DEPARTMENT OF OCEANOGRAPHY UNIVERSITY OF WASHINGTON

Technical Report No. 27

PROPAGATION AND DISSIPATION OF LONG INTERNAL WAVES

Office of Naval Research
Contract N8onr-520/III
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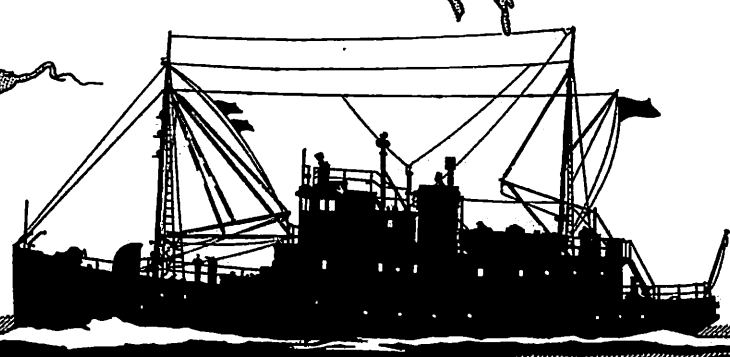
March 1954



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PROPAGATION AND DISSIPATION OF LONG INTERNAL WAVES

By

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for Richard H. Fleming
Executive Officer

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ABSTRACT

The effect of friction on the propagation of long internal waves in a rotating ocean is investigated theoretically for the simple case of a two-layer system subjected to a constant eddy viscosity. Waves of inertial period are excluded. Typical distances for a 50% decrease in amplitude are found to be 1,000 to 2,000 km.

LIST OF SYMBOLS

Symbol	Meaning
x, y, z:	Cartesian coordinates of position, with $z = 0$ at the mean interface level.
t:	Time.
u, v, w:	Components of water velocity.
ζ :	Elevation of upper surface of layer.
ζ_0:	Value of ζ at time $t = 0$ and position $x = 0$.
g:	Acceleration due to gravity.
ν :	Eddy viscosity.
ω :	Angular velocity of rotation of ocean area.
ρ:	Density of sea water.
$\Delta\rho$:	Density difference between two layers.
h:	Mean layer depth.
σ :	Frequency of the internal wave.
k:	Wave number of the internal wave.
γ :	Damping coefficient of the internal wave.
β_1:	$\sqrt{(\sigma + 2\omega)/2\nu}$.
β_2:	$\sqrt{(\sigma - 2\omega)/2\nu}$.
ε :	ζ_0'/ζ_0'' .
R:	$\dfrac{g\frac{\Delta\rho}{\rho}(\gamma + ik)^2 (h'+h'')}{\sigma^2 - 4\omega^2}$.
Θ :	$\dfrac{\varepsilon}{\Delta\rho/\rho}$.

Symbol	Meaning
B_1:	$\beta_1(h'+h'')$.
B_2:	$\beta_2(h'+h'')$.
p :	$1/2 - \frac{\omega}{\sigma}$.
q:	$1/2 + \frac{\omega}{\sigma}$.
S_0:	$-\frac{(h'+h'')^2}{h'h''}$.
S_1:	Real part of $\frac{R}{S_0} - 1$.
T_1:	Imaginary part of $\frac{R}{S_0} - 1$.
Φ_0:	$-\frac{h''}{h'+h''}$.
Φ_1:	Real part of $\frac{\Theta}{\Phi_0} - 1$.
Ψ_1:	Imaginary part of $\frac{\Theta}{\Phi_0} - 1$.
P:	$\frac{p}{2B_1}$.
Q:	$\frac{q}{2B_2}$.

INTRODUCTION

Evaluation of oceanographic data is at best uncertain where the influence of long internal waves cannot be determined. This is clearly shown by the California current where an apparent eddy structure is found which is caused by internal oscillations of diurnal period (Defant, 1950a). From the apparent wide distribution of such internal waves it is reasonable to suppose that the possibility of error exists in the interpretation of much similar data. A knowledge of the occurrence and behavior of long internal waves in the ocean becomes then of prime importance for the advance of oceanography.

Obtaining information on internal waves at sea is a difficult process which requires long series of observations for statistically valid results. It has been demonstrated that reported internal waves, in many cases, may be only harmonic representations of short series of random data (Haurwitz, 1952, 1953; Rudnick and Cochrane, 1951). In addition, in order to adequately describe the behavior of the internal waves it is necessary to take measurements simultaneously at three stations (Haurwitz, 1953).

The essential reason for the difficulty in obtaining internal wave data in the ocean is the lack of an adequate theory to guide the investigations. Even the presence of tidal internal waves is not well explained. The resonance theory (Haurwitz, 1950; Defant, 1950) is inadequate because of the large frictional damping and narrow belt of resonance for these waves. Since internal waves of tidal period have not generally been found in the open ocean but mainly in offshore regions, a mechanism of generation associated with the ocean boundaries is suggested (Rudnick and Cochrane, 1951). Model experiments have shown the generation of internal waves by the passage of surface waves over bottom irregularities (Zeilon, 1934). At that time it was felt this did not offer an explanation for the observed tidal internal waves since their amplitudes would be decreased rapidly by friction and divergence. However, internal waves generated by the action of surface tides over the continental shelves will not be subject to radial divergence as from a point source, but will undergo convergence on traveling seaward from a boundary source. It is proposed therefore to derive expressions for the effect of friction on internal wave motion for the case of a rotating earth.

