T of G. Since $\frac{1}{2}\mathbf{1} + D_T^*(\varphi) \leq C_{\varphi_0}$ and C_{φ_0} is a translate of $V(0)$, taking normal cones yields that

$$N_{C_{\varphi_0}}(\tfrac{1}{2}\mathbf{1} + D_T^*(\varphi)) = \operatorname{cone}(e_t \ : \ t \in D_T(\varphi)^0) = \operatorname{cone}(D_T(\varphi) - \xi).$$

Now, recalling that $\varphi - \frac{1}{2}\partial\mathbf{1} \in \operatorname{relint}(\partial D_T(\varphi))$ (e.g. by the definition of the cells $D_T(\varphi)$), we automatically also have that

$$\varphi - \varphi_0 = \varphi - \tfrac{1}{2}\partial\mathbf{1} - \partial\xi \in \operatorname{cone}(\partial D_T(\varphi) - \partial\xi) = \partial N_{C_{\varphi_0}}(\tfrac{1}{2}\mathbf{1} + D_T^*(\varphi)).$$

In particular, now applying Lemma 3.1.12 yields that $\operatorname{Del}_\varphi(G)|_{Z_{\varphi_0}}$ is ρ-coherent given by the functional $\varphi - \varphi_0$. □

For any orientable polarization φ, Theorem 5.5.1 now provides a means of associating to φ a ρ-coherent subdivision of Z_{φ_0}. Since Z_{φ_0} is itself a translate of the zonotope $Z := \rho V(0)$, this association also determines a ρ-coherent zonotopal subdivision of Z and therefore (by Bohne-Dress, see Theorem 2.3.11) a single-element lifting $\mathcal{M}_\varphi^*$ of the oriented matroid associated with Z.

Since the oriented matroid of Z is the cographic oriented matroid of G (uniquely determined up to reorientation as in Section 2.4), we can now associate φ to a single-element lift $\mathcal{M}_\varphi^*$ of the cographic oriented matroid $\mathcal{M}^*(G)$.

On the other hand, Proposition 5.4.6 and Remark 5.4.7 provide an alternative means of generating a (classical) matroid lift M_φ^* for each choice of numerical polarization $\psi = \partial^{-1}\varphi$ (recall that ∂ induces an isomorphism $\delta C_0(g;\mathbb{R}) \simeq \partial C_1(G;\mathbb{R})$), so the choice of ψ is equivalent to a choice of φ). The next result ensures that, for a choice of orientable φ, these two constructions are compatible.

Proposition 5.5.2. *Suppose that φ is an orientable polarization, so that it yields both a single-element oriented matroid lift $\mathcal{M}_\varphi^*$ of $\mathcal{M}^*(G)$ (by Theorem 5.5.1) and a single-element matroid lift M_φ^* of $M^*(G)$ (by Proposition 5.4.6). Then these lifts are compatible in the sense* *that M^*_φ is the underlying matroid of $\mathcal{M}^*_\varphi$:*

$$M^*_\varphi = \underline{\mathcal{M}}^*_\varphi.$$

Remark 5.5.3. The results above provide the motivation for referring to φ as *orientable* (Definition 5.3.22). Indeed, by Theorem 5.5.1 and the discussion following it, each orientable polarization φ gives rise to an oriented matroid whose underlying matroid is itself necessarily *orientable*. ♦

We now proceed to describe a purely combinatorial means of describing an orientable compactified Jacobian $\overline{J}_X(\varphi)$. Before we do so, we first establish a proposition describing the local structure of the hyperplane arrangement $\mathrm{Arr}(G)$ at certain vertices φ_0 of $\mathcal{A}(G)$.

Proposition 5.5.4. *For any $\varphi_0 \in \partial C_1(G;\mathbb{R})$ with $\frac{1}{2}\partial\mathbf{1} + \varphi_0 \in \partial C_1(G;\mathbb{Z})$, the poset of cells $A \in \mathcal{A}(G)$ with $\varphi_0 \leq A$ is canonically isomorphic to the face poset of the discriminantal arrangement over the cographic oriented matroid $\mathcal{M}$.*

Proof. Given such a φ_0, recall from Remark 5.3.11 that $\varphi_0 - \frac{1}{2}\partial\mathbf{1} \in \mathrm{Sk}_0(\mathcal{A}_T(G))$ for every spanning tree T of G. That is, the locus of hyperplanes in $\mathrm{Arr}(G)$ passing through $\varphi_0 - \frac{1}{2}\partial\mathbf{1}$ consists of one hyperplane of the form $H_{\omega,k}$ for each fundamental cut ω of G. By translation, we then see that the locus of hyperplanes passing through $\varphi_0 9\frac{1}{2}\partial\mathbf{1}$ is canonically identified with the set of hyperplanes $\{H_{\omega,0} : \omega \text{ is a fundamental cut of } G\}$.

On the other hand, recall that the circuits of $\mathcal{M} = \mathcal{M}^*(G)$ are precisely the (oriented) fundamental cuts of G. In other words, the normal vectors of the discriminantal arrangement over $\mathcal{M}$ are precisely these fundamental cuts (cf. Definition 3.2.6), so that $\{H_{\omega,0} : \omega \text{ is a fundamental cut of } G\}$ is canonically identified with the discriminantal arrangement. Since the poset of cells $A \in \mathcal{A}(G)$ with $\varphi_0 \leq A$ is isomorphic to the cones of the complex induced by the subarrangement

$$\{H_{\omega,k} : \varphi_0 - \tfrac{1}{2}\partial\mathbf{1} \in H_{\omega,k}\} \subset \mathrm{Arr}(G),$$

we therefore see the correspondence desired. $\square$

By analyzing the periodicity of the Namikawa subdivisions $\mathrm{Del}_\varphi(G)$ and the arrangement $\mathrm{Arr}(G)$, we now obtain a convenient alternative means by which to determine orientable Namikawa subdivisions. Indeed, the following result establishes that the choice of an orientable Namikawa subdivision (hence an orientable compactivied Jacobian $\overline{J}_\varphi(G)$) amounts to the choice of a vertex $\overline{\varphi_0} \in \mathcal{A}(G)/\partial\delta C_0(G;\mathbb{R})$ and a single-element lifint $\widehat{\mathcal{M}}$ of the cographic oriented matroid $\mathcal{M} = \mathcal{M}^*(G)$.

Corollary 5.5.5. *Each orientable Namikawa subdivision $Del_\varphi(G)$ is determined by specifying a pair $(\overline{\varphi_0}, \widehat{\mathcal{M}})$ consisting of a vertex $\overline{\varphi_0}$ of the quotient cell complex $\mathcal{A}(G)/\partial\delta C_0(G;\mathbb{R})$ and a (possibly trivial) single-element lifting $\widehat{M}$ of the cographic oriented matroid $\mathcal{M}$ of G which is realizable over $\mathbb{R}$.*

In particular, if $\varphi \in \mathrm{relint}(A)$ for some $A \in \mathcal{A}(G)$ with $\frac{1}{2}\partial\mathbf{1} + \varphi_0 \leq A$, then $\widehat{\mathcal{M}}$ is the single-element lift corresponding to the cone of the discriminantal arrangement over $\mathcal{M}$ specified by A.

Proof. Suppose that $\overline{\varphi_0} \in \mathcal{A}(G)/\partial\delta C_0(G;\mathbb{R})$ and that $\widehat{\mathcal{M}}$ is a single-element lifting of $\mathcal{M} = \mathcal{M}^*(G)$ which is realizable over $\mathbb{R}$. Then any choice of $\varphi_0 \in \overline{\varphi_0}$ determines a Namikawa subdivision which is a translate of the Voronoi subdivision of $H^1(G;\mathbb{R})$. Indeed, writing $\varphi_0 = \frac{1}{2}\partial\mathbf{1} + \partial\xi$ for some $\xi \in C_1(G;\mathbb{R})$, Proposition 5.3.19 gives that any such choice determines the Namikawa subdivision

$$\mathrm{Del}_{\varphi_0}(G) = \rho(\tfrac{1}{2}\mathbf{1} + \xi) + \mathrm{Vor}(H^1(G;\mathbb{R}), \rho H_1(G;\mathbb{Z})),$$

and the periodicity of $\mathcal{A}(G)$ ensured by Proposition 5.3.12 guarantees that this choice is independent of $\varphi_0 \in \overline{\varphi_0}$. Further, Proposition 5.3.2 implies that any polarization φ which is orientable with respect to φ_0 gives rise to a Namikawa subdivision $\mathrm{Del}_\varphi(G)$ which is a refinement of $\mathrm{Del}_{\varphi_0(G)}$.

By Theorem 5.5.1, the refinement $\mathrm{Del}_\varphi(G)$ is given by subdividing all full-dimensional cells $\gamma + Z_{\varphi_0} = \gamma + \rho C_{\varphi_0}$ of $\mathrm{Del}_{\varphi_0}(G)$ with the ρ-coherent subdivision $\mathcal{Z}_\varphi$ specified by the cell of $\mathcal{A}(G)$ containing φ in its relative interior. Thus it is clear that a refinement Del_φ of $\mathrm{Del}_{\varphi_0}(G)$ is specified by choosing a ρ-coherent subdivision of $Z_\varphi \simeq \rho V(0)$. By Propositions 3.2.3 and 5.5.4, choosing a realizable single-element lift $\widehat{\mathcal{M}}$ of $\mathcal{M}$ specifies such a subdivision. In particular, we now see that $(\varphi_0, \widehat{\mathcal{M}})$ specifies $\mathrm{Del}_\varphi(G)$ for any φ lying in the relative interior of the cell $A \in \mathcal{A}(G)$ corresponding to $\widehat{\mathcal{M}}$. □

5.6. Grothendieck Class of Orientable Compactifications

5.6.1 Grothendieck Ring of Varieties

Let $\mathbf{k}$ be a field. The category $\mathrm{Var}_\mathbf{k}$ of algebraic varieties over $\mathbf{k}$ has as objects the reduced separated schemes of finite type over $\mathbf{k}$ and has as morphisms the morphisms of such objects. Construct a group from this category by taking the free abelian group on isomorphism classes $[X]$ of varieties X in $\mathrm{Var}_\mathbf{k}$ and taking the quotient by relations of the form $[X \setminus Z] + [Z] = [X]$ whenever Z is a closed subvariety of X. This yields the *Grothendieck group* of the category $\mathrm{Var}_\mathbf{k}$, denoted by $K_0(\mathrm{Var}_\mathbf{k})$. This group also admits a ring structure defined by the product $[X][Y] = [(X \times_\mathbf{k} Y)_{\mathrm{red}}]$. This product makes $K_0(\mathrm{Var}_\mathbf{k})$ into a commutative ring with multiplicative unit $1 = [\mathrm{Spec}\ \mathbf{k}]$ and with additive identity $0 = [\varnothing]$. This ring is known as the **Grothendieck ring of varieties** over $\mathbf{k}$. Another important element of this ring is the **Lefschetz motive**, the isomorphism class of the affine line $\mathbb{L} := [\mathbb{A}^1_\mathbf{k}]$. Many important varieties can be expressed in terms of the Lefschetz motive.

The Grothendieck ring $K_0(\mathrm{Var}_\mathbf{k})$ is an important invariant of the category $\mathrm{Var}_\mathbf{k}$ and the study of its structure has yielded fruitful results in arithmetic and birational geometry. In particular, the rings $K_0(\mathrm{Var}_\mathbf{k})$ and $K_0(\mathrm{Var}_\mathbf{k})(\mathbb{L}^{-1})$ are of crucial importance in the theory of motivic integration. Further, several important invariants of varieties are known to define homomorphisms out of $K_0(\mathrm{Var}_\mathbf{k})$. For example, when $\mathbf{k} = \mathbb{F}_q$ is a finite field, taking the number of closed points of any representative of the class $[X]$ is an invariant which yields a homomorphism to the integers. Additionally, for $\mathbf{k} = \mathbb{C}$, the compactly-supported Euler characteristic defines another such homomorphism. Any ring homomorphism out of $K_0(\mathrm{Var}_{\mathbf{k}})$ is said to be a *motivic measure*. Perhaps unsurprisingly, the reason for this terminology comes from motivic integration.

Still, not much is known about the ring $K_0(\mathrm{Var}_{\mathbf{k}})$ in general. It is known at least that this ring is not a domain: Poonen showed in [48] that when char $\mathbf{k}$ is nonzero, $K_0(\mathrm{Var}_{\mathbf{k}})$ is not an integral domain. Further, it was shown in [13] that for $\mathbf{k} = \mathbb{C}$ the Lefschetz motive is a zero divisor:

$$([X_W] - [Y_W])(\mathbb{L}^2 - 1)(\mathbb{L} - 1)\mathbb{L}^7 = 0,$$

where X_W and Y_W are non-birational smooth Calabi-Yau threefolds. Still, the Grotherndieck ring carries a wealth of interesting information about varieties over $\mathbf{k}$.

As a result of the ring structure of $K_0(\mathrm{Var}_{\mathbf{k}})$, we have that classes $[X]$ are additive over locally closed stratifications and multiplicative over fibrations. That is, when $X = \bigsqcup X_i$ we have $[X] = \sum [X_i]$ and when X is a fibration over Y with fiber Z we have $[X] = [Y][Z]$. These properties are illustrated by the following examples.

Example 5.6.1. The class of the algebraic torus $\mathbb{G}_m$ over $\mathbf{k}$ is given by $[\mathbb{G}_m] = \mathbb{L} - 1$. To see twhy this is true, recall that the underlying topological space of $\mathbb{G}_m^*$ agrees with that of $\mathbf{k}^\times$ with the Zariski topology. ◊

Example 5.6.2. The class of affine n-space is given by

$$[\mathbb{A}^n_{\mathbf{k}}] = [\mathbb{A}^1_{\mathbf{k}} \times \cdots \times \mathbb{A}^1_{\mathbf{k}}] = \mathbb{L}^n.$$

Further, since projective n-space $\mathbb{P}^n_{\mathbf{k}}$ satisfies the recursive relation $\mathbb{P}^n_{\mathbf{k}} \setminus \mathbb{P}^{n-1}_{\mathbf{k}} = \mathbb{A}^n_{\mathbf{k}}$, its class in the Grothendieck ring can be expressed as $[\mathbb{P}^n_{\mathbf{k}}] = [\mathbb{A}^n_{\mathbf{k}}] + [\mathbb{P}^{n-1}_{\mathbf{k}}]$. Applying this relation inductively, this gives the expression

$$[\mathbb{P}^n_{\mathbf{k}}] = 1 + \mathbb{L} + \cdots + \mathbb{L}^n$$

◊

Finally we illustrate a further example which will be of use to us in the sequel. This example gives the class of a semiabelian variety as a product in terms of an abelian variety and the Lefschetz motive.

Example 5.6.3. Suppose that A is a semiabelian variety given as an extension of an abelian variety $\tilde{A}$ by an algebraic torus $\mathbb{G}_m^n$. That is, X sits in a short exact sequence of algebraic groups:

$$0 \longrightarrow \mathbb{G}_m^n \longrightarrow A \longrightarrow \tilde{A} \longrightarrow 0.$$

Then Zariski locally we have that A is a fiber bundle over $\tilde{A}$ with fiber $\mathbb{G}_m^n$. Using the examples above we may therefore write

$$[A] = (\mathbb{L} - 1)^n [\tilde{A}].$$

◊

5.6.2 The Class of an Orientable Compactified Jacobian

In this section we derive from Theorem 4.2.1 and Theorem 5.5.1 two corollaries that describe the structure of the compactifications $\overline{J}_X(\varphi)$ for orientable polarizations φ. Essentially, Theorem 4.2.1 allows us to compute the number of cells in orientable Namikawa complexes $\overline{\mathrm{Del}}_\varphi(G)$, which by Proposition 5.3.6 correspond to $\mathrm{Pic}^0(X)$-orbits in $\overline{J}_X(\varphi)$. From here, we recall from Subsection 5.6.1 that a stratification yields a sum decomposition in $K_0(\mathrm{Var}_{\mathbf{k}})$, so that a brief analysis of the geometry of Pic^0_X-orbits in $\overline{J}_X(\varphi)$ yields a formula for the class of the compactification in the Grothendieck ring.

Theorem 5.6.4. *Let φ be an orientable polarization for X and let $Del_\varphi(G)$ denote the corresponding Namikawa subdivision. Let Z_φ be the associated fundamental domain as described in Theorem 5.5.1. Then the number of i-dimensional faces of the Namikawa complex $\overline{Del}_\varphi(G)$ is given by*

$$f_i(\overline{Del}_\varphi(G)) = (-1)^{g-i} \sum_{\substack{R,S\in\mathcal{L}_\varphi \\ \dim S=0 \\ \dim F=g-i}} \mu(F,G),$$

where $\mathcal{L}_\varphi$ is the lattice of flats of the associated lifting $\mathcal{M}_\varphi$ of $\mathcal{M}^*(G)$ and $\mu_\varphi(\cdot,\cdot)$ its Möbius function.

Proof. This follows from Theorems 5.5.1 and 4.2.1. □

Corollary 5.6.5. *Let $Del_\varphi(G_X)$ be an orientable Namikawa subdivision associated with a polarization φ. Then the class of the compactified Jacobian $\overline{J}_X(\varphi)$ in the Grothendieck ring $K_0(Var_{\mathbf{k}})$ is given by the expression*

$$[\overline{J}_\varphi(X)] = [J_{\tilde{X}}] \cdot \sum_{k=0}^{g} (-1)^{g-k} \sum_{\substack{R,S\in\mathcal{L}_\varphi \\ \dim(S)=0 \\ \dim(R)=g-k}} \mu_\varphi(R,S)(\mathbb{L}-1)^{g_k}.$$

Here, $\tilde{X}$ is the normalization of X, $\mathcal{L}_\varphi$ is the lattice of flats of the oriented matroid lifting associated to φ, and g_k is the first Betti number of the subgraph of G obtained by deleting edges outside supp($\overline{N}$) for any cell $\overline{N} \in \overline{Del}_\varphi(G)$ with $\dim(\overline{N}) = k$.

Proof. Recall from Proposition 5.3.6 that each cell $\overline{N}_i$ of $\overline{\mathrm{Del}}_\varphi(X)$ corresponds to an orbit $O(\overline{N}_i)$ of $\overline{J}_X(\varphi)$ of complementary dimension. More precisely, we have that there are isomorphisms of schemes

$$O(\overline{N}_i) \simeq \mathrm{Pic}^{n_i}_{X(E-\mathrm{supp}(\overline{N}_i))},$$

where $n_i \in C_0(G;\mathbb{Z})$ is given by $n_i = \partial b(\overline{N}_i) - \frac{1}{2} d(\mathrm{supp}(\overline{N}_i))$. Further, we have the following equality relating dimensions of cells $\overline{N}_i$ and orbits $O(\overline{N}_i)$:

$$\dim(\overline{N}_i) + \dim(O(\overline{N}_i)) = \dim(\overline{J}_X(\varphi)).$$

For the duration of this proof, let $E_i = E - \mathrm{supp}(\overline{N}_i)$ for each i. Now, since $X(E_i)$ is itself a nodal curve, we can appeal to Proposition 5.3.9 to see that $\mathrm{Pic}^{n_i}_{X(E_i)}$ is a $\mathbb{G}_m \otimes H^1(G(E_i);\mathbb{Z})$-bundle over $\mathrm{Pic}^{n_i}_{\tilde{X}}$. Observe also that Remark 5.3.8 guarantees that the $\mathrm{Pic}^{n_i}_{\tilde{X}}$ is isomorphic to $\mathrm{Pic}^0_{\tilde{X}}$. From this, we see that in the Grothendieck ring $K_0(\mathrm{Var}_{\mathbf{k}})$ we have

$$\begin{aligned}
[O(\overline{N}_i)] &= [\mathrm{Pic}^{n_i}_{X(E_i)}] \\
&= [\mathbb{G}_m \otimes H^1(G(E_i);\mathbb{Z})][\mathrm{Pic}^{n_i}_{\tilde{X}}] \\
&= [\mathbb{G}_m]^{g(G(E_i))}[\mathrm{Pic}^{n_i}_{\tilde{X}}] \\
&= (\mathbb{L}-1)^{g(G(E_i))}[\mathrm{Pic}^{0}_{\tilde{X}}] \\
&= (\mathbb{L}-1)^{g(G(E_i))}[J_{\tilde{X}}].
\end{aligned}$$

Here, we have denoted by $g(G(E_i))$ the genus of the graph $G(E_i)$. To determine this genus and its dependence on the cell $\overline{N}_i$, recall from Proposition 5.3.9 that for each k we have that each $\mathrm{Pic}^k_{X(E_i)}$ has the structure of a $\mathbb{G}_m^{g(G(e_i))}$-bundle over $\mathrm{Pic}^k_{\tilde{X}}$, and that Pic^k_X is itself a $\mathbb{G}_m^{g(G_X)}$-bundle over $\mathrm{Pic}^k_{\tilde{X}}$. Since the dimension of a bundle is the sum of the dimensions of the base and the fiber, we deduce the following from the description of orbits given in Proposition 5.3.6:

$$\begin{aligned}
\dim(\overline{N}_i) + \dim(O(\overline{N}_i)) &= \dim(\overline{J}_X(\varphi)) \\
\dim(\mathrm{Pic}^0_{\tilde{X}}) + g(G(E_i)) + \dim(\overline{N}_i) &= \dim(\mathrm{Pic}^0_{\tilde{X}}) + g(G_X) \\
g(G(E_i)) &= g(G_X) - \dim(\overline{N}_i).
\end{aligned}$$

That is, the value $g(G(E_i))$ depends only on $\dim(\overline{N}_i)$. Note that in the above calculation we have used that $\dim(\overline{J}_X(\varphi)) = \dim(J_X)$, since the compactifications $\overline{J}_X(\varphi)$ are obtained by gluing on torus orbits of positive codimension. Combining the observations above and writing $g_k := g(G(E_i))$ for any cell $\overline{N}_i$ of dimension k, we then have that each orbit corresponding to a k-dimensional cell $\overline{N}$ satisfies

$$[O(\overline{N})] = (\mathbb{L} - 1)^{g_k}[J_{\tilde{X}}].$$

Note that under the usual toric dictionary, we have that $\dim(\overline{N}) = \dim(D) = \dim(\rho(D))$ for any $D \in K^{\circ}_{\varphi}(G)$ with $\rho(D) \in \overline{N}$. This implies in particular that the number of cells $\overline{N} \in \overline{\mathrm{Del}}_{\varphi}(G)$ with $\dim(\overline{N}) = k$ is the same as the number f_k of k-dimensional cells in the associated cell decomposition of the tropical Jacobian $J(G_X)$. Recalling the formula for f_k from Theorem 4.2.1, we then have the following.

$$[\overline{J}_X(\varphi)] = \sum_{k=0}^{g} f_k(\mathbb{L} - 1)^{g_k}[J_{\tilde{X}}] = [J_{\tilde{X}}] \cdot \sum_{k=0}^{g}(-1)^{g-k} \sum_{\substack{R,S\in\mathcal{L}_\varphi \\ \dim(S)=0 \\ \dim(R)=g-k}} \mu_\varphi(R,S)(\mathbb{L} - 1)^{g_k}.$$

□

Chapter 6

Categories of Pointed F-Matroids and Tropical Vector Bundles

6.1. The category F-$\mathbf{Mat}_\bullet$

6.1.1 Morphisms of Pointed F-Matroids

Fix a perfect idyll F. Hereafter, by a *matroid* we mean a pointed F-matroid unless otherwise indicated. We now recall morphisms of matroids over a perfect idyll F, following definitions introduced in [31].

Let $S = \tilde{S} \sqcup \{*_S\}$ and $T = \tilde{T} \sqcup \{*_T\}$ be pointed sets. A **submonomial matrix** over F is a matrix A with entries in F such that each row and column contains at most one nonzero entry. Given such a matrix with rows and columns indexed by $\tilde{T}$ and $\tilde{S}$, respectively, submonomiality ensures that each such A defines a mapping $A : F^{\tilde{S}} \to F^{\tilde{T}}$ via matrix-vector multiplication. The **underlying map** of a submonomial matrix A is the function $\underline{A} : S \to T$ defined by

$$\underline{A}(i) = \begin{cases} j & i \neq *_S \text{ and } A_{i,j} \neq 0_F \\ *_T & \text{else} \end{cases}.$$

In particular, $\underline{A}$ is an $\mathbb{F}_1$-linear map of pointed sets. It is clear from the definitions that the transpose A^t of such a submonomial matrix A is again submonomial and defines a map from $F^{\tilde{T}}$ to $F^{\tilde{S}}$ and an associated underlying map $\underline{A^t} : T \to S$. We refer to A^t as the **adjoint** or **dual** of A.

Definition 6.1.1. A **morphism** of (nonzero) F-matroids $f : M \to N$ is a submonomial matrix f over F indexed by $\tilde{E}_N$ and $\tilde{E}_M$ such that $f \cdot \mathcal{V}_M \subseteq \mathcal{V}_N$, where $\mathcal{V}_M$ and $\mathcal{V}_N$ are the sets of vectors of M and N, respectively. ♦

Remark 6.1.2. Recall that the matroid U_1^0 consisting of only a basepoint is indeed a pointed F-matroid. F-matroid morphisms $M \to N$ where either $N = U_1^0$ or $M = U_1^0$ do not make sense in the above definition, since they would be defined as empty matrices. Instead, note that in either of these cases we have natural notions of morphisms given by the only existing pointed maps on underlying pointed sets. It easily follows that $0 := U_1^0$ is the zero object of F-$\mathbf{Mat}_\bullet$. ♦

F-matroid morphisms are preserved under duality. Explicitly, a submonomial matrix f is a morphism from M to N if and only if f^t is a morphism from N^* to M^* ([31, Corollary 2.9]). In the sequel, we will use the following alternative characterizations of matroid morphisms.

Proposition 6.1.3 ([31], Theorems 2.8)**.** *Let $M = [\mu]$ and $N = [\nu]$ be F-matroids of ranks m and n, respectively, and let f be a submonomial F-matrix indexed by $\tilde{E}_N \times \tilde{E}_M$. Then the following are equivalent:*

1. *f is a morphisms from M to N*

2. *for all $\mathbf{x} = (x_0, \ldots, x_m) \in \tilde{E}_M^{m+1}$ and all $\mathbf{y} = (y_2, \ldots, y_n) \in \tilde{E}_N^{n-1}$, we have*

$$\sum_{k=0}^{m} (-1)^k \cdot \mu(\mathbf{x}_{\hat{k}}) \cdot f_{\underline{f}(x_k), x_k} \cdot \nu(\underline{f}(x_k), \mathbf{y}) \in N_F.$$

Remark 6.1.4. In [31], Proposition 6.1.3 is stated and proven in terms of morphisms of non-pointed matroids. The adaptation of this result to the pointed case is immediate and has been omitted. ♦

For each perfect idyll F, the collection of (pointed) F-matroids together with the morphisms above forms a category F-$\mathbf{Mat}_\bullet$. One can also define a category F-$\mathbf{Mat}$ of non-pointed F-matroids, but we do not study this category here.

Remark 6.1.5. Recall that to each F-matroid M is associated an underlying matroid $\underline{M}$. With the above definition of matroid morphisms, the assignment $M \mapsto \underline{M}$ is a functor from F-matroids to $\mathbb{K}$-matroids. More generally, for any morphism of perfect idylls $\Phi : F \to G$ we have that the base-change map $M \to \Phi_* M$ is a functor from F-$\mathbf{Mat}_\bullet$ to G-$\mathbf{Mat}_\bullet$. On morphisms, these functors assign to an F-matroid morphism $f : M \to N$ a G-matroid morphism $\Phi_* f : \Phi_* M \to \Phi_* N$ given by applying Φ entrywise to the submonomial matrix f. ♦

We now illustrate how the contraction, deletion, and restriction of F-matroids described in Section 2.4 can be seen as F-matroid morphisms. Let M be an F-matroid and let $S \subseteq E_M$.

(1) (Contraction) Associated to the contraction M/S we have a **contraction map** $c_S : M \to M/S$ given by the submonomial matrix indexed by $\tilde{E}_M \setminus S$ and $\tilde{E}$ with entries $(c_S)_{ij} := \delta_{ij}$, where δ_{ij} is the Kronecker delta. In other words, up to reordering of columns, c_S is a matrix whose first $|\tilde{E}_M \setminus S|$ columns form an identity matrix and whose other columns are zero. That this is a matroid morphism is guaranteed for example by [2, Proposition 4.4].

(2) (Restriction) Associated to the restriction $M|S$ we have a **restriction map** $r_S : M|S \to M$ given by the submonomial matrix indexed by $\tilde{E}_M$ and S with entries $(r_S)_{ij} = \delta_{ij}$. Thus in particular we have $r_S = c^*_{E_M \setminus S}$ and so r_S is also a matroid morphism.

These morphisms can be understood in terms of their underlying maps: the contraction $\underline{c_S}$ sends all elements of S to the basepoint of $M \setminus S$ and preserves all other elements, while the restriction $\underline{r_S}$ embeds S into E_M.

Morphisms of $\mathbb{K}$-matroids induce $\mathbb{F}_1$-linear strong maps between the associated classical matroids, in the sense that the underlying map is an $\mathbb{F}_1$-linear strong map of matroids. Similarly, morphisms of F-matroids provide an appropriate notion of $\mathbb{F}_1$-linear strong maps for $F = \mathbb{S}, \mathbb{T}$.

We now establish some useful facts about morphisms of matroids which will be of use to us in the sequel. In particular, we are able to characterize monomorphisms and epimorphisms completely.

Proposition 6.1.6. *A morphism $f : M \to N$ of F-matroids is a monomorphism if and only if $\underline{f}$ is injective. Similarly, f is an epimorphism if and only if $\underline{f}$ is surjective.*

Proof. This is established by [27, Corollary 3.8]. Its analog for pointed F-matroids is immediate. □

Note that this ensures that F-$\mathbf{Mat}_\bullet$ satisfies axiom (PA1) of a proto-abelian category (see Definition B.4.1).

