Brownian particles interacting with a Newtonian Barrier: Skorohod maps and their use in solving a PDE with free boundary, strong approximation, and hydrodynamic limits.

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Abstract

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Chair of the Supervisory Committee:
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In this thesis, we pioneer the use of Skorohod maps in establishing the hydrodynamic behavior of an interacting particle system. This technique has the benefit of using stochastic methods to show both existence and uniqueness of the resulting PDE with free boundary condition. In 2001, Frank Knight constructed a stochastic process modeling the one dimensional interaction of two particles, one being Newtonian in the sense that it obeys Newton's laws of motion, and the other particle being Brownian. In the first chapter we construct a multi-particle analog, using Skorohod map estimates in proving a propagation of chaos and characterizing the hydrodynamic limit as the solution to a PDE with free boundary condition. The resulting PDE is similar to the solution of the Stefan problem. As mentioned, both existence and uniqueness of the PDE are done using stochastic methods; the uniqueness is done using a novel, and new, coupling method.

In the second chapter, we give a strong approximation of Brownian motion with inert drift. We also determine the distribution of the maximum of the Newtonian particle via its Laplace transform.

In the third chapter, we consider a random walker on the nonnegative lattice, moving in continuous time, whose transition rate is a linear function of the time the walker spends at the origin. In this way the walker is a jump process with a stochastic and adapted intensity.

When Brownian scaling is introduced, such a process converges to Brownian motion with inert drift. This solves a conjecture of Burdzy and White in 2008. This convergence result is used to show two Brownian motions separated by an inert particle has a product stationary distribution on its state space where the velocity of the inert particle is Gaussian. This process of two Brownian motions separated by an inert particle was studied by White, [45, section 4], where the demonstration of existence for the process contains a nontrivial gap that we complete.

TABLE OF CONTENTS

LIST OF FIGURES

ACKNOWLEDGMENTS

First and foremost, *soli Deo gloria.* Whether or not you believe the big bang and "Let there be light" are mutually compatible, I thank you, dear reader, for being a member of the exclusive class who has opened this document.

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[1] In Proposition 3.9 of this thesis I required [6, Theorem 2.9] (a paper by Burdzy, Chen and Sylvester), but with weaker assumptions. Burdzy was out of the country when I discovered this, so I emailed him pointing out "[t]here may be a critical error which may ruin the entire hydrodynamic limit thing....I would need an extension of Theorem 2.9 in the one dimensional case. What do you think? I'm a bit worried that what I've done so far would be useless." The truth is that I was more than "a bit" worried. Burdzy assuaged my immediate fear. "Do not panic. Is there a mathematical reason why [the] theorem from BCS would not work for C^2 boundaries? If not, I guess that the theorem works for C^2. Examine the proof and see whether it works for C^2 (in 1-dim case)." He was correct. "Thanks, Professor. Who could know math is this emotional! You are correct in that the 1-dim case yields many blessings. I've checked the first few theorems and they stand. I'm less panicked now, though still wary. How has your trip been thus far?" The extension of [6, Theorem 2.9] is carried out in appendix 5 to the first chapter.

will cherish the memories made during this time.

At the beginning of my work with Chris, our meetings were joined by my academic brothers Tvrtko Tadić and Wai-Tong "Louis" Fan, both of whom were, and are, great role models. I appreciate the support of my academic brothers Mauricio Duarte and Sayan Banerjee whom I overlapped with briefly. I have the pleasure of catching up with them during conferences. Tvrtko is driven by an excellent work ethic. Thanks for being a great gym partner and for being an older brother in multiple ways. Andrey Sarantsev is a хороший и настоящий друг (good and true friend), and is always willing to share a meal, discuss life, or review papers I send him. Coincidentally, he was my grader for real analysis. I had the unique privilege of introducing him to peanut butter and jelly sandwiches. He agrees that it's a food item easily enjoyed on a regular basis. Tvrtko had the unique privilege of introducing me to octopus; a food item enjoyed less easily on a regular basis than peanut butter and jelly! During these early years I met Sudip Paul, my Bengali brother!

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To my friends, thanks for the hikes, the game nights, the late night rides listening to music, the discussions, the runs, the gym workouts, the Thanksgiving dinners, the companionship, and taking care of me when I'm in your city: The Gonzalez, Dailey, and Platter families, Josh Swanson, Stephen McKeown, Nick Reichert and family, Arya Nazari, Matt Junge, Jacob Richey, Manar Riman, Kristin Devleming, James Cameron, Sean McCurdy, Tim Mesikepp, Avi Levy, Jessica Merhej, Gordon Brown, Sid Mathur, Darcy Camargo, Carl-Erik Gauthier, Ben and Cami Palacios, Ben Farhner and family, Chloe Krakauer, Sudip Paul and family, Lucas and Anike Braune, Spencer Gibbs and family, Haik Kalantarian and family, Kevin and Guang Osborne, and, of course, Andy Johnson and his family. A shout out to my Bengali family for making me a plethora of wonderful dishes: Chintu and Monika Das and sons, Sudip Paul with Dupitka Paul and Ananya Paul, Nilanjana Laha, and Moumanti Podder. Yes, I listed Sudip more than once. To my extended family: the Barnes clan (every one!) and Dan, Rhoda, Tim and James Parrott, as well as Pete Erkeneff, Terri Baker and Mocha Man (Bill Martin), and Boots, Steve, Duncan, and Charlotte. I'm sure I missed someone.

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DEDICATION

To my family, Clayton, Priscilla, Carina, and Joy Barnes, and my grandparents, Walter and Mary (Erkeneff) Parrott

Chapter 1

HYDRODYNAMIC LIMIT OF BROWNIAN PARTICLES WITH A NEWTONIAN (INERT) BARRIER

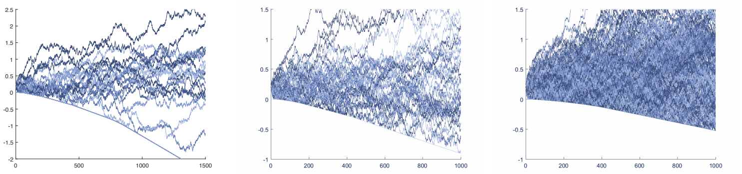

Figure 1.1: Simulations of 20, 40, and 200 Brownian particles reflecting from the Newtonian barrier.

1 *Introduction*

1.1 Description of the model

This chapter characterizes the hydrodynamic limit for a multiparticle generalization of a process originally introduced by Knight [33]. We start with an informal description of the model. Consider n Brownian particles $X_1^{(n)}(t), \dots, X_n^{(n)}(t)$ on the real line, reflecting from the same side of a moving barrier $Y^{(n)}(t)$. The moving barrier is "massive" in the sense that it is not Brownian but obeys Newton's laws of motion. By this we mean the barrier is modeled to have momentum, and that it experiences an impulse upon colliding with one of the Brownian particles. Impulse is equivalent to change in momentum, and in Newtonian physics is proportional to the change in velocity. In this way the Brownian particles drive the massive barrier by increasing (or decreasing, depending on sign) its velocity. We assume the Brownian particles have an equal "mass" of n^{-1} so the total mass of the system remains at unity, and we fix a constant $K \geq 0$, the *impulse constant*, which determines the strength of the Brownian particles' interaction with the massive barrier. Increasing K will give the Brownian particles more ability to increase the massive barrier's velocity. If $K = 0$ the Brownian particles have no influence on the barrier, the Brownian particles become independent reflecting Brownian motions while the barrier will travel with constant speed. If $K > 0$, however, the Brownian particles are dependent. This can be seen intuitively, for in the event that one Brownian particle happens to impart a large impulse to the massive barrier, it influences the barrier's trajectory and alters the region where the other Brownian particles are allowed to disperse themselves.

We now present a formal mathematical model describing the above scenario, and begin by assuming the standard setting: a filtered probability space $(\Omega, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ with the filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. Take i.i.d. $(\mathcal{F}_t)_{t\geq 0}$-Brownian motions $B^{(1)}, \ldots, B^{(n)}$, a coefficient $K \geq 0$ and an initial velocity $v \in \mathbb{R}$ for the massive particle. Consider continuous $\mathcal{F}_t$ adapted processes which satisfy the system of stochastic differential equations for $t \in [0, T]$ and $i = 1, \ldots, n$:

$$\begin{aligned}
&dX_i^{(n)} = dB^{(i)} + dL_i^{(n)}, \\
&dY^{(n)} = V^{(n)}(t)dt := \Big(v - \frac{K}{n}\sum_{i=1}^{n} L_i^{(n)}(t)\Big)dt, \\
&X_i^{(n)}(t) \geq Y^{(n)}(t), \text{ for all } t, \text{ almost surely,} \\
&L_i^{(n)} \text{is nondecreasing, and is flat off the set } \{s : X_i^{(n)}(s) = Y^{(n)}(s)\}.
\end{aligned} \tag{1.1}$$

By flat we mean

$$\int_{\mathbb{R}} 1(X_i^{(n)}(s) > Y^{(n)}(s))dL_i^{(n)}(s) = 0.$$

In other words, $L_i^{(n)}$ increases only when $X_i^{(n)}$ makes contact with $Y^{(n)}$. These conditions imposed on $L_i^{(n)}$ imply that $L_i^{(n)}$ is the local time of $X_i^{(n)}$ on $Y^{(n)}$, which we will see in the proof of Proposition 2.12. We can in fact let

$$L_i^{(n)}(t) = \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_0^t 1_{[0,\epsilon]}(X_i^{(n)}(s) - Y^{(n)}(s))ds \text{ for all } t \in [0, T], \text{ almost surely,}$$

which is our definition of the local time of $X_i^{(n)}$ on $Y^{(n)}$. We assume the initial conditions $Y^{(n)}(0) = 0$, $V^{(n)}(0) = v$ and that $X_i^{(n)}(0)$, $i = 1, \ldots, n$ are drawn from i.i.d. samples of an $L^p([0, \infty))$ random variable with $p \geq 1$. In this case we require $\mathcal{F}_0$ to be large enough to contain $\sigma\{X_i^{(n)}(0) : 1 \leq i \leq n\}$. See the figure above for sample path realizations. Existence of a strong solution to this system is proved in Proposition 2.12.

A system $(X_1^{(n)}, \dots, X_n^{(n)}, Y^{(n)}, V^{(n)})$ satisfying (3.1) above will be called a *system of Brownian particles reflecting from a massive barrier* with *impulse coefficient* K. The processes $X_1^{(n)}, \dots, X_n^{(n)}$ are the *Brownian particles,* $Y^{(n)}$ is the *reflecting barrier* with $V^{(n)}$ its *velocity.*

1.2 Free boundary problem

In Theorem 1.2 we characterize the hydrodynamic limit of the empirical process together with the random barrier. The hydrodynamic limit is the solution to a free boundary problem given as a pair $(p(t,x), y(t))$, both of which interact according to the PDE below. We think of $p(t,x)$ as the temperature at time-space location (t,x) and $y(t)$ as an insulating barrier. In our convention the heat is concentrated above the insulating barrier. So $p(t,\cdot)$ is supported in $[y(t),\infty)$. Our initial temperature distribution is given by π_0 which need not have a continuous density.

$$\begin{aligned}
&\frac{\partial p(t,x)}{\partial t} = \frac{1}{2}\Delta_x p, \;\; x > y(t), \\
&\frac{\partial^+ p(t,x)}{\partial x^+} = -2y'(t)p(t,x), \;\; x = y(t), \\
&y''(t) = -(K/2)p(s,y(s)), \, y(0) = 0, \, y'(0) = v \in \mathbb{R}, \\
&\lim_{t\downarrow 0} p(t,x)\mathrm{d}x = \pi_0(\mathrm{d}x),
\end{aligned} \tag{1.2}$$

The second condition is a one sided derivative on the positive side, and is mathematically equivalent to conservation of heat. That is, the function $y(t)$ acts as an insulating barrier. The third condition says the insulating barrier has an acceleration proportional to its temperature. This is contrasted with the Stefan problem mentioned in Section 1.4 in that the barrier reflects the heat back into the domain rather than absorbing it, and its acceleration is proportional to its temperature as opposed to the velocity being proportional to the heat flux. The unique solution will be one in which the equalities above hold in the classical sense. That is, $p(t,x)$ is a differentiable function in its domain $\{(t,x) : 0 \leq t \leq T, \, x \geq y(t)\}$, which is C^1 in time, C^2 in space, and where $y \in C^2([0,T],\mathbb{R})$.

1.3 Main results

Theorem 1.1. *There exists a unique classical solution to the free boundary problem* (1.2).

For the hydrodynamic limit we consider the empirical measure

$$\pi_t^{(n)} = \frac{1}{n} \sum_{i=1}^{n} \delta_{\{X_i^{(n)}(t)\}}.$$

For fixed $t \geq 0$, $\pi_t^{(n)}$ is a random variable with values in the space $\mathcal{P}(\mathbb{R})$. For a time horizon $T > 0$, $\{\pi_t^{(n)} : t \in [0, T]\}$ is a process with paths in the space $C([0, T], (\mathcal{P}, W_p))$ with metric

$$\|\nu' - \nu''\|_{[0,T]} := \max_{t \in [0,T]} W_p(\nu'(t), \nu''(t)).$$

That this process indeed has a.s. continuous paths is proved in Lemma 3.13. In other words, $\{\pi_t^{(n)} : t \in [0, T]\}$ is a continuous measure-valued process. As such, $\pi^{(n)}$ induces a probability measure on $C([0, T], (\mathcal{P}, W_p))$. The hydrodynamic limit characterizes this distribution for large n.

Theorem 1.2. *Assume that*

$$W_p(\pi_0^{(n)}, \pi_0) \longrightarrow 0, \; n \to \infty,$$

for $p \geq 1$ *and where* π_0 *has support in* $[0, \infty)$. *Then*

$$(\pi^{(n)}, Y^{(n)}) \longrightarrow (p(t, w)dw, y(t)), \; n \to \infty, \tag{1.3}$$

in the Prohorov metric on $C([0, T], (\mathcal{P}, W_p) \times \mathbb{R})$ *where* $y \in C^2([0, T], \mathbb{R})$, $p(t, x)$ *is a probability density supported in* $[y(t), \infty)$, *and with* $(p(t, x), y(t))$ *solving* (1.2).

The proof is at the end of Section 3.

The third result is the propagation of chaos, which means the dependence of any finite collection of tagged particles disappears as the number of particles tends to infinity.

Theorem 1.3. *Assume* $X_i^{(n)}(0) = \xi_i$, $i = 1, \ldots, n$, *where* $\xi_i, i \in \mathbb{N}$ *are i.i.d. samples of nonnegative integrable random variable. Fix positive integers* $i_1, \ldots, i_k$. *Then*

$$(X_{i_1}^{(n)}, \ldots, X_{i_k}^{(n)}) \to (X_{i_1}^{(\infty)}, \ldots, X_{i_k}^{(\infty)}) \ \text{ a.s. as } \ n \to \infty,$$

where the limit consists of independent processes $X_{i_1}^{(\infty)}, \ldots, X_{i_k}^{(\infty)}$.

The ξ_i are given so the processes have an initial condition which does not depend on n in the triangular array. This ensures that after $n \geq \max i_k$ the initial conditions for the $X_{i_1}^{(n)}, \ldots, X_{i_k}^{(n)}$ are all defined and unchanging with n. The proof is in Section 3.

1.4 Organization of paper

The paper is organized as follows. A short historical background for the origin of our model and a brief on related hyrodynamic limits is in Section 1.5 below.

In Section 2, we construct the processes $X_i^{(n)}$ *path-by-path* on any probability space supporting an infinite sequence of i.i.d. Brownian motions $B^{(1)}, B^{(2)}, \ldots, B^{(n)}$, as well as the initial random variables $X_i^{(n)}(0)$ for all $1 \leq i \leq n, n \in \mathbb{N}$. We do this by constructing a functional to which we apply pathwise to the n Brownian motions $B^{(1)}, \ldots, B^{(n)}$. In Proposition 2.12 we show this pathwise construction gives a system of processes satisfying (3.1). Such a method for reflected processes is called a Skorohod map, since Skorohod used the method to construct a reflected Brownian motion on the positive half-line $\mathbb{R}_+ := [0, \infty)$. For instance, if $B(t)$ is a standard Brownian motion and $m(t) = \sup_{0<s<t}[-B(s) \vee 0]$, then $B(t) + m(t)$ has the same distribution as X, where $dX = dB + dL$ and L is the semimartingale local time of X at zero; see [30, Section 3.6C]. Here $m(t)$ would be the Skorohod map which corresponds to reflected Brownian motion.

The proofs of Theorems 1.2 and 1.3 are in Section 3. We use the estimates derived in the second section to demonstrate almost sure convergence of the barrier $Y^{(n)}$ to a unique deterministic function y in the form of a functional strong law of large numbers; see Propositions 3.5 and 3.7. Here we introduce properties of the measure-valued process $\pi^{(n)}$ mentioned above. In Proposition 3.19, we prove uniform stochastic equicontinuity, which is stronger than the typical stochastic equicontinuity necessary for tightness of processes in some metric space. We conclude the paper with Section 4, where we use our stochastic tools to prove uniqueness of the free boundary problem described in Section 1.2.

1.5 Background

As previously mentioned, Knight introduced this model in the case of one Brownian particle where he studied density of the final velocity [33]. Later, White [45] generalized Knight's construction and studied several related processes. This inspired a higher dimensional version of a reflected process whose velocity vector is proportional to the boundary local time, and the stationary distribution of its position and velocity was studied by Bass, Burdzy, Chen, and Hairer [3]. For a history of hydrodynamic limits see [24] and [11]. The methods used to establish a hydrodynamic limit are varied. See [42], where Varadhan uses entropy methods to examine a spin system on a lattice when the mesh goes to zero. Entropy and relative entropy methods are general methods. However, these are not always feasible. For instance, see [11], where Chen and Fan study a system of particles reflecting from a separating interface. For an introductory reading on hydrodynamic limits, see the book [32] where Kipnis and Landim present a self contained treatment of hydrodynamic limits via the study of the generalized exclusion process and the zero-range process. Other hydrodynamic limit results have biological motivations in neuron modeling. See [38], [12], and [26, Chapter 4.3]. Hydrodynamic limits are related to the theory of partial differential equations since the empirical measure of the particles converge to a solution of a PDE or free boundary problem.

In [10] Chayes and Swindle study the one dimensional model of hot random walkers which are emitted by a source and which annihilate cold particles which remain stationary. When a Brownian scaling is introduced, the density of the hot particles together with the cold region converge to a solution of the Stefan problem. The Stefan problem is a free boundary problem modeling the melting of ice next to a heat source. The heat particles are killed upon reaching the ice boundary, i.e. a Dirichlet boundary condition is imposed at the ice barrier, while the melting of this ice barrier is proportional to the flux of heat across it. In this way the density of heat and the ice barrier interact, producing the free boundary effect. The hydrodynamic limit we study in this paper resembles that of the Stefan problem but with some distinctive features; see (1.2).

This article is the first in which continuity properties of Skorohod maps are used to demonstrate a hydrodynamic limit; see Section 2. By applying this method with a stochastic representation (Corollary 3.11) we prove existence and uniqueness of the free boundary problem without relying on existence and uniqueness theorems from the theory of PDEs. Properties of the transition density for Brownian motion reflected in a time varying domains is a key ingredient for a stochastic representation of the PDE with free boundary; see [7]. This is the first existence and uniqueness result for the free boundary problem we study, as it seems not to be subsumed by known results in the analysis literature; see [16], [17], [18] for existence and uniqueness of the Stefan problem.

Notation

For ease of reference we introduce notation which will be used along the paper. First let (E, d) be a metric space.

1.1. $C(E_1, E_2)$ is the space of continuous functions from (E_1, d_1) to (E_2, d_2).

1.2. $\mathcal{P}(E)$ is the space of probability measures on E. We may abbreviate $\mathcal{P}(\mathbb{R})$ as $\mathcal{P}$.

1.3. The *Prohorov metric* is the metrization of distributional convergence for the space $\mathcal{P}(E)$. This is also a metric on the space of E-valued random variables through their induced measure on E.

1.4. For $f \in C([0,T],\mathbb{R})$ and $[a,b] \subset [0,T]$

$$\|f\|_{[a,b]} := \max_{x\in[a,b]} |f(x)|.$$

1.5. For $f = (f_1,\ldots,f_n) \in C([0,T],\mathbb{R}^n)$ and $[a,b] \subset [0,T]$

$$\|f\|_{[a,b]} := \sum_{i=1}^{n} \|f_i\|_{[a,b]}.$$

1.6. For $p > 0$, $(\mathcal{P}(E), W_p)$ is the space of probability measures on $\mathbb{R}$ together with the *Wasserstein-p* distance

$$W_p(\mu,\nu) := \Big(\inf_{(X,Y)} \mathbb{E} d(X,Y)^p \Big)^{1/p}$$

where the infimum is taken over pairs (X,Y) defined on the same probability space, with $X \overset{d}{=} \mu$ and $Y \overset{d}{=} \nu$. If (E,d) is complete then so is $(\mathcal{P}(E), W_p)$. We consider $p \geq 1$. See [43].

1.7. For $f \in C([0,T],(E,d))$ and $\delta > 0$ we define the modulus of continuity for f by

$$\omega_{(E,d)}(f,\delta) := \sup_{\substack{0<s<t<T \\ |t-s|<\delta}} d(f(t), f(s)).$$

1.8. When $\nu_t \in C([0,T],(\mathcal{P},W_p))$ we let $\omega'(\nu,\delta) := \omega_{(\mathcal{P},W_p)}(\nu,\delta)$.

2 *Skorohod Map: Construction and Estimates*

In this section we construct the system given in (3.1) by applying a Skorohod map to the collection of Brownian paths. First we recall the classical Skorohod equation.

Lemma 2.1 (Skorohod, see [30])**.** *Let $f \in C([0,T],\mathbb{R})$ with $f(0) \geq 0$. There is a unique, continuous, nondecreasing function $m(t)$ such that*

$$x_f(t) = f(t) + m_f(t) \geq 0,$$
$$m_f(0) = 0,\ m_f(t) \textit{ is flat off } \{s : x_f(s) = 0\},$$

and is given by

$$m_f(t) = \sup_{0<s<t} [-f(s)] \vee 0.$$

Remark 2.2. As stated in the introduction, flatness off $\{z : x_f(z) = 0\}$ for m_f means $\int_0^t 1(x_f(s) > 0)dm_f(s) = 0$. The classical Lévy's theorem says when f is replaced by a Brownian motion, the corresponding process x_f is distributed as $|B|$.

Remark 2.3. The solution of the Skorohod equation has a time shift property: For any $0 \leq s \leq t \leq T$,

$$x_f(t) = x_g(t-s),$$

where $g(t) = x_f(s) + f(t) - f(s)$.

The following lemmas will be useful later when proving tightness of our processes; see Lemma 3.1.

Lemma 2.4. *Let $f, g \in C([0,T],\mathbb{R})$ and assume that $f \geq g$. Then*

$$m_f(t) \leq m_g(t), \textit{ for all } t \in [0,T].$$

Proof. From Lemma 1.2,

$$m_f(t) = \sup_{0<u<t} [-f(u)] \vee 0 \leq \sup_{0<u<t} [-g(u)] \vee 0 = m_g(t).$$

□

Lemma 2.5. *Let* $f, y_1, y_2 \in C([0,T],\mathbb{R})$ *and assume that* $y_1(0) = y_2(0)$, $f(0) + y_1(0) \geq 0$, *and*

$$y_1(t) - y_1(s) > y_2(t) - y_2(s) \text{ for all } 0 \leq s < t \leq T. \tag{2.1}$$

Then

$$m_{f+y_2}(t) - m_{f+y_2}(s) \geq m_{f+y_1}(t) - m_{f+y_1}(s), \text{ for all } 0 \leq s < t \leq T,$$

where $m_{f+y_i}, i = 1,2$ *correspond to the solution of the Skorohod problem provided by Lemma 1.2.*

Proof. We first show that $x_{f+y_1}(t) \geq x_{f+y_2}(t)$ for all $t \in [0,T]$. That this holds for $t = 0$ is guaranteed by the assumption on the initial conditions, which imply $x_{f+y_1}(0) = x_{f+y_2}(0)$. Assume for the sake of contradiction that there is some $t^* \in [0,T]$ such that $x_{f+y_2}(t^*) > x_{f+y_1}(t^*) \geq 0$. Let

$$\tau = \sup\{t < t^* : x_{f+y_2}(t) = 0\}$$

be the last zero of x_{f+y_2} before time t^*. It follows that m_{f+y_2} is flat on the interval $[\tau, t^*]$. In other words,

$$0 = m_{f+y_2}(t^*) - m_{f+y_2}(\tau) \leq m_{f+y_1}(t^*) - m_{f+y_1}(\tau). \tag{2.2}$$

By shifting the Skorohod solution by time τ as in Remark 1.4, using (2.2), the fact that $x_{f+y_1}(\tau) \geq 0 = x_{f+y_2}(\tau)$, and assumption (2.1),

$$\begin{aligned} x_{f+y_1}(t^*) &= x_{f+y_1}(\tau) + f(t^*) - f(\tau) + y_1(t^*) - y_1(\tau) + m_{f+y_1}(t^*) - m_{f+y_1}(\tau) \\ &\geq x_{f+y_2}(\tau) + f(t^*) - f(\tau) + y_2(t^*) - y_2(\tau) + m_{f+y_2}(t^*) - m_{f+y_2}(\tau) \\ &= x_{f+y_2}(t^*) \end{aligned}$$

which contradicts the definition of t^*. Therefore $x_{f+y_1}(t) \geq x_{f+y_2}(t)$ for all $t \in [0,T]$.

For a fixed $s \in [0,T]$ let

$$g_i(t) = x_{f+y_i}(s) + f(t) - f(s) + y_i(t) - y_i(s) \text{ for } s \leq t \leq T,$$

and $i = 1, 2$. The assumption (2.1) on y_i together with the fact that $x_{f+y_1} \geq x_{f+y_2}$ imply $g_1(t) \geq g_2(t)$. Apply Lemma 2.4 to g_1, g_2 and shift time by s as in Remark 1.4 to see

$$m_{f+y_1}(t) - m_{f+y_1}(s) = m_{g_1}(t-s) \leq m_{g_2}(t-s) = m_{f+y_2}(t) - m_{f+y_2}(s),$$

proving the result. $\square$

We construct a generalization of the Skorohod map for $f = (f_1, \ldots, f_n) \in C([0,T], \mathbb{R}^n)$, $v \in \mathbb{R}$ and $K \geq 0$. If any $f_i(0) < 0$, the velocity of our corresponding inert particle will immediately receive a negative jump of $\frac{1}{n}\sum_{i=1}^{n}(f_i(0) \wedge 0)$. Therefore by allowing any initial real velocity we may assume without loss of generality that $f_i(0)$ are nonnegative.

Theorem 2.6. *Corresponding to each* $f = (f_1, \cdots, f_n) \in C([0,T], \mathbb{R}^n)$, $v \in \mathbb{R}, K \geq 0$ *is a pair of continuous functions*

$$(I_f^{(n)}(t), V_f^{(n)}(t)) =: \Gamma_n f(t) \in C([0,T], \mathbb{R}^2)$$

satisfying

$$x_i(t) := f_i(t) + I_f^{(n)}(t) + m_i(t) \geq 0, \tag{2.3}$$

$$m_i(t) \textit{ is flat off } \{t : x_i(t) = 0\}, \tag{2.4}$$

$$V_f^{(n)}(t) = -v + \frac{K}{n}\sum_{i=1}^{n} m_i(t),\ v \in \mathbb{R}, \tag{2.5}$$

$$I_f^{(n)}(t) = \int_0^t V_f^{(n)}(s)ds, \tag{2.6}$$

for all $t \in [0,T]$.

Remark 2.7. It follows from the classical Skorohod equation that

$$m_i(t) = \sup_{0<s<t}[-(f_i(s) + I^{(n)}(s))] \vee 0.$$

This is used in the proof of Proposition 2.12 below.

Proof. Uniqueness: We prove continuity estimates which holds for any solutions of (2.3) - (3.4). That is, assume that (2.3) - (3.4) holds for two collections of functions $f = (f_1, \dots, f_n), g = (g_1, \dots, g_n) \in C([0,T], \mathbb{R}^n)$. Let $(I_f^{(n)}, V_f^{(n)}), (I_g^{(n)}, V_g^{(n)})$ correspond to solutions of the Skorohod problem. Since we are proving uniqueness we are assuming such solutions exist for f, g. By Remark 2.7, $m_i^f(t)$ is the running minimum of $f_i + I_f^{(n)}$ below zero until time t, and the same holds for $m_i^g(t)$. Hence

$$\|m_i^f - m_i^g\|_{[0,t]} \leq \|(f_i + I_f^{(n)}) - (g_i + I_g^{(n)})\|_{[0,t]}.$$

By the triangle inequality, (2.5), (3.4), and (2)

$$\begin{aligned}
\alpha(t) &:= \sum_{i=1}^{n} \|(f_i + I_f^{(n)}) - (g_i + I_g^{(n)})\|_{[0,t]} \\
&\leq \sum_{i=1}^{n} \Big(\|f_i - g_i\|_{[0,t]} \Big) + n \|I_f^{(n)} - I_g^{(n)}\|_{[0,t]} \\
&\leq \|f - g\|_{[0,t]} + K \int_0^t \sum_{i=1}^{n} |m_i^f(s) - m_i^g(s)| ds \qquad (2.7) \\
&\leq \|f - g\|_{[0,t]} + K \int_0^t \sum_{i=1}^{n} \|m_i^f - m_i^g\|_{[0,t]} \\
&\leq \|f - g\|_{[0,t]} + K \int_0^t \alpha(s) ds.
\end{aligned}$$

Now apply Grönwall's inequality to attain

$$\alpha(t) \leq \|f - g\|_{[0,t]} \exp(Kt).$$

Consequently,

$$\begin{aligned}
\|V_f^{(n)} - V_g^{(n)}\|_{[0,t]} &\leq \frac{K}{n} \sum_{i=1}^{n} |m_i^f(t) - m_i^g(t)| \qquad (2.8) \\
&\leq (K/n)\alpha(t) \leq (K\|f - g\|_{[0,t]}/n) \exp(Kt). \qquad (2.9)
\end{aligned}$$

This holds for any f, g and any two pairs $(I_f^{(n)}, V_f^{(n)}), (I_g^{(n)}, V_g^{(n)})$ solving the equations (2.3) - (3.4). Taking $g = f$ in (2.9) shows $(I_f^{(n)}, V_f^{(n)})$ is unique, and Γ_n is well defined granted existence.

Remark 2.8. The case $n = 1$ is in [45].

Existence: To demonstrate existence, we use a limiting procedure to construct the processes $I_f^{(n)}, V_f^{(n)}$ which in turn produce the map Γ_n. For a fixed $\epsilon > 0$, define the functions $I_{M\epsilon}^{\epsilon}, V_{M\epsilon}^{\epsilon}$ as recursively in the intervals $[0, \epsilon], [\epsilon, 2\epsilon], \dots, [(M-1)\epsilon, M\epsilon]$ as follows.

2.1. On the interval $[0, \epsilon]$, simply let $I_{\epsilon}^{\epsilon}(t) = vt$ and $V_{\epsilon}^{\epsilon} = v$.

2.2. Assume we are given $I_{M\epsilon}^{\epsilon}, V_{M\epsilon}^{\epsilon}$. Let

$$I_{(M+1)\epsilon}^{\epsilon}\big|_{[0,M\epsilon)} = I_{M\epsilon}^{\epsilon} \text{ and } V_{(M+1)\epsilon}^{\epsilon}\big|_{[0,M\epsilon)} = V_{M\epsilon}^{\epsilon}.$$

For $t \in [M\epsilon, (M+1)\epsilon)$ let

$$V_{(M+1)\epsilon}^{\epsilon}(t) = \frac{K}{n} \sum_{i=1}^{n} \max_{0 \le u \le M\epsilon} (-[f_i(u) + I_M^{\epsilon}(u)] \vee 0)$$

be the average of the running minimum below zero of $f_i + I_{M\epsilon}^{\epsilon}$ until time $M\epsilon$. Notice that $V_{(M+1)\epsilon}^{\epsilon}$ is piecewise constant on subintervals of $[0, T]$ of the form $[j\epsilon, (j+1)\epsilon), j \in \mathbb{N}$.

2.3. Extend $I_{(M+1)\epsilon}^{\epsilon}$ to $[M\epsilon, (M+1)\epsilon)$ linearly by giving it slope $V_{(M+1)\epsilon}^{\epsilon}$.

2.4. Set $I_f^{(n,\epsilon)}, V_f^{(n,\epsilon)}$ as the functions produced once the recursion covers the interval $[0, T]$.

A couple observations follow easily from this construction. First,

$$I^{(n,\epsilon)}(t) = \int_0^t V^{(n,\epsilon)}(s)ds.$$

Second, $V^{(n,\epsilon)}$ is monotonically increasing, and $I^{(n,\epsilon)}$ is differentiable and convex. By construction

$$\|V^{(n,\epsilon)}\|_{[0,T]} \le |v|T + \frac{K}{n} \sum_{i=1}^{n} \max_{0 \le u \le T} -(f_i(u) \vee 0) < \infty,$$

for every $\epsilon > 0$, and therefore $\{\|V^{(n,\epsilon)}\|_{[0,T]}\colon \epsilon > 0\}$ is a bounded set. Consequently the collection $\{I^{(n,\epsilon)} : \epsilon > 0\}$ is uniformly Lipschitz, and since $I^{(n,\epsilon)}(0) = 0$ for all $\epsilon > 0$ it is pointwise bounded as well. Hence the family $\{I^{(n,\epsilon)} : \epsilon > 0\}$ satisfies the Arzelà-Ascoli criterion. By taking a subsequence $\epsilon_k \to 0$ there is a continuous function $I^{(n)}$ such that

$$\int_0^t V^{(n,\epsilon_k)}(s)ds =: I^{(n,\epsilon_k)}(t) \longrightarrow I(t)$$

uniformly for t in $[0,T]$. By the construction of $V^{(n,\epsilon_k)}$, this implies

$$\begin{aligned} V^{(n,\epsilon_k)}(t) &= \frac{K}{n}\sum_{i=1}^{n} \max_{0\le u\le \lfloor t/\epsilon_k\rfloor \epsilon_k} (-[f_i(u) + I^{(n,\epsilon_k)}(u)] \vee 0) \\ &\longrightarrow \frac{K}{n}\sum_{i=1}^{n} \max_{0\le u\le t} (-[f_i(u) + I^{(n)}(u)] \vee 0) \end{aligned}$$

uniformly for t in $[0,T]$, as $\epsilon_k \to 0$. Set

$$m_i(t) = \max_{0\le u\le t} -[f_i(u) + I^{(n)}(u)] \vee 0,$$

so that

$$V^{(n)}(t) = v + \frac{K}{n}\sum_{i=1}^{n} m_i(t).$$

Then m_i is flat off $\{s : f_i(s) + I^{(n)}(s) + m_i(s) = 0\}$ by Skorohod's lemma 1.2. By the dominated convergence theorem,

$$I^{(n)}(t) = \int_0^t V^{(n)}(s)ds,$$

and clearly $f_i(t) + I^{(n)} + m_i(t) \ge 0$. Therefore $(I^{(n)}, V^{(n)})$ satisfy the equations (2.3)–(3.4).

□

We state the bounds attained in (2.9) as we have shown that $V^{(n)}$, as a map between function spaces $C([0,T],\mathbb{R}^n) \to C([0,T],\mathbb{R})$, is Lipschitz with Lipschitz constant $(K/n)\exp(KT)$.

Proposition 2.9. *(Lipschitz property of $V^{(n)}$) For any $v \in \mathbb{R}, K \ge 0$, take $f, g \in C([0,T],\mathbb{R}^n)$ such that $\|f - g\|_{[0,T]} < \eta$. We have*

$$\|V_f^{(n)} - V_g^{(n)}\|_{[0,T]} \le (K\eta/n)\exp(KT), \tag{2.10}$$

and consequently

$$\|I_f^{(n)} - I_g^{(n)}\|_{[0,T]} \leq (K\eta/n)T\exp(KT). \tag{2.11}$$

Remark 2.10. Clearly

$$\frac{1}{n}\|f-g\|_{[0,T]} = \frac{1}{n}\sum_{i=1}^{n}\|f_i - g_i\|_{[0,T]}$$

is the average distance between the f_i, g_i. Proposition 2.9 says that if this average distance is small, the difference in the drifts $V_g^{(n)}, V_f^{(n)}$ is small as well.

Proof. The first bound (2.10) is shown on (2.9), while (2.11) follows as

$$\begin{aligned}\|I_f^{(n)} - I_g^{(n)}\|_{[0,T]} &= \sup_{0\leq u\leq T}\left|\int_0^u V_f^{(n)}(s) - V_g^{(n)}(s)ds\right| \\ &\leq \sup_{0<u<T}\int_0^u |V_f^{(n)}(s) - V_f^{(n)}(s)|ds \\ &\leq \int_0^T \|V_f^{(n)} - V_g^{(n)}\|_{[0,T]}ds \\ &\leq T(K\eta/n)\exp(KT).\end{aligned}$$

□

Consider the above sequence $I_f^{(n,\epsilon)}$ defined above for a given $f = (f_1, \ldots, f_n) \in \mathbb{R}^n$. By Proposition (2.9), $I_f^{(n,\epsilon)}$ converges in the uniform norm on $C([0,T],\mathbb{R})$ to a unique continuous function. The Proposition below says this rate of convergence only depends on $\|f\|_{[0,T]}$.

Proposition 2.11. *Consider the sequence $I_f^{(n,\epsilon)}$ defined above for a given $f = (f_1, \ldots, f_n) \in C([0,T],\mathbb{R}^n)$. If $l < m$, then*

$$\|I_f^{(n,2^{-l})} - I_f^{(n,2^{-m})}\|_{[0,T]} \leq ((2+K)\|f\|_{[0,T]}/n)2^{-l}\exp(KT).$$

Proof. The proof is in a similar vein as that of Proposition 2.9. We make a couple notational of conveniences. For $j = l, m$ we will write I^j in place of $I_f^{(n,2^{-j})}$, and $I^j_{k2^{-j}}$ in place of $I^{(n,2^{-j})}_{f,k2^{-l}}$. Recall I^l is piecewise linear by definition. For fixed $l < m$, define

$$\begin{aligned} D(k) &:= \sup_{0<t<k2^{-l}} |I^l(t) - I^m(t)| = \frac{1}{n} \sum_{i=1}^{n} \sup_{0<t<k2^{-l}} |f_i(t) + I^l(t) - (f_i(t) + I^m(t))| \\ &= \|(f + I^l) - (f + I^m)\|_{[0,k2^{-l}]}/n. \end{aligned}$$

We develop bounds for $D(k)$ using a recursive argument. By construction $I^l \equiv 0$ on $[0, 2^{-l}]$. Due to nonegativity of I^m, for any $t \in [0, T]$

$$\begin{aligned} |V^{(n,2^{-m})}(t)| &\leq \frac{K}{n} \sum_{i=1}^{n} \sup_{0<u<T} [-f_i(u) - I^m] \vee 0 \\ &\leq \frac{K}{n} \sum_{i=1}^{n} \sup_{0<u<T} [-f_i(u)] \vee 0 \leq K\|f\|/n. \end{aligned}$$

Therefore I^m is piecewise linear with a slope not exceeding $K\|f\|/n$, and so

$$D(1) \leq (K\|f\|/n)2^{-l}. \tag{2.12}$$

Assume we are given $D(k)$. We wish to bound the difference between I^l and I^m on the interval $[0, (k + 1)2^{-l}]$. We know $I^l|_{[0,k2^{-l}]} = I^l_{k2^{-l}}$ and $I^m|_{[0,k2^{-l}]} = I^m_{k2^{-l}}$. Similarly the function I^l has constant slope on $[k2^{-l}, (k + 1)2^{-l})$, with its slope adjustment being at the end of this interval at $(k+1)2^{-l}$; so $I^l_{k2^{-l}} = I^l_{(k+1)2^{-l}}$ on $[k2^{-l}, (k+1)2^{-l}]$. On the other hand, I^m has a slope adjustment at each time $k2^{-l} + 2^{-m}, k2^{-l} + 2^{-m+1}, \ldots, (k + 1)2^{-l}$. Note that the difference in the slope between I^l, I^m at time $k2^{-l}$ is not more than $D(k)$, so that

$$\begin{aligned} \|I^l_{(k+1)2^{-l}} - I^m_{k2^{-l}}\|_{[0,(k+1)2^{-l}]} &\leq \|I^l_{(k+1)2^{-l}} - I^m_{k2^{-l}}\|_{[0,k2^{-l}]} + KD(k)2^{-l} \\ &= \|I^l_{k2^{-l}} - I^m_{k2^{-l}}\|_{[0,k2^{-l}]} + KD(k)2^{-l} \\ &= D(k) + KD(k)2^{-l}. \end{aligned}$$

By this and the triangle inequality,

$$\begin{aligned} D(k+1) &= \|I^l_{(k+1)2^{-l}} - I^m_{(k+1)2^{-l}}\|_{[0,(k+1)2^{-l}]} \\ &\le \|I^l_{(k+1)2^{-l}} - I^m_{k2^{-l}}\|_{[0,(k+1)2^{-l}]} + \|I^m_{k2^{-l}} - I^m_{(k+1)2^{-l}}\|_{[0,(k+1)2^{-l}]} \\ &\le D(k) + KD(k)2^{-l} + \|I^m_{k2^{-l}} - I^m_{(k+1)2^{-l}}\|_{[0,(k+1)2^{-l}]}. \end{aligned} \tag{2.13}$$

Let

$$\beta_k = \frac{1}{n}\sum_{i=1}^{n} \sup_{0<t<(k+1)2^{-l}} [-(f(t)+I^m(t))] - \frac{1}{n}\sum_{i=1}^{n} \sup_{0<t<k2^{-l}} [-(f(t)+I^m(t))],$$

so that $K\beta_k$ is the amount the velocity I^m increases in the interval $[k2^{-l},(k+1)2^{-l}]$. From this telescoping definition of β_k we see that

$$\sum_{k=1}^{\lceil T2^l\rceil} \beta_k = \frac{1}{n}\sum_{i=1}^{n} \sup_{0<s<T} [-(f(s)+I^m(s)) \vee 0] \le \frac{1}{n}\|f\|_{[0,T]}. \tag{2.14}$$

Combine with (2.13) above we have

$$D(k+1) \le D(k) + KD(k)2^{-l} + K\beta_k 2^{-l} = D(k) + K(D(k)+\beta_k)2^{-l}. \tag{2.15}$$

Set $A(k)$ to be recursively defined with the above inequality taken as equality. That is,

$$A(k+1) = A(k) + KA(k)2^{-l} + K\beta_k 2^{-l}.$$

We have $D(k) \le A(k)$ for all k. Note that $A(k)$ is maximized when all the mass of $\sum_1^{2^l} \beta_k$ is instead concentrated at β_1 because this allows the entire mass to be compounded from the beginning. Since the total sum of the β_k does not exceed $\|f\|_{[0,T]}/n$,

$$\begin{aligned} A(1) &= (K\|f\|_{[0,T]}/n)2^{-l}, \\ A(2) &= A(1)(1+K2^{-l}) + K\sum_{k=1}^{2^l} \beta_k 2^{-l} \le A(1)(1+K2^{-l}) \\ &\quad + (K\|f\|_{[0,T]}/n)2^{-l}, \\ A(k{+}1) &= A(k)(1+K2^{-l}), \end{aligned}$$

giving

$$\begin{aligned} D(\lceil T2^{-l}\rceil) &\le A(\lceil T2^{l}\rceil) \le (A(1)+A(2))(1+K2^{-l})^{\lceil T2^{l}\rceil} \\ &\le ((2+K)\|f\|_{[0,T]}/n)2^{-l}\exp KT \end{aligned}$$

which concludes the result.

