Flavors of the Fubini-Bruhat Order

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Abstract

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Fubini words are generalized permutations, allowing for repeated letters, and they are in one-to-one correspondence with ordered set partitions. Brendan Pawlowski and Brendon Rhoades extended permutation matrices to pattern matrices for Fubini words. Under a lower triangular action, these pattern matrices produce cells in projective space, specifically $(\mathbb{P}^{k-1})^n$. The containment of the cell closures in the Zariski topology gives rise to a poset which generalizes the Bruhat order for permutations. Unlike Bruhat order, containment is not equivalent to intersection of a cell with the closure of another cell. This allows for a refinement of the poset. It is additionally possible to define a weaker order, giving rise to a subposet containing all the elements. We call these orders, in order of decreasing strength, the espresso, medium roast, and decaf Fubini-Bruhat orders. Hence, the title "Flavors of the Fubini-Bruhat Order." The espresso and medium roast orders are not ranked in general. The decaf order is ranked by codimension of the corresponding cells. In fact, the decaf order has rank generating function given by a well-known q-analog of the Stirling numbers of the second kind. In this thesis, we give increasingly smaller sets of equations describing the cell closures, which lead to several different combinatorial descriptions for the relations in all three orders. We also describe a few classes of covering relations in each of the orders.

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Dedication

To J. Bruce White,

who first taught me how to think.

CHAPTER 1

Introduction

The purpose of this thesis is to study a recent generalization of the Bruhat order on permutations to Fubini words due to Pawlowski and Rhoades [**25**]. The Bruhat order on permutations arises out of Schubert varieties in Grassmannian varieties and flag varieties, which have been important for over 100 years [**19**, **20**]. Hilbert's fifteenth problem was to rigorously justify Schubert's work [**19**, **27**, **28**], and has incited tremendous mathematical progress in enumerative geometry. A nice overview of the history is given by Kleiman and Laksov [**20**]. Contributions to the theory of enumerative geometry have been made throughout the twentieth century, such as by Ehresmann [**13**], Severi [**29**], Demazure [**9**], and Weil [**33**]. Hilbert's fifteenth problem is now considered solved via contributions from Borel [**5**], Bernstein-Gelfand-Gelfand [**2**], Lascoux-Schützenberger [**22**], Macdonald [**24**], and others as described by Stanley [**30**, Notes to Ch. 7]. Many promising properties of Bruhat order extend to one or more of the three Fubini-Bruhat orders we define. In this chapter and the next one, we introduce corresponding pairs of definitions related to Bruhat order and Fubini-Bruhat order, in order to highlight the parallels between the two.

1. Motivation

This section gives definitions of permutations and Fubini words, and discusses the similarities and differences of the corresponding cell closure intersections. The differences in behavior of the cell closures are responsible for the differences in the resulting posets. After developing the necessary theorems in the meat of the thesis, we will see a few examples of Fubini-Bruhat posets and intervals of posets in Chapters 6 and 7.

Definition 1.1.1. *A* ***permutation*** $w \in S_n$ *represents a surjective map* $w : [n] \to [n]$*, where* $[n] = \{1, 2, \ldots, n\}$*. We denote a permutation by its* ***one-line notation****, an ordered list* $w = w_1 w_2 \cdots w_n$*, where* $w_i = w(i)$*.*

Observe that a surjective map $w : [n] \to [n]$ is a bijective map. Defining a permutation as a surjective map sets up a natural generalization to Fubini words. A Fubini word is a generalization of the one-line notation of a permutation, where we allow repeated letters.

Definition 1.1.2. *A* ***Fubini word*** $w = w_1 \cdots w_n$ *represents a surjective map* $w : [n] \to [k]$*, where* k *and* n *are positive integers with* $k \leq n$*. We denote a Fubini word by its* ***one-line notation****, an ordered list* $w = w_1 w_2 \cdots w_n$*, where* $w_i = w(i)$*. We denote by* $\mathcal{W}_{n,k}$ *the Fubini words of length* n *on the alphabet* $[k]$*.*

Observe that when $k = n$, a Fubini word $w \in \mathcal{W}_{n,n}$ is exactly a permutation in S_n, and the one-line notation for w is the same whether w is viewed as a Fubini word or a permutation. When $k < n$, the one-line notation for a Fubini word will have repeated letters. For example, $3132 \in \mathcal{W}_{4,3}$ represents the surjection

$$1 \to 3,$$

$$2 \to 1,$$

$$3 \to 3,$$

$$4 \to 2.$$

Representing permutations by matrices leads to combinatorial methods of studying the geometry of Schubert cells and the flag variety. We give a brief overview of the mechanics of this construction, with more detail and an example in Section 2.1. In this thesis, for concreteness, we work over the field $\mathbb{C}$. If unspecified, all matrices have entries in $\mathbb{C}$. Much of what we describe below can be done over an arbitrary field, but extra care needs to be taken over fields of nonzero characteristic.

Definition 1.1.3. *For every permutation* $w \in S_n$*, the* ***permutation matrix*** M_w *of* w *is the* $n \times n$ *matrix with 1's in the positions* (w_j, j) *for* $j \in [n]$*, and 0's everywhere else.*

Definition 1.1.4. *A* ***complete flag*** $F_\bullet = (F_1, \ldots, F_n)$ *in* $\mathbb{C}^n$ *is a sequence of complex vector spaces* $F_1 \subset \cdots \subset F_n$ *such that* $dim(F_i) = i$ *for* $i = 1, \ldots, n$*. Any complete flag* $F_\bullet$ *is determined by an* ***ordered basis*** $\langle f_1, \ldots, f_n \rangle$*, where* $F_i = span(f_1, \ldots, f_i)$*.*

Definition 1.1.5. *The* ***standard flag*** $E_\bullet = (E_1, \ldots, E_n)$ *is the flag that has ordered basis* $\langle e_1, \ldots, e_n \rangle$*, where* $e_i \in \mathbb{C}^n$ *has a 1 in the* i*th position and 0's everywhere else.*

Definition 1.1.6. *The* ***flag variety*** $Fl_n(\mathbb{C})$ *is the set of all complete flags in the vector space* $\mathbb{C}^n$*.*

Denote by $GL_n(\mathbb{C})$ the general linear group of invertible $n \times n$ matrices. Denote by B^- the subgroup of invertible lower triangular matrices, and by B^+ the subgroup of invertible upper triangular matrices. The Bruhat decomposition of the general linear group is $GL_n(\mathbb{C}) = \bigsqcup_{w \in S_n} B_- M_w B^+$ [**6**, Thm. 7.1]. Since a lower triangular action preserves the row span of the top i rows of a matrix in $GL_n(\mathbb{C})$, for each $i \in [n]$, the flag variety can be described by the cosets $B_- \setminus GL_n(\mathbb{C})$. Thus, the flag variety can be naturally decomposed into a disjoint union of sets of the form $C_w = B_- \setminus B_- M_w B^+$. These are the Schubert cells of the flag variety. Given any flag in $B_- \setminus GL_n(\mathbb{C})$ represented by a matrix M, one can determine which Schubert cell contains M via Gaussian elimination. The Schubert variety indexed by w is defined as the closure of the Schubert cell C_w, denoted $\overline{C}_w$. Since every Schubert cell has a natural B^+ action on the right, every Schubert variety does as well. Therefore, Schubert varieties naturally decompose as a union of Schubert cells. This decomposition determines a partial order $\leq$ on permutations via $\overline{C}_v = \bigcup_{v \leq w} C_w$. More specifically, we define the Bruhat order $(S_n, \leq)$ below.

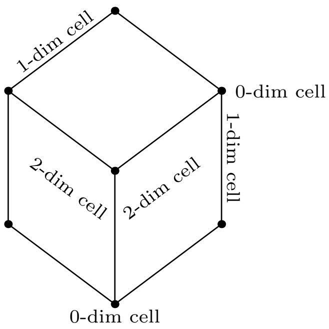

Figure 1.1.1. Cube visual of 3-dimensional "Schubert" variety with 0-, 1-, and 2-dimensional cells contained in boundary

Definition 1.1.7. *The* ***Bruhat order*** $(S_n, \leq)$ *is defined on permutations by* $v \leq w$ *if and only if one of the following equivalent statements is true:*

(1) $\overline{C}_v \supseteq C_w$*,*

(2) $\overline{C}_v \supseteq \overline{C}_w$*,*

(3) $\overline{C}_v \cap C_w \neq \emptyset$*.*

We can visualize a Schubert variety as a disjoint union of Schubert cells as follows. Though Schubert varieties are not at all cubical in shape, an easy visual to show the cell containment properties is to think of a Schubert cell as an open cube. The corresponding Schubert variety would then be the closed cube. The lower dimensional Schubert cells that are completely contained in the variety must all be contained in its boundary. The lower dimensional cells in this example would be the two-dimensional open squares, the one-dimensional open intervals along the boundaries of these squares, and the zero-dimensional vertices of the cube, which are both open and closed. See Figure 1.1.1 for some labeled examples of these cells. All these lower dimensional cells are contained in the closed cube representing the Schubert variety, and, in fact, they form a disjoint union of the closed cube.

In 2017 [**25**], Pawlowski and Rhoades defined an analog for the Bruhat decomposition on flag varieties in the context of **spanning line configurations**

$$X_{n,k} = \{l_\bullet = (l_1, \ldots, l_n) \in (\mathbb{P}^{k-1})^n \mid l_1 + \cdots + l_n = \mathbb{C}^k\}. \tag{1.1.1}$$

A spanning line configuration $l_\bullet = (l_1, \ldots, l_n) \in X_{n,k}$ consists of an ordered tuple of n 1-dimensional subspaces in $\mathbb{C}^k$. Therefore, each $l_\bullet$ can be represented by a full rank $k \times n$ matrix with no zero columns with entries in $\mathbb{C}$.

Definition 1.1.8. *Denote by $\mathcal{M}_{k\times n}(\mathbb{C})$ the set of full rank $k \times n$ matrices with no zero columns with entries in $\mathbb{C}$.*

For example, $((1,0,0),(0,1,0),(0,0,1),(1,2,0))$ are a spanning line configuration in $X_{4,3}$. This configuration can be represented by the full-rank matrix with no zero columns

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

If T represents diagonal matrices in $GL_n(\mathbb{C})$, then spanning line configurations in $X_{n,k}$ are in bijection with the cosets $\mathcal{M}_{k\times n}(\mathbb{C})/T$. This way, columns can be scaled by any complex number, and thus represent lines in $\mathbb{C}^k$. The full rank condition guarantees that the configuration is spanning. Observe that if $k < n$, the lines in the configuration will not be linearly independent. Thus the columns of a matrix representing the configuration will not be linearly independent.

Using the matrix representation, Pawlowski and Rhoades decompose $X_{n,k}$ as a union of cells. One can identify which cell contains any $M \in \mathcal{M}_{k\times n}(\mathbb{C})$ via a variation on Gaussian elimination, which is described in detail in Section 2.1. The cells in Pawlowski and Rhoades' decomposition are indexed by Fubini words. We will also denote these cells by C_w, this time with $w \in \mathcal{W}_{n,k}$. We call C_w the **Pawlowski-Rhoades cell** or **PR cell** indexed by $w \in \mathcal{W}_{n,k}$. The **PR variety** indexed by w is defined as the closure of the PR cell C_w in the Zariski topology. The PR variety indexed by w is denoted $\overline{C}_w$.

Pawlowski and Rhoades' main motivation for studying spanning line configurations was to find a variety whose cohomology ring can be represented algebraically as $R_{n,k}$ [**25**, Thm. 5.12], where

$$R_{n,k} = \mathbb{Z}[x_1, \ldots, x_n]/I_{n,k}$$

and

$$I_{n,k} = \langle x_1^k, \ldots, x_n^k, e_n, e_{n-1}, \ldots, e_{n-k+1} \rangle.$$

Here, e_i denotes the ith elementary symmetric polynomial in variables $x_1, \ldots, x_n$. The ring $R_{n,k}$ was introduced in [**18**] in connection with Macdonald polynomials and the Delta conjecture. In the $k = n$ case, $I_{n,n} = \langle x_1^n, \ldots, x_k^n, e_n, \ldots, e_1 \rangle$, but the generators $x_1^n, \ldots, x_k^n$ are redundant. Thus, $I_{n,n} = \langle e_n, \ldots, e_1 \rangle$, and $R_{n,n} = \mathbb{Z}[x_1, \ldots, x_k]/\langle e_n, \ldots, e_1 \rangle$, which is known as the coinvariant algebra. By Borel's Theorem [**5**], we know this is the cohomology ring for the flag variety $Fl_n(\mathbb{C})$.

As with Schubert cells, one can look at the containment of cell closures for PR cells. This led Pawlowski and Rhoades to define an order on Fubini words that extends the Bruhat order for permutations.

Definition 1.1.9. *[**25**, Sec. 9] The **medium roast Fubini-Bruhat order** $(\mathcal{W}_{n,k}, \leq)$ is defined on Fubini words by $v \leq w$ if and only if one of the following equivalent statements is true:*

(1) $\overline{C}_v \supseteq C_w$,

(2) $\overline{C}_v \supseteq \overline{C}_w$.

We sometimes refer to the medium roast Fubini-Bruhat order as medium roast order.

Observe that the definition for medium roast Fubini-Bruhat order on Fubini words is equivalent to Definition 1.1.7 for Bruhat order on permutations when $n = k$. However, the third equivalent condition from the definition for Bruhat order does not appear in the definition for medium roast Fubini-Bruhat order. For PR cells and varieties, the condition $\overline{C}_v \cap C_w \neq \emptyset$ is not always equivalent to the two equivalent conditions in Definition 1.1.9. Example 9.1 of [**25**] shows that there exist Fubini words v and w such that $\overline{C}_v \cap C_w \neq \emptyset$ but $C_w \not\subseteq \overline{C}_v$. This example is described in detail in Section 6.3. Since $\overline{C}_v \cap C_w \neq \emptyset$ is a weaker condition than $C_w \subseteq \overline{C}_v$, this suggests a refinement of the medium roast Fubini-Bruhat order, which we will denote by $\preceq$. Note that our notation for $\preceq$ is $\leq'$ in Pawlowski and Rhoades' notation. They use $\preceq$ for the dual order to $\leq$.

Definition 1.1.10. *For Fubini words $v, w \in \mathcal{W}_{n,k}$, C_v* ***touches*** *C_w, denoted $v \rightharpoonup w$, if $\overline{C}_v \cap C_w \neq \emptyset$.*

Pawlowski and Rhoades observe in [**25**, Sec. 9] that unlike $\leq$, the relation $\rightharpoonup$ on Fubini words suggested by $\overline{C}_v \cap C_w \neq \emptyset$ is not transitive. However, the lemma below allows a partial order on $\mathcal{W}_{n,k}$ to be defined using the transitive closure of the relation $\rightharpoonup$.

Lemma 1.1.11. *[**25**, Prop. 9.2] The transitive closure of the relation $v \rightharpoonup w$ does not contain any loops.*

Definition 1.1.12. *The* ***espresso Fubini-Bruhat order*** *$(\mathcal{W}_{n,k}, \preceq)$ is defined by taking the transitive closure of the relation v touches w. Thus, $v \rightharpoonup w$ implies $v \preceq w$, but the converse does not necessarily hold. We sometimes refer to espresso Fubini-Bruhat order as espresso order.*

Observe that for Fubini words $v, w \in \mathcal{W}_{n,k}$, $v \leq w$ implies $v \rightharpoonup w$, which implies $v \preceq w$. Thus, the medium roast Fubini-Bruhat poset is a subposet of the espresso Fubini-Bruhat poset.

Unlike for Schubert varieties, it is not true for PR varieties that $\overline{C}_v = \bigcup_{v \leq w} C_w$. See Section 7.1 for an example of a PR variety decomposition that contains a partial PR cell. Again, PR cells look nothing like cubes, but cubes can be a useful visualization tool. We can think of two PR cells such that $v \rightharpoonup w$ like a badly stacked tower of two cubical blocks. That way, some of the lower-dimensional cells, represented by open squares and intervals, only partially intersect the corresponding PR varieties, represented by the closed cubes. In Figure 1.1.2, we see the labeled two-dimensional open square cell partially intersects the boundary of the upper closed cube, which represents a three-dimensional variety.

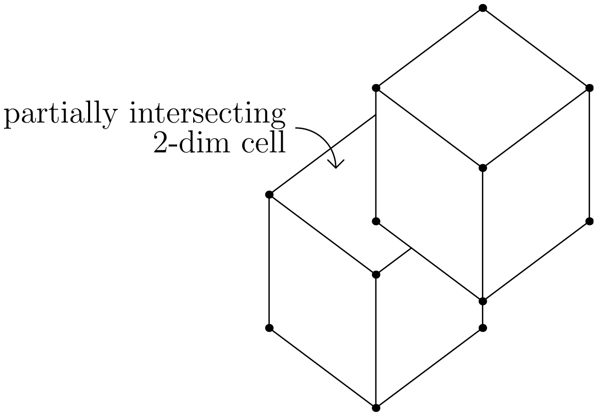

FIGURE 1.1.2. Stacked cube visual of 3-dimensional "PR" varieties with 2-dimensional cell partially intersecting

The geometry of $X_{n,k}$, PR cells, PR varieties, and the Fubini-Bruhat orders is more complicated than Schubert cells and varieties and the Bruhat order. However, the good news is that $X_{n,k}$ admits a cellular decomposition into PR cells. This cell decomposition is key to understanding all three flavors of the Fubini-Bruhat order.

Pawlowksi and Rhoades asked for a combinatorial description of the medium roast Fubini-Bruhat order. In this thesis, we address this problem by giving several increasingly more efficient sets of equations for a PR variety. Each of the sets of equations implies a test for comparing elements in the medium roast Fubini-Bruhat order. Some of these tests also extend to espresso Fubini-Bruhat order. We use these sets of equations to obtain combinatorial descriptions for the relations of one or more Fubini-Bruhat orders. We end by describing some of the covering relations for each of the flavors of the Fubini-Bruhat order. This specifically addresses [**25**, Prob. 9.5].

2. Main Results

In this section, we state our main results, while introducing the necessary definitions and background about permutations and Fubini words. When applicable, we first list the corresponding result for Bruhat order on permutations, in order to highlight the parallels between Bruhat order and one or more of our three flavors of Fubini-Bruhat order.

Definition 1.2.1. *In this thesis, an (I, J)-**minor** of a matrix $M \in \mathcal{M}_{k\times n}(\mathbb{C})$ denotes the square submatrix of M indexed by rows $I \subseteq [k]$ and columns $J \subseteq [n]$, with $|I| = |J|$. So the minor refers to the submatrix itself, not the determinant of said submatrix. We will often equate the term (I, J)-minor with just the ordered pair (I, J).*

Definition 1.2.2. *For $I \subseteq [k]$ and $J \subseteq [n]$, with $|I| = |J|$, denote by $\Delta_{I,J}(M)$ the **determinant** of the (I, J)-minor of $M \in \mathcal{M}_{k\times n}(\mathbb{C})$. We will use the convention that $\Delta_{\emptyset,\emptyset}(M) = 1$ for every matrix M.*

Definition 1.2.3. *Given a Fubini word $w \in \mathcal{W}_{n,k}$, the set of **truly vanishing minors of** C_w are the ordered pairs (I, J) such that $\Delta_{I,J}(A) = 0$ for every $k \times n$ matrix $A \in C_w$. The truly vanishing minors of $\overline{C}_w$, defined similarly, are the same as the truly vanishing minors of C_w due to $\overline{C}_w$ being the closure of C_w in the Zariski topology.*

In Chapter 3, we will show that a PR variety is defined as the set of solutions to the determinantal equations defined by the truly vanishing minors of C_w. Our first set of more efficient equations is based on what we call northerly minors of the $k \times n$ matrices in each cell C_w.

Definition 1.2.4. *Denote by $\binom{[n]}{[k]}$ the set of all subsets of $[n]$ of size less than or equal to k.*

Definition 1.2.5. *The **northerly minor** indexed by $J \in \binom{[n]}{[k]}$ of a matrix $M \in \mathcal{M}_{k\times n}(\mathbb{C})$ is the $([h], J)$-minor of M, where $|J| = h \in [k]$.*

Definition 1.2.6. *Given a Fubini word $w \in \mathcal{W}_{n,k}$, define S_w, T_w, and U_w by*

$$S_w = \{J \in \binom{[n]}{[k]} \mid |J| = h, \exists\, A \in C_w \text{ such that } \Delta_{[h],J}(A) = 0, \text{ and}$$
$$\exists\, B \in C_w \text{ such that } \Delta_{[h],J}(B) \neq 0\},$$

$$T_w = \{J \in \binom{[n]}{[k]} \mid |J| = h, \Delta_{[h],J}(A) = 0 \ \ \forall A \in C_w\}, \text{ and}$$
$$U_w = \{J \in \binom{[n]}{[k]} \mid |J| = h, \Delta_{[h],J}(A) \neq 0 \ \ \forall A \in C_w\}.$$

We call S_w *the set of* ***sometimes vanishing northerly minors*** *on* C_w*,* T_w *the set of* ***truly vanishing northerly minors*** *on* C_w*, and* U_w *the set of* ***unvanishing northerly minors*** *on* C_w*.*

Observe that for any Fubini word $w \in \mathcal{W}_{n,k}$, the set $\binom{[n]}{[k]}$ is a disjoint union of S_w, T_w, and U_w. The northerly minors yield our first two new results about the Fubini-Bruhat orders.

Theorem 1.2.7. *For any Fubini word* $w \in \mathcal{W}_{n,k}$*, we have*

$$\overline{C}_w = \{A \in \mathcal{M}_{k\times n}(\mathbb{C})/T \mid \Delta_{[h],J}(A) = 0 \,\forall\, J \in T_w, \text{ where } h = |J|\}.$$

Theorem 1.2.8. *For any two Fubini words* $v, w \in \mathcal{W}_{n,k}$*, we have*

(1) $v \leq w$ *in medium roast Fubini-Bruhat order if and only if* $T_v \subseteq T_w$*, or equivalently,* $T_v \cap (S_w \cup U_w) = \emptyset$*, and*

(2) $v \rightharpoonup w$ *if and only if* $T_v \subseteq (S_w \cup T_w)$*, or equivalently,* $U_w \subseteq (S_v \cup U_v)$*, or equivalently,* $T_v \cap U_w = \emptyset$*.*

Calculating determinants of northerly minors of C_w is more efficient than calculating determinants of all minors of C_w, but still cumbersome. However, we can use northerly minors to develop combinatorial tests for the Fubini-Bruhat orders, analogous to tests that exist for Bruhat order on permutations. In order to describe these tests, we define the following two partial orders on multisets. Note that the dominated order defined below induces a poset that corresponds with Young's lattice on partitions, not dominance order on partitions. It is what Ardila, Rincón, and Williams refer to as the Gale order in [**1**], based on [**17**].

Definition 1.2.9. *If* $\{a_1 \leq \cdots \leq a_m\}$ *and* $\{b_1 \leq \cdots \leq b_m\}$ *are multisets of the same size with their elements listed in increasing order, we say* $\{a_1 \leq \cdots \leq a_m\}$ *is*

dominated by $\{b_1 \leq \cdots \leq b_m\}$*, denoted* $\{a_1 \leq \cdots \leq a_m\} \trianglelefteq \{b_1 \leq \cdots \leq b_m\}$*, if and only if*

$$a_1 \leq b_1,\ a_2 \leq b_2,\ \ldots,\ a_m \leq b_m.$$

DEFINITION 1.2.10. *Multisets on the alphabet* $[n]$ *of the same size are in* ***lexicographic order****, denoted* $\{a_1 \leq \cdots \leq a_m\} <_L \{b_1 \leq \cdots \leq b_m\}$*, if the corresponding elements written in increasing order satisfy the usual lexicographic order, namely, if and only if there exists* $1 \leq i \leq m$ *such that*

$$a_1 = b_1,\ a_2 = b_2,\ \ldots,\ a_{i-1} = b_{i-1},\ \text{and}\ \ a_i < b_i.$$

Lexicographic order is a total order on multisets of a given size.

THEOREM 1.2.11. *[13]* ***The Ehresmann Tableau Criterion.*** *Given permutations* $v, w \in S_n$*. Then* $v \leq w$ *in Bruhat order if and only if* $\{v_1, \ldots, v_j\} \trianglelefteq \{w_1, \ldots, w_j\}$ *for each* $j \in [n]$*.*

Ehresmann introduced this criterion for detecting when one Schubert variety is contained in another in 1934, which determines Bruhat order by Definition 1.1.7. The following theorem of Björner and Brenti's make the tests of Ehresmann's Tableau Criterion [**3**] more efficient by only checking the rows of the monotone triangles corresponding to the permutations $v, w \in S_n$ at the descents of v. The **descents** of v are the indices $j \in [n]$ such that $v_j > v_{j+1}$.

THEOREM 1.2.12. *[3] Given permutations* $v, w \in S_n$*. Then* $v \leq w$ *in Bruhat order if and only if* $\{v_1, \ldots, v_j\} \trianglelefteq \{w_1, \ldots, w_j\}$ *whenever* $v_j > v_{j+1}$*.*

There have also been several other generalizations of Bruhat order. In particular, Chevalley generalized Bruhat order to all Coxeter groups [**7**]. Deodhar [**10**] generalized Ehresmann's Tableau Criterion to all Weyl groups. Lascoux and Schützenberger [**23**] embedded Bruhat order in a lattice on alternating sign matrices.

Definition 1.2.13. *For Fubini word* $w \in \mathcal{W}_{n,k}$*, denote by* $\alpha_i = \alpha_i(w)$ *the position of the initial* i *in* w *for each* $i \in [k]$*, where we will sometimes drop the* (w) *when it is clear from context. Observe that when* $k = n$*, this coincides with the notion of* $w^{-1}(i)$ *for* $w \in S_n = \mathcal{W}_{n,n}$*.*

Definition 1.2.14. *For Fubini word* $w \in \mathcal{W}_{n,k}$*, the* ***initial positions*** *of* w *are the set in*$(w) = \{\alpha_1, \dots, \alpha_k\}$*. A* ***redundant position*** *of* w *is any position that is not initial. An* ***initial letter*** *is a letter appearing in an initial position, and a* ***redundant letter*** *is a letter appearing in a redundant position.*

Definition 1.2.15. *For* $w \in \mathcal{W}_{n,k}$*, denote by* $\pi(w) \in S_k$ *the permutation in one-line notation formed by deleting the redundant letters from the one-line notation of* w*.*

Definition 1.2.16. *Denote by* $\alpha_J(w)$ *the multiset* $\{\alpha_{w(j)} \,|\, j \in J\}$*.*

Example 1.2.17. For Fubini word $w = 21231231 \in \mathcal{W}_{8,3}$, $\alpha_1 = 2$, $\alpha_2 = 1$, $\alpha_3 = 4$, $\mathrm{in}(w) = \{1, 2, 4\}$, and $\pi(w) = 213$. If $J = \{2, 4, 6, 7, 8\}$, $\alpha_J(w) = \{2, 4, 1, 4, 2\}$.

