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Pitch-class set multiplication in Boulez's "Le Marteau sans maître" with [Original composition]

Heinemann, Stephen John, D.M.A.

University of Washington, 1993

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Pitch-class Set Multiplication in
Boulez's *Le Marteau sans maître*

by
Stephen Heinemann

A dissertation submitted in partial fulfillment
of the requirements for the degree of

Doctor of Musical Arts

University of Washington

1993

Approved by  _____
(Chairperson of Supervisory Committee)

Program Authorized
to Offer Degree _____ School of Music

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Abstract

Pitch-class Set Multiplication in
Boulez's *Le Marteau sans maître*

by Stephen Heinemann

Chairperson of the Supervisory Committee:

Professor Diane Thome

School of Music

Since its 1955 première, Pierre Boulez's *Le Marteau sans maître* has engendered a great deal of discourse, most notably in analyses by Lev Koblyakov, related to both its compositional processes and its serialist aesthetics. The exact nature of the process of pitch-class set multiplication that generates the pitch classes of the first cycle of *Le Marteau* has never been fully explored, with the result that the process appears to have an ad hoc quality in which the transpositional type of a multiplicative result is fixed but its pitch-class content is not. Coinciding with this appearance is the question of why a composer of Boulez's skill would care to use the process at all. The theory presented here not only refutes the appearance of capriciousness, but also demonstrates the attractiveness that the

operation would have for a composer--an extension of traditional serialism that permits the generation of a large number of different, yet interrelated, unordered pitch-class sets.

This study formalizes pitch-class set multiplication in three configurations. The first, called "simple multiplication," involves the construction of one operand set's intervallic structure on each pitch class of another operand set; this operation is examined not only as it pertains to Boulez, but also as a compositional/analytical tool that parallels, in pitch-class-specific terms, Richard Cohn's theory of transpositional combination. The second, "compound multiplication," illustrates schemata for the transposition of a simple multiplicative product. The third, "complex multiplication," is the elegant operation that generates pitch-class sets in the first cycle of *Le Marteau*; like arithmetical multiplication, this process is commutative. The operation, the twelve-tone row to which it is applied, and the resulting pitch-class sets are then examined with respect to their implications for a process-based listening strategy.

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1. Raging Craftsmanship

In his controversial article "Schoenberg Is Dead,"¹ Pierre Boulez described his perception of the Viennese master's failure to grasp the larger implications of the harmonic revolution he had unleashed:

In Schoenberg's serial works. . . the confusion between theme and series is explicit enough to show his impotence to foresee the sound-world that the series demands. Dodecaphonism, then, consists of only a rigorous law for controlling chromatic writing; playing only the role of regulating instrument, the serial phenomenon itself was not, so to speak, perceived by Schoenberg.

What, then, was his ambition, once the chromatic synthesis had been established through the series, or in other words, once this coefficient of security had been adopted? To construct works of the same essence as that of those in the sound-universe he had just left behind, works in which the new technique of writing should "prove its worth."²

Schoenberg's thematic employment of the series, regarded here as oxymoronic, led Boulez (and his contemporaries) to the music of Webern, the composer he believed to hold a "privileged position among the three

¹Pierre Boulez, "Schoenberg Is Dead," *The Score* (London), May 1952. Retranslated in Boulez, *Notes of an Apprenticeship*, ed. Paule Thévenin, trans. Herbert Weinstock (New York: Alfred A. Knopf, 1968), pp. 268-276.

²*Ibid.*, pp. 271-272.

Viennese. . . [because he] reacted in the direction of rehabilitating the power of sound and against all inherited rhetoric."³ Having himself inherited Webern's rhetoric, Boulez sought to expand its implications. His path led him first to total serialism, as evidenced in his 1952 work *Structures*, Book I; by 1961, in *Pli selon pli*, elements of chance had insinuated themselves. Between these crucial pieces is the one that, for many, represents the consummate work of Boulez's compositional career--the chamber song cycle *Le Marteau sans maître*.

Le Marteau was largely completed in 1953 and 1954. The first performance was originally scheduled for the 1954 Donaueschingen Festival (site, four years earlier, of the stormy première of *Polyphonie X*), but was postponed due to an instrumentalist's illness. *Le Marteau* was not heard publicly until June 18, 1955 in Baden-Baden, where it was received with great enthusiasm. In the interim between the scheduled and actual premières, Boulez indulged an idiosyncrasy by revising the score--the most radical change being the addition of the ninth movement. Further revisions gave the score its final version in 1957.⁴

³Boulez, "Incipit," in *Notes of an Apprenticeship*, pp. 277-279.

⁴Dominique Jameux, *Pierre Boulez*, trans. Susan Bradshaw (Cambridge: Harvard University Press, 1991), pp. 73-74.

That the piece was composed according to certain tenets of serialism seems never to have been in doubt; Boulez had already established himself as a composer of serial works, and certain passages of *Le Marteau*, such as the first measures of the fifth movement, permit a traditional twelve-counting. There are, however, far more exceptions than adherences to mainstream serialism, and it is highly unlikely that any correct accounting of the work's pitch-class structure would have appeared were it not for Boulez's theoretical writings. The most important of these (at least as far as *Le Marteau* is concerned) is the chapter entitled "Musical Technique" which forms the bulk of the book *Boulez on Music Today*.⁵ Here the composer delineated certain aspects of an operation applied to pitch-class sets which he called "multiplication."⁶ Theorist Lev Koblyakov, apparently using *Boulez on Music Today* as his starting point, was able to give the first convincing account of the applicability of this operation to *Le Marteau*, specifically to the work's

⁵Boulez, *Boulez on Music Today*, trans. Susan Bradshaw and Richard Rodney Bennett (Cambridge: Harvard University Press, 1971), pp. 35-143. The title and intent of the chapter recall *Technique de mon langage musical* by Olivier Messiaen (Paris: Alphonse Leduc, 1944), Boulez's principal teacher.

⁶Ibid., pp. 39-40, 79-80. This operation differs greatly from the familiar M1, M5, M7 and M11 transforms, the "multiplicative operations" discussed in John Rahn, *Basic Atonal Theory* (New York and London: Longman, 1980).

first cycle--the three movements relating to René Char's surrealist poem "L'Artisanat furieux" ("Raging Craftsmanship").⁷

Koblyakov's contribution came in stages. He first published an English-language article in the (now-defunct) German journal *Zeitschrift für Musiktheorie* in 1977;⁸ in 1981 he completed his dissertation, "The World of Harmony of Pierre Boulez: Analysis of *Le marteau sans maître*" at the Hebrew University of Jerusalem. The 1977 article posed considerable obstacles, created in more or less equal parts by the novelty of the theory, by language difficulties, by the journal's poor printing quality which obscured many examples, and by Koblyakov's vagueness as to certain important matters of technique and terminology. Some of these problems were resolved, to an extent, in his dissertation, but even this did not gain currency until its publication, nine years later, as *Pierre Boulez: A World of Harmony*.⁹

The importance of Koblyakov's contributions to the theoretical literature on *Le Marteau* cannot be overempha-

⁷The nine movements of *Le Marteau* are organized as three overlapping cycles: the cycle of "L'Artisanat furieux," consisting of the first, third, and seventh movements; the cycle of "Bourreaux de solitude" ("Executioners of solitude"), the even-numbered movements; and the cycle of "Bel édifice et les pressentiments" ("Handsome building and forebodings"), the fifth and ninth movements.

⁸Lev Koblyakov, "P. Boulez 'Le marteau sans maître,' analysis of pitch structure," *Zeitschrift für Musiktheorie* 8/1 (1977), pp. 24-39.

⁹Idem, *Pierre Boulez: A World of Harmony* (New York: Harwood Academic Publishers, 1990).

sized, but his published analyses are by no means complete. He is especially vague--even evasive--with regard to the actual workings of multiplication, despite the significance he attaches to the operation;¹⁰ one must conclude that he was unable to decipher its methodology. Neither Boulez's nor Koblyakov's writings deal with the choices of *specific* pitch-class sets representing multiplicative results. These choices contain apparent contradictions, the foremost being that such results come from an operation that, like arithmetical multiplication, is commutative. These contradictions can, in turn, lead to these perceptions: the operation is arbitrary; nothing more specific than the T_n type of a multiplicative result can be predicted; the choice of the pc set that represents this T_n type is capricious. And underlying such perceptions is an aesthetically fundamental question: What compelled Boulez to use the operation at all? In this study, the process is detailed in such a way that all of these problematic areas can be addressed satisfactorily.

Pierre Boulez wrote:

The world of music today is a *relative* world, that is to say, one where structural relationships are not defined once and for all according to absolute criteria,

¹⁰Ibid., pp. 31-33. Koblyakov lists nine works following *Le Marteau* in which multiplication is applied: the Third Sonata, *Structures II*, *Don*, *Tombeau*, *Eclat*, *Eclat/Multiples*, *Figures-Doubles-Prismes*, *Domaines*, and *cummings ist der Dichter* (*Pierre Boulez: A World of Harmony*, p. 32).

but are organized instead according to various schemata.¹¹

One of these schemata is closely inspected here. Multiplication merits such an examination, not merely as the generating process for the pitch-class sets of the first cycle, but also as an embryonic form of serialism with its own attributes. With the theory presented herein, composers wishing to duplicate Boulez's precompositional process (that is, to generate the same pitch-class sets or, by extension, sets of their own design) will now be able to do so; music theorists can apply this work toward other pieces in Boulez's oeuvre.¹²

In this study, multiplication is first approached in a general way through transpositional combination. Richard Cohn's theory, among its many other virtues, provides a useful introduction to and explains the more abstract properties of Boulez's technique; see Chapter 3. The operation is then formalized in three configurations (the first two of which are delineated in Boulez's writings), here called: *simple multiplication*, wherein intervallic structures derived from one pitch-class set are constructed on each pc of another (Chapter 4); *compound multiplication*, in which pc sets resulting from simple multiplication are transposed

¹¹Boulez, *Boulez on Music Today*, p. 35.

¹²See note 10 above. Although the scope of this study is, like Koblyakov's, restricted primarily to *Le Marteau*, a sketch from the earlier (subsequently withdrawn) *Oubli signal lapidé* is examined in Chapter 6.

according to a given schema (Chapter 5); and *complex multiplication*, a special variety of compound multiplication which is the elegant, commutative generating operation of pitch classes in the first cycle of *Le Marteau* (Chapter 6). The specific properties of the twelve-tone row used by Boulez, their applications to pitch-class set multiplication, their realization in sections of the first movement, and their implications for a process-based listening strategy are explored in Chapter 7. Appendices A, B and C deal with issues raised in the body of the paper.

2. Terminology and Notation

The following terms, symbols, and conventions of notation, taken primarily from John Rahn's *Basic Atonal Theory* and cross-referenced where further discussion may be helpful, appear in this paper. Other conventions, symbols and definitions will be introduced as they become necessary.

Integer notation. When integer notation is used, C = 0 unless otherwise stated. The complete chromatic is 0,1,2,3, 4,5,6,7,8,9,10,11; the single-digit substitutes for 10 and 11, T and E, are used only in calculation matrices, since T is the symbol for transposition and E is a pitch name.

Staff notation. In the absence of a clef, treble clef is assumed.

Pitch names. C4 = Middle C.

Aggregate. The universe of 12 pc is abbreviated U.

Ordered set. Angled brackets enclose an ordered pc set: <2,9,6>. Ordered sets are read from left to right in pc notation and from left to right or from bottom to top, as appropriate, in staff notation.

Unordered set. Braces enclose an unordered pc set: $\{2,6,9\}$. Unordered sets are usually shown in normal form (*Basic Atonal Theory*, pp. 31-38); reasons for usages other than normal form are given as appropriate.

Partial ordering. A partially-ordered set is one in which some elements are ordered and other elements are unordered.¹

T_n -type set (*Basic Atonal Theory*, p. 75). Parentheses enclose a T_n -type set: $(0,4,7)$.

T_n/T_nI -type set (*Basic Atonal Theory*, pp. 76-77). Square brackets enclose a T_n/T_nI -type set: $[0,3,7]$. A T_n/T_nI -type set may also be referred to as a "set class."

Set element. \in indicates a set element: $2 \in \{2,6,9\}$. More abstractly, lower-case letters indicate elements of sets indicated by upper-case letters: $a \in A$; $b \in B$.

Cardinality. Reference to the number of elements in a set is indicated by vertical lines: $|\{2,6,9\}| = 3$. If set $A = \{2,6,9\}$, $|A| = 3$.

Subset. The algebraic symbol \supseteq indicates a superset/subset relationship: $\{2,6,9\} \supseteq \{2,6\}$; $\{2,6,9\} \supseteq \{2,6,9\}$. Distinctions between subset and proper subset (normally indicated by \supset) are unimportant to the present study.

¹Robert Morris, *Composition with Pitch-Classes: A Theory of Compositional Design* (New Haven and London: Yale University Press, 1987), p. 345.

Union. The union of two sets is shown by the symbol \cup : $\{2,6,9\} \cup \{7,11,2\} = \{6,7,9,11,2\}$.

Intersection. The intersection of two sets is shown by the symbol \cap : $\{2,6,9\} \cap \{7,11,2\} = \{2\}$.

If and only if. This condition is abbreviated IFF.

Ordered pitch interval (Basic Atonal Theory, pp. 20-21). $ip\langle a,b \rangle = b-a$ (for any two pitches a and b , the ordered pitch interval between a and b in that order equals $b-a$):
 $ip\langle 2,6 \rangle = 4$; $ip\langle 6,2 \rangle = -4$; $ip\langle 2,19 \rangle = 17$; $ip\langle 19,2 \rangle = -17$.

Unordered pitch interval (Basic Atonal Theory, p. 22). $ip(a,b)$ = the absolute value of $b-a$: $ip(2,6) = 4$; $ip(6,2) = 4$; $ip(2,19) = 17$; $ip(19,2) = 17$.

Ordered pitch-class interval (Basic Atonal Theory, pp. 25-26). $i\langle a,b \rangle = b-a$ (for any two pc a and b , the ordered pitch-class interval between a and b in that order equals $b-a \pmod{12}$): $i\langle 2,6 \rangle = 4$; $i\langle 6,2 \rangle = 8$.

Unordered pitch-class interval (Basic Atonal Theory, pp. 27-29). $i(a,b)$ = the smaller of $i\langle a,b \rangle$ and $i\langle b,a \rangle$:
 $i(2,6) = 4$; $i(6,2) = 4$. An unordered pitch-class interval may also be referred to as an "interval class."

Pitch-class (set) transposition (Basic Atonal Theory, pp. 42-43). For any pc x and any ordered pc interval n , $T_n(x) = x + n \pmod{12}$ --the pitch-class transposition of x by n semitones equals x plus $n \pmod{12}$: $T_3(8) = 11$; $T_{11}\{8,10,0\} = \{7,9,11\}$.

Transpositional equivalence. Any two sets A and B are transpositionally equivalent IFF $|A| = |B|$, and for each $a \in A$ and $b \in B$, there is a value for T_n such that $T_n(a) = b$. Transpositional equivalence of sets is indicated by the algebraic sign for congruence: $A \cong B$; $\{2,6,9\} \cong \{5,9,0\}$.

3. Transpositional Combination

Richard Cohn's theory of transpositional combination¹ (TC) deals almost exclusively with T_n - and T_n/T_nI -type sets. Certain elements of this theory provide a useful background for the workings of pitch-class set multiplication, and are delineated here. Cohn defines transpositional combination both as a property which certain sets possess and as an operation which generates such sets. He states:

Any pitch- or pc-set has the TC-property if it may be disunited into two or more transpositionally related subsets. . . . In its simplest general form, TC is a binary operation which takes as its operands two set-classes, and adds the value of each element in the prime form of the first operand to that of each element in the prime form of the second operand. The result is a larger set which bears the TC-property.²

¹Richard L. Cohn, "Transpositional Combination in Twentieth-Century Music" (Ph.D. dissertation, Eastman School of Music, University of Rochester, 1987). See also the more readily available work by Cohn, "Inversional Symmetry and Transpositional Combination in Bartók," *Music Theory Spectrum* 10 (1988), pp. 19-42, for an explication of many of the salient features of transpositional combination.

²Cohn, "Inversional Symmetry and Transpositional Combination in Bartók," pp. 23, 27-28. Cohn notes the similarity of TC to Boulezian multiplication here (p. 23n) and, in greater detail, in "Transpositional Combination in Twentieth-Century Music," pp. 47-50.

For example, a pc set $\{2,4,5,7\}$ can be "disunited" into the transpositionally equivalent subsets $\{2,4\}$ and $\{5,7\}$ or $\{2,5\}$ and $\{4,7\}$; see Figure 3.1.



Figure 3.1. $\{2,4,5,7\}$ and its T_n -equivalent subsets

The T_n/T_{nI} type of $\{2,4,5,7\}$, $[0,2,3,5]$, can be formed by the transpositional combination of T_n/T_{nI} -type operands $[0,2]$ and $[0,3]$. Such an operation may be informally understood as the construction of one operand on each element of the other.³

The operation of TC is symbolized by a star (*); in this example, the equation is shown as $[0,2] * [0,3] = [0,2,3,5]$ or as $[0,3] * [0,2] = [0,2,3,5]$.⁴ The addition of every element of one operand to every element of the other can be calculated on a matrix, the mechanical structure of which is shown in

³This will be more rigorously defined in Chapter 4.

⁴These are the only non-redundant ways in which $[0,2,3,5]$ will result from transpositional combination. $[0,2,3,5]$ also results redundantly from the identity operation $[0,2,3,5] * [0]$. Cohn calls the identity operation "trivially idempotent;" an idempotent operation is one in which the result duplicates one of the operands ("Transpositional Combination in Twentieth-Century Music," pp. 88, 583).

Figure 3.2 using hypothetical T_n/T_nI -type sets $[a,b,c]$ and $[d,e,f]$;⁵ the application of the matrix in the present example $[0,2] * [0,3] = [0,2,3,5]$ is shown in Figure 3.3. The transpositionally symmetrical subsets are found in each sum column and sum row.

$$\begin{array}{r|lll}
 c & (c+d) & (c+e) & (c+f) \\
 b & (b+d) & (b+e) & (b+f) \\
 a & (a+d) & (a+e) & (a+f) \\
 \hline
 & d & e & f
 \end{array}$$

Figure 3.2. TC calculation matrix:
 $[a,b,c] * [d,e,f]$

$$\begin{array}{r|ll}
 2 & 2 & 5 \\
 0 & 0 & 3 \\
 \hline
 & 0 & 3
 \end{array}$$

Figure 3.3. Calculation of $[0,2] * [0,3] = [0,2,3,5]$

The matrix shows the result of the transpositional combination in its sum area. These sum integers are analogous to pitch classes from which a T_n/T_nI -type set can be derived.⁶ In Fig. 3.3, the only required manipulation of the

⁵The cardinality of these propaedeutic sets can be reduced or enlarged at will.

⁶The sum integers do not represent *actual* pitch classes, since T_n - and T_n/T_nI -type sets are not specific with regard to pitch class content.

sum integers is their placement in ascending order. In Fig. 3.4, the ascending form of the sum integers is 0,4,5,9-- analogous to a pc set with the normal form {4,5,9,0}, the T_n type (0,1,5,8), and the T_n/T_nI type [0,1,5,8].

$$\begin{array}{r}
 4 \mid 4 \ 9 \\
 \hline
 0 \mid 0 \ 5 \\
 \hline
 \mid 0 \ 5
 \end{array}
 = \{4,5,9,0\} \rightarrow \\
 (0,1,5,8) \rightarrow \\
 [0,1,5,8]$$

Figure 3.4. Calculation of [0,4] * [0,5]

A compelling feature of TC is commutativity: reversing the order of operand sets will not change the resultant set.⁷

In many set-theoretical writings, the greater abstraction of the T_n/T_nI type is preferred to the T_n type. In transpositional combination (and in pc set multiplication), this preference is of limited value. Any T_n/T_nI -type set that is not inversionally symmetrical will yield two different T_n -type sets, and the results of TC operations using these different sets may differ from each other. For example, the T_n/T_nI -type set operation [0,1,2,6] * [0,1,5] can be performed in four ways with T_n -type sets, as neither

⁷Cohn, "Transpositional Combination in Twentieth-Century Music," p. 61. Commutativity will be a thorny but ultimately resolvable issue in subsequent chapters dealing with pitch-class set multiplication.

operand is inversionally symmetrical: $(0,1,2,6) * (0,1,5)$; $(0,1,2,6) * (0,4,5)$; $(0,4,5,6) * (0,1,5)$; and $(0,4,5,6) * (0,4,5)$. These operations are shown in Figures 3.5 through 3.8 respectively; they reveal results that differ even to the extent of their cardinalities--8 in Figs. 3.5 and 3.8, and 9 in Figs. 3.6 and 3.7. Because of "discrepancies" such as these, most operations in this study will employ T_n -type rather than T_n/T_nI -type sets.

$$\begin{array}{r}
 6| 6 7 E \\
 2| 2 3 7 \\
 1| 1 2 6 \\
 \hline
 0| 0 1 5 \\
 | 0 1 5
 \end{array}
 = \{11,0,1,2,3,5,6,7\} \rightarrow \\
 (0,1,2,3,4,6,7,8) \rightarrow \\
 [0,1,2,3,4,6,7,8]$$

Figure 3.5. Calculation of $(0,1,2,6) * (0,1,5)$

$$\begin{array}{r}
 6| 6 T E \\
 2| 2 6 7 \\
 1| 1 5 6 \\
 \hline
 0| 0 4 5 \\
 | 0 4 5
 \end{array}
 = \{10,11,0,1,2,4,5,6,7\} \rightarrow \\
 (0,1,2,3,4,6,7,8,9) \rightarrow \\
 [0,1,2,3,4,6,7,8,9]$$

Figure 3.6. Calculation of $(0,1,2,6) * (0,4,5)$

$$\begin{array}{r}
 6 | 6 \ 7 \ E \\
 5 | 5 \ 6 \ T \\
 4 | 4 \ 5 \ 9 \\
 \hline
 0 | 0 \ 1 \ 5 \\
 | 0 \ 1 \ 5
 \end{array}
 \quad = \begin{array}{l}
 \{4,5,6,7,9,10,11,0,1\} \rightarrow \\
 (0,1,2,3,5,6,7,8,9) \rightarrow \\
 [0,1,2,3,4,6,7,8,9]
 \end{array}$$

Figure 3.7. Calculation of $(0,4,5,6) * (0,1,5)$

$$\begin{array}{r}
 6 | 6 \ T \ E \\
 5 | 5 \ 9 \ T \\
 4 | 4 \ 8 \ 9 \\
 \hline
 0 | 0 \ 4 \ 5 \\
 | 0 \ 4 \ 5
 \end{array}
 \quad = \begin{array}{l}
 \{4,5,6,8,9,10,11,0\} \rightarrow \\
 (0,1,2,4,5,6,7,8) \rightarrow \\
 [0,1,2,3,4,6,7,8]
 \end{array}$$

Figure 3.8. Calculation of $(0,4,5,6) * (0,4,5)$

Cohn differentiates T_n -type sets by labeling inversionally asymmetrical T_n/T_nI -type sets as either "A" or "B" forms. The A form of a T_n -type set duplicates the integer content of its T_n/T_nI type; the B form is the inversion of the A form.⁸ For example, $(0,1,5)$ is the A form and $(0,4,5)$ the B form of $[0,1,5]$. Interestingly, the transpositional combination of any two A forms will always result in the same T_n/T_nI -type set as that of the two corresponding B forms (even if the resultant T_n types are different--compare Figs. 3.5 and 3.8); similarly, the transpositional combination of the A form of one set with the B form of another will always

⁸Ibid., pp. 66-68.

result in the same T_n/T_nI type as the TC of the B form of the first set with the A form of the second (compare Figs. 3.6 and 3.7).⁹ If either operand is inversionally symmetrical, all results will be of the same T_n/T_nI type.¹⁰

Because pitch-class set multiplication involves only two operand pc sets, a delineation of features of the transpositional combination of more than two operands, including recursion and associativity, is inappropriate here.

Cohn's theory deals elegantly with T_n - and T_n/T_nI -type sets; it is less acute in its brief consideration of sets comprising specific pitch classes. His definition of transpositional combination of pc sets essentially substitutes pc integers for those of set classes:

For pc sets A, B, the transpositional combination of A and B, denoted $A * B$, is a pc set C IFF

1. for all $a \in A$, $b \in B$, $(a + b) \in C$, and
2. for all $c \in C$, there is an $a \in A$, $b \in B$ such that $c = (a + b)$.¹¹

Thus, just as $[0,2] * [0,3] = [0,2,3,5]$, so also does $\{0,2\} * \{0,3\} = \{0,2,3,5\}$.

Although Cohn maintains that "the generation of less abstract entities [than set classes] such as pitch-set-types

⁹Ibid., p. 75.

¹⁰Ibid., pp. 74-75.

¹¹Ibid., p. 60.

is unproblematic,"¹² his definition of TC of pc sets errs both conceptually and formally. While it is easy to conceive of adding one interval to another to result in a third interval (transpositional combination of T_n - and T_n/T_nI -type sets) and of adding an ordered pitch-class interval to a pitch class to result in a pitch class (T_n), no criteria have been established whereby two pitch classes can be added to result in a third pitch class.¹³ Intuitively, $E + E^\flat =$ the dyad E/E^\flat ; yet, given $C = 0$, Cohn's definition suggests the equation $E * E^\flat = G: 4 + 3 = 7$.

Nor is the equation absolute. If, for example, $E^\flat = 0$ (a reasonable change in an analysis of *Le Marteau*, since E^\flat is the first pc of the row), then $E * E^\flat = E: 1 + 0 = 1$. Similarly, Figure 3.1 suggests $\{0,2\} * \{0,3\} = \{0,2,3,5\}$ when

¹²Cohn, "Inversional Symmetry and Transpositional Combination in Bartók," p. 27.

¹³John Rahn has shown that two pitch classes can be added to result in an interval--specifically, a value for n in T_nI : if $a + b = n$, then $T_nI(a) = b$ and $T_nI(b) = a$ (*Basic Atonal Theory*, p. 49). Robert Morris has extended this equation to pitch-class sets in order to simplify the calculation of the TICS vector, the number of common tones of a pc set under T_nI : the number of occurrences of n in an additive matrix (of the same type as that employed in transpositional combination) equals the number of common tones under T_nI . (Similarly, subtraction in the matrix will reveal the number of common tones between two pc sets A and B under T_n ; here, calculation must be done in two directions (since $a-b \neq b-a$) to provide two sets of T_n values--one to predict $|T_n(A) \cap B|$ and one to predict $|T_n(B) \cap A|$.) See Morris, *Composition With Pitch-Classes*, pp. 70-71; cf. *Basic Atonal Theory*, pp. 111-113. As with the TC of pc sets, a change of the pc equalling 0 changes the result, here the n of T_nI ; Rahn calls this "the intuitively absurd consequence" of the T_nI definition, but observes that this definition still "has its own advantages and its own kind of elegance" (*Basic Atonal Theory*, p. 57n).

$D = 0$, but when $C = 0$, $\{2,4\} * \{2,5\} \neq \{2,4,5,7\}$. What is actually being added is not operand pitch classes, but the ordered pitch-class intervals from whatever pc is designated as 0 to the operand pitch classes. As a result, the transpositional combination of a pc with itself cannot result in an odd-numbered pc (since $x + x = 2x$); for example, $C * C = C$ when $C = 0$, but there is no pc x such that $x * x = C$ when, for instance, $E^b = 0$.

Using this theorem, such counterintuitive results as the following can be obtained when pc names are substituted for integers ($C = 0$): $C * C = C$; $G^b * G^b = C$; $E^b * E^b = G^b$; $A * A = G^b$; but $x * x \neq E^b$ and $x * x \neq A$. "Counterintuitive" should not be parsed as meaning "useless"; doubtless, a compositional paradigm could be, and perhaps has been, constructed that utilizes just such relationships. As an analytical tool, however, this aspect of transpositional combination does not appear to have much relevance (no examples from the literature are presented), and, once exposed by Cohn, is not referenced again in his theory.