□

To construct our system (3.1) in the introduction from Proposition 2.6, we apply the map Γ_n pathwise with

$$(f_1,\ldots,f_n) = (B^{(1)} + X_1^{(n)}(0),\ldots,B^{(n)} + X_n^{(n)}(0)) =: B + X(0),$$

producing the pair of processes

$$\Gamma_n\left(B + X(0)\right) = \left(I^{(n)}_{B+X(0)}, \widetilde{V}^{(n)}_{B+X(0)}\right).$$

Set

$$X_i^{(n)} = X_i^{(n)}(0) + B^{(i)} + m_i^{(n)},\ V^{(n)} = -\widetilde{V}^{(n)}_{B+X(0)},\ Y^{(n)} = -I^{(n)}_{B+X(0)}. \tag{2.16}$$

Then

Proposition 2.12. $(X_1^{(n)},\ldots,X_n^{(n)},Y^{(n)},V^{(n)})$ *satisfies* (3.1)*, therefore giving a strong solution to that system of SDE's.*

Proof. We begin from (2.3) - (3.4) with the $f_i(t)$ replaced with $B^{(i)}(t)+X_i^{(n)}(0)$. The following holds almost surely:

$$X_i^{(n)}(t) = X_i^{(n)}(0) + B^{(i)}(t) + m_i^{(n)}(t) \ge Y^{(n)}(t),\ \textit{for all } 0 < t < T, \tag{2.17}$$

$$V^{(n)}(t) = v - \frac{K}{n}\sum_{i=1}^{n} m_i^{(n)}(t), \tag{2.18}$$

$$Y^{(n)}(t) = \int_0^t V^{(n)}(s)ds, \tag{2.19}$$

$$m_i^{(n)} \textit{ is flat off of } \{t : X_i^{(n)}(t) = Y^{(n)}(t)\}. \tag{2.20}$$

We take $v = 0$ for convenience. The fact that we have a strong solution of the system follows from the path-by-path construction. We apply a transformation of measure argument. As mentioned in Remark 2.7, for a fixed time $t \in [0, T]$,

$$V^{(n)}(t) = -\frac{K}{n} \sum_{i=1}^{n} \sup_{0<u<t} [-(B^{(i)}(u) + X_i^{(n)}(0) - Y^{(n)}(u)) \vee 0],$$

which, due to nonnegativity of $X_i^{(n)}(0)$ and the fact that $Y^{(n)} \leq 0$,

$$\sup_{u\in[0,T]} |V^{(n)}(u)| \leq \frac{K}{n} \sum_{i=1}^{n} \sup_{0<u<T} [-B^{(i)}(u) \vee 0].$$

This is equivalent to saying a continuous function plus a nonnegative drift has a running distance below zero less than that of the continuous function. It follows from continuity of the processes on $[0, T]$ that $\sup_{0<u<t} |V^{(n)}(u)| \leq |V^{(n)}(T)| < \infty$ almost surely. Therefore

$$Z(t) = \exp\left(\frac{K}{n} \sum_{i=1}^{n} \int_0^t V^{(n)}(s) dB^{(i)}(s) - nY^{(n)}(t) \right)$$

is a local martingale, and therefore there exists a collection of exhaustive stopping times $\tau_k \overset{a.s.}{\to} \infty$ such that $Z(t \wedge \tau_k)$ is a true martingale for each k. We will apply a Girsanov transformation of measure, see [30, Ch. 3.5]. Let $\mathbb{Q}$ be defined by $d\mathbb{Q}/d\mathbb{P} = Z(t \wedge \tau_k)$. Under $\mathbb{Q}$ each $\widetilde{B}^{(i)}(t \wedge \tau_k) := B^{(i)}(t \wedge \tau_k) - Y^{(n)}(t \wedge \tau_k)$ has the law of a Brownian motion, and the joint law of $(X_1^{(n)}, \dots, X_n^{(n)})$, when stopped at τ_k, has the same law as $\widetilde{X}_i(t \wedge \tau_k) := X_i^{(n)}(0) + \widetilde{B}^{(i)}(t \wedge \tau_k) + \tilde{m}_i(t \wedge \tau_k) \geq 0$, where $\tilde{m}_i(t) = \sup_{0<u<t} -[\widetilde{B}^{(i)}(u) + X_i^{(n)}(0)] \vee 0$. The $\tilde{m}_i$ are then equal to $m_i^{(n)} := \sup_{0<u<t} -[B^{(i)}(u) + X_i^{(n)}(0) - Y^{(n)}(u)] \vee 0$. Because $\tilde{m}_i$ is flat off $\{t : \widetilde{X}_i(t) = 0\}$, the classical Lévy's theorem [30, Chapter 3] shows that this system $(\widetilde{X}_1, \dots, \widetilde{X}_n)$ is equivalent in law to processes solving

$$d\widetilde{X}_i = d\widetilde{B}^{(i)} + d\widetilde{L}_i,\ \widetilde{X}_i(0) = X_i^{(n)}(0),$$

when stopped at τ_k, and where $\widetilde{L}_i$ is the local time at zero of $\widetilde{X}_i$. That is,

$$\begin{aligned}\widetilde{L}_i(t) &= \lim_{\epsilon\to 0} \frac{1}{2\epsilon} \int_0^t 1_{[0,\epsilon]}(\widetilde{X}_i(s)) ds \\ &= \lim_{\epsilon\to 0} \frac{1}{2\epsilon} \int_0^t 1_{[0,\epsilon]}(X_i^{(n)}(s) - Y^{(n)}(s)) ds =: L_i^{(n)}(t),\end{aligned}$$

for all t, almost surely. Additionally, $\tilde{m}_i$ is the local time of $\widetilde{X}_i^{(n)}$ at zero, which by definition is the local time of contact between $X_i^{(n)}$ and $Y^{(n)}$. Since $\tilde{m}_i = m_i^{(n)}$ this shows that $m_i^{(n)}(t) = L_i^{(n)}(t)$ for all t, almost surely. This means that under $\mathbb{P}$, as processes stopped at τ_k, solutions to (2.17)-(2.19) are solutions of

$$dX_i^{(n)} = dB^{(i)} + dL_i^{(n)},\ dY^{(n)} = -\frac{K}{n}\sum_{i=1}^{n} L_i^{(n)}(t)dt,$$

with the given initial conditions and where $L_i^{(n)}$ is the local time of $X_i^{(n)} - Y^{(n)}$ at zero. This latter process is the definition of (3.1). Since $\tau_k \to \infty$ almost surely, the equivalence in law holds as processes defined on $[0, T]$. □

Lemma 2.13. *Let* $(Y^{(n)}, V^{(n)})$ *be defined as in equation* (2.16)*, then*

$$\|V^{(n)}\|_{[0,T]} \leq v + K\left(vT + \frac{1}{n}\sum_{i=1}^{n} m_i(T)\right),$$

where $m_i(t) = \sup_{0<s<t}[-B^{(i)}(s)] \vee 0$.

Proof. Clearly $\sup_{0<s<T} V^{(n)}(s) \leq v$, which implies that $Y^{(n)}(t) \leq vt$ for $t \in [0, T]$. From Remark 2.7 and Lemma 2.5, we have

$$\begin{aligned}
\|V^{(n)}\|_{[0,T]} &= \sup_{0<s<T}\left(v - \frac{K}{n}\sum_{i=1}^{n}(\sup_{0<u<s}[-(B^{(i)}(u) - Y^{(n)}(u))] \vee 0)\right) \\
&\leq v + \frac{K}{n}\sum_{i=1}^{n}(\sup_{0<s<T}[-(B^{(i)}(s) - vs)] \vee 0) \\
&\leq v + K\left(vT + \frac{1}{n}\sum_{i=1}^{n}(\sup_{0<s<T}[-B^{(i)}(s)] \vee 0)\right).
\end{aligned}$$

□

3 *Hydrodynamic Limit and Propagation of Chaos*

In this section we assume the conditions in 1.2.

Lemma 3.1. *The collection* $\{(Y^{(n)}(t), V^{(n)}(t)) : t \in [0,T]\}_{n\geq 1}$ *is tight in the space of continuous functions.*

Proof. It suffices to show $V^{(n)}$ is tight, since $Y^{(n)}(t) = \int_0^t V^{(n)}(s)ds$. By our representation of $V^{(n)}$ as (2.5), together with Remark 2.7, we apply Lemma 2.5 with $y_1 = -Y^{(n)}$ and $y_2 = 0$ to show the increment $V^{(n)}(t+\delta) - V^{(n)}(t)$ is not more than the change of the running maximum of the Brownian paths, averaged over n. That is, letting $m_1(t), \ldots, m_n(t)$ denote the respective running minimum below zero of the $B^{(1)}, \ldots, B^{(n)}$,

$$0 \leq |V^{(n)}(t+\delta) - V^{(n)}(t)| \leq \frac{K}{n}\sum_{i=1}^{n}(m_i(t+\delta) - m_i(t)) \text{ for all } t \in [0, T-\delta]$$

for any positive delta, almost surely. Since $V^{(n)}$ and m_i are a.s. nondecreasing, we have the same inequality but for the modulus of continuity:

$$\omega(V^{(n)}, T, \delta) \leq \frac{1}{n}\sum_{i=1}^{n}\omega(m_i, T, \delta) \tag{3.1}$$

almost surely. By independence of the m_i and the strong law of large numbers, for each rational $q \in [0,T]$, $S_n(q) \to \mathbb{E}m_1(q) = \sqrt{2q/\pi}$ as $n \to \infty$. Note that S_n is monotone for each n, almost surely. It is known that if a sequence of continuous monotone functions converge to a continuous function pointwise on a dense subset of a compact set, the entire sequence converges uniformly. Hence $S_n(t) \to \sqrt{2t/\pi}$ uniformly in $t \in [0,T]$, almost surely. If $f_n \to f$ uniformly in $C([0,T],\mathbb{R})$, then $\sup_n \omega(f, T, \delta) \to 0$ as $\delta \to 0$. Therefore $\sup_n \omega(S_n, T, \delta) \to 0$ as $\delta \to 0$, almost surely. By (3.1) we know $\sup_n \omega(V^{(n)}, T, \delta) \to 0$ almost surely as well. Since $V^{(n)}(0) = v$, almost surely, this is sufficient for tightness of the $V^{(n)}$. □

We give another proof of tightness. In the following proof tightness of $Y^{(n)}$ is shown first, and is used show tightness of $V^{(n)}$. That this can be done is not surprising, since $V^{(n)}$ is the average of the Brownian paths running minimum below $Y^{(n)}$. Conseqently, if $Y^{(n)}$ converges to a deterministic function Y it is reasonable to think $V^{(n)}$ converges to the average of the Brownian paths running minimum below Y. However, this requires us to know the limit of $Y^{(n)}$ (or an arbitrary subsequence thereof) is deterministic. Consequently we require Proposition 3.5. In a sense, this shows tightness of $V^{(n)}$ and $Y^{(n)}$ is equivalent when we know subsequences of $Y^{(n)}$ converge to something deterministic.

Lemma 3.2. *The collection* $\{(Y^{(n)}(t), V^{(n)}(t)) : t \in [0,T]\}_{n\geq 1}$ *is tight in the space of continuous functions.*

Remark 3.3. We use Proposition 3.5 in the proof.

Proof. (Tightness of $Y^{(n)}$) For simplicity we take the initial velocity $v = 0$. We first show that $Y^{(n)}$ is tight. By our representation of $V^{(n)}$ as (2.5), together with Remark 2.7, the maximum velocity $\|V^{(n)}\|_{[0,T]}$ is bounded almost surely by the scaled running minimum of the Brownian paths below zero, averaged over n. In other words, letting $m_1(t), \ldots, m_n(t)$ denote the running minimum below zero of $B^{(1)}, \ldots, B^{(n)}$, Lemma 2.13 gives

$$\|V^{(n)}\|_{[0,T]}\leq \frac{K}{n}\sum_{i=1}^{n} m_i(T).$$

Consequently for any $0 < \delta < T$,

$$\omega(Y^{(n)}, \delta, T) := \sup_{\substack{0<t<T-\delta \\ t-\delta<s<t+\delta}} |Y^{(n)}(t) - Y^{(n)}(s)|\leq \delta\|V^{(n)}\|_{[0,T]}\leq \delta\frac{K}{n}\sum_{i=1}^{n} m_i(T)$$

almost surely. Taking expectations, we have

$$\mathbb{E}\,\omega(Y^{(n)}, \delta, T) \leq \delta K\mathbb{E}m_1(T) = \delta K\sqrt{2T/\pi}. \tag{3.2}$$

Fix $\epsilon > 0$ and apply Markov's inequality, we see

$$\sup_n \mathbb{P}(\omega(Y^{(n)}, \delta, T) > \epsilon) \leq \epsilon^{-1}\sup_n \mathbb{E}\,\omega(Y^{(n)}, \delta, T) \leq \epsilon^{-1}\delta K\sqrt{2T/\pi},$$

which implies

$$\lim_{\delta\to 0}\sup\sup_n \mathbb{P}(\omega(Y^{(n)}, \delta, T) > \epsilon) = 0.$$

Together with $\mathbb{P}(Y^{(n)}(0) = 0) = 1$, this is sufficient for tightness of the sequence of continuous processes $\{Y^{(n)}(t) : t \in [0,T]\}$.

(Tightness of $V^{(n)}$) Take any subsequence n' for which $Y^{(n')}$ converges to some process Y in distribution. By Proposition 3.5, Y is deterministic. Recall that the initial conditions $X_i^{(n)}(0)$, $1 \leq i \leq n$, are i.i.d. samples with distribution $\pi_0^{(n)}$ and that $W_p(\pi_0^{(n)}, \pi_0) \to 0$ by assumption. Let $X_i^{(\infty)}(0)$, $i \in \mathbb{N}$, be independent samples with distribution π_0. By definition of the W_p metric there is a probability space supporting all the processes $\{X_i^{(n)}(t) : t \in [0,T], 1 \leq n, n \in \mathbb{N}\}$ such that

$$\sup_{1\leq i\leq n} \mathbb{E}|X_i^{(n)}(0) - X_i^{(\infty)}(0)|^p \to 0, \text{ as } n \to \infty.$$

Hence,

$$\sup_{1\leq i\leq n} \mathbb{E}|X_i^{(n)}(0) - X_i^{(\infty)}(0)| \leq \sup_{1\leq i\leq n} \Big(\mathbb{E}|X_i^{(n)}(0) - X^{(\infty)}|^p\Big)^{1/p} \to 0, \text{ as } n \to \infty.$$

This enlarged probability space can be constructed by taking $\mathcal{F}_t \times \sigma\{X_i^{(n)}(0) : 1 \leq i \leq n,\ n \in \mathbb{N}\}$ as our new filtration. Then,

$$\frac{1}{n}\mathbb{E}\sum_{i=1}^{n}|X_i^{(n)}(0) - X_i^{(\infty)}(0)| \leq \sup_{1\leq i\leq n} \mathbb{E}|X_i^{(n)}(0) - X_i^{(\infty)}(0)| \longrightarrow 0,$$

and so

$$\frac{1}{n}\sum_{i=1}^{n}|X_i^{(n)}(0) - X_i^{(\infty)}(0)| \xrightarrow{P} 0.$$

Hence every sequence n' has a further subsequence n'_k with

$$\frac{1}{n'_k}\sum_{i=1}^{n'_k}|X_i^{(n'_k)}(0) - X_i^{(\infty)}(0)| \to 0 \tag{3.3}$$

almost surely. Without loss of generality we relabel such a sequence n'_k as n'. For $i = 1, \ldots, n'$, define

$$m_i^{(n)}(t) = \sup_{0<u<t} -[(B^{(i)}(u) + X_i^{(n)}(0)) - Y^{(n)}(u)] \vee 0,$$

$$\tilde{m}_i(t) = \sup_{0<u<t} -[(B^{(i)}(u) + X_i^{(\infty)}(0)) - Y(u)] \vee 0.$$

Since

$$V^{(n')}(t) = -\frac{K}{n}\sum_{i=1}^{n'} m_i^{(n')}(t),$$

we compute

$$\begin{aligned}\Big\|\frac{K}{n'}\sum_{i=1}^{n'}\tilde{m}_i - V^{(n')}\Big\|_{[0,T]} &\le \frac{K}{n'}\sum_{i=1}^{n'}\|\tilde{m}_i - m_i^{(n')}\|\\ &\le \frac{K}{n'}\sum_{i=1}^{n'}(|X_i^{(\infty)}(0) - X_i^{(n')}(0)|+\|Y^{(n')} - Y\|_{[0,T]}) \qquad (3.4)\\ &\longrightarrow 0\end{aligned}$$

almost surely. In words, $V^{(n')}$ and the average of the running minimum of the i.i.d. Brownian paths below the curve Y become arbitrarily close in the uniform distance. By the strong law of large numbers $\frac{1}{n'}\sum_{i=1}^{n'}\tilde{m}_i(t) \to \mathbb{E}\,\tilde{m}_i(t)$ almost surely for each t. That is,

$$\lim_{n'\to\infty} V^{(n')}(t) = -\lim_{n'\to\infty}\frac{K}{n'}\sum_{i=1}^{n'}\tilde{m}_i(t) = -K\mathbb{E}\,\tilde{m}_i(t), \qquad (3.5)$$

and by (3.7), $V^{(n')}$ converges in the uniform norm to $-K\mathbb{E}\,\tilde{m}_i(t)$. This implies tightness for $\{V^{(n)}(t) : t \in [0,T]\}$. □

We use Proposition 2.9 in Proposition 3.5 to show subsequential limits of $V^{(n)}$ converge to deterministic functions. Proposition 2.11 will be used to prove this limit is unique.

Remark 3.4. Tightness of $\{V^{(n)}(T) : t \in [0,T]\}$ implies there exists a subsequence $V^{(n')}$ which converges in distribution to some process V in $C([0,T],\mathbb{R})$. By the Skorohod representation theorem one can exhibit a probability space supporting an entire sequence of processes, $U^{(n')}$, and U such that $U^{(n')} \to U$ almost surely in the uniform norm on $C([0,T],\mathbb{R})$ and where $U^{(n')}$ (resp. U) has the same distribution as $V^{(n')}$ (resp. V). Consequently if U is deterministic V is also deterministic, and therefore the conclusion of the next proposition also holds when $V^{(n')}$ converges to V in distribution.

Proposition 3.5. *Let n' be some sequence such that*

$$V^{(n')} = \frac{K}{n'}\sum_{i=1}^{n'} L_i^{(n)} \longrightarrow_{a.s.} V,$$

where convergence holds uniformly on $[0, T]$*, on some probability space supporting underlying Brownian motions* $\{B^{(i)} : i \in \mathbb{N}\}$ *and* $\{X_i^{(n)}(0) : 1 \leq i \leq n, n \in \mathbb{N}\}$*. Then* V *is deterministic,*

$$Y^{(n')} \longrightarrow_{a.s.} Y := \int_0^{\cdot} V(s)ds,$$

and

$$V(t) \overset{a.s.}{=} v - K\mathbb{E}m(t),$$

where $m(t) = \sup_{0<s<t} -[(B^{(1)}(s) + X_1^{(\infty)}(0)) - Y(u)] \vee 0$ *and* $X_1^{(\infty)} \overset{d}{=} \pi_0(dx)$*.*

Remark 3.6. Since

$$Y^{(n')}(t) = \int_0^t V^{(n')}(s)ds$$

for all $0 < t < T$, almost surely, we see that $Y^{(n')}$ converges uniformly to some deterministic Y if the sequence n' is as given in the Proposition statement.

Proof. Consider a subsequence n' for which $Y^{(n')}$ converges to some process Y in distribution. Recall that the initial conditions $X_i^{(n)}(0)$, $1 \leq i \leq n$, are i.i.d. samples with distribution $\pi_0^{(n)}$ and that $W_p(\pi_0^{(n)}, \pi_0) \to 0$ by assumption. Let $X_i^{(\infty)}(0)$, $i \in \mathbb{N}$, be independent samples with distribution π_0. By definition of the W_p metric there is a probability space supporting all the processes $\{X_i^{(n)}(t) : t \in [0, T], 1 \leq n, n \in \mathbb{N}\}$ such that

$$\sup_{1 \leq i \leq n} \mathbb{E}|X_i^{(n)}(0) - X_i^{(\infty)}(0)|^p \to 0, \text{ as } n \to \infty.$$

Hence,

$$\sup_{1 \leq i \leq n} \mathbb{E}|X_i^{(n)}(0) - X_i^{(\infty)}(0)| \leq \sup_{1 \leq i \leq n} \Big(\mathbb{E}|X_i^{(n)}(0) - X_i^{(\infty)}|^p\Big)^{1/p} \to 0, \text{ as } n \to \infty.$$

This enlarged probability space can be constructed by taking $\mathcal{F}_t \times \sigma\{X_i^{(n)}(0) : 1 \leq i \leq n,\ n \in \mathbb{N}\}$ as our new filtration. Then,

$$\frac{1}{n}\mathbb{E}\sum_{i=1}^{n} |X_i^{(n)}(0) - X_i^{(\infty)}(0)| \leq \sup_{1 \leq i \leq n} \mathbb{E}|X_i^{(n)}(0) - X_i^{(\infty)}(0)| \longrightarrow 0,$$

and so

$$\frac{1}{n}\sum_{i=1}^{n} |X_i^{(n)}(0) - X_i^{(\infty)}(0)| \xrightarrow{P} 0.$$

By convergence in probability, every sequence n' has a further subsequence n'_k with

$$\frac{1}{n'_k}\sum_{i=1}^{n'_k}|X_i^{(n'_k)}(0) - X_i^{(\infty)}(0)|\to 0 \tag{3.6}$$

almost surely. Without loss of generality we relabel such a sequence n'_k as n'. From this and Proposition 2.9, we have:

$$\begin{aligned}&\|V^{(n')}_{(B^{(1)}+X_1^{(n')}(0),\ldots,B^{(n')}+X_{n'}^{(n')}(0))} - V^{(n')}_{(B^{(1)}+X_1^{(\infty)}(0),\ldots,B^{(n')}+X_{n'}^{(\infty)}(0))}\|_{[0,T]}\\ &\leq \frac{K}{n'}\sum_{i=1}^{n'}|X_i^{(n')}(0) - X_i^{(\infty)}(0)|\exp(KT)\to 0\end{aligned}$$

almost surely. Therefore it suffices to show $V^{(n')}(B^{(1)} + X_1^{(\infty)}(0), \ldots, B^{(n')} + X_{n'}^{(\infty)}(0))$ converges to a deterministic limit. For almost all ω in our probability space and each $k \geq 1$, there is a constant $C(\omega, k)$ such that $\|(B^{(1)} + X_1^{(\infty)}(0), \ldots, B^{(k)} + X_k^{(\infty)}(0))\|_{[0,T]}< C(\omega, k) < \infty$. This follows from continuity of the $B^{(i)}$ and the assumption that the initial samples $X_i^{(\infty)}(0)$ come from an almost surely finite random variable. Apply Proposition 2.9 with $f = (B^{(1)} + X_1^{(n')}(0), \ldots, B^{(n')} + X_{n'}^{(n')}(0))$, $g = (B^{(1)} + X_1^{(\infty)}(0), \ldots, B^{(n')} + X_{n'}^{(\infty)}(0))$ and $\eta = \|f - g\|_{[0,T]}< C(\omega, k)$ to give the almost sure bound

$$\begin{aligned}&\|V^{(n')}_{(B^{(1)}+X_1^{(\infty)}(0),\ldots,B^{(n')}+X_{n'}^{(\infty)}(0))} - V^{(n')}_{(0,\ldots0,B^{(k+1)}+X_{k+1}^{(\infty)}(0),\ldots,B^{(n')}+X_{n'}^{(\infty)}(0))}\|_{[0,T]}\\ &\leq (KC(\omega, k)/n')\exp(KT)\to 0 \text{ as } n'\to\infty.\end{aligned}$$

Therefore

$$V = \lim_{n'\to\infty} V^{(n')}(0, \ldots 0, B^{(k+1)} + X_{k+1}^{(\infty)}(0), \ldots, B^{(n')} + X_{n'}^{(\infty)}(0)) \in \mathcal{F}_T^{k+1,\infty}$$

where $\mathcal{F}_T^{k,\infty}$ is the sigma-field generated by $\{B^{(i)}(t) + X_i^{(\infty)}(0) : 0 < t < T, k \leq i\}$. By definition this means the continuous function V is adapted to the tail sigma-field of the infinite sequence of i.i.d. processes. Hence $\{V(t) : t \in [0,T]\}$ is adapted to a trivial sigma-field, so it is deterministic.

To prove the equalities in the proposition statement, we take $v = 0$ for simplicity. For $i = 1, \dots, n'$, define

$$\begin{aligned} m_i^{(n)}(t) &= \sup_{0<u<t} -[(B^{(i)}(u) + X_i^{(n)}(0)) - Y^{(n)}(u)] \vee 0, \\ \tilde{m}_i(t) &= \sup_{0<u<t} -[(B^{(i)}(u) + X_i^{(\infty)}(0)) - Y(u)] \vee 0. \end{aligned}$$

Since

$$V^{(n')}(t) = -\frac{K}{n} \sum_{i=1}^{n'} m_i^{(n')}(t),$$

we compute

$$\begin{aligned} \Big\| \frac{K}{n'} \sum_{i=1}^{n'} \tilde{m}_i - V^{(n')} \Big\|_{[0,T]} &\le \frac{K}{n'} \sum_{i=1}^{n'} \| \tilde{m}_i - m_i^{(n')} \| \\ &\le \frac{K}{n'} \sum_{i=1}^{n'} (|X_i^{(\infty)}(0) - X_i^{(n')}(0)| + \| Y^{(n')} - Y \|_{[0,T]}) \qquad (3.7) \\ &\longrightarrow_{a.s.} 0 \end{aligned}$$

almost surely. In words, $V^{(n')}$ and the average of the running minimum of the i.i.d. Brownian paths below the curve Y become arbitrarily close in the uniform distance. By the strong law of large numbers $\frac{1}{n'} \sum_{i=1}^{n'} \tilde{m}_i(t) \to \mathbb{E}\, \tilde{m}_i(t)$ almost surely for each t. That is,

$$\lim_{n' \to \infty} V^{(n')}(t) = -\lim_{n' \to \infty} \frac{K}{n'} \sum_{i=1}^{n'} \tilde{m}_i(t) = -K \mathbb{E}\, \tilde{m}_i(t), \text{ almost surely.} \qquad (3.8)$$

By (3.7), $V^{(n')}$ converges in the uniform norm to $-K\mathbb{E}\, \tilde{m}_i(t)$ almost surely. □

Proposition 3.7 (Uniqueness of Limit). *All subsequential limits given Proposition 3.5 are in fact the same.*

Proof. Similar to the proof of Proposition 3.5, it suffices to take the inital conditions of $X_i^{(n)}$ to be $X_i^{(\infty)}(0)$. Let Y^1, Y^2 be two limits associated with two subsequences n_k^1, n_k^2, so $\lim_{n_k^i \to \infty} Y^{(n_k^i)} = Y^i$ for $i = 1, 2$. By the construction given in Theorem 2.6

$$Y^i = -\lim_{n_k^i \to \infty} \lim_{2^{-l} \to 0} I^{(n_k^i, 2^{-l})}_{(B^{(1)} + X_1^{(\infty)}(0), \dots, B^{(n_k^i)} + X_{n_k^i}^{(\infty)}(0))}. \qquad (3.9)$$

It follows from the strong law of large numbers that $\|(B^{(1)}+X_1^{(\infty)}(0), \ldots, B^{(n)}+X_n^{(\infty)}(0))\|_{[0,T]}/n < C(\omega) < \infty$ for almost each ω. Applying Proposition 2.11 we see that

$$\|I^{(n_k^i,2^{-l})} - I^{(n_k^i,2^{-m})}\|_{[0,T]} \leq (2+K)C(\omega)2^{-l}\exp(KT).$$

Let $m \to \infty$ and we have

$$\|I^{(n_k^i,2^{-l})} - I^{(n_k^i)}\|_{[0,T]} \leq (2+K)C(\omega)2^{-l}\exp(KT).$$

In other words,

$$\sup_{n_k^i \geq 1, i=1,2} \|I^{(n_k,2^{-l})} - I^{(n_k^i)}\|_{[0,T]} \leq (2+K)C(\omega)2^{-l}\exp(KT),$$

and as $2^{-l} \to 0$ the convergence of $I^{(n_k^i,2^{-l})}$ to $I^{(n_k^i)}$ is uniform over $(n_k^i)_{k\geq 1}$, almost surely. By the Moore-Osgood theorem this guarantees an interchange of the limiting operations in (3.9). Hence,

$$Y^i = -\lim_{2^{-l}\to 0} \lim_{n_k^i \to \infty} I^{(n_k^i,2^{-l})}.$$

We will use the strong law of large numbers to show $\lim_{n_k^1\to\infty} I^{(n_k^1,2^{-l})} = \lim_{n_k^2\to\infty} I^{(n_k^2,2^{-l})}$. This can be seen by induction on $[0, N2^{-l}]$: By construction of the $I^{(n_k^i,2^{-l})}$ the two limits are identically zero on $[0, 2^{-l}]$. Assume the two limits agree on $[0, N2^{-l}]$. This induction hypothesis implies the slope of $I^{(n_k^1,2^{-l})}$ and the slope of $I^{(n_k^2,2^{-l})}$ become arbitrarily close as $k \to \infty$. Since the slope of $I^{(n_k^i,2^{-l})}$ on $[N2^{-l}, (N+1)2^{-l}]$ is the average of the positive part of the running minimums of the $B^{(1)} + I^{(n_k^i,2^{-l})}, \ldots, B^{(n_k^i)} + I^{(n_k^i,2^{-l})}$, and because the limit in the strong law of large numbers is independent on the subsequence chosen, the slopes of $I^{(n_k^1,2^{-l})}, I^{(n_k^2,2^{-l})}$ become arbitrarily close on $[0,(N+1)2^{-l}]$ as $k \to \infty$. This completes proves the induction step. $\square$

The previous two propositions imply the following corollary.

Corollary 3.8. *There are deterministic functions $(Y(t), V(t))$ defined on $t \in [0,T]$, with $dY/dt = V$, such that*

$$(Y^{(n)}, V^{(n)}) \overset{W_q}{\to} (Y, V), \text{ for any } q \geq 1.$$

Furthermore, for $t \in [0,T]$,

$$V(t) = v - K\mathbb{E}\, m(t),$$

where $m(t) = \sup_{0<u<t} -[B(u) + X_1^{(\infty)}(0) - Y(u)] \vee 0$ *is the running minimum of the Brownian motion with initial condition* $X_1^{(\infty)}(0)$ *under the curve* Y.

Proof. Convergence in W_q for any $q \geq 1$ is shown once we can establish that $Y^{(n)}$ and $V^{(n)}$ converge almost surely and in L_q to Y and V, respectively, in some probability space supporting a sequence of i.i.d. Brownian motions and the initial conditions $\{X_i^{(n)}(0) : 1 \leq i \leq n, m \in \mathbb{N}\}$. Convergence in distribution follows from Propositions 3.5 and 3.7. By Skorohod's representation there is a probability space where convergence holds almost surely. The convergence in L_q comes from the bound indicated in the proof of Proposition 2.12, that

$$|V^{(n)}(t)| \leq \frac{K}{n} \sum_{i=1}^{n} L_i'(t)$$

where the L^i are i.i.d. local times at zero of Brownian motion. Now use the fact that $\frac{1}{n}\sum_{i=1}^{n} L_i'$ converges almost surely and in L_q to its mean function, see [21], and apply the (generalized) dominated convergence theorem [20, Chapter 2.3]. □

We are now in a position to prove the propagation of chaos result.

Proof of Theorem 1.3. It suffices to prove the theorem for two particles $X_1^{(n)}, X_2^{(n)}$. The initial conditions ξ_1 and ξ_2 are independent by assumption. From the Skorohod representation theorem there is a probability space supporting all our processes such that $Y^{(n)}$ converges almost surely to Y in $C([0,T],\mathbb{R})$. For $l = 1,2$ set

$$m_l(t) = \sup_{0<u<t} -[B^{(l)}(u) + \xi_l - Y(u)] \vee 0,$$
$$m_l^{(n)}(t) = \sup_{0<u<t} -[B^{(l)}(u) + \xi_l - Y^{(n)}(u)] \vee 0.$$

By an argument similar to the one in Proposition 2.12, $\xi_l + B^{(l)} + m_l^{(n)}$ has the same distribution as $X_l^{(n)}$. Since $Y^{(n)} \to Y$ almost surely, $m_l^{(n)} \to m_l$ almost surely as well. Hence $\xi_l + B^{(l)} + m_l^{(n)} \to \xi_l + B^{(l)} + m_l$, almost surely. Clearly $\xi_1 + B^{(1)} + m_1$ and $\xi_2 + B^{(2)} + m_2$ are independent as each is a Brownian motion reflected from Y, driven by different independent Brownian motions with independent initial positions. □

In [7], Burdzy, Chen and Sylvester study the density of Brownian motion reflected inside a time dependent domain. They assume the boundary is C^3 in both time and space, see [7, Section 2]. In our case $n = 1$, and their results hold under the weaker assumption that the space-time boundary is C^2. Let $g(t) \in C^2([0,T],\mathbb{R})$ be a twice differentiable function with $g(0) = 0$. Given a Brownian motion $B(t)$ and $x \geq 0$, let $p(t,y)$ be the transition density of the reflected Brownian motion solving $dX(t) = dB(t) + dL(t)$, and the initial condition $X(0) = x$, where L is the local time of X on g. That is, for a given Borel set $A \subset [g(t),\infty)$,

$$\mathbb{P}_x(X(t) \in A) = \int_A p(t,y)dy.$$

Proposition 3.9 ([7], Theorem 2.9)**.** *The transition density $p(t,y)$ defined above solves the following heat equation in a time-dependent domain:*

$$\begin{aligned}
&\frac{\partial p(t,y)}{\partial t} = \frac{1}{2}\Delta_y p(t,y), \;\; y > g(t),\\
&\frac{\partial^+ p(t,y)}{\partial^+ y} = -2g'(t)p(t,y), \;\; y = g(t),\\
&\lim_{t\downarrow 0} p(t,y)dy = \delta_x(dy).
\end{aligned}$$

Remark 3.10. Here

$$\frac{\partial^+ p(t,y)}{\partial^+ y} = \lim_{h\downarrow 0} \frac{p(t,y+h) - p(t,y)}{h}$$

is the one sided derivative on the positive side. In particular, $p(t,y)$ has differentiability necessary for these statements to hold in the classical sense.

Corollary 3.11. *Let ξ be a random variable with law $\pi_0(dx)$, independent from the Brownian motion B, both supported on $(\Omega,\mathbb{P},\mathcal{F}_t)$. Let $g \in C^2([0,T],\mathbb{R})$ and*

$$X(t) = \xi + B(t) + m(t), \qquad m(t) = \sup_{0<s<t} -(\xi + B(s) - g(s)) \vee 0.$$

Then $p(t,x) := \mathbb{P}(X(t) = dx)$ *solves the PDE*

$$\begin{aligned}
&\frac{\partial p}{\partial t} = \frac{1}{2}\Delta_y p, \ \ y > g(t), \\
&\frac{\partial p}{\partial y} = -2g'(t)p, \ \ y = g(t), \\
&\lim_{t \downarrow 0} p(t,y)dy = \pi_0(dy).
\end{aligned} \tag{3.10}$$

Proof. As in the proof of Proposition 2.12, it follows from Lévy's theorem applied after a Girsanov change of measure that X is distributed as a Brownian motion reflected from the curve g. Now apply Proposition 3.9 after conditioning on ξ. □

For a given time $0 < t < T$ and fixed value of n, the definition of our interacting diffusions gives us n particles $X_1^{(n)}(t), \dots, X_n^{(n)}(t)$ which all lie in $[Y^{(n)}(t), \infty)$. Recall that

$$\pi_t^{(n)} = \frac{1}{n}\sum_{i=1}^n \delta_{\{X_i^{(n)}(t)\}} \tag{3.11}$$

denotes the empirical process of the arrangment of these particles. Similarly recall the definition of W_p in 1.6. The main property of W_p we will need is that $\mathcal{P}$ is separable and complete under W_p. Clearly $\pi_t^{(n)}$ is a random variable with state space $\mathcal{P}$. In this way $\{\pi_t^{(n)} : t \in [0,T]\}$ is a $(\mathcal{P}, W_p)-$valued stochastic process. By Proposition 3.13 below $\pi_t^{(n)}$ is continuous, and $(\pi_\cdot^{(n)}, Y^{(n)}(\cdot), V^{(n)}(\cdot))$ has the strong Markov property.

Lemma 3.12. *For any collection* $x_i, y_i \in \mathbb{R}, \ i = 1, \dots, n$ *we have*

$$W_p\Big(\frac{1}{n}\sum_{i=1}^n \delta_{\{x_i\}}, \frac{1}{n}\sum_{i=1}^n \delta_{\{y_i\}}\Big) \le \Big(\frac{1}{n}\sum_{i=1}^n |x_i - y_i|^p\Big)^{1/p}$$

Proof. This follows from coupling (X,Y) with

$$X \overset{d}{=} \frac{1}{n}\sum_{i=1}^n \delta_{\{x_i\}}, \ Y \overset{d}{=} \frac{1}{n}\sum_{i=1}^n \delta_{\{y_i\}}$$

so that X has mass on $\{x_i\}$ exactly when Y has mass on $\{y_i\}$. □

Proposition 3.13. *The pair* $\{(\pi_t^{(n)}, Y^{(n)}(t), V^{(n)}(t)) : 0 < t < T\}$ *is a continuous strong Markov process on* $\mathcal{P} \times \mathbb{R}^2$ *under the product metric* $W_p \times |\cdot|$.

Proof. The strong Markov property follows from the strong Markov property of $(X_1^{(n)}, \ldots, X_n^{(n)}, Y^{(n)}, V^{(n)})$. We need only show continuity of $\pi^{(n)}$ since $(Y^{(n)}, V^{(n)})$ is continuous. By Lemma 3.12,

$$W_p(\pi_t^{(n)}, \pi_s^{(n)}) \leq \Big(\frac{1}{n}\sum_{i=1}^{n} |X_i^{(n)}(t) - X_i^{(n)}(s)|^p\Big)^{1/p}, \tag{3.12}$$

and continuity follows from the continuity of the $X_i^{(n)}$. □

As $\pi^{(n)}$ is a continuous $\mathcal{P}-$valued process, it induces a probability measure on $C([0,T], (\mathcal{P}, W_p))$. We will abuse notation, which should be clear from context, by letting $\pi^{(n)}$ denote the measure on $C([0,T], \mathcal{P})$, and $\pi_t^{(n)}$ to denote either the stochastic process or the element in $\mathcal{P}$ when t is fixed. Let

$$\tilde{\pi}_t^{(n)} := \frac{1}{n}\sum_{i=1}^{n} \delta_{\{\widetilde{X}^{(i)}(t)\}},$$

where

$$d\widetilde{X}^{(i)} = dB^{(i)} + d\widetilde{L}^{(i)}, \qquad X_i^{(\infty)}(0) \overset{d}{=} \pi_0 \text{ for } i = 1, \ldots, n, \tag{3.13}$$

the $X_i^{(\infty)}(0)$ are i.i.d. and $\widetilde{L}^{(i)}$ is the local time of $\widetilde{X}_i^{(n)}$ on the function Y given in Corollary 3.8.

Proposition 3.14. *There is a probability space supporting* $\pi^{(n)}$, $\tilde{\pi}^{(n)}$ *for all* n *such that*

$$\sup_{0<t<T} W_p(\pi_t^{(n)}, \tilde{\pi}_t^{(n)}) \longrightarrow 0$$

almost surely.

Remark 3.15. This shows distributional convergence of $\pi^{(n)}$ and convergence of $\tilde{\pi}^{(n)}$ are equivalent. They will approach the same limiting measure should one (hence both) of them converge.

Proof. Consider the probability space supporting all the $\{B^{(i)}(t) : 0 < t < T\}$ together with the initial conditions $\{X_i^{(n)}(0) : 1 \leq i \leq n, n \in \mathbb{N}\}$. This space will then support $Y^{(n)}, Y$ as well. By Corollary 3.8 we may also assume $Y^{(n)} \to Y$ almost surely. As in the proof of Proposition 3.5, $\{X_i^{(\infty)}(0) : i \in \mathbb{N}\}$ are i.i.d. samples with distribution π_0. By our assumption that $\pi_0^{(n)} \to \pi_0$ in $(\mathcal{P}, W_p)$, we may further choose our probability space so that

$$\frac{1}{n}\sum_{i=1}^{n} |X_i^{(n)}(0) - X_i^{(\infty)}(0)|^p \to 0 \tag{3.14}$$

in probability. Using the Skorohod representation theorem we can find a supporting probability space where this holds almost surely. We use the same representation of our processes as in the proof of the propagation of chaos. That is,

$$X_i^{(n)}(t) = X_i^{(n)}(0) + B^{(i)}(t) + m_i^{(n)}(t), \tag{3.15}$$

$$\widetilde{X}^{(i)}(t) = X_i^{(\infty)}(0) + B^{(i)}(t) + \tilde{m}_i(t), \tag{3.16}$$

for $i = 1, \cdots, n$, and $t \in [0, T]$, where

$$m_i^{(n)}(t) = \sup_{0<u<t} -[(B^{(i)}(u) + X_i^{(n)}(0)) - Y^{(n)}(u)] \vee 0, \tag{3.17}$$

$$\tilde{m}_i(t) = \sup_{0<u<t} -[(B^{(i)}(u) + X_i^{(\infty)}(0)) - Y(u)] \vee 0. \tag{3.18}$$

By the triangle inequality

$$\|m_i^{(n)} - \tilde{m}_i\|_{[0,t]} \leq |X_i^{(n)}(0) - \widetilde{X}_i^{(\infty)}(0)| + \|Y^{(n)} - Y\|_{[0,t]} \tag{3.19}$$

for any $t \in [0, T]$. For any nonnegative numbers a and b, $(a+b)^p \leq (2(a \vee b))^p \leq 2^p(a^p + b^p)$. Using (3.19), Lemma 3.12, (3.14) and the fact that $\|Y^{(n)} - Y\|_{[0,T]} \to 0$ almost surely,

$$\begin{aligned}
\sup_{0<t<T} W_p(\pi_t^{(n)}, \tilde{\pi}_t^{(n)}) &\leq \sup_{0<t<T} \Big(\frac{1}{n}\sum_{i=1}^{n} |X_i^{(n)}(t) - \widetilde{X}^{(i)}(t)|^p\Big)^{1/p} \\
&\leq \sup_{0<t<T} \Big(\frac{1}{n}\sum_{i=1}^{n} \Big(|X_i^{(n)}(0) - X_i^{(\infty)}(0)| + \|m_i^{(n)} - \tilde{m}_i\|_{[0,t]}\Big)^p\Big)^{1/p} \\
&= \Big(\frac{1}{n}\sum_{i=1}^{n} 2^{p+1}|X_i^{(n)}(0) - X_i^{(\infty)}(0)|^p + 2^p\|Y^{(n)} - Y\|_{[0,T]}^p\Big)^{1/p} \\
&\longrightarrow 0,
\end{aligned}$$

almost surely. ☐

Recall the following notions of modulus of continuity. For $\gamma \in C([0,T], (\mathcal{P}, W_p))$,

$$\omega'(\gamma, T, \delta) = \sup_{\substack{0<t<T \\ |t-s|<\delta}} W_p(\gamma_t, \gamma_s),$$

and similarly for $f \in C([0,T], \mathbb{R})$,

$$\omega(f, T, \delta) = \sup_{\substack{0<t<T \\ |t-s|<\delta}} |f(t) - f(s)|.$$

In our method of showing tightness of the collection $\pi^{(n)}$ we utilize p-th moment bounds of $\omega(B, T, \delta)$ for a Brownian motion B. This is to be compared to Lévy's theorem on the modulus of continuity for Brownian motion which deals with the almost sure behavior of the modulus of continuity for small values of δ. We cite the article [19], where the authors Fischer and Nappo give a more general statement concerning moment bounds of $\omega(X, T, \delta)$, when X is an Ito process.

Theorem 3.16 ([19])**.** *Let $B(t)$ be a one dimensional Brownian motion and $T > \delta > 0$. For any $q > 0$ there exists a positive constant C_q independent of T and δ such that*

$$\mathbb{E}\,\omega(B, T, \delta)^q < C_q \left(\delta \log \frac{T}{\delta}\right)^{q/2}.$$

This leads directly to the following strong law of large numbers applied to the modulus of continuity $\omega(B^{(i)}, T, \delta)$.

