Invariants of Poisson Algebras, Poisson Enveloping Algebras, and Deformation Quantizations

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A dissertation
submitted in partial fulfillment of
the requirements for the degree of

Doctor of Philosophy

University of Washington

2024

Reading Committee:

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Program Authorized to Offer Degree:

Mathematics

University of Washington

Abstract

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Mathematics

The Shephard-Todd-Chevalley Theorem and the Watanabe Theorem are among the earliest results addressing the homological properties of invariant subalgebras. Initially studied in the context of polynomial algebras, these theorems have motivated researchers to generalize their applicability beyond the scope of commutative algebras. Notable instances include, but certainly are not limited to: Alev and Polo's studies on enveloping algebra of semisimple Lie algebras and Weyl algebras; Kirkman, Kuzmanovich, and Zhang's studies on skew polynomial rings, quantum matrix algebras, non-PI Sklyanin algebras and down up algebras; Gaddis, Veerapen, and Wang's studies on semiclassical limits (Poisson algebras) of several families of Artin-Schelter regular algebras. In this dissertation, we will continue Gaddis, Veerapen, and Wang's studies on Poisson algebras, a commutative algebra together with a non-commutative bracket. Our primary emphasis will be on quadratic Poisson structures on polynomial rings of three variables. Our objective is to prove variants of the Shephard-Todd-Chevalley Theorem for these Poisson algebras and their associated algebraic structures: Poisson enveloping algebras and deformation quantizations. Furthermore, we will prove a variant of the Watanabe Theorem for Poisson enveloping algebras arising from quadratic Poisson structures on an arbitrary polynomial ring.

TABLE OF CONTENTS

ACKNOWLEDGMENTS

I want to express my sincerest gratitude to Professor James J. Zhang, without whose support and guidance, this dissertation would not have progressed to even half of its current level of completion. Professor Zhang, I want you to know that I have greatly benefited from studying with you over the past three years. Your lectures, seminars, and weekly meetings have been of profound values. Thank you for your mentorship.

In addition, I would like to extend my gratitude to Professor Xingting Wang, Professor Jason Gaddis and Dr. Hongdi Huang for their help with my dissertation. And, of course, I must not forget to mention my fellows: Haocheng Cai, Baicheng Li, Yuqiao Li, Andrew Tawfeek, Kuan-Ting Yeh and Jiahao Zhang. The days and nights we spent studying together will be the moments I miss the most when I depart from the University of Washington.

As for my parents, it is impossible to describe the profound significance they have held in my academic journey with words. Their continued financial support throughout both my undergraduate and graduate studies has allowed me to focus on my studies. I am forever indebted to them for their faith in my education.

Lastly, but by no means least, I want to express my appreciation to my friends Haichao Cui, Porter Howland, and Luwen Qiu for their companionship throughout this journey, even when miles apart. Your presence has been a constant source of positivity, and I will forever cherish these memories.

— *Acknowledgement of AI assistance* —

I would like to acknowledge the use of ChatGPT-4 for assistance with grammar, clarity, and sentence structure in editing this dissertation. Rest assured, ChatGPT was not used to edit or generate any mathematical components of this dissertation.

DEDICATION

To Fangyu Yu,

People who embark on the Ph.D. journey are often
asked what led them down this path.
For anyone familiar with mathematics, it doesn't take
much to justify an interest in algebra due to its
abstractness, universality, elegant diagrams, and so on.
But for me, it may be simpler than that –
I met you in my first algebra class.

Chapter 1
INTRODUCTION

This dissertation extensively references two of my preprints, [Ma23a] and [Ma23b]. Significant portions of this dissertation have been submitted for publication.

1.1 Motivations and Basic Concepts

Throughout $\Bbbk$ is an algebraically closed field of characteristic 0.

Let $A = \Bbbk[x_1, \cdots, x_n]$ and let G be a finite subgroup of the graded automorphism group of A. The invariant subalgebra of A under the action of G is

$$A^G := \{a \in A : \phi(a) = a \text{ for all } \phi \in G\}.$$

It is natural to ask: what properties, in particular, what homological properties does the invariant subalgebra A^G satisfy? Two of the earliest answers are encapsulated in the Shephard-Todd-Chevalley Theorem and the Watanabe Theorem, stated as follows:

Theorem 1.1.1. (Shephard-Todd-Chevalley Theorem, [ST54], [Che55]) Let $A = \Bbbk[x_1, \cdots, x_n]$ and G be a finite subgroup of the graded automorphism group of A. Then the invariant subalgebra A^G is regular (or equivalently, $A^G \cong A$ as $\Bbbk$-algebras) if and only if G is generated by (pseudo-)reflections.

Theorem 1.1.2. (Watanabe Theorem, [Wat74]) Let $A = \Bbbk[x_1, \cdots, x_n]$ and G be a finite subgroup of the graded automorphism group of A containing no (pseudo-)reflections. Then the invariant subalgebra A^G is Gorenstein if and only if $\det(\phi\big|_{A_1}) = 1$ for all $\phi \in G$.

In the following decades, the Shephard-Todd-Chevalley Theorem and the Watanabe Theorem have prompted researchers to generalize their applicability beyond the scope of commutative algebras: if A is a non-commutative Artin-Schelter regular algebra and G is a finite subgroup of the graded automorphism group of A, under what conditions on G is the invariant subalgebra A^G Artin-Schelter regular or Artin-Schelter Gorenstein? Artin-Schelter regular algebras, initially introduced in [AS87], emerged as the "industry standard" when researchers sought for a non-commutative generalization of polynomial algebras because they satisfy a range of properties inherent to polynomial algebras. In a more formal manner:

Definition 1.1.3. A finitely generated $\Bbbk$-algebra A is called *Artin-Schelter regular* if

(1) A is connected $\mathbb{N}$-graded: A admits a $\Bbbk$-vector space decomposition $A = \bigoplus_{n\in\mathbb{N}} A_n$ such that $A_0 = \Bbbk$ and $A_iA_j \subseteq A_{i+j}$ for all $i, j \in \mathbb{N}$.

(2) A has finite Gelfand-Kirillov dimension: $\dim_{\Bbbk} A_n$ has polynomial growth.

(3) A has finite global dimension d.

(4) A is Gorenstein: $\operatorname{Ext}^i_A(\Bbbk, A) \cong \begin{cases} 0 & i \neq d \\ \Bbbk(l) & i = d \end{cases}$, for some $l \in \mathbb{Z}$.

Example 1.1.4. Let $A = \Bbbk\langle x, y\rangle/(yx - qxy)$ for some $q \neq 0$. Then A is Artin-Schelter regular and its minimal free resolution of the trivial module $\Bbbk_A$ is:

$$0 \to A(-2) \xrightarrow{\begin{bmatrix} -qy \\ x \end{bmatrix}} A(-1)^{\oplus 2} \xrightarrow{\begin{bmatrix} x & y \end{bmatrix}} A \to \Bbbk \to 0.$$

Example 1.1.5. Let $A = \Bbbk\langle x, y\rangle/(yx - xy - x^2)$. Then A is Artin-Schelter regular and its minimal free resolution of the trivial module $\Bbbk_A$ is:

$$0 \to A(-2) \xrightarrow{\begin{bmatrix} -y - x \\ x \end{bmatrix}} A(-1)^{\oplus 2} \xrightarrow{\begin{bmatrix} x & y \end{bmatrix}} A \to \Bbbk \to 0.$$

Artin-Schelter Gorensteinness can be described as the homological components of Artin-Schelter regularity and can be formally defined as follows:

Definition 1.1.6. A finitely generated $\Bbbk$-algebra A is called *Artin-Schelter Gorenstein* if

(1) A has finite injective dimension d.

(2) A is Gorenstein: $\operatorname{Ext}^i_A(\Bbbk, A) \cong \begin{cases} 0 & i \neq d \\ \Bbbk(l) & i = d \end{cases}$, for some $l \in \mathbb{Z}$.

Example 1.1.7. Let $A = \Bbbk[x]/(x^2)$. Baer's criterion implies that A is self-injective, which in turn implies that A is Artin-Schelter Gorenstein. Note that A is not Artin-Schelter regular. The $\Bbbk$-algebra A is a Noetherian local ring of Krull dimension 0. However, the minimal number of generators of its maximal ideal does not equal to its Krull dimension.

Returning to our discussion of the non-commutative Shephard-Todd-Chevalley question and the non-commutative Watanabe question. There are some established answers, including, but not limited to, universal enveloping algebra of semisimple Lie algebras and Weyl algebras in [AP95], non-PI Sklyanin algebras of global dimension ≥ 3 in [KKZ09], skew polynomial rings and quantum matrix algebras in [KKZ10], down-up algebras in [KKZ15]. Recently, Gaddis, Veerapen, and Wang have proposed an investigation into these questions for Poisson algebras. Broadly speaking, Poisson algebras are a family of commutative algebras endowed with a non-commutative bracket. In a more rigorous language:

Definition 1.1.8. A *Poisson algebra* is a commutative $\Bbbk$-algebra P together with a bracket:

$$\{-,-\} : P^{\otimes 2} \to P$$

such that

(1) $(P, \{-,-\})$ is a Lie algebra over $\Bbbk$, namely $\{-,-\}$ satisfies bilinearity, alternativity, anti-commutativity, and the Jacobi identity.

(2) $\{-,-\}$ satisfies the Leibniz rule: $\{a, bc\} = \{a, b\}c + b\{a, c\}$ for all $a, b, c \in P$.

Definition 1.1.9. Let $P = \Bbbk[x_1, \cdots, x_n]$ be a Poisson algebra under the standard grading.

- The Poisson algebra P is called *quadratic* if $\{P_1, P_1\} \subseteq P_2$.
- The *modular derivation* of P is

$$\underline{m}(f) := \sum_{i=1}^{n} \frac{\partial\{x_i, f\}}{\partial x_i}$$

  for all $f \in P$. The Poisson algebra P is called *unimodular* if its modular derivation $\underline{m}$ vanishes and is called *non-unimodular* otherwise.

Example 1.1.10. Let $P = \Bbbk[x, y]$ together with the following bracket:

$$\{f, g\} = \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial y}\right)qxy,$$

for some $q \neq 0$, for all $f, g \in P$. Then P is a quadratic Poisson algebra. However, P is non-unimodular: $\underline{m}(x) = \dfrac{\partial\{y, x\}}{\partial y} = -qx \neq 0$.

Example 1.1.11. Let $P = \Bbbk[x, y]$ together with the following bracket:

$$\{f, g\} = \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial y}\right)x^2,$$

for all $f, g \in P$. Then P is a quadratic Poisson algebra. However, P is non-unimodular: $\underline{m}(y) = \dfrac{\partial\{x, y\}}{\partial x} = 2x \neq 0$.

Poisson algebras originally emerged in classical mechanics and subsequently assume a role in mathematical physics. In recent decades, Poisson algebras have also attracted attention in pure mathematics due to their proximity to Artin-Schelter regular algebras. One instance of such proximity is Poisson enveloping algebras.

Definition 1.1.12. Let P be a Poisson algebra. A (The) *Poisson enveloping algebra* of P is a triple (U, α, β):

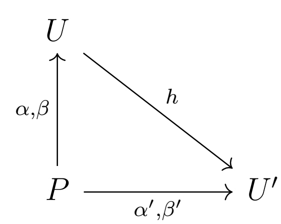
- U is a $\Bbbk$-algebra,
- $\alpha : (P, \cdot) \to U$ is an algebra homomorphism,
- $\beta : (P, \{-,-\}) \to U_L$ is a Lie algebra homomorphism,

subjecting to the following conditions:

(1) $\alpha(\{a, b\}) = \beta(a)\alpha(b) - \alpha(b)\beta(a)$ for all $a, b \in P$.

(2) $\beta(ab) = \alpha(a)\beta(b) + \alpha(b)\beta(a)$ for all $a, b \in P$.

(3) If (U', α', β') is another triple satisfying (1) and (2), then there exists a unique algebra homomorphism $h : U \to U'$ making the following diagram commutative:



If $P = \Bbbk[x_1, \cdots, x_n]$ is a quadratic Poisson algebra, then its Poisson enveloping algebra $U(P)$ satisfies a range of preferred properties, including being Artin-Schelter regular. This is one connection between Poisson algebras and Artin-Schelter regular algebras. Another connection lies in semiclassical limits and deformation quantizations, which are introduced in the following definitions:

Definition-Lemma 1.1.13. Let A be a $\Bbbk[[\hbar]]$-algebra such that $A/\hbar A$ is a commutative algebra. Let $f, g \in A$, $(f + \hbar A)(g + \hbar A) = (g + \hbar A)(f + \hbar A)$ and therefore $[f, g] = fg - gf =$

$\hbar\gamma(f,g)$ for some $\gamma(f,g) \in A$. Let $\overline{f}, \overline{g}$ be the projections of f, g on $A/\hbar A$, respectively. Define $\{-,-\} : (A/\hbar A)^{\otimes 2} \to A/\hbar A$ as follows:

$$\{\overline{f}, \overline{g}\} = \overline{\gamma(f,g)},$$

Then $(A/\hbar A, \{-,-\})$ is a Poisson algebra, called the *semiclassical limit* of A.