The technique of *multiplication* of pitch-class sets, invented by Boulez and formalized herein, offers an elegant alternative to Cohn's transpositional combination of pitch-class sets. Although the theories of transpositional combination of set classes and multiplication of pitch-class sets share many features (and Cohn's work is essential in proving

and extending Boulez's), many of the terms used below are drawn, metaphorically or otherwise, from arithmetical multiplication rather than from transpositional combination; for example, the multiplication of specific pitch-class sets will be signified by a circled symbol for arithmetical multiplication ($A \otimes B$) rather than by Cohn's star ($A * B$), since he has defined a different meaning for the latter operation. (Cohn's notation is retained for operations involving T_n - and T_n/T_nI -type sets.) This is further due not only to the fact that Boulez's work predates Cohn's, but also to distinctions of multiplication that either are not present in transpositional combination or are differently realized.¹⁴

¹⁴I am referring primarily to distinctions between operands. In Cohn's work, such distinctions are generally limited to horizontal and vertical realizations in a musical surface ("Transpositional Combination in Twentieth-Century Music," pp. 178-187 *et seq.*). In my work, operands are distinguished according to different criteria; see Chapter 4.

4. The Theory of Simple Multiplication

Lev Koblyakov, in *Pierre Boulez: A World of Harmony*, describes Boulez's technique of multiplication of pitch-class sets: "To obtain [the product of multiplying {3,5} by {3,5}, the set] is multiplied by itself, any repeated sounds appearing only once."¹ He follows this description with the illustration shown in Figure 4.1. A cursory examination of this illustration will reveal that the first "A," F-E♭, is not equal to the second "A," E♭-D; the set has not been "multiplied by itself." Rather, the ordered interval of one of the sets--it is unclear which one, since they are the same--has been constructed on each pc of the other. This is illustrated in Figure 4.2: the interval of 10 half-steps, represented by solid noteheads, is constructed on F and E♭; the union of the resulting sets is Koblyakov's product, {1,3,5}.

This is, at best, an ambiguous example. A reordering of the pc will change the interval from 10 to 2 half-steps, thereby changing the product; see Figure 4.3. It is there-

¹Koblyakov, *Pierre Boulez: A World of Harmony*, p. 5. The same example appears in Koblyakov, "P. Boulez 'Le marteau sans maître,'" analysis of pitch structure," p. 25. Lower-case letters in the original have been capitalized here.

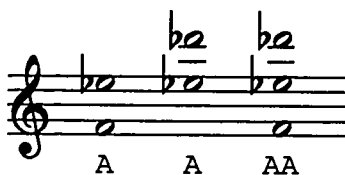


Figure 4.1. Koblyakov's illustration of pitch-class set multiplication
Used by permission of Harwood Academic Publishers

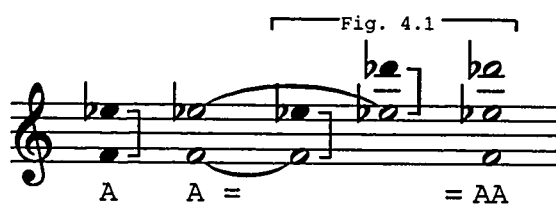


Figure 4.2. Interpretation of Figure 4.1

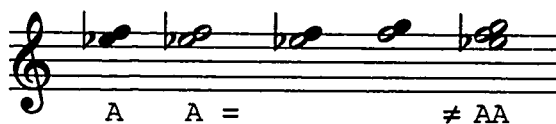


Figure 4.3. Counterexample to Figure 4.2



Figure 4.4. Different representation of interval of ten half-steps multiplied by F-E \flat



Figure 4.5. Non-commutativity (compare with Fig. 4.4)

fore apparent that some prior ordering is necessary to achieve a certain result. Furthermore, any interval of 10 half-steps will, when constructed on pc 5 and 3, yield the same result as in Fig. 4.2; see Figure 4.4, which indicates that the pc content of the first operand is of secondary importance relative to the ordered interval. Kobylakov's example also circumvents the question of commutativity; see Figure 4.5, in which the operands of Figure 4.4 are reversed. Even granting oneself the luxury of prior ordering (as no ordering criteria have been given; the derivation of the set, explained in Chapter 6, would suggest the ordering of Figure 4.3), this is, unlike arithmetical multiplication and transpositional combination of set classes, a non-commutative operation. It is not without value, however, for it is through this process, which I call *simple multiplication*, that Boulez's own process may be understood.²

In order to define simple multiplication of pitch-class sets, it will be necessary to expand on the concept of ordered pitch-class interval, as pc sets may of course have more than two members. In an ordered pc set $\langle a, b, c \rangle$, does one calculate the intervals from a to b and from b to c (as does Anatol Vieru³), or is it more useful to calculate the

²Two other incongruities of Kobylakov's example--apparent lack of transposition of the product, and the fact that a set equivalent in origin to this one is the only one which does not appear in *Marteau*--will be explored in Chapter 6.

³Anatol Vieru, "Modalism--A 'Third World,'" *Perspectives of New Music* 24/1 (Fall/Winter 1985), p. 65. Vieru also calculates the inter-

intervals from a to b and from a to c? Although either approach will accurately reflect the ordered interval content of a pc set, the latter method (modeled on T_n type) will simplify calculations of pc set multiplication, and is defined as follows:

Ordered pitch-class intervallic structure (OIS). For any ordered pc set $\langle a, b, c, \dots, k \rangle$, the OIS of that set is equal to $\langle (i\langle a, a \rangle), (i\langle a, b \rangle), (i\langle a, c \rangle), \dots, (i\langle a, k \rangle) \rangle$. (Note that this definition calculates ordered pc intervals from the first member of the set, including the interval from the first element to itself (i.e., 0).)

The symbol for an OIS is an underlining of the ordered pc intervals which are separated by commas. The OIS of an ordered pc set may be expressed as $OIS\langle a, b, c, \dots, k \rangle$, as $OIS(A)$, or as \underline{A} .

$$i\langle 2, 2 \rangle = 0$$

$$i\langle 2, 6 \rangle = 4$$

$$i\langle 2, 9 \rangle = 7$$

$$OIS = \underline{0, 4, 7}$$

Figure 4.6. Calculation of $OIS\langle 2, 6, 9 \rangle$

val from (in this example) c to a, so that the intervallic structure of $\langle 2, 5, 9 \rangle$ would be represented as 3 4 5. Since, in dealing with pitch-class sets in normal form or any rotation thereof, such an intervallic calculation will always sum to 12, this calculation from last to first pc seems unnecessary.

In this example, OIS resembles T_n type. The difference between the two becomes clear if the pc set is rotated (i.e., reordered). The pc content and the T_n type will not change, but the OIS usually will:⁴

$$\begin{aligned}i\langle 6, 6 \rangle &= 0 \\ i\langle 6, 9 \rangle &= 3 \\ i\langle 6, 2 \rangle &= 8 \\ \text{OIS} &= \underline{0, 3, 8}\end{aligned}$$

Figure 4.7. Calculation of OIS $\langle 6, 9, 2 \rangle$

$$\begin{aligned}i\langle 9, 9 \rangle &= 0 \\ i\langle 9, 2 \rangle &= 5 \\ i\langle 9, 6 \rangle &= 9 \\ \text{OIS} &= \underline{0, 5, 9}\end{aligned}$$

Figure 4.8. Calculation of OIS $\langle 9, 2, 6 \rangle$

⁴The word "usually" is chosen carefully. Transpositionally symmetrical pc-sets--those which map into themselves under all T_n where $n \neq 0$ --will replicate OIS under rotation. Transpositionally symmetrical pc sets belong to the following set classes: $[0, 6]$ and its complement $[0, 1, 2, 3, 4, 6, 7, 8, 9, 10]$; $[0, 4, 8]$ and $[0, 1, 2, 4, 5, 6, 8, 9, 10]$, the only such sets with an odd-numbered cardinality; $[0, 1, 6, 7]$ and $[0, 1, 2, 3, 6, 7, 8, 9]$; $[0, 2, 6, 8]$ and $[0, 1, 2, 4, 6, 7, 8, 10]$; $[0, 3, 6, 9]$ and $[0, 1, 3, 4, 6, 7, 9, 10]$; $[0, 1, 2, 6, 7, 8]$; $[0, 1, 3, 6, 7, 9]$ (the only such set that is not also inversionally symmetrical); $[0, 1, 4, 5, 8, 9]$; and $[0, 2, 4, 6, 8, 10]$. All of these except $[0, 4, 8]$, $[0, 1, 4, 5, 8, 9]$ and $[0, 1, 2, 4, 5, 6, 8, 9, 10]$ map into themselves under (at least) T_6 ; only these three exceptions appear as generating and/or domain sets in *Le Marteau*. The reasons for this will be explored in the Chapter 7 discussion of the properties of the *Marteau* row.

These results suggest an equivalent algorithm for determining ordered pitch-class intervallic structure:
 $OIS\langle a,b,c,\dots,k\rangle = T_{-a}\langle a,b,c,\dots,k\rangle$, expressed as an OIS (i.e., underlined):

$$T_8\langle 4,8,9,11\rangle = \langle 0,4,5,7\rangle \rightarrow \underline{0,4,5,7}$$

Figure 4.9. Calculation of $OIS\langle 4,8,9,11\rangle$

Boulez's (and Koblyakov's) examples in staff notation frequently show structures spanning more than an octave, generally to avoid the overcrowding of notes that occurs when large or closely-spaced sets are represented. Figure 4.10 shows two trichordal sets of the same pc content (and, therefore, T_n type):



Figure 4.10. Two pitch-class sets

If read as ordered pc sets, the first is $\langle 2,6,9\rangle$ and the second is $\langle 2,9,6\rangle$; the OIS of the first is $\underline{0,4,7}$ and the OIS of the second is $\underline{0,7,4}$. In OIS, as in the concept of pitch class, distinctions of octave are erased; the pc sets in

Figure 4.10 may therefore be considered as having equal OIS. (If this seems counterintuitive, it may be due to the difference between pitch and pitch class and the difficulty of distinguishing them in staff notation. Consider, for example, a set of pitches $\langle D4, C4 \rangle$. By definitions in *Basic Atonal Theory*, $ip\langle D4, C4 \rangle = -2$; $ip(D4, C4) = 2$; $i(D, C) = 2$; but $i\langle D, C \rangle = 10$.) This leads to further refinements of the OIS concept:

OIS equality. Any two ordered pitch-class intervallic structures are equal IFF they have exactly the same integer content, regardless of the order of the integers. The OIS of the sets in Figure 4.10 exemplify OIS equality.

OIS normal form. The "normal form" for the notation of an OIS shows integers in increasing order. For example, an OIS of 0,9,5 would be rewritten as 0,5,9. (As with pc set normal form, this is a convenience for sake of clarity and comparison, and not a prerequisite to achieving a result.) The normal forms of OIS of any OIS-equal sets will therefore be identical. OIS normal form will be used through the remainder of this paper.

OIS/pitch class construction. An OIS can be constructed on any pc to form a pc set. The construction of an OIS a,b,c,...m on a pc y is symbolized as a,b,c,...m $\otimes y$.⁵

⁵The use of the sign \otimes for OIS/pc construction and, below, for pc set multiplication is motivated by the metaphor of multiplication and to avoid confusion between the very different operations of pitch-class

The resulting pc set is calculated as $\{(a+y), (b+y), (c+y), \dots(m+y)\}$. For example, either pc set in Figure 4.10 can be constructed as $\underline{0,4,7} \otimes 2 = \{(0+2), (4+2), (7+2)\} = \{2,6,9\}$. Ordering and partial ordering of the result will be considered below.

OIS equivalence. Two equal or unequal OIS are equivalent IFF their construction on a pc results in pc sets A and B such that $A \cong B$. Put another way, two equal or unequal OIS \underline{A} and \underline{B} are equivalent IFF $|\underline{A}| = |\underline{B}|$, and for every $(a \in \underline{A})$ and $(b \in \underline{B})$, there is a value for n such that $a + n = b$. (Equal OIS are therefore always equivalent under $n = 0$.) This equivalence is expressed as $\underline{A} \cong \underline{B}$ or as $\text{OIS}(\underline{A}) \cong \text{OIS}(\underline{B})$.

An equivalent OIS is calculated by successive subtractions of an element of an OIS from each element of the same OIS, itself included. This is shown as $\underline{a,b,c,\dots,m} - \underline{x}$ (where $x \in \underline{a,b,c,\dots,m}$), and is calculated as $(a-x, b-x, c-x, \dots, m-x)$ to give an integer set which is put into OIS normal form. A complete tabulation of OIS equivalent to $\underline{a,b,c,\dots,m}$ includes $(\underline{a,b,c,\dots,m} - a)$, $(\underline{a,b,c,\dots,m} - b)$, $(\underline{a,b,c,\dots,m} - c)$, ... $(\underline{a,b,c,\dots,m} - m)$. For example, the OIS equivalent to $\underline{0,4,7}$ include: $\underline{0,4,7}$ $((0-0), (4-0), (7-0) = \underline{0,4,7})$; $\underline{0,3,8}$ $((0-4),$

set multiplication and arithmetical multiplication. Thus $|\underline{A} \otimes \underline{B}|$ involves pitch-class set multiplication (the cardinality of the product of multiplied pc sets), while $|\underline{A}| \times |\underline{B}|$ involves arithmetical multiplication (the multiplied cardinalities of two pc sets).

(4-4), (7-4) = 8,0,3); and 0,5,9 ((0-7), (4-7), (7-7) = 5,9,0). (See Figures 4.6 through 4.8.) The number of equivalent OIS of a pc set A therefore equals |A|.

Ordering and OIS equality. Part of the fallout of OIS equality is that any of the pc in an ordered set may actually appear in any order *except for the first listed pc*. Since all OIS calculations involve the distance from the first pc to each of the pc in the set, the only significant change that can be effected--that is, to an unequal OIS--is to change the first pc. This in turn means that total ordering of sets is unnecessary and that a special type of partially ordered set may be defined:

Initially-ordered pitch-class set (IO set). An initially-ordered pitch-class set is an otherwise unordered set in which the first pc is ordered. Such a set will be represented with a combination of ordered and unordered set symbols: in the initially-ordered pc set $\langle r, \{s, t, u\} \rangle$, r is fixed as the first element, while elements s , t and u are unordered with respect to each other but succeed r . The OIS of an initially-ordered pitch-class set is calculated by the same process as that of a completely ordered set.

The initially-ordered pitch-class set is not a novel concept. Consider the following example:

G: V_5^6 V_5^6 V_5^6 V_5^6 V_5^6 V_5^6

Figure 4.11. An IO set in tonal theory

The complete dominant seventh chord in G major is represented as V_5^6 when F# is in the bass--that is, when F# is ordered as the first pc. (This set could be listed as $\langle F\#, \{A, C, D\} \rangle$.) Only an alteration of the bass will change the analytical representation. The combination of roman numeral and figured bass (shown completely as V_3^5 in this example) is the tonal equivalent of OIS, showing diatonic intervals above the bass rather than semitones but again in an ascending normal form and ignoring octave distinctions.

Initial pitch class (r). An initial pitch class is the first pc in an initially-ordered pitch-class set. This is symbolized by the letter r: in the IO set $A = \langle 3, \{6, 11\} \rangle$, $r(A) = 3$.

IO set normal form. The normal form of an IO set is calculated by listing the set elements in ascending order,

rotating this list to begin with the initial pitch class r , and applying IO set symbols. An example: since the ascending order of $\{8,6,10,1\}$ is $\{1,6,8,10\}$, the normal form of the IO set where 8 is ordered as the first pc is $\langle 8, \{10,1,6\} \rangle$.⁶ IO set normal form will be used for the remainder of this paper.

Number of IO sets/unequal OIS. The number of different initially-ordered sets and, therefore, greatest number of unequal ordered pitch-class intervallic structures that can be constructed from a pc set $A = |A|$.

The value of defining initially-ordered pc sets and identifying OIS equality will be apparent in simple multiplicative calculations below, as these characteristics will eliminate redundant calculations. A pc set with a cardinality greater than 2 has more total orderings ($|A|!$) than initial orderings ($|A|$), but an ordered set will have an OIS equal to any of its permutations as long as its initial pc is unchanged.

⁶There is, therefore, an ordered mapping between the normal form of an IO set and the normal form of its OIS. Since one of the rotations of the ascending order will equal the normal form of the (unordered) pc set, the OIS of that rotation will equal the integer content of the set's T_n type. In the example given, the normal form of the IO set where $r = 6$ is $\langle 6, \{8,10,1\} \rangle$, with an OIS of 0.2.4.7; the normal form of the unordered pc set is $\{6,8,10,1\}$; the T_n type is $(0,2,4,7)$.

Definitions of three terms borrowed from arithmetical multiplication will pave the way for a definition of simple multiplication of pitch-class sets. They are:

Multiplicand, multiplier and product. In a multiplicative operation, the first operand (preceding the multiplication sign) is the multiplicand; the second is the multiplier. In the operation $A \otimes B$, A is the multiplicand, B the multiplier. If the operands are reversed, the function of each operand changes, so that in the operation $B \otimes A$, B is the multiplicand, A the multiplier. The result of multiplication is an unordered pc set, the product (here, AB or BA).

Simple multiplication of pc sets. Where A and B are pc sets, the simple multiplicative product of $A \otimes B$ is the union of all pc sets that result from constructing the OIS of IO set A on each pc of B. In accordance with formulae presented above, this may be expressed as follows: where A and B are pc sets and $B = \{b, c, d, \dots, m\}$, $A \otimes B = \{OIS(A) \otimes b\} \cup \{OIS(A) \otimes c\} \cup \{OIS(A) \otimes d\} \dots \cup \{OIS(A) \otimes m\}$.

This definition means that each element of the multiplier may serve as a T_n value applied to a pc set which duplicates the integer content of the OIS;⁷ the product consists of the union of these transposed sets. The reverse

⁷In a subtle distinction, T_n has no numeric effect on an OIS, since transposing an interval does not change the interval. A pc set is thus substituted for the OIS in Fig. 4.12.

is also true: each element of an OIS may be considered as a value for T_n which is applied to the multiplier set. These formulae are exemplified in Figures 4.12 and 4.13 respectively, using the operation $\langle 3, \{6, 11\} \rangle \otimes \{2, 4\}$.

$$\begin{aligned} \text{OIS } \langle 3, \{6, 11\} \rangle &= \underline{0, 3, 8} \\ T_2\{0, 3, 8\} &= \{2, 5, 10\} \\ T_4\{0, 3, 8\} &= \{4, 7, 0\} \\ \{2, 5, 10\} \cup \{4, 7, 0\} &= \{10, 0, 2, 4, 5, 7\} \end{aligned}$$

Figure 4.12. Calculation of $\langle 3, \{6, 11\} \rangle \otimes \{2, 4\}$, using multiplier elements as T_n values

$$\begin{aligned} \text{OIS } \langle 3, \{6, 11\} \rangle &= \underline{0, 3, 8} \\ T_0\{2, 4\} &= \{2, 4\} \\ T_3\{2, 4\} &= \{5, 7\} \\ T_8\{2, 4\} &= \{10, 0\} \\ \{2, 4\} \cup \{5, 7\} \cup \{10, 0\} &= \{10, 0, 2, 4, 5, 7\} \end{aligned}$$

Figure 4.13. Calculation of $\langle 3, \{6, 11\} \rangle \otimes \{2, 4\}$, using OIS elements as T_n values

These calculations suggest that a matrix of the type used by Cohn for transpositional combination (shown in Figure 3.2) may also be used for simple multiplication. Although simple multiplication *as an operation* is not commutative, the calculation of simple multiplication (i.e., once the OIS of the multiplicand has been substituted for an IO set) is com-

mutative. Calculations may also be shown in staff notation (as in Figures 4.2 through 4.5), which preserves the more intuitive aspects of the process. In the matrix, the OIS of the multiplicand will always be shown on the vertical axis and the pc of the multiplier (in normal form) on the horizontal;⁸ products are shown as unordered sets in normal form. Examples using both follow.

(See Figure 4.14) Calculate $\langle 2, \{6, 9\} \rangle \otimes \{4, 7\}$:

$$\text{OIS}\langle 2, \{6, 9\} \rangle = \underline{0, 4, 7}$$

$$\begin{array}{r|l} 7 & E \ 2 \\ 4 & 8 \ E \\ \hline 0 & 4 \ 7 \\ | & 4 \ 7 \end{array} = \{2, 4, 7, 8, 11\}$$

The next example demonstrates the non-commutativity of simple multiplication (compare with Fig. 4.14):

(See Figure 4.15) Calculate $\langle 4, \{7\} \rangle \otimes \{2, 6, 9\}$:

$$\text{OIS}\langle 4, \{7\} \rangle = \underline{0, 3}$$

$$\begin{array}{r|l} 3 & 5 \ 9 \ 0 \\ 0 & 2 \ 6 \ 9 \\ \hline | & 2 \ 6 \ 9 \end{array} = \{0, 2, 5, 6, 9\}$$

That the initial ordering of the multiplicand affects the product is shown in the next two calculations (compare with Figs. 4.15 and 4.14 respectively).

⁸This establishes a convention which has the advantage of paralleling calculations in staff notation (the vertical columns of the matrix, read from bottom to top, form IO sets coinciding with OIS constructed on multiplier pc on the staff); however, the multiplicative product would be unaffected by the reversal of the operands on the matrix.

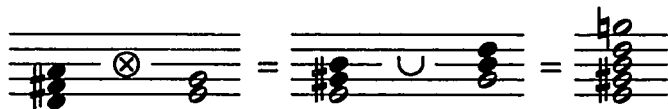


Figure 4.14. $\langle 2, \{6, 9\} \rangle \otimes \{4, 7\} = \{2, 4, 7, 8, 11\}$



Figure 4.15. $\langle 4, \{7\} \rangle \otimes \{2, 6, 9\} = \{0, 2, 5, 6, 9\}$



Figure 4.16. $\langle 7, \{4\} \rangle \otimes \{2, 6, 9\} = \{9, 11, 2, 3, 6\}$



Figure 4.17. $\langle 6, \{9, 2\} \rangle \otimes \{4, 7\} = \{10, 0, 3, 4, 7\}$



Figure 4.18. $\langle 10, \{1, 6\} \rangle \otimes \{4, 7\} = \{10, 0, 3, 4, 7\}$

(See Figure 4.16) Calculate $\langle 7, \{4\} \rangle \otimes \{2, 6, 9\}$:

$$\text{OIS}\langle 7, \{4\} \rangle = \underline{0, 9}$$

$$\begin{array}{r|l} 9 & E \ 3 \ 6 \\ \hline 0 & \underline{2 \ 6 \ 9} \\ & | \ 2 \ 6 \ 9 \end{array} = \{9, 11, 2, 3, 6\}$$

(See Figure 4.17) Calculate $\langle 6, \{9, 2\} \rangle \otimes \{4, 7\}$:

$$\text{OIS}\langle 6, \{9, 2\} \rangle = \underline{0, 3, 8}$$

$$\begin{array}{r|l} 8 & 0 \ 3 \\ 3 & \underline{7 \ T} \\ \hline 0 & \underline{4 \ 7} \\ & | \ 4 \ 7 \end{array} = \{10, 0, 3, 4, 7\}$$

Multiplicand redundancy. $AB = CB$ if $\text{OIS}(A) = \text{OIS}(C)$. The OIS of a multiplicand pc set has more influence on the product than does the pc content; compare Figs. 4.15 with 4.16 and 4.14 with 4.17. Figure 4.18 shows a multiplicand with a pc content different from that of Figure 4.17, yet, because the OIS of each multiplicand is the same, the products are equal.

(See Figure 4.18) Calculate $\langle 10, \{1, 6\} \rangle \otimes \{4, 7\}$:

$$\text{OIS}\langle 10, \{1, 6\} \rangle = \underline{0, 3, 8}$$

$$\begin{array}{r|l} 8 & 0 \ 3 \\ 3 & \underline{7 \ T} \\ \hline 0 & \underline{4 \ 7} \\ & | \ 4 \ 7 \end{array} = \{10, 0, 3, 4, 7\}$$

Multiplicand redundancy suggests that the abstraction of the OIS from specific pitch classes constitutes a principal value of simple multiplication. This idea is explored

further in the discussion of simple multiplication of lines in Appendix A.

Product cardinality. Given a multiplicand and a multiplier, the cardinality of the product may be determined by one of two methods. The first of these uses a theorem in Cohn (1987) which requires that a complex six-step process be employed in order to attain a result.⁹ The second method is empirical: it requires simply that the operands be multiplied and the elements of the product counted. Although the present study is more concerned with generalized theory than is Koblyakov's, it is less so than is Cohn's, and efficiency prescribes the second method.

The product of multiplying two pc sets A and B may have as many pitch classes as $(|A| \times |B|)$ or 12, whichever is fewer; it may have as few as the greater of $|A|$ and $|B|$.¹⁰ Compare these examples, in which pc sets of cardinality 3 are operands:

⁹Cohn, "Transpositional Combination in Twentieth-Century Music," pp. 103-107. An exception to the six-step process is described in Appendix A: if pc sets A and B have no interval classes in common, then $|A \otimes B| = |A| \times |B|$.

¹⁰See Cohn, "Transpositional Combination in Twentieth-Century Music," p. 102. The formula for determining the greatest product cardinality of multiplying a set by itself--AA--is $((|A| \times |A|) + |A|) + 2$ (see Cohn, p. 149, n2). Therefore, a set of cardinality 3 multiplied by itself may have at most 6 elements $((3 \times 3) + 3) + 2$, e.g.: $(0,1,3) * (0,1,3) = (0,1,2,3,4,6)$. This formula and some of its empirical ramifications will be considered in Chapter 6 (in the discussion of *Le Marteau* domain set cardinality) and in Appendix C.

$$\text{OIS}\langle 4, \{5, 7\} \rangle = \underline{0, 1, 3}$$

$$\begin{array}{r|l} 3 & 4 \ 8 \ 0 \\ 1 & 2 \ 6 \ 7 \\ \hline 0 & 1 \ 5 \ 9 \\ & | \ 1 \ 5 \ 9 \end{array} = \{0, 1, 2, 4, 5, 6, 8, 9, 10\}$$

Figure 4.19. Calculation of $\langle 4, \{5, 7\} \rangle \otimes \{1, 5, 9\}$:
 $|A \otimes B| = (|A| \times |B|)$

$$\text{OIS}\langle 3, \{7, 11\} \rangle = \underline{0, 4, 8}$$

$$\begin{array}{r|l} 8 & 9 \ 1 \ 5 \\ 4 & 5 \ 9 \ 1 \\ \hline 0 & 1 \ 5 \ 9 \\ & | \ 1 \ 5 \ 9 \end{array} = \{1, 5, 9\}$$

Figure 4.20. Calculation of $\langle 3, \{7, 11\} \rangle \otimes \{1, 5, 9\}$:
 $|A \otimes B| = |A|$ (or $|B|$)

Variety of products. The largest number of different products of multiplying two pc sets A and B equals $|A| + |B|$. The greatest number of unequal OIS in a set equals the cardinality of the set, since only changing the initial pc will change the OIS; when each pc has served as the first member of the set, the OIS possibilities have been exhausted. Since operands may be reversed, an OIS can be constructed from every operand pc.

The smallest number of different products of multiplying two pc sets A and B equals 2 (when $B \not\supseteq A$ and $A \not\supseteq B$) or one (when $B \supseteq A$ or $A \supseteq B$). For an example of the former situation, see Figure 4.20. Any reordering of the multiplier IO set will still produce an equal OIS; all products

will equal {1,5,9}. Reversing the operands will, for the same reasons, produce only {3,7,11}. For the latter situation, consider {1,5,9} multiplied by itself:

$$\text{OIS}\langle 1, \{5,9\} \rangle = \underline{0,4,8}$$

$$\begin{array}{r|l} 8 & 9 \ 1 \ 5 \\ 4 & 5 \ 9 \ 1 \\ \hline 0 & 1 \ 5 \ 9 \\ & | \ 1 \ 5 \ 9 \end{array} = \{1,5,9\}$$

Figure 4.21. Calculation of $\{1,5,9\} \otimes \{1,5,9\}$

Again, a reordering of the multiplicand pc will not change the OIS.¹¹

Boulez, in *Boulez on Music Today*, showed that he was aware of the maximum number of products: "If an object A of three notes is multiplied by an object E of two notes, five totally isomorphic objects will result: (AE)1,2,3 and (EA)1,2."¹² The illustration shown in Figure 4.22 follows this statement.