Corollary 3.17. *Consider a sequence of independent Brownian motions $\{B^{(i)} : i \in \mathbb{N}\}$ all defined on the same probability space. We have*

$$\frac{1}{n}\sum_{i=1}^{n} \omega(B^{(i)}, T, \delta)^q \xrightarrow{a.s.} \mathbb{E}\,\omega(B^{(i)}, T, \delta)^q < C_q \left(\delta \log \frac{T}{\delta}\right)^{q/2}$$

for every $q > 0$, every $\delta > 0$, and some positive constant C_q depending on q only.

Remark 3.18. Typically when X_n are continuous stochastic process on a complete and separable metric space (E, d), one demonstrates tightness of the measures induced on $C([0,T],E)$ by showing "stochastic equicontinuity"

$$\lim_{\delta \to 0} \sup_n \mathbb{P}(\omega(X_n, T, \delta) > \epsilon) = 0 \tag{3.20}$$

together with a compact containment condition for a countable dense set of times $[0,T]$: given any $\eta > 0$ one can find a relatively compact set $\Gamma_{t,\eta} \subset E$ such that

$$\inf_n \mathbb{P}(X_n(t) \in \Gamma_{t,\eta}) > 1 - \eta. \tag{3.21}$$

Consider (3.20) and the corresponding δ for $\epsilon = 1$. Repeated use of the triangle inequality between time increments of size δ can be used to bound $X_n(t)$ with high probability uniformly in n at each time t should X_n be bounded w.h.p. uniformly in n at a fixed time t_0. Since boundedness in $\mathbb{R}^d$ is equivalent to relative compactness, if E is Euclidean, (3.21) can be concluded from (3.20) provided there is some time t_0 such that $X_n(t_0)$ is bounded w.h.p. uniformly in n. If E is not Euclidean, finding compact sets may not be particularly easy, especially if E is not locally compact. Since our processes are $(\mathcal{P}, W_p)-$valued continuous processes, as shown in Lemma 3.13, and since $(\mathcal{P}, W_p)$ is not locally compact, we face similar issues. One can use the p-th moment bounds on $\omega(B^{(i)}, T, \delta)$ with a similar arguments in the proof of Proposition 3.19 to demonstrate (3.20). This would need to be paired with a compact containment condition as mentioned. We sidestep dealings with compact sets in $(\mathcal{P}, W_p)$ by establishing almost sure pointwise convergence of subsequential limits of $\pi_t^{(n)}$ together with a uniform stochastic equicontinuity result Proposition 3.19 below.

Proposition 3.19. *For every $\epsilon, \eta > 0$ there corresponds a $\delta > 0$ such that*

$$\mathbb{P}\left(\sup_n \omega'(\tilde{\pi}^{(n)}, T, \delta) \le \epsilon\right) > 1 - \eta.$$

Proof. Recall the role of v in (3.1). We first prove the case when $v \le 0$ so that Y is monotonically decreasing. The general case follows by applying the proof to each partition of $[0,T] = [0,t^*] \cup [t^*,T]$, where t^* is the unique zero of V. From Lemma 3.12 and the definitions of ω, ω' the following holds almost surely,

$$\begin{aligned}
\omega'(\tilde{\pi}^{(n)}, T, \delta) &:= \sup_{\substack{0<t<T \\ |t-s|<\delta}} W_p(\tilde{\pi}^{(n)}_s, \tilde{\pi}^{(n)}_t) \\
&\leq \sup_{\substack{0<t<T \\ |t-s|<\delta}} \Big(\frac{1}{n}\sum_{i=1}^n \big[|B^{(i)}(s) - B^{(i)}(t)| + |\tilde{m}_i(s) - \tilde{m}_i(t)|\big]^p\Big)^{1/p} \\
&\leq \Big(\frac{1}{n}\sum_{i=1}^n \sup_{\substack{0<t<T \\ |t-s|<\delta}} \big[|B^{(i)}(s) - B^{(i)}(t)| + |\tilde{m}_i(s) - \tilde{m}_i(t)|\big]^p\Big)^{1/p} \\
&\leq \Big(\frac{2^p}{n}\sum_{i=1}^n \sup_{\substack{0<t<T \\ |t-s|<\delta}} [|B^{(i)}(s) - B^{(i)}(t)|^p + |\tilde{m}_i(s) - \tilde{m}_i(t)|^p]\Big)^{1/p} \\
&\leq \Big(\frac{2^p}{n}\sum_{i=1}^n \sup_{\substack{0<t<T \\ |t-s|<\delta}} |B^{(i)}(t) - B^{(i)}(s)|^p + \sup_{\substack{0<t<T \\ |t-s|<\delta}} |\tilde{m}_i(t) - \tilde{m}_i(s)|^p\Big)^{1/p} \\
&= \Big(\frac{2^p}{n}\sum_{i=1}^n \omega(B^{(i)}, T, \delta)^p + \omega(\tilde{m}_i, T, \delta)^p\Big)^{1/p}.
\end{aligned}$$

Because $dY/dt \leq v$ is monotonically decreasing, $\omega(\tilde{m}_i, T, \delta) \leq v\delta + \omega(B^{(i)}, T, \delta)$. That is, the maximum change the Brownian path makes below Y, in the span of δ time, is bounded by the change made by the line vt in addition to the change of the Brownian path. This gives

$$\omega'(\tilde{\pi}^{(n)}, T, \delta) \leq \Big(2^p v\delta + \frac{2^{p+1}}{n}\sum_{i=1}^n \omega(B^{(i)}, T, \delta)^p\Big)^{1/p}$$

almost surely. For simplicity we take $v = 0$ in the remaining argument. Setting $I_\epsilon = (\epsilon^p/2^{p+1}, \infty)$,

$$\begin{aligned}
&\mathbb{P}(\sup_{n>N} \omega'(\tilde{\pi}^{(n)}, T, \delta) > \epsilon) \leq \mathbb{P}\Big(\sup_{n>N} \frac{1}{n}\sum_{i=1}^n \omega(B^{(i)}, T, \delta)^p > \frac{\epsilon^p}{2^{p+1}}\Big) \\
&= \mathbb{E}\, 1_{I_\epsilon}\Big\{\sup_{n>N} \frac{1}{n}\sum_{i=1}^n \omega(B^{(i)}, T, \delta)^p\Big\}.
\end{aligned}$$

By Corollary 3.17 and the dominated convergence theorem,

$$\begin{aligned}\lim_{N\to\infty} \mathbb{E}\, 1_{I_\epsilon}\Big\{\sup_{n>N} \frac{1}{n}\sum_{i=1}^{n} \omega(B^{(i)},T,\delta)^p\Big\} &= \mathbb{E}1_{I_\epsilon}\Big\{\mathbb{E}\,\omega(B^{(i)},T,\delta)^p\Big\} \\ &\le \mathbb{E}\, 1_{I_\epsilon}\Big\{C_p\Big(\delta\log\frac{T}{\delta}\Big)^{p/2}\Big\} \\ &= 1_{I_\epsilon}\Big\{C_p\Big(\delta\log\frac{T}{\delta}\Big)^{p/2}\Big\}.\end{aligned}$$

In other words,

$$\lim_{N\to\infty} \mathbb{P}(\sup_{n>N} \omega'(\tilde{\pi}^{(n)},T,\delta) > \epsilon) \le 1_{I_\epsilon}\Big\{C_p\Big(\delta\log\frac{T}{\delta}\Big)^{p/2}\Big\}, \tag{3.22}$$

which is 0 when δ satisfies

$$\delta\log\frac{T}{\delta} < \frac{\epsilon^2}{4^{(p+1)/p}\, C_p^{2/p}}.$$

With this chosen value of δ, take N large enough so that

$$\mathbb{P}(\sup_{n>N} \omega'(\tilde{\pi}^{(n)},T,\delta) > \epsilon) < \eta/2,$$

then appropriately shrink δ until

$$\sum_{i=1}^{N} \mathbb{P}(\omega'(\tilde{\pi}^{(i)},T,\delta) > \epsilon) < \eta/2,$$

to conclude that

$$\mathbb{P}(\sup_{n} \omega'(\tilde{\pi}^{(n)},T,\delta) > \epsilon) < \eta.$$

□

Corollary 3.20. *The collection* $\{\tilde{\pi}^{(n)}, n \ge 1\}$ *is equicontinuous on* $C([0,T],(\mathcal{P},W_p))$ *with probability 1.*

Proof. Apply Proposition 3.19 to decreasing sequences $\epsilon = 1/k$ and $\eta = 2^{-k}$ to yield a sequence $\delta_k \to 0$ with

$$\sum_{k=1}^{\infty} \mathbb{P}(\sup_{n} \omega'(\tilde{\pi}^{(n)},T,\delta_k) > 1/k) < \infty.$$

By Borel-Cantelli the probability that $\{\sup_n \omega'(\tilde{\pi}^{(n)}, T, \delta_k) > 1/k\}$ occurs infinitely often is zero. Almost surely, $A_k := \{\sup_n \omega'(\tilde{\pi}^{(n)}, T, \delta_k) \leq 1/k\}$ occurs all but finitely many times. This means for an almost sure set of ω in our probability space there is a finite integer $N(\omega)$ so that $\omega \in \bigcap_{k>N(\omega)} A_k$, which in turn implies the sequence $\tilde{\pi}^{(1)}(\omega)$, $\tilde{\pi}^{(2)}(\omega), \ldots,$ is equicontinuous. ☐

We quote one more theorem and present a lemma before proving the main result.

Theorem 3.21 ([21]). *For $p \geq 1$, let $\{\xi_i : i \in \mathbb{N}\}$ be i.i.d. samples of an L^p bounded random variable ξ with density f, all supported on the same probability space. Then*

$$\mathbb{E}\, W_p\Big(\frac{1}{n}\sum_{i=1}^{n} \delta_{\{\xi_i\}}, f\Big) \to 0, \ \ \textit{as } n \to \infty.$$

Lemma 3.22. *Let V be a continuous function, and X a solution to $dX = dB + Vdt + dL$ where L is the local time of X at zero. Then*

$$Z(t) = \exp\Big(- \int_0^t V_s dB_s - \frac{1}{2}\int_0^t V_s^2 ds\Big)$$

is a martingale with $Z(0) = 0$ and $\mathbb{E}[Z(t)^p] < \infty$ for any $p > 0$.

Proof. Since V is continuous, it is bounded, and so it follows from Novikov's condition that Z is a martingale. In fact, if $M(t)$ is a continuous local martingale, $Z' := \exp(M - \frac{1}{2}\langle M\rangle)$ is a local martingale from Ito's lemma. Because it is non-negative we may apply Fatou's lemma to an exhaustive sequence of local times $T_n \overset{a.s.}{\to} \infty$ to see $\mathbb{E}(Z'(t)|\mathcal{F}_s) \leq \lim_{n\to\infty} \mathbb{E}(Z'(t \wedge T_n)|\mathcal{F}_s) = \lim_{n\to\infty} Z'(s \wedge T_n) = Z'(s)$. That is, Z' is a supermartingale. Take any $p, q, q' > 0$ with $\frac{1}{q} + \frac{1}{q'} = 1$. Then

$$\mathbb{E}[Z(t)^p] = \mathbb{E}\Big[\exp\Big(- p\int_0^t V_s dB_S - \frac{qp^2}{2}\int_0^t V_s^2 ds\Big) \exp\Big(\frac{p(qp-1)}{2}\int_0^t V_s^2 ds\Big)\Big].$$

Now apply Holder's inequality with q, q'

$$\begin{aligned}\mathbb{E}[Z(t)^p] &\le \mathbb{E}\Big[\exp\Big(-pq\int_0^t V_s dB_s - \frac{q^2p^2}{2}\int_0^t V_s^2 ds\Big)\Big]^{1/q}\mathbb{E}\Big[\Big(\frac{pq'(qp-1)}{2}\int_0^t V_s^2 ds\Big)\Big]^{1/q'} \\ &\le 1\cdot\mathbb{E}\Big[\exp\Big(\frac{pq'(qp-1)}{2}\int_0^t V_s^2 ds\Big)\Big]^{1/q'} = \exp\Big(\frac{pq'(qp-1)}{2}\int_0^t V_s^2 ds\Big) < \infty.\end{aligned}$$

Here

$$\mathbb{E}\Big[\exp\Big(-pq\int_0^t V_s dB_s - \frac{q^2p^2}{2}\int_0^t V_s^2 ds\Big)\Big] \le 1,$$

since

$$M(t) = -pq\int_0^t V_s dB_s,\ \langle M\rangle(t) = q^2p^2\int_0^t V_s^2 ds$$

and because $\exp(M(t) - \langle M\rangle(t))$ is a supermartingale as explained above. □

We are now in a position to prove the hydrodynamic limit result, Theorem 1.2.

Proof of Theorem 1.2. We first show that $\pi^{(n)}$ converges in distribution to the measure induced by $p(t,\cdot)$. By Proposition 3.14 it suffices to show this for $\tilde{\pi}^{(n)}$. Take any subsequence n_k. For each rational $0 < t < T$ we have defined $\tilde{\pi}_t^{(n)}$ as an empirical measure of i.i.d. random variables with density $p(t,\cdot)$ taken from Corollary 3.11 by replacing g in the Corollary statement with Y. We show that $Y \in C^2([0,T],\mathbb{R})$ so the Corollary can be applied. By Theorem 3.21,

$$\mathbb{E}\, W_p(\tilde{\pi}_t^{(n_k)}, p(t,\cdot)) \to 0, \text{ for each } t \in [0,T].$$

For each rational $t \in [0,T]$ there is a subsequence n_k' such that

$$W_p(\tilde{\pi}_t^{(n_k')}, p(t,\cdot)) \to 0,$$

almost surely. By a Cantor diagonalization applied to each subsequence for an enumeration of the rationals, there exists a single subsequence n_k'' such that $W_p(\tilde{\pi}_t^{(n_k'')}, p(t,\cdot)) \to 0$ for each rational $t \in [0,T]$, almost surely. Apply the uniform equicontinuity given by Corollary 3.20, and follow the proof of Arzéla-Ascoli verbatim to see that the subsequence $\tilde{\pi}^{(n_k'')}$ is totally bounded in the space $C([0,T],(\mathcal{P},W_p))$, almost surely. See [20, Chapter 4.6]. Total boundedness in a metric space is equivalent to every sequence having a Cauchy subsequence.

Consequently, for almost every ω in the probability space, every subsequence of $\pi^{(n''_k)}(\omega)$ has a Cauchy subsequence in $C([0,T],(\mathcal{P},W_p))$. Because $C([0,T],(\mathcal{P},W_p))$ is complete, every subsequence of $\pi^{(n''_k)}(\omega)$ has a convergent subsequence. Since $\pi_t^{(n''_k)}(\omega)$ already converges to the continuous $p(t,\cdot)$ along rationals, every subsequence of $\pi^{(n''_k)}(\omega)$ has a further subsequence converging to $p(t,\cdot)$. Therefore $\pi_t^{(n''_k)}$ converges to $p(t,\cdot)$ in $C([0,T],(\mathcal{P},W_p))$ almost surely. This proves the claim that $\{\pi_t^{(n)} : t \in [0,T]\}$ converges in distribution to $p(t,\cdot)$. Next, we show

$$V(t) = -(K/2)\int_0^t p(s,Y(s))ds$$

and that $V \in C^1([0,T],\mathbb{R})$. This also demonstrates $Y \in C^2([0,T],\mathbb{R})$, which we took for granted above. We take $v = 0$ for simplicity. Let

$$m(t) = \sup_{0<u<t} -[B^{(1)}(u) + X_i^{(\infty)}(0) - Y(u)] \vee 0.$$

As in the proof of Proposition 2.12 we know $m(t)$ is distributed as $\widetilde{L}^{(1)}(t)$, the local time of

$$\widetilde{X}(t) := B^{(1)}(t) + X_1^{(\infty)}(0) + m(t)$$

on Y. From Corollary 3.8 we have, almost surely,

$$\begin{aligned} V(t) &= \mathbb{E}\, m_1(t) = \mathbb{E}\, \widetilde{L}^{(1)}(t) \\ &= \mathbb{E} \lim_{\epsilon\to 0} \frac{-K}{2\epsilon}\int_0^t 1_{[0,\epsilon]}(\widetilde{X}(s) - Y(s))ds \\ &= \lim_{\epsilon\to 0} \frac{-K}{2\epsilon}\mathbb{E}\int_0^t 1_{[0,\epsilon]}(\widetilde{X}(s) - Y(s))ds \\ &= \frac{-K}{2}\lim_{\epsilon\to 0}\int_0^t \frac{\mathrm{F}(s,\epsilon)}{\epsilon}ds, \end{aligned} \tag{3.23}$$

where $\mathrm{F}(s,\epsilon) = \mathbb{P}(0 \le \widetilde{X}(s) - Y(s) \le \epsilon)$, provided we justify the passing of the limit under the expectation. In the proof of Proposition 2.12 we saw that $\widetilde{X} - Y$ solves an SDE of the form $dW = dB + Vdt + dL$ for the continuous function V, and such processes have a continuous density $\phi(s,x) = p(s,Y(s)+x)$. That such processes have a continuous density is shown in [36]. Write

$$\frac{1}{\epsilon}\int_0^t \mathrm{F}(s,\epsilon)ds = \int_0^t \frac{1}{\epsilon}\int_0^\epsilon \phi(s,x)dxds = \int_0^t \phi(s,x^*)ds$$

for some $0 < x^* < \epsilon$ by the mean value theorem. For all $0 < \epsilon < 1$

$$\frac{1}{\epsilon}\int_0^t \mathrm{F}(s,\epsilon)ds \leq \sup_{0<x^*<1}\int_0^t \phi(s,x^*)ds \leq \int_0^t \sup_{\substack{0<x^*<1\\ 0<s<t}} \phi(s,x^*)ds < \infty$$

and the bounded convergence theorem justifies the passing of the limit inside the time integral,

$$\frac{-K}{2}\lim_{\epsilon\to 0}\int_0^t \frac{\mathrm{F}(s,\epsilon)}{\epsilon}ds = \frac{-K}{2}\int_0^t \lim_{\epsilon\to 0}\frac{\mathrm{F}(s,\epsilon)}{\epsilon}ds = \frac{-K}{2}\int_0^t \phi(s,0)ds.$$

That is,

$$V(t) = \frac{-K}{2}\int_0^t p(s,Y(s))ds.$$

We now justify the exchange of limit in (3.23) using the definition of local time to replace the time integral with a space integral. Let $\widetilde{L}^{(1)}(s,a)$ denote the local time of $\widetilde{X}-Y$ at level a and time s. We see

$$\frac{1}{\epsilon}\int_0^t 1_{[0,\epsilon]}(X(s)-Y(s))ds = \int_0^t \frac{1}{\epsilon}\int_0^\epsilon \widetilde{L}^{(1)}(s,z)dz\,ds \leq \int_0^t \sup_z\,[\widetilde{L}^{(1)}(s,z)]ds.$$

The Lebesgue dominated convergence theorem will justify (3.23) provided we show

$$\mathbb{E}\int_0^t \sup_z[\widetilde{L}^{(1)}(s,z)]ds \leq t\,\mathbb{E}\sup_z\,[\widetilde{L}^{(1)}(t,z)] < \infty.$$

We apply a Girsanov change of measure as in Lemma 3.22 which is justified because Y satisfies the Novikov condition. So

$$Z(t) = \exp\Big(-\int_0^t V_s dB_s - \frac{1}{2}\int_0^t V_s^2 ds\Big)$$

is an exponential martingale with $|B|$ having the same distribution as $\widetilde{X}-Y$ under the measure defined by the Girsanov transformation $d\mathbb{Q}/d\mathbb{P} = Z(t)$. Lemma 3.22 states $\mathbb{E}[Z(T)^2] = C < \infty$. From this, the change of measure formula, and Cauchy-Schwarz,

$$\begin{aligned}\mathbb{E}\,\sup_z[\widetilde{L}^{(1)}(t,z)] &= \mathbb{E}(Z(t)\sup_z L(t,z))\\ &\leq \mathbb{E}(Z^2(t))^{1/2}\,\mathbb{E}\,[(\sup_z L(t,z))^2]^{1/2}\\ &\leq C^{1/2}\,\mathbb{E}\,[(\sup_z L(t,z))^2]^{1/2}\end{aligned}$$

where $L(t,z)$ is the local time at level z of Brownian motion reflected from the origin. The main results in [2, Theorem 3.1] demonstrate bounds on the last term, where Barlow and Yor show the existence of a constant C_p such that

$$\mathbb{E}\left[(\sup_z L_t(z))^p\right] \leq C_p\, t^{p/2}.$$

It follows that

$$\mathbb{E}\sup_z[\widetilde{L}^{(1)}(t,z)] < \infty,$$

completing the proof. $\square$

4 *Uniqueness of the heat equation with free-boundary*

In this section we give existence and uniqueness for the PDE with free boundary condition $(p(t,\cdot), y(t))$ which is the solution of our hydrodynamic limit given by (1.2). If (p,y) is a solution and $p(t,\cdot)$ represents the distribution of heat, then the equation in Theorem 1.2 is interpreted as saying the acceleration of the moving barrier $y(t)$ is proportional to the current amount of heat on it. The hydrodynamic limit already yields existence of such a solution. In that statement of Theorem 1.2 $(\pi^{(n)}, Y^{(n)})$ converges in some sense to a solution of (1.2). Here we show this is the only solution by demonstrating uniqueness of this PDE with free boundary.

Remark 4.1. For any solution $(p,\ y)$ of (1.2) make a substitution $u(t,x) = p(t, x + y(t))$ and see $(u,\ y)$ is a classical solution to

$$u_t(t,x) = \frac{1}{2}u_{xx}(t,x) + y'(t)u_x(t,x), \text{ when } x > 0, \tag{4.1}$$

$$u_x(t,0) = -2y'(t)u(t,0), \tag{4.2}$$

$$y''(t) = -\frac{K}{2}u(t,0),\ y(0) = 0,\ y'(0) = v \in \mathbb{R},\ y'' \in C([0,T],\mathbb{R}), \tag{4.3}$$

$$\lim_{t\downarrow 0} u(t,x) = f(x)dx,\ \ f \in L^1(\mathbb{R}_+). \tag{4.4}$$

In this way the two problems are equivalent.

Theorem 4.2. *The PDE problem in* (4.1)-(4.3)*, and equivalently that in* (1.2)*, has a unique solution for any* $K \geq 0$.

Remark 4.3. The regularity of the boundary plays an important role because if y'' exists then the solution to (1.2) has a stochastic representation given from Corollary 3.11. We exploit this to show uniqueness.

Proof. Theorem 1.2 gives existence. To show uniqueness we will prove the corresponding barriers y_1, y_2 of any two solutions are in fact equal. Assume that $(p_1(t,\cdot), y_1(t)),\ (p_2(t,\cdot), y_2(t))$ are pairs solving the PDE with the given initial conditions. Following Corollary 3.11 above we know that the transition density $p_i(t,x)$ of Brownian motion reflecting from y_i satisfies the PDE

$$\frac{\partial p_i}{\partial t} = \frac{1}{2}\Delta_y p_i,\ \ y > y_i(t), \tag{4.5}$$

$$\frac{\partial p_i}{\partial y} = -2y_i'(t)p_i,\ \ y = y_i(t), \tag{4.6}$$

$$\lim_{t \downarrow 0} p_i(t,y)dy = f(y)dy \in L^1(\mathbb{R}_+). \tag{4.7}$$

Without loss of generality we assume $\int f(y)dy = 1$. Let $(\Omega, \mathcal{F}_t, \mathbb{P})$ be a probability space supporting a Brownian motion $B(t)$ and an independent random variable ξ with density f. As in the proof of Theorem 1.2 we know

$$y_i'(t) = -(K/2)\mathbb{E}\, m_i(t), \quad \text{where} \ \ m_i(t) = \max_{u \in [0,t]} -[B(u) + \xi - y_i(u)] \vee 0.$$

Linearity of expectation yields the following comparison between y_1', y_2' :

$$\begin{aligned} |y_1(t) - y_2(t)| &\leq \int_0^t |y_1'(s) - y_2'(s)| ds \\ &= \frac{K}{2}\int_0^t |\mathbb{E}\,(m_1(s) - m_2(s))| ds \\ &\leq \frac{K}{2}\int_0^t \|y_1 - y_2\|_{[0,s]} ds \leq \frac{K}{2} t \|y_1 - y_2\|_{[0,t]}. \end{aligned}$$

Consequently,

$$|y_1(t) - y_2(t)| \leq \frac{K}{2} t \|y_1 - y_2\|_{[0,x]} \leq (K/2)t\|y_1 - y_2\|_{[0,t]}.$$

Because the right hand is nondecreasing this inequality holds when the left hand is maximized across time,

$$\|y_1 - y_2\|_{[0,t]} \leq (K/2)t\|y_1 - y_2\|_{[0,t]}.$$

Therefore $\|y_1 - y_2\|_{[0,t]} \leq C\|y_1 - y_2\|_{[0,t]}$ for some $C < 1$ as long as $0 \leq t < t^* < 2/K$. As a result $\|y_1 - y_2\|_{[0,t^*]} = 0$ for all $t^* \in [0, \sqrt{2/K}]$. In other words, the barriers y_1 and y_2 are identical up until this fixed positive time. A renewal argument shows that y_1 and y_2 are identical across the entire interval $[0, T]$. $\square$

5 *Appendix*

In this appendix we prove 3.9 by taking the proofs from [7] and adapting them, ever so slightly, to our situation. Much of the following is close to verbatim from section 2. For $T \in (0, \infty]$, let $g \in C^2([0, T), \mathbb{R})$ and define $D = \{(t, x) : x > g(t),\ 0 \leq t < T\}$ and $D(s) = [g(s), \infty)$. So $\partial D(s) = \{g(s)\}$, and $D(s)$ has only one boundary point. We have $n(s, x) = (0, 1) \in \mathbb{R}^2$ is the unit inward normal of $D(s)$ at this boundary point. Let $\vec{\gamma} = (\gamma_1, \gamma_2)$ be the inward normal vector field on ∂D, which is the graph of g. At the location $(t, g(t)) \in \partial D$, we compute $\vec{\gamma}(t) := (\gamma_1(t), \gamma_2(t))$ which solve

$$(\gamma_1(t), \gamma_2(t)) = (-g'(t)\gamma_2(t), \gamma_2(t)), \text{ and } \gamma_1^2(t) + \gamma_2^2(t) = 1.$$

This equations is easy to derive: the first equation follows from the fact that the inward normal has slope $-1/g'(t)$, and the second equation guarantees it is a unit vector. One derives $\gamma_2^2(t) = 1/(1 + (g'(t))^2)$.

Theorem 5.1 (2.1, [7])**.** *Suppose that* $\vec{\gamma}(t) * n(t, x) = \gamma_2(t) \geq c_0 > 0$ *on* $\partial D \cap (0, T) \times \mathbb{R}$*. Let* B *be a standard one dimensional Brownian motion. Then for each* $(s, x) \in \overline{D}$ *with* $s < T$*, there is a unique pair of continuous processes* $(X^{s,x}, L^{s,x})$ *adapted to the minimal admissible filtration of* B*, such that:*

(i) $(t, X_t^{s,x}) \in \overline{D}$ *for* $t \in [s,T)$ *with* $X_s^{s,x} = x$,

(ii) $\{L_t^{s,x}, t \in [s,T)\}$ *is a nondecreasing process with* $L_s^{s,x} = 0$*, such that* $t \mapsto L_t^{s,x}$ *increases only when the process* (t, X_t) *is on the boundary of* D*. That is,* $L_t^{s,x} = \int_s^t 1_{\partial D}(r, X_r) dL_r^{s,x}$ *for* $s \leq t < T$.

(iii) $X_t^{S,x} = x + (B_t - B_s) + \int_s^t n(r, X_r^{s,x}) dL_r^{s,x}$ *for* $s \leq t < T$.

Proof. The authors of [7] reference Lions and Snitman (1984). In our case, the theorem follows from the Skorohod map construction of reflected Brownian motion. Since ∂D is the graph of g, one can see condition (ii) in the theorem statement above is our condition that L_t be flat away from the set $\{t : X_t^{s,x} = g(s)\}$, that is, flat away from the set of times $X_t^{s,x}$ contacts g. $\square$

Let $\mathbb{P}^{(s,x)}$ denote the law of $X^{s,x}$ induced on $C[0,\infty)$, the space of continuous functions equipped with the uniform topology on each compact time set. Let X be the canonical map on $C[0,\infty)$. The uniqueness of $X^{s,x}$ implies $X = (X, \mathbb{P}^{(s,x)}, (s,x) \in \overline{D})$ is a time-inhomogeneous strong Markov process. In our case, the strong Markov property is easily garnered from the time shift property of the Skorohod map described in Remark 1.4.

We will now prove the existence of the transition density for X and find some estimates for it, using parametric methods from the theory of partial differential equations [see. e.g., Itô (1957) or Hsu (1987)]. We work with the half Laplacian operator $\frac{1}{2}\Delta$.

Theorem 5.2. *There exists a fundamental solution* $p(s,x;t,y), (s,x), (t,y) \in D, s < t < T$*, for the for the following differential equation:*

$$\begin{cases} \frac{\partial p}{\partial s} + \frac{1}{2}\Delta_x p = 0, & for\ (s,x) \in D\ with\ s < t, \\ \frac{\partial p}{\partial n} = 0, & for\ (s,x) \in \partial D\ with\ s < t, \\ \lim_{s \uparrow t} p(s,x;t,y) dx = \delta_y(dx), & for\ (t,y) \in D. \end{cases} \tag{5.1}$$

Proof. We will use $|\cdot|$ for the Euclidean norm and d to denote the Euclidean distance. Let $\Gamma(s,x;t,y)$ be the fundamental solution for the heat equation $\frac{\partial u}{\partial s}+\frac{1}{2}\Delta_x u=0$ in $\mathbb{R}$; that is, $\Gamma(s,x;t,y)=(2\pi(t-s))^{-1/2}\exp(-|x-y|^2/(2(t-s)))$. For $(s,x)\in D$, let $x_0\in\partial D(s)$ be such that $|x-x_0|=d(x,D(s))$ and let $x_s^*=2x_0-x$, the point x_0. Since $\partial D(s)=g(s)$, we see $x_0=g(s)$ and $x_s^*=2g(s)-x$. Because $g\in C^2$, X_0 and x_s^* are uniquely determined by (s,x) and are C^2-smooth in (s,x). For fixed $T_0<T$, let $\phi\in C_c^\infty(\mathbb{R}_+\times\mathbb{R}^n)$ (the space of infinitely differentiable functions with compact support), $0\le\phi\le 1$ and such that $s\le T_0$,

$$\phi(s,x)=\begin{cases}1, & if\ d((sx),\partial D)\le\epsilon_0/2,\\ 0, & if\ d((s,x),\partial D)\ge\epsilon_0/2,\end{cases}$$

where ϵ_0 isa fixed small constant and $d((s,x),\partial D)$ is the Euclidean distance between (s,x) and the boundary of D in $\mathbb{R}_+\times\mathbb{R}$. where ϵ_0 is a fixed small constant and $d((s,x),\partial D)$ is the Euclidean distance between (s,x) and the boundary of D, which is the graph of g, in $\mathbb{R}_+\times\mathbb{R}$. As a first approximation of p, set $p_0(s,x;t,y)=\Gamma(s,x;t,y)+\phi(s,x)\Gamma(s,x_s^*;t,y)$. This function satisfies the boundary and terminal conditions in equation (5.1). The idea of the remaining part of the argument is to find a suitable function $f(s,x;t,y)$ so that if

$$p_1(s,x;t,y)=\int_s^t\left(\int_{D(r)}p_0(s,x;r,z)f(r,z;t,y)dz\right)dr,$$

then

$$p(s,x;t,y)=p_0(s,x;t,y)+p_1(s,x;t,y),\ \ s<t\le T_0, \tag{5.2}$$

is the fundamental solution for (5.1). Note that p defined in (5.2) satisfies the boundary and terminal condition in (5.1). We would like the function p defined in (5.2) to satisfy the heat equation, that is,

$$\left(\frac{\partial}{\partial s}+\frac{1}{2}\Delta_x\right)p_0+\left(\frac{\partial}{\partial s}+\frac{1}{2}\Delta_x\right)\int_s^t\left(\int_{D(r)}p_0(s,x;r,z)f(r,z;t,y)dz\right)dr=0.$$

This is equivalent to

$$\begin{aligned} f(s,x;t,y) = \left(\frac{\partial}{\partial s} + \frac{1}{2}\Delta_x\right) p_0(s,x;t,y) \\ + \int_s^t \left(\int_{D(r)} f(r,z;t,y) \left(\frac{\partial}{\partial s} + \frac{1}{2}\Delta_x\right) p_0(s,x;t,y) dz\right) dr. \quad (5.3) \end{aligned}$$

It remains to solve (5.3) for f. This is an integral equation of Volterra type, which can be solved by the method of iteration. Let

$$\begin{aligned} f_0(s,x;t,y) &= \left(\frac{\partial}{\partial s} + \frac{1}{2}\Delta_x\right) p_0(s,x;t,y), \\ f_k(s,x;t,y) &= \int_s^t \left(\int_{D(r)} f_0(s,x;r,z) f_{k-1}(r,z;t,y) dz\right) dr, \ k \geq 1, \\ f(s,x;t,y) &= \sum_{k=0}^{\infty} f_k(s,x;g,y). \end{aligned} \quad (5.4)$$

We will show that $\sum_{k=0}^{\infty} f_k(s,x;t,y)$ is absolutely convergent and solves (5.3).

Using induction, we can show that [27, page 375] for each fixed $l < T$, there are constants K_1, K_2 and C such that for all $(s,x),(t,y) \in \overline{D}$ with $s < t \leq l$ and $k \geq 1$,

$$|f_k(s,x;t,y)| \leq K_1 K_2^k \gamma \left(\frac{k+1}{2}\right)^{-1} (t-s)^{(k-2)/2} \exp\left(-\frac{c|x-y|^2}{(t-s)}\right), \quad (5.5)$$

where Γ is the Gamma function. Thus, $f(s,x;t,y) = \sum_{k=0}^{\infty} f_k(s,x;t,y)$ is well defined, continuous, and is easily seen to satisfy (5.3). □

For a fixed $t < T$, and a bounded continuous function ϕ on $\overline{D}(t)$, we see from Theorem 5.2 that

$$u(s,x) = \int_{D(t)} p(s,x;t,y)\phi(y)dy$$

is a solution of the following equation:

$$\begin{cases} \frac{\partial u}{\partial s} + \frac{1}{2}\Delta_x u = 0, & \text{for } (s,x) \in D \text{ with } s < t, \\ \frac{\partial u}{\partial n} = 0, & \text{for } (s,x) \in \partial D \text{ with } s < t, \\ \lim_{s \uparrow t} u(s,x) = \phi(x). \end{cases} \quad (5.6)$$

The following is a special case of the uniqueness result in Friedman [22, Theorem 15 in Chapter 2].

Theorem 5.3. *For fixed $(t, x) \in D$, their is a unique solution to the heat equation* (5.6).

Remark 5.4. See page 40 in Friedman for definitions of relevent objects in [22, Theorem 15]. We do not require the "inside strong sphere property" (ISSP) at $\{t = T\}$ since $t < T$. This is possible by modifying the operator L, in his notation, so that $b < 0$ from $[t + \epsilon, T]$ and therefore the ISSP condition will hold at $\{t = T\}$ for this modification.

Theorem 5.5. *The function $p(s, x; t, y)$ in Theorem 5.2 has the following properties:*

(i) *$p(s, x; t, y)$ is strictly positive and C^2-smooth on $\{(s, x, t, y) \in D \times D : s < t < T\}$.*

(ii) *For $s < t < T, (s, x) \in \overline{D}$,*

$$\int_{D(t)} p(s, x; t, y)dy = 1.$$

(iii) *The Chapman-Kolmogorov equations hold for any $0 \leq s < r < t < T$ and any $(s, x), (t, y) \in \overline{D}$,*

$$p(s, x; t, y) = \int_{D(r)} p(s, x; r, z)p(r, z; t, y)dz.$$

(iv) *For each fixed $0 < l < T$, there exist constants $K_l > 0$ and $C_l < \infty$ such that*

$$p(s, x; t, y) \leq C_l(t - s)^{-n/2} \exp\left(\frac{-K_l|x - y|^2}{t - s}\right)$$

for $s < t < l$ and $(s, x), (t, y) \in \overline{D}$.

(v) *Let $D_\epsilon = \{(t, x) \in D : d(x, \partial D(t) = g(t)) < \epsilon\}$. For each fixed $0 < l < T$, there are constants $\epsilon_l > 0$ and $C_l > 0$ such that for $0 < \epsilon < \epsilon_l, 0 < s < t \leq l$ and $(s, x) \in \overline{D}$,*

$$\frac{1}{\epsilon}\int_{D_\epsilon} p(s, x; t, y)dy \leq C_l/\sqrt{t - s}.$$

Proof. (i) The positivity of $p(s, x; t, y)$ is a consequence of a strong version of the maximum principle [22, Theorem 1 in Chapter 2] while the C^2 smoothness follows from (5.2) and (5.3). Assertions (ii) and (iii) follow from Theorem 5.3. Claim (iv) follows from the estimate (5.5) and (5.2), while (v) follows from (iv). $\square$

Theorem 5.6. *The function $p(s, x; t, y)$ in Theorem 5.2 is the transition density of the time-inhomogeneous RBM X on D defined in Theorem* (5.1)*. Therefore, X is a continuous Feller process and, hence, a strong Markov process.*

Proof. For any fixed $t < T$, and a bounded continuous function ϕ on $\overline{D}(t)$, let

$$u(s, x) = \int_{D(t)} p(s, x; t, y)\phi(y)dy.$$

The function $u(s, x)$ is a C^2-smooth solution to equation (5.6). For $(s, x) \in \overline{D}$, applying Itô's formula to $u(r, X_r^{s,x})$, we have

$$\begin{aligned} du(r, X_r^{s,x}) &= u_s(r, X_r^{s,x})dr + \frac{1}{2}\Delta u(r, X_r^{s,x})dr \\ &\quad + \nabla u(r, X_r^{s,x})dB_r + \frac{\partial u}{\partial n}(r, X_r^{s,x})dL_r^{s,x} \\ &= \nabla u(r, X_r^{s,x})dB_r. \end{aligned}$$

Hence,

$$u(s, x) = \mathbb{E}[u(t, X_t^{s,x})] = \mathbb{E}[\phi(X_t^{s,x})].$$

This shows that the distribution of $X_t^{s,x}$ is absolutely continuous with respect to the Lebesgue measure and its density function $p(s, x; t, y)$. From the continuity of p, we see that for bounded measurable functions ϕ on $\overline{D}(t)$, $u(s, x) = \mathbb{E}[\phi(X_t^{s,x})]$ is a continuous function in $\overline{D} \cap [0, t) \times \mathbb{R}^n$. This means that X is a Feller process. The Feller property together with the continuity of the sample paths implies that X is a strong Markov process. Note that an alternative way of proving the strong Markov property in our case was by using the time shift property of the Skorohod map, as indicated in Theorem 5.1 and the paragraph following it. $\square$

For $\epsilon > 0$, let $D_\epsilon = \{(s, x) \in D : d(x, \partial D(s) = g(s)) < \epsilon\}$ and let σ_r denote the surface area measure on $\partial D(r)$.

Theorem 5.7. *For* $(s,x) \in \overline{D}$ *and* $s < t < T$,

$$L_t^{s,x} = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_s^t 1_{D_\epsilon}(r, X_r^{s,x}) dr, \tag{5.7}$$

in L^2 *and a.s., uniformly on relatively compact sets of* t. *For each fixed* $0 < l < T$, *there is a constant* $c_l > 0$ *such that for* $(s,x) \in \overline{D}$ *and* $s < t \leq l$,

$$\mathbb{E}[L_t^{s,x}] = \frac{1}{2} \int_s^t \int \left(\int_{\partial D(r)} p(s,x;r,z) \sigma_r(dz) \right) ds \leq c_l \sqrt{t-s}. \tag{5.8}$$

Proof. For each fixed small constant $\epsilon > 0$, define $\psi_\epsilon(\delta) = (\epsilon - \delta)^2/2$ if $0 \leq \delta \leq \epsilon$ and 0 if $\delta > \epsilon$. Let

$$f_\epsilon(s,x) = \psi_\epsilon(d(x, D(s)^c)) = \psi(d(x, g(s))) = x - g(s).$$

Since g is C^2, f_ϵ is twice differentiable with bounded second derivative on $\{(t,x) \in D : t \leq l\}$ for each fixed $l < T$. Note that $0 \leq f_\epsilon \leq \epsilon^2$, $\frac{\partial f_\epsilon}{\partial s} \leq c_l \epsilon$, $|\nabla_x f_\epsilon| \leq c_l \epsilon$, $\nabla_x f_\epsilon(s,x) = -\epsilon n(s,x)$ for $(s,x) \in \partial D$, and $\Delta_x f_\epsilon = (1 + O(\epsilon)) 1_{D_\epsilon}$. By Itô's formula,

$$\begin{aligned} f_\epsilon(t, X_t^{s,x}) &= f_\epsilon(s,x) + \int_s^t \nabla f_\epsilon(r, X_r^{s,x}) dB_r \\ &\quad + \int_s^t \frac{\partial f_\epsilon}{\partial n}(r, X_r^{s,x}) dL_r^{s,x} + \frac{1}{2} \int_s^t \Delta f_\epsilon(r, X_r^{s,x}) dr. \end{aligned}$$

Dividing both sides by ϵ, we obtain

$$\begin{aligned} &L_t^{s,x} - \frac{1}{2\epsilon} \int_s^t 1_{D_\epsilon}(r, X_r^{s,x}) dr \\ &= \frac{1}{\epsilon} \int_s^t \nabla f_\epsilon(r, X_r^{s,x}) dB_r + O(\epsilon) + O(\epsilon) \int_s^t 1_{D_\epsilon}(r, X_r^{s,x}) dr \end{aligned} \tag{5.9}$$

By Doob's maximal inequality and Theorem 2.4(v),

$$\begin{aligned} \mathbb{E}\left[\sup_{s \leq t \leq l} \left| \frac{1}{\epsilon} \int_s^t \nabla f_\epsilon(r, X_r^{s,x}) dB_r \right|^2 \right] &\leq \frac{4}{\epsilon^2} \mathbb{E}\left[\int_s^l |\nabla f_\epsilon(r, X_r^{s,x})|^2 dr \right] \\ &\leq c_l \mathbb{E}\left[\int_s^l |1_{D_\epsilon}(r, X_r^{s,x})| dr \right] \\ &= c_l \int_s^l \left(\int_{D_\epsilon} p(s,x;r,y) dy \right) dr \\ &\leq C\epsilon \int_s^l (r-s)^{-1/2} dr = C\epsilon \sqrt{l-s}. \end{aligned} \tag{5.10}$$

This and (5.9) imply that

$$\lim_{\epsilon\downarrow 0}\mathbb{E}\left[\sup_{s\le t\le l}\left|L_t^{s,x}-\frac{1}{2\epsilon}\int_s^t 1_{D_\epsilon}(r,X_r^{s,x})dr\right|^2\right]=0,$$

that is, (5.7) holds in the L^2-sense. From (5.10) and Chebyshev's inequality we see that

$$\sum_{k=1}^{\infty}\mathbb{P}\left(\sup_{s\le t\le l}\left|k^4\int_s^t \nabla f_{1/k^4}(r,X_r^{s,x})dB_r\right|\ge\frac{1}{k}\right)\le C\sum_{k=1}^{\infty}\frac{1}{k^2}<\infty.$$

By the Borel-Cantelli lemma, with probability 1,

$$\lim_{k\to\infty}\sup_{s\le t\le l}k^4\left|\int_s^t \nabla f_{1/k^4}(r,X_r^{s,x})dB_r\right|=0.$$

This implies that, a.s.,

$$\lim_{k\to\infty}\sup_{s\le t\le l}\left|L_t^{s,x}-\frac{k^4}{2}\int_s^t 1_{D_{1/k^4}}(r,X_r^{s,x})dr\right|=0. \tag{5.11}$$

For $0<\epsilon<1$, let $k_\epsilon\ge 1$ be the integer such that $1/k^4<\epsilon\le 1/(k\epsilon-1)^4$. Since $D_{1/k^4}\subset D_\epsilon\subset D_{1/(k_\epsilon-1)^4}$, we have

$$\begin{aligned}\frac{(k_\epsilon-1)^4}{2}\int_s^t 1_{D_{1/k_\epsilon^4}}(r,X_r^{s,x})dr &\le \frac{1}{2\epsilon}\int_s^t 1_{D_\epsilon}(r,X_r^{s,x})dr\\ &\le \frac{k_\epsilon^4}{2}\int_s^t 1_{D_{1/(k_\epsilon-1)^4}}(r,X_r^{s,x})dr.\end{aligned}$$

This, together with (5.11), implies that, a.s.,

$$\lim_{k\to\infty}\sup_{s\le t\le l}\left|L_t^{s,x}-\frac{1}{2\epsilon}\int_s^t 1_{D_\epsilon}(r,X_r^{s,x})dr\right|=0.$$

Inequality (5.8) follows from (5.7) and Theorem 5.5(v). □

Lemma 5.8. *For each fixed $\alpha<\infty$ and $0<l<T$,*

$$\sup_{\substack{(s,x)\in\overline{D}\\ s<t\le l}}\mathbb{E}[\exp(\alpha L_t^{s,x})]<\infty.$$

Proof. It follows from Theorem 5.7 that there is a $\delta > 0$ such that

$$\sup_{\substack{(s,x)\in D, s<r\leq l \\ |s-r|<\delta}} \mathbb{E}[\alpha L_r^{s,x}] < 1/2,$$

and therefore, by Khasminskii's inequality [13, page 231],

$$\sup_{\substack{(s,x)\in D, s<r\leq l \\ |s-r|<\delta}} \mathbb{E}[\exp(\alpha L_r^{s,x})] < 2.$$

Let $k \geq 1$ be such that $l/k < \delta$. Then by the Markov property of X and the additivity of local time L, we have for any $0 \leq s < t \leq l$ and $(s,x) \in \overline{D}$,

$$\mathbb{E}[\exp(\alpha L_t^{s,x})] \leq \left(\sup_{\substack{(s,x)\in D, s<r<l \\ |s-r|<\delta}} \mathbb{E}[\exp(\alpha L_r^{s,x})] \right)^k \leq 2^k.$$

□

Theorem 5.9. *Firx some $t > 0$. Let $f(s,x)$ be a bounded function defined on ∂D and ϕ be a continuous function on $\overline{D(t)}$. Suppose $u(s,x) \in C^2(D) \cap C^1(\overline{D})$ is a C^2-smooth solution for*

$$\begin{cases} \frac{\partial u}{\partial s} + \frac{1}{2}\delta_x u = 0, & \text{for } (s,x) \in D, \text{ with } s \leq t, \\ \frac{\partial u}{\partial n} + f(s,x)u = 0, & \text{for } (s,x) \in \partial D, \text{ with } s < t, \\ \lim_{s\uparrow t} u(s,x) = \phi(x). \end{cases} \tag{5.12}$$

Then for $(s,x) \in \overline{D}$ with $s < t$,

$$u(s,x) = \mathbb{E}\left[\exp\left(\int_s^t f(r, X_r^{s,x}) dL_r^{s,x}\right) \phi(X_t^{s,x})\right]. \tag{5.13}$$

Conversely, if $f(s,x)$ is a bounded continuous function on ∂D, then the function $u(s,x)$ defined by (5.13) is continuous on $\overline{D}$ for $s \leq t$, it is continuously differentiable in $s \in (0,t)$, it belongs to class $C^2(D(s)) \cap C^1(\overline{D(s)})$ as a function of x, and it solves the equation (5.12).

