The Inverse Problem of Thermoacoustic Tomography in Attenuating Media

Benjamin P Palacios

A dissertation
submitted in partial fulfillment of the
requirements for the degree of

Doctor of Philosophy

University of Washington

2018

Reading Committee:

Gunther Uhlmann, Chair

Kenneth Bube

Hart Smith

Program Authorized to Offer Degree:
Department of Mathematics

University of Washington

Abstract

The Inverse Problem of Thermoacoustic Tomography in Attenuating Media

Benjamin P Palacios

Chair of the Supervisory Committee:
Professor Gunther Uhlmann
Department of Mathematics

Thermoacoustic tomography is a developing medical imaging technique that combines the propagation of electromagnetic and ultrasound waves with the purpose of producing a high contrast and high resolution internal image of a specific part of the body. The underlying physics involved in this technique naturally divides its theoretical analysis into two different problems corresponding to each type of waves. One of them is the inverse thermoacoustic tomography problem which studies the propagation of ultrasound across biological tissues and aims to obtain internal information related with the initial source of the acoustic waves, from data acquired on transducers placed outside the body of interest.

A natural difficulty in the analysis of ultrasound propagation is the attenuation effect caused by different tissues, in particular in the case of biological bodies. Its study is of course of great interest for real applications of this technology and it has received considerable attention in the last few years. In this thesis, I review some previous results for the thermoacoustic problem in the absence of acoustic dissipation, and in the second part, I analyze the inverse problem of thermoacoustic tomography in the presence of two types of attenuation models, local and nonlocal in time, and provide reconstruction formulas for both cases under some assumptions on the geometry of the problem. New results on uniqueness and stability of the inverse problem in the latter case are also established.

TABLE OF CONTENTS

Page

List of Figures . . . ii

Glossary . . . iii

Chapter 1: Introduction . . . 1

Chapter 2: The Classical TAT Inverse Problem . . . 7
- 2.1 Uniqueness: Carleman Estimates . . . 7
- 2.2 Stability: Propagation of Singularities and Control Theory . . . 12
- 2.3 Reconstruction . . . 19

Chapter 3: Attenuated TAT with Complete Data in Free Space . . . 37
- 3.1 The Inverse Problem . . . 39
- 3.2 Preliminaries . . . 42
- 3.3 Uniqueness . . . 44
- 3.4 Stability . . . 60
- 3.5 Reconstruction . . . 62

Bibliography . . . 84

Appendices . . . 89

Appendix A: Some Tools from Microlocal Analysis . . . 90
- A.1 Symbols, ΨDO's and FIO's . . . 90
- A.2 Garding's inequality . . . 91

Appendix B: Well-Posedness of (3.5) . . . 93

LIST OF FIGURES

Figure Number — Page

3.1 Foliation by level sets of $\mathrm{dist}(x, \partial\Omega)$ 46
3.2 Foliation and sub-characteristic unique continuation in Theorem 3.3.3 58
3.3 Unique continuation from $\{x \in \mathbb{R}^n \backslash \bar{\Omega} : \mathrm{dist}_e(x, \partial\Omega) > T/2\}$ 82

GLOSSARY

TAT/PAT	: thermoacoustic/photoacoustic tomography;
TR	: time reversal;
IVP	: initial value problem;
IBVP	: initial boundary value problem;
WF	: wave-front set;
ΨDO	: pseudo-differential operator;
FIO	: fourier integral operator;
$C^m(U)$	: space of m-times continuously differentiable functions in U;
$C_0^\infty(U)$	: space of infinitely differentiable functions supported inside U;
$L^2(U;\mu)$	: space of square integrable function in U, for the measure μ;
$L^2(U)$	: space of square integrable function in U for the Lebesgue measure;
$H^m(U)$	: space of functions in $L^2(U)$, such that $(1+\|\xi\|^2)^{m/2}\hat{u}(\xi) \in L^2$, where $\hat{}$ stands for the Fourier transform;
$H_0^1(U)$	: space of functions $u \in H^1(U)$, such that $u = 0$ on $\partial\Omega$;
H^1_{loc}	: space of functions for which there exists $\phi \in C_0^\infty(\mathbb{R}^n)$ such that $\phi u \in H^1(\mathbb{R}^n)$;
$\mathcal{L}(H)$	: space of linear operators from H into itself;
$T^*\mathbb{R}^n$	: cotangent space of $\mathbb{R}^n$;
$T^*_U\mathbb{R}^n\backslash\{0\}$	: subset of $T^*\mathbb{R}^n$ containing all cotangent vectors (x,ξ) with base point x in $U \subset \mathbb{R}^n$ and positive phase direction, i.e. $\xi > 0$;
S^*U:	cosphere bundle of $U \subset \mathbb{R}^n$;
$\mathcal{D}'(U)$	: space of distributions in U;

$\mathcal{E}'(U)$: space of distributions compactly supported in U;

$\Psi^0(U)$: space of pseudo-differential operators of order zero in U.

ACKNOWLEDGMENTS

I would like to thanks in the first place my advisor, Professor Gunther Uhlmann, for being such an amazing mentor throughout all these years as a PhD student. I will always be grateful for his friendship, for trusting in my capabilities and for the countless things I learned from him. I feel very fortunate and proud for had having him as an advisor, of course, because of his recognized talent for mathematics, but mostly because of his great quality as a person.

My sincere gratitude to Professors Kenneth Bube, Hart Smith and Anne Greenbaum for accepting being part of my defense committee.

I would like to personally thanks my colleagues and friends, Kaloyan Marinov, Yang Yang, Ru-Yu Lai, Yernat Assylbekov, Carlos Montalto, Sebastian Acosta, Francois Monard and Karthik Iyer, from whom I have learned many things and for making my PhD years joyful and unforgettable.

I would also like to express my infinite gratitude and love to all my friends in Seattle, especially the Chilean and Mexican community, for giving me tons of love, laughter and memorable moments. Also, many thanks to the great Husky Triathlon Team and all its members, for being an amazing company in the fun and suffering of triathlon training and racing.

Finally, and most importantly, I would like to express my deepest love to my family for all their immeasurable support, and specially to my amazing wife for being the greatest partner in my life, for making me a happier, better and wiser person, and for encouraging me to travel the world by her side.

DEDICATION

to my two favorite people in the world, my amazing wife *Camila Tobar* and my beloved brother *Lucas Palacios*

Chapter 1
INTRODUCTION

This thesis is intended to be a revision of my work on the inverse problem of thermoacoustic tomography (TAT) as a graduate student in the Department of Mathematics at University of Washington. The problem of TAT in attenuating media was the first task that my advisor, Professor Gunther Uhlmann, suggested me to work on and which gave rise to two published papers [35] and [3], the later in a joint work with S. Acosta from Baylor College of Medicine, Texas, US. Before start talking about TAT, I have decided to give a brief and general introduction to the field of inverse problem.

We call inverse problem to the exercise of collecting some internal information about an specific object or media where we don't have full access. The idea is to illuminate it by applying a particular type of waves (e.g. electromagnetic, elastic or sound waves) and measure the behavior of these when they interact with the media. Given this information, the goal is the reconstruction of some physical parameter in a given region of interest. Many of the applications of such problems belong to medical imaging where the objective is to obtain good resolution and good contrast pictures of the inside of the human body. For instance, in X-ray Computerized Tomography (X-ray CT) the image corresponds to the electromagnetic absorption coefficient of the biological tissues and it is obtained by probing the body with high energy electromagnetic rays.

In mathematical terms, an inverse problem refers to the inversion of the forward or measurement operator $\mathcal{M} : p \mapsto u\big|_\Gamma$, where provided a parameter p of the media being analyzed, we measure the its interaction with an emitted field of waves in a particular region Γ. More specifically, we seek to answer the following four questions:

- *Uniqueness:* $\mathcal{M}(p^1) = \mathcal{M}(p^2) \ \Rightarrow\ p^1 = p^2$. (injectivity of $\mathcal{M}$)

This property means that for every set of data there is a unique value of p that gives such measurements. Provided the inyectivity of $\mathcal{M}$ is now reasonable to think about its inverse $\mathcal{M}^{-1}$. Then we can address the problems of:

- *Stability:* $\mathcal{M}(p^1) \approx \mathcal{M}(p^2) \;\Rightarrow\; p^1 \approx p^2$. In other words, we would like to find the modulus of continuity of $\mathcal{M}^{-1}$, i.e., an increasing function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$, $\omega(0) = 0$, such that
$$\|p^1 - p^2\| \leq \omega\big(\|\mathcal{M}(p^1) - \mathcal{M}(p^2)\|\big).$$
for suitable norms.

This property tells us how the error in the measurements translates into error in the reconstruction of the parameters and of course depends on how this function ω is.

Finally we have:

- *Reconstruction:* Find an explicit formula for $\mathcal{M}^{-1}$.

- *Implementation:* Provide an efficient algorithm to compute $\mathcal{M}^{-1}$.

In most of the practical cases, using only one type of waves results in images that either have good contrast and bad resolution or the other way around. However, in the last decade many new imaging methods involving two types of waves have being developed, which rely in different physical principle that respectively link the different waves. They receive the name of coupled physics modalities.

Thermoacoustic and Photoacoustic tomography are coupled physics medical imaging techniques that consist in the application of a harmless electromagnetic pulse to some target tissue which causes an slightly increment in the local temperature, and makes the tissue to expand and produce pressure waves which are then recorded and used to reconstruct optical parameters of the region of interest. Their difference lies in the frequency employed for the illumination pulse being in the microwave range for TAT and in the infrared range for PAT. The goal behind the integration of two different types of waves is to take advantage of the good contrast of the electromagnetic absorption parameters and the good resolution the ultrasound data provides, to finally obtain a tomogram endowed with both desired features.

This method is naturally divided into two steps due to the different time-scales of the electromagnetic and sound propagation. The first one is the inversion of the ultrasound measurements to get some internal information related with the absorption of the electromagnetic radiation (same problem for both TAT and PAT), and the second step, also called *quantitative TAT/PAT*, consists in the inversion of such internal data to produce the actual image. We will focus on the first step and we refer the reader to [7] to learn more about QTAT/QPAT.

The problem of thermoacoustic tomography has been broadly studied by many authors. Several reconstruction methods have been proposed for homogeneous media [16, 31, 15, 29, 56, 34, 36, 18], and also for heterogeneous media [5, 24, 23, 41, 42, 52, 26, 25, 1, 46, 30, 12, 46]. See also the reviews [4, 28, 6] for additional references. One of the most remarkable techniques employed to reconstruct the initial condition is the so-called *time reversal* (TR) method. The theoretical analysis of TR has gained considerable attention in the past few years, mainly due to the work of Stefanov and Uhlmann in [41, 42]. In its initial formulation, the time reversal technique gives an approximate solution that converges to the exact one as the observation time increases. The problem of recovering the initial source for optimally short measurement time was solved in [41] for variable sound speed employing techniques from microlocal analysis. This work led to a very fruitful collection of articles seeking to extend the original ideas to different practical setting.

Recently, the focus of the mathematical analysis has been placed on extensions in the following two areas. First, there is the problem of accounting for attenuating media. Homan in [20] gave a first extension of Stefanov and Uhlmann's work in this direction by considering the damped wave equation with sufficiently small damping coefficients and considering an straightforward extension of the time reversal method. In the complete data case, those results were extended to more general damping coefficients in my article [35]. In [2] the authors addressed the TAT problem with thermoelastic attenuation. Second, recent publications have addressed the TAT problem in enclosed domains to model the interaction of acoustic waves with reflectors and sensors. The advantages of working with this setting is that it naturally allows to consider partial data and the inverse problem is closely related with Boundary Control Theory. See for instance [1, 46, 30, 12].

The mathematical formulation of the inverse TAT problem is as follows. We denote by $f(x)$ a source of ultrasound waves that is related with the absorbed EM energy and the absorption coefficient of the body. Because of the different time-scales between the EM energy absorption and the sound propagation, f is considered to act only at the initial time $t = 0$. Therefore, the ultrasound waves represented by $u(t, x)$ satisfy the system

$$\partial_t^2 u - c^2(x)\Delta u = f(x)\delta'(t), \quad u(t, x) = 0 \text{ for } t < 0,$$

where $c(x)$ is the (variable) sound speed of the media and δ' is the derivative of the Dirac delta distribution in time. It can be shown that this problem is equivalent to solve an initial value problem (IVP) which we introduce next. Let U and Ω be two open sets in $\mathbb{R}^n$ with the latter being bounded. We are interested in two cases: $U = \mathbb{R}^n$, and $U = \Omega$. The IVP studied in TAT is the system

$$\begin{cases} Pu = 0, & (t, x) \in (0, \infty) \times U \\ u(0, x) = f(x), & x \in U \\ \partial_t u(0, x) = 0, & x \in U, \end{cases} \tag{1.1}$$

with supp $f \subseteq \overline{\Omega}$ and P is a hyperbolic operator of the form

$$Pu = \partial_t^2 u - c^2(x)\Delta u, \tag{1.2}$$

where all the parameters of the system (f and c) live in suitable functional spaces and we assume $P = \partial_t^2 u - \Delta u$ outside Ω. If $U = \Omega$ we also consider boundary conditions of the form

$$\partial_\nu u = \lambda(x)\partial_t u, \quad \text{on } [0,T] \times \partial\Omega \tag{1.3}$$

where $\lambda \geq 0$ is assumed to be smooth. In the case $\lambda \equiv 0$ we say that the problem has a *perfectly reflecting boundary*, otherwise it has *partially reflecting boundary.* To carried out the analysis below it is necessary to place the above functions in a suitable functional setting. We introduce the following spaces, $H_D(U)$ and $\mathcal{H}(U)$, as the completions of $C_0^\infty(U)$ and $C_0^\infty(U) \times C_0^\infty(U)$ for the respective norms

$$\|f\|^2_{H_D(U)} = \int_U |\nabla f|^2 dx \quad \text{and} \quad \|\mathbf{f}\|^2_{\mathcal{H}(U)} = \|f_1\|^2_{H_D(U)} + \|f_2\|^2_{L^2(U)}$$

with $\mathbf{f} = (f_1, f_2)$ and $L^2(U) = L^2(U; c^{-2}\mathrm{d}x)$, the space of L^2 functions for the measure $c^{-2}\mathrm{d}x$. Then, $\mathcal{H}(U)$ can be identified with the space $H_D(U) \oplus L^2(U)$.

Under the hypothesis that the sound speed of the media is known, our goal is to analyze and hopefully invert the forward operator

$$\mathcal{M} : f \mapsto u\big|_\Sigma,$$

where $\Sigma = [0,T] \times \Gamma$, $\Gamma \subseteq \partial\Omega$ open, and $0 < T < \infty$. Since u depends linearly on f (i.e. $\mathcal{M}$ is linear), to prove uniqueness of the inverse problem, or equivalently the inyectivity of the operator $\mathcal{M}$, it is enough to show that

$$\mathcal{M}(f) = 0 \;\Rightarrow\; f = 0.$$

As you will see soon, the linearity of $\mathcal{M}$ would be very helpful in the stability problem and reconstruction as well.

Notice in the case $U = \mathbb{R}^n$, u is a solution in the whole space so we assume the measurements on the boundary do not affect its propagation. On the contrary, the case $U = \Omega$ with corresponding boundary conditions, models the interaction between the acoustic waves and the ultrasound transducers.

The rest of the thesis is divided into two main parts. Chapter 2 reviews the initial formulation of the thermoacoustic inverse problem, this is when media does not attenuate the propagating waves, which I call the *classical TAT inverse problem.* It contains three subsections devoted to respectively answer the first three main questions presented above, namely, uniqueness, stability and reconstruction. The most recent results on each topic are stated, and most of the proofs, or at least the ideas behind them, are provided as well. On the other hand, Chapter 3 consists in a detailed exposition of my two articles on the TAT problem in attenuating media, where I seek to answer the same three main questions already mentioned.

Chapter 2
THE CLASSICAL TAT INVERSE PROBLEM

Most of the recent advances in TAT are a consequence of a seminal paper of P. Stefanov and G. Uhlmann [41] in 2009, where the authors considered the case of variable sound speed $c = c(x)$, partial data on $\Gamma \subset \partial\Omega$, and finite observation time $0 < T < \infty$. Before [41] the research was mostly based on the assumption of constant wave speed and infinite measurement time, setting that even though is fair useful for applications it is not precisely what one encounters in practice. Indeed, in some cases as for example in brain imaging, due to the presence of the skull, a more accurate model for the sound speed is to consider $c(x)$ as a piecewise smooth function. This problem was addressed in a subsequent paper of the same authors in 2011 [42]. In the rest of this section we review the questions of uniqueness, stability and reconstruction based on the analysis developed in both articles [41] and [42].

2.1 Uniqueness: Carleman Estimates

The uniqueness question is intimately related with the existence/non-existence of pathological solutions of the equation in matter. In the context of boundary observation this means we have uniqueness for the inverse problem whenever there is no non-trivial solution leaving no trace on the measurement region. Such condition is guaranteed by Carleman estimates. These are weighted inequalities which serve as magnifying glasses near the boundary and help to ensure there is no pathological solutions invisible from the boundary, and consequently we are able to acquire all the necessary information to solve the inverse problem. The main result of this subsection is the next uniqueness theorem from [41]. Its proof is a generalization of the method applied in [15] for the constant sound speed case, by using a more powerful unique continuation result which of course is based on the existence of Carleman estimates (see [50]). The theorem reads as follows.

Theorem 2.1.1 (Stefanov & Uhlmann, 2009, [41])**.** *Assume that* $P = \partial_t^2 - \Delta$ *outside* Ω *and* $\partial\Omega$ *is strictly convex. Under the assumption that* $f \in H_D(\Omega)$ *with* $\text{supp} f \subseteq \mathcal{K}$*, for some compact* $\mathcal{K} \subset \Omega$ *satisfying*

$$\forall x \in \mathcal{K},\ \exists z \in \Gamma \text{ so that } \text{dist}(x, z) < T,$$

if $u|_{[0,T]\times\Gamma} = 0$*, then* $f \equiv 0$ (dist$(\cdot,\cdot)$ *stands for the distance under the metric* $c^{-2}\mathrm{d}x^2$).

Remark. In the full data case, uniqueness is guaranteed as long as we have $\text{supp} f \subset \bar{\Omega}$ and $T > T_0(\Omega) = \sup_{x\in\Omega} d(x, \partial\Omega)$, with $d(x, \partial\Omega)$ the infimum of the lengths of curves with respect to $c^{-2}\mathrm{d}x^2$, starting at x and ending at $\partial\Omega$.

The proof is based on the fact that we can extend in a even way the system (1.1) to negative times, and the next.

Proposition 2.1.1. *If* $u \in H^1_{\text{loc}}$ *satisfies* $Pu = 0$ *and* $u = 0$ *in a neighborhood of* $[-T, T] \times \{q\}$*, with some* $T > 0$*,* $q \in \mathbb{R}^n$*, then* $u = 0$ *in the set* $\{(t, x) \ :\ \text{dist}(x, q) + |t| \le T\}$,

This unique continuation for hyperbolic equations is a consequence of the next more general result due to D. Tataru and the subsequent remark. Let's first recall the following definition.

Definition 2.1.1. Let S be a C^2-hypersurface, this is, $S = \{x : \varphi(x) = 0\}$ for some $\psi \in C^2(\mathbb{R}^n)$ such that $\nabla\varphi(x) \neq 0$ in S. We say that S is *strongly pseudoconvex* with respect to P at $x_0 \in S$ on a closed conic subset $\Gamma \subset T^*\mathbb{R}^n$ if,

$$\text{Re}\{\overline{p}, \{p, \varphi\}\}(x_0, \xi) > 0 \quad \text{for all } \xi \in \Gamma_{x_0}\backslash\{0\} \quad \text{such that} \quad p(x_0, \xi) = \{p, \varphi\}(x_0, \xi) = 0, \tag{2.1}$$

and

$$\begin{gathered}\frac{1}{\mathrm{i}\tau}\{\overline{p}(x, \overline{\zeta}), p(x, \zeta)\}(x_0, \xi) > 0 \quad \text{for all } \xi \in \Gamma_{x_0},\ \tau > 0 \quad \text{such that} \\ p(x_0, \zeta) = \{p(x_0, \zeta), \varphi\}(x_0, \xi) = 0, \quad \text{with } \zeta = \xi + \mathrm{i}\tau\nabla\varphi(x_0).\end{gathered} \tag{2.2}$$

Theorem 2.1.2 (Tataru, [50]). *Let's write $x = (x_a, x_b) \in \mathbb{R}^n$. Assume that P has coefficients analytic with respect to x_a and let $S = \{\varphi = 0\}$ be an oriented hypersurface which is strongly pseudoconvex in the subset $\Gamma = \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n : \xi_a = 0\} \subseteq T^*\mathbb{R}^n$. Assume that either of the following two conditions holds:*

a) P is elliptic in Γ.

b) P has real symbol on Γ and Γ is invariant with respect to the null bicharacteristic flow of P (i.e. $\partial_{x_a} p = 0$ on $\{p = 0, \xi_a = 0\}$).

Then, if u is solution to $P(x,D)u = 0$ near $x^0 \in S$ so that $\operatorname{supp} u \subseteq \{\varphi \leq \varphi(x^0) = 0\}$*, then $x^0 \notin$* $\operatorname{supp} u$.

Remark. Let $P = \partial_t^2 - g^{ij}(x)\partial_i\partial_j$ with $(g^{ij}(x))_{ij}$ positive definite matrix-function, where here and the rest of the thesis, we use the Einstein summation convention for repeated indices. Its symbol is given by

$$p(t,x,\xi,s) = -s^2 + g^{ij}(x)\xi_i\xi_j.$$

Since P has time independent coefficient they are analytic with respect to the time variable t, therefore $\Gamma = \{s = 0\}$ and the principal symbol becomes $p = g^{ij}(x)\xi_i\xi_j$ in Γ, i.e. P is elliptic in Γ. On the other hand, $\{p(x_0, \xi + \mathrm{i}\tau\nabla\varphi(x_0)), \varphi\} \neq 0$ whenever $\{\varphi = \varphi(x^0)\}$ is noncharacteristic (i.e. $p(t,x,\nabla_{t,x}\varphi) \neq 0$), so the strongly pseudoconvexity condition is satisfied by default in Γ. Consequently, P has the unique continuation property across any noncharacteristic surface.

The underlying principle behind Theorem 2.1.2 is a generalization of the classical Carleman estimates that accounts for the lost of pseudo-convexity in the whole cotangent bundle. In the simpler case where an hypersurface $S = \{\varphi = 0\}$ is strongly pseudoconvex with respect to P for all $(x,\xi) \in \mathbb{R}^{2n}$, Hörmander in [21] shown the existence of a constant $C > 0$ where roughly speaking

$$\|e^{\tau\varphi}u\| \leq C\|e^{\tau\varphi}Pu\|$$

for all u supported near a point on S and for all values of $\tau > \tau_0$ with τ_0 large. Then, if we are given a solution to $Pu = 0$, by localizing and using the above inequality it is possible to obtain the unique continuation property across S. This result applies for instance to general elliptic operators and any smooth hypersurface S, and to the wave operator with constant coefficients and strictly-convex hypersurfaces.

For more general differential operators and hypersurfaces S, the later might not be strongly pseudoconvex with respect to P for all covectors in $T^*_S\mathbb{R}^n \backslash \{0\}$, with $\{0\}$ the zero section of the cotangent space. This is in fact the case of the operator in (1.2). If we let now P to be as in Theorem 2.1.2 it is still possible to obtain an estimate of Carleman type by microlocalizing first on Γ. This is achieved by applying the pseudo-differential operator with symbol $e^{-\frac{\epsilon}{2\tau}|\xi_a|^2}$, which it is nothing but a Gaussian function in phase space, cutting off a small neighborhood of Γ. Then, denoting $D_a = \frac{1}{\mathrm{i}}\partial_{x_a}$, we get the inequality

$$\tau^{-1}\|e^{-\frac{\epsilon}{2\tau}|D_a|^2}e^{\tau\varphi}u\|^2_{m,\tau} \leq C\left(\|e^{-\frac{\epsilon}{2\tau}|D_a|^2}e^{\tau\varphi}Pu\|^2_0 + \|e^{\tau(\varphi-d\epsilon)}Pu\|^2_0 + \|e^{\tau(\varphi-d\epsilon)}u\|^2_{m-1,\tau}\right),$$

where

$$\|u\|^2_{m,\tau} = \sum_{|\alpha|+|\beta|\leq m} \tau^{2(m-|\alpha|-|\beta|)}\|D^\alpha_{x_a}D^\beta_{x_b}u\|^2_{L^2(\mathbb{R}^n)} \quad \text{and} \quad \|\cdot\|_0 = \|\cdot\|_{L^2(\mathbb{R}^n)}.$$

Notice that in this case we have extra terms in the right hand side which account for lost of strongly pseudoconvexity in the whole cotangent bundle, therefore, if u solves $Pu = 0$, and recalling that $\varphi = 0$ on the boundary, we deduce the inequality

$$\|e^{-\frac{\epsilon}{2\tau}|D_a|^2}e^{\tau\varphi}u\|^2_0 \leq Ce^{\tau\gamma}, \quad \gamma < 0,$$

on a sufficiently small neighborhood of a given point $x_0 \in S$. The conclusion that u vanishes on the other side of S follows from this inequality and [50, Theorem 4].

Let's go back for a moment to Proposition 2.1.1. The remark following Theorem 2.1.2

tells us that we can uniquely extend a solution to $Pu = 0$ across a non-characteristic surface. By considering a suitable family of (non-characteristic) hypersurfaces exhausting the double cone $\{\text{dist}(x, q) + |t| \leq T\}$, we can iteratively extend the zero set of u until we hit the characteristic set $\{\text{dist}(x, q) + |t| = T\}$. The initialization of this inductive process follows from the fact that there exists one of this hypersurfaces contained in the neighborhood of $[-T, T] \times \{q\}$ where $u = 0$.

We are now ready to give the idea behind the proof of the uniqueness Theorem 2.1.1.

Sketch of the proof. The first thing to notice is that by extending u in an even way, we obtain a solution to $Pu = 0$ for times in $[-T, T]$. Secondly, u is a solution of $Pu = 0$ in the whole space and in particular it satisfies the exterior problem

$$\begin{cases} Pu = 0, & x \in \mathbb{R}^n \backslash \bar{\Omega},\ t \in \mathbb{R}, \\ u = u_t = 0, & x \in \mathbb{R}^n \backslash \bar{\Omega}, \\ u = h, & x \in \partial\Omega,\ t \in \mathbb{R} \end{cases}$$

with $h = 0$ on $(-T, T) \times \Gamma$. Therefore, the finite speed of propagation property for solutions of this type of equations [16, Proposition 2] implies that $u = 0$ on a exterior neighborhood of Γ, and for times $t \in (-T, T)$. For an arbitrary $x \in \mathcal{K}$, there exists p in such neighborhood so that $\text{dist}(x, p) < T$, thus by applying Proposition 2.1.1 on p get that $f(x) = 0$. □

In the case of an enclosure, the uniqueness is also a consequence of Proposition 2.1.1 and the proof follows almost in the same way as above. The extra step we have to take is the extension of the problem outside Ω close to the observable region Γ. We denote this set by $\tilde{\Omega}$ and we extend the sound speed smoothly outside Ω, thus since u has null Cauchy data on Γ, we can set u equal to zero on $\tilde{\Omega} \backslash \Omega$ and it still satisfies the wave equation in the larger domain. Since $u = 0$ for all $(t, x) \in (-T, T) \times (\tilde{\Omega} \backslash \Omega)$ we can apply Proposition 2.1.1 as in the proof of Theorem 2.1.1 and obtain the uniqueness. We then have.