¹¹This is due, of course, to the symmetrical properties of [0,4,8]. Cohn calls this a "multiplicative-idempotent operation"--an operation in which the operands are identical, in which the operands have a cardinality greater than 1, and in which the product is equal to the operand. Operands belonging to [0,6], [0,3,6,9] and [0,2,4,6,8,10] will achieve similar results, multiplied either by themselves or by any of their proper subsets. See Cohn, "Transpositional Combination in Twentieth-Century Music," p. 88.

¹²Boulez, *Boulez on Music Today*, p. 79. Lower-case letters in the original figure (and quote) have been capitalized here. "Totally isomorphic" means "transpositionally equivalent." According to Fig. 4.22 and to the theory herein, Boulez's statement should read more precisely to the effect that set A multiplied by set B will produce three transpositionally equivalent sets (=|A|) and that set B multiplied by set A will produce two transpositionally equivalent sets

The figure shows musical notation for two chords, A and E, and their products. The top staff shows chord A (trichord) and chord E (dyad). The bottom staff shows five products: AE, 1; AE, 2; AE, 3; EA, 1; and EA, 2. Each product is represented by a chord on a staff with a key signature of one flat.

Figure 4.22. From *Boulez on Music Today*:
trichord, dyad \rightarrow five products
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Given pc sets $A = \{7, 10, 0\}$ and $E = \{6, 9\}$, correspondences between Figure 4.22 and the theory of simple multiplication as described here are:

$$AE, 1: \langle 7, \{10, 0\} \rangle \otimes \{6, 9\} = \underline{0, 3, 5} \otimes \{6, 9\}:$$

$$\begin{array}{r|l} 5 & E \ 2 \\ 3 & 9 \ 0 \\ \hline 0 & 6 \ 9 \\ | & 6 \ 9 \end{array} = \{6, 9, 11, 0, 2\}$$

$$AE, 2: \langle 0, \{7, 10\} \rangle \otimes \{6, 9\} = \underline{0, 7, 10} \otimes \{6, 9\}:$$

$$\begin{array}{r|l} T & 4 \ 7 \\ 7 & 1 \ 4 \\ \hline 0 & 6 \ 9 \\ | & 6 \ 9 \end{array} = \{1, 4, 6, 7, 9\}$$

(=|B|). The property of transpositional equivalence will be considered below.

$$AE, 3: \langle 10, \{0,7\} \rangle \otimes \{6,9\} = \underline{0,2,9} \otimes \{6,9\}:$$

$$\begin{array}{r|l} 9 & 3 \ 6 \\ 2 & 8 \ E \\ \hline 0 & 6 \ 9 \\ & | \ 6 \ 9 \end{array} = \{3,6,8,9,11\}$$

$$EA, 1: \langle 6, \{9\} \rangle \otimes \{7,10,0\} = \underline{0,3} \otimes \{7,10,0\}:$$

$$\begin{array}{r|l} 3 & T \ 1 \ 3 \\ 0 & 7 \ T \ 0 \\ \hline & | \ 7 \ T \ 0 \end{array} = \{7,10,0,1,3\}$$

$$EA, 2: \langle 9, \{6\} \rangle \otimes \{7,10,0\} = \underline{0,9} \otimes \{7,10,0\}:$$

$$\begin{array}{r|l} 9 & 4 \ 7 \ 9 \\ 0 & 7 \ T \ 0 \\ \hline & | \ 7 \ T \ 0 \end{array} = \{4,7,9,10,0\}$$

Multiplier replication. Given two pc sets A and B, $AB \supseteq B$. This theorem is perhaps self-evident, given the foregoing examples and the definition of simple multiplication: for every $(b \in B)$, $b \in AB$; therefore, $AB \supseteq B$. Alternatively, since each element of a multiplicand OIS may be considered as a T_n value for the multiplier, and any OIS by definition contains $\underline{0}$, then $T_0(B) = B$.¹³ (This is why the bottom row of the product in every calculation matrix duplicates the multiplier.)

Although this discussion of simple multiplication has been relatively value- and context-free, it is worthwhile to point out that a sequence of n sets produced from a single multiplier will state each multiplier pc n times. Unless the

¹³Variety of products and multiplier replication are formalized somewhat differently in Appendix B.

products are relatively large, the multiplier pc must inevitably assert themselves, via repetition, as more heavily weighted than other pc. A composer striving to avoid such a weighting (Boulez, for an apparent example) will likely attempt to circumvent this feature of simple multiplication through one method or another, such as those described below in Chapters 5 and 6.

Transpositional equivalence of products. Pitch-class set multiplication is a type of transpositional combination. An operand pc set will not change T_n type regardless of initial ordering, OIS, or transposition. Cohn has proven that the transpositional combination of two T_n -type sets will always result in a particular T_n type.¹⁴ Neither the transposition nor the OIS of the operand sets will have any effect on the T_n type of the result.¹⁵ This characteristic of product transpositional equivalence, expressible as $AB \cong BA$, has been demonstrated in each of Figs. 4.14 through 4.18: one operand has been of the T_n type (0,4,7) while the other has been (0,3), and the product has been (0,2,5,6,9). The T_n

¹⁴This result will usually be a T_n type different from that of either operand; see Fig. 4.20, however, for an example of T_n type replication. See also Cohn, "Transpositional Combination in Twentieth-Century Music," pp. 88, 100, 133-148 for a fuller discussion of idempotent relations such as Fig. 4.21.

¹⁵Cohn, "Transpositional Combination in Twentieth-Century Music," pp. 63-64.

type of all products in the cited examples is predictable via a TC matrix, even if the pc content is not:

$$\begin{array}{r|l} 7 & 7 \text{ T} \\ 4 & 4 \text{ 7} \\ \hline 0 & 0 \text{ 3} \\ | & 0 \text{ 3} \end{array} = \{10,0,3,4,7\} \rightarrow (0,2,5,6,9)$$

Figure 4.23. Calculation of $(0,4,7) * (0,3)$

Similarly, Boulez's products shown in Fig. 4.22 are all of the T_n type $(0,3,5,6,8)$, since one operand = $(0,3,5)$ and the other = $(0,3)$:

$$\begin{array}{r|l} 5 & 5 \text{ 8} \\ 3 & 3 \text{ 6} \\ \hline 0 & 0 \text{ 3} \\ | & 0 \text{ 3} \end{array} = \{0,3,5,6,8\} \rightarrow (0,3,5,6,8)$$

Figure 4.24. Calculation of $(0,3,5) * (0,3)$

It is worth considering the property of transpositional equivalence within the scheme of theory presented to this point. Simple multiplication of pc sets is not a particularly elegant process. It is not commutative; $AB \neq BA$. Because of variety of products, it does not even guarantee that AB necessarily equals AB --the OIS of the multiplicand is changeable under most circumstances. The pc content of the multiplicand is hierarchically secondary to its various forms of OIS; even if $A \neq C$, AB may equal CB . Initial ordering of the multiplicand is a requirement for

achieving a product, but no preferential criteria for initial ordering have been presented; there is, therefore, no reason to prefer one product to another. The multiplier continually replicates itself. The product is an unordered set only by definition, since one of its generating operands has been initially ordered. In fact, if the theorem of simple multiplication were considered as a "black box" for production of pc sets, it would be a balky black box indeed.¹⁶ But transpositional equivalence of products is an unquestionably elegant feature of simple multiplication, operative regardless of problematic areas such as ordering of or within operands. Without transpositional equivalence, the other types of multiplication described herein (compound and complex) would probably not exist, as this elegance is carried forward and built on until it reaches its peak in complex multiplication.

¹⁶For a description and evaluation of an earlier (1952) Boulez "machine," see György Ligeti, "Pierre Boulez: Decision and Automatism in *Structures Ia*," *Die Reihe* 4, trans. Leo Black (Bryn Mawr: Theodore Presser, 1960), pp. 36-62 *passim*.

5. Compound Multiplication

It was explained in the previous chapter that the multiplier must be a subset of the product in any simple multiplication of pc sets. The more subjective observation was made that this property would lead to a hierarchization of any pc contained within the multiplier. Whether this is the rationale for the avoidance of simple multiplication as the generating process in *Le Marteau's* pc domains is not really the question at hand; however, the fact that these domains consist of the products of some commutative process means that some modification of noncommutative simple multiplication has been implemented. Intuitively, this could be the result of defining simple multiplication as commutative, and substituting AB for BA whenever the latter operation is called for; yet a multiplier *must* be a subset of the product, and one counterexample will prove that simple multiplication alone has not been employed.

Le Marteau's pc domains have not yet been explored, but the following example from one of them will illustrate the point. The first domain includes a set, {8,10,11,1,2},

that apparently has been produced in one of two ways.¹ The first is that it is the product of two operand sets segmented from the original row (P_0), {9,0} and {4,6,7}. The second is that it is the product of two operand sets from P_3 , {0,3} and {7,9,10}.² Yet the product cannot be the result of simple multiplication of any of these operands, since none is a subset of {8,10,11,1,2}. Nor is it the result of Cohn's pitch-class set transpositional combination as described in Chapter 3. This example is paradigmatic of Cohn's statement that "it is unclear in what way multiplication motivates the choice of the particular [domain set],"³ as only the T_n -type of the product, (0,2,3,5,6), can be predicted from any of the theory presented to this point.

The most logical conclusion to be drawn is that a product of simple multiplication has been transposed. This is not only theoretically logical (the product under consideration is transpositionally equivalent to the product of simple multiplication) but also *conceptually* logical, for at least two reasons: first, simple multiplication is itself

¹This set will be identified as either 1-CE or 1-EC.

²Koblyakov believes that this is the case, stating that "each domain is based on the transposition of its derived series" (Koblyakov, *Pierre Boulez: A World of Harmony*, p.5), and that P_3 is the basis of the first pc domain. (This will be discussed further in Chapter 6.) Sets {0,3} and {7,9,10} do appear in this domain as sets 1-CD (= 1-DC) and 1-DE (= 1-ED).

³Cohn, "Transpositional Combination in Twentieth-Century Music," pp. 49-50.

a series of transpositions; second, transposition of the product will in most cases eliminate the problematic appearance of the multiplier as a product subset.⁴ *Compound multiplication* is an operation whereby a product of simple multiplication is transposed according to a given schema.

Given these observations, it now becomes necessary to provide reasonable criteria for the selection of transposition values. One possibility involves selecting a single pc from each operand, calculating a T_n value between them, and applying that value to the simple multiplicative product. The theory presented to this point suggests that the most logical pc to isolate is the initial pitch class from an IO set. (Recall that the initial pc is designated with the letter r; the initial pc in each of IO sets A and B will appear as r(A) and r(B) respectively.)

If the first equation above, $\{4,6,7\} \otimes \{9,0\} = \{8,10,11,1,2\}$, is desired, an algorithm involving an IO set multiplier might operate like this:

$OIS(A) \otimes B = \text{simple } AB$ (the simple multiplicative product)

$T_n(\text{simple } AB) = AB$, where $n = i\langle r(A), r(B) \rangle$

This algorithm works as follows:

⁴ T_0 will replicate the multiplier; see Chapter 4, note 4 for possibilities of multiplier-as-subset "reappearance" due to transpositional symmetry.

$$\begin{aligned}
 r\langle 7, \{4, 6\} \rangle &= 7; \quad r\langle 9, \{0\} \rangle = 9 \\
 i\langle 7, 9 \rangle &= 2 \\
 \underline{0, 9, 11} \otimes \langle 9, \{0\} \rangle &= \{6, 8, 9, 11, 0\} \\
 T_2\{6, 8, 9, 11, 0\} &= \{8, 10, 11, 1, 2\}
 \end{aligned}$$

Figure 5.1. Calculation of $\langle 7, \{4, 6\} \rangle \otimes \langle 9, \{0\} \rangle$

Reversing the operands and reordering of one of the IO sets results in the same product:

$$\begin{aligned}
 r\langle 0, \{9\} \rangle &= 0; \quad r\langle 7, \{4, 6\} \rangle = 7 \\
 i\langle 0, 7 \rangle &= 7 \\
 \underline{0, 9} \otimes \langle 7, \{4, 6\} \rangle &= \{1, 3, 4, 6, 7\} \\
 T_7\{1, 3, 4, 6, 7\} &= \{8, 10, 11, 1, 2\}
 \end{aligned}$$

Figure 5.2. Calculation of $\langle 0, \{9\} \rangle \otimes \langle 7, \{4, 6\} \rangle$

This seems to be, on its face, a logical method. However, despite the identical products, it is not a commutative process, as one of the operands has had to be reordered. Literal reversal of the operands of Ex. 5.1 gives this result:

$$\begin{aligned}
 r\langle 9, \{0\} \rangle &= 9; \quad r\langle 7, \{4,6\} \rangle = 7 \\
 i\langle 9, 7 \rangle &= 10 \\
 \underline{0,3} \otimes \langle 7, \{4,6\} \rangle &= \{4,6,7,9,10\} \\
 T_{10}\{4,6,7,9,10\} &= \{2,4,5,7,9\}
 \end{aligned}$$

Figure 5.3. Calculation of $\langle 9, \{0\} \rangle \otimes \langle 7, \{4,6\} \rangle$

This product is not merely different from the original, it is as far removed from it (T_6) as possible. Other orderings of the operands will produce still different results, so that one of the problematic areas of simple multiplication--variety of products--again manifests itself. The algorithm has also been applied in a procrustean way, tailored to fit the desired result. Finally, initial ordering of the multiplier, unnecessary in simple multiplication, is required of this method.

Having set up and knocked down a straw man, let us consider another approach, reproduced here as Figure 5.4, invented by Boulez and described by him in this way:

If the ensemble of all the complexes [i.e., pc sets in Line 1] is multiplied by a given complex, this will result in a series of complexes of mobile density [i.e., cardinality], of which, in addition, certain constituents will be irregularly reducible; although *multiple* and *variable*, these complexes are deduced from one another in the most functional way possible, in that they obey a logical, coherent structure.⁵

⁵Boulez, *Boulez on Music Today*, pp. 39-40. Sets inside the dashed box in Figure 5.4 are implied but not literally given by Boulez. The line designations 1,2,3 and 4 have been added here for reference purposes. This example probably provided Koblyakov with valuable clues

The figure displays four staves of musical notation. Staff 1 contains a single melodic line with a circled notehead and a box around a group of notes. Staves 2, 3, and 4 are multi-measure rests, each with a circled notehead and a bracket indicating a duration. A dashed box encloses the circled noteheads on staves 2, 3, and 4. Arrows and numbers (1, 2, 3) indicate relationships between the notes and rests.

Figure 5.4. Compound multiplication in
Boulez on Music Today

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Boulez's staff notation shows invented noteheads--circles with dots inside--placed to represent all of the

in his detective work: Line 1 consists of sets partitioned from the *Marteau* row, and Lines 1 and 2 appear in the fifth pc domain as fields VE and C respectively; see Chapter 6.

Line 1. Three arrows connect each of the first three noteheads with the fourth; the pairs of notes thus linked are replicated individually as the rightmost figures of Lines 2, 3 and 4 respectively.

Figure 5.4 may be interpreted as presenting different schemata for compound multiplication, explored here.

Examining Line 1, an inference may be made that the D in the first set is the multiplicand IO set's initial element and that the third set, shown in a solid box, is the multiplier:

$$\begin{array}{r}
 3 | E 0 2 3 \\
 1 | 9 T 0 1 \\
 \hline
 0 | 8 9 E 0 \\
 | 8 9 E 0
 \end{array} = \{8, 9, 10, 11, 0, 1, 2, 3\}$$

Figure 5.5. Calculation of $\langle 2, \{3, 5\} \rangle \otimes \{8, 9, 11, 0\}$

The product of simple multiplication matches the first pc set of Line 2. But an extension of the conjecture falls short, as can be seen in Line 3. The noteheads suggest changing the multiplicand IO set, and therefore the product:

$$\begin{array}{r}
 T | 6 \ 7 \ 9 \ T \\
 9 | 5 \ 6 \ 8 \ 9 \\
 \hline
 0 | 8 \ 9 \ E \ 0 \\
 | 8 \ 9 \ E \ 0
 \end{array} = \{5, 6, 7, 8, 9, 10, 11, 0\}$$

Figure 5.6. Calculation of $\langle 5, \{2, 3 \} \rangle \otimes \{8, 9, 11, 0\}$

This does not match the first pc set of Line 3, which is $\{11, 0, 1, 2, 3, 4, 5, 6\}$. Predictably, the two are transpositionally equivalent; less predictably, each is T_6 of the other. As will frequently be the case in compound multiplication, the multiplier is no longer a subset of the product.

Line 4 suggests the following:

$$\begin{array}{r}
 E | 7 \ 8 \ T \ E \\
 2 | T \ E \ 1 \ 2 \\
 \hline
 0 | 8 \ 9 \ E \ 0 \\
 | 8 \ 9 \ E \ 0
 \end{array} = \{7, 8, 9, 10, 11, 0, 1, 2\}$$

Figure 5.7. Calculation of $\langle 3, \{5, 2 \} \rangle \otimes \{8, 9, 11, 0\}$

The first set of Line 4, $\{9, 10, 11, 0, 1, 2, 3, 4\}$, is T_2 of the simple multiplicative product.

None of the foregoing calculations seems obscure, but none is predictable, either. Reading the intervals shown on the far right of Lines 2, 3, and 4 as transposition determinants provides no assistance. For example, the T_6 value of Line 2 (D-A \flat) is only used in Line 3. There is, however, an interesting relationship among each of the first sets of Lines 2 through 4, pc 2, and the original simple multiplica-

tive product $\langle 2, \{3, 5\} \rangle \otimes \{8, 9, 11, 0\}$, expressible through the following observations:

1. The first product in Line 2, where $r(A) = 2$ (the invented notehead D), = T_0 of the original simple multiplicative product; $T_0(2) = 2$.

2. The first product in Line 3, where $r(A) = 5$ (the invented notehead F), = T_3 of the original simple multiplicative product; $T_3(2) = 5$.

3. The first product in Line 4, where $r(A) = 3$ (the invented notehead E), = T_1 of the original simple multiplicative product; $T_1(2) = 3$.

In this schema, multiplicand pc 2, in tandem with the initial pc of the multiplicand, determines transposition levels in each of Lines 2 through 4. This corresponds with Boulez's identification of $\{8, 9, 11, 0\}$ as the multiplier in Fig. 5.4.

If, however, $\{8, 9, 11, 0\}$ is taken to be the multiplicand in each operation with A designated as the initially-ordered pitch class (represented as $\langle 8, \{9, 11, 0\} \rangle$), the invented note-head figures in each of Lines 2 through 4, read right to left, provide T_n values for each simple multiplicative product: Line 2, T_6 ; Line 3, T_9 ; and Line 4, T_7 . Thus, the products shown in Lines 2 through 4 (labeled here respectively by line number and horizontal position as A through E) can be calculated as follows:

$$A. \langle 8, \{9, 11, 0\} \rangle \otimes \{2, 3, 5\} =$$

$$\underline{0, 1, 3, 4} \otimes \{2, 3, 5\} = \{2, 3, 4, 5, 6, 7, 8, 9\}$$

$$T_6\{2, 3, 4, 5, 6, 7, 8, 9\} = \{8, 9, 10, 11, 0, 1, 2, 3\} \text{ (Product 2A)}$$

$$T_9\{2, 3, 4, 5, 6, 7, 8, 9\} = \{11, 0, 1, 2, 3, 4, 5, 6\} \text{ (Product 3A)}$$

$$T_7\{2, 3, 4, 5, 6, 7, 8, 9\} = \{9, 10, 11, 0, 1, 2, 3, 4\} \text{ (Product 4A)}$$

$$B. \langle 8, \{9, 11, 0\} \rangle \otimes \{10, 1\} =$$

$$\underline{0, 1, 3, 4} \otimes \{10, 1\} = \{10, 11, 1, 2, 4, 5\}$$

$$T_6\{10, 11, 1, 2, 4, 5\} = \{4, 5, 7, 8, 10, 11\} \text{ (Product 2B)}$$

$$T_9\{10, 11, 1, 2, 4, 5\} = \{7, 8, 10, 11, 1, 2\} \text{ (Product 3B)}$$

$$T_7\{10, 11, 1, 2, 4, 5\} = \{5, 6, 8, 9, 11, 0\} \text{ (Product 4B)}$$

$$C. \langle 8, \{9, 11, 0\} \rangle \otimes \{8, 9, 11, 0\} =$$

$$\underline{0, 1, 3, 4} \otimes \{8, 9, 11, 0\} = \{8, 9, 10, 11, 0, 1, 2, 3, 4\}$$

$$T_6\{8, 9, 10, 11, 0, 1, 2, 3, 4\} = \{2, 3, 4, 5, 6, 7, 8, 9, 10\} \text{ (Product 2C)}$$

$$T_9\{8, 9, 10, 11, 0, 1, 2, 3, 4\} = \{5, 6, 7, 8, 9, 10, 11, 0, 1\} \text{ (Product 3C)}$$

$$T_7\{8, 9, 10, 11, 0, 1, 2, 3, 4\} = \{3, 4, 5, 6, 7, 8, 9, 10, 11\} \text{ (Product 4C)}$$

$$D. \langle 8, \{9, 11, 0\} \rangle \otimes \{4, 7\} =$$

$$\underline{0, 1, 3, 4} \otimes \{4, 7\} = \{4, 5, 7, 8, 10, 11\}$$

$$T_6\{4, 5, 7, 8, 10, 11\} = \{10, 11, 1, 2, 4, 5\} \text{ (Product 2D)}$$

$$T_9\{4, 5, 7, 8, 10, 11\} = \{1, 2, 4, 5, 7, 8\} \text{ (Product 3D)}$$

$$T_7\{4, 5, 7, 8, 10, 11\} = \{11, 0, 2, 3, 5, 6\} \text{ (Product 4D)}$$

$$E. \langle 8, \{9, 11, 0\} \rangle \otimes \{6\} =$$

$$\underline{0, 1, 3, 4} \otimes \{6\} = \{6, 7, 9, 10\}$$

$$T_6\{6, 7, 9, 10\} = \{0, 1, 3, 4\} \text{ (Product 2E)}$$

$$T_9\{6, 7, 9, 10\} = \{3, 4, 6, 7\} \text{ (Product 3E)}$$

$$T_7\{6, 7, 9, 10\} = \{1, 2, 4, 5\} \text{ (Product 4E)}$$

In either reading offered here, Boulez appears to present a solution to the problem of multiplier replication, yet there is still no rationale for preferring the products of any of Lines 2 through 4 to the others. And until only a single product can be derived from two operands, questions of commutativity--and, further, of elegance--must remain unanswered. Behind the opacity of Boulez's prose (or that of his translators) lies a riddle: what is the "most functional way possible" of arriving at a single, elegant solution?

**6. The Theory of Complex Multiplication and the
Generation of Pitch-class Domains**

Boulez's delineation of pitch-class domain structure¹ represents multiplicative products on a *domain matrix* (as distinct from a TC matrix). Five multiplicand sets, here denoted generally by *Y* and more specifically by *A* through *E*, form the vertical axis; the same sets, functioning as multipliers and denoted generally by *Z*, form the horizontal axis:

	Z=A	Z=B	Z=C	Z=D	Z=E
Y=A	AA	AB	AC	AD	AE
Y=B	BA	BB	BC	BD	BE
Y=C	CA	CB	CC	CD	CE
Y=D	DA	DB	DC	DD	DE
Y=E	EA	EB	EC	ED	EE

Figure 6.1. Domain matrix
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¹Boulez, *Boulez on Music Today*, pp. 79-80. The original, like Figure 4.22, has pc-sets represented by lower-case letters; upper-case letters are used here to conform with usage elsewhere in this paper.

Koblyakov's analyses of the first cycle and his representation of pc domains show the products as commutatively generated: $AB = BA$, etc. Within each domain, therefore, fifteen unique pc sets are possible. The domain matrix is mirror-symmetrical with respect to the AA-EE diagonal.

Table 6.1 shows the five pitch-class domains used in the first cycle of *Le Marteau sans maître*. The top row, or *field*, of each domain strays from the model in Figure 6.1 in that the field generated by $Y=A$ has been jettisoned² in favor of five *V-sets*--unordered pc sets³ partitioned from the untransposed original form of the *Marteau* 12-tone row:

$$P_0 = \langle 3, 5, 2, 1, 10, 11, 9, 0, 8, 4, 7, 6 \rangle$$

Figure 6.2. The *Marteau* row

The partitioning into *V-sets* is accomplished according to the sequence of cardinalities 2/4/2/1/3. This sequence--but not the row--is rotated to the left (i.e., the first cardinality of one rotation becomes the last cardinality of the next) to produce five different fields of *V-sets*. The original

²The replacement of the $Y=A$ field is noted but not explained by Koblyakov in *Pierre Boulez: A World of Harmony*, amending his analysis in "P. Boulez 'le marteau sans maître'--Analysis of pitch structure." One explanation of this replacement will be offered below.

³The term "field" and the "V" designation follow Koblyakov, "P. Boulez 'le marteau sans maître'--Analysis of pitch structure." Koblyakov changes the designation of *V-sets* (to a bracketing system) in *Pierre Boulez: A World of Harmony*, but, for reasons given below, I prefer his original system and use it here.

Table 6.1. Pitch-class domains for the first cycle of
Le Marteau sans maître

<i>Domain 1</i>					
	A	B	C	D	E
VD	{3,5}	{10,11,1,2}	{9,0}	{8}	{4,6,7}
B	{8,9,10,11,0,1,2}	{3,4,5,6,7,8,9,10,11}	{2,3,5,6,8,9}	{1,2,4,5}	{9,10,11,0,1,2,3,4}
C	{7,9,10,0}	{2,3,5,6,8,9}	{1,4,7}	{0,3}	{8,10,11,1,2}
D	{6,8}	{1,2,4,5}	{0,3}	{11}	{7,9,10}
E	{2,4,5,6,7}	{9,10,11,0,1,2,3,4}	{8,10,11,1,2}	{7,9,10}	{3,5,6,7,8,9}
<i>Domain 2</i>					
	A	B	C	D	E
VC	{1,2,3,5}	{10,11}	{9}	{0,4,8}	{6,7}
B	{9,10,11,0,1,2}	{6,7,8}	{5,6}	{0,1,4,5,8,9}	{2,3,4}
C	{8,9,10,0}	{5,6}	{4}	{3,7,11}	{1,2}
D	{3,4,5,7,8,9,11,0,1}	{0,1,4,5,8,9}	{3,7,11}	{2,6,10}	{0,1,4,5,8,9}
E	{5,6,7,8,9,10}	{2,3,4}	{1,2}	{0,1,4,5,8,9}	{10,11,0}
<i>Domain 3</i>					
	A	B	C	D	E
VB	{3,5}	{2}	{10,11,1}	{9,0}	{4,6,7,8}
B	{0,2}	{11}	{7,8,10}	{6,9}	{1,3,4,5}
C	{8,9,10,11,1}	{7,8,10}	{3,4,5,6,7,9}	{2,3,5,6,8}	{9,10,11,0,1,2,3,4}
D	{7,9,10,0}	{6,9}	{2,3,5,6,8}	{1,4,7}	{8,10,11,0,1,2,3}
E	{2,4,5,6,7,8}	{1,3,4,5}	{9,10,11,0,1,2,3,4}	{8,10,11,0,1,2,3}	{3,5,6,7,8,9,10,11}
<i>Domain 4</i>					
	A	B	C	D	E
VA	{3}	{1,2,5}	{10,11}	{8,9,0,4}	{6,7}
B	{1,2,5}	{11,0,1,3,4,7}	{8,9,10,0,1}	{6,7,8,10,11,2,3}	{4,5,6,8,9}
C	{10,11}	{8,9,10,0,1}	{5,6,7}	{3,4,5,7,8,11,0}	{1,2,3}
D	{8,9,0,4}	{6,7,8,10,11,2,3}	{3,4,5,7,8,11,0}	{1,2,3,5,6,9,10}	{11,0,1,3,4,7,8}
E	{6,7}	{4,5,6,8,9}	{1,2,3}	{11,0,1,3,4,7,8}	{9,10,11}
<i>Domain 5</i>					
	A	B	C	D	E
VE	{2,3,5}	{10,1}	{8,9,11,0}	{4,7}	{6}
B	{10,11,1,2,4}	{6,9,0}	{4,5,7,8,10,11}	{0,3,6}	{2,5}
C	{8,9,10,11,0,1,2,3}	{4,5,7,8,10,11}	{2,3,4,5,6,7,8,9,10}	{10,11,1,2,4,5}	{0,1,3,4}
D	{4,5,7,8,10}	{0,3,6}	{10,11,1,2,4,5}	{6,9,0}	{8,11}
E	{6,7,9}	{2,5}	{0,1,3,4}	{8,11}	{10}

sequence produces the V-sets of Domain 1; rotation 1 produces the V-sets of Domain 2, and so on. This is illustrated in Figure 6.3 where, for sake of clarity, V-sets are shown not in normal form but in P_0 order.