MATHEMATICAL FORMULATION

For mathematical simplicity the problem considered is the propagation of long internal waves through a two-layer fluid on a rotating disk subject to constant eddy viscosity. A right-handed coordinate system is used with the origin at the mean level of the interface, the z-axis vertical, positive upwards, and the x-axis positive in the direction of wave travel. A primed quantity refers to the upper layer and a double-primed quantity to the lower layer. The following assumptions are made: the vertical accelerations are negligible permitting use of a hydrostatic pressure distribution;

all quantities are independent of position in the y direction; the water is incompressible; and u, v, w and variations in ρ and p are sufficiently small that terms of higher order can be neglected.

The equations of motion then become, for the upper layer:

$$\frac{\partial u'}{\partial t} - 2\omega v' - \nu \frac{\partial^2 u'}{\partial z^2} = - g \frac{\partial \zeta'}{\partial x} , \qquad (1)$$

$$\frac{\partial v'}{\partial t} + 2\omega u' - \nu \frac{\partial^2 v'}{\partial z^2} = 0 , \qquad (2)$$

and for the lower layer:

$$\frac{\partial u''}{\partial t} - 2\omega v'' - \nu \frac{\partial^2 u''}{\partial z^2} = - g \frac{\partial \zeta'}{\partial x} - g \frac{\Delta\rho}{\rho} \frac{\partial \zeta''}{\partial x} , \qquad (3)$$

$$\frac{\partial v''}{\partial t} + 2\omega u'' - \nu \frac{\partial^2 v''}{\partial z^2} = 0 . \qquad (4)$$

The equation of continuity is

$$\int_{\zeta''}^{h' + \zeta'} \frac{\partial u'}{\partial x} dz = \frac{\partial \zeta''}{\partial t} - \frac{\partial \zeta'}{\partial t} \qquad (5)$$

for the upper layer, and

$$\int_{-h''}^{\zeta''} \frac{\partial u''}{\partial x} dz = - \frac{\partial \zeta''}{\partial t} \qquad (6)$$

for the lower layer.

The boundary conditions are:

at the free surface there is no tangential stress, or

$$\text{at } z = h', \qquad \frac{\partial u'}{\partial z} = \frac{\partial v'}{\partial z} = 0 \; ; \qquad (7)$$

at the interface, the velocities and tangential stresses are continuous, or

$$\text{at } z = 0, \qquad u' = u'', \quad v' = v'',$$

$$\frac{\partial u'}{\partial z} = \frac{\partial u''}{\partial z} , \quad \frac{\partial v'}{\partial z} = \frac{\partial v''}{\partial z} \; ; \qquad (8)$$

at the bottom there is no slippage, or

$$\text{at } z = -h'', \qquad u'' = v'' = 0. \qquad (9)$$

Equations (1) through (9) are solved in a manner similar to that applied to tidal waves on the Siberian shelf by Sverdrup (Sverdrup, 1926).

SOLUTION

A solution is found by assuming

$$\zeta' = \zeta_0' \, e^{-\gamma x} \, e^{i(\sigma t - kx)} , \qquad (10)$$

$$\zeta'' = \zeta_0'' \, e^{-\gamma x} \, e^{i(\sigma t - kx)} , \qquad (11)$$

where ζ_0', ζ_0'', γ, σ, and k are all constants characteristic of the motion. The general solutions of equations (1) through (4) are then

$$u' = - \frac{ig \, \zeta_0' \, (\gamma + ik) e^{-\gamma x} \, e^{i(\sigma t - kx)}}{\sigma^2 - 4\omega^2} \Big\{ \sigma + A_1' \, e^{(1+i)\beta_1 z} + A_2' \, e^{-(1+i)\beta_1 z} - A_3' \, e^{(1+i)\beta_2 z} - A_4' \, e^{-(1+i)\beta_2 z} \Big\} , \qquad (12)$$

$$v' = - \frac{g \, \zeta_0' \, (\gamma + ik) e^{-\gamma x} \, e^{i(\sigma t - kx)}}{\sigma^2 - 4\omega^2} \Big\{ -2\omega + A_1' \, e^{(1+i)\beta_1 z} + A_2' \, e^{-(1+i)\beta_1 z} + A_3' \, e^{(1+i)\beta_2 z} + A_4' \, e^{-(1+i)\beta_2 z} \Big\} , \qquad (13)$$