Theorem 1.2.18. *Suppose* $w \in \mathcal{W}_{n,k}$ *and* $J \in \binom{[n]}{[k]}$ *with* $|J| = h$*. Then*

(1) $J \in S_w$ *if and only if* $\{\alpha_1, \dots, \alpha_h\} \underset{\neq}{\trianglelefteq} \alpha_J(w)$*,*
(2) $J \in T_w$ *if and only if* $\{\alpha_1, \dots, \alpha_h\} \ntrianglelefteq \alpha_J(w)$*, and*
(3) $J \in U_w$ *if and only if* $\{\alpha_1, \dots, \alpha_h\} = \alpha_J(w)$*.*

Theorem 1.2.19. *Let* $v, w \in W_{n,k}$*. Then* $v \rightharpoonup w$ *if and only if for each* $J \in \binom{[n]}{[k]}$ *with* $|J| = h \leq k$ *such that*

$$\{\alpha_1(w), \dots, \alpha_h(w)\} = \alpha_J(w) \tag{1.2.2}$$

we also have

$$\{\alpha_1(v), \dots, \alpha_h(v)\} \trianglelefteq \alpha_J(v). \tag{1.2.3}$$

In particular, if $v \rightharpoonup w$*, then* $\{\alpha_1(v), \dots, \alpha_h(v)\} \trianglelefteq \{\alpha_1(w), \dots, \alpha_h(w)\}$ *for all* $1 \leq h \leq k$*.*

THEOREM 1.2.20. *Let* $v, w \in \mathcal{W}_{n,k}$. *Then* $v \leq w$ *in medium roast Fubini-Bruhat order if and only if for each* $J \in \binom{[n]}{[k]}$ *with* $|J| = h \leq k$ *such that*

$$\{\alpha_1(w), \ldots, \alpha_h(w)\} \trianglelefteq \alpha_J(w) \tag{1.2.4}$$

we also have

$$\{\alpha_1(v), \ldots, \alpha_h(v)\} \trianglelefteq \alpha_J(v). \tag{1.2.5}$$

In particular, if $v \leq w$, *then* $v \rightharpoonup w$. *Thus,* $\{\alpha_1(v), \ldots, \alpha_h(v)\} \trianglelefteq \{\alpha_1(w), \ldots, \alpha_h(w)\}$ *for all* $1 \leq h \leq k$.

A different improvement over computing determinants of all northerly minors of C_w comes from considering efficient sets of rank conditions for matrices in C_w. We summarize in Section 4 of Chapter 2 the relationship between rank conditions and Schubert cells, and we describe in Chapter 3 results about the relationship between rank conditions and PR cells. In search of efficient sets of rank conditions, we turn to Fulton, who characterized one such set for Schubert cells. In [**15**], Fulton showed it suffices to use only the equations determined by rank conditions for the essential set of a permutation to define the Schubert varieties in the flag variety.

DEFINITION 1.2.21. *The* ***Rothe diagram*** *of a permutation* $w \in S_n$ *is the subset of* $[n] \times [n]$ *in matrix coordinates given by*

$$D(w) = \{(w_j, i) \,|\, i < j \text{ and } w_i > w_j\}. \tag{1.2.6}$$

DEFINITION 1.2.22. *[**15**] The* ***essential set*** *$Ess(w)$ of a permutation* $w \in S_n$ *is defined as the set of southeast corners of the connected components of the Rothe diagram of the permutation. Equivalently,*

$$Ess(w) = \{(i, j) \in D(w) \,|\, (i, j+1), (i+1, j) \notin D(w)\}.$$

Definition 1.2.23. *For every permutation $w \in S_n$, define its **rank function** to be the map $r_w : [n] \times [n] \to \mathbb{Z}_{\geq 0}$ that sends (i,j) to the rank of the northwest submatrix of M_w with row set $[i]$ and column set $[j]$.*

Definition 1.2.24. *[**14**] The **ranked essential set** of a permutation $w \in S_n$ is the essential set with ranks included. We will denote ranked essential set elements by $(i, j, r_w(i,j))$, where $(i,j) \in Ess(w)$. We will abuse notation and also denote the ranked essential set by $Ess(w)$.*

For example, in Figure 1.2.1, for $w = 35421$, each dot to the east and south of each pair (w_j, j) for $j = 1,2,3,4,5$ has been crossed out. The dots that have not been crossed out form the Rothe diagram $D(35421)$, and the boxed dots indicate the essential set elements for $35421 \in S_5$. Therefore,

$$D(35421) = \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(4,2)\},$$

and

$$Ess(35421) = \{(1,4),(2,3),(4,2)\}.$$

The ranked essential set is useful for keeping track of which rank conditions are necessary to determine a Schubert cell. In our example, the ranked essential set of 35421 is

$$Ess(35421) = \{(1,4,0),(2,3,0),(4,2,1)\}.$$

Theorem 1.2.25. *[**15**, Lem. 3.10(b)] A permutation $w \in S_n$ is determined by the restriction of its rank function r_w to $Ess(w)$.*

Fulton showed in the above theorem that only the rank conditions at essential set elements are necessary to determine a permutation. The remaining rank conditions follow from the essential set rank conditions. In fact, given the essential set of an unknown permutation w, Eriksson and Linusson give an algorithm to reconstruct the rank function r_w, and thus the permutation w [**14**]. See Section 2.4 for more information on rank conditions for Schubert cells.

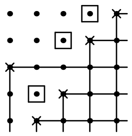

FIGURE 1.2.1. 35421 Essential Set

DEFINITION 1.2.26. *As in [**25**, (5.19)], the **standardization** of a Fubini word* $w \in \mathcal{W}_{n,k}$*, denoted* $std(w) \in S_n$*, is obtained by replacing the* $n-k$ *redundant letters of* w *with* $k+1, k+2, \ldots, n$ *from left to right.*

DEFINITION 1.2.27. *A Fubini word* $w \in \mathcal{W}_{n,k}$ *is called **convex** if* $h < j$ *and* $w_h = w_j$ *implies that* $w_i = w_j$ *for every* i *such that* $h < i < j$*. Then the **convexification** of* w*, denoted by* $conv(w)$*, is the unique convex word such that* $\pi(conv(w)) = \pi(w)$ *and the letters of* $conv(w)$ *as a multiset are the same as the letters of* w*.*

In Section 5.3, we define the analogous essential set and ranked essential set for a Fubini word $w \in \mathcal{W}_{n,k}$. The construction is based on the essential set for $\mathrm{std}(\mathrm{conv}(w))$, and the sets $\beta_i(w)$, which are defined in Chapter 5. We also prove the following theorems in Chapter 5, giving our most efficient set of equations for PR cells and another test for medium roast Fubini-Bruhat order.

THEOREM 1.2.28. *A matrix* $A \in \mathcal{M}_{k\times n}(\mathbb{C})$ *is in the PR variety* $\overline{C}_w$ *if and only if the rank of the top* h *rows of* A *in the columns* $\beta_i(w)$ *is at most* r *for each* $(h, \beta_i(w), r) \in Ess(w)$*.*

Theorem 1.2.28 is a Fubini word analog of Fulton's theorem for permutations, Theorem 1.2.25. As with other results in this thesis, each theorem about the structure of PR cells and varieties gives rise to a test about one or more of the Fubini-Bruhat orders. The following theorem is an alternative test for medium roast Fubini-Bruhat order, which only involves comparing essential set elements of two Fubini words. The number of checks required to determine if $v \leq w$ is $|Ess(v)|$. The proof of Lemma 6.4.3 is one example of the power of this theorem.

THEOREM 1.2.29. *Suppose* $v, w \in \mathcal{W}_{n,k}$. *Then* $v \leq w$ *if and only if for every* $(m, \beta_j(v), s) \in Ess(v)$, *there exists an* $(h, \beta_i(w), r) \in Ess(w)$ *such that*

$$max(0, m-h) + |\beta_j(v) \setminus \beta_i(w)| \leq s - r. \tag{1.2.7}$$

To address Pawlowski and Rhoades' question about covering relations in medium roast Fubini-Bruhat order, we recall the Transposition Rule. The transitive closure of the Transposition Rule is perhaps the simplest combinatorial description of Bruhat order. It describes all the covering relations for Bruhat order on S_n.

DEFINITION 1.2.30. *A* ***transposition*** *is a permutation* $t_{ij} \in S_n$ *with* $i \neq j$, $i, j \in [n]$. *The surjection* $t_{ij} : [n] \to [n]$ *is defined by*

$$t_{ij}(x) = \begin{cases} x \text{ if } x \neq i, j, \\ j \text{ if } x = i, \text{ and} \\ i \text{ if } x = j. \end{cases}$$

The transposition t_{ij} *may be denoted* $t_{i,j}$ *when the comma is needed for clarity. The* ***adjacent transposition*** $t_{i,i+1}$ *is sometimes denoted by* s_i.

THEOREM 1.2.31. *[4, Ch. 2]* ***The Transposition Rule.*** *Suppose* $w \in S_n$ *and* $i < j$. *Then* $w < t_{ij}w$ *if and only if* $w_i^{-1} < w_j^{-1}$. *Furthermore,* $t_{ij}w$ *covers* w *if and only if there does not exist an* m *such that* $i < m < j$ *and* $w_i^{-1} < w_m^{-1} < w_j^{-1}$.

In Section 6.1, we prove an analogous Transposition Rule for Fubini words. However, due to redundant letters in Fubini words, this rule does not tell the whole story of covering relations for either the medium roast or the espresso Fubini-Bruhat order. In fact, only taking the transitive closure of the Transposition Rule would lead to a disconnected poset on elements in $\mathcal{W}_{n,k}$.

THEOREM 1.2.32. ***The Transposition Rule.*** *If* $w \in \mathcal{W}_{n,k}$ *and* $i < j$, *then* $w < t_{ij}w$ *in medium roast Fubini-Bruhat order if and only if* $\alpha_i(w) < \alpha_j(w)$.

Furthermore, $t_{ij}w$ *covers* w *in medium roast Fubini-Bruhat order if and only if* $\pi(t_{ij}w)$ *covers* $\pi(w)$ *in Bruhat order on* S_k*.*

The Pushback Rule gives many of the remaining covering relations in the Fubini-Bruhat orders. The pushback operator works by modifying a single redundant letter for a Fubini word $w \in \mathcal{W}_{n,k}$. Taking the transitive closure of the Transposition Rule and the Pushback Rule leads to a connected ranked poset on $\mathcal{W}_{n,k}$. The pushback operator is a special case of the superpushback operator.

DEFINITION 1.2.33. *Suppose* $w = w_1 \cdots w_n \in \mathcal{W}_{n,k}$*,* $i \in [k-1]$*, and* $j \in [n]$ *such that* $w_j = \pi_i$ *is a redundant letter in* w*. Define the* ***superpushback operator*** $\Phi(w, j, p) \in \mathcal{W}_{n,k}$ *to be*

$$\Phi(w, j, p) = w_1 \cdots w_{j-1} \pi_{i+p} w_{j+1} \cdots w_n. \tag{1.2.8}$$

Define the ***pushback operator*** $\varphi(w, j) \in \mathcal{W}_{n,k}$ *to be* $\varphi(w, j) = \Phi(w, j, 1)$*.*

THEOREM 1.2.34. ***The Pushback Rule.*** *Suppose* $w = w_1 \cdots w_n \in \mathcal{W}_{n,k}$*,* $i \in [k-1]$*, and* $j \in [n]$ *such that* $w_j = \pi_i$ *is a redundant letter in* w*. If* $v = \varphi(w, j)$*, then* $v < w$ *in medium roast Fubini-Bruhat order. Furthermore,* w *covers* v *in medium roast Fubini-Bruhat order.*

DEFINITION 1.2.35. *Define the* ***decaf Fubini-Bruhat order*** $(\mathcal{W}_{n,k}, \ll)$ *by taking the transitive closure of the Transposition Rule and the Pushback Rule. We sometimes refer to the decaf Fubini-Bruhat order as the decaf order.*

The decaf Fubini-Bruhat order, the third and final flavor of Fubini-Bruhat order we will define, produces a ranked subposet of the medium roast Fubini-Bruhat order that contains all the elements of $\mathcal{W}_{n,k}$. It is the only order of the three flavors of Fubini-Bruhat order we have defined that is ranked for all values of n and k. In fact, the rank-generating function of the decaf order is closely related to a q-analog of the Stirling numbers of the second kind, which will be discussed further in Chapter 2.

The Superpushback Rule is an extension of the Pushback rule that describes relations in the espresso Fubini-Bruhat order. It is the only set of relations that we know of that are relations in the espresso Fubini-Bruhat order but not necessarily in the medium roast Fubini-Bruhat order.

THEOREM 1.2.36. ***The Superpushback Rule.*** *Suppose* $w = w_1 \cdots w_n \in \mathcal{W}_{n,k}$*,* $i \in [k-1]$*, and* $j \in [n]$ *such that* $w_j = \pi_i$ *is a redundant letter in* w*.. If* $i + p \leq k$ *and* $v = \Phi(w, j, p)$*, then* $v \rightharpoonup w$*.*

Many authors reference the Lifting Property for Bruhat order. We will show that the following special case of the Lifting Property generalizes to Fubini-Bruhat order.

THEOREM 1.2.37. *[**4**, Prop. 2.2.7]* ***The Lifting Property.*** *Suppose* $v, w \in S_n$ *are permutations,* $v \leq w$ *in Bruhat order, and* $i \in [n-1]$*. Suppose that in both* v *and* w*,* $i+1$ *precedes* i*. Then* $s_i v \leq s_i w$ *in Bruhat order.*

THEOREM 1.2.38. ***The Lifting Property.*** *Suppose* $v, w \in \mathcal{W}_{n,k}$*,* $i \in [k-1]$*,* $\alpha_{i+1}(v) < \alpha_i(v)$*, and* $\alpha_{i+1}(w) < \alpha_i(w)$*. Then*

1. *if* $v \leq w$ *in medium roast Fubini-Bruhat order, then* $s_i v \leq s_i w$*, and*
2. *if* $v \rightharpoonup w$*, then* $s_i v \rightharpoonup s_i w$*.*

The subsequent chapters are organized as follows. In Chapter 2, we give some background on permutations, Schubert varieties, and Bruhat order, along with parallel information previously known about Fubini words, PR varieties, and the Fubini-Bruhat orders. In Chapter 3, we restate with more detail the results about minors determining PR varieties and Fubini-Bruhat orders from [**25**], to establish notation and for use in future chapters. In Chapter 4, we prove our results about northerly minors, leading to a more efficient set of equations describing PR cells. These results then lead to combinatorial descriptions for the medium roast and espresso Fubini-Bruhat orders given by Theorems 1.2.18, 1.2.19, and 1.2.20. Chapter 5 introduces monotone triangles for Fubini words, as well as our extension of

Fulton's essential set to Fubini words, leading to proofs of Theorems 1.2.28 and 1.2.29. Chapter 6 uses the theorems of Chapters 4 and 5 to establish some covering relations and properties of the Fubini-Bruhat orders, including proving the Transposition Rule, the Pushback Rule, the Superpushback Rule, and the Lifting Property for the Fubini-Bruhat orders. In Chapter 7, we give examples of the properties and relations of Chapter 6, examples illustrating the difficulty of the study of the Fubini-Bruhat orders, and some open problems.

CHAPTER 2

Background

In this chapter, we describe in more depth some of the parallels between permutations and Fubini words, Schubert and PR cells and varieties, and Bruhat order and the Fubini-Bruhat orders. Pawlowski-Rhoades' notion of the pattern matrix in $\mathcal{M}_{k\times n}(\mathbb{C})$ describing PR cells [**25**] is similar to finding canonical representations of Schubert cells in $GL_n(\mathbb{C})$. The number of $\star$'s in the pattern matrices for either permutations or Fubini words are related to the dimension of the corresponding Schubert or PR cell. Monotone triangles for permutations have something of an analog for Fubini words that is developed in Chapter 5. The analog to PR varieties and Fubini-Bruhat order of the rank condition definition for Schubert varieties and Bruhat order, which we alluded to in Chapter 1 when discussing Fulton's essential set, will be a recurring theme that gets developed throughout this thesis. We assume the reader is familiar with basic properties of linear algebra, matrix group actions, varieties, and posets. For more background on these topics, see [**8**], [**11**], and [**31**].

1. Permutation Matrices and Pattern Matrices

Recall that a permutation w in the symmetric group S_n is a surjection $w : [n] \to [n]$. Observe that a surjection $[n] \to [n]$ is also a bijection. Observe that S_n acts on itself on the left or the right. That is, for permutations $v, w \in S_n$, the equation $M_{vw} = M_v M_w$ holds for the permutation matrices of v, w, and vw.

Recall that the **flag variety** $Fl_n(\mathbb{C})$ is the set of complete flags in the vector space $\mathbb{C}^n$. For each flag $F_\bullet \in Fl_n(\mathbb{C})$, its basis $\langle f_1, \ldots, f_n\rangle$ can be represented by an $n \times n$ invertible matrix M with respect to the standard basis $\langle e_1, \ldots, e_n\rangle$, where the ith row of M consists of the coefficients of $f_i = c_1 e_1 + \cdots + c_n e_n$. Observe that for any lower triangular invertible matrix A, AM replaces each row with a nonzero linear combination of itself and the rows above it. Thus, M and AM represent the same flag. Hence, the points in the flag variety are in bijection with the cosets in $B_- \setminus GL_n(\mathbb{C})$, where B_- is the set of all invertible lower triangular matrices.

Definition 2.1.1. *The* ***Schubert cell*** $C_w \subset Fl_n(\mathbb{C})$ *corresponding to* $w \in S_n$ *is the set of all flags that can be represented by matrices of the form* M_wA*, for some* $A \in B^+$*, where* B^+ *is the set of all invertible upper triangular matrices. Since* C_w *is represented by matrix cosets in* $B_- \setminus GL_n(\mathbb{C})$*, we can consider only the matrices of the form* M_wA *with* $A \in U^+$*, where* U^+ *is the subgroup of* $GL_n(\mathbb{C})$ *of upper triangular matrices with 1's on the diagonal.*

Example 2.1.2. Consider $w = 2431 \in S_4$. Then M_wU^+ is the set of all matrices of the form

$$M_{2431}U^+ = \left\{ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & \star & \star & \star \\ 0 & 1 & \star & \star \\ 0 & 0 & 1 & \star \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & \star & \star & \star \\ 0 & 0 & 1 & \star \\ 0 & 1 & \star & \star \end{pmatrix} \right\} \subset GL_n(\mathbb{C}),$$

where each $\star$ can be replaced by any complex number. Now for each of these matrices, we can find a canonical representative in its coset of $B_- \setminus GL_n(\mathbb{C})$ by performing Gaussian elimination on the rows. Start with the topmost row and subtract multiples of this row from the other rows below it until every entry below the 1 is 0. Repeat this process with each row, moving from top to bottom, so that every $\star$ below a 1 can be eliminated to 0. In our example, the resulting set of canonical matrices representing points of C_w are all of the form

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & \star & \star & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

where each $\star$ can be replaced by any complex number.

Pawlowski and Rhoades define the **pattern matrix** of a Fubini word in $\mathcal{W}_{n,k}$ [**25**]. Letting $n = k$, this definition can be applied to permutations in S_n.

Definition 2.1.3. *The* ***pattern matrix*** P_w *of a permutation* $w = w_1 \cdots w_n \in S_n$ *is an* $n \times n$ *matrix with entries* 0, 1, *or* $\star$. *Obtain* P_w *by starting with* M_w *and replacing a* 0 *in position* (w_i, j) *with a* $\star$ *whenever* $w_i < w_j$ *and* $i < j$. *An* $n \times n$ *matrix* A *is said to* ***fit the pattern of*** w *if that matrix can be obtained by replacing the* $\star$*'s in the pattern matrix of* w *with complex numbers.*

For every $w \in S_n$, the set of canonical matrices representing points of C_w all fit the pattern of w. This can be observed in Example 2.1.2 for $w = 2431$, where the final matrix obtained after performing Gaussian elimination is P_{2431}.

Recall from Chapter 1 that a Fubini word $w = w_1 \cdots w_n \in \mathcal{W}_{n,k}$ is a surjection $w : [n] \to [k]$. The surjection is given by $w_j = w(j)$ for $j \in [n]$.

Definition 2.1.4. *For each Fubini word* $w = w_1 \cdots w_n \in \mathcal{W}_{n,k}$, *there is a corresponding matrix* $M_w \in \mathcal{M}_{k\times n}(\mathbb{C})$ *defined exactly like a permutation matrix, where the entries in positions* (w_j, j) *are 1, and every other entry is 0.*

Observe that the symmetric group S_k acts on the left of $\mathcal{W}_{n,k}$ by composition, where if $v \in S_k$ and $w \in \mathcal{W}_{n,k}$, v acts on w on the left by permuting the values of w. For example, if $w = 21231231 \in \mathcal{W}_{8,3}$ and $v = s_1 \in S_3$, $vw = 12132132$. Similarly, S_n acts on the right by composition, where if $x \in S_n$ and $w \in \mathcal{W}_{n,k}$, x acts on w on the right by permuting the positions of w. For example, if $w = 21231231 \in \mathcal{W}_{8,3}$ and $x = s_1 \in S_8$, $wx = 12231231$. Also observe that if $v \in S_k$, $w \in \mathcal{W}_{n,k}$, and $x \in S_n$, then $M_{vwx} = M_v M_w M_x$.

More generally, Pawlowski and Rhoades define the **pattern matrix** P_w of $w \in \mathcal{W}_{n,k}$ in [**25**]. When $k = n$, this definition is identical to Definition 2.1.3, the definition for the pattern matrix of a permutation.

Definition 2.1.5. *The* ***pattern matrix*** P_w *of* $w = w_1 \cdots w_n \in \mathcal{W}_{n,k}$ *is a* $k \times n$ *matrix in* $\mathcal{M}_{k\times n}(\mathbb{C})$ *with entries* 0, 1, *or* $\star$. *Obtain* P_w *by starting with* M_w *and replacing the* 0 *by a* $\star$ *in each position* (w_i, j) *whenever* $i \in in(w)$, $i < \alpha_{w(j)}$, *and either*

- $j \in in(w)$ *and* $w_i < w_j$, *or*
- $j \notin in(w)$.

A matrix $A \in \mathcal{M}_{k\times n}(\mathbb{C})$ *is said to* ***fit the pattern of*** w *if that matrix can be obtained by replacing the* $\star$*'s in the pattern matrix of* w *with complex numbers.*

Example 2.1.6. The pattern matrix of $w = 21231231$ is

$$P_{21231231} = \begin{pmatrix} 0 & 1 & 0 & \star & 1 & 0 & \star & 1 \\ 1 & 0 & 1 & \star & \star & 1 & \star & \star \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The matrix

$$\begin{pmatrix} 0 & 1 & 0 & 3 & 1 & 0 & \frac{1}{5} & 1 \\ 1 & 0 & 1 & i+1 & -7 & 1 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

fits the pattern of 21231231.

Recall that **spanning line configurations** in $X_{n,k}$ are ordered n-tuples of lines through the origin in $\mathbb{C}^k$ that span the whole space $\mathbb{C}^k$. A line in $\mathbb{C}^k$ can be thought of as a point in complex projective space $\mathbb{P}^{k-1}$. So spanning line configurations are also elements of $(\mathbb{P}^{k-1})^n$. Recall that complex projective space $\mathbb{P}^{k-1}$ can be thought of as a collection of nonzero vectors $\mathbf{z} = (z_1, \ldots, z_k) \in \mathbb{C}^k$ modulo the equivalence relation $\mathbf{z} \sim \lambda\mathbf{z}$ for any nonzero $\lambda \in \mathbb{C}$. Thus, a spanning line configuration can be represented by a full rank $k \times n$ matrix with no zero columns, modulo scaling columns by nonzero complex numbers. Recall that $\mathcal{M}_{k\times n}(\mathbb{C})$ denotes the set of such full rank $k \times n$ matrices. If T denotes the subgroup of invertible diagonal matrices of $GL_n(\mathbb{C})$, then points in $X_{n,k}$ are in bijection with the cosets in $\mathcal{M}_{k\times n}(\mathbb{C})/T$. If U^- denotes the subgroup of $GL_k(\mathbb{C})$ of lower triangular matrices with 1's on the diagonal, then by [**25**, Prop 3.2], every matrix $M \in \mathcal{M}_{k\times n}(\mathbb{C})$ can be written uniquely as the product of a lower triangular matrix in U^-, a matrix A that fits the pattern of w for some $w \in \mathcal{W}_{n,k}$, and an invertible diagonal matrix T. Thus $X_{n,k}$ can be decomposed as a union of cells corresponding to Fubini words, as follows.

Definition 2.1.7. *The* ***Pawlowski-Rhoades cell*** *or* ***PR cell*** $C_w \in X_{n,k}$ *of a Fubini word* $w \in \mathcal{W}_{n,k}$ *is the set of all spanning line configurations that can be represented by matrices of the form* MA*, for some* $M \in U_-$ *and* A *fitting the pattern of* w*. The* ***PR variety*** *of* $w \in \mathcal{W}_{n,k}$ *is* $\overline{C}_w \subset X_{n,k}$*, the closure of the PR cell* C_w *in the Zariski topology on* $X_{n,k}$*.*

Consider the Fubini word $w = 21231231$ from Example 2.1.6. Its PR cell $C_{21231231}$ is all spanning line configurations that can be represented by matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ \star & 1 & 0 \\ \star & \star & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & \star & 1 & 0 & \star & 1 \\ 1 & 0 & 1 & \star & \star & 1 & \star & \star \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

In this thesis, we will occasionally abuse terminology and discuss matrices in C_w, when in reality we mean cosets of $\mathcal{M}_{k\times n}(\mathbb{C})/T$ in C_w, as each such coset corresponds to a spanning line configuration.

Another way to describe PR varieties is via their equations. For positive integers, k and n with $k \leq n$, we can associate with $\mathcal{M}_{k\times n}(\mathbb{C})$ the underlying ring $\mathbb{C}[x_{ij} \,|\, i \in [k], j \in [n]]$. For every possible minor (I, J) on matrices in $\mathcal{M}_{k\times n}(\mathbb{C})$, we can associate its determinant $\Delta_{I,J}$ with a polynomial in $\mathbb{C}[x_{ij} \,|\, i \in [k], j \in [n]]$. For example, if $I = \{1, 3\}$ and $J = \{2, 5\}$, then

$$\Delta_{I,J} = x_{12}x_{35} - x_{32}x_{15}.$$

For a Fubini word $w \in \mathcal{W}_{n,k}$, the corresponding PR variety is the variety that vanishes on the ideal in $\mathbb{C}[x_{ij} \,|\, i \in [k], j \in [n]]$ generated by the determinants of the truly vanishing minors of C_w. In Chapter 3, we will use these ideals to study PR varieties.

2. Dimension and Generating Functions

In this section, we describe the generating function for inversions of permutations in S_n, as well as the relationship between the number of inversions of a permutation and the dimension of the corresponding Schubert cell. We then summarize the work of Pawlowski-Rhoades in [**25**] of connecting q-Stirling numbers of the second kind, ordered set partitions, and a generating function for what they define as the dimension of a Fubini word. Since we are working over $\mathbb{C}$ in this thesis, all references to dimension refer to complex dimension.