<u>Domain</u>	<u>Partition</u>	<u>V-sets</u>
1.	2/4/2/1/3	{3,5} {2,1,10,11} {9,0} {8} {4,7,6}
2.	4/2/1/3/2	{3,5,2,1} {10,11} {9} {0,8,4} {7,6}
3.	2/1/3/2/4	{3,5} {2} {1,10,11} {9,0} {8,4,7,6}
4.	1/3/2/4/2	{3} {5,2,1} {10,11} {9,0,8,4} {7,6}
5.	3/2/4/2/1	{3,5,2} {1,10} {11,9,0,8} {4,7} {6}

Figure 6.3. Formation of *Le Marteau* V-sets

Individual V-sets are identified by a general model VXY as follows:

1. The letter V designates set VXY as a V-set.
2. The X changes in each domain and is specified by the letters A, B, C, D and E. These letters represent, in order, the first through fifth positions of the monad in any partition of the row. Thus, V-sets from Domain 1 are identified generally as VDY, since the monad occupies the fourth position in the 2/4/2/1/3 partition sequence.

3. The Y refers to each column and is again represented by A through E, where A is the first column, B the second, etc.

4. These designations provide identification from the abstract ($VXY = \text{any V-set}$) to the generalized ($VCY = \text{any V-set in Domain 2}$; $VXA = \text{the first V-set in any domain}$) to the specific ($VBC = \{10,11,1\}$ in Domain 3).

5. When paired with a reference to VXZ , VXY refers to a multiplicand, while VXZ refers to a multiplier. The Z of VXZ is represented by the letters A through E in exactly the same manner as Y. The Y and Z correspond, therefore, with the domain matrix in Fig. 6.1.

A *domain set* is any set within a domain that has been generated by a multiplicative operation. Such a set is identified in Table 6.1 according to its position within the domain matrix of Fig. 6.1, with the multiplicand designation first, the multiplier second. Thus, in Domain 1, domain set $CA = \{7,9,10,0\}$. Since every domain has a CA position, this set may be identified specifically by domain number and grid position, separated by a hyphen. $\{7,9,10,0\}$ can be designated in this way as 1-CA. However, since domains are more often considered individually, this hyphenate is not always used.⁴

⁴The foregoing information in this chapter is freely adapted and re-formalized from Koblyakov's writings.

It is the position of this study that the domain sets of *Le Marteau* are generated entirely by V-sets via a process I call *complex multiplication*. According to this process, two V-sets VXY and VXZ will be, respectively, the multiplicand and multiplier in complex multiplication to produce a domain set, YZ.

This position differs significantly from that of Koblyakov, who apparently believes that the five domains are generated by sets partitioned from, respectively, P₃, P₇, P₉, P₀, and P₄ by some kind of multiplication which he does not describe.⁵ There is evidently support for this conclusion. In Table 6.1, the V-sets are designated by the letters V and one of A through E as explained above. The corresponding field A through E within each domain shows transpositions of the V-sets by, respectively, T₃, T₇, T₉, T₀, and T₄. (E.g., Domain 1: T₃(field VD) = field D.)

There is also the matter of apparent inconsistency with regard to products. If only V-sets are used as

⁵Koblyakov is actually self-contradictory on the matter. His illustration of multiplication (reproduced above as Fig. 4.1) shows domain set 1-AA as the product of VXA ⊗ VXA--{3,5} ⊗ {3,5}--which corresponds to the theory presented herein. But he writes that "Each of the five marked sounds [i.e., V-sets] could become the basis of a transposition for its derived series. *Boulez however starts by an additional pitch transposition of the derived series* [meaning that 1-AA is the product of {6,8} ⊗ {6,8}], with the exception of derived series IV [i.e., V-sets in Domain 4], which keeps the E flat transposition." (Emphasis added. From *Pierre Boulez: A World of Harmony*, p. 3.) It is difficult to interpret his statement as agreeing with the position I have taken, just as it is difficult to justify the *particular* transpositions (including T₀ in Domain 4) without recourse to the present theory.

operands, the domains in Table 6.1 will reveal the operations shown in Figure 6.4. In the first pair of operations, V-sets of the same pc content produce identical products, but in the second pair, the same operands produce different products.

1. Domain 1: $VDC \otimes VDA = \{9,0\} \otimes \{3,5\} = CA = \{7,9,10,0\}$
2. Domain 3: $VBD \otimes VBA = \{9,0\} \otimes \{3,5\} = DA = \{7,9,10,0\}$
3. Domain 2: $VCE \otimes VCB = \{6,7\} \otimes \{10,11\} = EB = \{2,3,4\}$
4. Domain 4: $VAE \otimes VAC = \{6,7\} \otimes \{10,11\} = EC = \{1,2,3\}$

Figure 6.4. Selected domain operations

This seems contradictory, but, in fact, similarities and differences such as these are accounted for in the theory of complex multiplication as detailed below.⁶

The previous chapter's discussion of Figure 5.4 and its compound multiplicative method introduced a new consideration: the possible importance of a pc within a multiplicand other than its OIS-producing function. Pitch class 8, the initial pc of the multiplicand, was paired with each element

⁶Such differences may have convinced Koblyakov that V-sets were not the true domain set generators. My speculation is that, given an understanding of compound (not complex) multiplication principles, he was able to predict the T_n types of domain sets, segment pc-set representatives of these T_n types in *Marteau*, and show these representatives on a domain matrix. He may then have assumed that they were products of transposed V-sets, and that the original V-sets were simply substituted for the A field in each domain--hence the bracketing of these sets in his analysis in *Pierre Boulez: A World of Harmony* (see note 3 above). This paper demonstrates that the "transposed V-sets" are actually products of complex multiplication.

of {2,3,5} to provide transposition levels for each set in Lines 2 through 4. An extension of this consideration might include the use of a transposition-determining pc which remains constant and a part of the operation *even if it is not an element of either operand*. Not coincidentally, this is exactly the "missing link" which disposes of every problematic feature of simple and compound multiplication.

Transposition-determining constant (k). The transposition-determining constant is a pitch class, chosen according to some schema, that participates in a complex multiplicative operation or group of operations. This constant, designated k , may or may not be an element of the multiplicand, multiplier, or product. A V-set in which k is an element may be referred to as a k -set where this feature is important.

Complex multiplication employs the principles of simple and compound multiplication, and is unified and catalyzed by the transposition-determining constant.

Complex multiplication of pc sets. Where k is the transposition-determining constant and A and B are pc sets, the formula for the complex multiplication of A by B is:

$$\text{OIS}(A) \otimes B = \text{simple } AB; T_n(\text{simple } AB) = AB,$$

where $n = i\langle k, r(A) \rangle$

This theorem is in two steps:

Step 1: The simple multiplication, as described in Chapter 4, of the OIS of IO set A by multiplier B, resulting in a simple multiplicative product AB (here again called "simple AB");

Step 2: The transposition of this simple multiplicative product by the ordered pitch-class interval from the transposition-determining constant to the initial pitch class of the multiplicand.⁷

The following hypothetical examples will demonstrate the mechanics and some of the more intriguing features of complex multiplication. There are many ways that a transposition-determining constant could be chosen--the discussion of the *Marteau* pc domains will demonstrate one algorithm for this--but for purposes of these examples, $k = 5$. The first set of examples borrows the operation $\{7,10,0\} \otimes \{6,9\}$ from *Boulez on Music Today*, p. 80 (cited above as Fig. 4.22).

(See Fig. 6.5) Where $k = 5$, calculate $\langle 7, \{10,0\} \rangle \otimes \{6,9\}$:

(Step 1) $\underline{0,3,5} \otimes \{6,9\} = \{6,9,11,0,2\}$

(Step 2) (since $i\langle 5,7 \rangle = 2$.) $T_2\{6,9,11,0,2\} = \{8,11,1,2,4\}$

⁷Alternatively, T_n may be performed on the multiplier prior to multiplication, so that $OIS(A) \otimes (T_n(B)) = AB$. This corollary is proven in Appendix B, and will be referenced below to demonstrate k-set co-operand replication.

Initial ordering of multiplicand. No preferential criteria for initial ordering have been given, because in complex multiplication no such criteria are necessary: any initial ordering of the multiplicand will result in the same product as long as k is not changed. Figures 6.6 and 6.7 show the other OIS of $\{7,10,0\}$ and the product $\{8,11,1,2,4\}$.

(See Fig. 6.6) Calculate $\langle 0, \{7,10\} \rangle \otimes \{6,9\}$:

$$\text{(Step 1) } \underline{0,7,10} \otimes \{6,9\} = \{1,4,6,7,9\}$$

$$\text{(Step 2) } (\text{since } i\langle 5,0 \rangle = 7,) T_7\{1,4,6,7,9\} = \{8,11,1,2,4\}$$

(See Fig. 6.7) Calculate $\langle 10, \{0,7\} \rangle \otimes \{6,9\}$:

$$\text{(Step 1) } \underline{0,2,9} \otimes \{6,9\} = \{3,6,8,9,11\}$$

$$\text{(Step 2) } (\text{since } i\langle 5,10 \rangle = 5,) T_5\{3,6,8,9,11\} = \{8,11,1,2,4\}$$

Initial ordering of the multiplicand remains a necessary part of the process, but just *which* initial ordering is chosen is irrelevant; proof of this is given in Appendix B. Unlike in simple and compound multiplication, initial ordering takes place inside the "black box" in complex multiplication.

Commutativity. Further, and even more elegantly, complex multiplication is commutative. Figures 6.8 and 6.9 show the operands reversed from the preceding figures and both possible initial orderings of the new multiplicand:

(See Fig. 6.8) Calculate $\langle 6, \{9\} \rangle \otimes \{7,10,0\}$:

$$\text{(Step 1) } \underline{0,3} \otimes \{7,10,0\} = \{7,10,0,1,3\}$$

$$\text{(Step 2) } (\text{since } i\langle 5,6 \rangle = 1,) T_1\{7,10,0,1,3\} = \{8,11,1,2,4\}$$

Figure 6.5. $\langle 7, \{10, 0\} \rangle \otimes \{6, 9\} = \{8, 11, 1, 2, 4\}$

Figure 6.6. $\langle 0, \{7, 10\} \rangle \otimes \{6, 9\} = \{8, 11, 1, 2, 4\}$

Figure 6.7. $\langle 10, \{0, 7\} \rangle \otimes \{6, 9\} = \{8, 11, 1, 2, 4\}$

Figure 6.8. $\langle 6, \{9\} \rangle \otimes \{7, 10, 0\} = \{8, 11, 1, 2, 4\}$

Figure 6.9. $\langle 9, \{6\} \rangle \otimes \{7, 10, 0\} = \{8, 11, 1, 2, 4\}$

(See Fig. 6.9) Calculate $\langle 9, \{6\} \rangle \otimes \{7, 10, 0\}$:

(Step 1) $0, 9 \otimes \{7, 10, 0\} = \{4, 7, 9, 10, 0\}$

(Step 2) (since $i\langle 5, 9 \rangle = 4$,) $T_4\{4, 7, 9, 10, 0\} = \{8, 11, 1, 2, 4\}$

Proof of the commutativity of complex multiplication is given in Appendix B.

Multiplicand pc sets. The pc content of the multiplicand now takes on an importance aside from its OIS. Figures 4.17 and 4.18 showed an apparent conceptual incongruity of simple multiplication: although $A \neq C$, $AB = CB$ where $OIS(A) = OIS(C)$. This is not the case in complex multiplication. A change of the pc content will alter the T_n value between k and the initial pc; the simple multiplicative products AB and CB will therefore be transposed differently, forming unequal complex multiplicative products.

Since any initial ordering of the multiplicand may be used without changing the product, most calculations below will use an IO set derived from the normal form of the multiplicand: the first listed element in normal form is the initial pitch class.⁸

Determination of k in Le Marteau. The selection of the transposition-determining constant can be done by any means the composer deems satisfactory. In *Le Marteau*, the

⁸This is a convenience, not a requirement. Part of the pleasure of working with complex multiplication includes observing the T_n values change as the initial ordering of the multiplicand changes, while the product remains the same.

following method obtains: k is the rightmost of the first three pc in the original row-- $\langle 3,5,2 \rangle$ --that appears in set VXA. (It is probably no accident that these three pc duplicate the invented noteheads in the first set of Figure 5.4, or that D is the lowest notated pitch of the three.) Thus, in Domains 1 and 3, where VDA and VBA = $\{3,5\}$, $k = 5$; in Domain 2, where VCA = $\{3,5,2,1\}$ and in Domain 5, where VEA = $\{3,5,2\}$, $k = 2$; and in Domain 4, where VAA = $\{3\}$, $k = 3$.

There is no apparent reason that k should not equal 1 in Domain 2, where it is the rightmost pc of P_0 in set VCA; Table 6.2 shows this hypothetical domain. The decision to choose only the first three pc of P_0 seems therefore to have been a precompositional one (actually, pre-precompositional).

Table 6.2. Domain 2 (hypothetical: $k = 1$)

	A	B	C	D	E
VC	{1,2,3,5}	{10,11}	{9}	{0,4,8}	{6,7}
B	{10,11,0,1,2,3}	{7,8,9}	{6,7}	{1,2,5,6,9,10}	{3,4,5}
C	{9,10,11,1}	{6,7}	{5}	{0,4,8}	{2,3}
D	{0,1,2,4,5,6,8,9,10}	{1,2,5,6,9,10}	{0,4,8}	{3,7,11}	{1,2,5,6,9,10}
E	{6,7,8,9,10,11}	{3,4,5}	{2,3}	{1,2,5,6,9,10}	{11,0,1}

The only change within the complex multiplication equation that will effect an alteration of the product of two operand pc sets is a change of the transposition-determining

constant. Each semitone that k is raised will lower the value of n by 1, thus effectively transposing the product down by a semitone. This accounts for both similarities and differences in products when equal V-sets are multiplied in different domains as noted above (Fig. 6.4). Identical results are therefore obtained from the following--

$$1. \text{ Domain 1 (k = 5): } \text{VDC} \otimes \text{VDA} = \langle 9, \{0\} \rangle \otimes \{3, 5\}: \\ \underline{0, 3} \otimes \{3, 5\} = \{3, 5, 6, 8\}; i\langle 5, 9 \rangle = 4 \rightarrow T_4\{3, 5, 6, 8\} = \\ \{7, 9, 10, 0\}$$

$$2. \text{ Domain 3 (k = 5): } \text{VBD} \otimes \text{VBA} = \langle 9, \{0\} \rangle \otimes \{3, 5\}: \\ \underline{0, 3} \otimes \{3, 5\} = \{3, 5, 6, 8\}; i\langle 5, 9 \rangle = 4 \rightarrow T_4\{3, 5, 6, 8\} = \\ \{7, 9, 10, 0\}$$

--while different results are obtained from these operations, as k is different in the cited domains:

$$3. \text{ Domain 2 (k = 2): } \text{VCE} \otimes \text{VCB} = \langle 6, \{7\} \rangle \otimes \{10, 11\}: \\ \underline{0, 1} \otimes \{10, 11\} = \{10, 11, 0\}; i\langle 2, 6 \rangle = 4 \rightarrow T_4\{10, 11, 0\} = \\ \{2, 3, 4\}$$

$$4. \text{ Domain 4 (k = 3): } \text{VAE} \otimes \text{VAC} = \langle 6, \{7\} \rangle \otimes \{10, 11\}: \\ \underline{0, 1} \otimes \{10, 11\} = \{10, 11, 0\}; i\langle 3, 6 \rangle = 3 \rightarrow T_3\{10, 11, 0\} = \\ \{1, 2, 3\}$$

Does this process accurately reflect Boulez's own precompositional process? It is impossible to say with complete certainty, but it should be noted that through complex multiplication as defined here (including the derivation of V-sets and the algorithm for determining k),

the domain sets in Table 6.1 are the only possible products. (Given recent writings on *Le Marteau* that find significance in as little as 71% congruence, this complete agreement is not merely refreshing but very nearly positive proof of its use.⁹) Also, while this process is complex, it is not particularly complicated; one can easily imagine the composer experimenting with structures such as those in Figures 6.5 through 6.9 and delighting in the results.

Replication of k-set co-operand. When $k \in A$, $AB \supseteq B$. This is complex multiplication's equivalent of multiplier replication in simple multiplication. When k is an element of one operand, the other operand will be a subset of the product since an ordering will exist where k is the initial pc of the multiplicand. If $k = r(A)$, $i\langle k, r \rangle = 0$; $T_0(B) = B$.¹⁰ The identity operation T_0 makes the product of complex multiplication equal to the product of simple multiplication, of which the multiplier must be a subset.

This property is demonstrated in each of the *Marteau* pc domains (Table 6.1): where $Y = Z$, $AZ \supseteq VXY$, since $k \in VXA$. For example, in Domain 3, where $k = 5$, $VXA = \{3, 5\}$, VXE

⁹See, for example, Steven D. Winick, "Symmetry and Pitch-Duration Associations in Boulez's *Le Marteau sans maître*," *Perspectives of New Music* 24/2 (Spring/Summer 1986), pp. 280-321, and Wayne C. Wentzel, "Dynamic and Attack Associations in Boulez's *Le Marteau sans maître*," *Perspectives of New Music* 29/1 (Winter 1991), pp. 142-170.

¹⁰See note 7 above.

= {4,6,7,8}, and $n = i\langle 5,5 \rangle = 0$, domain set AE can be calculated as follows:

$$\text{(Step 1) } \langle 5, \{3\} \rangle \otimes \{4,6,7,8\} = \underline{0,10} \otimes \{4,6,7,8\} = \{2,4,5,6,7,8\};$$

$$\text{(Step 2) } T_0\{2,4,5,6,7,8\} = \{2,4,5,6,7,8\}.$$

$$\{2,4,5,6,7,8\} \supseteq \{4,6,7,8\}.$$

Any reordering of or within operands will achieve the same result, even as the T_n value changes;¹¹ for example,

$$\text{(Step 1) } \langle 4, \{6,7,8\} \rangle \otimes \{3,5\} = \underline{0,2,3,4} \otimes \{3,5\} = \{3,5,6,7,8,9\};$$

$$\text{(Step 2) } T_{11}\{3,5,6,7,8,9\} = \{2,4,5,6,7,8\} \text{ (since } i\langle 5,4 \rangle = 11).$$

$$\{2,4,5,6,7,8\} \supseteq \{4,6,7,8\}.$$

This property provides a reasonable explanation for the replacement of field A by V-sets in each domain. The usage of V-sets compositionally (that is, on the musical surface along with domain sets) establishes a literal connection with the original row and is thus conceptually rational. However, k-set co-operand replication requires that each V-set appear in connection with its domain three times--once as VXY, once as a subset of domain set YA, and once as a subset of domain set AZ.¹² By omitting field A,

¹¹Because complex multiplication is commutative, the set in which k is an element may be either multiplicand or multiplier--hence the use of the term "co-operand" here.

¹²Set VXA is the exception to this threefold appearance, appearing as VXA and as a subset of AA.

hierarchization of the V-sets via repetition is avoided. Lost in the transaction is set AA in each domain¹³ (except, by equality, in Domain 4: $VXA = AA = \{3\}$).

Greatest cardinality of domain sets. Koblyakov makes this statement regarding the cardinality of products: "When dividing the general series into five groups, as in the first cycle of *Le Marteau*, the number of sounds in a group cannot in fact be more than 10."¹⁴ No explanation for deriving this maximum cardinality is offered, and he is vague as to whether this is a general principle of multiplication or an assessment specific to *Le Marteau*. Since Koblyakov's statement is not empirically based--no domain set in *Le Marteau* has a cardinality greater than 9--it may reasonably be assumed that he believes this greatest cardinality to be a general principle of multiplication. Given the cardinalities that "divide the general series," 2/4/2/1/3, the largest products will presumably result either from multiplying the tetrachord by itself (a process here called *set squaring*) or from multiplying the tetrachord and the trichord. In theory, according to the formula for unique pairings of n objects $\frac{n^2 + n}{2}$ used to

¹³The elimination of domain set AA makes Koblyakov's choice of set 1-AA as his prototype for multiplication (Fig. 4.1) even more questionable. This set is also the only one in which T_0 can be applied to the simple multiplicative product regardless of the order of operands, creating the illusion that this product has not been subjected to transposition.

¹⁴Koblyakov, *Pierre Boulez: A World of Harmony*, p. 5.

predict greatest cardinality, tetrachordal set squaring can produce a decachord. In practice, however, tetrachordal set squaring will never generate a set with a cardinality exceeding 9; this is demonstrated in Appendix C. Or Koblyakov may have been influenced by an illustration in Boulez's *Notes of an Apprenticeship*, reproduced here as Figure 6.10, which shows the simple multiplication of a trichord by a tetrachord to produce a decachord.¹⁵



Figure 6.10. From *Notes of an Apprenticeship*:
 $\langle 3, \{4, 1\} \rangle \otimes \{6, 9, 10, 2\} = \{6, 7, 8, 9, 10, 11, 0, 2, 3, 4\}$

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However, this example is not the final word on the subject. Since it demonstrates that the multiplication of a

¹⁵Boulez, *Notes of an Apprenticeship*, p. 168. The inversions of each of the T_n types represented here can be found in the *Marteau* row, their multiplication producing a decachord of the same T_n type as Fig. 6.10: $\{3, 5, 2\} \otimes \{9, 0, 8, 4\} \rightarrow (0, 1, 3) * (0, 1, 4, 8) = (0, 1, 2, 3, 4, 5, 6, 8, 9, 10)$. A restructuring of the cardinality sequence would be required for this operation to occur in *Le Marteau*. A charitable reading of Koblyakov's original statement on cardinality could regard this operation as its rationalization.

trichord and a tetrachord can produce a larger set than can tetrachordal set squaring, it invites the inspection of other 3- and 4-note sets. There are twelve different pairings of a trichord and a tetrachord in which transpositional combination will produce the aggregate,¹⁶ as shown in Table 6.3.

Table 6.3. Trichord * tetrachord = aggregate

[0,1,2]	*	[0,3,6,9]	=	U
[0,1,5]	*	[0,3,6,9]	=	U
[0,2,4]	*	[0,1,6,7]	=	U
[0,2,4]	*	[0,3,6,9]	=	U
[0,2,7]	*	[0,3,6,9]	=	U
[0,4,8]	*	[0,1,2,3]	=	U
[0,4,8]	*	[0,1,2,7]	=	U
[0,4,8]	*	[0,1,3,6]	=	U
[0,4,8]	*	[0,1,6,7]	=	U
[0,4,8]	*	[0,2,3,5]	=	U
[0,4,8]	*	[0,2,5,7]	=	U
[0,4,8]	*	[0,3,6,9]	=	U

Every tetrachord in Table 6.3 is transpositionally combined with at least [0,4,8], the trichord which divides the aggregate into thirds and contains only interval class 4; the tetrachords form a complete list of those lacking this interval class. Similarly, every trichord is combined with

¹⁶Cohn, "Transpositional Combination in Twentieth-Century Music," pp. 563-566. In accordance with the theory presented in Chapter 3, T_n/T_nI types can be employed instead of T_n types in Tables 6.3 and 6.4 because each operation involves at least one inversionally symmetrical operand.

at least $[0,3,6,9]$, the tetrachord which divides the aggregate into quarters and contains only interval classes 3 and 6; the trichords form a complete list of those lacking these interval classes. In accordance with the common-tone theorem given in Appendix A, Table 6.3 lists all pairs of trichords and tetrachords with no interval classes held in common.

An examination of the *Marteau* row (Fig. 6.2) will reveal three V-sets, actual or potential, that fit any of the T_n/T_nI -type sets in Table 6.3: $\{10,11,9\} = [0,1,2]$; $\{0,8,4\} = [0,4,8]$; $\{10,11,9,0\} = [0,1,2,3]$. The last two of these appear as paired operands in aggregate production ($[0,4,8] * [0,1,2,3] = U$). However, the pc sets cannot appear in the same domain: because of row partitioning, V-sets are non-intersecting, but $\{0,8,4\} \cap \{10,11,9,0\} = \{0\}$. In fact, the same characteristic is true of every operand pair in Table 6.3. Where pc set A is a trichord and pc set B is a tetrachord, and $|A \otimes B| = 12$, then $|A \cap B| = 1$. (For example, choosing $\{0,2,3,5\}$ as a pc set to represent $[0,2,3,5]$: each pc set representative of the aggregate-producing co-operand $[0,4,8]$ -- $\{0,4,8\}$, $\{1,5,9\}$, $\{2,6,10\}$, and $\{3,7,11\}$ --intersects $\{0,2,3,5\}$ by one element.) Therefore, it is impossible to produce the aggregate via multiplication of non-intersecting pc sets of cardinalities 3 and 4.¹⁷

¹⁷A restructuring of the V-set cardinality sequence to include two tetrachords would permit aggregate production--for example, $(0,2,5,8) * (0,1,2,6) = U$, representable in non-intersecting pc-sets as

Producing sets of cardinality 11 (hypothetical Forte set class¹⁸ 11-1) is another matter. There are eight ways in which a trichord and a tetrachord can be transpositionally combined with the result $[0,1,2,3,4,5,6,7,8,9,10]$,¹⁹ and all eight can be represented by non-intersecting pc sets. These combinations and examples of representative pc sets are shown in Table 6.4.

$\{0,2,5,8\} \otimes \{9,10,11,3\} = U$. (Inverting either, but not both, of these T_n types produces an undecachord.) However, such a restructuring would necessitate the omission of either a dyad or a trichord from the cardinality sequence, since $(4+4+3+2) > 12$. Aggregate production also trivializes the transposition-determining constant, since $T_n(U) = U$.

¹⁸Allen Forte, *The Structure of Atonal Music* (New Haven: Yale University Press, 1973), pp. 11-13, 179-181. Forte's well-known (but not always-used) taxonomy employs a two-number hyphenate, of which the first shows cardinality. Only sets of cardinalities 3 through 9 inclusive are labeled by Forte. Any eleven-pc set must have a T_n/T_nI type of $[0,1,2,3,4,5,6,7,8,9,10]$, which would be 11-1 in his system. (Its monadic complement $[0]$ would be 1-1.) See also Cohn, "Transpositional Combination in Twentieth-Century Music," pp. 54 (n1) and 598-599.

¹⁹Cohn, "Transpositional Combination in Twentieth-Century Music," pp. 563-566.

Table 6.4. Trichord * tetrachord = undecachord;
non-intersecting pc set representatives

$[0,1,2] * [0,2,5,8] = 11-1; \{0,1,2\} \otimes \{3,5,8,11\}$
 $[0,1,2] * [0,3,5,8] = 11-1; \{0,1,2\} \otimes \{3,6,8,11\}$
 $[0,1,4] * [0,2,5,7] = 11-1; \{0,1,4\} \otimes \{3,5,8,10\}$
 $[0,2,4] * [0,1,4,7] = 11-1; \{0,2,4\} \otimes \{6,7,10,11\}$
 $[0,2,4] * [0,1,5,6] = 11-1; \{0,2,4\} \otimes \{5,6,10,11\}$
 $[0,2,7] * [0,1,3,4] = 11-1; \{0,2,7\} \otimes \{5,6,8,9\}$
 $[0,2,7] * [0,2,3,6] = 11-1; \{0,2,7\} \otimes \{3,5,6,9\}$
 $[0,3,7] * [0,1,2,3] = 11-1; \{0,3,7\} \otimes \{8,9,10,11\}$

Thus the greatest product cardinality that may result from multiplication of non-intersecting operands of cardinalities 3 and 4 is 11.

Variations on complex multiplication in Boulez's writings and sketches. Figure 5.4 above, exemplifying compound multiplication in *Boulez on Music Today*, can also be regarded as containing either of two variations on the technique, but not the basic principles, of complex multiplication. The first variation involves the selection of k , the transposition-determining constant. Line 2 will be generated if $k = 2$; Line 3, if $k = 11$; Line 4, if $k = 1$. The variation here is that pitch classes 11 and 1 are not elements of the first set (that corresponding to VXA); the algorithm for selecting k must be altered to accommodate these choices.