Remark 6.1.7. From this we see that any epi-monic $f : M \to N$ of F-matroids has underlying map a bijection, but the inverse map on sets is not necessarily a morphism of F-matroids. Instead, we see that $f : M \to N$ is an isomorphism of F-matroids if and only if the submonomial matrix associated to f is invertible and the inverse matrix f^{-1} defines an F-matroid morphism from N to M. ♦

Proposition 6.1.8. *A morphism $\alpha : M \to N$ is a matroid isomorphism if and only if its adjoint $\alpha^t : N^* \to M^*$ is also an isomorphism.*

Proof. Considering the F-matrices associated to α and to α^{-1}, note that

$$(\alpha^{-1})_{ij} = \begin{cases} \alpha_{ji}^{-1} & \alpha_{ji} \neq 0 \\ 0 & \text{else} \end{cases}.$$

On the other hand, the morphism $\alpha^t : N^* \to M^*$ by definition has the F-matrix given by $(\alpha^t)_{ij} = \alpha_{ji}$ and the adjoint of α^{-1} has the F-matrix given by

$$((\alpha^{-1})^t)_{ij} = (\alpha^{-1})_{ji} = \begin{cases} \alpha_{ij}^{-1} & \alpha_{ij} \neq 0_F \\ 0_F & \text{else} \end{cases}.$$

Thus as F-matrices, $(\alpha^{-1})^t = (\alpha^t)^{-1}$. Finally the fact that $\alpha^{-1} : N \to M$ is a morphism implies immediately that $(\alpha^{-1})^t : M^* \to N^*$ is also a morphism, and since $(\alpha^{-1})^t = (\alpha^t)^{-1}$ the result follows. □

Proposition 6.1.9. *Let $\alpha : M \to N$ be a matroid isomorphism, let $A \subseteq E_N$, and $S \subseteq E_M$. Then α induces isomorphisms $M|\alpha^{-1}(A) \cong N|A$ and $M/S \cong N/\alpha(S)$.*

Proof. Let $M = [\mu]$ and $N = [\nu]$ be matroids with ranks m and n. We begin with the contraction statement. Denote the associated contractions $M/S = [\mu'']$ and $N/\alpha(S) = [\nu'']$. Assume also that M/S has rank m'' and $N/\alpha(S)$ has rank n''. Then by definition α induces a bijection on the ground sets of M/S and $N/\alpha(S)$. We seek to show that this induced bijection is a matroid morphism. That is, we would like that for all $\mathbf{x} = (x_0, \ldots, x_{m''}) \in (E_M \setminus S)^{m''+1}$ and all $\mathbf{y} \in (E_N \setminus \alpha(S))^{n''-1}$ we have

$$\sum_{k=0}^{m''} (-1)^k \mu''(\mathbf{x}_{\hat{k}}) \alpha_{\underline{\alpha}(x_k), x_k} \nu''(\underline{\alpha}(x_k), \mathbf{y}) \in N_F. \tag{6.1}$$

Since α is an isomorphism, we know that for all $\mathbf{z} \in E_M^{m+1}$ and all $\mathbf{w} \in E_N^{n-1}$ we have

$$\sum_{k=0}^{m} (-1)^k \mu(\mathbf{z}_{\hat{k}}) \alpha_{\underline{\alpha}(z_k), z_k} \nu(\underline{\alpha}(z_k), \mathbf{w}) \in N_F. \tag{6.2}$$

Fix (ordered) bases $\mathbf{a}$ of $\underline{M}|S$ and $\mathbf{d}$ of $\underline{N}|\alpha(S)$. Then taking $\mathbf{z} = (\mathbf{x}, \mathbf{a})$ and $\mathbf{w} = (\mathbf{y}, \mathbf{d})$, we see that Expressions (6.1) and (6.2) differ only by the following terms:

$$\sum_{k>m''+1} (-1)^k \mu(\mathbf{x}, \mathbf{a}_{\hat{k}}) \alpha_{\underline{\alpha}(a_k), a_k} \nu(\underline{\alpha}(a_k), \mathbf{y}, \mathbf{d}). \tag{6.3}$$

Considering each summand in Expression (6.3), note that by Proposition 2.4.5 we see that the each factor $\nu(\underline{\alpha}(a_k), \mathbf{y}, \mathbf{d})$ is nonzero only when $|\mathbf{y}| \cup |\mathbf{d}| \cup \{\underline{\alpha}(a_k)\}$ is a basis of $\underline{N}$. But since $\underline{\alpha}(a_k) \in \alpha(S)$ by definition, we see that necessarily $|\mathbf{d}| \cup \{\underline{\alpha}(a_k)\}$ is dependent in $\underline{N}|\alpha(S)$, as $\mathbf{d}$ was chosen to be a basis. This then implies that $|\mathbf{y}| \cup |\mathbf{d}| \cup \{\underline{\alpha}(a_k)\}$ cannot be a basis of N and so $\nu(\underline{\alpha}(a_k), \mathbf{y}, \mathbf{d}) = 0_F$. Thus Expressions (6.1) and (6.2) express the same element of F and so α induces a morphism $M/S \to N/\alpha(S)$. Finally, because α is an isomorphism, the above argument applies also to α^{-1}, so that $M/S \cong N/\alpha(S)$.

For the restriction statement, we appeal to duality. In particular note that by Proposition 6.1.8 α induces an isomorphism $\alpha^t : N^* \to M^*$. For $A \subseteq E_N$ the above argument establishes an isomorphism $N^*/A \cong M^*/\alpha^t(A)$, which is equivalent to the desired $N|A \cong M|\alpha^{-1}(A)$ since $\underline{\alpha^t} = \underline{\alpha^{-1}}$. □

6.1.2 (Co)kernels of F-matroid morphisms

We now show that each morphism of F-matroids admits both a kernel and cokernel, each of which is again a morphism of F-matroids.

Proposition 6.1.10. *Let $M = [\mu]$ and $N = [\nu]$ be matroids of ranks m and n, respectively, and let $f : M \to N$ be a morphism of F-matroids. Then the restriction morphism $M|\underline{f}^{-1}(*_N) \rightarrowtail M$ is a kernel of f and the contraction morphism $N \twoheadrightarrow N/\underline{f}(E_M)$ is a cokernel of f.*

Proof. Let $K := \underline{f}^{-1}(*_N) \setminus \{*_M\} \subseteq E_M$ and $C := \underline{f}(\tilde{E}_M) \subseteq E_N$ and suppose that $r_{\underline{M}}(K) = k$

and $r_{\underline{N}}(C) = c$. Note that we then have the following commutative squares in F-$\mathbf{Mat}_\bullet$:

$$\begin{tikzcd} M|K \arrow[r, tail] \arrow[d, two heads] & M \arrow[d, "f"] \\ 0 \arrow[r, tail] & N \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "f"] \arrow[d, two heads] & N \arrow[d, two heads] \\ 0 \arrow[r, tail] & N/C \end{tikzcd}$$

We seek to show that the first square is Cartesian and the second is co-Cartesian. To this end, suppose that $R = [\rho]$ is a matroid of rank r for which the solid arrows of following diagram commute:

$$\begin{tikzcd} R \arrow[dr, dashed, "h"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right] & & \\ & M|K \arrow[r, tail, "r_K"] \arrow[d, two heads] & M \arrow[d, "f"] \\ & 0 \arrow[r, tail] & N \end{tikzcd} \tag{6.4}$$

We now construct the morphism h which makes the full diagram commute. Noting that g is given by an $|\tilde{E}_M| \times |\tilde{E}_R|$ submonomial matrix, let h be the $|K| \times |\tilde{E}_R|$ submatrix obtained from g by deleting all rows associated to elements of $\tilde{E}_M \setminus K$.

To see that the resulting matrix h is a matroid morphism, recall first that since g is a matroid morphism, for all $\mathbf{y} \in E_R^{r+1}$ and all $\mathbf{x} \in E_M^{m-1}$ we have

$$\sum_{i=0}^{r} (-1)^i \rho(\mathbf{y}_{\hat{i}}) \cdot g_{\underline{g}(y_i),y_i} \cdot \mu(\underline{g}(y_i), \mathbf{x}) \in N_F.$$

In order for h to be a morphism from R to $M|K$, we thus need that for each $\mathbf{y} \in E_R^{r+1}$ and each $\mathbf{z} \in K^{r_{\underline{M}}(K)-1}$ we have

$$\sum_{i=0}^{r} (-1)^i \rho(\mathbf{y}_{\hat{i}}) \cdot h_{\underline{h}(y_i),y_i} \cdot (\mu|K)(\underline{h}(y_i), \mathbf{z}) \in N_F.$$

However, recalling the definition of $\mu|K$ we see that we can take $\mathbf{x} = (\mathbf{z}, \mathbf{b})$ for any basis $\mathbf{b}$ of $\underline{M}/K$ to obtain

$$(\mu|K)(\underline{h}(y_i), \mathbf{z}) = \mu(\underline{h}(y_i), \mathbf{x}).$$

Moreover, by the construction, for each $y_i \in K$ we have $g_{\underline{g}(y_i),y_i} = h_{\underline{h}(y_i),y_i}$. In particular, we thus see that

$$\sum_{i=0}^{r}(-1)^i\rho(\mathbf{y}_{\hat{i}}) \cdot h_{\underline{h}(y_i),y_i} \cdot (\mu|K)(\underline{h}(y_i), \mathbf{z}) = \sum_{i=0}^{r}(-1)^i\rho(\mathbf{y}_{\hat{i}}) \cdot g_{\underline{g}(y_i),y_i} \cdot \mu(\underline{g}(y_i), \mathbf{x}) \in N_F.$$

Observe further that any such morphism h must be unique. To see this, note that for any two such morphisms h and h' making the diagram commute, the fact that r_K is a monomorphism implies that $h = h'$.

The proof that N/C is a cokernel of f is completely analogous and so has been omitted. $\square$

The above thus illustrates that F-**Mat**$_\bullet$ is a pointed category with kernels and cokernels. We have already seen that it satisfies axiom (PA1) of Definition B.4.1.

6.1.3 Proto-exactness of F-Mat$_\bullet$

We now prove the main structural result about the category of (pointed) matroids over a perfect idyll F.

Theorem 6.1.11. *For each perfect idyll F, the category F-**Mat**$_\bullet$ is proto-exact.*

Combining this theorem with Propositions B.4.4, 6.1.6, and 6.1.10, we obtain the following.

Corollary 6.1.12. *For each perfect idyll F, F-**Mat**$_\bullet$ is proto-abelian.*

We begin by introducing the classes $\mathfrak{M}$ and $\mathfrak{E}$ of admissible monomorphisms and epimorphisms.

Definition 6.1.13. Let $f : M \to N$ be a morphism in F-**Mat**$_\bullet$.

1. f is an admissible monomorphism if $f = r_A \circ \alpha$, where $A \subseteq E_N$ and $r_A : M|A \to M$ is an inclusion map, and α is an isomorphism of F-matroids:

$$M \xrightarrow[\sim]{\alpha} N|A \xrightarrow{r_A} N,$$

Denote by $\mathfrak{M}$ the class of admissible monomorphisms.

2. f is an admissible epimorphism if $f = \beta \circ c_B$, where $B \subseteq E_M$ and $c_B : M \to M/B$ is a contraction map and β is an isomorphism of F-matroids:

$$M \xrightarrow{c_B} M/B \xrightarrow[\sim]{\beta} N.$$

Denote by $\mathfrak{E}$ the class of admissible epimorphisms.

♦

By the characterization in the previous section of submonomial matrices corresponding to restrictions and contractions, note that, up to reordering ground set elements, any $f \in \mathfrak{M}$ is represented by an $|\tilde{E}_N| \times |\tilde{E}_M|$ submonomial matrix whose topmost $|\tilde{E}_M| \times |\tilde{E}_M|$ submatrix is invertible (in fact this submatrix is the matrix of α). Likewise, up to reordering, any $g \in \mathfrak{E}$ is represented by a submonomial matrix whose leftmost full submatrix is invertibe, and is the matrix of β. Note that from Proposition 6.1.8, we have that $\mathfrak{M}^t = \mathfrak{E}$ and vice versa.

Further, by Proposition 6.1.10 we see that $\mathfrak{M}$ consists of isomorphisms followed by kernels and $\mathfrak{E}$ consists of cokernels followed by isomorphisms.

With the above characterizations of the maps in $\mathfrak{M}$ and $\mathfrak{E}$, we are ready to verify the axioms of a proto-exact category.

Proposition 6.1.14. *The classes $\mathfrak{M}$ and $\mathfrak{E}$ contain all isomorphisms.*

Proof. Let α be an isomorphism of matroids. Observe that we can write $\alpha = I \cdot \alpha = \alpha \cdot I$, where I is the identity F-matrix. These factorizations and the fact that α is an isomorphism illustrate that α is in both $\mathfrak{M}$ and $\mathfrak{E}$. □

Proposition 6.1.15. *The classes $\mathfrak{M}$ and $\mathfrak{E}$ are closed under composition.*

Proof. Let $f, g \in \mathfrak{M}$ be a pair of composable admissible monomorphisms, so that we have

$f = r_A \circ \alpha : M \to N$ and $g = r_B \circ \beta : N \to R$. Thus we have

$$M \xrightarrow[\sim]{\alpha} N|A \xrightarrow{r_A} N \xrightarrow[\sim]{\beta} R|B \xrightarrow{r_b} R.$$

Now define $h := g \circ f = r_B \circ \beta \circ r_A \circ \alpha$. Then $h : M \to R$ is a $|\tilde{E}_R| \times |\tilde{E}_M|$ F-matrix with entries

$$h_{ij} = \begin{cases} \beta_{\underline{\beta}(\underline{\alpha}(j)),\underline{\alpha}(j)} \cdot \alpha_{\underline{\alpha}(j),j} & i \in \underline{g \circ f}(\tilde{E}_M) \text{ and } i = \underline{g \circ f}(j) \\ 0_F & i \notin \underline{g \circ f}(\tilde{E}_M) \end{cases}.$$

We can now illustrate the appropriate factorization of h by constructing an isomorphism γ by applying $\beta \circ \alpha$ to $\underline{\alpha}^{-1}(A) = E_M$. Define an $|\tilde{E}_M| \times |\tilde{E}_M|$ F-matrix γ with entries

$$\gamma_{ij} := \begin{cases} \beta_{\underline{\beta}(\underline{\alpha}(j)),\underline{\alpha}(j)} \cdot \alpha_{\underline{\alpha}(j),j} & i = \underline{g \circ f}(j) \\ 0 & \text{else} \end{cases}.$$

Thus by construction we have the factorization $h = r_{\underline{g \circ f}(\tilde{E}_M)} \circ \gamma$ as F-matrices. Finally, note that γ is an isomorphism of matroids since it is a composition of (restrictions of) isomorphisms.

We now show the corresponding result for $\mathfrak{E}$ by appealing to adjoint morphisms. Supposing now that $f, g \in \mathfrak{E}$ are composable admissible epimorphisms with associated factorizations $f = \alpha \circ c_A : M \to N$ and $g = \beta \circ c_B : N \to R$, we can write

$$M \xrightarrow{c_A} M/A \xrightarrow[\sim]{\alpha} N \xrightarrow{c_B} N/B \xrightarrow[\sim]{\beta} R \ .$$

Let $h := g \circ f$. Taking duals of everything in sight, we now have that $g^t : R^* \to N^*$ and $f^t : N^* \to M^*$ are composable morphisms in $\mathfrak{M}$. Thus by the above argument we have that the composition $h^t = f^t \circ g^t$ is also in $\mathfrak{M}$, so that $h \in \mathfrak{E}$. Alternatively, one can easily reproduce an analog of the above argument for $\mathfrak{E}$. □

We now prove an auxiliary result regarding biCartesian squares in F-**Mat**$_\bullet$, following [22, Lemma 4.9].

Lemma 6.1.16. *Let N be a pointed F-matroid and let $B \subseteq A \subseteq \tilde{E}_N$. The following square is biCartesian in F-**Mat**$_\bullet$:*

$$\begin{tikzcd} N|A \arrow[r, tail, "f"] \arrow[d, two heads, "h"] & N \arrow[d, two heads, "k"] \\ (N|A)/B \arrow[r, tail, "g"] & N/B \end{tikzcd} \tag{6.5}$$

where $f = r_A$ and $g = r_{(A \setminus B)}$.

Proof. Note first that this diagram is commutative since by Proposition 2.4.7 we have that $(N|A)/B = (N/B)|A$. We proceed by illustrating that the square (6.5) is coCartesian. Suppose that we have the following diagram in F-**Mat**$_\bullet$:

$$\begin{tikzcd} N|A \arrow[r, tail, "f"] \arrow[d, two heads, "h"] & N \arrow[d, two heads, "k"] \arrow[ddr, "\beta"] & \\ (N|A)/B \arrow[r, tail, "g"] \arrow[drr, "\alpha"'] & N/B & \\ & & R \end{tikzcd} \tag{6.6}$$

Now define an F-matrix γ given by removing from β all columns associated to elements of B. Doing so preserves submonomiality and therefore defines a map $\underline{\gamma} : E_{N/B} \to E_R$ and gives a factorization $\gamma k = \beta$ as F-matrices. It remains now to show that γ is an F-matroid map, which we check using Proposition 6.1.3.

Let $M = [\mu]$, $N = [\nu]$, and $R = [\rho]$, and suppose that $\mathrm{rk}_{\underline{N}}(B) = \ell$. In order to show that γ is a morphism, we must establish that for all choices of $\mathbf{x} = (x_0, \ldots, x_{r_N - \ell}) \in (\tilde{E}_N \setminus B)^{r_N - \ell + 1}$ and all choices of $\mathbf{y} = (y_2, \ldots, y_{r_R}) \in (\tilde{E}_R)^{r_R - 1}$ we have

$$\sum_{j=0}^{r_N - \ell} (-1)^j (\nu/B)(\mathbf{x}_{\hat{j}}) \gamma_{\underline{\gamma}(x_j), x_j} \rho(\underline{\gamma}(x_j), \mathbf{y}) \in N_F, \tag{6.7}$$

where r_N is the rank of N. We can re-express (6.7) by using the definition of the Grassmann-Plücker function ν/B. Let $\mathbf{b} = (b_1, \ldots, b_\ell) \in (\tilde{E}_N)^\ell$ be a vector where $b_1, \ldots, b_\ell$ is a maximal independent set of $\underline{N}|B$. Then we are interested in the sum

$$\sum_{j=0}^{r_N-\ell}(-1)^j\nu(\mathbf{x}_{\hat{j}},\mathbf{b})\gamma_{\underline{\gamma}(x_j),x_j}\rho(\underline{\gamma}(x_j),\mathbf{y}). \tag{6.8}$$

However, recalling that $\beta : N \to R$ is a morphism, we are guaranteed that

$$\sum_{j=0}^{r_N-\ell}(-1)^j\nu(\mathbf{x}_{\hat{j}},\mathbf{b})\beta_{\underline{\beta}(x_j),x_j}\rho(\underline{\beta}(x_j),\mathbf{y}) + \sum_{j=r_N-\ell+1}^{r_N}(-1)^j\nu(\mathbf{x},\mathbf{b}_{\hat{j}})\beta_{\underline{\beta}(b_j),b_j}\rho(\underline{\beta}(b_j),\mathbf{y}) \in N_F. \tag{6.9}$$

Noting that f acts as the identity on elements of B, we see that by the commutativity of Diagram (6.6) we obtain

$$\beta(B) = \beta(f(B)) = \alpha(h(B)) = *_R, \tag{6.10}$$

in particular for all b_j we have $\beta_{\underline{\beta}(b_j),b_j} = 0_F$. This illustrates that the second sum in (6.9) collapses to 0_F, so that the first sum must be in N_F. But by the construction of γ, for all $x \notin B$ we have $\gamma_{\underline{\gamma}(x),x} = \beta_{\underline{\beta}(x),x}$, hence the first sum is equal to the sum (6.8). This establishes that $\gamma : N/B \to R$ is indeed a morphism of F-matroids.

For the uniqueness of γ, suppose that $\delta : N/B \to R$ is another morphism making Diagram (6.6) commute, so that $\beta = \delta k = \gamma k$. By Proposition 6.1.6, k is an epimorphism, hence $\delta = \gamma$. This establishes that Diagram (6.5) is coCartesian.

To see that our diagram is also Cartesian, we could reproduce a similar argument to the above, which involves changing little other than the morphisms involved and the order of some compositions. Instead, we apply duality for a swifter proof. Consider the following pair of dual diagrams:

$$
\begin{tikzcd}
P \arrow[drr, "\beta"] \arrow[ddr, "\alpha"'] & & \\
& N|A \arrow[r, tail, "f"'] \arrow[d, two heads, "h"] & N \arrow[d, two heads, "k"] \\
& (N|A)/B \arrow[r, tail, "g"] & N/B
\end{tikzcd}
\qquad
\begin{tikzcd}
(N/B)^* \arrow[r, "k^t"] \arrow[d, "g^t"] & N^* \arrow[d, "f^t"'] \arrow[ddr, "\beta^t"] & \\
((N|A)/B))^* \arrow[r, "h^t"] \arrow[drr, "\alpha^t"] & (N|A)^* & \\
& & P^*
\end{tikzcd}
\tag{6.11}
$$

We would like to produce a morphism $P \to N|A$ making the left-hand diagram commute and then prove its uniqueness. In view of duality, this is equivalent to producing such a morphism $(N|A)^* \to P^*$ which commutes with the right-hand diagram. By Proposition 2.4.7, the right-hand diagram above can be re-expressed as follows:

$$
\begin{tikzcd}
N^*|(E_N \setminus B) \arrow[r, "k^t"] \arrow[d, "g^t"] & N^* \arrow[d, "f^t"'] \arrow[ddr, "\beta^t"] & \\
(N^*|(E_N \setminus B))/(E_N \setminus A) \arrow[r, "h^t"] \arrow[drr, "\alpha^t"] & N^*/(E_N \setminus A) & \\
& & P^*
\end{tikzcd}
$$

But we have just seen in the above that this diagram admits a unique morphism $N^*/(E_N \setminus A) \to P^*$ making the diagram commute. Thus taking duals again we see that Diagram (6.5) is Cartesian.

Alternatively, one can establish directly that Diagram (6.5) is Cartesian using a proof totally analogous to the coCartesian case. This approach has the benefit of circumventing duality, and so applies also in the context of simple F-matroids. $\square$

Proposition 6.1.17. *Any diagram in F-**Mat**$_\bullet$ of the form $M \rightarrowtail R \twoheadleftarrow N$ can be completed to a bi-Cartesian square.*

Proof. Let $i : M \rightarrowtail R$ and $j : N \twoheadrightarrow R$. Using the definition of the class $\mathfrak{E}$, we can write $j = \beta \circ c_A$, where β is an isomorphism and $c_A : N \to N/A$ is a contraction for some

$A \subseteq E_N$. Likewise, we can write $i = r_B \circ \alpha$, where α is an isomorphism and $r_B : R|B \to R$ is a restriction for some $B \subseteq E_R$. Letting $C := \underline{j}^{-1}(B) \subseteq E_N$, one can see that $A \subseteq C$. Hence, we obtain the following diagram:

$$
\begin{tikzcd}
 & N|C \arrow[r, tail] \arrow[d, two heads] \arrow[ddl] & N \arrow[d, two heads, "c_A"] \\
 & (N/A)|C \arrow[r, tail] \arrow[d, "\beta|_C"] & N/A \arrow[d, "\beta"] \\
M \arrow[r, "\alpha"] & R|B \arrow[r, "r_B"] & R
\end{tikzcd}
\tag{6.12}
$$

Note that Proposition 6.1.9 gives that $\beta|_C$ is an isomorphism. Since α is an isomorphism, we have the following commutative square:

$$
\begin{tikzcd}
N|C \arrow[r, tail] \arrow[d, two heads] & N \arrow[d, two heads] \\
M \arrow[r, tail] & R
\end{tikzcd}
\tag{6.13}
$$

We claim that (6.13) is the bi-Cartesian completion of $M \rightarrowtail R \twoheadleftarrow N$. By Lemma 6.1.16 the square in the following diagram is biCartesian:

$$
\begin{tikzcd}
 & N|C \arrow[r, tail] \arrow[d, two heads] & N \arrow[d, two heads] \\
(N|C)/A \arrow[r, equal] & (N/A)|C \arrow[r, tail] & N/A
\end{tikzcd}
\tag{6.14}
$$

Here the equality in the bottom left is given by Proposition 2.4.7. Suppose now that we have diagrams

$$
\begin{tikzcd}
P \arrow[drr] \arrow[ddr, "f"'] & & \\
 & N|C \arrow[r, tail] \arrow[d, two heads] & N \arrow[d, two heads] \\
 & M \arrow[r, tail] & R
\end{tikzcd}
\qquad
\begin{tikzcd}
N|C \arrow[r, tail] \arrow[d, two heads] & N \arrow[d, two heads] \arrow[ddr] & \\
M \arrow[r, tail] \arrow[drr, "g"'] & R & \\
 & & Q
\end{tikzcd}
\tag{6.15}
$$

Then appealing to the fact that $\beta|_C$ is an isomorphism, we now have

$$\begin{tikzcd}
P \arrow[drr] \arrow[ddr, "(\beta|_C)^{-1}\alpha f"'] & & \\
& N|C \arrow[r, tail] \arrow[d, two heads] & N \arrow[d, two heads] \\
& (N/A)|C \arrow[r, tail] & N/A
\end{tikzcd}
\qquad
\begin{tikzcd}
N|C \arrow[r, tail] \arrow[d, two heads] & N \arrow[d, two heads] \arrow[ddr] & \\
(N/A)|C \arrow[r, tail] \arrow[drr, "g\alpha^{-1}(\beta|_C)"'] & N/A & \\
& & Q
\end{tikzcd}
\tag{6.16}$$

In particular, the bi-Cartesian property of (6.14) now guarantees the existence of unique maps $P \to N|C$ and $R \cong N/A \to Q$. This establishes that the completion is biCartesian. □

Proposition 6.1.18. *Any diagram in F-**Mat**$_\bullet$ of the form $M \twoheadleftarrow R \rightarrowtail N$ can be completed to a biCartesian square.*

Proof. This proof follows immediately from Proposition 6.1.17 by applying duality as in the proof of Lemma 6.1.16. However, alternatively one can follow analogous steps to the proof of Proposition 6.1.17 as would be played out in the opposite category. □

Proposition 6.1.19. *Any square of the following form in F-**Mat**$_\bullet$ is Cartesian if and only if it is coCartesian:*

$$\begin{tikzcd}
M \arrow[r, tail] \arrow[d, two heads] & N \arrow[d, two heads] \\
P \arrow[r, tail] & R
\end{tikzcd}$$

Proof. Suppose first that the square is Cartesian. Then by Proposition 6.1.17, $P \rightarrowtail R \twoheadleftarrow N$ can be extended to the following commutative square:

$$\begin{tikzcd}
M' \arrow[r, tail] \arrow[d, two heads] & N \arrow[d, two heads] \\
P \arrow[r, tail] & R
\end{tikzcd}$$

Since our original square was assumed Cartesian, we obtain that $M \cong M'$ by the uniqueness of pullbacks up to isomorphism. In particular, this means that our square is coCartesian as well. The other direction is analogous, instead considering the biCartesian completion of $P \twoheadleftarrow M \rightarrowtail N$. $\square$

Putting together Propositions 6.1.14, 6.1.15,6.1.17, 6.1.18 and 6.1.19 we obtain Theorem 6.1.11 and therefore also Corollary 6.1.12. Observe also that the functor $\Phi_* : F$-$\mathbf{Mat}_\bullet \to G$-$\mathbf{Mat}_\bullet$ of Remark 6.1.5 given by a morphism $\Phi : F \to G$ of perfect idylls is an exact functor, owing to the definition of admissibles.

We now characterize the sum and intersection of strict subobjects in F-$\mathbf{Mat}_\bullet$ as described in Subsection B.5. Note that from Proposition 6.1.10, the notion of admissible subobjects in F-$\mathbf{Mat}_\bullet$ (viewed as a proto-exact category) agree with the notion of strict subobjects in F-$\mathbf{Mat}_\bullet$ (viewed as a proto-abelian category). Hence, we use the same symbol $\rightarrowtail$ to denote strict subobjects.

Let M be an F-matroid with strict subobjects $M_1, M_2 \rightarrowtail M$. Without loss of generality, we write $M_1 = M|A_1$ and $M_2 = M|A_2$ for some $A_1, A_2 \subseteq E_M$. In analogy with Diagram (B.7) we then draw the diagram

$$\begin{tikzcd}
(M|A_1) + (M|A_2) \arrow[dr, tail] & M|A_1 \arrow[l, tail] \arrow[d, tail] & \\
M|A_2 \arrow[u, tail] \arrow[r, tail] & M \arrow[r, two heads] \arrow[d, two heads] & M/A_2 \arrow[d, two heads] \\
& M/A_1 \arrow[r, two heads] & M'
\end{tikzcd}$$

where as in Diagram B.7 the bottom right square is a pushout and the map $(M|A_1) + (M|A_2) \rightarrowtail M$ is the kernel of the composition $M \to M'$. By a straightforward argument similar to the proof of Lemma 6.1.16, it follows that M' can be identified with the matroid $M/(A_1 \cup A_2)$, so that Proposition 6.1.10 yields $(M|A_1) + (M|A_2) = M|(A_1 \cup A_2)$. Likewise, one verifies analogously that $(M|A_1) \cap (M|A_2) = M|(A_1 \cap A_2)$. This establishes the following proposition.

Proposition 6.1.20. *Let $M_1, M_2 \rightarrowtail M$ be strict subobjects of M in F-$\mathbf{Mat}_\bullet$ given as restric-*

tions $M_1 = M|A_1$ and $M_2 = M|A_2$. Then the sum and intersection of the subobjects M_1 and M_2 are given by

$$(M|A_1) + (M|A_2) = M|(A_1 \cup A_2)$$

$$(M|A_1) \cap (M|A_2) = M|(A_1 \cap A_2).$$

We close this subsection with a verification of Artinian and Noetherian conditions in F-**Mat**$_\bullet$.

Proposition 6.1.21. *Let M be an F-matroid and let $\cdots \rightarrowtail M_{i-1} \rightarrowtail M_i \rightarrowtail \cdots$ be an infinite chain of strict subobjects of M. Then for some $n_-, n_+ \in \mathbb{Z}$ we have that $M_{i-1} \cong M_i$ for $i \leq n_-$ and $M_j \cong M_{j+1}$ for $j \geq n_+$.*

Proof. Since each M_i is a strict subobject of M, we may write $M_i = M|A_i$ for some $A_i \subseteq E_M$. In particular, since E_M is a finite set and $M_{i-1} \rightarrowtail M_i$ implies $A_{i-1} \subseteq A_i$, we must have that the sets A_i stabilize as $i \to \infty$ or $i \to -\infty$. This completes the proof. □

6.1.4 Further Structure on F-Mat$_\bullet$

In this subsection we establish directly that the categories F-**Mat**$_\bullet$ are combinatorial and have direct sum and duality in the sense of Subsection B.2. We continue to work with a perfect idyll F.

Theorem 6.1.22. *The category F-**Mat**$_\bullet$ is a combinatorial proto-exact category with duality and with exact direct sum.*

We prove this theorem in three parts as Propositions 6.1.24, 6.1.26, and 6.1.25 which we separate for convenience. The reader may wish to refer back to Definitions B.2.1 and B.2.2.