$$u'' = - \frac{ig \, (\zeta_0' + \frac{\Delta\rho}{\rho} \, \zeta_0'') (\gamma + ik) e^{-\gamma x} \, e^{i(\sigma t - kx)}}{\sigma^2 - 4\omega^2} \Big\{ \sigma + A_1'' \, e^{(1+i)\beta_1 z} + A_2'' \, e^{-(1+i)\beta_1 z} - A_3'' \, e^{(1+i)\beta_2 z} - A_4'' \, e^{-(1+i)\beta_2 z} \Big\} , \qquad (14)$$

$$v'' = -\frac{g(\zeta_0' + \frac{\Delta\rho}{\rho}\zeta_0')(\gamma + ik)e^{-\gamma x}\, e^{i(\sigma t - kx)}}{\sigma^2 - 4\omega^2}\left\{-2\omega + A_1'' e^{(1+i)\beta_1 z}\right.$$

$$\left. + A_2'' e^{-(1+i)\beta_1 z} + A_3'' e^{(1+i)\beta_2 z} + A_4'' e^{-(1+i)\beta_2 z}\right\} . \qquad (15)$$

β_1 and β_2 are constants given by $\sqrt{\frac{\sigma + 2\omega}{2\nu}}$ and $\sqrt{\frac{\sigma - 2\omega}{2\nu}}$, respectively. Relations for the constant coefficients A, determined from the boundary conditions, after some rearranging are:

$$\varepsilon\left(\frac{\sigma - 2\omega}{2} + A_1' + A_2'\right) = \left(\varepsilon + \frac{\Delta\rho}{\rho}\right)\left(\frac{\sigma - 2\omega}{2} + A_1'' + A_2''\right) , \qquad (16)$$

$$\varepsilon\left(\frac{\sigma + 2\omega}{2} - A_3' - A_4'\right) = \left(\varepsilon + \frac{\Delta\rho}{\rho}\right)\left(\frac{\sigma + 2\omega}{2} - A_3'' - A_4''\right) , \qquad (17)$$

$$\varepsilon\left(A_1' - A_2'\right) = \left(\varepsilon + \frac{\Delta\rho}{\rho}\right)\left(A_1'' - A_2''\right) , \qquad (18)$$

$$\varepsilon\left(A_3' - A_4'\right) = \left(\varepsilon + \frac{\Delta\rho}{\rho}\right)\left(A_3'' - A_4''\right) , \qquad (19)$$

$$A_1' e^{(1+i)\beta_1 h'} = A_2' e^{-(1+i)\beta_1 h'} , \qquad (20)$$

$$A_3' e^{(1+i)\beta_2 h'} = A_4' e^{-(1+i)\beta_2 h'} , \qquad (21)$$

$$\frac{\sigma - 2\omega}{2} + A_1'' e^{-(1+i)\beta_1 h''} + A_2'' e^{(1+i)\beta_1 h''} = 0 , \qquad (22)$$

$$\frac{\sigma + 2\omega}{2} - A_3'' e^{-(1+i)\beta_2 h''} - A_4'' e^{(1+i)\beta_2 h''} = 0 , \qquad (23)$$

where $\varepsilon = \frac{\zeta_0'}{\zeta_0''}$, a small quantity for internal waves. Solution of these equations yields the following expressions for the coefficients:

$$\frac{2\varepsilon}{\sigma - 2\omega} \left(e^{-2(1+i)\beta_1 h''} + e^{2(1+i)\beta_1 h'}\right) A_1' = \frac{1}{2}\frac{\Delta\rho}{\rho}\left(1 + e^{-2(1+i)\beta_1 h''}\right) - \left(\varepsilon + \frac{\Delta\rho}{\rho}\right) e^{-(1+i)\beta_1 h''} , \qquad (24)$$

$$\frac{2\left(\varepsilon + \frac{\Delta\rho}{\rho}\right)}{\sigma - 2\omega} \left(e^{-2(1+i)\beta_1 h''} + e^{2(1+i)\beta_1 h'}\right) A_1'' = \frac{1}{2}\frac{\Delta\rho}{\rho}\left(1 - e^{2(1+i)\beta_1 h'}\right) - \left(\varepsilon + \frac{\Delta\rho}{\rho}\right) e^{-(1+i)\beta_1 h''} , \qquad (25)$$

$$\frac{2\varepsilon}{\sigma - 2\omega} \left(e^{-2(1+i)\beta_1 h'} + e^{2(1+i)\beta_1 h''}\right) A_2' = \frac{1}{2}\frac{\Delta\rho}{\rho}\left(1 + e^{2(1+i)\beta_1 h''}\right) - \left(\varepsilon + \frac{\Delta\rho}{\rho}\right) e^{(1+i)\beta_1 h''} , \qquad (26)$$

$$\frac{2(\varepsilon + \frac{\Delta\rho}{\rho})}{\sigma - 2\omega} (e^{-2(1+i)\beta_1 h'} + e^{2(1+i)\beta_1 h''}) A_2'' = \frac{1}{2} \frac{\Delta\rho}{\rho} (1 - e^{-2(1+i)\beta_1 h'})$$