The second variation is more provocative: the complex multiplication formula itself is altered. In this case, the transposition-determining constant is chosen from the first set, as is usual; however, the calculation of n in T_n reverses the order of k and the initial pitch class of the multiplicand: $n = i\langle r(A), k \rangle$. (This change reintroduces variety of products and non-commutativity, and may constitute a red herring of Boulez's device.) With this variation, Line 2 will be generated if $k = 2$; Line 3, if $k = 5$; Line 4, if $k = 3$. Since these three values for k correspond with the invented noteheads from the first generating set $\{2,3,5\}$ that appear to the right of each line of products, it is possible that this figure is a hint from Boulez as to the workings of complex multiplication.

A much longer example of a variation on complex multiplication is from one of Boulez's compositional sketches held by the Paul Sacher Foundation's Pierre Boulez Collection, reproduced in Figure 6.11 as shown in *From Pierrot to Marteau*,²⁰ where it is identified as "Row table for *Le Marteau sans Maître*." This identification is apparently spurious. It has the same general appearance as Koblyakov's

²⁰*From Pierrot to Marteau* (tenth-anniversary *Festschrift* of the Arnold Schoenberg Institute) (Los Angeles: Arnold Schoenberg Institute, 1990), p. 21. Treble clef is implied in the sketch.

representations of *Le Marteau's* pc domains,²¹ but is rather, according to Koblyakov, from an earlier (1952), subsequently withdrawn work, *Oubli signal lapidé*.²² The domains are shown in integer pitch-class normal form in Table 6.5.

²¹Koblyakov, "P. Boulez 'le marteau sans maître'--Analysis of pitch structure," pp. 33-34, and, in a slightly different form, *Pierre Boulez: A World of Harmony*, p. 137.

²²Koblyakov to author, Oct. 18, 1990. His talks with Boulez and access to the Sacher Foundation's holdings have convinced him that all sketch materials for *Le Marteau* have been lost.

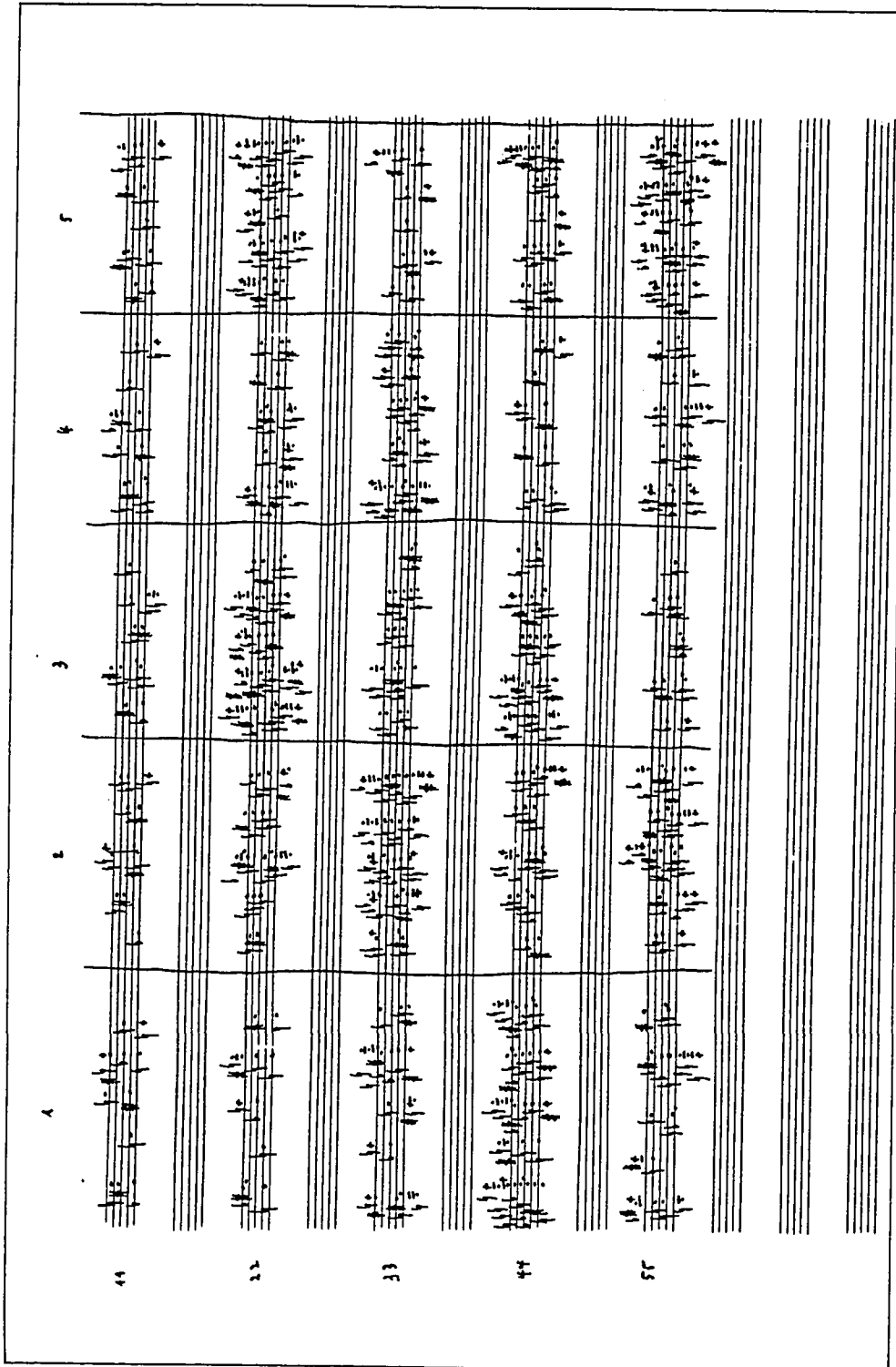


Figure 6.11. Boulez's notation of pc domains for *Oubli signal lapidé*
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Table 6.5. Pitch-class domains for *Oubli signal lapidé*

<i>Domain 1</i>					
	A	B	C	D	E
VB	{1,3,4}	{5}	{7,8}	{6,9,10,2}	{11,0}
B	{3,5,6}	{7}	{9,10}	{8,11,0,4}	{1,2}
C	{5,6,7,8,9}	{9,10}	{11,0,1}	{10,11,1,2,3,6,7}	{3,4,5}
D	{6,7,8,9,10,11,0,2,3,4}	{8,11,0,4}	{10,11,1,2,3,6,7}	{0,1,3,4,5,8,9}	{2,3,5,6,7,10,11}
E	{9,10,11,0,1}	{1,2}	{3,4,5}	{2,3,5,6,7,10,11}	{7,8,9}
<i>Domain 2</i>					
	A	B	C	D	E
VA	{3}	{1,4}	{5,7,8,9}	{6,10}	{11,0,2}
B	{1,4}	{11,2,5}	{3,5,6,7,8,9,10}	{4,7,8,11}	{9,10,0,1,3}
C	{5,7,8,9}	{3,5,6,7,8,9,10}	{7,9,10,11,0,1,2,3}	{8,10,11,0,2,3,4}	{1,2,3,4,5,6,7,8}
D	{6,10}	{4,7,8,11}	{8,10,11,0,2,3,4}	{1,5,9}	{2,3,5,6,7,9}
E	{11,0,2}	{9,10,0,1,3}	{1,2,3,4,5,6,7,8}	{2,3,5,6,7,9}	{7,8,9,10,11,1}
<i>Domain 3</i>					
	A	B	C	D	E
VE	{1,3}	{4,5,7,8}	{6,9}	{10,11,2}	{0}
B	{2,3,4,5,6,7,8}	{5,6,7,8,9,10,11,0,1}	{7,8,10,11,1,2}	{11,0,1,2,3,4,6,7}	{1,2,4,5}
C	{4,6,7,9}	{7,8,10,11,1,2}	{9,0,3}	{1,2,4,5,8}	{3,6}
D	{8,9,10,11,0,2}	{11,0,1,2,3,4,6,7}	{1,2,4,5,8}	{5,6,7,9,10,1}	{7,8,11}
E	{10,0}	{1,2,4,5}	{3,6}	{7,8,11}	{9}
<i>Domain 4</i>					
	A	B	C	D	E
VD	{1,3,4,5}	{7,8}	{6,9,10}	{2}	{11,0}
B	{5,6,7,8,9,10}	{11,0,1}	{10,11,1,2,3}	{6,7}	{3,4,5}
C	{4,6,7,8,9,10,11,0}	{10,11,1,2,3}	{9,0,1,3,4,5}	{5,8,9}	{2,3,5,6,7}
D	{0,2,3,4}	{6,7}	{5,8,9}	{1}	{10,11}
E	{9,10,11,0,1,2}	{3,4,5}	{2,3,5,6,7}	{10,11}	{7,8,9}
<i>Domain 5</i>					
	A	B	C	D	E
VC	{1,3}	{4,5,8}	{7}	{6,9}	{10,11,0,2}
B	{2,3,4,5,6,8}	{5,6,7,9,10,1}	{8,9,0}	{7,8,10,11,2}	{11,0,1,2,3,4,5,7}
C	{5,7}	{8,9,0}	{11}	{10,1}	{2,3,4,6}
D	{4,6,7,9}	{7,8,10,11,2}	{10,1}	{9,0,3}	{1,2,3,4,5,6,8}
E	{8,9,10,11,0,2}	{11,0,1,2,3,4,5,7}	{2,3,4,6}	{1,2,3,4,5,6,8}	{5,6,7,8,9,10,11,1}

An examination of these domains reveals interesting parallels with the domains from *Le Marteau* and with the theory presented herein:

1. The generating row, shown in Figure 6.12, is the inversion of that of *Le Marteau*.²³

$$P_0 = \langle 3, 1, 4, 5, 8, 7, 9, 6, 10, 2, 11, 0 \rangle$$

Figure 6.12. The *Oubli signal lapidé* row

2. The sequence of cardinalities partitioning the row is the retrograde of that of *Le Marteau*, 3/1/2/4/2. As in *Le Marteau*, this sequence is rotated to the left,²⁴ resulting in V-sets as shown in Figure 6.13 (like Fig. 6.3, given in P_0 order rather than in normal form).

²³"The general series of [*Le M*]art^eau was first used by Boulez . . . in his choral composition *Oubli, Signal, Lapidé* (1952), which has remained unpublished." (Koblyakov, *Pierre Boulez: A World of Harmony*, p. 32n. Punctuation and capitalization of the title herein are taken from Jameux, *Pierre Boulez*, p. 32.)

²⁴Reversing the direction of rotation would not affect the domains' pc content, merely their labeling--Domain 5 would become Domain 2, 4 would become 3, etc.

<u>Domain</u>	<u>Partition</u>	<u>V-sets</u>
1.	3/1/2/4/2	{3,1,4} {5} {8,7} {9,6,10,2} {11,0}
2.	1/2/4/2/3	{3} {1,4} {5,8,7,9} {6,10} {2,11,0}
3.	2/4/2/3/1	{3,1} {4,5,8,7} {9,6} {10,2,11} {0}
4.	4/2/3/1/2	{3,1,4,5} {8,7} {9,6,10} {2} {11,0}
5.	2/3/1/2/4	{3,1} {4,5,8} {7} {9,6} {10,2,11,0}

Figure 6.13. Formation of *Oubli* V-sets

3. Field A in each domain is again replaced by V-sets.

4. As stated earlier in this chapter, the selection of the transposition-determining constant can be done by any means the composer deems satisfactory. In *Le Marteau*, the transposition-determining constant is the rightmost of $\langle 3,5,2 \rangle$ appearing in set VXA. In *Oubli*, the *leftmost* pc in the row has been chosen. Thus, in every domain, $k = 3$. (The principal effect of this choice is that "discrepancies" such as those shown in Fig. 6.4 will not occur; this effect may be viewed, however, as resulting in too many identical sets.)

5. While the method for choosing k has changed, the complex multiplication algorithm itself is the same as that shown for *Le Marteau*.

6. The decachord-producing operation shown in Fig. 6.10 appears in Domain 1: $VBA \otimes VBD = \langle 3, \{4,1\} \rangle \otimes \{6,9,10,2\} = \text{set AD} = \{6,7,8,9,10,11,0,2,3,4\}$. Because $(k = r(A)) \rightarrow$

T_0 , the simple multiplicative product in Fig. 6.10 duplicates the complex multiplicative product AD.

7. The V-sets of Domain 1 appear in *Notes of an Apprenticeship* as the first half of Example XII, p. 167. The Domain 1 sets themselves, with field A omitted as in the sketch, are shown as Example XIII, p. 168. (The second half of Example XII shows the *Marteau* row, but partitioned according to the *Oubli* sequence 3/1/2/4/2. Example XIIIa is the above-referenced Fig. 6.10.)

8. In many cases, the domain sets appear to have been calculated individually rather than simply copied from a commutatively equal set. In Domain 1, for example, set ED, $\{2, 3, 5, 6, 7, 10, 11\}$, is represented on the staff as F3, B3, D4, G4, B^b4, E^b5, F[#]5. Set DE, however, is represented as F4, G4, B^b4, E^b5, F[#]5, B5, D6. Like Figures 6.5 through 6.9, different calculations in staff notation will usually result in products with different "spacings." This method of individual calculation provides a buffer against errors.

However, there are two mistakes in the sketch. In Domain 2, set BC appears to have an E[#] rather than the correct E^b. The other, larger, error is in Domain 5, in which set BB ($\{4, 5, 8\} \otimes \{4, 5, 8\}$) is shown as A3, B3, F4, G4, C5, E^b5, F[#]5, B^b5--a set of cardinality 8, with the pc normal form $\{3, 5, 6, 7, 9, 10, 11, 0\}$ and the T_n type $(0, 2, 3, 4, 6, 7, 8, 9)$. A correct calculation will give a set of cardinality 6, with

the pc normal form {5,6,7,9,10,1} and the T_n type (0,1,2,4,5,8). It is not surprising to find such an error when set squaring is involved, as there is no corresponding commutatively generated set with which to check calculations.²⁵ (Both of the cited errors have been corrected in Table 6.5.)

The complex multiplication shortcut. This simplified method for calculating complex multiplicative products grew directly out of the proofs shown in Appendix B. It is not offered as a representation of Boulez's own process, which is, I believe, more accurately reflected in the staff notation examples of Figs. 6.5 through 6.9. The shortcut, proven in Appendix B, first involves complex multiplication of pitch classes, and can then be extended to involve complex multiplication of pitch-class sets.

Theorem: Where a , d and k are pitch classes (expressed as integers) and k is the transposition-

²⁵It would be interesting to see if and how these errors were manifested in *Oubli signal lapidé*, or if errors of pitch class discovered by Koblyakov in *Le Marteau* resulted from incorrect domain calculations. Since *Oubli* is withdrawn and the sketches for *Le Marteau* are missing, such research is not possible.

This is not to say, however, that these errors would have mattered a great deal to a composer who wrote:

"I can only really conceive of the musical world from the point of view of more or less limited fields; this is why I have never exaggerated the importance of the complete elimination of error in diagrams. The field in which a written pitch can be played is extremely narrow, and the risk of error is practically nil. . ." (Boulez on *Music Today*, p. 41).

determining constant, the complex multiplicative product $a \otimes d = a + d - k$.

For example, the monad domain sets in *Le Marteau* are produced by the squaring of monad V-sets (refer to Table 6.1):

Domain 1 ($k = 5$): $VDD \otimes VDD = \{8\} \otimes \{8\} = 8 + 8 - 5 = DD = \{11\}$.

Domain 2 ($k = 2$): $VCC \otimes VCC = \{9\} \otimes \{9\} = 9 + 9 - 2 = CC = \{4\}$.

Domain 3 ($k = 5$): $VBB \otimes VBB = \{2\} \otimes \{2\} = 2 + 2 - 5 = BB = \{11\}$.

Domain 4 ($k = 3$): $VAA \otimes VAA = \{3\} \otimes \{3\} = 3 + 3 - 3 = AA = \{3\}$.²⁶

Domain 5 ($k = 2$): $VEE \otimes VEE = \{6\} \otimes \{6\} = 6 + 6 - 2 = EE = \{10\}$.

Because complex multiplication of pitch-class sets involves the multiplication of every multiplicand pc (as represented within the OIS) by every multiplier pc, a Cohn matrix can again be employed. The k value can be subtracted from every matrix sum (equivalent to $(a + d) - k$) or from every element of either operand prior to adding the elements in the matrix (equivalent to $a + (d - k)$ or $(a - k) + d$). The following examples from Domain 3 of *Le Marteau* will

²⁶As explained above, sets 4-AY duplicate sets VAY. Technically, $\{3\}$ is VAA and not 4-AA, but the principle under consideration still applies.

demonstrate each of these methods, showing the multiplication of VBC by VBD to produce domain set CD (and, commutatively, domain set DC): when $k = 5$, $\{10,11,1\} \otimes \{9,0\} = \{2,3,5,6,8\}$.

$$\begin{array}{r|l} 1 | T 1 \\ E | 8 E \\ \hline T | 7 T \\ | 9 0 \end{array} = \{7,8,10,11,1\} - 5 = \{2,3,5,6,8\}$$

Figure 6.14. Calculation of $\{10,11,1\} \otimes \{9,0\}$, subtracting ($k = 5$) from each ($a + d$)

$$\begin{array}{r|l} 1 | 5 8 \\ E | 3 6 \\ \hline T | 2 5 \\ | 4 7 \end{array} = \{2,3,5,6,8\}$$

Figure 6.15. Calculation of $\{10,11,1\} \otimes \{9,0\}$, subtracting ($k = 5$) from every multiplier element

$$\begin{array}{r|l} 8 | 5 8 \\ 6 | 3 6 \\ \hline 5 | 2 5 \\ | 9 0 \end{array} = \{2,3,5,6,8\}$$

Figure 6.16. Calculation of $\{10,11,1\} \otimes \{9,0\}$, subtracting ($k = 5$) from every multiplicand element

The superficial similarity of this shortcut to Cohn's theorem for transpositional combination of pc sets and pitch classes (in which $a * d = a + d$) is offset by crucial differ-

ences--the most obvious being the use of the transposition-determining constant in complex multiplication. Others include:

1. In transpositional combination, no criteria were established whereby two pc could be added to result in a third pc. As Appendix B demonstrates, the complex multiplication shortcut is an algebraic simplification of the pitch-class set theory presented herein.

2. The product of a complex multiplicative operation is unaffected by which pc is designated as 0. (This is true regardless of the form of the algorithm, be it the original theorem, the corollary, or the shortcut; see proof in Appendix B.) By way of example, the operations shown in Figures 6.5 through 6.9 could be expressed in this way: where F is the transposition-determining constant, the complex multiplication of operands $\{G, B, C\}$ and $\{F, A\}$ results in the product $\{G, B, C, D, E\}$. If, for instance, $E = 0$, this product should be $\{5, 8, 10, 11, 1\}$. The original theorem shows the calculation of Fig. 6.5 as:

$$\begin{aligned} &\text{Where } E = 0 \text{ and } k = 2, \langle 4, \{7, 9\} \rangle \otimes \{3, 6\} = \\ \text{(Step 1)} \quad &\underline{0, 3, 5} \otimes \{3, 6\} = \{3, 6, 8, 9, 11\} \\ \text{(Step 2)} \quad &(\text{since } i \langle 2, 4 \rangle = 2,) T_2\{3, 6, 8, 9, 11\} = \{5, 8, 10, 11, 1\} \end{aligned}$$

The shortcut method likewise gives the correct result, as shown in Figure 6.17.

$$\begin{array}{r|l}
 9 & T \ 1 \\
 7 & 8 \ E \\
 \hline
 4 & 5 \ 8 \\
 | & 1 \ 4
 \end{array} = \{5, 8, 10, 11, 1\}$$

Figure 6.17. Calculation of $\{4, 7, 9\} \otimes \{3, 6\}$, subtracting ($k = 2$) from every multiplier element

The complex multiplication shortcut is an operation unique in pc set theory. While it is possible to subtract one pitch class from another to obtain an interval ($i\langle a, b \rangle = b - a$) and to add an interval to a pitch class to obtain a pitch class ($T_n(a) = a + n$), the complex multiplication shortcut is the only theoretically proven operation which permits the addition and subtraction of pc to result in another pc. Ultimately, this works because an interval is added to a pitch class: $a + d - k =$ either $(a + i\langle k, d \rangle)$ or $(d + i\langle k, a \rangle)$. However, such an interval is not a part of the original operation, which calculates intervals to or from the initial pitch class of the multiplicand.²⁷

Complex multiplicative products as unordered pc sets.

The five operations in Figures 6.5 through 6.9 show four different orderings of the product. If operand spacings are changed (as in Fig. 4.22, where $\langle 7, \{10, 0\} \rangle$ is represented as $\langle 7, 0, 10 \rangle$), still other orderings are possible. Any product

²⁷The complex multiplication shortcut also suggests a compositional paradigm whereby Cohn's theory of transpositional combination of pitch classes can be employed: the composer can simply designate the pitch class that equals 0 as the transposition-determining constant.

pitch class can be shown to precede or succeed any other. Each operation validates the others, and none deserves any preference. These are truly *unordered* sets by nature of their generation, and Boulez may have found this to be one of complex multiplication's most appealing characteristics:

There is in fact a very clear and very strict element of control [in *Le Marteau*], but starting from this strict control and the work's overall discipline there is also room for what I call *local indiscipline*: . . . a freedom to choose, to decide and to reject.²⁸

The "very strict element of control" to which Boulez refers may be the row as manipulated by the complex multiplication operation; his "local indiscipline" may include both the precompositionally unordered domain sets and (less locally) the selection for inclusion or exclusion and the imposition of ordering of these sets as reflected in the matrix tracings (shown in Chapter 7). Through complex multiplication, Boulez was able to order set elements to his liking; the domains offer significantly more freedom than does a conventional 12-tone row chart. (And, as has been amply demonstrated in the history of scholarly works dealing with *Le Marteau*, they make the analyst's task significantly more difficult.)

In his provocative article "Cognitive Constraints on Compositional Systems," Fred Lerdahl wrote:

²⁸Pierre Boulez, *Conversations with Célestin Deliège* (London: Eulenberg Books, 1976), p. 66.

Within an avant-garde aesthetic it became possible to believe that one's own new system was the wave of the future. Boulez's generation was the last to believe this. To a younger generation these systems have come to seem merely arbitrary.²⁹

But, until now, a composer wishing to duplicate Boulez's "system" would have been unable to do so. Boulez never intended complex multiplication to be "the wave of the future"--otherwise, he would not have concealed its workings. Whether he was being pragmatic in this concealment (in 1954, he would not have had the theoretical tools at his disposal to prove any claims of, for instance, commutativity) or was simply unwilling to reveal everything about his own technique is now unimportant--at least as far as this *particular* technique is concerned. After all, there is, as of this writing, no analysis of the second and third cycles of *Le Marteau* that accounts for their pitch classes as satisfactorily as does the theory of complex multiplication for those of the first cycle. One suspects that this study is not the last of precompositional process in *Le Marteau sans maître*.

²⁹Fred Lerdahl, "Cognitive Constraints on Compositional Systems," in *Generative Processes in Music: The Psychology of Performance, Improvisation, and Composition*, ed. John Sloboda (Oxford: Oxford University Press, 1988), pp. 231-259. This quote appears on pp. 235-236.

7. The Row and the Musical Surface

In "Cognitive Constraints on Compositional Systems," Fred Lerdahl states that Boulez's pitch-class set multiplication is "a system that could just produce a quantity of musical material having a certain consistency."¹ He does not describe the form that this "consistency" takes (intervallic relationships? dissonance? unity? contrast?), so the reader is unsure whether to agree or disagree. Consistency of intervallic relationships and compositional process can be found with effective results, as this chapter will demonstrate. Here, the generators and results of pitch-class set multiplication in *Le Marteau* will be explored (in tandem with an overview of characteristic compositional techniques) in order to posit a listening strategy for selected sections of the first movement.

Since operands are partitioned from the original row according to a rotating cardinality sequence, and because multiplication preserves the unordered pc intervallic content of, but not between, operands--the T_n type of one operand is replicated no fewer times than the cardinality of its co-

¹Lerdahl, p. 232.

operand²--the row itself bears inspection as to its unordered pc intervallic content. (The perceived flaws that forced Boulez to withdraw *Oubli signal lapidé* must not have included the intervallic construction of the generating row, since the *Marteau* row is the inversion of the *Oubli* row.)

Because the largest set in the partitioning sequence is a tetrachord, two pc in the row must be located within a four-note span to appear in the same V-set. A pair of pitch classes so located is here called a *proximate dyad*. Since it is possible to reorder the partitioning sequence, whether through rotation (as in *Le Marteau*), through a combination of retrograde and rotation (as in *Oubli signal lapidé*), or through some other schema, all proximate dyads in the row related by each interval class will be inspected for general information; comparisons are then made to actual occurrences within V-sets in *Le Marteau*. (Since any permutation of the original partitioning sequence contains just one tetrachord, proximate dyad pc will not necessarily share an operand; only five of the nine possible tetrachords will be manifested when the sequence has been rotated.) Figure 7.1, which beams pc related by interval class 6, demonstrates that the row is

²It may appear more often, as in the operation $0.2.4 \otimes \{0,4\} = \{0,2,4,6,8\}$ (the multiplicand T_n type is replicated three times in the product: $\{0,2,4\}$, $\{2,4,6\}$, and $\{4,6,8\}$).

course; the interval will be formed when intervals from each operand sum to six.

Interval class 5 is similarly, but not completely, avoided; see Figure 7.2. Here, and in Figures 7.3 through 7.6, proximate dyad pc are joined by two beams.



Figure 7.2. Interval class 5 in the row

Of the three proximate dyads related by interval class 5-- $\{5,10\}$, $\{4,9\}$ and $\{7,0\}$ --only $\{4,9\}$ actually occurs in a V-set in *Le Marteau*, and this only within set VAD. Perhaps the tonal implications of this interval class led Boulez to bar its extensive occurrence in V-sets.

The remaining interval classes occur frequently in V-sets as follows:

The six proximate dyads related by interval class 4 can be derived from Figure 7.3. Five of these occur a total of eleven times in V-sets; only $\{9,1\}$ does not occur.

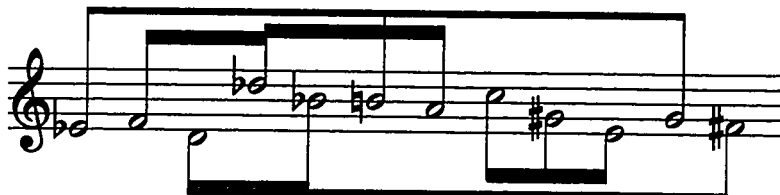


Figure 7.3. Interval class 4 in the row

Interval class 4 is the only interval class with consecutive occurrences in the row. Row elements $\langle 0, 8, 4 \rangle$ all appear in VAD (with pc 9) and, as a trichord, in VCD; these occurrences account for six of the eleven pairings. The use of $\{0, 4, 8\}$ as an operand in Domain 2 provides the opportunity for construction of domain sets which are transpositionally symmetrical under T_n where $n \neq 6$, including $[0, 4, 8]$ (CD/DC), $[0, 1, 4, 5, 8, 9]$ (BD/DB, DE/ED), and $[0, 1, 2, 4, 5, 6, 8, 9, 10]$ (DA). The whole-tone collection, which is transpositionally symmetrical under six values for T_n , does not appear, since none of the other VCY sets is of the T_n/T_nI types $[0, 2]$ or $[0, 2, 4]$ (the types not containing a tritone which will combine with $[0, 4, 8]$ to produce $[0, 2, 4, 6, 8, 10]$).

Interval class 3, shown in Figure 7.4, appears in six proximate dyads, all of which are realized in V-sets from one

to four times ($\{11,2\}$ in VDB and $\{8,11\}$ in VEC; $\{9,0\}$ in VAD, VBD, VDC, and VEC)--a total of fifteen occurrences.



Figure 7.4. Interval class 3 in the row

This interval class provides the most interesting distribution, appearing five times in four V-sets in Domain 5. Some ramifications of this will be explored presently.

Figure 7.5 shows interval class 2 in the row. The seven proximate dyads are realized from zero to four times: $\{10,0\}$ does not occur; $\{3,5\}$ occurs in VBA, VCA, VDA, and VEA.