Lemma 6.1.23. *Let $M_1 = [\mu_1]$ and $M_2 = [\mu_2]$ be pointed F-matroids of ranks r_1 and r_2 on ground sets E_{M_1} and E_{M_2} and let $A_i \subseteq E_{M_i}$ for $i = 1, 2$. Then we have the following* *equalities:*

$$(M_1/A_1) \oplus (M_2/A_2) = (M_1 \oplus M_2)/(A_1 \sqcup A_2)$$
$$(M_1|A_1) \oplus (M_2|A_2) = (M_1 \oplus M_2)|(A_1 \sqcup A_2)$$

Proof. We verify only the first equality above, as the other follows analogously. For $i = 1, 2$, suppose that A_i satisfies $r_{\underline{M_i}}(A_i) = s_i$ and let $\mathbf{a}^{(i)}$ be a basis of the classical matroid $\underline{M_i}|A_i$. Then for $\mathbf{x} \in (E_{M_1} \setminus A_1)^{r_1 - s_1}$ and $\mathbf{y} \in (E_{M_2} \setminus A_2)^{r_2 - s_2}$ we see the following.

$$\begin{aligned}
((\mu_1/A_1) \oplus (\mu_2/A_2))(\mathbf{x}, \mathbf{y}) &= \mu_1(\mathbf{x}, \mathbf{a}^{(1)}) \cdot \mu(\mathbf{y}, \mathbf{a}^{(2)}) \\
((\mu_1 \oplus \mu_2)/(A_1 \sqcup A_2))(\mathbf{x}, \mathbf{y}) &= (\mu_1 \oplus \mu_2)(\mathbf{x}, \mathbf{y}, \mathbf{a}^{(1)}, \mathbf{a}^{(2)}) \\
&= \mu_1(\mathbf{x}, \mathbf{a}^{(1)}) \cdot \mu_2(\mathbf{y}, \mathbf{a}^{(2)}).
\end{aligned}$$

We thus see that the associated Grassmann-Plücker functions are equal and so must be the associated matroids. □

Proposition 6.1.24. *The operation $\oplus$ is an exact direct sum on the category F-**Mat**$_\bullet$.*

Proof. By the definition of the operation $\oplus$ for pointed F-matroids, the axiom (DS1) is immediate. To see that $\oplus : F\text{-}\mathbf{Mat}_\bullet \times F\text{-}\mathbf{Mat}_\bullet \to F\text{-}\mathbf{Mat}_\bullet$ is exact, suppose that we have a short exact sequence

$$(M_1, M_2) \rightarrowtail (X_1, X_2) \twoheadrightarrow (N_1, N_2).$$

This means that $M_i \rightarrowtail X_i \twoheadrightarrow N_i$ is a short exact sequence for $i = 1, 2$, and so we may assume without loss of generality that there are subsets $A_i \subseteq E_{X_i}$ for which $M_i = X_i|A_i$ and $N_i = X_i/A_i$. We would like to show that the image of this sequence under $\oplus$ is a short exact sequence in F-$\mathbf{Mat}_\bullet$. That is, we would like to see that $M_1 \oplus M_2 \cong (X_1 \oplus X_2)|(A_1 \sqcup A_2)$ and $N_1 \oplus N_2 \cong (X_1 \oplus X_2)/(A_1 \sqcup A_2)$. These isomorphisms in fact hold with equality by Lemma

6.1.23. This establishes that (DS2) holds.

To verify axiom (DS3), let M, N, R be F-matroids on ground sets E_M, E_N, and E_R, respectively. Denote by $\iota_M : M \rightarrowtail M \oplus N$ and $\pi_M : M \oplus N \twoheadrightarrow M$ the associated inclusion and projection, and similar for ι_N, ι_R, π_N, and π_R. We illustrate that the following map is an inclusion:

$$\mathrm{Hom}(M \oplus N, R) \longrightarrow \mathrm{Hom}(M, R) \times \mathrm{Hom}(N, R)$$
$$f \longmapsto (f \circ \iota_M, f \circ \iota_N)$$

The second property regarding $\mathrm{Hom}(R, M \oplus N)$ follows analogously and so we do not include the proof here. Suppose that $f, g : M \oplus N \to R$ are morphisms of F-matroids for which $f \circ \iota_M = g \circ \iota_M$ and $f \circ \iota_N = g \circ \iota_N$. Noting that ι_M is in fact the map $r_{E_M} : (M \oplus N)|E_M \rightarrowtail M \oplus N$, we note that it is represented by a block matrix $\begin{bmatrix} I & 0 \end{bmatrix}^T$, where I is the $\tilde{E}_M \times \tilde{E}_M$ identity matrix and 0 is the $\tilde{E}_M \times \tilde{E}_N$ zero matrix. Likewise, $\iota_N = r_{E_N}$ can be represented by the block matrix $\begin{bmatrix} 0 & I \end{bmatrix}^T$ with blocks of the appropriate size.

The morphisms f and g are both represented by $\tilde{E}_R \times (\tilde{E}_M \sqcup \tilde{E}_N)$ matrices. Say f has matrix $\begin{bmatrix} F_M & F_N \end{bmatrix}$ and g has matrix $\begin{bmatrix} G_M & G_N \end{bmatrix}$, where F_M and G_M are of size $\tilde{E}_R \times \tilde{E}_M$ while F_N and G_N are $\tilde{E}_R \times \tilde{E}_N$. Then $f \circ \iota_M = g \circ \iota_M$ has matrix

$$\begin{bmatrix} F_M & F_N \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} F_M & 0 \end{bmatrix} = \begin{bmatrix} G_M & G_N \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix}.$$

Similarly, we conclude that $f \circ \iota_N = g \circ \iota_N$ has matrix

$$\begin{bmatrix} F_M & F_N \end{bmatrix} \begin{bmatrix} 0 \\ I \end{bmatrix} = \begin{bmatrix} 0 & F_N \end{bmatrix} = \begin{bmatrix} G_M & G_N \end{bmatrix} \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

Thus we see that $F_M = G_M$ and $F_N = G_N$, so that $f = g$. This shows that the above map of Hom-sets is an injection. Together with the analogous property for $\mathrm{Hom}(R, M \oplus N) \to$

$\mathrm{Hom}(R, M) \times \mathrm{Hom}(R, N)$, this establishes axiom (DS3).

For (DS4), suppose that $M \rightarrowtail X \twoheadrightarrow N$ is a short exact sequence in F-$\mathbf{Mat}_\bullet$ with injection ι and projection π, for which we have a section s of π giving the following diagram in F-$\mathbf{Mat}_\bullet$.

$$\begin{tikzcd}
& X & \\
M \arrow[ur, tail, "\iota"] \arrow[r, tail, "\iota_M"'] & M \oplus N \arrow[u, dashed, "f"] & N \arrow[ul, tail, "s"'] \arrow[l, tail, "\iota_N"]
\end{tikzcd}$$

We seek to find an isomorphism $f : M \oplus N \to X$ making the diagram above commute and to show that it is unique. As earlier, we assume without loss of generality that $M = X|A$ and $N = X/A$ for some $A \subseteq E_X$. Then $\pi = c_A$ is represented by the $(\tilde{E}_X \setminus A) \times \tilde{E}_X$ matrix $\begin{bmatrix} 0 & I \end{bmatrix}$, where I is an $(\tilde{E}_X \setminus A) \times (\tilde{E}_X \setminus A)$ identity matrix. Similarly, $\iota = r_A$ has $\tilde{E}_X \times A$ matrix $\begin{bmatrix} I & 0 \end{bmatrix}^T$ with first block an $A \times A$ identity matrix.

The section $s : X/A \to X$ is represented by an $\tilde{E}_X \times (\tilde{E}_X \setminus A)$ matrix $\begin{bmatrix} 0 & S \end{bmatrix}^T$ where S is an $(\tilde{E}_M \setminus A) \times (\tilde{E}_M \setminus A)$ matrix. This means that $\pi \circ s$ has matrix

$$\begin{bmatrix} 0 & I_{E_X \setminus A} \end{bmatrix} \begin{bmatrix} 0 \\ S \end{bmatrix} = \begin{bmatrix} S \end{bmatrix}.$$

Since $\pi \circ s = \mathrm{id}_N$, we see that s has matrix $\begin{bmatrix} 0 & I_{E_X \setminus A} \end{bmatrix}^T$, hence $s = r_{E_X \setminus A}$ is an injection. We can now restate our diagram as follows:

$$\begin{tikzcd}
& X & \\
X|A \arrow[ur, tail, "r_A"] \arrow[r, tail, "r_A"'] & (X|A) \oplus (X/A) \arrow[u, dashed, "f"] & X/A \arrow[ul, tail, "r_{E_X \setminus A}"'] \arrow[l, tail, "r_{E_X \setminus A}"]
\end{tikzcd}$$

Clearly one can choose f to have the $\tilde{E}_X \times \tilde{E}_X$ identity matrix. This is in fact the only choice.

To see this, suppose that f is represented by the block matrix

$$\begin{bmatrix} F_{00} & F_{01} \\ F_{10} & F_{11} \end{bmatrix},$$

where F_{00} is $A \times A$ and F_{11} is $(\tilde{E}_X \setminus A) \times (\tilde{E}_X \setminus A)$. Then the two equations

$$\begin{bmatrix} I_A \\ 0 \end{bmatrix} = \begin{bmatrix} F_{00} & F_{01} \\ F_{10} & F_{11} \end{bmatrix} \begin{bmatrix} I_A \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ I_{E_X \setminus A} \end{bmatrix} = \begin{bmatrix} F_{00} & F_{01} \\ F_{10} & F_{11} \end{bmatrix} \begin{bmatrix} 0 \\ I_{\tilde{E}_X \setminus A} \end{bmatrix}$$

imply that the only choice is $F_{00} = I_A$ and $F_{11} = I_{\tilde{E}_X \setminus A}$ with all other entries being zero. The argument regarding retractions follows similarly. $\square$

Proposition 6.1.25. *F-**Mat**$_\bullet$ is a combinatorial proto-exact category with respect to the exact direct sum $\oplus$.*

Proof. Let M_1, M_2, and N be F-matroids on ground sets E_{M_1}, E_{M_2}, and E_N. Given an admissible mono $g : N \rightarrowtail M_1 \oplus M_2$, we would like to exhibit admissible monos $\iota_k : N_k \rightarrowtail M_k$ for $k = 1, 2$ for which there is an isomorphism $f : N \longrightarrow N_1 \oplus N_2$ with $(\iota_1 \oplus \iota_2) \circ f = g$.

As before, assume without loss of generality that $N = (M_1 \oplus M_2)|(A_1 \sqcup A_2)$ for some subsets $A_k \subseteq E_{M_k}$. We define $N_k := N|A_k = (M_1 \oplus M_2)|A_k = M_k|A_k$. Then by Lemma 6.1.23 we see

$$\begin{aligned} N_1 \oplus N_2 &= (M_1|A_1) \oplus (M_2|A_2) \\ &\cong (M_1 \oplus M_2)|(A_1 \sqcup A_2) = N \end{aligned}$$

and the equation $(\iota_1 \oplus \iota_2) \circ f = g$ immediately follows. $\square$

Proposition 6.1.26. *F-matroid duality $M \mapsto M^*$ endows F-**Mat**$_\bullet$ with the structure of a proto-exact category with duality.*

Proof. Verification of the axioms (D1) and (D2) are immediate. Indeed, the zero element of

F-**Mat**$_\bullet$ is the matroid $U(0,1)$, whose self-duality establishes (D1). Further, (D2) follows from the fact that dual morphisms are given by transposing the corresponding matrices and the form of matrices representing admissible monos and epis.

To verify (D3), suppose that we have a biCartesian square in F-**Mat**$_\bullet$. Without loss of generality we may assume that it is of the following form for some matroid M and disjoint subsets $A, B \subseteq E_M$.

$$\begin{tikzcd} M|A \arrow[r, tail] \arrow[d, two heads] & M \arrow[d, two heads] \\ (M|A)/B \arrow[r, tail] & M/B \end{tikzcd} \tag{6.17}$$

Recall that for any $S \subseteq E_M$ we have $(M|S)^* = M^*/S$ and $(M/S)^* = M^*|S$. Together with Proposition 2.4.7 we then see

$$((M|A)/B)^* = (M|A)^*|B = (M^*/A)|B \cong (M^*|B)/A.$$

So that applying duality to the above square yields

$$\begin{tikzcd} M^*/A & M^* \arrow[l, two heads] \\ (M^*|B)/A \arrow[u, tail] & M^*|B \arrow[l, two heads] \arrow[u, tail] \end{tikzcd} \tag{6.18}$$

and we know this to be biCartesian by Lemma 6.1.16. □

Proposition 6.1.27. *The category F-**Mat**$_\bullet$ is not split. In particular, for each perfect idyll F there exist short exact sequences $M \rightarrowtail N \twoheadrightarrow P$ for which N is not isomorphic to $M \oplus P$.*

Proof. Let C_4 be the cycle graph with edges a, b, c, d and let $S = \{a\}$. Then, denoting by M the (simple) graphic matroid of C_4, we have that M is an $\mathbb{F}_1^{\pm}$-matroid (i.e. a regular matroid) and we have the following short exact sequence in $\mathbb{F}_1^{\pm}$-**Mat**$_\bullet$:

$$M|S \rightarrowtail M \twoheadrightarrow M/S.$$

Here $M|S$ is the graphic matroid of the path of length 1 and M/S is the graphic matroid of the cycle C_3. Again, both matroids are simple. Since direct sums of graphic matroids correspond to disjoint union of the associated graphs, it is clear that M is not isomorphic to $(M|S) \oplus (M/S)$.

For an arbitrary perfect idyll F, we can push the above example along the unique idyll morphism $\mathbb{F}_1^{\pm} \to F$ to realize this example in F-$\mathbf{Mat}_\bullet$. □

This establishes that F-$\mathbf{Mat}_\bullet$ is neither split nor uniquely split for any perfect idyll F.

6.2. The category F-SMat$_\bullet$

Recall that an F-matroid M is said to be simple precisely when its underlying matroid $\underline{M}$ is. The collection of all simple matroids with strong maps forms a reflective subcategory of the category of matroids with strong maps. Indeed, the assignment $M \mapsto \mathrm{si}(M)$ is a functor from matroids to simple matroids, and this functor is left adjoint to the inclusion.

These statements also hold for pointed matroids by using the pointed simplification $\mathrm{si}_\bullet$ of a pointed matroid M. In short, one forgets the distinguished loop of M, takes the non-pointed simplification, and takes the direct sum with $U(0,1)$ to recover a pointed matroid. See [27, Section 7] for more details on pointed and non-pointed simplification.

In our setting, the analogous statements are immediately verified: the collection of all pointed simple F-matroids together with F-matroid morphisms forms a reflective subcategory of F-$\mathbf{Mat}_\bullet$. This follows because simplicity is defined purely in terms of the underlying matroid and because $\mathbb{K}$-$\mathbf{Mat}_\bullet$ can be identified with the subcategory of $\mathbf{Mat}_\bullet$ consisting of $\mathbb{F}_1$-linear morphisms.

Observe that all restrictions of simple matroids are necessarily again simple. In light of Proposition 6.1.10 this observation establishes that F-$\mathbf{SMat}_\bullet$ has all kernels. Since the simplification functor $\mathrm{si}_\bullet$ is a left adjoint, we also see that it preserves colimits and so F-$\mathbf{SMat}_\bullet$ also has cokernels.

Remark 6.2.1 (Kernels and cokernels)**.** In fact, we can say more regarding kernels and cokernels in F-**SMat**$_\bullet$. Recall that the kernel of a morphism $f : M \to N$ in F-**Mat**$_\bullet$ is $M|\underline{f}^{-1}(*_N)$. If both M and N are simple, then by definition this means that $*_N$ is a flat of $\underline{N}$. Thus, by the fact that underlying map $\underline{f} : \underline{M} \to \underline{N}$ is strong, we see that $\underline{f}^{-1}(*_N)$ must be a flat of $\underline{M}$. This implies that any kernel in F-**SMat**$_\bullet$ is the restriction morphism associated to some flat $F \in \mathcal{L}(M)$.

With regards to cokernels, observe that given a simple F-matroid M and contraction morphism $c_A : M \to M/A$ in F-**Mat**$_\bullet$, applying $\mathrm{si}_\bullet$ yields $\mathrm{si}_\bullet(M) \to \mathrm{si}_\bullet(M/A)$. A routine exercise in matroid theory yields that the underlying matroid of $\mathrm{si}_\bullet(M/A)$ is $\underline{M}/\langle A\rangle$, where $\langle A\rangle \in \mathcal{L}(M)$ is the flat of $\underline{M}$ obtained as the closure of the subset A. Thus all contractions (and therefore all cokernels) in F-**SMat**$_\bullet$ are contractions at flats. ♦

As a reflective subcategory of F-**Mat**$_\bullet$, it is also immediate that F-**SMat**$_\bullet$ satisfies axiom (PA1) of proto-abelian categories (Subsection B.4).

Proposition 6.2.2. *The category F-**SMat**$_\bullet$ admits all kernels and cokernels of morphisms. Moreover, it satisfies axiom (PA1) of a proto-abelian category.*

As a departure from the proto-exact structure on F-**Mat**$_\bullet$, we define the classes of admissible monomorphisms and epimorphisms yielding the proto-exact structure on F-**SMat**$_\bullet$ as follows.

Definition 6.2.3. Let $f : M \to N$ be a morphism in F-**SMat**$_\bullet$.

1. f is an admissible monomorphism if $f = r_F \circ \alpha$, where $F \in \mathcal{L}(N)$ and $r_F : M|F \to M$ is an inclusion map, and α is an isomorphism of F-matroids:

$$M \xrightarrow[\sim]{\alpha} N|F \xrightarrow{r_F} N,$$

Denote by $\mathfrak{M}_s$ the class of admissible monomorphisms in F-**SMat**$_\bullet$.

2. f is an admissible epimorphism if $f = \beta \circ c_G$, where $G \in \mathcal{L}(M)$ and $c_G : \mathcal{E} \to \mathcal{E}/G$ is a contraction map and β is an isomorphism of F-matroids:

$$M \xrightarrow{c_G} M/G \xrightarrow[\sim]{\beta} N.$$

Denote by $\mathfrak{E}_s$ the class of admissible epimorphisms in F-**SMat**$_\bullet$.

♦

Remark 6.2.4 (Admissible morphisms)**.** Observe that $\mathfrak{M}_s$ and $\mathfrak{E}_s$ are strictly smaller than the intersection of the classes $\mathfrak{M}$ and $\mathfrak{E}$ with the morphisms of the subcategory F-**SMat**$_\bullet$. For example, if M is a simple matroid and there is some subset $A \subseteq E_M$ for which $M|A$ is simple but A is not a flat of M, then the inclusion map $r_A : M|A \to M$ is in $\mathfrak{M}$ but is not in $\mathfrak{M}_s$.

Despite this difference, however, note that $\mathfrak{M}$ and $\mathfrak{M}_s$ both consist of isomorphisms followed by kernels of morphisms in their respective categories. Likewise, $\mathfrak{E}$ and $\mathfrak{E}_s$ consist of cokernels of morphisms followed by isomorphisms. A similar characterization of admissibles will be used for the proto-exact structure on tropical reflexive sheaves in Section 6.4. ♦

Observe that Propositions 6.1.14 and 6.1.15, as well as Lemma 6.1.16, hold in F-**SMat**$_\bullet$. The latter then gives that Propositions 6.1.17, and 6.1.19 hold as well. Lemma 6.1.16 is true on the nose with arrows coming from $\mathfrak{M}_s$ and $\mathfrak{E}_s$ (so that A and B should be flats), while the other propositions follow essentially from the observation that if N, $N|A$, and N/B are simple matroids, so must be $(N|A)/B$. Together with Proposition B.4.4 and Proposition 6.2.2, we obtain the following corollary.

Corollary 6.2.5. *For each perfect idyll F, the category F-**SMat**$_\bullet$ is proto-abelian.*

Sums and intersections of subobjects in F-**SMat**$_\bullet$ depart slightly from the characterization of those in F-**Mat**$_\bullet$ as given in Proposition 6.1.20. Instead, we have the following:

Proposition 6.2.6. *Let $M_1, M_2 \rightarrowtail M$ be strict subobjects of M in F-**SMat**$_\bullet$ given as restrictions $M_1 = M|F_1$ and $M_2 = M|F_2$ at flats $F_1, F_2 \in \mathcal{L}(M)$. Then the sum and intersection of the subobjects M_1 and M_2 are given by*

$$(M|F_1) + (M|F_2) = M|(F_1 \vee F_2)$$
$$(M|F_1) \cap (M|F_2) = M|(F_1 \wedge F_2).$$

The characterization of the intersection is identical to F-**Mat**$_\bullet$, as the intersection of F_1 and F_2 is precisely $F_1 \wedge F_2$. The characterization of the sum follows by reasoning identical to that regarding contractions of simple matroids. In particular, when contracting $F_1 \cup F_2$ one must instead contract $\langle F_1 \cup F_2 \rangle = F_1 \vee F_2$ in order to obtain a simple F-matroid.

Remark 6.2.7 (No duality)**.** Since the dual of a simple matroid is not necessarily simple, we see that F-**SMat**$_\bullet$ is not proto-exact with duality as F-**Mat**$_\bullet$ is. Indeed, one easily finds simple planar graphs whose planar duals are not simple. Taking the associated graphic matroids then yields two dual $\mathbb{F}_1$-matroids (i.e. regular matroids), one of which is simple and the other of which is not. Since $\mathbb{F}_1$ is initial in the category of idylls, this counterexample can be realized in all categories F-**SMat**$_\bullet$. ♦

Remark 6.2.8 (Artinian and Noetherian conditions)**.** Observe that F-**SMat**$_\bullet$ inherits the Artinian and Noetherian properties of F-**Mat**$_\bullet$ described by Proposition 6.1.21. This is an immediate consequence of the fact that F-**SMat**$_\bullet$ is a subcategory of F-**Mat**$_\bullet$. ♦

6.3. Hall algebras and K-theory of F-matroids

Proposition 6.3.1. *For a finite perfect idyll F, the category F-**Mat**$_\bullet$ is finitary. In particular, $\mathbb{K}$-**Mat**$_\bullet$, $\mathbb{S}$-**Mat**$_\bullet$, and $\mathbb{F}_1^\pm$-**Mat**$_\bullet$ are finitary categories.*

Proof. We show this directly, by illustrating that for any two F-matroids M and N that the sets $\mathrm{Hom}(M, N)$ and $\mathrm{Ext}^1(M, N)$ are finite. Let M and N be F-matroids on ground sets E_M

and E_N, and let $f : M \to N$ be a morphism. Then f is given by a submonomial matrix with entries in F, hence $\mathrm{Hom}(M, N)$ is necessarily finite by the finiteness of F.

Suppose now that P is an extension of N by M of rank r_P. Then that we have a short exact sequence

$$M \rightarrowtail P \twoheadrightarrow N.$$

By the exactness of the above sequence, we see that P is necessarily a matroid on a ground set E_P of cardinality $|\tilde{E}_M \sqcup \tilde{E}_N \sqcup \{*_P\}|$, so that we can obtain the desired finiteness by illustrating that there are only finitely many such F-matroids. But this is necessarily the case since the number of such matroids $P = [\pi]$ is bounded by the number of Grassmann-Plücker maps $\pi : E_P^{r_P} \to F$. The common domain and codomain of these maps is finite, hence there are only finitely many such maps and so finitely many such F-matroids P. □

The above proposition now establishes that F-$\mathbf{Mat}_\bullet$ is finitary for $F = \mathbb{K}, \mathbb{S}$, and $\mathbb{F}_1^\pm$. However for $F = \mathbb{T}$ the situation is slightly more delicate. In particular, $\mathbb{T}$ is not finite, and so the above argument does not apply.

From Theorem 6.1.11, one obtains the Hall algebra $H_{F\text{-}\mathbf{Mat}_\bullet}$ for the category F-$\mathbf{Mat}_\bullet$.

Example 6.3.2. One can easily check that $H_{\mathbb{K}\text{-}\mathbf{Mat}_\bullet}$ is the Hopf dual of matroid-minor Hopf algebra. In fact, in [22], it is shown that the matroid-minor Hopf algebra is the Hopf dual of the Hall algebra $H_{\mathbf{Mat}_\bullet}$. ◊

6.3.1 K-Theory of F-$\mathbf{Mat}_\bullet$

Fix a perfect idyll F. In this subsection we compute the Grothendieck group $K_0(F\text{-}\mathbf{Mat}_\bullet)$ and show that it coincides with the Grothendieck group $K_0(\mathbf{Mat}_\bullet)$ of pointed (classical) matroids, as computed in [22]. This illustrates in particular that the Grothendieck group is not sensitive to the coefficients used to define the associated category of matroids.

Proposition 6.3.3. *We have $K_0(F\text{-}\mathbf{Mat}_\bullet) \cong \mathbb{Z} \oplus \mathbb{Z}$.*

Proof. Given $M = [\mu] \in F\text{-}\mathbf{Mat}_\bullet$, fix some non-loop element $e \in E_M$. Then we have the following short exact sequence:

$$M|e \rightarrowtail M \twoheadrightarrow M/e.$$

Thus in $K_0(F\text{-}\mathbf{Mat}_\bullet)$ we have the equation $[M] = [M|e] + [M/e]$. Note that $M|e$ is now an F-matroid of rank 1 on a ground set consisting of a non-loop and the basepoint. Evaluating the associated Grassmann-Plücker function, we then see

$$(\mu|e)(e') = \begin{cases} \alpha & e' = e \\ 0_F & e' = *_M \end{cases}$$

for some $\alpha \in F^\times$, so that in particular we have an isomorphism $M|e \cong C$, where C is the matroid consisting of one non-loop and the basepoint. We would like to perform this process $\mathrm{rk}(\underline{M})$ times. To this end, let $B \subseteq \tilde{E}_M$ be a basis of $\underline{M}$ and perform the above process for each $e \in B$. This yields the following decomposition:

$$[M] = \mathrm{rk}(\underline{M})[C] + [M/B].$$

On the other hand, when we contract all elements of B, we see that the resulting matroid consists only of loops and the basepoint $*_M$. However, appealing to the definition of the direct sum on $F\text{-}\mathbf{Mat}_\bullet$, we then see that

$$M/B \cong L \oplus \cdots \oplus L,$$

where L is the matroid with two loops, one distinguished, and the number of summands is equal to the corank (or nullity) of M. Denoting this number by $\mathrm{null}(M)$, we then see that

$$[M] = \mathrm{rk}(\underline{M})[C] + \mathrm{null}(M)(\underline{M})[L].$$

Since an F-matroid can have arbitrary rank and an arbitrary number of loops, this establishes the desired result. □

Proposition 6.3.4. *Let $\Phi : F \to G$ be a morphism of idylls. Then, for each $n \in \mathbb{N}$, Φ induces a morphism*

$$\Phi_* : K_n(F\text{-}\boldsymbol{Mat}_\bullet) \to K_n(G\text{-}\boldsymbol{Mat}_\bullet).$$

Proof. By Remark 6.1.5 we have that Φ induces a functor from F-$\mathbf{Mat}_\bullet$ to G-$\mathbf{Mat}_\bullet$. By the definition of restrictions and contractions we then immediately see that this functor preserves short exact sequences, hence induces a morphism on K_n for each n. □

Remark 6.3.5. Proposition 6.3.3 suggests that to see any difference between K-theory of F-$\mathbf{Mat}_\bullet$, one may have to go higher K-theory. In fact, from the adjunction between $\mathbf{Mat}_\bullet$ and $\mathbf{FinSet}_\bullet$, one obtains the following:

$$\pi_n^s(\mathbf{S}) \cong K_n(\mathbf{FinSet}_\bullet) \hookrightarrow K_n(\mathbf{Mat}_\bullet) \text{ for all } n \in \mathbb{N},$$

where $\pi_n^s(\mathbf{S})$ denotes the stable homotopy groups of the sphere spectrum (see [22] for further details). So, it would be interesting to compute even $K_1(\mathbf{Mat}_\bullet)$ to see whether or not the above inclusion is proper. ♦

6.4. The category TRS$_\bullet$

We now examine the categorical structure of tropical toric reflexive sheaves. In this section, all tropical toric reflexive sheaves, $\mathbb{T}$-matroids, and morphisms thereof are assumed to be pointed. Let L be a free abelian group of finite rank and fix a complete rational fan Σ in a vector space $L_\mathbb{R}$ and let $\Lambda = \mathrm{Hom}(L, \mathbb{Z})$ be the dual lattice. Denote by X_Σ the associated complete toric variety and $\mathrm{trop}(X_\Sigma)$ the tropical toric varierty with fan Σ as in [34] or [47].

Definition 6.4.1. Let $\mathcal{E} = (M, \{F_\bullet^\rho\}_\rho)$ and $\mathcal{F} = (N, \{G_\bullet^\rho\}_\rho)$ be tropical toric reflexive sheaves on $\mathrm{trop}(X_\Sigma)$ (Section 3.4.1). A **morphism** $f : \mathcal{E} \to \mathcal{F}$ is a pair $f = (f_\mathbb{T}, u)$, where $f_\mathbb{T}$ is a morphism of simple $\mathbb{T}$-matroids $f_{\mathbb{T}} : M \to N$ and $u \in \Lambda$ is a vector such that for each ray $\rho \in \Sigma(1)$ and each $j \in \mathbb{Z}$, we have

$$f(F^\rho_j) \subseteq G^\rho_{j+u\cdot v_\rho} \tag{6.19}$$

Recall that $v_\rho \in L$ is the primitive lattice vector associated to the ray ρ. ♦

It is clear that the tropical toric reflexive sheaves form a category, and for morphisms $f = (f_{\mathbb{T}}, u) : \mathcal{E} \to \mathcal{F}$ and $g = (g_{\mathbb{T}}, w) : \mathcal{F} \to \mathcal{G}$ the composition is given by $g \circ f = (g_{\mathbb{T}} \circ f_{\mathbb{T}}, u + w)$. We denote the resulting category by $\mathbf{TRS}_\bullet$. Note that the definition of morphisms in $\mathbf{TRS}_\bullet$ yields a forgetful functor $\mathbf{TRS}_\bullet \to \mathbb{T}\text{-}\mathbf{Mat}_\bullet$ which simply forgets the filtrations associated to objects and the translations u associated to morphisms.

Under this definition of morphism, observe that two tropical toric reflexive sheaves are isomorphic precisely when they are isomorphic in the sense of [38].