Definition 1.1.14. Let P be a Poisson algebra. A *(graded) deformation quantization* of P is a graded $\Bbbk[[\hbar]]$-algebra $P_\hbar$ satisfying the following conditions:

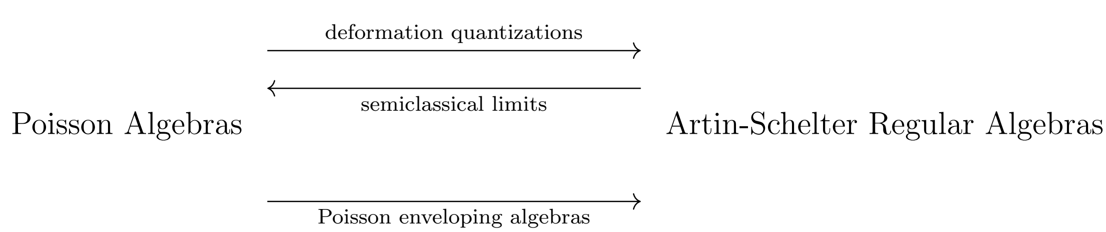
(1) $P_\hbar \cong P[[\hbar]]$ as graded $\Bbbk[[\hbar]]$-modules.

(2) $P_\hbar/\hbar P_\hbar \cong P$ as graded $\Bbbk$-algebras. In particular, for all $f, g \in P_\hbar$, $(f+\hbar P_\hbar)(g+\hbar P_\hbar) = (g + \hbar P_\hbar)(f + \hbar P_\hbar)$ and therefore $[f,g] = fg - gf = \hbar\gamma(f,g)$ for some $\gamma(f,g) \in P_\hbar$.

(3) Let $f, g \in P_\hbar$ and let $\overline{f}, \overline{g}$ be the projections of f, g on $P_\hbar/\hbar P_\hbar \cong P$, respectively. Then $\{\overline{f}, \overline{g}\} = \overline{\gamma(f,g)}$.

The Poisson algebra P is called *quantizable* if P admits a deformation quantization $P_\hbar$. It is possible for a Poisson algebra to admit multiple non-isomorphic deformation quantizations; nevertheless, within the scope of this dissertation, our primary focus lies on a variant of the deformation quantizations constructed in [DML98].

As previously mentioned, Poisson algebras are associated with Artin-Schelter regular algebras through semiclassical limit, deformation quantizations and Poisson enveloping algebras:



These associations have established a strong correlation between investigations of the Shephard-Todd-Chevalley question and the Watanabe question for Poisson algebras, and investigations of the same questions for Artin-Schelter regular algebras. In their study [GVW23], Gaddis, Veerapen, and Wang provided a partial answer to the Shephard-Todd-Chevalley question by investigating multiple Poisson structures arising from the semiclassical limits of specific families of Artin-Schelter regular algebras. Additionally, they offered some insights on the Watanabe question for Poisson enveloping algebras under induced actions. Building upon their research, we shall further investigate these questions for a broader range of Poisson structures, with a primary emphasis on quadratic Poisson structures on the polynomial ring of three variables $\Bbbk[x_1, x_2, x_3]$.

In the remaining part of this section, we will introduce a set of concepts necessary for understanding the statements of our main results. To start, we will define Poisson homomorphisms and Poisson ideals.

Definition 1.1.15. Let P, Q be Poisson algebras.

- A map $\phi : P \to Q$ is called a *Poisson homomorphism* if ϕ is an algebra homomorphism and a Lie algebra homomorphism.

- A subset $I \subseteq P$ is called a *Poisson ideal* if I is an ideal and a Lie ideal. An element $f \in P$ is called *Poisson normal* if $\{f, P\} \subseteq fP$, or equivalently, the principal ideal (f) is a Poisson ideal.

Subsequently, we will review the key concepts in non-commutative invariant theory, both in the context of Poisson algebras and in the context of Artin-Schelter regular algebras.

Definition 1.1.16. Let $P = \Bbbk[x_1, \cdots, x_n]$ be a Poisson algebra under the standard grading.

- A *graded Poisson automorphism* of P is a bijective Poisson homomorphism $\phi : P \to P$ such that $\phi(P_i) = P_i$ for all $i \geq 0$. The graded Poisson automorphism group of P will be denoted as $\mathrm{PAut}_{\mathrm{gr}}(P)$.

- A *Poisson reflection* of P is a finite-order graded Poisson automorphism $\phi : P \to P$ such that $\phi\big|_{P_1}$ has eigenvalues $\underbrace{1, \cdots, 1}_{n-1}, \xi$ for some primitive root of unity ξ. The set of Poisson reflections of P will be denoted as $\mathrm{PR}(P)$.

Definition 1.1.17. Let A be a connected $\mathbb{N}$-graded $\Bbbk$-algebra that is locally finite: $\dim_{\Bbbk} A_i < \infty$ for all $i \in \mathbb{N}$. Let ϕ be a graded automorphism of A. The *trace series* of ϕ is

$$\mathrm{Tr}_A(\phi, t) = \sum_{i=0}^{\infty} \mathrm{tr}(\phi\big|_{A_i}) t^i.$$

In particular, if ϕ is the identity of A, then we recover the *Hilbert series* of A:

$$h_A(t) = \mathrm{Tr}_A(\mathrm{id}_A, t) = \sum_{i=0}^{\infty} (\dim_{\Bbbk} A_i) t^i.$$

Suppose that $A = \Bbbk[x_1, \cdots, x_n]$ and ϕ is a graded automorphism of A. Let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of $\phi\big|_{A_1}$. The trace series

$$\mathrm{Tr}_A(\phi, t) = \frac{1}{\det(I_n - t\phi\big|_{A_1})} = \frac{1}{(1 - \lambda_1 t) \cdots (1 - \lambda_n t)}.$$

If in particular ϕ is a reflection: $\lambda_1 = \cdots = \lambda_{n-1} = 1$ and $\lambda_n = \xi$ for some primitive root of unity ξ, then the trace series

$$\mathrm{Tr}_A(\phi, t) = \frac{1}{(1 - t)^{n-1}(1 - \xi t)}.$$

With this calculation as motivation, Kirkman, Kuzmanovich, and Zhang generalized the concept of "reflection" to non-commutative settings [KKZ09].

Definition 1.1.18. Let A be an Artin-Schelter regular algebra with Hilbert series $h_A(t) = \dfrac{1}{(1-t)^n f(t)}$ for some polynomial $f(t)$ satisfying $f(1) \neq 0$. A *quasi-reflection* of A is a finite-order graded automorphism $\phi : A \to A$ such that $\mathrm{Tr}_A(\phi, t) = \dfrac{1}{(1-t)^{n-1} g(t)}$ for some polynomial $g(t)$ satisfying $g(1) \neq 0$. The set of quasi-reflections of A will be denoted as $\mathrm{QR}(A)$.

Let A be an Artin-Schelter regular algebra with Hilbert series $h_A(t) = \dfrac{1}{(1-t)^n f(t)}$ for some polynomial $f(t)$ satisfying $f(1) \neq 0$. A graded automorphism $\phi : A \to A$ is called a *classical reflection* if $\phi\big|_{A_1}$ has an eigenvalue 1 of multiplicity $n-1$ and another eigenvalue ξ of multiplicity 1 for some primitive root of unity ξ. The following two examples illustrate that a classical reflection is not necessarily a quasi-reflection, and conversely, a quasi-reflection is not necessarily a classical reflection.

Example 1.1.19. [Kir15, Example 1.8] Let $A = \Bbbk\langle x, y\rangle/(yx + xy)$ and ϕ be the graded automorphism of A satisfying $\phi\big|_{A_1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The graded automorphism ϕ is a classical reflection because its eigenvalues are ± 1. However, $\phi(x^i y^j) = y^i x^j = (-1)^{ij} x^j y^i$ for all $i, j \geq 0$ and $\mathrm{Tr}_A(\phi, t) = 1 - t^2 + t^4 - t^6 + t^8 - \cdots = \dfrac{1}{1+t^2}$. By definition, ϕ is not a quasi-reflection.

Example 1.1.20. [Kir15, Example 1.8] Let $A = \Bbbk\langle x, y\rangle/(yx + xy)$ and ϕ be the graded automorphism of A satisfying $\phi\big|_{A_1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The graded automorphism ϕ is not a classical reflection because its eigenvalues are $\pm i$. Nevertheless, $\phi(x^i y^j) = (-1)^i y^i x^j = (-1)^{i(j+1)} x^j y^i$ for all $i, j \geq 0$ and $\mathrm{Tr}_A(\phi, t) = 1 + t^2 + t^4 + t^6 + t^8 + \cdots = \dfrac{1}{(1-t)(1+t)}$. By definition, ϕ is a quasi-reflection.

Definition 1.1.21. A *quantum polynomial ring* is a Noetherian Artin-Schelter regular domain of global dimension n, with Hilbert series $h_A(t) = \dfrac{1}{(1-t)^n}$.

A quasi-reflection ϕ of a quantum polynomial ring A has a structured form, being either a classical reflection or a *mystic reflection*: ϕ has order 4 and $\phi\big|_{A_1}$ has eigenvalues $\underbrace{1, \cdots, 1}_{n-2}, i, -i$ [KKZ09, Theorem 3.1]. Note that the graded automorphism ϕ in Example 1.1.20 is a mystic reflection.

Finally, we introduce the homological determinant.

Definition-Lemma 1.1.22. Let A be an Artin-Schelter regular algebra of global dimension n and let ϕ be a graded automorphism of A. The trace series $\mathrm{Tr}_A(\phi, t) = \dfrac{1}{1 + c_1 t + \cdots + c_l t^l}$ for some $c_i \in \Bbbk$, $1 \le i \le l$ [JZ97, Theorem 2.3]. The *homological determinant* of ϕ is

$$\mathrm{hdet}(\phi) = (-1)^n c_l \text{ [JZ00, Lemma 2.6] [JZ00, Proposition 4.2].}$$

It is possible to define the homological determinant of a graded automorphism of an Artin-Schelter Gorenstein algebra. However, within the scope of this dissertation, the algebraic structures we will investigate are all Artin-Schelter regular. Consequently, we will adopt the above, simpler definition.

It is noteworthy that the homological determinant $\mathrm{hdet} : \mathrm{Aut}_{\mathrm{gr}}(A) \to \Bbbk^\times$ is a group homomorphism [JZ00, Proposition 2.5]. Additionally, if $A = \Bbbk[x_1, \cdots, x_n]$, the homological determinant $\mathrm{hdet}(\phi)$ coincides with the determinant $\det(\phi\big|_{A_1})$ [JZ00, page 322].

1.2 Main Results

The primary contribution of our research is the following variants of the Shephard-Todd-Chevalley Theorem and the Watanabe Theorem for quadratic Poisson structures on the polynomial ring $\Bbbk[x_1, x_2, x_3]$ and their associated Artin-Schelter regular algebras.

Theorem 1.2.1. (Shephard-Todd-Chevalley Theorem) Let $P = \Bbbk[x_1, x_2, x_3]$ be a unimodular quadratic Poisson algebra and let $G \subseteq \mathrm{PAut}_{\mathrm{gr}}(P)$ be a finite subgroup. Then the invariant subalgebra P^G is isomorphic to P as Poisson algebras if and only if G is trivial.

Theorem 1.2.2. (Shephard-Todd-Chevalley Theorem) Let $P = \Bbbk[x_1, x_2, x_3]$ be a non-unimodular quadratic Poisson algebra and let $G \subseteq \mathrm{PAut}_{\mathrm{gr}}(P)$ be a finite subgroup. If $\{-, -\}$ satisfies the following conditions on its coefficients:

- *Non-unimodular 1.* $p, q, r \neq 0$.
- *Non-unimodular 2.* $p, q \neq 0$, $p \neq q$, $4p^2 + q^2 \neq 0$.

- *Non-unimodular 3.* $p \neq 0$.
- *Non-unimodular 4.* $p, q \neq 0$.
- *Non-unimodular 5.* $p \neq 0, \frac{1}{2}$.
- *Non-unimodular 6.* $p \neq 0$.
- *Non-unimodular 7.* $p, q \neq 0$, $2p + r \neq 0$, $(2p + r)^2 + q^2 \neq 0$.
- *Non-unimodular 8.* $p \neq 0$, $p + q \neq 0$.
- *Non-unimodular 9.* $p \neq 0$.
- *Non-unimodular 10.* $p \neq 0, -\frac{1}{4}, -\frac{1}{3}, -\frac{1}{2}$.
- *Non-unimodular 11.* $p \neq 0, -\frac{1}{2}, -\frac{1}{4}$, $q \neq 0$.
- *Non-unimodular 12.* $p \neq 0, -\frac{1}{2}, -\frac{1}{3}$, $q = 0$.
- *Non-unimodular 13.* $p \neq 0, -\frac{1}{2}$, $q = 0$, $r = 0$.