Proof. Assume that $u(s,x)$ solves (5.12). By Itô's formula,

$$\begin{aligned}
&d\left(\exp\left(\int_s^r f(v, X_v^{s,x})dL_v^{s,x}\right)u(r, X_r^{s,x})\right)\\
&=\exp\left(\int_s^r f(v, X_v^{s,x})dL_v^{s,x}\right)\\
&\quad\times\left(u(r, X_r^{s,x})f(r, X_r^{s,x})dL_r^{s,x} + \left(\frac{\partial}{\partial r} + \frac{1}{2}\Delta_x\right)u(r, X_r^{s,x})dr + \nabla u(r, X_r^{s,x})dB_r + \frac{\partial u}{\partial n}(r, X_r^{s,x}dL_r^{s,x}\right)\\
&= \exp\left(\int_s^r f(v, X_v^{s,x})dL_v^{s,x}\right)\nabla u(r, X_r^{s,x})dB_r.
\end{aligned}$$

Hence, $\{\exp(\int_s^r f(v, X_v^{s,x})dL_v^{s,x})u(r, X^{s,x}), s \leq r \leq t\}$ is a local martingale. By Lemma 5.8, it is, in fact, a martingale since u and f are bounded. This implies

$$\begin{aligned}
u(s,x) &= \mathbb{E}\left[\exp\left(\int_s^t f(r, X_r^{S,x})dL_r^{s,x}\right)u(t, X_t^{s,x})\right]\\
&= \mathbb{E}\left[\exp\left(\int_s^t f(v, X_r^{s,x})dL_r^{s,x}\right)\phi(X_t^{s,x})\right],
\end{aligned}$$

and, hence, proves (5.13).

Now suppose that $f(s,x)$ is a bounded continuous function on ∂D and U is a function defined by (5.13). Clearly, $\lim_{s\uparrow t} u(s,x) = \phi(x)$. We have

$$\begin{aligned}
u(s,x) &= \mathbb{E}[\phi(X_t^{s,x})] + \mathbb{E}\left[\left(\exp\left(\int_s^t f(r, X_r^{s,x})dL_r^{s,x}\right) - 1\right)\phi(X_t^{s,x})\right]\\
&= \mathbb{E}[\phi(X_t^{s,x})]\\
&- \exp\left[\int_s^t f(r, X_r^{s,x})\exp\left(\int_r^t f(v, X_v^{s,x})dL_v^{s,x}\right)\phi(X_t^{s,x})dL_r^{s,x}\right]\\
&= \mathbb{E}[\phi(X_t^{s,x})] - \mathbb{E}\left[\int_s^t f(r, X_r^{s,x})u(r, X_r^{s,x})dL_r^{s,x}\right] \qquad (5.14)\\
&= \int_{D(t)} p(s,x;t,y)\phi(y)dy\\
&- \frac{1}{2}\int_s^t\left(\int_{\partial D(r)} p(s,x;r,z)f(r,z)u(r,z)\sigma_r(dz)\right).
\end{aligned}$$

From (5.14), we see that $u \in C^2(D)\cap C^1(\overline{D})$ for $s < t$. By Theorem 5.2, u satisfies $\frac{\partial u}{\partial s} + \frac{1}{2}\Delta_x u = 0$ in D. To show that u satisfies the boundary conditions in (5.12), we adapt an approach from Hsu [27], Proposition 3.2. Applying Itô's formula, we have for $s < r < t$,

$$u(r, X_r^{s,x}) - u(s,x) = \int_s^l \nabla u(v, X_v^{s,x}) dB_v + \int_s^l \frac{\partial u}{\partial n}(v, X_v^{s,x}) dL_v^{s,x}. \tag{5.15}$$

On the other hand,

$$\begin{aligned} u(r, X_r^{s,x}) &= \mathbb{E}\left[\exp\left(\int_r^t f(v, X_v^{r,X_r^{s,x}}) dL_v^{r,X_r^{s,x}}\right) \phi(X_t^{r,X_r^{s,x}}) \Big| \mathcal{F}_{s,r}\right] \\ &= \exp\left(-\int_s^r f(v, X_v^{S,x}) dL_v^{s,x}\right) \\ &\quad \times \mathbb{E}\left[\exp\left(\int_s^t f(v, X_v^{s,x}) dL_v^{s,x}\right) \phi(X_t^{s,x}) \Big| \mathcal{F}_{s,r}\right], \end{aligned}$$

where $\mathcal{F}_{s,r}$ is the $\sigma - field$ generated by $X_v^{s,x}$ for $v \in [s, r]$. Let

$$M_r = \mathbb{E}\left[\exp\left(\int_s^t f(v, X_v^{S,x}) dL_v^{s,x}\right) \phi(X_t^{s,x}) \Big| \mathcal{F}_{s,r}\right].$$

In view of Lemma 5.8, M_r is a martingale so by Itô's formula,

$$\begin{aligned} & u(r, X_r^{s,x}) - u(s,x) \\ &\quad = \int_s^r \exp\left(-\int_s^\theta f(v, X_v^{s,x}) dL_v^{s,x}\right) dM_\theta \\ &\quad - \int_s^r f(\theta, X_\theta^{s,x}) \exp\left(-\int_s^\theta f(v, X_v^{s,x}) dL_v^{s,x}\right) M_\theta dL_\theta^{s,x} \\ &\quad = \int_s^r \exp\left(-\int_s^\theta f(v, X_v^{s,x}) dL_v^{s,x}\right) dM_\theta \\ &\quad - \int_s^r f(\theta, X_\theta^{s,x}) u(\theta, X_\theta^{s,x}) dL_\theta^{s,x}. \end{aligned} \tag{5.16}$$

From (5.15) and (5.16), we see that the bounded variation process

$$\int_s^r \left(\frac{\partial u}{\partial n}(v, X_v^{s,x}) + f(v, X_v^{s,x}) u(v, X_v^{s,x})\right) dL_v^{s,x}$$

is a continuous martingale and, therefore, it must be identically zero. Were $\frac{\partial u}{\partial n} \neq -fu$ on ∂D, say $\frac{\partial u}{\partial n}(s,x) + f(s,x)u(s,x) > 0$ for some $(s,x) \in \partial D$, there would be a neighborhood U of (s,x) such that $\frac{\partial u}{\partial n}(s,x) + f(s,x)u(s,x) \geq \epsilon_0 > 0$ on $U \cap \partial D$. Let $\tau = \int\{r \geq s :$

$(r, X_r^{s,x}) \in \partial D \setminus U\}$. Clearly, $\tau > 0$ almost surely and, therefore, there is a $t_0 > 0$ such that $P^{s,x}(\tau > t_0) > 0$. Then on $\{\tau > t_0\}$,

$$0 = \int_s^{t_0} \left(\frac{\partial u}{\partial n}(v, X_v^{s,x}) + f(v, X_v^{s,x}) u(v, X_v^{s,x}) \right) dL_v^{s,x} \geq \epsilon_0 dL_{t_0}^{s,x}.$$

This is impossible as (s, x) is a regular point of D for the space-time Brownian motion because $L_{t_0}^{s,x} > 0$, $\mathbb{P}s, x$-almost surely. This is true for one dimensional Brownian motion reflected from the origin, and gor $g \in C^2$ holds for reflected Brownian motion with drift (the solution Z of $dZ = dB + gdt + dL_Z$ where Z is the local time at zero of Z). Therefore $\frac{\partial u}{\partial n} = fu$ on ∂D. □

Remark 5.10. Uniqueness of C^2-smooth solutions to (5.12) is a by-product of the probabilistic representation (5.13).

The equation in (5.1) is the "backward partial differential equation" for the transition density function $p(s, xlt, y)$ of X, in variables s and x. Our next result is concerned with $p(s, x; t, y)$ as a function of t and y. If we view $L = \frac{\partial}{\partial s} + \frac{1}{2}\Delta_x$ as an operator in D together with its zero Neumann boundary condition given in (5.1), and we let L^* be its formal adjoint operator in $L^2(D)$, then the function $(t, y) \to p(s, x; t, y)$ is in the domain $\mathcal{D}(L^*)$ of L^* and it satisfies the differential equation $L^* p = 0$ [41, pages 2-3]. The following is a result of the divergence formula in $\mathbb{R}^{n+1}$. Recall that $\vec{\gamma} = (-g'(t)/(1 + (g'(t))^2), 1/(1 + (g'(t))^2))$ is the unit inward normal vector on the boundary of D.

Theorem 5.11. *The function $p(s, x; t, y)$ satisfies the following forward differential equation in (t, y) for each fixed $(s, x) \in \overline{D}$:*

$$\begin{aligned} &\frac{\partial p}{\partial t} - \frac{1}{2}\Delta_y p = 0, \text{ for } (t, y) \in D \text{ with } s < t, \\ &\frac{\partial p}{\partial n} - \frac{2\gamma_1}{\vec{\gamma} \cdot n} p = 0, \text{ for } (t, y) \in \partial D \text{ with } s < t, \\ &\lim_{t \downarrow s} p(s, x; t, y) dy = \delta_{\{x\}}(dy), \text{ for } (s, x) \in D. \end{aligned} \tag{5.17}$$

Proof. A function ψ belongs to $\mathcal{D}(L^*) \subset L^2(D)$ if and only if there is a $\phi \in L^2(D)$ such that

$$\int_D \psi L u \; dt \; dx = \int_D \phi u \; dt \; dx$$

for any $u \in \mathcal{D}(L)$, and in this case $L^*\psi = \phi$. In view of the remarks about the function $(t, y) \to p(s, x; t, y)$ preceding the theorem, it will suffice to show that $\phi \in \mathcal{D}(L^*)$ if and only if $L^*\psi = (-\frac{\partial}{\partial t} + \frac{1}{2}\Delta)(\psi)$ and $\frac{\partial p}{\partial n} - \frac{2\gamma_1}{\vec{\gamma}n}\psi = 0$.

For any test function $\psi \in C_c^\infty((0, T) \times \mathbb{R}^n)$ and $u \in \mathcal{D}(L)$,

$$\begin{aligned}
&\int_D \psi \left(\frac{\partial}{\partial t} + \frac{1}{2}\Delta\right) u \; dt \; dx \\
&\quad= \int_D \left(\frac{\partial(u\psi)}{\partial t} - u\frac{\partial\psi}{\partial t}\right) \; dt \; dx + \frac{1}{2}\int_0^T \int_{D(t)} \psi\Delta u \; dx \; dt \\
&\quad= \int_D \left(\frac{\partial(u\psi)}{\partial t} - u\frac{\partial\psi}{\partial t}\right) \; dt \; dx + \frac{1}{2}\int_0^T \int_{D(t)} u\Delta\psi \; dx \; dt \\
&\quad+ \frac{1}{2}\int_0^T \int_{\partial D(t)} u\frac{\partial\psi}{\partial n}\sigma_t(dx)dt \\
&\quad= \int_D \left(\frac{\partial(u\psi)}{\partial t} - u\frac{\partial\psi}{\partial t}\right) \; dt \; dx + \frac{1}{2}\int_0^T \int_{D(t)} u\Delta\psi \; dx \; dt \\
&\quad+ \frac{1}{2}\int_0^T \int_{D(t)} \operatorname{div}_{\mathbb{R}^n}(-u\nabla\psi) \; dx \; dt \\
&\quad= \int_D u\left(-\frac{\partial}{\partial t} + \frac{1}{2}\Delta\right)\psi \; dt \; dx + \int_D \operatorname{div}_{\mathbb{R}^{n+1}}\left(u\psi, -\frac{1}{2}u\nabla_x\psi\right) dt \; dx \\
&\quad= \int_D u\left(-\frac{\partial}{\partial t} + \frac{1}{2}\Delta\right)\psi \; dt \; dx + \int_{\partial D} u\vec{\gamma}\cdot\left(-\psi, \frac{1}{2}\nabla_x\psi\right) d\sigma.
\end{aligned}$$

Therefore, $\phi \in D(L^*)$ if and only if $\vec{\gamma}\cdot(-\psi, \frac{1}{2}\nabla_x\psi) = 0$ on ∂D and $L^* = (-\frac{\partial}{\partial t} + \frac{1}{2}\Delta)\psi$. This is equivalent to $L^*\psi = (-\frac{\partial}{\partial t} + \frac{1}{2}\Delta)(\psi)$ and $\frac{\partial p}{\partial n} - \frac{2\gamma_1}{\vec{\gamma}n}\psi = 0$. □

Chapter 2

STRONG APPROXIMATION AND DISTRIBUTION OF THE MAXIMUM

1 Introduction

In the previous chapter we constructed a multi-particle whose original motivation and study of Brownian motion with inert-drift began with F. Knight in his 2001 paper [34]. In his paper, Knight proves the existence of two particles X, Y, with $X \geq Y$, where X is the Brownian particle and Y is the massive barrier with inert-drift.

In this chapter we prove several results in the same scenario of one Brownian particle X interacting with the inert particle Y. In Section 2 we calculate the distribution of the maximum of Y for an arbitrary initial velocity in terms of its Laplace transform; we provide an algorithm for *strong approximation* of the process (X, Y) in Section 3; in Section 4 we show the inert-particle cannot trap the Brownian particle at a reflecting wall. As mentioned earlier, the following theorem of Knights is the first to establish existence of the process (X, Y, V).

Theorem 1.1 ([34] Knight)**.** *Let $B(t)$ be a standard Brownian motion. For a fixed $K \in \mathbb{R}$, there exists process (X, Y, V) adapted to the Brownian filtration where the following holds a.s. for $t \geq 0$:*

(i) *$V(0) - v, V(t)$ is continuous, non-decreasing,*

(ii) *$X(t)$ is Brownian reflected below $Y(t) := \int_0^t V(s)ds$, for $t \geq 0$,*

(iii) *$V(t) = v + KL(t)$, where L is the occupation local time of X on Y.*

Here the occupation local time of X on Y is

$$L(t, \omega) := \lim_{\epsilon \to 0^+} (2\epsilon)^{-1} \int_0^t 1_{(0,\epsilon]}(X(s) - Y(s))ds.$$

By recentering Y as the new origin, we have an equivalent formulation of the above theorem but in terms of one particle and its drift. Namely, there exists a process X adapted to the Brownian filtration such that almost surely

(i) $X(t) = B(t) + L(t) + \int_0^t KL(s)ds,$

(ii) $X(t) \geq 0,$

(iii) $L(t) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^t 1_{[0,\epsilon)} X(s)ds.$

It is this process X which we term *Brownian motion with inert-drift.* David White proves the existence of a more general form of the above by proving Theorem 1.5 below, which can be viewed as a variant of the Skorokhod lemma. We recall the Skorokhod lemma and a few remarks from Lemma 1.2 in the first chapter.

Lemma 1.2 (Skorohod, see [30]). *Let $f \in C([0,T],\mathbb{R})$ with $f(0) \geq 0$. There is a unique, continuous, nondecreasing function $m(t)$ such that*

$$x_f(t) = f(t) + m_f(t) \geq 0,$$

$$m_f(0) = 0,\ m_f(t) \text{ is flat off } \{s : x_f(s) = 0\},$$

and is given by

$$m_f(t) = \sup_{0<s<t} [-f(s)] \vee 0.$$

Remark 1.3. Flatness off $\{z : x_f(z) = 0\}$ for m_f means $\int_0^t 1(x_f(s) > 0)dm_f(s) = 0$. The classical Lévy's theorem says when f is replaced by a Brownian motion, the corresponding process x_f is distributed as $|B|$.

Remark 1.4. The solution of the Skorohod equation has a time shift property: For any $0 \leq s \leq t \leq T$,

$$x_f(t) = x_g(t-s),$$

where $g(t) = x_f(s) + f(t) - f(s)$.

In [40], Skorohod introduces the above solution and applies it to the construction of reflected Brownian motion in the domain $[0, \infty)$. By replacing the $f(t)$ with a Brownian sample path $B(t)$, $x(t)$ becomes Brownian motion reflected at 0, and $L(t)$ is the corresponding local time. In a similar vein, applying Theorem 1.5 below to each Brownian path yields a more general construction of Brownian motion with inert-drift, as taking $\mu(l) = Kl$ for some positive K will yield Knights original construction.

Theorem 1.5. *[45, Theorem 3] Let $f(t)$ be a continuous function with $f(0) \geq 0$, and let $\mu(l) \geq 0$ be a continuous monotone function with*

$$\lambda(l) = \sup_{a<b<l} \frac{|\mu(a) - \mu(b)|}{b - a} < \infty$$

for each l. There is a unique continuous function $L(t)$ which satisfies

(i) $x(t) = f(t) + L(t) + \int_0^t \mu \circ L(s) ds \geq 0$,

(ii) $L(0) = 0, L(t)$ *is nondecreasing,*

(iii) $L(t)$ *is flat off* $\{t : x(t) = 0\}$, *that is,* $\int_0^\infty 1_{\{x(s)>0\}} dL(s) = 0$.

Remark 1.6. The Skorohod lemma implies that $L(t)$ is the running minimum of $B(t)$ below $\int_0^t \mu \circ L(s)ds$. See Remark 2.7. See the proof of Theorem 1.5.

A proof of Theorem 1.5 will be reproduced here as we will need the details of it later in this chapter. The contruction is very similar to that done in chapter one in Proposition 3.2.

Proof of Theorem 1.5. We will first show uniqueness. Assume that we have two functions $L(t), \tilde{L}(t)$ which both satisfy the assumptions of theorem 3. Let $Q = \inf\{t > 0 : L(t) \neq \tilde{L}(t)\}$ be the first time these two functions differ. Suppose that $Q < \infty$. Define $M(t) := \lambda(L(t) \wedge \tilde{L}(t))(t - Q)$; this is an increasing function as both factors are increasing. Notice $M(Q) = 0$, so let us choose an $R > Q$ with $M(R) < 1$. By continuity of L and $\tilde{L}$, there is a T between Q and R, and a positive δ, such that $|L(t) - \tilde{L}(t)| < \delta$ for $t < T$, whereas $|L(T) - \tilde{L}(T)| = \delta$.

We show that the case $L(T) - \tilde{L}(T) = \delta$ gives a contradiction. Reversing L and $\tilde{L}$ will then prove uniqueness. Since $L(T) - \tilde{L}(T) = \delta$ and $L(t) - \tilde{L}(t) < \delta$ for $t < T$. We have

$$\begin{aligned} x(T) - \tilde{x}(T) &= (x(T) - x(Q)) - (\tilde{x}(T) - \tilde{x}(Q)) \\ &= (L(t) - L(Q) + \int\limits_Q^T \mu \circ L(s)ds) \\ &\quad - (\tilde{L}(T) - \tilde{L}(Q) + \int\limits_Q^T \mu \circ \tilde{L}(s)ds) \\ &= (L(T) - \tilde{L}(T)) + \int\limits_Q^T (\mu \circ L(s) - \mu \circ \tilde{L}(s))ds, \end{aligned}$$

whence

$$\begin{aligned} |\int\limits_Q^T (\mu \circ L(s) - \mu \circ \tilde{L}(s))ds| &\leq (T - Q)\lambda(L(T) \wedge \tilde{L}(T)) \sup_{Q \leq t \leq T} (L(t) - \tilde{L}(t)) \\ &= M(T)(L(T) - \tilde{L}(T)). \end{aligned}$$

Combining these two equations, we have

$$x(T) - \tilde{x}(T) \geq (1 - M(T))(L(T) - \tilde{L}(T)) > 0.$$

Let $S = \sup\{t < T : x(t) = \tilde{x}(t)\}$. Then for $S < t \leq T$, $x(t) > \tilde{x}(t) \geq 0$, and by the hypothesis that L ($\tilde{L}$) is flat away from the zero set of x ($\tilde{x}$) we have $L(t) - L(S)$; $L(T) = L(S)$ as well. Then for such values of t, $L(t) - \tilde{L}(t) = L(T) - \tilde{L}(t)$. As $\tilde{L}(t)$ is nondecreasing, $L(T) - \tilde{L}(t)$ is nonincreasing and we have $L(S) - \tilde{L}(S) \geq L(T) - \tilde{L}(T) = \delta$. This contradicts how we chose T. Hence $L(t) = \tilde{L}(t)$ for all t.

Now we construct the function $L(t)$ for a given μ and f. We do this by finding, for every $\epsilon > 0$, an L^ϵ and I^ϵ such that $L^\epsilon(t) = L(f + I^\epsilon)(t)$ and $\frac{d}{dt}I^\epsilon(t) = \mu(n\epsilon)$ when $n\epsilon \leq L^\epsilon(t) \leq (n+1)\epsilon$. This is constructed recursively by initializing $I_0^\epsilon = 0$, $L_0^\epsilon(t) = Lf(t)$, and

$T_0^\epsilon = \inf\{t > 0 : L_0^\epsilon(t) = \epsilon\}$. The recursion is then defined by setting

$$I_{n+1}^\epsilon(t) = \begin{cases} I_n^\epsilon(t) & \text{for } t \leq T_n^\epsilon, \\ I_n^\epsilon(T_n^\epsilon) + \mu(n\epsilon)(t - T_n^\epsilon) & \text{for } t \geq T_n^\epsilon, \end{cases} \tag{1.1}$$

$$L_{n+1}^\epsilon(t) = L(f + I_{n+1}^\epsilon),\ T_{n+1}^\epsilon = \inf\{t > T_n : L_{n+1}(t) = (n+1)\epsilon\}. \tag{1.2}$$

In this way I_n^ϵ is piecewise linear and increasing, with its increases in slope determined by μ. If $m \geq n$ we see that $I_m^\epsilon(t) = I_n^\epsilon(t)$ whenever $t \leq T_n^\epsilon$. Furthermore, for a fixed $t \geq 0$, we have $I_m^\epsilon(t) \geq I_n^\epsilon(t)$ as I_m has more opportunities for an increase of slope. Whence $L_m^\epsilon(t) \leq L_n^\epsilon(t)$ and consequently $L^\epsilon(t) := \lim_{m\to\infty} L_m^\epsilon(t)$ is well defined, as is $I^\epsilon(t) := \lim_{n\to\infty} I_n^\epsilon(t)$. Observe also that

$$\begin{aligned} L^\epsilon(t) - L^\epsilon(s) &= L(f + I^\epsilon)(t) - L(f + I^\epsilon)(s) \\ &\leq \max_{s \leq r \leq t}\{(f + I^\epsilon)(s) - (f + I^\epsilon)(r)\} \\ &\leq \max_{s \leq r \leq t}\{f(s) - f(r)\} \end{aligned}$$

is a uniform upper bound, showing the family of functions $\{L^\epsilon(t) : \epsilon > 0\}$ is equicontinuous. The family is pointwise bounded as well, since nonnegativity of I^ϵ implies $L^\epsilon(t) = L(f + I^\epsilon)(t) \leq Lf(t) < \infty$. The requirements of Arzela-Ascoli are now verified, so that for any sequence $\epsilon_n \to 0$ we have a subsequence ϵ_n' such that $L^{\epsilon_n'}$ converges uniformly on compact sets to a function to a function we call $L(t)$. Once this $L(t)$ is shown to satisfy the properties in the theorem statement the existence portion of this proof will be complete. □

Whites proof is more general than the case we consider because we take $\mu(l) = Kl$ for some constant $K > 0$ as in the original setting of Knight. It would be interesting to extend some these results in the more general setting.

2 *Global Maximum of the Inert-Particle*

In this section we consider the model where the Brownian particle X dominates the inert-particle Y. We assume the inert particle begins at the origin with velocity $V(0)$ independent of the initial position of the Brownian particle $X(0)$. In other words, we assume

$$Y(0) = 0,\ Y'(0) = V(0) \perp X(0). \tag{2.1}$$

We know the velocity $V(t)$ will change sign from positive to negative, almost surely on $V(0) > 0$. To see this, note that the velocity $V(t)$ is proportional to the maximum distance $B(t)$ has traveled below $Y(t)$. That is, $V(t) = v - Km(t)$, where

$$m(t) = \max_{s\in[0,t]}[Y(s) - (X(0) + B(t))] \vee 0,$$

which follows from Remark 1.6 and Theorem 1.5. The construction of the process is done in Chapter 1, Theorem 2.12 for $n = 1$. If we suppose that $Y(t)$ is ever increasing, then $V(t)$ is ever positive. But a Brownian path has a liminf of negative infinity, almost surely, so that $V(t)$ must become negative because $m(t)$ would otherwise become unbounded, leading to a contradiction. Consequently $Y(t)$ reverses direction almost surely on the event $V(t) > 0$.

We condition $X(0) = x \geq 0$ and $V(0) = \alpha > 0$, and define

$$T := \inf\{t > 0 : V(t) < 0\}$$

as the time the inert particle reverse direction. Clearly the inert particle reaches its maximum value at this time T, and so

$$S := Y(T)$$

is the random variable $\max_{t>0} Y(t)$. I.e., S is the global maximum of the inert-particle given our condition $X(0) = x, V(0) = v$. See Figure 2.1 for a sample path showing X, Y, S and T.

For $\mu, a > 0$, define

$$T_a^{(\mu)} = \inf\{t > 0 : B(t) + \mu t = a\}$$

as the first time that Brownian motion with drift μ reaches level a. It is well known that the density of $T_a^{(\mu)}$ is given as

$$\mathbb{P}(T_a^{(\mu)} \in dt) = \frac{a}{\sqrt{2\pi t^3}} e^{\frac{-(a-\mu t)^2}{2t}} dt,$$

see, for instance, [31]. We have the following moment calculations.

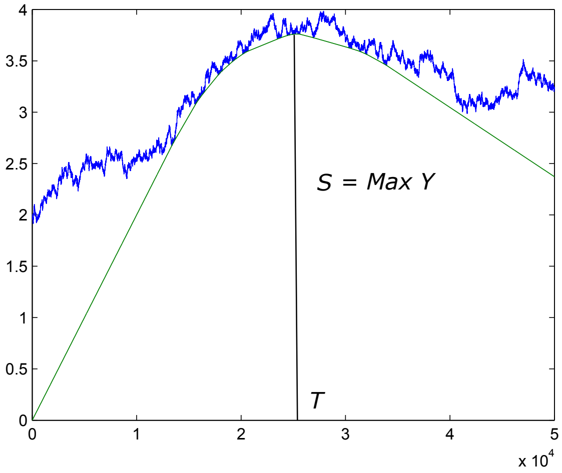

Figure 2.1: A sample path representation showing the Brownian particle X, the inert particle Y and its maximum S occurring at time T.

Lemma 2.1. *For $a, \mu > 0$, and $T_a^{(\mu)}$ as above, we have*

(i) $\mathbb{E}(T_a^{(\mu)}) = \dfrac{a}{\mu}$.

(ii) $\mathbb{E}(T_a^{(\mu)})^2 = \dfrac{a}{\mu^3} + \dfrac{a^2}{\mu^2}$, *and*

(iii) $\mathbb{E}(T_a^{(\mu)})^{\frac{3}{2}} = \sqrt{\dfrac{2}{\pi}} \dfrac{a^2 e^{a\mu}}{\mu} \mathrm{BesselK}[1, a\mu]$.

Where BesselK$[n, z]$ *is the well studied modified Bessel function.*

Let $\alpha > 0$ and consider the standard inert-particle triple (X, Y, V) with $Y(t) \leq X(t)$ for all time t, as mentioned in the beginning of this section. We initiate the process with $X(0) \geq 0 = Y(0)$ and $V(0) = \alpha$. Let $n \in \mathbb{N}$. We use the algorithmic construction of Y contained in the proof of Theorem 1.5 by considering the piecewise linear process when

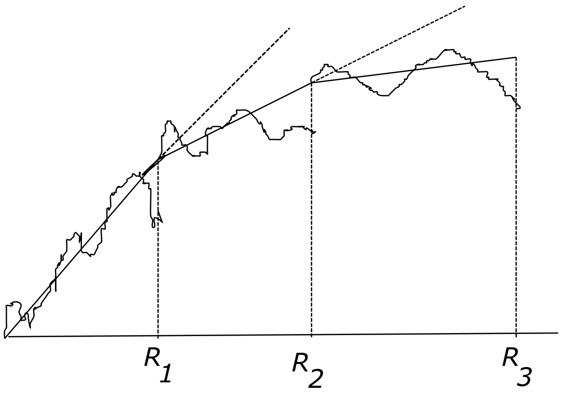

Figure 2.2: An example of the stopping times R_j, which are the times the piecewise linear process Y^n changes slope. The time changes occur when the recentered Brownian path reaches α/n below Y^n.

ϵ contained in the theorem's proof is set to α/n. For this fixed n, inductively define the following stopping times by setting $R_0^n = 0$, and

$$R_j^n := \inf\left\{ t > R_{i-1}^n : \alpha\left(\frac{n-j+1}{n}\right) t - [B(t) - B(R_{j-1}^n)] = \frac{\alpha}{nK} \right\}, \tag{2.2}$$

$$Y^n(t) = Y^n(R_{i-1}^n) + \alpha\left(\frac{n-j+1}{n}\right) t, \text{ for } t \in [R_{j-1}^n, R_j^n], \tag{2.3}$$

$$S_n := \max_{t \geq 0} Y^n(t). \tag{2.4}$$

Recall that K in in our definition of $\mu(l) = Kl$ is used in (2.2). In other words, $Y^n(t)$ is a line of slope $\alpha\frac{n-j+1}{n}$ for $t \in [R_{j-1}^n, R_j^n]$, while R_j^n is the next time the recentered Brownian path reaches a distance of $\alpha/(nK)$ below this line. In that sense See Figure 2.2 for an example of the stopping times R_j shown with the process Y^n and the recentered Brownian paths.

Proposition 2.2. *In the above setting, with* $X(0) = 0 = Y(0)$,

(i) $S_n = \displaystyle\sum_{j=1}^{n} \alpha\left(\frac{n-j+1}{n}\right)(R_i^n - R_{i-1}^n)$,

(ii) $S_n \xrightarrow{a.s.} S$,

(iii) $\mathbb{E}\, S_n = \alpha$ *for all* n.

Proof. (i) Observe that $Y^n(t)$ has positive slope until R_n^n. Because the slope of $Y^n(t)$ is $\alpha\dfrac{n-j+1}{n}$ for $t \in [R_{j-1}^n, R_j^n]$, it necessarily follows that

$$\begin{aligned} S_n &:= \max_{t \geq 0} Y^n(t) = Y^n(R_n^n) \\ &= \sum_{j=1}^{n} \alpha \frac{n-j+1}{n}(R_j^n - R_{i-1}^n). \end{aligned}$$

(ii) This follows from the proof of Theorem 1.5, since it is known that $Y^n(t) \to Y(t)$ uniformly on compact time sets, almost surely.

(iii) It follows from definition of the R_j^n, (2.2), that the gaps $R_1^n, R_2^n - R_1^n, \ldots, R_j^n - R_{j-1}^n$ are independent by the strong Markov property of Brownian motion and that

$$R_j^n - R_{j-1}^n \overset{d}{=} T_{a_n}^{(\mu_j)},$$

where

$$a_n = \frac{\alpha}{nK}, \text{ and } \mu_j = \alpha(n-j+1)/n.$$

Using Lemma 2.1 for the moment computation, by (i) in Proposition 2.2 proved above gives

$$\begin{aligned}
\mathbb{E}S_n &= \sum_{j=1}^{n} \alpha \left(\frac{n-j+1}{n}\right) \mathbb{E}(R_j^n - R_{j-1}^n) \\
&= \sum_{j=1}^{n} \alpha \left(\frac{n-j+1}{n}\right) \mathbb{E}(T_{a_n}^{(\mu_j)}) \\
&= \sum_{j=1}^{n} \alpha \frac{n-j+1}{nK} \cdot \frac{1}{n-j+1} \\
&= \frac{\alpha}{K} \sum_{j=1}^{n} \frac{1}{n} \\
&= \frac{\alpha}{K}.
\end{aligned}$$

□

Remark 2.3. In light of Proposition 2.2, we have $\mathbb{E}(S) \geq \alpha$ by Fatou's lemma. In order to prove $\mathbb{E}(S) = \alpha/K$ we will need to control the tails of S_n for which it suffices to show $\sup_n \mathbb{E}|S_n|^p < C < \infty$.

The following is a useful theorem, which we quote as the lemma below, and is found in the 1969 paper of von Bahr and Esseen [44]. Note that the hypotheses assume the random variables are independent, not necessarily i.i.d.

Lemma 2.4 ([44])**.** *Let X_i be independent with mean 0 and choose $1 \leq p \leq 2$. There is a positive constant C such that for any n,*

$$\mathbb{E}\Big|\sum_{j=1}^{n} X_j\Big|^p \leq C \sum_{j=1}^{n} \mathbb{E}|X_j|^p.$$

Proposition 2.5. *For each $\alpha > 0$ there is a constant $C < \infty$ not depending on α such that for all n,*

$$\mathbb{E}|S_n - \alpha/K|^{\frac{3}{2}} \leq (a^{5/2} + a^{3/2})(C/a^2 + 1),$$

where $a = \alpha/K$.

Proof. We apply Lemma 2.4 to the independent, mean zero, random variables

$$\mu_j \left(R_j^n - R_{j-1}^n\right) - a_n,$$

where $\mu_j = \alpha(n-j+1)/n$ and $a_n = \alpha/(nK)$. As in the proof of Proposition 2.2, we know

$$\mu_j \left(R_j^n - R_{j-1}^n\right) - a_n \overset{d}{=} \mu_j T_{a_n}^{(\mu_j)} - a_n,$$

so these are indeed mean zero by Lemma 2.1. That they are independent is mentioned in the proof of Proposition 2.2 as well, it follows from the strong Markov property of Brownian motion. In the following calculation we take $K = 1$. Replace α with α/K. Using 2.1 in our computation for the 3/2 moment, by Lemma 2.4 there is a positive C such that

$$\begin{aligned}
\mathbb{E}|S_n - \alpha/K|^{\frac{3}{2}} &\leq C\sum_{j=1}^{n} \mathbb{E}|\mu_j T_{a_n}^{(\mu_j)} - a_n|^{\frac{3}{2}} \\
&\leq C\sum_{j=1}^{n} \mathbb{E}\max(\mu_j T_{a_n}^{(\mu_j)}, a_n)^{3/2} \\
&\leq C\sum_{j=1}^{n} \left(\mathbb{E}|\mu_j T_{a_n}^{(\mu_j)}|^{\frac{3}{2}} + \frac{\alpha^{3/2}}{n^{3/2}}\right), \\
&= C\sum_{j=1}^{n} \left(\mu_j^{3/2} \frac{\alpha^2/n^2}{\mu_j} \text{BesselK}\left[1, \frac{\alpha^2(n-j+1)}{n^2}\right]\right) + \alpha^{3/2} C \frac{1}{\sqrt{n}} \\
&\leq (\alpha^{5/2} + \alpha^{3/2}) C \left(\sum_{j=1}^{n} \sqrt{\frac{n-j+1}{n}} \frac{1}{n^2} \text{BesselK}\left[1, \frac{\alpha^2(n-j+1)}{n^2}\right] + \frac{1}{\sqrt{n}}\right) \\
&\leq (\alpha^{5/2} + \alpha^{3/2}) C \left(\sum_{j=1}^{n} \sqrt{\frac{n-j+1}{n}} \frac{1}{n^2} \frac{n^2}{\alpha^2(n-j+1)} + C_\alpha^2\right) \\
&= (\alpha^{5/2} + \alpha^{3/2}) C \left(\frac{1}{\sqrt{n}} \sum_{j=1}^{n} \frac{1}{\sqrt{j}} + C_\alpha^2\right) \\
&\leq (\alpha^{5/2} + \alpha^{3/2})(C/\alpha^2 + 1) \\
&< \infty
\end{aligned}$$

□

Corollary 2.6. *The collection* $\{S_n\}_{n=1}^{\infty}$ *is uniformly integrable.*

Proof. This follows by the triangle inequality, because for each n we have

$$\mathbb{E}|S_n|^{\frac{3}{2}} \leq \left((\mathbb{E}|S_n - \alpha/K|^{\frac{3}{2}})^{\frac{2}{3}} + \alpha/K\right)^{\frac{3}{2}} < C_{\alpha,K} < \infty.$$

$\square$

Corollary 2.7. *Recall S is the maximum of the inert particle with initial slope α. When and $X(0) = Y(0) = 0$, we have*

$$\mathbb{E}S = \alpha/K.$$

Proof. Combining the previous corollary with (2.2), we see that $S_n \to S$ in L_1. Since $\mathbb{E}(S_n) = \alpha$ for each n, $\mathbb{E}(S) = \alpha$ as well. $\square$

Using a similar analysis we can characterize the distribution of S by calculating its Laplace transform. The Laplace transform of $T_a^{(\mu)}$ is well known, for example, in [31].

Lemma 2.8. *For $\lambda, a, \mu > 0$, we have*

$$\mathbb{E}(e^{-\lambda T_a^{(\mu)}}) = e^{a\mu - a\sqrt{\mu^2+2\lambda}}$$

Theorem 2.9. *Let S the maximum of the inert particle with initial velocity $\alpha > 0$ in the setting with $X(0) = 0 = Y(0)$. The Laplace transform of S is given by*

$$\phi(\lambda) := \mathbb{E}(e^{-\lambda S}) = \exp\left[\frac{\alpha^2}{2K} - \frac{\alpha}{K}\int_0^1 \sqrt{\alpha^2x^2 + 2\lambda\alpha x}dx\right],$$

where $\lambda > 0$.

Proof. For each fixed n we compute the Laplace transform, $\phi_n(\lambda)$, of S_n. Since $S_n = \sum_{j=1}^n \alpha(\frac{n-j+1}{n})(R_j^n - R_{j-1}^n)$ by (2.2), we will find the Laplace transform of each summand and take their product as our result. For an individual j, we have

$$\mathbb{E}(\exp(-\lambda\alpha\frac{n-j+1}{n}(R_j^n - R_{j-1}^n))) = \mathbb{E}(\exp(-\lambda\mu_j T_{\alpha/(nK)}^{\mu_j})),$$

where $\mu_j = \alpha(\frac{n-j+1}{n})$. By a straightforward substitution into the previous lemma, this becomes

$$\exp\Big[\alpha^2\Big(\frac{n-j+1}{n}\Big) - \frac{\alpha}{n}\sqrt{\alpha^2\Big(\frac{n-j+1}{n}\Big)^2 + 2\lambda\alpha\Big(\frac{n-j+1}{n}\Big)}\Big].$$

Thus,

$$\begin{aligned}
\phi_n(\lambda) &= \prod_{j=1}^{n} \exp\left[\frac{\alpha^2}{K}\Big(\frac{n-j+1}{n}\Big) - \frac{\alpha}{Kn}\sqrt{\alpha^2\Big(\frac{n-j+1}{n}\Big)^2 + 2\lambda\alpha\Big(\frac{n-j+1}{n}\Big)}\right] \\
&= \exp\left[\sum_{j=1}^{n} \frac{\alpha^2}{Kn}\Big(\frac{n-j+1}{n}\Big) - \frac{\alpha}{Kn}\sqrt{\alpha^2\Big(\frac{n-j+1}{n}\Big)^2 + 2\lambda\alpha\Big(\frac{n-j+1}{n}\Big)}\right] \\
&= \exp\left[\frac{\alpha^2}{2K}\left(\frac{n+1}{n}\right) - \sum_{j=1}^{n} \frac{\alpha}{Kn}\sqrt{\alpha^2\Big(\frac{n-j+1}{n}\Big)^2 + 2\lambda\alpha\Big(\frac{n-j+1}{n}\Big)}\right]
\end{aligned}$$

where the last equality is given from the formula $1+2+\cdots+n = n(n+1)/2$. Observe that the summation in the last term above is a Riemann sum of the function $\alpha\sqrt{\alpha^2x^2 + 2\lambda\alpha x}$ over the interval $[0,1]$. Thus

$$\lim_{n\to\infty} \sum_{j=1}^{n} \frac{\alpha}{Kn}\sqrt{\alpha^2\Big(\frac{n-j+1}{n}\Big)^2 + 2\lambda\alpha\Big(\frac{n-j+1}{n}\Big)} = \frac{\alpha}{K}\int_0^1 \sqrt{\alpha^2x^2 + 2\lambda\alpha x}\,dx,$$

and consequently we have

$$\lim_{n\to\infty} \phi_n(\lambda) = \exp\left[\frac{\alpha^2}{2K} - \frac{\alpha}{K}\int_0^1 \sqrt{\alpha^2x^2 + 2\lambda\alpha x}\,dx\right] =: \phi(\lambda)$$

so $\phi(\lambda)$ is the Laplace transform of S, as we know $S_n \Rightarrow S$. $\square$

Remark 2.10. This gives us another way to compute the expectation of S. Namely

$$\begin{aligned}
\mathbb{E}(S) &= -\lim_{\lambda\to 0} \phi'(\lambda) \\
&= -\lim_{\lambda\to 0} \phi(\lambda)\Big(-\frac{\alpha}{K}\int_0^1 \frac{\alpha x}{\sqrt{\alpha^2x^2 + 2\lambda\alpha x}}dx\Big) \\
&= \frac{\alpha}{K}
\end{aligned}$$

2.1 Adding a Reflecting Wall

In this subsection we add a reflecting wall at the origin so that Y will reflect from this wall with its angle of incidence. In other words, we have $0 \leq Y(t) \leq X(t)$ for all t, and the inert particle Y reflects at $y = 0$.