Definition 2.2.1. *The* ***dimension*** *of a permutation* $w \in S_n$*, denoted dim*(w)*, is the number* $\star$*'s in its pattern matrix* P_w*.*

Definition 2.2.2. *An* ***inversion*** *of a permutation* $w = w_1 \cdots w_n \in S_n$ *is a pair* (i, j)*,* $1 \leq i < j \leq n$*, such that* $w_i > w_j$*. We denote by inv*(w) *the number of inversions of* w*.*

For $i < j$, observe that $\mathrm{inv}(w) < \mathrm{inv}(t_{ij}w)$ if and only if i occurs before j in the one-line notation for w. Furthermore, $\mathrm{inv}(w) = m$ if and only if there exists a minimal length expression $w = s_{a_1} \cdots s_{a_m}$ [**4**].

The **codimension** of a Schubert cell C_w is the number of inversions of the permutation $w \in S_n$, because a $\star$ in the (w_i, j) position of the pattern matrix P_w represents that w_i and w_j are not inverted in w. In Example 2.1.2, for $w = 2431$, observe that there are $\star$'s in positions $(2, 2)$ and $(2, 3)$ of P_w, representing that $w_1 = 2$ is not inverted with either $w_2 = 4$ or $w_3 = 3$. Every other pair of letters in $w = 2431$ is inverted. Therefore, for any $w \in S_n$, the codimension is $\mathrm{codim}(w) = \mathrm{inv}(w)$, and the dimension of C_w is $\dim(w) = \binom{n}{2} - \mathrm{inv}(w)$. Observe also that the dimension of $Fl_n(\mathbb{C})$ is $\binom{n}{2} = \dim(w) + \mathrm{codim}(w)$.

Definition 2.2.3. *The* q***-analog of*** k *is* $[k]_q = q^{k-1} + q^{k-2} + \cdots + 1$*. The* q***-analog of*** k ***factorial*** *is* $[k]!_q = [k]_q[k-1]_q \cdots [1]_q$*.*

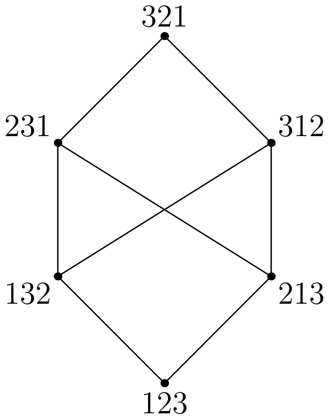

FIGURE 2.2.1. S_3 Bruhat Order

The generating function for the codimension, equivalently number of inversions, of permutations in S_n is

$$\sum_{w\in S_n} q^{\operatorname{codim}(w)} = \sum_{w\in S_n} q^{\operatorname{inv}(w)} = [n]!_q. \tag{2.2.9}$$

Since the polynomial $[n]!_q$ is palindromic of degree $\binom{n}{2}$, then the generating function for dimension of Schubert cells indexed by permutations in S_n is also

$$\sum_{w\in S_n} q^{\dim(w)} = [n]!_q. \tag{2.2.10}$$

For example, when $n = 3$, $[3]!_q = (q^2 + q + 1)(q + 1)(1) = q^3 + 2q^2 + 2q + 1$. Therefore, S_3 has 1 permutation with dimension 3, 2 permutations with dimension 2, 2 permutations with dimension 1, and 1 permutation with dimension 0. The Bruhat order on permutations is ranked by number of inversions, or equivalently codimension [**4**, Ch. 2]. This can be seen in Figure 2.2.1. The identity permutation $\mathrm{id} = 12\cdots n$ is the unique minimal element in the Bruhat order on S_n, and $w_0 = n(n-1)\cdots 1$ is the unique maximal element in the Bruhat order on S_n.

Pawlowski and Rhoades define the dimension of a Fubini word in [**25**, Eq. (3.6)]. As with permutations, the dimension of a Fubini word is given by the number of $\star$'s in its pattern matrix.

Definition 2.2.4. *The* ***dimension*** *of a Fubini word* $w \in \mathcal{W}_{n,k}$*, denoted* $\dim(w)$*, is the number* $\star$*'s in its pattern matrix* P_w*. By [**25**, Obs. 3.3],*

$$\dim(w) = \binom{k}{2} - inv(\pi(w)) + \sum_{i \notin in(w)} (\pi^{-1}_{w(i)} - 1).$$

As a cell, C_w *can be parametrized by* $\mathbb{C}^d$*, where* $d = \dim(w) + \binom{k}{2}$*.*

Observe that the dimension of C_w as a Schubert cell works differently from the dimension of C_w as a PR cell. The dimension of a permutation was equal to the dimension of the corresponding Schubert cell. The dimension of a Fubini word in $\mathcal{W}_{n,k}$ and the dimension of the corresponding PR cell differ by $\binom{k}{2}$. This is due to the fact that in the $k = n$ case, the flags of $Fl_n(\mathbb{C})$ are in bijection with $B_- \backslash GL_n(\mathbb{C})$, whereas the spanning line configurations in $X_{n,n}$ are in bijection with $T \setminus \mathcal{M}_{n\times n}(\mathbb{C})$ which is the same as $T \setminus GL_n(\mathbb{C})$. The subgroup B_- has dimension $\binom{n+1}{2}$ and the subgroup T has dimension n. The difference of these is $\binom{n}{2}$, which, when $n = k$, is exactly $\binom{k}{2}$.

Remark 2.2.5. The unique minimal element of $\mathcal{W}_{n,k}$ in Fubini-Bruhat order is $123\cdots kkk\cdots k$ [**25**, Sec. 9], which has dimension $\binom{k}{2} + (n-k)(k-1)$. Thus, the dimension of $X_{n,k} = \overline{C}_{123\cdots kkk\cdots k}$ is $2\binom{k}{2} + (n-k)(k-1) = n(k-1)$. So $X_{n,k}$ has the same dimension as $(\mathbb{P}^{k-1})^n$. Note, however that $X_{n,k} \subsetneq (\mathbb{P}^{k-1})^n$, because $k \times n$ matrices representing $(\mathbb{P}^{k-1})^n$ may not be full rank. A minimum dimension PR cell is indexed by a Fubini word w with $\dim(w) = 0$. If w has dimension 0, then $\pi(w) = w_0 \in S_k$, and all redundant letters in w are equal to $\pi_1(w) = k$. Note that $\dim(w) = 0$ implies that the dimension of C_w is $\binom{k}{2}$.

It will be useful for us to consider the codimension of C_w. Recall from [**8**] that the **codimension** of a subvariety X of a variety Y is the dimension of Y minus the dimension of X. Since the dimension of $X_{n,k}$ is $n(k-1)$, the codimension of C_w, denoted codim(w), is

$$\begin{aligned}\operatorname{codim}(w) = n(k-1) - \dim(C_w) = n(k-1) - \binom{k}{2} - \dim(w)\\ = \binom{k}{2} + (n-k)(k-1) - \dim(w).\end{aligned}$$

For example, $\operatorname{codim}(123\cdots kkk\cdots k) = 0$ and $\operatorname{codim}(k(k-1)(k-2)\cdots 1kkk\cdots k) = \binom{k}{2} + (n-k)(k-1)$.

One of the main achievements of [**25**] is connecting the Betti numbers of the cohomology ring of $X_{n,k}$ with the combinatorics of q-Stirling numbers. We recall some relevant definitions before stating one of Pawlowski-Rhoades' main theorems.

Definition 2.2.6. *The q-Stirling numbers $Stir_q(n,k)$ are defined by initial conditions $Stir_q(0,0) = 1$ and $Stir_q(n,k) = 0$ if $k < 0$ or $n < k$, and the recurrence relation*

$$Stir_q(n+1,k) = Stir_q(n,k-1) + [k]_q Stir_q(n,k). \tag{2.2.11}$$

We consider $[0]_q$ to be 0.

The above initial conditions and recurrence are enough to derive $\mathrm{Stir}_q(n,k)$ for all $n, k \geq 0$. For reference, the table below shows some of the smallest values. Not in the table are $\mathrm{Stir}_q(0,0) = 1$, $\mathrm{Stir}_q(0,k) = 0$ if $k \neq 0$, and $\mathrm{Stir}_q(n,0) = 0$ if $n \neq 0$.

	$k=2$	$k=3$	$k=4$	$k=5$
$n=0$	0	0	0	0
$n=1$	0	0	0	0
$n=2$	1	0	0	0
$n=3$	$q+2$	1	0	0
$n=4$	q^2+3q+3	q^2+2q+3	1	0
$n=5$	q^3+4q^2+6q+4	$q^4+3q^3+7q^2+8q+6$	q^3+2q^2+3q+4	1

Definition 2.2.7. *The rev_q of a q-polynomial reverses the coefficients of the polynomial. For example, $rev_q(q^3+2q^2+3) = 3q^3+2q+1$.*

Theorem 2.2.8. *[**25**, Sec. 1, Prop 3.4] If*

$$f(q) = \sum_{w \in \mathcal{W}_{n,k}} q^{dim(w)} = [k]!_q \cdot Stir_q(n,k), \tag{2.2.12}$$

then the Poincaré polynomial for $H^*(X_{n,k},\mathbb{Z})$ *is determined by* $rev_q(f(q^2))$.

For example, when $n = 5$ and $k = 3$,

$$[3]!_q\mathrm{Stir}_q(5,3) = (q^2+q+1)(q+1)(1)(q^4+3q^3+7q^2+8q+6)$$

$$= q^7+5q^6+15q^5+29q^4+39q^3+35q^4+20q+6.$$

Thus, there is 1 Fubini word in $\mathcal{W}_{5,3}$ with dimension 7, 5 Fubini words with dimension 6, 15 with dimension 5, and so on. Observe that for every (n,k) such that $0 < k \leq n$, $\mathrm{Stir}_q(n,k)$ has leading coefficient 1, as does $[k]!_q$, confirming the uniqueness of the maximum dimension Fubini word $123\cdots kkk\cdots k \in \mathcal{W}_{n,k}$ found in Remark 2.2.5. Observe that its dimension is $\binom{k}{2} + (n-k)(k-1)$, which we calculated by counting the $\star$'s in its pattern matrix in Remark 2.2.5. This agrees with the degree of $[k]!_q \cdot \mathrm{Stir}_q(n,k)$.

The Stirling numbers of the second kind $\mathrm{Stir}(n,k)$ count set partitions of $[n]$ into k blocks. Therefore, $k!\,\mathrm{Stir}(n,k)$ counts ordered set partitions. There is a natural bijection from ordered set partitions to Fubini words, described as follows.

Definition 2.2.9. *An* ***ordered set partition*** *of size n with k blocks is an ordered list of k disjoint subsets whose union is $[n]$. We denote an ordered set partition by $\sigma = (A_1|A_2|\cdots|A_k)$, where $\bigcup_{i=1}^k A_i = [n]$. We denote the family of ordered set partitions of size n with k blocks by $\mathcal{OP}_{n,k}$.*

One can biject $\sigma = (A_1|A_2|\cdots|A_k) \in \mathcal{OP}_{n,k}$ to $w \in \mathcal{W}_{n,k}$ as follows. If $j \in A_i$, then set $w_j = i$. This also means that A_i is the set of positions in w with value i. Given $w \in \mathcal{W}_{n,k}$, denote the corresponding $\sigma \in \mathcal{OP}_{n,k}$ by $\sigma(w)$, and for $\sigma \in \mathcal{OP}_{n,k}$, denote the corresponding $w \in \mathcal{W}_{n,k}$ by $w(\sigma)$. Using $w = 21231231$ from Example 2.1.6, $\sigma(21231231) = (258|136|47) \in \mathcal{OP}_{8,3}$. Observe that in the case of $n = k$, $\mathcal{W}_{n,n} = S_n$, and for $w \in \mathcal{W}_{n,n} = S_n$, writing $\sigma(w) \in \mathcal{OP}_{n,n}$ without parentheses and bars gives w^{-1} in one-line notation. Steingrímsson [**32**], Remmel-Wilson [**26**], and Ehrenborg-Readdy [**12**] discuss other statistics on ordered set partitions.

If $v < w$ in medium roast Fubini-Bruhat order, then, as with Bruhat order, $\operatorname{codim}(v) < \operatorname{codim}(w)$. This is equivalent to saying that the dimension of $\overline{C}_v$ is strictly greater than the dimension of $\overline{C}_w$, which holds since $\overline{C}_w$ is a proper subvariety of $\overline{C}_v$ [**8**, Prop 9.4.10]. Pawlowski and Rhoades observe in [**25**] that the espresso Fubini-Bruhat order is poorly behaved with respect to dimension. Namely, they note that $1323 \prec 1123$ in $\mathcal{W}_{4,3}$, but $\dim(1323) = \dim(1123) = 3$. There are even worse examples. As we discuss in Chapter 6, $24313 \preceq 24413$, but $\dim(24313) = 4$ and $\dim(24413) = 5$. We will see an example in Chapter 7 that shows that even medium roast Fubini-Bruhat order is not ranked by codimension. In Chapter 6, we discuss the decaf Fubini-Bruhat order, which is the only Fubini-Bruhat order we define that is ranked by codimension, so its rank generating function is given by Equation (2.2.12).

3. Monotone Triangles

Among others, Kobayashi used monotone triangles to study permutations. The Macneille completion of the Bruhat order to a lattice can be seen via alternating sign matrices, or, as used in Kobayashi's work, monotone triangles. Notably, in [**21**], Kobayashi used monotone triangles to generalize Lascoux and Schützenberger's result that join-irreducibles in Bruhat order are precisely the permutations with exactly one element in their essential set [**23**]. Though this result does not extend to any of the Fubini-Bruhat posets, the monotone triangles for a Fubini word that we define in Chapter 5 are still closely related to the essential set of a Fubini word, defined later in the same chapter. We review the background for these concepts here.

Definition 2.3.1. *A* ***monotone triangle*** *x of order n is a triangular array of numbers $(x_{ab} \mid 1 \leq b \leq a \leq n-1)$ such that $1 \leq x_{ab} \leq n$, $x_{ab} < x_{a,b+1}$, $x_{ab} \geq x_{a+1,b}$, and $x_{ab} \leq x_{a+1,b+1}$ for all a, b.*

We will right-justify our monotone triangles in this thesis. For example, if $n = 4$, a generic monotone triangle is shown in Figure 2.3.1. The entries are weakly

$$\begin{array}{cccc} & & & x_{11} \\ & & x_{21} & x_{22} \\ & x_{31} & x_{32} & x_{33} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{array}$$

FIGURE 2.3.1. Generic monotone triangle when $n = 4$

increasing from top to bottom in each column, strictly increasing from left to right in each row, and weakly increasing along each southwest to northeast diagonal.

DEFINITION 2.3.2. *The* ***monotone triangle corresponding to a permutation*** $w = w_1 w_2 \cdots w_n \in S_n$ *is given by* $(x_{ab} \,|\, 1 \leq b \leq a \leq n-1)$*, with*

$$\{x_{a1} < x_{a2} < \cdots < x_{aa}\} = \mathit{sort}(\{w_1, w_2, \ldots, w_a\}).$$

Observe that the conditions of the Ehresmann Tableau Criterion compare the entries of the monotone triangles corresponding to two permutations $v, w \in S_n$. Thus, a restatement of the Ehresmann Tableau Criterion is that $v \leq w$ in Bruhat order if and only if the monotone triangle for v is entrywise smaller than the monotone triangle for w.

For example, Figure 2.3.2 gives Bruhat order for S_3 with monotone triangles written in place of their corresponding permutations. There exists one monotone triangle of order 3 that does not correspond to any permutation in S_3, but can be inserted into the poset, as shown in Figure 2.3.3. An extension of a Bruhat order poset on S_n is known as its **Macneille completion**, which can be defined as the partial order on all monotone triangles of order n, where a monotone triangle S is less than a monotone triangle T if it is entrywise smaller.

4. Rank Conditions

An alternative way of defining Schubert cells is given by rank functions. Recall the rank function from Definition 1.2.23. Observe that

$$r_w : (i, j) \mapsto |\{1 \leq h \leq j \,|\, w_h \leq i\}|. \tag{2.4.13}$$

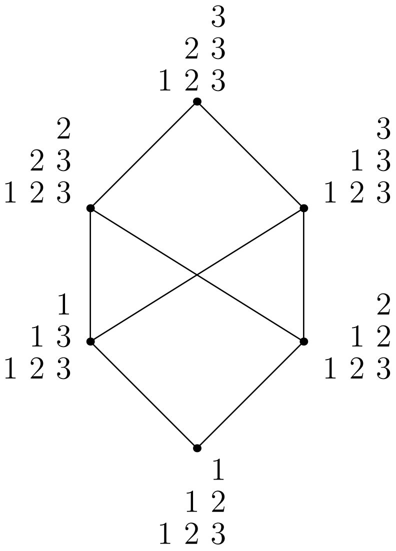

FIGURE 2.3.2. S_3 Bruhat order with permutations as monotone triangles

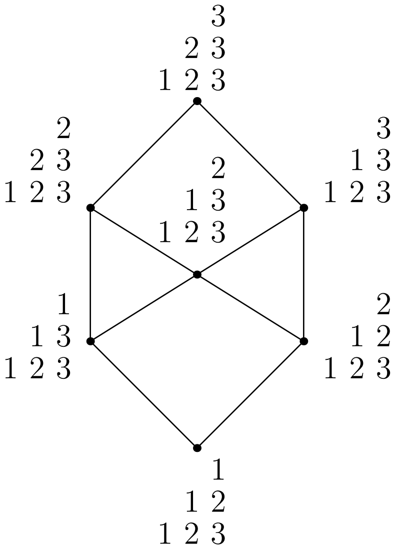

FIGURE 2.3.3. Macneille completion of S_3 Bruhat order as monotone triangles

The Schubert cell C_w can be equivalently described as the set of all flags in $Fl_n(\mathbb{C})$ that can be represented by some matrix M such that the rank of the submatrix of $M \in GL_n(\mathbb{C})$ indexed by rows $[i]$ and columns $[j]$ is $r_w(i,j)$ for all $(i,j) \in [n] \times [n]$.

Every matrix in the same coset of $B_- \backslash GL_n(\mathbb{C})$ will give rise to the same northwest rank conditions $r_w(i,j)$ for all $(i,j) \in [n] \times [n]$.

Recall that the Schubert variety $\overline{C}_w$ is the closure of C_w under the Zariski topology. Since the Schubert cell C_w can be defined by rank conditions, $\overline{C}_w$ is the set of all flags in $Fl_n(\mathbb{C})$ that can be represented by some matrix M such that the rank of the submatrix of $M \in GL_n(\mathbb{C})$ indexed by rows $[i]$ and columns $[j]$ is at most $r_w(i,j)$ for all $(i,j) \in [n] \times [n]$. Observe that if $r_w(i,j) = a$ and $I \subset [i]$ and $J \subset [j]$ and $|I| = |J| = a+1$, then $\Delta_{I,J}(M) = 0$ for all $M \in C_w$. One can show that this is equivalent to

$$C_w = \{M \in B_- \setminus GL_n(\mathbb{C}) \,|\, \forall\,(i,j) \in [n]^2, \mathrm{rank}(M_{([i],[j])}) = r_w(i,j)\} \tag{2.4.14}$$

and

$$\overline{C}_w = \{M \in B_- \setminus GL_n(\mathbb{C}) \,|\, \forall\,(i,j) \in [n]^2, \mathrm{rank}(M_{([i],[j])}) \leq r_w(i,j)\}. \tag{2.4.15}$$

For more information on Schubert varieties see [**16**, Ch. 10], but note that we have used a different representation of the canonical matrices and related Gaussian elimination.

Recall that Bruhat order $\leq$ on permutations $v, w \in S_n$ is defined by

$$v \leq w \Leftrightarrow \overline{C}_w \subseteq \overline{C}_v \Leftrightarrow C_w \subseteq \overline{C}_v \Leftrightarrow \overline{C}_v \cap C_w \neq \emptyset.$$

Therefore, Bruhat order can be equivalently defined via rank conditions

$$v \leq w \Leftrightarrow r_v(i,j) \geq r_w(i,j)\ \forall\,(i,j) \in [n] \times [n]. \tag{2.4.16}$$

For example, by Theorem 1.2.31, $2431 < 2134$. The rank tables of these permutations can be derived by taking the ranks of the northwest submatrices of either their permutation matrices or their pattern matrices. Thus, the rank tables are

$$r_{2431} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} \text{ and } r_{2134} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

We can observe that r_{2431} is entrywise less than r_{2134}. This is another way to verify that $2431 < 2134$.

Pawlowski and Rhoades show in Section 5 of [**25**] that rank conditions fully describe PR varieties for convex Fubini words. They also describe the importance of the set $\{v \in \mathcal{W}_{n,k} \mid \mathrm{conv}(v) = w,\, \mathrm{in}(v) = \mathrm{in}(w)\}$ for a convex Fubini word w. They describe a cellular decomposition of $X_{n,k}$, which can be restated as rank conditions of matrices in $\mathcal{M}_{k\times n}(\mathbb{C})$. In Chapter 3, we continue to develop properties of PR varieties in analogy to existing results about Schubert varieties. Thus, Chapter 3 serves as a bridge to our newer results about PR cells and varieties and the Fubini-Bruhat orders.

CHAPTER 3

Defining Equations for PR Varieties

In this section, we use results from [**25**] to prove that a PR variety $\overline{C}_w$ is defined by the set of truly vanishing minors on the corresponding cell C_w. This allows us to give an alternate characterization of medium roast Fubini-Bruhat order, where $v \leq w$ if and only if the truly vanishing minors on C_v are contained in the truly vanishing minors on C_w by regular subset inclusion.

We recall some notation for minors and their determinants. Recall that $\Delta_{I,J}(A)$ is the determinant of the minor of the matrix $A \in \mathcal{M}_{k\times n}(\mathbb{C})$ using rows $I \subseteq [k]$ and columns $J \in \binom{[n]}{[k]}$, where I and J have the same size. Recall that a minor (I, J) is truly vanishing if $\Delta_{I,J}(A) = 0$ is zero for all matrices $A \in C_w$.

1. Consequences from Convex Fubini Words

Convex Fubini words and convexifications of Fubini words will be central to the results of this chapter. Deduce from Definition 1.2.27 that two Fubini words $v, w \in \mathcal{W}_{n,k}$ have the same convexification, $\mathrm{conv}(v) = \mathrm{conv}(w)$, if and only if $\pi(v) = \pi(w)$ and they have the same multiset of letters. Recall that $\mathrm{in}(w)$ refers to the set of k initial positions of the letters in a Fubini word $w \in \mathcal{W}_{n,k}$.

Definition 3.1.1. *Let $v, w \in \mathcal{W}_{n,k}$. Define v and w to be* ***column equivalent via*** *$x \in S_n$ if $conv(v) = conv(w)$ and $v = wx$ for some permutation $x \in S_n$ such that x maps $in(v)$ to $in(w)$.*

For example, $v = 32312$ and $w = 32231$ are column equivalent via $x = 12453$. Indeed, $\mathrm{conv}(v) = \mathrm{conv}(w) = 33221$, and x maps $\mathrm{in}(v) = \{1, 2, 4\}$ to $\mathrm{in}(w) = \{1, 2, 5\}$. The following lemma states parts of [**25**, Obs 5.10, Prop 5.11] in more detail.

Lemma 3.1.2. *Let $v, w \in \mathcal{W}_{n,k}$. If v and w are column equivalent via $x \in S_n$, then the following hold.*

(i) The pattern matrices of v and w are related by $P_v = P_w M_x$.

(ii) The dimension of C_v equals the dimension of C_w.

(iii) The permutation x bijectively maps matrices in C_v to C_w by matrix multiplication by M_x on the right. Explicitly,

$$A \in C_v \iff AM_x \in C_w.$$

(iv) For all $k \times n$ matrices A, $I \subset [k]$, and $J \in \binom{[n]}{[k]}$ of the same size as I, we have $\Delta_{I,J}(A) = \pm\Delta_{I,x(J)}(AM_x)$.

(v) The truly vanishing minors of C_v and C_w are in bijection given by $(I, J) \to (I, x(J))$. The unvanishing minors of C_v and C_w are in bijection given by $(I, J) \to (I, x(J))$.

Proof. We have $\pi(v) = \pi(w)$, $v = wx$, and $j \in [n]$ is an initial position for w if and only if $x(j)$ is an initial position for v, by Definition 3.1.1. Thus, Part ((i)) holds. Since P_v is obtained from P_w by applying x to the columns while preserving initial columns, we observe both pattern matrices have the same number of $\star$'s, so part ((ii)) holds. Thus, since for any $u \in \mathcal{W}_{n,k}$ the dimension of C_u is the number of $\star$'s in P_u plus $\binom{k}{2}$ by Definition 2.2.4. Thus, Part ((iii)) holds since multiplication on the left by invertible $k \times k$ matrices commutes with multiplication by the permutation matrix M_x on the right.

For Part ((iv)), observe that $\Delta_{I,J}(A) = \pm\Delta_{I,x(J)}(AM_x)$ by the definition of determinant. Part ((v)) follows from Part ((iv)) since a minor $\Delta_{I,J}(A) = 0$ for all on $A \in C_w$ if and only if $\Delta_{I,x(J)}(AM_x) = 0$, and a minor $\Delta_{I,J}(A) \neq 0$ for all on $A \in C_w$ if and only if $\Delta_{I,x(J)}(AM_x) \neq 0$. □

Definition 3.1.3. *For every Fubini word $w \in \mathcal{W}_{n,k}$, define its* ***rank function*** *to be the map $r_w : [k] \times [n] \to \mathbb{Z}_{\geq 0}$ that sends (i, j) to the rank of the northwest submatrix of M_w with row set $[i]$ and column set $[j]$.*

This generalizes the rank function on permutations from Section 4 of Chapter 2. Observe that any matrix A fitting the pattern of w has the same northwest rank conditions as M_w. Pawlowski and Rhoades note in [**25**, Def 5.1, Lemma 5.8] that C_w for a convex word w is completely determined by the rank conditions on P_w, which is equivalent in the $n = k$ case to Schubert cells. More generally, we can extend Definition 3.1.3 to get a corollary to Lemma 3.1.2 that is analogous to Equations (2.4.14) and (2.4.15) for permutations.