Lemma 6.4.2. *Let $\mathcal{E} = (M, \{F^\rho_\bullet\}_\rho)$ and $\mathcal{F} = (N, \{G^\rho_\bullet\}_\rho)$ be tropical toric reflexive sheaves on trop(X_Σ) Suppose that $f = (f_{\mathbb{T}}, u) : \mathcal{E} \to \mathcal{F}$ and $g = (f_{\mathbb{T}}^{-1}, w) : \mathcal{F} \to \mathcal{E}$ are morphisms in $\boldsymbol{TRS}_\bullet$ for which $f_{\mathbb{T}}^{-1} = (f_{\mathbb{T}})^{-1}$ in $\mathbb{T}$-$\boldsymbol{Mat}_\bullet$. Then $w = -u$.*

Proof. Note that for all $\rho \in \Sigma(1)$ and all $j \in \mathbb{Z}$ we have

$$\begin{aligned} f_{\mathbb{T}}(F^\rho_j) &\subseteq G^\rho_{j+u\cdot v_\rho} \\ f_{\mathbb{T}}^{-1}(G^\rho_j) &\subseteq F^\rho_{j+w\cdot v_\rho}. \end{aligned}$$

Then we may write

$$\begin{aligned} F^\rho_j &= f_{\mathbb{T}}^{-1}(f_{\mathbb{T}}(F^\rho_j)) \subseteq f_{\mathbb{T}}^{-1}(G_{j+u\cdot v_\rho}) \subseteq F^\rho_{j+(u+w)\cdot v_\rho} \\ G^\rho_j &= f_{\mathbb{T}}(f_{\mathbb{T}}^{-1}(G^\rho_j)) \subseteq G^\rho_{j+(u+w)\cdot v_\rho}. \end{aligned}$$

Now suppose for the sake of contradiction that $u + w \neq 0$. Then by the assumption that

Σ is a complete fan, there is some ray $\rho_+ \in \Sigma(1)$ with primitive generator v_+ for which $(u+w) \cdot v_+ > 0$. Then since $F_\bullet^{\rho_+}$ is a decreasing chain of flats we see that $F_{j+(u+w)\cdot v_+}^{\rho_+} \subseteq F_j^{\rho_+}$ as well, hence $F_j^{\rho_+} = F_{j+(u+w)\cdot v_+}^{\rho_+}$ for each $j \in \mathbb{Z}$. But then the chain $F_\bullet^{\rho_+}$ is constant, violating the definition of a tropical toric reflexive sheaf. Thus it cannot be that $u + w \neq 0$. $\square$

Proposition 6.4.3. *Two tropical reflexive sheaves $\mathcal{E} = (M, \{F_\bullet^\rho\}_\rho)$ and $\mathcal{F} = (N, \{G_\bullet^\rho\}_\rho)$ are isomorphic in the sense of Definition 3.4.4 if and only if there is an isomorphism $f : \mathcal{E} \to \mathcal{F}$ in **TRS**$_\bullet$.*

Proof. If $\mathcal{E}$ and $\mathcal{F}$ are isomorphic in the sense of Definition 3.4.4, then there is a $\mathbb{T}$-matroid isomorphism $f_\mathbb{T} : M \to N$ and a vector $u \in \Lambda$ for which

$$f_\mathbb{T}(F_j^\rho) = G_{j+u\cdot v_\rho}^\rho$$

for every $\rho \in \Sigma(1)$ and every $j \in \mathbb{Z}$. Denote by $(f_\mathbb{T})^{-1}$ the inverse of $f_\mathbb{T}$ in $\mathbb{T}$-**Mat**$_\bullet$. Then by definition $f := (f_\mathbb{T}, u)$ and $f^{-1} := ((f_\mathbb{T})^{-1}, -u)$ are morphisms in **TRS**$_\bullet$ which are mutually inverse.

Conversely, let $f = (f_\mathbb{T}, u)$ be an isomorphism from $\mathcal{E}$ to $\mathcal{F}$ in **TRS**$_\bullet$. Then by definition there is an inverse morphism in **TRS**$_\bullet$ which by Lemma 6.4.2 we can write as $f^{-1} = (f_\mathbb{T}^{-1}, -u)$. That is, we have

$$f_\mathbb{T}(F_j^\rho) \subseteq G_{j+u\cdot v_\rho}^\rho$$
$$f_\mathbb{T}^{-1}(G_{j+u\cdot v_\rho}^\rho) \subseteq F_{j+(-u+u)\cdot v_\rho}^\rho = F_j^\rho.$$

Thus by applying $f_\mathbb{T}$ to both sides of the second equality we see that in fact $f_\mathbb{T}(F_j^\rho) = G_{j+u\cdot v_\rho}^\rho$, establishing that $\mathcal{E}$ and $\mathcal{F}$ are isomorphic in the sense of Definition 3.4.4. $\square$

This illustrates that isomorphism in **TRS**$_\bullet$ and isomorphism in the sense of Definition 3.4.4 are the same.

Definition 6.4.4. [38, Definitions 7.2 and 7.17] Let $\mathcal{E} = (M, \{E^\rho_\bullet\}_\rho)$ be a tropical toric reflexive sheaf and take $F \in \mathcal{L}(M)$.

1. The restriction of $\mathcal{E}$ to F is the tropical reflexive sheaf $\mathcal{E}|F = (M|F, \{F^\rho_\bullet\}_\rho)$, where

$$F^\rho_j := F \wedge E^\rho_j \in \mathcal{L}(M).$$

2. The contraction (or quotient) of $\mathcal{E}$ by F is the tropical reflexive sheaf $\mathcal{E}/F = (M/F, \{G^\rho_\bullet\}_\rho)$, where $G^\rho_j \in \mathcal{L}(M/F)$ is the flat such that under the lattice isomorphism $\varphi : \mathcal{L}(M/F) \to [F, E_M]$ we have

$$\varphi(G^\rho_j) = F \vee E^\rho_j \in \mathcal{L}(M).$$

♦

Definition 6.4.5. With the same notation as above, for $A \subseteq E_M$, we let $\mathcal{E}|A$ (resp. $\mathcal{E}/A$) be $\mathcal{E}|\langle A\rangle$ (resp. $\mathcal{E}/\langle A\rangle$), where $\langle A\rangle$ is the flat generated by A.[1] ♦

Lemma 6.4.6. *Let $\mathcal{E} = (M, \{E^\rho_\bullet\}_\rho)$ be a tropical toric reflexive sheaf. Let $F \in \mathcal{L}(M)$. Then the following hold:*

1. *The restriction map $r_F : M|F \to M$ induces a morphism $\iota = (r_F, 0) : \mathcal{E}|F \to \mathcal{E}$ of tropical toric reflexive sheaves.*

2. *The contraction map $c_F : M \to M/F$ induces a morphism $\pi = (c_F, 0) : \mathcal{E} \to \mathcal{E}/F$ of tropical toric reflexive sheaves.*

Proof. (1): Let $\mathcal{E}|F = (M|F, \{F^\rho_\bullet\})$. Note that for $F_1, F_2 \in \mathcal{L}(M)$, we have $F_1 \cap F_2 = F_1 \wedge F_2$. It follows that

$$\underline{r_F}(E^\rho_j) = F^\rho_j = F \wedge E^\rho_j \subseteq E^\rho_j,$$

showing that $\iota = (r_F, 0)$ is a morphism.

[1] This definition makes sense since the intersection of two flats is again a flat.

(2): Recall that for $F \subseteq E_M$, with $\widetilde{F} := F \backslash \{*_M\}$, one has

$$\mathcal{L}(M/F) = \{A \backslash \widetilde{F} \mid \widetilde{F} \subseteq A \in \mathcal{L}(M)\} \cong \{A \vee F \mid A \in \mathcal{L}(M)\}.$$

Let $\mathcal{E}/F = (M/F, \{G_\bullet^\rho\})$. From Definition 6.4.4, we have that G_j^ρ can be identified with $E_j^\rho \vee F$. We thus see

$$\underline{c_F}(E_j^\rho) = G_j^\rho,$$

showing that $\pi = (c_F, 0)$ is a morphism of tropical toric reflexive sheaves. □

As in the category $\mathbb{T}$-**SMat**$_\bullet$, restrictions and contractions of tropical reflexive sheaves at flats furnish **TRS**$_\bullet$ with kernels and cokernels.

Proposition 6.4.7. *Let $\mathcal{E} = (M, \{E_\bullet^\rho\}_\rho)$ and $\mathcal{F} = (N, \{F_\bullet^\rho\}_\rho)$ be tropical reflexive sheaves in **TRS**$_\bullet$ and let $f : \mathcal{E} \to \mathcal{F}$ be a morphism. Then the inclusion $\mathcal{E}|\underline{f_\mathbb{T}}^{-1}(*_N) \to \mathcal{E}$ is a kernel of f and the contraction $\mathcal{F} \to \mathcal{F}/\langle \underline{f_\mathbb{T}}(E_M)\rangle$ is a cokernel of f.*

Proof. Applying the forgetful functor to $f : \mathcal{E} \to \mathcal{F}$ we obtain the map $f_\mathbb{T} : M \to N$ in $\mathbb{T}$-**SMat**$_\bullet$, where we find a kernel $M|\underline{f_\mathbb{T}}^{-1}(*_N)$ of $f_\mathbb{T}$. By virtue of Definition 6.4.4, the restriction has a natural tropical reflexive sheaf structure which is immediately verified to satisfy the universal property of kernels. The proof for cokernels is similar and has been omitted. □

Appealing to Proposition 6.2.2 with $F = \mathbb{T}$, we see that **TRS**$_\bullet$ also satisfies axiom (PA1) from Subsection B.4.

Proposition 6.4.8. ***TRS**$_\bullet$ has all kernels and cokernels and satisfies axiom (PA1) of a proto-abelian category.*

We now define admissible monomorphisms $\mathfrak{M}_t$ and admissible epimorphisms $\mathfrak{E}_t$, which we later see furnish **TRS**$_\bullet$ with a proto-exact structure.

Definition 6.4.9. Let $f : \mathcal{E} = (M, \{E_\bullet^\rho\}_\rho) \to \mathcal{F} = (N, \{F_\bullet^\rho\}_\rho)$ be a morphism in **TRS**$_\bullet$.

1. f is an admissible monomorphism if $f = r_F \circ \alpha$, where $F \in \mathcal{L}(N)$ and $r_F : \mathcal{F}|F \to \mathcal{F}$ is an inclusion map, and α is an isomorphism of tropical toric reflexive sheaves. Denote by $\mathfrak{M}_t$ the class of admissible monomorphisms.

2. f is an admissible epimorphism if $f = \beta \circ c_G$, where $G \in \mathcal{L}(M)$ and $c_G : \mathcal{E} \to \mathcal{E}/G$ is a contraction map and β is an isomorphism of tropical toric reflexive sheaves. Denote by $\mathfrak{E}_t$ the class of admissible epimorphisms.

♦

Remark 6.4.10. As in F-**Mat**$_\bullet$ and F-**SMat**$_\bullet$, the classes of admissible monomorphisms and admissible epimorphisms consist of kernels and cokernels of **TRS**$_\bullet$ composed with isomorphisms. However, note that as with F-**SMat**$_\bullet$ these classes are only defined in terms of the flats of the underlying matroid, rather than with arbitrary subsets. ♦

The following are straightforward.

Lemma 6.4.11. *The following hold for* $(\textbf{\textit{TRS}}_\bullet, \mathfrak{M}_t, \mathfrak{E}_t)$.

1. ***TRS***$_\bullet$ *is pointed.*

2. *The classes* $\mathfrak{M}_t$ *and* $\mathfrak{E}_t$ *contain all isomorphisms.*

3. *The classes* $\mathfrak{M}_t$ *and* $\mathfrak{E}_t$ *are closed under composition.*

Proof. (1): One readily check that the zero object is the tropical toric reflexive sheaf whose $\mathbb{T}$-matroid is U_1^0 with the trivial filtration.

(2): Let $f : \mathcal{E} = (M, \{E_\bullet^\rho\}) \to \mathcal{F} = (N, \{F_\bullet^\rho\})$ be an isomorphism. For $\mathfrak{M}_t$, one can easily see that $f = r_{E_N} \circ f$. For $\mathfrak{E}_t$, likewise, one can see that $f = f \circ c_\emptyset$.

(3): This directly follows from Proposition 6.1.15 and the definition of compositions in **TRS**$_\bullet$. □

Lemma 6.4.12. *Let $\mathcal{E} = (M, \{E^\rho_\bullet\})$ be a tropical toric reflexive sheaf and let $G \leq F \in \mathcal{L}(M)$. The following square is biCartesian in* $\textbf{\textit{TRS}}_\bullet$*:*

$$\begin{tikzcd} \mathcal{E}|F \arrow[r, tail, "f"] \arrow[d, two heads, "h"] & \mathcal{E} \arrow[d, two heads, "k"] \\ (\mathcal{E}|F)/G \arrow[r, tail, "g"] & \mathcal{E}/G \end{tikzcd} \tag{6.20}$$

where $f = r_F$ and $g = r_{(F\setminus G)}$.

Proof. We first prove that (6.20) is co-Cartesian. Suppose that we have the following diagram in $\mathbf{TRS}_\bullet$ with $\mathcal{F} = (N, \{F^\rho_\bullet\}_\rho)$.

$$\begin{tikzcd} \mathcal{E}|F \arrow[r, "f"] \arrow[d, "h"] & \mathcal{E} \arrow[d, "k"] \arrow[ddr, "\beta"] & \\ (\mathcal{E}|F)/G \arrow[r, "g"] \arrow[drr, "\alpha"] & \mathcal{E}/G & \\ & & \mathcal{F} \end{tikzcd} \tag{6.21}$$

Since all morphisms in (6.21) are morphisms in $\mathbb{T}$-$\mathbf{SMat}_\bullet$, from Lemma 6.1.16, we have a unique morphism $\gamma_{\mathbb{T}} : M/G \to N$ which makes (6.21) commute at the level of $\mathbb{T}$-matroids. We only have to check that $\gamma_{\mathbb{T}}$ defines a morphism in $\mathbf{TRS}_\bullet$. In other words, we need to show that there is some $w \in \Lambda$ with

$$\underline{\gamma_{\mathbb{T}}}(E^\rho_j \vee G) \subseteq F^\rho_{j+w\cdot v_\rho}$$

But, since $\beta = (\beta_{\mathbb{T}}, u)$ is a morphism of tropical toric reflexive sheaves, we have

$$\underline{\gamma_{\mathbb{T}}}(E^\rho_j \vee G) = \underline{\gamma_{\mathbb{T}}}(k_{\mathbb{T}}(E^\rho_j)) = \underline{\beta_{\mathbb{T}}}(E^\rho_j) \subseteq F^\rho_{j+u\cdot v_\rho},$$

showing that $\gamma = (\gamma_{\mathbb{T}}, u)$ is a morphism in $\mathbf{TRS}_\bullet$.

The case for Cartesian is similar, but we just include it here for the completeness. Suppose that we have the following diagram in **TRS$_\bullet$** with $\mathcal{G} = (R, \{G^\rho_\bullet\}_\rho)$.

$$\begin{tikzcd}
\mathcal{G} \arrow[rrd, "\beta"] \arrow[rdd, "\alpha"'] & & \\
& \mathcal{E}|F \arrow[r, "f"'] \arrow[d, "h"] & \mathcal{E} \arrow[d, "k"] \\
& (\mathcal{E}|F)/G \arrow[r, "g"] & \mathcal{E}/G
\end{tikzcd} \tag{6.22}$$

Again, from Lemma 6.1.16, we have a unique morphism $\gamma_{\mathbb{T}} : R \to M|F$ which makes (6.22) commute at the level of $\mathbb{T}$-matroids. We only have to check that $\gamma_{\mathbb{T}}$ defines a morphism in **TRS$_\bullet$**. We need to show that there is some $w \in \Lambda$ with

$$\gamma(G^\rho_j) \subseteq F^\rho_{j+w\cdot v_\rho} = E^\rho_{j+w\cdot v_\rho} \wedge F.$$

But, since $r_F \circ \gamma_{\mathbb{T}} = \beta_{\mathbb{T}}$ and $\beta = (\beta_{\mathbb{T}}, u)$ is a morphism in **TRS$_\bullet$**, we have

$$\underline{f_{\mathbb{T}}}(\underline{\gamma_{\mathbb{T}}}(G^\rho_j)) = \underline{\beta_{\mathbb{T}}}(G^\rho_j) \subseteq E^\rho_{j+u\cdot v_\rho}.$$

It follows that

$$\underline{\gamma_{\mathbb{T}}}(G^\rho_j) \subseteq \underline{f_{\mathbb{T}}}^{-1}(E^\rho_{j+u\cdot v_\rho}) = E^\rho_{j+u\cdot v_\rho} \wedge F = F^\rho_{j+u\cdot v_\rho},$$

showing that $\gamma = (\gamma_{\mathbb{T}}, u)$ is a morphism in **TRS$_\bullet$**. □

Proposition 6.4.13. *Any diagram in* ***TRS$_\bullet$*** *of the form $\mathcal{E} \rightarrowtail \mathcal{G} \twoheadleftarrow \mathcal{F}$ can be completed to a biCartesian square.*

Proof. The same argument as in the proof of Proposition 6.1.17 can be used to prove this by using Lemma 6.4.12. □

Proposition 6.4.14. *Any diagram in* ***TRS$_\bullet$*** *of the form $\mathcal{E} \twoheadleftarrow \mathcal{G} \rightarrowtail \mathcal{F}$ can be completed to a biCartesian square.*

Proof. As with F-**SMat**$_\bullet$, we cannot appeal to duality here as we did in F-**Mat**$_\bullet$, but as described after Proposition 6.1.18 one readily follows steps in analogy with the proof of Proposition 6.1.17. □

Proposition 6.4.15. *Any square of the following form in* ***TRS***$_\bullet$ *is Cartesian if and only if it is coCartesian:*

$$\begin{tikzcd} \mathcal{E} \arrow[r, tail] \arrow[d, two heads] & \mathcal{F} \arrow[d, two heads] \\ \mathcal{G} \arrow[r, tail] & \mathcal{H} \end{tikzcd}$$

Proof. As in the setting F-matroids, this statement directly follows from Propositions 6.4.13 and 6.4.14 along with the uniqueness of pushout and pullback. □

By combining the above propositions, we obtain the following.

Theorem 6.4.16. *The category* ***TRS***$_\bullet$ *is proto-exact with* $\mathfrak{M}_t$ *and* $\mathfrak{E}_t$ *as in Definition 6.4.9.*

Alongside Proposition 6.4.8 and Proposition B.4.4, we have the following corollary.

Corollary 6.4.17. *The category* ***TRS***$_\bullet$ *is proto-abelian.*

Say that a tropical toric reflexive sheaf $\mathcal{E} = (M, \{F^\rho_\bullet\}_\rho)$ is *modular* if the underlying matroid $\underline{M}$ is itself modular, and denote the full subcategory of **TRS**$_\bullet$ consisting of modular tropical toric reflexive sheaves by **MTRS**$_\bullet$. Since restrictions and contractions of modular matroids at flats are again modular, and since U^0_1 is modular, the proto-exact and proto-abelian structures on **TRS**$_\bullet$ descends to **MTRS**$_\bullet$.

Corollary 6.4.18. *The category* ***MTRS***$_\bullet$ *is proto-abelian, with underlying proto-exact structure coming from the restriction of the classes* $\mathfrak{M}_t$ *and* $\mathfrak{E}_t$ *to* ***MTRS***$_\bullet$*.*

Before moving to the question of stability in **MTRS**$_\bullet$, we give a characterization of sums and intersections of strict subobjects in **TRS**$_\bullet$. As expected, the characterization follows immediately from Proposition 6.2.6 with $F = \mathbb{T}$ since restrictions and contractions of tropical reflexive sheaves are determined by the restrictions and contractions of their underlying $\mathbb{T}$-matroids.

Proposition 6.4.19. *Let $\mathcal{E}_1, \mathcal{E}_2 \rightarrowtail \mathcal{E}$ be strict subobjects of $\mathcal{E} = (M, \{E_j^\rho\}_\rho)$ in* ***TRS$_\bullet$*** *(resp.* ***MTRS$_\bullet$****) given as restrictions $\mathcal{E}_1 = \mathcal{E}|F_1$ and $\mathcal{E}_2 = \mathcal{E}|F_2$ at flats $F_1, F_2 \in \mathcal{L}(M)$. Then the sum and intersection of the subobjects $\mathcal{E}_1$ and $\mathcal{E}_2$ are given by*

$$(\mathcal{E}|F_1) + (\mathcal{E}|F_2) = \mathcal{E}|(F_1 \vee F_2)$$

$$(\mathcal{E}|F_1) \cap (\mathcal{E}|F_2) = \mathcal{E}|(F_1 \wedge F_2).$$

6.4.1 Stability for tropical toric reflexive sheaves

The proto-abelian structure on the category **MTRS$_\bullet$** of modular tropical toric reflexive sheaves on $\mathrm{trop}(X_\Sigma)$ can be used to study slope stability phenomena in the context of tropical geometry. Following [38], we define the slope of a modular tropical toric reflexive sheaf as the ratio of the degree and rank functions (see Subsection 3.4.1). We then illustrate that the existence and uniqueness of the Harder-Narasimhan filtration of [38] in the category **MTRS$_\bullet$** follows from the results of [4] and [41].

Let X_Σ be a smooth toric variety equipped with a constant polarization $h = (c, c, \cdots, c) \in \mathbb{Z}^{\Sigma(1)}$. For a tropical toric reflexive sheaf $\mathcal{E} = (M, \{F_\bullet^\rho\}_\rho)$, recall that its rank $\mathrm{rk}(\mathcal{E})$ is defined to be the rank of the $\mathbb{T}$-matroid M and that the degree of $\mathcal{E}$ is defined to be the following quantity

$$\deg(\mathcal{E}) = \sum_\rho h_\rho \sum_{j \in \mathbb{Z}} j \cdot (r_{\underline{M}}(F_j^\rho) - r_{\underline{M}}(F_{j+1}^\rho)). \tag{6.23}$$

We now define the **slope** of $\mathcal{E}$ to be $\mu(\mathcal{E}) = \mathrm{rk}(\mathcal{E})/\deg(\mathcal{E})$. Say that $\mathcal{E}$ is **stable** (resp. **semistable**) with respect to μ if for every nonzero strict subobject $\mathcal{F}$ of $\mathcal{E}$, we have $\mu(\mathcal{F}) < \mu(\mathcal{E})$ (resp. $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$).

Using the above machinery, [38] constructs a Harder-Narasimhan filtration for tropical toric reflexive sheaves in analogy with the usual case of vector bundles on curves. However, in their construction they heavily employ the notion of modularity of various involved flats. They establish the following using methods from tropical geometry and direct investigation of the structure of tropical reflexive sheaves.

Theorem 6.4.20 ([38], Theorem 7.25)**.** *Let $\mathcal{E} = (M, \{F_\bullet^\rho\}_\rho)$ be a tropical toric reflexive sheaf on the tropical toric variety trop(X_Σ). Assume that all F_j^ρ are modular flats of M. Then there is an increasing filtration in **TRS$_\bullet$** as follows:*

$$0 = \mathcal{F}_0 \rightarrowtail \mathcal{F}_1 \rightarrowtail \cdots \rightarrowtail \mathcal{F}_k = \mathcal{E}.$$

The successive quotients $\mathcal{F}_i/\mathcal{F}_{i-1}$ are semistable and satisfy

$$\mu(\mathcal{F}_1/\mathcal{F}_0) > \mu(\mathcal{F}_2/\mathcal{F}_1) > \cdots > \mu(\mathcal{F}_k/\mathcal{F}_{k-1}).$$

Under the assumption that each $\mathcal{F}_i$ is of the form $\mathcal{E}|F_i$ for some modular flat F_i of M, this filtration is unique.

Remark 6.4.21. Note that in case not all involved flats are modular, the filtration of Theorem 6.4.20 may fail to be unique. This is in contrast to the classical situation, where the Harder-Narasimhan filtration of a vector bundle is unique up to isomorphism. However, in case the underlying matroid M of a tropical toric reflexive sheaf is modular, uniqueness is guaranteed. In particular, the Harder-Narasimhan filtrations of objects in **MTRS$_\bullet$** are unique. ♦

We now illustrate that the slope function μ on **MTRS$_\bullet$** defines a categorical slope filtration on **MTRS$_\bullet$**, which immediately yields that the Harder-Narasimhan filtrations of Theorems B.5.3 and 6.4.20 coincide.

Observe that by virtue of the additivity of the rank of $\mathbb{T}$-matroids on short exact sequences $M \rightarrowtail N \twoheadrightarrow R$, we immediately see that for every a short exact sequence $\mathcal{E} \rightarrowtail \mathcal{F} \twoheadrightarrow \mathcal{G}$ in **MTRS$_\bullet$** we have

$$\mathrm{rk}(\mathcal{E}) + \mathrm{rk}(\mathcal{G}) = \mathrm{rk}(\mathcal{F}).$$

Additionally, if $\mathrm{rk}(\mathcal{E}) = 0$ for some tropical reflexive sheaf $\mathcal{E}$ then necessarily the underlying (pointed simple) matroid of $\mathcal{E}$ must have rank zero, so that the underlying matroid is the zero object of $\mathbb{T}$-$\mathbf{SMat}_\bullet$. In particular, we must have that $\mathcal{E}$ is the zero object of $\mathbf{MTRS}_\bullet$.

The additivity of the degree function on short exact sequences is guaranteed by the following result.

Proposition 6.4.22 ([38], Remark 7.19 and Lemma 7.20)**.** *Let $\mathcal{E} = (M, \{E_\bullet^\rho\}_\rho)$ be a modular tropical reflexive sheaf and let $F \in \mathcal{L}(M)$ be a flat. Then*

$$\deg(\mathcal{E}|F) + \deg(\mathcal{E}/F) = \deg(\mathcal{E}).$$

Proof. This statement is established by the first half of the proof of Lemma 7.20 in [38]. In our context, this proof is guaranteed to hold as written by the modularity assumption on M. □

Lemma 6.4.23. *If $\mathcal{E}$ and $\mathcal{F}$ are isomorphic tropical reflexive sheaves then we have* $\deg(\mathcal{E}) = \deg(\mathcal{F})$.

Proof. By Proposition 6.4.3, we know that $\mathcal{F} = \mathcal{E} \otimes \mathcal{L}(\chi^u)$, where $\mathcal{L}(\chi^u)$ is the tropical line bundle corresponding to the principal divisor of some character χ^u of X_Σ. We know that $\mathcal{L}(\chi^u)$ corresponds to a vector $(a_\rho) \in \mathbb{Z}^{\Sigma(1)}$ which by principality of the associated divisor must satisfy $\sum_{\rho\in\Sigma(1)} a_\rho = 0$. We then compute the degree of $\mathcal{F}$ using the relation (3.3) as in Remark 3.4.5:

$$\begin{aligned}
\deg(\mathcal{F}) &= \deg(\mathcal{E} \otimes \mathcal{L}(\chi^u)) \\
&= \deg(\mathcal{E}) + \mathrm{rk}(\underline{M}) \cdot \deg(\mathcal{L}(\chi^u)) \\
&= \deg(\mathcal{E}) + \mathrm{rk}(\underline{M}) \cdot \sum_\rho h_\rho \cdot a_\rho \\
&= \deg(\mathcal{E}) + \mathrm{rk}(\underline{M}) \cdot c \cdot \sum_\rho a_\rho \\
&= \deg(\mathcal{E}).
\end{aligned}$$

Note that here we have used that the polarization h on X_Σ is of the form $h = (c, c, \cdots, c)$. □

From the above two results it immediately follows that deg is additive on short exact sequences in $\mathbf{MTRS}_\bullet$. Thus the functions deg and rk on $\mathbf{MTRS}_\bullet$ satisfy the necessary properties of degree and rank functions as in Subsection B.5. In particular, in order to establish that the slope function μ of [38] is a slope function in the sense of [41] we need only to verify that μ satisfies the strong slope inequality (B.9) in $\mathbf{MTRS}_\bullet$. This follows by using the characterization of sums and intersections of subobjects as given in Proposition 6.4.19.

Proposition 6.4.24. *The slope function μ satisfies the strong slope inequality on $\mathbf{MTRS}_\bullet$. In particular, given a modular tropical reflexive sheaf $\mathcal{E}$ with strict subobjects $\mathcal{F}, \mathcal{G}$ given as restrictions $\mathcal{F} = \mathcal{E}|F$ and $\mathcal{G} = \mathcal{E}|G$ at flats $F, G \in \mathcal{L}(M)$, we have*

$$\mu\left(\frac{\mathcal{F}}{\mathcal{F} \cap \mathcal{G}}\right) \leq \mu\left(\frac{\mathcal{F} + \mathcal{G}}{\mathcal{G}}\right).$$

Proof. We have

$$\begin{aligned} \frac{\mathcal{F}}{\mathcal{F} \cap \mathcal{G}} &= \frac{\mathcal{E}|F}{\mathcal{E}|(F \wedge G)} = (\mathcal{E}|F)/(F \wedge G) \\ \frac{(\mathcal{F} + \mathcal{G})}{\mathcal{G}} &= \frac{\mathcal{E}|(F \vee G)}{\mathcal{E}|F} = (\mathcal{E}|(F \vee G))/F = (\mathcal{E}/F)|(F \vee G) \end{aligned}$$

The desired inequality then follows from [38, Lemma 7.24]. □

This establishes that μ is a categorical slope function on $\mathbf{MTRS}_\bullet$ in the sense of [41]. In particular, since $\mathbf{TRS}_\bullet$ and $\mathbf{MTRS}_\bullet$ inherit the Artinian and Noetherian conditions from $\mathbb{T}$-$\mathbf{SMat}_\bullet$ guaranteed by Remark 6.2.8, Theorem B.5.3 guarantees that to μ there exists a categorical slope filtration on $\mathbf{MTRS}_\bullet$ which satisfies all properties of the slope filtration given in Theorem 6.4.20. The uniqueness of such a filtration as guaranteed by both theorems then yields the following.

Corollary 6.4.25. *The Harder-Narasimhan filtrations of Theorem 6.4.20 associated to modular tropical reflexive sheaves are given by the categorical slope filtration on $\mathbf{MTRS}_\bullet$ given* *by Theorem B.5.3. Stated differently, the Harder-Narasimhan filtration of [38] for modular tropical reflexive sheaves is a categorical slope filtration in the sense of [41].*

This illustrates that the proto-abelian structure of $\mathbb{T}$-matroids ultimately yields an analogue of the classical Harder-Narasimhan filtration of vector bundles in the context of tropical geometry. As noted in Section 8 of [38], this machinery can also be used to give Harder-Narasimhan filtrations of (modular) tropical vector bundles on those tropical varieties which can be embedded into tropical toric varieties $\mathrm{trop}(X_\Sigma)$.