Theorem 2.11. *For every $K > 0, v \in \mathbb{R}, x \geq y$, and any filtered probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$ supporting a Brownian motion $B(t) \in \mathcal{F}_t$, there exist stopping times T_n and a triple of continuous $\mathcal{F}_t$ adapted processes (X, Y, V) defined on the interval $[0, T = \lim_{n\to\infty} T_n]$ such that*

$$X(t) = B(t) + L(t) + x \geq Y(t), \text{ for all } t \in [0,T], \text{ almost surely,} \tag{2.5}$$

$$Y(t) = y + \int_0^t V(s)ds, \text{ for all } t \in [0,T], \text{ almost surely,} \tag{2.6}$$

$$V(t) \text{ is RCLL such that } \lim_{t\to T_n^+} V(t) = -\lim_{t\to T_n^-} V(t) \text{ for all } n \text{ and } V(0) = v, \tag{2.7}$$

$$T_n \text{ are the consecutive times } Y_n \text{ hits zero.} \tag{2.8}$$

Proof. Clearly the process (X, Y, V) in the theorem statement is the same as the standard one dimensional definition of Knights with our initial conditions up until the first time Y hits zero. We appeal to our Skorohod map construction given in the first Chapter, Section 2, so that the process is defined on the same probabily space $(\Omega, \mathcal{F}_t)$. Let $\tau_1 = \inf\{t > 0 : Y(t) = 0\}$ be the first time Y hits the origin. (From Propositions 5.4 and 5.6 we know that $X(\tau_1) > 0$, however, this is not necessary for our construction-only that $|V(\tau_1)| < \infty$. Indeed $\sup_{t\in\mathbb{R}}|V(t)| < \infty$ as shown in [33]). Call this process (X_1, Y_1, V_1). Repeat the above Skorohod map construction for Brownian motion with inert drift to construct the process (X_2, Y_2, V_2), satisfying the inert-Brownian interactions, on $(\Omega, \mathcal{F}_{t+\tau_1})$ whose initial conditions are $X_2(0) = X_1(\tau_1), Y_2(0) = 0, V_2(\tau_1) = -V_1(\tau_1)$. Let $\tau_2 = \inf\{t > 0 : Y_2(t) = 0\}$ be the first time Y_2 hits zero. Continue this recursion to get a sequence of processes (X_n, Y_n, V_n) and stopping times τ_n such that $X_n(\tau_n) = X_{n+1}(0), -V_n(\tau_n) = V(\tau_{n+1}), Y_n(\tau_n) = 0$. Set

$T_n = \sum_{i=1}^{n} \tau_i$, and $T_0 = 0$. For $t \in [T_n, T_{n+1})$ define

$$(X(t), Y(t), V(t)) = (X_{n+1}(t), Y_{n+1}(t), V_{n+1}(t)).$$

The dynamics of the processes in (2.5)-(2.8) follow from the dynamics of the classical Knights construction together with the definition of Y and V at the times τ_n. □

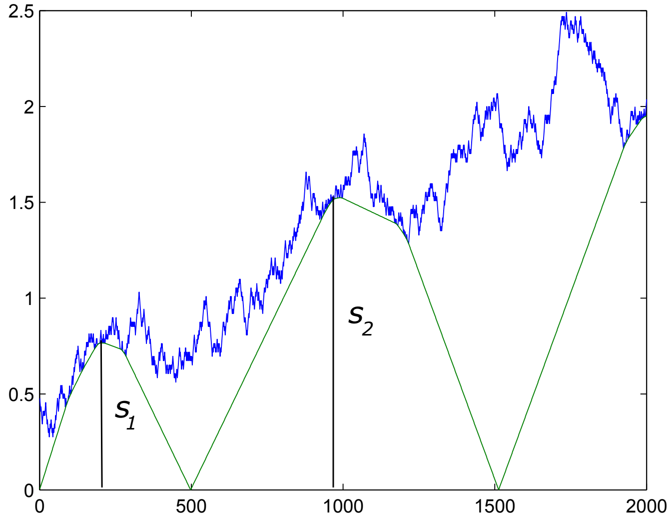

Figure 2.3: The Brownian particle and the inert particle when a reflecting barrier is introduced at the origin. In this case we have a sequence of maximums.

Call the sequence of local maximums S_n. See Figure 2.1 for an example. It is clear that S_n exists for all $n \in \mathbb{N}$ from Theorem 2.11 because there are an infinite number of reflections, hence an infinite number of maximums.

Theorem 2.12. *The sequence of local maximums* $(S_n)_{n\geq 1}$ *is a submartingale Markov chain adapted to* $\mathcal{F}_{T_n}$

Without loss of generality we let $K = 1$. Set τ as the next time of reflection after reaching S_1. The proof is conditional on $S_1 = \beta$. Consider the discrete approximation of S_2 given by the construction of the proof of Theorem 1.5. This is the same discretization given in (2.2) and Proposition (2.2). Set

$$Y_n := \sum_{j=1}^{n} \alpha_j \Delta T_j,$$

where ΔT_i are the times between which the piecewise process Y_n decreases slope by an amount of $1/n$. Because $\Delta T_i \overset{d}{=} T_{1/n}^{(-\frac{n-i+1}{n})}$, the hitting time of Brownian motion in Lemma 2.1, we have

$$(Y_n | Y_n < \infty) \overset{d}{=} \left(\sum_{j=1}^{n} \alpha_j \Delta R_j \middle| Y_n < \infty \right) =: Y_n^{(2)}$$

Let $T = \inf\{t > \tau : X_n(t) = Y_n(t)\}$ be the first time after τ when the Brownian particle reflected from Y_n hits $Y_n(t)$. Set $h_1 := \beta - Y_n, h_2 = Y(T)$. The bound

$$\mathbb{E}(h_1 | V(\tau+) = \alpha) \leq \mathbb{E}(h_2 | V_n(\tau+) = \alpha)$$

follows from the optional stopping theorem, because for any $h, -\alpha > 0$ we have

$$\mathbb{E}\left(B\left(\rho \wedge \frac{h}{\alpha}\right) 1_{\left\{\frac{h}{\alpha} \geq T\right\}} \right) < h,$$

This implies

$$\mathbb{E}\left(B\left(\rho \wedge \frac{h}{\alpha}\right) 1_{\left\{\frac{h}{\alpha} < T\right\}} \right) > 0.$$

where ρ is the first time Brownian motion hits $1/n$ below the line $h + \alpha t$. (In other words, since Y_n will not change a slope beyond α the Brownian path will stay away from $y(t) - 1/n$ between times $\tau - h/|\alpha|$ and τ. This will make the expected value of $X(t)$ larger).

We condition on $V_n(\tau+) = \alpha$ and compute

$$\begin{aligned}
\beta &= \mathbb{E}(Y_{n_\alpha} | V_n(\tau+) = \alpha) + \mathbb{E}(h_1 | V_n(\tau+) = \alpha) \\
&= \frac{1}{\mathbb{P}(V_n(\tau+) = \alpha | Y_{n_\alpha} < \infty)} \int_0^\beta x\mathbb{P}(Y_{n_\alpha} \in dx | Y_{n_\alpha} < \infty) + \mathbb{E}(h_1 | V_n(\tau+) = \alpha) \\
&\leq \frac{1}{\mathbb{P}(V_n(\tau+) = \alpha | Y_{n_\alpha} < \infty)} \int_0^\beta x\mathbb{P}(Y_{n_\alpha} \in dx | Y_{n_\alpha} < \infty) + \mathbb{E}(h_2 | V_n(\tau+) = \alpha) \\
&= \frac{1}{\mathbb{P}(V_n(\tau+) = \alpha | Y_{n_\alpha} < \infty)} \int_0^\beta x\mathbb{P}\left(\sum_{j=1}^{n_\alpha} \alpha_j \Delta R_j \in dx \middle| Y_{n_\alpha} < \infty \right) + \mathbb{E}(h_2 | V_n(\tau+) = \alpha) \\
&\leq \mathbb{E}\left(\sum_{j=1}^{n_\alpha} \alpha_j \Delta R_j \middle| V_n(\tau+) = \alpha \right) + \mathbb{E}(h_2 | V_n(\tau+) = \alpha) \\
&\leq \mathbb{E}(S_{2,n} | V_n(\tau+) = \alpha) + \mathbb{E}(h_2 | V_n(\tau+) = \alpha).
\end{aligned}$$

Now taking expectation with respect to $\mathbb{P}(V_n(\tau+) \in d\alpha | S_{1,n} = \beta)$, we have

$$\beta \leq \mathbb{E}(S_{2,n} | S_{1,n} = \beta).$$

We know by Proposition 2.2 and the construction of $S_{1,n}$, that $S_{1,n} = \beta$ is the same event as $S_1 = \beta$ simply because the maximum of the discrete process when $V(0) = 0$ is the same as the limit. From Proposition 2.5, we have a uniformly 3/2 moment for $S_{2,n}$. Hence we may pass to the limit to get

$$\beta \leq \mathbb{E}(S_2 | S_1 = \beta).$$

It follows from induction that $(S_n)_{n\geq 1}$ is a submartingale.

3 Strong approximation

There are many different notions of approximating a given stochastic process or random variable. For instance, if Z is a random variable with probability space $(\Omega, \mathbb{P})$ and state space $(d, \mathcal{X})$, one may approximate Z in law by finding another distribution on $\mathcal{X}$ within a given proximity to Z. Here, proximity could be described by a particular metric on $\mathcal{P}(\mathcal{X})$, the space of probability measures on $\mathcal{X}$. For instance, one could take the total variation metric, the Wasserstein distance, or the Prohorov distance that is metrizes convergence in

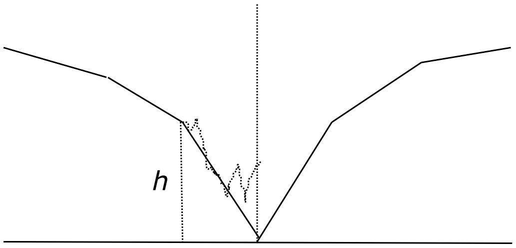

Figure 2.4: A sample path of X showing h for some fixed value n. Here τ is the the time marked with the vertical dash.

distribution. The Skorohod representation theorem says a sequence Z_n converges to Z in distribution if and only if there is a probability space $(\Omega', \mathbb{P}')$ supporting random variables Z'_n, Z' with $Z'_n \overset{d}{=} Z_n,\ Z' \overset{d}{=} Z$ and

$$Z'_n \longrightarrow Z', \text{ almost surely.}$$

A strong approximation result quantifies the distance between processes explicitly by giving an explicity construction of Z_n and Z on the same probability space such that

$$\mathbb{P}(d(Z, Z_n) < \epsilon_n) > 1 - \delta_n,$$

where ϵ_n and δ_n depend on n explicitly with $\epsilon_n, \delta_n \to 0$ as $n \to \infty$. In this Section we give an algorithm for a pair of processes (X, Y_n) that strongly approximate the pair (X, Y) of the Brownian and inert-particle. The idea is relatively simple. A strong approximation for Brownian motion with random walk is well established [35]. This gives explicit ϵ, δ depending on n such that $\|B - S_n\| < \epsilon$ with probability at least $1 - \delta$, for a Brownian motion B and random walk process S_n. At this point we apply our Skorohod map path-by-path to the random walk S_n. Lipshitz properties of the Skorohod map then give the strong approximation with ϵ, δ sufficiently altered. Let

$$S_n = X_1 + \cdots + X_n$$

be a one-dimensional simple random walk, and consider its path-wise linear interpolation,

$$S_t = S_k + (t - k)(S_{k+1} - S_k),\ k \le t \le k + 1.$$

Let $W^{(n)}(t) := n^{-1/2}S_{tn}$; properly scaled so its law converges to that of Brownian motion.

Theorem 3.1 (Ch. 3, [35])**.** *Let $(\Omega, \mathbb{P})$ supporting a Brownian motion $B(t)$. There is a continuous time random walk $W^{(n)}$ supported on $(\Omega, \mathbb{P})$ and positive numbers a and csuch that*

$$\mathbb{P}\left(\sup_{s\in[0,1]}|B(s) - W^{(n)}(s)| \geq rn^{-1/4}\sqrt{\log n}\right) \leq ce^{-ra}$$

whenever $r \leq n^{1/4}$.

Recall the Skorohod map constructed and used in the first Chapter. We use $n = 1$ since we have one Brownian particle.

Theorem 3.2. *Corresponding to each $f = (f_1, \cdots, f_n) \in C([0,T], \mathbb{R}^n)$, $v \in \mathbb{R}, K \geq 0$ is a pair of continuous functions*

$$(I_f^{(n)}(t), V_f^{(n)}(t)) =: \Gamma_n f(t) \in C([0,T], \mathbb{R}^2)$$

satisfying

$$x_i(t) := f_i(t) + I_f^{(n)}(t) + m_i(t) \geq 0, \tag{3.1}$$

$$m_i(t) \textit{ is flat off } \{t : x_i(t) = 0\}, \tag{3.2}$$

$$V_f^{(n)}(t) = -v + \frac{K}{n}\sum_{i=1}^{n} m_i(t), \;\; v \in \mathbb{R}, \tag{3.3}$$

$$I_f^{(n)}(t) = \int_0^t V_f^{(n)}(s)ds, \tag{3.4}$$

for all $t \in [0,T]$.

Remark 3.3. It follows from the classical Skorohod equation that

$$m_i(t) = \sup_{0<s<t}[-(f_i(s) + I^{(n)}(s))] \vee 0.$$

This is used in the proof of Proposition 2.12 below.

Also recall the Lipschitz property of the Skorohod map given in Proposition 2.11.

Proposition 3.4. *(Lipschitz property of $V^{(n)}$) For any $v \in \mathbb{R}, K \geq 0$, take $f, g \in C([0,T], \mathbb{R}^n)$ such that $\|f-g\|_{[0,T]} < \eta$. We have*

$$\|V_f^{(n)} - V_g^{(n)}\|_{[0,T]} \leq (K\eta/n)\exp(KT), \tag{3.5}$$

and consequently

$$\|I_f^{(n)} - I_g^{(n)}\|_{[0,T]} \leq (K\eta/n)T\exp(KT). \tag{3.6}$$

Theorem 3.5. *Any probability space $(\Omega, \mathbb{P})$ supporting a Brownian motion $B(t)$ also supports random process $V_n(t)$ and $Y_n(t) := \int_0^t V(s)ds$ such that*

$$\mathbb{P}\left(\sup_{s\in[0,1]} |Y(s) - Y_n(s)| \geq Ke^K rn^{-1/4}\sqrt{\log n}\right) \leq Ce^{-ra},$$

$$\mathbb{P}\left(\sup_{s\in[0,1]} |V(s) - V_n(s)| \geq Ke^K rn^{-1/4}\sqrt{\log n}\right) \leq Ce^{-ra},$$

$$\mathbb{P}\left(\sup_{s\in[0,1]} |X(s) - X_n(s)| \geq 3Ke^K rn^{-1/4}\sqrt{\log n}\right) \leq Ce^{-ra}$$

where Y is the inert particle with constant $K > 0$, for some positive constants a, c and where $r \leq n^{1/4}$.

Proof. By Theorem 3.1 we have

$$\mathbb{P}\left(\sup_{s\in[0,1]} |B(s) - W^{(n)}(s)| \geq rn^{-1/4}\sqrt{\log n}\right) \leq ce^{-ra}.$$

Here $T = 1$. Replaceing f in either (3.5) or (3.6) pathwise with $B(t)$, and f_δ with $W^{(n)}(t)$ yields

$$\mathbb{P}\left(\sup_{s\in[0,1]} |Y(s) - Y_{W^{(n)}}(s)| \geq Krn^{-1/4}\sqrt{\log n}\right) \leq Ce^{-ra},$$

and

$$\mathbb{P}\left(\sup_{s\in[0,1]}|V(s)-V_{W^{(n)}}(s)|\geq Krn^{-1/4}\sqrt{\log n}\right)\leq Ce^{-ra}.$$

To approximate the Brownian particle $X(t)$, use Remark 3.3 in the fact that

$$X(t)=B(t)+m(t),\ m(t)=\sup_{s\in[0,t]}(Y(s)-B(s))\vee 0.$$

Setting

$$X_n(t):=W^{(n)}(t)+m_n(t),\ m_n(t)=\sup_{s\in[0,t]}(Y_n(s)-W^{(n)}(s))\vee 0,$$

yields the proper bound by applying the above to the triangle inequality with Y_n estimating Y, m_n estimating m, and $W^{(n)}$ estimating B. □

Recall 2.11 from the first chapter. (We remove the dependence on the number of particles which is n in that statement.)

Proposition 3.6. *Consider the sequence $I_f^{(\epsilon)}$ defined above for a given $f\in C([0,T],\mathbb{R})$. If $l<m$, then*

$$\|I_f^{(2^{-l})}-I_f^{(2^{-m})}\|_{[0,T]}\leq((2+K)\|f\|_{[0,T]})2^{-l}\exp(KT).$$

Taking $m\to\infty$ yields

$$\|I_f^{(2^{-l})}-I_f\|_{[0,T]}\leq((2+K)\|f\|_{[0,T]})2^{-l}\exp(KT). \tag{3.7}$$

Together with Theorem 3.5 this will provide a numerical algorithm for discrete approximation to the inert particle through the definition of $I^{(2^{-l})}$ in the existence portion of Theorem 2.6.

Theorem 3.7. *There exists a probability space $(\Omega,\mathbb{P})$ supporting a Brownian motion, and an algorithmically constructed process $Y_n^{(l)}$ such that*

$$\mathbb{P}\left(\sup_{s\in[0,1]}|Y(s)-Y_n^{(l)}(s)|\geq(2+K)e^K\sqrt{n}2^{-l}+Ke^Krn^{-1/4}\sqrt{\log n}\right)\leq Ce^{-ra}.$$

for $n,l\in\mathbb{N}$, $r\leq n^{-1/4}$ and the same positive constants C,a in Theorem 3.5.

Proof. We work on some space $(\Omega, \mathbb{P})$ with a random walk $W^{(n)}$ strongly approximating a supported Brownian motion according to Theorem 3.5. Use (3.7) by setting $f = W^{(n)}$ and $Y_n^{(l)} := I_{W^{(n)}}^{(2^{-l})}$. We know that $\|W^{(n)}\|_{[0,1]} \leq \sqrt{n}$, and applying the triangle inequality

$$|Y_n^{(l)} - Y| \leq |Y_n - Y_n^{(l)}| + |Y_n - Y|$$

with Theorem 3.5 proves the result. □

Once we have the above result for $Y_n^{(l)}$, we can use the approximation of $B(t)$ with $W^{(n)}$ to get similar statements for

$$V_n^{(l)}(t) := v + K \sup_{s \in [0,t]} (Y_n^{(l)}(s) - W^{(n)}(s)) \vee 0.$$

In fact, in this way $V_n^{(l)}$ and $Y_n^{(l)}(t) = \int_0^t V_n^{(l)}(s)ds$ can be obtained from one another. Below is the algorithm for constructing $V_n^{(l)}$ for $K \geq 0, v \in \mathbb{R}$. These are obtained by applying steps 2.1 - 2.4 in the existence portion of the proof of Theorem 2.6 to each sample path of $W^{(n)}$.

First, let $S_k = n^{-1/2} \sum_{i=1}^{k} W_i$ where $\mathbb{P}(W_i = \pm 1) = 1/2$ are i.i.d. and set $n^{1/2} W^{(n)}(t) = S_k + (t-k)(S_{k+1}$ for $k \leq tn < k+1$.

```
Result: Functions $Y_n^{(l)}$ and $V_n^{(l)}$ defined on $[0,1]$ with referenced in Theorem 3.7
        which approximates the inert-particle uniformly on $[0,1]$.
Set $j = 1$ and $V_n^{(l)}(t) = v$ for $t \in [0, 2^{-l}]$;
Set $Y_n^{(l)} = vt$ for $t \in [0, 2^{-l}]$;
for $k = 1$ to $2^l$ do
    Set $m = \sup_{s \in [0, k2^{-l}]} (Y_n^{(l)}(s) - W^{(n)}(s)) \vee 0$;
    $V_n^{(l)}(t) = v - km$ for $t \in [k2^{-l}, (k+1)2^{-l}]$;
    $Y_n^{(l)}(t) = Y_n^{(l)}(k2^{-l}) + (t - k2^{-l}) V_n(t)$ for $t \in [k2^{-l}, (k+1)2^{-l}]$.
end
```

4 Brownian motion with inert-drift and a boundary

In this section we consider a regime of the Brownian particle X with inert particle Y where $0 \leq X(t) \leq Y(t)$ and where we add a reflecting barrier for X at the origin. We construct our model via a sequence of stopping times and concatenating paths of reflected Brownian motion with this process for which there is a Skorohod map.

Lemma 4.1. *Let* $y > 0, K \geq 0, v \in \mathbb{R}$ *and let* $f(t)$ *be a continuous function with* $0 \leq f(0) \leq y$*. There are unique continuous, monotonically increasing functions* $L_0(t), L_1(t)$ *such that*

(i) $V(t) := KL(t) + v,$

(ii) $Y(t) := y + \displaystyle\int_0^t V(s)ds,$

(iii) $0 \leq f(t) + L_0(t) - L_1(t) \leq Y(t),$

(iv) $L_0(0) = 0 = L_1(0).$

(v) L_0 *is flat away from* $X^{-1}(0),$

(vi) L_1 *is flat away from* $\{X = Y\}.$

The above holds for all times $0 \leq t \leq T := \inf\{t \geq 0 : Y(t) = 0\}.$ *Under the above hypotheses we have* $T > 0$*. If additionally we have* $v \geq 0$*, then* $T = \infty$ *and the above functions are defined for all time.*

When applying this pathwise to the Brownian sample paths, we have processes X, Y, V, L_0, L_1 with the following properties.

(i) $X(t) = B(t) + L_0(t) - L_1(t)$

(ii) $V(t) = KL_1(t) - v_0,$ with $v_0 > 0$

(iii) $Y(t) = \int_0^t V(s)ds + y_0,$ with $y_0 > 0$

(iv) $0 \leq X(t) \leq Y(t)$ a.s.

It follows from definition that for each sample path, $(X, L_0 - L_1)$ solves the Skorokhod problem for Brownian motion reflected in the time-dependent domain $[0, Y(t)]$. See [8]. In such a situation we refer to L_1 as the top constraining process.

In order to show that the above processes are defined for all time, we show $T = \infty$ for all initial v. This yields the following

Theorem 4.2. *For fixed* $K > 0, v \in \mathbb{R}$, *and* $y_0 > 0$, *there exists on* $(\Omega, \mathcal{F}_t^B, \mathbb{P}^x, B(\cdot))$ *a pair of processes* $(X(t), V(t))$ *defined for* $t \geq 0$ *such that with probability 1 (under* $\mathbb{P}^0$*),*

(i) $V(0) = v$, *V(t) is continuous, non-decreasing, and* $\mathcal{F}_t^B -$ *measurable,*

(ii) $X(0) = 0, X(t)$ *is* $\mathcal{F}_t^B$ *measurable and is Brownian motion reflected between 0 and* $S(t) := \int_0^t V(s)ds + y_0$,

(iii) $V(t) = v + KL_1(t), 0 \leq t$, *where* L_1 *is the occupation local time of* Y *on S.*

In the following we will need a sequence of positive number a_k, decreasing to 0, such that

$\sum_{i=1}^{\infty} a_i$ converges, but

$\sum_{i=1}^{\infty} \sqrt{a_i - a_{i+1}}$ diverges.

There are, in fact, sequences with the above properties, and one can construct a particular example by defining $a_1 = 1$ and $\sqrt{a_i - a_{i+1}} = 1/(i \log(i+1))$. Notice that this immediately gives the second property above. To show the infinite sum converges, notice that there is a positive constant C such that

$$a_n - a_{n+1} = \frac{1}{n^2 \log^2(k+1)} \geq C(\frac{1}{n^{\frac{3}{2}}} - \frac{1}{(n+1)^{\frac{3}{2}}}).$$

It follows that we have

$$\frac{1}{n^{3/2}} = 1 - \sum_{k=1}^{n-1}(\frac{1}{k^{3/2}} - \frac{1}{(k+1)^{3/2}}) \geq 1 - \sum_{k=1}^{n-1}(a_k - a_{k+1}) = a_n > 0,$$

from which one can easily see that $\sum_{i=1}^{\infty} a_i < \infty$. We will now define several stopping times. Let $T_a = \inf\{t : Y(t) = a\}$ denote the first time the inert particle reaches level $a > 0$, and $S = \inf\{t : V(t) = 0\}$ the first time the velocity of the inert particle reaches zero. Since V and Y are both adapted to the Brownian filtration, T_a and S are both stopping times. Furthermore, since $V(t) = KL_1(t) - v_0$ is monotonically increasing, $V(t) < 0$ for $t < S$, and $V(t) \geq 0$ for $t \geq S$. Consequently $Y(t)$ is monotonically decreasing on $(0, S)$ and monotonically increasing on (S, T). So if $T_{a_k} < \infty$, we have $T_{a_k} \leq S$. Similarly because the a_k are decreasing, $T_{a_1} < T_{a_2} \cdots < T_{a_k}$ if they are all finite. If $T_{a_k} = \infty$, let $i^* = \min\{n : T_{a_n} = \infty\}$ be the first index where T_{a_k} is infinite. Define $\tilde{T}_{a_k}$ to be T_{a_k} if it is finite. Otherwise, define it to be $S + \frac{1}{v_0}\sum_{m=i^*}^{k}(a_{m-1} - a_m)$. Geometrically speaking we are setting the gaps between the T_{a_k} after one becomes infinite. It is clear that $\tilde{T}_{a_k}$ is a stopping time as well. The first result will show that the gaps between $\tilde{T}_{a_k}$ and $\tilde{T}_{a_{k+1}}$ are bounded below.

Lemma 4.3. *$\tilde{T}_{a_{k+1}} - \tilde{T}_{a_k} \geq (a_k - a_{k+1})/v_0$ almost surely.*

Proof. Notice we have three cases depending whether $T_{a_k}, T_{a_{k+1}}$ are both finite, both infinite, or if the first is finite and the other is infinite.

To see the lower bound when T_{a_k} and $T_{a_{k+1}}$ are both finite, we compute

$$\begin{aligned} a_k - a_{k+1} &= |Y(T_{a_k}) - Y(T_{a_{k+1}})| \\ &\leq \int_{T_{a_k}}^{T_{a_{k+1}}} |V(s)| ds \\ &\leq v_0(T_{a_{k+1}} - T_{a_k}) \\ &= v_0(\tilde{T}_{a_{k+1}} - \tilde{T}_{a_k}) \end{aligned}$$

as $|V(s)|= V(s)$ is nonincreasing for the interval $0 < s \leq T_{a_{k+1}} \leq S$.

If T_{a_k} is finite but $T_{a_{k+1}}$ is infinite, we have $i^* = k+1$ by definition. Thus

$$\tilde{T}_{a_{k+1}} - \tilde{T}_{a_k} = S + (a_k - a_{k+1})/v_0 - T_{a_k} \geq (a_k - a_{k+1})/v_0$$

since $S \geq T_{a_k}$.

When T_{a_k} and $T_{a_{k+1}}$ are both infinite, $\tilde{T}_{a_{k+1}} - \tilde{T}_{a_k} = (a_k - a_{k+1})/v_0$. □

Proposition 4.4. *We have*

$$\sum_{k=1}^{\infty} \max_{[\tilde{T}_{a_k}, \tilde{T}_{a_{k+1}}]} (B(t) - B(\tilde{T}_{a_k}) - a_k) = \infty$$

almost surely.

Proof. Recall the sequence a_k has the property that $\sum_{k=1}^{\infty} a_k$ converges, so it suffices to show that

$$\sum_{k=1}^{\infty} \max_{[\tilde{T}_{a_k}, \tilde{T}_{a_{k+1}}]} (B(t) - B(\tilde{T}_{a_k})) \geq \sum_{k=1}^{\infty} \max_{[\tilde{T}_{a_k}, \tilde{T}_{a_k}+\alpha_k]} (B(t) - B(\tilde{T}_{a_k})) = \infty$$

almost surely, by the previous lemma, where $\alpha_k = (a_k - a_{k+1})/v_0$. To do this, let

$$X_k := \max_{[\tilde{T}_{a_k}, \tilde{T}_{a_k}+\alpha_k]} (B(t) - B(\tilde{T}_{a_k})) \overset{d}{=} \max_{[0,\alpha_k]} B(s) \tag{4.1}$$

where the equality in distribution follows by the strong Markov property. It is clear that the X_i are independent, and therefore the three-series theorem says $\sum\limits_{k=1}^{\infty} X_k = \infty$ almost surely if $\sum\limits_{k=1}^{\infty} \mathbb{E}[X_k 1_{\{|X_k| \leq A\}}] = \infty$ for some positive A. Using (4.1) above and the fact that

$\max_{[0,\alpha_k]} B(s)$ is nonnegative, we compute

$$\mathbb{E}[X_k 1_{\{|X_k|\leq A\}}] = \mathbb{E}[\max_{[0,\alpha_k]} B(s) 1_{\left\{0<\max_{[0,\alpha_k]} B(s)\leq A\right\}}] \tag{4.2}$$

$$= \int\limits_0^\infty \mathbb{P}(\max_{[0,\alpha_k]} B(s) 1_{\left\{0<\max_{[0,\alpha_k]} B(s)\leq A\right\}} > y)dy \tag{4.3}$$

$$= \int\limits_0^A \mathbb{P}(y < \max_{[0,\alpha_k]} < A)dy \tag{4.4}$$

$$= \int\limits_0^A \mathbb{P}(\max_{[0,\alpha_k]} B(s) > y) - \mathbb{P}(\max_{[0,\alpha_k]} B(s) > A)dy \tag{4.5}$$

$$= 2\int\limits_0^A \mathbb{P}(B(\alpha_k) > y)dy - 2A\mathbb{P}(B(\alpha_k) > A). \tag{4.6}$$

We use the upper bound in the inequalities

$$\frac{x}{x^2+1} < \sqrt{2\pi}\exp(x^2/2)\mathbb{P}(N > x) < \frac{1}{x}, \tag{4.7}$$

for a standard normal variable N, to bound the negative term in (4.6) with

$$-2A\mathbb{P}(B(1) > \frac{A}{\sqrt{\alpha_k}}) \geq -2A\frac{\sqrt{\alpha_k}}{\sqrt{2\pi}}(\exp(-\frac{A^2}{2\alpha_k})).$$

By definition $\alpha_k = 1/(k^2\log^2(k+1))$, so this gives

$$-2A\mathbb{P}(B(\alpha_k) > A) \geq \frac{C}{(k^2\log^2(k+1))}(k+1)^{-\log(k+1)k^2} \tag{4.8}$$

for a $C > 0$. The same pair of inequalities on the standard normal distribution gives

$$2\int\limits_0^A \mathbb{P}(B(\alpha_k) > y)dy \geq 2\int\limits_0^A \mathbb{P}(B(1) > \frac{y}{\sqrt{\alpha_k}})dy \tag{4.9}$$

$$\geq \frac{2}{\sqrt{2\pi}}\int\limits_0^A \frac{y/\sqrt{\alpha_k}}{\frac{y^2}{\alpha_k}+1}\exp(\frac{-y^2}{2\alpha_k})dy \tag{4.10}$$

$$= \frac{2}{\sqrt{2\pi}}\alpha_k \int\limits_0^{A/\sqrt{\alpha_k}} \frac{u}{u^2+1}\exp(-u^2)du, u = y/\sqrt{\alpha_k} \tag{4.11}$$

$$\geq N\sqrt{\alpha_k} \tag{4.12}$$

For some positive constant N, since the fact that $A/\sqrt{\alpha_k}$ is bounded below implies the integral at (4.11) does not degenerate. Combining the previous inequalities shows

$$\mathbb{E}[X_k 1_{\{|X_k| \le A\}}] \ge N\sqrt{\alpha_k} - \frac{C}{(k^2 \log^2(k+1))}(k+1)^{-\log(k+1)k^2}$$

from which it follows that

$$\begin{aligned}\sum_{k=1}^{\infty} \mathbb{E}[X_k 1_{\{|X_k| \le A\}}] &\ge \sum_{k=1}^{\infty} (N\sqrt{\alpha_k} - \frac{C}{(k^2 \log^2(k+1))}(k+1)^{-\log(k+1)k^2}) \\ &\ge (\sum_{k=1}^{\infty} N\sqrt{\alpha_k}) - C' = \infty\end{aligned}$$

by the properties of our sequence a_n. ☐

Corollary 4.5. *The Brownian particle is never trapped. That is, the event*

$$\{Y(t) = 0 \text{ for some } t > 0\}$$

has probability zero.

Proof. In the event that $T_{a_k} < \infty$ for all k, lemma 1 shows that

$$\sum_{k=1}^{\infty} L_1([T_{a_k}, T_{a_{k+1}}]) \ge \sum_{k=1}^{\infty} \max_{t \in (\tilde{T}_{a_k}, \tilde{T}_{a_{k+1}})} (B(t) - B(\tilde{T}_{a_k}) - a_k) = \infty.$$

by the above proposition. ☐

We have just shown that the inert particle Y is positive almost surely. As such, there are several natural questions regarding its behavior. For instance, what bounds can one gather on the event $\{Y \ge a_n\}$, or what happens when the initial velocity gets extremely negative, i.e. when $v \to -\infty$. For clarification let us denote the inert particle with initial velocity $-v$ as Y_v. (We take v to be positive). One would expect that for a fixed level $a > 0$, the inert particle will reach level a with high probability as the initial velocity goes to negative infinity. That is, one would expect $\mathbb{P}(Y_v(t) < a \text{ for some } t) \to 1$ as $v \to \infty$. This is shown below.

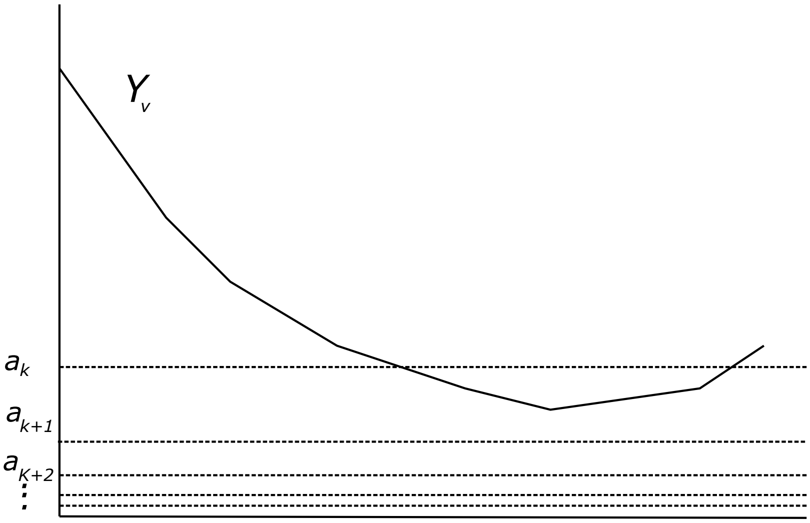

Figure 2.5: A sample path in the event $\{Y_v(t) < a_k \text{ for some } t\}$.

Proposition 4.6. *For a fixed $a > 0$, we have $\mathbb{P}(Y_v(t) < a$ for some $t) \to 1$ as $v \to \infty$.*

Proof. Recall that the time in which the velocity V of Y_v reaches 0 is the same instant in which Y_v turns around. Therefore if $Y_v(t)$ reaches level a_N, that is the same as saying the total push on Y from the Brownian particle, by time T_{a_N}, is less than v. Choosing $a_N < a$, we compute

$$\mathbb{P}(Y_v(t) < a \text{ for some } t) \geq \mathbb{P}(Y_v(t) < a_N \text{ for some } t) \tag{4.13}$$

$$= \mathbb{P}(T_{a_N} < \infty) \tag{4.14}$$

$$= \mathbb{P}(\sum_{i=1}^{N-1} M^{Y_v}_{[T_{a_i}, T_{a_{i+1}}]} < v) \tag{4.15}$$

$$= 1 - \mathbb{P}(\sum_{i=1}^{N-1} M^{Y_v}_{[T_{a_i}, T_{a_{i+1}}]} \geq v) \tag{4.16}$$

$$\geq 1 - \mathbb{P}(\sum_{i=1}^{N-1} \max_{[T_{a_i}, T_{a_{i+1}}]} (B(t) - B(T_{a_i}) + a_i - a_{i+1}) \geq v) \tag{4.17}$$

$$\geq 1 - \mathbb{P}(\sum_{i=1}^{N-1} \max_{[T_{a_i}, T_{a_{i+1}}]} B^i(t) \geq v + \sum_{i=1}^{N-1} (-a_i + a_{i+1})) \tag{4.18}$$

where the B^i are independent Brownian motions and the last inequality follows from the second lemma. The last term (4.18) clearly approaches 1 as $v \to \infty$. $\square$

5 *The inert particle is never trapped*

Before discussing the proof we will point out the three unique ways in which three stochastic processes $W_1 \leq W_2 \leq W_3$ may collide. Assume it is known that W_2 and W_3 meet at a (random) time T. If T is to be a triple collision between all three particles, we have two cases:

5.1. There is an $\epsilon > 0$ with W_1 having no contact with W_2 in $[T - \epsilon, T)$.

5.2. For every $\epsilon > 0$, W_1 makes contact with W_2 in $(T - \epsilon, T)$.

There is one more case that needs to be covered, though the three particles do not actually make contact.

5.3. There are two increasing sequences of stopping times T_n, T_n' where W_2 and W_3 make contact on T_n and W_1 makes contact with W_2 on T_n', and $T_n \leq T_n' \leq T_{n+1}$ with T_n bounded.

Lemma 5.1. *The first case of triple collision does not occur, almost surely, for* $W_1 = X, W_2 = Y$ *and* $W_3 \equiv 1$.

Proof. Let T be the first time that Y hits 1. For a given $\epsilon > 0$, define

$$T_1 = \inf \{t \geq 0 : X(s) \neq Y(s) \text{ for } s \in [t - \epsilon, t]\},$$

$$T_n = \inf \{t \geq T_{n-1} : X(s) \neq Y(s) \text{ for } s \in [t - \epsilon, t]\}.$$

and let A be the event $\{T_n < T\}$, which has positive probability for some ϵ small enough, and B be the event $\left\{B(t)+X(T_n)=1-Y(T_n) \text{ at time } t=\frac{1-Y(t_n)}{V(T_n)}\right\}$. Then we have

$$\begin{aligned}\mathbb{E}(1_A 1_B) &= \mathbb{E}(\mathbb{E}(1_A 1_B|\mathcal{F}_{T_n})\\ &= \mathbb{E}(1_A \mathbb{E}(1_B|\sigma(Y(T_n),V(T_n),X(T_n)))=0\end{aligned}$$

□

Because $X \leq Y \leq 1$ this says that there is not a triple collision between these two particles with the line. Let $T_0=\inf\{t \geq 0 : Y(t)=1\}$ be the first time that Y reaches the upper barrier. We will be considering a piecewise stochastic process for Y that has slope between two positive numbers $v_u > v_l > 0$. Designate L_a as the first time V reaches a, we define $\widetilde{Y}(t)$ as $Y(t)$ for $t \in [0,T]$ if $L_{v_l} \geq T \geq L_{v_u}$, otherwise if $L_{v_u} < T$ we let $\widetilde{Y}(t)=Y(t)$ for $t \in [0,L_{v_u}]$, and $\widetilde{Y}(t)=(t-L_{v_u})v_u+Y(L_{v_u})$ when $t \geq L_{v_u}$ until this line hits 1, if $L_{v_l} > T$ we let $\widetilde{Y}(t)=v_l t$ for $t \in [0,1/v_l]$. (These definitions are strictly for convenience, in the end we will obtain almost sure pathwise properties for $\widetilde{Y}$ while letting $v_l \to 0$ and $v_u \to \infty$ will give the same pathwise properties for Y). Now define the following stopping times:

$$Z_1=\inf\{t \geq 0 : V(t)=v_l\},\ \tau_1=\frac{v_u}{1-Y(Z_1)},$$

$$Z_i=\inf\{t \geq Z_{n-1}+\tau_{n-1} : X(t)=Y(t)\},\ \tau_n=\frac{v_u}{1-Y(Z_n)}.$$

Lemma 5.2. *Let $B(t)$ be standard Brownian motion reflected under the line $v_l t$. Let $\epsilon > 0$ and let $C > 0$ be some constant. Then the probability of excursions with $\max_{[0,\epsilon/v_l]}|B(t)| < C$ does not contact the line $v_l t$ during times in the interval $[\frac{\epsilon}{v_u},\frac{\epsilon}{v_l}]$ has probability greater than some $\alpha > 0$ that does not depend on ϵ.*

Proof. Let $\mathcal{F}_t^B$ be the filtration of a standard Brownian motion. Consider the change of measure $\mathbb{P}^{(-v_l)}(\cdot)$, the Girsanov transform, given by $\mathbb{P}^{(-v_l)}(A)=\mathbb{E}(1_A Z_t)$ when $A \in \mathcal{F}_t^B$ and where $Z_t=\exp(-v_l B(t)-.5v_l^2 t)$. Then $B(t)$ under $P^{(-v_l)}$ is Brownian motion with drift

$-v_l t$. Therefore the excursions in question refer to the event

$$\begin{aligned}
A_\epsilon &= \left\{0 < |B(t)|< C \text{ for } \frac{\epsilon}{v_u} < t < \frac{\epsilon}{v_l}\right\} \\
&= \left\{0 < |\frac{1}{\sqrt{\epsilon}}B(t)|< \frac{C}{\sqrt{\epsilon}} \text{ for } \frac{1}{v_u} < \frac{t}{\epsilon} < \frac{1}{v_l}\right\} \\
&= \left\{0 < |\frac{1}{\sqrt{\epsilon}}B(\epsilon t)|< \frac{C}{\sqrt{\epsilon}} \text{ for } \frac{1}{v_u} < t < \frac{1}{v_l}\right\} \\
&\overset{d}{=} \left\{0 < |B(t)|< \frac{C}{\sqrt{\epsilon}} \text{ for } \frac{1}{v_u} < t < \frac{1}{v_l}\right\} =: B
\end{aligned}$$

where the equality in distribution is under $\mathbb{P}$. Notice that A_ϵ and B are both contained in $\mathcal{F}^B_{1/v_l}$ and that these two events have the same probability under $\mathbb{P}$. Because $\mathbb{P}$ and $\mathbb{P}^{(-v_u)}$ are equivalent on $\mathcal{F}^B_{1/v_u}$, this shows that $\mathbb{P}^{(-v_u)}(A_\epsilon)$ does not degenerate with ϵ. □

Corollary 5.3. *The second case of triple collision does not occur, almost surely, for* $W_1 = X, W_2 = Y$ *and* $W_3 \equiv 1$.

Proof. Define the stopping times T_n to be the first time that X hits $\widetilde{Y}$ after it reaches a height of $1 - \frac{1}{n}$, and let A_n be the event where $T_n < T$ Brownian motion restarted at T_n has an excursion behaving as in the previous lemma. Here T is the first time $\widetilde{Y}$ reaches 1. It follows geometrically by observing individual excursions that the A_n are are disjoint. Their probability is bounded below by α in the previous lemma, whence they occur only a finite number of times. Taking the upper bounding slope to infinity, $v_u \to \infty$, and the lower bounding slope to zero, $v_l \to 0$, gives the result. □

5.1 *Trapping the Inert Particle*

The main result in Section 4 could be viewed as a proof showing triple collisions between the inert particle Y, the Brownian particle X, and the reflecting wall at the origin collide almost never in the regime $Y \geq X \geq 0$. Put another way, Y does not trap the process X. In this subsection we give a partial result showing the Brownian particle cannot trap the inert particle when $X \geq Y \geq 0$ and Y reflects from the origin. The construction of this regime is

Theorem 2.11. The Brownian particle $X(t)$ could "trap" the inert particle $Y(t)$ at the origin in a number of ways. Consider the stopping time

$$T := \inf\{t > 0 : \lim_{s \to t^+} X(s) = 0\}. \tag{5.1}$$

It is clearly adapted to the filtration of $B(t)$ since $\{T < t\} \in \sigma\{X(s) : s < t\} \subset \mathcal{F}_t$. We say the Brownian particle X *traps* the inert particle Y at T if $T < \infty$ and $\lim_{t \to T^+} Y(t) = 0$. Trapping can be separated into three disjoint events:

T1: There is an $\epsilon > 0$ such that $X(s) > Y(s)$ for $s \in (T - \epsilon, T]$.