$$- (\varepsilon + \frac{\Delta\rho}{\rho}) e^{(1+i)\beta_1 h''} , \qquad (27)$$

$$\frac{2\varepsilon}{\sigma + 2\omega} (e^{-2(1+i)\beta_2 h''} + e^{2(1+i)\beta_2 h'}) A_3' = - \frac{1}{2} \frac{\Delta\rho}{\rho} (1 + e^{-2(1+i)\beta_2 h''})$$

$$+ (\varepsilon + \frac{\Delta\rho}{\rho}) e^{-(1+i)\beta_2 h''} , \qquad (28)$$

$$\frac{2(\varepsilon + \frac{\Delta\rho}{\rho})}{\sigma + 2\omega} (e^{-2(1+i)\beta_2 h''} + e^{2(1+i)\beta_2 h'}) A_3'' = - \frac{1}{2} \frac{\Delta\rho}{\rho} (1 - e^{2(1+i)\beta_2 h'})$$

$$+ (\varepsilon + \frac{\Delta\rho}{\rho}) e^{-(1+i)\beta_2 h''} , \qquad (29)$$

$$\frac{2\varepsilon}{\sigma + 2\omega} (e^{-2(1+i)\beta_2 h'} + e^{2(1+i)\beta_2 h''}) A_4' = - \frac{1}{2} \frac{\Delta\rho}{\rho} (1 + e^{2(1+i)\beta_2 h''})$$

$$+ (\varepsilon + \frac{\Delta\rho}{\rho}) e^{(1+i)\beta_2 h''} , \qquad (30)$$

$$\frac{2(\varepsilon + \frac{\Delta\rho}{\rho})}{\sigma + 2\omega} (e^{-2(1+i)\beta_2 h'} + e^{2(1+i)\beta_2 h''}) A_4'' = - \frac{1}{2} \frac{\Delta\rho}{\rho} (1 - e^{-2(1+i)\beta_2 h'})$$

$$+ (\varepsilon + \frac{\Delta\rho}{\rho}) e^{-(1+i)\beta_2 h''} . \qquad (31)$$

The equations of continuity can now be written

$$\int_{0}^{h'} \zeta_0'(\Upsilon^2 + 2ik\Upsilon - k^2) f'(z)dz = i\sigma (\zeta_0'' - \zeta_0') , \qquad (32)$$

and

$$\int_{-h''}^{0} (\zeta_0' + \frac{\Delta\rho}{\rho} \zeta_0'')(\Upsilon^2 + 2ik\Upsilon - k^2) f''(z)dz = -i\sigma \zeta_0'' , \qquad (33)$$

where

$$f'(z) = \frac{ig}{\sigma^2 - 4\omega^2} (\sigma + A_1' e^{(1+i)\beta_1 z} + A_2' e^{-(1+i)\beta_1 z} - A_3' e^{(1+i)\beta_2 z} - A_4' e^{-(1+i)\beta_2 z}) ,$$

and

$$f''(z) = \frac{ig}{\sigma^2 - 4\omega^2} (\sigma + A_1'' e^{(1+i)\beta_1 z} + A_2'' e^{-(1+i)\beta_1 z} - A_3'' e^{(1+i)\beta_2 z} - A_4'' e^{-(1+i)\beta_2 z}) .$$

Integration of the equations of continuity and substitution of the values for the constants A, yields the following two expressions in $(\Upsilon + ik)$ and ε :

$$\frac{2(\sigma^2 - 4\omega^2)\sigma(1-\varepsilon)}{g(\gamma + ik)^2} = \frac{(\sigma - 2\omega)\frac{\Delta\rho}{\rho}\sinh[(1+i)\beta_1 h']\left\{\cosh[(1+i)\beta_1 h''] + 1\right\}}{(1+i)\beta_1 \cosh[(1+i)\beta_1(h'+h'')]}$$

$$+ \frac{(\sigma + 2\omega)\frac{\Delta\rho}{\rho}\sinh[(1+i)\beta_2 h']\left\{\cosh[(1+i)\beta_2 h''] + 1\right\}}{(1+i)\beta_2 \cosh[(1+i)\beta_2(h'+h'')]}$$

$$+ \varepsilon\left\{2\sigma h' + \frac{(\sigma - 2\omega)\sinh[(1+i)\beta_1 h']}{(1+i)\beta_1\cosh[(1+i)\beta_1(h'+h'')]} - \frac{(\sigma + 2\omega)\sinh[(1+i)\beta_2 h']}{(1+i)\beta_2\cosh[(1+i)\beta_2(h'+h'')]}\right\}, \quad (34)$$

$$\frac{2(\sigma^2 - 4\omega^2)\sigma\varepsilon}{g(\gamma + ik)^2} = -2\sigma h''\frac{\Delta\rho}{\rho}$$

$$- \frac{(\sigma - 2\omega)\frac{\Delta\rho}{\rho}\left\{\sinh[(1+i)\beta_1 h'] + \sinh[(1+i)\beta_1(h'+h'')]\right\}}{(1+i)\beta_1\cosh[(1+i)\beta_1(h'+h'')]}$$