Theorem 2.1.3 (Theorem 1 in [46]). *$\Lambda f = 0$ for some $f \in H_D(\Omega)$ implies $f(x) = 0$ for* $\text{dist}(x, \Gamma) < T$. *In particular, if $T \geq T_0$, then $f = 0$.*

2.2 Stability: Propagation of Singularities and Control Theory

The stability in the recovery of the initial source depends strongly in our ability of acquiring information of the high frequency features of such function. In mathematical terminology those high frequency features are called singularities. In this section we will see that in order to answer the question of whether or not there is a stable recovery in the inverse problem, it is necessary to study how singularities of the initial condition propagate through the media and hopefully be able to detect everyone of them. We say $(x, \xi) \in T^*\mathbb{R}^n \setminus \{0\}$ (recall that $\{0\}$ stands for the zero section) is a singularity of a distribution u if (x, ξ) is an element of its *Wave Front Set*, $WF(u)$:

Definition 2.2.1. Let $X \subset \mathbb{R}^n$ open. For $u \in \mathcal{D}'(X)$, $(x_0, \xi_0) \notin WF(u)$ if and only if there exists $\varphi \in C_0^\infty(X)$ with $\varphi(x_0) \neq 0$, and a conic neighborhood Γ of ξ_0, such that $\widehat{\varphi u}(\xi)$, the Fourier transform of φu, is of rapid decrease in Γ (i.e. for every $N > 0$, we have $|\widehat{\varphi u}(\xi)| \leq C_N(1 + |\xi|)^{-N}, \xi \in \Gamma$).

In what follows we will use the following characterization of the wave front set.

Proposition 2.2.1.

$$(x_0, \xi_0) \notin WF(u) \quad \textit{iff} \quad \exists P \in \Psi^0(X), \ \textit{elliptic at } (x_0, \xi_0) \ \textit{s.t.} \ Pu \in C^\infty(X),$$

where $\Psi^m(X)$ stands for the set of pseudo-differential operators (ΨDO) of order m in X (see Definition A.1.2).

We will focus on the free space case and then give some ideas behind the analysis for the case of an enclosure.

2.2.1 Stability in Free Space

The stability analysis is based on the idea that provided uniqueness for the inverse problem and assuming we can recover all the singularities of the initial condition, the linear character of the inverse problem gives us the stability. Such approach is based in a corollary of the following well known functional analysis result [51, Proposition V.3.1].

Theorem 2.2.1. *Let $T : E \to X$ be a closed linear operator between Banach spaces, and suppose $K : E \to Y$ is compact. If for every u in the domain of T*

$$\|u\|_E \leq C\|Tu\|_X + C\|Ku\|_Y,$$

then T has closed range.

Corollary 2.2.1. *Under the same hypothesis of the previous theorem, if in addition T is injective then*

$$\|u\|_E \leq C\|Tu\|_X,$$

for every u in the domain of T.

Recall we are trying to recover the initial source f in (1.1) by collecting Dirichlet boundary data $h = u|_{\partial\Omega\times(0,T)}$, where Ω is a bounded set whose boundary does not perturb the outgoing waves.

Propagation of Singularities

The analysis of singularities seek to determine the dynamic of singularities of solutions to differential equations. We will see soon that for the wave equation we are able to explicitly identify the paths that singularities follow, and assuming some conditions on the geometry of the domain we can guarantee that they eventually reach $\partial\Omega$, living the domain when the boundary is invisible for the wave propagation, or being detected and/or reflected whenever $\partial\Omega$ is a reflecting boundary. A first result and rather elemental is obtained by writing (1.1) as a non-homogeneous initial value problem. By extending u to zero for negative times, this is by replacing u with $H(t)u(x,t)$, where H stands for the Heaviside function, we get a solution to the problem

$$Pu = \delta' f \text{ in } \mathbb{R}^{n+1}, \quad u = 0 \text{ for } t < 0, \tag{2.3}$$

where δ is the Dirac's delta distribution. In what follows we denote by u the extended solution. Considering P as a pseudo-differential operator, the next proposition gives us a first idea of the behavior of singularities.

Proposition 2.2.2. *For any pseudo-differential operator P and $u \in \mathcal{D}'(\mathbb{R}^n)$,*

$$WF(Pu) \subseteq WF(u) \subseteq WF(Pu) \cup \mathrm{char}P,$$

where char$P$ is the characteristic set of P defined as

$$\mathrm{char}P := \{(x;\xi) \in T^*(\mathbb{R}^n) \setminus \{0\} : p(x;\xi) = 0\},$$

with p its principal symbol.

Let's see what we can deduce from the above. If u solves $Pu = F$, then all the singularities of F are contained in the wave front set of u. On the other hand, the singularities of u are either singularities coming from the source F, or they are produced by the dynamic generated by P and therefore they belong to its characteristic set. In the problem we are studying, $p(t,x;\tau,\xi) = \tau^2 - c^2(x)|\xi|^2$ and we have

$$WF(u) \subseteq WF(\delta' f) \cup \{(t,x;\tau,\xi) \in T^*(\mathbb{R}^n) \setminus \{0\} : \tau^2 = c^2(x)|\xi|^2\}.$$

We can go further tracking singularities thanks to the next sharp result due to Hörmander (see for instance [22]). Recall Definition A.1.3 for the set Ψ^m_{cl} of classical ΨDO's of order m and Definition A.1.4 of operators of real principal type. We recall also that a null-bicharacteristic strip $\gamma = (x(s), \xi(s))$ of a ΨDO with symbol $p(x,\xi)$ is a curve in phase space such that

$$p(x(s), \xi(s)) = 0 \quad \text{and} \quad \frac{dx}{ds} = \frac{\partial p}{\partial \xi}(x,\xi), \;\; \frac{d\xi}{ds} = -\frac{\partial p}{\partial x}(x,\xi).$$

Theorem 2.2.2. *Let $P \in \Psi^m_{cl}(X)$ of real principal type, $X \subseteq \mathbb{R}^n$ open, and let $\gamma : [a,b] \to T^*X \setminus 0$ be a null bicharacteristic strip with $-\infty < a < b < \infty$. Let $u \in \mathcal{D}'(X)$ satisfy*

$\gamma([a,b]) \cap WF(Pu) = \emptyset$. *Thein either* $\gamma([a,b]) \subseteq WF(u)$ *or* $\gamma([a,b]) \cap WF(u) = \emptyset$.

Let's u be a solution to (2.3). Given a singularity $\rho_1 = (t_1, x_1; \tau_1, \xi_1)$ of u for $t_1 > 0$, we can parametrize the null bicharacteristic γ passing through ρ_1 with parameter given by the time variable, and track the singularity back in time until it reaches $t = 0$. If $\gamma|_{t=0} = (0, x_0; \tau_0, \xi_0) \notin WF(\delta' f)$, then we can apply again the theorem and conclude that $\gamma(t) \in WF(u)$ for $t < 0$ close to 0. However, u is smooth for negative times which contradicts the previous, thus we must have that $(0, x_0; \tau_0, \xi_0) \in WF(\delta' f)$ and then $(x_0, \xi_0) \in WF(f)$. We finally get that

$$WF(u) = \{\gamma_\pm(s) : \text{null bicharacteristic s.t.} \gamma_\pm(0) = (0, x; \pm c(x)|\xi|, \xi) \text{ for some } (x, \xi) \in WF(f)\}.$$

In the particular case of the wave equation, the projection $x(s)$ of null bicharacteristics $\gamma \subseteq T^*(\mathbb{R}^{n+1})$ to $\mathbb{R}^n$ are precisely the geodesics for the metric $c^{-2}\mathrm{d}x^2$ and consequently the singularities propagate along geodesics. We say that $(\Omega, c^{-2}\mathrm{d}x^2)$ is *non-trapping* if all singularities inside Ω have finite length or in other words, fixing an initial time, they all leave the domain in finite time.

We are now ready to state and prove the main result of this section. We first introduce the stability time T_1 for which the condition $T > T_1$ guarantees that all singularities leave a trace on $\partial\Omega$ and we are able to detect them. We set

$$T_1(\Omega) := \sup\{|\gamma|_g : \gamma \text{ geodesic for } g = c^{-2}\mathrm{d}x^2 \text{ on } \bar{\Omega}\}.$$

The condition of $(\Omega, c^{-2}\mathrm{d}x^2)$ being non-trapping translates to $T_1 < \infty$.

Theorem 2.2.3 (Stability)**.** *Assuming that* $T_1 < T < \infty$*, then*

$$\|f\|_{H_D(\Omega)} \leq C\|\Lambda f\|_{H^1(\partial\Omega\times(0,T))}.$$

Remark. This theorem is not sharp in the sense that it has been proven that stability still holds when considering $T > T_1/2$ (see next Theorem 2.2.4). The final form of the stability theorem follows from a more detailed microlocal analysis that we present in §2.3. Notice that in a non-trapping geometry, $T_0 < T_1/2$, where T_0 is the uniqueness time defined above.

Remark. If we a-priori know that supp$f \subset \mathcal{K}$ for some compact set $\mathcal{K} \subset \Omega$, then the stability is guaranteed as long as we have $T > T_1(\mathcal{K})$, with the latter time defined as $T_1(\Omega)$ but with the supremum taken over all the geodesic passing through $\mathcal{K}$.

Proof. Let's denote by $A : H^1_{(0)}([0,T] \times \partial\Omega) \to H^1_0(\Omega) \times L^2(\Omega)$ the *Time Reversal operator* that back propagates the Dirichlet data with null final conditions. Let χ be a cut-off function such that $\chi = 1$ in $(0, T-\delta)$ for some small $\delta > 0$, and $\chi(T) = 0$. If we call $h = u|_{[0,T]\times\partial\Omega}$ the data from the TAT problem then $A\chi h$ is an approximation of f. It turns out that by linearity and uniqueness of solutions for the IBVP, their difference satisfies that $f - A\chi\Lambda f = w(\cdot, 0)$ with w solution of

$$\begin{cases} (\partial_t^2 - c^2(x)\Delta)w = 0, & (t,x) \in (0,T) \times \Omega \\ (w, w_t)|_{t=T} = (u, u_t)|_{t=T}, & x \in \Omega \\ w = (1-\chi)h, & (x,t) \in (0,T) \times \partial\Omega. \end{cases} \tag{2.4}$$

By the above propagation of singularities result and since we are assuming $T > T_1(\Omega)$, all the singularities of u are outside Ω at time T thus the operator $f \mapsto (u, u_t)|_{t=T}$ maps $H_D(\Omega)$ to $C^\infty(\Omega) \times C^\infty(\Omega)$. Then, since there is compatibility to arbitrary order between the boundary condition and the final condition, we get that $w|_{t=0}$ is also smooth, therefore the operator $R : f \mapsto w|_{t=0}$, maps $H_D(\Omega)$ to $C^\infty(\Omega)$ and we can write $f = A\chi\Lambda f - Rf$. From the last relation we get

$$\|f\|_{H_D} \leq \|A\chi\Lambda f\|_{H_D} + \|Rf\|_{H_D},$$

with A bounded from $H^1(\partial\Omega)$ to $H_D(\Omega)$, thus we in fact have

$$\|f\|_{H_D} \leq C\left(\|\Lambda f\|_{H^1} + \|Rf\|_{H^1}\right).$$

Recalling that Λ is injective due to the uniqueness result of Section 2.1, and that R is smoothing thus compact from H_D to H^1, we conclude the proof by applying Corollary 2.2.1. □

Remark. A similar proof works when we assume the waves travel along an attenuating media by satisfying the damped wave equation in $\mathbb{R}^n$,

$$P_a u := (\partial_t^2 + a(x)\partial_t - c^2(x)\Delta)u = 0. \tag{2.5}$$

Singularities propagate in the same way since lower order terms does not alter their behavior, so they still flow along geodesics for the metric $c^{-2}\mathrm{d}x^2$.

In [42] the authors improved the stability result by only requiring the observation time T to be larger than $T_1/2$. This is a direct consequence of a reconstruction formula obtained in the same article and which we present in the next section (see Theorem 2.3.2). For a compact $\mathcal{K} \subset \Omega$ as in the second remark following Theorem 2.2.3, we have.

Theorem 2.2.4 (Improved stability)**.** *Assuming that* $T_1(\mathcal{K})/2 < T < \infty$*, then there exists* $C = C(\mathcal{K}) > 0$ *such that*

$$\|f\|_{H_D(\mathcal{K})} \leq C\|\Lambda f\|_{H^1((0,T)\times\partial\Omega)}.$$

2.2.2 Stability in an Enclosure

In the case we deal with an enclosure, the singularities are reflected in the boundary and they remain inside the domain for all times. We don't have anymore that the solution inside the domain becomes smoother after some time which was a crucial ingredient in the proof of stability for the free space case. However, we will see soon that due to such reflection of singularities the case of partial data arises naturally and we are able to solve it by requiring a geometrical condition on the acquisition region.

The stability for the inverse problem in an enclosure has been widely study by the Control Theory community under the name of *boundary observability.* The observability property of a system like (1.1)-(1.3) means in words that enough energy coming from the initial condition is detected in the accessible part of the boundary. In fact, this property is equivalent to the detection of singularities in the observable region since it is a well-known fact that the majority of the energy is carried by singularities.

Definition 2.2.2. We say the system (1.1)-(1.3) is observable from $\Gamma \subseteq \partial\Omega$ if there exists $C > 0$ so that

$$E_\Omega(u(0)) := \int_\Omega |\nabla u(0)|^2 dx + \int_\Omega |u_t(0)|^2 c^{-2} dx \leq C \int_{(0,T)\times\Gamma} |\partial_t u|^2 dt dx. \tag{2.6}$$

The term in the left hand side is the initial energy of the system.

In 1992, Bardos, Lebeau and Rauch in [8] proved a sharp condition on the boundary acquisition region as well as in the observation time for which (2.6) holds. This is the so-called *Geometric Control Condition* (GCC):

Condition 2.2.1. *The geodesic flow of* $(\Omega, c^{-2}\mathrm{d}x^2)$ *reaches* Γ *after possible reflections on* $\partial\Omega\backslash\Gamma$ *in finite time* τ*, and in addition, if* $\lambda \not\equiv 0$ *on system* (1.1)-(1.3), $\{\lambda > 0\} \subseteq \Gamma$*. In other words, there exists* $0 < \tau < \infty$ *such that any geodesic ray, originating from any point in* Ω *at* $t = 0$*, eventually reaches* Γ *in a non-diffractive manner (after possible geometrical reflections on* $\partial\Omega \setminus \Gamma$*) before time* $t = \tau$*. The infimum of such* τ*'s is called exact controllability time.*

We of course can reformulate the above condition in the case we a-priori know that $\mathrm{supp} u(0)$ is contained in a given compact set $\mathcal{K}$ by requiring the GCC to hold only for the geodesics passing through such set. In the particular case that $\Gamma = \partial\Omega$, then $\tau = T_1(\mathcal{K})$.

The proof of the sufficiency of Condition 2.2.1 is based on microlocal analysis techniques. Bardos, Lebeau and Rauch studied how the energy of the solution is deposited in the boundary, and concluded that violating the GCC there exists solutions concentrated near geodesics so that most of their energy is never detected in the accessible region of the boundary. It then follows directly from [8] that assuming the GCC there is stability for the inverse problem in closed domains.

Theorem 2.2.5. *Assuming the GCC for some compact set* $\mathcal{K} \subset \Omega$ *and* $\tau < T$*, there exists* $C = C(\mathcal{K}, \Gamma) > 0$ *such that*

$$\|f\|_{H^1(\mathcal{K})} \leq C\|\Lambda f\|_{H^1((0,T)\times\Gamma)}.$$

2.3 Reconstruction

2.3.1 Free Space: Microlocal Energy Estimates

Most of the resent work in this area has been focus on the search of reconstruction precedures via iterative formulas. These methods consist in iterative processes where one uses the boundary data to solve wave equations, forward and backward in time, to refine the reconstruction on every step. The principle exploited in this approach is the time reversibility of the wave equation. In [23] it was proved that back propagating the Dirichlet data into inside the domain, by solving the wave equation backward in time with null final conditions, gives a good approximate solution to the inverse problem, which gets better as we consider larger measurement times. This time reversal technique is the natural thing to do due to the already mentioned time-reversibility. In fact, if we had access to the final state of the wave (i.e. at time T) it would be possible to perfectly reconstruct the initial source. In TAT/PAT this is no the case though. The novel idea of Stefanov and Uhlmann introduced in [41] was to use the boundary data at time T, and construct a final condition for the backward problem that makes the error between the initial condition and this new time reversal technique to be a contraction. This results in the convergence of a Neumann series which is nothing but an iterative process like the one described above. In fact, choosing a measurement time $T_1(\Omega) < T < \infty$, they obtained that such error operator was compact.

Theorem 2.3.1 (Stefanov-Uhlmann, 2009, [41])**.** *Let* $T_1(\Omega) < T < \infty$*. Then* $A\Lambda = Id - K$*, where* K *is compact in* $H_D(\Omega)$*, and* $\|K\|_{H_D(\Omega)} < 1$*. In particular,* $Id - K$ *is invertible on*

$H_D(\Omega)$ *and the inverse thermoacoustic problem has an explicit solution of the form*

$$f = \sum_{m=0}^{\infty} K^m A h, \quad h := \Lambda f.$$

Proof. If we denote $w = u - v$ with u the forward wave and v the time reversed one satisfying

$$\begin{cases} (\partial_t^2 - c^2(x)\Delta)v = 0, & (t,x) \in (0,T) \times \Omega \\ (v, v_t)|_{t=T} = (\phi, 0), & x \in \Omega \\ v = h, & (t,x) \in (0,T) \times \partial\Omega, \end{cases} \tag{2.7}$$

with ϕ the harmonic extension of $h(T)$, it is easy to see that $Kf = w(0,\cdot)$, where w satisfies

$$\begin{cases} (\partial_t^2 - c^2(x)\Delta)w = 0, & (t,x) \in (0,T) \times \Omega \\ (w, w_t)|_{t=T} = (u|_{t=T} - \phi, u_t|_{t=T}), & x \in \Omega \\ w = 0, & (t,x) \in (0,T) \times \partial\Omega. \end{cases} \tag{2.8}$$

The proof is then basically energy estimates for w and u. We have that

$$\begin{aligned} \|Kf\|_{H_D}^2 = E_\Omega(w,T) &= \|u|_{t=T} - \phi\|_{H_D}^2 + \|u_t|_{t=T}\|_{L^2}^2 \\ &= \|u|_{t=T}\|_{H_D}^2 + \|u_t|_{t=T}\|_{L^2}^2 - \|\phi\|_{H_D}^2 \\ &\leq E_\Omega(u,T), \end{aligned} \tag{2.9}$$

thus by energy preservation in $\mathbb{R}^n$,

$$\|Kf\|_{H_D}^2 \leq E_\Omega(u,T) \leq E_{\mathbb{R}^n}(u,T) = E_{\mathbb{R}^n}(u,0) = \|f\|_{H_D}^2.$$

We claim that the previous inequality is strict. If there is equality for some f, then all the above inequalities are equalities which means that $u = 0$ in $\mathbb{R}^n \setminus \Omega$. By extending u in an even way to negative times, the unique continuation property for the wave operator in Proposition 2.1.1 (se also [41, Theorem 4], [50]) implies that $f \equiv 0$.

We conclude now that $\|K\|_{\mathcal{L}(H_D)} < 1$ by noticing that K is compact since it is a composition of a compact and and a bounded operator, similarly as the case of the operator R in the proof of Theorem 2.2.3 (i.e. by propagation of singularities). Then, if λ_1 denotes the largest eigenvalue of K^*K we have that $\|K\| = \sqrt{\lambda_1}$ and

$$\|Kf\|_{H_D}^2 = (K^*Kf, f)_{H_D} = \lambda_1 \|f\|_{H_D}^2,$$

thus by the above we obtain $\sqrt{\lambda_1} < 1$. $\square$

The previous result was improved in [42] by just requiring the time T to be larger than $T_1(\Omega)/2$. Furthermore, to make the result work even in the case where the sound speed is piecewise smooth, it is necessary to assume the initial condition is supported in a compact set $\mathcal{K} \subset \Omega$ where all the information generated there reaches the boundary in less than the measurement time (probably after being transmitted and reflected several times). Such condition is the following:

Condition 2.3.1 (Reconstruction geometric condition)**.** *For $\mathcal{K} \subset \Omega$ compact set with smooth boundary, we assume that $S^*\mathcal{K} \subseteq \mathcal{U}$ (the cosphere bundle of $\mathcal{K}$), with*

$$\mathcal{U} = \{(x,\xi) \in S^*\Omega; \textit{there is a path of the geodesic issued from either } (x,\xi) \\ \textit{or } (x,-\xi) \textit{ at } t=0 \textit{ never tangent to } \partial\Omega, \textit{ that crosses } \partial\Omega \textit{ at time } t<T\}.$$

Recall we are assuming $c = 1$ in $\mathbb{R}^n \backslash \Omega$ therefore those geodesics that cross the boundary transversally never return to Ω.

Since the time reversed wave at time $t = 0$ might not have support in $\mathcal{K}$ we need to project it, so we introduce the orthogonal projection map $\Pi_{\mathcal{K}} : H_D(\Omega) \to H_D(\mathcal{K})$

$$\Pi_{\mathcal{K}} f = f|_{\mathcal{K}} - P_{\partial\mathcal{K}}(f|_{\partial\mathcal{K}}),$$

where $P_{\partial\mathcal{K}}$ is the Poisson operator of harmonic extension in $\mathcal{K}$. The main idea behind this sharp result is that we can separate the analysis of the singularities to those which travel respectively with positive and negative sound speed. The previous is a consequence of the fact that for the wave operator $P = \partial_t^2 - c^2\Delta$, we are able to orthogonally project the initial data (up to smooth terms) to functions that generate only singularities traveling under the positive or negative sound speed, this is $\mathbf{f} = \mathbf{\Pi}_+\mathbf{f} + \mathbf{\Pi}_-\mathbf{f}$ (see [42, §4.2] for more details). The solution then can be split into two parts, u_+ and u_-, up to smooth terms. It will be important to remember later that this property breaks when considering some perturbations of the wave operator such as adding a damping $-a(x)\partial_t u$ or an integro-differential term $\int_0^t q(t-s,x)u(s)ds$.

Theorem 2.3.2 (Stefanov-Uhlmann, 2011, [42])**.** *Let $\mathcal{K}$ satisfy condition 2.3.1. Then $\Pi_{\mathcal{K}}A\Lambda = Id - K$ in $H_D(\mathcal{K})$, with $\|K\|_{H_D(\mathcal{K})} < 1$. In particular, $Id - K$ is invertible on $H_D(\mathcal{K})$, and Λ restricted to $H_D(\mathcal{K})$ has an explicit left inverse of the form*

$$f = \sum_{m=0}^{\infty} K^m \Pi_{\mathcal{K}} A h, \quad h := \Lambda f.$$

Let's point out first a few thing on the non-trapping case. By taking $T > T_1(\Omega)/2$ we have that $\mathcal{U}$ equals the whole cosphere bundle $S^*\Omega$, so condition 2.3.1 is satisfied for any compact set in $\bar{\Omega}$. In addition, we saw in §2.2.1 that when $T > T_1(\Omega)$, the error operator K is compact, however, for observation times in $(T_1(\Omega)/2, T_1(\Omega))$, K might fail to be since there could still be singularities inside the domain. In order to overcome the non-compactness issue it is necessary to prove that measuring only one of the signals emanating from each singularity is enough to achieve reconstruction. This is done by carrying out a careful microlocal analysis close to the boundary and get microlocal energy estimates, this is, energy estimates for the high-frequency part of the solution. As a direct consequence of the reconstruction formula we get stability for $T \in (T_1(\Omega)/2, T_1(\Omega))$ in the non-trapping case. Even though we are considering smooth sound speeds, the next reconstruction result as well as uniqueness and stability, holds for piece-wise smooth speeds [42].

The purpose of the next two subsections is to construct approximate solutions in the sense that they satisfy (1.1) up to smooth terms and they contain all the singularities (high frequency features) of the exact solution. We first construct one from the initial data and after that we do a similar construction but this time from the boundary and obtain an integral representation of the measuring map Λ (up to smoothing operator). For a more general and detailed exposition of the parametrix construction for hyperbolic equations we refer the reader to [17] and [55]

Geometric Optics

Let's denote by P the wave operator with smooth sound speed $c(x)$,

$$P = \partial_t^2 - c^2(x)\Delta,$$

which has principal symbol $p(t, x; \tau, \xi) = -\tau^2 + c^2(x)|\xi|^2$. We want to construct an operator E acting on $\mathbf{f} = [f_1, f_2]$ such that $P(E\mathbf{f}) = 0$ modulo smooth terms, or equivalently $P(E\mathbf{f}) \in C^\infty(\mathbb{R}^n)$. Such E must also satisfies, up to smooth term, initial conditions

$$E\mathbf{f}|_{t=0} = f_1, \quad \partial_t E\mathbf{f}|_{t=0} = f_2, \quad f_1, f_2 \in \mathcal{D}'.$$

Of course we are particularly interested in the case $f_1 = f$, $f_2 = 0$, however, we do the computations for the general case since they will be helpful later. For simplicity we first assume $f_i \in C^\infty(\mathbb{R}^n)$ with $\text{supp} f_i \subset \overline{\Omega}$ and then we can extend the result to distributions.

Let's start by considering E of the form

$$E(f_1, f_2)(t) = \frac{1}{(2\pi)^n} \sum_{\sigma=\pm} \int e^{i\phi_\sigma(t,x;\xi)} \left(a_{1,\sigma}(t, x; \xi)\hat{f}_1(\xi) + |\xi|^{-1} a_{2,\sigma}(t, x; \xi)\hat{f}_2(\xi) \right) d\xi, \quad (2.10)$$

where $\hat{f}_1$ and $\hat{f}_2$ are the respective Fourier transforms of f_1, f_2. The reason why we consider two terms on E is because of the fact that the symbol p of the wave operator can be written as $p = (\tau - c(x)|\xi|)(\tau + c(x)|\xi|)$, so there are two solutions associated respectively with the positive and negative wave speed $c|\xi|$ and $-c|\xi|$.