Chapter 7

Torsor Structures on Spanning Trees

7.1. Introduction

The number of spanning trees of a finite loopless graph G is equal to the determinant of the reduced combinatorial Laplacian of G; this is the celebrated Kirchhoff matrix-tree theorem. In addition to counting spanning trees, this determinant also counts the number of elements of a certain group, known as the (degree 0) **Picard group**, critical group, or sandpile group of the graph. This group, denoted $\mathrm{Pic}^0(G)$, is defined as the additive group of chip-firing equivalence classes on the collection of divisors on the graph.

In addition to equality in cardinality of the set $\mathcal{T}(G)$ of spanning trees of G and the Picard group $\mathrm{Pic}^0(G)$, it is known that $\mathcal{T}(G)$ admits at least two distinct torsor structures for the Picard group; the Picard group of a graph acts simply and transitively on the set $\mathcal{T}(G)$. Two such actions, namely the Bernardi action β_q and the rotor-routing action ρ_q, have been the subject of study in recent years [9, 14] (see also [26, 29, 43]). To define these torsors, we work with graphs endowed with a **ribbon structure**, a cyclic ordering of the edges incident to each vertex. By giving a ribbon structure to a finite graph, we determine an orientable surface in which the graph can be considered as embedded; for this reason, ribbon structures were originally termed *combinatorial embeddings* of graphs [10].

After fixing a ribbon structure on a graph G, the Bernardi torsor β_q and the rotor-routing torsor ρ_q rely on the choice of a base vertex vertex q of the graph. Chan, Church, and Grochow prove that the torsor ρ_q is independent of the initial data q if and only if the ribbon graph is planar [14]. Separately, Baker and Wang prove that the Bernardi torsor β_q is also independent of the initial data q if and only if the ribbon graph is planar. Moreover, Baker and Wang prove in [9] that in the case of a planar ribbon graph, the two torsor structures coincide; the Bernardi and rotor-routing processes produce the same simply transitive group action. Baker and Wang include the following conjecture:

Conjecture 7.1.1 (Baker and Wang [9])**.** Given a nonplanar ribbon graph G with no multiple edges and no loops, there exists a vertex q for which $\rho_q \neq \beta_q$.

We resolve this conjecture and extend the result to a class of graphs having multiple edges but no loops. In particular, it is proven here that any nonplanar graph with or without multiple edges admits a vertex q for which $\rho_q \neq \beta_q$ when endowed with any ribbon structure. The criterion for the presence of such a vertex q in a nonplanar ribbon graph with multiple edges is the presence of what we call a **proper witness pair**. In Theorem 7.6.2, we prove that the presence of a proper witness pair in a nonplanar ribbon graph is sufficient to ensure the presence of a vertex q for which $\rho_q \neq \beta_q$. Further, we show in Proposition 7.3.3 that any nonplanar graph, when endowed with any ribbon structure, must contain a proper witness pair. In particular, any nonplanar graph endowed with any ribbon structure contains a vertex at which the torsor structures disagree. We prove Theorem 7.7.1, the Baker-Wang conjecture for loopless nonplanar ribbon graphs with no multiple edges admitting no proper witness pair. Finally, we consider two examples and some associated computations; in particular we exhibit a nonplanar ribbon graph having distinct vertices p and q such that $\rho_p = \beta_p$ while $\rho_q \neq \beta_q$.

In an independent work, Changxin Ding [18] also resolves this conjecture. As in the current paper, Ding considers multiple cases, but the approach taken there utilizes decompositions of graphs to facilitate study of the difference between the two torsors. In the present document, we work instead with classes of graphs depending on the presence or absence of certain subgraphs, referred to here as **witness pairs**.

7.2. Background and terminology

By a **graph** we mean a finite connected multigraph with no loops; for a graph G, we denote by $V(G)$ its vertex set and by $E(G)$ its collection of edges, given as unordered pairs of vertices. Occasionally we use V and E to denote the vertices and edges of a graph G when the graph is unambiguous. Unless otherwise specified, multiple parallel edges are allowed. A **ribbon graph** is a finite graph together with a cyclic ordering of the edges around each vertex, referred to as a **ribbon structure**. Note that a ribbon structure on a graph induces a natural ribbon structure on any graph minor. Given two edges e_0, e_1 incident to a given vertex v, we will write $e_0 \prec e_1$ if e_1 immediately succeeds e_0 in the order around v. If a vertex v has two edges e_0 and e_1, the **interval** $[e_0, e_1]$ between e_0 and e_1 is the collection of edges $\{e_0, f_1, \ldots, f_k, e_1\}$, where

$$e_0 \prec f_1 \prec \cdots \prec f_k \prec e_1.$$

Analogously, we define the open interval $(e_0, e_1) = \{f_2, \ldots, f_k\}$. The **length** of an interval is its cardinality as a set.

A finite ribbon graph is equivalent to an embedding of the underlying graph into a closed oriented surface such that the ribbon structure agrees with the orientation of the surface (see, e.g. [54, Theorem 3.7]). In all figures in this paper, the ribbon structure is assumed to be given by the counterclockwise orientation of the underlying surface.

A **path** P in a ribbon graph G is the image of a mapping $P_k \to G$ of the length-k path into G that is injective on edges but not necessarily on vertices. A **cycle** in G is the image of a mapping $C_k \to G$ of the length-k cycle into G that is injective on both edges and vertices. We sometimes refer to paths and cycles as sets of edges rather than as subgraphs. Let C be a cycle and endow C with an orientation. A vertex v of C is incident to precisely two edges of C, e_{in} and e_{out}, where the labeling of these edges agrees with the orientation of C. An edge e incident to v is said to be to the *left* of C if $e \in (e_{\text{out}}, e_{\text{in}})$ and to the *right* of C if $e \in (e_{\text{in}}, e_{\text{out}})$.

Given a group Γ, we consider a group action of Γ on a set X as a function

$$\Gamma \times X \longrightarrow X\,.$$

We say that a group action is **regular** if, for every $\gamma \in \Gamma$ and every $x \in X$, the restricted mappings $\{\gamma\} \times X \to X$ and $\Gamma \times \{x\} \to X$ are isomorphisms of sets. Stated differently, an action of Γ on X is regular if Γ acts simply and transitively on X. If $X = \{1, \ldots, N\}$, we can consider a group action as a homomorphism from Γ to the symmetric group on N symbols $\Gamma \to S_N$. We therefore sometimes work with the image of such a mapping, representing group elements as permutations. All multiplication of permutations will be from right to left, so that for example we have $(123)(34) = (1234)$.

7.3. Cycles and witnesses

Here we are concerned in particular with nonplanar graphs. Following [14], we discuss planarity and nonplanarity of ribbon graphs in terms of the absence or presence of nonseparating cycles.

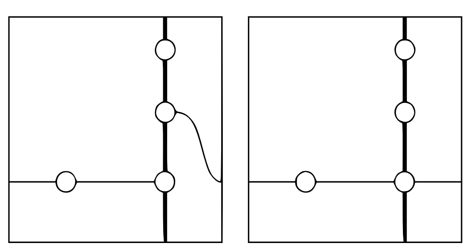

Figure 7.1: Nonseparating cycles (thickened) in two nonplanar ribbon graphs on the torus. The left image portrays a proper witness while the right image portrays an improper witness.

Definition 7.3.1. A cycle C is **nonseparating** if for any orientation of C there exists a path P that intersects C only in the endpoints of P, such that the first edge of P is on the left of C, and the last edge of P is on the right of C. We will call P a **witness** for the nonseparating cycle C and refer to the ordered pair (C, P) as a **witness pair**. ♦

Definition 7.3.2. Given a nonseparating cycle C, we will say that a witness P for C is a **proper witness** if the endpoints of P are distinct vertices of C. In this case, we call the pair (C, P) a **proper witness pair** (see Figure 7.1). ♦

We say that a connected ribbon graph G is **nonplanar** if and only if it contains a nonseparating cycle. Note that this definition of nonplanarity is not equivalent to the underlying graph G being nonplanar in the typical sense, but all nonplanar graphs yield nonplanar ribbon graphs when endowed with any ribbon structure. A planar graph G can be endowed with a ribbon structure such that the resulting ribbon graph is nonplanar.

We have the following structural property of nonplanar graphs endowed with a ribbon structure.

Proposition 7.3.3. *A connected nonplanar graph G endowed with any ribbon structure admits a* proper *witness pair.*

Proof. Starting from an arbitrary ribbon graph G, first note that the operations of edge deletion and edge contraction cannot create a proper witness in G. Thus, by Kuratowski's theorem, it is enough to illustrate that K_5 and $K_{3,3}$ necessarily admit a proper witness pair when endowed with any ribbon structure.

Given any ribbon structure on K_5, there necessarily exists a nonseparating cycle C; if some witness path P is of length 1, then the fact that K_5 is loopless ensures that P must be proper since the endpoints of P in C are distinct. Further, if P is of length 2, then the fact that K_5 has no multiple edges ensures P must be proper. Now, C either includes all vertices or excludes at least one vertex. In the case that C includes all vertices, any witness path must be of length 1 and so must be proper.

Suppose then that C excludes at least one vertex and that all witness paths have length greater than 1. Then since C is necessarily of length 3 or 4, any witness path has length either 2 or 3. Let P be some such witness path. If P is proper, we are done. If P is improper, then P has length 3 and we can write $P = \{\{a, b\}, \{b, c\}, \{c, a\}\}$ and $C = \{\{a, d\}, \{d, e\}, \{e, a\}\}$. Suppose without loss of generality that $\{a, b\}$ is on the left of C. Then either the edges $\{d, b\}, \{e, b\}, \{d, c\}$, and $\{e, c\}$ are on the left of C or one is on the right. In the former case we have a proper witness $P' = \{\{d, b\}, \{b, c\}, \{c, a\}\}$, and in the latter we have some proper witness P' by replacing either $\{c, a\}$ or $\{b, c\}, \{c, a\}$ with the edge on the right of C (see Figure 7.2).

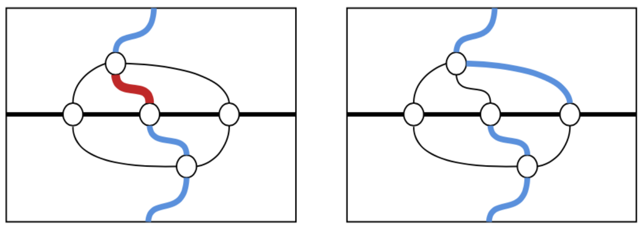

Figure 7.2: The construction of a proper witness in K_5 as described in the proof of Proposition 7.3.3. At the left, we see a thickened cycle of length 3 with an improper witness with edges in red and blue. After replacing the red edge (extra thickened), we have a proper witness in blue, depicted at the right.

Consider now $K_{3,3}$. Since $K_{3,3}$ is nonplanar, any ribbon structure admits a nonseparating cycle C. In the case that C contains all vertices, the fact that $K_{3,3}$ contains no loops ensures that any witness path P is a proper witness. Suppose C excludes at least one vertex. Since all cycles in bipartite graphs are of even length, any witness path P must have length 2. But since there are no multiple edges in $K_{3,3}$, the endpoints of P in C must be distinct, hence P is proper (see Figure 7.3). $\square$

We next prove a similar result describing ribbon graphs admitting K_4 as a minor in such a way that the inherited ribbon structure is nonplanar.

Proposition 7.3.4. *Suppose that G is a graph containing the complete graph K_4 as a minor. If G is endowed with a ribbon structure making the K_4 minor with the induced ribbon structure*

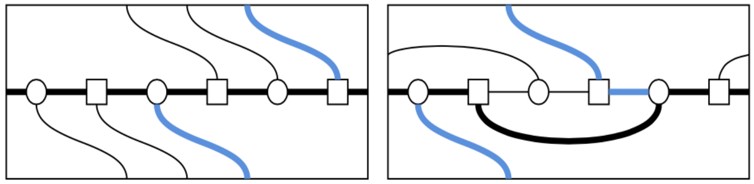

Figure 7.3: Two torus embeddings of $K_{3,3}$, each admitting proper witness pairs. In each, the two partition classes of vertices are represented by circles and squares. At the left, the thickened nonseparating cycle contains all 6 vertices; on the right, the thickened cycle contains only 4. Again, in each case the embedding admits a proper witness pair.

nonplanar, then G admits a proper witness pair.

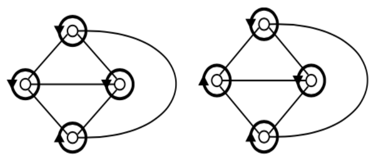

Figure 7.4: Two ribbon structures on K_4 yielding a nonplanar ribbon graph.

Proof. As in the proof of Proposition 7.3.3, we begin with the observation that the presence of a nonseparating cycle with proper witness in the graph K_4 is sufficient to guarantee the presence of such a cycle with proper witness in any graph G containing K_4 as a minor. It is therefore sufficient to show that each nonplanar ribbon structure on K_4 admits a proper witness pair.

Since each vertex in K_4 is of degree 3, there are only two choices of cyclic ordering of the edges around each vertex (see Figure 7.4). In particular, we can fix a planar embedding of K_4 and enumerate all ribbon structures by choosing either a clockwise or counter-clockwise orientation of edges about each vertex.

Moreover, by symmetry, we only need to consider the number of vertices oriented in either direction, rather than their relative positions. We refer to the ribbon structure in which i vertices are oriented clockwise and j are oriented counter-clockwise as being of type (i, j).

By reversing the orientation of the plane, it is clear that the presence of a proper witness pair in the ribbon structure of type (i, j) ensures the presence of such a pair in the structure of type (j, i). Now, since the ribbon structures of type $(4, 0)$ and $(0, 4)$ give rise to planar ribbon graph structures, we have only two choices of ribbon structures yielding a nonplanar ribbon graph K_4, arising respectively from the pairs in types $(3, 1)$ and $(2, 2)$. For each of these two ribbon structures, one can directly find the associated surface of minimal genus and verify that there exists a proper witness (see Figure 7.5). □

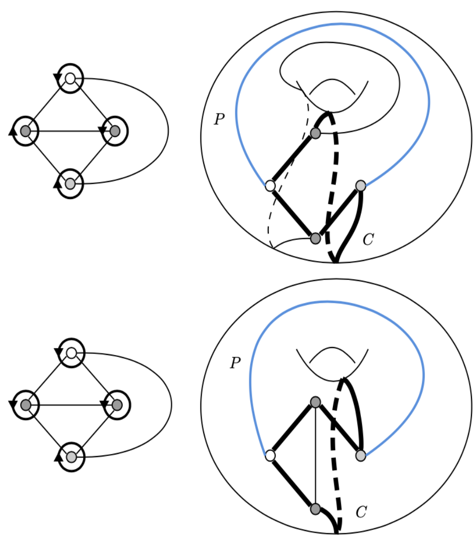

Figure 7.5: Proper witness pairs (C, P) in each of the nonplanar ribbon structures on K_4. In each case, the cycle C and its proper witness P are given.

In the case of a nonplanar ribbon graph which does not admit a proper witness pair, there is another kind of witness pair which will be of use to us.

Definition 7.3.5. Let G be a nonplanar ribbon graph that does not admit a proper witness pair. Suppose that G admits a witness pair (C, P) in G such that C and P share a unique vertex z. We say that (C, P) is a **tight** witness pair if the edges e and f incident to z in C are such that the interval $[e, f]$ in the ribbon structure about z has minimal length among all cycles C passing through z that have P as a witness path. ♦

The tightness of a witness pair (C, P) ensures that there are relatively few edges incident to z on one side of C. More precisely, this property means that for an oriented cycle C, the number of edges containing z on one side of C is as small possible. This definition is used in particular in the proof of Theorem 7.7.

Every finite nonplanar ribbon graph having no proper witness pair has at least one tight witness pair. This follows from the fact that G is a finite graph, and so we are taking a minimum over a finite collection of witness pairs.

7.4. Divisors on graphs

In this section we review the basic notions of the divisor theory of graphs, providing a setting in which to discuss the Picard group of a graph as the group of chip-firing equivalence classes. Alternative presentations of this material can be found in Chapter 4 of [39] or in Chapter 2 of [15]. Given a graph G, let $\mathrm{Div}(G)$ denote the free abelian group with formal generators corresponding to the vertices of G. Elements of $\mathrm{Div}(G)$ are called **divisors** on G; a given divisor D has the form

$$D = \sum_{v \in V} a_v (v) \,,$$

where $a_v \in \mathbb{Z}$ for all $v \in V$ and (v) is the formal generator corresponding to v. It is often convenient to think of a divisor as an arrangement of **chips** placed at the vertices of the graph G. The **degree** of a divisor D is the sum $\sum_{v \in V} a_v$. The collection of degree d divisors is denoted by $\mathrm{Div}^d(G)$. Note that, given any vertex q of G, the group $\mathrm{Div}^0(G)$ is generated by the divisors $\{(v) - (q) : \ v \in V\}$.

Denote by $\mathcal{M}(G)$ the collection of functions $f : V \to \mathbb{Z}$. The **combinatorial Laplacian** on G is the mapping $\Delta : \mathcal{M}(G) \to \mathrm{Div}^0(G)$ given by

$$\Delta f := \sum_{v \in V(G)} \Bigg(\sum_{e = \{v, w\}} [f(v) - f(w)] \Bigg) (v) \,.$$

The image of Δ is the set of **principal** divisors on G; such divisors form a subgroup $\mathrm{Prin}(G)$ of $\mathrm{Div}^0(G)$. We say that two divisors D and D' are **linearly equivalent** if they differ by a principal divisor, so that $D' = D + \Delta f$ for some $f \in \mathcal{M}(G)$. The (degree 0) **Picard group** of G is the quotient group

$$\mathrm{Pic}^0(G) = \frac{\mathrm{Div}^0(G)}{\mathrm{Prin}(G)}.$$

In general, we write $\mathrm{Pic}^d(G) = \mathrm{Div}^d(G)/\mathrm{Prin}(G)$; this is the collection of linear equivalence classes of degree-d divisors. $\mathrm{Pic}^0(G)$ admits a regular action on $\mathrm{Pic}^d(G)$ for any d given by addition of divisor classes. It is known that the cardinality of the Picard group of G is equal to the number of spanning trees of the graph G (see [8] and references therein).

For a graph G, the **genus** of G is the first Betti number $g = |E(G)| - |V(G)| + 1$; any spanning tree of G has exactly $|E(G)| - g$ edges. Given a spanning tree $T \in \mathcal{T}(G)$ excluding edges $e_1, \ldots e_g$, a divisor $B \in \mathrm{Div}^g(G)$ is referred to as a **break divisor** for T if B is of the form $B = \sum_{i=1}^{g} s(e_i)$, where $s(e_i)$ is one vertex of the edge e_i. Note that there are many possible distinct break divisors associated to a given tree. We have the following property of break divisors:

Proposition 7.4.1 (Theorem 1.1, [1]). *Each divisor class of* $\mathrm{Pic}^g(G)$ *contains a unique break divisor.*

Consequently, we can use break divisors as canonical representatives of the equivalence classes in $\mathrm{Pic}^g(G)$.

7.5. Bernardi and rotor-routing torsors

In addition to having the same cardinality as $\mathcal{T}(G)$, the Picard group $\mathrm{Pic}^0(G)$ also admits at least two regular group actions, or **torsor structures**, on $\mathcal{T}(G)$. One is derived from a process due to Bernardi and the other arises from a process referred to as rotor-routing ([10], [29]). We deal with torsor structures that depend on a choice of base vertex in the graph G. We remark that Kálmán, Lee, and Tóthmérész have recently shown in [35] that, using the machinery of planar trinities, it is possible to give a canonical definition of the Bernardi action on planar graphs without an initial choice of a base vertex. Moreover, this work extends these results to the situation of balanced plane digraphs.

7.5.1 Bernardi Torsor

The first torsor structure considered in this work arise from a family of combinatorial maps, collectively referred to as the Bernardi process. As mentioned above, $\mathrm{Pic}^0(G)$ is in bijective correspondence with the set of spanning trees $\mathcal{T}(G)$. The works of Bernardi and Baker–Wang establish a family of bijections $\beta_{(q,e)} : \mathcal{T}(G) \to \mathrm{Pic}^g(G)$ witnessing this correspondence by mapping spanning trees to break divisors ([10], [9]). These maps are parametrized by pairs (q, e) consisting of a vertex q and an edge e incident to q.

The Bernardi process uses the data (q, e) and a spanning tree T to perform a **tour** $\tau_{(q,e)}(T)$ of the graph G. This tour can be thought of as a special walk in G beginning and ending at q, traversing each edge of T twice, and excluding each edge outside T. More specifically, the tour $\tau_{(q,e)}(T)$ takes the form $\tau_{(q,e)}(T) = (v_0, e_1, v_1, \ldots, e_k, v_k)$, where $v_0 = v_k = q$ and $e = e_1 = \{v_0, v_1\}$.

Given e_i and v_i, the edge e_{i+1} is chosen to be the first edge of T succeeding e_i in the ribbon structure about v_i; the process can be thought of as originating with the edge e_0 immediately preceding e in the ribbon structure about q. We think of this process as starting with the initial data, traversing each edge of T to the other endpoint, and cycling through the ribbon structure until finding another edge of T, ignoring each edge f not included in T. In such a tour we ignore g many edges $f_1, \ldots, f_g \notin T$, each being passed over two distinct times, once from each endpoint.

In performing such a tour of G, we construct a divisor on G by placing a chip at our feet the first time we ignore a given edge $f \notin T$. In this case, we say that f is **cut** at the first endpoint from which we ignore it. The divisor obtained in this fashion is necessarily a break

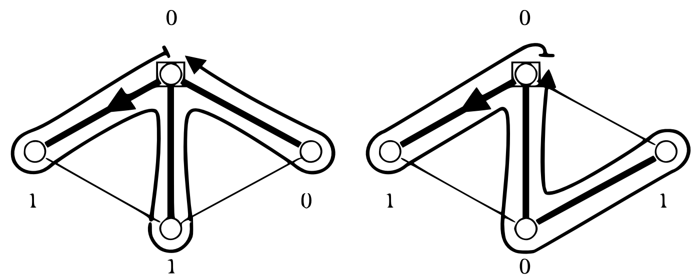

Figure 7.6: Bernardi tours and break divisors associated to two spanning trees of the given graph. Spanning trees are denoted by thickened edges. In each case, the initial data (q, e) is represented by a boxed vertex and an arrow along the initial edge.

divisor for the tree T. The mapping $\beta_{(q,e)} : \mathcal{T}(G) \to B(G)$ is given by mapping a spanning tree T to the resulting break divisor.

The process of deriving the break divisor $\beta_{(q,e)}(T)$ from T is illustrated graphically in Figure 7.6. Since, by Proposition 7.4.1, each divisor class of $\mathrm{Pic}^g(G)$ contains a unique break divisor, it follows that the maps $\beta_{(q,e)}$ induce explicit combinatorial bijections between spanning trees and divisor classes of degree g.

From these bijections, Baker and Wang define a group action $\beta_q : \mathrm{Pic}^0(G) \times \mathcal{T}(G) \to \mathcal{T}(G)$ using the natural action of $\mathrm{Pic}^0(G)$ on $\mathrm{Pic}^g(G)$ [9]. Fix initial data (q, e). Given any divisor class $[D] \in \mathrm{Pic}^0(G)$, define

$$\beta_q([D], T) := \beta^{-1}_{(q,e)}([D] + [\beta_{(q,e)}(T)])\,. \tag{7.1}$$

A central result of the paper of Baker and Wang states that the action (7.1) is in fact independent of the chosen edge e used in defining the bijection $\beta_{(q,e)}$ (see [**?**, Theorem 4.1]). As a result of this, one can define a group action of $\mathrm{Pic}^0(G)$ on $\mathcal{T}(G)$ after fixing only a vertex of G, the **base vertex** of the action.

7.5.2 Rotor-Routing Torsor

Another regular action is derived from the **rotor-routing** process on G, introduced in [49] and studied further in [29] and [28]. The rotor-routing process is a discrete-time dynamical system derived from G. The state space consists of **rotor configurations** on G: pairs (σ, v) where v is a vertex of G, interpreted as the location of a chip, and $\sigma : V(G) \to E(G)$ is a **rotor function** assigning to each vertex w of G an edge $\sigma(w)$ containing w. The edge $\sigma(w)$ is referred to as the **rotor at w**.

A single step of the rotor-routing process takes the rotor configuration (σ, v) to (σ', w), where σ' is the rotor configuration for which $\sigma'(u) = \sigma(u)$ for all $u \neq v$, $\sigma'(v)$ is the successor of $\sigma(v)$ in the ribbon structure about v, and w is the other vertex of $\sigma'(v)$. Informally, we move the rotor at v once according to the ribbon structure about v and move the chip along the new edge to its other vertex w.

For the purposes of defining the action of $\mathrm{Pic}^0(G)$ on $\mathcal{T}(G)$, we choose a vertex q of G as a sink of the system and ignore the rotor at q. Given a spanning tree T of G and a vertex q of G, we can obtain a rotor function σ_T such that $\sigma_T(v)$ is the first edge on the unique path in T from v to q. Conversely, given a rotor function σ, we can obtain a subgraph H_σ of G, whose only edges are the rotors of σ.

Given any vertex q of G and any spanning tree T of G, iterating the rotor-routing process from any configuration (σ_T, v) will eventually yield a configuration (τ, q); the chip will necessarily reach the vertex q ([28, Lemma 3.6]). Moreover, it can be shown that the graph H_τ derived from τ will be a spanning tree of G ([28, Lemma 3.10]).

Using this observation, we can construct a group action ρ_q of $\mathrm{Div}^0(G)$ on $\mathcal{T}(G)$ by defining the action of generators $\{(v) - (q) : v \in V(G)\}$ and extending linearly. Beginning from the rotor configuration (σ_T, v), iterate the process until reaching some (τ, q) and let T' be the spanning tree given by the final rotor function. This action is trivial on any divisor in $\mathrm{Prin}(G)$ and so descends to an action by $\mathrm{Pic}^0(G)$. The rotor-routing action $\rho_q : \mathrm{Pic}^0(G) \times \mathcal{T}(G) \to \mathcal{T}(G)$ is then given by setting

$$\rho_q([(v) - (q)], T) = T' \,. \tag{7.2}$$

See Figure 7.7 for an illustration of the rotor-routing action. In the investigation of [14], the following definition is of crucial use to establish the main findings.

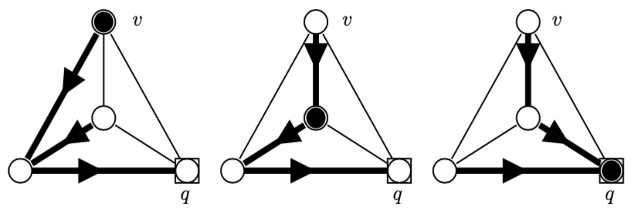

Figure 7.7: Rotor-routing process applied to a spanning tree of K_4. The original spanning tree T, indicated at the left by thickened edges, is mapped to the rightmost spanning tree $\rho_q([(v) - (q)])(T)$. At each stage, the location of the chip is indicated by a filled vertex.

Definition 7.5.1. A rotor configuration (σ, v) is called a **unicycle** if the image of the rotor function σ contains a unique directed cycle and v is a vertex of this cycle, or if v has some neighbor w such that $\sigma(v) = \sigma(w)$. ♦

This definition will be of particular use to us due to the interaction between the rotor routing action and unicycles.

Lemma 7.5.2 (Lemma 4.9, [29])**.** *Let (σ, v) be a unicycle on a graph with m edges. Then in iterating the rotor-routing process $2m$ times from (σ, v), the chip traverses each edge of G exactly once in each direction, each rotor makes exactly one full rotation, and the final state is again (σ, v).*

The following lemma is an easy consequence of Lemma 7.5.2:

Lemma 7.5.3. *Suppose that G is a ribbon graph, (σ, w) a unicycle on G, and z a vertex satisfying $\sigma(z) = \{z, w\}$. Then, for all neighbors $v \neq w$ of z satisfying $\sigma(v) = \{v, z\}$, the rotor at z reaches $\{z, v\}$ before the rotor at v completes a full rotation.*

Proof. Let v be a neighbor of z such that $\sigma(v) = \{v, z\}$. Suppose for the sake of contradiction that in iterating the rotor-routing process from (σ, z), the rotor at v completes a full rotation before the rotor at z reaches $\{z, v\}$. Then in order for the rotor at z complete a full rotation, we pass through some rotor configuration (ρ, v) such that $\rho(z) = \{z, v\}$. Then iterating the rotor-routing process one additional time the rotor at v has completed more than one full turn before the rotor at z returns to its starting position, contradicting Lemma 7.5.2. $\square$

7.5.3 Comparison of Torsor Structures

The rotor-routing and Bernardi processes described in the previous subsections give rise to regular group actions of $\mathrm{Pic}^0(G)$ on $\mathcal{T}(G)$, yielding two torsor structures on $\mathcal{T}(G)$ via (7.1) and (7.2). Each of the two structures is dependent on a choice of some vertex of G as base vertex for the action. These torsor structures have been the subject of much study in recent years ([**?**, 14, 35, 43]).

Fix some vertex $q \in V(G)$ and consider ρ_q and β_q. To see that the two homomorphisms ρ_q and β_q agree, we must have that

$$\rho_q([D], T) = \beta_q([D], T)$$

for each $T \in \mathcal{T}(G)$ for each divisor class $[D] \in \mathrm{Pic}^0(G)$. In particular, since divisors of the form $(v) - (q)$ generate $\mathrm{Div}^0(G)$, by (7.1) and (7.2), it is sufficient to verify that for each $v \in V(G)$ we have the equality

$$\beta_{(q,e)}(\rho_q([(v) - (q)], T)) = [(v) - (q)] + \beta_{(q,e)}(T)\,.$$

In contrast, to show that $\rho_q \neq \beta_q$ we need only find a single tree $T \in \mathcal{T}$ for which the above equality does not hold; doing so immediately yields that, as permutations of $\mathcal{T}(G)$,

$$\rho_q([(v) - (q)],\ \cdot\ ) \neq \beta_q([(v) - (q)],\ \cdot\ )$$

and therefore ρ_q and β_q cannot be equal.

We have the following theorem about the rotor-routing torsor structure on $\mathcal{T}(G)$:

Theorem 7.5.4 (Theorem 2, [14]). *Let G be a connected ribbon graph. The action ρ_q of $Pic^0(G)$ on $\mathcal{T}(G)$ is independent of the base vertex q if and only if G is a planar ribbon graph.*

The work of Baker and Wang established an analogous result for the torsor structure derived from the Bernardi process:

Theorem 7.5.5 (Theorems 5.1 and 5.4, [9]). *Let G be a connected ribbon graph. The action β_q of $Pic^0(G)$ on $\mathcal{T}(G)$ is independent of the base vertex q if and only if G is a planar ribbon graph.*

In the case of a planar ribbon graph G, neither the rotor-routing action nor the Bernardi action is dependent on the choice of base vertex; both actions are in this sense canonical.