$$- \frac{(\sigma + 2\omega)\frac{\Delta\rho}{\rho}\left\{\sinh[(1+i)\beta_2 h'] - \sinh[(1+i)\beta_2(h'+h'')]\right\}}{(1+i)\beta_2\cosh[(1+i)\beta_2(h'+h'')]}$$

$$+ \varepsilon\left\{\frac{(\sigma + 2\omega)\sinh[(1+i)\beta_2(h'+h'')]}{(1+i)\beta_2\cosh[(1+i)\beta_2(h'+h'')]}\right.$$

$$\left. - \frac{(\sigma - 2\omega)\sinh[(1+i)\beta_1(h'+h'')]}{(1+i)\beta_1\cosh[(1+i)\beta_1(h'+h'')]} - 2\sigma(h'+h'')\right\}. \quad (35)$$

These equations in their present form are somewhat intractable. In order to determine the relative importance of the various terms, the dimensionless quantities

$$R = \frac{g \frac{\Delta\rho}{\rho} (\gamma + ik)^2 (h' + h'')}{\sigma^2 - 4\omega^2} , \quad \Theta = \frac{\varepsilon}{\Delta\rho/\rho} , \tag{36}$$

$$B_1 = \beta_1(h' + h'') , \quad B_2 = \beta_2(h' + h''), \tag{37}$$

$$p = \frac{1}{2} - \omega/\sigma , \quad q = \frac{1}{2} + \omega/\sigma, \tag{38}$$

are introduced. Equations 34 and 35 can now be written in the form

$$\frac{(1 - \frac{\Delta\rho}{\rho}\Theta)}{R} = \frac{p \sinh[(1+i)\beta_1 h'] \left\{\cosh[(1+i)\beta_1 h'' + 1\right\}}{(1+i)B_1 \cosh[(1+i)\beta_1(h'+h'')]}$$

$$+ \frac{q \sinh[(1+i)\beta_2 h'] \left\{\cosh[(1+i)\beta_2 h''] + 1\right\}}{(1+i)B_2 \cosh[(1+i)\beta_2(h' + h'')]}$$

$$+ \Theta\left\{\frac{h'}{h' + h''} + \frac{p \sinh[(1+i)\beta_1 h']}{(1+i)B_1 \cosh[(1+i)\beta_1(h'+ h'')]} - \frac{q \sinh[(1+i)\beta_2 h']}{(1+i)B_2 \cosh[(1+i)\beta_2(h'+h'')]}\right\}, \tag{39}$$

$$\frac{\frac{\Delta\rho}{\rho}\,\Theta}{R} = -\frac{h''}{h'+h''} - \frac{p\left\{\sinh[(1+i)\beta_1 h'] + \sinh[(1+i)\beta_1(h'+h'')]\right\}}{(1+i)B_1 \cosh[(1+i)\beta_1(h'+h'')]}$$

$$-\frac{q\left\{\sinh[(1+i)\beta_2 h'] - \sinh[(1+i)\beta_2(h'+h'')]\right\}}{(1+i)B_2 \cosh[(1+i)\beta_2(h'+h'')]}$$

$$+\ \Theta\left\{\frac{q \sinh[(1+i)\beta_2(h'+h'')]}{(1+i)B_2 \cosh[(1+i)\beta_2(h'+h'')]} - \frac{p \sinh[(1+i)\beta_1(h'+h'')]}{(1+i)B_1 \cosh[(1+i)\beta_1(h'+h'')]} - 1\right\}. \quad (40)$$

These equations can be solved for any given values of σ, ω, ν, h', h'', and $\Delta\rho/\rho$. However, a particular numerical solution will not indicate the dependence of k, Υ, ζ_0', and ζ_0'' on the above independent variables. For the most important range of variables it is possible to approximate the relations to give functional relationships between the above quantities.

The following ranges of variables will be considered, using c.g.s. units:

$$h' \geqslant 5 \times 10^3 ,$$

$$5 \times 10^5 \geqslant h'' \geqslant 10^5 ,$$

$$1 \leqslant \nu \leqslant 10^3 ,$$

$$\frac{\Delta\rho}{\rho} \leqslant 10^{-3} ,$$

$$\omega \leqslant .725 \times 10^{-4} ,$$

$$0.7 \times 10^{-4} \leqslant \sigma \leqslant 3 \times 10^{-4} ,$$

with the additional restriction that

$$\sqrt{\frac{\sigma - 2\omega}{2\nu}}\; h' \geqslant 1.2 . \quad (40a)$$

The derived constants then have the following ranges:

$$\beta_1 h', \beta_2 h' \gg 1.2 \quad ,$$

$$\beta_1 h'' \quad \beta_2 h'' \gg 24 \quad ,$$

$$B_1 , B_2 \gg 25 \quad ,$$

$$|p|, |q| \leqslant 1 \quad .$$

Under these conditions the following approximations are valid (to within 10 per cent):

$$\sinh [(1+i)\beta_1 h'] = \cosh [(1+i)\beta_1 h'] = \frac{1}{2} e^{\beta_1 h'} (\cos \beta_1 h' + i \sin \beta_1 h'), \text{ etc.}$$