Then the invariant subalgebra P^G is isomorphic to P as Poisson algebras if and only if G is trivial.

Let $P = \Bbbk[x_1, \cdots, x_n]$ be a Poisson algebra and let $U(P)$ be its Poisson enveloping algebra. If ϕ is a graded Poisson automorphism of P, it is possible to construct a unique graded automorphism $\widetilde{\phi}$ of $U(P)$ such that

$$\begin{CD} P @>{\phi}>> P \\ @V{\alpha,\beta}VV @VV{\alpha,\beta}V \\ U(P) @>>{\widetilde{\phi}}> U(P) \end{CD}$$

is a commutative diagram.

Theorem 1.2.3. Let $P = \Bbbk[x_1, \cdots, x_n]$ be a quadratic Poisson algebra and $U(P)$ be its Poisson enveloping algebra. Let G be a finite subgroup of the graded Poisson automorphism group of P and let $\widetilde{G} = \{\widetilde{\phi} : \phi \in G\}$ be the corresponding finite subgroup of the graded automorphism group of $U(P)$. The invariant subalgebra $U(P)^{\widetilde{G}}$ is Artin-Schelter regular if and only if G is trivial.

Theorem 1.2.4. (Watanabe Theorem) Let $P = \Bbbk[x_1, \cdots, x_n]$ be a quadratic Poisson algebra and let $U(P)$ be its Poisson enveloping algebra. Let G be a finite subgroup of the graded Poisson automorphism group of P and let $\widetilde{G} = \{\widetilde{\phi} : \phi \in G\}$ be the corresponding finite subgroup of the graded automorphism group of $U(P)$. If G is generated by graded Poisson automorphisms $\phi_1, \cdots, \phi_m$ such that $\det(\phi_i\big|_{P_1}) = \pm 1$ for all $1 \leq i \leq m$, then $U(P)^{\widetilde{G}}$ is Artin-Schelter Gorenstein.

Let $P = \Bbbk[x_1, x_2, x_3]$ be a quadratic Poisson algebra, with its Poisson structure written as $\{x_i, x_j\} = \sum_{k,l} c_{i,j}^{k,l} x_k x_l$, $1 \leq i < j \leq 3$. Donin and Makar-Limanov proved that P is quantizable, admitting the following deformation quantization [DML98]:

$$P_\hbar = \Bbbk\langle y_1, y_2, y_3\rangle / ([y_i, y_j] = \frac{\hbar}{2} \sum_{k,l} c_{i,j}^{k,l}(y_k y_l + y_l y_k))_{1 \leq i,j \leq 3},$$

for some $0 \neq \hbar \in \Bbbk$. It is possible to prove that $P_\hbar$ is a quantum polynomial ring and $\mathrm{PAut}_{\mathrm{gr}}(P) \cong \mathrm{Aut}_{\mathrm{gr}}(P_\hbar)$. If in addition P is unimodular, then $\mathrm{PR}(P) \cong \mathrm{QR}(P_\hbar)$.

Theorem 1.2.5. (Shephard-Todd-Chevalley Theorem) Let $P = \Bbbk[x_1, x_2, x_3]$ be a unimodular quadratic Poisson algebra. Let G be a finite subgroup of the graded Poisson automorphism group of P, and $G_\hbar$ be the corresponding finite subgroup of the graded automorphism group of $P_\hbar$ under the isomorphism $\mathrm{PAut}_{\mathrm{gr}}(P) \cong \mathrm{Aut}_{\mathrm{gr}}(P_\hbar)$. The following are equivalent:

(1) G is generated by Poisson reflections.

(2) $G_\hbar$ is generated by quasi-reflections.

(3) $P_\hbar^{G_\hbar}$ is Artin-Schelter regular.

Theorem 1.2.6. Let $P = \Bbbk[x_1, x_2, x_3]$ be a unimodular quadratic Poisson algebra. Let G be a finite subgroup of the graded Poisson automorphism group of P and let $G_\hbar$ be the corresponding finite subgroup of the graded automorphism group of $P_\hbar$ under the isomorphism

$\mathrm{PAut}_{\mathrm{gr}}(P) \cong \mathrm{Aut}_{\mathrm{gr}}(P_\hbar)$. Define $Q_\hbar := P_\hbar^{G_\hbar}$, with $\hbar$ viewed as a formal parameter (as opposed to a scalar value). Then

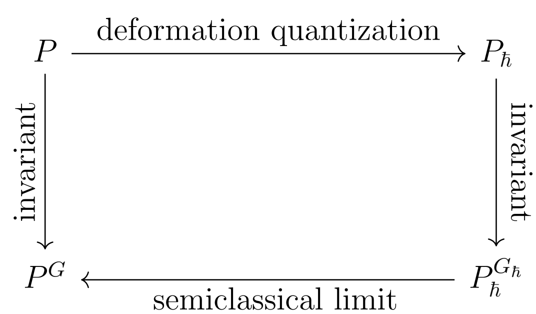
(1) $Q_\hbar/(\hbar) \cong P^G$ as $\Bbbk$-algebras.

(2) For all $f, g \in Q_\hbar$, $fg - gf = \hbar\pi_1(f,g)$ for some $\pi_1(f,g) \in Q_\hbar$.

(3) $Q_\hbar/(\hbar)$ together with the following Poisson bracket:

$$\{\overline{f}, \overline{g}\} = \overline{\pi_1(f,g)},$$

where $\overline{(\ )}$ denotes the image under the natural projection $Q_\hbar \to Q_\hbar/(\hbar)$, is isomorphic to P^G as Poisson algebras.



Chapter 2

QUADRATIC POISSON STRUCTURES ON $\mathbb{k}[X_1, X_2, X_3]$

In this chapter, we delve into the classification of quadratic Poisson structures on the polynomial ring of three variables $\mathbb{k}[x_1, x_2, x_3]$, along with their associated algebraic structures, namely, Poisson enveloping algebras and deformation quantizations.

2.1 Classification of Quadratic Poisson Structures

Let P be the polynomial ring of three variables $\mathbb{k}[x_1, x_2, x_3]$. Dufour and Haraki, Donin and Marka-Limanov, and Liu and Xu, have independently classified all quadratic Poisson structures on P into 13 + 1 classes [DH91], [DML98], [LX92]:

Case	$\{x_1, x_2\}$	$\{x_2, x_3\}$	$\{x_3, x_1\}$
1	px_1x_2	qx_2x_3	rx_1x_3
2	$p(x_1^2 + x_2^2)$	$2px_1x_3 - qx_2x_3$	$qx_1x_3 + 2px_2x_3$
3	x_1^2	$2x_1x_3 - px_2x_3$	px_1x_3
4	px_1x_2	$x_1^2 + qx_2x_3$	px_1x_3
5	px_1^2	$(2p+1)x_1x_3 + x_2x_3$	$-x_1x_3$
6	$-\frac{1}{2}x_1^2$	px_2x_3	$-px_1x_3$
7	$p(x_1^2 + x_2^2)$	$(2p+r)x_1x_3 + qx_2x_3$	$-qx_1x_3 + (2p+r)x_2x_3$
8	$\frac{p+q}{2}x_1^2 + \frac{p+q}{2}x_2^2 \pm x_3^2$	px_1x_3	px_2x_3
9	$-\frac{1}{3}x_1^2$	$px_1^2 - \frac{1}{3}x_2^2 + \frac{1}{3}x_1x_3$	$\frac{1}{3}x_1x_2$

Case	$\boldsymbol{\{x_1, x_2\}}$	$\boldsymbol{\{x_2, x_3\}}$	$\boldsymbol{\{x_3, x_1\}}$
10	$-(2p+1)x_1^2$	$px_2^2 - (4p+1)x_1x_3$	$(2p+1)x_1x_2$
11	$px_1^2 + qx_3^2$	$(2p+1)x_1x_3$	0
12	$px_1^2 + qx_3^2$	$x_1^2 + (2p+1)x_1x_3$	0
13	$px_1^2 + qx_3^2 + 2x_1x_3$	$rx_1^2 + x_3^2 + (2p+1)x_1x_3$	0
14	$\frac{\partial \Omega}{\partial x_3}$	$\frac{\partial \Omega}{\partial x_1}$	$\frac{\partial \Omega}{\partial x_2}$

In Case 5, $p \neq \frac{1}{2}$. In Case 14, Ω is a homogeneous polynomial in x_1, x_2, x_3 of degree 3, called the *superpotential* of the Poisson structure. For $P = \Bbbk[x_1, x_2, x_3]$, Luo, Wang, and Wu have established that unimodularity of a Poisson structure is equivalent to the existence of a superpotential Ω [LWW15, Proposition 2.6]. In other words, Case 1 - Case 13 consist of all non-unimodular Poisson structures on P, assuming that the coefficients p, q, r are placed in general positions; Case 14 consists of all unimodular Poisson structures on P.

There are multiple references of the classification of the homogeneous polynomials in x_1, x_2, x_3 of degree 3, for example [BW79]. It should be pointed out that every unimodular Poisson structure on $P = \Bbbk[x_1, x_2, x_3]$, or equivalently, every Poisson structure on $P = \Bbbk[x_1, x_2, x_3]$ derived from a superpotential Ω, is isomorphic to the Poisson structure derived from one of the following homogeneous polynomials [HWZ23, Theorem 3.4]:

Case	$\boldsymbol{\Omega}$
14-1	x_1^3
14-2	$x_1^2x_2$
14-3	$2x_1x_2x_3$
14-4	$x_1^2x_2 + x_1x_2^2$

Case	Ω
14-5	$x_1^3 + x_2^2 x_3$
14-6	$x_1^3 + x_1^2 x_3 + x_2^2 x_3$
14-7	$\frac{1}{3}(x_1^3 + x_2^3 + x_3^3) - \lambda x_1 x_2 x_3, \lambda^3 \neq 1$
14-8	$x_1^3 + x_1^2 x_2 + x_1 x_2 x_3$
14-9	$x_1^2 x_3 + x_1 x_2^2$

Plugging the classification of Ω into Case 14 yields the classification of unimodular Poisson structures on $\Bbbk[x_1, x_2, x_3]$ as follows:

Case	$\{x_1, x_2\}$	$\{x_2, x_3\}$	$\{x_3, x_1\}$
14-1	0	$3x_1^2$	0
14-2	0	$2x_1x_2$	x_1^2
14-3	$2x_1x_2$	$2x_2x_3$	$2x_1x_3$
14-4	0	$2x_1x_2 + x_2^2$	$x_1^2 + 2x_1x_2$
14-5	x_2^2	$3x_1^2$	$2x_2x_3$
14-6	$x_1^2 + x_2^2$	$3x_1^2 + 2x_1x_3$	$2x_2x_3$
14-7	$x_3^2 + \lambda x_1x_2$	$x_1^2 + \lambda x_2x_3$	$x_2^2 + \lambda x_1x_3$
14-8	x_1x_2	$3x_1^2 + 2x_1x_2 + x_2x_3$	$x_1^2 + x_1x_3$
14-9	x_1^2	$2x_1x_3 + x_2^2$	$2x_1x_2$

2.2 *Poisson Twistings*

Let $P = \Bbbk[x_1, \cdots, x_n]$ be a non-unimodular $\mathbb{Z}$-graded Poisson algebra such that $\sum_{i=1}^{n} \deg(x_i) \neq 0$. It is possible to transform the Poisson structure of P into a unimodular Poisson structure using Poisson twists, the Poisson analog of the Zhang twists [Zha96] of associative algebras.

Definition-Lemma 2.2.1. [TWZ22, Theorem 3.8] Let $P = \Bbbk[x_1, \cdots, x_n]$ be a $\mathbb{Z}$-graded Poisson algebra: $\deg(x_i) = d_i$ for all $1 \leq i \leq n$, such that $d = \sum_{i=1}^{n} d_i \neq 0$. Let $\pi = \{-,-\}$ be its Poisson structure and let $\underline{m}$ be its modular derivation. Define

$$\begin{aligned}\langle -,- \rangle : P^{\otimes 2} \to & P \\ f \otimes g \mapsto & \{f, g\} - \frac{1}{d}\deg(f) f \underline{m}(g) + \frac{1}{d}\deg(g) g \underline{m}(f),\end{aligned}$$

for all homogeneous elements $f, g \in P$. After extending bilinearly, $\langle -,- \rangle$ is a unimodular Poisson structure on P. In future discussions, the Poisson algebra $(P, \langle -,- \rangle)$ will be denoted as $P^{-\frac{1}{d}\underline{m}}$, with its Poisson structure written as $-\frac{1}{d} E \wedge \underline{m}$.

