COALESCENCE OF SKEW BROWNIAN MOTIONS

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The purpose of this short note is to prove almost sure coalescence of two skew Brownian motions starting from different initial points, assuming that they are driven by the same Brownian motion. The result is very simple but we would like to record it in print as it has already become the foundation of a research project of Burdzy and Chen (1999). Our theorem is a by-product of an investigation of variably skewed Brownian motion, see Barlow et al. (1999). We use exccursion theory in a manner similar to that in a paper on "perturbed Brownian motion" by Perman and Werner (1997). See also other articles on perturbed Brownian motion by Chaumont and Doney (1999), Doney (1998), Doney, Warren and Yor (1998) and Werner (1995).

Suppose that B_t is the standard Brownian motion with $B_0 = 0$ and consider the equation

$$X_t^x = x + B_t + \beta L_t^x, \qquad t \geq 0, \tag{1}$$

where X_t^x satisfies the initial condition $X_0^x = x$. Here β is a fixed number in $[-1, 1]$ and L_t^x is the symmetric local time of X_t^x at 0. Harrison and Shepp (1981) proved that (1) has a unique strong solution, which is skew Brownian motion. One way to define skew Brownian motion in the case $\beta \geq 0$ is to start with a standard Brownian motion B_t' and flip every excursion of B_t' below 0 to the positive side with probability β, independent of what happens to other excursions. See Itô and McKean (1965) or Walsh (1978) for more information.

Theorem. *If X_t^x and X_t^y are solutions of (1) with the same $\beta \in [-1, 1] \setminus \{0\}$, relative to the same Brownian motion B_t, then $X_t^x = X_t^y$ for some $t < \infty$, a.s.*

Proof. For simplicity assume that $\beta > 0$ and $0 = x < y$. Let $\widehat{L}_t^0 = \beta L_t^0$, $\widehat{L}_t^y =$

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1. Research partially supported by an NSERC (Canada) grant.
2. Research partially supported by NSF grant DMS-9700721.
3. Research partially supported by the Fund for the Promotion of Research at the Technion.

$y + \beta L_t^y$, and

$$
\begin{aligned}
T_0 &= 0,\\
S_k &= \inf\{t > T_k : -B_t = \widehat{L}_t^y\}, \quad k \geq 0,\\
T_k &= \inf\{t > S_{k-1} : -B_t = \widehat{L}_t^0\}, \quad k \geq 1,\\
W_k &= \frac{\widehat{L}^y_{S_{k-1}} - \widehat{L}^0_{S_{k-1}}}{\widehat{L}^y_{T_{k-1}} - \widehat{L}^0_{T_{k-1}}}, \quad k \geq 1,\\
V_k &= \frac{\widehat{L}^y_{T_k} - \widehat{L}^0_{T_k}}{\widehat{L}^y_{S_{k-1}} - \widehat{L}^0_{S_{k-1}}}, \quad k \geq 1,\\
M_k &= \widehat{L}^y_{T_k} - \widehat{L}^0_{T_k}, \quad k \geq 0.
\end{aligned}
$$

We will first find the distributions of W_k's and V_k's using excursion theory. Recall the fundamentals of excursion theory for the standard Brownian motion from, e.g., Karatzas and Shreve (1991). The Brownian excursions from 0 form a Poisson point process whose clock can be identified with the local time of Brownian motion at 0. The intensity of excursions on the positive side of 0 whose height is greater than h is equal to $1/(2h)$.

The stopping time S_0 may be described as the first time when an excursion of $-B_t - \widehat{L}_t^0$ above 0 hits the level $y - \widehat{L}_t^0$. These excursions can be identified with the excursions of the skew Brownian motion X_t^0 below 0. They form a Poisson point process $\mathcal{P}$ similar to the Poisson point process of excursions of the standard Brownian motion from 0. The intensity of $\mathcal{P}$-excursions above 0 with height greater than h is equal to $(1-\beta)/(2h)$. Note the extra factor $1-\beta$ as compared to the analogous formula for the excursions of the standard Brownian motion. The factor can be explained using the excursion flipping construction of skew Brownian motion mentioned in the introduction—in a sense, the fraction of excursions flipped to the other side is equal to $\beta/2$. When the clock L_t^0 for the Poisson point process $\mathcal{P}$ takes a value u then the instantaneous intensity of excursions with height greater than $y - \widehat{L}_t^0$ is equal to $(1-\beta)/(2(y-\beta u))$. We have $\widehat{L}^y_{S_0} - \widehat{L}^0_{S_0} < a$ if no $\mathcal{P}$-excursion with height greater than $y - \widehat{L}_t^0$ occurs before the time s when $y - \widehat{L}_s^0 = a$, i.e., when $L_s^0 = (y-a)/\beta$. Thus excursion theory enables us to write the probability of this event using Poisson probabilities as follows,