Definition 3.1.4. *Given any column vector $\mathbf{a} = (a_1, \dots, a_k)^t$ define the* ***h*-subcolumn** *of $\mathbf{a}$ to be $(a_1, \dots, a_h)^t$ for $1 \leq h \leq k$. The h-subcolumn is a projection of $\mathbf{a}$ to h-dimensional space. Similarly, given a collection J of columns in a matrix $A = (a_{ij})$, define the* ***h*-subcolumns indexed by** *J to be the collection of vectors $\{(a_{1j}, a_{2j}, \dots, a_{hj})^t) \mid j \in J\}$.*

Definition 3.1.5. *Define the* ***h*-rank of *J* for** *w, denoted $rank_w^{(h)}(J)$, to be the maximum rank of the h-subcolumns indexed by J over all matrices $A \in C_w$. Equivalently, $rank_w^{(h)}(J)$ is the largest value r such that there exist subsets $I \subset [h]$ and $J' \subset J$ such that $r = |I| = |J'|$ and $\Delta_{I,J'}(A) \neq 0$. If no such pair (I, J') exists, we say the rank is 0.*

Definition 3.1.6. *For every matrix $M \in \mathcal{M}_{k\times n}(\mathbb{C})$, define its* ***rank function*** *to be the map $r_M : [k] \times \binom{[n]}{[k]} \to \mathbb{Z}_{\geq 0}$ that sends (h, J) to the rank of the submatrix of M with row set $[h]$ and column set J.*

Corollary 3.1.7. *For any $w \in \mathcal{W}_{n,k}$, C_w and $\overline{C}_w$ are completely determined by the rank conditions on subsets of columns of P_w. That is,*

$$C_w = \{M \in \mathcal{M}_{n\times k}(\mathbb{C}) \mid \forall\, i \in [k], \forall J \in \binom{[n]}{[k]}, r_M(i, J) = rank_w^{(i)}(J)\}, \tag{3.1.17}$$

and

$$\overline{C}_w = \{M \in \mathcal{M}_{n\times k}(\mathbb{C}) \mid \forall\, i \in [k],\, \forall\, J \in \binom{[n]}{[k]}, r_M(i,J) \leq rank_w^{(i)}(J)\}. \tag{3.1.18}$$

Lemma 3.1.8. *Given $w \in \mathcal{W}_{n,k}$, the PR variety $\overline{C}_w$ is exactly the set of matrices in $\mathcal{M}_{k\times n}(\mathbb{C})$ that vanish for every polynomial in the ideal generated by the determinants of all truly vanishing minors of C_w.*

Proof. First, assume w is a convex Fubini word. For convex words, $\overline{C}_w$ is determined by northwest rank conditions on $k \times n$ matrices with entries in $\mathbb{C}$ [**25**, Def 5.1, Lemma 5.8]. The northwest rank conditions are equivalent to saying the following.

(1) All minors (I, J) such that $I \subset [i]$, $J \subset [j]$, $|I| = |J| = r_w(i,j) + 1$ are truly vanishing on $\overline{C}_w$.
(2) There exists some minor (I, J) such that $I \subset [i]$, $J \subset [j]$, and $|I| = |J| \leq r_w(i,j)$ and some matrix $A \in C_w$ such that $\Delta_{I,J}(A) \neq 0$.

Thus, for a convex word $w \in \mathcal{W}_{n,k}$, the determinants of the minors described in (1) generate an ideal in the ring $\mathbb{C}[x_{ij} \mid i \in [k], j \in [n]]$ that vanishes on $\overline{C}_w$.

Let $v \in \mathcal{W}_{n,k}$ be any nonconvex word. By Lemma 3.1.2, the minimal length permutation x such that $v = \mathrm{conv}(v)x$ gives rise to a bijection from the truly vanishing minors on $C_{\mathrm{conv}(v)}$ to the truly vanishing minors on C_v. Hence, the variety defined by the truly vanishing minors on C_v is $\overline{C}_v$, just as with $C_{\mathrm{conv}(v)}$. □

Corollary 3.1.9. *Suppose $t, u, v, w \in W_{n,k}$ are Fubini words.*

(1) The relation $v \leq w$ in medium roast Fubini-Bruhat order holds if and only if the truly vanishing minors of C_v are contained in the truly vanishing minors of C_w.
(2) If $v \leq w$, then for each subset $J \subset [n]$ and each integer $h \in [k]$,

$$rank_v^{(h)}(J) \geq rank_w^{(h)}(J).$$

(3) *Suppose* $x \in S_n$ *is any permutation such that* t *and* u *are column equivalent via* x*, and* v *and* w *are also column equivalent via* x*. Then* $t \leq v$ *in medium roast Fubini-Bruhat order if and only if* $u \leq w$*. Furthermore,* $\dim(u) = \dim(t)$ *and* $\dim(v) = \dim(w)$*, so the dimension difference is maintained under the action of* x*.*

Proof. Part (1) follows from Lemma 3.1.8, since the medium roast Fubini-Bruhat poset is defined by $v \leq w$ if and only if $\overline{C}_w \subset \overline{C}_v$. Part (2) follows from the rank conditions on the convex word case by permuting columns.

For Part (3), we know that the truly vanishing minors of C_u and C_w are in bijection via x to the truly vanishing minors of C_t and C_v, respectively, by Lemma 3.1.2. Thus the question of whether or not one set of truly vanishing minors is contained in another can be realized equivalently before or after the bijection. The dimension statement holds by Lemma 3.1.2 Part ((ii)). ☐

There are some immediate combinatorial implications of Corollary 3.1.9 to medium roast Fubini-Bruhat order. However, redundant letters make broad generalizations difficult without further results. Part (2) of Corollary 3.1.9 implies $v_1 \leq w_1$ whenever $v \leq w$. Furthermore, if $v \leq w$ and $v_1, v_2, \ldots, v_h$ are distinct and $w_1, w_2, \ldots, w_h$ are distinct, then

$$\{v_1, v_2, \ldots, v_h\} \trianglelefteq \{w_1, w_2, \ldots, w_h\},$$

as in Ehresmann's Tableau Criterion for Bruhat order. On the other hand, if say $w_1 = w_2$, then $v_1 \leq w_1$, but v_2 can be anything. Thus, the medium roast Fubini-Bruhat order and the Bruhat order on the corresponding initial permutations are not correlated.

Example 3.1.10. In $\mathcal{W}_{4,3}$, we note $v = 2311 < 2213 = w$, but $231 = \pi(v) > \pi(w) = 213$ in Bruhat order. Furthermore, note that $\pi(w) = \pi(2113) = \pi(2133)$, but v is incomparable with 2113, and $v > 2133$ in medium roast Fubini-Bruhat order.

Due to redundant letters, the northwest rank conditions of M_w do not determine a unique PR cell. However, taking ranks of subsets of h-subcolumns of M_w suffices.

THEOREM 3.1.11. *[**25**, Thm. 5.12 (2)] Each matrix* $M \in \mathcal{M}_{k\times n}(\mathbb{C})$ *has rank conditions equal to the rank conditions of subsets of* h*-subcolumns, for all* $h \in [k]$*, of* M_w *for a unique Fubini word* w*.*

CHAPTER 4

Northerly Vanishing Minors on PR Varieties

In this chapter, we develop results to allow us to focus on only northerly minors instead of all possible minors of $k \times n$ matrices. Instead of checking the vanishings of all possible minors on C_v and C_w to decide if v and w are related in the medium roast and espresso Fubini-Bruhat orders, Theorem 1.2.8 gives us a way to check only the northerly minors for matrices fitting the pattern of v and w. Theorem 1.2.18 gives a characterization of the truly vanishing, sometimes vanishing, and unvanishing northerly minors in terms of the $\trianglelefteq$ order on multisets from Definition 1.2.9. Theorems 1.2.19 and 1.2.20 follow from Theorems 1.2.8 and 1.2.18, and give combinatorial tests for the medium roast and espresso Fubini-Bruhat orders. We will prove all of these theorems in this chapter.

1. Defining Equations for PR Varieties Using Northerly Minors

The northerly minors have some nice properties in terms of the spanning line configurations. The determinant of every northerly minor is invariant under left multiplication by unipotent lower triangular matrices. Thus, if we know a northerly minor vanishes for every matrix fitting the pattern of some $w \in \mathcal{W}_{n,k}$, then we know that northerly minor is truly vanishing on C_w. The analogous statement is not generally true for all minors.

The truly vanishing northerly minors for w do not generate the same ideal as the truly vanishing minors for w as shown below in Example 4.1.3. We can, however, get enough information from the northerlies to determine w.

Definition 4.1.1. *For a matrix $A \in \mathcal{M}_{k\times n}(\mathbb{C})$, the set of northerly minors that vanish on A is defined by*

$$T(A) = \{J \in \binom{[n]}{[k]} \mid |J| = h \leq k, \Delta_{[h],J}(A) = 0\}.$$

Lemma 4.1.2. *Given $A \in \mathcal{M}_{k\times n}(\mathbb{C})/T$, the PR cell C_w containing A is determined by $T(A)$, the set of northerly minors that vanish on A.*

Proof. The determinant of every northerly minor is invariant under left multiplication by unipotent lower triangular matrices in U^-, so we only need to consider northerly minors matrices A that fit the pattern of w. The entire algorithm below for recovering w remains valid under a left U^- action, so we will be able to apply it for any matrix in C_w, as shown in Example 4.1.3.

Note that α_1, the position of the initial 1 in w, is determined by the leftmost column j of A such that $\{j\} \notin T(A)$, which must exist since A is full rank and thus has a nonvanishing $k \times k$ minor. For $h \geq 1$, assume that $\alpha_1, \ldots, \alpha_h$ are known. Observe from the definition of the pattern matrix that the position of the initial $h+1$ is the leftmost column j of A such that $\{j, \alpha_1, \ldots, \alpha_h\} \notin T(A)$, which also must exist. After k iterations, this procedure recovers $\alpha_1, \ldots, \alpha_k$, from which we can deduce $\pi(w) = \pi_1\pi_2\cdots\pi_k$.

Now assume $j \in [n] \setminus \{\alpha_1, \ldots, \alpha_k\}$. Note that if $w_j = \pi_i$, then any $k \times k$ minor containing columns $\alpha_{\pi_1}, \ldots, \alpha_{\pi_i}$ and j must vanish on A by definition of the pattern matrix, since those $i+1$ columns only span an i dimensional space. Therefore, we can determine w_j from $T(A)$ as follows. Loop for a from k to 1, setting $J_a = \{j, \alpha_1, \ldots, \alpha_k\} \setminus \{\alpha_{\pi_a}\}$. If $J_a \notin T(A)$, return $w_j = \pi_a$ and break out of the loop. □

EXAMPLE 4.1.3. Consider the 4×5 matrix

$$A = \begin{pmatrix} 0 & 1 & 2 & 0 & 1 \\ 0 & 3 & 6 & 7 & 3 \\ 1 & 2 & 6 & 7 & 2 \\ 3 & 5 & 17 & 13 & 6 \end{pmatrix}.$$

We will recover the Fubini word $w \in \mathcal{W}_{5,4}$ for which $A \in C_w$.

The northerly minors that vanish on A are

$$\begin{aligned} T(A) = & \{\{1\},\{4\},\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,5\},\{3,5\}, \\ & \{1,2,3\},\{1,2,5\},\{1,3,5\},\{2,3,5\},\{2,4,5\},\{1,2,3,5\}\}. \end{aligned}$$

Running through the algorithm above, first note $\alpha_1 = 2, \alpha_2 = 4, \alpha_3 = 1$, and $\alpha_4 = 3$, so the initial permutation of w is $\pi = 3142$. Since $5 \notin \mathrm{in}(w)$, we need to determine the value of w_5. Run the loop starting at $i = 4$. Since $J_4 = \{1,2,3,4,5\} \setminus \{\alpha_2\} = \{1,2,3,5\} \in T(A)$, we continue. For $i = 3$, $J_3 = \{1,2,3,4,5\} \setminus \{\alpha_4\} = \{1,2,4,5\} \notin T(A)$, so break out of the loop, and $w_5 = \pi_3 = 4$. Thus, $w = 31424$.

Note that T_{31424}, the set of truly vanishing northerly minors of 31424, does not contain $\{4\}$ even though $T(A)$ does. Any instance of a matrix $A \in C_w$ may vanish on additional northerly minors not in T_w, but $T_w \subseteq T(A)$. However, the minors used in the algorithm to construct w from the complement of $T(A)$ can never vanish on all of C_w, given the structure of P_w under a left action by U^-.

A truly vanishing minor of C_w can appear in the ideal generated by the truly vanishing northerly minors of C_w, but this is not always the case. For example, the $(124,123)$-minor vanishes on all of C_{31424} including on A. Observe that $\Delta_{124,123} = x_{4,1}\Delta_{12,23} - x_{4,2}\Delta_{12,13} + x_{4,3}\Delta_{12,12}$, and $(12,23)$,$(12,13)$, and $(12,12)$ are truly vanishing northerly minors on C_{31424}. On the other hand, the minor $(2,1)$ vanishes on all of C_{31424} including on A, but $(2,1)$ is not northerly and has degree 1, so it is not in the ideal generated by truly vanishing northerly minors of C_{31424}. The set of truly vanishing northerly minors of C_{31424} does tell us a different way that $(2,1)$ vanishes on all of C_{31424}. Namely, every 2×2 northerly minor of the form $(12, 1j)$ vanishes on all of C_{31424}.

Corollary 4.1.4. *The set T_w of truly vanishing northerly minors on the PR cell C_w determines $w \in \mathcal{W}_{n,k}$, and therefore the rank conditions defining $\overline{C}_w$, conv(w), and the minimum length permutation $x \in S_n$ such that $w = $ conv$(w)x$.*

Lemma 4.1.5. *For $I \subseteq [k]$ and $J \in \binom{[n]}{[k]}$ of the same size, suppose (I,J) is a truly vanishing minor on the PR cell C_w for some Fubini word $w \in \mathcal{W}_{n,k}$. Then (H,J) is a truly vanishing minor on C_w for any H such that $|H| = |I|$ and $H \leq_L I$. In particular, $\Delta_{\{1,2,\ldots,|I|\},J}$ is a truly vanishing minor on C_w, so $J \in T_w$.*

Proof. If $I = \{1,2,\ldots,|I|\}$, then I is minimal in lex order so the statement holds. Furthermore, (I,J), or simply J, is a truly vanishing northerly minor for w, so $J \in T_w$.

If $H <_L I$, then there exist h and i with $1 \leq h < |I|$ and $h < i \leq \max(I)$ such that $h \in H \setminus I$ and $i \in I \setminus H$. Via the lower triangular action, we can add row h to row i in any matrix $A \in C_w$ to get another matrix $A' \in C_w$. Let the entry in A in row i and column j be a_{ij}. Then expanding $\Delta_{I,J}$ along row i before and after this row addition yields expressions

$$0 = \Delta_{I,J}(A) = (-1)^{\#\{c \in I \,|\, c<i\}} \sum_{j \in J} (-1)^{\#\{b \in J \,|\, b<j\}} a_{ij} \Delta_{I\setminus\{i\},J\setminus\{j\}}(A) \tag{4.1.19}$$

and

$$0 = \Delta_{I,J}(A') = (-1)^{\#\{c \in I \,|\, c<i\}} \sum_{j \in J} (-1)^{\#\{b \in J \,|\, b<j\}} (a_{ij} + a_{hj}) \Delta_{I\setminus\{i\},J\setminus\{j\}}(A). \tag{4.1.20}$$

Subtracting (4.1.19) from (4.1.20) yields

$$0 = (-1)^{\#\{c \in I \,|\, c<i\}} \sum_{j \in J} (-1)^{\#\{b \in J \,|\, b<j\}} a_{hj} \Delta_{I\setminus\{i\},J\setminus\{j\}}(A) = \pm \Delta_{I',J}(A) \tag{4.1.21}$$

where $I' = I \cup \{h\} \setminus \{i\}$, so $\Delta_{I',J}(A) = 0$ for all $A \in C_w$. Continue these swaps until the rows are H, and the proof of the first claim is complete. In the case that H is as small as possible in lexicographic order, $H = \{1, 2, \ldots, |I|\}$, so (H, J) is a truly vanishing minor on C_w. Equivalently, J is a truly vanishing northerly minor on C_w, so $J \in T_w$. $\square$

LEMMA 4.1.6. *Suppose $w \in \mathcal{W}_{n,k}$ is a Fubini word, $J \subset [n]$, and h is a positive integer such that $1 \leq h \leq k$. The following conditions are equivalent.*

(1) The rank of the h-subcolumns indexed by J is bounded on C_w by $\text{rank}_w^{(h)}(J) < |J|$.

(2) For every $I \subseteq [h]$ such that $|I| = |J|$, the minor (I, J) is truly vanishing on C_w.

(3) For all subsets $K \subseteq [n]$ such that $J \subseteq K$ and $|K| = h$, $K \in T_w$.

PROOF. The equivalence of the first two statements follows directly from the definition of the rank of the h-subcolumns indexed by J of a matrix. To see the first two statements imply the third, suppose A is any matrix in C_w. Then by (1), the rank of the h-subcolumns indexed by J in A is strictly less than $|J|$. We know that the h-subcolumns indexed by any $K \subseteq [n]$ such that $J \subseteq K$ and $|K| = h$ must have rank strictly less than $|K|$ since rank goes up by at most one with each additional vector. Therefore, $\Delta_{[h],K}(A) = 0$. Since this is true for every $A \in C_w$, $K \in T_w$. So the first two conditions imply the third.

Conversely, we will show that the third condition implies the first condition. Suppose there exists a matrix $A \in C_w$ with the rank of the h-subcolumns indexed by J equal to $|J|$. Since $A \in C_w$, we also have $A \in \mathcal{M}_{n,k}(\mathbb{C})$. So we know the rank of A is k and the rows of A seen as vectors in $\mathbb{C}^n$ are linearly independent. Therefore, the rank of the first h rows of A is h, which implies the h-subcolumns of A span the vector space $\langle e_1, \ldots, e_h \rangle$. By assumption, the h-subcolumns of A indexed by J are independent, so they can be extended to a basis of $\langle e_1, \ldots, e_h \rangle$ by choosing additional columns to create a subset $K \supseteq J$, $K \subseteq [n]$. Then $|K| = h$

and $\Delta_{[h],K}(A) \neq 0$. Hence, the third condition does not hold. We conclude that the third condition implies the first condition.

□

COROLLARY 4.1.7. *Suppose* $I \subseteq [k]$ *and* $J \subseteq [n]$ *are sets of the same size, and* $h = \max(I)$*. If the* (I, J)*-minor is truly vanishing on* C_w *for a Fubini word* $w \in \mathcal{W}_{n,k}$*, then at least one of the following hold.*

1. *For every* $j \in J$*, the* $(I \setminus \{h\}, J \setminus \{j\})$*-minor is truly vanishing on* C_w*.*
2. *For all subsets* K *such that* $J \subseteq K \subseteq [n]$ *and* $|K| = h$*,* $K \in T_w$*.*

PROOF. The (I, J)-minor is truly vanishing on C_w if and only if the dimension of the row span of every matrix $A \in C_w$ when restricted to rows in I and columns in J is at most $|I| - 1$. Similarly, the row span of every matrix $A \in C_w$ restricted to rows in $I \setminus \{h\}$ and columns in J has dimension at most $|I| - 2$ if and only if (1) holds. If (1) holds, we are done.

Assume (1) does not hold, so there exists a column $c \in J$ such that

$$\Delta_{I\setminus\{h\},J\setminus\{c\}}(A) \neq 0$$

for some matrix $A \in C_w$. The row span of A restricted to rows in $I \setminus \{h\}$ and columns in J has dimension at most $|I| - 1$. Call this subspace V_A. By hypothesis, $\Delta_{I,J}(A) = 0$. Hence, the row span of A restricted to rows in I and columns in J is also V_A. In particular, the row vector $(A(h, j) \,|\, j \in J)$ belongs to the subspace V_A.

For any $1 \leq a < h$, we can add any complex multiple of row a to row h in A to get another matrix $A' \in C_w$ via the lower triangular action while holding $\{A(i, j) \,|\, i \in I \setminus \{h\}, j \in J\}$ constant. Suppose there exists an $a \in [h] \setminus I$ such that $(A(a, j) \,|\, j \in J) \notin V_A$. Then $(A(h, j) + A(a, j) \,|\, j \in J) \notin V_A$, and so the corresponding A' would have a row span of dimension $|I|$ when restricted to rows in I and columns in J contradicting the hypothesis that (I, J) is truly vanishing on C_w. Therefore, $(A(a, j) \,|\, j \in J) \in V_A$ for all $a \in [h]$, and the row span of A

restricted to rows in $[h]$ and columns in J is V_A. Thus, $\Delta_{H,J}(A) = 0$ for all $H \subset [h]$ with $|H| = |I|$. By Lemma 4.1.6, we conclude that (2) holds. $\square$

We are now ready to prove Theorem 1.2.7. This gives us a characterization of PR varieties in terms of the truly vanishing northerly minors of the corresponding Fubini word.

Proof of Theorem 1.2.7. We know the truly vanishing northerly minors on C_w are a subset of all truly vanishing minors on C_w. Therefore,

$$\overline{C}_w \subset \{A \in \mathcal{M}_{k\times n}(\mathbb{C})/T \mid \Delta_{[h],J}(A) = 0 \,\forall\, J \in T_w, \text{ where } h = |J|\}.$$

Thus, we only need to prove the other direction.

It suffices to consider any matrix $A \in \mathcal{M}_{k\times n}(\mathbb{C})$ with the property that $\Delta_{[|J|],J}(A) = 0$ for all $J \in T_w$. Every truly vanishing minor on C_w involving only row 1, say $(I, J) = (\{1\}, \{j\})$, is also a northerly minor, so $\Delta_{I,J}(A) = 0$ by hypothesis. Assume by induction that for every truly vanishing minor (I', J') for w with $I' \subset [h-1]$, we have $\Delta_{I',J'}(A) = 0$, and consider a truly vanishing minor (I, J) for w with $\max(I) = h$. We need to show $\Delta_{I,J}(A) = 0$ as well in order to prove $A \in \overline{C}_w$.

Since the (I, J)-minor is truly vanishing on C_w and $\max(I) = h$, we know from Corollary 4.1.7 that either

1. the $(I \setminus \{h\}, J \setminus \{j\})$-minor is truly vanishing on C_w for every $j \in J$, or
2. every $K \subset [n]$ such that $J \subseteq K \subseteq [n]$ and $|K| = h$ is in T_w.

If Condition (1) holds, then $\Delta_{I,J}(A) = 0$ via cofactor expansion along row h. If Condition (2) holds, we claim the h-subcolumns indexed by J of A must be dependent, so $\Delta_{I,J}(A) = 0$. This claim holds since $A \in \mathcal{M}_{k\times n}(\mathbb{C})$, which implies the top h rows of A are independent as vectors in $\mathbb{C}^n$. Therefore, the h-subcolumns indexed by $[n]$ of A span $\mathbb{C}^h$. If the h-subcolumns indexed by J of A were independent, then they could be extended to a basis of $\mathbb{C}^h$ using some column set $K \supset J$ such that $|K| = h$. However, this would contradict Condition (2), since

$\Delta_{[h],K}(A)$ would have to be nonzero. We conclude that the h-subcolumns indexed by J of A are dependent, proving the claim. $\square$

Lemma 4.1.8. *Suppose $w \in \mathcal{W}_{n,k}$ is a Fubini word and $J \in \binom{[n]}{[k]}$. Let $h = |J|$. Then $J \in U_w$ if and only if $\Delta_{[h],J}(A) = \pm 1$ for every matrix $A \in C_w$.*

Proof. It is clear that if for every matrix $A \in C_w$ we have $\Delta_{[h],J}(A) = \pm 1$, then $J \in U_w$. Conversely, suppose $J \in U_w$, so $\Delta_{[h],J}(M_w) \neq 0$. Observe that minors of M_w can only have determinants -1, 0, or 1, so $\Delta_{[h],J}(M_w) = \pm 1$, and the $([h], J)$-minor of M_w is a permutation matrix. Thus the leftmost column of the $([h], J)$-minor of P_w has one 1, say in position (w_j, j).

Now suppose w is convex. For a such that $w_a \neq w_j$ and $w_a \in [h]$, if position (w_a, j) is a $\star$ in P_w, then w_a occurs to the left of w_j in the Fubini word w. Since w is convex, then all the occurrences of w_a appear to the left of all the occurrences of w_j in w. Thus, the w_a row in the $([h], J)$-minor of M_w is all 0's, which contradicts that $\Delta_{[h],J}(M_w) \neq 0$. Thus, the other entries in the leftmost column of the $([h], J)$-minor of P_w are all 0's. Expanding the determinant along that column yields $\Delta_{[h],J}(P_w) = \pm\Delta_{[h]\setminus\{w_j\},J\setminus\{j\}}(P_w)$. Now by the same argument, the leftmost column of the $([h] \setminus \{w_j\}, J \setminus \{j\})$-minor of P_w has one 1 and the rest 0's. Continuing in this fashion, we conclude by induction that $\Delta_{[h],J}(P_w) = \pm 1$. Multiplication by a unipotent matrix does not change the determinant, so $\Delta_{[h],J}(A) = \pm 1$ for every $A \in C_w$.

Now, let $w \in \mathcal{W}_{n,k}$ be any nonconvex word. By Lemma 3.1.2, the minimal length permutation x such that $w = \operatorname{conv}(w)x$ gives rise to a bijection from the unvanishing minors on $C_{\operatorname{conv}(w)}$ to the unvanishing minors on C_w. The bijection on minors preserves determinants. Hence, $\Delta_{[h],J}(A) = \pm 1$ for every $A \in C_w$ also holds when w is nonconvex. $\square$

Corollary 4.1.9. *Suppose $w \in \mathcal{W}_{n,k}$ is a Fubini word and $J \in \binom{[n]}{[k]}$. Let $h = |J|$. Then $J \in U_w$ if and only if $\Delta_{[h],J}(M_w) \neq 0$.*

PROOF. First suppose $J \in U_w$. Then by Lemma 4.1.8, $\Delta_{[h],J}(M_w) = \pm 1$, so we are done.

Conversely, if $\Delta_{[h],J}(M_w) \neq 0$, then $\Delta_{[h],J}(M_w) = \pm 1$, so the minor $([h], J)$ of M_w is a permutation matrix. Suppose w is convex. Then in the $([h], J)$-minor of P_w, the $\star$'s can only appear to the right of the 1's, which means they can be eliminated via columns operations, by subtracting multiples of columns from columns to their right, starting with the leftmost column. Determinants are unchanged under such column operations, so $\Delta_{[h],J}(P_w) = \Delta_{[h],J}(M_w) \neq 0$. Multiplication by a unipotent matrix also does not change the determinant, so $\Delta_{[h],J}(A) \neq 0$ for every $A \in C_w$. Thus, $J \in U_w$.

Now let $w \in \mathcal{W}_{n,k}$ be any nonconvex word. By Lemma 3.1.2, the minimal length permutation x such that $w = \text{conv}(w)x$ gives rise to a bijection from the unvanishing minors on $C_{\text{conv}(w)}$ to the unvanishing minors on C_w. Hence, $\Delta_{[h],J}(A) \neq 0$ for every $A \in C_w$ also holds when w is nonconvex. Thus, $J \in U_w$ also when w is nonconvex. □

COROLLARY 4.1.10. *Suppose* $w \in \mathcal{W}_{n,k}$ *is a Fubini word and* $J \in \binom{[n]}{[k]}$. *Let* $h = |J|$. *Then if* $J \in S_w$, $\Delta_{[h],J}(M_w) = 0$.

We are now ready to prove Theorem 1.2.8. This theorem will allow us to only work with northerly minors when dealing with medium roast and espresso Fubini-Bruhat orders, instead of calculating determinants of all possible minors on C_w.