Example 2.2.2. Let $P = \Bbbk[x_1, x_2, x_3]$ be a non-unimodular quadratic Poisson algebra. Then P can be transformed into the following unimodular Poisson algebras:

Case	**Superpotential**
1	$\frac{1}{3}(p+q+r)x_1x_2x_3$
2	$px_1^2x_3 + px_2^2x_3$
3	$x_1^2x_2$
4	$\frac{1}{3}(2p+q)x_1x_2x_3 + \frac{1}{3}x_1^3$
5	$(p+\frac{1}{3})x_1^2x_3$
6	$-\frac{1}{6}x_1^2x_3$
7	$(p+\frac{1}{3}r)x_1^2x_3 + (p+\frac{1}{3}r)x_2^2x_3$
8	$\frac{1}{6}(3p+q)x_1^2x_3 + \frac{1}{6}(3p+q)x_2^2x_3 \pm \frac{1}{3}x_3^3$
9	$\frac{1}{3}px_1^3$
10	$\frac{1}{3}(3p+1)x_1x_2^2 - \frac{2}{3}(3p+1)x_1^2x_3$

Case	Superpotential
11	$\frac{1}{3}(3p+1)x_1^2x_3 + \frac{1}{3}qx_3^3$
12	$\frac{1}{3}x_1^3 + \frac{1}{3}(3p+1)x_1^2x_3 + \frac{1}{3}qx_3^3$
13	$\frac{1}{3}rx_1^3 + \frac{1}{3}(3p+1)x_1^2x_3 + x_1x_3^2 + \frac{1}{3}qx_3^3$

2.3 Poisson Enveloping Algebras

Let $P = \Bbbk[x_1, \cdots, x_n]$ be a Poisson algebra. Its Poisson enveloping algebra $U(P)$ is necessarily unique and can be described by an explicit set of generators and relations as follows:

Theorem 2.3.1. [OPS06, 2.4], [Bav21, Theorem 2.2] Let $P = \Bbbk[x_1, \cdots, x_n]$ be a Poisson algebra. The Poisson enveloping algebra $U(P)$ is the free $\Bbbk$-algebra generated by $x_1, \cdots, x_n, y_1, \cdots, y_n$ subjecting to the following relations:

(1) $[x_i, x_j] = 0$,

(2) $[y_i, y_j] = \sum_{k=1}^{n} \frac{\partial\{x_i, x_j\}}{\partial x_k} y_k$,

(3) $[y_i, x_j] = \{x_i, x_j\}$,

for all $1 \leq i, j \leq n$, and α, β are defined as the follows:

$$\begin{aligned} \alpha : P \to & U(P), & \beta : P \to U(P) \\ f \mapsto & f, & f \mapsto \sum_{k=1}^{n} \frac{\partial f}{\partial x_k} y_k, \end{aligned}$$

for all $f \in P$.

Remark 2.3.2. Since α is the identity map on P, we will abuse notation slightly by omitting α when referring to the elements $\alpha(f) \in U(P)$ for all $f \in P$.

If in addition $P = \Bbbk[x_1, \cdots, x_n]$ is quadratic, then the Poisson enveloping algebra $U(P)$ is Artin-Schelter regular [LWZ17, Corollary 1.5] and satisfies a range of preferred qualities:

- $U(P)$ is Noetherian [Oh99, Proposition 9].
- $U(P)$ admits a Poincaré-Birkhoff-Witt basis:

$$\{x_1^{i_1} \cdots x_n^{i_n} y_1^{j_1} \cdots y_n^{j_n} : i_r, j_s \geq 0\} \text{ [OPS06, Theorem 3.7].}$$

- $U(P)$ has global dimension $2n$ [BZ18, Proposition 2.1].
- The Hilbert series $h_{U(P)}(t) = \dfrac{1}{(1-t)^{2n}}$ [GVW23, Lemma 5.4].

Example 2.3.3. Let $P = \Bbbk[x_1, x_2]$ be the Poisson algebra $\{f, g\} = \left(\frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_2} - \frac{\partial g}{\partial x_1}\frac{\partial f}{\partial x_2}\right) q x_1 x_2$, for some $q \neq 0$, for all $f, g \in P$. Its Poisson enveloping algebra $U(P)$ is the $\Bbbk$-algebra $\Bbbk\langle x_1, x_2, y_1, y_2\rangle$ subjecting to the following relations:

- $x_2 x_1 = x_1 x_2$,
- $y_2 y_1 = y_1 y_2 - q x_2 y_1 - q x_1 y_2$,
- $y_1 x_1 = x_1 y_1$,
- $y_1 x_2 = x_2 y_1 + q x_1 x_2$,
- $y_2 x_1 = x_1 y_2 - q x_1 x_2$,
- $y_2 x_2 = x_2 y_2$.

Example 2.3.4. Let $P = \Bbbk[x_1, x_2]$ be the Poisson algebra $\{f, g\} = \left(\frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_2} - \frac{\partial g}{\partial x_1}\frac{\partial f}{\partial x_2}\right) x_1^2$, for all $f, g \in P$. Its Poisson enveloping algebra $U(P)$ is the $\Bbbk$-algebra $\Bbbk\langle x_1, x_2, y_1, y_2\rangle$ subjecting to the following relations:

- $x_2 x_1 = x_1 x_2$,

- $y_2y_1 = y_1y_2 - 2x_1y_1$,
- $y_1x_1 = x_1y_1$,
- $y_1x_2 = x_2y_1 + x_1^2$,
- $y_2x_1 = x_1y_2 - x_1^2$,
- $y_2x_2 = x_2y_2$.

Let $P = \Bbbk[x_1, \cdots, x_n]$ be a quadratic Poisson algebra. Suppose that ϕ is a graded Poisson automorphism of P. It is natural to ask: does ϕ induce a unique graded automorphism $\widetilde{\phi}$ on the Poisson enveloping algebra $U(P)$ that is natural in the sense that the following diagram is commutative:

$$\begin{array}{ccc} P & \xrightarrow{\alpha,\ \beta} & U(P) \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\widetilde{\phi}} \\ P & \xrightarrow[\alpha,\ \beta]{} & U(P) \end{array}$$

The answer is affirmative.

Lemma 2.3.5. Let $P = \Bbbk[x_1, \cdots, x_n]$ be a quadratic Poisson algebra. Suppose that ϕ is a graded Poisson automorphism of P. Then there exists a unique graded automorphism $\widetilde{\phi}$ on the Poisson enveloping algebra $U(P)$ such that

$$\alpha \circ \phi = \widetilde{\phi} \circ \alpha, \quad \beta \circ \phi = \widetilde{\phi} \circ \beta.$$

Proof. Define $\widetilde{\phi} : U(P) \to U(P)$ as follows: $\widetilde{\phi}(x_i) = \phi(x_i)$ and $\widetilde{\phi}(y_i) = \sum_{j=1}^{n} \frac{\partial \phi(x_i)}{\partial x_j} y_j$, for all $1 \leq i \leq n$. For commutativity, $\alpha \circ \phi = \widetilde{\phi} \circ \alpha$;

$$\beta \circ \phi(x_i) = \sum_{j=1}^{n} \frac{\partial \phi(x_i)}{\partial x_j} y_j, \quad \widetilde{\phi} \circ \beta(x_i) = \widetilde{\phi}(y_i) = \sum_{j=1}^{n} \frac{\partial \phi(x_i)}{\partial x_j} y_j,$$

for all $1 \leq i \leq n$, and therefore $\beta \circ \phi = \widetilde{\phi} \circ \beta$. Suppose that $\widetilde{\widetilde{\phi}} : U(P) \to U(P)$ is another graded automorphism such that $\alpha \circ \phi = \widetilde{\widetilde{\phi}} \circ \alpha$ and $\beta \circ \phi = \widetilde{\widetilde{\phi}} \circ \beta$. From the commutativity $\alpha \circ \phi = \widetilde{\widetilde{\phi}} \circ \alpha$, we have $\widetilde{\widetilde{\phi}}(x_i) = \phi(x_i) = \widetilde{\phi}(x_i)$, for all $1 \leq i \leq n$. From the commutativity $\beta \circ \phi = \widetilde{\widetilde{\phi}} \circ \beta$, we have

$$\widetilde{\widetilde{\phi}}(y_i) = \widetilde{\widetilde{\phi}} \circ \beta(x_i) = \beta \circ \phi(x_i) = \widetilde{\phi} \circ \beta(x_i) = \widetilde{\phi}(y_i),$$

for all $1 \leq i \leq n$. Since $\widetilde{\widetilde{\phi}}$ and $\widetilde{\phi}$ agree on the generators of $U(P)$, the graded automorphism $\widetilde{\widetilde{\phi}}$ coincides with the graded automorphism $\widetilde{\phi}$. □

Retain the above notations. Suppose that G is a subgroup of the graded Poisson automorphism of P. By Lemma 2.3.5, we can construct a subgroup $\widetilde{G} = \{\widetilde{\phi} : \phi \in G\}$ of the graded automorphism group of $U(P)$. It is natural to ask: is $\widetilde{G}$ isomorphic to G as groups? Once again, the answer is affirmative.

Lemma 2.3.6. Let $P = \Bbbk[x_1, \cdots, x_n]$ be a quadratic Poisson algebra. Suppose that G is a subgroup of the graded Poisson automorphism group of P and $\widetilde{G} = \{\widetilde{\phi} : \phi \in G\}$ is the corresponding subgroup of the graded automorphism group of $U(P)$. Then $\widetilde{G}$ is isomorphic to G as groups.

Proof. Define $G \to \widetilde{G} : \phi \mapsto \widetilde{\phi}$. First, we claim that this mapping is a group homomorphism. Let $\phi_1, \phi_2 \in G$. It is clear that $\widetilde{\phi_1}\widetilde{\phi_2}$ and $\widetilde{\phi_1\phi_2}$ agree on the generators $x_1, \cdots, x_n$. In the meantime, on the generators $y_1, \cdots, y_n$,

$$\begin{aligned}\widetilde{\phi_1}\widetilde{\phi_2}(y_i) =& \widetilde{\phi_1}(\widetilde{\phi_2}(y_i)) \\ =& \widetilde{\phi_1}\left(\sum_{j=1}^{n} \frac{\partial \phi_2(x_i)}{\partial x_j} y_j\right) \\ =& \sum_{j=1}^{n} \phi_1\left(\frac{\partial \phi_2(x_i)}{\partial x_j}\right)\left(\sum_{k=1}^{n} \frac{\partial \phi_1(x_j)}{\partial x_k} y_k\right)\end{aligned}$$

$$
\begin{aligned}
&=\sum_{j=1}^{n}\sum_{k=1}^{n}\phi_1\left(\frac{\partial\phi_2(x_i)}{\partial x_j}\right)\left(\frac{\partial\phi_1(x_j)}{\partial x_k}y_k\right)\\
&=\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial\phi_1(\phi_2(x_i))}{\partial\phi_1(x_j)}\frac{\partial\phi_1(x_j)}{\partial x_k}y_k\\
&=\sum_{k=1}^{n}\frac{\partial\phi_1(\phi_2(x_i))}{\partial x_k}y_k\\
&=\widetilde{\phi_1\phi_2}(y_i),
\end{aligned}
$$

for all $1\leq i\leq n$, in which the fifth equality follows from the commutativity of the following diagram:

$$
\begin{array}{ccc}
\mathbb{k}[x_1,\cdots,x_n] & \xrightarrow{\frac{\partial}{\partial x_j}} & \mathbb{k}[x_1,\cdots,x_n]\\
\downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_1}\\
\mathbb{k}[x_1,\cdots,x_n] & \xrightarrow[\frac{\partial}{\partial\phi_1(x_j)}]{} & \mathbb{k}[x_1,\cdots,x_n]
\end{array}
$$

For injectivity, suppose that $\widetilde{\phi_1}=\widetilde{\phi_2}$. On the generators $x_1,\cdots,x_n$, $\phi_1(x_i)=\widetilde{\phi_1}(x_i)=\widetilde{\phi_2}(x_i)=\phi_2(x_i)$, for all $1\leq i\leq n$. Consequently, $\phi_1=\phi_2$. Finally, surjectivity follows easily from the construction. □

2.4 Deformation Quantizations

Let $P=\mathbb{k}[x_1,x_2,x_3]$ be a quadratic Poisson algebra, with its Poisson structure written as $\{x_i,x_j\}=\sum_{k,l}c_{i,j}^{k,l}x_kx_l$, $1\leq i<j\leq 3$. Donin and Makar-Limanov proved that P is quantizable, admitting a deformation quantization:

$$P_\hbar=\mathbb{k}[[\hbar]]\langle y_1,y_2,y_3\rangle/([y_i,y_j]=\frac{\hbar}{2}\sum_{k,l}c_{i,j}^{k,l}(y_ky_l+y_ly_k))_{1\leq i,j\leq 3}.$$

In practice, we will work with a variant of the above deformation quantizations, modified by replacing the coefficient ring $\Bbbk[[\hbar]]$ with an algebraically closed field $\mathbb{F}$ of characteristic 0. In a more formal manner, we can replace $P_\hbar$ with $P_\hbar \otimes_{\Bbbk[[\hbar]]} \overline{\Bbbk((\hbar))}$, or equivalently, we can replace the coefficient ring $\Bbbk[[\hbar]]$ by $\mathbb{F} = \overline{\Bbbk((\hbar))}$, the Puiseux series of $\hbar$ over $\Bbbk$. Given that our attention rarely extends to scalars, we will assume that the formal parameter $\hbar$ has been replaced by a non-zero scalar in $\Bbbk$ and that $\mathbb{F} = \Bbbk$. For the remainder of the dissertation, we will refer this variant as the *standard deformation quantization* of P.