Figure 7.5. Interval class 2 in the row

Figure 7.6 shows interval class 1 in the row. The eight proximate dyads are realized from zero to four times in a total of seventeen occurrences: {9,10} does not occur; {10,11} occurs in VAC, VBC, VCB, and VDB; {6,7} occurs in VAE, VBE, VCE, and VDE.



Figure 7.6. Interval class 1 in the row

The most homogeneous distribution of an interval class is found in Domain 5. Interval class 3 occurs five times in

four V-sets; only the monad VEE does not contain ic 3. As shown in Figure 7.7, only pc 3 and 6 do not join with another pc to form ic 3 within a V-set. (That these "isolated" pc are also related by ic 3 is noted as interesting (and inevitable) but unimportant, since multiplication does not preserve unordered pc intervals between operands.) In Fig. 7.7, pc related by ic 3 are shown as beamed solid noteheads, while "isolated" pc are shown as open noteheads. Partitioning of the row into V-sets is indicated by barlines.



Figure 7.7. Interval class 3 in Domain 5 V-sets

Since only VEA and VEE contain "isolated" pc, each pc in every domain set will pair with one or two other pc related by interval class 3 except where VEA and VEE are operands with each other or with themselves--in domain sets AE, EA, AA or EE. Since the A field is jettisoned in all domains, only domain sets EA and EE (which turn out to be

transpositions of VEA and VEE) will contain "isolated" pc.⁴ This is demonstrated in Figure 7.8, which shows Domain 5 sets in pc normal form on the staff (compare with Table 6.1); ic 3-related pc are bracketed, and "isolated" pc are again shown as open noteheads.

The figure consists of five staves of musical notation, labeled A, B, C, D, and E at the top. The staves are labeled VE, B, C, D, and E on the left. The notation includes notes, stems, and accidentals, with some notes enclosed in brackets to indicate interval class 3 relationships. The 'isolated' pc are shown as open noteheads.

Figure 7.8. Interval class 3 in Domain 5

⁴By commutativity, the missing AE duplicates EA; the missing AA would be {2,3,4,5,6,8}, with pc 4 "isolated" in relation to interval class 3.

This diagram suggests two possibilities for a process-based listening strategy. The first is derived from the intervallic structures of the generating V-sets: interval class 3 may be clearly audible in *Le Marteau*, particularly in sections of the first cycle where pc content is determined by Domain 5. The second is derived from the intervallic structures and pc content of the V-sets and domain sets: with two exceptions, CA and CC, the sets shown in Fig. 7.8 are subsets of the octatonic T_n/T_nI -type set, $[0,1,3,4,6,7,9,10]$. In every such set with four or more pc, in trichords VEA and EA, and (trivially) in monads VEE and EE, the subset relationship is reinforced by virtue of the fact that they are octatonic scale segments; other subsets are of the T_n/T_nI -type sets $[0,3]$ or $[0,3,6]$. Each of these two possibilities has its merits, as explored below, and both are sufficiently intertwined that the emphasis of either does not preclude reference to the other.

The importance of interval class 3 in *Le Marteau* is noted by at least one writer without reference to Koblyakov's work: "[A] feature of certain works of this period [of Boulez's career] is the use of preponderant intervals. . . ; thus, much of *Le Marteau sans maître* shows a preponderance of minor 3rds. . . ." ⁵ This refutes Lerdahl's claim that, prior

⁵The *New Grove Dictionary of Music and Musicians*, 1980 ed., s.v. "Boulez, Pierre," by G. W. Hopkins. Although this article was published after Koblyakov's, Hopkins gives Koblyakov no bibliographic

to the appearance of Koblyakov's 1977 *Zeitschrift für Musiktheorie* article, "listeners made what sense they could of the piece in ways unrelated to its construction."⁶ The observation of "a preponderance of minor 3rds" is *directly* related to the generation of domain sets as described herein, even if the specifics of that generation were unknown to the observer.

The octatonic approach may be particularly fruitful. The ubiquity of octatonic organization in twentieth-century music--particularly in the music of Stravinsky, as demonstrated by Pieter van den Toorn⁷--has made the "octatonic sound" a familiar one; as such, it embodies a prerequisite for any listening strategy--a basis in prior listening experience.⁸ In fact, the theory of octatonic organization has

mention. Koblyakov does not deal with intervallic structure, or any other listening strategy, to any appreciable extent.

⁶Lerdahl, p. 231.

⁷Pieter van den Toorn, *The Music of Igor Stravinsky* (New Haven and London: Yale University Press, 1983), and *Stravinsky and The Rite of Spring: The Beginnings of a Musical Language* (Berkeley and Los Angeles: University of California Press, 1987). See also David Lewin, *Generalized Musical Intervals and Transformations* (New Haven and London: Yale University Press, 1987), pp. 251-253, for the application of other transformational operations to the octatonic collection.

⁸I make no claim that Boulez would approve of either listening strategy presented here. It seems unlikely that he would, given his assertion (since disproved by Koblyakov) that *Le Marteau* was immune to analysis, as well as his aesthetic stance that "there is no escape from the knowledge of our own culture, nor nowadays from meeting the cultures of other civilizations--but how imperious a duty we have to volatilize them! Praise be to amnesia!" ("*Stravinsky: Style or Idea?*" in *Orientations: Collected Writings by Pierre Boulez*, ed. Jean-Jacques Nattiez, trans. Martin Cooper [Cambridge: Harvard University Press, 1986], p. 359.) The "amnesia" issue is something of a red herring,

progressed to the point that the three pc sets representing the octatonic T_n/T_{nI} type have been labeled by van den Toorn as shown in Figure 7.9.

Collection I 

Collection II 

Collection III 

Figure 7.9. Three octatonic pitch-class collections

Since van den Toorn does not describe how he chose this taxonomy, it is interesting to note that Collections I, II, and III can be constructed respectively on pc 1, 2, and 3 as follows:

Collection I: $\underline{0,1,3,4,6,7,9,10} \otimes 1 = \{1,2,4,5,7,8,10,11\}$

Collection II: $\underline{0,1,3,4,6,7,9,10} \otimes 2 = \{2,3,5,6,8,9,11,0\}$

Collection III: $\underline{0,1,3,4,6,7,9,10} \otimes 3 = \{3,4,6,7,9,10,0,1\}$

however; if a person were to lose all memory, any piece of music, tonal or atonal, would be regarded as opaque.

V-sets and domain sets in Domain 5 are subsets of Collections I, II, and/or III as shown in Figure 7.10.

	A	B	C	D	E
VE	II	I, III	II	I, III	II, III
B	I	II, III	I	II, III	I, II
C	\emptyset	I	\emptyset	I	III
D	I	II, III	I	II, III	I, II
E	III	I, II	III	I, II	I, III

Figure 7.10. Domain 5 octatonic subset matrix

Figure 7.10 suggests that Collections I and II may become the primary foci of the octatonic listening strategy. Only sets CA, CC, CE, EA, and EC are not subsets of these collections. Koblyakov's analysis through matrix tracings demonstrates that sets CA and EC are not used.⁹ The remaining sets, CC, CE, and EA, could easily have been realized in such a way that they could not be interpreted as belonging to either Collections I or II--but such an interpretation will prove entirely reasonable, as will be shown.

⁹Koblyakov, *Pierre Boulez: A World of Harmony*, p. 13.

Figures 7.11 and 7.12 are reductions of, respectively, measures 11 through 20 and 53 through the downbeat of 60 in the first movement of *Le Marteau*. These figures illustrate the aspects of Domain 5 discussed above. The following notational conventions apply to these figures:

1. The four instrumental parts--alto flute, vibraphone, guitar, and viola--are combined into a two-staff system, written at pitch. Accidentals apply only to the notes they immediately precede. The fourth note in measure 57 (Figure 7.12) appears in the original score as an E5 (a mistransposed A5 in the alto flute¹⁰) and is corrected to D5 here.

2. As in Figures 7.7 and 7.8, barlines signify the segmentation of sets from Domain 5; these sets are identified directly beneath the system.¹¹ Note that one or two sets are used per measure, except for the three in m. 55, and that no set extends beyond a metrical downbeat (although a pitch may be tied over as a common tone between two sets).

3. Rhythms are not directly notated, but can be inferred from the placement of noteheads at their points of attack relative to the notated meter (shown below each system

¹⁰Koblyakov, "P. Boulez 'Le marteau sans maître,' analysis of pitch structure," p. 32n.

¹¹Koblyakov, *Pierre Boulez: A World of Harmony*, p. 13. I agree with Koblyakov's analysis here, which amends that in his *Zeitschrift für Musiktheorie* article.

as small, evenly-spaced eighth notes; meter signatures and measure numbers are also given here).

4. Pitches which are joined by beams, either horizontally with stems or vertically with direct connections, are related by interval class 3. "Isolated" pc are again shown as open noteheads. To avoid overcrowding, only the relationships appearing within V-sets and domain sets are shown (see Figures 7.7 and 7.8); other relationships can be easily found, such as that between the "isolated" F#5 at the end of measure 12 and the A3 at the beginning of measure 13.

5. Octatonic collections I, II, and III are represented by the graphs above each system. A thick line on a graph indicates that a significant subset of that octatonic collection appears in the music. Unlike the analysis of interval class 3 relationships, the subset analysis extends beyond segmented V-sets and domain sets; for example, measure 11 shows set CE not only as a subset of Collection III (see Fig. 7.10) but also as a continuation of Collection II, represented by the dyad E♭5-C6, followed by a change to Collection I, represented by the dyad E2-C#3. A graph's thick line ends at the attack point of any of the four pitch classes not contained within that collection.

I
II
III

DE CE BE VEE DD ED

♩=208 5/8 3/4 6/8

11 12 13

I
II
III

DC CC BC VEC DB CB

2/4 2/4 2/4 2/4

14 15 16 17

I
II
III

BB VEB DA EA

3/8 3/8 3/4

18 19 20

poco rit. -----

Figure 7.11. Reduction/analysis of *Le Marteau sans maître*, first movement, measures 11-20

Pierre Boulez, *Le Marteau sans maître*. © Copyright 1954 by Universal Edition (London) Ltd., London. Final Version: © Copyright 1957 by Universal Edition (London) Ltd., London. © Copyrights renewed. All rights for the USA and Canada owned exclusively by European American Distributors Corporation. Used by permission.

I
II
III

VEE BE BD CD DD ED

$\text{♩} = 208$ $\frac{3}{4}$ $\frac{2}{4}$ $\frac{5}{8}$

53 54 55

I
II
III

BC VEC EB DB CB

$\frac{7}{8}$ $\frac{2}{4}$ $\frac{3}{8}$

56 57 58

*this D5 shown as E5 in score (see text, p. 106)

I
II
III

BB VEA BA

$\frac{6}{8}$ $\frac{2}{4}$

59 60

presser

Figure 7.12. Reduction/analysis of *Le Marteau sans maître*, first movement, measures 53-60

Pierre Boulez, *Le Marteau sans maître*. © Copyright 1954 by Universal Edition (London) Ltd., London. Final Version: © Copyright 1957 by Universal Edition (London) Ltd., London. © Copyrights renewed. All rights for the USA and Canada owned exclusively by European American Distributors Corporation. Used by permission.

These figures validate the conjecture regarding the emphasis on Collections I and II. Of the five sets not contained within these collections, none is used in measures 53-60. Three appear in measures 11-20, but they are realized in such a way that they can easily and legitimately be interpreted as subsets of Collections I and II. This is especially striking in the case of measure 15: set CC, a chromatic nonachord containing the pitch classes D through B \flat and not a subset of any of the octatonic collections, could have been composed in many different ways (the set permits $9! = 362,880$ different orderings, most impervious to the octatonic analysis here); yet the set is realized in such a way that the first four pc belong to Collection I and the last five pc belong to Collection II. Similarly, set EA in measure 20 could easily have been composed so that an analysis of Collection III alone was sensible (for example, switching the G4 and A2); but the G4 continues the previously established Collection I, and the subsequent A2 and F \sharp 3 also belong to Collection II. Finally, as described above, set CE in measure 11 is realized not simply as Collection III, but as a continuation of Collection II moving to Collection I. Although all Collection III subsets are shown as such in Figures 7.11 and 7.12, the relationship is a subsidiary one; the "momentum" of the other octatonic collections enables one to ignore III as an important organizing factor.

Figures 7.11 and 7.12 provide a clear illustration of the characteristic organization of V-sets and domain sets in the musical surface. One of Koblyakov's principal contributions to an understanding of the organization of *Le Marteau* is the *matrix tracing* (evolved from the domain matrix, Fig. 6.1), which shows the succession of these sets. The matrix tracings of *Le Marteau* parallel the 12-counting of traditional dodecaphonic music. Koblyakov's matrix tracings of measures 11 through 20 and 53 through 60 are shown in Figures 7.13 and 7.14 respectively.¹²

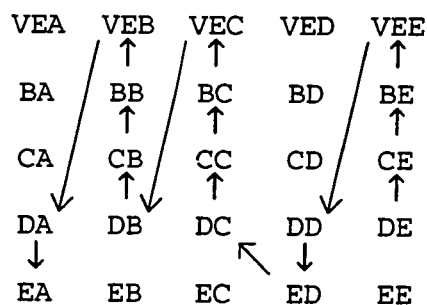


Figure 7.13. Koblyakov's matrix tracing of mm. 11-20
Used by permission of Harwood Academic Publishers

¹²Koblyakov, *Pierre Boulez: A World of Harmony*, p. 13. Lower-case letters in the original have been capitalized here.

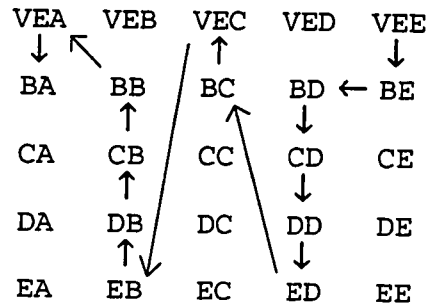


Figure 7.14. Koblyakov's matrix tracing of mm. 53-60
Used by permission of Harwood Academic Publishers

Octave avoidance. In addition to the ic 3 and octa-
tonic segmentations discussed here, Figures 7.11 and 7.12
demonstrate a compositional tenet of the serialist tradition:
Boulez exhibits caution in introducing pitches in octaves
other than those previously established for the pitch class.
A survey of Figure 7.11 will reveal the following: Two pc
appear in only one octave--pc 2 (realized as four occurrences
of D₄) and pc 7 (five occurrences of G₅). Pitch class 8 is
realized as G₅ (or as A₅) in the first six of its seven
occurrences; after an absence spanning a duration of five
eighth-notes, the other occurrence of pc 8 is realized as
A₃. The first three occurrences of pc 6 are realized as F₅;
after an absence spanning a duration of twelve eighth-notes,
the remaining four occurrences of pc 6 are realized as F₃.
Of the four occurrences of pc 3, the first is realized as

E♭5; a duration of more than twenty eighth-notes passes before the second, realized as E♭6; seven eighth-notes later, E♭5 reappears twice in close succession. Only pc 1 appears in a different octave with each occurrence--but there are a mere three occurrences, and a substantial absence is required before a new octave is introduced: C♯3 (duration of more than 14 eighth-notes), C♯5 (more than 16 eighth-notes), C♯4.

Pitch class repetition. Measures 11-20 and 53-60 represent the only sections of the first movement in which the repetition of pitch classes within a V-set or domain set is a significant compositional element; other sections tend to articulate pc only once per set. The repetitions are handled consistently: dual occurrences of a pitch class within a V-set or domain set are realized at the same pitch (see above). This is true both of consecutive and non-consecutive iterations; for an example of the latter, see the composing of sets DE, BE, and VEB in Fig. 7.11. In each realization of these dyadic sets, a pitch is followed by an upward skip (of ic 3) to the other element of the set, which is either itself repeated (m. 12) or not; this is followed by a descent to the original pitch. In measures 53-60, only set EB (m. 57) uses non-consecutive iterations, realized as a repetition of the figure F4—D5.

Characteristic intervals. The importance of ic 3 here has already been examined. Of nearly equal importance is the

ways in which this interval class is realized as an unordered pitch interval (upi); consistency in this regard may be as close to motivic repetition as one can expect in *Le Marteau*. Interval class 3 can be realized as unordered pitch intervals 3, 9, 15, 21. . . . The most exposed realizations in measures 11-20 emphasize upi 9; see, for example mm. 11-12, where each ic 3 beam also denotes upi 9.

Figures 7.11 and 7.12 also demonstrate another feature of unordered pitch interval composition that characterizes *Le Marteau*: a strong avoidance of upi 1, the (literal) semitone.¹³ The frequent realization of ic 1 as the ordered pitch interval +13 is most evident in, again, measures 11 and 12. Measure 11 begins B4—G#5—B4—C6, a line of ordered pitch intervals (+9)—(-9)—(+13); measure 12 is composed similarly as F4—D5—D5—F4—F#5, a line of ordered pitch intervals (+9)—(0)—(-9)—(+13).

Despite these consistencies, and despite the listening strategies presented herein, *Le Marteau* remains an exceptionally difficult piece in which to discern systematic organization. The other domains do not parse as easily as Domain 5, and a good deal of experimentation and analysis are yet to be done. However, armed with an understanding of pitch-class set multiplication and the conviction that *Le Marteau* is

¹³Koblyakov, *Pierre Boulez: A World of Harmony*, p. 25.

worth the effort, future theoretical works may reveal much more than this one.

Composer-percussionist William Kraft, who performed in the American premiere of *Le Marteau* (March 11, 1957 in Los Angeles, with Boulez conducting), wrote of the experience:

The newness and originality of Boulez's style, and the consequent challenge it placed on the average listener, prompted me to ask him if he was at all concerned about public acceptance. His answer was, "Yes." I went on, "How long do you think it will take?" to which he replied, "Eighty years."¹⁴

Given the current state of "public acceptance," even this estimate might seem overly optimistic. But we have not quite reached its halfway point. As long as composers continue to challenge our capabilities rather than stay safely within what they perceive these to be, we can hope that today's difficulties will become tomorrow's achievements.

¹⁴From *Pierrot to Marteau*, pp. 55-57.

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Appendix A: Simple Multiplication of Lines

Although this appendix relates somewhat tangentially to Boulez's process of pitch-class set generation, the theory of simple multiplication would be incomplete without a consideration of simple multiplication of lines. Cohn's dissertation provides numerous examples of transpositional combination as realized in musical surfaces, many of which can be described in terms of simple multiplication of lines of pitches or of pitch classes. An example which could have been included in his survey is a seminal, systematic variation on simple multiplication of pc sets predating Boulez's work: Nicolas Slonimsky's *Thesaurus of Scales and Melodic Patterns*.¹ Most of this speculative theoretical work involves the construction of multiplicand *line ordered pitch-class intervallic structures* (line OIS) on pc within multiplier ordered sets which divide the octave into equal parts (and are thus both transpositionally and inversionally symmetrical). Figure A.1 shows examples from the *Thesaurus* which will be discussed in terms of simple multiplication.

¹Nicolas Slonimsky, *Thesaurus of Scales and Melodic Patterns* (New York: Schirmer Books/Macmillan, 1987). The *Thesaurus* was first published (by Charles Scribner's Sons) in 1947.



Figure A.1. Slonimsky's patterns 196 and 395
 From Nicolas Slonimsky: *Thesaurus of Scales and Melodic Patterns*.
 Used by permission of Schirmer Books/Macmillan, Inc.

The theory of simple multiplication of pc sets can be easily modified to construct patterns such as these; all that is required is an adaptation of the OIS to account for linearity. A combination of Rahn's line equivalence class notation² and the OIS notation used herein would represent ordered pc intervals (again, from an initial pc) separated by dashes and underlined. These line OIS are multiplied by ordered pc sets (or pc lines) by constructing the line OIS on each pc of the multiplier pc set, thus resulting in lines of pc. The construction of the first half of Pattern 196 in Fig. A.1 would be shown as $\underline{0-5-9} \otimes \langle 0, 4, 8 \rangle = 0-5-9-4-9-1-8-1-5$, and the first half of Pattern 395 as $\underline{0-5} \otimes \langle 0, 3, 6, 9 \rangle = 0-5-3-8-6-11-9-2$. (An ordered reading of the product in a Cohn matrix--bottom to top and left to

²Rahn, *Basic Atonal Theory*, p. 139.

right--will show these lines.) Further refinements of pitch and ordering would account for the octave displacement and line retrograde that characterize Slonimsky's patterns, but the essential qualities of each pattern are adequately described through the present schema.

As these patterns are lines rather than unordered sets, their characteristic use is in a compositional surface; in this context, their appearance becomes somewhat problematic. Multiplier replication is an audibly obvious feature, particularly since the multipliers employed by Slonimsky, $\langle 0,6 \rangle$, $\langle 0,4,8 \rangle$, $\langle 0,3,6,9 \rangle$, $\langle 0,2,4,6,8,10 \rangle$, and various permutations thereof, are so familiar. Since each multiplier is based on pitch class C, every pattern contains (and, moreover, begins and ends with) that pitch class, thus forming a C paratonality.³ (The transposition of a pattern, suggested by Slonimsky, will avoid this; such a transposition may be regarded as part of the compound multiplication of a line.)

³Therefore, a pc analysis of any pattern would likely designate $C = 0$. Here, Cohn's theorem of transpositional combination of pc sets (see Chapter 3) will predict the pc content of a pattern (for example, Pattern 395: $\{0,5\} * \{0,3,6,9\} = \{2,3,5,6,8,9,11,0\}$); however, this occurs because the multiplicand duplicates the integer content of the line OIS. Any representative segment of the line will show the same product ($\{0,5\} * \{0,3,6,9\} = \{3,8\} * \{0,3,6,9\} = \{6,11\} * \{0,3,6,9\} = \{9,2\} * \{0,3,6,9\} = \{2,3,5,6,8,9,11,0\}$). The theorem thus explains neither the mechanics of the operation nor the inevitability of multiplier replication. Yet Cohn's theory as it relates to set classes achieves some of its most important insights when applied to compositional surfaces such as these; here it develops criteria for making distinctions between heretofore interchangeable operands. The theory of simple multiplication of line OIS as presented here extends Cohn's work by providing a means of specifying a line's pc content, not simply its intervallic structure.

Further, many patterns will repeat pc; line OIS/pc constructions, unlike the union of IO set OIS/pc constructions, do not omit repetitions. Pattern 196 in Fig. A.1, for example, repeats pitch classes 9, 1 and 4: pc 9, as a result of both $\underline{2} + 0$ and $\underline{5} + 4$; pc 1, from $\underline{2} + 4$ and $\underline{5} + 8$; and pc 5, from $\underline{2} + 8$ and $\underline{5} + 0$. (Such repetitions are explored below.)

Another problematic area is a compositionally aesthetic one, that of melodic sequence: at what point does a pattern become overly predictable? Tonal practice would suggest that Pattern 196 be broken off after the seventh note, the start of the third statement of the sequence--the pattern starts to wear thin at the one-third point of its ascent. Atonal practice is less clear on this issue; Slonimsky demonstrates that rhythmic variation within any pattern will, to a degree, compensate for melodic sequential predictability.⁴ Another solution is shown in Figure A.2: the upper line is a brief oboe passage from Witold Lutosławski's *Concerto for Orchestra* (1956), and the lower line illustrates the two multiplicative operations $\underline{0-2-5-3} \otimes \langle 10, 11, 1 \rangle = 10-0-3-1-11-1-4-2-1-3-6-4$ and $\underline{0-11-10}$

⁴Slonimsky, p. iv. Any future studies of the possibilities and problems of Slonimsky's work might profitably inspect free jazz improvisations recorded since the *Thesaurus* achieved unexpected popularity with performers influenced by saxophone virtuoso John Coltrane, who used it as a practice book (J. C. Thomas, *Chasin' The Trane* [New York: Da Capo, 1976], p. 102).

$\otimes \langle 1,7,4 \rangle = 1-0-11-7-6-5-4-3-2$ which motivate the passage.⁵

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$\langle 10,11,1 \rangle$

$\otimes \langle 0-2-5-3 \rangle$

0-11-10

$\otimes \langle 1,7,4 \rangle$

Figure A.2. Lutosławski, *Concerto for Orchestra*,
mm. 471-478, oboes; analysis of line multiplication
Used by permission of Kunzelmann Editions

Here, sequential predictability is broken in three ways:
first, through the juxtaposition of two different multiplica-
tive operations in parallel symmetry;⁶ second, through the
use of the asymmetrical multiplier $\langle 10,11,1 \rangle$ in the first

⁵The second operation could also be shown as $2-1-0 \otimes \langle 11,5,2 \rangle$; I have chosen the temporally first rather than the lowest pitch in each group to represent the initial pc of the line OIS, as I believe this more accurately represents the grouping inferred in the listening experience.

In his very kind private response to a draft of this paper, Richard Cohn observed that a possible advantage of the more abstract transpositional combination of set classes is that it is not required to "privilege" any pc in describing an operation such as this one.

⁶Parallel symmetry is detailed by Jonathan W. Bernard in "Space and Symmetry in Bartók," *Journal of Music Theory* 30/2 (1986), pp. 185-201 *passim*. The relationship of parallel symmetry to transpositional combination is delineated by Cohn in "Transpositional Combination in Twentieth-Century Music," pp. 50-51, and in "Inversional Symmetry and Transpositional Combination in Bartók," p. 19n.

operation; and third, through rhythmic variation of line segments in the second operation.

Repetition and pairings of pitch classes. As noted above, line OIS/pc constructions do not omit repetitions. Such repetitions can be predicted in the form of *pairings*. (A single occurrence of a pc cannot be paired with itself. A twice-occurring pc, x^1 and x^2 , forms one pairing, x^1/x^2 . A thrice-occurring pc, x^1 , x^2 and x^3 , forms three pairings, x^1/x^2 , x^1/x^3 , and x^2/x^3 . The number of pairings of n occurrences = $((n \times n) - n) + 2$, so that 4 occurrences result in 6 pairings, 5 occurrences in 10 pairings, etc.) With one exception, an interval class that appears in both operands will result in one pairing in the product, including $\underline{0-1} \otimes \langle 0,1 \rangle = 0-1-1-2$, $\underline{0-2} \otimes \langle 0,2 \rangle = 0-2-2-4$, and so on until the exception, ic 6, which results in two pairings: $\underline{0-6} \otimes \langle 0,6 \rangle = 0-6-6-0$. Pairings can therefore be predicted via the interval vectors of the operands: *the sum of the arithmetical multiplicative products of the quantities of like interval classes (with the product of ic 6 doubled) equals the number of pairings.* Examples of the workings of this theorem follow.

In Slonimsky's Pattern 196, the interval vector of the multiplicand $\underline{0-5-9}$ is $\langle 0,0,1,1,1,0 \rangle$, and the interval vector of the multiplier $\langle 0,4,8 \rangle$ is $\langle 0,0,0,3,0,0 \rangle$:

$$\begin{array}{r} \langle 0, 0, 1, 1, 1, 0 \rangle \\ \times \langle \underline{0, 0, 0, 3, 0, 0} \rangle \\ \hline 0+0+0+3+0+0 = 3 \end{array}$$

Interval class 4 appears once in the multiplicand--i(5-9)-- and thrice in the multiplier--i(0,4), i(4,8) and i(8,0)--thus resulting in three pairings.

By way of comparison, Fig. 4.21 suggests 0-4-8 \otimes $\langle 1, 5, 9 \rangle = 1-5-9-5-9-1-9-1-5$ (each of the three pc occurs three times, or nine pairings):

$$\begin{array}{r} \langle 0, 0, 0, 3, 0, 0 \rangle \\ \times \langle \underline{0, 0, 0, 3, 0, 0} \rangle \\ \hline 0+0+0+9+0+0 = 9 \end{array}$$

An example of operands with no interval classes held in common is 0-1-3-6 \otimes $\langle 0, 4, 8 \rangle = 0-1-3-6-4-5-7-10-8-9-11-2$ (no pairings):

$$\begin{array}{r} \langle 1, 1, 2, 0, 1, 1 \rangle \\ \times \langle \underline{0, 0, 0, 3, 0, 0} \rangle \\ \hline 0+0+0+0+0+0 = 0 \end{array}$$

An example of operands with three interval classes held in common is 0-1-2-5 \otimes $\langle 0, 1, 4 \rangle = 0-1-2-5-1-2-3-6-4-5-6-9$ (four pairings: 1/1, 2/2, 5/5, 6/6):

$$\begin{array}{r} \langle 2, 1, 1, 1, 1, 0 \rangle \\ \times \langle \underline{1, 0, 1, 1, 0, 0} \rangle \\ \hline 2+0+1+1+0+0 = 4 \end{array}$$

As described above, the arithmetical multiplicative product of ic 6 must be doubled. The next example shows a tetrachord containing a tritone multiplied by itself.