PROOF OF THEOREM 1.2.8. First we prove (1). The two statements are equivalent by set theory and the fact that all northerly minors of C_w are a disjoint union of S_w, T_w, and U_w. We will prove that $v \leq w$ if and only if $T_v \subseteq T_w$. By Corollary 3.1.9 (1), $v \leq w$ in medium roast Fubini-Bruhat order implies $T_v \subseteq T_w$. We just need to prove the other direction. Suppose $T_v \subseteq T_w$. We will show that all truly vanishing minors on C_v are also truly vanishing minors on C_w, and we will then have $v \leq w$ by Corollary 3.1.9 (1). Suppose $\Delta_{I,J} = 0$ on C_v for some $I \subset [k]$ and

$J \subset [n]$ of the same size. Let $h = \max(I)$. We proceed by induction on the size of I.

If $|I| = 1$, then $I = \{h\}$. Say $J = \{j\}$. Since the minor $(\{h\}, \{j\})$ is truly vanishing on C_v, we know $(\{i\}, \{j\})$ is also truly vanishing on C_v for all i such that $1 \leq i \leq h$ by Lemma 4.1.5. Therefore, for every matrix $A \in C_v$, we have $A(i, j) = 0$ if $1 \leq i \leq h$, which implies every northerly minor indexed by K such that $j \in K$ and $|K| \leq h$ is truly vanishing on C_v. Since $T_v \subset T_w$, this implies that every northerly minor indexed by K such that $j \in K$ and $|K| \leq h$ is also truly vanishing on C_w. Hence, by Lemma 4.1.6, the minor $(\{i\}, \{j\})$ is truly vanishing on C_w for every $1 \leq i \leq h$. In particular, (I, J) is truly vanishing on C_w.

If $|I| = |J| > 1$, then assume (I', J') being truly vanishing on C_v implies that (I', J') is truly vanishing on C_w for all subsets $I' \subseteq [k]$ and $J' \in \binom{[n]}{[k]}$ of size $|I'| = |J'|$ strictly smaller than $|I|$. Since (I, J) is truly vanishing on C_v, we know by Corollary 4.1.7 that one of the following two conditions hold. In the first case, every minor of the form $(I \setminus \{h\}, J \setminus \{j\})$ is truly vanishing on C_v. In the second case, for all subsets K such that $J \subset K \subset [n]$ and $|K| = h$, $K \in T_v$. In the first case, every $(I \setminus \{h\}, J \setminus \{j\})$ is truly vanishing on C_w by induction, so (I, J) is truly vanishing on C_w by cofactor expansion along row h. In the second case, for all subsets K such that $J \subset K \subset [n]$ and $|K| = h$, $K \in T_w$ since $T_v \subseteq T_w$. Thus, (I, J) is truly vanishing on C_w by Lemma 4.1.6.

Now we prove (2). The three statements are equivalent by set theory and the fact that all northerly minors of C_w are a disjoint union of S_w, T_w, and U_w. So we only need to prove each direction for one of the three statements. Recall also that $v \rightharpoonup w$ if and only if $\overline{C}_v \cap C_w \neq \emptyset$.

First suppose $T_v \subseteq S_w \cup T_w$. Since $M_w \in C_w$, then $S_w \cup T_w$ are the vanishing northerly minors for M_w by Corollaries 4.1.9 and 4.1.10. Thus, T_v are all vanishing northerly minors for M_w. Thus, by Theorem 1.2.7, $M_w \in \overline{C}_v$. So $\overline{C}_v \cap C_w \neq \emptyset$.

Conversely, suppose $\overline{C}_v \cap C_w \neq \emptyset$. Then we may choose a matrix $A \in \overline{C}_v \cap C_w$. Suppose $J \in T_v \cap U_w$, and let $h = |J|$. Then we have both $\Delta_{[h],J}(A) = 0$ and $\Delta_{[h],J}(A) \neq 0$, a contradiction. Thus, $\overline{C}_v \cap C_w \neq \emptyset$ implies $T_v \cap U_v = \emptyset$. $\square$

Corollary 4.1.11. *Suppose $v, w \in \mathcal{W}_{n,k}$ are Fubini words. Then $v \rightharpoonup w$ if and only if $M_w \in \overline{C}_v$.*

Proof. First, suppose $v \rightharpoonup w$. By the proof of Theorem 1.2.8 (2), $M_w \in \overline{C}_v \cap C_w$, so $M_w \in \overline{C}_v$.

Conversely, suppose $M_w \in \overline{C}_v$. By definition of C_w, $M_w \in C_w$. Thus, $M_w \in \overline{C}_v \cap C_w$, which means $\overline{C}_v \cap C_w \neq \emptyset$. Thus, by definition of $\rightharpoonup$, $v \rightharpoonup w$. $\square$

2. Combinatorial Test for Fubini-Bruhat Orders

Theorem 1.2.8 finally allows us to prove Theorem 1.2.18, a test for determining whether the northerly minor indexed by $J \subseteq [n]$ is in S_w, T_w, or U_w based on combinatorial properties of the Fubini word $w \in \mathcal{W}_{n,k}$. This leads to our first combinatorial tests for the medium roast and espresso Fubini-Bruhat orders, Theorems 1.2.19 and 1.2.20, which we also prove in this section.

Proof of Theorem 1.2.18. Part (3) follows from Corollary 4.1.9, since $J \in U_w$ if and only if the northerly minor of M_w indexed by columns J is a permutation matrix. We will prove Part (2) below. Part (1) will then follow from the fact that S_w, T_w, and U_w are a disjoint union of all possible northerly minors on $k \times n$ matrices.

To see Part (2), let $A = \{a_1 < \cdots < a_h\}$ be defined such that $\{a_1, \ldots, a_h\} = \{\alpha_1(w), \ldots, \alpha_h(w)\}$, and let $B = \{b_1 \leq \cdots \leq b_h\}$ be defined such that $\{b_1, \ldots, b_h\} = \alpha_J(w)$. Suppose $A \trianglelefteq B$. Since $a_g \leq b_g$ for every $1 \leq g \leq h$, there exists a bijective map $f : J \to [h]$ such that

(1) if $j \in J \cap A$, then $f(j) = w(j)$, and
(2) if $j \in J \setminus A$, then $\alpha_{f(j)} \leq \alpha_{w(j)}$.

If $j \in J \cap A$, then j is an initial position for w, and the entry $(f(j), j)$ is 1 in the pattern matrix P_w. On the other hand, if $j \in J \setminus A$, either $w(j) > h$ or j indexes a redundant letter in w. Either way, for $j \in J \setminus A$, the column indexed by j in P_w contains a 1 or a $\star$ in every row indexed by $g \in [h]$ such that $\alpha_g \leq \alpha_{w(j)}$, by definition of the pattern matrix. Therefore, for each $j \in J$, the entry in P_w in position $(f(j), j)$ is either a 1 or a $\star$ by property (2) of the bijection. We can obtain a matrix $M \in C_w$ from the pattern matrix P_w by setting $M(f(j), j) = 1$ for each $j \in J$ and setting every other $\star$ in P_w to be 0.

We claim $\Delta_{[h],J}(M) \neq 0$. To see this, observe that by definition of the bijection f, every row and every column of the submatrix of M restricted to positions in $[h] \times J$ has at least one 1. The columns indexed by $A \cap J$ have exactly one 1. Use these columns to clear out additional 1's to their right by Gaussian elimination on columns. The resulting matrix M' when restricted to positions in $[h] \times J$ is a permutation matrix whose determinant is ± 1. Furthermore, $\Delta_{[h],J}(M) = \Delta_{[h],J}(M') = \pm 1$, which means $J \notin T_w$.

Conversely, consider when $A \ntrianglelefteq B$. There must exist a c, $1 \leq c \leq h$, such that $a_c > b_c \geq \cdots \geq b_1$. Then for $j \in J$ such that $\alpha_{w(j)} \in \{b_1, \ldots, b_c\}$, there will be 0's in position (i, j) of P_w whenever $\alpha_i > \alpha_{w(j)}$. In particular, $P_w(i, j) = 0$ for every j such that $\alpha_{w(j)} \in \{b_1, \ldots, b_c\}$ and every i such that $\alpha_i \in \{a_c, a_{c+1}, \ldots, a_h\}$. Let $J_c = \{j \,|\, j \in J, \alpha_{w(j)} \leq b_c\}$, and let L be any matrix fitting the pattern of w. Then the submatrix of L with rows indexed by $[h]$ and columns indexed by J_c will have all-zero rows in row positions $\{i \,|\, \alpha_i \in \{a_c, a_{c+1}, \ldots, a_h\}\}$. That is, there will be $h - c + 1$ all-zero rows. So the h-subcolumns indexed by J_c of L have rank at most $c - 1$. Since adding a new column adds at most 1 to the rank of the column set, the h-subcolumns indexed by J of L have rank at most $h - 1$. Therefore, $\Delta_{[h],J}(A) = 0$ for any L fitting the pattern of w. Since a left lower triangular action does not change the vanishing of northerly minors, $\Delta_{[h],J}(L) = 0$ for every $L \in C_w$. Thus, $J \in T_w$.

□

We are now ready to prove Theorems 1.2.19 and 1.2.20. They follow directly from Theorems 1.2.8 and 1.2.18.

PROOF OF THEOREM 1.2.19. By Theorem 1.2.8, $v \rightharpoonup w$ if and only if $U_w \subset (S_v \cup U_v)$. The unvanishing and sometimes vanishing northerly minors on C_v and C_w are completely determined by $\trianglelefteq$ order using the inequalities (1.2.2) and (1.2.3) by Theorem 1.2.18. Thus, the condition on the α sets stated in the theorem is equivalent to testing $U_w \subset (S_v \cup U_v)$.

To prove the in particular statement, suppose $J = \{\alpha_1(w), \ldots, \alpha_h(w)\}$. Then the minor $[h], J$ of M_w is a permutation matrix, so by Corollary 4.1.9, $J \in U_w$. If $v \rightharpoonup w$, then $J \in S_v \cup U_v$ by Theorem 1.2.8, which implies $\{\alpha_1(v), \ldots, \alpha_h(v)\} \trianglelefteq \alpha_J(v)$ by Theorem 1.2.18. Observe that $\alpha_{v(j)}(v) \leq j$ for every j such that $1 \leq j \leq n$, so $\alpha_J(v) \trianglelefteq J$. Thus,

$$\{\alpha_1(v), \ldots, \alpha_h(v)\} \trianglelefteq \alpha_J(v) \trianglelefteq J = \{\alpha_1(w), \ldots, \alpha_h(w)\}.$$

□

PROOF OF THEOREM 1.2.20. By Theorem 1.2.8, $v \leq w$ if and only if $T_v \subseteq T_w$, which is set theoretically equivalent to $(S_w \cup U_w) \subseteq (S_v \cup U_v)$. The unvanishing and sometimes vanishing northerly minors on C_v and C_w are completely determined by $\trianglelefteq$ order using the inequalities (1.2.4) and (1.2.5) by Theorem 1.2.18. Thus, the condition on the α sets stated in the theorem is equivalent to testing $(S_w \cup U_w) \subset (S_v \cup U_v)$.

To see the in particular statement, observe that $v \leq w$ implies $v \rightharpoonup w$. So this follows from the in particular statement of Theorem 1.2.19. □

Observe that the converses of the in particular statements of Theorems 1.2.19 and 1.2.20 do not hold. For example, consider Fubini words $3213, 3231 \in \mathcal{W}_{4,3}$. Then $\alpha_1(3213) = 3$, $\alpha_2(3213) = 2$, $\alpha_3(3213) = 1$, $\alpha_1(3231) = 4$, $\alpha_2(3231) = 2$, and $\alpha_3(3231) = 1$, so we can check that

$$\{\alpha_1(3213), \ldots, \alpha_h(3213)\} \triangleleft \{\alpha_1(3231), \ldots, \alpha_h(3231)\}$$

for $h = 1, 2, 3$. However, it is not true that $3213 \leq 3231$ in medium roast Fubini-Bruhat order, as they are both Fubini words of dimension 0, and thus maximal elements in the medium roast Fubini-Bruhat poset on $\mathcal{W}_{4,3}$.

Corollary 4.2.1. *Given Fubini word $w \in \mathcal{W}_{n,k}$, for each $1 \leq h \leq k$, the set*

$$\{\alpha_1, \alpha_2, \dots, \alpha_h\}$$

is the lexicographically smallest set in $S_w \cup U_w$.

Proof. Observe that for every $J \subseteq [n]$ such that $|J| = h$, $\alpha_J(w) \trianglelefteq J$. If $J <_L \{\alpha_1, \alpha_2, \dots, \alpha_h\}$, then $\{\alpha_1, \dots, \alpha_h\} \ntrianglelefteq J$. Thus, $\{\alpha_1, \dots, \alpha_h\} \ntrianglelefteq \alpha_J(w)$. So by Theorem 1.2.18, $J \in T_w$. Furthermore, observe that $\{\alpha_1, \dots, \alpha_h\} \trianglelefteq \{\alpha_1, \dots, \alpha_h\}$, so $\{\alpha_1, \dots, \alpha_h\} \in S_w \cup U_w$. □

While the converses of the in particular statements of Theorems 1.2.19 and 1.2.20 do not hold, adding another condition allows us to form sufficient conditions for medium roast Fubini-Bruhat order. Recall that Fubini words $\mathcal{W}_{n,k}$ are in bijection with ordered set partitions $\mathcal{OP}_{n,k}$.

Definition 4.2.2. *For $\sigma = (A_1|\cdots|A_k) \in \mathcal{OP}_{n,k}$, define its **sort** to be $sort(\sigma) \in \mathcal{OP}_{n,k}$ where the blocks are arranged in order of minimum element. For example, $sort(258|136|47) = (136|258|47)$. For a Fubini word $w \in \mathcal{W}_{n,k}$, define its **sorted word** to be $s(w) = w(sort(\sigma(w)))$.*

For example, $s(21231231) = 12132132$. Observe that $\pi(s(w)) = \mathrm{id}$ for every $w \in \mathcal{W}_{n,k}$, and $s(w)_i = s(w)_j$ if and only if $w_i = w_j$.

Corollary 4.2.3. *If the following two conditions are true, then $v \leq w$ in medium roast Fubini-Bruhat order.*

1. *For all $j \in [n]$, $s(v)_j \geq s(w)_j$.*
2. *For all $h \in [k]$, $\{\alpha_1(v), ..., \alpha_h(v)\} \trianglelefteq \{\alpha_1(w), ..., \alpha_h(w)\}$.*

PROOF. Suppose $J \subset [n]$ has size $h \in [k]$ and $J \notin T_w$. By Theorem 1.2.18, we have

$$\{\alpha_1(w), \ldots, \alpha_m(w)\} \trianglelefteq \alpha_J(w).$$

Condition (1) says that

$$\alpha_{v(j)}(v) \geq \alpha_{w(j)}(w)$$

for all $j \in J$, so

$$\alpha_J(w) \trianglelefteq \alpha_J(v).$$

Applying Condition (2) also, we get

$$\{\alpha_1(v), ..., \alpha_h(v)\} \trianglelefteq \{\alpha_1(w), ..., \alpha_h(w)\} \trianglelefteq \alpha_J(w) \trianglelefteq \alpha_J(v).$$

Therefore, by Theorem 1.2.18, $J \notin T_v$, so $T_v \subset T_w$. Then by Theorem 1.2.8, $v \leq w$. $\square$

CHAPTER 5

Monotone Triangles and the Essential Set

Monotone triangles, Bruhat order, and Fulton's essential set are related concepts from permutations that can be extended to Fubini words. Some results from Bruhat order cannot be generalized to any of the Fubini-Bruhat orders. For example, the join-irreducibles in any of the three Fubini-Bruhat orders do not coincide with the set of Fubini words whose essential sets have exactly one element. However, both monotone triangles and the essential set give new tests for the medium roast Fubini-Bruhat order.

1. Alpha and Beta Triangles

An extension of the $\trianglelefteq$ order on multisets and Ehresmann's criterion for Bruhat order on permutations naturally lead to the construction of monotone triangles associated with permutations. Since Fubini words have redundant letters, two triangles are required to fully describe each Fubini word. We call them the α **triangle** and the β **triangle**. The construction of these triangles leads to another combinatorial test for T_w given a Fubini word $w \in \mathcal{W}_{n,k}$ in Theorem 5.1.7, and a necessary condition for medium roast Fubini-Bruhat order in Theorem 5.1.11.

Definition 5.1.1. *If $A = \{a_1 \leq \cdots \leq a_m\}$ and $B = \{b_1 \leq \cdots \leq b_l\}$ are multisets of not necessarily the same size with their elements listed in increasing order, we say A is* ***dominated by*** *B, denoted $A \trianglelefteq B$, if and only if $|B| = l \leq m = |A|$ and*

$$a_1 \leq b_1,\ a_2 \leq b_2,\ \ldots,\ a_l \leq b_l.$$

Observe that this definition can be thought of as equivalent to Definition 1.2.9 if we think of padding B with infinities, so that $B = \{b_1 \leq \cdots \leq b_l, \infty, \ldots, \infty\}$, and $|B| = m = |A|$. Thus, we can check the conditions of Definition 1.2.9 via the comparisons

$$a_1 \leq b_1,\ a_2 \leq b_2,\ \ldots,\ a_l \leq b_l, a_{l+1} < \infty, \ldots, a_m < \infty.$$

LEMMA 5.1.2. *Let*

$$A = \{a_1 < \cdots < a_t\}$$

and

$$B = \{b_1 < \cdots < b_s\}$$

be subsets of $[n]$. *Then the following are equivalent:*

(1) $A \trianglelefteq B$, *and*

(2) $|[j] \cap A| \geq |[j] \cap B|$ *for* $j = 1, \ldots, n$.

PROOF. To see that (1) implies (2), recall from Definition 5.1.1 that (1) is equivalent to $s \leq t$ and $a_i \leq b_i$ for $i \in [s]$. Fix some $j \in [n]$. Then $b_1 < b_2 < \cdots < b_m \leq j < b_{m+1} < \cdots < b_s$ for some $m \in [t]$, so $|[j] \cap B| = m$. We have $a_m \leq b_m$, so $|[j] \cap A| \geq m$.

To see that (2) $\implies$ (1), observe that (2) implies that $|[a_i] \cap A| \geq |[a_i] \cap B|$ for $i \in [s]$. Also, $|[a_i] \cap A| = i$, so $|[a_i] \cap B| \leq i$. Thus, $a_i \leq b_i$ for $i \in [s]$. Furthermore, $|[n] \cap A| = |A|$ and $|[n] \cap B| = |B|$, so $t \geq s$. By Definition 5.1.1, $A \trianglelefteq B$. ☐

DEFINITION 5.1.3. *The* α ***triangle for*** $w \in \mathcal{W}_{n,k}$ *is a* $\binom{k+1}{2}$*-tuple* $(\alpha_{ij}(w) \mid 1 \leq j \leq i \leq k)$, *where we may drop the* (w) *when it is clear from context, given by*

$$\{\alpha_1(w), \ldots, \alpha_i(w)\} = \{\alpha_{i1}(w) < \alpha_{i2}(w) < \cdots < \alpha_{ii}(w)\}.$$

Thus, $1 \leq \alpha_{ij} \leq n$, $\alpha_{ij} < \alpha_{i,j+1}$, $\alpha_{ij} \geq \alpha_{i+1,j}$, *and* $\alpha_{ij} \leq \alpha_{i+1,j+1}$.

DEFINITION 5.1.4. *For* $w \in \mathcal{W}_{n,k}$, *suppose* $\sigma(w) = (A_1|A_2|\cdots|A_k)$ *is the ordered set partition corresponding to* w. *Consider the associated ordered set partition*

$$(B_1(w)|B_2(w)|\cdots|B_k(w))$$

$$\begin{array}{cccc} & & & 3 \\ & & 3 & 5 \\ & 2 & 3 & 5 \\ 1 & 2 & 3 & 5 \end{array}$$

Figure 5.1.1. α triangle for 431324

such that $\{A_1, \dots, A_k\} = \{B_1, \dots, B_k\}$ *and*

$$\mathit{min}\ B_1 < \mathit{min}\ B_2 < \cdots < \mathit{min}\ B_k.$$

We may drop the (w) *when it is clear from context. Now let* $\beta_i(w) = \sqcup_{h=1}^{i} B_i(w)$, *with the convention that* $\beta_0(w) = \emptyset$ *for every Fubini word* w. *Then the* i*th row of the* β ***triangle for*** w *is composed of the elements of* β_i *written in ascending order. Thus,* $\beta_{ij} < \beta_{i,j+1}$, $\beta_{ij} \geq \beta_{i+1,j}$, *and* $\beta_{ij} \leq \beta_{i+1,j+1}$.

Observe that the length of the ith row of the β triangle for w is given by $|\beta_i(w)|$ for all $i \in [k]$. Thus, the β triangle for w is an m-tuple, where $m = \sum_{i=1}^{k} |\beta_i(w)|$, organized into k rows. The exact size of m depends on w, but $\frac{k(k+1)}{2} + n - k \leq m \leq \frac{(2n-k+1)k}{2}$. Observe that the last row of the α triangle for w always has entries $\{\alpha_1(w), \alpha_2(w), \dots, \alpha_k(w)\}$ appearing in increasing order from left to right. Observe that the last row of the β triangle for w always has entries $1, 2, \dots, n$.

The sets $B_i(w)$ and $\beta_i(w)$ can be alternatively defined by $s(w)$, as in Definition 4.2.2. Observe that $(B_1(w)|B_2(w)|\cdots|B_k(w)) = \sigma(s(w))$. Then

$$B_i(w) = \{j \in [n] \mid s(w)_j = i\}, \text{ and } \beta_i(w) = \{j \in [n] \mid s(w)_j \leq i\}. \tag{5.1.22}$$

For example, if $w = 431324 \in \mathcal{W}_{6,4}$, then $(\alpha_1(w), \alpha_2(w), \alpha_3(w), \alpha_4(w)) = (3, 5, 2, 1)$. Also, $s(w) = 123241$, so $\sigma(s(w)) = (16|24|3|5)$. Thus, the α triangle for w is shown in Figure 5.1.1. The β triangle for w is shown in Figure 5.1.2.

Lemma 5.1.5. *Suppose* $w \in \mathcal{W}_{n,k}$ *is a Fubini word and* $J \subseteq [n]$ *with* $|J| = h \in [k]$. *Then* $J \in T_w$ *if and only if there exists a* $j \in [h]$ *such that*

$$j \leq |[\alpha_{hj} - 1] \cap \alpha_J(w)|.$$

$$\begin{array}{cccccc}
 & & & & 1 & 6 \\
 & & 1 & 2 & 4 & 6 \\
 & 1 & 2 & 3 & 4 & 6 \\
1 & 2 & 3 & 4 & 5 & 6
\end{array}$$

FIGURE 5.1.2. β triangle for 431324

PROOF. Suppose $J \in T_w$. By Theorem 1.2.18, $\{\alpha_1, \ldots, \alpha_h\} \ntrianglelefteq \alpha_J(w)$, where $|J| = h$. That is, $\{\alpha_{h1} < \cdots < \alpha_{hh}\} \ntrianglelefteq \alpha_J(w)$. By Lemma 5.1.2, there exists an $a \in [n]$ such that

$$|[a] \cap \{\alpha_{h1} < \cdots < \alpha_{hh}\}| < |[a] \cap \alpha_J(w)|. \tag{5.1.23}$$

Now suppose

$$|[\alpha_{hj} - 1] \cap \{\alpha_{h1} < \cdots < \alpha_{hh}\}| \geq |[\alpha_{hj} - 1] \cap \alpha_J(w)|$$

for every $j \in [h]$. Now if we take the convention that $\alpha_{h0} = 0$, then for any $a \in [n]$, we can choose $j \in [h]$ such that $\alpha_{h0} < \alpha_{h1} < \cdots < \alpha_{h,j-1} \leq a < \alpha_{hj} < \cdots < \alpha_{hh}$. However, this gives us

$$\begin{aligned}
|[a] \cap \{\alpha_{h1} < \cdots < \alpha_{hh}\}| &= |[\alpha_{hj} - 1] \cap \{\alpha_{h1} < \cdots < \alpha_{hh}\}| \\
&\geq |[\alpha_{hj} - 1] \cap \alpha_J(w)| \\
&\geq |[a] \cap \alpha_J(w)|,
\end{aligned}$$

which contradicts Equation (5.1.23). Thus, our supposition was incorrect, and there exists a $j \in [h]$ such that

$$j \leq |[\alpha_{hj} - 1] \cap \alpha_J(w)|.$$

Conversely, suppose for every $j \in [h]$,

$$j > |[\alpha_{hj} - 1] \cap \alpha_J(w)|.$$

Then for any $a \in [n]$, we can choose $j \in [h]$ such that $\alpha_{h0} < \alpha_{h1} < \cdots < \alpha_{h,j-1} \leq a < \alpha_{hj} < \cdots < \alpha_{hh}$, again using the convention that $\alpha_{h0} = 0$. Then

$$\begin{aligned} |[a] \cap \{\alpha_{h1} < \cdots < \alpha_{hh}\}| &= |[\alpha_{hj} - 1] \cap \{\alpha_{h1} < \cdots < \alpha_{hh}\}| \\ &= j \\ &> |[\alpha_{hj} - 1] \cap \alpha_J(w)| \\ &\geq |[a] \cap \alpha_J(w)|. \end{aligned}$$

So by Lemma 5.1.2, $\{\alpha_{h1} < \cdots < \alpha_{hh}\} \trianglelefteq \alpha_J(w)$, which is equivalent to $\{\alpha_1, \ldots, \alpha_h\} \trianglelefteq \alpha_J(w)$. Thus, by Theorem 1.2.18, $J \notin T_w$. □

LEMMA 5.1.6. *Suppose $w \in \mathcal{W}_{n,k}$ is a Fubini word and $J \subseteq [n]$. Then for any $l \in [k]$,*

$$|[\alpha_{kl} - 1] \cap \alpha_J(w)| = |J \cap \beta_{l-1}(w)|.$$

PROOF. The left-hand side counts the number of $j \in J$ such that $\alpha_{w(j)}(w)$ appears strictly to the left of α_{kl}. If this holds for some $j \in J$, then $j \in \beta_{l-1}(w)$. The $j \in J$ with this property is what is counted by the right-hand side. □

THEOREM 5.1.7. *Suppose $w \in \mathcal{W}_{n,k}$ is a Fubini word, and $J \subseteq [n]$ with $|J| = h \in [k]$. Then $J \in T_w$ if and only if there exists a $j \in [h]$ such that $|J \cap \beta_{l-1}(w)| \geq j$, where l is determined by $\alpha_{hj}(w) = \alpha_{kl}(w)$.*

PROOF. Suppose $J \in T_w$. Then by Lemma 5.1.5, there exists a $j \in [h]$ such that

$$j \leq |[\alpha_{hj} - 1] \cap \alpha_J(w)|.$$

Pick $l \in [k]$ such that $\alpha_{hj} = \alpha_{kl}$. Thus, we have

$$j \leq |[\alpha_{kl} - 1] \cap \alpha_J(w)| = |J \cap \beta_{l-1}(w)|$$

by Lemma 5.1.6, or equivalently,

$$j \leq |J \cap \beta_{l-1}(w)|.$$

Conversely, suppose for every $j \in [h]$, $|J \cap \beta_{l-1}(w)| < j$, where l is determined by $\alpha_{hj}(w) = \alpha_{kl}(w)$. Then

$$|[\alpha_{kl} - 1] \cap \alpha_J(w)| = |J \cap \beta_{l-1}(w)| < j,$$

so

$$|[\alpha_{hj} - 1] \cap \alpha_J(w)| < j.$$

Then by Lemma 5.1.5, $J \notin T_w$. $\square$

We extend some of the standard vocabulary from linear algebra to subsets of columns in matrices in a PR cell. This helps us state a necessary condition for the medium roast Fubini-Bruhat order in terms of the sets B_i.