Theorem 2.4.1. [DML98, Theorem 3.1] Let $P = \Bbbk[x_1, x_2, x_3]$ be a quadratic Poisson algebra, with its Poisson structure written as $\{x_i, x_j\} = \sum_{k,l} c_{i,j}^{k,l} x_k x_l$, $1 \le i < j \le 3$. Then the Poisson algebra P is quantizable, admitting the standard deformation quantization:

$$P_\hbar = \Bbbk\langle y_1, y_2, y_3 \rangle / ([y_i, y_j] = \frac{\hbar}{2} \sum_{k,l} c_{i,j}^{k,l} (y_k y_l + y_l y_k))_{1 \le i,j \le 3},$$

for some $\hbar \in \Bbbk^\times$.

Working with the standard deformation quantization offers two benefits:

(1) It is imperative to set $A_0 = \Bbbk$, as Artin-Schelter regularity necessitates a connected $\mathbb{N}$-grading. In the later part of this section, we will prove that the deformation quantization in Theorem 2.4.1 satisfies Artin-Schelter regularity.

(2) It is computationally (and psychologically) easier to work with scalars from a field rather than with scalars from a formal power series ring.

Example 2.4.2. Let $P = \Bbbk[x_1, x_2, x_3]$ be a unimodular quadratic Poisson algebra. The standard deformation quantization of P is the free $\Bbbk$-algebra generated by y_1, y_2, y_3 subjecting to the following relations:

Case	$\boldsymbol{y_2 y_1 =}$	$\boldsymbol{y_3 y_2 =}$	$\boldsymbol{y_3 y_1 =}$
14-1	$y_1 y_2$	$y_2 y_3 - 3\hbar y_1^2$	$y_1 y_3$
14-2	$y_1 y_2$	$y_2 y_3 - 2\hbar y_1 y_2$	$y_1 y_3 + \hbar y_1^2$

Case	$\boldsymbol{y_2y_1} =$	$\boldsymbol{y_3y_2} =$	$\boldsymbol{y_3y_1} =$
14-3	$\frac{1-\hbar}{1+\hbar}y_1y_2$	$\frac{1-\hbar}{1+\hbar}y_2y_3$	$\frac{1+\hbar}{1-\hbar}y_1y_3$
14-4	y_1y_2	$y_2y_3 - 2\hbar y_1y_2 - \hbar y_2^2$	$y_1y_3 + \hbar y_1^2 + 2\hbar y_1y_2$
14-5	$y_1y_2 - \hbar y_2^2$	$y_2y_3 - 3\hbar y_1^2$	$(y_1 + 2\hbar y_2)y_3 - 3\hbar^2 y_1^2$
14-6	$y_1y_2 - \hbar y_1^2 - \hbar y_2^2$	$(-\frac{2\hbar}{1+\hbar^2}y_1 + \frac{1-\hbar^2}{1+\hbar^2}y_2)y_3 - \frac{3\hbar}{1+\hbar^2}y_1^2$	$(\frac{1-\hbar^2}{1+\hbar^2}y_1 + \frac{2\hbar}{1+\hbar^2}y_2)y_3 - \frac{3\hbar^2}{1+\hbar^2}y_1^2$
14-7	$\frac{2-\lambda\hbar}{2+\lambda\hbar}y_1y_2 - \frac{2\hbar}{2+\lambda\hbar}y_3^2$	$\frac{2-\lambda\hbar}{2+\lambda\hbar}y_2y_3 - \frac{2\hbar}{2+\lambda\hbar}y_1^2$	$\frac{2+\lambda\hbar}{2-\lambda\hbar}y_1y_3 + \frac{\hbar}{2-\lambda\hbar}y_2^2$
14-8	$\frac{2-\hbar}{2+\hbar}y_1y_2$	$\frac{2-\hbar}{2+\hbar}y_2y_3 - \frac{6\hbar}{2+\hbar}y_1^2 - \frac{8\hbar}{(2+\hbar)^2}y_1y_2$	$\frac{2+\hbar}{2-\hbar}y_1y_3 + \frac{2\hbar}{2-\hbar}y_1^2$
14-9	$y_1y_2 - \hbar y_1^2$	$(-2\hbar y_1 + y_2)y_3 + \hbar^3 y_1^2 - 2\hbar^2 y_1y_2 - \hbar y_2^2$	$y_1y_3 - \hbar^2 y_1^2 + 2\hbar y_1y_2$

Example 2.4.3. Let $P = \Bbbk[x_1, x_2, x_3]$ be a non-unimodular quadratic Poisson algebra. The standard deformation quantization of P is the free $\Bbbk$-algebra generated by y_1, y_2, y_3 subjecting to the following relations:

Case	$\boldsymbol{y_2y_1} =$	$\boldsymbol{y_3y_2} =$	$\boldsymbol{y_3y_1} =$
1	$\frac{2-p\hbar}{2+p\hbar}y_1y_2$	$\frac{2-q\hbar}{2+q\hbar}y_2y_3$	$\frac{2+r\hbar}{2-r\hbar}y_1y_3$
2	$y_1y_2 - p\hbar y_1^2 - p\hbar y_2^2$	$(-\frac{8p\hbar}{4+(4p^2+q^2)\hbar^2-4q\hbar}y_1 + \frac{4-(4p^2+q^2)\hbar^2}{4+(4p^2+q^2)\hbar^2-4q\hbar}y_2)y_3$	$(\frac{4-(4p^2+q^2)\hbar^2}{(4+(4p^2+q^2)\hbar^2-4q\hbar)}y_1 + \frac{8p\hbar}{(4+(4p^2+q^2)\hbar^2-4q\hbar)}y_2)y_3$
3	$y_1y_2 - \hbar y_1^2$	$(-\frac{8\hbar}{(2-p\hbar)^2}y_1 + \frac{2+p\hbar}{2-p\hbar}y_2)y_3$	$\frac{2+p\hbar}{2-p\hbar}y_1y_3$
4	$\frac{2-p\hbar}{2+p\hbar}y_1y_2$	$\frac{2-q\hbar}{2+q\hbar}y_2y_3 - \frac{2\hbar}{2+q\hbar}y_1^2$	$\frac{2+p\hbar}{2-p\hbar}y_1y_3$
5	$y_1y_2 - p\hbar y_1^2$	$(-\frac{4(2p+1)\hbar}{(2+\hbar)^2}y_1 + \frac{2-\hbar}{2+\hbar}y_2)y_3$	$\frac{2-\hbar}{2+\hbar}y_1y_3$
6	$y_1y_2 + \frac{\hbar}{2}y_1^2$	$\frac{2-p\hbar}{2+p\hbar}y_2y_3$	$\frac{2-p\hbar}{2+p\hbar}y_1y_3$

Case	$y_2y_1 =$	$y_3y_2 =$	$y_3y_1 =$
7	$y_1y_2 - p\hbar y_1^2 - p\hbar y_2^2$	$(-\frac{4(2p+r)\hbar}{4+((2p+r)^2+q^2)\hbar^2+4q\hbar}y_1 + \frac{4-((2p+r)^2+q^2)\hbar^2}{4+((2p+r)^2+q^2)\hbar^2+4q\hbar}y_2)y_3$	$(\frac{4-((2p+r)^2+q^2)\hbar^2}{4+((2p+r)^2+q^2)\hbar^2+4q\hbar}y_1 + \frac{4(2p+r)\hbar}{4+((2p+r)^2+q^2)\hbar^2+4q\hbar}y_2)y_3$
8	$y_1y_2 - \frac{(p+q)\hbar}{2}y_1^2 - \frac{(p+q)\hbar}{2}y_2^2 \mp \hbar y_3^2$	$(-\frac{4p\hbar}{4+p^2\hbar^2}y_1 + \frac{(2+p\hbar)^2}{4+p^2\hbar^2}y_2)y_3$	$(\frac{(2+p\hbar)(2-p\hbar)}{4+p^2\hbar^2}y_1 + \frac{4p\hbar}{4+p^2\hbar^2}y_2)y_3$
9	$y_1y_2 + \frac{\hbar}{3}y_1^2$	$(-\frac{\hbar}{3}y_1 + y_2)y_3 - (p + \frac{\hbar^2}{108})\hbar y_1^2 - \frac{\hbar^2}{18}y_1y_2 + \frac{\hbar}{3}y_2^2$	$y_1y_3 + \frac{\hbar^2}{18}y_1^2 + \frac{\hbar}{3}y_1y_2$
10	$y_1y_2 + (2p+1)\hbar y_1^2$	$((4p+1)\hbar y_1 + y_2)y_3 + \frac{(4p+1)(2p+1)^2\hbar^3}{4}y_1^2 + \frac{(4p+1)(2p+1)\hbar^2}{2}y_1y_2 - p\hbar y_2^2$	$y_1y_3 + \frac{(2p+1)^2\hbar^2}{2}y_1^2 + (2p+1)\hbar y_1y_2$
11	$y_1y_2 - p\hbar y_1^2 - q\hbar y_3^2$	$y_2y_3 - (2p+1)\hbar y_1y_3$	y_1y_3
12	$y_1y_2 - p\hbar y_1^2 - q\hbar y_3^2$	$y_2y_3 - \hbar y_1^2 - (2p+1)\hbar y_1y_3$	y_1y_3
13	$y_1y_2 - p\hbar y_1^2 - q\hbar y_3^2 - 2\hbar y_1y_3$	$y_2y_3 - r\hbar y_1^2 - (2p+1)\hbar y_1y_3 - \hbar y_3^2$	y_1y_3

Lemma 2.4.4. Let $P = \Bbbk[x_1, x_2, x_3]$ be a quadratic Poisson algebra. Then its standard deformation quantization $P_\hbar$ is a quantum polynomial ring of global dimension 3.

Proof. In Case 1 to 7, 9 to 13, 14-1 to 14-6, 14-8, and 14-9, the standard deformation quantization is an Ore extension of either the skew polynomial ring of two variables or the Jordan plane, and therefore takes the form of an iterated Ore extension $\Bbbk[y_1][y_2; \sigma_2, \delta_2][y_3; \sigma_3, \delta_3]$, where appropriate selections of graded automorphisms σ_2 and σ_3, along with σ_2-derivation δ_2 and σ_3-derivation δ_3, are made. This can be verified by applying the following fact: the graded automorphisms of the skew polynomial ring of two variables is $\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \neq 0 \right\}$ when the relation is written as $y_2y_1 - \lambda y_1y_2$ ($\lambda \neq 0, \pm 1$) and

$\left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} : a, b \neq 0 \right\}$ when the relation is written as $y_2y_1 - y_1y_2 + \lambda y_1^2 + \lambda y_2^2$ ($\lambda \neq 0, \pm 1$), and the graded automorphisms of the Jordan plane is $\left\{ \begin{bmatrix} a & 0 \\ b & a \end{bmatrix} : a \neq 0 \right\}$ when the relation is written as $y_2y_1 - y_1y_2 - \lambda y_1^2$ ($\lambda \neq 0$). These iterated Ore extensions are Noetherian Artin-Schelter regular domains, as shown in [AST91, Proposition 2] and [GRBW04, Corollary 2.7, Exercise 2O]. For the Hilbert series, the standard deformation quantization $P_\hbar$ is isomorphic to P as graded $\Bbbk$-vector spaces, with the isomorphism being established by substituting $\hbar$ with a non-zero scalar as described in Definition 1.1.14 (1). Consequently, the Hilbert series $h_{P_\hbar}(t) = \dfrac{1}{(1-t)^3}$.

In Case 14-7, the standard deformation quantization is the three-dimensional Sklyanin algebra, a family of Artin-Schelter regular algebras initially studied in [AS87] and [ATVdB07]. The proof of the remaining properties: Noetherian, domain, and Hilbert series, can be found in a number of references, for example [Wal12, Theorem 2.14].