We are looking for phase functions $\phi_\pm \in C^\infty(\mathbb{R}_t \times \mathbb{R}^n_x \times (\mathbb{R}^n \setminus 0)_\xi)$, positive homogeneous of degree 1 in ξ and such that $\nabla_{x,t,\xi}\phi_\pm \neq 0$, and amplitude functions $a_{i,\pm}$ given asymptotically by

$$a_{i,\pm} \sim \sum_{j=0}^{\infty} a_{i,\pm}^{(j)}(t,x;\xi), \quad i = 1,2, \tag{2.11}$$

with $a_{i,\pm}^{(j)}$ positive homogeneous of degree $-j$. The notation above means that $a_{i,\pm} - \sum_{j=0}^{m} a_{i,\pm}^{(j)} \in S^{m+1}(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n)$ for all $m \geq 0$ (see Definition A.1.1 and Lemma A.1.1 in the Appendix). The construction is carried out iteratively by applying P to $E\mathbf{f}$ and choosing $\phi_\pm$ and $a_{i,\pm}^{(j)}$ such that they kill the homogeneous terms of every degree. By applying P we get

$$\begin{aligned} PE\mathbf{f} &= \frac{1}{(2\pi)^n}\sum_{\sigma=\pm}\int P\left[e^{i\phi_\sigma(t,x;\xi)}\Big(a_{1,\sigma}(t,x;\xi)\hat{f}_1(\xi) + |\xi|^{-1}a_{2,\sigma}(t,x;\xi)\hat{f}_2(\xi)\Big)\right]d\xi \\ &\sim \frac{1}{(2\pi)^n}\sum_{\sigma=\pm}\int e^{i\phi_\sigma}\Big\{\left(-|\partial_t\phi_\sigma|^2 + c^2(x)|\nabla_x\phi_\sigma|^2\right)\Big(a_{1,\sigma}\hat{f}_1 + |\xi|^{-1}a_{2,\sigma}\hat{f}_2\Big) \\ &\quad + \sum_{k=1,2}\sum_{j=0}^{\infty}\left[2i\left(\partial_t\phi_\pm\partial_t - c^2\nabla_x\phi_\pm\cdot\nabla_x + C_\pm\right)a_{k,\pm}^{(j)} + Pa_{k,\pm}^{(j-1)}\right]|\xi|^{-(k-1)}\hat{f}_k\Big\}d\xi \end{aligned}$$

with $2C_\pm = P\phi_\pm$ and $a_{k,\pm}^{(-1)} = 0$ for $k = 1,2$. We also impose the initial conditions $E(f_1,f_2)(0) = f_1$ and $\partial_t E(f_1,f_2)(0) = f_2$, therefore we need

$$\phi_\pm(x,0;\xi) = x\cdot\xi, \tag{2.12}$$

$$a_{1,+}(0) + a_{1,-}(0) = 1, \quad a_{2,+}(0) + a_{2,-}(0) = 0, \tag{2.13}$$

$$i\left[\partial_t\phi_+(0)a_{1,+}(0) + \partial_t\phi_-(0)a_{1,-}(0)\right] + \partial_t(a_{1,+} + a_{1,-})(0) = 0, \tag{2.14}$$

$$i\left[\partial_t\phi_+(0)a_{2,+}(0) + \partial_t\phi_-(0)a_{2,-}(0)\right] + \partial_t(a_{2,+} + a_{2,-})(0) = |\xi|. \tag{2.15}$$

To get rid of the terms homogeneous of degree 2, we choose $\phi_\pm$ to be solutions of the *eikonal* *equation* plus initial conditions:

$$\partial_t \phi_\pm = \pm c(x)|\nabla_x \phi_\pm|, \quad \phi_\pm(x,0;\xi) = x \cdot \xi.$$

Notice that then $\partial_t \phi_\pm(0) = \pm c(x)|\xi|$ and we get two solutions associated with the wave speeds $\pm c(x)|\xi|$ respectively.

To deal with the terms homogeneous of degree 1 we choose $a^{(0)}_{1,\pm}$ such that they satisfy the *transport equations*

$$2i\left(\partial_t \phi_\pm \partial_t - c^2 \nabla_x \phi_\pm \cdot \nabla_x + C_\pm\right) a^{(0)}_{1,\pm} = 0. \tag{2.16}$$

By the method of characteristics we can solve it as long as the eikonal equation is solvable and (from (2.13) and (2.14)) considering the initial conditions

$$a^{(0)}_{1,+}(0) + a^{(0)}_{1,-}(0) = 1, \quad ic(x)|\xi|(a^{(0)}_{1,+}(0) - a^{(0)}_{1,-}(0)) = 0.$$

Next, for the terms homogeneous of degree $1-j$ with $j \geq 1$, we proceed by induction so we choose $a^{(j)}_{1,\pm}$ as follows, with initial conditions from (2.13) and (2.14),

$$\begin{aligned} 2i\left(\partial_t \phi_\pm \partial_t - c^2 \nabla_x \phi_\pm \cdot \nabla_x + C_\pm\right) a^{(j)}_{1,\pm} &= -(\partial_t^2 - c^2 \Delta) a^{(j-1)}_{1,\pm}, \\ a^{(j)}_{1,+}(0) + a^{(j)}_{1,-}(0) &= 0, \\ ic(x)|\xi|(a^{(j)}_{1,+}(0) - a^{(j)}_{1,-}(0)) &= -\partial_t(a^{(j-1)}_{1,+} + a^{(j-1)}_{1,-})(0), \end{aligned} \tag{2.17}$$

and $a_{2,\sigma}^{(j-1)}$, with initial conditions from (2.13) and (2.15), satisfying

$$2i|\xi|^{-1}\left(\partial_t\phi_\pm\partial_t - c^2\nabla_x\phi_\pm\cdot\nabla_x + C_\pm\right)a_{2,\pm}^{(j-1)} = \begin{cases} 0 & j=1,\\ -|\xi|^{-1}(\partial_t^2 - c^2\Delta)a_{2,\pm}^{(j-2)} & j\geq 2,\end{cases}$$

$$a_{2,+}^{(j-1)}(0) + a_{2,-}^{(j-1)}(0) = 0, \quad j\geq 1,$$

$$ic(x)|\xi|(a_{2,+}^{(j-1)}(0) - a_{2,-}^{(j-1)}(0)) = \begin{cases} 0 & j=1,\\ \left(|\xi| - \partial_t(a_{1,+}^{(j-1)} + a_{1,-}^{(j-1)})(0)\right) & j\geq 2.\end{cases}$$

Given the functions $\{a_{i,\pm}^{(j)}\}$ obtained above, by Borel's Lemma A.1.1, we define the amplitudes $a_\pm$ as in (2.11), which belong to $S^1(\mathbb{R}_t\times\mathbb{R}_x^n\times\mathbb{R}_\xi^n)$.

Remark. It's important to point out that in general the eikonal equation can only be solved locally for small intervals of time. We refer the reader to [17, §5] for a brief exposition on Hamilton-Jacobi theory. To get then a parametrix in the whole interval $[0,T]$, by compactness it is just necessary to repeat the construction a finite number of times. More precisely, we split such interval into small subintervals (t_i, t_{i+1}) so that we can solve the eikonal equation there and considering initial conditions at times $t = t_i$. The global parametrix then follows by gluing the local solutions through a (finite) partition of unity.

Remark. In the case we have $c\equiv 1$, the phase function reduces to $\phi_\pm(t,x;\xi) = x\cdot\xi \pm t|\xi|$, thus $(\partial_t^2 - c^2(x)\Delta)\phi_\pm = 0$, and the transport equations simplify to

$$2i\left(\pm c(x)|\xi|\phi_\pm\partial_t - c^2(x)\xi\cdot\nabla_x\right)a_{k,\pm}^{(j)} = -(\partial_t^2 - c^2\Delta)a_{k,\pm}^{(j-1)}.$$

Then, their solutions are

$$a_{1,+}^{(0)} = a_{1,+}^{(0)} = \frac{1}{2}, \quad a_{1,\pm}^{(j)} = 0 \text{ for } j = 1,2,\ldots$$

$$a_{2,+}^{(0)} = -a_{2,+}^{(0)} = -\frac{i}{2}, \quad a_{2,\pm}^{(j)} = 0 \text{ for } j = 1,2,\ldots$$

which are globally defined, and we get the simpler formula

$$E(f_1, f_2)(t) = \frac{1}{(2\pi)^n} \sum_{\sigma=\pm} \frac{1}{2} \int e^{i(x\cdot\xi+\sigma t|\xi|)} \left(\hat{f}_1(\xi) + i|\xi|^{-1}\hat{f}_2(\xi)\right) d\xi.$$

The operator $E(t) : C_0^\infty \times C_0^\infty \to C^\infty((-\epsilon_0, \epsilon_0) \times \mathbb{R}^n)$ as in (2.10) is called a *parametrix* for the wave operator an it is an example of a *Fourier Integral Operator* (FIO). See definition A.1.6. Moreover, it can be extended to a linear operator from $\mathcal{E}'(\mathbb{R}^n) \times \mathcal{E}'(\mathbb{R}^n)$ to $C^\infty((-\epsilon_0, \epsilon_0); \mathcal{D}'(\mathbb{R}^n))$. Then, for all $\mathbf{f} = [f_1, f_2] \in \mathcal{E}'(\mathbb{R}^n) \times \mathcal{E}'(\mathbb{R}^n)$, $u = E(t)\mathbf{f}$ is such that

$$Pu = K\mathbf{f}, \quad u|_{t=0} = f_1,\ \partial_t u|_{t=0} = f_2, \tag{2.18}$$

where $K : C_0^\infty \times C_0^\infty \to \mathcal{D}'((-\epsilon_0, \epsilon_0) \times \mathbb{R}^n)$ has C^∞ Schwartz kernel thus $K : \mathcal{E}'(\mathbb{R}^n) \times \mathcal{E}'(\mathbb{R}^n) \to C_0^\infty((-\epsilon_0, \epsilon_0) \times \mathbb{R}^n)$ is smoothing. We recall that there are two parametrix solutions, $u_+ = E_+(t)\mathbf{f}$ and $u_- = E_-(t)\mathbf{f}$, corresponding to the positive and negative sound speed respectively. By restricting $E\mathbf{f}$ to the boundary we obtain a representation of $\Lambda\mathbf{f}$ as an oscillatory integral,

$$\Lambda\mathbf{f} \cong \frac{1}{(2\pi)^n} \sum_{\sigma=\pm} \int e^{i\phi_\sigma(x,t;\xi)} \left(a_{1,\sigma}(t, x; \xi)\hat{f}_1(\xi) + |\xi|^{-1}a_{2,\sigma}(t, x; \xi)\hat{f}_2(\xi)\right)\Big|_{\partial\Omega} d\xi$$

where $\cong$ means equality up to a smooth function. It can be shown (see [41]) that Λ is an elliptic FIO of order zero microlocally for covectors $(x, \xi) \in S^*\mathcal{K}$ provided condition 2.3.1 holds. A similar estatement is true for partial data on a subset of the boundary as long as such set satisfies an analogous visibility condition.

Another representation of the parametrix of the wave equation is given by the semigroup operator associated to the wave equation. By letting

$$\mathbf{P} = \begin{pmatrix} 0 & I \\ c^2\Delta & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{u} = (u, u_t),$$

the semigroup is denoted by $e^{t\mathbf{P}}$ and when applied to an initial condition $\mathbf{f}$ gives $\mathbf{u} = e^{t\mathbf{P}}\mathbf{f}$, the solution to the IVP (1.1). For Q a pseudo-differential square roots of $-c^2\Delta$, i.e. it satisfies $Q^2 = -c^2\Delta$ up to a smoothing operator, we can microlocally split the semigroup by doing $e^{t\mathbf{P}} = e^{\mathrm{i}tQ}\mathbf{\Pi}_+ + e^{-\mathrm{i}tQ}\mathbf{\Pi}_-$, where

$$\mathbf{\Pi}_+ = \frac{1}{2}\begin{pmatrix} 1 & -\mathrm{i}Q^{-1} \\ \mathrm{i}Q & 1 \end{pmatrix}, \quad \mathbf{\Pi}_- = \frac{1}{2}\begin{pmatrix} 1 & \mathrm{i}Q^{-1} \\ -\mathrm{i}Q & 1 \end{pmatrix}.$$

Then, the parametrix solution constructed above satisfy that $\mathbf{u}_+ = e^{\mathrm{i}tQ}\mathbf{\Pi}_+\mathbf{f}$ and $\mathbf{u}_- = e^{\mathrm{i}tQ}\mathbf{\Pi}_-\mathbf{f}$.

Boundary Parametrix

The idea is to get an approximate representation of the Dirichlet-to-Neumann map using microlocal analysis. Close to the boundary we choose boundary normal coordinates $x = (x', x^n)$, this is $x^n = \mathrm{dist}(x, \partial\Omega)$ is the *distance to the boundary function* (for the metric $c^{-2}\mathrm{d}x^2$) and (x') are local coordinates on $\partial\Omega$. Then, the boundary is characterized by $x^n = 0$ and $x^n > 0$ in $\mathbb{R}^n \setminus \Omega$.

Consider a compactly supported distribution h on $\mathbb{R} \times \partial\Omega$ with $WF(h)$ contained in a small conic neighborhood of some $(t_1, x_1, 1, (\xi^1)') \in T^*(\mathbb{R} \times \partial\Omega)$, where $(\xi^1)'$ is the tangential projection of ξ^1 onto the boundary and (t_1, x_1) is the exit time and point of a geodesic γ_{x_0,ξ_0} issued from some $(x_0, \xi_0) \in S^*\mathcal{K}$. Notice that we will do the parametrix construction microlocally near $(t_1, x_1, 1, (\xi^1)')$ and later we will set $h = \Lambda\mathbf{f}$. As in [42] we can get parametrix solutions for the transmitted wave outside Ω with Dirichlet boundary data h. Since $\partial\Omega$ is an invisible boundary that does not perturb the propagation of waves, if we set $h = \Lambda\mathbf{f}$ the transmitted waves will be the same as the initial ones (up to a smooth error) originated from $\mathbf{f}$ at time $t = 0$. We want a transmitted solutions of the form

$$u_\sigma^T = (2\pi)^{-n} \int e^{i\varphi_\sigma(t,x,\tau,\xi')} b_\sigma(t, x, \tau, \xi') \hat{h}_\sigma(\tau, \xi') d\tau d\xi', \quad \sigma = \pm, \tag{2.19}$$

where $\hat{h}_\pm = \int_{\mathbb{R}\times\mathbb{R}^{n-1}} e^{-i(\mp t\tau + x'\cdot\xi')} h(t, x') dt dx'$. The phase functions $\varphi_\pm$ must satisfy the eikonal equation plus boundary conditions on $x^n = 0$:

$$\pm\partial_t \varphi_\pm + c(x)|\partial_x \varphi_\pm| = 0, \quad \varphi_\pm|_{x^n=0} = \mp t\tau + x' \cdot \xi'; \tag{2.20}$$

while the amplitude functions satisfy equations similar to (2.16) and (2.17), and in particular $b_0^{(0)} = 1$. In what follows we only show the argument in the case of the positive sound speed, this is $\sigma = +$, since the other case is analogous.

Let's denote

$$P_T : h \mapsto u_+^T|_{\mathbb{R}\times\partial\Omega},$$

which because of (2.20) is a ΨDO of order zero with constant principal symbol $b_+^{(0)} = 1$. You can see this by plugging the boundary condition in (2.20) into (3.37). Near the intersection point (t_1, x_1) between γ_{x_0,ξ_0} and $\partial\Omega$, we locally define the outgoing Dirichlet-to-Neumann map as the following ΨDO of order 1

$$N_{\text{out}} : u_+^T\Big|_{\mathbb{R}\times\partial\Omega} \mapsto \frac{\partial u_+^T}{\partial x^n}\Big|_{\mathbb{R}\times\partial\Omega}.$$

From (3.37) we deduce that its principal symbol is

$$\sigma(N_{\text{out}}) = \mathrm{i}\frac{\partial \varphi_+}{\partial x^n}\Big|_{\mathbb{R}\times\partial\Omega} = \mathrm{i}\sqrt{c^{-2}\tau^2 - |\xi'|^2},$$

thus it is elliptic in the hyperbolic conic set $c^{-1}|\tau| > |\xi'|$. Notice that the microsupport of u_+^T lies in this set due to the strictly convexity of $\partial\Omega$ and because it is concentrated near the null bicharacteristic associated to the geodesic issued from (x_0, ξ^0), therefore near such null bicharacteristic

$$c^{-2}\tau^2 - |\xi'|^2 \approx |\xi_n|^2 > 0.$$

Remark. It's important to point out that when $h = \Lambda\mathbf{f}$ both approximate solutions, for the Cauchy problem and for the boundary value problem, coincide up to a smooth function. Indeed, if (t_1^+, x_1^+) is the point where the geodesic issued from (x_0, ξ_0) at time $t = 0$ hits the boundary, by denoting $\mathbf{u}_+^T$ the above solution when $h = \Lambda\mathbf{f}$, and by $\mathbf{u}_+$ the first parametrix constructed for the Cauchy problem, both solutions are equal near (t_1^+, x_1^+) up to a smooth operator applied to $\mathbf{f}$, this is $\mathbf{u}_+^T = \mathbf{u}_+ + R\mathbf{f}$ with R smoothing. Moreover, for $t > t_1^+$ close enough, we have the estimate

$$\|\mathbf{u}_+^T(t)\|_{H^1\oplus L^2} \leq C\|\mathbf{u}_+(t)\|_{H^1\oplus L^2} + C\|\mathbf{f}\|_{L^2\oplus H^{-1}}. \tag{2.21}$$

Proof of Theorem 2.3.2

Recall that for any open set $U \subset \mathbb{R}^n$ and $t' < t''$, the solution of the wave equation in the whole space satisfies the following energy equality

$$E_U(\mathbf{u}, t'') = E_U(\mathbf{u}, t') + 2\Re \int_{[t',t'']} u_t \partial_\nu \bar{u} dt dS, \tag{2.22}$$

where ν stands for the outward unit normal vector to ∂U. In particular, if $U = B\backslash\overline{\Omega}$, with B a sufficiently large ball containing Ω, and $U = \Omega$, the solution $\mathbf{u}$ to (1.1) satisfies respectively

$$E_{\Omega^c}(\mathbf{u}, T) = -2\Re \int_{[0,T]} u_t \partial_\nu \bar{u} dt dS \quad \text{and} \quad E_\Omega(\mathbf{u}, 0) = E_\Omega(\mathbf{u}, T) - 2\Re \int_{[0,T]} u_t \partial_\nu \bar{u} dt dS \tag{2.23}$$

with ν the exterior unit normal to $\partial\Omega$.

Assume for a moment that $WF(\mathbf{f})$ is contained in a conic neighborhood of a given $(x_0, \xi^0) \in S^*\mathcal{K}$ and set $h = \Lambda\mathbf{f}$. With out lost of generality we can assume that h is supported near (t_1, x_1) since otherwise we localize it with a smooth cut-off function. The wave front set of u follows the bicharacteristic issued from $(0, x_0; 1, \xi^0)$ which hits the boundary at time t_1, therefore u is smooth near the boundary except for times close to t_1. Then, near (t_1, x_1) we can approximate u by u_T as constructed above, and away from there we approximate it using u_+ which by the assumption on the wave front set of $\mathbf{f}$ it becomes a smooth function.

We can now replace the right hand side of the first equality in (2.23) as follows

$$E_{\Omega^c}(\mathbf{u}, T) \cong -2\Re \int_{[0,T]\times\partial\Omega} \partial_t u_T \partial_\nu \overline{u}_T dt dS$$

where here $\cong$ means equality up to smoothing operator applied to h. On the other hand, since the parametrix obtained for the Cauchy problem u_+ satisfies (1.1) up to a smoothing term depending on $\mathbf{f}$, it satisfies an equality analogous to the second one in (2.23), of course again up to smoothing operator applied to f. Since $h = \Lambda\mathbf{f}$, the error can be expressed in terms of a compact operator applied to h. Furthermore, due to the last remark above, the boundary integral can be replaced by a similar one containing instead the second parametrix u_T. Therefore,

$$E_{\Omega}(\mathbf{u}_+, 0) \cong -2\Re \int_{[0,T]\times\partial\Omega} \partial_t u_T \partial_\nu \overline{u}_T dt dS,$$

where there is no term at time T since by propagation of singularities and condition 2.3.1, $\mathbf{u}$ and therefore $\mathbf{u}_+$ are smooth inside Ω at such time (here $\cong$ again means equality up to smoothing operator applied to h). Then, for any $\mu \in (0,1)$ we get

$$E_{\Omega^c}(\mathbf{u}, T) - \mu E_{\Omega}(\mathbf{u}_+, 0) = \Re(Mh, h),$$

with M a ΨDOs of order 2, with principal symbols given by

$$\sigma(M) = -2(1-\mu)\sigma(P_T^* N_{\mathrm{out}}^* P_t P_T) = 2(1-\mu)\tau\sqrt{c^{-2}\tau^2 - |\xi'|^2}.$$

In the hyperbolic region $c^2(x)\tau^2 - |\xi'|^2 > 0$ on the boundary it satisfies $\sigma(M) > 0$, consequently, since we are assuming h compactly supported (near (t_1, x_1)) and with wave front set contained in such region, we apply Garding's inequality and get

$$E_{\Omega^c}(\mathbf{u}, T) - \mu E_{\Omega}(\mathbf{u}_+, 0) \geq -C\|h\|^2_{H^{1/2}(\mathbb{R}\times\partial\Omega)}, \tag{2.24}$$

for constant $C > 0$ depending on $\mathcal{K}$.

Consider now an arbitrary $\mathbf{f} \in \mathcal{H}(\Omega)$ and set $h = \Lambda\mathbf{\Pi}_+\mathbf{X}\mathbf{f}$, where $\mathbf{X} = [X, X]$ and X a ΨDO of order zero with essential support contained in a small conic neighborhood of $(x_0, \xi_0) \in S^*\mathcal{K}$. From [41], we know the map $\Lambda : \mathbf{f} \mapsto h$ is a FIO of order 0 with canonical relation of graph type, therefore it is continuous from $H^{1/2}(\Omega) \oplus H^{-1/2}(\Omega)$ to $H^{1/2}(\mathbb{R} \times \partial\Omega)$ and we get

$$E_\Omega(\mathbf{u}_+, 0) \leq CE_{\Omega^c}(\mathbf{u}, T) + C\|\mathbf{\Pi}_+\mathbf{X}\mathbf{f}\|^2_{H^{1/2}(\Omega)\oplus H^{-1/2}(\Omega)}.$$

In addition, by construction of the parametrix, $\mathbf{u}_+$ now satisfies that $\mathbf{u}_+(0) = \mathbf{\Pi}_+\mathbf{X}\mathbf{f}$ which leads to

$$\|\mathbf{\Pi}_+\mathbf{X}\mathbf{f}\|^2_{\mathcal{H}(\Omega)} \leq C\|\mathbf{u}(T)\|^2_{H^1(\Omega^c)\oplus L^2(\Omega^c)} + C\|\mathbf{f}\|^2_{H^{1/2}(\Omega)\oplus H^{-1/2}(\Omega)},$$

where here $\mathbf{u}$ stands for the real solution of (1.1) with initial conditions $\mathbf{\Pi}_+\mathbf{X}\mathbf{f}$, i.e. $\mathbf{u} = e^{t\mathbf{P}}\mathbf{\Pi}_+\mathbf{X}\mathbf{f}$. By restricting the initial conditions to functions of the form $\mathbf{f} = [f, 0]$ with $f \in H_D(\mathcal{K})$, we have that $\mathbf{\Pi}_+\mathbf{X}\mathbf{f} = \frac{1}{2}[Xf, \mathrm{i}QXf]$, thus since Q is an elliptic ΨDO of order 1, elliptic regularity implies

$$\|Xf\|_{H_D(\Omega)} \leq C\|\mathbf{u}(T)\|_{H^1(\Omega^c)\oplus L^2(\Omega^c)} + C\|f\|_{H^{1/2}(\Omega)}.$$

We glue all the microlocal estimates obtained above using a partition of unity. Indeed, by compactness of $S^*\mathcal{K}$ we can take a finite pseudo-differential partition of unity of symbols, $1 = \sum_j \chi_j$, of ΨDO's X_j microlocalizing in conical neighborhoods of a finite number of covectors $(x_j, \xi^j) \in S^*\mathcal{K}$. Then, $f = \sum_j X_j f + (Id - \sum_j X_j)f$ and and moreover $WF(f) \cap WF(Id - \sum_j X_j)f = \emptyset$, i.e. all the singularities of f are contained in the term $\sum_j X_j f$. We have

$$\|f\|_{H_D(\Omega)} \leq \sum_j \|X_j f\|_{H_D(\Omega)} \leq \sum_j C\|e^{tP}\mathbf{\Pi}_{\sigma(j)}\mathbf{X}_j\mathbf{f}\big|_{t=T}\|_{H^1(\Omega^c)\oplus L^2(\Omega^c)} + C\|f\|_{H^{1/2}(\Omega)},$$

with $\sigma(j) = +$ or $-$, depending on which branch of the geodesic issued from (x_j, ξ^j) hits the boundary first. To flip the order of the operators in the first term of the right hand side we use Egorov's Theorem (see [17, Theorem 10.1]), thus there exists a collection $\{\tilde{\mathbf{X}}_j\}$ of ΨDO's of order 1 such that $e^{tP}\mathbf{\Pi}_{\sigma(j)}\mathbf{X}_j = \tilde{\mathbf{X}}_j e^{tP}$ up to smoothing operators. Therefore, denoting $\mathbf{u} = e^{t\mathbf{P}}\mathbf{f}$ the exact solution to (1.1) we get

$$\|f\|_{H_D(\mathcal{K})} = \|f\|_{H_D(\Omega)} \leq C\|\mathbf{u}(T)\|_{H^1(\Omega^c)\oplus L^2(\Omega^c)} + C\|f\|_{H^{1/2}(\Omega)}.$$

Notice that the inclusion $H_D(\mathcal{K}) \mapsto H^{1/2}(\Omega)$ is compact, and also as a consequence of domain of dependance for the exterior problem and unique continuation we have that

$$H_D(\mathcal{K}) \ni f \mapsto \mathbf{u}(T) \in H^1(\Omega^c) \oplus L^2(\Omega^c)$$

is an injective bounded map (see [42] for more details on this), therefore Corollary 2.2.1 leads to

$$\|f\|_{H_D(\mathcal{K})} \leq C\|\mathbf{u}(T)\|_{H^1(\Omega^c)\oplus L^2(\Omega^c)}. \tag{2.25}$$

This is an observability inequality that says that at time T most of the initial energy has propagated across the boundary, and therefore it has been measured and lies outside Ω.

The finite speed of propagation for the wave equation allows us to apply Poincare's inequality and get

$$\|f\|^2_{H_D(\mathcal{K})} \leq CE_{\Omega^c}(\mathbf{u}(T)). \tag{2.26}$$

We now conclude the proof of the theorem by noticing that since $K = \Pi_{\mathcal{K}}(I - A\Lambda)$, then we get a similar inequality as in (2.9),

$$\|Kf\|^2_{H_D(\mathcal{K})} \leq E_\Omega(\mathbf{w}(0)) = E_\Omega(\mathbf{w}(T)) \leq E_\Omega(\mathbf{u}(T)),$$

nevertheless, from the observability estimate for the error operator (2.26), it follows

$$\begin{aligned}\|Kf\|^2 &\leq E_\Omega(\mathbf{u}(T)) = E_{\mathbb{R}^n}(\mathbf{u}(T)) - E_{\Omega^c}(\mathbf{u}(T))\\ &= E_{\mathcal{K}}(\mathbf{u}(0)) - E_{\Omega^c}(\mathbf{u}(T))\\ &\leq (1 - C^{-1})\|f\|^2_{H_D(\mathcal{K})}.\end{aligned}$$

2.3.2 Enclosure: Stabilization of Waves

The reconstruction method based on Neumann series, as you could see in the previous section, relies on having good energy (observability) estimates. More precisely, it is expected to have those estimates if either the forward problem is energy-decreasing or the error function satisfies an energy-decreasing backward problem. The latter can be achieved by modifying accordingly the time reversal technique. Here again the advantage of considering a domain with reflecting boundaries comes from Control Theory, in particular from the problem of *stabilization of waves* which is exactly the study of energy estimates for energy decreasing hyperbolic systems. Of course, as in the stability section we will also need the GCC 2.2.1.