and $e^{\beta_1 h''}$, $e^{\beta_2 h''} \gg 1$,

and equations (39) and (40) can be approximated by:

$$\frac{(1 - \frac{\Delta\rho}{\rho}\Theta)}{R} = \frac{p}{2B_1} + \frac{q}{2B_2} - i \left[\frac{p}{2B_1} + \frac{q}{2B_2} \right]$$

$$+ \Theta \left\{ \frac{h'}{h' + h''} + \frac{p}{2B_1} \left(\frac{\cos \beta_1 h'' - \sin \beta_1 h''}{e^{\beta_1 h''}} \right) \right. \qquad (41)$$

$$\left. - \frac{q}{2B_2} \left(\frac{\cos \beta_2 h'' - \sin \beta_2 h''}{e^{\beta_2 h''}} \right) - i \frac{p}{2B_1} \left(\frac{\cos \beta_1 h'' + \sin \beta_1 h''}{e^{\beta_1 h''}} \right) + i \frac{q}{2B_2} \left(\frac{\cos \beta_2 h'' + \sin \beta_2 h''}{e^{\beta_2 h''}} \right) \right\}$$

$$\frac{\frac{\Delta\rho}{\rho}\,\Theta}{R} = -\frac{h''}{h' + h''} + i\left[\frac{p}{2B_1} - \frac{q}{2B_2}\right] - \left[\frac{p}{2B_1} - \frac{q}{2B_2}\right]$$

$$+ \Theta\left\{-1 - \frac{p}{2B_1} + \frac{q}{2B_2} + i\ \frac{p}{2B_1} - \frac{q}{2B_2}\right\} . \qquad (42)$$

The unknown quantities are now split into real and imaginary parts:

$$R = S + i\,T = S_0\,(1 + S_1 + i\,T_1)\ , \qquad (43)$$

$$\Theta = \Phi + i\Psi = \Phi_0\,(1 + \Phi_1 + i\Psi_1)\ , \qquad (44)$$

where $S_0 = -\frac{(h' + h'')^2}{h'h''}$ and $\Phi_0 = -\frac{h''}{h' + h''}$, the limiting values for the unknowns when $\nu \rightarrow 0$. Upon substitution of these expressions into equations (41) and (42) and replacement of $\frac{p}{2B_1}$ by P, $\frac{q}{2B_2}$ by Q, the result is:

$$\frac{1 + \frac{\Delta\rho}{\rho}\frac{h''}{h' + h''}(1 + \Phi_1 + i\,\Psi_1)}{-\frac{(h' + h'')^2}{h'h''}(1 + S_1 + i\,T_1)} = P + Q - i(P + Q) - \frac{h''}{h' + h''}(1 + \Phi_1 + i\Psi_1)$$

$$\left\{\frac{h'}{h' + h''} + P\,\frac{\cos\beta_1 h'' - \sin\beta_1 h''}{e^{\beta_1 h''}} - Q\,\frac{\cos\beta_2 h'' - \sin\beta_2 h''}{e^{\beta_2 h''}}\right.$$

$$\left. - iP\,\frac{\cos\beta_1 h'' + \sin\beta_1 h''}{e^{\beta_1 h''}} + iQ\,\frac{\cos\beta_2 h'' + \sin\beta_2 h''}{e^{\beta_2 h''}}\right\} , \qquad (45)$$

$$\frac{-\frac{\Delta\rho}{\rho}\frac{h''}{h'+h''}(1+\Phi_1+i\Psi_1)}{-\frac{(h'+h'')^2}{h'h''}(1+S_1+iT_1)} = -\frac{h''}{h'+h''} + i(P-Q) - (P-Q) + \frac{h''}{h'+h''}(1+\Phi_1+i\Psi_1)$$

$$\left\{1 + P - Q - i(P-Q)\right\} . \qquad (46)$$

These equations can be solved by approximation in a straightforward manner if S_1, T_1, Φ_1, and Ψ_1 are all sufficiently small so that powers higher than the first can be neglected, to give

$$(S_1 + iT_1) = \frac{\Delta\rho}{\rho}\frac{h''}{h'+h''}\left(1 - \frac{h'}{h'+h''}\right) + \left\{\frac{(h'+h'')^2}{h'h''} - \frac{h'}{h''}\right\} P + \left\{\frac{(h'+h'')^2}{h'h''} + \frac{h'}{h''}\right\} Q$$

$$- i\left\{\frac{(h'+h'')^2}{h'h''} + \frac{h'}{h''}\right\} P - i\left\{\frac{(h'+h'')^2}{h'h''} - \frac{h'}{h''}\right\} Q , \qquad (47)$$

$$(\Phi_1 + i\Psi_1) = \frac{\Delta\rho}{\rho}\frac{h'h''}{(h'+h'')^2} + \frac{h'}{h''}(P-Q) - i\frac{h'}{h''}(P-Q) - (P+Q)\frac{\Delta\rho}{\rho} + i\frac{\Delta\rho}{\rho}(P+Q). \qquad (48)$$