T2: There is an $\epsilon > 0$ such that for all $0 < \eta < \epsilon$, there is a $\tau \in (T - \eta, T]$ such that $X(\tau) = Y(\tau)$.

T3: For all $\epsilon > 0$ there is an $s \in (T - \epsilon, T]$ such that $Y(s) = 0$.

Condition T3 is saying Y has an infinite number of reflections in a finite time T for which $\lim_{t \to T^+} Y(t) = 0$.

Proposition 5.4. *The event T1 occurs with probability zero.*

Proof. We prove this for an arbitrary initial condition $X(0) = x > y = Y(0)$ and $V(0) = -v < 0$. In other words, the inert particle travels toward the origin with the Brownian particle above it. Consider note that $X(t) - Y(t)$ is equal to $B(t) + vt + x - y$ until the first time X collides with Y. Let $\tau = \inf\{t > 0 : B(t) + vt + x - y = 0\}$. Notice the event T1 is the same as $\{\tau = y/v\}$, which clearly has probability zero. $\square$

To show the event T2 occurs with probability zero we use the following lemma.

Lemma 5.5. *Fix $v_u, v_l > 0$. Let $B(t)$ be standard Brownian motion reflected under the line $v_l t$. Let $\epsilon > 0$ and let $C > 0$ be some constant. The probability a Brownian path with $\max_{[0,\epsilon/v_l]}|B(t)|< C$ does not contact the line $v_l t$ during times in the interval $[\frac{\epsilon}{v_u}, \frac{\epsilon}{v_l}]$ has probability greater than some $\alpha > 0$ when ϵ varies in a bounded set.*

Proof. Let $\mathcal{F}_t^B$ be the filtration of a standard Brownian motion. Consider the change of measure $\mathbb{P}^{(-v_l)}(\cdot)$, the Girsanov transform, given by $\mathbb{P}^{(-v_l)}(A) = \mathbb{E}(1_A Z_t)$ when $A \in \mathcal{F}_t^B$ and where $Z_t = \exp(-v_l B(t) - .5v_l^2 t)$. Then $B(t)$ under $\mathbb{P}^{(-v_l)}$ is Brownian motion with drift $-v_l t$. It is known that $|B(t) - v_l t|$ is distributed as the process of reflected Brownian motion, see [25]. It suffices to consider the event

$$\begin{aligned}
A_\epsilon &= \left\{0 < |B(t)|< C \text{ for } \frac{\epsilon}{v_u} < t < \frac{\epsilon}{v_l}\right\} \\
&= \left\{0 < |\frac{1}{\ }\sqrt{\epsilon}B(t)|< \frac{C}{\sqrt{\epsilon}} \text{ for } \frac{1}{v_u} < \frac{t}{\epsilon} < \frac{1}{v_l}\right\} \\
&= \left\{0 < |\frac{1}{\sqrt{\epsilon}}B(\epsilon t)|< \frac{C}{\sqrt{\epsilon}} \text{ for } \frac{1}{v_u} < t < \frac{1}{v_l}\right\} \\
&\overset{d}{=} \left\{0 < |B(t)|< \frac{C}{\sqrt{\epsilon}} \text{ for } \frac{1}{v_u} < t < \frac{1}{v_l}\right\} =: B_\epsilon
\end{aligned}$$

where the equality in distribution is under $\mathbb{P}$. Notice that A_ϵ and B are both contained in $\mathcal{F}^B_{1/v_l}$ and that these two events have the same probability under $\mathbb{P}$. Clearly $\mathbb{P}(B_\epsilon)$ does not degenerate as $\epsilon \to 0^+$. Because $\mathbb{P}$ and $\mathbb{P}^{(-v_u)}$ are equivalent on $\mathcal{F}^B_{1/v_u}$, this shows that $\mathbb{P}^{(-v_u)}(A_\epsilon)$ because equivalent measure have the same null sets. But $\mathbb{P}^{(-v_l)}(A_\epsilon)$ is the probability of the event in question, proving the lemma. □

Proposition 5.6. *The event T2 occurs with probability zero.*

Proof. Suppose out initial conditions are $X(0) = x > y = Y(0)$, and $V(0) = v \in \mathbb{R}$. Clearly Y will reach its maximum and reverse direction, almost surely, as studied in the beginning of Section 2. So it suffices to take $V(0) = -v < 0$ after applying the strong Markov property to a stopping time after the Y has reached its maximum. For instance, one could take the stopping time to be the first time Y hits a repeated value and $V < 0$. Now, we know that V

decreases, so Y hits the origin sometimes in the interval $[0, y/v]$. Let $\tau = \inf\{t < 0 : Y(t) = 0\}$ denote this time. Since the local time of X on Y is stochastically bounded above by the running maximum of $B(t)$ below the horizontal line x, we see that $|V(\tau)|$ is finite almost surely. Fix $C > 0$. We alter the process (X, Y, V) given in Theorem 1.1 to get a new process (X_C, Y_C, V_C) where the process is the same up until the stopping time

$$\tau_C := \inf\{t > 0 : |V_C(t)| > C\}.$$

From $t \in [\tau_C, \infty]$ we declare $V_C(t) = V(\tau_C) = -C$. In other words, $(X_C(t), Y_C(t), V_C(t)) = (X(t), Y(t), V(t))$ for $t \in [0, \tau_C]$, while for $t \in [\tau_C, \infty]$, $Y_C(t) = Y(\tau_C) - Ct$ and $X_C(t)$ behave as Brownian motion reflected from this line Y_C such that $X_C(\tau_C) = X(\tau_C)$. We show the event T2 corresponding to (X_C, Y_C, V_C) is a null event. Define the stopping times recursively as

$$\begin{aligned} T_1 &= \inf\{t > 0 : X_C(t) = Y_C(t) < 1\} \\ T_{n+1} &= \inf\{t > T_n + Y(T_n)/(C+1) : X_C(t) = Y_C(t)\}. \end{aligned}$$

Now define the events

$$A_n = \{X_C(t) > Y_C(t) : t \in [Y_C(T_n)/(C+1), Y_C(T_n)/v]\}.$$

One can see that

$$A_n \supset \{X_C(t) > Y_C(T_n) - v(t - T_n) : t \in [Y_C(T_n)/(C+1), Y_C(T_n)/v]\},$$

By Lemma 5.5,

$$\begin{aligned} &\mathbb{P}(A_n | (X_C(T_n), Y_C(T_n), V_C(T_n))) \\ &\geq \mathbb{P}(\{X_C(t) > Y_C(T_n) - v(t - T_n) : t \in [Y_C(T_n)/(C+1), Y_C(T_n)/v]\} | (X_C(T_n), Y_C(T_n), V_C(T_n))) \\ &\geq \alpha. \end{aligned}$$

In other words, this lower bound holds when conditioning on any of the value $(X_C(T_n), Y_C(T_n), V_C(T_n))$. Let $T = \inf\{t > 0 : Y_C(t) = 0\}$ be the first time Y_C hits the origin. Clearly $A_n^c \in \mathcal{F}_{T_{n+1}}$

because T_{n+1} is the next time X_C and Y_C make contact after time $Y_C(T_n)/(C+1)$. By the strong Markov property and the above, we compute

$$
\begin{aligned}
&\mathbb{P}(T_{n+1} < T) \leq \mathbb{P}(A_1^c \cap \ldots \cap A_n^c)\\
&\leq \int \mathbb{P}(A_1^c \cap \ldots \cap A_n^c | (X_C, Y_C, V_C)(T_n) = (x,y,z)) \mathbb{P}((X_C, Y_C, V_C)(T_n) \in (dx, dy, dz))\\
&\leq \int \mathbb{P}(A_1^c \cap \ldots \cap A_{n-1}^c | (X_C, Y_C, V_C)(T_n) = (x,y,z))\\
&\cdot \mathbb{P}(A_n^c | (X_C, Y_C, V_C)(T_n) = (x,y,z)) \mathbb{P}((X_C, Y_C, V_C)(T_n) \in (dx, dy, dz))\\
&\leq (1-\alpha) \int \mathbb{P}(A_1^c \cap \ldots \cap A_{n-1}^c | (X_C, Y_C, V_C)(T_n) = (x,y,z)) \mathbb{P}((X_C, Y_C, V_C)(T_n) \in (dx, dy, dz))\\
&= (1-\alpha) \mathbb{P}(A_1^c \cap \ldots \cap A_{n-1}^c).
\end{aligned}
$$

By induction we see

$$\mathbb{P}(A_1^c \cap \ldots \cap A_n^c) \leq (1-\alpha)^n,$$

and so

$$\mathbb{P}(T_{n+1} < T) \leq (1-\alpha)^n$$

as well. We see that

$$\mathbb{P}(\text{Event T2 occurs for } (X_C, Y_C, V_C)) \leq \mathbb{P}(\cap_{n=1}^{\infty} \{T_n < T\}) = \lim_{n \to \infty} \mathbb{P}(T_n < T) = 0.$$

Going back to the original process (X, Y, V), we know that $(X_C, Y_C, V_C) \longrightarrow (X, Y, V)$ almost surely on compact sets by our definition of (X_C, Y_C, V_C) and because $\sup_{t \in [0, y/|v|]} |V| < \infty$ almost surely. Therefore, if the event T2 occured for (X, Y, V) with positive probability, it must also occur with positive probability for (X_C, Y_C, V_C) for some large C. Because we showed this does not happen, the result follows. $\square$

Chapter 3

APPROXIMATING BROWNIAN MOTION WITH INERT DRIFT BY POINT PROCESSES WITH STOCHASTIC INTENSITY

1 *Introduction*

The first and second chapters reference Knights study of Brownian motion with inert drift [33]. White constructed a multidimensional analog in [45] that was also studied in [3] by Bass, Burdzy, Chen and Hairer. The multidimensional analog is a pair of processes (X, K) where X is a diffusion reflecting inside a sufficiently smooth domain D in $\mathbb{R}^n$, and K is the drift. This drift is the inward normal integrated against the local time. That is,

$$\begin{aligned} X(t) &= B(t) + \int_0^t \eta(X(s))dL(s) + \int_0^t K(s)ds, \\ K(t) &= K_0 + \int_0^t \eta(X(s))dL(s). \end{aligned} \tag{1.1}$$

Here $\eta(x)$ is the inward unit normal for $x \in \partial D$ and $t \to L(t)$ is a nondecreasing continuous function flat off of ∂D. By this we mean L increases only on $X^{-1}(\partial D)$.

The authors show (X, K) has a stationary distribution of $\mu \times \gamma$, where μ is the uniform distribution on D and γ is the standard Gaussian distribution on $\mathbb{R}^n$. This is interesting in part because the stationary distribution of the drift does not depend on D. Notice the marginal processes are not Markovian. This is clear, since X has a drift that is dependent on its past. When X is one dimensional and $D = [0, \infty)$ the process X is called one dimensional reflected Brownian motion with inert drift which is essentially the same as the process introduced by Knight.

In [9], Burdzy and White attempt to answer this question from the discrete point of view. Let (X, L) be a pair of processes with state space $\mathscr{L} \times \mathbb{R}^d$. Here $\mathscr{L}$ is a finite set so without loss of generality we consider $\mathscr{L} = \{1, \ldots, N\}$. Burdzy and White consider such processes where the transition rate of X to move between two states depends on L, which depends on the time X has spent on previous states. We call this class of processes $\mathcal{C}$ and describe it in more detail in the next section. The authors find necessary and sufficient conditions for such a process (X, L) to have stationary distribution $\mu \times \gamma$ where in this situation μ is the uniform measure on $\mathscr{L}$ and γ is the Gaussian distribution on $\mathbb{R}^d$. The Gaussian stationarity for the velocity term does not seem to result from a central limit theorem. The authors conjecture that one dimensional Brownian motion with inert drift can be approximated by processes in the class $\mathcal{C}$ and suggest this as one explanation for both this Gaussian behavior of the velocity and the stationary distribution of (X, L) decoupling into a product form. The first contribution of this chapter is the proof of this conjecture.

This is just one example where a process with memory has a Gaussian stationary distribution. See [23], where Gauthier studies diffusions whose drift is also dependant on the diffusions past. He shows the displacement obtains a Gaussian stationary distribution under rescaling, as time approaches infinity.

The second contribution of this chapter concerns a pair of reflected Brownian motions (X_1, X_2) separated by a Newtonian/inert particle Y whose drift is proportional to the difference of its local time of contact between X_1 and X_2. For a pair of independent Brownian motions (B_1, B_2) both adapted to a continuous filtration $\mathcal{F}_t$, while (X_1, Y, X_2, V) solves the following system with initial conditions (x_1, y, x_2, v),

$$\begin{aligned}
&dX_1 = dB_1 - dL_1,\ dX_2 = dB_2 + dL_2\\
&\frac{dY(t)}{dt} = v + K(L_2(t) - L_1(t)) =: V(t),\\
&L_i(0) = 0, L_i \text{ is nondecreasing continuous, and flat off of } \{s : X_i(s) = Y(s)\},\\
&X_1(0) = x_1 < y = Y(0) < x_2 = X_2(0),\\
&X_1(t) \geq Y(t) \geq X_2(t) \text{ for all } t \in [0, T], \text{ almost surely.}
\end{aligned}$$

See example sample paths in Figure 1. Strong existence of this system was studied in [45], unfortunately the proof contains a nontrivial gap that we complete. Moreover, we use the discrete approximation scheme of processes in class $\mathcal{C}$, described above, to find an invariant measure of (X_1, Y, X_2, V).

2 *Markov Processes with Memory*

Before introducing the class of processes $\mathcal{C}$, we introduce common facts of Poisson processes and point process with stochatic intensity. We reference [5, Chapter 2].

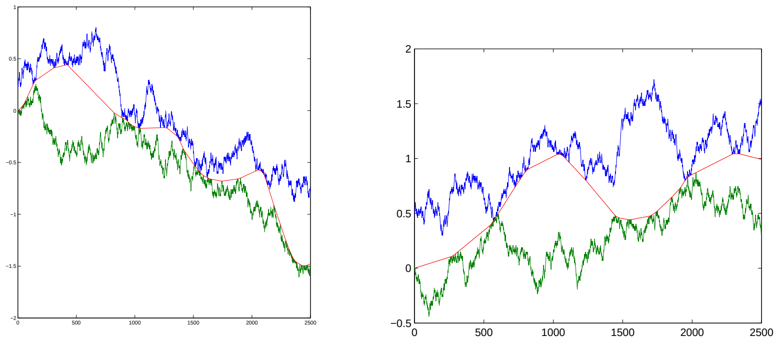

Figure 3.1: Sample path simulations of the previous system with $X_1 \leq Y \leq X_2$. Notice that X_1, X_2 have different initial values between the figures.

2.1 Nonhomogeneous Poisson processes

The Poisson distribution of mean λ is a random variables Z with probability mass function

$$\mathbb{P}(Z = n) = e^{-\lambda}\frac{\lambda^n}{n!}.$$

A homogeneous Poisson process $N(t), t \geq 0$ with state space $\mathbb{N}$ and intensity (or rate) λ is a process such that

(i) $N(0) = 0$,

(ii) N has independent increments,

(iii) The number of jumps in an interval (a, b) is Poisson r.v. with mean $\lambda(b - a)$.

Typically we denote $N(a, b]$ is the number of jumps in the interval $(a, b]$. A nonhomogeneous Poisson process with a nonnegative locally integrable rate (or intensity) function $\lambda(t)$ is defined similarly, where $\Lambda(a, b) = \int_a^b \lambda(s)ds$ and

(i) $N(0) = 0$, a.s.

(ii) N has independent increments,

(iii) N is RCLL, a.s.

(iv) $N(a,b] \overset{d}{=} \text{Poisson}(\Lambda(a,b))$.

If we let $T = \inf\{t : N(t) > 0\}$ be the first jump time, then

$$\mathbb{P}(T > t) = \mathbb{P}(N(t) = 0) = e^{-\Lambda(t)} = e^{-\int_0^t \lambda(s)ds}.$$

Lemma 2.1. *Let N_1, N_2 be two independent Poisson processes with rate functions λ_1, λ_2. Then $N_1 + N_2$ is a Poisson process with rate $\lambda_1 + \lambda_2$.*

Proof. Recall that the sum of two independent Poisson random variables of rates a, b is itself a Poisson random variable of rate $a + b$. Using this fact, the items (i)-(iv) are easily verified for $N_1 + N_2$ with a rate function of $\lambda_1 + \lambda_2$. □

Lemma 2.2. *Let $N_1(t), N_2(t), t \geq 0$ be two independent Poisson processes with rate functions λ_1, λ_2, respectively. Let T_1, T_2 be the first jump time of N_1, N_2 respectively. Then $T_1 \wedge T_2$ is the first jump time of a Poisson process with rate function $\lambda_1 + \lambda_2$.*

Proof. Clearly T_1 and T_2 are independent. Let $\Lambda_i(0,t) = \int_a^b \lambda_i(s)ds$ for $i = 1, 2$. For any $t > 0$,

$$\begin{aligned}\mathbb{P}(T_1 \wedge T_2 > t) &= \mathbb{P}(T_1 > t, T_2 > t) \\ &= e^{-\Lambda_1(0,t)} e^{-\Lambda_2(0,t)} \\ &= e^{-(\Lambda_1(0,t) + \Lambda_2(0,t))} \\ &= e^{-\int_0^t (\lambda_1(s) + \lambda_2(s))ds},\end{aligned}$$

proving the lemma. □

Lemma 2.3. *Let λ_1, λ_2 be two rate functions such that $\lambda_1(t) \leq \lambda_2(t)$ for all $t \geq 0$ and let T_1, T_2 be the first jump time of their corresponding Poisson process. Then T_1 stochastically dominates T_2.*

Proof. One could prove this simply from the explicit expression of the tail distribution. Another way is the use Lemma 2.2 above. Define $\lambda_0 = \lambda_2 - \lambda_1$ and let T_0 be the first jump of a Poisson process with rate function λ_0 independent from T_1. By the Lemma above, $T_0 \wedge T_1 \overset{d}{=} T_2$ so there is a coupling of (T_1, T_2) such that $T_1 \geq T_2$ almost surely. Hence T_1 stochastically dominates T_2. $\square$

In [5, Chapter II], Brémaud discusses the notion of point processes adapted to a filtration $\mathcal{F}_t$ whose intensity $\lambda(s)$ is not a deterministic function but rather a process adapted to $\mathcal{F}_t$ with certain conditions.

Definition 2.4. *[5, II] Let N_t be a point process adapted to the filtration $\mathcal{F}_t$ and let λ_t be a nonnegative $\mathcal{F}_t$-progressive process such that $\int_0^t \lambda_s ds < \infty$ almost surely for each $t \in [0, T]$. If for all nonnegative $\mathcal{F}_t$-predictable processes C_t,*

$$\mathbb{E}\left(\int_0^\infty C_s dN_s\right) = \mathbb{E}\left(\int_0^\infty C_s \lambda_s ds\right), \tag{2.1}$$

then we say N_t has stochastic intensity λ_t.

Remark 2.5. In the proofs of later results we will refer to point processes with a given intensity or jump/step size. By a point process of jump/step size a and (stochastic) intensity λ we mean a process aN_t where N_t is a point process with (stochastic) intensity λ. It may occur that a is negative.

In this context, we have the following extension of Lemmas 2.1 and 2.3 to processes with stochastic intensities.

Lemma 2.6. *Let N_1, N_2 be two independent point processes with stochastic intensities λ^1, λ^2 adapted to filtrations $\mathcal{F}^1, \mathcal{F}^2$ respectively. Then $N_1 + N_2$ is a point process with stochastic intensity $\lambda^1 + \lambda^2$, adapted to $\mathcal{F}_t := \sigma\left(\mathcal{F}_t^1, \mathcal{F}_t^2\right)$.*

Proof. The fact that N_1, N_2 are independent implies the two processes do not have common jumps, so that (N_1, N_2) is a multivariate point process. The result follows from [5, T15, Chapter II.2]. $\square$

Lemma 2.7. *Let N_1 be a point processes with stochastic intensity $\lambda^1 \geq \lambda$, almost surely, for some $\lambda \in (0,\infty)$. Then we can enlarge the probability space to support a Poisson point process N_2 with constant intensity λ and a point process N_3 of stochastic intensity $\lambda^3 = \lambda^1 - \lambda$ such that N_3 is independent of N_2 and $N_1 \overset{d}{=} N_2 + N_3$ almost surely.*

Remark 2.8. It is relatively clear that one can generate a Poisson point process N_2 with constant intensity λ which is independent of N_2. It is most likely the case that Lemma 2.7 could be generalized to the case when N_2 also has stochastic intensity dominated by λ^1 under additional assumptions. However, we don't require this and go through the details only in the case N_2 has constant intensity.

Proof. Enlarge the probability space to support two independent processes N_2', N_3' where N_2' is a Poisson point process of rate λ and N_3' is a point process with stochastic rate $\lambda^3 = \lambda^1 - \lambda$. By Lemma 2.6 $N_1 \overset{d}{=} N_2 + N_3$, and the processes are adapted to the filtration generated by (N_1, N_2, N_3). $\square$

2.2 Class $\mathcal{C}$ of Markov Processes with Memory

As mentioned above, Burdzy and White study continuous time Markov processes (X, L) on $\mathscr{L} \times \mathbb{R}^d$ where $\mathscr{L} = \{1, 2, \ldots, N\}$ is a finite set. For each $j \in \mathscr{L}$ we associate a vector $v_j \in \mathbb{R}^d$ and define $L_j(t) = Leb(0 < s < t : X(s) = j)$ as the time X has spent at location j until time t. We also define

$$L(t) = \sum_{j \in \mathscr{L}} v_j L_j(t)$$

as the accumulated time X spends at each location. The transition rates of X will depend on $L(t)$. More preciseley, define RCLL functions

$$a_{ij} : \mathbb{R}^d \to \mathbb{R}.$$

The functions a_{ij} are the rate functions for the Poisson process defining the transition of X from i to j. Conditional on $X(t_0) = i, L(t_0) = l$, the jump rate of X transitioning from i to j is $a_{ij}(l + [t - t_0]v_i)$ with $t \geq t_0$. In other words, for a fixed time t_0 we condition on $L(t_0) = l$

and $X(t_0) = i$, let $N_j(t)$ be a Poisson process with rate function $a_{ij}(l + [t - t_0]v_i), t \geq t_0$ such that N_j are mutually independent for $j \neq i$. Let $T_{i,j}$ be the first jump time of $N_j(t)$. By Section 2.1 above,

$$\mathbb{P}(T_{i,j} > t + t_0 | X(t_0) = i, L(t_0) = l) = \exp\left(- \int_0^t a_{ij}(l + sv_i)ds \right),$$

for all $t > 0$. Pick j' such that $T_{i,j'} = \min_{j \neq i} T_{i,j}$, and define the first transition of X after time t_0 to be location j' and occur at time $T_{i,j'}$. The pair (X, L) is a strong Markov process with generator

$$Af(j,l) = v_j \cdot \nabla_l f(j,l) + \sum_{i \neq j} a_{ji}(l)[f(i,l) - f(j,l)], \ j = 1, \dots, N, \ l \in \mathbb{R}^d.$$

Burdzy and White assume (X, L) is irreducible in the sense that there is some $\{j_0\} \times U \subset \mathscr{L} \times \mathbb{R}^d$ such that

$$\mathbb{P}((X(t), L(t)) \in \{j_0\} \times U | X(0) = i, L(0) = l) > 0, \quad \text{for all } (i,l) \in \mathscr{L} \times \mathbb{R}^d.$$

We denote $\mathcal{C}$ as the class of such processes. See Brémaud's description of a doubly-stochastic point process in [5, Chapter 2]. Here X has a stochastic intensity function adapted to right continuous filtration generated by X.

Define a distribution on $\mathscr{L} \times \mathbb{R}^d$ given by $h(j,l) = p_j g(l)$, where the p_j is the probability mass at $j \in \mathscr{L}$, and $q(l)$ is a measure on $\mathbb{R}^d$. This gives a measure π on $\mathscr{L} \times \mathbb{R}$ via $\pi(i,l) = p_i q(l)$.

Theorem 2.9. *(Burdzy and White, [9]) Consider a Markov process $(X(t), L(t))$ with values in $\mathscr{L} \times \mathbb{R}^d$. When $L_s = l$, $X(s)$ transitions from $i \to j$ with rate $a_{ij}(l)$ which is continuous with l. With π as above, $0 < s < t$ and $f \in C_c^1(\mathscr{L} \times \mathbb{R}^d)$,*

$$\mathbb{E}_\pi(f(X(t), L(t))) - \mathbb{E}_\pi(f(X(s), L(s))) = \int_s^t \sum_{j \in \mathscr{L}} \int_{\mathbb{R}^d} u_z(j,l)(A^* h)(j,l) dl dz$$

where

$$u_z(j,l) = \mathbb{E}_{(j,l)} f(X(z), L(z)), \qquad A^* h(j,l) = -p_j v_j \nabla q(l) + \sum_{i \neq j} [p_i a_{ij}(l) - p_j a_{ji}(l)] q(l).$$

Consequently, the distribution π is invariant if and only if $A^* h = 0$.

3 *Equivalent formulation of Brownian motion reflecting from Newtonian boundary*

In the first chapter of this thesis we described the triple (X, Y, V) where X is Brownian motion reflecting from the inert particle Y. We recall the setup, which is a probability space $(\Omega, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ with the filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions supporting a Brownian motion B. Take an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion $B(t)$ and a coefficient $K \geq 0$ and an initial velocity $v \in \mathbb{R}$ for the massive particle. Consider continuous $\mathcal{F}_t$ adapted processes (X, Y, V) which satisfy:

$$\begin{aligned}
&dX = dB + dL, \\
&dY = V(t)dt := (v - KL(t))dt, \\
&X(t) \geq Y(t), \text{ for all } t, \text{ almost surely,} \\
&L \text{ is nondecreasing, and is flat away from the set } \{s : X(s) = Y(s)\}.
\end{aligned} \tag{3.1}$$

By flat we mean

$$\int_{\mathbb{R}} 1_{\{z:X(z)>Y(z)\}}(s)dL(s) = 0.$$

See the first chapter for existence and uniqueness. As mentioned previously, Burdzy and White give conjectures essentially asserting the triple (X, Y, V) can be approximated by processes in the class $\mathcal{C}$. To do this we consider an equivalent formulation of (X, Y, V).

3.1 Equivalent Formulation

The equivalent formulation of the system (3.1) is given by considering the gap process $G(t) = X(t) - Y(t)$. Obviously $G \geq 0$, almost surely, and from (3.1) one can see

$$dG = dB + dL + KLdt, \tag{3.2}$$

where L is continuous nondecreasing, and flat off $U^{-1}(0)$. That is, G is a reflected diffusion whose drift is proportional to it's local time at zero. We consider a pair of processes (U, V)

adapted to $\mathcal{F}_t$ such that for fixed $K \geq 0, v \in \mathbb{R}$

$$\begin{aligned} U(t) &= B(t) + \int_0^t V(x)dx, \\ V(t) &= -v + KM^U(t), \\ M^U(t) &= \sup_{0<s<t} [-U(s)] \vee 0. \end{aligned} \tag{3.3}$$

In the system above, it is clear from the definition of M^U that $U + M^U \geq 0$ and it follows from the Skorohod lemma 1.2 that M^U is flat off the set $\{s : U(s) + M^U(s) = 0\}$. It follows that $B(t) + M^U \geq -\int_0^t V(s)ds$ and M^U is flat off of $\{s : B(t) + M^U = -\int_0^t V(s)ds\}$. Consequently

$$\left(B(t) + M^U(t), -\int_0^t V(s)ds, -V(t) \right)$$

satisfies the original equation (3.1), and one can similarly go from a solution of (3.3) to (3.1). This demonstrates the equivalence of the two systems (3.1) and (3.3).

As mentioned in the first chapter, existence of a solution to (3.1) first began by Knight in [33], and later a strong solution to a slightly more general process was attained via the employment of a Skorohod map by David White in [45] that we give below.

Theorem 3.1 (White, [45])**.** *For every $f \in C([0,T], \mathbb{R}), K \geq 0, v \in \mathbb{R}$ there is a unique pair of continuous functions (I, V) such that*

$$\begin{aligned} x(t) &:= f(t) + I(t), \\ V(t) &= v + Km(t), \\ I(t) &= \int_0^t V(s)ds, \\ m(t) &= \sup_{0<s<t} [-x(s) \vee 0]. \end{aligned} \tag{3.4}$$

Skorohod's Lemma implies that m is the unique monotonically increasing, continuous function which is flat off of the level set $\{s : x(s) + m(s) = 0\}$ such that $x + m$ is nonnegative.

4 *Discrete Approximation*

4.1 Definition of Processes

The reflected diffusion (3.2) is a process whose drift depends on the local time of the diffusion at zero. If we are to approximate this diffusion with a Markov process on a fine lattice $2^{-n}\mathbb{N}$ one would intuitively want the upward "velocity" to depend on the accumulated time spent at zero. This drift will be a Poisson process whose intensity function is stochastic and depends on the accumulated time the process spends at zero. This is exactly the setting described by the class $\mathcal{C}$ in the above subsection, where we use the same notation to describe one candidate for such a discrete process. Such a process (X_n, V_n^X) is in class $\mathcal{C}$ defined on the lattice $2^{-n}\mathbb{N} \times \mathbb{R}$ such that $v_j = 0$ for all $j \neq 0, v_0 = K2^n$. (We may the dependence on n for convenience). In other words, the rate will only depend on the accumulated time at zero. For an initial "velocity" $v \in \mathbb{R}$, we define

$$V_n^X(t) = -v + K2^n L_0(t) = -v + K2^n \cdot Leb(0 < s < t : X_n(s) = 0).$$

The rate functions $a_{ij} : \mathbb{R} \to \mathbb{R}$ are defined as

$$\begin{aligned} a_{i(i+\mathrm{sign}(l)2^{-n})}(l) &= 2^{2n} + 2^n(l) = 2^{2n} + 2^n V_n^X(t), \\ a_{i(i-\mathrm{sign}(l)2^{-n})}(l) &= 2^{2n}, \end{aligned} \tag{4.1}$$

when $l = V_n^X(t)$, except when $i = 0$ where we do not allow a downward transition. Intuitively, by Lemma 2.2, the point process X_n can be decomposed into a sum of independent processes S_n, Z_n whose rate functions sum to that of X_n.

Definition 4.1. *For a process $Q(t)$ we define $M^Q(t)$ as the signed running minimum below zero of Q. That is,*

$$M^Q(t) = \max_{0<s<t}[-Q(s)] \vee 0.$$

Consider the processes (S_n, Z_n, V_n) on $(2^{-n}\mathbb{Z})^2 \times \mathbb{R}$ where

(i) S_n is a continuous time simple random walk on $2^{-n}\mathbb{Z}$ with upward/downward jump rate 2^{2n}.

(ii) Z_n is a point process with upward (resp. downward) stochastic and adapted jump rate of $2^n|V_n|$ when $V_n > 0$ (resp. $V_n < 0$), with jump sizes of 2^{-n} (resp. -2^{-n})

(iii) We have

$$\begin{aligned} &V_n(t) = -v + K2^n \cdot Leb(0 < s < t : U_n(s) = -M_n(s)), \\ &U_n = S_n + Z_n, \\ &M_n(t) := M^{U_n}(t). \end{aligned}$$

Point processes with adapted intensity functions are discussed in [5, Chapter 2]. In our situation V_n is adapted to the right continuous filtration $\mathcal{F}_t$ generated by the pair (S_n, Z_n). With these definitions $(U_n + M_n, V_n)$ has the same law as (X_n, V_n^X) in that it is of class $\mathcal{C}$ and satisfies (4.1). To see this, first note that $U_n + M_n = S_n + Z_n + M_n$ is a nonnegative process on $2^{-n}\mathbb{N}$. By Lemma 2.2, $S_n + Z_n$ has a jump rate function of $2^{2n} + 2^n V_n(t)$ where

$$\begin{aligned} V_n(t) &= -v + K2^n \cdot Leb\{0 < s < t : U_n(s) = -M_n(s)\} \\ &= -v + K2^n \cdot Leb\{0 < s < t : U_n(s) + M_n(s) = 0\}, \end{aligned}$$

$(U_n + M_n, V_n)$ is one realization of the process (X_n, V_n^X) in class $\mathcal{C}$ given by (4.1). In the proofs of the following subsections we define

$$L_n(t) = 2^n \cdot Leb\{0 < s < t : U_n(s) + M_n(s) = 0\}.$$

4.2 Theorem Statement

Compare the equivalent formulation of Brownian motion with Inert Drift (BMID) in (3.3) to the definition of the processes (S_n, Z_n, V_n, U_n) in the previous subsection item (iii). One would expect that (S_n, V_n, Z_n, U_n) converges in some appropriate sense to $(B, V, \int_0^\cdot V, U)$. This is the main result of this chapter.

Theorem 4.2. *For $K \geq 0$ and $v \in \mathbb{R}$, which is the initial condition of $-V$, let (S_n, Z_n, V_n, U_n) be defined by (i)-(iii) in the previous subsection. Then*

$$(S_n, V_n, Z_n, U_n) \longrightarrow_d (B, V, \int_0^{\cdot} V, U)$$

in the Skorohod topology $D([0,T], \mathbb{R}^4)$, where $(B, V, \int_0^{\cdot} V, U)$ is a quadruple of continuous processes adapted to the Brownian filtration of the first coordinate B with the following holding for all $t \in [0,T]$ almost surely:

$$U(t) = B(t) + \int_0^t V(x)dx,$$
$$V(t) = -v + KM^U(t),$$

where M^U is the running minimum given in Definition 4.1.

Remark 4.3. Let $D([0,T], \mathbb{R})$ denote the space of RCLL paths equipped with the Skorohod metric d [15, Chapter 3 Section 5]. If a process W_n with paths in $D([0,T], \mathbb{R})$ converges weakly to W then according to the Skorohod representation, [15, Theorem 3.1.8], we can place W_n, W on the same probability space such that

$$d(W_n, W) \longrightarrow 0,$$

almost surely. If the limiting process W is continuous almost surely, then we also have

$$\|W_n - W\|_{[0,T]} := \sup_{0<s<T} |W_n(s) - W(s)| \longrightarrow 0,$$

almost surely. See [15, Chapter 3 Section 5] and [15, Chapter 3 Section 10] (Ethier and Kurtz), and [4, Chapter 3] (Billingsley).

4.3 Proof of Theorem 4.2

The rest of this section will be devoted to the proof of Theorem 4.2. We split the proof into a number of lemmata. The first of which gives tightness of the sequence.

Lemma 4.4. *The collection of processes* $\{(S_n, Z_n, V_n) : n \in \mathbb{N}\}$ *is tight in* $D([0,T], \mathbb{R}^2) \times C[0,T] \subset D([0,T], \mathbb{R}^3)$. *Because* $U_n = S_n + Z_n$, *it follows that* $\{(S_n, V_n, Z_n, U_n) : n \in \mathbb{N}\}$ *is also tight in* $D([0,T], \mathbb{R}) \times C[0,T] \times D([0,T], \mathbb{R}^2)$. *Furthermore, all limiting processes are continuous.*

We prove Lemma 4.4 at the end of the section. Assuming Lemma 4.4 holds, it remains to show there is a unique limit.

Lemma 4.5. *Consider a subsequence* n_k *with processes* $(S_{n_k}, Z_{n_k}, V_{n_k}, U_{n_k})$ *converging to* (S, Z, V, U) *in* $D([0,T], \mathbb{R}^4)$. *Then* (S, Z, V, U) *is continuous and satisfies the equivalent formulation for BMID given in* (3.3). *That is,*

(i) $U(t) = S(t) + Z(t)$,

(ii) $S(t)$ *is a Brownian motion,*

(iii) $V(t) = KL(t) - v$, *where* $L(t) = \max_{0<s<t}[-U(s)] \vee 0$,

(iv) $Z(t) = \int_0^t V(s)ds$.

Remark 4.6. Since this equivalent formulation of BMID is unique in law, Lemma 4.4 characterizes the subsequential limits of (S_n, Z_n, V_n). See the first chapter and [45] where existence (and uniqueness) of such a system is proved. Consequently Lemma 4.5 shows convergence of the entire sequence to this equivalent formulation of BMID.

4.4 Proof of Lemma 4.5

We separately prove items (i)-(iv) given in 4.5. The proof of (iii) was inspired by the proof of Lévy's theorem given in [37, Chapter 6] where the authors essentially note the equivalence of the processes $(U_n + M_n, V_n)$ and (X_n, V_n^X) described in subsection 4.1 above for the case $K = 0$. Lévy's theorem is the statement that $(L, |B|)$ and $(M^B, B + M^B)$, yield the same distribution on $C([0,T],\mathbb{R}^2)$. Here L is the local time of $|B|$ at zero. See [30, Chapter 3.6] for detailed statement.

Proof of (i):

This follows trivially from the definition of $U_n = S_n + Z_n$.

Proof of (ii):

Recall $S_n(t)$ is a continuous time scaled simple random walk. Since S_{n_k} converges to S, S is a Brownian motion by Donsker's theorem.

Proof of (iii):

We give a brief outline of the proof using heuristics. Recall $L_n = 2^n \cdot Leb(0 < s < t : U_n(s) = -M_n(s))$ and $V_n = -v + KL_n$. Each time M_n increases, U_n will make approximately a Geometric(1/2) number of visits to this new minimum value before M_n increases again. Also, U_n will spend approximately an Exp(2^{2n}) amount of time at each one of these visits. Therefore L_n, which is the total amount of time U_n spends on $-M_n$ scaled by 2^n, is approximately

$$\sum_{i=1}^{2^n M_n} 2^n Exp(2^{2n}) = \sum_{i=1}^{2^n M_n} Exp(2^n),$$

where $Exp(2^n)$ indicates independent exponential random variables. If you suppose this sum is concentrated around its expectation, then L_n is approximately

$$\sum_{i=1}^{2^n M_n} 2^{-n} = M_n.$$

If M_n converges almost surely to a process M in $C[0,T]$, then one expects L_n to converge to M as well.

We now begin the formal proof by making the above explanation rigorous. By tightness, and without loss of generality, assume $L_{n_k} \longrightarrow L$ and $U_{n_k} \to U$ almost surely in the uniform norm of continuous functions. See Remark 4.3. We will use a localization argument to show that $L = M^U$ almost surely. For a positive constant $C > |v|$, define

$$T_C^{(n_k)} = \inf\{t > 0 : L_{n_k} > C\}.$$

For each n_k, consider a modifincation of $(S_{n_k}, Z_{n_k}, U_{n_k})$, denoted $(S_{n_k}^C, Z_{n_k}^C, U_{n_k}^C)$, solving the system (i)-(iii) in subsection 4.1 but replacing (iii) with

$$\begin{aligned}
&V_n^C(t) = -v + K[2^n Leb(0 < s < t : U_n^C(s) = -M_n^C(s)) \wedge C],\\
&U_n^C = S_n^C + Z_n^C,\\
&M_n^C(t) := M^{U_n^C}(t).
\end{aligned}$$

Heuristically speaking we are stopping L_n when it reaches C while keeping the other dynamics the same. Note that $(S_{n_k}^C, Z_{n_k}^C, U_{n_k}^C)$ is the same as $(S_{n_k}, Z_{n_k}, U_{n_k})$ on the interval $[0, T_C^{(n)}]$, while on $[T_C^{(n)}, \infty)$ we stop L_n^C at C. Here L_n^C Fix $m > 0$ and let

$$\tau_{m,i}^{(n_k)} = \inf\{t > \tau_{m,i}^{(n_k)} : U_{n_k}^C(t) = -m = -M_{n_k}^C\},\ \tau_{m,0}^{(n_k)} = 0$$

be the consecutive times $U_{n_k}^C$ arrives at $-m$ when $M_{n_k}^C = m$. Let

$$z_j^+ = \inf\{s > 0 : U_{n_k}^C(\tau_{m,j}^{(n_k)} + s) > U_{n_k}^C(\tau_{m,j}^{(n_k)})\}$$

and

$$z_j^- = \inf\{s > 0 : U_{n_k}^C(\tau_{m,j}^{(n_k)} + s) < U_{n_k}^C(\tau_{m,j}^{(n_k)})\}$$

be the amount of time until the next positive, respectively negative, jump of $U_{n_k}^C$ after $\tau_{m,j}^{(n_k)}$. Then $z_j^+ \wedge z_j^-$ is the time $U_{n_k}^C$ spends on $-m$ during it's j^{th} visit to m when $M_{n_k}^C = m$. Since $|L_n^C| \leq C$, the stochastic intensity of $Z_{n_k}^C$ is $2^{n_k}|V_{n_k}^C|$ is bounded below by $2^{2n_k} - v$ and above by $2^{2n_k} + C2^{n_k}$. We set $v = 0$ in the remaining computations for convenience, so the positive jump times of $U_{n_k}^C = S_{n_k}^C + Z_{n_k}^C$ have intensity $2^{2n_k} + 2^{n_k}|V_{n_k}^C|$ and the negative jump times arrive with intensity 2^{2n_k}. In other words, $z_j^- \overset{d}{=} \text{Exp}(2^{2n_k})$.

By Lemma 2.3 we may enlarge the probability space to contain two independent sequences of i.i.d. exponential $\{\nu_i\}_{i\in\mathbb{N}}$, $\{e_i\}_{i\in\mathbb{N}}$ that are also independent of z_j^- , with rates 2^{2n_k} and

$C2^{n_k}$ respectively, and where

$$\nu_j \wedge e_j \leq z_j^+ \leq \nu_j,$$

almost surely. Consequently,

$$z_j^- \wedge \nu_j \wedge e_i \leq z_j^- \wedge z_j^+ \leq z_j^- \wedge \nu_j, \text{ almost surely.} \tag{4.2}$$

Define

$$A_j := 1_{\{z_j^- \wedge \nu_j = \nu_j\}}, \; B_j := 1_{\{z_j^- \wedge z_j^+ = z_j^+\}}, \; C_j := 1_{\{z_j^- \wedge \nu_j \wedge e_j = \nu_j \wedge e_j\}}.$$

By construction A_j, B_j, C_j are Bernoulli random variables and are coupled so that

$$A_j \leq B_j \leq C_j, \text{ almost surely.}$$

The definition of A_j, C_j depend on n_k, which is hidden from notation. While the sequence (B_j) is not an i.i.d. sequence, and is not a sequence of independent random variables since the jump rate changes with time, both (A_j) and (B_j) are i.i.d. sequences of Bernoulli(1/2), Bernoulli($[1 + C2^{-n_k}]/[2 + C2^{-n_k}]$), respectively. Consider the first index j such that A_j (resp. B_j) is zero. For each $0 < i2^{-n_k} < m$ denote Q_i as the number of visits to $-i2^{-n_k}$ by U_{n_k} while $M_{n_k}^C = i2^{-n_k}$. This is the number of visits $U_{n_k}^C$ makes to $-i2^{-n_k}$ when $-i2^{-n_k}$ is the running minimum. We will sandwich Q_i above and below geometric random variables. Clearly,

$$Q_i = \inf\{j \geq 1 : B_j = 0\}$$

the initial sequence of $B_1, \ldots, B_{Q_i - 1}$ represents the number of times $U_{n_k}^C$ jumps up after arriving to it's (signed) running minimum. But for each of these upward jumps $U_{n_k}^C$ must make another visit to the signed running minimum $-i2^{-n_k}$. Consider

$$W_i = \inf\{j \geq 1 : A_j = 0\}, V_i = \inf\{j \geq 1 : C_j = 0\}.$$

Because $(A_j), (C_j)$ are each i.i.d. sequences of Bernoulli random variables, W_i, V_i are geometric random variables and we have

$$\begin{aligned}&W_i \leq Q_i \leq V_i, \text{ almost surely,}\\&\mathbb{P}(W_i = k) = \Big(\frac{1}{2}\Big)^k,\\&\mathbb{P}(V_i = k) = \Big(\frac{1}{2 + C/2^{n_k}}\Big)\Big(\frac{1 + C/2^{n_k}}{2 + C/2^{n_k}}\Big)^{k-1}.\end{aligned}$$

That is, W_i, V_i are geometrically distributed with parameters $1/2$, $[1 + C2^{-n_k}]/[2 + C2^{-n_k}]$ respectively.