Neumann series reconstruction formulas have been found for partially reflecting boundaries and perfectly reflecting boundaries in [1] and [30] respectively. In the later case, there is a different approach introduced in [47], which through an averaging process of the Time Reversal technique they can prove the error operator in their case is compact and consequently obtain a Neumann series formula for the reconstruction. On the other hand, the method used in both [1, 30] are based on the stabilization result in [8] which reads as follows:

Proposition 2.3.1. *Let u be a solution of* (1.1)-(1.3) *with $\lambda \not\equiv 0$. Assume that $\Gamma = \{\lambda > 0\}$ satisfies the GCC 2.2.1 with τ the controllability time. Then, the following estimate holds for some $\alpha > 0$ and for every $T > \tau$*

$$E_\Omega(\mathbf{u}(T)) \leq e^{-\alpha T} E_\Omega(\mathbf{u}(0)).$$

In the case of partially reflecting boundary the forward wave u satisfies boundary conditions with $\lambda \not\equiv 0$ (Robin boundary conditions), thus the energy is lost through $\Gamma = \{\lambda > 0\}$, and u satisfies an inequality as in the previous proposition. By considering the back projected wave given by

$$\begin{cases} (\partial_t^2 - c^2(x)\Delta)v = 0, & (t,x) \in (0,T) \times \Omega \\ (v, v_t)|_{t=T} = (0,0), & x \in \Omega \\ \partial_\nu v = -\lambda \partial_t \Lambda f, & (t,x) \in (0,T) \times \partial\Omega, \end{cases} \tag{2.27}$$

the error function $w = u - v$ which defines the error operator $Kf = w(\cdot, 0)$, satisfies the energy preserving system

$$\begin{cases} (\partial_t^2 - c^2(x)\Delta)w = 0, & (t,x) \in (0,T) \times \Omega \\ (w, w_t)|_{t=T} = (u, u_t)|_{t=T}, & x \in \Omega \\ \partial_\nu w = 0, & (t,x) \in (0,T) \times \partial\Omega. \end{cases} \tag{2.28}$$

Then, in some suitable norms (see [1]), from the usual energy inequalities and the stabilization of waves we get

$$\|Kf\|^2 = E_\Omega(\mathbf{w}(0)) = E_\Omega(\mathbf{w}(T)) = E_\Omega(\mathbf{u}(T)) \leq \mu(T) E_\Omega(\mathbf{u}(0)) = \mu(T)\|f\|^2,$$

for some $\mu(T) \in (0,1)$ independent of f. We then have proved the error operator to be a contraction so $A\Lambda = I - K$ has a left inverse given by a Neumann series.

For Ω having perfectly reflecting boundary we only have to modify the time reversed problem. Here $\lambda \equiv 0$, so we instead consider a new smooth function $\gamma \geq 0$ defined on $\partial\Omega$ and let v be a solution of

$$\begin{cases} (\partial_t^2 - c^2(x)\Delta)v = 0, & (t,x) \in (0,T) \times \Omega \\ (v, v_t)|_{t=T} = (0,0), & x \in \Omega \\ \partial_\nu v - \gamma \partial_t v = -\gamma \partial_t \Lambda f, & (t,x) \in (0,T) \times \partial\Omega. \end{cases} \tag{2.29}$$

Such function γ helps us to absorb energy during the back projection process making the system satisfied by the error function to be energy decreasing when solved backward in time. In fact, $w = u - v$ solves

$$\begin{cases} (\partial_t^2 - c^2(x)\Delta)w = 0, & (t,x) \in (0,T) \times \Omega \\ (w, w_t)|_{t=T} = (u, u_t)|_{t=T}, & x \in \Omega \\ \partial_\nu w - \gamma \partial_t w = 0, & (t,x) \in (0,T) \times \partial\Omega. \end{cases} \tag{2.30}$$

Assuming that Ω and $\Gamma = \{\gamma > 0\}$ satisfies the GCC 2.2.1, we apply Proposition 2.3.1 to (2.30) and similarly as in the previous case, for some $\mu'(T) \in (0,1)$, we obtain

$$\|Kf\|^2 = E_\Omega(\mathbf{w}(0)) \leq \mu'(T) E_\Omega(\mathbf{w}(T)) = \mu'(T) E_\Omega(\mathbf{u}(T)) = \mu'(T) E_\Omega(\mathbf{u}(0)) = \mu'(T)\|f\|^2.$$

Consequently, there is a Neumann series reconstruction formula.

Chapter 3

ATTENUATED TAT WITH COMPLETE DATA IN FREE SPACE

The effect of ultrasound attenuation produced by biological tissue has been observed and studied by groups of researchers working in the experimental side of the thermoacoustic technology (see for instance [37]). In fact, standard reconstruction techniques in some cases generate poor images due to the presence of attenuation. Consequently, in order for this medical imaging technique to reach real applications in hospitals, it becomes necessary to compensate the attenuation effect for instance by including it in the modeling of the problem as well as developing suitable reconstruction procedures.

In the literature we can find many works on this area using different approaches to tackle this issue as well as different models used to represent the attenuation phenomenon. Even though there is no consensus about the precise model for attenuation in biological bodies, it is known that such effect is frequency dependent, which then leads to consider fractional or non-local models (see [27] and references therein). The first article that followed the method of Time Reversal and iterative reconstruction formulas (explained in the first part of this thesis) was [20], where the author considers the simpler model of the damped wave equation, this is, the attenuation is represented by a term of the form $a(x)\partial_t$. It is a critical assumption on that work the smallness of the damping coefficient $a(x)$ because then the reconstruction formula is obtained from a continuity argument with respect to the attenuation term and using an straightforward extension of the TR method employed in [41]. My first paper published as a Ph.D student [35] is a follow-up on this problem, where I was able to remove the restriction of $a(x)$ being small, by applying the microlocal energy estimate argument used in the proof of Theorem 2.3.2. My second paper on this topic is a joint work with Sebastián

Acosta [3]. In this article we studied the more accurate model of an integro-differential wave equation with attenuation expressed as a memory term (convolution in time) and which is motivated by some of the fractional models found in the literature. The novelty in this work is that we allow the memory kernel to depend on both space and time, and as far as we know this has not been fully considered in the literature from an analytical point of view. From a heuristic perspective, some advances have been made though. For the case of constant wave speed and constant coefficient of attenuation, Modgil et al. [33] designed a method based on relating the unattenuated wave field to the attenuated wave field via an integral operator and its subsequent inversion using a singular value decomposition. Treeby et al. [53, 54] proposed a reconstruction based on time reversal and the k-space computational method. Attenuation compensation was achieved by separating the absorbing and dispersion terms in the wave equation, and reversing the sign of the absorbing coefficient during the time reversal. This method was modified in [25] where the coefficient of attenuation was allowed to vary within the region of interest, but the exponent of the power-law attenuation was still assumed to be constant. A general class of integro-differential models was considered in the recent paper [19], where a different iterative method is used to achieve reconstruction and which is based on solving an adjoint problem. That work though still requires the assumption of constant sound speeds and power-law exponent. However, in some practical settings such as in the presence of bone and soft-tissue, the domain exhibits regions of varying power-law exponents. An appropriate method needs to be devised to avoid blurring and distortions in the reconstruction. Our work is a step in that direction, where the coefficients a, b, c and the kernel Φ in (3.1)-(3.2) are allowed to vary, which effectively accounts for power-law attenuation of spatially varying exponent.

Considering attenuation terms of integral type brings some difficulties to the analysis on the propagation of waves. In particular, the equation is no longer reversible and local in time, and consequently, it is not possible to use techniques such as Tataru's unique continuation to get uniqueness, at least not in a direct way. Moreover, the microlocal properties of this type of integro-differential operators are not well understood. Nevertheless, it is possible to exploit the fact that an integral term of the sort considered here only presents a compact perturbation of the differential operator.

The rest of this thesis is devoted to present with details the results obtained in both articles [35, 3]. As in the first part we split this section in three subsections corresponding to the problems of uniqueness, stability and reconstruction.

3.1 The Inverse Problem

As it was mentioned above, it is well known that for biological tissues the attenuation of acoustic waves is frequency-dependent. One way to model this attenuation is to use fractional time derivatives and consequently the representation of the propagation of ultrasound waves by integro-differential equations. Examples of this modeling are frequency power-law attenuation or fractional Szabo models (see for instance [48, 11, 39, 53, 38, 25, 27]) where the traveling wave may be assumed to satisfy an equation of the form

$$c^{-2}\partial_t^2 u - \Delta u + \beta\partial_t^{k+\alpha}u = F(t,x), \quad \text{for some} \quad \alpha \in (0,1),\ k = 1,2,$$

and where the fractional derivative term can be written as a convolution in time

$$\beta(x)\partial_t^{k+\alpha}u = \int_{-\infty}^{t} \Phi_\alpha(t-s,x)\partial_s^{k+1}u(s,x)ds.$$

Assuming, as in thermoacoustics, that the wave field vanishes for negative times, and provided that the kernel is bounded and regular enough, we can perform integration by parts and write the previous integral as a convolution of u with a different kernel, plus time-derivatives of u up to order two. In the case $k = 2$, the sound speed is perturbed resulting in a different speed $\tilde{c}^{-2} = c^{-2} + \beta\Phi_\alpha(0)$, which requires conditions on β and Φ_α in order to get an effective wave speed $\tilde{c} > 0$. We point out there is a recent definition for derivatives of fractional order which employs such continuous and bounded kernels [10].

The inverse problem in consideration is then the determination of the initial source f in an attenuating medium, provided boundary data $u|_{[0,T]\times\partial\Omega}$ and where the acoustic wave u is assumed to satisfy the system

$$\begin{cases} \partial_t^2 u - c^2\Delta u + bu + \mathcal{A}u(t,x) = 0, & \in \mathbb{R}\times\mathbb{R}^n \\ u(t,x) = 0, & t<0, \end{cases} \tag{3.1}$$

where $\mathcal{A}$ is assume to be one of the next attenuation processes,

$$\mathcal{A}_\Phi^k u(t,x) = a\partial_t u + \int_0^t \Phi(t-s,x)\partial_t^k u(s,x)ds, \quad \alpha\in(0,1),\ k=0,1,2,3. \tag{3.2}$$

To alleviate the notation, when there is no confusion on the kernel in consideration we will just write $\mathcal{A}^k$. Here a is a strong (and local) damping coefficient, while Φ represents a memory kernel which depends on the history of the system and varies in space. We suppose $a,b,c\in C^\infty(\mathbb{R}^n)$, $\Phi\in C^2(\mathbb{R}^{n+1})$, $a,b\geq 0$, $c_0^{-1}\geq c\geq c_0>0$, and for a fixed open bounded set $\Omega\subset\mathbb{R}^n$ with smooth boundary, we suppose $a=b=c-1=\Phi=0$ in $\mathbb{R}^n\backslash\bar{\Omega}$. This last condition can be interpreted as knowing those coefficients outside the region of interest Ω. Recall that as it was mentioned before, due to the continuity of the kernel and since $u=0$ for negative times, it is enough to analyze (3.1) with $k=0$, so from now on we assume this. We shall use the following notation throughout this section:

$$P_{a,\Phi} := \partial_t^2 - c^2\Delta + b + \mathcal{A}^0, \quad \Phi * u = \int_0^t \Phi(t-s,x)u(s,x)ds.$$

In particular $P_{a,\Phi} = P := \partial_t^2 - c^2\Delta$ outside Ω. The Cauchy problem associated with (3.1) is

$$\begin{cases} P_{a,\Phi}u = 0, & (t,x)\in(0,\infty)\times\mathbb{R}^n, \\ u|_{t=0} = f, \\ \partial_t u|_{t=0} = -af, \end{cases} \tag{3.3}$$

since any solution of (3.3) extended by zero to $(-\infty,0)\times\mathbb{R}^n$ is a solution of (3.1). Indeed, given a smooth solution u of (3.3) we consider $H(t)u(x,t)$ where $H(t)$ is the Heaviside function. Then, we can pull out the Heaviside function from the convolution since it integrates on the interval $(0,t)$, thus we get

$$P_{a,\Phi}(Hu) = u\delta' + 2(\partial_t u)\delta + au\delta + (P_{a,\Phi}u)H$$

with the last term vanishing because $P_{a,\Phi}u = 0$.

For an arbitrary test function $\phi \in C_c^\infty(\mathbb{R}^{n+1})$ we have the following,

$$\begin{aligned}\langle P_{a,\Phi}(Hu), \phi\rangle &= \int_{\mathbb{R}^n} \big[-(\partial_t u)\phi - u(\partial_t \phi) + 2(\partial_t u)\phi + au\phi \big]\big|_{t=0} dx \\ &= -\int_{\mathbb{R}^n} u\partial_t \phi|_{t=0} dx \\ &= \langle f\delta', \phi\rangle,\end{aligned}$$

which is the same as problem (3.1).

The thermoacoustic tomography problem in a medium with convolution-type attenuation can be modeled by the following (a bit more general) initial value problem (IVP):

$$\begin{cases} P_{a,\Phi}u(t,x) = 0, \quad (t,x) \in (0,T) \times \mathbb{R}^n \\ u|_{t=0} = f_1, \\ \partial_t u|_{t=0} = f_2, \end{cases} \tag{3.4}$$

where we aim to recover the initial source $\mathbf{f} = (f_1, f_2)$ from boundary measurements $u|_{(0,T)\times\partial\Omega}$, and assuming the waves propagate freely in space, that is, we suppose the boundary of Ω does not interact with the outgoing waves. Notice we have decided to state the problem for general initial conditions (f_1, f_2), nevertheless, in applications we are interested in the case $\mathbf{f} = (f, -af)$. When $\Phi = 0$, so there is no memory kernel, the equation in (3.4) reduces to the damped wave equation and we replace $P_{a,\Phi}$ with the operator $P_a = \partial_t^2 - c^{-2}\Delta + a\partial_t + b$.

3.2 Preliminaries

3.2.1 Direct Problem

Let $U \subset \mathbb{R}^n$ be an open bounded set with smooth boundary, $u_0 \in H_0^1(U)$, $u_1 \in L^2(U)$ and $F \in L^2([0,T]; L^2(U))$. We say u is a generalized solution of

$$P_{a,\Phi} u = F \text{ in } [0,T] \times U, \quad u|_{[0,T]\times\partial U} = 0, \quad u(0) = u_0,\ u_t(0) = u_1, \tag{3.5}$$

if $u \in L^2([0,T]; H_0^1(U))$, $u_t \in L^2([0,T]; L^2(U))$, $u_{tt} \in L^2([0,T]; H^{-1}(U))$ and

$$\langle c^{-2} u_{tt}, \varphi \rangle + B(u, \varphi) = (c^{-2} f, \varphi) \quad \forall \varphi \in C_0^\infty(U) \text{ and for a.e. } t \in [0,T] \tag{3.6}$$

where $\langle \cdot, \cdot \rangle$ and $(\cdot, \cdot)$ stand for the duality product of H^{-1} and H_0^1, and the L^2 inner product respectively, and $B(\cdot, \cdot)$ is the bilinear form given by

$$B(u, \varphi) = (\nabla u, \nabla \varphi) + (ac^{-2} u_t, \varphi) + (bc^{-2} u, \varphi) + (c^{-2} \Phi * u, \varphi).$$

The well-posedness follows from Theorems 2.1 and 2.2 in [13]. We refer to the appendix for a complete proof. In our case, by finite speed of propagation we can take U to be a large ball containing Ω to ensure we have null Dirichlet conditions.

3.2.2 Energy Space and Positive-Definite Kernels

Given a domain $U \subseteq \mathbb{R}^n$ and a scalar function $u(t,x)$, we define the local energy of $\mathbf{u} = [u, u_t]$ at time t as

$$E_U(\mathbf{u}(t)) = \int_U (|\nabla_x u|^2 + b|u|^2 + c^{-2}|u_t|^2) dx.$$

In order to give problem (3.4) a physical sense we need to assume some conditions on the attenuation terms since it is natural to request the system to be energy decreasing over time due to the attenuation. We recall once again that all the attenuation processes are linked through the system (3.1), so the next condition on $\mathcal{A}$ is such that the above energy functional for solutions of (3.4) with $\mathcal{A} = \mathcal{A}_1$ decreases in time.

Condition 3.2.1.

$$a(x) \geq 0 \quad \textit{and} \quad (-1)^j \partial_t^j \Phi(t,x) \geq 0,\ \forall t \geq 0,\ x \in \mathbb{R}^n,\ j = 0, 1, 2. \tag{3.7}$$

The previous condition guarantees the positive-definiteness of the kernel as shown in [32] and [40]. Moreover, if we define

$$\Psi(t,x) := -\int_t^\infty \Phi(s,x)ds, \tag{3.8}$$

it turns out that $-\Psi$ is also a positive-definite kernel since it satisfies the same condition as Φ.

Remark. An example of a kernel satisfying Condition 3.2.1 is $\Phi(t,x) = q(x)e^{-\alpha(x)t}$, for some positive functions $q, \alpha \in C(\mathbb{R}^n)$. In the recent article [10], the authors introduce a new definition for fractional derivatives whose kernel is of the form $e^{-\alpha t}$. As a consequence, the analysis carried out here might be applied to fractional models of wave propagation following this new definition of fractional derivatives.

Under Condition 3.2.1 we define an extended energy functional at time $\tau > 0$, analogously as in [35], to be

$$\mathcal{E}_U(u,\tau) = E_U(\mathbf{u}(\tau)) + 2\int_{[0,\tau]\times U} ac^{-2}|u_t|^2 dxdt + 2\int_{[0,\tau]\times U} c^{-2}\left(\Phi * u_t\right) u_t dxdt, \tag{3.9}$$

where the last two terms take into account the portion of the energy that is lost due to the attenuation process. This functional is the natural one from (3.1) with attenuation process $\mathcal{A}_1$ (i.e. $k = 1$). If we set $U = \mathbb{R}^n$, or by finite propagation speed we take U equal to any sufficiently large ball, in the interval $[0,T]$ the unattenuated energy functional E_U is non-increasing since we get

$$\frac{d}{dt}E_U(\mathbf{u}(t)) = -2\int_{[0,\tau]\times U} ac^{-2}|u_t|^2 dxdt - 2\int_{[0,\tau]\times U} c^{-2}\left(\Phi * u_t\right) u_t dxdt \leq 0,$$

and integrating in time we deduce that the extended functional is conserved.

We adopt the same functional framework as in previous articles related to thermoacoustic tomography. The energy space $\mathcal{H}(U)$ of initial conditions is defined to be the completion of $C_0^\infty(U) \times C_0^\infty(U)$ under the energy norm

$$\|\mathbf{f}\|^2_{\mathcal{H}(U)} = \int_U (|\nabla_x f_1|^2 + c^{-2}|f_2|^2) dx.$$

with $\mathbf{f} = (f_1, f_2)$. We also let $H_D(U)$ denote the completion of $C_0^\infty(U)$ under the norm

$$\|f\|^2_{H_D(U)} = \int_U |\nabla_x f|^2 dx.$$

Notice that $\mathcal{H}(U) = H_D(U) \oplus L^2(U; c^{-2}dx)$ with the latter space denoting the L^2 functions under the weight $c^{-2}dx$.

Denoting by Ω the region of interest and $\Sigma = [0,T] \times \partial\Omega$, we introduce the measurement operator

$$\Lambda : \mathcal{H}(\Omega) \ni \mathbf{f} \mapsto u|_\Sigma \in H^1(\Sigma),$$

where u satisfies (3.4). We might also write Λf, in which case it is implicit that we are taking $\mathbf{f} = (f, -af)$, with $f \in H_D(\Omega)$.

3.3 Uniqueness

There is a simple uniqueness result for the purely damped wave equation, this is when $\Phi = 0$, that follows in the same fashion as the unattenuated case and the reader can find its proof in [20]. We only point out that it is based on Proposition 2.1.1, and since the damping term breaks the reversibility in time of the equation it is not possible to extend the problem to $(-T,0)$, thus we are forced to take a larger time-observation window. It is worth to mention that such unique continuation result can not be applied to the integro-differential case because we lose locality of the equation.

Theorem 3.3.1 (Theorem 3.1 in [20]). *Assume that* Ω *is strictly convex and that* $T > 2T_0(\Omega)$. *If* $\Lambda\mathbf{f} = 0$, *then* $\mathbf{f} = 0$.

In the presence of a memory integral the new tool that comes in play is a foliation condition. In the rest of this subsection it is more convenient to work with the more general hyperbolic operator

$$\mathcal{P}_{a,\Phi}u = u_{tt} - \partial_j(g^{ij}(x)\partial_i u) + \langle A(x), u'\rangle + b(x)u + \Phi * u, \tag{3.10}$$

associated to a Riemannian metric g. Here $u' = (u_x, u_t)$, $A = (A_1, \ldots, A_n, a)$, and we assume the vector-function A, the scalar-function b and the kernel Φ to be merely continuous.

Foliation conditions were introduced in [44] in the context of an inverse source problem for hyperbolic equations. In the field of travel time tomography in seismology, where one aims to recover the inner structure of the Earth by measuring travel times of seismic waves, at the beginning of the 20th century, Herglotz, and Weichert and Zoeppritz considered a special assumption on the isotropic wave speed that turned out to be a particular instance of Stefanov and Uhlmann's foliation condition (see [45, §6] and reference therein). These type of assumptions seem to be the natural conditions under which one could expect to propagate information from the exterior towards the interior of the domain. In particular, it has been applied before in the thermoacoustic setting to prove uniqueness for the inverse problem of recovering a sound speed when the initial source is known [44, §3].

Theorem 3.3.2 is a direct consequence of [9, Theorem 1], a boundary unique continuation result for hyperbolic equations with a memory term, where, for the sake of simplicity, the authors restricted their analysis to the Euclidean case. Nevertheless, in our work we need the full strength of such unique continuation so we have included a brief proof in the case of waves traveling in a general Riemannian setting. In few words, the idea introduced in [9]

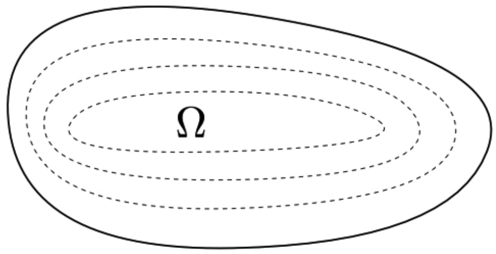

Figure 3.1: Foliation by the strictly convex levels sets of the distance to the boundary function $\mathrm{dist}(x, \partial\Omega)$.

works as follows. For a solution with null Cauchy data at the boundary, it can be shown by compactness that it vanishes near the boundary and for small times, and then extend its zero set up to the characteristic time but still in a neighborhood of the boundary. The proof is then complete by noticing that assuming the levels sets of the distance to the boundary function are strictly convex, one can repeat the same process in a layer stripping fashion (see Figure 3.1). The utilization of this strategy then makes necessary to consider full boundary data and the convexity assumption on the distance to the boundary function, which indeed gives us the internal foliation of the domain. This is the main reason of our hypotheses in Theorem 3.3.2. We point out that a similar method was also applied in [44] to obtain uniqueness for partial data and more general foliations. It was of fundamental importance in such proof the possibility of using a partial boundary unique continuation result independent of the foliation (see Proposition 2.1 in that article).

Theorem 3.3.2 (Theorem 1 in [3])**.** *Let Ω be a bounded open subset of $\mathbb{R}^n$ with $\partial\Omega$ smooth and strictly convex for a Riemannian metric g. Let $T > 0$ be such that $x^n = \mathrm{dist}(x, \partial\Omega)$ is a smooth function in Ω with non-zero differential for $0 \leq x^n \leq T$ and its level surfaces $\{x^n = s\}$, for $0 \leq s \leq T$, are strictly convex for the metric g as well. If $f \in H_D(\Omega)$ is such that $\Lambda\mathbf{f} = 0$ with $\mathbf{f} = (f, -af)$, then $f = 0$ in $\{x \in \Omega : \mathrm{dist}(x, \Omega) < T\}$. In particular, if $T \geq T_0(\Omega) := \sup_{x\in\Omega} \mathrm{dist}(x, \partial\Omega)$, then $f \equiv 0$.*

Remark. Under the above hypothesis, this result presents an improvement in the condition imposed on T for uniqueness in the damped wave equation ($T > 2T_0(\Omega)$ in [20, Theorem

3.1]).

Let $Q = (0,T)\times\Omega$ and $x^n = \mathrm{dist}(x,\partial\Omega)$ the signed distance function defined in a neighborhood of the boundary and such that Ω and $\partial\Omega$ are characterized respectively by $x^n > 0$ and $x^n = 0$. We define the following weight function

$$\varphi(x,t) = (R - x^n) - \alpha t^2 - r^2, \tag{3.11}$$

which is invariantly defined for any local coordinates $(x^1, ..., x^{n-1})$ in $\partial\Omega$. Here $\alpha = \alpha(\Omega, g) > 0$ is sufficiently small and $R, r > 0$ will be chosen large and close to each other. For $\epsilon \geq 0$ we also consider the sets

$$Q(\epsilon) = \{(t,x) \in Q : \varphi(x,t) > \epsilon\}, \tag{3.12}$$

$$\Omega(\epsilon) = \{x \in \Omega : (R - x^n)^2 > r^2 + \epsilon\}. \tag{3.13}$$

By taking r close to R, the set $Q(0)$ is a small neighborhood of $\{0\}\times\partial\Omega$ inside Q.

We recall that in boundary normal coordinates, a Riemannian metric g takes the form

$$\tilde{g}_{\alpha,\beta}(x', x^n)\mathrm{d}x^\alpha \mathrm{d}x^\beta + (\mathrm{d}x^2)^2, \tag{3.14}$$

for $\alpha, \beta \leq n-1$. We denote $\tilde{g} = (\tilde{g}_{\alpha\beta}(x))$. Moreover, the strictly convexity of the level surfaces $\{x^n = s\}$ translates into

$$\Pi(v,v) = \left(-\frac{1}{2}\frac{\partial \tilde{g}_{\alpha\beta}}{\partial x^n}\right) v^\alpha v^\beta \geq \kappa_s |v|^2_{\tilde{g}}, \quad \forall v \in T\{x^n = s\},$$

with $\kappa_s > 0$ the smallest eigenvalue (principal curvature) of the second fundamental form Π in $\{x^n = s\}$, where $R_s = \kappa_s^{-1}$ can be think as the largest curvature radius of $\{x^n = s\}$. The analogous condition for convectors follows from the natural isomorphism $\xi_i = g_{ij}(x)v^j$ and reads

$$\Pi(\xi,\xi) = \left(\frac{1}{2}\frac{\partial \tilde{g}^{\alpha\beta}}{\partial x^n}\right)\xi_\alpha \xi_\beta \geq \kappa_s |\xi|^2_{\tilde{g}}, \quad \forall \xi \in T^*\{x^n = s\}. \tag{3.15}$$

Remark. The next two lemmas also hold if the coefficients A and b are analytic functions in t.