Now the following expressions can be written for the unknowns:

$$\frac{g\frac{\Delta\rho}{\rho}(h'+h'')}{\sigma^2 - 4\omega^2}(\gamma + ik)^2 = -\frac{(h'+h'')^2}{h'h''}(1 + S_1 + iT_1) , \qquad (49)$$

and

$$\frac{\varepsilon}{\Delta\rho/\rho} = -\frac{h''}{h'+h''}(1 + \Phi_1 + i\Psi_1) . \qquad (50)$$

In most cases of interest the terms in equations (47) and (48) containing $\frac{\Delta\rho}{\rho}$ will be negligibly small and $\frac{h'}{h''} \ll \frac{(h'+h'')^2}{h'h''}$. Equations (47) and (48) then can be written:

$$(S_1 + iT_1) = \frac{(h'+h'')^2}{h'h''} (P + Q)(1 - i) , \qquad (51)$$

$$(\Phi_1 + i\Psi_1) = \frac{h'}{h''} (P - Q)(1 - i) . \qquad (52)$$

Approximate solutions of these equations for the wave number, K, and the damping coefficient, γ, of the internal waves are:

$$k = \sqrt{\frac{(\sigma^2 - 4\omega^2)(h'+h'')}{g\, h'\, h''\, \Delta\rho/\rho}} \left\{1 + \frac{(h'+h'')\ \nu^{1/2}\left[(\sigma - 2\omega)^{3/2} + (\sigma + 2\omega)^{3/2}\right]}{4\sqrt{2}\ \ h'h''\ \ \sigma(\sigma^2 - 4\omega^2)^{1/2}}\right\}, \qquad (53)$$

$$\gamma = \left(\frac{\nu}{2g\,\Delta\rho/\rho}\right)^{1/2} \left(\frac{h'+h''}{h'h''}\right)^{3/2} \left\{\frac{(\sigma - 2\omega)^{3/2} + (\sigma + 2\omega)^{3/2}}{4\sigma}\right\} . \qquad (54)$$

The solution for ε is then:

$$\varepsilon = -\frac{\Delta\rho}{\rho}\frac{h''}{h'+h''}\left[1 - \frac{h'}{h''}(1-i)\frac{(2\nu)^{1/2}\left\{(\sigma + 2\omega)^{3/2} - (\sigma - 2\omega)^{3/2}\right\}}{4\sigma(h'+h'')\ \ (\sigma^2 - 4\omega^2)^{1/2}}\right] . \qquad (55)$$

DISCUSSION OF RESULTS

The wave number, k, is increased, thus wave length and velocity are decreased by the effect of eddy viscosity. The correction term in the expression for the wave number varies as the square root of the eddy viscosity and inversely as the depth of the shallower layer. Its magnitude is small unless the restriction of equation (40a) is violated by approach to the inertial period.

The damping coefficient, γ, varies directly as the square root of the eddy viscosity, and inversely as both the square root of the density difference and the three-halves power of the depth of the shallower layer. It increases approximately as the square root of the frequency but has only small dependence on latitude in the range under consideration.

The ratio of free surface to interface displacement is decreased in magnitude by friction with a decrease in time lag for the free surface motion. Again the correction terms vary directly as the square root of the eddy viscosity and are small unless the inertial period is approached.

Representative values for k and γ under various typical conditions are given in Table 1.

The distribution of internal waves will depend very strongly on the value of the damping coefficient γ. This quantity varies from a maximum of 6.0×10^{-3} to a minimum of 0.045×10^{-3} for the conditions considered. That is, wave amplitudes would be decreased by 50% in a distance of 120 km. and 15,000 km., respectively. The actual distance of wave travel will depend strongly on the prevailing oceanographic conditions and the wave period but not on the latitude within the restrictions of equation (40a).

Let us consider a typical example. Since the depth of maximum stability is that of maximum velocity shear, the effective eddy viscosity is taken to be 1 $cm.^2 sec^{-1}$. An average value of 100 m. is used for the upper layer depth and 10^{-3} $gms./cm.^3$ for the density difference across the interface. Then at latitude 15° N, k and γ will be, respectively, .15 km^{-1}. and 0.50×10^{-3} km^{-1}. for a 12 hr. period and .055 km^{-1}. and 0.32×10^{-3} km^{-1}. for a 24 hr. period. The 12 hr. and 24 hr. waves have, respectively, wave lengths of 42 km. and 110 km. and distances for a 50% amplitude decrease of 1,400 km. and 2,100 km. Figure 1 shows the variation of γ with the depths of the upper and lower layers. These results are the right order of magnitude to give regions of strong internal waves surrounding bottom irregularities and yet not large enough to give standing waves over the entire ocean.

TABLE 1

Values of wave number and damping coefficient under varying conditions.