0-1-5-7 \otimes $\langle 0, 1, 5, 7 \rangle = 0-1-5-7-1-2-6-8-5-6-10-0-7-8-0-2$ (nine pairings: 0/0/0, 1/1, 2/2, 5/5, 6/6, 7/7, 8/8):

$$\begin{array}{r}
 \langle 1,1,0,1,2,1 \rangle \\
 \times \langle 1,1,0,1,2,1 \rangle \\
 \hline
 1+1+0+1+4+1 \times 2 = 1+1+0+1+4+2 = 9
 \end{array}$$

Just as other common-tone theorems have their limitations, this method will predict neither the types of pairings (e.g., whether three pairings are realized as three twice-occurring pc or as one thrice-occurring pc) nor, except as noted below, the cardinality of the pc set comprising the union of resulting line pc. The next example inverts the multiplier of the previous example, which does not change that operand's interval vector: $\underline{0-1-5-7} \otimes \langle 0,2,6,7 \rangle = 0-1-5-7-2-3-7-9-6-7-11-1-7-8-0-2$. Nine pairings still result (0/0, 1/1, 2/2, 7/7/7/7), but the cardinality of the resultant set is greater by 1.

This method does suggest, however, that two operand sets A and B with no interval classes in common will produce a set with a cardinality equal to $|A| \times |B|$, since no pc are duplicated. (This was exemplified above in the operation $\underline{0-1-3-6} \otimes \langle 0,4,8 \rangle = 0-1-3-6-4-5-7-10-8-9-11-2$, a cardinality of $4 \times 3 = 12$; for similar examples, see Table 6.3.) Further, when one pairing is predicted (as in the case of every operation in Table 6.4), it must be realized as one twice-occurring pc, so that $|A \otimes B| = (|A| \times |B|) - 1$; when two pairings are predicted, they must be realized as two twice-occurring pc, so that $|A \otimes B| = (|A| \times |B|) - 2$. (Since three pairings may be realized either as three twice-

occurring pc or as one thrice-occurring pc, the cardinality of a product with three or more pairings cannot be predicted with this theorem.) Conversely, when $|A| \times |B| > 12$ (as is the case when one tetrachord is multiplied by another), the operands must have at least one interval class in common.

(This provides one explanation of why no tetrachord is devoid of both ic 3 and ic 6, since $[0,3,6,9]$, the sole tetrachord limited to two interval classes, contains only those interval classes. There are instances of every other possible pair of interval classes missing from at least one T_n/T_nI -type tetrachord.)

Appendix B: Complex Multiplication Proofs

In Chapter 6, the following claims were made for the complex multiplication process:

1. Although initial ordering of the multiplicand is a prerequisite to attaining a product, any initial ordering may be used.
2. Complex multiplication is a commutative operation.
3. The corollary for complex multiplication--application of the calculated T_n value to the multiplier prior to multiplication, rather than to the product--will result in the same product as applying T_n to the product of simple multiplication. (Because of its clarity, this corollary will be used in the proof matrices below.)
4. Where a , d and k are pitch classes and k is the transposition-determining constant, $a \otimes d = a + d - k$.
5. The product of a complex multiplicative operation will remain unaffected by a change of the pitch class which equals 0.

The proofs begin with an algebraic formalization of simple multiplication. According to the discussions of ordered pitch-class intervallic structure and initially-

ordered pitch-class sets (Chapter 4), a pc-set $\{a,b,c\}$ can generate three different IO sets and corresponding OIS:

$\langle a, \{b,c\} \rangle$, with an OIS of $\underline{(a-a), (b-a), (c-a)}$;

$\langle b, \{c,a\} \rangle$, with an OIS of $\underline{(a-b), (b-b), (c-b)}$; and

$\langle c, \{a,b\} \rangle$, with an OIS of $\underline{(a-c), (b-c), (c-c)}$.

While the OIS numeric values are put into OIS normal form in practice, the proofs will be clarified by leaving them in the orders shown. The simple multiplication of the IO sets by a multiplier $\{d,e,f\}$ will demonstrate both variety of products and multiplier replication, as shown in Figures B.1 through B.3. (Although sets of cardinality 3 are used here, the size of each propaedeutic set could be reduced or enlarged at will.)

$$\begin{array}{rcccc}
 c-a| & ((c-a)+d) & ((c-a)+e) & ((c-a)+f) \\
 b-a| & ((b-a)+d) & ((b-a)+e) & ((b-a)+f) \\
 a-a| & d & e & f \\
 \hline
 | & d & e & f
 \end{array}$$

Figure B.1. Simple multiplication matrix of $\langle a, \{b,c\} \rangle \otimes \{d,e,f\}$

$$\begin{array}{rcccc}
 c-b| & ((c-b)+d) & ((c-b)+e) & ((c-b)+f) \\
 b-b| & d & e & f \\
 a-b| & ((a-b)+d) & ((a-b)+e) & ((a-b)+f) \\
 \hline
 | & d & e & f
 \end{array}$$

Figure B.2. Simple multiplication matrix of $\langle b, \{c,a\} \rangle \otimes \{d,e,f\}$

$$\begin{array}{r|cccc}
 c-c| & d & e & f \\
 b-c| & ((b-c)+d) & ((b-c)+e) & ((b-c)+f) \\
 a-c| & ((a-c)+d) & ((a-c)+e) & ((a-c)+f) \\
 \hline
 | & d & e & f
 \end{array}$$

Figure B.3. Simple multiplication matrix of $\langle c, \{a, b\} \rangle \otimes \{d, e, f\}$

Complex multiplication differs from simple multiplication in that every element of the simple multiplicative product is transposed by the ordered pitch-class interval from the transposition-determining constant (k) to the initial pc of the multiplicand IO set. This transposition may take place after simple multiplication (as in Figures 6.6 through 6.10) or as a result of transposing the multiplier prior to simple multiplication (the complex multiplication corollary). In Figure B.1, this interval is $i\langle k, a \rangle = (a-k)$; the transposition of the simple multiplicative product element $((c-a)+d)$ is therefore equal to $T_{(a-k)}((c-a)+d) = (((c-a)+d) + (a-k))$. Transposition of the multiplier is accomplished by the same interval: $T_{(a-k)}\{d, e, f\} = \{(d+(a-k)), (e+(a-k)), (f+(a-k))\}$. In Figure B.2, the interval of transposition is $(b-k)$; in Figure B.3, this interval is $(c-k)$. In Figures B.4 through B.6, the multiplier has been transposed by the appropriate interval so that the product may be seen more clearly as the result of an additive procedure. A comparison of Figures B.1 through B.3 with, respectively, Figures B.4 through B.6 will demonstrate the similarities and differences between simple and complex multiplication.

$$\begin{array}{l|ccc}
 c-a & ((c-a)+(d+(a-k))) & ((c-a)+(e+(a-k))) & ((c-a)+(f+(a-k))) \\
 b-a & ((b-a)+(d+(a-k))) & ((b-a)+(e+(a-k))) & ((b-a)+(f+(a-k))) \\
 a-a & ((a-a)+(d+(a-k))) & ((a-a)+(e+(a-k))) & ((a-a)+(f+(a-k))) \\
 \hline
 | & d+(a-k) & e+(a-k) & f+(a-k)
 \end{array}$$

Figure B.4. Complex multiplication matrix of $\langle a, \{b, c\} \rangle \otimes \{d, e, f\}$

$$\begin{array}{l|ccc}
 c-b & ((c-b)+(d+(b-k))) & ((c-b)+(e+(b-k))) & ((c-b)+(f+(b-k))) \\
 b-b & ((b-b)+(d+(b-k))) & ((b-b)+(e+(b-k))) & ((b-b)+(f+(b-k))) \\
 a-b & ((a-b)+(d+(b-k))) & ((a-b)+(e+(b-k))) & ((a-b)+(f+(b-k))) \\
 \hline
 | & d+(b-k) & e+(b-k) & f+(b-k)
 \end{array}$$

Figure B.5. Complex multiplication matrix of $\langle b, \{c, a\} \rangle \otimes \{d, e, f\}$

$$\begin{array}{l|ccc}
 c-c & ((c-c)+(d+(c-k))) & ((c-c)+(e+(c-k))) & ((c-c)+(f+(c-k))) \\
 b-c & ((b-c)+(d+(c-k))) & ((b-c)+(e+(c-k))) & ((b-c)+(f+(c-k))) \\
 a-c & ((a-c)+(d+(c-k))) & ((a-c)+(e+(c-k))) & ((a-c)+(f+(c-k))) \\
 \hline
 | & d+(c-k) & e+(c-k) & f+(c-k)
 \end{array}$$

Figure B.6. Complex multiplication matrix of $\langle c, \{a, b\} \rangle \otimes \{d, e, f\}$

When the products of each of these operations are algebraically simplified, they are shown to be identical from one operation to the next, proving the first claim. For example, the simplification of the element referenced above is as follows: $((c-a) + (d+(a-k))) = ((c+d-k) + (a-a)) = (c+d-k)$. The initial pitch class designation r may therefore be used in lieu of a , b or c where appropriate (a substitution possible but not particularly helpful in simple multi-

plication, due to the variability of r): although r is variable (its only restriction being $r \in$ the multiplicand), it cancels itself out in the process of algebraic simplification.

$$\begin{array}{r|lll}
 c-r & (c+d-k) & (c+e-k) & (c+f-k) \\
 b-r & (b+d-k) & (b+e-k) & (b+f-k) \\
 \hline
 a-r & (a+d-k) & (a+e-k) & (a+f-k) \\
 | & d+(r-k) & e+(r-k) & f+(r-k)
 \end{array}$$

Figure B.7. Simplification of Figures B.4 through B.6

Having established that r may be substituted for the initial pc of the multiplicand, Figure B.8 shows another abstraction of Figures B.4 through B.6, with the product elements in their entirety (i.e., prior to simplification).

$$\begin{array}{r|lll}
 c-r & ((c-r)+(d+(r-k))) & ((c-r)+(e+(r-k))) & ((c-r)+(f+(r-k))) \\
 b-r & ((b-r)+(d+(r-k))) & ((b-r)+(e+(r-k))) & ((b-r)+(f+(r-k))) \\
 \hline
 a-r & ((a-r)+(d+(r-k))) & ((a-r)+(e+(r-k))) & ((a-r)+(f+(r-k))) \\
 | & d+(r-k) & e+(r-k) & f+(r-k)
 \end{array}$$

Figure B.8. Generalized complex multiplication matrix of $\{a,b,c\} \otimes \{d,e,f\}$

Figure B.7 demonstrates the fourth claim, the complex multiplication shortcut, which is shown and proven here in a more explicit form:

$$\begin{aligned}
a \otimes d &= \\
(a-r) + (d+(r-k)) &= \\
(a+(-r)) + (d+r+(-k)) &= \\
(a+d+(-k)) + (r+(-r)) &= \\
(a+d-k) + (r-r) &= \\
a + d - k.
\end{aligned}$$

In each of the complex multiplication examples, the transposition of the product has been shown as an additive procedure: the multiplicand OIS values are added to the transposed multiplier. The original complex multiplication theorem of transposing the product of simple multiplication (that is, using the untransposed multiplier) will still yield the same result:

$$\begin{aligned}
a \otimes d &= ((a-r)+d) \text{ (the simple multiplicative product);} \\
T_{i<k,r}&((a-r)+d) = \\
((a-r)+d) + (r-k) &= \\
(a+(-r)+d) + (r+(-k)) &= \\
(a+d+(-k)) + (r+(-r)) &= \\
(a+d-k) + (r-r) &= \\
a + d - k.
\end{aligned}$$

With the first, third and fourth claims proven, the proof of commutativity, the second claim, becomes quite uncomplicated. The product of $a \otimes d = a + d - k$, as proven. Reversing the operands works as follows:

$$\begin{aligned}
d \otimes a &= \\
(d-r) + (a+(r-k)) &= \\
(d+(-r)) + (a+r+(-k)) &= \\
(d+a+(-k)) + (r+(-r)) &= \\
(d+a-k) + (r-r) &= \\
d + a - k &= a + d - k.
\end{aligned}$$

This is shown for pc-sets in Figure B.9 (compare with Fig. B.8), and is algebraically simplified in Figure B.10. The conceptual difference is that r is now an element of $\{d, e, f\}$, the new multiplicand; however, the initial pc continues to cancel itself out in algebraic simplification. A comparison of Figures B.7 and B.10 will show the products to be equal and the operation to be commutative.

$$\begin{array}{r|ccc}
f-r & ((f-r)+(a+(r-k))) & ((f-r)+(b+(r-k))) & ((f-r)+(c+(r-k))) \\
e-r & ((e-r)+(a+(r-k))) & ((e-r)+(b+(r-k))) & ((e-r)+(c+(r-k))) \\
d-r & ((d-r)+(a+(r-k))) & ((d-r)+(b+(r-k))) & ((d-r)+(c+(r-k))) \\
\hline
& a+(r-k) & b+(r-k) & c+(r-k)
\end{array}$$

Figure B.9. Generalized complex multiplication matrix of $\{d, e, f\} \otimes \{a, b, c\}$

$$\begin{array}{r|ccc}
f-r & (f+a-k) & (f+b-k) & (f+c-k) \\
e-r & (e+a-k) & (e+b-k) & (e+c-k) \\
d-r & (d+a-k) & (d+b-k) & (d+c-k) \\
\hline
& a+(r-k) & b+(r-k) & c+(r-k)
\end{array}$$

Figure B.10. Simplification of Figure B.9

One of the objections to Cohn's theorem of transpositional combination of pc-sets concerned the alteration of a result if the pitch class equalling 0 were changed (see Chapter 3). Such an alteration will not occur in complex multiplication if a change of 0 by $-m$ semitones will add m not only to each operation pc a , d , and k , but also to the complex multiplicative product pc $(a + d - k)$. It happens that this is the case, proving the fifth claim:

Since $(a \otimes d) = (a + d - k)$, then:

$$(a + m) \otimes (d + m) =$$

$$(a + m) + (d + m) - (k + m) =$$

$$(a + d + 2m) - (k + m) =$$

$$(a + d - k) + (2m - m) =$$

$$(a + d - k) + m.$$

This can be verified in practice by resetting 0 in any of the calculations of complex multiplication (for example, Figures 6.5 through 6.9). The product of each operation will remain unaffected by such a resetting.

Appendix C: Tetrachordal Set Squaring

In Chapter 6, the possibility was raised that the largest product cardinality will result from the largest generating sets--that is, from the tetrachord multiplied by itself. The process of multiplying a set by itself is here called *set squaring*.¹ Figure C.1, a Cohn matrix utilizing a hypothetical T_n/T_n^I -type set $[a,b,c,d]$, illustrates the potential cardinality of tetrachordal set squaring.

d		(a+d)	(b+d)	(c+d)	(d+d)
c		(a+c)	(b+c)	(c+c)	(d+c)
b		(a+b)	(b+b)	(c+b)	(d+b)
a		(a+a)	(b+a)	(c+a)	(d+a)
		a	b	c	d

Figure C.1. Matrix of tetrachordal set squaring

Ten unique sums can be derived: $(a+a)$, $(a+b)$, $(a+c)$, $(a+d)$, $(b+b)$, $(b+c)$, $(b+d)$, $(c+c)$, $(c+d)$, and $(d+d)$. This corresponds with the formula for unique pairings of n objects, $\frac{n^2 + n}{2}$ (see Chapter 4, note 11).

¹Cohn calls this a "multiplicative operation" ("Transpositional Combination in Twentieth-Century Music," p. 86), a term avoided here for obvious reasons.

Interestingly, however, this greatest cardinality is not achieved in practice. This is illustrated in Table C.1, which shows the T_n/T_{nI} types of the operands, results and result cardinalities of all possible tetrachordal set squarings.² As demonstrated here, the transpositional combination of a tetrachord with itself will never produce a set with a cardinality greater than 9. An empirically-based theorem would thus state that where $|A| = 4$, then $4 \leq |AA| \leq 9$, and that $|AA| \neq 5$.

²It is naturally assumed that both operands are not only of the same T_n/T_{nI} type but also of the same T_n type. For example, the operation $[0,1,5,7] * [0,1,5,7]$ could be expressed as $(0,1,5,7) * (0,1,5,7)$ or as $(0,2,6,7) * (0,2,6,7)$, but not as $(0,1,5,7) * (0,2,6,7)$.

Table C.1. Tetrachordal set squaring

<u>A</u>	*	<u>A</u>	=	<u>AA</u>	<u> AA </u>
[0,1,2,3]	*	[0,1,2,3]	=	[0,1,2,3,4,5,6]	7
[0,1,2,4]	*	[0,1,2,4]	=	[0,1,2,3,4,5,6,8]	8
[0,1,2,5]	*	[0,1,2,5]	=	[0,1,2,3,4,5,6,7,9]	9
[0,1,2,6]	*	[0,1,2,6]	=	[0,1,2,3,4,6,7,8]	8
[0,1,2,7]	*	[0,1,2,7]	=	[0,1,2,3,4,7,8,9]	8
[0,1,3,4]	*	[0,1,3,4]	=	[0,1,2,3,4,5,6,7,8]	9
[0,1,3,5]	*	[0,1,3,5]	=	[0,1,2,3,4,5,6,8,10]	9
[0,1,3,6]	*	[0,1,3,6]	=	[0,1,2,3,4,6,7,9]	8
[0,1,3,7]	*	[0,1,3,7]	=	[0,1,2,3,4,6,7,8,10]	9
[0,1,4,5]	*	[0,1,4,5]	=	[0,1,2,4,5,6,8,9,10]	9
[0,1,4,6]	*	[0,1,4,6]	=	[0,1,2,3,4,6,7,8,10]	9
[0,1,4,7]	*	[0,1,4,7]	=	[0,1,2,3,5,6,8,9]	8
[0,1,4,8]	*	[0,1,4,8]	=	[0,1,2,4,5,8,9]	7
[0,1,5,6]	*	[0,1,5,6]	=	[0,1,2,3,4,7,8,9]	8
[0,1,5,7]	*	[0,1,5,7]	=	[0,1,2,3,5,7,8,9]	8
[0,1,5,8]	*	[0,1,5,8]	=	[0,1,2,4,5,6,8,9,10]	9
[0,1,6,7]	*	[0,1,6,7]	=	[0,1,2,6,7,8]	6
[0,2,3,5]	*	[0,2,3,5]	=	[0,1,2,3,4,5,6,8,10]	9
[0,2,3,6]	*	[0,2,3,6]	=	[0,1,3,4,5,6,7,9]	8
[0,2,3,7]	*	[0,2,3,7]	=	[0,1,2,3,4,5,7,8,10]	9
[0,2,4,6]	*	[0,2,4,6]	=	[0,2,4,6,8,10]	6
[0,2,4,7]	*	[0,2,4,7]	=	[0,1,2,3,5,6,8,10]	8
[0,2,4,8]	*	[0,2,4,8]	=	[0,2,4,6,8,10]	6
[0,2,5,7]	*	[0,2,5,7]	=	[0,1,3,5,6,8,10]	7
[0,2,5,8]	*	[0,2,5,8]	=	[0,1,2,4,5,7,8,10]	8
[0,2,6,8]	*	[0,2,6,8]	=	[0,2,4,6,8,10]	6
[0,3,4,7]	*	[0,3,4,7]	=	[0,1,2,4,5,6,8,9,10]	9
[0,3,5,8]	*	[0,3,5,8]	=	[0,1,2,3,5,6,7,8,10]	9
[0,3,6,9]	*	[0,3,6,9]	=	[0,3,6,9]	4

Appendix D

Stephen Heinemann

Piano Variations

Musical score for piano, measures 13-25. The score is written in two staves (treble and bass clef) and includes various dynamic markings and articulations. The key signature is B-flat major (two flats). The time signature is 7/8, which changes to 6/8 at measure 14 and 22, and 2/4 at measure 23. The score is divided into five systems, each with a measure number on the left. Dynamic markings include *pp*, *mp*, *sfz*, and *p*. There are also slurs and accents throughout the piece.

13 *pp* *pp*

16 *mp* *sfz pp* *sfz pp*
p *pp* *sfz*

19 *sfz pp* *sfz pp* *sfz*
pp *sfz pp* *sfz pp* *sfz*

22 *pp sfz pp sfz pp sfz* *pp sfz pp sfz* *pp*
pp *sfz pp sfz* *pp sfz pp* *sfz* *pp*
2 4 5

25 *sfz pp* *sfz pp* *sfz pp*
sfz *pp*
11 3

28 *sfz pp ff*

sfz pp sfz pp ff

This system contains measures 28, 29, and 30. The music is written for piano in a key with three sharps (F#, C#, G#). It features a complex texture with multiple voices in both hands, including chords and moving lines. Dynamic markings *sfz*, *pp*, and *ff* are used to indicate changes in volume and intensity.

31 *mf dim. mp*

Red......

This system contains measures 31, 32, and 33. Measure 31 begins with a *mf dim.* marking. The music continues with *mp* dynamics. A *Red.* (ritardando) marking is indicated with a dotted line under the bass staff. The texture is dense with many notes.

34 *mf*

This system contains measures 34, 35, and 36. The music is characterized by a steady eighth-note pattern in the right hand and a more sparse bass line. A *mf* dynamic marking is present in measure 35.

37 *f mp f mf*

This system contains measures 37, 38, and 39. The music features a mix of dynamics, starting with *f* in measure 37, moving to *mp* in measure 38, and alternating between *f* and *mf* in measure 39.

40 *f p sub. cresc. molto f*

42 *2 3*

44 *2 4*

This system contains measures 40, 41, 42, 43, and 44. Measure 40 starts with *f*. Measure 41 has *p sub.* (piano subito). Measure 42 has *cresc. molto*. Measure 43 has *f*. Measure 44 has *f*. There are also some numerical markings: *2 3* above measure 42 and *2 4* above measure 44.

Meno mosso A tempo

43 (*f*) *dim.* *mp*

47 *mf* *mp*

50 *mf* *dim.*

53 (*mf*) *dim.*

56 *p*

1 2 3 4 5 1 2 3 4 5

Coda

58 *f sub.* *p*

60 *accel. e cresc.*

62 *(mp)* *(mf)*

65 *(f)* *allargando* *ff* *loco* *J = 52* *dim.*

69 *accel.** *J = 160* *pp* *dim.* *p* *(p)*

*Repeat ad lib. and
accelerate to tempo:
pedal gradually off

74

System 1: Measures 74-78. Treble clef, bass clef. The music consists of eighth-note patterns in both hands.

79 *cresc.* *f* *p* *cresc.*

System 2: Measures 79-83. Treble clef, bass clef. Dynamics include *cresc.*, *f*, *p*, and *cresc.*

84 *f* *p* *f*

System 3: Measures 84-88. Treble clef, bass clef. Dynamics include *f*, *p*, and *f*.

89

System 4: Measures 89-93. Treble clef, bass clef. Features sustained chords in the treble and eighth-note patterns in the bass.

94 *p*

System 5: Measures 94-98. Treble clef, bass clef. Dynamics include *p*.

99 *mp* *f* *(p)*

First system of musical notation, measures 99-102. The right hand features a melodic line with slurs and accents, while the left hand plays a steady eighth-note accompaniment. Dynamics range from *mp* to *f*, with a *(p)* marking in measure 101.

103 *mp* *f* *(p)*

Second system of musical notation, measures 103-106. Similar to the first system, it features a melodic line in the right hand and an eighth-note accompaniment in the left hand. Dynamics include *mp*, *f*, and *(p)*.

107 *p*

Third system of musical notation, measures 107-110. The right hand contains a triplet of eighth notes, and the left hand continues with an eighth-note accompaniment. The dynamic is marked *p*.

111 *mp* *f* *(p)*

Fourth system of musical notation, measures 111-114. The right hand features a melodic line with slurs and accents, and the left hand has an eighth-note accompaniment. Dynamics include *mp*, *f*, and *(p)*.

115 *mp* *f* *p*

Fifth system of musical notation, measures 115-118. The right hand contains a melodic line with slurs and accents, and the left hand has an eighth-note accompaniment. Dynamics include *mp*, *f*, and *p*.

119

System 1: Measures 119-122. The right hand features a melodic line with slurs and accents, while the left hand plays a steady eighth-note accompaniment. Fingering '5' is indicated for the left hand in measures 120, 121, and 122.

123

System 2: Measures 123-126. Continuation of the eighth-note accompaniment in the left hand and the melodic line in the right hand.

127

System 3: Measures 127-130. Continuation of the eighth-note accompaniment in the left hand and the melodic line in the right hand.

131

System 4: Measures 131-134. Continuation of the eighth-note accompaniment in the left hand and the melodic line in the right hand.

135

System 5: Measures 135-138. Measure 135 includes a dynamic marking of *mf*. Measures 136-138 show a change in texture with sustained chords in the left hand and a melodic phrase in the right hand. A tempo marking of $J = 52$ is present above measure 137. The system concludes with a dynamic marking of *p*.

Musical score for measures 141-145. The piece is in 4/4 time with a key signature of one flat (B-flat major or D minor). The music features a complex texture with multiple voices in both hands, including sixteenth-note runs and chords. Measure 145 ends with a fermata and a *pp* dynamic marking.

Musical score for measures 146-148. The music consists of a single melodic line in the right hand, primarily using a seven-fingered scale. It begins with a *dim.* (diminuendo) marking and a *p* (piano) dynamic. Measure 148 ends with a fermata.

Musical score for measures 149-152. The music features a seven-fingered scale in the right hand and a bass line in the left hand. The dynamics range from *pp* (pianissimo) to *p* (piano). Measure 152 ends with a fermata.

Musical score for measures 153-157. The music is in 4/4 time and features a bass line in the left hand and a right hand with triplets and a *cresc.* (crescendo) marking. A *sost. pedal until* instruction is present. Measure 157 ends with a fermata.

Musical score for measures 158-162. The music features a bass line in the left hand and a right hand with triplets and a *pp* (pianissimo) dynamic. Measure 162 ends with a fermata.

163 *mf*

3 3 3

Ped.

168

3 3 *mp* 3 3 *mf*

172

3 *mp* 3 5 5 5 *p*

176 *mf*

3 3 *cresc.*

Ped.

179 *f dim. poco a poco*

5 5 5 5

Musical score system 1, measures 184-188. The system consists of two staves. The upper staff contains a melodic line with triplets and a *pp* dynamic marking. The lower staff contains a bass line with triplets and a *pp* dynamic marking. A *sost. pedal until* instruction is present in the lower staff. The system concludes with a *mp* dynamic marking.

Musical score system 2, measures 189-192. The system consists of two staves. The upper staff features a melodic line with triplets. The lower staff contains a bass line with a *legato* instruction and two sixteenth-note chords marked with a '6'. The system concludes with a *mp* dynamic marking.

Musical score system 3, measures 193-196. The system consists of two staves. The upper staff contains a melodic line with triplets. The lower staff contains a bass line with a sixteenth-note chord marked with a '6' and a *pp* dynamic marking. A *Red.* instruction is present in the lower staff. The system concludes with a *p* dynamic marking.

Musical score system 4, measures 197-200. The system consists of two staves. The upper staff contains a melodic line with triplets and a *mp* dynamic marking. The lower staff contains a bass line. The system includes tempo markings: *rit.*, *accel.*, *sempre cresc.*, and *rit.*.

Musical score system 5, measures 201-204. The system consists of two staves. The upper staff contains a melodic line with triplets and a *3* marking. The lower staff contains a bass line with a *3* marking. The system includes tempo markings: *accel.* and *f in tempo*. The system concludes with a *3* marking.

a)

203

allargando

ff a tempo p

Red.

206

sf

(p)

R.H.

Red.

211

pp

lunga

♩. = 132

p staccato

Red.