Lemma 3.3.1. *Let Ω and T be as in Theorem 2.1. Let $f \in L^2(\Omega)$ and $u \in H^2(Q)$ be a solution of*

$$\begin{cases} \mathcal{P}_\Phi u = 0 & \text{in } (0,T)\times\Omega, \\ u|_{t=0} = 0 & \text{in } \Omega, \\ \partial_t u|_{t=0} = f & \text{in } \Omega. \end{cases} \tag{3.16}$$

If $u = \partial_\nu u = 0$ on $\partial Q(0) \cap \partial\Omega$, then

$$u = 0 \text{ in } Q(0), \text{ and in particular } f = 0 \text{ in } \Omega(0).$$

Proof. Given a point $y = (y', 0) \in \partial\Omega$, let's consider local coordinates $(U, (x^1, ..., x^{n-1}))$ in the boundary near y'. For $\epsilon \geq 0$ we define the sets

$$Q_y(\epsilon) = \{(t,x) \in Q : \varphi(x,t) > \epsilon, \ x' \in U\}, \tag{3.17}$$

$$\Omega_y(\epsilon) = \{x \in \Omega : (R - x^n)^2 > r^2 + \epsilon, \ x' \in U\}. \tag{3.18}$$

In what follows we take $r = R - \delta$, for some $\delta > 0$ small enough, therefore $x^n \in [0, \delta)$ in the set $Q(0)$.

Let's first consider an arbitrary function $u \in C^\infty(\overline{Q}_y(0))$ such that $u = \partial_\nu u = 0$ on $\overline{Q}_y(0) \cap \partial\Omega$, and let $\tilde{u}(t,x) = \chi(x')u(t,x)$, with $\chi \in C_0^\infty(U)$. The idea is to obtain a well known local Carleman estimate for $\tilde{u}$ and later use it, along with a partition of unity, to get an analogous estimate in $Q(0)$.

Let's denote $\mathcal{P} = u_{tt} - \partial_j(g^{ij}(x)\partial_i u)$, the principal part of $\mathcal{P}_\Phi$. By analyzing the conjugate operator $\mathcal{P}_\tau = e^{\tau\varphi}\mathcal{P}e^{-\tau\varphi}$, it is possible to deduce (after long computations) a pointwise estimate for $v = e^{\tau\varphi}\tilde{u}$ of the form:

$$C|\mathcal{P}_\tau v|^2 \geq \tau(|v_t|^2 + |v_n|^2) + \tau^3|v|^2 + \mathrm{div}_x(Y) + \partial_t Z \\ + 4\tau(R-\delta)^2\left(\frac{1}{2}\partial_n \tilde{g}^{kl} v_k v_l\right) - 2\tau\gamma|v_x|^2_{\tilde{g}} \tag{3.19}$$

for some constant $\gamma > 0$ depending on the parameter α which is chosen small enough, and with (Y, Z) a vector-valued function depending on lower order derivatives of v and vanishing in $\partial Q_y(0)\backslash\{\varphi = 0\}$. In fact, the previous follows by decomposing $\mathcal{P}_\tau v$ as the sum of two operators,

$$\mathcal{P}_+ v = v_{tt} - \partial_j(g^{ij}\partial_i v) + \tau^2\Phi v, \quad \Phi = \varphi_t^2 - |\varphi_x|^2_g$$

and

$$\mathcal{P}_- v = 2\tau\big(\langle\varphi_x, v_x\rangle_g - \varphi_t v_t\big) + \tau\Psi v, \quad \Psi = \partial_j(g^{ij}\partial_i\varphi) - \varphi_{tt},$$

and bounding from below the inequality

$$|\mathcal{P}_\tau v|^2 \geq |\mathcal{P}_+ v|^2 + 2(\mathcal{P}_+ v)(\mathcal{P}_- v).$$

Here we apply the convexity condition on the level surfaces $\{x^n = s\}$ in (3.15). By choosing then R large enough and δ small, we arrive to the estimate

$$\tau^3|v|^2 + \tau(|v_t|^2 + |v_x|^2_g) \leq C\big(e^{\tau\varphi}|\mathcal{P}\tilde{u}|^2 - \mathrm{div}_x(Y) - \partial_t Z\big).$$

Because $v = e^{\tau\varphi}\tilde{u}$, we can bound from below the left hand side of the previous inequality by similar terms but involving now the function u (and the exponential weight function). Integration over $Q_y(0)$ and the Gauss-Ostrogradskiĭ formula give us that

$$\tau\int_{Q_y(0)} e^{2\tau\varphi}\big(\tau^2|\tilde{u}|^2 + |\tilde{u}_t|^2 + |\tilde{u}_x|^2_g\big)dxdt \\ \leq C\int_{Q_y(0)} e^{2\tau\varphi}|\mathcal{P}\tilde{u}|^2 dxdt + C\int_{\Gamma_y(0)}\big(\langle X_1 u', u'\rangle + \langle X_2, u'\rangle u + X_3|u|^2\big)dS, \tag{3.20}$$

where dS denotes the surface measure on $\Gamma_y(0) = Q_y(0) \cap \{\varphi = 0\}$, and the matrix-function $X_1(x,t)$, the vector-function $X_2(x,t)$, and the scalar-function $X_3(x,t)$ are some continuous functions depending on φ and $Q_y(0)$. Using the continuity of the coefficients in the lower order terms (l.o.t) of $\mathcal{P}_\Phi$ and noticing that

$$|\mathcal{P}\tilde{u}|^2 \leq 2|\mathcal{P}_\Phi \tilde{u}|^2 + 2|(\text{l.o.t of } \mathcal{P}_\Phi)\tilde{u}|^2,$$

we can choose τ_0 larger if necessary and absorb the second summand in the right hand side above with the left hand side of (3.20). Then

$$\begin{aligned} &\tau \int_{Q_y(0)} e^{2\tau\varphi}\big(\tau^2|\tilde{u}|^2 + |\tilde{u}_t|^2 + |\tilde{u}_x|_g^2\big)dxdt \\ &\leq C\int_{Q_y(0)} e^{2\tau\varphi}|\mathcal{P}_\Phi\tilde{u}|^2 dxdt + C\int_{\Gamma_y(0)} \big(\langle X_1 u', u'\rangle + \langle X_2, u'\rangle u + X_3|u|^2\big)dS, \end{aligned} \tag{3.21}$$

The analogous inequality in the larger set $Q(0)$ is obtained by considering a partition of unity and using the compactness of $\partial\Omega$. More precisely, let now $u \in C^\infty(\overline{Q}(0))$ and let $\{U_i\}_i$ be a finite covering of the boundary such that on each U_i we can define boundary local coordinates, and let $\{\chi_i\}_i$ be a finite smooth partition of unity subordinate to $\{U_i\}_i$. We also consider a collection of points $y_i \in U_i$. Then, denoting $u_i = \chi_i^{1/2}u$, and the measure

$d\sigma = dtd\text{Vol}(x)$ on Q, from the previous estimates we get

$$\begin{aligned}
&\tau \int_{Q(0)} e^{2\tau\varphi}\big(\tau^2|u|^2 + |u_t|^2 + |u_x|_g^2\big)d\sigma \\
&\quad = \tau \sum_i \int_{Q_{y_i}(0)} e^{2\tau\varphi}\chi_i\big(\tau^2|u|^2 + |u_t|^2 + |u_x|_g^2\big)dxdt \\
&\quad \leq \tau \sum_i \int_{Q_{y_i}(0)} e^{2\tau\varphi}\big(\tau^2|u_i|^2 + |(u_i)_t|^2 + |(u_i)_x|_g^2\big)dxdt \\
&\qquad\qquad + C\tau \int_{Q(0)} e^{2\tau\varphi}|u|^2 d\sigma \\
&\quad \leq C\Bigg(\int_{Q(0)} e^{2\tau\varphi}|\mathcal{P}_\Phi u|^2 d\sigma + \int_{\Gamma(0)} \big(\langle X_1 u', u'\rangle + \langle X_2, u'\rangle u + X_3|u|^2\big)dS \\
&\qquad\qquad + \sum_i \int_{Q(0)} e^{2\tau\varphi}|[\mathcal{P}_\Phi, \chi_i]u|^2 dxdt + \tau \int_{Q(0)} e^{2\tau\varphi}|u|^2 d\sigma\Bigg),
\end{aligned}$$

where notice $[\mathcal{P}_\Phi, \chi_i]$ are differential operators of order 1. We absorb the interior integrals with lower order derivatives of u using the left hand side and get

$$\begin{aligned}
&\tau \int_{Q(0)} e^{2\tau\varphi}\big(\tau^2|u|^2 + |u_t|^2 + |u_x|_g^2\big)d\sigma \\
&\quad \leq C\int_{Q(0)} e^{2\tau\varphi}|\mathcal{P}_\Phi u|^2 d\sigma + C\int_{\Gamma(0)} \big(\langle X_1 u', u'\rangle + \langle X_2, u'\rangle u + X_3|u|^2\big)dS.
\end{aligned} \tag{3.22}$$

It follows from a density argument that the previous estimate also holds for functions in $H^2(Q)$ with null Cauchy data in $\partial Q(0) \cap \partial\Omega$.

Let u be as in the hypothesis of the lemma. Then, u satisfies an inequality of the form (3.22), without the interior integral in the right hand side. Noticing that the boundary integral does not depends on τ, we let τ goes to infinity and conclude that $u = 0$ in $Q(0)$. □

The aim of the second lemma is to extend the time for which u is zero. Based again on Carleman estimates we will be able to succeed until we hit the characteristic surface associated to the principal part of $\mathcal{P}_\Phi$, this is the surface $\{(t, x) : T - t = \text{dist}(x, \partial\Omega)\}$.

Lemma 3.3.2. *Let Ω and T be as in Theorem 2.1. If $u \in H^2(Q)$ is a solution of* (3.16) *such that* $u = \partial_\nu u = 0$ *on* $(0,T)\times\partial\Omega$, *then*

$$u = 0 \quad in \quad \{(t,x)\in Q : \mathrm{dist}(x,\partial\Omega) < \epsilon,\ 0 < t < T - \mathrm{dist}(x,\partial\Omega)\}$$

for some $0 < \epsilon \le T$.

Proof. From Lemma 3.3.1, $u = 0$ in some neighborhood $\{(t,x)\in Q : (R - x^n)^2 > \alpha t^2 + r^2\}$ for appropriate constants α, R, r. It is clear that for sufficiently small $\epsilon_1, \epsilon_2 > 0$, the previous set contains $[0,\epsilon_1]\times\{x\in\Omega : \mathrm{dist}(x,\partial\Omega) < \epsilon_2\}$.

In a neighborhood of $\partial\Omega$ we define

$$\psi(t,x) := (\epsilon_2 - x^n)(T - t - x^n), \tag{3.23}$$

and for $\gamma > 0$ we consider the sets

$$Q^{\epsilon_2}_\gamma := \{(t,x)\in Q \mid \psi(t,x) > \gamma,\ x^n < \epsilon_2\},$$

which exhaust $Q^{\epsilon_2} = \{(t,x)\in Q \mid x^n < \epsilon_2,\ 0 < t < T - x^n\}$, this is $Q^{\epsilon_2} = \bigcup_{\gamma>0} Q^{\epsilon_2}_\gamma$. Moreover, there exists $\gamma_0 > 0$ such that

$$\emptyset \neq Q^{\epsilon_2}_{\gamma_0} \subset [0,\epsilon_1]\times\{x\in\Omega : x^n < \epsilon_2\}.$$

We denote by $B(t_0,x_0;r)$ the ball centered at (t_0,x_0) and radius r for the euclidean metric. Given the following

Claim. *Suppose that for* $(t_0,x_0)\in Q^{\epsilon_2}$, u *vanishes below the level surface* $\{\psi(x,t) = \psi(t_0,x_0)\}$ *near* (t_0,x_0), *this is in* $Q^{\epsilon_2}_{\psi(t_0,x_0)} \cap B(t_0,x_0;r)$ *for some* $r > 0$. *Then,* $u = 0$ *in a neighborhood of* (x_0,t_0).

the proof of the lemma is complete by the next argument. Let's assume that $\mathrm{supp} u \cap Q^{\epsilon_2} \neq \emptyset$.

We can find $0 < \gamma^* \leq \gamma_0$ such that

$$\mathrm{supp} u \cap Q^{\epsilon_2}_\gamma = \emptyset,\ \forall \gamma > \gamma^* \quad \text{and} \quad \mathrm{supp} u \cap \{(t,x) \in Q^{\epsilon_2} : \psi(t,x) = \gamma^*\} \neq \emptyset.$$

The application of the claim on every contact point $(t^*, x^*) \in \mathrm{supp} u \cap \{(t,x) \in Q^{\epsilon_2} : \psi(t,x) = \gamma^*\}$, contradicts the choice of γ^*. Consequently, we deduce that $u = 0$ on every $Q^{\epsilon_2}_\gamma$, $\gamma > 0$, and therefore $u = 0$ in Q^{ϵ_2}.

It only remains to show the previous claim. Here is where Carleman estimates play a fundamental role, and as before we will consider a particular choice of weight function which needs to fulfill a pseudo-convex condition with respect to $\mathcal{P} = \partial_t^2 - \partial_{x^j}\big(g^{ij}\partial_{x^i}\,\cdot\,\big)$, in the set $\{(0,\xi) \in T^*_{(t_0,x_0)}\Omega\}$. Moreover, we will take it to be linear and non-increasing in time. Provided the above, it is possible to apply a pseudo-differential Carleman estimate introduced in [49] and conclude that u vanishes near (t_0, x_0).

Let's consider local coordinates in $\partial\Omega$ near some $y \in \partial\Omega$ such that in those coordinates $y = (x_0', 0)$. For some $\delta > 0$ to be appropriately chosen, we define the following weight function

$$\varphi(t,x) = \psi(t,x) - \psi(t_0,x_0) - \frac{1}{2}\delta|x - x_0|^2$$

where here $|\cdot|$ stands for the euclidean norm and ψ as in (3.23). Denoting the principal symbol of $\mathcal{P}$ by $p(t,x;\theta,\xi) = -\theta^2 + |\xi|_g^2$, where $|\xi|_g^2 = g^{ij}(x)\xi_i\xi_j$ is the norm on covectors induced by g, the pseudo-convexity condition requires to show that φ satisfies

(1) $\mathrm{Re}\{\bar{p}, \{p,\varphi\}\}(t_0,x_0;0,\xi) > 0$ for all $\xi \neq 0$ such that $p(t_0,x_0;0,\xi) = 0$,

(2) $\frac{1}{i\tau}\{\bar{p}_\varphi, p_\varphi\}(t_0,x_0;0,\xi;\tau) > 0$ for all $\xi \neq 0,\ \tau > 0$
such that $p_\varphi(t_0,x_0;0,\xi,\tau) = 0$.

Here $p_\varphi(t_0,x_0;0,\xi;\tau) = p(t,x;\theta + \mathrm{i}\tau\varphi_t, \xi + \mathrm{i}\tau\varphi_x)$ and $\{\cdot,\cdot\}$ is the Poisson bracket

$$\{f,h\} = \sum_{j=1}^n \frac{\partial f}{\partial \xi_j}\frac{\partial h}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial h}{\partial \xi_j} + \frac{\partial f}{\partial \theta}\frac{\partial h}{\partial t} - \frac{\partial f}{\partial t}\frac{\partial h}{\partial \theta}.$$

Recall that we are working in boundary normal coordinates hence the metric g takes the form (3.14). The first condition is trivially fulfilled since the principal symbol p is elliptic in the set $\{\theta = 0\}$. Let's use the following notation: the variable appearing in the subindex means we are differentiating with respect to such variable, for instance $\varphi_{x'} = \partial_{x'}\varphi$ and $\varphi_{tx^n} = \partial_t\partial_{x^n}\varphi$. To verify the second condition we notice first that $\varphi_{x'}(t_0, x_0) = \psi_{x'}(t_0, x_0) = 0$, $\varphi_{x^n}(t_0, x_0) = \psi_{x^n}(t_0, x_0) = -\alpha$ and $\varphi_t(t_0, x_0) = \psi_t(t_0, x_0) = -\beta$ where $\alpha > \beta > 0$. In fact,

$$\alpha = (\epsilon_2 - x_0^n) + (T - t_0 - x_0^n), \quad \text{and} \quad \beta = \epsilon_2 - x_0^n.$$

Also, denoting δ_{ij} the Kronecker delta,

$$\varphi_{tt} = 0, \quad \varphi_{tx^i} = \delta_{in}, \quad \varphi_{x^ix^j} = 2\delta_{in}\delta_{jn} - \delta \cdot \delta_{ij}.$$

Secondly, it is easy to check that $p_\varphi(t_0, x_0; 0, \xi, \tau) = 0$ is equivalent to $\xi_n = 0$ and $|\xi'|^2_{\tilde{g}} = \tau^2(\alpha^2 - \beta^2)$. Then, after some tedious computations, in the set of points $(t_0, x_0; 0, \xi; \tau)$ such that $p_\varphi = 0$, we get

$$\begin{aligned}\frac{1}{i\tau}\{\bar{p}_\varphi, p_\varphi\} &= \frac{1}{\tau}\{\mathrm{Re}p_\varphi, \mathrm{Im}p_\varphi\} \\ &= 8\tau^2(\alpha^2 - \alpha\beta) + 4\alpha\left(\frac{1}{2}\partial_n\tilde{g}^{ij}\right)\xi_i'\xi_j' - \delta M,\end{aligned}$$

with $M = 4\tau^2\alpha^2 + 4\left(\tilde{g}^{jk}\xi_j'\right)\left(\tilde{g}^{ik}\xi_i'\right)$ such that, for some $C > 0$,

$$M \leq 4\tau^2(\alpha^2 + C(\alpha^2 - \beta^2)).$$

Let's recall the positive-definiteness of the second fundamental form in (3.15), and denote $\kappa = \min_{s\in[0,\epsilon_2]} \kappa_s$. By choosing $\delta > 0$ small enough we obtain that

$$\frac{1}{i\tau}\{\bar{p}_\varphi, p_\varphi\} \geq 8\tau^2\alpha(\alpha - \beta)\left(1 + \frac{\kappa}{2}(\alpha + \beta)\right) - 4\delta\tau^2(\alpha^2 + C(\alpha^2 - \beta^2)) > 0,$$

therefore φ satisfies the second condition of pseudo-convexity. It follows from [50, Theorem 3] that there exists $\eta, C, d > 0$ such that any function v supported inside $B(t_0, x_0; \eta)$ (we of course choose $0 < \eta < r$), for which the RHS of the next inequality is finite, satisfies the pseudo-differential Carleman estimate

$$\tau^{-1}\|Ee^{\tau\varphi}v\|^2_{(2,\tau)} \leq C\left(\|Ee^{\tau\varphi}Pv\|^2 + e^{-d\epsilon\tau}\|e^{\tau\varphi}Pv\|^2 + e^{-d\epsilon\tau}\|e^{\tau\varphi}v\|^2_{(1,\tau)}\right), \tag{3.24}$$

for the weighted norms

$$\|v\|^2_{(m,\tau)} := \sum_{|\alpha|+j\leq m} \tau^{2(m-|\alpha|-j)}\|D^\alpha D_t^j v\|^2_{L^2(\mathbb{R}^{n+1})}, \quad \tau > 0; \quad \|\cdot\| := \|\cdot\|_{(0,\tau)},$$

and the pseudo-differential operator $E := e^{\frac{\epsilon}{2\tau}|D_t|^2}$. This operator can also be considered as the convolution operator

$$Ev(x,t) = \left(\frac{\tau}{2\pi\epsilon}\right)^{1/2}\int e^{-\frac{\tau|t-s|^2}{2\epsilon}}v(x,s)ds.$$

We would like to apply the above Carleman estimate to u and eventually deduce that u vanishes near (t_0, x_0). With that in mind we need first to localize it near (t_0, x_0). As in [9], in $(\psi'(t_0, x_0))^\perp = \{(\theta, \xi) : \langle\psi'(t_0, x_0), (\theta, \xi)\rangle_{e\otimes g} = 0\}$ we see that $|\theta| \leq C_1|\xi|_g$, hence

$$\langle(\psi - \varphi)''(\theta, \xi), (\theta, \xi)\rangle_g = \delta|\xi|^2_g \geq c_2|(\theta, \xi)|^2_{e\otimes g}.$$

Therefore, by choosing $l_1 < 0$ small enough in magnitude, the set $\{\varphi(t, x) > l_1\} \cap \{\psi(t, x) < \psi(t_0, x_0)\}$ is contained in a sufficiently small vicinity of (t_0, x_0). We then localize u by multiplying it with a function of the form $\chi(\varphi(t, x))$ with $\chi \in C^\infty(\mathbb{R})$ a nondecreasing function such that

$$\chi(s) = \begin{cases} 0 & \text{for } s < l_1, \\ 1 & \text{for } s > l_2, \end{cases}$$

where $l_1 < l_2 < 0$ are small enough in magnitude, then

$$\operatorname{supp}\big[u(t,x)\chi(\varphi(t,x))\big] \subset B(t_0, x_0; \eta).$$

In what follows we write χ meaning the composition $\chi \circ \varphi$. Consequently, $v = \chi u$ satisfies the inequality (3.24). We include the integral term in the estimates by noticing that

$$\mathcal{P}(\chi u) = \chi \mathcal{P} u + [\mathcal{P}, \chi] u = \chi \mathcal{P}_\Phi u - \chi \Phi * u + \mathcal{P}_1 u,$$

where $\mathcal{P}_1$ is a differential operator of order 1 with coefficients supported in $\{(t,x) | \varphi(t,x) < l_2\}$. Consequently

$$\begin{aligned} \tau^{-1} \| E e^{\tau\varphi} (\chi u) \|^2_{(2,\tau)} \leq c \Big(& \| E e^{\tau\varphi} \mathcal{P}_1 (\chi u) \|^2 + \| E e^{\tau\varphi} \chi (\Phi * u) \|^2 \\ & + e^{-d\epsilon\tau} \| e^{\tau\varphi} \mathcal{P}(\chi u) \|^2 + e^{-d\epsilon\tau} \| e^{\tau\varphi} (\chi u) \|^2_{(1,\tau)} \Big) . \end{aligned} \tag{3.25}$$

The idea in what remains of the proof is to estimate $\| E e^{\tau\varphi} (\chi u) \|_{(2,\tau)}$ by a term of the form $e^{l\tau}$, with $l < 0$, and use [49, Proposition 4.1] to conclude that $\chi u = 0$ in $\{(t,x) | \varphi(t,x) > l\}$. Such estimate is obtained in exactly the same way as in the proof of Lemma 6 in [9], where everything reduces to estimate the term with the convolution since the other terms in the right hand side of the last inequality are easily bounded. For the arguments needed to conclude the claim we refer the reader to [9]. $\square$

Proof of Theorem 2.1. Let u be a solution of $P_{a,\Phi} u = 0$ with initial conditions $[f, -af]$ and such that $\Lambda f = 0$. Due to our assumption on the coefficients of $P_{a,\Phi}$, u solves $(\partial_t^2 - \Delta) u = 0$ in $(0,T) \times (\mathbb{R}^n \setminus \overline{\Omega})$ with null initial and Dirichlet boundary data. Then, for any $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$, u vanishes in $(0,T) \times V$ for some small neighborhood V of x_0 such that $\overline{V} \cap \overline{\Omega} = \emptyset$. The previous is a consequence of a sharp domain of dependence for the wave operator in the exterior problem (see [16, Proposition 2]). Then $u = 0$ in $(0,T) \times (\mathbb{R}^n \setminus \Omega)$ which implies null Neumann data, $\frac{\partial u}{\partial \nu}\big|_{(0,T)\times\partial\Omega} = 0$.

Let's set
$$\bar{u}(t,x) = \int_0^t u(s,x)ds \quad \text{and} \quad \Psi(t,x) = -\int_t^\infty \Phi(s,x)ds. \tag{3.26}$$
Note that $\bar{u}_t(t,x) = u(t,x)$ and $\partial_t \Psi = \Phi$. Moreover
$$\partial_t\Big(\int_0^t \Psi(t-s,x)\bar{u}_s(s,x)ds\Big) = \Psi(0,x)\bar{u}_t(t,x) + \int_0^t \Phi(t-s,x)\bar{u}_s(s,x)ds,$$
which, since $\bar{u}(0,x) = 0$ and integration by parts, implies
$$\int_0^t \Big(\int_0^\tau \Phi(\tau - s,x)u(s,x)ds\Big) \;= \int_0^t \Phi(t-s,x)\bar{u}(s,x)ds. \tag{3.27}$$
We integrate equation (3.4) on the interval $(0,t)$ for any $t > 0$. It follows from the previous computations that $\bar{u}$ solves a system of the form (3.16) with vanishing Cauchy data. In addition, notice that $\bar{u}_{tt} = u_t \in L^2(Q)$, so using equation (3.16) we get $c^2\Delta\bar{u} \in L^2(Q)$, which by elliptic regularity implies $\bar{u} \in H^2(Q)$. We can now apply Lemma 3.3.2 on $\bar{u}$ and conclude that $u = 0$ in a set of the form $\{(t,x) \in Q : x^n < \epsilon\ 0 < t < T - x^n\}$. This implies we have reduced the problem to the smaller domain $[0, T-\epsilon] \times \{x \in \Omega : x^n > \epsilon\}$. If $\epsilon = T$ we are done, otherwise we can apply again Lemma 3.3.2 in the new domain. Iterating this process we conclude the result. ■

Let's consider now the isotropic case where the metric takes the form $g = c^{-2}\mathrm{d}x^2$. There is a common condition appearing in the literature of Carleman estimates and inverse problems related to the wave equation with variable sound speed, which assumes the existence of some $x_0 \in \mathbb{R}^n$ for which
$$(x - x_0)\cdot \partial_x c(x) < c(x) \quad \forall x \in \mathbb{R}^n. \tag{3.28}$$
In geometric terms, (3.28) says that the spheres with center at x_0 are strictly convex for the metric $c^{-2}dx^2$ [44, §3]. Such collection of spheres can then be used to foliate the domain Ω and, as you will see in the next theorem (see also Figure 3.2), it allows us to prove unique

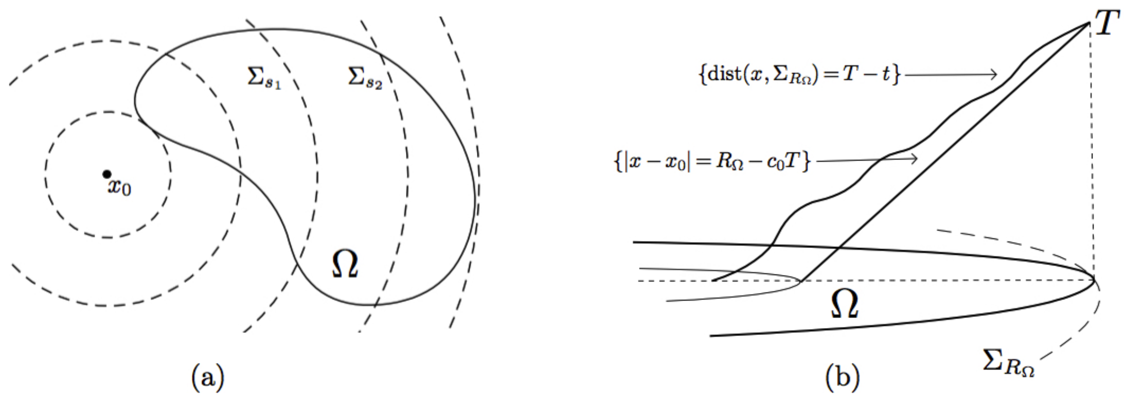

Figure 3.2: (a) Foliation of Ω by Euclidean spheres $\{\Sigma_s\}_s$ centered at x_0. (b) Sub-characteristic unique continuation under condition (3.28).

continuation and consequently uniqueness for the inverse problem without the assumption of Ω and the level surfaces of the distance function, $\mathrm{dist}(\cdot, \partial\Omega)$, being strictly convex. The price we pay by removing the convexity requirement on Ω is the lost of sharpness in the bound of T that guarantee uniqueness.