L (degrees)	T (hrs)	h' (m)	h" (m)	ν ($cm^2 sec^{-1}$)	$\Delta\rho$ ($gms.cm^{-3}$)	k (km^{-1})	γ (km^{-1})
50	12	100	1000	100	10^{-3}	.10	6.0×10^{-3}
30	12	100	1000	100	10^{-3}	.14	5.4×10^{-3}
15	12	100	1000	100	10^{-3}	.15	5.0×10^{-3}
15	24	100	1000	100	10^{-3}	.058	3.2×10^{-3}
15	24	100	1000	10	10^{-3}	.056	1.0×10^{-3}
15	24	100	1000	1	10^{-3}	.055	0.32×10^{-3}
15	24	500	1000	1	10^{-3}	.027	0.045×10^{-3}
15	24	100	1000	1	2×10^{-3}	.039	0.23×10^{-3}
0	12	100	1000	1	10^{-3}	.15	0.49×10^{-3}
0	24	100	1000	1	10^{-3}	.076	0.35×10^{-3}

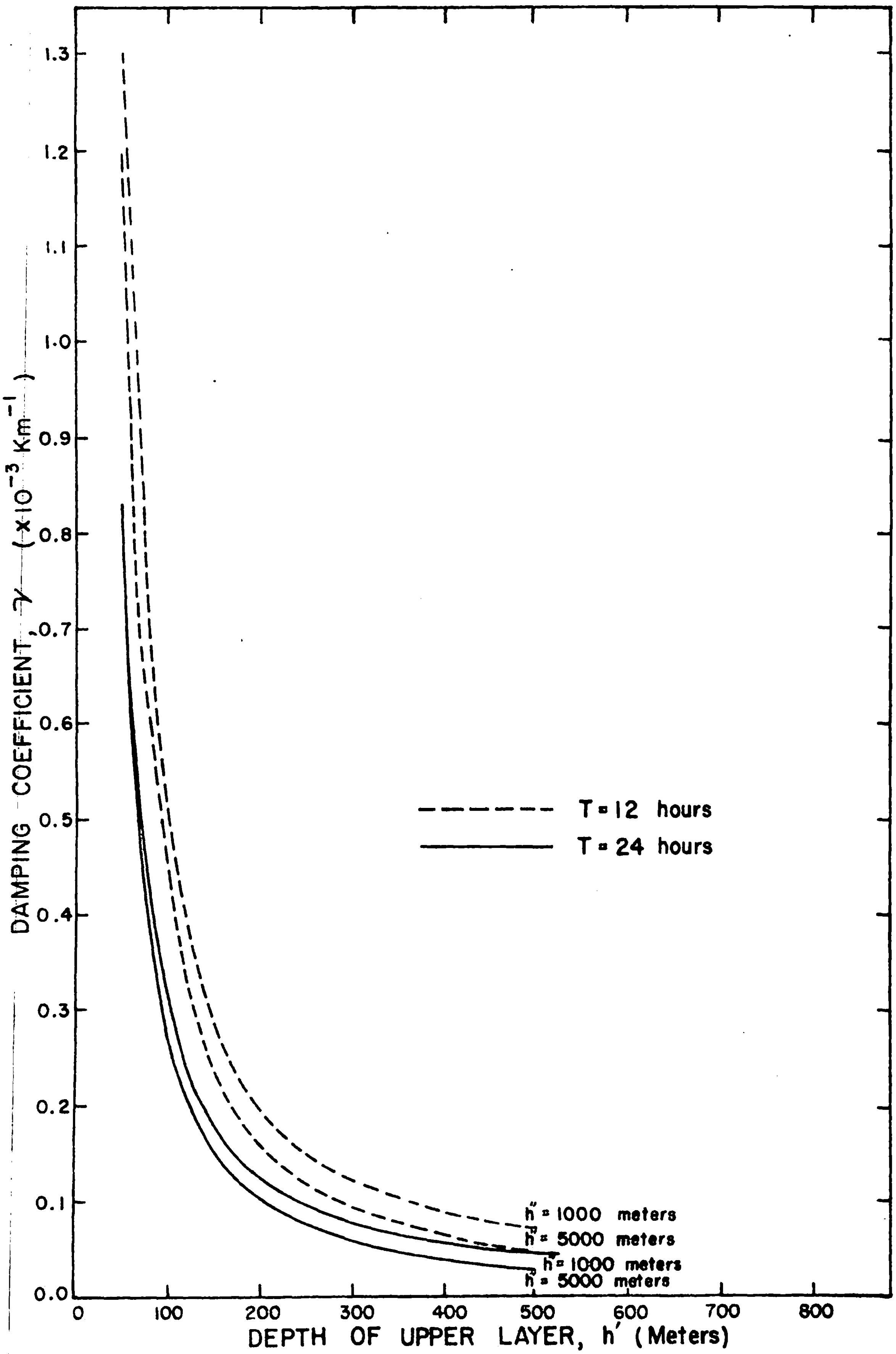

FIGURE 1. Damping Coefficient Variation with Layer Depth ($L = 15^{\circ}$, $\nu = 1$ cm.^{2} sec.$^{-1}$, $\Delta\rho = 10^{-3}$ gms. cm.$^{-3}$).

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