In Case 8, the standard deformation quantization $P_\hbar$ admits a $\Bbbk$-linear basis $\{y_1^i y_2^j y_3^k : i, j, k \geq 0\}$ [DML98, page 254]. This $\Bbbk$-linear basis leads to two implications. Firstly, it implies that the Hilbert series is $h_{P_\hbar}(t) = \dfrac{1}{(1-t)^3}$. Secondly, it implies that y_3 is a non-zero-divisor. Suppose that $fy_3 = 0$ for some $f \in P_\hbar$. Rewrite f with respect to this $\Bbbk$-linear basis: $f = \sum_{(i,j,k)} c_{i,j,k} y_1^i y_2^j y_3^k$. The element $fy_3 = 0$ is a non-trivial linear combination of the $\Bbbk$-linear basis $\{y_1^i y_2^j y_3^k : i, j, k \geq 0\}$, a contradiction. Similarly, one can demonstrate that $y_3 f \neq 0$ by using the relations provided in Example 2.4.3. This establishes a sufficient condition for applying [RZ10, Corollary 1.2]. It is worth noting that while the corollary necessitates a domain, its proof, specifically the Rees Lemma $\mathrm{Ext}^i_{P_\hbar}(\Bbbk, P_\hbar) \cong \mathrm{Ext}^{i-1}_{P_\hbar/(y_3)}(\Bbbk, P_\hbar/(y_3))$, only requires y_3 to be a non-zero-divisor. The quotient algebra

$$P_\hbar/(y_3) \cong \Bbbk\langle y_1, y_2\rangle / (y_2y_1 = y_1y_2 - \frac{(p+q)\hbar}{2} y_1^2 - \frac{(p+q)\hbar}{2} y_2^2)$$

can be identified with the skew polynomial ring of two variables. Then $P_\hbar$ is an Artin-Schelter regular algebra of global dimension 3, possessing 3 generators and 3 relations. Such algebras are Noetherian, as demonstrated, for instance, in [Ste96, Theorem 3.1]. $\square$

Chapter 3

CLASSIFICATIONS OF GRADED POISSON AUTOMORPHISMS OF $P = \Bbbk[X_1, X_2, X_3]$

In this chapter, we delve into the classification of graded Poisson automorphisms for all quadratic Poisson structures on $\Bbbk[x_1, x_2, x_3]$, as well as the classification of graded automorphisms for their standard deformation quantizations.

3.1 Classification Techniques

Let $P = \Bbbk[x_1, \cdots, x_n]$ be a Poisson algebra under the standard grading. Suppose that $\phi \in \text{PAut}_{\text{gr}}(P)$. The graded Poisson automorphism ϕ is uniquely determined by its action on $P_1 = \bigoplus_{i=1}^{n} \Bbbk x_i$, allowing it to be represented as an invertible $n \times n$ matrix: $\phi = \begin{bmatrix} a_{ij} \end{bmatrix}_{1 \le i,j \le n}$ for some $a_{ij} \in \Bbbk$. By calculation,

$$\phi(\{x_i, x_j\}) = \{\phi(x_i), \phi(x_j)\} = \left\{ \sum_{k=1}^{n} a_{ik} x_k, \sum_{j=1}^{n} a_{jl} x_l \right\} = \sum_{1 \le k < l \le n} (a_{ik} a_{jl} - a_{il} a_{jk}) \{x_k, x_l\},$$

for all $1 \le i \le j \le n$. In particular, when $n = 3$, the above calculation can be summarized as the following lemma:

Lemma 3.1.1. Let $P = \Bbbk[x_1, x_2, x_3]$ be a quadratic Poisson algebra and let $\phi \in \text{PAut}_{\text{gr}}(P)$. Then ϕ can be uniquely represented as an invertible 3×3 matrix over $\Bbbk$: $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ satisfying the following equations:

(1) $\phi(\{x_1, x_2\}) = (a_{11}a_{22} - a_{12}a_{21})\{x_1, x_2\} + (a_{12}a_{23} - a_{13}a_{22})\{x_2, x_3\} + (a_{13}a_{21} - a_{11}a_{23})\{x_3, x_1\}$,

(2) $\phi(\{x_2,x_3\}) = (a_{21}a_{32}-a_{22}a_{31})\{x_1,x_2\}+(a_{22}a_{33}-a_{23}a_{32})\{x_2,x_3\}+(a_{23}a_{31}-a_{21}a_{33})\{x_3,x_1\}$,

(3) $\phi(\{x_3,x_1\}) = (a_{12}a_{31}-a_{11}a_{32})\{x_1,x_2\}+(a_{13}a_{32}-a_{12}a_{33})\{x_2,x_3\}+(a_{11}a_{33}-a_{13}a_{31})\{x_3,x_1\}$.

In some instances, computing the solution set of the equations stated in Lemma 3.1.1 can be exceedingly challenging for both humans and machines. Consequently, we will use the following techniques for further simplification before proceeding with our computation.

Lemma 3.1.2. Let $P = \Bbbk[x_1, \cdots, x_n]$ be a Poisson algebra.

(1) If $f \in P$ satisfies: $\{f, x_i\} \subseteq fP$ for all $1 \le i \le n$, then f is Poisson normal.

(2) Let $\phi : P \to Q$ be a graded surjective Poisson homomorphism. If $f \in P$ is Poisson normal, so is $\phi(f)$.

Proof.

(1) Let $p \in P$. $\{f, p\} = \sum_{i=1}^{n} \frac{\partial p}{\partial x_i}\{f, x_i\}$. If $\{f, x_i\} \subseteq fP$ for all $1 \le i \le n$, then $\{f, p\} \in fP$.

(2) Let $q \in Q$. $\{\phi(f), q\} = \{\phi(f), \phi(p)\} = \phi(\{f, p\}) = \phi(fp') = \phi(f)\phi(p') \in \phi(f)Q$ for some $p, p' \in P$.

□

Lemma 3.1.3. Let $P = (\Bbbk[x_1, \cdots, x_n], \{-,-\}_P)$ and $Q = (\Bbbk[x_1, \cdots, x_n], \{-,-\}_Q)$ be Poisson algebras. If $\{x_i, x_j\}_Q = \lambda\{x_i, x_j\}_P$ for some $\lambda \in \Bbbk$ for all $1 \le i \le j \le n$, then the graded Poisson automorphism group of P is isomorphic to the graded Poisson automorphism group of Q as groups: $\mathrm{PAut}_{\mathrm{gr}}(P) \cong \mathrm{PAut}_{\mathrm{gr}}(Q)$.

Proof. Let ϕ be a graded Poisson automorphism of P such that $\phi\big|_{P_1} = \begin{bmatrix} a_{ij} \end{bmatrix}_{1\le i,j\le n}$. It is sufficient to demonstrate that ϕ induces a graded Poisson automorphism of Q, as the reverse direction is symmetric. Consider the graded endomorphism of Q that is determined by $\begin{bmatrix} a_{ij} \end{bmatrix}_{1\le i,j\le n}$. To establish that φ preserves the Poisson structure of Q,

$$\varphi(\{x_i, x_j\}_Q) = \varphi(\lambda\{x_i, x_j\}_P) = \lambda\{\varphi(x_i), \varphi(x_j)\}_P = \{\varphi(x_i), \varphi(x_j)\}_Q,$$

for all $1 \leq i \leq j \leq n$. $\square$

Lemma 3.1.4. [HTWZ23, Lemma 3.3] Let $P = \Bbbk[x_1, x_2, x_3]$ be a $\mathbb{Z}$-graded unimodular Poisson algebra determined by some homogeneous superpotential Ω such that $\deg(\Omega) > \max\{\deg(x_1), \deg(x_2), \deg(x_3)\}$. If ϕ is a graded automorphism of the $\Bbbk$-algebra P, then ϕ is a graded Poisson automorphism of the Poisson algebra P if and only if

$$\phi(\Omega) = \det \begin{bmatrix} \frac{\partial\phi(x_1)}{\partial x_1} & \frac{\partial\phi(x_1)}{\partial x_2} & \frac{\partial\phi(x_1)}{\partial x_3} \\ \frac{\partial\phi(x_2)}{\partial x_1} & \frac{\partial\phi(x_2)}{\partial x_2} & \frac{\partial\phi(x_2)}{\partial x_3} \\ \frac{\partial\phi(x_3)}{\partial x_1} & \frac{\partial\phi(x_3)}{\partial x_2} & \frac{\partial\phi(x_3)}{\partial x_3} \end{bmatrix} \Omega.$$

In particular, if $P = \Bbbk[x_1, x_2, x_3]$ is equipped with the standard grading: $\deg(x_1) = \deg(x_2) = \deg(x_3) = 1$, then for a graded Poisson automorphism ϕ of the Poisson algebra P, the image of the superpotential $\phi(\Omega)$ and the superpotential Ω differ by a nonzero constant $\det\left[\frac{\partial(\phi(x_1),\phi(x_2),\phi(x_3))}{\partial(x_1,x_2,x_3)}\right] \in \Bbbk$.

Lemma 3.1.2 and Lemma 3.1.4 provide us means to simplify the equations stated in Lemma 3.1.1. because a graded Poisson automorphism ϕ necessarily permutes the set of linear Poisson normal elements of P and fixes the superpotential up to a scalar. Furthermore, Lemma 3.1.3 enables us to re-scale the Poisson structure in cases where its coefficients p, q, r or its linear Poisson normal elements pose computational challenges.

3.2 Classification for Unimodular Poisson Algebras

In this section, our objective is to provide a classification of graded Poisson automorphisms for unimodular Poisson structures on $\Bbbk[x_1, x_2, x_3]$.

Unimodular 1. $\{x_1, x_2\} = 0$, $\{x_2, x_3\} = 3x_1^2$, $\{x_3, x_1\} = 0$.

Let $\phi \in \mathrm{PAut}_{\mathrm{gr}}(P)$. By Lemma 3.1.1, we have the following system of equations, with redundant equations omitted:

(1) $a_{11}^2 = a_{22}a_{33} - a_{23}a_{32}$.

(2) $a_{12}^2 = 0$.

(3) $a_{13}^2 = 0$.

These relations simplify to $a_{11} = \pm\sqrt{a_{22}a_{33} - a_{23}a_{32}} \neq 0$, $a_{12} = a_{13} = 0$. In conclusion,

$$\mathrm{PAut}_{\mathrm{gr}}(P) = \left\{ \begin{bmatrix} \pm\sqrt{bf - ce} & 0 & 0 \\ a & b & c \\ d & e & f \end{bmatrix} : bf \neq ce \right\}.$$

Unimodular 2. $\{x_1, x_2\} = 0$, $\{x_2, x_3\} = 2x_1x_2$, $\{x_3, x_1\} = x_1^2$.

Let $\phi \in \mathrm{PAut}_{\mathrm{gr}}(P)$. By Lemma 3.1.1, we have the following system of equations, with redundant equations omitted:

(1) $a_{13}a_{21} = a_{11}a_{23}$.

(2) $2a_{11}a_{21} = a_{23}a_{31} - a_{21}a_{33}$.

(3) $a_{11}^2 = a_{11}a_{33} - a_{13}a_{31}$.

(4) $a_{12}^2 = 0$.

(5) $a_{13}^2 = 0$.

By (4) and (5), $a_{12} = a_{13} = 0$ and $a_{11} \neq 0$. By substituting the variables in (1) and (3), $a_{23} = 0$ and $a_{33} = a_{11}$. Finally, it can be deduced from (2) that $a_{21} = 0$. In conclusion,

$$\mathrm{PAut}_{\mathrm{gr}}(P) = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ c & d & a \end{bmatrix} : a, b \neq 0 \right\}.$$

Unimodular 3. $\{x_1, x_2\} = x_1x_2$, $\{x_2, x_3\} = x_2x_3$, $\{x_3, x_1\} = x_1x_3$.

Let $\phi \in \mathrm{Aut}_{\mathrm{gr}}(P)$. By Lemma 3.1.1, we have the following system of equations, with redundant equations omitted:

(1) $a_{11}a_{21} = 0$.

(2) $a_{12}a_{22} = 0$.

(3) $a_{13}a_{23} = 0$.

(4) $a_{11}a_{22} + a_{12}a_{21} = a_{11}a_{22} - a_{12}a_{21}$.

(5) $a_{11}a_{23} + a_{13}a_{21} = a_{13}a_{21} - a_{11}a_{23}$.

(6) $a_{12}a_{23} + a_{13}a_{22} = a_{12}a_{23} - a_{13}a_{22}$.

(7) $a_{11}a_{31} = 0$.

(8) $a_{12}a_{32} = 0$.

(9) $a_{13}a_{33} = 0$.

(10) $a_{11}a_{32} + a_{12}a_{31} = a_{12}a_{31} - a_{11}a_{32}$.

(11) $a_{11}a_{33} + a_{13}a_{31} = a_{11}a_{33} - a_{13}a_{31}$.

(12) $a_{12}a_{33} + a_{13}a_{32} = a_{13}a_{32} - a_{12}a_{33}$.

We conduct a case-by-case examination:

- Suppose that $a_{11} \neq 0$. First, (1), (5), (7), (10) lead to $a_{21} = a_{23} = a_{31} = a_{32} = 0$ and $a_{22}, a_{33} \neq 0$. Expanding upon this, (2) and (9) imply that $a_{12} = a_{13} = 0$.