Notice that $M^C_{n_k}$ visits all sites between 0 and m up until time $\tau^{(n_k)}_m$. (Also notice that S_n is identical to S^C_n since it is a continuous time random walk not depending on L_n). Here $M^C_{n_k} = M^{U^C_{n_k}}$. Since the size of each step is 2^{-n_k} this means that until time $\tau^{(n_k)}_m$, $M^C_{n_k}$ has visited between $2^{n_k}m - 1$ and $2^{n_k}m + 1$ sites.

Let T_i be the time that $\{M_{n_k}(s) : s \in [0, \tau^{(n_k)}_{m,1}]\}$ spends at that vertex $i2^{-n_k}$, where $0 \leq i2^{-n_k} \leq m$. Since $|L^C_n| \leq C$, the stochastic intensity of $Z^C_{n_k}$ is $2^{n_k}|V^C_{n_k}|$, which is bounded below by $2^{2n_k} - v$ and above by $2^{2n_k} + C2^{n_k}$. We set $v = 0$ in the remaining computations for convenience, so the jump times of $U^C_{n_k} = S^C_{n_k} + Z^C_{n_k}$ have intensity $2^{2n_k} + 2^{n_k}|V^C_{n_k}|$, which is bounded below by 2^{2n_k} and above by $2^{2n_k} + C2^{n_k}$. By Lemma definition of ν_j, e_j, z_j

$$\Phi'^i_{n_k} = \sum_{j=1}^{W_i} z^-_{j,i} \wedge \nu_{j,i} \wedge e_{j,i} \leq T_i \leq \sum_{j=1}^{W_i} z^-_{j,i} \wedge \nu_{j,i} + \sum_{j=W_i}^{V_i} z^-_{j,i} \wedge \nu_{j,i} =: \Phi^i_{n_k} + \sum_{j=W_i}^{V_i} z^-_{j,i} \wedge \nu_{j,i},$$

almost surely. We add the additional subscripts j, i are added to make it clear the collection of random variables depend on i, e.g. $z^-_{j,i}$ is a double array of i.i.d. $\mathrm{Exp}(2^{2n_k})$ random variables. By Lemma 4.15, $z^-_j \wedge \nu_j$, is an $\mathrm{Exp}(2^{2n+1})$ random variable independent from A_j. Because $z^-_j \wedge \nu_j \wedge e_j$ is a measurable function of $z^-_j \wedge \nu_j$ and e_j, both of which are independent from A_j, hence $z^-_j \wedge \nu_j \wedge e_j$ is independent from A_j as well. Consequently $z^-_j \wedge \nu_j \wedge e_j$ is independent from W_j. Since we defined

$$L^C_{n_k}(s) = \sum_{i=1}^{2^{n_k} M^C_{n_k}(s)} 2^{n_k} T_i,$$

we have

$$\begin{aligned}
&\sum_{i=1}^{2^{n_k}M^C_{n_k}(s)-1} 2^{n_k}\Phi'^{i}_{n_k} \le L^C_{n_k}(s)\\
&\le \sum_{i=1}^{2^{n_k}M^C_{n_k}(s)+1} 2^{n_k}\Phi^{i}_{n_k} + \sum_{i=1}^{2^{n_k}M^C_{n_k}(s)+1} \sum_{j=W_i}^{V_i} z^-_{j,i}\wedge \nu_{j,i}\\
&=: R_{n_k}(s).
\end{aligned} \tag{4.3}$$

The sum of a Geometric(p) number of independent exponentials of rate λ is exponential with rate $p\lambda$. Since W_i, V_i are geometric and independent of μ'_i, μ_i, it follows that $\Phi'^{i}_{n_k}$ is distributed as an exponential of rate $(2^{2n_k+1}+C2^{n_k})/2$, and similarly $\Phi^{i}_{n_k}$ has exponential rate 2^{2n_k}. We think of $2^{n_k}\Phi'^{i}_{n_k}$ as an exponential random variable with rate approximately 2^{n_k}, while in fact $2^{n_k}\Phi^{i}_{n_k}$ is an exponential with rate exactly 2^{2n_k}. Our assumption that $U^C_{n_k}(\cdot)$ converges uniformly on $[0,T]$ to a process U^C, almost surely, implies have $M^C_{n_k}(\cdot)$ converges uniformly on $[0,T]$ to M^{U^C}. We will show that the left hand and right hand sides of (4.3) converges almost surely to $M^{U^C}(s)$ for each fixed $s\in[0,T]$. We go through the details for the left hand side, and the right hand is similar. Notice that

$$\begin{aligned}
&\left|\sum_{i=1}^{2^{n_k}M^C_{n_k}(s)} 2^{n_k}\Phi'^{i}_{n_k} - \sum_{i=1}^{2^{n_k}M^{U^C}(s)} 2^{n_k}\Phi'^{i}_{n_k}\right|\\
&\le \sum_{i=2^{n_k}[M^C_{n_k}(s)\wedge M^{U^C}(s)]}^{2^{n_k}[M^C_{n_k}(s)\vee M^{U^C}(s)]} 2^{n_k}\Phi'^{i}_{n_k} \le \frac{1}{2^{n_k}} \sum_{i=1}^{2^{n_k}|M^C_{n_k}(s)-M^{U^C}(s)|} e_i,
\end{aligned} \tag{4.4}$$

almost surely, where e_i are i.i.d. exp(1) that are independent from $M^C_{n_k}$. The last inequality comes from Lemma 2.3 and the fact that $2^{n_k}\Phi'^{i}_{n_k} \overset{d}{=} \exp(2^{n_k}+C/2)$. Since $|M^C_{n_k}(s)-M^{U^C}(s)|\to 0$, almost surely, the strong law of large numbers implies that

$$\frac{1}{2^{n_k}} \sum_{i=1}^{2^{n_k}|M^C_{n_k}(s)-M^{U^C}(s)|} e_i \longrightarrow 0, \text{ almost surely.}$$

We can express $2^{n_k}\Phi'^{i}_{n_k}$ as $2^{-n_k}u_i^k$ where $u_i \overset{d}{=} \exp(1 + C2^{-(n_k+1)})$, and so

$$\sum_{i=1}^{2^{n_k}M^{U^C}(s)} 2^{n_k}\Phi'^{i}_{n_k} = \frac{1}{2^{n_k}} \sum_{i=1}^{2^{n_k}M^{U^C}(s)} u_i^k.$$

Conditional on $M^{U^C}(s)$ we compute

$$\begin{aligned} \mathrm{Var}\left(\frac{1}{2^{n_k}} \sum_{i=1}^{2^{n_k}M^{U^C}(s)} u_i^k \middle| M^{U^C}(s)\right) &= \frac{1}{2^{2n_k}} 2^{n_k} M^{U^C}(s) \frac{1}{(1 + C2^{-(n_k+1)})^2} \\ &= \frac{M^{U^C}(s)}{2^{n_k} + C + C^2 2^{-(n_k+2)}}, \end{aligned}$$

which approaches zero. The expectation conditional on $M^{U^C}(s)$ is

$$\mathbb{E}\left(\frac{1}{2^{n_k}} \sum_{i=1}^{2^{n_k}M^{U^C}(s)} u_i^k \middle| M^{U^C}(s)\right) = M^{U^C}(s) \frac{1}{1 + C2^{-(n_k+1)}}.$$

Consequently

$$\sum_{i=1}^{2^{n_k}M^{U^C}(s)} 2^{n_k}\Phi'^{i}_{n_k} = \frac{1}{2^{n_k}} \sum_{i=1}^{2^{n_k}M^{U^C}(s)} u_i^k \longrightarrow M^{U^C}(s),$$

almost surely, and by (4.4),

$$\sum_{i=1}^{2^{n_k}M^{C}_{n_k}(s)} 2^{n_k}\Phi'^{i}_{n_k} \longrightarrow M^{U^C}(s), \tag{4.5}$$

in probability. Therefore we can find a subsequence n'_k where the above converges almost surely. Similarly one can show

$$\sum_{i=1}^{2^{n_k}M^{C}_{n_k}(s)} 2^{n_k}\Phi^{i}_{n_k} \longrightarrow M^{U^C}(s), \tag{4.6}$$

in probability. The right side of (4.3) will then converges to M^{U^C} after showing

$$\sum_{i=1}^{2^{n_k}M^{C}_{n_k}(s)+1} \sum_{j=W_i}^{V_i} z^-_{j,i} \wedge \nu_{j,i}$$

converges to zero in probability. But this follows from two applications of Wald's equation, the second of which uses the filtration generated (for fixed i) by $z^-_{j,i}, \nu_{j,i}$ and $e_{j,i}$ to compute

$$\begin{aligned}\mathbb{E}\left(\sum_{j=W_i}^{V_i} z^-_{j,i} \wedge \nu_{j,i}\right) &= \mathbb{E}(V_i - W_i)\mathbb{E}(z^-_{j,i} \wedge \nu_{j,i}) \\ &= \left(2 - \frac{2 + C2^{-n_k}}{1 + C2^{-n_k}}\right) \cdot \frac{1}{2^{2n_k+1} + C2^{n_k}} =: a_n,\end{aligned}$$

which clearly approaches zero. Now, using Lemma 4.14 the moment bound for the number of sums in Wald's equation is satisfied, we compute

$$\mathbb{E}\left(\sum_{i=1}^{2^{n_k} M^C_{n_k}(s)+1} \sum_{j=W_i}^{V_i} z^-_{j,i} \wedge \nu_{j,i}\right) = \mathbb{E}(2^{n_k} M^C_{n_k}) \cdot a_n,$$

which also approaches zero. Consequently

$$\sum_{i=1}^{2^{n_k} M^C_{n_k}(s)+1} \sum_{j=W_i}^{V_i} z^-_{j,i} \wedge \nu_{j,i}$$

converges to zero in probability and as mentioned, and therefore R_{n_k} converges to M^{U^C}.

Consequently we can find some common subsequence n'_k where (4.5) and (4.6) both occur simultaneously, almost surely for fixed s. We relabel n_k as n'_k. By (4.3) and the squeeze theorem $L^C_{n_k}(s) = L_{n_k}(T^{(n_k)}_C \wedge s)$ converges to $M^{U^C}(s)$ for each $s \in [0, T]$, almost surely. By a Cantor diagonalization one can find a subsequence n_k such that $L^C_{n_k}(s) \longrightarrow M^{U^C}(s)$ for each rational $s \in [0, T]$. Since we assumed $L^C_{n_k}$ converges uniformly to some continuous process L^C, it follows that $L^C = M^{U^C}$ almost surely. We will now extend this to show the unlocalized processes L is equal to M^U almost surely. Notice that for any $\epsilon > 0$,

$$\sup_{0<s<T} |L_{n_k}(s) - M^U(s)| \longrightarrow 0$$

holds almost surely on the event $\{T_{C-\epsilon} > T\}$ where

$$T_C = \inf\{s > 0 : M^U(s) > C\}$$

is the first time the limit function M^U passes $C - \epsilon$. To see this, recall the dynamics of $(S^C_{n_k}, Z^C_{n_k}, U^C_{n_k}) \overset{a.s.}{=} (S_{n_k}, Z_{n_k}, U_{n_k})$ on $[0, T^C_{n_k}]$. Since $L_{n_k} \longrightarrow L$ by our assumption of tightness and L, L_{n_k} are nondecreasing, we know that $T_{C-\epsilon} \leq \limsup_{n_k} T_{n_k}^{C-\epsilon}$. Therefore $M^{U^C} := \lim_{n_k} L_{n_k}^C = M^U$ on $[0, T_{C-\epsilon} \wedge T] \subset [0, \limsup_{n_k} T_{n_k}^C]$. As C approaches infinity, we see that $M^U = \lim_{n_k} L_{n_k}$ uniformly on $[0, T]$, almost surely. This completes the proof of (iii).

Proof of (iv):

As in the other proofs, Remark 4.3 and Lemma 4.4 give a subsequence $(S_{n_m}, Z_{n_m}, V_{n_m}, L_{n_m})$ such that

$$(S_{n_m}, Z_{n_m}, V_{n_m}, L_{n_m}) \longrightarrow (S, Z, V, L) \tag{4.7}$$

in the uniform norm on $C([0,T])$; we have $U_{n_m} \to U$ on $C([0,T])$ as well. In the previous proof we showed $L(t) \overset{a.s}{=} M^U(t)$ while here we wish to show

$$Z(t) = \int_0^t V(x)dx, \text{ for } t \in [0,T]. \tag{4.8}$$

where $V = KM^U - v$. We take $v = 0$ for the time being and reduce to this case using a strong Markov property. It suffices to demonstrate

$$Z_{n'_m}(s) \xrightarrow{a.s.} \int_0^s V(x)dx \tag{4.9}$$

for fixed $s \in [0,T]$, for some subsequence n'_m of n_m. This implies the finite dimensional distribution of $Z(\cdot)$ are the same as those of $\int_0^{\cdot} M^U(s)ds$. The two processes will then agree on $[0,T]$, almost surely, because both processes are continuous. For a given n,

$$\widetilde{Z}_n(s) := 2^n Z_n(s)$$

counts the number of jumps of Z_n by time s. Alternatively this counts the number of inter arrival times $\{u_k : k \geq 1\}$ between jumps given by the process Z_n. (We hide the dependence on n for convenience).

Let

$$\tau_C^{(n)} = \inf\{s > 0 : V_n(s) > C\} \text{ for a given } C > 0,$$

$$\overline{\alpha}_k = \sup_{s \in [u_k, u_{k+1}]} V_n(s), \ \underline{\alpha}_k = \inf_{s \in [u_k, u_{k+1}]} V_n(s).$$

Assume for the time being that for every fixed $\delta > 0$, with aribtrarily high probability

$$\sup\{(u_{i+1} - u_i) : \widetilde{Z}_{n_m}(\delta \wedge s) \leq i \leq \widetilde{Z}_{n_m}(s)\} \longrightarrow 0. \tag{4.10}$$

We use the gap times $u_{k+1} - u_k$ as the time step in a Riemann sum approximation of the integral in (4.9). By the definition of $\overline{\alpha}_k, \underline{\alpha_k}$ and the inert clock, one has a sequence μ_k of i.i.d. $\text{Exp}(2^n)$ random variables such that

$$\underline{\alpha}_k(u_{k+1} - u_k) \leq \mu_k \leq \overline{\alpha}_k(u_{k+1} - u_k),$$

therefore

$$\sum_{k=\widetilde{Z}_{n_m}(\delta\wedge s)}^{\widetilde{Z}_{n_m}(s)} \underline{\alpha}_k(u_{k+1} - u_k) \leq \sum_{k=\widetilde{Z}_{n_m}(\delta\wedge s)}^{\widetilde{Z}_{n_m}(s)} \mu_k \leq \sum_{k=\widetilde{Z}_{n_m}(\delta\wedge s)}^{\widetilde{Z}_{n_m}(s)} \overline{\alpha}_k(u_{k+1} - u_k) \tag{4.11}$$

where we define the left and right sums to be zero should the set of such indices $\widetilde{Z}_{n_m}(\delta \wedge s) \leq k \leq \widetilde{Z}_{n_m}(s)$ be empty.

From (4.10) together with (4.7) and Riemann integrability of the limiting function gives

$$\lim_{n'_m\to\infty} \sum_{k=\widetilde{Z}_{n_m}(\delta\wedge s)}^{\widetilde{Z}_{n_m}(s)} \underline{\alpha}_k(u_{k+1} - u_k) = \int_{\delta\wedge s}^{s} V(x)dx = \lim_{n'_m\to\infty} \sum_{k=\widetilde{Z}_{n_m}(\delta\wedge s)}^{\widetilde{Z}_{n_m}(s)} \overline{\alpha}_k(u_{k+1} - u_k) \tag{4.12}$$

with high probability, where convergence holds uniformly on $[0, T]$. By the squeeze theorem

$$\sum_{k=\widetilde{Z}_{n_m}(\delta\wedge s)}^{\widetilde{Z}_{n_m}(s)} \mu_k \longrightarrow \int_{\delta\wedge s}^{s} V(x)dx \tag{4.13}$$

with high probability as well. Since the μ_k are i.i.d. exponential of rate 2^{n_m} and $\widetilde{Z}_{n_m}(s) = 2^{n_m} Z_{n_m}(s)$ with $Z_{n_m}(\cdot) \to Z(\cdot)$ almost surely, the law of large numbers implies

$$\sum_{k=\widetilde{Z}_{n_m}(\delta\wedge s)}^{\widetilde{Z}_{n_m}(s)} \mu_k \xrightarrow{a.s.} Z(s) - Z(\delta \wedge s),$$

for each $s \in [0, T]$. Therefore

$$Z(s) - Z(\delta \wedge s) = \int_{\delta\wedge s}^{s} V(x)dx \tag{4.14}$$

for each s in $[0,T]$, with arbitrarily high probability. Hence (4.14) occurs with probability one. Since $Z(\delta) \to 0$ almost surely and $\int_0^\delta V(x)dx \to 0$, as $\delta \to 0$, this gives

$$Z(s) = \int_0^s V(x)dx$$

as wanted.

To demonstrate (4.10), recall the jump process Z_n determining its interarrival times $u_{i+1} - u_i$ has intensity process bounded below by ϵ/K on the interval $[\tau_\epsilon^{(n_k)}, \infty)$. Heuristically, the intensity cannot be too small so the interarrival times are not too large. By Lemma 2.3 there exists an i.i.d. sequence v_i of exponentials with rate $\mu = \epsilon/K$ and which stochastically dominate $u_{i+1} - u_i$. As $V_{n_m} \to V$ almost surely, and $v \geq 0$ so that V_{n_m} and V are monotone, we have

$$\limsup_{n_m \to \infty} \tau_\epsilon^{(n_m)} \leq T_\epsilon := \inf\{s > 0 : V(s) > \epsilon\}, \tag{4.15}$$

almost surely. We first show $\sup\{u_{i+1} - u_i : \widetilde{Z}_{n_m}(\delta) \leq i \leq \widetilde{Z}_{n_m}(t)\} \xrightarrow{P} 0$. For $0 < \eta < 1, C > 0$,

$$\begin{aligned}
&\mathbb{P}(\sup\{(u_{i+1} - u_i) : \widetilde{Z}_{n_m}(\delta) \leq i \leq \widetilde{Z}_{n_m}(t)\} > \eta) \\
&\leq \mathbb{P}(\sup\{(u_{i+1} - u_i) : \widetilde{Z}_{n_m}(\tau_\epsilon^{(n)}) \leq i \leq \widetilde{Z}_{n_m}(t)\} > \eta, \tau_\epsilon^{(n_m)} \leq \delta) + \mathbb{P}(\tau_\epsilon^{(n_m)} > \delta) \\
&\leq \mathbb{P}(\sup\{v_i : 1 \leq i \leq \widetilde{Z}_{n_m}(t)\} > \eta) + \mathbb{P}(\tau_\epsilon^{(n_m)} > \delta) \\
&\leq \mathbb{P}(\sup\{v_i, 1 \leq i \leq C2^n\} > \eta, \widetilde{Z}_{n_m}(t) \leq C2^n) + \mathbb{P}(\widetilde{Z}_{n_m}(t) > C2^n) + \mathbb{P}(\tau_\epsilon^{(n_m)} > \delta) \\
&\leq \mathbb{P}(v_i > \eta : \text{ some } 1 \leq i \leq C2^n) + \mathbb{P}(Z_n(t) > C) + \mathbb{P}(\tau_\epsilon^{(n_m)} > \delta) \\
&\leq C2^n \mathbb{P}(v_i > \eta) + \mathbb{P}(\tau_\epsilon^{(n_m)} > \delta) \\
&\leq C2^n \exp(-\eta\mu 2^n) + \mathbb{P}(Z_n(t) > C) + \mathbb{P}(\tau_\epsilon^{(n_m)} > \delta).
\end{aligned}$$

Taking lim sup with respect to n_m on both sides and applying (4.15) and the assumption that $Z_n \to Z$ almost surely,

$$\limsup_{n \to \infty} \mathbb{P}(\sup\{(u_{i+1} - u_i) : \widetilde{Z}_{n_m}(\delta) \leq i \leq \widetilde{Z}_{n_m}(t)\} > \eta) \leq \mathbb{P}(Z(t) > C) + \mathbb{P}(T_\epsilon > \delta).$$

Since $C, \epsilon > 0$ are arbitrary

$$\limsup_{n \to \infty} \mathbb{P}(\sup\{(u_{i+1} - u_i)\widetilde{Z}_{n_m}(\delta) \leq i \leq \widetilde{Z}_{n_m}(t)\} > \eta) = 0,$$

for every fixed $\delta > 0$. Consequently there exists a subsequence n'_m where (4.10) holds which is sufficient for our proof of (iv).

To show the case $v < 0$ reduces to the above, notice that

$$T_{-3\epsilon/2} \leq \liminf_{n_m \to \infty} \tau_{-\epsilon}^{(n_m)} \leq \limsup_{n_m \to \infty} \tau_{\epsilon}^{(n_m)} \leq T_{3\epsilon/2}.$$

For almost each ω in our probability space there is an $N(\omega)$ such that V_{n_m} is monotone on the intervals $[0, \liminf_{n_m \to \infty} \tau_{-\epsilon}^{(n_m)}]$ and $[\limsup_{n_m \to \infty} \tau_{\epsilon}^{(n_m)}, T]$ for all $n_m \geq N(\omega)$. With this fact we can apply the above proof to show

$$Z(t) = \int_0^t V(x)dx \text{ for } t \in [0, T_{-3\epsilon/2} \wedge T] \cup [T_{3\epsilon/2} \wedge T, T].$$

In addition to this,

$$\int_{T_{-3\epsilon/2} \wedge T}^{T_{3\epsilon/2} \wedge T} |V(x)|dx \leq (3\epsilon/2)T, \text{ almost surely,}$$

which goes to zero as $\epsilon \to 0$. It follows that $Z(s) = \int_0^s V(x)dx$ for $s \in [0, T]$ in the case $v < 0$ as well. □

4.5 Lemma 4.4: Tightness of (S_n, Z_n, V_n)

Recall that our process (S_n, Z_n, V_n) are in $D([0, T], \mathbb{R}^3)$, the space of RCLL paths with the Skorohod topology defined by the product metric $d \times d \times d$ where d is the Skorohod metric, see [4]. The following definition is inspired from notes by Ruth Williams on tightness of stochastic process.

Definition 4.7. *A sequence of processes $\{X_i : i \in \mathbb{N}\}$ in $D([0, T], \mathbb{R})$ is said to be C-tight if $\{X_i : i \in \mathbb{N}\}$ is tight on $D([0, T], \mathbb{R})$ and all limiting processes are continuous.*

See Remark 4.3. The proof that (S_n, Z_n, V_n) is tight is broken into a number of lemmas. Recall that $L_n(t) = 2^n Leb(0 < s < t : U_n(s) = -M_n(s))$ where $U_n = S_n + Z_n$ and $M_n = M^{U_n}$, and that $V_n = -v + KL_n$.

Definition 4.8. *For $f \in D([0,T],\mathbb{R})$, let*

$$\omega(f,\delta) := \sup_{\substack{0<s<t<T \\ |t-s|<\delta}} |f(t)-f(s)|.$$

be the modulus of continuity of f.

Recall that $\|f\|_{[a,b]} = \sup_{a<x<b}|f(x)|$.

Lemma 4.9. *[29, Proposition VI.3.26] A sequence of processes X_n in $D([0,T],\mathbb{R})$ is C-tight if and only if for every $\epsilon > 0$,*

(i) $\lim_{C\to\infty} \limsup_{n\to\infty} \mathbb{P}(\|X_n\|_{[0,T]} \geq C) = 0$

(ii) $\lim_{\delta\to 0} \limsup_{n\to\infty} \mathbb{P}(\omega(X_n,\delta) > \epsilon) = 0.$

Remark 4.10. Notice that (i) follows from (ii) and $\lim_{C\to\infty} \limsup_{n\to\infty} \mathbb{P}(|X_n(0)| > C) = 0$. to see this, take $\delta = 1$ and $\epsilon = C/2$ so that by definition of the modulus of continuity

$$\|X_n\|_{[0,T]} \leq |X_n(0)| + \sum_{i=1}^{T} \omega(X_n, 1).$$

Consequently, the triangle inequality gives

$$\begin{aligned}\mathbb{P}(\|X_n\|_{[0,T]} > C) &\leq \mathbb{P}(|X_n(0)| + \omega(X_n,1) > C) \\ &\leq \mathbb{P}(|X_n(0)| > C/2) + \mathbb{P}(\omega(X_n,1) > C/2),\end{aligned}$$

from which it is clear that (ii) and $\lim_{C\to\infty} \limsup_{n\to\infty} \mathbb{P}(|X_n(0)| > C) = 0$ imply (i).

Lemma 4.11. *Assume that the sequences $(X_n), (X_n'), (X_n'')$ are in $D([0,T],\mathbb{R})$ and*

$$X_n'(t) - X_n'(s) \leq X_n(t) - X_n(s) \leq X_n''(t) - X_n''(s), \quad 0 \leq s \leq t.$$

almost surely. If both (X_n') and (X_n'') are C-tight, then (X_n) is also C-tight.

Proof. For any $C > 0$, the triangle inequality gives

$$\mathbb{P}(\|X_n\|_{[0,T]} > C) \leq \mathbb{P}(\|X'_n\|_{[0,T]} > C/2) + \mathbb{P}(\|X''_n\|_{[0,T]} > C/2),$$

which verifies condition (i) in the statement of Lemma 4.9 by taking $\lim_{C\to\infty} \limsup_{n\to\infty}$ of both sides. Similarly, for every $\delta, \epsilon > 0$,

$$\mathbb{P}(\omega(X_n, \delta) > \epsilon) \leq \mathbb{P}(\omega(X'_n, \delta) > \epsilon/2) + \mathbb{P}(\omega(X''_n, \delta) > \epsilon/2),$$

and taking $\lim_{\delta\to 0} \limsup_{n\to\infty}$ on both sides verifies condition (ii) in the statement of Lemma 4.9. $\square$

Lemma 4.12. *There is a probability space* $(\Omega, \mathbb{P})$ *supporting* (S_n, Z_n, V_n) *that also support processes*

$$\begin{aligned} U'_n &= S_n + Z'_n, \\ M'_n &= M^{U'_n}, \\ L'_n(t) &= 2^n Leb(0 < s < t : U'_n = -M'_n). \end{aligned}$$

where Z'_n *is a Poisson point process* Z'_n *of intensity* $|v2^n|$ *and jump size* $\mathrm{sign}(v)2^{-n}$ *such that*

$$0 \leq Z'_n(t) - Z'_n(s) \leq Z_n(t) - Z_n(s), \tag{4.16}$$

for all $0 \leq s \leq t \leq T$*, almost surely, and*

$$0 \leq L_n(t) - L_n(s) \leq L'_n(t) - L'_n(s), \tag{4.17}$$

for all $0 \leq s \leq t \leq T$*, almost surely.*

Proof. By definition $V_n \geq -v$ almost surely. Recall that Z_n is a point process with stochastic intensity $|2^n V_n|$ and jump size $\mathrm{sign}(V_n)2^{-n}$, so by Lemma 2.7 we assume the probability space included a process Z'_n with downward stochastic jump intensity $|v2^n|$ and step size 2^{-n} such that

$$Z'_n(t) - Z'_n(s) \leq Z_n(t) - Z_n(s)$$

for all $0 \leq s \leq t \leq T$, almost surely. This demonstrates (4.16). It follows that

$$U_n := S_n + Z_n \geq S_n + Z'_n =: U'_n,$$

almost surely. Notice both $U_n + M_n^U, U'_n + M_n^{U'}$ are equivalent to reflected continuous time walks which may transition from zero to zero. For instance, a transition of U_n from its running minimum corresponds to a transition from zero for the walk $U_n + M_n$. Both nonnegative walks $U_n + M_n, U'_n + M'_n$ transition as a continuous time (nonnegative) random walk but with an additional "drift" process of Z_n, Z'_n respectively. Since $Z_n \geq Z'_n$ the process of $U_n + M_n^U$ dominates that of $U'_n + M_n^{U'}$. That is,

$$U_n + M_n \geq U'_n + M'_n \geq 0, \text{ almost surely.} \tag{4.18}$$

In other words, $U'_n + M'_n$ is zero whenever $U_n + M_n$ is zero. Consequently, $\{s < z < t : U'_n(z) = -M'_n(z)\} \subset \{z : s < z < t, U_n(z) = -M_n(z)\}$ for every $(s,t) \subset [0,T]$, almost surely. Therefore

$$0 \leq L_n(t) - L_n(s) \leq L'_n(t) - L'_n(s), \tag{4.19}$$

for every $(s,t) \subset [0,T]$, almost surely, demonstrating (4.17). □

Lemma 4.13. *For each $n \in \mathbb{N}$ let N_n be a Poisson process with intenstiy $C2^n$. Then $2^{-n}N_n(t), t \in [0,T]$ converges in $D([0,T],\mathbb{R})$ to the line $g(t) = Ct$. In particular $\{2^{-n}N_n(t) : t \in [0,T], n \in \mathbb{N}\}$ is C-tight.*

Proof. This is in fact classical, but we give a proof. $N_n(t)$ is a Poisson process with rate $C2^n$, so $N_n(t) - C2^n t = N_n(t) - 2^n g(t)$ is a martingale. Consequently $2^{-n}(N_n(t) - 2^n g(t)) = 2^{-n}N_n(t) - g(t)$ is also a martingle. By Doob's martingale inequality and Cauchy-Schwarz, for any $\epsilon > 0$

$$\begin{aligned}
\mathbb{P}(\sup_{0<s<T} |2^{-n}N_n(t) - g(t)| > \epsilon) &\leq \epsilon^{-1}\mathbb{E}|2^{-n}N_n(T) - g(T)| \\
&\leq \epsilon^{-1}\sqrt{\text{Var}(2^{-n}N_n(T))} \\
&= \epsilon^{-1}2^{-2n}C2^n \\
&\longrightarrow 0.
\end{aligned}$$

Thus $2^{-n}N_n(t)$ converges weakly in $D([0,T],\mathbb{R})$ to g because convergence in the uniform norm implies convergence in the Skorohod topology on $D([0,T],\mathbb{R})$ [4]. ☐

Lemma 4.14. *For every* $T>0$, $\mathbb{E}(M_n(T)) \le \mathbb{E}(M'_n(T)) \le 2\sqrt{2T+|v|T+|v|2^{-n}}$ *where* v *is the initial value of* V_n *and* M'_n *is defined as in Lemma 4.12.*

Proof. According to Lemma 4.12 we enlarge our probability space to contain (S_n, Z_n, V_n) as well as the Z'_n given in that lemma statement. Consequently,

$$U_n := S_n(t) + Z_n(t) \ge S_n(t) + Z'_n(t) =: U'_n,$$

for all $t \in [0,T]$, almost surely, which implies

$$M_n(T) := M^{U_n}(T) \le M^{U'_n}(T) =: M'_n(T),$$

almost surely. By Doob's Martingale inequality, and the fact that $Z'_n(T)$ is distributed as $-2^{-n}\text{Poisson}(|v2^nT|)$, we compute

$$\begin{aligned}\mathbb{E}(M_n(T)) &\le \mathbb{E}(M'_n(T))\\ &\le 2\sqrt{\mathbb{E}(|S_n(T)+Z'_n(T)|^2)}\\ &\le 2\sqrt{\mathbb{E}(S_n^2(T))+|v|2^{-n}+|v|T}.\end{aligned}$$

We can express the continuous time random walk S_n as $2^{-n}(N_1(t)-N_2(t))$ where N_i are independent Poisson processes of rate 2^{2n}, and consequently $\mathbb{E}(S_n(T)^2) = \text{Var}(S_n(T)^2) = 2^{-2n}(\text{Var}(N_1(T)) + \text{Var}(N_2(T))) = 2T$. Substituting this into the previous string of inequalities, we have

$$\mathbb{E}(M_n(T)) \le 2\sqrt{2T+|v|T+|v|2^{-n}}.$$

☐

Lemma 4.15. *Let* $X \overset{d}{=} \text{Exp}(\lambda), Y \overset{d}{=} \text{Exp}(\mu)$ *be independent. Then* $X \wedge Y \overset{d}{=} \text{Exp}(\lambda+\mu)$ *is independent from* $W = 1_{\{X\wedge Y = X\}}$. *Furthermore* $\mathbb{P}(W=1) = \lambda/(\lambda+\mu)$. *In other words, the minimum of two independent exponential random variables is independent from which exponential r.v. occured first.*

Proof. Because X, Y are independent we can write their joint pdf as $f(x,y) = \lambda\mu\exp(-\lambda x - \mu y)$. With this one can compute, for any $z > 0$,

$$\begin{aligned}\mathbb{P}(X \wedge Y > z | W = 1) &= \mathbb{P}(X \wedge Y > z | X \wedge Y = X) \\ &= \int_z^\infty \int_x^\infty f(x,y)dydx \cdot \frac{1}{\mathbb{P}(X \wedge Y = X)} \\ &= \frac{\lambda}{\mu + \lambda} e^{-(\lambda+\mu)z} \cdot \frac{\mu + \lambda}{\lambda} \\ &= e^{-(\lambda+\mu)z} \\ &= \mathbb{P}(X \wedge Y > z).\end{aligned}$$

□

Lemma 4.16. *For each $T > 0$, L'_n converges in distribution to $M^{B^{-v}}$, where $B^{-v}(t) = B(t) - vt$ where B is a Brownian motion. In particular $\{L'_n : n \in \mathbb{N}\}$ is C-tight. Furthermore,* $\sup_n \mathbb{E}L_n(T) \leq \sup_n \mathbb{E}L'_n(T) < \infty$.

Proof of lemma. We begin by showing the weak convergence for which we use a similar technique to that used in the proof of Lemma 4.5 (iii). We record the amount of time U'_n spends at each maximum, and express $L'_n(t)$ as the sum of these. By Lemma 4.13, $Z'_n(t) = -2^{-n}N_n(t)$ converges in probability to $g(t) = -vt$ in distribution on $D([0,T], \mathbb{R})$. By Donsker's theorem S_n converges in distribution to a Brownian motion B, and consequently $U'_n := S_n + Z'_n$ converges in distribution to $B + g =: U'$. This implies $M'_n := M^{U'_n}$ converges to $M^{U'}$ in distribution because $f \mapsto M^f$ is continuous in the Skorohod topology. Note that $2^n M_n^{U'}(t) + 1$ is the number of levels the running minimum of U'_n has reached by time t. Let $\tau^{(j)} = \inf\{t > 0 : U'_n(t) = -j2^{-n}\}$, so that $\tau^{(j)}$ is the first time M'_n reaches 2^{-n}. Define

$$T_j := Leb\{s \geq \tau^{(j)} \mid -U'_n(s) = M'_n(s) = j2^{-n}\}.$$

Then,

$$2^n \sum_{j=0}^{2^n M'_n(t)} T_j \leq L'_n(t) \leq 2^n \sum_{j=0}^{2^n M'_n(t)+1} T_j, \tag{4.20}$$

for all $[s,t) \subset [0,T]$, almost surely. The process U'_n will jump up upon the arrival of an $\mathrm{Exp}(2^{2n})$ random variable, call it $\mu_k^{(j)}$, while it will jump down upon the arrival of an $\mathrm{Exp}(2^{2n} + |v2^n|)$ random variable $\mu_k^{(j)'}$. Consider the pair $(\mu_k^{(j)} \wedge \mu_k^{(j)'}, V_j)$ where $V_j = 1_{\mu_k^{(j)} \wedge \mu_k^{(j)'} = \mu_k^{(j)}}$. By Lemma 4.15, V_j is independent from all $\mu_k^{(j)} \wedge \mu_k^{(j)'}$. Then $W_j = \inf\{k : V_k = 0\}$ is the number of times U'_n visits $-j2^{-n}$ while M'_n is $j2^{-n}$. Because $V_j \overset{d}{=} \mathrm{Bernoulli}(p)$ with p given below, W_j is the first time this sequence of Bernoulli's is zero. Thus, W_j is a Geometric(p) random variable independent of $\mu_i^{(j)}, \mu_i^{(j)'}$. Thus

$$T_j = \sum_{i=1}^{W_j} \mu_i^{(j)} \wedge \mu_i^{(j)'}$$

is a Geometric sum of i.i.d. exponential random variables from which it is independent. Such a sum is exponential of rate $p\lambda$, where

$$p = \frac{2^{2n}}{2^{2n+1} + v2^n}, \quad \lambda = 2^{2n+1} + v2^n.$$

That is, $T_j \overset{d}{=} \mathrm{Exp}(p\lambda) = \mathrm{Exp}(2^{2n})$. We show the left hand side of (4.20) is tight, while the proof of the right hand side being tight is essentially identical. Clearly

$$\sup_{s\in[0,T]} \left| \sum_{i=1}^{2^n M'_n(s)} 2^{-n} - M'_n(s) \right| \leq 2^{-n}.$$

Without loss of generality, we may assume M'_n converges almost surely to M' by the Skorohod representation theorem and the fact shown above that M'_n converges to M' in distribution. Therefore

$$\sum_{i=1}^{2^n M'_n(s)} 2^{-n} \xrightarrow{P} M'$$

To show $2^n \sum_{i=1}^{2^n M'_n(s)} T_j$ converges in probability to M' it suffices to show

$$2^n \sum_{i=1}^{2^n M'_n(s)} T_j - \sum_{i=1}^{2^n M'_n(s)} 2^{-n} \xrightarrow{P} 0.$$

Because $T_j \overset{d}{=} \text{Exp}(2^{2n})$, we know $z_j := 2^n T_j \overset{d}{=} \text{Exp}(2^n)$. Hence

$$2^n \sum_{i=1}^{2^n M_n'(s)} T_j - \sum_{i=1}^{2^n M_n'(s)} 2^{-n} = \sum_{i=1}^{2^n M_n'(s)} (z_j - 2^{-n})$$

Note that $z_j - 2^{-n}$ are i.i.d. mean zero random variables with variance 2^{-2n}, so by Kolmogorov's maximal inequality, for each $C, \epsilon > 0$ we have

$$\begin{aligned}
&\mathbb{P}\left(\sup_{s\in[0,T]}\left|\sum_{i=1}^{2^n M_n'(s)} (z_j - 2^{-n})\right| > \epsilon\right) \\
&\leq \mathbb{P}\left(\sup_{1\leq k\leq 2^n C}\left|\sum_{i=1}^{k}(z_j - 2^{-n})\right| > \epsilon\right) + \mathbb{P}(M_n'(T) > C) \\
&\leq \mathbb{P}\left(\sup_{1\leq k\leq 2^n C}\sum_{i=1}^{k}|z_j - 2^{-n}|> \epsilon\right) + \mathbb{P}(M_n'(T) > C) \\
&\leq \epsilon^{-2}\text{Var}\left(\sum_{i=1}^{2^n C}(z_j - 2^{-n})\right) + \mathbb{P}(M_n'(T) > C) \\
&\leq \epsilon^{-2} 2^n C 2^{-2n} + \mathbb{P}(M_n'(T) > C).
\end{aligned}$$

Because M_n' converges to M' almost surely,

$$\begin{aligned}
&\limsup_{n\to\infty} \mathbb{P}\left(\sup_{s\in[0,T]} \sum_{i=1}^{2^n M_n'(s)} (z_j - 2^{-n}) > \epsilon\right) \\
&\leq \limsup_{n\to\infty} \mathbb{P}(M_n'(T) > C) \\
&= \mathbb{P}(M'(T) > C),
\end{aligned}$$

where $C > 0$ is arbitrary. Because M' is a continuous process C can be chosen so $\mathbb{P}(M'(T) > C)$ is arbitrarily small. Hence

$$\sum_{i=1}^{2^n M_n'(s)} (z_j - 2^{-n}) \xrightarrow{P} 0,$$

and the left hand side of (4.20) converges in probability to M'. The convergence in probability for the right hand side is similar, and because (4.20) is an almost sure bound,

$$\begin{aligned}&\mathbb{P}(\|L'_n - M'\| > \epsilon)\\ &\leq \mathbb{P}\left(\sup_{s\in[0,T]}\left|2^n \sum_{j=0}^{2^n M'_n(s)} T_j - M'\right| > \epsilon/2\right) + \mathbb{P}\left(\sup_{s\in[0,T]}\left|2^n \sum_{j=0}^{2^n M'_n(s)+1} T_j - M'\right| > \epsilon/2\right),\end{aligned}$$

and taking $\limsup_{n\to\infty}$ on both sides shows

$$\limsup_{n\to\infty} \mathbb{P}(\|L'_n - M'\| > \epsilon) = 0,$$

for every $\epsilon > 0$. Thus L'_n converges in probability to M'. That L'_n converges to a continuous process implies the sequence $\{L'_n : n \in \mathbb{N}\}$ is C-tight.

Corollary 4.17. *The collection of processes* $\{L_n : n \in \mathbb{N}\}$ *is* C*-tight.*

Proof. This follows directly from (4.19), the fact that $\{L'_n : n \in \mathbb{N}\}$ is C-tight by Lemma 4.16 (and that the zero process is trivially C-tight, and Lemma 4.9. □

Lemma 4.18. *The collection of processes* $\{Z_n : n \in \mathbb{N}\}$ *is* C*-tight.*

We will use a localization argument by stopping the stochastic intensity of Z_n when it becomes large. Recall that $|2^n V_n|$ is stochastic intensity of Z_n. For $C > v \geq 0$ let

$$T^n_C = \inf\{t > 0 : V_n(t) > C\} = \inf\left\{t > 0 : L_n(t) > \frac{v + C}{K}\right\}.$$

Define a process $\widetilde{Z}_n$ such that $\widetilde{Z}_n|_{[0,T^n_C]} = Z_n|_{[0,T^n_C]}$, almost surely, while after time T^n_C let $\widetilde{Z}_n$ have intensity $C2^n$ and jump size 2^{-n}. By Lemma 2.7 we can stochastically dominate the number of transition made by $\widetilde{Z}_n$ in a given time interval by the number of transitions made by a point process $Z^1_n(t)$ of intensity $C2^n$ and jump size 2^{-n} (in the same time interval). More precisely, $Z^1_n(t) = 2^{-n}N(C2^n t)$ where N is Poisson process of unit intensity. So

$$0 \leq |Z_n(t) - Z_n(s)| \leq |Z^1_n(t) - Z^1_n(s)| \tag{4.21}$$

for every interval $(s, t) \subset [0, T^{(n)}_C]$, almost surely. By monotonicity of Z^1_n and Doob's maximal inequality, for fixed $\epsilon, \delta, C > 0$ we have

$$
\begin{aligned}
&\mathbb{P}\Big(\sup_{t\in[0,T-\delta]}\sup_{t<u,v<t+\delta}|Z_n(u)-Z_n(v)|>\epsilon\Big)\\
&\le \mathbb{P}\Big(\sup_{t\in[0,T-\delta]}\sup_{t<u,v<t+\delta}|\widetilde{Z}_n(u)-\widetilde{Z}_n(v)|>\epsilon, T_C^{(n)}>T\Big)+P(T_C^{(n)}<T)\\
&\le \mathbb{P}\Big(\sup_{t\in[0,T-\delta]}\sup_{t<u,v<t+\delta}|Z_n^1(u)-Z_n^1(v)|>\epsilon\Big)+P(T_C^{(n)}<T)\\
&= \mathbb{P}\Big(\sup_{t\in[0,T-\delta]}|Z_n^1(t+\delta)-Z_n^1(t)|>\epsilon\Big)+P(L_n(T)>C)\\
&\le \mathbb{P}\Big(\sup_{t\in[0,T-\delta]}|Z_n^1(t+\delta)-Z_n^1(t)|>\epsilon\Big)+\frac{\sup_n\mathbb{E}(L_n(T))}{C}
\end{aligned}
$$

By Lemma 4.13 and Lemma 4.9 we know

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\mathbb{P}\Big(\sup_{t\in[0,T-\delta]}|Z_n^1(t+\delta)-Z_n^1(t)|>\epsilon\Big)=0.$$

Therefore, by Lemma 4.16 and applying Chebyshev's inequality, there exists a constant A independent of C,ϵ,n such that

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\mathbb{P}\Big(\sup_{t\in[0,T-\delta]}\sup_{t<u,v<t+\delta}|Z_n(u)-Z_n(v)|>\epsilon\Big)\le\frac{A}{C}.$$

By choosing C arbitrarily large, this shows

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\mathbb{P}\Big(\sup_{t\in[0,T-\delta]}\sup_{t<u,v<t+\delta}|Z_n(u)-Z_n(v)|>\epsilon\Big)=0.$$

Hence Z_n satisfies (ii) of Lemma 4.9. Condition (i) follows from Remark 4.10 and the fact that $Z_n(0)=0$ almost surely, successfully verifying the conditions of C-tightness for $\{Z_n : n\in\mathbb{N}\}$ given in Lemma 4.9. ☐

Corollary 4.19. *The collection of processes* $\{(S_n,Z_n,V_n) : n\in\}$ *are* C*-tight.*

Proof. The collection $\{(S_n,Z_n,V_n) : n\in\}$ is C-tight in the product metric on $D([0,T],\mathbb{R})\times D([0,T],\mathbb{R})\times D([0,T],\mathbb{R})$ which will follow from C-tightness of each marginal sequence. Note that S_n is C-tight by Donsker's theorem while $V_n=v-KL_n$ and Z_n are C-tight from Corollary 4.17 and Lemma 4.18. ☐

5 Two Brownian motions separated by an inert particle

In this section we consider a system of processes (X_1, Y, X_2, V) on $C(\mathbb{R}, \mathbb{R}^4)$. The processes X_1, X_2 are the lower and upper, respectively, Brownian particles separated by an inert particle Y. The velocity of Y is V. In [45, Section 4], White gives existence of the system up until a triple collision of the processes X_1, X_2, Y. The processes exist for all real times, almost surely, once one shows that triple collisions occur with probability zero. Unfortunately, the proof of [45, Theorem 4.1] contains an oversight in the upper bound demonstrating satisfaction of Novikov's condition for V. We remedy this by apply a localization argument to V followed by a Grönwall inequality that applies pathwise. Because the inequality applies pathwise, we are able to obtain similar bounds when undoing the localization. We begin by recalling White's theorem of the Skorohod map used to construct the system of one Brownian particle reflecting from an inert particle.