Definition 5.1.8. *For Fubini word $w \in \mathcal{W}_{n,k}$, the set $J \subseteq [n]$ is* ***dependent for*** *w if and only if for every $I \subseteq [k]$ such that $|J| = |I|$, the (I, J)-minor is truly vanishing on C_w. If there exists an (I, J)-minor with $|I| = |J|$ that is not truly vanishing on C_w, then J is* ***independent for*** *w.*

Lemma 5.1.9. *Suppose $w \in \mathcal{W}_{n,k}$ is a Fubini word and $J \subseteq [n]$. Then J is dependent for w if and only if there exists an $i \in [k]$ such that $|J \cap \beta_i(w)| > i$. In particular, every set $J \subseteq \beta_i(w)$ such that $|J| > i$ is dependent for w.*

Proof. Suppose there exists an $i \in [k]$ such that $|J \cap \beta_i(w)| > i$. The only nonzero entries in P_w in columns in $J \cap \beta_i(w)$ are in rows $\pi_1(w), \ldots, \pi_i(w)$. So P_w restricted to columns $J \cap \beta_i(w)$ has rank i. Applying a lower triangular action does not change the rank of a subset of columns. Therefore, any matrix A in C_w restricted to columns $J \cap \beta_i(w)$ has rank i. Adding each column in $J \setminus \beta_i(w)$ can increase the rank by at most 1. Thus, any matrix A in C_w restricted to columns J has rank at most $i + |J| - |J \cap \beta_i(w)| < |J|$, since $|J \cap \beta_i(w)| > i$. So by Lemma 4.1.6, for every $I \subset [k]$ such that $|J| = |I|$, the (I, J)-minor is truly vanishing on C_w. Thus, J is dependent for w.

Conversely, suppose for every $i \in [k]$, $|J \cap \beta_i(w)| \leq i$. Then $|J| \leq k$, since $\beta_k(w) = [n]$. Also observe that if $|J \cap B_i(w)| = 0$, then $|J \cap \beta_i(w)| = |J \cap \beta_{i-1}(w)| \leq i - 1$.

Construct K as follows. Let $J_0 = J$. For $l = k, \ldots, 1$, if $|J \cap B_l(w)| = 0$ and $|J_{k-l}| < k$, let $J_{k-l+1} = J_{k-l} \cup \{\alpha_{\pi_l(w)}\}$. Otherwise, let $J_{k-l+1} = J_{k-l}$. Observe that at each step, $|J_l \cap \beta_i(w)| \leq i$ is preserved for every $i \in [k]$. Finally, let $K = J_k$. Then $|K \cap \beta_i(w)| \leq i$ for every $i \in [k]$. Therefore, $\{\alpha_1, \ldots, \alpha_k\} \trianglelefteq K$, and by Theorem 1.2.18, $K \notin T_w$. So by Lemma 4.1.6, J is independent for w. □

LEMMA 5.1.10. *Suppose $v, w \in \mathcal{W}_{n,k}$ are Fubini words, and $v \leq w$ in medium roast Fubini-Bruhat order. If J is dependent for v, then J is dependent for w.*

PROOF. By definition of dependence, for every subset $I \subseteq [k]$ such that $|J| = |I|$, the (I, J)-minor is truly vanishing on C_v. By Corollary 3.1.9 (1), the (I, J)-minor also vanishes on C_w. Thus, J is dependent for w. □

THEOREM 5.1.11. *Suppose $v, w \in \mathcal{W}_{n,k}$ are Fubini words with $v \leq w$ in medium roast Fubini-Bruhat order. Then $\beta_i(w) \trianglelefteq \beta_i(v)$ for $i = 1, \ldots, k$.*

PROOF. Observe that $\beta_k(v) = \beta_k(w) = [n]$, so the statement is trivially true when $i = k$. Suppose for $i < k$ that $\beta_i(w) \ntrianglelefteq \beta_i(v)$. Then by Lemma 5.1.2, there exists a $j \in [n]$ such that

$$|[j] \cap \beta_i(w)| < |[j] \cap \beta_i(v)|. \tag{5.1.24}$$

If $j \notin \beta_i(v)$, then $|[j-1] \cap \beta_i(w)| \leq |[j] \cap \beta_i(w)| < |[j] \cap \beta_i(v)| = |[j-1] \cap \beta_i(v)|$. So we can decrease j until $j \in \beta_i(v)$. If $j \in \beta_i(v)$ and $j \in \beta_i(w)$, then we can remove j from the sets on both sides of Equation (5.1.24) to get $|[j-1] \cap \beta_i(w)| < |[j-1] \cap \beta_i(v)|$. So we can decrease j until $j \in \beta_i(v)$ and $j \notin \beta_i(w)$. Thus, $|[j-1] \cap \beta_i(w)| \leq |[j-1] \cap \beta_i(v)|$, and

$$|[j-1] \cap \beta_i(w) \setminus \beta_i(v)| \leq |[j-1] \cap \beta_i(v) \setminus \beta_i(w)|. \tag{5.1.25}$$

If j is an initial position for v, then since $j \in \beta_i(v)$, we must have $j = \alpha_{\pi_a(v)}(v)$ for some $a \in [i]$. Then by the in particular statement of Theorem 1.2.20, $\alpha_{\pi_a(w)}(w) \geq \alpha_{\pi_a(v)}(v) = j$. Therefore, $[j] \subseteq \beta_i(w)$. Thus $|[j] \cap \beta_i(w)| = j \geq |[j] \cap \beta_i(v)|$, which contradicts Equation (5.1.24).

So j must be a redundant position for v. Since $j \notin \beta_i(w)$,

$$\{\alpha_{\pi_1(w)}(w), \ldots, \alpha_{\pi_i(w)}(w)\} \subseteq [j-1].$$

Let

$$A = \{\alpha_{\pi_1(w)}(w), \ldots, \alpha_{\pi_i(w)}(w)\} \cap \beta_i(v)$$

and let

$$B = \{\alpha_{\pi_1(w)}(w), \ldots, \alpha_{\pi_i(w)}(w)\} \setminus \beta_i(v) \subseteq [j-1] \cap \beta_i(w) \setminus \beta_i(v).$$

Then by Equation (5.1.25), there exists a set $C \subseteq [j-1] \cap \beta_i(v) \setminus \beta_i(w)$ such that $|B| = |C|$. Observe that $C \cap \beta_i(w) = \emptyset$, so $A \cap C = \emptyset$. Thus, $|A \cup C| = |A| + |C| = |A| + |B| = i$. Now let $K = A \sqcup C \sqcup \{j\}$. Then $K \subseteq \beta_i(v)$ and $|K| = i+1$, so by Lemma 5.1.9, K is dependent for v. Now for $c \leq i$, by construction, $|K \cap \beta_c(w)| = c$. For $c > i$, $|K \cap \beta_c(w)| \leq |K| = i+1 \leq c$. Therefore, by Lemma 5.1.9, K is independent for w. However, this means the set K provides a contradiction to Lemma 5.1.10. Thus K cannot exist, which means our supposition that $\beta_i(w) \not\trianglelefteq \beta_i(v)$ was incorrect. □

DEFINITION 5.1.12. *Let* $w \in \mathcal{W}_{n,k}$ *and* $m \in [n]$. *Define* $\pi_{[m]}(w) = \{w_j \mid 1 \leq j \leq m\}$, *as a set, not a multiset.*

LEMMA 5.1.13. *Suppose* $v, w \in \mathcal{W}_{n,k}$ *are Fubini words. If* $v \leq w$ *in medium roast Fubini-Bruhat order, then* $\pi_{[m]}(v) \trianglelefteq \pi_{[m]}(w)$ *for* $m \in [n]$.

PROOF. By Lemma 5.1.2, $\pi_{[m]}(v) \trianglelefteq \pi_{[m]}(w)$ is equivalent to $|[h] \cap \pi_{[m]}(v)| \geq |[h] \cap \pi_{[m]}(w)|$ for $h = 1, \ldots, n$. Observe that $|[h] \cap \pi_{[m]}(v)| = \mathrm{rank}_v^{(h)}([m])$ and $|[h] \cap \pi_{[m]}(w)| = \mathrm{rank}_w^{(h)}([m])$. Thus, $\pi_{[m]}(v) \trianglelefteq \pi_{[m]}(w)$ is equivalent to $\mathrm{rank}_v^{(h)}([m]) \geq$

$\mathrm{rank}_w^{(h)}([m])$. Since $v \leq w$, this is true by Corollary 3.1.9 (2) for every $h \in [k]$ and every $m \in [n]$. Thus, $\pi_{[m]}(v) \trianglelefteq \pi_{[m]}(w)$ for every $m \in [n]$. □

2. The Linear Algebra of Column Sets

In this section, we develop results about the h-rank of a set of columns J on matrices in a PR cell C_w by extending Definition 5.1.8 to more general linear algebra terms defined for subsets $J \subseteq [n]$. Recall that since left multiplication by any unitriangular matrix does not change the rank of the h-subcolumns of any set of columns in a matrix, we only need to consider the pattern matrix of w when determining $\mathrm{rank}_w^{(h)}(J)$.

Definition 5.2.1. *We say J is an h-**dependent set** for $w \in \mathcal{W}_{n,k}$ if the corresponding h-subcolumns of every matrix in C_w have the same property. We say J is an h-**independent set** for w there exists some matrix $A \in C_w$ such that the h-subcolumns of A indexed by J are independent.*

By Corollary 3.1.9, we know $v \leq w$ in Fubini-Bruhat order if and only if

$$\mathrm{rank}_v^{(h)}(J) \geq \mathrm{rank}_w^{(h)}(J)$$

for every $1 \leq h \leq k$ and $J \subset [n]$. Hence our goal is to identify the smallest possible collection of pairs (h, J) on which we need to test these rank conditions to guarantee $v \leq w$ and a way to compute $\mathrm{rank}_w^{(h)}(J)$ directly from w. This leads us to introduce the h-**projection** of a Fubini word and the **essential set** of a Fubini word.

Definition 5.2.2. *A **Fubini word with zeros** is a word of a length n with alphabet $\{0, 1, 2, \ldots, k\}$, with each letter in $\{1, 2, \ldots, k\}$ appearing at least once. Denote the Fubini words with zeros of length n with alphabet $\{0, 1, 2, \ldots, k\}$ by $\widetilde{\mathcal{W}}_{n,k}$.*

Definition 5.2.3. *The h-**projection** of a Fubini word $w \in \mathcal{W}_{n,k}$, denoted $s^{(h)}(w) = (s^{(h)}(w)_1, s^{(h)}(w)_2, \ldots, s^{(h)}(w)_n)$, is defined by*

$$s^{(h)}(w)_j = |\{g \leq h \mid \alpha_g(w) \leq \alpha_{w(j)}(w)\}|$$

for $j = 1, 2, \ldots, n$*, where the right hand side measures the size of a set not a multiset. That is, for each* $j \in [n]$*, find the initial position of the letter* w_j *in* w *and count the number of distinct letters less than or equal to* h *that appear in that position or to its left.*

Note that $\alpha_1(w), \alpha_2(w), \ldots, \alpha_h(w)$ are also the initial columns containing 1,2,…,h for $s^{(h)}(w)$, though possibly in a different order, because the initial permutation for $s^{(h)}(w)$ is always the identity permutation. For example, if $w = 44253136541 \in \mathcal{W}_{11,6}$ and $h = 3$, then $s^{(3)}(w) = 00112323103$. Observe that $s^{(h)}(w)_j$ is the number of nonzero entries in rows $1, 2, \ldots, h$ and column j in the pattern matrix for w. Also observe that $s^{(k)}(w) = s(w)$, as in Definition 4.2.2, the definition of the sort of w.

For a Fubini word $w \in \mathcal{W}_{n,k}$, recall the definition for $\beta_i(w)$ given by Equation (5.1.22). The pattern matrix for w in columns in $\beta_i(w)$ has nonzero values only in rows $\pi_1(w), \ldots, \pi_i(w)$, and setting all $\star$'s to be 0 produces a permutation matrix on these rows and columns $\alpha_{\pi_1(w)}(w), \ldots, \alpha_{\pi_i(w)}(w)$. Hence, $\text{rank}_w^{(k)}(\beta_i(w)) = i$.

Definition 5.2.4. *Similarly to Equation (5.1.22), define* $B_i^{(h)}(w)$ *and* $\beta_i^{(h)}(w)$ *to be*

$$B_i^{(h)}(w) = \{j \in [n] \mid s^{(h)}(w)_j = i\} \text{ and } \beta_i^{(h)}(w) = \{j \in [n] \mid s^{(h)}(w)_j \leq i\} \tag{5.2.26}$$

for $i \in [0, h]$.

Here $\beta_0^{(h)}(w) = B_0^{(h)}(w)$ is the set of positions of zeros in $s^{(h)}(w)$. A Fubini word does not have any zeros, but we could still define $\beta_0(w) = \beta_0^{(k)}(w) = \emptyset$. Observe that $\beta_i^{(h)}(w) = \beta_j(w)$ for some $j \geq i$. Recall from our earlier example that if $w = 44253136541 \in \mathcal{W}_{11,6}$, then $s^{(3)}(w) = 00112323103$. Thus $\beta_2^{(3)}(w) = \{1, 2, 3, 4, 5, 7, 9, 10\} = \beta_4(w)$.

Definition 5.2.5. *The* h***-initial permutation of*** w*, denoted* $\pi^{(h)}(w)$*, is given by the initial letters of* w *less than or equal to* h*, taken from left to right. So* $\pi^{(h)}(w)$ *is a permutation in* S_h*.*

For example, the 3-initial permutation of $w = 44253136541 \in \mathcal{W}_{11,6}$ is $\pi^{(3)}(44253136541) = 231 \in S_3$. The pattern matrix for w in rows $[h]$ and columns $B_i^{(h)}(w)$ has nonzero values only in rows $\pi_1^{(h)}(w), \ldots, \pi_i^{(h)}(w)$. Setting all $\star$'s to be 0 produces a permutation matrix on these rows and columns $\alpha_{\pi_1^{(h)}(w)}(w), \ldots, \alpha_{\pi_i^{(h)}(w)}(w)$. Hence, $\mathrm{rank}_w^{(h)}(\beta_i^{(h)}(w)) = i$.

LEMMA 5.2.6. *Given Fubini word $w \in \mathcal{W}_{n,k}$, subset $J \subset [n]$, and $h \in [k]$, then the following are equivalent.*

1. *The set J is h-independent for w.*
2. *For each $i \in [0, h]$, we have $|J \cap \beta_i^{(h)}(w)| \leq i$.*
3. *As multisets, $[h] \trianglelefteq \{s^{(h)}(w)_j \,|\, j \in J\}$.*

PROOF. To see that (1) implies (2), suppose J is h-independent for w. Recall that $\mathrm{rank}_w^{(h)}(\beta_i^{(h)}(w)) = i$. Suppose $|J \cap \beta_i^{(h)}(w)| > i$. Then $J \cap \beta_i^{(h)}(w)$ is h-dependent, which implies J is h-dependent. This contradicts that J is h-independent.

To see that (2) implies (3), observe that by Equation (eq:B.beta.h.eq.def), for all $i \in [0, h]$,

$$|J \cap \beta_i^{(h)}(w)| = |[i] \cap \{s^{(h)}(w)_j \,|\, j \in J\}|.$$

Then by (2),

$$|[i] \cap \{s^{(h)}(w)_j \,|\, j \in J\}| \leq i = |[i] \cap [h]|$$

for $i \in [h]$. By Lemma 5.1.2, this is equivalent to $[h] \trianglelefteq \{s^{(h)}(w)_j \,|\, j \in J\}$, which proves (3).

To see that (3) implies (1), let

$$\{s^{(h)}(w)_j \,|\, j \in J\} = \{b_1 \leq b_2 \leq \cdots \leq b_{|J|}\}.$$

Then by (3), $|J| \leq h$ and $b_i \geq i$ for each $i = 1, 2, \ldots, |J|$. Since $s^{(h)}(w)_j = b_i$ is the number of nonzero entries in rows $1, 2, \ldots, h$ and column j in P_w, and these nonzero entries are in rows $\pi_1^{(h)}(w), \pi_2^{(h)}(w), \ldots, \pi_{b_i}^{(h)}(w)$. Thus, since $i \leq b_i$, this means that there exists a matrix $A \in C_w$ in which columns $b_1, b_2, \ldots, b_i$ and rows $\pi_1^{(h)}(w), \pi_2^{(h)}(w), \ldots, \pi_i^{(h)}(w)$ have rank i. Thus, columns $b_1, \ldots, b_{|J|}$ and rows $\pi_1^{(h)}(w), \pi_2^{(h)}(w), \ldots, \pi_{|J|}^{(h)}(w)$ have rank $|J|$. Since $|J| \leq h$, this means J in h-independent for w. □

Definition 5.2.7. *For set $J \subseteq [n]$ and Fubini word $w \in \mathcal{W}_{n,k}$, an h-**basis for** J **and** w is a subset $G \subseteq J$ such that there exists a matrix $A \in C_w$ such that the h-subcolumns indexed by G in A form a basis for the subspace spanned by the h-subcolumns indexed by J in A.*

Lemma 5.2.8. *Given integer $h \in [k]$, subset $J \subseteq [n]$, and Fubini word $w \in \mathcal{W}_{n,k}$ as input, the following **Basis Finder Algorithm** returns an h-basis for J and w.*

Input: *integer $h \in [k]$, subset $J \subset [n]$, Fubini word $w \in \mathcal{W}_{n,k}$*

Output: *$Z(h, J, w)$, a maximal size h-independent subset of J for w*

Algorithm:

```
(1) Set $Y = (y_1, \ldots, y_h)$, with each $y_i = 0$ initially.
(2) Compute $s^{(h)}(w) = (s^{(h)}(w)_1, \ldots, s^{(h)}(w)_n)$.
(3) For each $j \in J$, check if there exists some $y_d = 0$ with $1 \leq d \leq s^{(h)}(w)_j$.
    If so, choose $d$ to be the maximum such value. Set $y_d = j$.
(4) Return $Z(h, J, w) = \{y_i \in Y \mid y_i > 0\}$.
```

Proof. By construction, $|Z(h, J, w) \cap \beta_i^{(h)}(w)| \leq i$ for $i = 0, 1, \ldots, h$. So by Lemma 5.2.6, $Z(h, J, w)$ is h-independent for w. To see that the h-subcolumns of $Z(h, J, w)$ span the h-subcolumns of J, consider the matrix O_w obtained by setting all the $\star$'s in the pattern matrix P_w to 1. The algorithm guarantees that if row $a \leq h$ is not all zeros in the submatrix of O_w restricted to columns J, then row a is not all zeros in the submatrix of O_w restricted to columns $Z(h, J, w)$. Since the two submatrices of O_w obtained by restricting to column sets J and $Z(h, J, w)$ have the same number of nonzero rows in the row set $[h]$, and the columns of the second submatrix are h-independent, then the ranks of these submatrices in row set $[h]$ must be equal. □

Observe that the value $\mathrm{rank}_w^{(h)}(J)$, as defined by Definition 3.1.5, is the size of the output of the Basis Finder Algorithm on (h, J, w). Also observe that if $|J \cap \beta_i(w)| = i$, we can add any element in $\beta_i(w) \setminus J$ to J and not change $\mathrm{rank}_w^{(h)}(J)$.

Corollary 5.2.9. *A Fubini word $w \in \mathcal{W}_{n,k}$ is completely determined by the collection of words $s^{(h)}(w)$ for $h \in [k]$ or equivalently by the sets $\beta_i^{(h)}(w)$ for $h \in [k]$ and $i \in [h]$.*

Proof. The sets $\beta_i^{(h)}(w)$ are determined by $s^{(h)}(w)$ for $i \in [h]$. The Basis Finder Algorithm only depends on the values in $s^{(h)}(w)$, so the rank conditions defining C_w can be determined. These rank conditions define C_w uniquely by Theorem 3.1.11. $\square$

3. The Essential Set

In this section, we extend Fulton's essential set for permutations to Fubini words. Recall that Fulton's essential set for permutations gives a more efficient set of rank conditions than the set of all possible rank conditions. This gives a more efficient rank condition test for the Bruhat order on permutations. We define the essential set for a Fubini word w based on the standardization of the convexification of w. This gives us a smaller set of rank conditions that determine the PR variety $\overline{C}_w$ via Theorem 1.2.28, and an alternative test for medium roast Fubini-Bruhat order via Theorem 1.2.29. We prove both of these theorems in this section.

Recall the Rothe diagram from Equation (1.2.6), and the definition of the ranked essential set from Definition 1.2.24. We now use the standardization $\mathrm{std}(w)$ of $w \in \mathcal{W}_{n,k}$ from Definition 1.2.26 to extend these concepts to Fubini words.

Definition 5.3.1. *Given Fubini word $w \in \mathcal{W}_{n,k}$, define the* ***diagram*** *of w to be $D(std(conv(w)))$.*

Observe that $D(\mathrm{std}(\mathrm{conv}(w))) \subset [k] \times [n]$, as none of the bottom $n - k$ rows will contribute any elements to $D(\mathrm{std}(\mathrm{conv}(w)))$. Thus, the diagram of a Fubini word in $\mathcal{W}_{n,k}$ can be drawn as a $k \times n$ grid of dots. For each pair (w_j, j) with

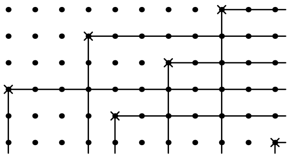

FIGURE 5.3.1. Diagram of 44253136541

$j \in \{\alpha_1(\mathrm{conv}(w)), \ldots, \alpha_k(\mathrm{conv}(w))\}$, cross out the dot (w_j, j) and all dots to the south and east. For example, the diagram for $w = 44253136541 \in W_{11,6}$ is $D(\mathrm{std}(44425533116))$, the top 6 rows of which are shown in Figure 5.3.1.

LEMMA 5.3.2. *For a convex Fubini word $w \in \mathcal{W}_{n,k}$, the southeast corners of the connected components of $D(std(w))$ occur in cells $(h, |\beta_i(w)|)$ for $h, i \in [k]$ such that*

- *$\pi_i > h$ or the value π_i occurs more than once in w,*
- *$\pi_{i+1} \leq h$,*
- *$\alpha_h(w) > \alpha_{h+1}(w)$ or $h = k$, and*
- *$\alpha_{h+1}(w) \leq |\beta_i(w)| < \alpha_h(w)$ if $h < k$.*

PROOF. The first condition implies the $(h, |\beta_i|)$ cell is in $D(\mathrm{std}(w))$. The second condition guarantees that the cell in the position directly to the right of $(h, |\beta_i|)$ is not in $D(\mathrm{std}(w))$. The third and fourth conditions guarantee that the cell directly below $(h, |\beta_i|)$ is not in $D(\mathrm{std}(w))$. Hence, $(h, |\beta_i|)$ is a southeast corner of $D(\mathrm{std}(w))$. The converse follows similarly. □

DEFINITION 5.3.3. *Given any Fubini word $w \in \mathcal{W}_{n,k}$, define its* ***rank function*** *to be the map $r_w : [k] \times [k] \to \mathbb{Z}_{\geq 0}$ that sends (h, i) to $rank_w^{(h)}(\beta_i(w))$.*

DEFINITION 5.3.4. *Given any Fubini word $w \in \mathcal{W}_{n,k}$, define the* ***essential set*** *of w to be*

$$Ess(w) = \{(h, \beta_i(w), r) \mid (h, |\beta_i(w)|) \in D(std(conv(w)))\}$$

where $r = r_w(h, i) = rank_w^{(h)}(\beta_i(w))$.

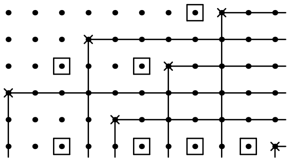

FIGURE 5.3.2. Essential set any w such that of $\mathrm{conv}(w) = 44425533116$

For example, the essential set of $w = 44253136541 \in \mathcal{W}_{11,6}$ is

$$\begin{aligned} Ess(44253136541) = \{&(3, \{1, 2, 10\}, 0),\\ &(6, \{1, 2, 10\}, 1),\\ &(3, \{1, 2, 3, 4, 9, 10\}, 1),\\ &(6, \{1, 2, 3, 4, 9, 10\}, 3),\\ &(1, \{1, 2, 3, 4, 5, 7, 9, 10\}, 0),\\ &(6, \{1, 2, 3, 4, 5, 7, 9, 10\}, 4),\\ &(6, \{1, 2, 3, 4, 5, 6, 7, 9, 10, 11\}, 5)\}. \end{aligned}$$

The essential set elements are boxed in $D(\mathrm{std}(\mathrm{conv}(44253136541)))$, as shown in Figure 5.3.2. In fact, these boxed elements describe the h, i, and r for the essential set elements in every word $w \in \mathcal{W}_{n,k}$ such that $\mathrm{conv}(w) = \mathrm{conv}(44253136541)$. The inclusion of the entire set $\beta_i(w)$ in the notation for the essential set elements distinguishes between essential sets of different Fubini words that have the same convexification. When $n < 10$, we may drop the set braces when listing $\beta_i(w)$ in the essential set. For example, we may write $Ess(3132) = \{(2, 13, 0)\}$ instead of $Ess(3132) = \{(2, \{1, 3\}, 0)\}$.

LEMMA 5.3.5. *For a convex Fubini word $w \in \mathcal{W}_{n,k}$, the ranks of the northwest submatrices $[i] \times [j]$ of the pattern matrix for w are constant in any connected component of the diagram of w.*

PROOF. The lemma follows from Equation (2.4.13). $\square$

As with Schubert cells, not all rank conditions are necessary to determine a PR cell. In fact, only the rank conditions given by the essential set are necessary. We prove Theorem 1.2.28 to show this.

Proof of Theorem 1.2.28. By definition of the variety $\overline{C}_v$, a matrix $A \in \mathcal{M}_{k\times n}(\mathbb{C})$ is in $\overline{C}_v$ if and only if for every subset $J \subset [n]$ and every $h \in [k]$, the rank of the h-subcolumns of J in A is at most $\mathrm{rank}_v^{(h)}(J)$. Recall from Chapter 3 that $\mathrm{rank}_v^{(h)}(J)$ is determined by the rank of corresponding h-subcolumns in $\mathrm{conv}(v)$. Applying the Basis Finder Algorithm to these h-subcolumns in $\mathrm{conv}(v)$, the rank of this matrix is determined by the intersections $|J \cap \beta_i(w)|$. By Lemma 5.3.5, the most binding rank conditions are exactly those such that $J = \beta_i(w)$ and $(h, \beta_i(w), r) \in Ess(w)$. □

Corollary 5.3.6. *For $v, w \in \mathcal{W}_{n,k}$, $v \leq w$ in medium roast Fubini-Bruhat order if and only if*

$$rank_w^{(h)}(\beta_i(v)) \leq r$$

for all $(h, \beta_i(v), r) \in Ess(v)$.