- Suppose that $a_{11} = 0$ and $a_{12} \neq 0$. First, (2), (4), (8), (12) lead to $a_{21} = a_{22} = a_{32} = a_{33} = 0$ and $a_{23}, a_{31} \neq 0$. Expanding upon this, (3) implies that $a_{13} = 0$.

- Suppose that $a_{11} = 0$ and $a_{12} = 0$. First, the invertibility of ϕ implies that $a_{13} \neq 0$. The equations (3), (6), (9), (11) result in $a_{22} = a_{23} = a_{31} = a_{33} = 0$ and $a_{21}, a_{32} \neq 0$.

In conclusion,

$$\mathrm{PAut_{gr}}(P) = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}, \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & a \\ b & 0 & 0 \\ 0 & c & 0 \end{bmatrix} : a, b, c \neq 0 \right\}.$$

Unimodular 4. $\{x_1, x_2\} = 0$, $\{x_2, x_3\} = 2x_1x_2 + x_2^2$, $\{x_3, x_1\} = x_1^2 + 2x_1x_2$.

Let $\phi \in \mathrm{Aut_{gr}}(P)$. By Lemma 3.1.1, we have the following system of equations, with redundant equations omitted:

(1) $2a_{11}a_{21} + a_{21}^2 = a_{23}a_{31} - a_{21}a_{33}$.

(2) $2a_{12}a_{22} + a_{22}^2 = a_{22}a_{33} - a_{23}a_{32}$.

(3) $2a_{13}a_{23} + a_{23}^2 = 0$.

(4) $a_{11}a_{22} + a_{12}a_{21} + a_{21}a_{22} = a_{22}a_{33} - a_{23}a_{32} + a_{23}a_{31} - a_{21}a_{33}$.

(5) $a_{11}^2 + 2a_{11}a_{21} = a_{11}a_{33} - a_{13}a_{31}$.

(6) $a_{12}^2 + 2a_{12}a_{22} = a_{13}a_{32} - a_{12}a_{33}$.

(7) $a_{13}^2 + 2a_{13}a_{23} = 0$.

It can be deduced from (3) and (7) that $a_{13} = a_{23} = 0$ and $a_{33} \neq 0$. Next, we will proceed with a case-by-case discussion.

- Suppose that $a_{21} \neq 0$ and $a_{11} = 0$. From invertibility, $a_{12} \neq 0$. From (1), $a_{21} = -a_{33}$. If $a_{22} \neq 0$, the combination of (2) and (6) leads to $a_{12} = -a_{22} = a_{33}$ and the remaining equations are nullified. If $a_{22} = 0$, (4) states that $a_{12} = -a_{33}$ and the remaining equations are nullified. Consequently, we have two possible forms for ϕ: $\begin{bmatrix} 0 & a & 0 \\ -a & -a & 0 \\ b & c & a \end{bmatrix}$, $\begin{bmatrix} 0 & -a & 0 \\ -a & 0 & 0 \\ b & c & a \end{bmatrix}$, for some $a \neq 0$.

- Suppose that $a_{21} \neq 0$ and $a_{11} \neq 0$. First, a combination of (1) and (5) leads to $a_{11} = -a_{21} = -a_{33}$. If $a_{22} \neq 0$, a combination of (2) and (4) implies $a_{12} = 0$ and $a_{22} = a_{33}$ and the remaining equations are nullified. If $a_{22} = 0$, (4) states that $a_{12} = -a_{33}$ and the remaining equations are nullified. Consequently, we have two possible forms for ϕ: $\begin{bmatrix} -a & 0 & 0 \\ a & a & 0 \\ b & c & a \end{bmatrix}$, $\begin{bmatrix} -a & -a & 0 \\ a & 0 & 0 \\ b & c & a \end{bmatrix}$, for some $a \neq 0$.

- Suppose that $a_{21} = 0$. From invertibility, $a_{22} \neq 0$. If $a_{12} \neq 0$, a combination of (2) and (6) results in $a_{12} = -a_{22} = a_{33}$, (4) results in $a_{11} = a_{33}$, and the remaining equations are nullified. If $a_{12} = 0$, (2) and (4) imply $a_{11} = a_{22} = a_{33}$ and the remaining equations are nullified. Consequently, we have two possible forms for ϕ: $\begin{bmatrix} a & a & 0 \\ 0 & -a & 0 \\ b & c & a \end{bmatrix}$, $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ b & c & a \end{bmatrix}$, for some $a \neq 0$.

In conclusion,

$$\mathrm{PAut}_{\mathrm{gr}}(P) = \left\{ \begin{bmatrix} 0 & a & 0 \\ -a & -a & 0 \\ b & c & a \end{bmatrix}, \begin{bmatrix} 0 & -a & 0 \\ -a & 0 & 0 \\ b & c & a \end{bmatrix}, \begin{bmatrix} -a & 0 & 0 \\ a & a & 0 \\ b & c & a \end{bmatrix}, \right.$$

$$\left.\begin{bmatrix} -a & -a & 0 \\ a & 0 & 0 \\ b & c & a \end{bmatrix}, \begin{bmatrix} a & a & 0 \\ 0 & -a & 0 \\ b & c & a \end{bmatrix}, \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ b & c & a \end{bmatrix} : a \neq 0 \right\}.$$

Unimodular 5. $\{x_1, x_2\} = x_2^2$, $\{x_2, x_3\} = 3x_1^2$, $\{x_3, x_1\} = 2x_2x_3$.

Let $\phi \in \mathrm{Aut_{gr}}(P)$. By Lemma 3.1.1, we have the following system of equations, with redundant equations omitted:

(1) $a_{21}^2 = 3a_{12}a_{23} - 3a_{13}a_{22}$.

(2) $a_{23}^2 = 0$.

(3) $a_{11}^2 = a_{22}a_{33} - a_{23}a_{32}$.

(4) $a_{13}^2 = 0$.

(5) $a_{11}a_{12} = 0$.

(6) $2a_{22}a_{32} = a_{12}a_{31} - a_{11}a_{32}$.

(7) $a_{21}a_{32} + a_{22}a_{31} = 0$.

(8) $a_{22}a_{33} + a_{23}a_{32} = a_{11}a_{33} - a_{13}a_{31}$.

It is immediate from (2) and (4) that $a_{13} = a_{23} = 0$ and $a_{33} \neq 0$. Expanding upon these, (1) and (8) suggest that $a_{21} = 0$ and $a_{11} = a_{22} \neq 0$. Continuing further, (3) and (5) imply that $a_{11} = a_{33}$ and $a_{12} = 0$. Finally, (6) and (7) necessitate that $a_{31} = a_{32} = 0$. In conclusion,

$$\mathrm{PAut_{gr}}(P) = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} : a \neq 0 \right\}.$$

Unimodular 6. $\{x_1, x_2\} = x_1^2 + x_2^2$, $\{x_2, x_3\} = 3x_1^2 + 2x_1x_3$, $\{x_3, x_1\} = 2x_2x_3$.

Let $\phi \in \mathrm{Aut}_{\mathrm{gr}}(P)$. By Lemma 3.1.1, we have the following system of equations, with redundant equations omitted:

(1) $a_{13}^2 + a_{23}^2 = 0$.

(2) $3a_{11}^2 + 2a_{11}a_{31} = a_{21}a_{32} - a_{22}a_{31} + 3a_{22}a_{33} - 3a_{23}a_{32}$.

(3) $3a_{12}^2 + 2a_{12}a_{32} = a_{21}a_{32} - a_{22}a_{31}$.

(4) $3a_{13}^2 + 2a_{13}a_{33} = 0$.

(5) $3a_{11}a_{12} + a_{11}a_{32} + a_{12}a_{31} = 0$.

(6) $3a_{11}a_{13} + a_{11}a_{33} + a_{13}a_{31} = a_{22}a_{33} - a_{23}a_{32}$.

(7) $3a_{12}a_{13} + a_{12}a_{33} + a_{13}a_{32} = a_{23}a_{31} - a_{21}a_{33}$.

(8) $2a_{21}a_{31} = a_{12}a_{31} - a_{11}a_{32} + 3a_{13}a_{32} - 3a_{12}a_{33}$.

(9) $a_{23}a_{33} = 0$.

(10) $a_{21}a_{32} + a_{22}a_{31} = 0$.

Suppose that $a_{33} = 0$. Equation (1) and (4) necessitate that $a_{13} = a_{23} = 0$, a contradiction to the invertibility. Therefore, it follows that $a_{33} \neq 0$, and subsequently, according to equations (1) and (9), $a_{13} = a_{23} = 0$. From (6) and (7), $a_{11} = a_{22}$ and $a_{12} = -a_{21}$.

Let us assume that $a_{12} = 0$ (implying $a_{21} = 0$ implicitly). From (5), (10), and the invertibility, $a_{31} = a_{32} = 0$. Finally, from (2), $a_{11} = a_{22} = a_{33}$ and the remaining equations are nullified. This results in one possible form of ϕ: $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$ for some $a \neq 0$. Now, let us consider the alternative scenario $a_{12} \neq 0$. From a combination of (3) and (10), we can derive that $a_{32} = -\frac{3}{4}a_{12}$. Substituting our results into (5), we obtain that $a_{31} = -\frac{9}{4}a_{11}$. Examining (8), we observe that $a_{11} = -\frac{1}{2}a_{33}$. Lastly, from (10), $a_{12} = \pm\sqrt{3}a_{11}$ and the remaining equations are nullified. This results in one possible form of ϕ: $\begin{bmatrix} -\frac{1}{2}a & \pm\frac{\sqrt{3}}{2}a & 0 \\ \mp\frac{\sqrt{3}}{2}a & -\frac{1}{2}a & 0 \\ \frac{9}{8}a & \mp\frac{3\sqrt{3}}{8}a & a \end{bmatrix}$ for some $a \neq 0$. In conclusion,

$$\mathrm{PAut}_{\mathrm{gr}}(P) = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}, \begin{bmatrix} -\frac{1}{2}a & \pm\frac{\sqrt{3}}{2}a & 0 \\ \mp\frac{\sqrt{3}}{2}a & -\frac{1}{2}a & 0 \\ \frac{9}{8}a & \mp\frac{3\sqrt{3}}{8}a & a \end{bmatrix} : a \neq 0 \right\}.$$

Unimodular 7. $\{x_1, x_2\} = x_3^2 + \lambda x_1 x_2$, $\{x_2, x_3\} = x_1^2 + \lambda x_2 x_3$, $\{x_3, x_1\} = x_2^2 + \lambda x_1 x_3$.

This instance has been addressed in [MLTU09, Theorem 1]. The graded Poisson automorphism group

$$\mathrm{PAut}_{\mathrm{gr}}(P) = \left(\langle \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} : a \neq 0 \rangle \times \langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b^2 \end{bmatrix} : b = \xi_3, \xi_3^2 \rangle \right) \rtimes \langle \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \rangle.$$

Unimodular 8. $\{x_1, x_2\} = x_1 x_2$, $\{x_2, x_3\} = 3x_1^2 + 2x_1 x_2 + x_2 x_3$, $\{x_3, x_1\} = x_1^2 + x_1 x_3$.

Let $\phi \in \mathrm{Aut}_{\mathrm{gr}}(P)$. By Lemma 3.1.1, we have the following system of equations, with redundant equations omitted:

(1) $a_{11}a_{21} = 3a_{12}a_{23} - 3a_{13}a_{22} + a_{13}a_{21} - a_{11}a_{23}$.

(2) $a_{12}a_{22} = 0$.

(3) $a_{13}a_{23} = 0$.

(4) $a_{11}a_{23} + a_{13}a_{21} = a_{13}a_{21} - a_{11}a_{23}$.

(5) $3a_{11}^2 + a_{21}a_{31} + 2a_{11}a_{21} = 3a_{22}a_{33} - 3a_{23}a_{32} + a_{23}a_{31} - a_{21}a_{33}$.

(6) $3a_{12}^2 + a_{22}a_{32} + 2a_{12}a_{22} = 0$.

(7) $3a_{13}^2 + a_{23}a_{33} + 2a_{13}a_{23} = 0$.

(8) $6a_{11}a_{12} + a_{21}a_{32} + a_{22}a_{31} + 2a_{11}a_{22} + 2a_{12}a_{21} = a_{21}a_{32} - a_{22}a_{31} + 2a_{22}a_{33} - 2a_{23}a_{32}$.

(9) $a_{11}a_{32} + a_{12}a_{31} + 2a_{11}a_{12} = a_{12}a_{31} - a_{11}a_{32} + 2a_{13}a_{32} - 2a_{12}a_{33}$.