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded subset with $\partial\Omega$ smooth, and $T > 0$. We assume the sound speed $c(x)$ satisfies condition (3.28) and assume the constant $c_0 > 0$ is a lower bound for the sound speed. Let's denote

$$R_\Omega = \max_{r>0}\{|x - x_0| : x \in \partial\Omega\},$$

$$r_\Omega = \begin{cases} \min\limits_{r>0}\{|x - x_0| : x \in \partial\Omega\}, & \text{if } x_0 \in \mathbb{R}^n \backslash \bar{\Omega} \\ 0, & \text{otherwise,} \end{cases}$$

and $D_\Omega = R_\Omega - r_\Omega$.

Theorem 3.3.3 (Theorem 2 in [3])**.** *Assume Ω, T and c are as above, and as in the TAT problem, we assume $P_{a,\Phi} = \partial_t^2 - \Delta$ outside Ω. If $u \in H^2(Q)$ is a solution of* (3.16) *such that $u = \partial_\nu u = 0$ on $(0, T) \times \partial\Omega$, then*

$$u = 0 \quad in \quad \{(t, x) \in Q : 0 < t < T - c_0^{-1}(R_\Omega - |x - x_0|)\}.$$

As a consequence, in the thermoacoustic problem, if $f \in H_D(\Omega)$ *is such that* $\Lambda\mathbf{f} = 0$*, with* $\mathbf{f} = [f, -af]$*, then*

$$f = 0 \quad in \quad \{x \in \Omega : |x - x_0| > R_\Omega - c_0 T\},$$

and in particular, $f \equiv 0$ *when* $T \geq c_0^{-1} D_\Omega$.

Remark. From [43, Proposition 7.1], the uniqueness time defined in Theorem 2.1 satisfies $T_0 < c_0^{-1} D_\Omega$.

Proof. Let's extend u to be zero outside Ω in the interval $[0, T]$. Due to the null Cauchy data, finite speed of propagation and the well-posedness of the exterior problem, u solves (3.16) in the whole space. Notice that in particular, $u = \partial_\nu u = 0$ on the Euclidean sphere $\{x \in \mathbb{R}^b : |x - x_0| = R_\Omega\}$, for all $t \in [0, T]$. ∂_ν stands for a generic exterior normal derivative.

We denote $\Sigma_r = \{x \in \mathbb{R}^b : |x - x_0| = r\}$ the sphere of center x_0 and radius r, and we set $r_0 = \max\{0, c_0 T - D_\Omega\}$. By hypothesis, Σ_r with $r \in [r_0, R_\Omega]$ are strictly convex surfaces for the metric $c^{-2}dx^2$ that foliate Ω (see Figure 3.2(a)). For a given $r \in [r_0, R_\Omega]$, let's assume that

$$u = \partial_\nu u = 0 \quad \text{on} \quad [0, T - c_0^{-1}(R_\Omega - r)] \times \Sigma_r.$$

Since Σ_r is strictly convex we can apply Lemma 3.3.2 with Ω replaced by $B(x_0, r)$, the Euclidean ball of center x_0 and radius r, and deduce that $u = 0$ in

$$\{(t, x) \in (0, T) \times B(x_0, r) : \text{dist}(x, \Sigma_r) < \epsilon,\ t < T - c_0^{-1}(R_\Omega - r) - \text{dist}(x, \Sigma_r)\},$$

for some $\epsilon > 0$. Recalling that c_0 is a lower bound for c, we have that

$$\text{dist}(x, \Sigma_r) < c_0^{-1}(r - |x - x_0|) \quad \forall x \in B(x_0, r),$$

therefore we can find $r_1 \in (0, r)$ such that u vanishes in the smaller set

$$\{(t, x) : r_1 < |x - x_0| < r,\ 0 < t < T - c_0^{-1}(R_\Omega - |x - x_0|)\}.$$

Moreover, u has null Cauchy data on Σ_{r_1} for all $t \in (0, T - c_0^{-1}(R_\Omega - r_1))$ (see Figure 3.2(b)).

If we denote by s the infimum of the radius $r \geq r_0$ for which u has vanishing Cauchy data in $(0, T - c_0^{-1}(R_\Omega - r)) \times \Sigma_r$, by the first paragraph and the previous argument we know $s < R_\Omega$ (since $T > 0$). Moreover, if $s > r_0$, it must also satisfies the same property, this is, $u = \partial_\nu u = 0$ in Σ_s for all $t \in (0, T - c_0^{-1}(R_\Omega - s))$. Consequently, we can still apply the arguments in the paragraph above which leads us to conclude $s = r_0$.

Let now $f \in \mathcal{H}_{D,a}(\Omega)$ be as in the hypothesis, and u solution of (3.4). Analogously to the proof of Theorem 2.1, the function $\bar{u}$ defined in (3.26) satisfies a system of the form (3.16) with null Cauchy data. The result then follows directly from the previous. □

3.4 Stability

A fundamental reason behind the fact that the stability property holds in the classical setting for observation times larger than $T_1(\Omega)/2$, is the possibility of (microlocally) split the dynamic of singularities to those following respectively the positive and negative sound speed $\pm c(x)$. It will be clear in the next subsection that this is no longer possible when an strong attenuation process is in action, however, if we let $T > T_1(\Omega)$ we still hope to obtain stability. Indeed, this is the case. As it was mentioned in the remark following the proof of Theorem 2.2.3, the local and lower order perturbation of the wave operator $P = \partial_t^2 - \ c^2\Delta$ given by the damping coefficient do not perturb the propagation of singularities, and they still travel along geodesics for the metric $c^{-2}\mathrm{d}x^2$. In consequence, the stability property for the purely damped wave equation (i.e. $\Phi = 0$) follows from Theorem 2.2.3. In the case there is no damping and instead the attenuation is given uniquely by a memory term, we recover the optimal stability time $T_1(\Omega)/2$. We will see that in terms of stability, the integral (memory) perturbation is in fact milder than the local one (damping term).

From now on, we restrict our analysis to the conformal case of $g = c^{-2}\mathrm{d}x^2$, with c the wave speed of the media. The next theorem establish the stability property for the two cases treated in the previous subsection, this is, when the distance to the boundary function has strictly convex level sets, and when the sound speed satisfies condition (3.28). They are related through the inequality $T_1/2 \leq (R_\Omega - r_\Omega)/(\alpha c_0)$ with (see [43, Proposition 7.1])

$$\alpha = \min_{x \in \bar{\Omega}} (1 - c^{-1}(x - x_0) \cdot \partial_x c) > 0. \tag{3.29}$$

Theorem 3.4.1. *Let Ω be strictly convex for the metric $g = c^{-2}dx^2$. Assume that Ω and T are as in Theorem 2.1 (or as in Theorem 3.3.3). In addition, assume $T_1(\Omega) < T < \infty$ if $a \neq 0$ and $\frac{1}{2}T_1(\Omega) < T < \infty$ otherwise (resp. $2\alpha^{-1}c_0^{-1}D_\Omega < T < \infty$ and $\alpha^{-1}c_0^{-1}D_\Omega < T < \infty$). Then there exists $C > 0$ such that for all $f \in H_D(\Omega)$,*

$$\|f\|_{H_D(\Omega)} \leq C\|\Lambda f\|_{H^1((0,T)\times\partial\Omega)}.$$

Remark. In the case $\Phi = 0$ so there is only local attenuation, there is a slightly stronger stability result which follows in the same way as Theorem 2.2.4 (see the remarks following it). Indeed,

$$\|\mathbf{f}\|_{\mathcal{H}_D(\Omega)} \leq C\|\Lambda \mathbf{f}\|_{H^1((0,T)\times\partial\Omega)}, \quad \forall \mathbf{f} \in \mathcal{H}(\Omega).$$

Proof. The idea is to compare the observation map Λ with its analogous for the undamped ($a = \Phi = 0$) and damped case ($a \neq 0$, $\Phi = 0$), which here we respectively denote by Λ_0 and Λ_a. From §2.2.1, these last two operators are known to be stable maps whenever $T > T_1(\Omega)/2$ and $T > T_1(\Omega)$ respectively. Furthermore, from the results of the previous subsection §3.3, we know Λ is injective. The proof then reduces to show that the respective error operators are compact. We only show this for the case $a \equiv 0$, the proof when there is a damping coefficient is obtained analogously.

From Theorem 2.2.4 it follows there is a constant $C > 0$ such that

$$\|f\|_{H_D} \leq C\|\Lambda_0 f\|_{H^1} \leq C\|\Lambda f\|_{H^1} + C\|(\Lambda - \Lambda_0)f\|_{H^1}.$$

Let's denote $R = \Lambda - \Lambda_0$ and $u(t, x)$ the attenuated wave related with Λ. Then, R maps

$f \in H_D(\Omega)$ to the boundary data $w|_{(0,T)\times\partial\Omega}$ of the system

$$\begin{cases} (\partial_t^2 - c^2\Delta + b)w = -\Phi * u, & (t,x) \in (0,T)\times\mathbb{R}^n \\ w|_{t=0} = 0, \\ w_t|_{t=0} = 0. \end{cases} \tag{3.30}$$

By finite propagation speed we can work in a larger domain Ω' such that $w = u = 0$ on its boundary and outside Ω'. Due to the higher regularity theorem in [14, §7.2.3 Theorem 5], since $F(t,x) = -[\Phi * u](t,x)$ satisfies $F, F_t \in L^2((0,T); L^2(\Omega'))$, we obtain that $w \in C((0,T); H^2(\Omega'))$ and $w_t \in C((0,T); H^1(\Omega'))$, and consequently the trace of w in $\partial\Omega$ belongs to $H^{3/2}((0,T)\times\partial\Omega)$, with the latter space compactly embedded in $H^1((0,T)\times\partial\Omega)$.

The stability inequality is obtained by recalling the injectivity of Λ from Theorem 2.1 (respectively Theorem 3.3.3) and applying the classical result [51, Proposition V.3.1]. □

3.5 Reconstruction

We aim to construct a Neumann series that allow us to recover **f** in (3.4) from boundary measurements as it was done in §2.3.1 for the classical setting. A formula of this kind was obtained for the damped wave equation in [20], under the hypothesis of the damping coefficient being sufficiently small. For the Time Reversal step, the author considered the system

$$\begin{cases} (\partial_t^2 - c^2(x)\Delta + a\partial_t)v = 0, & (x,t) \in \Omega\times(0,T) \\ (v, v_t)|_{t=T} = (\phi, 0), & x \in \Omega \\ v = h, & (x,t) \in \partial\Omega\times(0,T), \end{cases}$$

which has the problem of being energy increasing when solved backward, from the final time T to the initial one $t = 0$. One possible way of controlling the energy in the iterations of the Neumann series is by means of the smallness assumption on the damping coefficient. However, it is also possible to modify the TR system in such a way that the energy in that process is controlled. We call this new system *dissipative Time Reversal* and was firstly introduced by myself in [35] for the damped wave equation and subsequently for the integro-differential equation in (3.4) in my second article [3].

3.5.1 Local Attenuation

Let's denote by $\mathbf{A}_a : H^1_{(0)}([0,T] \times \partial\Omega) \to \mathcal{H}(\Omega)$, the dissipative TR operator for the thermoacoustic problem with local damping, which is defined as $\mathbf{A}_a h = (v(0,\cdot), v_t(0,\cdot))$, for v solution to

$$\begin{cases} (\partial_t^2 - c^2(x)\Delta + b - a\partial_t)v = 0, & (x,t) \in \Omega \times (0,T) \\ (v, v_t)|_{t=T} = (\phi, 0), & x \in \Omega \\ v = h, & (x,t) \in \partial\Omega \times (0,T). \end{cases}$$

We then have the next result.

Theorem 3.5.1. *Let Ω be strictly convex for the metric $g = c^{-2}dx^2$ and non-trapping, and assume $T > T_1(\Omega)$. Then $\mathbf{A}_a\Lambda = \mathbf{Id} - \mathbf{K}$ with $\|\mathbf{K}\|_{\mathcal{L}(\mathcal{H}(\Omega))} < 1$, and for any $\mathbf{f} \in \mathcal{H}(\Omega)$, the thermoacoustic inverse problem with local attenuation has a reconstruction formula given by*

$$\mathbf{f} = \sum_{m=0}^{\infty} \mathbf{K}^m \mathbf{A}_a h, \quad h = \Lambda \mathbf{f}.$$

Proof. We follow the same approach as the proof of Theorem 2.3.2, so let's recall the underlying idea behind such demonstration. We want to construct two approximate solutions, or parametrix solutions, in the sense that they contain all the singularities of the original solution to (3.4), however they satisfy the damped wave equation up to a smooth function which depends on the initial source $\mathbf{f}$. For this, we employ an analogous geometric optic construction as in §2.3.1, for the initial value problem and for the boundary value problem with data $\Lambda\mathbf{f}$. Those microlocal approximations are used to obtain an observability type estimate, which together with energy estimates implies the error operator is a contraction, and consequently the convergence of the Neumann series. One of the main implications of considering an attenuation is the lack of (microlocal) projection operators to split the analysis of singularities for the positive and negative wave speeds. You will see that for every singularity in the initial source, we are forced to track both branches of the respective bicharacteristic. Then, for a singularity to be visible at the boundary it is necessary to measure the Dirichlet data for times larger than $T_1(\Omega)$.

1. Parametrix for the Cauchy problem. The next parametrix construction was introduced in [20] for the case $\mathbf{f} = (f, -af)$, following the standard argument presented in §2.3.1. Its generalization to initial conditions $\mathbf{f} = (f_1, f_2)$ is straightforward and the main difference resides in the transport equations (3.33).

We are looking for a solution of the form

$$u(t,x) = (2\pi)^{-n} \sum_{\sigma=\pm} \int e^{i\phi^\sigma(t,x,\eta)} \big(A_1^\sigma(t,x,\eta)\hat{f}_1(\eta) + |\eta|^{-1} A_2^\sigma(t,x,\eta)\hat{f}_2(\eta) \big) d\eta,$$

where $\hat{f}_i$ stands for the Fourier transform of f_i. To find the equations that $\phi^\pm$ and $A_j^\pm$, j=1,2, must satisfy we first compute:

$$P_{a,0}u = (2\pi)^{-n} \sum_{\sigma=\pm} \int e^{i\phi^\sigma} \big([I_{1,0}^\sigma + I_{1,1}^\sigma + I_{1,2}^\sigma]\hat{f}_1 + |\eta|^{-1}[I_{2,0}^\sigma + I_{2,1}^\sigma + I_{2,2}^\sigma]\hat{f}_2 \big) d\eta$$

where

$$\begin{aligned} I_{j,2}^\sigma &= -A_j^\sigma((\partial_t\phi^\sigma)^2 - c^2|\nabla_y\phi^\sigma|^2), \\ I_{j,1}^\sigma &= 2i[(\partial_t\phi^\sigma)(\partial_t A_j^\sigma) - c^2\nabla_y\phi^\sigma \cdot \nabla_y A_j^\sigma] + iA_j^\sigma \Box_a \phi^\sigma, \\ I_{j,0}^\sigma &= P_{a,0}A_j^\sigma. \end{aligned}$$

Considering classical amplitude functions given by the asymptotic expansions

$$A_j^\sigma(t,x,\eta) \sim \sum_{k\geq 0} A_{j,k}^\sigma(t,x,\eta), \quad \sigma = \pm,$$

with $A_{j,k}^\sigma$ homogeneous of degree $-k$ in η, we would like to find those $A_{j,k}^\sigma$ so that u solves

$P_{a,0}u = 0$ up to a smooth terms. We then choose ϕ^σ and $A^\sigma_{j,k}$ so that the terms of same order of homogeneity in η cancel each other. The phase functions must satisfy the eikonal equation which takes into account the second order terms $I^\sigma_{j,2}$, and we endow it with initial conditions

$$\begin{cases} \mp\partial_t\phi^\pm & = & c|\nabla_x\phi^\pm| \\ \phi^\pm|_{t=0} & = & x\cdot\eta. \end{cases} \tag{3.31}$$

Consequently, we obtain $I^\sigma_{j,2} = 0$.

To get rid of the next terms with less order of homogeneity we have to solve transport equations. We define the vector field

$$X^\sigma = 2(\partial_t\phi^\sigma)\partial_t - 2c^2\nabla_x\phi^\sigma\cdot\nabla_x. \tag{3.32}$$

Then, the coefficients of both amplitude functions must satisfy

$$X^\sigma A^\sigma_{j,0} + A^\sigma_{j,0}P_{a,0}\phi^\sigma = 0, \quad \text{and} \quad X^\sigma A^\sigma_{j,k} + A^\sigma_{j,k}P_{a,0}\phi^\sigma = -P_{a,0}A^\sigma_{j,k-1} \text{ for } k\geq 1. \tag{3.33}$$

Since we must have that $u|_{t=0} = f_1$, this is

$$f_1(x) = (2\pi)^{-n}\int e^{ix\cdot\eta}\big((A_1^+ + A_1^-)|_{t=0}\hat{f}_1(\eta) + |\eta|^{-1}(A_2^+ + A_2^-)|_{t=0}\hat{f}_2(\eta)\big)d\eta,$$

we need the condition

$$A_1^+ + A_1^- = 1, \quad A_2^+ + A_2^- = 0, \quad \text{at } t=0.$$

Analogously, from $u_t|_{t=0} = f_2$ and since ϕ^σ satisfies (3.31), we have

$$f_2(x) = (2\pi)^{-n} \int e^{ix\cdot\eta}\big([ic|\eta|(-A_1^+ + A_1^-) + \partial_t(A_1^+ + A_1^-)]|_{t=0}\hat{f}_1(\eta) \\ + [ic(-A_2^+ + A_2^-) + |\eta|^{-1}\partial_t(A_2^+ + A_2^-)]|_{t=0}\hat{f}_2(\eta)\big)d\eta,$$

thus, at $t = 0$ we require

$$ic|\eta|(-A_1^+ + A_1^-) + \partial_t(A_1^+ + A_1^-) = 0, \quad ic(-A_2^+ + A_2^-) + |\eta|^{-1}\partial_t(A_2^+ + A_2^-) = 1.$$

We then consider initial conditions given by the following system at $t = 0$, which can be solved iteratively:

$$\begin{cases} A_{1,0}^+ + A_{1,0}^- &= 1 \\ A_{1,k}^+ + A_{1,k}^- &= 0,\ k \geq 1 \end{cases} \quad \begin{cases} A_{1,0}^+ - A_{1,0}^- &= 0 \\ A_{1,k}^+ - A_{1,k}^- &= ic^{-1}|\eta|^{-1}\partial_t(A_{1,k-1}^+ + A_{1,k-1}^-),\ k \geq 1, \end{cases} \tag{3.34}$$

$$\begin{cases} A_{2,0}^+ + A_{2,0}^- &= 0 \\ A_{2,k}^+ + A_{2,k}^- &= 0,\ k \geq 1 \end{cases} \quad \begin{cases} A_{2,0}^+ - A_{2,0}^- &= i/c \\ A_{2,k}^+ - A_{2,k}^- &= ic^{-1}|\eta|^{-1}\partial_t(A_{2,k-1}^+ + A_{2,k-1}^-),\ k \geq 1. \end{cases} \tag{3.35}$$

We solve the transport equations in (3.33) on integral curves of X^σ as longs as the eikonal equation (3.31) is solvable, and imposing the initial conditions from (3.34) and (3.35). In particular, at $t = 0$, the leading term are given by $A_{1,0}^+ = A_{1,0}^- = \frac{1}{2}$, and $A_{2,0}^+ = -A_{2,0}^- = \frac{i}{2c}$.

We can iteratively use the previous construction by solving the eikonal equation in small increments on time and via a finite partition of unity, get parametrices u_+, u_- defined on $[0, T]$. In addition, by assuming the wave front set of $\mathbf{f}$ lies inside a small conic neighborhood of some $(x_0, \xi_0) \in S^*\bar{\Omega}$, their supports can be assumed to be contained in small neighborhoods of the respective branches of the geodesics issued from (x_0, ξ_0), and their wave front sets contained in small neighborhoods of the bicharacteristics issued from $(0, x_0, 1, \xi_0)$ and $(0, x_0, 1, -\xi_0)$ respectively.

If we restrict the previous solution to the boundary we obtain an approximate representation (up to smooth term) of the measurement operator Λ_a, as a sum of Fourier Integral Operators, this is $\Lambda_a \mathbf{f} \cong \Lambda_a^+ \mathbf{f} + \Lambda_a^- \mathbf{f}$, where

$$\begin{aligned}[\Lambda_a^{\pm}\mathbf{f}](t,x) &= [\Lambda_{a,1}^{\pm} f_1 + \Lambda_{a,2}^{\pm} f_2](t,x) \\ &= (2\pi)^{-n} \int e^{i\phi^{\pm}(t,x,\eta)} \big(A_1^{\pm}(t,x,\eta)\hat{f}_1(\eta) + |\eta|^{-1} A_2^{\pm}(t,x,\eta)\hat{f}_2 \big) d\eta \Big|_{\partial\Omega}.\end{aligned}$$

It follows from the same arguments as in [20] and [41] that their canonical relations are of graph type. In consequence, since $\Lambda_{a,1}^{\pm}$ are FIO's of order 0 and $\Lambda_{a,2}^{\pm}$ are of order -1, writing $h = \Lambda_a \mathbf{f}$ we get the estimate

$$\|h\|_{H^0([0,T]\times\partial\Omega)} \leq C\|\mathbf{f}\|_{H^0(\Omega)\times H^{-1}(\Omega)}. \tag{3.36}$$

Remark. From the assumption on the wave front set of $\mathbf{f}$, we can assume that $\Lambda_a^+\mathbf{f}$ and $\Lambda_a^-\mathbf{f}$ have disjoint wave front set which is a consequence of the fact that bicharacteristics don't self intersect.

2. Parametrix at the boundary. We now want to have a pseudodifferential representation of the Dirichlet-to-Neumann map. Let's pick any $(x_0, \xi_0) \in S^*\bar{\Omega}$ and denote by $(t_1, x_1, 1, \xi_1) \in T^*(\mathbb{R} \times \Omega)$ the point in phase space where the bicharacteristic issued from $(0, x_0, 1, \xi_0)$ hits the boundary. We also denote by $(t_1, x_1, 1, (\xi_1)')$ its projection onto $T^*(\mathbb{R} \times \partial\Omega)$. Close to the boundary and in a neighborhood of x_1 we choose local coordinates $x = (x', x^n)$ such that $\partial\Omega$ is given by $x^n = 0$ and $x^n > 0$ in $\mathbb{R}^n \setminus \Omega$. The following analysis can also be done for the other bicharacteristic issued from $(0, x_0, 1, -\xi_0)$.

Consider a compactly supported distribution h on $\mathbb{R} \times \partial\Omega$ with $WF(h)$ contained in a small conic neighborhood of $(t_1, x_1, 1, (\xi^1)')$. As in [42] we can get an approximate solution to (3.4) outside Ω for the transmitted wave with positive wave speed $c(x)|\xi|$, as an FIO applied to h. Since $\partial\Omega$ is an invisible boundary for the forward wave, such transmitted wave would be the same (up to smooth function) as the solution u_+ defined above. The transmitted wave related to the positive wave speed has the form

$$u_T^+ = (2\pi)^{-n} \int e^{i\varphi_+(t,x,\tau,\xi')} b_+(t,x,\tau,\xi') \hat{h}_+(\tau,\xi') d\tau d\xi', \tag{3.37}$$

where $\hat{h}_+ = \int_{\mathbb{R}\times\mathbb{R}^{n-1}} e^{-i(-t\tau + x'\cdot\xi')} h(t,x') dt dx'$. In particular, the phase functions φ_+ must satisfy the eikonal equation plus boundary conditions on $x^n = 0$:

$$\partial_t \varphi_+ + c(x)|\nabla_x \varphi_+| = 0, \quad \varphi_+|_{x^n=0} = -t\tau + x'\cdot\xi', \tag{3.38}$$

where the choice of sign in the eikonal equation is such there is compatibility with ϕ^+ in (3.31). In the case of the negative sound speed, the transmitted wave u_T^- has the same form (3.37) but interchanging $\hat{h}_+$ with $\hat{h}_- = \int_{\mathbb{R}\times\mathbb{R}^{n-1}} e^{-i(+t\tau + x'\cdot\xi')} h(t,x') dt dx'$, and φ_+ with φ_- which satisfies

$$-\partial_t \varphi_- + c(x)|\nabla_x \varphi_-| = 0, \quad \varphi_-|_{x^n=0} = t\tau + x'\cdot\xi'. \tag{3.39}$$

As in the previous construction, we consider the amplitude functions to be classical, this is $b_\pm \sim \sum_{k\geq 0} b_k^\pm$ with $b_k^\pm$ homogeneous of degree $-k$ in τ and ξ'. It then follows that $b_\pm$ satisfy analogous equations to (3.33) with boundary conditions

$$b_0^\pm = 1, \quad b_k^\pm = 0,\ k \geq 1, \quad \text{at } x^n = 0.$$

We use the previous to locally define the Dirichlet-to-Neumann (DN) maps for the positive and negative wave speed as the following ΨDO's of order 1 (this is due to (3.38))

$$N_\pm : h \mapsto \frac{\partial u_T^\pm}{\partial x^n}\Big|_{\mathbb{R}\times\partial\Omega}.$$

This definition is local since it depends on the choice of local boundary coordinates. From (3.37) we get their principal symbols are the same and given by

$$\sigma(N_\pm) = \mathrm{i}\frac{\partial \varphi_\pm}{\partial x^n}\Big|_{\mathbb{R}\times\partial\Omega} = \mathrm{i}\sqrt{c^{-2}\tau^2 - |\xi'|^2},$$

thus they are elliptic in the hyperbolic conic set $c^{-1}|\tau| > |\xi'|$. In particular, it is positive because the symbol is supported near a null bicharacteristic for the symbol of $\partial_t^2 - c^2\Delta$, where $c^{-2}\tau^2 - |\xi'|^2 \approx |\xi_n|^2 > 0$. Notice that in contrast with [42], since the boundary doesn't perturb the propagation of the wave, there is no distinction between the incoming and outgoing DN maps.

Remark. The previous parametrix agrees up to smooth terms with the approximate solutions for the Cauchy problem u_+ and u_-, in a neighborhood of the boundary, when we take $h = \Lambda_a^{\pm}\mathbf{f}$ respectively.