Proof. By Theorem 1.2.28 a matrix $A \in \mathcal{M}_{k\times n}(\mathbb{C})$ is in $\overline{C}_v$ if and only if for every $(h, \beta_i(v), r) \in Ess(v)$, the rank of the h-subcolumns indexed by $\beta_i(v)$ in A is at most $\mathrm{rank}_v^{(h)}(\beta_i(v))$. Thus, $\overline{C}_v$ contains all matrices in C_w if and only if

$$\mathrm{rank}_w^{(h)}(\beta_i(v)) \leq r$$

for all $(h, \beta_i(v), r) \in Ess(v)$. The proof now follows since $v \leq w$ if and only if $\overline{C}_v \supseteq C_w$, by Definition 1.1.9. □

Corollary 5.3.7. *For $v, w \in W_{n,k}$, we have $v \leq w$ if and only if*

$$|Z(h, \beta_i(v), w)| \;\leq\; |\{\pi_1(v), \pi_2(v), \ldots, \pi_i(v)\} \cap [h]|$$

for all $(h, \beta_i(v), r) \in Ess(v)$.

Proof. By Corollary 5.3.6, we know $v \leq w$ if and only if

$$\mathrm{rank}_w^{(h)}(\beta_i(v)) \leq \mathrm{rank}_v^{(h)}(\beta_i(v))$$

for all $(h, \beta_i(v), r) \in Ess(v)$. From the pattern matrix P_v and the definition of $Ess(v)$, it is straightforward to observe that

$$r = \mathrm{rank}_v^{(h)}(\beta_i(v)) = |\{\pi_1(v), \pi_2(v), \ldots, \pi_i(v)\} \cap [h]|$$

for every $(h, \beta_i(v), r) \in Ess(v)$. To determine $\mathrm{rank}_w^{(h)}(\beta_i(v))$, one uses the Basis Finder Algorithm on the input $h, J = (\beta_i(v)), w$ to find an h-basis for the columns J. Taking the size of the output determines the rank. $\square$

The corollary below serves two purposes. The first part is a weaker statement than Corollary 5.3.6, and is only used for the proof of the next corollary. The in fact statement is logically in between Corollary 5.3.6 and the first part of Corollary 5.3.8. We state it as an analog to Björner and Brenti's improved Tableau Criterion for permutations, Theorem 1.2.12.

Corollary 5.3.8. *For Fubini words $v, w \in \mathcal{W}_{n,k}$, $v \leq w$ in medium roast Fubini-Bruhat order if and only if*

$$rank_w^{(h)}(\beta_i(v)) \leq rank_v^{(h)}(\beta_i(v))$$

for all $(h, i) \in [k] \times [k]$.

In fact, $v \leq w$ in medium roast Fubini-Bruhat order if and only if

$$rank_w^{(h)}(\beta_i(v)) \leq rank_v^{(h)}(\beta_i(v))$$

for all $i \in [k]$ and $h \in [k]$ such that $\alpha_h(v) > \alpha_{h+1}(v)$.

Proof. By Corollary 3.1.9, we know $v \leq w$ if and only if

$$\mathrm{rank}_v^{(h)}(J) \geq \mathrm{rank}_w^{(h)}(J)$$

for every $1 \leq h \leq k$ and $J \subseteq [n]$. By Corollary 5.3.6, $v \leq w$ if and only if

$$\operatorname{rank}_w^{(h)}(\beta_i(v)) \leq \operatorname{rank}_v^{(h)}(\beta_i(v))$$

for all $(h, \beta_i(v), r) \in Ess(v)$. The condition that

$$\operatorname{rank}_w^{(h)}(\beta_i(v)) \leq \operatorname{rank}_v^{(h)}(\beta_i(v))$$

for all $(h, i) \in [k] \times [k]$ is logically between the conditions of Corollary 3.1.9 and Corollary 5.3.6, so it must also be equivalent to $v \leq w$.

The in fact statement is stronger. It follows from Lemma 5.3.2 and Corollary 5.3.6. □

Corollary 5.3.9. *Suppose $v, w \in \mathcal{W}_{n,k}$ are Fubini words. If*

1. *for $i \in [k]$, $\beta_i(v) = \beta_i(w)$, and*
2. *for $h \in [k]$, $\{\alpha_1(v), \ldots, \alpha_h(v)\} \trianglelefteq \{\alpha_1(w), \ldots, \alpha_h(w)\}$,*

then $v \leq w$ in medium roast Fubini-Bruhat order.

Proof. By (1), for any pair $(h, i) \in [k] \times [k]$,

$$\operatorname{rank}_w^{(h)}(\beta_i(v)) \leq \operatorname{rank}_v^{(h)}(\beta_i(v))$$

if and only if

$$\operatorname{rank}_w^{(h)}(\beta_i(w)) \leq \operatorname{rank}_v^{(h)}(\beta_i(v)).$$

By (2) and Lemma 5.1.2, the second equation is true for all $(h, i) \in [k] \times [k]$. So by Corollary 5.3.8, $v \leq w$. □

Lemma 5.3.10. *Given Fubini word $w \in \mathcal{W}_{n,k}$ and $j, m \in [k]$,*

$$rank_w^{(m)}(\beta_j(w)) = min(m, j, max(m-h, 0) + max(j-i, 0) + r)$$

for some $(h, \beta_i(w), r) \in Ess(w)$.

Proof. By definition of $\operatorname{rank}_w^(m)(J)$, $\operatorname{rank}_w^{(m)}(\beta_j(w)) \leq m$. Since $\operatorname{rank}_w^{(k)}(\beta_j(w)) = j$, then $\operatorname{rank}_w^{(m)}(\beta_j(w)) \leq j$.

If $\operatorname{rank}_w^{(m)}(\beta_j(w)) < m$ and $\operatorname{rank}_w^{(m)}(\beta_j(w)) < j$, then there must be a limiting rank condition on w that prevents this upper bound from being achieved. The most binding rank conditions come from $Ess(w)$, so suppose the rank condition in question is $(h, \beta_i(w), r) \in Ess(w)$. If $m \leq h$ and $j \leq i$, this means $\operatorname{rank}_w^{(m)}(\beta_j(w))$ is at most r. If $m > h$, each additional row of the m-subcolumns indexed by $\beta_i(w)$ of w can contribute at most 1 more to $\operatorname{rank}_w^{(m)}(\beta_i(w))$. If $j > i$, each additional set of m-subcolumns $B_l(w)$ with $i < l \leq j$ can contribute at most 1 more to $\operatorname{rank}_w^{(m)}(\beta_j(w))$, by definition of the pattern matrix of w and the β-triangle for w. If we pick $(h, \beta_i(w), r)$ to be an element of $Ess(w)$ that most limits $\operatorname{rank}_w^{(m)}(\beta_j(w))$, these maximum contributions will be achieved. Thus,

$$\operatorname{rank}_w^{(m)}(\beta_j(w))(w) = \max(m - h, 0) + \max(j - i, 0) + r.$$

□

Lemma 5.3.10 is not sufficient by itself to prove Theorem 1.2.29. We introduced the extended essential set below solely for the purpose of the proof of Theorem 1.2.29. This will give us Lemma 5.3.12, which is similar to Lemma 5.3.10, but removes j from the min term on the right-hand side of the equation. This removal is necessary to prove Theorem 1.2.29.

Definition 5.3.11. *For $w \in \mathcal{W}_{n,k}$, define the **extended essential set** of w to be*

$$\overline{Ess}(w) = Ess(w) \cup \{(k, \emptyset, 0)\}.$$

Lemma 5.3.12. *Given Fubini word $w \in \mathcal{W}_{n,k}$ and $j, m \in [k]$,*

$$rank_w^{(m)}(\beta_j(w)) = min(m, max(m - h, 0) + max(j - i, 0) + r)$$

for some $(h, \beta_i(w), r) \in \overline{Ess}(w)$.

Proof. By Lemma 5.3.10, we already have that

$$\operatorname{rank}_w^{(m)}(\beta_j(w)) = \min(m, j, \max(m - h, 0) + \max(j - i, 0) + r)$$

for some $(h, \beta_i(w), r) \in Ess(w)$. So we just need to verify that $j = \max(m-h, 0) + \max(j-i, 0) + r$ when $(h, \beta_i(w), r) = (k, \emptyset, 0)$. Observe that this means $i = 0$, since $\beta_0(w) = \emptyset$. Thus, $\max(m-h, 0) = 0$, $\max(j-i, 0) = j$, and $r = 0$, and the equation is verified. $\square$

Definition 5.3.13. *Given subset $J \subseteq [n]$, integer $h \in [k]$, and Fubini word $w \in \mathcal{W}_{n,k}$, suppose $Y = (y_1, \ldots, y_h)$ is the result of the Basis Finder Algorithm for h, J, and w before removing zeros. Define $\gamma(h, J, w)$ to be the smallest possible index such that $z_{\gamma(h,J,w)+1} = 0$. If there are no zeros in $Y = (y_1, \ldots, y_h)$, then we say $\gamma(h, J, w) = h$.*

Lemma 5.3.14. *Given subset $J \subseteq [n]$, integer $h \in [k]$, and Fubini word $w \in \mathcal{W}_{n,k}$. Then*

$$rank_w^{(h)}(J) = \gamma(h, J, w) + |J \setminus \beta^{(h)}_{\gamma(h,J,w)}(w)|.$$

Proof. Suppose $Y = (y_1, \ldots, y_h)$ is the result of the Basis Finder Algorithm for h, J, and w before removing zeros. Since $y_1, y_2, \ldots, y_{\gamma(h,J,w)}$ are nonzero, we know that

$$\text{rank}_w^{(h)}(J \cap \beta^{(h)}_{\gamma(h,J,w)}(w)) = \text{rank}_w^{(h)}(\beta^{(h)}_{\gamma(h,J,w)}(w)) = \gamma(h, J, w).$$

Since $y_{\gamma(h,J,w)+1} = 0$, we know that the h-subcolumns $J \setminus \beta^{(h)}_{\gamma(h,J,w)}(w)$ are linearly independent. Also, these columns are all independent from the span of the h-subcolumns $\beta^{(h)}_{\gamma(h,J,w)}(w)$. Therefore,

$$\begin{aligned}\text{rank}_w^{(h)}(J) &= \text{rank}_w^{(h)}(J \cap \beta^{(h)}_{\gamma(h,J,w)}(w)) + \text{rank}_w^{(h)}(J \setminus \beta^{(h)}_{\gamma(h,J,w)}(w)) \\ &= \gamma(h, J, w) + |J \setminus \beta^{(h)}_{\gamma(h,J,w)}(w)|.\end{aligned}$$

$\square$

Corollary 5.3.15. *Given subset $J \subseteq [n]$, and Fubini word $w \in \mathcal{W}_{n,k}$, then*

$$rank_w^{(h)}(J) = min_{i \in [0,h]} \left(i + |J \setminus \beta_i^{(h)}(w)| \right).$$

Proof. Since $\operatorname{rank}_w^{(h)}(\beta_i^{(h)}(w)) = i$ for any $i \in [h]$ and each additional column of J adds at most 1 to $\operatorname{rank}_w^{(h)}(J)$, we have that

$$\operatorname{rank}_w^{(h)}(J) \leq i + |J \setminus \beta_i^{(h)}(w)|$$

for every $i \in [k]$. By Lemma 5.3.14, equality is achieved when $i = \gamma(h, J, w)$ and possibly other values of i. $\square$

We are now at a point where we can determine the h-rank of any column set J for a Fubini word $w \in \mathcal{W}_{n,k}$, given its essential set. We also have some ways via $Ess(v)$ to relate the h-ranks of various column sets for Fubini words $v, w \in \mathcal{W}_{n,k}$ whenever $v \leq w$. We are thus ready to prove Theorem 1.2.29, the culminating proof for this chapter.

Proof of Theorem 1.2.29. Suppose for every $(m, \beta_j(v), s) \in Ess(v)$, there exists an $(h, \beta_i(w), r) \in Ess(w)$ such that

$$\max(0, m-h) + |\beta_j(v) \setminus \beta_i(w)| \leq s - r.$$

Thus,

$$\begin{aligned} s &\geq r + \max(0, m-h) + |\beta_j(v) \setminus \beta_i(w)| \\ &= \operatorname{rank}_w^{(h)}(\beta_i(w)) + \max(0, m-h) + |\beta_j(v) \setminus \beta_i(w)|, \end{aligned}$$

since $\operatorname{rank}_w^{(h)}(\beta_i(w)) = r$. But $\operatorname{rank}_w^{(m)}(\beta_i(w)) \leq \operatorname{rank}_w^{(h)}(\beta_i(w)) + \max(0, m-h)$, so

$$s \geq \operatorname{rank}_w^{(m)}(\beta_i(w)) + |\beta_j(v) \setminus \beta_i(w)|.$$

Finally, $\operatorname{rank}_w^{(m)}(\beta_j(v)) \leq \operatorname{rank}_w^{(m)}(\beta_i(w)) + |\beta_j(v) \setminus \beta_i(w)|$, so

$$s \geq \operatorname{rank}_w^{(m)}(\beta_j(v)).$$

Then by Corollary 5.3.6, $v \leq w$.

Conversely, suppose $v \leq w$ and $(m, \beta_j(v), s) \in Ess(v)$. Then $s < m$ and $s < j$. By Corollary 5.3.6,

$$s = \operatorname{rank}_v^{(m)}(\beta_j(v)) \geq \operatorname{rank}_w^{(m)}(\beta_j(v)). \tag{5.3.27}$$

By Lemma 5.3.14,

$$\operatorname{rank}_w^{(m)}(\beta_j(v)) = \gamma(m, \beta_j(v), w) + |\beta_j(v) \setminus \beta^{(m)}_{\gamma(m,\beta_j(v),w)}(w)|,$$

and so

$$\operatorname{rank}_w^{(m)}(\beta_j(v)) = \operatorname{rank}_w^{(m)}(\beta^{(m)}_{\gamma(m,\beta_j(v),w)}(w)) + |\beta_j(v) \setminus \beta^{(m)}_{\gamma(m,\beta_j(v),w)}(w)|. \tag{5.3.28}$$

But $\beta^{(m)}_{\gamma(m,\beta_j(v),w)}(w) = \beta_x(w)$ for some $x \geq \gamma(m, \beta_j(v), w)$, so

$$s \geq \operatorname{rank}_w^{(m)}(\beta_j(v)) = \operatorname{rank}_w^{(m)}(\beta_x(w)) + |\beta_j(v) \setminus \beta_x(w)|.$$

By Lemma 5.3.12,

$$\operatorname{rank}_w^{(m)}(\beta_x(w)) = \begin{cases} m, \text{ or} \\ \max(m-h, 0) + \max(x-i, 0) + r, \end{cases}$$

for some $(h, \beta_i(w), r) \in \overline{Ess}(w)$. In the first case, $s \geq m + |\beta_j(v) \setminus \beta_x(w)| \geq m$. This is not possible, since $s < m$.

In the second case, by Equation (5.3.27),

$$s \geq \operatorname{rank}_w^{(m)}(\beta_j(v)).$$

Thus, by Equation (5.3.28),

$$\begin{aligned} s &\geq \max(m-h, 0) + \max(x-i, 0) + r + |\beta_j(v) \setminus \beta_x(w)| \\ &= \operatorname{rank}_w^{(h)}(\beta_x(v)) + \max(m-h, 0) + |\beta_j(v) \setminus \beta_x(w)|. \end{aligned}$$

By the final statement of Corollary 5.3.15,

$$\operatorname{rank}_w^{(h)}(\beta_x(w)) = \operatorname{rank}_w^{(h)}(\beta_i(w)) + |\beta_x(w) \setminus \beta_i(w)|.$$

Therefore,

$$\begin{aligned} s &\geq \operatorname{rank}_w^{(m)}(\beta_j(v)) \\ &= \operatorname{rank}_w^{(h)}(\beta_i(v)) + |\beta_x(v) \setminus \beta_i(w)| + \max(m-h, 0) + |\beta_j(v) \setminus \beta_x(w)| \\ &\geq r + \max(m-h, 0) + |\beta_j(v) \setminus \beta_i(w)|. \end{aligned}$$

If $(h, \beta_i(w), r) \in Ess(w)$, then we are done. If $(h, \beta_i(w), r) = (k, \emptyset, 0)$, then $r + \max(m-h, 0) + |\beta_j(v) \setminus \beta_i(w)| = |\beta_j(v)|$, which is impossible, since $(m, \beta_j(v), s) \in Ess(v)$ implies $s < j \leq |\beta_j(v)|$.

$\square$

CHAPTER 6

Covering Relations and Properties

In this section, we prove some families of combinatorially described covering relations for the medium roast and espresso Fubini-Bruhat orders, giving a partial answer to Problem 9.5 in [**25**]. We also discuss useful properties of the decaf and espresso Fubini-Bruhat orders, and finish the chapter by proving the Lifting Property, which generalizes from the property of the same name for Bruhat order on permutations.

1. The Transposition Rule

The transitive closure of the Transposition Rule for Bruhat order is one of the nicest descriptions of Bruhat order. A description via a single rule seems impossible for Fubini-Bruhat order due to the redundant letters and phenomena like Example 3.1.10. However, the Transposition Rule does hold for medium roast Fubini-Bruhat order, as we stated it in Theorem 1.2.32. We prove this theorem below.

Proof of Theorem 1.2.32. Suppose $i < j$ and $\alpha_i(w) < \alpha_j(w)$. Observe that there is a relationship on the sorted words $s(w) = s(t_{ij}w)$ from Definition 4.2.2, so Condition (1) of Corollary 4.2.3 is satisfied. Also, $\alpha_l(w) = \alpha_l(t_{ij}w)$ for $l \neq i$ and $l \neq j$. Since $\alpha_i(w) < \alpha_j(w)$, we have $\alpha_i(t_{ij}w) > \alpha_j(t_{ij}w)$. Since $i < j$, this means Condition (2) of Corollary 4.2.3 is satisfied. Therefore, $w < t_{ij}w$ by Corollary 4.2.3.

Conversely, suppose $w < t_{ij}w$. Then by Theorem 1.2.20, $\{\alpha_1(w), \dots, \alpha_h(w)\} \trianglelefteq \{\alpha_1(t_{ij}w), \dots, \alpha_h(t_{ij}w)\}$ for all $1 \leq h \leq k$. Since $\alpha_h(w) = \alpha_h(t_{ij}(w))$ for $h = 1, \dots, i-1$, this means $\alpha_i(w) < \alpha_i(t_{ij}(w)) = \alpha_j(w)$.

To see the furthermore statement, notice that upon applying the transposition t_{ij}, the following changes occur to the pattern matrix.

1. Rows i and j of the redundant columns of P_w and $P_{t_{ij}w}$ will swap, leaving the same number of $\star$'s in these columns.
2. The initial columns of P_w will be identical to the columns of $P_{\pi(w)}$, and the initial columns of $P_{t_{ij}w}$ will be identical to the columns of $P_{\pi(t_{ij}w)}$.

If $\pi(t_{ij}w)$ covers $\pi(w)$ in Bruhat order on S_k, then the number of $\star$'s in $P_{\pi(w)}$ is one more than the number of $\star$'s in $P_{\pi(t_{ij}w)}$. From (1) above, adding in the redundant columns to get P_w and $P_{t_{ij}w}$ preserves the dimension difference. Therefore, $\dim(w) = \dim(t_{ij}w)+1$, so $t_{ij}w$ covers w, since 1 is the smallest possible dimension jump between any variety and one of its proper subvarieties.

Conversely, if $t_{ij}w$ covers w in $\mathcal{W}_{n,k}$, then $\alpha_i(w) < \alpha_j(w)$, so $\pi_i^{-1}(w) < \pi_j^{1}(w)$. So $\pi(w) < t_{ij}\pi(w) = \pi(t_{ij}w)$ in Bruhat order on S_k by Theorem 1.2.31. Suppose there exists a $v \in S_k$ such that $\pi(w) < v < \pi(t_{ij}w)$. Then we can construct $v' \in \mathcal{W}_{n,k}$ such that $\pi(v') = v$ and $s(v') = s(w)$. Then, by Corollary 4.2.3, $w < v' < t_{ij}w$, contradicting the existence of v'. Therefore, $\pi(t_{ij}w)$ covers $\pi(w)$ in Bruhat order on S_k. $\square$

2. The Pushback and Superpushback Rules

The Pushback and Superpushback Rules from Theorems 1.2.34 and 1.2.36 interact with the redundant letters in Fubini words, and how changing one of these letters affects medium roast and espresso Fubini-Bruhat orders, respectively. The pushback operator from Definition 1.2.33 always produces a covering relation in medium roast Fubini-Bruhat order.

Proof of Theorem 1.2.34. Since $v_i = w_i$ for all $i \neq j$, then $s(v)_i = s(w)_i$ for all $i \neq j$. In the case of j, and $s(v)_j = i + 1 > i = s(w)_j$ by Definition 1.2.33. Thus, Condition (1) of Corollary 4.2.3 is satisfied. Observe that $\pi(v) = \pi(w)$, so we may refer to π without ambiguity. Either v_j is redundant or initial in v. In the first case, $\alpha_l(v) = \alpha_l(w)$ for $l = 1, \ldots, k$. In the second case, by constuction, we know that

$$\alpha_{\pi_{i+1}}(v) = j < \alpha_{\pi_{i+1}}(w).$$

Furthermore, $\alpha_l(v) = \alpha_l(w)$ for $l \neq \pi_{i+1}$. Also, $\alpha_{\pi_1(v)}(v) < \cdots < \alpha_{\pi_k(v)}(v)$ and $\alpha_{\pi_1(w)}(w) < \cdots < \alpha_{\pi_k(w)}(w)$. In either case, Condition (2) of Corollary 4.2.3 is satisfied. Therefore, by Corollary 4.2.3, $v \leq w$.

To see the dimension statement, observe again that v_j is either redundant or initial in v. In the first case, column j of P_v has exactly one more $\star$ than column j of P_w. That star is in row π_{i+1}. In the second case, column since $\pi(v) = \pi(w)$, column $j = \alpha_{\pi_{i+1}}(v)$ of P_v has the same number of $\star$'s as column $\alpha_{\pi_{i+1}}(w)$ of P_w. Also, column $\alpha_{\pi_{i+1}}(w)$ of P_v has exactly one more $\star$ than column j of P_w. In either case, $\dim(v) = \dim(w) + 1$. □

PROOF OF THEOREM 1.2.36. Suppose $J \in \binom{[n]}{[k]}$ with $|J| = h \leq k$, such that $\alpha_J(w) = \{\alpha_1(w), \ldots, \alpha_h(w)\}$. If $j \notin J$, then $\{v_l \,|\, l \in J\} = \{w_l \,|\, l \in J\}$. Thus, $\alpha_J(v) = \{\alpha_1(v), \ldots, \alpha_h(v)\}$. Then by Theorem 1.2.19, we are done.

Now suppose $j \in J$. Then either v_j is redundant or initial in v. If v_j is redundant in v, then $\alpha_h(v) = \alpha_h(w)$ for $h \in [k]$, $\alpha_{\{j\}}(v) > \alpha_{\{j\}}(w)$, and $\alpha_{J\setminus\{j\}}(v) = \alpha_{J\setminus\{j\}}(w)$. So $\alpha_J(w) \trianglelefteq \alpha_J(v)$. Thus, $\{\alpha_1(v), \ldots, \alpha_h(v)\} = \alpha_J(w) \trianglelefteq \alpha_J(v)$, and by Theorem 1.2.19, we are done.

If v_j is initial in v, then $\alpha_h(v) = \alpha_h(w)$ for $h \neq \pi_{i+p}(w)$, and $\alpha_{\pi_{i+p}(w)}(v) = j < \alpha_{\pi_{i+p}(w)}(w)$. Also, $\alpha_{\{j\}}(v) = j > \alpha_{\{j\}}(w)$. If $h < \pi_{i+p}(w)$, then since $\alpha_J(w) = \{\alpha_1(w), \ldots, \alpha_h(w)\}$, $\{w_l \,|\, l \in J \setminus \{j\}\}$ does not contain $\pi_{i+p}(w)$. Then $\alpha_{J\setminus\{j\}}(v) = \alpha_{J\setminus\{j\}}(w)$. So $\alpha_J(w) \trianglelefteq \alpha_J(v)$. Also, $\{\alpha_1(w), \ldots, \alpha_h(w)\} = \{\alpha_1(v), \ldots, \alpha_h(v)\}$. So $\{\alpha_1(v), \ldots, \alpha_h(v)\} = \{\alpha_1(w), \ldots, \alpha_h(w)\} = \alpha_J(w) \trianglelefteq \alpha_J(v)$, so by Theorem 1.2.19, we are done.

If $h \geq \pi_{i+p}(w)$, then since $\alpha_J(w) = \{\alpha_1(w), \ldots, \alpha_h(w)\}$, $\{w_l \,|\, l \in J \setminus \{j\}\}$ contains $\pi_{i+p}(w)$ exactly once. Suppose $w_m = \pi_{i+p}(w)$. Then $\alpha_{\{m\}}(v) = j > \alpha_{\{m\}}(w)$, and $\alpha_{J\setminus\{j,m\}}(v) = \alpha_{J\setminus\{j,m\}}(w)$. Since $\alpha_{\{j\}}(v) = j > \alpha_{\{j\}}(w)$, $\alpha_J(w) \trianglelefteq \alpha_J(v)$. Since $\alpha_{\pi_{i+p}(w)}(v) = j < \alpha_{\pi_{i+p}(w)}(w)$, then $\{\alpha_1(v), \ldots, \alpha_h(v)\} \trianglelefteq \{\alpha_1(w), \ldots, \alpha_h(w)\}$.

Thus, $\{\alpha_1(v), \ldots, \alpha_h(v)\} \trianglelefteq \{\alpha_1(w), \ldots, \alpha_h(w)\} = \alpha_J(w) \trianglelefteq \alpha_J(v)$, so by Theorem 1.2.19, we are done. □

3. The Decaf Fubini-Bruhat Order

Now that we have proved the Transposition Rule and the Pushback Rule, we have a full characterization of covering relations in the decaf Fubini-Bruhat order. Recall that the decaf order is defined as the transitive closure of the Transposition Rule and the Pushback Rule. In fact, we can show that the decaf order is a direct product of two subposets arising from these covering relations. Thus, the decaf order is ranked by codimension of Fubini words.

Lemma 6.3.1. *The decaf Fubini-Bruhat order is a direct product of two ranked posets. In particular, the decaf order is ranked by codimension.*

Proof. Let $\mathcal{A}$ be the poset on Fubini words $w \in \mathcal{W}_{n,k}$ for which $\pi(w) = id \in S_k$ with covering relations given by the Pushback Rule, and let $\mathcal{B}$ be Bruhat order on S_k. We can define a map $f : \mathcal{A} \times \mathcal{B} \to \mathcal{W}_{n,k}$ by

$$f(a, b) = w$$

if and only if $s(w) = s(a)$ and $\pi(w) = b \in S_k$. Note that this subposet contains all the elements of $\mathcal{W}_{n,k}$. For various choices of $a \in \mathcal{A}$, $s(w) = s(a)$ gives all possible $\beta_1(w), \ldots, \beta_k(w)$. The choice of $b \in S_k$ gives all possible $\pi(w)$, and Bruhat order can be described as the transitive closure of all transpositions. Also, the Transposition Rule and the Pushback Rule commute. Thus, the subposet of $\mathcal{W}_{n,k}$ given by $f(\mathcal{A} \times \mathcal{B})$ is isomorphic to $(\mathcal{W}_{n,k}, \ll)$, the decaf Fubini-Bruhat order.