216

mp

(4)

a)

3

3

3

221 *cresc.* *mf pp* *mp*

226 *cresc.* *mf* *mp*

231 *mf* *cresc.* *f p legato* *cresc. poco a poco*

236 (*mp*) (*mf*)

240 (*f*) *ff* *R.H.* *L.H.*

245 *staccato* *(ff)*

250 *p* *f pp* *f* *mp* *f*

255 *f* *f* *mp* *p cresc. molto*

260 *f*

Musical score for measures 264-267. The system consists of two staves. Measure 264 is marked with a forte *(f)* dynamic. The music features a complex texture with sixteenth-note runs in the right hand and a more melodic line in the left hand. Dynamics shift to *ff* and then *pp* in the later measures. A *rit.* marking is present at the end of the system.

Musical score for measures 268-272. Measure 268 is marked with *ff* and *poco rit.*. A tempo marking of $\text{♩} = 48$ is shown above the staff. The right hand has a melodic line with *pp* dynamics, while the left hand plays a rhythmic accompaniment. A *L.H.* section is indicated, and the instruction *una corda* is written below the left hand staff. The system concludes with a *rit.* marking.

Musical score for measures 273-275. Measure 273 features a triplet of eighth notes in the right hand. The dynamics are marked *p* and *pp*. The left hand provides a steady accompaniment with sustained notes. The system ends with a *rit.* marking.

Musical score for measures 276-279. Measure 276 is marked with a piano *p* dynamic. The right hand contains a melodic line with a *mp* dynamic in measure 277. The left hand features a complex accompaniment with a quintuplet of eighth notes (marked '5') in measure 277 and a triplet (marked '3') in measure 279. The system concludes with a *rit.* marking.

Musical score system 1, measures 279-281. The system consists of two staves. The right-hand staff (treble clef) contains a melodic line with various ornaments and fingerings, including a 6th and a 10th. The left-hand staff (bass clef) contains a bass line with some ornaments. The tempo/mood marking is *tres corde accel. e cresc.*

Musical score system 2, measures 282-284. The system consists of two staves. The right-hand staff (treble clef) features a melodic line with dynamics *mp*, *p*, and *pp*, and includes a *una corda* marking. The left-hand staff (bass clef) has a bass line with a triplet of eighth notes. The tempo marking is *a tempo*.

Musical score system 3, measures 285-287. The system consists of two staves. The right-hand staff (treble clef) has a melodic line with a dynamic marking of *pp* and a *tres corde* marking. The left-hand staff (bass clef) has a bass line with a triplet of eighth notes. The tempo/mood marking is *pp tres corde*.

Musical score system 4, measures 288-290. The system consists of two staves. The right-hand staff (treble clef) has a melodic line with a triplet of eighth notes. The left-hand staff (bass clef) has a bass line with a triplet of eighth notes. The tempo/mood marking is *pp*.

$\text{♩} = 84$

290 *mf* *p*
p *acc.*

294 *mf* *p* *dim.*

298 *pp* *mf* *mp*

302 *dim.* *pp* *mp* *poco piu mosso*

305 *rit.* *f sub.*

Red...

308

(C)

(A)

312

p

f

p

$\text{♩} = 140$

(B)

315

cresc.

p

319

p

cresc.

p

Musical score for measures 324-328. The piece is in 3/4 time. Measure 324 starts with a treble clef and a key signature of one sharp (F#). The melody in the right hand features eighth and sixteenth notes. The left hand has a bass clef and plays a steady eighth-note accompaniment. Dynamic markings include *mf* and *p*. There are also markings for *v* (accents) and *volo* (trills).

Musical score for measures 329-332. The key signature changes to one flat (Bb) in measure 329. The right hand continues with a melodic line, and the left hand provides accompaniment. A *cresc.* (crescendo) marking is present in measure 329. The time signature changes to 6/8 in measure 330 and back to 3/4 in measure 332.

Musical score for measures 333-336. The piece is in 3/4 time. The right hand has a melodic line with eighth notes. The left hand has a bass clef and plays a steady eighth-note accompaniment. A *f sempre* (forte sempre) marking is present in measure 333.

Musical score for measures 337-340. The piece is in 3/4 time. The right hand has a melodic line with eighth notes. The left hand has a bass clef and plays a steady eighth-note accompaniment. A *sim.* (sforzando) marking is present in measure 337.

Musical score for measures 341-345. The piece is in 3/4 time. The right hand has a melodic line with eighth notes. The left hand has a bass clef and plays a steady eighth-note accompaniment. Fingerings are indicated with numbers 1-5 above and below notes. A *v* (accent) marking is present in measure 345.

This musical score consists of five systems of piano music, each with a treble and bass clef staff. The key signature is three sharps (F#, C#, G#). Measure numbers 346, 353, 359, 367, and 374 are indicated at the beginning of their respective systems. The score includes various musical notations such as slurs, ties, and dynamic markings. Performance instructions include *loco*, *rit.*, *A tempo*, *p cresc.*, *f*, *mp*, and *ff*. There are also markings for *V* (vibrato) and *8* (octave). Measure 346 features a wavy line in the treble staff. Measure 353 has a *pp.* marking. Measure 359 includes a *mp* marking and a *rit.* instruction. Measure 367 is marked *A tempo* and *p cresc.*. Measure 374 starts with a *f* marking and includes *mp* and *ff* markings.

Appendix E











Stephen Heinemann

Inscriptions
for orchestra

Instrumentation

Piccolo
 2 Flutes
 2 Oboes
 English Horn
 2 Clarinets in B \flat
 Bass Clarinet in B \flat
 2 Bassoons
 Contrabassoon
 4 Horns in F
 3 Trumpets in B \flat
 2 Trombones
 Bass Trombone
 Tuba

Percussion (4 players):

Timpani 
 Snare Drum (snares on) 
 Tenor Drum 
 Bass Drum 
 Suspended cymbal 
 Crotales 
 Tam-tam 
 Glockenspiel 
 Vibraphone 
 Xylophone 

Mallets: hard  
 medium  
 soft  
 Sticks:  

Harp

Piano

Strings

Inscriptions

Stephen Heinemann

Score for *Inscriptions* by Stephen Heinemann. The score is in common time (C) and begins with a tempo marking of *J = 88*. The instrumentation includes:

- Flute
- Picc.
- Oboe
- English Horn
- B♭ Clarinet
- B♭ Bass Clarinet
- Saxophone 1
- Contrabass
- Horn 1 & 2 (Horn in F)
- B♭ Trumpet
- Trumpet 1
- Bass Trumpet
- Tuba
- Percussion (4 parts)
- Key
- Piano
- Viola 1 & 2
- Violin
- Cello
- Bass

The score is divided into several systems. The first system includes Flute, Picc., Oboe, English Horn, B♭ Clarinet, B♭ Bass Clarinet, Saxophone 1, and Contrabass. The second system includes Horn 1 & 2, B♭ Trumpet, Trumpet 1, Bass Trumpet, and Tuba. The third system includes Percussion (4 parts). The fourth system includes Key and Piano. The fifth system includes Viola 1 & 2, Violin, Cello, and Bass. The score contains various musical notations including notes, rests, and dynamic markings such as *f* and *mf*.

This page of a musical score, numbered 163, contains a complex arrangement of instruments. The score is organized into several systems of staves. The top system includes staves for Flute (Fl.), Oboe (Ob.), Clarinet (Cl.), Bassoon (Fg.), Trumpet (Tr.), Trombone (Tbn.), and Cymbal (Cym.). The second system features the Violin I and II sections, Viola, and Cello/Double Bass. The third system is for Percussion, with four individual staves. The fourth system is for Piano (P). The fifth system is for Violoncello (Vcllo) and Double Bass (Cb.). The sixth system is for Violins I and II. The seventh system is for Violoncello (Vcllo) and Double Bass (Cb.). The score includes various musical notations such as notes, rests, and dynamic markings like *mf* and *f*. A rehearsal mark 'J. m' is present at the top of the page, and another 'J. m' appears above the Violoncello/Double Bass staff in the lower section. The page is divided into measures by vertical bar lines.

This page of musical score, numbered 164, features a large ensemble of instruments and voices. The parts are arranged as follows from top to bottom:

- Flute (Fl):** Two staves with melodic lines.
- Oboe (Ob):** Two staves with melodic lines.
- Clarinet (Cl):** Two staves with melodic lines.
- Bassoon (Bn):** Two staves with melodic lines.
- Trumpet (Tr):** Three staves with melodic lines.
- Trombone (Tbn):** Three staves with melodic lines.
- Percussion (Perc):** Four staves with rhythmic patterns.
- Piano (P):** Two staves with complex accompaniment.
- Violin (Vn):** Two staves with melodic lines.
- Viola (Va):** Two staves with melodic lines.
- Cello (Vc):** Two staves with melodic lines.
- Double Bass (Cb):** Two staves with melodic lines.

The score is written in a complex, multi-measure format, with various dynamics and articulations. The music is arranged in a large ensemble format, typical of a symphony or concert band score. The notation includes various musical symbols such as notes, rests, and dynamic markings.

10 J-66

This page contains a musical score for measures 10 through 13. The score is organized into several systems of staves. The top system includes Flute (Fl), Piccolo (Pic), Oboe (Ob), Clarinet in B-flat (Cl Bb), Bassoon (Bsn), and Contrabass (Cb). The second system includes Trumpet (Tr), Trombone (Tbn), and Tuba (Tu). The third system includes four Horns (Hr 1-4). The fourth system includes French Horn (Fr), Trombone (Tbn), and Tuba (Tu). The fifth system includes Violin I (Vn I), Violin II (Vn II), Viola (Va), and Double Bass (Db). The music features various dynamics such as *p* (piano) and *f* (forte), and includes articulation marks like accents and slurs. The notation includes notes, rests, and complex rhythmic patterns.

This page of a musical score, numbered 166, contains several systems of staves. The top system includes a vocal line with lyrics and a piano accompaniment. The piano part features a complex rhythmic pattern of sixteenth notes in the right hand and a bass line in the left hand. The middle section of the page consists of several systems of empty staves, likely for additional instruments or parts. The bottom section features a grand staff with piano accompaniment, including a treble clef with a melodic line and a bass clef with a bass line. The score is written in black ink on a white background.

This page of musical score, numbered 167, is divided into six systems of staves. The first system (measures 21-24) features woodwinds (flute, oboe, clarinet, bassoon) and strings. The second system (measures 25-28) includes brass (trumpets, trombones) and strings. The third system (measures 29-32) contains percussion (snare, cymbals) and strings. The fourth system (measures 33-36) features woodwinds and strings. The fifth system (measures 37-40) includes woodwinds and strings. The sixth system (measures 41-44) features woodwinds and strings. The score contains various musical notations such as notes, rests, and dynamic markings like 'pp' and 'f'. There are also some handwritten annotations and a 'J-128' marking in the top right corner.

This page of a musical score, numbered 168, contains six systems of music. The first system (measures 1-4) features a melody in the first violin with a long slur. The second system (measures 5-8) continues the melodic line. The third system (measures 9-12) shows a change in texture with more active parts. The fourth system (measures 13-16) is mostly empty staves. The fifth system (measures 17-20) shows a new melodic entry in the first violin. The sixth system (measures 21-24) continues the piece with various rhythmic patterns.

Fl
Pic
Ob
E.Hr
Cl
B.Cl
Bsn
Cb

Tr
Tbn
Tu

Perc
1
2
3
4

Hp
Tm

Va
1
2
Via
Vcl
Cb

This page of a musical score, numbered 170, contains the following instruments and parts:

- Flute (Fl.)**: Part 1, starting at measure 51.
- Oboe (Ob.)**: Part 1, starting at measure 51.
- Clarinet (Cl.)**: Part 1, starting at measure 51.
- Bassoon (Fg.)**: Part 1, starting at measure 51.
- Trumpet (Tr.)**: Parts 1, 2, and 3, starting at measure 51.
- Trombone (Tbn.)**: Parts 1, 2, and 3, starting at measure 51.
- Drum (Dr.)**: Part 1, starting at measure 51.
- Percussion (Perc.)**: Part 1, starting at measure 51.
- Piano (P)**: Part 1, starting at measure 51.
- Violin (Va.)**: Parts 1 and 2, starting at measure 51.
- Viola (Vo.)**: Part 1, starting at measure 51.
- Cello (Vc.)**: Part 1, starting at measure 51.
- Double Bass (Cb.)**: Part 1, starting at measure 51.

The score is written in a standard musical notation style, with various notes, rests, and dynamic markings. The woodwinds and strings play melodic lines, while the brass and percussion provide harmonic support. The piano part is also visible, contributing to the overall texture of the music.

This page of a musical score, numbered 171, contains the following parts and markings:

- Flute (Fl):** Part 1, marked with a first ending bracket (1.2).
- Oboe (Ob):** Part 1, marked with a first ending bracket (1.2).
- Clarinet (Cl):** Part 1, marked with a first ending bracket (1.2).
- Bassoon (Bs):** Part 1, marked with a first ending bracket (1.2).
- Trumpet (Tr):** Part 1, marked with a first ending bracket (1.2).
- Trombone (Tbn):** Part 1, marked with a first ending bracket (1.2).
- Percussion (Perc):** Four parts (1, 2, 3, 4) with rhythmic patterns.
- Piano (Pf):** Part with a first ending bracket (1.2).
- Violin (Va):** Parts 1 and 2, marked with a first ending bracket (1.2).
- Viola (Vo):** Part 1, marked with a first ending bracket (1.2).
- Double Bass (Cb):** Part 1, marked with a first ending bracket (1.2).

The score is written in 2/4 time and includes various musical notations such as notes, rests, and first ending brackets. The page concludes with a double bar line and repeat signs.

This page of musical score, numbered 172, is arranged for a string quartet and piano accompaniment. It consists of six systems of staves. The first system contains four staves for the string quartet (Violin I, Violin II, Viola, and Violoncello) and two staves for the piano accompaniment. The second system contains four staves for the string quartet. The third system contains two staves for the piano accompaniment. The fourth system contains two staves for the piano accompaniment. The fifth system contains two staves for the piano accompaniment. The sixth system contains four staves for the string quartet. The score includes various musical notations such as notes, rests, and dynamic markings. A rehearsal mark 'A' is present at the end of the first system. The piano part features a complex, rhythmic accompaniment in the lower register.

This page of a musical score, numbered 173, contains several systems of staves. The top system includes staves for Flute (Fl), Oboe (Ob), Clarinet (Cl), Bassoon (Fg), Horn (Hr), Trumpet (Tr), Trombone (Tbn), and Tuba (Tu). The second system includes staves for Violin I (Vn I), Violin II (Vn II), Viola (Vla), and Cello (Vcl). The third system includes staves for Violoncello (Vcl) and Double Bass (Cb). The fourth system includes staves for Piano (P) and Harp (Hr). The bottom system includes staves for Violin I (Vn I), Violin II (Vn II), Viola (Vla), and Cello (Vcl). The score is written in a standard musical notation with various clefs and time signatures. The bottom system shows more detailed musical notation, including notes, rests, and dynamic markings.

Fl
Ob
Cl
Fg
Hr
Tp
Tbn
Tu
Vn I
Vn II
Vla
Vcl
Cb
P
Vn I
Vn II
Vla
Vcl

This page of a musical score, numbered 175, contains several systems of staves. The top system includes parts for Flute (Fl), Piccolo (Pic), Oboe (Ob), English Horn (E.Hr), Clarinet in B-flat (Cl Bb), Bass Clarinet (Bb Cl), Bassoon (Bsn), and Contrabassoon (Cb). The second system contains parts for Horns (Hr), Trumpets (Trp), Trombones (Tbn), and Tuba (Tu). The third system is for Percussion (Perc), and the fourth is for Harp (Hr). The fifth system is for Piano (P). The bottom system includes parts for Violins (Vn), Violas (Va), Cellos (Vc), and Double Basses (Cb). The score features complex musical notation, including various note values, rests, and dynamic markings such as *f* and *legato*. The woodwind and string parts are particularly active, with many notes beamed together and some passages marked with slurs.

This page of a musical score, numbered 176, contains a complex arrangement of instruments. The score is organized into several systems of staves. The top system includes staves for Flute (Fl.), Oboe (Ob.), Clarinet (Cl.), Bassoon (Fg.), Trumpet (Tr.), Trombone (Tbn.), and Tuba (Tub.). The second system features a string section with Violin I (Vn. I), Violin II (Vn. II), Viola (Vla.), Violoncello (Vcllo), and Contrabasso (Cb.). The third system is for the Piano (P). The fourth system is for the Percussion (Perc.). The fifth system is for the Harp (Arpa). The sixth system is for the Double Bass (Cb.). The seventh system is for the Double Bass (Cb.). The eighth system is for the Double Bass (Cb.). The score includes various musical notations such as notes, rests, slurs, and dynamic markings. A rehearsal mark '12' is present in the first system. The page is oriented vertically on the page.

This page of a musical score, numbered 177, contains the following parts and markings:

- Woodwinds:** Flute (Fl), Oboe (Ob), Clarinet in B-flat (Cl), Bassoon (Bs), and Contrabassoon (Cb). The Clarinet and Bassoon parts include a *p dim* marking.
- Brass:** Trumpets (Trp) and Trombones (Tbn). The Trombone parts include a *p dim* marking.
- Percussion:** Four parts (1-4) with various rhythmic patterns and dynamics like *p*.
- Harmonica (Hp):** Part with a *p* marking.
- Piano (Pn):** Part with a *p dim* marking.
- Violins (Vn):** First and Second Violin parts.
- Violas (Va):** Part.
- Celli (Vc):** Part.
- Double Basses (Cb):** Part.

Tempo and performance markings include *J. = 22* and *p* (piano). The score is written in a standard orchestral format with multiple staves per instrument.

The musical score on page 178 is organized into four systems. The first system features woodwinds (Flute, Clarinet, Bassoon, Oboe, Cor Anglais, Bass Clarinet, Contrabassoon) and strings (Violin I, Violin II, Viola, Violoncello, Double Bass). The second system includes Percussion (Timpani, Snare Drum, Cymbals, Tom-toms, Triangle, Gong, Chimes, Bells, Tambourine, Castanets, Maracas, Shakers, Tambour, Conga, Bongos, Djembe, Conga, Bongos, Djembe). The third system includes Percussion (Timpani, Snare Drum, Cymbals, Tom-toms, Triangle, Gong, Chimes, Bells, Tambourine, Castanets, Maracas, Shakers, Tambour, Conga, Bongos, Djembe, Conga, Bongos, Djembe). The fourth system includes Percussion (Timpani, Snare Drum, Cymbals, Tom-toms, Triangle, Gong, Chimes, Bells, Tambourine, Castanets, Maracas, Shakers, Tambour, Conga, Bongos, Djembe, Conga, Bongos, Djembe).

154

This page of a musical score, numbered 179, contains measures 154 through 158. The score is arranged in a multi-system format. The first system includes staves for Flute (Fl.), Piccolo (Pic.), Oboe (Ob.), Clarinet in B-flat (Cl. Bb.), Bassoon (Fg.), and Bassoon in C (Fg. C). The second system includes staves for Horn in F (Hr. F), Horn in E-flat (Hr. Eb), Trumpet in D (Tr. D), Trumpet in C (Tr. C), Trombone in B-flat (Tbn. Bb.), Trombone in C (Tbn. C), and Tuba (Tu.). The third system features four staves for Violins (V. 1, V. 2), Violas (V. 3), Cellos (V. 4), and Double Basses (V. 5). The fourth system contains staves for Flute (Fl.), Clarinet in B-flat (Cl. Bb.), Bassoon (Fg.), and Bassoon in C (Fg. C). The fifth system includes staves for Violins (V. 1, V. 2), Violas (V. 3), Cellos (V. 4), and Double Basses (V. 5). The sixth system contains staves for Flute (Fl.), Clarinet in B-flat (Cl. Bb.), Bassoon (Fg.), and Bassoon in C (Fg. C). The seventh system includes staves for Violins (V. 1, V. 2), Violas (V. 3), Cellos (V. 4), and Double Basses (V. 5). The eighth system contains staves for Flute (Fl.), Clarinet in B-flat (Cl. Bb.), Bassoon (Fg.), and Bassoon in C (Fg. C). The score is written in a standard musical notation style, with various note values, rests, and articulation marks. The key signature and time signature are not explicitly shown but are implied by the notation.

This page of a musical score, numbered 180, contains ten systems of staves. The instruments and their parts are as follows:

- Flute (Fl):** One staff, mostly empty.
- Clarinet (Cl):** One staff, containing a melodic line in the first measure.
- Trumpet (Tr):** Three staves, mostly empty.
- Trombone (Tbn):** Two staves, mostly empty.
- Percussion (Perc):** Four staves, containing rhythmic patterns and drum notation.
- Harmonica (Hr):** One staff, mostly empty.
- Piano (P):** Two staves, containing a complex piano accompaniment with arpeggiated figures and chords.
- Violin (Va):** Two staves, containing a melodic line.
- Cello (Cb):** One staff, mostly empty.

The score is written in a standard musical notation style with various clefs, time signatures, and dynamic markings. The first system is marked with a '180' at the beginning.

146

Flu
Fl 1
Ob 1
Bln
Cl 1
B.C.
Tr 1
Cm

Hr
Trp
Trb
B.Trb
Tr

1
2
3
4

Pf

Vn
Va
Vc

ppp

182

Fl

Ob

Cl B \flat

Fg

Hr F

Tb B \flat

Tr B \flat

Tr C

Tb C

Tb B \flat

Perc

Hr

P

Vn I

Vn II

Va

Cb

The musical score on page 184 is organized into five main systems of staves. The first system (measures 1-4) features woodwinds (Flute, Oboe, Clarinet, Bassoon) and strings (Violin I, Violin II, Viola, Violoncello, Double Bass). The second system (measures 5-8) includes brass instruments (Trumpets 1-3, Trombones 1-3, Tuba). The third system (measures 9-12) is for Percussion (1-4). The fourth system (measures 13-16) includes Harp and Piano. The fifth system (measures 17-20) includes Violins I and II, Viola, and Cello/Double Bass. The score contains various musical notations such as notes, rests, dynamics (pp, p), and articulation marks.

Fl

Ob

Cl Bb

Fg

Cb

1
Tp

2
3

1
Tb

2

Tu

1
P

2
3
4

Hp

1
Vn

2

Va

Vc

Cb

174

Flu
Pic
Ob
Eng
Cl
B Cl
Bsn
Cb
Hr
Trp
Tbn
Tuba
Perc
Hr
Pn
Vn
Vla
Vc
Cb

Detailed description of the musical score: The score is arranged in a standard orchestral layout. The top system contains Flute, Piccolo, Oboe, English Horn, Clarinet, Bass Clarinet, Bassoon, and Contrabassoon. The second system contains Horns (1-4), Trumpets (1-3), Trombones (1-3), and Tuba. The third system contains Percussion (1-4) and Harp. The fourth system contains Piano. The fifth system contains Violins (1-2), Viola, and Cello. The Piano part is the only one with musical notation, showing a complex texture with many sixteenth and thirty-second notes. The Violin parts have some notation in the later measures, including slurs and dynamics like *p* and *f*. The other parts are mostly blank staves.

180 J=60

Flu
Ob
Cl
B Cl
Trp
Tbn
Tuba
Sn
Cym
Tri
Harp
Vn I
Vn II
Va
Vc
Cb

1
2
3
4

Violin I
Violin II
Viola
Cello/Double Bass

p *cresc.* *decresc.* *p*

This page of a musical score, numbered 189, contains several systems of staves. The top two systems are for woodwinds and brass instruments, including Flute (Fl.), Oboe (Ob.), Clarinet (Cl.), Bassoon (Bs.), Trumpet (Tr.), Trombone (Tb.), and Tuba (Tu). The third system is for strings, with staves for Violin I (Vn. I), Violin II (Vn. II), Viola (Va.), and Violoncello (Vcl.). The bottom system is for the vocal ensemble, with staves for Soprano (S), Alto (A), Tenor (T), and Bass (B). The vocal parts feature melodic lines with lyrics and dynamic markings such as *p* and *pp*. The instrumental parts are mostly blank, indicating that the music for these instruments begins on a subsequent page.

215

This page of a musical score, numbered 215, contains the following staves from top to bottom:

- Flute** (Fl): Two staves with musical notation.
- Oboe** (Ob): Two staves with musical notation.
- Bassoon** (Bb): Two staves with musical notation.
- Clarinet** (Cl): Two staves with musical notation.
- Bass Clarinet** (Bb): Two staves with musical notation.
- Trumpet** (Tr): Three staves with musical notation.
- Trombone** (Tb): Three staves with musical notation.
- Piano** (P): Two staves with musical notation.
- Violin** (Va): Two staves with musical notation.
- Viola** (Vo): Two staves with musical notation.
- Double Bass** (Cb): Two staves with musical notation.

The score includes various musical notations such as notes, rests, and dynamic markings. The woodwind and string sections are active throughout the page, while the brass section has some rests. The piano part features a complex, rhythmic accompaniment.

This page of a musical score, numbered 192, features a variety of instruments. The top section includes woodwinds: Flute (Fl), Flute II (Fl 2), Oboe (Ob), English Horn (E.Hr), Clarinet in C (Cl 3), Clarinet in Bb (Cl 2), Bassoon (Ba), and Contrabassoon (Cb). The middle section contains brass instruments: Horns (Hr) in F major, Trumpets (Trp) in F major (1, 2, 3), Trombones (Tbn) in Bb major (1, 2), and Tuba (Tu). Below the brass are four Percussion (Perc) staves, a Harp (Hr), and a Piano (P). The bottom section features strings: Violins (Vn) in G major (1, 2), Violas (Va), and Cellos (Cb). The score is written in a standard musical notation with various dynamics and articulations. The piano part is particularly active, with dense sixteenth-note passages in both hands. The woodwinds and brass parts have several measures of rests, indicating they are not playing during those sections.

236

Perc
Fl 1
Fl 2
Ob
Bbn
Cl
B Cl
Bn 2
Cb

1
2
3
4
Trp 1
Trp 2
Tbn 1
Tbn 2
Tbn 3
Tbn 4

1
2
3
4
Pn

Hp

1
2
Vi 1
Vi 2
Va
Cb

237

Flute
Oboe
Clarinet
Bassoon
Trumpet
Trombone
Percussion
Violin
Viola
Cello/Double Bass

mus. instr.
mus. instr.
mus. instr.
mus. instr.
mus. instr.
mus. instr.
mus. instr.
mus. instr.
mus. instr.
mus. instr.

mus. instr.
mus. instr.
mus. instr.
mus. instr.

mus. instr.
mus. instr.
mus. instr.
mus. instr.

This page of musical score, numbered 195, contains the following instruments and parts:

- Woodwinds:** Flute (Fl.), Oboe (Ob.), Clarinet (Cl.), Bassoon (Fg.), and Contrabassoon (Cb.).
- Strings:** Violin I (Vn. I), Violin II (Vn. II), Viola (Vla.), Violoncello (Vcl.), and Double Bass (Cb.).
- Brass:** Trumpet (Tr.), Trombone (Tbn.), and Tuba (Tub.).

The score includes various musical notations such as notes, rests, and dynamics. A circled section of music is visible in the lower right quadrant.

This page of a musical score, numbered 197, contains a complex arrangement of instruments and voices. The score is organized into several systems of staves:

- Woodwinds:** The top system includes staves for Flute (Fl.), Piccolo (Pic.), Oboe (Ob.), Bassoon (Fag.), Clarinet in B-flat (Cl.), Bass Clarinet (Cl. B.), Bassoon in C (Fag. C), and Contrabassoon (Fag. Cb.).
- Strings:** The second system includes staves for Violin I (Vn. I), Violin II (Vn. II), Viola (Vla.), Violoncello (Vcl.), and Contrabass (Cb.).
- Percussion:** The third system includes staves for Timpani (Timp.), Snare Drum (Tr.), and Cymbals (Cym.).
- Other Instruments:** The fourth system includes staves for Harp (Arp.), Piano (P.), and Organ (Org.).
- Vocal Parts:** The bottom system includes staves for Soprano (S.), Alto (A.), Tenor (T.), and Bass (B.).

The score is written in a common time signature (C) and features a variety of musical notations, including notes, rests, and dynamic markings. The vocal parts have lyrics written below the notes. The overall layout is dense and professional, typical of a full orchestral score.

Vita

Stephen Heinemann was born in Evanston, Illinois on June 13, 1952. He received his Bachelor of Music degree in 1979 and Master of Arts degree in 1982, both from San Francisco State University. He was Instructor of Music at Northland Pioneer College from 1982 to 1988. In 1991, he joined the faculty of the Department of Music at Bradley University. He and his wife Mary work and play in Peoria, where they live with their triplet daughters Robin, Lindsay, and Karen.