3. Observability type estimate. We use the above approximate solutions to track the energy propagated near the singularities of the solution. This results in the next.

Proposition 3.5.1. *Let* $\mathbf{u}$ *be a solution of* (3.4) *with* $\Phi = 0$ *and initial condition in* $\mathcal{H}(\Omega)$. *There exists* $C > 1$ *so that for all* $\mathbf{f} \in \mathcal{H}(\Omega)$,

$$\|\mathbf{f}\|^2_{\mathcal{H}(\Omega)} \leq CE_{\Omega^c}(\mathbf{u}(T)).$$

Recall that for any bounded domain U with smooth boundary and for $t' \leq t \leq t''$ with $t' < t''$, if $\mathbf{u}$ is a solution of the damped wave equation then

$$\mathcal{E}_U(\mathbf{u}, t'') = \mathcal{E}_U(\mathbf{u}, t') + 2\Re \int_{[t',t'']\times\partial U} u_t \frac{\partial \overline{u}}{\partial \nu} dtdS, \tag{3.40}$$

where ν is the outward normal unit-vector to ∂U.

Let's assume for a moment that $WF(\mathbf{f})$ is contained in a conical neighborhood of some $(x_0, \xi_0) \in S^*\bar{\Omega}$ and as before we denote by $(t_1^{\pm}, x_1^{\pm})$ the times and points where the respective branches of the geodesic issued from (x_0, ξ_0) make contact with the boundary. We now want to estimate the energy transmitted outside Ω up to compact operator applied to $\mathbf{f}$, and at time $t = T$. For a large ball B, the energy of the solution $\mathbf{u} = e^{t\mathbf{P}_a}\mathbf{f}$ of (3.4) in $U = B \setminus \Omega$ is given by

$$E_{\Omega^c}(\mathbf{u}, t_2) = 2\Re \int_{[0,t_2]\times\partial\Omega} \frac{\partial u}{\partial t}\frac{\partial \overline{u}}{\partial \nu} dtdS. \tag{3.41}$$

where there is no term at time $t = 0$ since $\mathbf{u}(0)$ vanishes outside Ω. Here ν stands for the interior unit normal vector to $\partial\Omega$. Notice we used the hypothesis on the support of a in order to have the equality $E_{\Omega^c}(\mathbf{u}(t_2)) = \mathcal{E}_{\Omega^c}(\mathbf{u}, t_2)$.

Let's denote by h the Dirichlet data on $\partial\Omega$ given by $h = h_+ + h_-$, with $h_\pm = \Lambda_a^\pm \mathbf{f}$, and recall Remark 3.5.1. We can use the construction near the boundary from the previous section and get a representation of the transmitted wave, $\mathbf{u}_T = \mathbf{u}_T^+ + \mathbf{u}_T^-$, which satisfies that $\mathbf{u}_T \cong \mathbf{u}$, with $\cong$ meaning equality up to a smoothing operator applied to h. Notice this parametrix solutions are only constructed in neighborhoods of $(t_1^\pm, x_1^\pm)$. For times outside this neighborhoods, we can approximate $\mathbf{u}$ using the FIOs of section 4.1. Nevertheless, for such intervals of time we know the solution is smooth, therefore the more energetic part of $\mathbf{u}$ is contained precisely in $\mathbf{u}_T^+$ and $\mathbf{u}_T^-$. Then we can estimate the RHS of (3.41) and get, modulo compact operators applied to $h_\pm$,

$$\begin{aligned} E_{\Omega^c}(\mathbf{u}, T) &\cong 2\Re \int_{[0,T]\times\partial\Omega} \frac{\partial u_T}{\partial t}\frac{\partial \overline{u}_T}{\partial \nu} dtdS \\ &\cong 2\Re(P_t h, -(N_+ h^+ + N_- h^-)) \\ &\cong \sum_{\sigma=\pm} \Re(-2N_\sigma^* P_t h_\sigma, h_\sigma), \end{aligned} \tag{3.42}$$

where $(\cdot,\cdot)$ stands for the inner product in $L^2(\mathbb{R}\times\mathbb{R}^{n-1})$. Notice there is no cross terms between the functions $h_\pm$ in the right hand side. This is because the wave front set of $N_\pm^*$ is contained in a small neighborhood of $(t_1^\pm, x_1^\pm, 1, (\xi_1^\pm)')$, while the wave front set of $h_\mp$ lies close to $(t_1^\mp, x_1^\mp, 1, (\xi_1^\mp)')$. Since both bicharacteristics don't intersect each other the above wave front sets are disjoint and consequently $N_\pm^* P_t h_\mp$ are smooth functions. Therefore, those terms involve compact operators applied to the functions $h_\pm$.

From the definition of the DN map and the above energy relation we deduce that

$$E_{\Omega^c}(\mathbf{u}(T)) = \sum_{\sigma=\pm} \Re(M_\sigma h_\sigma, h_\sigma)$$

with $M_\pm$ two ΨDOs of order 2 with the same principal symbol

$$\begin{aligned}\sigma_p(M_\pm) = \sigma_p(-N_\pm^* P_t) &= -2 \cdot \overline{\mathrm{i}\sqrt{c^{-1}\tau^2 - |\xi'|^2}} \cdot (-\mathrm{i}\tau)\\ &= 2\tau\sqrt{c^{-2}\tau^2 - |\xi'|^2}.\end{aligned}$$

Notice that $h_\pm$ are compactly supported and as we mentioned before, their wave front sets are respectively contained in neighborhoods of the points $(t_1^\pm, x_1^\pm, 1, (\xi_1^\pm)')$ where the respective bicharacteristic through $(0, x_0, 1, \pm\xi_0)$ hits the boundary. Furthermore, their essential supports lie in the hyperbolic region $c|\xi'| < \tau$ due to the strictly convexity of Ω which makes the bicharacteristics to cross $\partial\Omega$ non-tangentially. Then, $\sigma(M_\pm) \geq C|(\tau, \xi')|^2$ and we can apply Garding's inequality to obtain

$$\begin{aligned}E_{\Omega^c}(\mathbf{u}(T)) &\geq C_1 \sum_{\sigma=\pm} \|h_\sigma\|^2_{H^1(\mathbb{R}\times\partial\Omega)} - C_2 \sum_{\sigma=\pm} \|h_\sigma\|^2_{H^0(\mathbb{R}\times\partial\Omega)}\\ &\geq C_1\|h\|^2_{H^1(\mathbb{R}\times\partial\Omega)} - C_2 \sum_{\sigma=\pm} \|h_\sigma\|^2_{H^0(\mathbb{R}\times\partial\Omega)}.\end{aligned} \tag{3.43}$$

Let $\mathbf{X} = \mathrm{diag}(X, X)$, with X a zero order ΨDO with essential support in a conic neighborhood of $(x_0, \xi_0) \in S^*\bar{\Omega} \setminus 0$, and such that $\mathbf{Xf}$ vanishes outside Ω. From the last inequality and by choosing $h = \Lambda_a \mathbf{Xf} = \Lambda_a^+ \mathbf{Xf} + \Lambda_a^- \mathbf{Xf}$, the continuity (3.36) and stability of the measurement operator (see the remark following Theorem 3.4.1) give us that

$$\|\mathbf{Xf}\|^2_{\mathcal{H}(\Omega)} \leq C E_{\Omega^c}(\mathbf{u}(T)) + C\|\mathbf{Xf}\|^2_{H^0(\Omega)\times H^{-1}(\Omega)}. \tag{3.44}$$

By compactness of $WF(\mathbf{f}) \cap S^*\bar{\Omega}$, in a conic neighborhood of such set we can consider a finite pseudo-differential partition of unity $1 = \sum \chi_j$ of symbols of ΨDO's X_j, localizing in conic neighborhoods of a finite number of points $(x_j, \xi^j) \in WF(\mathbf{f}) \cap S^*\bar{\Omega}$. Then, $\mathbf{f} = (I - \sum \mathbf{X}_j)\mathbf{f} + \sum \mathbf{X}_j\mathbf{f}$, where $WF(\mathbf{f}) \cap WF(I - \sum \mathbf{X}_j) = \emptyset$, thus from the inequality above we get

$$\|\mathbf{f}\|^2_{\mathcal{H}(\Omega)} \leq C \sum_j E_{\Omega^c}(e^{t\mathbf{P}_a}\mathbf{X}_j\mathbf{f}(T)) + C\|\mathbf{f}\|^2_{H^0(\Omega)\times H^{-1}(\Omega)}.$$

We can substitute $e^{t\mathbf{P}_a}\mathbf{X}_j\mathbf{f}$ by $Q_j\mathbf{X}_j\mathbf{f}$ in the right hand side, with Q_j denoting the parametrix operator for the wave equation localized near (x_j, ξ_j), since both are equal up to a compact operator applied to $\mathbf{f}$ (see [42, §4.10]). By means of Egorov's Theorem (see for instance [17, Theorem 10.1]) there exist zero order ΨDO's, $\tilde{\mathbf{X}}_j$, such that $Q_j\mathbf{X}_j = \tilde{\mathbf{X}}_jQ_j$ modulo a smoothing operator, therefore the exact solution $\mathbf{u} = e^{t\mathbf{P}_a}\mathbf{f}$ of (3.4) satisfies

$$\|\mathbf{f}\|_{\mathcal{H}(\Omega)} \leq C\|\mathbf{u}(T)\|_{H^1(\Omega^c)\oplus L^2(\Omega^c)} + C\|\mathbf{f}\|_{H^0(\Omega)\times H^{-1}(\Omega)}. \tag{3.45}$$

To get rid of the second term in the right hand side we use a classical argument that requires to show

$$\mathcal{H}(\Omega) \ni \mathbf{f} \mapsto \mathbf{u}(T) \in H^1(\mathbb{R}^n \setminus \Omega) \oplus L^2(\mathbb{R}^n \setminus \Omega)$$

is an injective bounded map. The continuity follows from [20, Proposition 1] when the domain is a large ball containing $\text{supp}(\mathbf{u}(T))$, which we know exists by finite speed of propagation. Assume there is $\mathbf{f} \in \mathcal{H}(\Omega)$ such that

$$u(T, x) = 0, \ \forall x \in \mathbb{R}^n \setminus \Omega.$$

By finite domain of dependence of the damped wave equation, u must vanish in $\{(t, x) \in (0, \infty) \times \mathbb{R}^n : \text{dist}(x, \partial\Omega) > |T - t|\}$, and since $\mathbf{u}(0, \cdot) = \mathbf{f}$ with $\text{supp}\mathbf{f} \subset \bar{\Omega}$, by the same reason $u = 0$ in $\{(t, x) \in (0, \infty) \times \mathbb{R}^n : \text{dist}(x, \partial\Omega) > t\}$. Intersecting both light cones we in fact have that u vanishes in $[0, 3T/2] \times \{x \in \mathbb{R}^n : \text{dist}(x, \partial\Omega) > T/2\}$. By Tataru's unique continuation [49, Theorem 4] and since we assume $T > T_0(\Omega) > 2T_1(\Omega)$, with

$$T_1(\Omega) = \sup_{x\in\Omega} d(x, \partial\Omega),$$

and $d(x, \partial\Omega)$ the infimum of the lengths of curves with respect to $c^{-2}dx^2$ starting at x and ending at $\partial\Omega$, we obtain $u = 0$ in a neighborhood of $\{3T/4\} \times \mathbb{R}^n$. From Proposition 1 in [20], it then follows by solving the initial value problem backward from $t = 3T/4$ to $t = 0$, that $u = 0$ in $[0, 3T/4] \times \Omega$, so in particular $\mathbf{f} \equiv 0$.

On the other hand, the inclusion $\mathcal{H}(\Omega) \hookrightarrow H^0(\Omega) \times H^{-1}(\Omega)$ is compact, so by Corollary 2.2.1, there is some $C > 1$ which depends on a, such that

$$\|\mathbf{f}\|_{\mathcal{H}(\Omega)} \le C\|\mathbf{u}(T)\|_{H^1(\mathbb{R}^n\setminus\Omega)\oplus L^2(\mathbb{R}^n\setminus\Omega)}.$$

To obtain the energy $E_{\Omega^c}(\mathbf{u}(T))$ in the right hand side of the last estimate and conclude the proof, we apply Poincare's inequality in $B \setminus \Omega$ with B a large ball containing Ω and such that $\mathbf{u}(T)$ vanishes in its complement (exists by finite speed of propagation).

4. Energy estimates. The error operator $\mathbf{K}$ can be characterized by $\mathbf{Kf} = (w(0,\cdot), w_t(0,\cdot))$, where $w = u - v$ is the error function satisfying the system

$$\begin{cases} (\partial_t^2 - c^2(x)\Delta + b(x))w = -a(x)(u_t + v_t)0, & (x,t) \in \Omega \times (0,T) \\ (w, w_t)|_{t=T} = (u|_{t=T} - \phi, u_t|_{t=T}), & x \in \Omega \\ w = 0, & (x,t) \in \partial\Omega \times (0,T). \end{cases}$$

Multiplying the above equation by $w_t = (u_t - v_t)$ and integrating on $(0,T) \times \Omega$, we get that the local energy of w satisfies

$$\begin{aligned} E_\Omega(\mathbf{w}(0)) &= E_\Omega(\mathbf{w}(T)) + 2\int_{[0,T]\times\Omega} ac^{-2}(u_t+v_t)(u_t-v_t)dtdx \\ &= E_\Omega(\mathbf{w}(T)) + 2\int_{[0,T]\times\Omega} ac^{-2}|u_t|^2 dtdx - 2\int_{[0,T]\times\Omega} ac^{-2}|v_t|^2 dtdx \\ &\le \|[u|_{t=T}-\phi, u_t|_{t=T}]\|^2_{\mathcal{H}(\Omega)} + 2\int_0^T\int_\Omega ac^{-2}|u_t|^2 dtdx. \end{aligned}$$

Moreover, since ϕ is harmonic and equal to $u|_{t=T}$ on $\partial\Omega$, we have

$$\|[u|_{t=T}-\phi, u_t|_{t=T}]\|^2_{\mathcal{H}(\Omega)} = E_\Omega(\mathbf{u}(T)) - \|\phi\|^2_{H_D(\Omega)}.$$

Then, gluing the above things together we get

$$E_\Omega(\mathbf{w}(0)) \le \mathcal{E}_\Omega(\mathbf{u}, T). \tag{3.46}$$

On the other hand, the previous proposition implies

$$\begin{aligned} \mathcal{E}_\Omega(\mathbf{u},T) = \mathcal{E}_{\mathbb{R}^n}(\mathbf{u},T) - \mathcal{E}_{\Omega^c}(\mathbf{u},T) &\le \|\mathbf{f}\|^2_{\mathcal{H}(\Omega)} - \|\mathbf{f}\|^2_{\mathcal{H}(\Omega)}/C \\ &\le (1-1/C)\|f\|^2_{\mathcal{H}(\Omega)}, \end{aligned}$$

where we used that the extended energy is preserved in $\mathbb{R}^n$, i.e. $\mathcal{E}_{\mathbb{R}^n}(\mathbf{u},T) = \mathcal{E}_{\mathbb{R}^n}(\mathbf{u},0)$. Finally, by recalling (3.46), we conclude that

$$\|\mathbf{Kf}\|^2_{\mathcal{H}(\Omega)} = E_\Omega(\mathbf{w}(0)) \le \mu\|\mathbf{f}\|^2_{\mathcal{H}(\Omega)},$$

with $0<\mu<1$, therefore $\mathbf{K}$ is a contraction in the norm induced by $\|\cdot\|_{\mathcal{H}(\Omega)}$ and there is convergence of the Neumann series. □

3.5.2 Integro-Differential Attenuation

For this case we restrict our analysis to the natural thermoacoustic initial conditions, this is, functions of the form $\mathbf{f} = (f, -af)$ for any $f \in H_D(\Omega)$. This choice is based on technical reasons and the problem of reconstructing a general initial source (f_1, f_2) remains open. The strategy is again to consider a dissipative TR system in such a way that the initial energy of the error function is bounded by the total energy (kinetic, potential and energy lost by attenuation) of the forward wave, inside the domain and at time T, analogously to (3.46). Such total energy in the whole space has the attribute of being conserved in time, fact that allows us to reduce the proof to an observability-type estimate (as in Proposition 3.5.1) involving the norm of the initial source and the energy of the forward wave outside Ω. The estimate says that at time T, a significant portion of the energy lies outside the domain.

Let's introduce the following convolution-type operator

$$[\Phi \tilde{*} v](s, x) = \int_s^T \Phi(t - s, x) v(t, x) dt, \tag{3.47}$$

which is just the adjoint operator of $\Phi * (\cdot)$ under the L^2 inner product in $(0, T)$, this is, for any L^2-functions u, v,

$$\langle \Phi * u, v \rangle_{L^2(0,T)} = \langle u, \Phi \tilde{*} v \rangle_{L^2(0,T)}. \tag{3.48}$$

Indeed, denoting by χ_I the indicator function in the interval $I \subset \mathbb{R}$,

$$\begin{aligned}
\int_0^T [\Phi * u](t) v(t) dt &= \int \int \chi(t)_{[0,T]} \chi(s)_{[0,t]} \Phi(t - s) u(s) v(t) ds dt \\
&= \int \int \chi(s)_{[0,T]} \chi(t)_{[s,T]} \Phi(t - s) u(s) v(t) ds dt \\
&= \int_0^T [\Phi \tilde{*} v](s) u(s) ds.
\end{aligned}$$

In the same way as in the proof of uniqueness, instead of working with u we set

$$\bar{u}(t,x) = \int_0^t u(s,x)ds,$$

and $\Psi(t,x)$ as in (3.8). Recalling our notation regarding the different attenuation processes in (3.2), we have $\mathcal{A}^1_\Psi = a\partial_t + \Psi * \partial_t$, therefore $\bar{u}$ satisfies

$$\begin{cases} \partial_t^2\bar{u} - c^2\Delta\bar{u} + p\bar{u} + \mathcal{A}^1_\Psi\bar{u} = 0 & \text{in } (0,T)\times\mathbb{R}^n, \\ \bar{u}|_{t=0} = 0 & \text{in } \mathbb{R}^n \\ \partial_t\bar{u}|_{t=0} = f & \text{in } \mathbb{R}^n \end{cases} \tag{3.49}$$

with $p(x) = b(x) - \Psi(x,0) \geq 0$. Notice we do not use (3.27) to obtain an equation as in (3.16) and we keep a derivative inside the convolution. Let's denote by $\bar{\Lambda}$ the observation operator corresponding to (3.49), this is

$$\bar{\Lambda} : L^2(\Omega; c^{-2}dx) \to H^1((0,T)\times\partial\Omega), \quad \bar{\Lambda}f = \bar{u}|_{(0,T)\times\partial\Omega}.$$

By well-posedness of the direct problem we have the following relation,

$$\bar{\Lambda}f = \int_0^t [\Lambda f](t)dt, \quad \forall f \in H_D(\Omega).$$

For the data $\bar{h} = \bar{\Lambda}f$ and denoting

$$\mathcal{B}^1_\Psi = -a\partial_t - \Psi\tilde{*}\partial_t,$$

we consider the solution v of the following dissipative TR system

$$\begin{cases} (\partial_t^2 - c^2\Delta + p + \mathcal{B}_\Psi^1)v &= 0 \text{ in } (0,T)\times\Omega, \\ v|_{t=T} &= \phi, \\ v_t|_{t=T} &= 0, \\ v|_{(0,T)\times\partial\Omega} &= \bar{h}, \end{cases} \tag{3.50}$$

with ϕ the harmonic extension of $\bar{h}(T,\cdot)$ in Ω. Notice that problem (3.50) is well-posed. This is due to the convolution term that involves values of v in the interval (t,T), thus by doing the change of variables $t \to T-t$ we get an IBVP of the form (3.49) which is uniquely solvable. The associated dissipative Time Reversal operator is then given by

$$A : H^1_{(0)}([0,T]\times\partial\Omega) \to L^2(\Omega; c^{-2}dx), \quad A\bar{h} = v_t(0,\cdot),$$

and we denote by K the error operator defined as

$$K : L^2(\Omega; c^{-2}dx) \to L^2(\Omega; c^{-2}dx), \quad K = \mathrm{Id} - A\bar{\Lambda},$$

and characterized by $Kf = w_t(0,\cdot)$, with $w = \bar{u} - v$, the error function that solves problem (3.51) below.

We recall that the reconstruction problem is in some sense a consequence of the injectivity of the inverse problem and the visibility of all the singularities from the boundary. Then, it is at least necessary to suppose the domain Ω is non-trapping (i.e. $T_1(\Omega) < \infty$) and measure the system until we receive information from every singularity. The result is the following.

Theorem 3.5.2. *Let Ω be strictly convex for the metric $g = c^{-2}dx^2$ and non-trapping. Assume that Ω and T are as in Theorem 2.1 (or as in Theorem 3.3.3). In addition, assume $T_1(\Omega) < T < \infty$ if $a \neq 0$ and $\frac{1}{2}T_1(\Omega) < T < \infty$ otherwise (resp. $2\alpha^{-1}c_0^{-1}D_\Omega < T < \infty$ and $\alpha^{-1}c_0^{-1}D_\Omega < T < \infty$, with α as in (3.29)). Then $A\bar{\Lambda} = Id - K$ with $\|K\|_{\mathcal{L}(L^2(\Omega;c^{-2}dx))} < 1$, and for any initial condition of (3.4) of the form $\mathbf{f} = (f, -af)$ with $f \in H_D(\Omega)$, the* *thermoacoustic inverse problem has a reconstruction formula given by*

$$f = \sum_{m=0}^{\infty} K^m A\bar{h}, \quad \bar{h} = \bar{\Lambda} f.$$

Proof. Notice the error function $w = \bar{u} - v$ satisfies the equation

$$\begin{cases} (\partial_t^2 - c^{-2}\Delta + p)w &= -a\bar{u}_t - av_t - \Psi * \partial_t \bar{u} - \Psi \tilde{*} \partial_t v \ \text{ in } (0,T)\times\Omega, \\ w|_{t=T} &= \bar{u}^T - \phi, \\ w_t|_{t=T} &= \bar{u}_t^T, \\ w|_{(0,T)\times\Gamma} &= 0. \end{cases} \tag{3.51}$$

with $\bar{u}^T = \bar{u}(T,\cdot)$ and $\partial_t \bar{u}^T = \partial_t \bar{u}(T,\cdot)$. Moreover, we can write

$$Kf = f - A\bar{h} = w_t(0), \quad \text{with} \quad \bar{h} = \bar{\Lambda}_\Psi f.$$

We want to estimate the norm of Kf, hence we need to compute the energy of w. Multiplying (3.51) by $2c^{-2}w_t$ and integrating over $(0,T)\times\Omega$ we obtain

$$\begin{aligned} E_\Omega(w,0) = E_\Omega(w,T) &+ 2\int_{[0,T]\times\Omega} ac^{-2}\bar{u}_t w_t dxdt + 2\int_{[0,T]\times\Omega} ac^{-2} v_t w_t dxdt \\ &+ 2\int_{[0,T]\times\Omega} c^{-2}(\Psi * \partial_t \bar{u}) w_t dxdt + 2\int_{[0,T]\times\Omega} c^{-2}\big(\Psi \tilde{*} \partial_t v\big) w_t dxdt \\ = E_\Omega(w,T) &+ 2\int_{[0,T]\times\Omega} ac^{-2}|\bar{u}_t|^2 dxdt - 2\int_{[0,T]\times\Omega} ac^{-2}|v_t|^2 dxdt \\ &+ 2\int_{[0,T]\times\Omega} c^{-2}(\Psi * \partial_t \bar{u})\partial_t \bar{u} dxdt - 2\int_{[0,T]\times\Omega} c^{-2}(\Psi \tilde{*} \partial_t v)\partial_t v dxdt \\ &- 2\int_{[0,T]\times\Omega} c^{-2}\left(\Psi * \partial_t \bar{u}\right)\partial_t v dxdt + 2\int_{[0,T]\times\Omega} c^{-2}\big(\Psi \tilde{*} \partial_t v\big)\partial_t \bar{u} dxdt. \end{aligned}$$

Neglecting the integration in the spatial variable in the last two terms for a moment, we can use the identity (3.48) which makes them cancel each other out. Furthermore, it follows from the same identity and Condition (3.7) on the kernels (which guarantees positive-definiteness) that

$$\int_{[0,T]\times\Omega} c^{-2}(\Psi\tilde{*}\partial_t v)\partial_t v dxdt = \int_{[0,T]\times\Omega} c^{-2}(\Psi * \partial_t v)\partial_t v dxdt \geq 0.$$

In consequence we get

$$\begin{aligned} E_\Omega(w,0) \leq E_\Omega(w,T) + 2\int_{[0,T]\times\Omega} ac^{-2}|\bar{u}_t|^2 dxdt \\ + 2\int_{[0,T]\times\Omega} c^{-2}(\Psi * \partial_t \bar{u})\partial_t \bar{u} dxdt. \end{aligned} \tag{3.52}$$

The choice of the time reversal system (3.51) helps to minimize the total energy in the dynamic satisfied by the error function w in a similar way as the functions ϕ helps to minimize the energy of w at time T. Indeed, by integration by parts we have that

$$(\bar{u}^T - \phi, \phi)_{H_D(\Omega)} = -\int_\Omega (\bar{u}^T - \phi)\Delta\phi dx + \int_{\partial\Omega} (\bar{u}^T - \phi)\partial_\nu \phi dS = 0,$$

therefore

$$E_\Omega(w(T)) = \|\bar{u}^T - \phi\|^2_{H_D(\Omega)} + \|\bar{u}^T_t\|^2_{L^2(\Omega)} = E_\Omega(\bar{u}(T)) - \|\phi\|^2_{H_D(\Omega)}. \tag{3.53}$$

From the above relations (3.52) and (3.53), we deduce

$$\|Kf\|^2_{L^2(\Omega;c^{-2}dx)} \leq E_\Omega(w,0) \leq \mathcal{E}_\Omega(\bar{u},T), \tag{3.54}$$

where recall the term in the right hand side is the extended energy functional associated to (3.4) and defined in (3.9). By conservation of the extended energy in $\mathbb{R}^n$,

$$\mathcal{E}_\Omega(\bar{u},T) = \mathcal{E}_{\mathbb{R}^n}(\bar{u},T) - E_{\Omega^c}(\bar{u},T) = \|f\|^2_{L^2(c^{-2}dx)} - E_{\Omega^c}(\bar{u},T). \tag{3.55}$$

The conclusion of the theorem follows from the next proposition which is known to hold when there is no integral term.

Proposition 3.5.2. *There is $C > 0$ so that for all $f \in L^2(\Omega; c^{-2}dx)$ and $\bar{u}$ solutions of*

(3.49),

$$\|f\|^2_{L^2(\Omega;c^{-2}dx)} \leq CE_{\Omega^c}(\bar{u}, T).$$

We again remark that in the absence of damping coefficient, the analysis of singularities can be decoupled to those following the positive and negative sound speed. The time needed then for the estimate to hold equals the time needed to get at least one signal from each singularity of the initial condition, this is $T > \frac{1}{2}T_1(\Omega)$. In contrast, the appearance of a damping term makes no longer possible such microlocal decoupling, and therefore it becomes necessary to wait until both signals, issued from every singularity of the initial condition, reach the boundary, or in other words $T > T_1(\Omega)$.