To see the in particular statement, since $\mathcal{A}$ and $\mathcal{B}$ both describe covering relations with a dimension difference of 1, a ranking of $\mathcal{A} \times \mathcal{B}$ by codimension is natural. Thus, the decaf Fubini-Bruhat order is ranked by codimension.

□

4. The Medium Roast Fubini-Bruhat Order

In general, it has been quite difficult to tame all the relations in medium roast and espresso Fubini-Bruhat orders. In general, when $n \geq 3$ and $2 < k < n$, there are more relations in the medium roast Fubini-Bruhat order than in the decaf Fubini-Bruhat order. For example, in $\mathcal{W}_{4,3}$ there are 6 more relations, in $\mathcal{W}_{5,3}$ there are 33 more relations, and in $\mathcal{W}_{5,4}$ there are 64 more relations. However, there are a few easy cases. In the case of $k = 1$, $\mathcal{W}_{n,1}$ contains the single Fubini word $111 \cdots 1$. In the case of $k = n$, PR cells are equivalent to Schubert cells. Thus, all three Fubini-Bruhat orders are equivalent to the Bruhat order on S_n. When $k = 2$, the following lemma shows that the decaf and medium roast Fubini-Bruhat orders are identical.

Lemma 6.4.1. *For $k = 2$ and $n \geq 2$, the decaf and medium roast Fubini-Bruhat orders are identical.*

Proof. We will show that the Transposition Rule and the Pushback Rule determine all of the covering relations for the medium roast Fubini-Bruhat order when $k = 2$. Then we will be done by definition of the decaf order.

We proceed by contradiction. Suppose $v < w$ and $k = 2$, and v and w are not related via a series of applications of the Transposition Rule and the Pushback Rule. Then suppose the first letter in which v and w differ is the jth letter. If $j \notin in(w)$, $\varphi(w, j)$ agrees with v in the jth letter, so $v \leq \varphi(w, j) < w$. So assume that we have already applied the pushback operator as many times as possible, and $j \in in(w)$. In particular, $w_{j-1} \neq w_j$, $v_{j-1} = w_{j-1}$, and $v_j \neq w_j$. Since the only possible letters in v and w are 1 and 2, then $v_{j-1} = v_j$. Also, note that since $j \in in(w)$, $w_1 = w_2 = \cdots = w_{j-1}$. Since v and w agree in the first $j-1$ letters and $v_j \neq w_j$, then $v_1 = v_2 = \cdots = v_j$.

Consider $J = \{j-1, j\}$. Then since $w_{j-1} \neq w_j$, the minor $\{1, 2\}, J$ is unvanishing on C_w. Since the first j letters of v are identical, the minor $\{1, 2\}, J$ is truly vanishing on C_v. So $J \in T_v$ and $J \notin T_w$. Thus, $T_v \not\subset T_w$, which contradicts $v \leq w$ by Theorem 1.2.8. $\square$

Recall that the Pushback Rule and the covering relations for the Transposition Rule produce relations of dimension difference 1. Thus, for $k = 1, 2$, or n, the medium roast Fubini-Bruhat order is ranked by codimension.

REMARK 6.4.2. For $n \geq 5$, there may be covering relations in the medium roast Fubini-Bruhat order $(\mathcal{W}_{n,k}, \leq)$ with a dimension difference of 2 or more, causing the medium roast Fubini-Bruhat order to be unranked in general. For example, in $\mathcal{W}_{5,4}$, 44312 covers 41321, but 44312 has dimension 1, and 41321 has dimension 3. The espresso Fubini-Bruhat order contains many covering relations between Fubini words of the same dimension, like the one described in [**25**, Ex. 9.1]. Pawlowski and Rhoades observe that in $\mathcal{W}_{4,3}$, $1323 \prec 1123$, which we can verify via the Superpushback Rule, but both 1323 and 1123 have dimension 3. In $\mathcal{W}_{5,4}$, we find a yet worse example. By the Superpushback Rule, $24313 \prec 24413$, but 24313 has dimension 4 and 24413 has dimension 5, thus they are in the wrong order by codimension.

The following lemma gives a family of covering relations missed by the Transposition Rule and the Push Back Rule. These are 2 of the 6 missing covering relations in $\mathcal{W}_{4,3}$. The remaining 4 can be generated from these by applying Theorem 1.2.38, which we prove later in this chapter.

LEMMA 6.4.3. *When $n = k + 1$, consider the words $w = (k, k, 1, 2, \cdots, k-1)$, $v = (k, 1, 2, \cdots, k-1, 1)$, and $u = (k, 2, 1, 3, 4, \cdots, k-1, 1)$. Then $v \leq w$ and $u \leq w$. Furthermore, both of these are covering relations.*

PROOF. We can compute the essential sets of w, v, and u to be

$$Ess(w) = \{(k-1, 12, 0)\},$$

$$Ess(v) = \{(k-1, 1, 0), (k-1, 12k, 1)\},$$

and

$$Ess(u) = \{(k-1,1,0),(1,12,0),(k-1,123k,2)\}.$$

We can verify that each element of $Ess(v)$ and $Ess(u)$ satisfy Equation (1.2.7) when paired with $(k-1,12,0)$, the only element in $Ess(w)$. Thus, by Theorem 1.2.29, $v \leq w$ and $u \leq w$.

To see the furthermore statement, compute the dimension of each of u, v, and w. By definition of the pattern matrix, P_u has no $\star$'s in the first 3 columns. Then, P_u has $2,3,\ldots,k-2$ $\star$'s, respectively, in each of the next $k-3$ columns, and 2 $\star$'s in the last column. Thus, $\dim(u) = \binom{k-1}{2}+1$. By definition of the pattern matrix, P_v has no $\star$'s in the first two columns. Then, P_v has $1,2,\ldots,k-2$ $\star$'s, respectively, in each of the next $k-2$ columns, and 1 $\star$ in the last column. Thus, $\dim(v) = \binom{k-1}{2}+1$. Finally, by definition of the pattern matrix, P_w has no $\star$'s in the first three columns. Then, P_w has $1,2,\ldots,k-2$ $\star$'s in the last $k-1$ columns. Thus, $\dim(w) = \binom{k-1}{2}$. Since 1 is the smallest possible dimension jump between any variety and one of its proper subvarieties, w covers both u and v in medium roast Fubini-Bruhat order. □

5. The Espresso Fubini-Bruhat Order

As we saw in Remark 6.4.2, the espresso Fubini-Bruhat order is, in some ways, the most irregular of our three Fubini-Bruhat orders. Recall that Pawlowski and Rhoades noticed that $\rightharpoonup$ was not even transitive. There are some partial transitivity results, however.

Lemma 6.5.1. *If $v \leq w$ in medium roast Fubini-Bruhat order, and $w \rightharpoonup z$, then $v \rightharpoonup z$.*

Proof. By Theorem 1.2.18, $T_v \subset T_w \subset (S_z \cup T_z)$. By Theorem 1.2.18, this means $v \rightharpoonup z$. □

Corollary 6.5.2. *If $v \leq w$ and $w \preceq z$, then $v \preceq z$.*

Remark 6.5.3. It is not true in general that if $v \preceq w$ and $w \leq z$, then $v \preceq z$. For example, in $\mathcal{W}_{4,3}$, $3212 \preceq 3312$ by Theorem 1.2.36, and $3312 \leq 3321$ by Lemma 6.4.3, but $\overline{C}_{3212} \cap C_{3321} = \emptyset$.

Remark 6.5.4. It is also not true in general that if $v \leq w$ and $w \preceq z$, then $v \leq z$. For example, in $\mathcal{W}_{5,3}$, $32123 \preceq 33123$ by Theorem 1.2.36, and $32122 \leq 32123$ by Theorem 1.2.34. However,

$$Ess(33123) = \{(2, 125, 0)\},$$

and

$$Ess(32122) = \{(2, 1, 0), (1, 1245, 0)\},$$

and $(2, 125, 0)$ and $(1, 1245, 0)$ do not satisfy Equation (1.2.7). Thus, by Theorem 1.2.29, $32122 \not\leq 33123$.

Remark 6.5.5. We reiterate the proof of [**25**, Prop. 9.2], which states that $X_{n,k} = X_0 \supset X_1 \supset X_2 \supset \cdots \supset X_m = \emptyset$, where $m = |\mathcal{W}_{n,k}|$ and each difference $X_i - X_{i+1}$ is a single cell C_v for some $v \in \mathcal{W}_{n,k}$. Also, from [**25**, Def. 4.2], each X_i is a closed subvariety of $X_{n,k}$. That means that if $C_v = X_{i+1} - X_i$, $\overline{C}_v \subseteq X_i$. So for any C_w such that $v \rightharpoonup w$, $C_w = X_{j+1} - X_j$ for $j \geq i$. That is, the espresso Fubini-Bruhat order is a subposet of a total order on $\mathcal{W}_{n,k}$ induced by a cellular decomposition of $X_{n,k}$. That is, it is possible to find a linear extension of the espresso Fubini-Bruhat order that describes a cellular decomposition of $X_{n,k}$. This will, of course, also be a linear extension of the weaker medium roast Fubini-Bruhat order.

6. The Lifting Property

A version of the Lifting Property for Bruhat order on permutations extends to both the medium roast and espresso Fubini-Bruhat orders. We give a few definitions before proving the lemmas that lead to the Lifting Property for Fubini-Bruhat order, which we stated as Theorem 1.2.38. Recall that $s_i = t_{i,i+1}$ is the adjacent transposition swapping i and $i + 1$.

DEFINITION 6.6.1. *For some matrix A with at least $i+1$ rows and some $a \in \mathbb{C}$, define*

$$\phi_i(A,a) = E_{i,i+1}(a)A \tag{6.6.29}$$

where $E_{i,i+1}(a)$ is the lower triangular matrix with 1's on the diagonal, a in the $(i+1,i)$ position, and 0's everywhere else. Observe that $\phi_i(A,a)$ adds a times the ith row of A to the $(i+1)$th row of A.

LEMMA 6.6.2. *Suppose $w \in \mathcal{W}_{n,k}$ is a Fubini word, $i \in [k-1]$, and $\alpha_{i+1}(w) < \alpha_i(w)$. Then for any A fitting the pattern of w, $s_i\phi_i(A,a)$ fits the pattern of s_iw. Furthermore, if B is any matrix fitting the pattern of s_iw, then $B = s_i\phi_i(A,a)$ for some A fitting the pattern of w.*

PROOF. Consider the transformation of P_w to P_{s_iw} in rows i and $i+1$. Since the first occurrence of $i+1$ precedes the first occurrence of i, there are only 6 possibilities for a 2×1 submatrix in the ith and $(i+1)$th rows in any column of P_w. These get sent to the 2×1 submatrices in the corresponding rows of P_{s_iw} given by

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{6.6.30}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{6.6.31}$$

$$\begin{pmatrix} 0 \\ \star \end{pmatrix} \to \begin{pmatrix} \star \\ 0 \end{pmatrix}, \tag{6.6.32}$$

$$\begin{pmatrix} \star \\ \star \end{pmatrix} \to \begin{pmatrix} \star \\ \star \end{pmatrix}, \tag{6.6.33}$$

$$\begin{pmatrix} 1 \\ \star \end{pmatrix} \to \begin{pmatrix} \star \\ 1 \end{pmatrix}, \tag{6.6.34}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} \star \\ 1 \end{pmatrix}. \tag{6.6.35}$$

All of these transformations can be described as first adding a multiple of the top row to the bottom row, then swapping rows. Thus, for any A fitting the pattern of w, $s_i\phi_i(A,a)$ fits the pattern of s_iw.

To see the furthermore statement, observe that (6.6.35) occurs in the column indexed by $\alpha_i(w)$. Let B be any matrix fitting the pattern of s_iw, and let a be the value of the $\star$ in (6.6.35). Then $B = s_i\phi_i(A,a)$ for some A fitting the pattern of w. □

Lemma 6.6.3. *Suppose $w \in \mathcal{W}_{n,k}$ is a Fubini word. If $\alpha_{i+1}(w) < \alpha_i(w)$, then $T_{s_iw} \subseteq T_w$ and $T_w \setminus T_{s_iw}$ contains only northerly minors indexed by sets of size i.*

Proof. By the Transposition Rule, $s_iw \leq w$. By Theorem 1.2.8, $T_{s_iw} \subseteq T_w$. To show $T_w \setminus T_{s_iw}$ contains only northerly minors indexed by sets of size i, recall that any northerly minor $[|J|], J$ truly vanishes on C_w if and only if $\Delta_{[|J|],J}(A) = 0$ for all matrices A fitting the pattern of w. Therefore, we focus on the pattern matrices P_w and P_{s_iw}. Observe that P_w and P_{s_iw} are identical in rows 1 through $i-1$, so northerly minors of size less than i truly vanish on both or neither of C_w and C_{s_iw}. Thus, we only need to show that all minors of size strictly larger than i that vanish on matrices fitting the pattern of w also vanish on matrices fitting the pattern of s_iw.

By Lemma 6.6.2, if B is any matrix fitting the pattern of s_iw, then $B = s_i\phi_i(A,a)$ for some A fitting the pattern of w. If a northerly minor of size strictly greater than i vanishes for every A fitting the pattern of w, then it vanishes for every $B = s_i\phi_i(A,a) = s_iE_{i,i+1}(a)A$, since s_i and $E_{i,i+1}(a)$ are invertible. Thus, the same northerly minor vanishes for every B fitting the pattern of s_iw. So $T_w \setminus T_{s_iw}$ contains only northerly minors indexed by sets of size i. □

Lemma 6.6.4. *Suppose $w \in \mathcal{W}_{n,k}$ is a Fubini word with $\alpha_{i+1}(w) < \alpha_i(w)$, $J \subset [n]$, and $|J| = i$. Then $J \in T_{s_iw}$ if and only if the minor (I,J) is truly vanishing on C_w, where $I = \{1,2,\cdots,i-1,i+1\}$.*

Proof. Suppose the minor (I,J) is truly vanishing on C_w. By Lemma 4.1.5, $J \in T_w$. To show that $J \in T_{s_iw}$, we need only show that $\Delta_{[i],J}(B) = 0$ for matrices B fitting the pattern of s_iw. By Lemma 6.6.2, if B is any matrix fitting the pattern of s_iw, then $B = s_i\phi_i(A,a)$ for some A fitting the pattern of w. Recall that the $(i+1)$th row of $\phi_i(A,a)$ is a times the ith row of A plus the $(i+1)$th row of A. By linearity of the determinant,

$$\Delta_{[i],J}(\phi_i(A,a)) = a\Delta_{[i],J}(A) + \Delta_{I,J}(A) = 0. \tag{6.6.36}$$

Then since determinant distributes over multiplication,

$$\Delta_{[i],J}(B) = \Delta_{[i],J}(s_i\phi_i(A,a)) = \Delta_{[i],J}(s_i)\Delta_{[i],J}(\phi_i(A,a)) = 0. \tag{6.6.37}$$

Thus, $J \in T_{s_iw}$.

Conversely, suppose $J \in Ts_iw$. Observe that $s_iw \leq w$ by the Transposition Rule, so $J \in T_w$ by Theorem 1.2.8. Any matrix in C_w can be written in the form MA, where $M \in U^-$ and A fits the pattern of w. Applying (6.6.36) to MA and rearranging,

$$\Delta_{I,J}(MA) = \Delta_{[i],J}(\phi_i(MA,a)) - a\Delta_{[i],J}(UA). \tag{6.6.38}$$

Since $J \in T_w$, $\Delta_{[i],J}(MA) = 0$. Observe that

$$\phi_i(MA,a) = E_{i,i+1}(a)MA = M'E_{i,i+1}(a)A = M'\phi_i(A,a),$$

where $M' = E_{i,i+1}(a)M(E_{i,i+1}(a))^{-1} \in U^-$. By Lemma 6.6.2, if A is any matrix fitting the pattern of w, $\phi_i(A,a)$ fits the pattern of s_iw. Therefore, $M'\phi_i(A,a) \in C_{s_iw}$. Since $J \in T_{s_iw}$,

$$\Delta_{[i],J}(\phi_i(MA,a)) = \Delta_{[i],J}(M'\phi_i(A,a)) = 0.$$

Then by (6.6.38), $\Delta_{I,J}(MA) = 0$. $\square$

Proof of Theorem 1.2.38. To see (1), suppose $J \notin T_{s_iw}$. Then either $J \notin T_w$ or $J \in T_w$. Suppose first that $J \notin T_w$. Then by the Transposition Rule, $s_iv \leq v \leq w$, so $J \notin T_{s_iv}$ by Theorem 1.2.8.

Now suppose $J \in T_w$. Then $J \in T_w \setminus T_{s_iw}$. By Lemma 6.6.3, $T_w \setminus T_{s_iw}$ has only truly vanishing northerly minors of size i, so $|J| = i$. Since $J \notin T_{s_iw}$, by Lemma 6.6.4, if $I = \{1, 2, \cdots, i-1, i+1\}$, the minor (I, J) is not a truly vanishing minor for C_w. Therefore, by Corollary 3.1.9, the minor (I, J) is not a truly vanishing minor for C_v. Again by Lemma 6.6.4, $J \notin T_{s_iv}$.

In both cases, $J \notin T_{s_iw}$ implies $J \notin T_{s_iv}$. Thus, by Theorem 1.2.8, $s_iv \leq s_iw$.

To see (2), suppose $J \notin (S_{s_iw} \cup T_{s_iw})$. Then either $J \notin (S_w \cup T_w)$ or $J \in (S_w \cup T_w)$. Suppose first that $J \notin (S_w \cup T_w)$. Then by the Transposition Rule, $s_iv \leq v \rightharpoonup w$, so $s_iv \rightharpoonup w$ by Lemma 6.5.1. Thus, $J \notin T_{s_iv}$ by Theorem 1.2.8.

Now suppose $J \in T_w$. This part of the proof is identical to the corresponding part of the proof for (1), and concludes with $J \notin T_{s_iv}$.

In both cases, $J \notin (S_{s_iw} \cup T_{s_iw})$ implies $J \notin T_{s_iv}$. Thus, by Theorem 1.2.8, $s_iv \rightharpoonup s_iw$.

$\square$

CHAPTER 7

Examples and Conclusion

In this chapter, we provide some examples illustrating the relations and properties of Chapter 6 put into practice. These examples also illustrate the remaining difficulties in finishing describing the covering relations for the medium roast and espresso Fubini-Bruhat orders. This leads us to some open problems described in Section 7.**??**.

1. The Fubini-Bruhat Poset on $\mathcal{W}_{4,3}$

While we did not fully answer Pawlowski and Rhoades' question of combinatorially describing all the covering relations for medium roast Fubini-Bruhat order, we have described enough of them to fully understand all three Fubini-Bruhat orders on $\mathcal{W}_{4,3}$. The different types of covering relations on $\mathcal{W}_{4,3}$ are labeled with different colors in Figure 7.1.1.

The lines indicate medium roast Fubini-Bruhat order, and the blue $\rightharpoonup$'s indicate the extra relations in espresso Fubini-Bruhat order. Each color represents a subposet with covering relations given by the Transposition Rule. Thus, these subposets are isomorphic to Bruhat order on S_3. The black lines signify covering relations in medium roast Fubini-Bruhat order determined by the Pushback Rule.

The red lines are the 6 relations missed by both the Transposition Rule and the Pushback Rule, but can be described by the 2 relations given by Lemma 6.4.3, and Theorem 1.2.38 applied to these relations. Observe that these relations do not satisfy Condition (1) of Corollary 4.2.3.

Deleting the red lines and the blue $\rightharpoonup$'s from Figure 7.1.1 gives the decaf Fubini-Bruhat order on $\mathcal{W}_{4,3}$. Including the red lines gives the medium roast Fubini-Bruhat order. Including the blue $\rightharpoonup$'s in addition gives the espresso Fubini-Bruhat order.

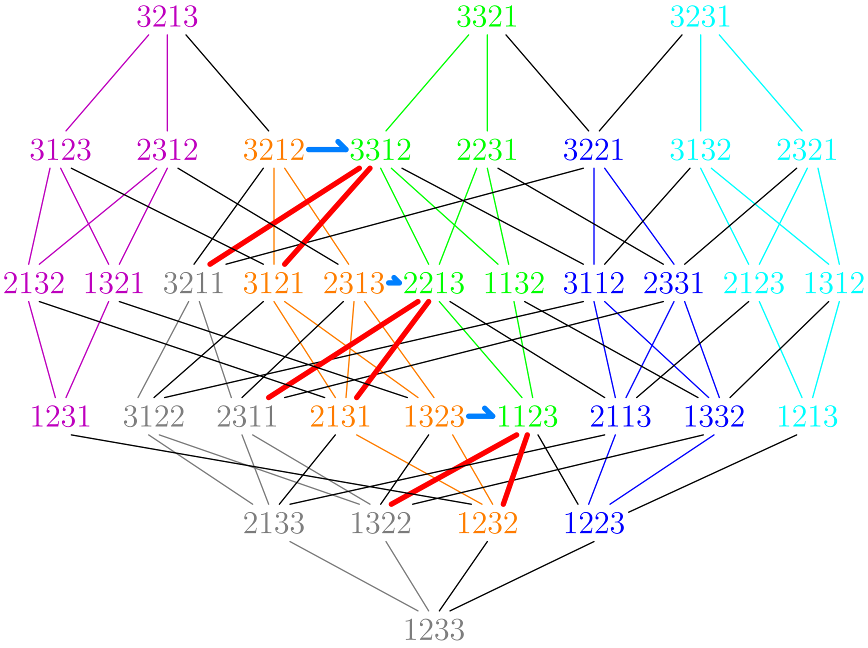

FIGURE 7.1.1. Fubini-Bruhat order on $\mathcal{W}_{4,3}$

There are many choices for a total order with the properties mentioned in Remark 6.5.5. All such choices must respect all three Fubini-Bruhat orders. One such possibility would be to read the rows of Figure 7.1.1 from left to right, bottom to top, giving

$$X_1 = \overline{C}_{1233},$$

$$X_2 - X_1 = \overline{C}_{2133},$$

$$X_3 - X_2 = \overline{C}_{1322},$$

$$X_4 - X_3 = \overline{C}_{1232},$$

$$X_5 - X_4 = \overline{C}_{1223},$$

$$X_6 - X_5 = \overline{C}_{1231},$$

$$\vdots$$

$$X_{36} - X_{35} = \overline{C}_{3231}.$$

Recall from Section 1.1 that a Schubert variety $\overline{C}_v$ is the disjoint union of Schubert cells C_w such that $v \leq w$ in Bruhat order on permutations. Recall that this is not true for Fubini words due to [**25**, Ex. 9.1]. We use a similar example to [**25**, Ex. 9.1] to explicitly write down the not quite cell decomposition of the PR variety $\overline{C}_{3212}$.

The pattern matrix for 3212 is

$$P_{3212} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & \star \end{pmatrix}.$$

By Theorem 1.2.34, $3213 \leq 3212$ in medium roast Fubini-Bruhat order. Hence, $C_{3213} \subset \overline{C}_{3212}$. We can also think of this as replacing the $\star$ in P_{3212} by t. Then we can scale the last column of the matrix by a factor of $\frac{1}{t}$ to get

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{t} \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Letting $t \to \infty$, we get the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix},$$

which is the pattern matrix for 3213. We also trivially have that $C_{3212} \subset \overline{C}_{3212}$. If PR cells and varieties behaved like Schubert cells and varieties, we would expect these two cells to decompose the entire PR variety $\overline{C}_{3212}$, since 3213 and 3212 are the only Fubini words greater than or equal to 3212 in medium roast Fubini-Bruhat order.

However, there is also the lower triangular action, which can add s times the second row to the third row of P_{3212}. Still replacing the $\star$ by t, this gives the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & s & 0 & s+t \end{pmatrix}.$$

Scaling the second column this time by a factor of $\frac{1}{s}$, we get

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & \frac{1}{s} & 0 & 1 \\ 1 & 1 & 0 & s+t \end{pmatrix}.$$

Setting $t = u - s$, then letting $s \to \infty$ gives

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & u \end{pmatrix}.$$

This matrix is in the PR cell C_{3312}. To see this, take the pattern matrix for 3312,

$$\begin{pmatrix} 0 & 0 & 1 & \star \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix},$$

set the $\star$ equal to 0, and add u times the second row to the third row via the lower triangular action. However, the cell C_{3312} also contains matrices not in $\overline{C}_{3212}$, namely the ones which come from the case where the $\star$ P_{3312} is not 0. Thus

$$\overline{C}_{3212} = C_{3212} \cup C_{3213} \cup \text{ part of } C_{3312}.$$

2. Unranked Interval in $\mathcal{W}_{5,4}$

In $\mathcal{W}_{5,4}$, the medium roast Fubini-Bruhat order is no longer ranked, as illustrated by the following example. Observe the interval $[41231, 44312]$ in Figure 7.2.1.

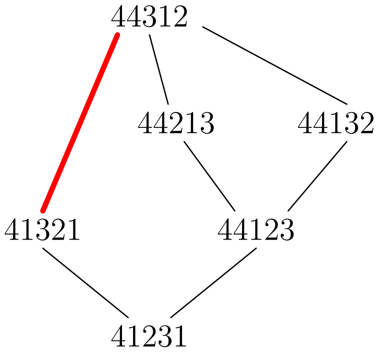

FIGURE 7.2.1. Unranked interval in medium roast Fubini-Bruhat order on $\mathcal{W}_{5,4}$

One may hope that the espresso poset on $\mathcal{W}_{5,4}$ will be ranked when the medium roast poset is not. In fact, it appears that this may be the case. Observe that by Theorem 1.2.36, $42312 \preceq 44312$, and by Theorem 1.2.32, $41321 \leq 42312$. Thus, by Corollary 6.5.2, $41321 \preceq 44312$. Again, by the Lemma 6.4.3, it so happens that we also have $41321 \leq 44312$. Inserting 42312 into the picture, we get Figure 7.2.2.

However, it is not, in general, true that espresso poset on $\mathcal{W}_{5,4}$ will be ranked when the medium roast poset is not. We have already seen in Remark 6.4.2 that this is not nearly the case.

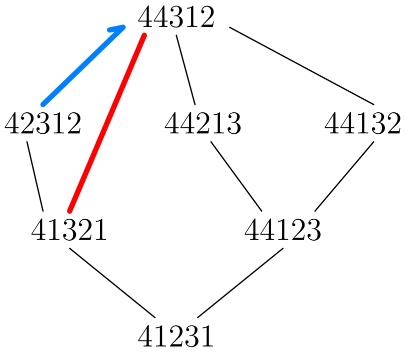

FIGURE 7.2.2. Unranked interval in medium roast Fubini-Bruhat order on $\mathcal{W}_{5,4}$

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