$$
P(\widehat{L}^y_{S_0} - \widehat{L}^0_{S_0} < a) = \exp\left(-\int_0^{(y-a)/\beta} \frac{1-\beta}{2(y-\beta u)} du\right) = \left(\frac{a}{y}\right)^{(1-\beta)/(2\beta)}.
$$

Recall that $\widehat{L}^y_{T_0} - \widehat{L}^0_{T_0} = y$. We have

$$
P(W_1 y < a) = P(W_1(\widehat{L}^y_{T_0} - \widehat{L}^0_{T_0}) < a) = P(\widehat{L}^y_{S_0} - \widehat{L}^0_{S_0} < a) = (a/y)^{(1-\beta)/(2\beta)}.
$$

By changing the variable we obtain for $w \in (0,1)$,

$$
P(W_1 < w) = w^{(1-\beta)/(2\beta)}.
$$

By the strong Markov property, $P(W_k < w) = w^{(1-\beta)/(2\beta)}$ for $w \in (0,1)$ and every $k \geq 1$.

A totally analogous argument shows that $P(V_k > v) = v^{-(1+\beta)/(2\beta)}$ for $v \geq 1$ and $k \geq 1$.

Note that, by the strong Markov property, all random variables $V_k, W_k, k \geq 1$, are jointly independent.

Next we will show that the process M_k is a martingale and converges to 0. First, note that $M_k = M_{k-1} W_k V_k$. It is elementary to check that $EW_k = (1-\beta)/(1+\beta)$ and $EV_k = (1+\beta)/(1-\beta)$. By the joint independence of W_k's and V_k's,

$$E(M_k \mid M_{k-1}, M_{k-2}, \ldots) = M_{k-1} EW_k EV_k = M_{k-1},$$

which shows that M_k is a martingale. As a positive martingale, the process M_k must converge with probability 1 to a random variable M_∞. Since for every k, M_k is the product of M_{k-1} and an independent random variable $W_k V_k$, the limit M_∞ can take only the values 0 or ∞. By Fatou's Lemma, $EM_\infty \leq EM_0 = y$, so $M_\infty = 0$ a.s.

On every interval $[T_k, S_k]$ the process $\widehat{L}^y_t - \widehat{L}^0_t$ is non-increasing but it is non-decreasing on intervals of the form $[S_k, T_{k+1}]$. Thus

$$\sup_{t \in [T_k, T_{k+1}]} \widehat{L}^y_t - \widehat{L}^0_t \leq \max(M_k, M_{k+1}).$$

In view of convergence of M_k to 0, we must have a.s. convergence of $\widehat{L}^y_t - \widehat{L}^0_t$ to 0 when $t \to \infty$. It remains to show that the convergence does not take an infinite amount of time.

Let $T_\infty = \lim_{k\to\infty} T_k$. In view of the remarks in the last paragraph, it is not hard to see that the value of $\widehat{L}^0_{T_\infty}$ is bounded by $\sum_{k=1}^\infty M_k$. Since $\widehat{L}^0_\infty = \infty$, it will suffice to show that $\sum_{k=1}^\infty M_k < \infty$ in order to conclude that $T_\infty < \infty$. We have for $k \geq 1$,

$$M_k = y \prod_{j=1}^k W_j V_j.$$

We can write

$$y \prod_{j=1}^k W_j V_j = \exp\left(\log y + \sum_{j=1}^k [\log W_j + \log V_j] \right).$$

One can directly check that the distribution of $-\log W_j$ is exponential with mean $2\beta/(1-\beta)$, while the distribution of $\log V_j$ is exponential with mean $2\beta/(1+\beta)$. Thus, $E(\log W_j + \log V_{j+1}) < 0$. It follows that for some $a > 0$, we eventually have

$$\sum_{j=1}^k [\log W_j + \log V_j] \leq -ak.$$

Hence, for some random c_1 and all k we have $M_k \le c_1 e^{-ak}$ and so $\sum_{k=1}^{\infty} M_k < \infty$, a.s. □

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