Let's prove the above proposition. Denote by $\bar{U}(x,t)$ the solution of

$$\begin{cases} (\partial_t^2 + a\partial_t - c^2\Delta + b)\bar{U}(t,x) = 0, \quad (t,x) \in (0,T) \times \mathbb{R}^n \\ \bar{U}|_{t=0} = 0, \\ \bar{U}_t|_{t=0} = f. \end{cases} \tag{3.56}$$

Denoting $\mathbf{f} = [0, f] \in \mathcal{H}(\Omega)$, in the presence of a damping ($a \neq 0$), it follows from Proposition 3.5.1 that there is $C > 0$ so that

$$\|f\|^2_{L^2(\Omega;c^{-2}dx)} = \|\mathbf{f}\|^2_{\mathcal{H}(\Omega)} \leq CE_{\Omega^c}(U, T),$$

whenever $T > T_1(\Omega)$. The same conclusion can be achieved in the absence of a damping by using instead the analogous to inequality (2.25) for initial conditions of the form $(0, f)$.

Furthermore, defining $\bar{W} = \bar{U} - \bar{u}$ we obtain

$$\|f\|^2_{L^2(\Omega;c^{-2}dx)} \leq C\left(E_{\Omega^c}(\bar{u}, T) + E_{\Omega^c}(\bar{W}, T)\right)$$

and letting $\bar{\mathbf{u}}(t) = [\bar{u}(t), \bar{u}_t(t)]$, $\bar{\mathbf{W}}(t) = [\bar{W}(t), \bar{W}_t(t)]$, the previous inequality implies

$$\|f\|_{L^2(\Omega;c^{-2}dx)} \leq C\|\bar{\mathbf{u}}(T)\|_{H^1(\Omega^c)\otimes L^2(\Omega^c)} + C\|\bar{\mathbf{W}}(T)\|_{H^1(\Omega^c)\otimes L^2(\Omega^c)},$$

where the error function $\bar{W}$ satisfies the IVP

$$\begin{cases} (\partial_t^2 + a\partial_t - c^2\Delta + b)\bar{W} = \Psi * \partial_t \bar{u}, \quad (t,x) \in (0,T) \times \mathbb{R}^n \\ \bar{W}|_{t=0} = 0, \\ \bar{W}_t|_{t=0} = 0. \end{cases} \tag{3.57}$$

We claim the bounded map $L^2(\Omega; c^{-2}dx) \ni f \mapsto \bar{\mathbf{u}}(T) \in H^1(\Omega^c) \otimes L^2(\Omega^c)$ is injective. In fact, it can be decomposed as the composition of two injective bounded maps, the first one being the observation operator $\bar{\Lambda}$, which is injective since (3.49) is equivalent (following the computation in (3.27)) to a system of the form (3.16) where the method used to prove Theorem 2.1 (resp. Theorem 3.3.3) can be applied, and our choice of $T > \frac{1}{2}T_1 \geq T_0$ (resp. $T > \alpha^{-1}c_0^{-1}R_\Omega \geq \frac{1}{2}T_1$). The second map is the exterior IBVP map that takes Dirichlet boundary data $\bar{h} \in H^1_{(0)}([0,T] \times \partial\Omega)$ to $\bar{\mathbf{v}}(T) \in H^1(\Omega^c) \otimes L^2(\Omega^c)$, where $\bar{v}$ solves:

$$\begin{cases} (\partial_t^2 - c^2\Delta)\bar{v}(t,x) = 0, \quad (t,x) \in (0,T) \times \mathbb{R}^n \setminus \overline{\Omega} \\ \bar{v}|_{t=0} = 0, \\ \partial_t \bar{v}|_{t=0} = 0, \\ \bar{v}|_{[0,T]\times\partial\Omega} = \bar{h}. \end{cases} \tag{3.58}$$

To see the injectivity of the latter map, consider $\bar{h} \in H^1_{(0)}([0,T] \times \partial\Omega)$ such that $\bar{v}(T) = \bar{v}_t(T) = 0$, with $\bar{v}$ solution of (3.58). By domain of dependence and reversibility in time of the exterior problem, we have that $\bar{v}$ vanishes in $\{(t,x) \in (0,\infty) \times \mathbb{R}^n \backslash \bar{\Omega} : \text{dist}_e(x,\partial\Omega) > t\}$ and also in $\{(t,x) \in (0,\infty) \times \mathbb{R}^n \backslash \bar{\Omega} : \text{dist}_e(x,\partial\Omega) > |T-t|\}$. Therefore

$$\bar{v} = 0 \quad \text{in} \quad \{(t,x) \in (0, 3T/2) \times \mathbb{R}^n \backslash \bar{\Omega} : \text{dist}_e(x,\partial\Omega) > T/2\}.$$

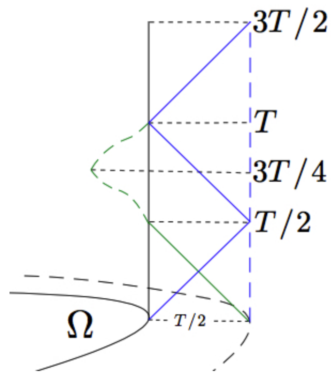

Figure 3.3: Unique continuation from points in $\{x \in \mathbb{R}^n \backslash \bar{\Omega} : \text{dist}_e(x, \partial\Omega) > T/2\}$ implies null Cauchy data on $(T/2, T) \times \partial\Omega$.

Applying Tataru's unique continuation theorem on any $p \in \{\text{dist}_e(x, \partial\Omega) > T/2\}$, we deduce that

$$\bar{v} = 0 \quad \text{in} \quad (\mathbb{R}^n \backslash \bar{\Omega}) \cap \{(t, x) \in (0, \infty) \times \mathbb{R}^n : |x - p| + |t - 3T/4| < 3T/4\},$$

which implies that $\bar{h}$ vanishes for $t \in (T/2, T)$ (see Figure 3.3). We can now apply the same argument replacing T by $T/2$ and get that $\bar{h}$ is null in the interval $(T/4, T/2)$. Iterating this process we finally conclude that $\bar{h} = 0$ for all $t \in (0, T)$.

Our second claim is that the map

$$L^2(\Omega; c^{-2}dx) \ni f \mapsto \bar{\mathbf{W}}(T) \in H^1(\Omega^c) \otimes L^2(\Omega^c)$$

is compact. It is in fact a composition of the bounded maps

$$L^2(\Omega; c^{-2}dx) \ni f \mapsto \bar{u}_t \in L^2((0, T); L^2(\mathbb{R}^n)), \quad \bar{u}_t \mapsto \bar{\mathbf{W}}(T) \in H^2(\Omega^c) \otimes H^1(\Omega^c)$$

and the compact embedding

$$H^2(\Omega^c) \otimes H^1(\Omega^c) \hookrightarrow H^1(\Omega^c) \otimes L^2(\Omega^c).$$

The continuity of the second map for those Sobolev spaces is due to [14, §7.2.3 Theorem 5] since denoting $F := \Psi * \bar{u}_t$, then $F, F_t \in L^2((0,T); L^2(\Omega^c))$. It follows from Corollary 2.2.1 that for a different constant

$$\|f\|_{L^2(\Omega;c^{-2}dx)} \leq C\|\bar{\mathbf{u}}(T)\|_{H^1(\Omega^c)\otimes L^2(\Omega^c)}.$$

The proposition is then proved by recalling the finite speed of propagation and applying Poincare's inequality on a large ball minus Ω.

We conclude the proof of Theorem 3.5.2 by joining (3.54), (3.55) and Proposition 3.5.2. $\square$

BIBLIOGRAPHY

[1] S. Acosta and C. Montalto. Multiwave imaging in an enclosure with variable wave speed. *Inverse Problems*, 31(6):065009, 12, 2015.

[2] S. Acosta and C. Montalto. Photoacoustic imaging taking into account thermodynamic attenuation. *Inverse Problems*, 32(11):115001, 2016.

[3] S. Acosta and B. Palacios. Thermoacoustic tomography for an integro-differential wave equation modeling attenuation. *J. Differential Equations*, 264(3):1984–2010, 2018.

[4] M. Agranovsky, P. Kuchment, and L. Kunyansky. *On reconstruction formulas and algorithms for the thermoacoustic and photoacoustic tomography*. Photoacoustic imaging and spectroscopy. CRC Press, 2009.

[5] M. A. Anastasio, J. Zhang, D. Modgil, and P. J. La Rivière. Application of inverse source concepts to photoacoustic tomography. *Inverse Problems*, 23(6):S21, 2007.

[6] G. Bal. Hybrid inverse problems and internal functionals. *Inverse Problems and Applications: Inside Out II*, 60:325–368, 2011.

[7] G. Bal, K. Ren, G. Uhlmann, and T. Zhou. Quantitative thermo-acoustics and related problems. *Inverse Problems*, 27(5):055007, 2011.

[8] C. Bardos, G. Lebeau, and J. Rauch. Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. *SIAM J. Control Optim.*, 30(5):1024–1065, 1992.

[9] A. L. Bukhgeim, G. V. Dyatlov, and G. Uhlmann. Unique continuation for hyperbolic equations with memory. *J. Inverse Ill-Posed Probl.*, 15(6):587–598, 2007.

[10] M. Caputo and M. Fabrizio. A new definition of fractional derivative without singular kernel. *Progr. Fract. Differ. Appl*, 1(2):1–13, 2015.

[11] W. Chen and S. Holm. Modified Szabo's wave equation models for lossy media obeying frequency power law. *The Journal of the Acoustical Society of America*, 114(5):2570–2574, 2003.

[12] O. Chervova and L. Oksanen. Time reversal method with stabilizing boundary conditions for photoacoustic tomography. *Inverse Problems*, 32(12):125004, 16, 2016.

[13] C. Dafermos. An abstract Volterra equation with applications to linear viscoelasticity. *J. Differential Equations*, 7:554–569, 1970.

[14] L. C. Evans. Partial differential equations. *Graduate Studies in Mathematics*, 19, 1998.

[15] D. Finch, M. Haltmeier, and Rakesh. Inversion of spherical means and the wave equation in even dimensions. *SIAM J. Appl. Math.*, 68(2):392–412, 2007.

[16] D. Finch, S. Patch, and Rakesh. Determining a function from its mean values over a family of spheres. *SIAM J. Math. Anal.*, 35(5):1213–1240, 2004.

[17] A. Grigis and J. Sjöstrand. *Microlocal analysis for differential operators*, volume 196 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1994. An introduction.

[18] M. Haltmeier. Universal inversion formulas for recovering a function from spherical means. *SIAM J. Math. Anal.*, 46(1):214–232, 2014.

[19] M. Haltmeier, R. Kowar, and L.V. Nguyen. Iterative methods for photoacoustic tomography in attenuating acoustic media. *Inverse Problems*, 33(11):115009, 2017.

[20] A. Homan. Multi-wave imaging in attenuating media. *Inverse Problems and Imaging*, 7(4):1235–1250, 2013.

[21] L. Hörmander. A uniqueness theorem for second order hyperbolic differential equations. *Comm. Partial Differential Equations*, 17(5-6):699–714, 1992.

[22] Lars Hörmander. On the existence and the regularity of solutions of linear pseudo-differential equations. *Enseignement Math. (2)*, 17:99–163, 1971.

[23] Y. Hristova. Time reversal in thermoacoustic tomography—an error estimate. *Inverse Problems*, 25(5):055008, 14, 2009.

[24] Y. Hristova, P. Kuchment, and L. Nguyen. Reconstruction and time reversal in thermoacoustic tomography in acoustically homogeneous and inhomogeneous media. *Inverse Problems*, 24(5):055006, 2008.

[25] C. Huang, L. Nie, R. Schoonover, L. Wang, and M. Anastasio. Photoacoustic computed tomography correcting for heterogeneity and attenuation. *Journal of Biomedical Optics*, 17(6):061211–1–061211–5, 2012.

[26] C. Huang, K. Wang, L. Nie, L.V. Wang, and M. Anastasio. Full-wave iterative image reconstruction in photoacoustic tomography with acoustically inhomogeneous media. *Medical Imaging, IEEE Transactions on*, 32(6):1097–1110, 2013.

[27] R. Kowar and O. Scherzer. Attenuation models in photoacoustics. *Mathematical modeling in biomedical imaging II: Optical, ultrasound, and opto-acoustic tomographies*, 2035:85–129, 2012.

[28] P. Kuchment and L. Kunyansky. Mathematics of photoacoustic and thermoacoustic tomography. In Otmar Scherzer, editor, *Handbook of Mathematical Methods in Imaging*, pages 817–865. Springer New York, 2011.

[29] L. Kunyansky. Reconstruction of a function from its spherical (circular) means with the centers lying on the surface of certain polygons and polyhedra. *Inverse Probl.*, 27(2):025012, 2011.

[30] L. Kunyansky L. V. Nguyen. A dissipative time reversal technique for photo-acoustic tomography in a cavity. *SIAM Journal on Imaging Sciences*, 9(2):748–769, 2016.

[31] P. Kuchment M. Agranovsky and L. Kunyansky. On reconstruction formulas and algorithms for the thermoacoustic and photoacoustic tomography. *Photoacoustic imaging and spectroscopy. CRC Press*, chapter 8:89–101, 2009.

[32] R. C. MacCamy and J. S. Wong. Stability theorems for some functional equations. *Trans. Amer. Math. Soc.*, 164:1–37, 1972.

[33] D. Modgil, B. Treeby, and P. La Rivière. Photoacoustic image reconstruction in an attenuating medium using singular-value decomposition. *Journal of biomedical optics*, 17(6):0612041–0612048, 2012.

[34] F. Natterer. Photo-acoustic inversion in convex domains. *Inverse Probl. Imaging*, 6(2):315–320, 2012.

[35] B. Palacios. Reconstruction for multi-wave imaging in attenuating media with large damping coefficient. *Inverse Problems*, 32(12):125008, 15, 2016.

[36] V. Palamodov. A uniform reconstruction formula in integral geometry. *Inverse Problems*, 28(6):065014, 2012.

[37] S.K. Patch and M. Haltmeier. Thermoacoustic tomography-ultrasound attenuation artifacts. In *Nuclear Science Symposium Conference Record, 2006. IEEE*, volume 4, pages 2604–2606. IEEE, 2006.

[38] H. Roitner and P. Burgholzer. Efficient modeling and compensation of ultrasound attenuation losses in photoacoustic imaging. *Inverse Problems*, 27(1):015003, 15, 2011.

[39] M. Haltmeier S. K. Patch. Thermoacoustic tomography - ultrasound attenuation effects. *IEEE Nuclear Science Symposium Conference Record*, 4:2604–2606, 2006.

[40] L. Šeliga and M. Slodička. An inverse source problem for a damped wave equation with memory. *J. Inverse Ill-Posed Probl.*, 24(2):111–122, 2016.

[41] P. Stefanov and G. Uhlmann. Thermoacoustic tomography with variable sound speed. *Inverse Problems*, 25(7):075011, 16, 2009.

[42] P. Stefanov and G. Uhlmann. Thermoacoustic tomography arising in brain imaging. *Inverse Problems*, 27(4):045004, 26, 2011.

[43] P. Stefanov and G. Uhlmann. Multi-wave methods via ultrasound. *Inverse Problems and Applications, Inside Out II, MSRI Publications*, 60:271–323, 2012.

[44] P. Stefanov and G. Uhlmann. Recovery of a source term or a speed with one measurement and applications. *Trans. Amer. Math. Soc.*, 365(11):5737–5758, 2013.

[45] P. Stefanov, G. Uhlmann, and A. Vasy. Boundary rigidity with partial data. *J. Amer. Math. Soc.*, 29(2):299–332, 2016.

[46] P. Stefanov and Y. Yang. Multiwave tomography in a closed domain: averaged sharp time reversal. *Inverse Problems*, 31(6):065007, 23, 2015.

[47] P. Stefanov and Y. Yang. Multiwave tomography in a closed domain: averaged sharp time reversal. *Inverse Problems*, 31(6):065007, 23, 2015.

[48] T. Szabo. Time domain wave equations for lossy media obeying a frequency power law. *The Journal of the Acoustical Society of America*, 96(1):491–500, 1994.

[49] D. Tataru. Unique continuation for solutions to PDE's; between Hörmander's theorem and Holmgren's theorem. *Comm. Partial Differential Equations*, 20(5-6):855–884, 1995.

[50] D. Tataru. Unique continuation for operators with partially analytic coefficients. *J. Math. Pures Appl. (9)*, 78(5):505–521, 1999.

[51] M. Taylor. Pseudodifferential operators. *(Princeton mathematical series ; 34). Princeton, N.J.: Princeton University Press.*, 1981.

[52] J. Tittelfitz. Thermoacoustic tomography in elastic media. *Inverse Problems*, 28(5):055004, 2012.

[53] B. Treeby and B. T. Cox. Modeling power law absorption and dispersion for acoustic propagation using the fractional Laplacian. *The Journal of the Acoustical Society of America*, 127(5):2741–2748, 2010.

[54] B. Treeby, E. Zhang, and B. Cox. Photoacoustic tomography in absorbing acoustic media using time reversal. *Inverse Problems*, 26(11):115003, 2010.

[55] F. Trèves. *Introduction to pseudodifferential and Fourier integral operators. Vol. 2.* Plenum Press, New York-London, 1980. Fourier integral operators, The University Series in Mathematics.

[56] K. Wang and M. Anastasio. A simple Fourier transform-based reconstruction formula for photoacoustic computed tomography with a circular or spherical measurement geometry. *Physics in Medicine and Biology*, 57(23):N493, 2012.

Appendices

Appendix A

SOME TOOLS FROM MICROLOCAL ANALYSIS

A.1 Symbols, ΨDO's and FIO's

Definition A.1.1 (Space of symbols of order m)**.** For any $X \subseteq \mathbb{R}^d$ open and $m \in \mathbb{R}$, we define $S^m(X \times \mathbb{R}^N)$ as the Fréchet space of all smooth functions $a \in C^\infty(X \times \mathbb{R}^N)$ such that for all compact $K \subseteq X$ and for all $\alpha \in \mathbb{N}^d$, $\beta \in \mathbb{N}^N$, there is a constant $C = C_{K,\alpha,\beta}$ such that

$$|\partial_x^\alpha \partial_\theta^\beta a(x,\xi)| \leq C(1+|\xi|)^{m-|\beta|}, \quad (x,\xi) \in K \times \mathbb{R}^N.$$

We also define $S^{-\infty}(X \times \mathbb{R}^N) = \cap_{m\in\mathbb{R}} S^m(X \times \mathbb{R}^N)$.

Lemma A.1.1 (Borel's lemma)**.** *Let $\{m_k\}_{k\geq 0} \subset \mathbb{R}$ be a sequence decreasing to $-\infty$. Given $p_k \in S^{m_k}(X \times \mathbb{R}^N)$ for $k \geq 0$, there exists a unique $p \in S^{m_0}(X \times \mathbb{R}^N)$ modulo $S^{-\infty}(X \times \mathbb{R}^N)$ such that $p \sim \sum_{k=0}^\infty p_k$, this is, $p - \sum_{k=0}^n p_k \in S^{m_{n+1}}(X \times \mathbb{R}^N)$ for all $n \geq 0$.*

Definition A.1.2 (Pseudodifferential operators of order m)**.** Let $X \subset \mathbb{R}^n$ open and $m \in \mathbb{R}$. Given a symbol $a(x,\xi) \in S^m(X \times \mathbb{R}^N)$, we define a pseudodifferential operator $A : C_0^\infty(X) \to \mathcal{D}'(X)$ of order m, as

$$Au(x) = \frac{1}{(2\pi)^n} \int e^{\mathrm{i}x\cdot\xi} a(x,\xi)\hat{u}(\xi) d\xi,$$

where $\hat{u}$ stands for the Fourier transform of u. The set of all pseudodifferential operators of order m is denoted by $\Psi^m(X)$, and $\Psi^{-\infty}(X) = \cap_{m\in\mathbb{R}} \Psi^m(X)$.

Definition A.1.3 (Classical ΨDO's)**.** We denote by $\Psi^m_{cl}(X)$ the set of all operators in $\Psi^m(X)$ whose symbols have the asymptotic expansion

$$a(x,\xi) \sim \chi(\xi) \sum_{j=0}^\infty a_{m-j}(x,\xi),$$

for some cut-off function $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ such that $\chi(\xi) = 0$ near $\xi = 0$ and $\chi(\xi) = 1$ for $|\xi| \geq 1$, and where $a_{m-j}(x, \xi) \sin S^{m-j}(X \times \mathbb{R}^N)$ are positively homogeneous in ξ of order $m - j$, this is,

$$a_{m-j}(x, \lambda\xi) = \lambda^{m-j} a_{m-j}(x, \xi), \quad \forall\, x \in X,\ \xi \in \mathbb{R}^N \backslash 0.$$

Definition A.1.4. A pseudodifferential operator A with principal symbol $a(x, \xi)$ is called of real principal type in $X \subset \mathbb{R}^n$ if, $a(x, \xi)$ is real-valued and for all $(x, \xi) \in T^*X\backslash 0$, the form $d_{x,\xi} a(x, \xi)$ is not proportional to the form ξdx.

Definition A.1.5 (Phase function)**.** Let $X \subset \mathbb{R}^n$ open. A function $\varphi(x, \xi) \in C^\infty(X \times (\mathbb{R}^N \backslash 0))$ is called a phase function if for all $(x, \xi) \in X \times (\mathbb{R}^N \backslash 0)$:

- $\mathfrak{Im}\varphi(x, \xi) \geq 0$,
- $\varphi(x, \lambda\xi) = \lambda\varphi(x, \xi)$ for all $\lambda > 0$,
- $d\varphi \neq 0$.

Definition A.1.6 (Fourier Integral Operators)**.** Let $X \subset \mathbb{R}^{n_X}$, $Y \subset \mathbb{R}^{n_Y}$ and $m \in \mathbb{R}$. Given a phase function φ on $X \times Y \times (\mathbb{R}^N \backslash 0)$ and $a \in S^m(X \times Y \times \mathbb{R}^N)$, we say that $A : C_0^\infty(Y) \to \mathcal{D}'(X)$ is a Fourier Integral Operator if it is given by

$$Au(x) = \int\int e^{\mathrm{i}\varphi(x,y,\theta)} a(x, y, \theta) u(y) dy d\theta, \quad u \in C_0^\infty(Y).$$

A.2 Garding's inequality

Let $X \subset \mathbb{R}^n$ open and $K \subset X$ compact. Let $A \in \Psi_{cl}^{2m}(X)$ properly supported, $m \geq 0$, with principal symbol $a(x, \xi)$ satisfying

$$\mathfrak{Re}\, a(x, \xi) \geq 0.$$

Then, there is a constant $C(K)$ such that

$$\mathfrak{Re}(Au, u) \geq -C\|u\|^2_{H^{m-1/2}},$$

for all smooth functions compactly supported in K.

In the case that for some $C_0 > 0$

$$\mathfrak{Re}\, a(x,\xi) \geq C_0|\xi|^{2m},$$

then, there is $C(K)$ such that

$$\mathfrak{Re}(Au,u) \geq \frac{1}{C}\|u\|_{H^m}^2 - C\|u\|_{H^0}^2.$$

Appendix B

WELL-POSEDNESS OF (??)

For the existence of solutions we follows the proof of [13, Theorem 2.1]. Let's assume without lost of generality that $u_0 = 0$. For a fixed $t_0 \in (0, T]$ let

$$\mathcal{E}_{t_0} = \{v(t)|v(t) \in C^\infty([0, t_0]; H_0^1(U)), v(0) = 0\},$$

with two inner product given by

$$(v, w)_1 := \int_0^{t_0} \Big\{(v_t(t), w_t(t))\rangle + (\nabla v(t), \nabla w(t))\Big\} dt$$

and

$$(v, w)_2 := (v, w)_1 + t_0(v_t(0), w_t(0)),$$

and respective norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Let F_{t_0} be the completion of $\mathcal{E}_{t_0}$ under the norm $\|\cdot\|_1$. It can be proved, for instance by Stone-Weierstrass, that $u \in F_{t_0}$ is a generalized solution in the interval $[0, t_0]$ if and only if

$$\mathcal{B}(u, v) = \mathcal{D}(f, v) + t_0(c^{-2}u_1, v_t(0))_{L^2(U)}, \quad \forall v \in \mathcal{E}_{t_0}, \tag{B.1}$$

where

$$\begin{aligned}\mathcal{B}(u,v) &= \int_0^{t_0}(t-t_0)\Big[(c^{-2}u_t(t),v_{tt}(t)) - (\nabla u(t),\nabla v(t)) - (c^{-2}au_t(t),v_t(t))\\ &\quad -(c^{-2}bu(t),v_t(t)) - \int_0^t (c^{-2}\Phi(t-\tau)u_\tau(\tau),u_t(t))d\tau\Big]dt\\ &\quad +\int_0^{t_0}(c^{-2}u_t(t),v_t(t))dt,\\ \mathcal{D}(f,v) &= -\int_0^{t_0}(t-t_0)(c^{-2}f(t),v_t(t))dt.\end{aligned}$$

where (B.1) is obtained by using the test function $(t-t_0)v_t(t)$ with $v\in\mathcal{E}_{t_0}$, in (3.6). Notice that applying integration by parts we get that the bilinear form $\mathcal{B}$ satisfies that for all $v\in\mathcal{E}_{t_0}$ (recall $v(0)=0$),

$$\begin{aligned}\mathcal{B}(v,v) =&\frac{1}{2}\int_0^{t_0}\left[(c^{-2}v_t(t),v_t(t)) + (\nabla v(t),\nabla v(t)) + (c^{-2}bv(t),v(t))\right]dt\\ &-\int_0^{t_0}(t-t_0)\Big[(c^{-2}av_t(t),v_t(t)) - (c^{-2}\Phi(0)v(t),v(t))\\ &\qquad -\int_0^t (c^{-2}\Phi(t-s)v(s),v(t))ds\Big]dt\\ &+\frac{t_0}{2}(c^{-2}v_t(0),v_t(0)).\end{aligned}$$

Therefore, recalling that $0<c_0\le c\le c_0^{-1}$, we bound from bellow and choosing $t_0>0$ small enough and using Poincare's inequality we get

$$\mathcal{B}(v,v) \ge\frac{1}{2}\min\{1,c_0^2\}\|v\|_2^2 - Ct_0\big([\|a\|_\infty + \|\Phi(0)\|_\infty + t_0\|\Phi\|_\infty\big)\|v\|_1^2 \ge \delta\|v\|_2^2.$$

for some $\delta>0$.

On the other hand,

$$|\mathcal{D}(f,v)| \le t_0c_0^{-2}\|f\|_{L^2}\|v\|_2$$

$$|t_0(c^{-2}u_1,v_t(0))| \le t_0^{1/2}c_0^{-2}\|u_1\|_{L^2}\|v\|_2$$

Then, similarly as in [**?**, Chap. III, Theorem 1.1], we get the existence of weak solutions on the interval $[0, t_0]$. Iterating this argument for the intervals $[t_0, 2t_0]$, $[2t_0, 3t_0]$ etc, we conclude the existence on $[0, T]$. The uniqueness follows the same ideas as in [13, Theorem 2.2].