Theorem 7.5.6 (Theorem 7.1, [9]). *For a planar ribbon graph G, the Bernardi and rotor-routing processes define the same $Pic^0(G)$-torsor structure on $\mathcal{T}(G)$.*

However, in the case of a non-planar G the situation was not as clear. The work [9] concludes with the following conjecture:

Conjecture 7.5.7 (Conjecture 7.2, [9]). Let G be a nonplanar ribbon graph with no multiple edges and no loops. Then there exists a vertex q of G such that $\rho_q \neq \beta_q$.

7.6. Baker-Wang Conjecture and proper witnesses

We now prove the Baker-Wang conjecture in the case that the graph G admits a proper witness pair. This will be done by first proving the conjecture in a more restricted situation.

Lemma 7.6.1. *Suppose that G is a ribbon graph with proper witness pair (C, P) such that P has endpoints x and z. If the two edges e_0 and e_1 of C incident to z satisfy $e_0 \prec e_1$, then there exists a vertex q of G such that $\rho_q \neq \beta_q$.*

Proof. Suppose that $e_1 = \{z, q\}$, so that q is the other vertex of e_1 in C. Now, let $e' = \{u, v\}$ be an edge of P and extend the collection of edges $(C \setminus e_1) \cup (P \setminus e')$ to a spanning tree T of G. We will show that

$$\rho_q([(z) - (q)], T) \neq \beta_q([(z) - (q)], T)\,.$$

In particular, denoting by T' the spanning tree $\rho_q([(z) - (q)], T)$, we show that the divisor classes $[(z) - (q)] + [\beta_{q,e_1}(T)]$ and $[\beta_{q,e_1}(T')]$ have distinct break divisor representatives. By the definition of the Bernardi action β_q, we immediately obtain $\rho_q \neq \beta_q$.

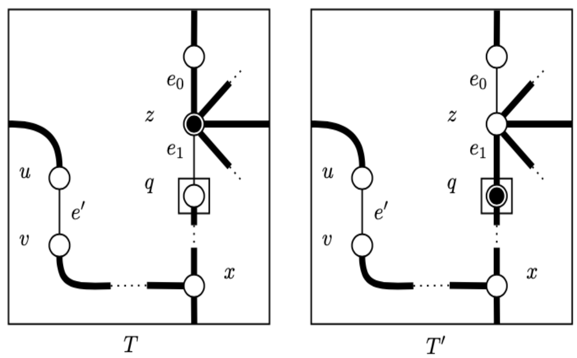

Figure 7.8: Rotor-routing process applied to the spanning tree T, thickened at the left, as in the proof of Lemma 7.6.1. After one step the chip is moved from z to q and the process is terminated, yielding the thickened tree T' at the right.

Acting on T with $\rho_q([(z) - (q)])$ entails only a single step of the rotor routing process: changing the rotor at z from e_0 to e_1 and moving the chip to q, terminating the process. As a result, the spanning trees T and T' differ only by a single edge. In particular, $T' = (T \setminus e_0) \cup e_1$. This situation is depicted in Figure 7.8.

Consider the divisors $(z) - (q) + \beta_{(q,e_1)}(T)$ and $\beta_{(q,e_1)}(T')$. By the definition of the Bernardi bijection, we know that $\beta_{(q,e_1)}(T')$ is a break divisor for T'. On the other hand, the divisor $(z) - (q) + \beta_{(q,e_1)}(T)$ is also a break divisor for T', distinct from $\beta_{(q,e_1)}(T')$. Since a break divisor for a spanning tree is a formal sum of endpoints of edges outside that tree, we can see the following:

$$(z) - (q) + \beta_{(q,e_1)}(T) = (z) - (q) + (q) + \sum_{\substack{e \notin T \\ e \neq e_1}} s(e) = (z) + \sum_{\substack{e \notin T' \\ e \neq e_0}} s(e)\,.$$

Recall from Section 7.4 that $s(e)$ denotes either endpoint of the edge e. Since z is an endpoint of e_0, this shows that $(z) - (q) + \beta_{(q,e_1)}(T)$ is a break divisor for T'.

We are reduced to showing that $(z) - (q) + \beta_{(q,e_1)}(T)$ and $\beta_{(q,e_1)}(T')$ are distinct break divisors. By the construction of T, the collection of edges $P \cap T$ consists of two connected components P_z and P_x, containing z and x, respectively. In the Bernardi tour of T, P_x is traversed before P_z, while the opposite is true for the tour of T'. This results in different placement of the chip associated to the edge e' in the two tours: without loss of generality, the T tour places a chip at v while the T' tour places a chip at u; this difference is depicted in Figure 7.9.

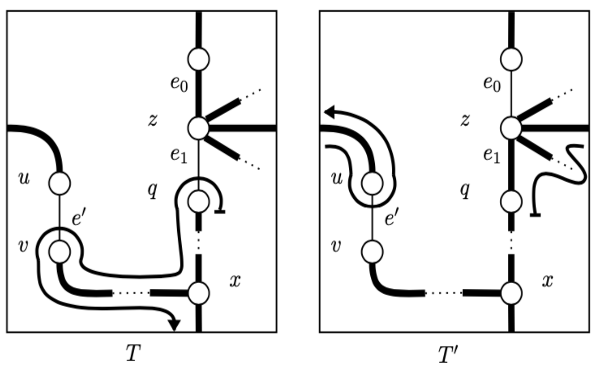

Figure 7.9: Bernardi process with initial data (q, e_1) applied to the spanning trees T and T' as in the proof. For simplicity, only the relevant parts of the tours are included in the figure. At the left, the tour cuts e' at u, while at the right e' is cut at v.

It now remains to show that the difference in placement of the chip associated to e' yields a difference between $(z) - (q) + \beta_{(q,e_1)}(T)$ and $\beta_{(q,e_1)}(T')$ as divisors. Suppose for the sake of contradiction that they are the same despite their difference in choice of $s(e')$. Since e' is cut at v in the tour of T, the equality of $(z) - (q) + \beta_{(q,e_1)}(T)$ and $\beta_{(q,e_1)}(T')$ at u implies that there must be some other edge $f_0 = \{u, w_1\}$, excluded from both T and T', which is cut at u in the tour of T and at its other endpoint w_1 in the tour of T'. Similarly, for $(z)-(q)+\beta_{(q,e_1)}(T)$ and $\beta_{(q,e_1)}(T')$ to agree at w_1, we see that there must be some other edge $f_1 = \{w_1, w_2\}$, excluded from both T and T', which is cut at w_1 in the tour of T and is cut at its other endpoint w_2 in the tour of T'.

Continuing in this fashion, we see that there must exist a cycle D in G having vertices $u = w_0, w_1, \ldots, w_m = v$ and edges $f_0, \ldots, f_{m-1}, e'$, where $f_i = \{w_i, w_{i+1}\}$, such that all edges f_i are excluded from both trees T and T'. Moreover, orienting D such that e' is oriented from v to u, we see that in the tour of T each edge of D must be cut at its tail, while in the tour of T' each edge of D must be cut at its head. This situation is depicted in Figure 7.10.

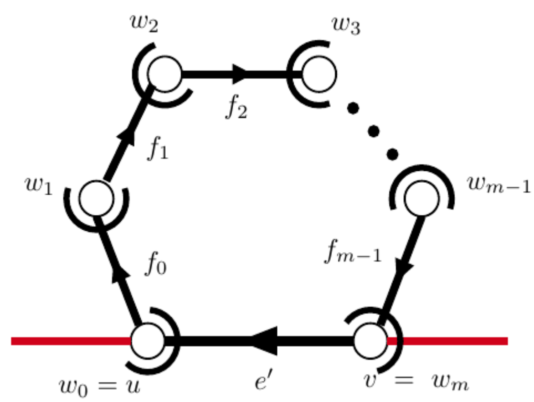

Figure 7.10: A depiction of the cycle D mentioned in the proof of Lemma 7.6.1. Here, the arcs surrounding vertices indicate that for this cycle all edges are cut at their heads, in accordance with the described tour of the tree T'. The depicted edges outside the cycle are the edges of T' incident to u and v.

We show that the existence of such a cycle D with edges being cut in this way is impossible. To see this we orient the cycle C in a fixed direction and partition the vertices w_i of D into two classes L and R according to whether the first edge outside C on the unique path in T from z to w_i falls on the left or right of the cycle C. Since T and T' are trees, if some vertex $w_i \in L$ is visited in the tour of T before some other vertex $w_j \in R$ (or vice versa), then all edges outside T incident to w_i will be cut before the tour visits w_j.

Consider now the tour of T. Since v is visited before u in this tour, e' is cut at its tail and the edge f_{m-1} must be cut before u is visited. Thus to avoid contradiction, it must be that

w_{m-1} is in the same partition class as v and f_{m-1} is cut at its tail w_{m-1}. Now, applying the same reasoning, it must be that w_{m-2} is in the same partition class as w_{m-1} and that f_{m-2} is cut at its tail w_{m-2}. Following this pattern around the cycle D, it must be that $w_0 = u$ is in the same class as w_1, and that the edge f_0 is cut at its tail $w_0 = u$. But this is impossible since, by construction, u and v cannot be in the same partition class.

Thus it cannot be that $(z) - (q) + \beta_{(q,e_1)}(T) = \beta_{(q,e_1)}(T')$. Since the divisors $(z) - (q) + \beta_{(q,e_1)}(T)$ and $\beta_{(q,e_1)}(T')$ are necessarily distinct break divisors, they cannot be representative of the same element of $\mathrm{Pic}^g(G)$ and so $\rho_q \neq \beta_q$. □

For a ribbon graph G admitting a proper witness pair, we either have z, e_0, and e_1 as in Lemma 7.6.1 above, or there are some edges $f_1, \ldots, f_k$ incident to z such that

$$e_0 \prec f_1 \prec \cdots \prec f_k \prec e_1 \,.$$

We now prove that in the latter case, we can still find vertices q' and z' and an edge e for which the break divisors $\beta_{(q',e)}(T')$ and $(z') - (q') + \beta_{(q',e)}(T)$ still differ as described.

Theorem 7.6.2. *For any nonplanar ribbon graph G admitting a proper witness pair* (C, P)*, there exists a vertex q for which* $\rho_q \neq \beta_q$*.*

Proof. Suppose that the proper witness pair (C, P) such that the path P intersects C in vertices x and z. If this witness pair satisfies $e_0 \prec e_1$ in the ribbon structure about z, then we are finished. Suppose also that the pair (C, P) fails to meet the conditions of Lemma 7.6.1. In particular, there exists an edge $f = \{z, v\}$ such that $f \in [e_0, e_1]$ and $f \neq e_0, e_1$. Let G' be the subgraph of G obtained by deleting the vertex z and all edges to which it is incident. Denote by F the connected component of the subgraph G' containing v. See Figure 7.11. Note in particular that F may intersect the cycle C or the path P.

If the component F meets neither C nor P, then one can ignore the edge f, applying the same reasoning as in Lemma 7.6.1: the rotor-routing process carried out in the subgraph F will not influence the rotors on the path P, so the break divisors $(z) - (q) + \beta_{q,e_1}(T)$ and

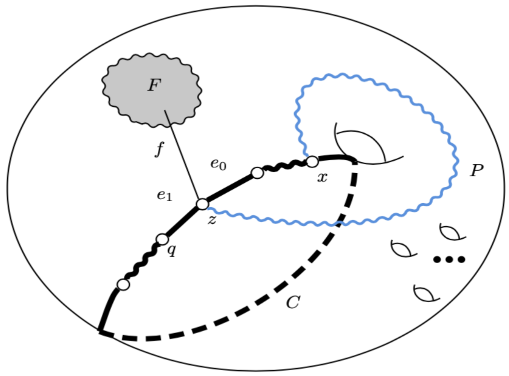

Figure 7.11: Proper witness pair (C, P) in a graph embedded on a surface of positive genus. Supposing that there is an edge f incident to z with $e_0 \prec f \prec e_1$, we can identify the component F of $G \setminus z$ and consider its possible intersection with the cycle C and the path P. Wavy lines represent paths that may contain additional vertices and edges.

$\beta_{q,e_1}(T')$ will still differ as described in the proof. We therefore assume that there is no such edge f with $F \cap C$ and $F \cap P$ empty.

Suppose that the interval (e_0, e_1) about z contains some positive number of edges f. We illustrate a method of finding a new witness pair (C', P') and reassigning the roles of e_0, e_1, z, and q such that the new interval I serving as the replacement of (e_0, e_1) has strictly fewer edges than before. This means that we can eventually find a witness pair (C'', P'') such that the interval between the two edges of C'' incident to z is empty and therefore Lemma 7.6.1 can be applied.

Suppose that there is some edge in (e_0, e_1) that can be extended to a path from z to a vertex of P. Let f be the maximal such edge and suppose it is extended to a path from $Q_{z,x'}$ through F connecting z to some vertex $x' \in V(P)$. Then, necessarily, there is a path $C_{z,x}$ from z to x in C which does not contain q and there is a path $P_{x,x'}$ from x to x' in P. In this case, we replace the nonseparating cycle C by the new cycle $C' = (C \setminus C_{z,x'}) \cup Q_{z,x'} \cup P_{x,x'}$ and the path P by the new path $P' = P \setminus P_{x,x'}$. Following this, we repeat this process, replacing e_0 with f and replacing x with x'. Doing so, we see that the new interval (f, e_1) has cardinality strictly less than that of the original interval (e_0, e_1). This process can be seen in Figure 7.12.

Suppose that there is some edge in (e_0, e_1) that can be extended to a path from z to a vertex of C. Let f be the minimal such edge and suppose it is extended to a path $Q_{z,c}$ through F connecting z to some vertex $c \in V(C)$. We have the following two cases:

1. If $c = x$ or if c lies on the path from z to x in C, denote by $C_{z,c}$ the path from z to c in C which avoids q. Replace the nonseparating cycle C by the new cycle $C' = (C \setminus C_{z,c}) \cup Q_{z,c}$ and leave the path P unchanged. We now let f fill the role of e_0 and see that the interval (f, e_1) has cardinality strictly less than that of (e_0, e_1).

2. If the vertex c lies on the path from z to x in C which passes through q or if $c = q$, let $C_{z,c}$ denote the path from z to c in C passing through (or stopping at) q. Replace the nonseparating cycle C by the new cycle $C' = (C \setminus C_{z,c}) \cup Q_{z,c}$ and leave the path P unchanged. We now let f fill the role of e_1 and see that the interval (e_0, f) has cardinality strictly less than that of (e_0, e_1). This process can be seen in Figure 7.13.

Using the above approaches, it follows that in the presence of a proper witness pair (C, P) in G we can always recover a witness pair (C'', P'') which either satisfies the conditions of Lemma 7.6.1 or such that no edges incident to z inhibit the proof of Lemma 7.6.1. Thus we conclude that if G is a nonplanar ribbon graph admitting a proper witness pair, then there is some vertex q of G such that $\rho_q \neq \beta_q$.

□

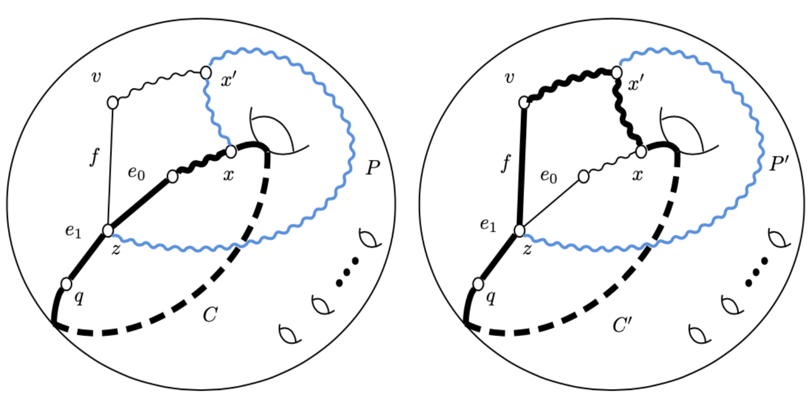

Figure 7.12: Situation in the proof of Theorem 7.6.2 where the intruding edge f leads to a path to P. The new proper witness pair (C', P) is shown in the right figure.

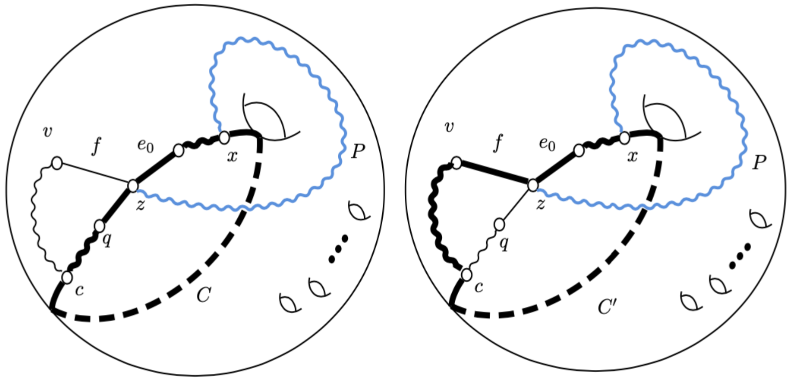

Figure 7.13: Situation in the proof of Theorem 7.6.2 where the intruding edge f leads to a path to C. The new proper witness pair (C', P) is shown in the image at the right.

Considering Proposition 7.3.3 alongside Theorem 7.6.2, we immediately obtain the following corollary:

Corollary 7.6.3. *For a nonplanar graph G endowed with any ribbon structure, there exists some vertex q of G such that $\rho_q \neq \beta_q$.*

7.7. Simple graphs without proper witnesses

We next show that for nonplanar ribbon graphs G without multiple edges or loops that do not contain a proper witness pair, there is some vertex q for which $\rho_q \neq \beta_q$. This completes the proof of the Baker-Wang conjecture.

Theorem 7.7.1. *Let G denote a nonplanar ribbon graph with no multiple edges and no loops such that G does not admit a proper witness pair. Then there is some vertex q of G such that $\rho_q \neq \beta_q$.*

Proof. As in the proof of Lemma 7.6.1, we will construct a spanning tree T such that the Bernardi and rotor-routing torsors disagree on T.

In order to do so, we must first fix some terminology regarding our construction. It is recommended that the reader reference Figure 7.14 while we introduce the required terminology.

1. Since G is a nonplanar ribbon graph with no proper witness pair, there exists at least one tight witness pair (Definition 7.3.5). Suppose (C, P) is one such pair and that z is the unique vertex in the intersection of P and C.

2. Suppose that q is a neighbor of z in C. Orient $\{z, q\}$ from q to z and extend this orientation to all of C. We may assume that the interval on the left of C is the minimal interval described in the definition of a tight witness pair. Let e be the edge $\{z, q\}$ and let e' be the edge immediately succeeding e in the ribbon structure about q.

3. Partition all neighbors y of z outside the cycle C into two classes L and R so that $y \in L$ if and only if $\{y, z\}$ is on the left of C and $y \in R$ if and only if $\{y, z\}$ is on the right of C.

4. Let H be the subgraph of G induced by all witness paths for C that intersect C at z. Further, let $L_H \subseteq L$ and $R_H \subseteq R$ denote the vertices of H which lie in L and R, respectively.

5. Let $Z \subseteq E(G)$ denote the collection of edges incident to z with endpoints in L or R. We can extend the collection of edges $C \cup Z \setminus \{e\}$ to a spanning tree T so that all edges incident to z except e are included in the tree T and such that all vertices in R_H have degree one in the subgraph $T \cap H$. This is because, by definition, all neighbors of z in R_H lie on paths connecting to z through vertices of L_H. Note that the vertices in R_H may have degree greater than one in T, but this situation arises only when there are other vertices of G whose only path to z passes through a vertex of R_H.

6. Let w_0 denote the vertex in L_H such that $\{z, w_0\}$ is the first edge in H to the left of C. Likewise, let v_0 denote the vertex in R_H such that $\{z, v_0\}$ is the first edge to the right of C.

7. Consider the vertices of $w \in L \setminus L_H$. As in the proof of Theorem 7.6.2, let G' denote the subgraph of G obtained by removing the vertex z and all edges incident to z. Let W

denote the connected component of G' containing w. Since (C, P) is a tight witness pair, W does not intersect the cycle C.

Indeed, if W intersects C then there are some vertices $w, w' \in W$ and $c \in C$ with $c \neq z$ such that $\{w, z\}$ and $\{w', c\}$ are edges in G. In this case, we may form a new cycle C' which agrees with C on the (oriented) path from z to c in C, but which contains the edges $\{w, z\}$ an $\{w', c\}$. Doing so then yields a witness pair (C', P) such that the length of the ribbon structure interval about z on the left of C' is strictly less than the length of the ribbon structure interval about z on the left of C. This is precisely what the tightness of the witness pair (C, P) prohibits, and so such a situation is impossible.

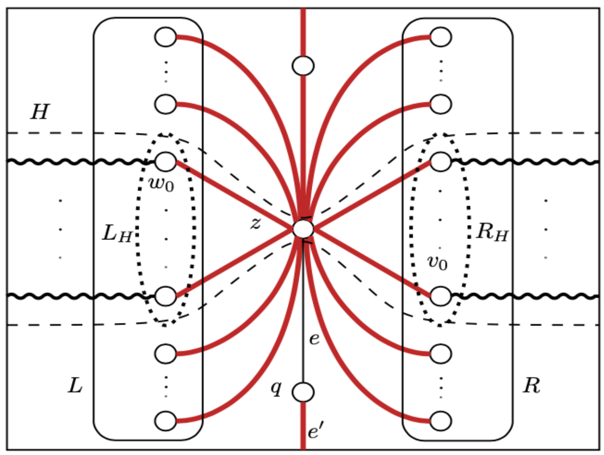

Figure 7.14: The situation in the proof of Theorem 7.7.1. The tree T is indicated with thickened edges, while the subgraph H of G is indicated using a dashed line. The vertex sets L_H and R_H are shown within the dotted regions within H. For simplicity, we have depicted the edge e' as an edge of the cycle C.

Consider the rotor-routing process taking T to $T' := \rho_q([(z) - (q)], T)$. At some point, the rotor at z shifts to the edge $\{z, w_0\}$, forming a unique cycle in the subgraph of H defined by the rotors at vertices of H. Restricting the rotor configuration to the graph H, we obtain a unicycle $U = (\sigma, z)$ in H. Further, by the assumption that G has no proper witness, we see that all paths from any vertex of H other than z to any vertex on the path C must pass through z. In particular, combining this with the fact that (C, P) is a tight witness pair ensures that no rotors outside of H or the subgraphs W will move until the rotor at z shifts to e.

Consider now the rotor routing process in the graph H beginning from starting configuration U. By Lemma 7.5.3, for all neighbors v of z in R_H the rotor at z shifts to $\{z, v\}$ before the rotor at any $v \in R_H$ completes a full rotation. Considering the implication of this statement in the graph G, we see that the rotor at z must shift to e before the rotors at any $v \in R_H$ complete a full rotation, and so the entire rotor routing process taking T to T' terminates before the rotor of any vertex in R_H completes a full rotation.

Now, either the rotor-routing process associated to $\rho_q([(z) - (q)], T)$ never deposited a chip at any vertex in R_H and so all edges from z to vertices in R_H lie in the tree T', or this rotor-routing process deposited a chip at some vertex of R_H and so the edge joining z to this vertex lies outside the tree T'.

In the former case, since no $v \in R_H$ received a chip, all vertices of R_H have degree 1 in the subgraph $T' \cap H$. This means in particular that on the Bernardi tour associated to T', all vertices of R_H receive at least one chip from their neighbors in H in addition to whatever chips they receive from edges to neighbors outside H. However, on the Bernardi tour associated to T all edges of H incident to vertices in R_H will be cut at their other endpoint. Therefore no vertex of R_H will receive any chips from neighbors in H on this tour, while each such vertex receives the same number of chips as in the tour of T' from neighbors outside H. This means in particular that the break divisors $\beta_{(q,e)}(T')$ and $(z) - (q) + \beta_{(q,e)}(T)$ differ at all vertices of R_H, and so cannot be representative of the same element of $\mathrm{Pic}^g(G)$.

In the latter case, suppose that the rotor routing process on T deposited a chip at the vertex v and so $\{z, v\}$ lies outside T'. Then in this case the chip must have returned to z across an edge connecting z to some vertex ℓ in L_H. Indeed, all edges incident to z on the right of C are included in the tree T, so in order for the chip to return to z across an edge connecting z to a vertex in R_H the rotor at that vertex must complete a full rotation, which is impossible by Lemma 7.5.3. Further, by appealing to this same lemma we also see that the edge $\{\ell, z\}$ across which the chip returned to z must be contained in T'. This follows from the fact that $\{\ell, z\}$ is in the tree T by construction, and so 7.5.3 ensures that the rotor at z reaches $\{\ell, z\}$

before the chip returns across this edge. In particular, the rotor at z continues to the next edge in the ribbon structure at z and so the rotor at ℓ remains $\{\ell, z\}$ at the termination of the process.

Thus for all $v \in R_H$ which received a chip during the rotor routing process, the unique path from v to z in T' passes through a vertex of L_H. Note in particular that this need not be the case if G has multiple edges. Indeed, in the event that there is some $v \in R_H$ such that there are parallel edges between z and v, then v may receive a chip in rotor-routing process and eventually terminate in a configuration such that one of the edges $\{v, z\}$ is still in the tree T'.

This means that in the Bernardi tour associated to T', the edge $\{z, v\}$ will be cut at z and so z will receive a chip from an edge to at least one neighbor in R_H. Considering also that z receives a chip from an edge in the cycle C, we see that z must have a coefficient of at least two in the break divisor $\beta_{(q,e)}(T')$. On the other hand all edges incident to z except for e are included in T, so that in the divisor $\beta_{(q,e)}(T)$, the vertex z has coefficient zero. Thus the break divisors $(z) - (q) + \beta_{(q,e)}(T)$ and $\beta_{(q,e)}(T')$ necessarily differ at z and cannot be representative of the same element of $\mathrm{Pic}^g(G)$.

Combining the above, we conclude that in either case the rotor-routing and Bernardi torsors disagree on the tree T, so that $\rho_q \neq \beta_q$. □

In view of Theorem 7.6.2 and Theorem 7.7.1 we obtain the following Corollary, verifying the Baker-Wang conjecture.

Corollary 7.7.2 (Baker-Wang Conjecture)**.** *Given a nonplanar ribbon graph with no multiple edges and no loops, there exists a vertex q for which $\rho_q \neq \beta_q$.*

7.8. Examples

We conclude with two examples to address the following questions

1. Is the assumption of no multiple edges necessary for the Baker–Wang Conjecture?

2. For nonplanar ribbon graphs, is the *difference* between the two torsor structures independent of the base vertex?

Our examples show that the answer to question (1) is yes, while the answer to question (2) is no.

Fix a labeling of the N spanning trees of the underlying graph and to consider the actions ρ_q and β_q as homomorphisms

$$\rho_q : \mathrm{Pic}^0(G) \longrightarrow S_N\,,$$
$$\beta_q : \mathrm{Pic}^0(G) \longrightarrow S_N\,.$$

The *difference* between the two torsors *on a given divisor class* $[D]$ can be computed as $\rho_q([D])^{-1}\beta_q([D]) \in S_N$. The difference between the two torsor structures is dependent on the base vertex if there exist two vertices p and q and a divisor class $[D]$ such that

$$\rho_p([D])^{-1}\beta_p([D]) \neq \rho_q([D])^{-1}\beta_q([D])\,.$$

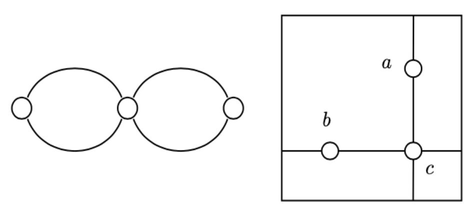

Figure 7.15: Rounded bowtie graph with the relevant torus embedding. As in prior figures, the ribbon structure is given by the counter-clockwise orientation of the torus.

Example 7.8.1. Consider the graph depicted in Figure 7.15, which we call the *rounded bowtie graph*. For the rounded bowtie graph, we have that $\rho_q = \beta_q$ for all vertices q.

We compute all Bernardi and rotor-routing torsor structures on the graph. This graph has only has four spanning trees, which yields a particularly straightforward collection of computations.

For a given spanning tree T and a given choice of vertex q and incident edge $e = \{v, q\}$, we first compute the rotor routing action, yielding a permutation $\rho_q([(v) - (q)]) \in S_n$. Following this, we calculate the permutation associated to the Bernardi action by computing break divisors $\beta_{(q,e)}(T)$ for all spanning trees T. Break divisors will be written as row vectors with coordinates in the order (a, b, c). With these in hand, we translate the break divisors by the divisor $(v) - (q)$, yielding new divisors

$$(v) - (q) + \beta_{(q,e)}(T)\,. \tag{7.3}$$

For each spanning tree T. Finally, we can check for linear equivalence between these divisors and the other break divisors $\beta_{(q,e)}(S)$ for other spanning trees S. In particular, if some pair of spanning trees S and T satisfy

$$(v) - (q) + \beta_{(q,e)}(T) \sim \beta_{(q,e)}(S)\,,$$

then the permutation $\beta_q([(v)-(q)])$ sends the spanning tree T to S. Calculating these divisors (7.3) and performing these comparisons for all spanning trees then yields a permutation representation for $\beta_q([(v) - (q)])$.

In order to verify that $\rho_q = \beta_q$ for all vertices q of the graph in Figure 7.15, we perform this process for all choices of vertices q and v of the graph in order to understand the action of divisor classes of $\mathrm{Div}^0(G)$ generators on the trees. There are three non-equivalent choices for q and v: one in which the target vertex q has degree four and the vertex v has degree two, one in which q and v both have degree two, and one in which q has degree two and v has degree four. In Figure 7.16, we have included a depiction of the four spanning trees of the rounded bowtie graph, together with their associated Bernardi tours and break divisors for this choice of $q = c$ and $v = a$.