Theorem 5.1. *(White, [45, Theorem 2.1]) Let $f(t)$ be a continuous function with $f(0) \geq 0$, and let $\mu(l) \geq 0$ be a continuous monotone function with*

$$\sup_{a<b<l} \frac{|\mu(a) - \mu(b)|}{b-a} < \infty$$

for each l. There is a unique continuous function $L(t)$ which satisfies

(i) $x(t) = f(t) + L(t) + \int\limits_0^t \mu \circ L(s) ds \geq 0,$

(ii) $L(0) = 0, L(t)$ *is nondecreasing,*

(iii) $L(t)$ *is flat off* $\{t : x(t) = 0\}$, *that is,* $\int\limits_0^\infty 1_{\{x(s)>0\}} dL(s) = 0.$

Remark 5.2. The statement of Theorem 5.1 is more general than is needed to construct Brownian motion with inert drift as described in the the first chapter of this thesis. In particular, setting $\mu(l) = Kl$ for $K \geq 0$ is equivalent to the Skorohod map given in Theorem 2.6 when $n = 1$ of the first chapter.

Theorem 5.3. *Let $(\Omega, \mathbb{P}, \mathcal{F}_t)$ be a filtered probability space supporting two indepenent Brownian motions B_1, B_2. Here $\mathcal{F}_t$ can be taken to be $\sigma(\mathcal{F}_t^1, \mathcal{F}_t^2)$ where $\mathcal{F}_t^i$ is the continuous filtration generated by $B_i, i = 1, 2$. Fix $K \geq 0$ and real initial conditions x_1, x_2, v, y such that $x_1 < y < x_2$. There exists a stopping time T_∞ and continuous processes (X_1, Y, X_2, V) adapted to $\mathcal{F}_{t \wedge T_\infty}$ such that*

$$\begin{aligned}
&X_1(t) \leq Y(t) \leq X_2(t), \ \text{for all } t \in [0, T_\infty], \text{ almost surely,} \\
&X_1 = x_1 + B_1 - L_1, \\
&X_2 = x_2 + B_2 + L_2, \\
&V = v + K(L_1 - L_2), \qquad (5.1) \\
&\frac{dY}{dt} = V, \ Y(0) = y, \\
&L_i(0) = 0 \text{ and } L_i \text{ is a continuous and nondecreasing,} \\
&L_i \text{ is flat off } \{s : X_i(s) = Y(s)\}.
\end{aligned}$$

Regarding the stopping time T_∞, consider the recursively defined stopping times

$$T_i' = \inf\{t > T_i'' : Y_t = X_t^1\}, T_0 = 0,$$

$$T_i'' = \inf\{t > T_i' : Y_t = X_t^2\}, T_0 = 0,$$

and relabel them all under one index as T_i. Then

$$T_\infty = \lim_{n \to \infty} T_n. \qquad (5.2)$$

For every fixed t, $(X(t), Y(t), V(t))$ takes values in $\mathcal{W} \times \mathbb{R}$ where $S = \{(x, y) \in \mathbb{R}^2 : x \geq y\}$. In this section we show that the invariant measure for the two particle system given above has an invariant measure $\mu \times \gamma_K$ for (X, Y, V) on $\mathcal{W} \times \mathbb{R}$.

Remark 5.4. A single inert-particle interacting with one Brownian particle has a strong solution provided by a Skorohod map that is applied pathwise on the probability space as described in the first chapter.

Proof. For a given interval $[T_j, T_{j+1}]$ the process Y interacts with one X^i exactly as one Brownian motion and the inert particle. Therefore we define the system by relying on the the Skorohod map construction as mentioned in Remark 5.2for the one particle interaction, which we apply path-by-path on $[0, T_1]$, and recursively to $[T_i, T_{i+1}]$. This gives strong existence until the first time of a triple collision between X_1, X_2, Y, which we define as

$$T := \lim_{n\to\infty} T_n. \tag{5.3}$$

□

5.1 Motivation and relation to similar systems

To complete the construction of the system 5.3 and correct the oversight in [45], we must show triple collisions occur with probability zero, that is $T = \infty$.

Theorem 5.5. *We have* $\mathbb{P}(T = \infty) = 1$*. As a result, there exists a pathwise unique strong solution to the system* (5.1) *with processes* (X_1, Y, X_2, V) *defined for all time, almost surely.*

Theorem 5.5 follows from Corollary 5.8 of the following lemma.

Lemma 5.6. *For a given* $t > 0$*, there are positive constants* β *depending on* K, t*, and* α *depending on* t, K, x_1, x_2*, such that for* $\lambda > \alpha$

$$\mathbb{P}\left(\sup_{0<s<t\wedge T_\infty} |V(s)| > \lambda\right) < e^{-\lambda^2/\beta}.$$

Proof. For $C > 0$, define

$$\tau_C = \inf\{0 < t < T_\infty : |V(t)| > C\}$$

as the first time the process $|V|$ reaches C, so that $V(s \wedge \tau_C)$ is a bounded process. Recall the definition of the T_i given in the statement of Theorem 5.3. Also recall the inert particle

Y ineracts with only one Brownian particle in the time interval $[T_i, T_{i+1}]$ and we define (X_1, Y, X_2, V) using the Skorohod map definition for one Brownian particle interaction per Remark 5.2. By (5.1), the following occurs for all $t \in [0, T_\infty \wedge \tau_C]$, almost surely under $\mathbb{P}$,

(i) L_i is nondecreasing and flat off of $\{B^i_t - Y_t = 0\}$,

(ii) $B^1(t) - Y(t) + L_1(t) \geq 0$, $Y(t) - B_2(t) + L_2(t) \geq 0$.

The Novikov condition is satisfied for $V(s \wedge \tau_C)$ because it is bounded. Consequently a Girsanov transformation gives the existence of a probability measure $\mathbb{Q}$ equivalent (i.e. mutually absolutely continuous) to $\mathbb{P}$ such that the joint processes

$$(\widetilde{B}_1(t), \widetilde{B}_2(t)) := (B_1(t) - Y(t), Y(t) - B_2(t))$$

stopped at $\tau_C \wedge T$, are distributed as a pair of two stopped independent Brownian motions under $\mathbb{Q}$, [30, Theorem 5.1]. Consider the processes $(X^1, Y, X^2, v)_{s \wedge \tau_C \wedge T}$ under this $\mathbb{Q}$. Since $\mathbb{Q}$ and $\mathbb{P}$ are mutually absolutely continuous, almost sure events under $\mathbb{P}$ are almost sure events under $\mathbb{Q}$. Therefore (i) and (ii) above translate to the following holding for $t \in [0, T_\infty \wedge \tau_C]$, almost surely under $\mathbb{Q}$,

(i) $\widetilde{B}_1(t) - L_1(t) \leq 0$, and $\widetilde{B}_2(t) + L_2(t) \geq 0$,

(ii) L_i is nondecreasing and flat off of $\{t : \widetilde{B}_i(t) = 0\}$.

It follows from uniqueness of the classical Skorohod map that

$$L_1(t) = \max_{0<s<t}(\widetilde{B}_1(s) \vee 0), \text{ and } L_2(t) = \max_{0<s<t}(-\widetilde{B}_2(s) \vee 0),$$

for all $t \in [0, \tau_C \wedge T_\infty]$, almost surely under $\mathbb{Q}$. Lévy's theorem says $(\widetilde{B}^i_t + L^i_t, 2L^i_t)$ has the same distribution as $(|\widetilde{B}^i_t|, \Lambda_t(0))$, reflected Brownian motion and its local time at zero. Since

$\mathbb{P} \ll \mathbb{Q}$, and by definition of $\widetilde{B}_i(t)$, the functions L_1 (resp. L_2) is path-by-path the running maximum of $B_1 + x_1$ above Y (resp. the running minimum of $B_2 + x_2$ below Y) and has the same distribution as the local time X_i spends on Y. This enables us to complete the following calculation necessary for a Grönwall inequality bound. For $0 \leq t \leq \tau_C \wedge T$, the following holds almost surely under $\mathbb{P}$,

$$\begin{aligned}
&|V(t)|= |K(L_1(t) - L_2(t))|\leq KL_1(t) + KL_2(t) \\
&\leq K(x_2 + \max_{0\leq s<t}|Y(s)|+ \max_{0\leq s\leq t}|B_1(s)|) + K(x_2 + \max_{0\leq s<t}|Y(s)|+ \max_{0\leq s\leq t}|B_2(s)|) \\
&= K(x_2 - x_1 + \max_{0\leq s\leq t}|B_2(s)|+ \max_{0\leq s\leq t}|B_1(s)|) + 2K \max_{0<s<t}|Y(s)| \\
&\leq K(x_2 - x_1 + \max_{0\leq s\leq t}|B_2(s)|+ \max_{0\leq s\leq t}|B_1(s)|)) + 2K \int\limits_0^t \max_{0<x<s}|V(x)|ds
\end{aligned}$$

Since the right hand is nondecreasing, the inequality holds when taking sup of the left hand side. Therefore

$$\max_{0<s<t\wedge T_\infty}|V(t)|\leq K(x_2 - x_1 + \max_{0\leq s\leq t}|B_2(s)|+ \max_{0\leq s\leq t}|B_1(s)|) + 2K \int\limits_0^t \max_{0\leq x\leq s}|V(x)|ds. \tag{5.4}$$

For u, α, β continuous on $[0, b)$ with the negative part of α locally integrable, Grönwall's inequality states that $u(s) \leq \alpha(s) + \int\limits_0^s \beta(z)u(z)dz$ implies

$$u(s) \leq \alpha(s) + \int\limits_0^t \alpha(z)\beta(z) \exp\left(\int_z^s \beta(r)dr\right).$$

Applying this to the bound (5.4) yields the following, almost surely,

$$\begin{aligned}
\max_{0\le s\le t\wedge T_\infty}|V(t)| &\le K(x_2 - x_1 + \max_{0\le s\le t}|B_2(s)|+\max_{0\le s\le t}|B_1(s)|) \\
&\quad + K\int_0^t (x_2 - x_1 + \max_{0\le s\le t}|B_2(s)|+\max_{0\le s\le t}|B_1(s)|)\exp 2K|t-s|ds \\
&\le (1+te^{2Kt})K(x_2 - x_1 + \max_{0\le s\le t}|B_2(s)|+\max_{0\le s\le t}|B_1(s)|) \\
&\le \alpha + \beta(\max_{0\le s\le t}|B_2(s)|+\max_{0\le s\le t}|B_1(s)|) \\
&=: \alpha + \beta(M_2(t)+M_1(t)),
\end{aligned}$$

for some positive constants α, β that only depend on t, K and the initial positions x_1, x_2 and where $M_i(t) = \max_{0\le s\le t}(-B_i(s)\vee 0)$. We have the following set inclusions holds almost surely,

$$\left\{\max_{0\le s\le t\wedge T_\infty}|V(s)|> w\right\} \subset \{\tau_w \le t\wedge T_\infty\} \subset \{\alpha + \beta(M^1(t)+M^2(t)) \ge w\}.$$

For $w > \alpha$,

$$\begin{aligned}
&\mathbb{P}\left(\max_{0<s<t\wedge T_\infty}|V(s)|> w\right) \\
&\le \mathbb{P}(\tau_w \le t\wedge T_\infty) \\
&\le \mathbb{P}\left(\alpha + \beta\left(\max_{0\le s\le t}|B_2(s)|+\max_{0\le s\le t}|B_1(s)|\right) > w\right) \\
&\le \mathbb{P}\Big(M^1 + M^2 > \frac{w-\alpha}{\beta}\Big) \\
&\le 2\mathbb{P}\Big(M^1 > \frac{w-\alpha}{2\beta}\Big) \\
&\le 2\mathbb{P}\Big(|W_t|> \frac{w-\alpha}{2\beta}\Big) \\
&= 2\mathbb{P}\Big(W_t > \frac{w-\alpha}{2\beta}\Big) + 2\mathbb{P}\Big(W_t < -\frac{w-\alpha}{2\beta}\Big) \\
&\le 8e^{-\left(\frac{w-\alpha}{2\beta}\right)^2/2t} \\
&\le 8e^{-\frac{w^2}{8\beta^2 t}}
\end{aligned}$$

where we get a tail bound for $\max_{0\le s\le t}|B_i(s)|$ from the fact that for fixed t, $\max_{0\le s\le t}|B_i(s)|\overset{d}{=}|N(0,t)|$ where $N(0,t)$ is a normal random variable of variance t, [30]. Relabel $8\beta^2 t$ as β in the statement of the lemma. □

We have two corollary's from 5.6.

Corollary 5.7. *$\tau_C \wedge T_\infty \longrightarrow T_\infty$ as $C\to\infty$.*

Proof. By Lemma 5.6, for any $C, t > 0$,

$$\mathbb{P}(\tau_C \wedge T_\infty < t\wedge T_\infty) \le \mathbb{P}\left(\sup_{0<s<t\wedge T_\infty}|V(s)|> C\right),$$

which approaches zero as $C\to\infty$. □

Corollary 5.8. *For every $C>0$, $\tau_C \le T_\infty$ almost surely and $X_1(T_i) < X_2(T_i)$ for all $i\in\mathbb{N}$. Furthermore $T_\infty = \infty$ almost surely.*

Proof. Fix $t>0$. From the previous corollary we know that $|V|$ is almost surely finite on $[0, t\wedge\tau_C\wedge T_\infty]$. Furthermore $\tau_C\to\infty$ in probability. Recall $\tau_C = \inf\{s>0 : |V(s)|= C\}$ is the first time $|V|$ reaches C. Upon reaching τ_C stop the value of V while keeping the same dynamics of the other processes. Denote this altered version as X_1^C, Y^C, X_2^C, V^C. In other words (X_1^C, X_2^C, Y^C, V^C) satisfy the system (5.3) but with V^C fixed at $\text{sign}(V(\tau_C))C$ for times after τ_C. (Consequently $Y^C(s) = Y^C(\tau_C) + \text{sign}(V(\tau_C))C(s-\tau_C))$ for $s>\tau_C$, and X_i^C are relecting Brownian motions from Y^C.) Because V^C is bounded and defined for all time, we have $\mathbb{E}\left(\exp\int_0^t |V^C(s)|ds\right) < \infty$. Consequently V^C satisfies Novikov's condition, allowing us to apply a Girsanov transform, as done in the proof of Lemma 5.6, to see that $(X_1^C - Y^C, Y^C - X_2^C)$ behaves as a pair of independent Brownian motions each reflecting from zero under a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$. Since two dimensional reflected Brownian motion hitting the origin at some time in $[0,t]$ is a null event, the same probability both $X_1^C(s) = X_2^C(s)$ for some $s\in[0,t]$ also has probability zero. But $(X_1^C, X_2^C, Y^C, V^C) = (X_1, Y, X_2, V)$ on the interval $[0, t\wedge\tau_C\wedge T_\infty]$. Hence X_1, X_2 do not collide on the interval

$[0, t \wedge \tau_C \wedge T_\infty]$, almost surely. This implies that $t \wedge \tau_C \leq T_\infty$, almost surely, for each $t > 0$. Consequenlty $\tau_C \leq T_\infty$, almost surely. By this fact and Lemma 5.6, for a fixed $t_0 > 0$,

$$\begin{aligned}\mathbb{P}(\tau_C \leq t_0) &= \mathbb{P}(\tau_C \leq t_0 \wedge T_\infty)\\ &= \mathbb{P}\left(\max_{0 \leq s < t_0 \wedge T_\infty} |V(s)| > C\right),\end{aligned}$$

which approaches zero as as C approaches infinity. Consequently $\tau_C > t_0$ with arbitrarily high probability as C approaches infinity, for every fixed t_0. Thus $t_0 < \tau_C < T$ with probability approaching 1. Therefore $\tau_C \to \infty$ almost surely, guaranteeing that X_1, X_2 collide with probability zero and completing the proof. □

6 *Discrete Approximation and an Invariant Measure of Two Brownian motions separated by an inert particle*

In this section we establish an invariant measure of (X_1, Y, X_2, V) by rigorously applying the procedure conjectured by Burdzy and White in [9]. The idea is the following: We approximate the system (X_1, Y, X_2, V) by a system of other three particles jumping on a discrete lattice $2^{-n}\mathbb{Z}$ of class $\mathcal{C}$. Then apply Theorem 2.9 to the processes of class $\mathcal{C}$ to obtain the invariant measure $\pi^{(n)}$ for each n so that passing to the limit gives an invariant measure of the system (X_1, Y, X_2, V). This process behaves as a continuous-time Markov process whose intensities of jumps can depend on the time the particles spend at the same location (i.e. a collision). We say "invariant measure" because it obtains infinite mass and is no longer a probability space.

6.1 Definition of Processes and Discrete Convergence

For each $n \in \mathbb{N}$, consider a process

$$(X_1^n(t), Y^n(t), X_2^n(t)), \text{ for } t \in \mathbb{R}_{\geq 0} := [0, \infty)$$

that for a fixed time lies in the state space of a discrete wedge

$$\mathcal{W}_n := \{(x_1, y, x_2) \in 2^{-n}\mathbb{Z}^3 : x_1 \leq y \leq x_2\}, \tag{6.1}$$

whose trajectories are right-continuous with left limits. We wish that its dynamics emulate that of (X_1, Y, X_2) given in (5.1). We do so by specifying a system of point processes very similar to that of (i)-(iii) of Section 4.1. Let

$$X_1^n(0) = x_1^n,\ Y^n(0) = y^n,\ X_2^n(0) = x_2^n.$$

Define the scaled times of collisions as

$$L_i^n(t) := 2^n Leb(s \in [0,t] : X_i^n(t) = Y_2^n(t)),\ \text{for } i = 1, 2.$$

For $i = 1, 2$ let $S_i^n(t)$ be a simple continuous time random walk with interarrival jump times of i.i.d. $\mathrm{Exp}(2^{2n})$ with jump size of 2^{-n}. Then (X_1^n, Y^n, X_2^n, V^n) is a point process defined by

(i) $X_1^n = S_1^n + M_1^n$ and $X_2^n = S_2^n - M_2^n$, where

$$M_1(t) = M^{Y^n - S_1^n}(t),\ \text{and } M_2(t) = M^{S_2^n - Y^n}(t)$$

is the running maximum of S_1 above Y^n, the running minimum of S_2^n below Y^n, respectively.

(ii) $V^n(t) = v_n - KL_2^n(t) + KL_1^n(t)$, where

$$L_i^n(t) := 2^n Leb(s \in [0,t] : X_i^n(t) = Y_2^n(t)),\ \text{for } i = 1, 2.$$

(iii) Y^n is a point process of stochastic intensity $|2^n V^n|$ with step size $\mathrm{sign}(V^n)2^{-n}$.

Such a process exists because it falls into the class $\mathcal{C}$ of processes. As in the previous sections we denote the Skorohod space of functions $D([0,T], \mathbb{R}^d)$ with the (complete) Skorohod metric, and $C([0,T], \mathbb{R}^d)$ the space of continuous functions with the uniform norm.

Theorem 6.1. *Fix $t_0 > 0$ and consider the system (X^1, Y, X^2) from (5.1) with the initial conditions satisfying*

$$x_2^n \to x_2,\ x_2^n \to x_2,\ y^n \to y,\ v_n \to v_0.$$

Then

$$\Xi_n := (X_1^n, Y^n, X_2^n, V^n) \longrightarrow \Xi := (X_1, Y^n, X_2, V)$$

weakly in $D([0,t_0],\mathbb{R}^3) \times C([0,t_0],\mathbb{R})$.

Proof. The idea is to construct a probability space supporting the sequence of processes Ξ_n that converge to the limiting system Ξ uniformly on $[0,t_0]$, almost surely. We do this inductively between the hitting times $[0,T_i)$ where T_i is as (5.3). We start from initial conditions such that $X_1(0) < Y(0)$ and by assumption the initial conditions of Ξ_n converges to the initial condition of Ξ. We first show there is a probability space supporting Ξ_n, Ξ such that Ξ_n converges almost surely in the uniform norm to Ξ on $[0,T_1)$. By the Skorohod representation theorem, Donsker's theorem, and Lemma 4.13 there exists a probability space $(\Omega_1, \mathbb{P}_1)$ supporting two independent continuous time simple random walks S_1^n, S_2^n, independent Brownian motions B_1, B_2, and a Poisson point process Z^n of intensity $|v2^n|$ and steps of $\mathrm{sign}(v)2^{-n}$ such that

$$\begin{aligned} \|S_1^n - B_1\|_{[0,t_0]} &\longrightarrow 0, \text{ almost surely,} \\ \|S_2^n - B_2\|_{[0,t_0]} &\longrightarrow 0, \text{ almost surely,} \\ \|Z^n - g\|_{[0,t_0]} &\longrightarrow 0, \text{ almost surely.} \end{aligned} \tag{6.2}$$

where $g(t) = vt$. Let $T_1^{(n)} = \inf\{t > 0 : S_i(t) + x_i^n = Z^n(t) + y^n, i = 1,2\}$ be the first time one of $S_1^n + x_1^n$ or $S_2^n + x_2^n$ contacts $Z^n + y^n$. By (6.2), $T_1^{(n)}$ converges, almost surely, to T_1 where $T_1 := \inf\{t > 0 : B_i(t) + x_i = g(t) + y\}$. Because $T_!$ is finite almost surely Because this is the first time the Brownian particles contact Y, $X_i(t) = B_i(t) + x_i$ and $Y(t) = g(t) + y$ on $[0,T_1)$. Furthermore,

$$\|X_i^n - X_i\|_{[0,T_1^{(n)} \wedge T_1 \wedge t_0]} \longrightarrow 0,$$

almost surely. This is the base case construction of the probability space $(\Omega_1, \mathbb{P})$. By Corollary 5.8 $\lim_n X_1^n(T_1^{(n)}) = X_1(T_1) < X_2(T_2) = \lim_n X_1^n(T_2^{(n)}$. For a fixed element in the probability space Ω_1 we have either $X_1^n(T_1^{(n)}) = Y^n(T_1^{(n)})$ or $X_2^n(T_1^{(n)}) = Y^n(T_1^{(n)})$. Let

$$I_1 = \{i : X_I^n(T_1^{(n)}) = Y^n(T_1^{(n)}),\ i = 1,2\}$$

indicate which process makes contact with Y^n at this stopping time. We will now "patch" new processes from a new probability space conditional on I. Create another probability space $(\Omega_2, \mathbb{P}_2)$ supporting independent continuous time random walks $S^n_{1,2}$ and $S^n_{2,2}$ that converge uniformly on $[0, t_0]$, almost surely, to independent Brownian motions $B_{1,2}, B_{2,2}$. Assume $I_1 = 2$, the case $I = 1$ is similar. By Theorem 4.2 and Skorohod's representation theorem we may assume Ω_2 also supports $(X^n_{2,2}, Y^n_2, V^n_2)$ with initial conditions $(X_{2,1}(T_1^{(n)}, Y_1^n(T_1^{(n)}, V_1(T_1^{(n)}))$, that converge uniformly on $[0, t_0]$, almost surely, to $(X_{2,2}, Y_2, V_2)$ solving (3.1) with the initial condition $(X_{2,1}(T_1), Y_1(T_1), V_1(T_1))$. Define $X_{1,1} = S_{1,2} + X_{1,1}(T_1^{(n)})$. (Should $I = 1$, we construct a similar process $(X^n_{1,2}, Y^n_2, V^n_2)$ but with $X_{1,2}$ reflecting below Y^n_2). Define $T^n_2 = \inf\{t > 0 : X_{i,2} = Y^n_2\}$ and $I_2 = \{i : X^n_{I,2}(T_2^{(n)}) = Y^n(T_2^{(n)}\}$. Continue this by induction. For a given $k \in \mathbb{N}$ with probability spaces $\Omega_1, \dots, \Omega_k$ and sequences of processes $(X^n_{1,1}, Y^n_1, X^n_{2,1}, V^n_1), \dots, (X^n_{1,k}, Y^n_k, X^n_{2,k}, V^n_k)$ along with stopping times $T^n_1, \dots, T^n_k$ and indicators of contact $I_1, \dots, I_k$, construct a probability space Ω_{k+1} independent of $\Omega_i, 1 \leq i \leq k$ supporting independent continuous time random walks $S^n_{1,k+1}$ and $S^n_{2,k+1}$ that converge uniformly on $[0, t_0]$, almost surely, to independent Brownian motions $B_{1,k+1}, B_{2,k+1}$ adapted to $\mathcal{F}_t^{1,k+1}, \mathcal{F}_t^{2,k+1}$ respectively. We also assume Ω_{k+1} supports $(X^n_{2,k+1}, Y^n_{k+1}, V^n_{k+1})$ with initial conditions $(X_{I_k,k}(T_{I_k}^{(n)}, Y_k^n(T_k^{(n)}, V_k(T_k^{(n)}))$, that converge uniformly on $[0, t_0]$, almost surely, to $(X_{I_k,k+1}, Y_{k+1}, V_{k+1})$ solving (3.1) but with $X_{I_k,k+1}$ reflecting above Y_{k+1} if $I_k = 2$, and with the initial condition $(X_{I_k,k}(T_k), Y_k(T_k), V_k(T_k))$. This completes the induction step giving existence of a sequence of probability spaces $\Omega_i, i \in \mathbb{N}$ supporting the processes mentioned. We consider the infinite product as a probability space,

$$(\Omega, \mathbb{P}) = \left(\prod_{i=1}^{\infty} \Omega_i, \prod_{i=1}^{\infty} \mathbb{P}_i \right),$$

and the times

$$T'_i := \sum_{j=1}^{i} T_i,$$

which represent the times between collisions of the "patched" process

$$\Xi' := (X'_1, X'_2, Y', V').$$

For $t \in [T'_k, T'_{k+1}]$, we define (X'_1, Y', X'_2, V') on $(\Omega, \mathbb{P})$ by

$$\Xi'(t) := (X'_1, Y', X'_2, V')(t) = (X_{1,k+1}, Y_{k+1}, X_{2,k+1}, V_{k+1})(t - T_k).$$

Similarly define two Brownian motions by

$$B'_1(t) = B_{1,k+1}(t - T_k),\ \ B'_2(t) = B_{2,k+1}(t - T_k)$$

for $t \in [T'_k, T'_{k+1}]$. The B'_i are adapted to the filtration $\mathcal{F}^i_t := \sigma(\mathcal{F}^{i,k+1}_{T_1}, \ldots, \mathcal{F}^{i,k+1}_{t-T_k})$ for $i = 1, 2$, which implies B'_1, B'_2 are independent. By construction (X'_1, X'_2, Y', V') solve (5.1) and is defined until $T'_\infty = \lim_{i\to\infty} T'_i$, which is infinite almost surely by Corollary 5.8. Similarly define the discrete approximations as follows. For fixed $n, k \in \mathbb{N}$ define

$$T^{(n)'}_k = \sum_{i=1}^{k} T^{(n)}_i$$

and for $t \in [T^{(n)'}_k, T^{(n)'}_{k+1}]$ let

$$\Xi'_n(t) := (X^{(n)'}_1, Y^{(n)'}, X^{(n)'}_2, V^{(n)'})(t) = (X^n_{1,k+1}, Y^n_{k+1}, X^n_{2,k+1}, V^n_{k+1})(t - T^{(n)}_k).$$

By construction

$$\|\Xi'_n - \Xi'\|_{[T^{(n)'}_k \wedge T'_k, T^{(n)'}_{k+1} \vee T'_{k+1}]} \longrightarrow 0$$

almost surely on $(\Omega, \mathbb{P})$. Because $T^{(n)'}_k \longrightarrow T'_k$ almost surely, this implies

$$\|\Xi'_n - \Xi\|_{[0,T'_k]} \longrightarrow 0,$$

almost surely on $(\Omega, \mathbb{P})$, for every $k \in \mathbb{N}$. By Corollary 5.8 $\mathbb{P}(T'_k > t_0) \to 1$ as k goes to infinity. Consequently $\|\Xi'_n - \Xi'\|_{[0,t_0]} \to 0$ almost surely. Because $\Xi'_n \overset{d}{=} \Xi_n$ and $\Xi' \overset{d}{=} \Xi$, this completes the proof. □

6.2 Invariant measure

In this last section we apply Burdzy and White's Theorem 2.9 to the processes Ξ_n to establish an invariant measure for the process $\Xi = (X_1, Y, X_2, V)$ of the previous section. now piece together the discrete approximation given in the first section by patching together the convergence between $[T_i, T_{i+1}]$. Intuitively we think of the Brownian particles X_i as "light" and the particle Y as "heavy." We define the space in which $\mathcal{X} = (X^1, Y, X^2)$ takes place to be

$$\mathcal{W} = \{(x, y, z) : x \leq y \leq z\}$$

and the discrete space

$$\mathcal{W}^n = \mathcal{W} \cap 2^{-n}\mathbb{Z} \times 2^{-n}\mathbb{Z} \times 2^{-n}\mathbb{Z}.$$

At time t, the process $\mathcal{X}^n := (X_1^n, Y^n, X_2^n)$ will transition from one state to another according to a point process with the intensities given below. See Sections **??** and 4.1. If $(i, j, k) \in \mathcal{W}^n$ with no coordinates adjacent:

$$\begin{aligned}(i, j, k) &\xrightarrow{2^{2n}} (i + 2^{-n}, j, k),\\ (i, j, k) &\xrightarrow{2^{2n}} (i, j + 2^{-n}, k).\end{aligned}$$

The second component of $\mathcal{X}^n$ is Y^n, which transitions either upward or downward depending on the sign of V^n.

$$(i, j, k) \overset{V^n(t)2^n}{\longrightarrow} (i, j + 2^{-n}, k), \text{ if } V(t) \geq 0 \tag{6.3}$$

$$(i, j, k) \overset{-V^n(t)2^n}{\longrightarrow} (i, j - 2^{-n}, k), \text{ otherwise.} \tag{6.4}$$

Recall in Section 6.1 (ii), that

$$V^n(t) := K2^n(Leb\{0 < s < t : X_1^n(s) = Y^n(s)\} - Leb\{0 < s < t : X_2^n(s) = Y^n(s)\}). \tag{6.5}$$

This definition is the same as our discrete drift d in the previous section except we have more than one particle which its value depends.

The process $(\mathcal{X}^n, V^n)$ converges to (5.1) by the results in the first section. A similar convergence result holds when considering an altered version of $\mathcal{X}^n$ to be defined on $\mathcal{W}^n$ and reflecting the particles upon reaching either n or $-n$, and we will relabel $\mathcal{X}^n$ as this alteration. The convergence of this altered process to the solution of (5.1) follows from Theorem 6.1

because the paths of $\mathcal{X}$ are bounded almost surely. The process $\mathcal{X}_t^n$ is a Markov process whose transition rate to the next state depends on the time spent at previous locations, and so falls into the class $\mathcal{C}$ of processes studied by Burdzy and White in [9] as we mentioned in Section 2.2.

The main application of their paper will allow us to calculate an invariant *measure* for (X_1, Y, X_2, V). We define this to mean a measure ν on $\mathcal{W} \times \mathbb{R}$ such that

$$\int \mathbb{E}(f(Z_t)|Z_0 = (y,l))d\nu(y,l) = \int f(y,l)d\nu(y,l), \tag{6.6}$$

for each $t > 0$ and every compactly supported continuous real function $f \in C_c(\mathcal{W} \times \mathbb{R})$. Once we know the measure is $\sigma-$finite it is only required to show such an equality for $f = \mathbb{1}_K$ where $K \in \mathcal{B}(\mathcal{W} \times \mathbb{R})$ because such a measure is defined by its values on compacta. In this case ν may not a probability measure. For example one uses the same definition to see that Lebesgue measure on $\mathbb{R}$ is a stationary measure with respect to one dimension Brownian motion W_t. This is easy to see because for any $a < b$

$$\int \mathbb{E}(\mathbb{1}_{(a,b)}(W_t)|W_0 = y)dy = \int \int\limits_{a-y}^{b-y} \frac{1}{\sqrt{2\pi t}} \exp(-x^2/2t)dxdy = b - a$$

so that the measure on $\mathbb{R}$ given by beginning with Lebesgue measure and running Brownian motion is the same as Lebesgue measure itself. By Tonelli's theorem this also holds for any spatially invariant stochastic process.

Theorem 6.2. *Let μ be uniform measure on $\mathcal{W}$ and $\gamma_K \overset{d}{=} N(0,K)$ be a normal distribution with variance K. Then (X^1, Y, X^2, V) is stationary under the measure $\nu := \mu \times \gamma_K$.*

Proof. Fix a compact set $A \subset \mathcal{W} \times \mathbb{R}$. We consider the measure $\nu_n := \mu_n \times \gamma_K$ on $\mathcal{W}^n \times \mathbb{R}$ where the first measure is uniform on $\mathcal{W}^n \cap [-n,n]^3$ and the second is $N(0,K)$. Clearly

$$\lim_{n\to\infty} \alpha_n \nu_n(A) = \nu(A). \tag{6.7}$$

where $\alpha_n = \nu([-n,n]^3) \approx n^3$.

Recall the processes (X_1^n, Y^n, X_2^n, V^n) are defined so X_i^n, Y^n cannot transition above/below $\pm n$. For $t > 0$ define

$$\begin{aligned} g_t^n(y,l) &:= \mathbb{E}\left(\mathbb{1}_A\left(\mathcal{X}_t^n, V_t^n\right)|\mathcal{X}_0^n = (y_n, l)\right), \\ g_t(y,l) &:= \mathbb{E}\left(\mathbb{1}_A\left(X_t^1, Y_t, X_t^2, V(t)\right)|(X_0^1, Y_0, X_0^2, V(0)) = (y,l)\right) \end{aligned}$$

where y_n is the element in $\mathcal{W}^n$ wherein each projection $R_i y_n$ is the smallest element in $2^{-n}\mathbb{Z}$ larger than $R_i y$, $i = 1, 2, 3$; this is defined for large enough n. By Theorem 6.1 $\lim\limits_{n\to\infty} |g_t(y,l) - g_t^n(y,l)| \longrightarrow 0$ for all $(y,l) \in \mathcal{W} \times \mathbb{R}$. This discrete process follows the framework of the previous proposition with $\mathscr{L} = \mathcal{W}^n \cap [-n,n]^3$ and $d = 1$. For $z_1 = (i_1, j_1, k_1), z_2 = (i_2, j_2, k_2) \in \mathcal{W}^n$ we see $v_z = 0$ unless $i_1 = j_1$, or $j_1 = k_1$. Similarly $a_{z_1 z_2}(l) = 0$ unless one of their components are adjacent. Let $B \subset \mathcal{W} \times \mathbb{R}$,

$$\begin{aligned} &\Big|\int_{\mathcal{W}\times\mathbb{R}} g_t(y,l)d\nu(y,l) - \int_{S\times\mathbb{R}} g_t^n(y,l)\alpha_n d\nu_n(y,l)\Big| \\ &\quad \le \Big|\int_B g_t d\nu - \int_B g_t^n \alpha_n d\nu_n\Big| + \int_{(S\times\mathbb{R})\setminus B} g_t^n \alpha_n d\nu_n + \int_{(\mathcal{W}\times\mathbb{R})\setminus B} g_t d\nu \end{aligned} \tag{6.8}$$

By pointwise convergence of g_t^n to g_t, along with (6.7) and uniform boundedness of g_t^n, shows the first term $\left|\int_B g_t d\nu - \int_B g_t^n d\nu_n\right| \longrightarrow 0$ as $n \to \infty$. Recall the definition of g depends on a fixed compact set $A \subset \mathcal{W}$. The space $\mathcal{W}$ is a cone in $\mathbb{R}^3$. This set A is where the path (X^1, Y, X^2), must land in at time t, so in particular X_t^1 and X_t^2 must be in $R_1 A$, $R_2 A$ respectively. If $B \supset A$ is sufficiently large, then the chance of this occurring with an initial location of $y \in S \setminus B$ will decrease as $||y|| \to \infty$. More precisely, by the definition of $\mathcal{W}$, if $(y_1, y_2, y_3) := y \in \mathcal{W}$, and $||y|| = r > 0$, then one of $|y_1|, |y_3| \ge r/(1+\sqrt{2})$. Otherwise $|y_2| < r/(1+\sqrt{2})$ since $y_3 < y_2 < y_1$, and the triangle inequality shows

$$\begin{aligned} ||y|| \le ||(0, y_2, 0)|| + ||(y_1, 0, y_2)|| &< \frac{r}{1+\sqrt{2}} + \sqrt{\frac{r^2}{(1+\sqrt{2})^2} + \frac{r^2}{(1+\sqrt{2})^2}} \\ = \frac{r}{1+\sqrt{2}} + \frac{\sqrt{2}r}{1+\sqrt{2}} = r \end{aligned}$$

would give a contradiction. The regular Brownian paths $B^1 < X^1$ and $B^2 > X^2$, g will be bounded by a proportional of the chance that B^1 beginning from $R_1 y$, and B^2 beginning from R_2y, will reach R_1A, (resp. R_2A) by time t for $y \in S \setminus B$. This probability decreases exponentially with R_1A, R_2A. Without loss of generality we may assume $R_iA = [a_i, b_i]$. Symbolically, there is a compact set B and constant $D > 0$ such that

$$0 \leq g(y) \leq D\left(P\left(B_t^1 \in [a_1, b_1]\Big| |B_0^1| = \frac{||y||}{2}\right) + P\left(B_t^2 \in [a_3, b_3]\Big| |B_0^2| = \frac{||y||}{2}\right)\right)$$
$$\leq \widetilde{D}e^{-||y||^2/2t}$$

This shows that for a given $\epsilon > 0$ we can choose a large enough set B such that $\int_{S\setminus B} g_t d\nu < \epsilon$. Similarly we can choose B so that $\int_{S\setminus B} g_t^n d\alpha_n \nu_n < \epsilon$ by adapting the previous argument by replacing the Brownian paths B^1, B^2 replaced with continuous time random walk that depend on n. Therefore

$$\left| \int g(y,l) d\nu(y,l) - \int g_t^n(y,l)\alpha_n d\nu_n(y) \right| \longrightarrow 0$$

as $n \to \infty$. We calculate $\int g_t^n \alpha_n d\nu_n(y,l)$ via (2.9) knowing that for $i, j \in \mathcal{W}^n \cap [-n,n]$ and $V^n(s) = l$. In this case $g(l)$ is the density of a $N(0,K)$ random variable. In their notation

$$v_i = \pm K2^n, \; p_i = |\mathcal{W}^n \cap [-n,n]^3| \sim n2^{-3n}$$

where the sign of v_i depends on whether $X_1^n(s)$ or $X_2^n(s)$ is contacting Y. That is, if $R_1 i = R_2 i$ then the sign is negative. The transition rates are the same as the process $(X_1^n(s), Y^n(s), X_2^n(s))$ given $V^n(s)$ as in (6.4), yet our process X_1^n (resp. X_2^n) reflects upon n (resp. $-n$.).

By plugging in these values we see that

$$A^*h(j,l) = -p_j v_j \nabla q(l) + \sum_{i \neq j} [p_i a_{ij}(l) - p_j a_{ji}(l)] q(l) = 0$$

for all j. We give one example of this calculation when $(X_1^n, Y^n, X_2^n) = (i,j,j) = z$ with i sufficiently below j. This means that the inert particle Y is contacting the top random walker X_2^n. Let's say $V^n = l > 0$, so the inert particle Y^n is pushing against X_2^n. In this case

$$v_z = -K2^n, \; \triangledown q(l) = -\frac{q(l)}{K}.$$

Now consider the possible path connections to z, i.e the configurations which can travel to and from z:

$$z_1^{\pm} = (j \pm 2^{-n}, i, i),\ z_2 = (j, i - 2^{-n}, i),\ z_3^{\pm} = (j, i, i \pm 2^{-n}).$$

The process (X_1^n, Y^n, X_2^n) may transition from z_i to z (indicated by $\longrightarrow$) or to z_i from z ($\longleftarrow$) with the following rates

$$z_1 \xleftrightarrow{2^{2n}} z,\ z_2 \xrightarrow{l2^n} z,\ z_3 \xleftrightarrow{2^{2n}} z.$$

(Note that z may not transition to z_2 since the drift l is positive, so we say this transition rate is zero. Also, these are the only transition locations to and from z since Y^n cannot move past X_2^n). All the p_{z_i}, p_z are the same, so

$$\begin{aligned}
&A^*h(z,l) = p\left(-v_z \bigtriangledown q(l) + [a_{z_1z}(l) - a_{zz_1}(l) + a_{z_2z}(l) - a_{zz_2}(l) + a_{z_3z}(l) - a_{zz_3}(l)]q(l)\right) \\
&= p\left(-lq(l)2^n + [2^{2n} - 2^{2n} + l2^n + 2^{2n} - 2^{2n}]q(l)\right) \\
&= p(-lq(l)2^n + lq(l)2^n) = 0.
\end{aligned}$$

One can check see this also holds for various other configurations. Hence in the previous notation in (2.9) that for $0 < s < t$

$$\begin{aligned}
&\int_{\mathcal{W}\times\mathbb{R}} g_t(y,l)d\nu(y,l) - \int_{\mathcal{W}\times\mathbb{R}} g_s(y,l)d\nu(y,l) \\
&= \lim_{n\to\infty} \int_{S\times\mathbb{R}} g_t^n(y,l)\alpha_n d\nu(y,l) - \int_{\mathcal{W}\times\mathbb{R}} g_s^n(y,l)\alpha_n d\nu(y,l) \\
&= \lim_{n\to\infty} \alpha_n \left(\mathbb{E}_\pi(g(\mathcal{X}_t^n, V_t^n)) - \mathbb{E}_\pi(g(\mathcal{X}_s^n, V_s^n))\right) \\
&= \lim_{n\to\infty} \alpha_n \int_s^t \sum_{j\in\mathcal{W}^n\cap[-n,n]^3} \int_{\mathbb{R}^d} u_z(j,l)(A^*h)(j,l)dldz \\
&= 0,
\end{aligned}$$

and so

$$\int_{\mathcal{W}\times\mathbb{R}} g_t(y,l)d\nu(y,l) = \int_{\mathcal{W}\times\mathbb{R}} g_s(y,l)d\nu(y,l)$$

for any $0 < s < t$. Thus $\nu = \mu \times \gamma_k$ satisfies (6.6), and is consequently an invariant measure for (X^1, Y, X^2, V). □

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