In calculating the Bernardi action we make use of the Bernardi bijection $\beta_{(c,e)}$, where $e = \{c, a\}$. Consider the divisor $(a) - (c) + \beta_{(c,e)}(T_1) = (1, 1, 0)$. This is the break divisor

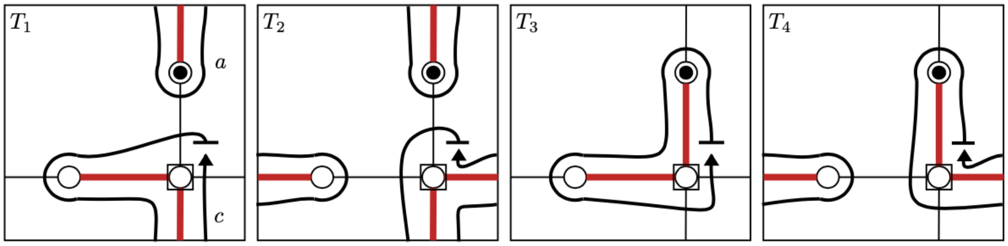

Figure 7.16: Bernardi and rotor-routing actions on spanning trees of the rounded bowtie graph. The spanning trees T_i are indicated in each of the figures (1) through (4) with thickened edges. The Bernardi tours depicted are associated to the bijection $\beta_{(c,e)}$, where $e = \{a, c\}$. The associated break divisors $\beta_{(c,e)}(T_i)$ are given below each tree.

$\beta_{(c,e)}(T_3)$, so we conclude that $\beta_c([(a) - (c)])$ maps T_1 to T_3. Similarly, we see

$$(a) - (c) + \beta_{(c,e)}(T_3) = (2, 1, -1) \sim (0, 1, 1) = \beta_{(c,e)}(T_1)\,,$$

$$(a) - (c) + \beta_{(c,e)}(T_2) = (1, 0, 1) = \beta_{(c,e)}(T_4)\,,$$

$$(a) - (c) + \beta_{(c,e)}(T_4) = (2, 0, 0) \sim (0, 0, 2) = \beta_{(c,e)}(T_2)\,.$$

Together, this implies that we have a permutation representation $\beta_c([(a) - (c)]) = (13)(24)$. Likewise, checking the rotor-routing action $\rho_c([(a) - (c)])$ yields that $\rho_c([(a) - (c)]) = (13)(24) = \beta_c([(a)-(c)])$. By the symmetry of the rounded bowtie graph, it follows that $\rho_c = \beta_c$. Other cases follow similarly and so the associated work is not included. $\Diamond$

We now turn to the issue of the discrepancy between base vertices, illustrating in particular a graph, depicted in Figure 7.17, having vertices p and q such that $\rho_p = \beta_p$ while $\rho_q \neq \beta_q$.

Example 7.8.2. Consider the graph depicted in Figure 7.17, which we call the *pointed bowtie graph*. For the pointed bowtie graph and the two indicated vertices p and q we have $\rho_p = \beta_p$ while $\rho_q \neq \beta_q$.

We deal with divisor classes of the form $[(z) - (q)]$ for the sake of their simplicity. In Figure 7.17 all vertices are labeled, and in addition we have included all spanning trees of the graph, together with the labeling to be used in the cycle representation of our actions. We will show that

$$\rho_q([(a)-(p)])^{-1}\beta_q([(a)-(p)]) \neq \rho_p([(a)-(p)])^{-1}\beta_p([(a)-(p)]).$$

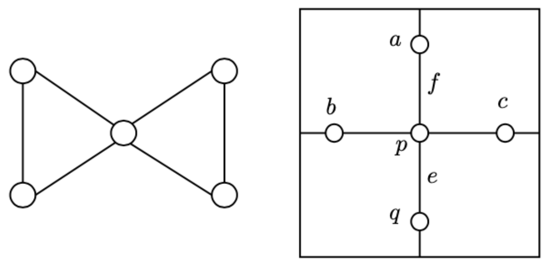

Figure 7.17: Pointed bowtie graph with the relevant torus embedding. Vertices are labeled a, b, c, p, q and edges e, f are relevant in Example 7.8.2.

Before proceeding, note that by Theorem 7.7.1, we are guaranteed that $\rho_q \neq \beta_q$ for this choice of q. We compute now the action of the divisor $[(a)-(p)]$ on each spanning tree using the actions ρ_q and β_q. Since the base vertex of these actions is q and not p, we must first rewrite $[(a)-(p)] = [(a)-(q)] - [(p)-(q)]$, after which one can compute the actions of $[(a)-(q)]$ and $[(p)-(q)]$ under both ρ_q and β_q. Finally,

$$\rho_q([(a)-(p)]) = \rho_q([(a)-(q)]\left[\rho_q([(p)-(q)])\right]^{-1},$$
$$\beta_q([(a)-(p)]) = \beta_q([(a)-(q)]\left[\beta_q([(p)-(q)])\right]^{-1}.$$

With these permutations in hand, we are able to form the product $\rho_q([(a)-(p)]^{-1}\beta_q([(a)-(p)])$. To start this process, we have produced in Figure 7.18 the Bernardi tours and break divisors associated to each of the spanning trees. For this example, break divisors will be written as row vectors with coordinates in the order (a, b, p, c, q).With these in hand, we can now compute the cycle representation of the permutation $\beta_q([(p)-(q)])$ using the Bernardi bijection $\beta_{(q,e)}$, where e is the edge $\{p, q\}$.

Beginning from the spanning tree T_1, computation of the translates $[(p)-(q)] + [\beta_{(q,e)}(T_i)]$ and their break divisor representatives yields that the permutation $\beta_q([(p)-(q)])$

has cycle representation (193)(278)(456). On the other hand, it can be verified from performing rotor-routing on the trees $T_1, \ldots, T_9$ that $\rho_q([(p) - (q)]) = (123)(459)(678)$. We have

$$\beta_q([(p) - (q)]) = (193)(278)(456)\,,$$

$$\rho_q([(p) - (q)]) = (123)(459)(678)\,.$$

Following this same reasoning, we can obtain cycle representations

$$\beta_q([(a) - (q)]) = (139)(287)(465)\,,$$

$$\rho_q([(a) - (q)]) = (132)(495)(687)\,.$$

combining the above expressions, it follows that

$$\rho_q([(a) - (p)])^{-1}\beta_q([(a) - (p)]) = (158)(269)\,.$$

It remains to see that the torsor actions of $\beta_p([(a)-(p)])$ and $\rho_p([(a)-(p)])$ coincide. This is most easily done by taking the same approach as above, utilizing the Bernardi bijection $\beta_{(p,f)}$. These computations are more straightforward than those above, so we leave the work as an exercise. One obtains

$$\beta_p([(a) - (p)]) = (163)(245)(789)\,,$$

$$\rho_p([(a) - (p)]) = (163)(245)(789)\,.$$

In particular, $\rho_p([(a) - (p)]) = \beta_p([(a) - (p)])$. Now, note that by the symmetry of the pointed bowtie graph, the coincidence of ρ_p and β_p on $[(a) - (p)]$ ensures their coincidence on all generators of $\mathrm{Div}^0(G)$. We now have $\rho_q \neq \beta_q$ while $\rho_p = \beta_p$.

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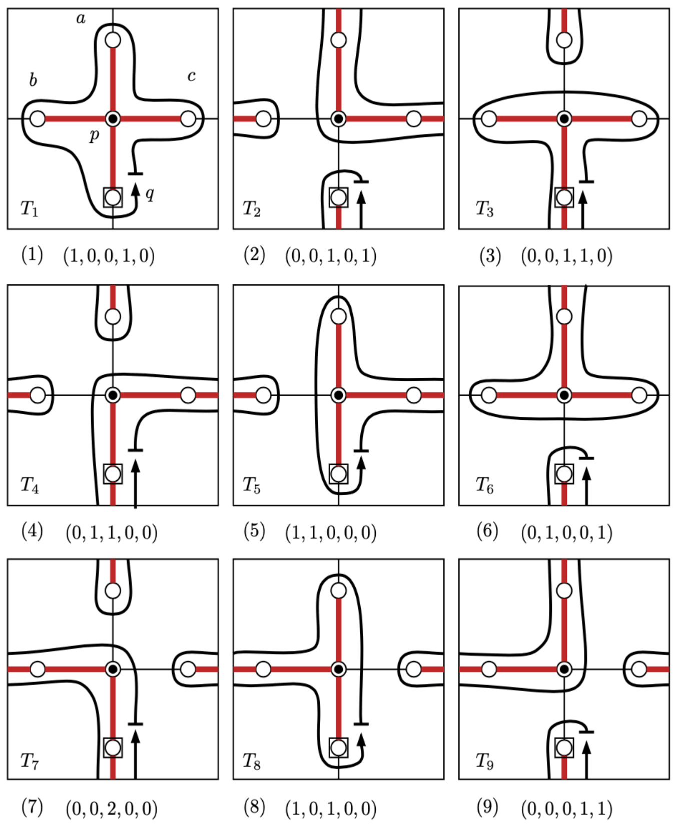

Figure 7.18: Bernardi and rotor-routing actions on spanning trees of the pointed bowtie graph. The spanning trees T_i are indicated in each of the figures in red. Denoting $e = \{p, q\}$, the break divisors $\beta_{(q,e)}(T_i)$ are given below the corresponding tree with the vertex ordering (a, b, p, c, q).

Appendix A

Simplicial Topology of Graphs

Let $G = (V, E)$ be a finite graph with $|V| = n$ and $|E| = m$. For simplicity, we assume that G is connected. Endow G with an orientation by giving functions $s, t : E \to V$ which specify for each edge $e \in E$ a **source** $s(e)$ and a **target** $t(e)$. We denote $e^+ := t(e)$ and $e^- := s(e)$. All considerations in this appendix are independent of the chosen orientation on G. We also fix a length function $\ell : E \to \mathbb{R}_{>0}$, although our interest in the present document is with the case $\ell(e) = 1$ for all $e \in E$.

Consider the free abelian groups $C_0(G;\mathbb{Z}) \simeq \mathbb{Z}^n$ and $C_1(G;\mathbb{Z}) \simeq \mathbb{Z}^m$, referred to as the groups of 0-chains and 1-chains with integral coefficients. Changing coefficients, we also have real vector spaces $C_0(G;\mathbb{R}) = C_0(G;\mathbb{Z}) \otimes \mathbb{R}$ and $C_1(G;\mathbb{R}) = C_1(G;\mathbb{Z}) \otimes \mathbb{R}$. The groups $C_0(G;\mathbb{Z})$ and $C_1(G;\mathbb{Z})$ are connected by the following homomorphisms, which each extend linearly to the real vector spaces above.

Definition A.0.1. The **boundary homomorphism** $\partial : C_1(G;\mathbb{Z}) \to C_0(G;\mathbb{Z})$ is defined by declaring $\partial(e) = e^+ - e^-$ and extending linearly. The **coboundary homomorphism** is the map $\delta : C_0(G;\mathbb{Z}) \to C_1(G;\mathbb{Z})$ defined by linearly extending the formula

$$\delta(v) = \sum_{e:v=e^-} e - \sum_{f:v=f^+} f.$$

Abusing notation, we also denote the scalar extension of the boundary and coboundary maps to $C_1(G;\mathbb{R})$ and $C_0(G;\mathbb{R})$ by the symbols ∂ and δ, respectively.

The **first homology group** of G (with coefficients in $\mathbb{R}$) is the subspace $H_1(G;\mathbb{R}) := \ker\partial \leq C_1(G;\mathbb{R})$, and the homology with $\mathbb{Z}$ coefficients is denoted by $H_1(G;\mathbb{Z}) = H_1(G;\mathbb{R}) \cap C_1(G;\mathbb{Z})$.

We also write $H^0(G;\mathbb{R})$ for $\ker\delta \leq C_0(G;\mathbb{R})$ and $H^1(G;\mathbb{R})$ for the dual space $H_1(G;\mathbb{R})^\vee$, referring to them as the and refer to it as the 0th and 1st **cohomology groups** of G with real coefficients. Fix also a linear isometry $\rho : H_1(G;\mathbb{R}) \to H^1(G;\mathbb{R})$. ♦

The vector space $C_0(G;\mathbb{R})$ comes equipped with the standard inner product, given by $\langle v_i, v_j\rangle = \delta_{ij}$, where δ_{ij} is understood to be the Kronecker delta. The choice of a length function ℓ or G moreover specifies an inner product on $C_1(G;\mathbb{R})$ defined by

$$\langle e_i, e_j\rangle = \begin{cases} \ell(e_i) & i = j \\ 0 & \text{else} \end{cases}.$$

Remark A.0.2. Note that by construction the groups $C_1(G;\mathbb{Z})$ is a Euclidean lattice in $C_1(G;\mathbb{R})$. Likewise, $H_1(G;\mathbb{Z})$ and $\rho H_1(G;\mathbb{Z})$ are lattices in $H_1(G;\mathbb{R})$ and $H^1(G;\mathbb{R})$, respectively, as is $\partial C_1(G;\mathbb{Z})$ in $\partial C_1(G;\mathbb{R})$. ♦

Proposition A.0.3. *If $\ell(e) = 1$ for all $e \in E$, then the linear maps ∂ and δ on the spaces of real 0-chains and 1-chains of G are mutually adjoint.*

Proof. This follows from noticing that the formula for $\delta(v)$ can be written

$$\delta(v) = \sum_{e\in E}\langle v, \partial e\rangle \cdot e.$$

In particular, we see that for $v \in V$ and $e \in E$ we have

$$\begin{aligned}\langle \delta v, e\rangle &= \sum_{e' \in E} \langle \langle v, \partial e'\rangle e', e\rangle \\ &= \sum_{e' \in E} \langle v, \partial e'\rangle \langle e, e'\rangle \\ &= \langle v, \partial e\rangle.\end{aligned}$$

Thus we obtain that for $x \in C_0(G;\mathbb{R})$ and $y \in C_1(G;\mathbb{R})$ we have $\langle \delta x, y\rangle = \langle x, \partial y\rangle$. $\square$

In this case, we see that we obtain direct sum splittings

$$\begin{aligned}C_0(G;\mathbb{R}) &= H^0(G;\mathbb{R}) \oplus \partial C_1(G;\mathbb{R}) \\ C_1(G;\mathbb{R}) &= H_1(G;\mathbb{R}) \oplus \delta C_0(G;\mathbb{R})\end{aligned}$$

The summands $H_1(G;\mathbb{R})$ and $\delta C_0(G;\mathbb{R})$ are important objects of their own right in graph theory, owing to the fact that they each admit bases of combinatorial significance.

Let T be a spanning tree of G and take any $e \in E - T$. Then $T \cup e$ contains a unique cycle, referred to as the **fundamental cycle** of T with respect to e. By extending the orientation of e to an orientation of the cycle, we may then write this cycle as a linear combination of its edges $e' \in E$, with each edge having coefficient either 1 or -1 in accordance to whether the orientation of e' agrees with that of the cycle. In this way, each pair (T, e) defines an element $\gamma_{T,e} \in C_1(G;\mathbb{R})$. Note that by definition we have $\partial \gamma_{T,e} = 0$, so that $\gamma_{T,e} \in H_1(G;\mathbb{Z})$ for all such pairs (T, e) with $e \notin T$. The collection of edges appearing in $\gamma_{T,e}$ with nonzero coefficient is immediately seen to be a cycle of the graph G.

let T again be a spanning tree of G and now take any $f \in T$. Then the removal of f from T yields two connected components, having vertex sets $W_1 \sqcup W_2 = V$, where we declare that W_1 is the set of vertices containing f^-. Then, viewing W_1 as an element of $C_0(G;\mathbb{Z})$, we immediately see that $\delta W_1 \in \delta C_0(G;\mathbb{R})$. Denote this 1-chain by $\omega_{T,f}$ and call it the

fundamental cut of T with respect to f. Again by definition, $\omega_{T,f} \in \delta C_0(G;\mathbb{Z})$ for all pairs (T, f) with $f \in T$. The collection of edges appearing in $\omega_{T,f}$ with nonzero coefficient is immediately seen to be a cut of the graph G.

Proposition A.0.4. *For any spanning tree T of G, the set $\{\gamma_{T,e} : e \notin T\}$ is a basis for $H_1(G;\mathbb{R})$ and the set $\{\omega_{T,f} : f \in T\}$ is a basis for $deltaC_0(G;\mathbb{R})$.*

Proof. □

Definition A.0.5. We also refer to $H_1(G;\mathbb{R})$ as the **cycle space** of G and to $\delta C_0(G;\mathbb{R})$ as the **cut space** of G, in accordance with the above-mentioned bases. ♦

Appendix B

Non-Additive Homological Algebra

B.1. Proto-exact categories

In this subsection, we recall the notion of proto-exact categories, introduced by Dyckerhoff and Kapranov [20].

Definition B.1.1. A **proto-exact** category is a category $\mathcal{C}$ with zero object equipped with two distinguished classes $\mathfrak{M}$ and $\mathfrak{E}$ of morphisms, called **admissible monomorphisms** and **admissible epimorphisms**, respectively. We denote morphisms in $\mathfrak{M}$ by $\rightarrowtail$ and morphisms in $\mathfrak{E}$ by $\twoheadrightarrow$. The classes $\mathfrak{M}$ and $\mathfrak{E}$ are required to satisfy the following conditions:

(PE1) Every morphism $0 \to M$ is in $\mathfrak{M}$, and every morphism $M \to 0$ is in $\mathfrak{E}$.

(PE2) The classes $\mathfrak{M}$ and $\mathfrak{E}$ are closed under composition and contain all isomorphisms.

(PE3) A commutative square of the following form is Cartesian if and only if it is co-Cartesian:

$$\begin{tikzcd} M \arrow[r, tail] \arrow[d, two heads] & N \arrow[d, two heads] \\ M' \arrow[r, tail] & N' \end{tikzcd} \tag{B.1}$$

(PE4) Every diagram $M' \rightarrowtail N' \twoheadleftarrow N$ can be completed to a bi-Cartesian square

(B.1).

(PE5) Every diagram $M' \twoheadleftarrow M \rightarrowtail N$ can be completed to a bi-Cartesian square (B.1).

♦

In a proto-exact category, a commutative diagram of the following form is said to be an admissible short exact sequence:

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, two heads] & B \arrow[d, two heads] \\ 0 \arrow[r, tail] & C \end{tikzcd} \tag{B.2}$$

We denote a short exact sequence by $A \rightarrowtail B \twoheadrightarrow C$. In this case, we also write $C = B/A$ and call C a quotient object.

Two admissible monomorphisms $f : A \rightarrowtail B$ and $g : A' \rightarrowtail B$ are isomorphic if there exists an isomorphism $\phi : A \to A'$ such that $g \circ \phi = f$. The isomorphism classes in $\mathfrak{M}$ are said to be **admissible subobjects**. We denote admissible subobjects by $A \subseteq B$.

Example B.1.2.

(1) Any Quillen exact category is also proto-exact, with the same classes of admissible monomorphisms and epimorphisms. In particular, any abelian category is proto-exact, with $\mathfrak{M}$ consisting of all monomorphisms and $\mathfrak{E}$ consisting of all epimorphisms.

(2) The category $\mathbf{Set}_\bullet$ of pointed sets and basepoint-preserving functions is proto-exact with $\mathfrak{M}$ consisting of all pointed injections and $\mathfrak{E}$ consisting of all pointed surjections. The full subcategory of finite pointed sets is also proto-exact, as is the subcategory $\mathbf{Vect}_{\mathbb{F}_1}$ of $\mathbb{F}_1$-vector spaces, which finite pointed sets with $\mathbb{F}_1$-linear functions. A function f is $\mathbb{F}_1$-linear if $f^{-1}(x)$ has cardinality 1 whenever x is not the basepoint. In the latter two cases, $\mathfrak{M}$ and $\mathfrak{E}$ consist of the restrictions of admissibles in $\mathbf{Set}_\bullet$ to these subcategories.

(3) The category $\mathbf{Mat}_\bullet$ of pointed matroids is finitary and proto-exact, as established in [22].

$\lozenge$

B.2. Duality and Direct Sum for Proto-Exact Categories

The proto-exact categories which we encounter in the sequel possess additional structures which behave well with respect to exact sequences. In this section we describe these structures, namely direct sums and duality, in the context of proto-exact categories. For a more detailed account of these concepts, see [21].

Definition B.2.1. Let $\mathcal{C}$ be a proto-exact category with classes $\mathfrak{M}$ and $\mathfrak{E}$ of admissible monomorphisms and epimorphisms, respectively. Say that $\mathcal{C}$ has **exact direct sum** if there is a symmetric monoidal product $\oplus$ satisfying the following properties:

(DS1) The monoidal unit is the zero object 0.

(DS2) $\oplus : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is an exact functor.

For $U, V \in \mathcal{C}$, denote by $\iota_U : U \to U \oplus V$ and $\pi_U : U \oplus V \to U$ the corresponding inclusion and projection, and similar for V.

(DS3) The following maps are injections:

$$\mathrm{Hom}(U \oplus V, W) \longrightarrow \mathrm{Hom}(U, W) \times \mathrm{Hom}(V, W)$$
$$f \longmapsto (f \circ \iota_U, f \circ \iota_V)$$
$$\mathrm{Hom}(W, U \oplus V) \longrightarrow \mathrm{Hom}(W, U) \times \mathrm{Hom}(W, V)$$
$$f \longrightarrow (\pi_U \circ f, \pi_V \circ f)$$

(DS4) Let $U \xrightarrow{\iota} X \xrightarrow{\pi} V$ be a short exact sequence. Then for all sections s of π, there is a unique isomorphism φ for which have the diagram

$$\begin{tikzcd} & X & \\ U \arrow[ur, tail, "\iota"] \arrow[r, tail, "\iota_U"'] & U \oplus V \arrow[u, "\varphi"] & V \arrow[ul, tail, "s"'] \arrow[l, tail, "\iota_V"] \end{tikzcd}$$

Further, for all retracts r of ι, there is a unique isomorphism ψ for which we have the diagram

$$\begin{tikzcd} & X \arrow[dl, "r"'] \arrow[dr, two heads, "\pi"] & \\ U & U \oplus V \arrow[u, "\psi"] \arrow[l, two heads, "\pi_U"] \arrow[r, two heads, "\pi_V"'] & V \end{tikzcd}$$

♦

We say that a proto-exact category $\mathcal{C}$ with exact direct sum is *combinatorial* if, for any admissible monomorphism $g : X \to U_1 \oplus U_2$ we have admissible subobjects $\iota_k : X_k \rightarrowtail U_k$ for $k = 1, 2$ for which there is an isomorphism $f : X \to X_1 \oplus X_2$ satisfying $(\iota_1 \oplus \iota_2) \circ f = g$. Additionally, we say that $\mathcal{C}$ is *uniquely split* if every short exact sequence splits in a unique way.

Definition B.2.2. Let $\mathcal{C}$ be a proto-exact category. Say that $\mathcal{C}$ is a proto-exact-category **with duality** if there is a functor $P : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ and a natural isomorphism $\Theta : \mathrm{id}_{\mathcal{C}} \to P \circ P^{\mathrm{op}}$ such that for all $U \in \mathcal{C}$ we have $P(\Theta_U) \circ \Theta_{P(U)} = \mathrm{id}_{P(U)}$ and for which P and Θ satisfy the following properties.

(D1) $P(0) \simeq 0$

(D2) $f \in \mathfrak{M}$ if and only if $P(f) \in \mathfrak{E}$

(D3) If either of the following squares is biCartesan, then the other is as well.

$$\begin{tikzcd} U \arrow[r, tail] \arrow[d, two heads] & X \arrow[d, two heads] \\ W \arrow[r, tail] & V \end{tikzcd} \qquad \begin{tikzcd} P(U) & P(X) \arrow[l, two heads] \\ P(W) \arrow[u, tail] & P(V) \arrow[l, two heads] \arrow[u, tail] \end{tikzcd}$$

♦

B.3. Hall algebras of finitary proto-exact categories

Given an abelian category $\mathcal{A}$ satisfying suitable conditions, one can associate to it the Hall algebra $H_{\mathcal{A}}$. Whether $H_{\mathcal{A}}$ carries a natural Hopf algebra structure depends delicately on the properties of $\mathcal{A}$. In his celebrated work [36], Kapranov studied the case where $\mathcal{A}$ is the category of coherent sheaves $\mathrm{Coh}(X)$ on a smooth projective curve X defined over a finite field $\mathbb{F}q$. He showed that when $X = \mathbb{P}^1$, a certain subalgebra of $H_{\mathrm{Coh}(X)}$ is isomorphic to the positive part of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}_2})$.

The definition of $H_{\mathcal{A}}$ can be naturally generalized to the case of proto-exact categories $\mathcal{C}$, and one may study the proto-exact structure of $\mathcal{C}$ through the lens of the associated Hall algebra $H_{\mathcal{C}}$. See [33] for some backgrounds on Hall algebras of proto-exact categories along with a combinatorial approach to study the Hall algebra of the category of coherent sheaves on $\mathbb{P}^2$ whose structure is rarely known.

Now, we recall the definition of $H_{\mathcal{C}}$. Say that a proto-exact category $\mathcal{C}$ is **finitary** if for all objects M and N of $\mathcal{C}$ we have that the sets $\mathrm{Hom}(M, N)$ and $\mathrm{Ext}^1(M, N)$ are finite.[1] Given a finitary proto-exact category $\mathcal{C}$ and a field k of characteristic zero, the **Hall algebra** $H_{\mathcal{C}}$ of $\mathcal{C}$ over k is defined to be

$$H_{\mathcal{C}} := \{f : \mathrm{Iso}(\mathcal{C}) \to k \mid f \text{ has finite support}\}.$$

[1] For a proto-exact cateogry, as in the classical setting, $\mathrm{Ext}^1(M, N)$ is defined as the set of equivalence classes of admissible short exact sequences.

Here, $\mathrm{Iso}(\mathcal{C})$ is the set of isomorphism classes of objects of $\mathcal{C}$. The Hall algebra is an associative k-algebra with product given by

$$(f * g)([M]) = \sum_{N \subseteq M} f([M/N]) \cdot g([N]). \tag{B.3}$$

The Hall algebra $H_{\mathcal{C}}$ is equipped with a basis of functions $\{\delta_{[M]} \mid [M] \in \mathrm{Iso}(\mathcal{C})\}$, defined in analogy with Dirac measures, given by

$$\delta_{[M]}([N]) = \begin{cases} 1 & M \simeq N, \\ 0 & \text{else.} \end{cases} \tag{B.4}$$

The function $\delta_{[0]}$ is the unit of $H_{\mathcal{C}}$ and multiplication of basis elements is given by

$$\delta_{[M]} * \delta_{[N]} = \sum_{[R] \in \mathrm{Iso}(\mathcal{C})} g^R_{M,N} \cdot \delta_{[R]}. \tag{B.5}$$

Here the constants $g^R_{M,N} \in \mathbb{N}$ are given by

$$g^R_{M,N} = \#\{R' \mid R' \subseteq R,\ R' \simeq N,\ R/R' \simeq M\}. \tag{B.6}$$

Note that finitarity ensures that the $g^R_{M,N}$ are natural numbers. When $\mathcal{C}$ is combinatorial and has a direct sum $\oplus$, the Hall algebra $H_{\mathcal{C}}$ can be promoted to a bialgebra by equipping it with *Green's comultiplication*, which is given by

$$\begin{aligned} \Delta : H_{\mathcal{C}} &\longrightarrow H_{\mathcal{C}} \otimes H_{\mathcal{C}} \\ \Delta(f)([M],[N]) &:= f([M \oplus N]). \end{aligned}$$

B.4. Proto-abelian categories

In this subsection, we recall the notion of proto-abelian categories, introduced by André [4]. Given a category $\mathcal{C}$ with zero object, a **kernel** of a morphism $f : M \to N$ is a morphism $k : K \to M$ for which the following square is Cartesian:

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d] & M \arrow[d, "f"] \\ 0 \arrow[r] & M \end{tikzcd}$$

Dually, a **cokernel** of f is a morphism $c : N \to C$ for which the following square is co-Cartesian:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d] & N \arrow[d, "c"] \\ 0 \arrow[r] & C \end{tikzcd}$$

A kernel is necessarily a monomorphism and a cokernel is necessarily an epimorphism. Say that a morphism in $\mathcal{C}$ is a **strict monomorphism** if it is the kernel of another morphism in $\mathcal{C}$. Likewise, a **strict epimorphism** is by definition the cokernel of another morphism. If $M \to N$ is a strict monomorphism (resp. epimorphism), say that M is a **strict subobject** of N (resp. N is a **strict quotient** of M). For the duration of this section, we denote strict monomorphisms by $\rightarrowtail$ and strict epimorphisms by $\twoheadrightarrow$.

By abuse of terminology, we refer to the domain of a kernel of f as a kernel of f and refer to the codomain of a cokernel of f as a cokernel of f. These objects are unique up to isomorphism, and it will always be clear from context whether we are speaking of a morphism or an object.

For a category $\mathcal{C}$ with kernels and cokernels, given an object M and two strict subobjects N_1 and N_2 of M we can construct a **sum** $N_1 + N_2$ and **intersection** $N_1 \cap N_2$ of N_1 and N_2 under M. These objects are again strict subobjects of M and are constructed as follows.

The intersection $N_1 \cap N_2$ is simply the pull-back of the diagram $N_1 \rightarrowtail M \leftarrowtail N_2$, which exists in $\mathcal{C}$ by [4, p. 9]. Let M/N_1 and M/N_2 be the cokernels of the inclusions $N_i \rightarrowtail M$.

Then one forms the pushout M' of $M/N_1 \twoheadleftarrow M \twoheadrightarrow M/N_2$, which is again guaranteed to exist by [4, p. 9]. The sum of N_1 and N_2 is the kernel $N_1 + N_2$ of the resulting strict epimorphism $M \to M'$. Diagrammatically, this can be written as follows:

$$\begin{tikzcd}
N_1 + N_2 \arrow[drr, tail] & & N_1 \arrow[ll, tail] \arrow[d, tail] & \\
N_2 \arrow[u, tail] \arrow[rr, tail] & & M \arrow[r, two heads] \arrow[d, two heads] & M/N_2 \arrow[d] \\
& & M/N_1 \arrow[r] & M'
\end{tikzcd} \tag{B.7}$$

It is shown in [41] that N_1 and N_2 are strict subobjects of $N_1 + N_2$ and that $N_1 \cap N_2$ is a strict subobject of both N_1 and N_2, so that we can take quotients $(N_1 + N_2)/N_i$ and $N_i/(N_1 \cap N_2)$ for $i = 1, 2$. By [41, Proposition 3.3], the sum and intersection of subobjects are related by the following Cartesian square:

$$\begin{tikzcd}
N_1 \cap N_2 \arrow[r, tail] \arrow[d, tail] & N_1 \arrow[d, tail] \\
N_2 \arrow[r, tail] & N_1 + N_2
\end{tikzcd} \tag{B.8}$$

Definition B.4.1. A **proto-abelian** category is a pointed category $\mathcal{C}$ which has all kernels and cokernels and additionally satisfies the following conditions:

(PA1) Any morphism with zero kernel (resp. cokernel) is a monomorphism (resp. epimorphism)

(PA2) The pullback of a strict epi along a strict mono is a strict epi, and the pushout of a strict mono along a strict epi is a strict mono.

♦

Each proto-abelian category carries a natural proto-exact structure, with $\mathfrak{M}$ consisting of strict monomorphisms and $\mathfrak{E}$ of strict epimorphisms. In proto-abelian categories, a **short**

exact sequence is a diagram of the form (B.2), where vertical arrows are strict epimorphisms and horizontal arrows are strict monomorphisms.

Remark B.4.2. Proto-abelian categories are sometimes defined in a different but related way to the above (see [20], [33]). In those treatments, a category is said to be proto-abelian if it is proto-exact and the classes $\mathfrak{M}$ and $\mathfrak{E}$ consist of all monomorphisms and all epimorphisms, respectively. Such categories are not necessarily proto-abelian in our sense, and neither notion of proto-abelianness immediately implies the other. ♦

Example B.4.3.

(1) Any abelian category is proto-abelian. Conversely, if a proto-abelian category has all finite products and coproducts, and if any morphism which is both monic and epic is an isomorphism, then the category is abelian (see [4]).

(2) The categories of finite-dimensional Euclidean and Hermitian vector spaces with linear maps of norm at most 1 is proto-abelian. This category has all finite coproducts and pushouts, but does not have all products or pullbacks.

(3) The category of Euclidean lattices with additive maps of norm at most 1 is proto-abelian (see [4], [41]).

◊

We can also promote a proto-exact category to a proto-abelian category, so long as it contains kernels and cokernels and satisfies the first axiom of proto-abelian categories above.

Proposition B.4.4. *Let $\mathcal{C}$ be a category with kernels and cokernels satisfying axiom* (PA1). *Suppose that $\mathcal{C}$ is proto-exact with $\mathfrak{M}$ being compositions of isomorphisms with strict monomorphisms and $\mathfrak{E}$ being the collection of strict epimorphisms composed with isomorphisms in $\mathcal{C}$. Then $\mathcal{C}$ is proto-abelian.*

Proof. By the definition of $\mathfrak{M}$ and $\mathfrak{E}$ we see that axioms (PE4) and (PE5) of a proto-exact category immediatly ensure that axiom (PA2) holds. □

B.5. Stability and slope in proto-abelian categories

Proto-abelian categories possess sufficient structure to define useful filtrations of their objects using slope functions, generalizing for example the Harder-Narasimhan filtration for vector bundles on curves. See [4], [51], and [41] for a broader account. We now introduce that part of this theory which will be of use to us in the sequel.

Suppose we are given **rank** and **degree** functions which are constant on isomorphism classes and additive on short exact sequences, so that they give additive maps out of the Grothendieck group of $\mathcal{C}$:

$$\mathrm{rk} : K_0(\mathcal{C}) \to \mathbb{N}$$

$$\deg : K_0(\mathcal{C}) \to \mathbb{R}$$

Suppose also that $r_M(E_M) = 0$ implies that M is the zero object. We then define a **slope** function $\mu : \mathrm{Iso}(\mathcal{C}) \setminus \{0\} \to \mathbb{R}$ on $\mathcal{C}$ by the formula

$$\mu(M) = \frac{\deg(M)}{\mathrm{rk}(M)}$$

Such a function μ is said to satisfy the **strong slope inequality** if for any object M and any strict subobjects N_1 and N_2 of M we have

$$\mu\left(\frac{N_1}{N_1 \cap N_2}\right) \leq \mu\left(\frac{(N_1 + N_2)}{N_2}\right). \tag{B.9}$$

Say that an object M of $\mathcal{C}$ is **semistable** (resp. **stable**) if we have $\mu(N) \leq \mu(M)$ (resp. $\mu(N) < \mu(M)$) for every strict subobject $N \rightarrowtail M$.

Definition B.5.1. A *slope filtration* of M is a flag of strict subobjects

$$0 = N_0 \rightarrowtail N_1 \rightarrowtail \cdots \rightarrowtail N_r = M \tag{B.10}$$

such that each quotient N_{i+1}/N_i is semistable and for which $\mu(N_i/N_{i-1}) > \mu(N_{i+1}/N_i)$. A **slope filtration** on $\mathcal{C}$ is a functorial assignment of a slope filtration to every object of $\mathcal{C}$. ♦

Remark B.5.2. The paper [4] establishes a framework in which slope functions are used to define slope filtrations on proto-abelian categories. The slope functions there are required to be non-decreasing under morphisms which are both monic and epic. Later, this framework was generalized by [41] to include the broader class of slope functions satisfying the strong slope inequality (B.9). The theory in [41] also departs from [4] in a handful of other subtle ways. However, since we do not need the level of generality attained there, we proceed with slope functions as defined above. ♦

An object M of $\mathcal{C}$ is said to be **Noetherian** (resp. **Artinian**) if it satisfies the Noetherian (resp. Artinian) property on chains of strict subobjects of M. In [41], Li establishes that given a function $\mu : \mathrm{Iso}(\mathcal{C}) \setminus \{0\} \to \mathbb{R}$ which satisfies the strong slope inequality, one obtains for each nonzero Noetherian and Artinian object of $\mathcal{C}$ a unique slope filtration associated to μ as above (see [41] Theorems 5.2, 5.3, 5.4, and the discussion in Section 5.3).

Theorem B.5.3 ([41], Theorems 5.2, 5.3, 5.4)**.** *Let $\mathcal{C}$ be a small proto-abelian category in which all objects are both Artinian and Noetherian and let $\mu : Iso(\mathcal{C}) \setminus \{0\} \to \mathbb{R}$ be a function satisfying the strong slope inequality (B.9). Then for each nonzero object M of $\mathcal{C}$ there exists a filtration*

$$0 = N_0 \rightarrowtail N_1 \rightarrowtail \cdots \rightarrowtail N_r = M$$

by strict subobjects, unique up to isomorphism, such that for all $i = 1, \ldots, r$ we have that N_i/N_{i-1} is semistable and satisfy

$$\mu(N_1/N_0) > \mu(N_2/N_1) > \cdots > \mu(N_r/N_{r-1}).$$

We call any filtration obtained in this way a **categorical slope filtration**. This establishes in particular that functions satisfying the strong slope inequality are a suitable generalization of the slope functions of [4] for the purpose of constructing filtrations.

Appendix C

Bands and Idylls

Bands and idylls are generalizations of rings and fields, respectively, which are of fundamental interest to us. Following [6] and [31], we introduce matroids over idylls and their pointed versions, as well as a handful of concepts promoted from classical matroid theory to matroids over idylls. The theory of matroids we encounter here follows the works [6] and [31]. For more details on bands, idylls, and relations to other partial algebraic structures, see [7].

A **pointed monoid** is a (multiplicative) commutative monoid $B = B' \sqcup \{0\}$ such that B' is a submonoid and 0 is an absorbing element, i.e., $0 \cdot b = 0$ for all $b \in B$. To each pointed monoid B is associated an **ambient semiring** $B^+ = \mathbb{N}[B]/\langle 0 \rangle$, where $\langle 0 \rangle$ is the semiring ideal generated by 0. A **band** is a pointed monoid equipped with a **null set** N_B, which is a proper ideal of B^+ satisfying

(B0) $0 \in N_B$

(B1) For every $a \in B$ there exists a unique element $b \in B$ such that $a + b \in N_B$. We denote this element by $-a$. In particular, note that each band contains an element -1 satisfying $1 + (-1) \in N_B$, where 1 is the multiplicative unit of B.

We think of the null set of a band B as capturing a notion of linear dependence over $\mathbb{N}$. An **idyll** is a band F such that $F^\times = F \setminus \{0\}$. A morphism of bands $\Phi : B \to C$ is a morphism of monoids with $\Phi(0_B) = 0_C$ such that $\sum_i \Phi(b_i) \in N_C$ for all $\sum_i b_i \in N_B$. With this notion of morphisms, bands form a category and idylls form a subcategory thereof.

Example C.0.1. 1. Any commutative ring R is a band with $N_R = \{\sum_i a_i : \sum_i a_i = 0\}$.

2. The *Krasner hyperfield* is the idyll $\mathbb{K}$ with underlying set $\{0, 1\}$ whose multiplicative structure is inherited from $\mathbb{Z}$ and whose nullset is $\mathbb{N} \setminus \{1\}$. This is the final object in the category of idylls.

3. The *regular partial field* is the idyll $\mathbb{F}_1^{\pm}$ with underlying set $\{-1, 0, 1\}$ whose multiplicative structure is inherited from $\mathbb{Z}$ and whose null set is $N_{\mathbb{F}_1^{\pm}} = \langle 1 + (-1) \rangle$.

4. The *tropical hyperfield* is the idyll $\mathbb{T}$ with underlying set $\mathbb{R} \sqcup \{\infty\}$ and multiplicative structure given by the additive group $\mathbb{R}$ and with additive structure defined by $a + b := \max\{a, b\}$. The null set is given by $N_{\mathbb{T}} = \{\sum_i a_i : \text{the maximum is attained at least twice}\}$.

5. The *hyperfield of signs* is the idyll $\mathbb{S}$ with underlying set $\{-1, 0, 1\}$ whose multiplicative structure is again inherited from $\mathbb{Z}$ but whose null set is defined by $N_{\mathbb{S}} = \{m \cdot 1 + n \cdot (-1) : m = n = 0 \text{ or } m \neq 0 \neq n\}$.

$\Diamond$

We will be particularly interested in the case where $B = F$ is a **perfect** idyll. In short, an idyll F is perfect precisely when the vectors of any F-matroid are orthogonal to the covectors of that F-matroid. The idylls $\mathbb{F}_1^{\pm}$, $\mathbb{K}$, $\mathbb{T}$, and $\mathbb{S}$ are perfect, as are all fields $\mathbf{k}$.

For ordered tuples of elements of a set, we use the following notation. Let E be a set and take some $n \in \mathbb{N}$. Then for a vector $\mathbf{x} \in E^n$, write $\mathbf{x}_k$ for the kth entry of $\mathbf{x}$ and write $\mathbf{x}_{\hat{k}}$ for the vector of length $n - 1$ obtained from $\mathbf{x}$ by deleting the kth entry. Write $|\mathbf{x}|$ for the **support** of $\mathbf{x}$, which is the set of nonzero elements of F appearing as entries of $\mathbf{x}$.

Appendix D

Two Kinds of Tropical Vector Bundles

The works [37] and [38] independently introduced two parallel notions of tropical vector bundles on tropical toric varieties X_Σ, using different axiomatic frameworks. In particular, [37] define vector bundles as matroids satisfying some compatibility with the fan Σ phrased in terms of the Bergman fan of the matroid. This definition is introduced in part to emphasize a connection with the theory of affine buildings and tropical linear spaces. On the other hand, [38] define their vector bundles as valuated matroids satisfying a different sort of compatibility with the fan Σ dealing with flags of flats in the underlying matroid.

In both cases, the definition of a tropical toric vector bundle is inspired by Klyachko's classification of torus-equivariant vector bundles on toric varieties, and in [38] the inspiration can be clearly seen from the definition of tropical toric vector bundles introduced there. As remarked in both relevant papers, the definitions of tropical toric vector bundles are equivalent in content but differ in terms of their technical details. The purpose of the present document is to illustrate the equivalence of these two definitions in the case where the toric variety X_Σ is smooth.

For the entirety of the discussion here, we fix a smooth toric variety X_Σ over a field K. Denote by M and N the corresponding character and cocharacter lattices. X_Σ is defined by the complete polyhedral fan Σ in the vector space $N_\mathbb{R}$. In what follows, we denote by Σ_M the

Bergman fan of a matroid M.

D.1. Tropical Toric Vector Bundles (A)

We first recall the definition of a tropical toric vector bundle used in [38]. We refer to this as the (A) definition of tropical toric vector bundles.

Definition D.1.1. A **tropical toric sheaf** of rank r on $\mathrm{trop}(X_\Sigma)$ is a tuple $\mathcal{E} = (\mathcal{M}, E, \{F_j^\rho\})$, where $\mathcal{M}$ is a valuated matroid of rank r on the set E and for each $\rho \in \Sigma(1)$ we have a flag of flats $\{F_j^\rho : j \in \mathbb{Z}\}$ for which $F_{j+1}^\rho \supseteq F_j^\rho$ and such that $F_j^\rho = 0$ for j sufficiently large and $F_j^\rho = E$ for j sufficiently small. ♦

Definition D.1.2. A tropical toric reflexive sheaf $\mathcal{E}$ is a **tropical toric vector bundle** (ttvb(A)) if for every cone $\sigma \in \Sigma$ there is some basis B_σ of $\underline{\mathcal{M}}$ for which F_j^ρ is a join of elements of B_σ for every $\rho \in \sigma(1)$ and every $j \in \mathbb{Z}$. ♦

D.2. Tropical toric vector bundles (B)

We now recall the definition of a tropical toric vector bundle used in [37]. We refer to this as the (B) definition of tropical toric vector bundles. For each matroid $M = (E_M, \mathcal{L}_M)$ we denote by P_M its associated matroid polytope and by Σ_M its Bergman fan in the space $\mathbb{R}^{E_M}$. Given a subset $S \subseteq E_M$, denote by $\mathbf{e}_S$ the indicator vector of S in $\mathbb{R}^{E_M}$. Identifying $\mathbb{R}^{E_M}$ with its dual $(\mathbb{R}^{E_M})^*$, we consider the normal fan $\mathcal{N}(P_M)$ as a fan in $\mathbb{R}^{E_M}$. The maximal cones of $\mathcal{N}(P_M)$ are denoted by σ_B and correspond to bases $B \in \mathcal{B}_M$.

Definition D.2.1. Let $B \in \mathcal{B}_M$ be a basis of M. Then the **apartment** of M associated to B is the subfan A_B of the Bergman fan defined by

$$A_\sigma = \Sigma_M \cap \sigma_B,$$

where σ_B is the maximal cone of $\mathcal{N}(P_M)$ determined by B. ♦

Definition D.2.2. A **tropical toric vector bundle** (ttvb(B)) of rank r on X_Σ is a pair (M, Φ). Here, M is a matroid of rank r on E_M and $\Phi : |\Sigma| \to \Sigma_M$ is a piecewise-linear map such that for every $\sigma \in \Sigma$ there is some basis B_σ of M for which $\Phi(\sigma) \subseteq A_{B_\sigma}$. Moreover, we require that for each such σ we have that $\Phi|_\sigma : \sigma \to A_{B_\sigma}$ is the restriction of an $\mathbb{R}$-linear map $\Phi_\sigma : \mathrm{span}(\sigma) \to A_{B_\sigma} \cong \mathbb{R}^r$. ♦

D.3. Equivalence of (A) and (B)

D.3.1 (A) Gives (B)

We recall first two auxiliary results in [37] which assist us in constructing a ttvb(B) from each ttvb(A).

Lemma D.3.1. *([37], Lemma 3.4) Each decreasing $\mathbb{R}$-filtration $\{F_r : r \in \mathbb{R}\}$ of flats determines a point $w \in \Sigma_M$ via declaring*

$$w_e = \sup\{r \in \mathbb{R} : e \in F_r\}.$$

Proposition D.3.2. *([37], Proposition 3.7(b)) Let $B \in \mathcal{B}_M$ be a basis of M with corresponding apartment A_B. Then A_B is piecewise-linearly isomorphic to $\mathbb{R}^r$ via an explicit homeomorphism α_B.*

In particular, note that Proposition D.3.2 furnishes each apartment A_B of a matroid M of rank r with an explicit homeomorphism α_B witnessing $A_B \cong \mathbb{R}^r$. These homeomorphisms α_B will be of essential use to us in assigning a ttvb(B) to ttvb(A).

Proposition D.3.3. *Each ttvb(A) determines a ttvb(B).*

Proof. Suppose that $(\mathcal{M}, E_M, \{F_j^\rho\})$ is a ttvb(A). Then for each $\rho \in \sigma(1)$ we obtain by

Lemma D.3.1a point $w_\rho \in \Sigma_M$ via

$$(w_\rho)_e = \sup\{k \in \mathbb{Z} : e \in F_k^\rho\}.$$

We use this assignment to produce the desired map Φ defining a ttvb(B). In particular, for each cone $\sigma \in \Sigma$ we first build a linear map $\varphi : \mathrm{span}(\sigma) \to \mathbb{R}^r$ and define the map to A_{B_σ} by composition with the homeomorphism α_σ.

Given a maximal cone $\sigma \in \Sigma$, let $\rho \leq \sigma$ be a ray of σ and let v_ρ be a primitive integer vector in its direction. Then declare $\varphi_\sigma(v_\rho) = \alpha_\sigma^{-1}(w_\rho)$. Since X_Σ was assumed to be smooth, we have that the cone σ is simplicial and so the assignment $\varphi_\sigma(v_\rho) = \alpha_\sigma^{-1}(w_\rho)$ can be uniquely extended to a linear transformation $\varphi_\sigma : \mathrm{span}(\sigma) \to \mathbb{R}^r$. We then define the piecewise-linear map $\Phi : |\Sigma| \to \Sigma_M$ by declaring $\Phi|_\sigma$ to be the map $\alpha_\sigma \circ \varphi_\sigma$.

To see that Φ then defines a piecewise-linear map on $|\Sigma|$, it is then enough to show that for any pair of cones $\sigma, \sigma' \in \Sigma$ meeting in a common face τ, we have that $(\Phi|_\sigma)|_\tau = (\Phi|_{\sigma'})|_\tau$. □

D.3.2 (B) Gives (A)

Lemma D.3.4. *Suppose that $F \in \mathcal{L}_M$ is a flat of M and that $B \in \mathcal{B}_M$ is a basis of M for which $\mathbf{e}_F \in \sigma_B$. Then F is a join of elements of B.*

Proof. Since $\mathbf{e}_F \in \sigma_B$, we have that

$$\langle \mathbf{e}_B, \mathbf{e}_F \rangle = \#F \cap B = \max_{\mathbf{v} \in P_M} \langle \mathbf{v}, \mathbf{e}_F \rangle.$$

In particular, for all other $b' \in \mathcal{B}_M$, we have

$$\#F \cap B = \langle \mathbf{e}_B, \mathbf{e}_F \rangle \geq \langle \mathbf{e}_{B'}, \mathbf{e}_F \rangle = \#F \cap B'.$$

Since $\mathcal{L}_M$ is a geometric lattice we can write $F = a_1 \vee \cdots \vee a_{r(F)}$, where $a_i \in \mathcal{L}_M$ is an atom for all i and $r(F)$ is the rank of F. Here, we have assumed in particular that $I_F := \{a_1, \ldots, a_{r(F)}\}$

is independent. Then letting $B' \in \mathcal{B}_M$ be any basis containing the subset I_F, we necessarily have that

$$\#F \cap B \geq \#F \cap B' = r(F),$$

so that in particular $\#F \cap B = r(F)$. This implies that there are elements $b_1, \ldots, b_{r(F)} \in F \cap B$ and so necessarily $F = b_1 \vee \cdots \vee b_{r(F)}$. This expresses F as a join of elements of B. $\square$

Proposition D.3.5. *Each ttvb(B) determines a ttvb(A) up to the choice of valuated matroid structure.*

Proof. Suppose that (M, Φ) is a ttvb(B). Then by the coherence criterion in Definition D.2.2 we know that for all $\sigma \in \Sigma$ there is some basis B_σ with $\Phi(\sigma) \subseteq A_{B_\sigma}$. In particular, for $\rho \in \Sigma(1)$, we obtain a basis B_ρ with $\Phi(\rho) \subseteq A_{B_\rho}$. Denoting by v_ρ the primitive integer vector in direction ρ, we have that $\Phi(v_\rho)$ lies in some cone $\sigma_{\mathcal{F}} \in \Sigma_M$ with $\sigma_{\mathcal{F}} \subseteq \sigma_B$. Since each such cone is associated to a flag $\mathcal{F} = \{F_\bullet\} \subseteq \mathcal{L}_M$, we thus obtain a way to associate a flag $\mathcal{F}$ to each ray ρ of Σ. Extending each such flag to a $\mathbb{Z}$-flag by appending 0 and E_M sufficiently many times, we therefore obtain flags $\{F_j^\rho : j \in \mathbb{Z}\}$ for each $\rho \in \Sigma(1)$. This construction then produces for each valuated matroid $\mathcal{M}$ over M a tropical toric reflexive sheaf $(\mathcal{M}, E_M, \{F_j^\rho\})$ for each ttbv(B).

To see that $(\mathcal{M}, E_M, \{F_j^\rho\})$ is a ttbv(A), we need only check the coherence criterion from Defintion D.1.2. In particular, given any cone $\sigma \in \Sigma$ containing a ray $\rho \leq \sigma$ we would like to see that each flat F_j^ρ is a join of elements of B, where B is a basis such that $\Phi(\sigma) \subseteq A_B$. By $v_\rho \in \sigma$ we trivially have that $\Phi(v_\rho) \in A_B$ and, since A_{B_σ} is a subfan of Σ_M, this implies that we have an inclusion of cones $\sigma_{\mathcal{F}} \subseteq \sigma_B$. In particular, for each flat $F_j \in \mathcal{F}$ we have $\mathbf{e}_F \in \sigma_{\mathcal{F}}$ and so $\mathbf{e}_{F_j} \in \sigma_B$. Finally, by Lemma D.3.4 we see that F_j is a join of elements of B as desired. $\square$

Bibliography

[1] Yang An, Matthew Baker, Greg Kuperberg, and Farbod Shokrieh. Canonical representatives for divisor classes on tropical curves and the matrix-tree theorem. *Forum Math. Sigma*, 2:Paper No. e24, 25, 2014.

[2] Laura Anderson. Vectors of matroids over tracts. *Journal of Combinatorial Theory, Series A*, 161:236–270, 2019.

[3] Laura Anderson. *Oriented matroids*, volume 216 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2025.

[4] Yves André. Slope filtrations. *Confluentes Math.*, 1(1):1–85, 2009.

[5] Roland Bacher, Pierre de la Harpe, and Tatiana Nagnibeda. The lattice of integral flows and the lattice of integral cuts on a finite graph. *Bull. Soc. Math. France*, 125(2):167–198, 1997.

[6] Matthew Baker and Nathan Bowler. Matroids over partial hyperstructures. *Adv. Math.*, 343:821–863, 2019.

[7] Matthew Baker, Tong Jin, and Oliver Lorscheid. New building blocks for $\mathbb{F}_1$-geometry: bands and band schemes. *J. Lond. Math. Soc. (2)*, 111(4):Paper No. e70125, 62, 2025.

[8] Matthew Baker and Farbod Shokrieh. Chip-firing games, potential theory on graphs, and spanning trees. *J. Combin. Theory Ser. A*, 120(1):164–182, 2013.

[9] Matthew Baker and Yao Wang. The Bernardi process and torsor structures on spanning trees. *Int. Math. Res. Not. IMRN*, (16):5120–5147, 2018.

[10] Olivier Bernardi. Tutte polynomial, subgraphs, orientations and sandpile model: new connections via embeddings. *Electron. J. Combin.*, 15(1):Research Paper 109, 53, 2008.

[11] Louis J. Billera and Bernd Sturmfels. Fiber polytopes. *Ann. of Math. (2)*, 135(3):527–549, 1992.

[12] Anders Björner, Michel Las Vergnas, Bernd Sturmfels, Neil White, and Günter M. Ziegler. *Oriented matroids*, volume 46 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 1999.

[13] Lev A. Borisov. The class of the affine line is a zero divisor in the Grothendieck ring. *J. Algebraic Geom.*, 27(2):203–209, 2018.

[14] Melody Chan, Thomas Church, and Joshua A. Grochow. Rotor-routing and spanning trees on planar graphs. *Int. Math. Res. Not. IMRN*, (11):3225–3244, 2015.

[15] Scott Corry and David Perkinson. *Divisors and sandpiles*. American Mathematical Society, Providence, RI, 2018. An introduction to chip-firing.

[16] Henry H. Crapo. Single-element extensions of matroids. *J. Res. Nat. Bur. Standards Sect. B*, 69B:55–65, 1965.

[17] Henry H. Crapo. The Möbius function of a lattice. *J. Combinatorial Theory*, 1:126–131, 1966.

[18] Changxin Ding. The rotor-routing torsor and the Bernardi torsor disagree for every non-planar ribbon graph. *Electron. J. Combin.*, 28(4):Paper No. 4.1, 17, 2021.

[19] Andreas Dress and Walter Wenzel. Grassmann-Plücker relations and matroids with coefficients. *Advances in Mathematics*, 86(1):68–110, 1991.

[20] Tobias Dyckerhoff and Mikhail Kapranov. *Higher Segal Spaces*. Springer International Publishing, 2019.

[21] Jens Niklas Eberhardt, Oliver Lorscheid, and Matthew B. Young. Group completion in the K-theory and Grothendieck-Witt theory of proto-exact categories. *J. Pure Appl. Algebra*, 226(8):Paper No. 107018, 32, 2022.

[22] Chris Eppolito, Jaiung Jun, and Matt Szczesny. Proto-exact categories of matroids, Hall algebras, and K-theory. *Math. Z.*, 296(1-2):147–167, 2020.

[23] Marco Fava, Nicola Pagani, and Filippo Viviani. A complete theory of smoothable compactified jacobians of nodal curves, 2025.

[24] Marco Fava, Nicola Pagani, and Filippo Viviani. A new class of compactified jacobians for families of reduced curves, 2025.

[25] Marco Fava, Nicola Pagani, and Filippo Viviani. A complete classification of modular compactifications of the universal jacobian, 2026.

[26] Ankan Ganguly and Alex McDonough. Rotor-routing induces the only consistent sandpile torsor structure on plane graphs. *Sém. Lothar. Combin.*, 86B:Art. 73, 12, 2022.

[27] Chris Heunen and Vaia Patta. The category of matroids. *Applied Categorical Structures*, pages 1–33, 2017.

[28] Alexander E. Holroyd, Lionel Levine, Karola Mészáros, Yuval Peres, James Propp, and David B. Wilson. Chip-firing and rotor-routing on directed graphs. In *In and out of equilibrium. 2*, volume 60 of *Progr. Probab.*, pages 331–364. Birkhäuser, Basel, 2008.

[29] Alexander E. Holroyd and James Propp. Rotor walks and Markov chains. In *Algorithmic probability and combinatorics*, volume 520 of *Contemp. Math.*, pages 105–126. Amer. Math. Soc., Providence, RI, 2010.

[30] Birkett Huber, Jörg Rambau, and Francisco Santos. The Cayley trick, lifting subdivisions and the Bohne-Dress theorem on zonotopal tilings. *J. Eur. Math. Soc. (JEMS)*, 2(2):179–198, 2000.

[31] Manoel Jarra, Oliver Lorscheid, and Eduardo Vital. Quiver matroids – matroid morphisms, quiver grassmannians, their euler characteristics and $\mathbb{F}_1$-points, 2024.

[32] Jaiung Jun, Alex Sistko, and Cameron Wright. Proto-exact categories of matroids over idylls and tropical toric reflexive sheaves, 2025.

[33] Jaiung Jun and Matt Szczesny. Hall Lie algebras of toric monoid schemes. *Selecta Mathematica*, 30(2):22, 2024.

[34] Takeshi Kajiwara. Tropical toric geometry. In *Toric topology*, volume 460 of *Contemp. Math.*, pages 197–207. Amer. Math. Soc., Providence, RI, 2008.

[35] Tamás Kálmán, Seunghun Lee, and Lilla Tóthmérész. The sandpile group of a trinity and a canonical definition for the planar Bernardi action. *Combinatorica*, 42(suppl. 2):1283–1316, 2022.

[36] MM Kapranov. Eisenstein series and quantum affine algebras. *Journal of Mathematical Sciences*, 84(5):1311–1360, 1997.

[37] Kiumars Kaveh and Christopher Manon. Tropical vector bundles and matroids, 2024.

[38] Bivas Khan and Diane Maclagan. Tropical vector bundles, 2024.

[39] Caroline J. Klivans. *The mathematics of chip-firing*. Discrete Mathematics and its Applications (Boca Raton). CRC Press, Boca Raton, FL, 2019.

[40] A. A. Klyachko. Equivariant vector bundles on toric varieties and some problems of linear algebra. In *Topics in algebra, Part 2 (Warsaw, 1988)*, volume 26, Part 2 of *Banach Center Publ.*, pages 345–355. PWN, Warsaw, 1990.

[41] Yao Li. Categorification of Harder-Narasimhan theory via slope functions valued in totally ordered sets. *Manuscripta Math.*, 173(3-4):1233–1271, 2024.

[42] Diane Maclagan and Bernd Sturmfels. *Introduction to tropical geometry*, volume 161 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2015.

[43] Alex McDonough. Genus from sandpile torsor algorithm. *Sém. Lothar. Combin.*, 80B:Art. 82, 12, 2018.

[44] Tadao Oda and C. S. Seshadri. Compactifications of the generalized Jacobian variety. *Trans. Amer. Math. Soc.*, 253:1–90, 1979.

[45] James Oxley. *Matroid theory*, volume 3. Oxford University Press, USA, 2006.

[46] Sam Payne. Moduli of toric vector bundles. *Compos. Math.*, 144(5):1199–1213, 2008.

[47] Sam Payne. Analytification is the limit of all tropicalizations. *Math. Res. Lett.*, 16(3):543–556, 2009.

[48] Bjorn Poonen. The Grothendieck ring of varieties is not a domain. *Math. Res. Lett.*, 9(4):493–497, 2002.

[49] V. B. Priezzhev, Deepak Dhar, Abhishek Dhar, and Supriya Krishnamurthy. Eulerian walkers as a model of self-organized criticality. *Physical Review Letters*, 77(25):5079–5082, Dec 1996.

[50] Jürgen Richter-Gebert and Günter M. Ziegler. Zonotopal tilings and the Bohne-Dress theorem. In *Jerusalem combinatorics '93*, volume 178 of *Contemp. Math.*, pages 211–232. Amer. Math. Soc., Providence, RI, 1994.

[51] Alexei Rudakov. Stability for an abelian category. *J. Algebra*, 197(1):231–245, 1997.

[52] Farbod Shokrieh and Cameron Wright. On torsor structures on spanning trees. *SIAM J. Discrete Math.*, 37(3):2126–2147, 2023.

[53] Daniel Slilaty and Thomas Zaslavsky. Cobiased graphs: Single-element extensions and elementary quotients of graphic matroids, 2024.

[54] Carsten Thomassen. Embeddings and minors. In *Handbook of combinatorics, Vol. 1, 2*, pages 301–349. Elsevier Sci. B. V., Amsterdam, 1995.

[55] Thomas Zaslavsky. Facing up to arrangements: face-count formulas for partitions of space by hyperplanes. *Mem. Amer. Math. Soc.*, 1:vii+102, 1975.