Suppose that $a_{12} \neq 0$. By combining (2) and (6), we can deduce that $a_{12} = 0$, a contradiction. Consequently, it follows that $a_{12} = 0$. From (3), (4), and the invertibility, $a_{23} = 0$. Immediately, (7) translates to $a_{13} = 0$ and $a_{11}, a_{33} \neq 0$. Subsequently, it can be inferred from (1) and (9) that $a_{21} = a_{32} = 0$ and $a_{22} \neq 0$. Based on (5) and (8), it follows that $a_{22} = \frac{a_{11}^2}{a_{33}}$ and $a_{31} = a_{33} - a_{11}$, respectively. In conclusion,

$$\text{PAut}_{\text{gr}}(P) = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & \frac{a^2}{b} & 0 \\ b-a & 0 & b \end{bmatrix} : a \neq 0 \right\}.$$

Unimodular 9. $\{x_1, x_2\} = x_1^2$, $\{x_2, x_3\} = 2x_1x_3 + x_2^2$, $\{x_3, x_1\} = 2x_1x_2$.

Let $\phi \in \text{Aut}_{\text{gr}}(P)$. By Lemma 3.1.1, we have the following system of equations, with redundant equations omitted:

(1) $a_{11}^2 = a_{11}a_{22} - a_{12}a_{21}$.

(2) $a_{12}^2 = a_{12}a_{23} - a_{13}a_{22}$.

(3) $a_{13}^2 = 0$.

(4) $a_{21}^2 + 2a_{11}a_{31} = a_{21}a_{32} - a_{22}a_{31}$.

(5) $a_{22}^2 + 2a_{12}a_{32} = a_{22}a_{33} - a_{23}a_{32}$.

(6) $a_{23}^2 + 2a_{13}a_{33} = 0$.

(7) $2a_{11}a_{21} = a_{12}a_{31} - a_{11}a_{32}$.

The initial step is straightforward. Equations (3), (6), (2) imply $a_{12} = a_{13} = a_{23} = 0$ and $a_{11}, a_{33} \neq 0$. Simplify the remaining equations. From (1) and (5), we conclude that $a_{11} = a_{22} = a_{33}$. From (7), we deduce that $a_{32} = -2a_{21}$. Finally, by examining (4), we ascertain that $a_{31} = -\frac{a_{21}^2}{a_{11}}$. In conclusion,

$$\mathrm{PAut}_{\mathrm{gr}}(P) = \left\{ \begin{bmatrix} a & 0 & 0 \\ b & a & 0 \\ -\frac{b^2}{a} & -2b & a \end{bmatrix} : a \neq 0 \right\}.$$

3.3 Classification for Non-unimodular Poisson Algebras

In this section, our objective is to provide a classification of graded Poisson automorphisms for non-unimodular Poisson structures on $\Bbbk[x_1, x_2, x_3]$. In the classification, we will impose certain restrictions on the coefficients associated with the Poisson structures. The primary motivation for imposing these conditions is to guarantee that whenever the product of a coefficient and a variable equals zero: $cx_i = 0$, the variable itself must also be equal to zero: $x_i = 0$.

Non-unimodular 1. $\{x_1, x_2\} = px_1x_2$, $\{x_2, x_3\} = qx_2x_3$, $\{x_3, x_1\} = rx_1x_3$.

Restriction on coefficients: $\neg(p = q = r)$.

The linear Poisson normal elements of P are scalar multiples of x_1, x_2, x_3 [GW20, page

1260]. This forces ϕ to take one of the following forms:

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}, \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & b \\ 0 & c & 0 \end{bmatrix}, \begin{bmatrix} 0 & a & 0 \\ b & 0 & 0 \\ 0 & 0 & c \end{bmatrix}, \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & a \\ b & 0 & 0 \\ 0 & c & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{bmatrix},$$

for some $a, b, c \neq 0$. The 1st matrix is a graded Poisson automorphism. However, the 2nd-6th matrices are not graded Poisson automorphisms. As an illustration, consider the 2nd matrix ϕ_2:

$$\begin{aligned} \phi_2(\{x_1, x_2\}) =& pabx_1x_3, & \{\phi_2(x_1), \phi_2(x_2)\} =& rabx_1x_3, \\ \phi_2(\{x_2, x_3\}) =& qbcx_2x_3, & \{\phi_2(x_2), \phi_2(x_3)\} =& -qbcx_2x_3, \\ \phi_2(\{x_3, x_1\}) =& racx_1x_2, & \{\phi_2(x_3), \phi_2(x_1)\} =& -pacx_1x_2. \end{aligned}$$

A concurrent satisfaction of all three equalities necessitates setting $p = q = r = 0$, a contradiction to our imposed constraints on coefficients. In conclusion,

$$\mathrm{PAut}_{\mathrm{gr}}(P) = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} : a, b, c \neq 0 \right\}.$$

Non-unimodular 2. $\{x_1, x_2\} = p(x_1^2 + x_2^2)$, $\{x_2, x_3\} = 2px_1x_3 - qx_2x_3$,
$\{x_3, x_1\} = qx_1x_3 + 2px_2x_3$.

Restriction on coefficients: $p, q \neq 0$, $p \neq q$, $4p^2 + q^2 \neq 0$.

A straightforward calculation that solves the equation $(ax_1+bx_2+cx_3)(d_ix_1+e_ix_2+f_ix_3) = \{ax_1 + bx_2 + cx_3, x_i\}$, $1 \leq i \leq 3$, shows that the linear Poisson normal elements of P are scalar multiples of $\pm ix_1 + x_2$ and x_3. Let $\phi \in \mathrm{PAut}_{\mathrm{gr}}(P)$. By Lemma 3.1.2, there are two scenarios:

- If $\phi(x_3)$ is a scalar multiple of $\pm ix_1 + x_2$, then $a_{31} = \pm ia_{32} \neq 0$ and $a_{33} = 0$.

- If $\phi(x_3)$ is a scalar multiple of x_3, then $a_{31}, a_{32} = 0$ and $a_{33} \neq 0$.

By Lemma 3.1.1, ϕ satisfies the following system of equations, with redundant equations omitted:

(1) $a_{13}^2 + a_{23}^2 = 0$.

(2) $2pa_{13}a_{33} = qa_{23}a_{33}$.

(3) $2pa_{11}a_{33} + 2pa_{13}a_{31} - qa_{21}a_{33} - qa_{23}a_{31} = 2pa_{22}a_{33} - 2pa_{23}a_{32} + qa_{23}a_{31} - qa_{21}a_{33}$.

(4) $2pa_{12}a_{33} + 2pa_{13}a_{32} - qa_{22}a_{33} - qa_{23}a_{32} = 2pa_{23}a_{31} - 2pa_{21}a_{33} - qa_{22}a_{33} + qa_{23}a_{32}$.

(5) $qa_{12}a_{33} + qa_{13}a_{32} + 2pa_{22}a_{33} + 2pa_{23}a_{32} = 2pa_{11}a_{33} - 2pa_{13}a_{31} - qa_{13}a_{32} + qa_{12}a_{33}$.

Suppose that $a_{31} = \pm ia_{32} \neq 0$ and $a_{33} = 0$. Equation (5) can be translated as $a_{23} = (\mp i - \frac{q}{p})a_{13}$. If we substitute this equality into (1): $(\pm 2ip + q)a_{13}^2 = 0$, a contradiction to the constraint $4p^2 + q^2 \neq 0$ unless $a_{13} = a_{23} = 0$. However, $a_{13} = a_{23} = a_{33} = 0$ contradicts the invertibility of ϕ.

Suppose that $a_{31} = a_{32} = 0$ and $a_{33} \neq 0$. A combination of (1) and (2) necessitates that $a_{13} = a_{23} = 0$. Simplify the remaining equations. Equation (3) and (4) imply $a_{11} = a_{22}$ and $a_{12} = -a_{21}$. In conclusion,

$$\mathrm{PAut}_{\mathrm{gr}}(P) = \left\{ \begin{bmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & c \end{bmatrix} : a, b, c \neq 0 \right\}.$$

Non-unimodular 3. $\{x_1, x_2\} = x_1^2$, $\{x_2, x_3\} = 2x_1x_3 - px_2x_3$, $\{x_3, x_1\} = px_1x_3$.

Restriction on coefficients: $p \neq 0$.

Let $\phi \in \mathrm{Aut}_{\mathrm{gr}}(P)$. By Lemma 3.1.1, we have the following system of equations, with redundant equations omitted:

(1) $a_{11}^2 = a_{11}a_{22} - a_{12}a_{21}$.

(2) $a_{12}^2 = 0$.

(3) $a_{13}^2 = 0$.

(4) $2a_{11}a_{13} = 2a_{12}a_{23} - 2a_{13}a_{22} + pa_{13}a_{21} - pa_{11}a_{23}$.

(5) $2a_{12}a_{32} - pa_{22}a_{32} = 0$.

(6) $2a_{11}a_{32} + 2a_{12}a_{31} - pa_{21}a_{32} - pa_{22}a_{31} = 0$.

Immediately, from (2) and (3), we can deduce that $a_{12} = a_{13} = 0$ and $a_{11} \neq 0$. Therefore, (1) and (4) imply that $a_{11} = a_{22}$ and $a_{23} = 0$, respectively. Lastly, (5) and (6) lead to $a_{32} = 0$ and $a_{31} = 0$. In conclusion,

$$\mathrm{PAut}_{\mathrm{gr}}(P) = \left\{ \begin{bmatrix} a & 0 & 0 \\ b & a & 0 \\ 0 & 0 & c \end{bmatrix} : a, b, c \neq 0 \right\}.$$

Non-unimodular 4. $\{x_1, x_2\} = px_1x_2$, $\{x_2, x_3\} = x_1^2 + qx_2x_3$, $\{x_3, x_1\} = px_1x_3$.

Restrictions on coefficients: $p, q \neq 0$.

A straightforward calculation that solves the equation $(ax_1+bx_2+cx_3)(d_ix_1+e_ix_2+f_ix_3) = \{ax_1 + bx_2 + cx_3, x_i\}$, $1 \leq i \leq 3$, shows that the linear Poisson normal elements of P are scalar multiples of x_1. Let $\phi \in \mathrm{PAut}_{\mathrm{gr}}(P)$. By Lemma 3.1.2, $a_{12} = a_{13} = 0$ and $a_{11} \neq 0$. By Lemma 3.1.1, ϕ satisfies the following system of equations, with redundant equations omitted:

(1) $a_{21} = 0$.

(2) $a_{23} = 0$.

(3) $a_{11}^2 = a_{22}a_{33}$.

(4) $a_{32} = 0$.

(5) $a_{31} = 0$.

Simplification is straightforward: (1), (2), (5) imply that $a_{21} = a_{23} = a_{31}$ and $a_{22} \neq 0$. Next, (3) implies that $a_{11} = \pm\sqrt{a_{22}a_{33}}$ and (4) implies that $a_{32} = 0$. In conclusion,

$$\mathrm{PAut}_{\mathrm{gr}}(P) = \left\{ \begin{bmatrix} \pm\sqrt{ab} & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix} : a, b \neq 0 \right\}.$$

Non-unimodular 5. $\{x_1, x_2\} = px_1^2$, $\{x_2, x_3\} = (2p+1)x_1x_3 + x_2x_3$, $\{x_3, x_1\} = -x_1x_3$.

Restrictions on coefficients: $p \neq 0, \frac{1}{2}$.

Let $\phi \in \mathrm{Aut}_{\mathrm{gr}}(P)$. By Lemma 3.1.1, we have the following system of equations, with redundant equations omitted:

(1) $pa_{11}^2 = pa_{11}a_{22} - pa_{12}a_{21}$.

(2) $pa_{12}^2 = 0$.

(3) $pa_{13}^2 = 0$.

(4) $2pa_{11}a_{13} = (2p+1)a_{12}a_{23} - (2p+1)a_{13}a_{22} - a_{13}a_{21} + a_{11}a_{23}$.

(5) $(2p+1)a_{12}a_{32} + a_{22}a_{32} = 0$.

(6) $(2p+1)a_{11}a_{32} + (2p+1)a_{12}a_{31} + a_{21}a_{32} + a_{22}a_{31} = 0$.

Simplification is straightforward. Equation (2) and (3) indicate that $a_{12} = a_{13} = 0$ and $a_{11} \neq 0$. Subsequently, (1) implies that $a_{11} = a_{22}$, (4) implies that $a_{23} = 0$, (5) implies that

$a_{32} = 0$, and (6) implies that $a_{31} = 0$. In conclusion,

$$\mathrm{PAut}_{\mathrm{gr}}(P) = \left\{ \begin{bmatrix} a & 0 & 0 \\ b & a & 0 \\ 0 & 0 & c \end{bmatrix} : a, c \neq 0 \right\}.$$

Non-unimodular 6. $\{x_1, x_2\} = -\frac{1}{2}x_1^2$, $\{x_2, x_3\} = px_2x_3$, $\{x_3, x_1\} = -px_1x_3$.

Restrictions on coefficients: $p \neq 0$.

Let $\phi \in \mathrm{Aut}_{\mathrm{gr}}(P)$. By Lemma 3.1.1, we have the following system of equations, with redundant equations omitted:

(1) $-\frac{1}{2}a_{11}^2 = -\frac{1}{2}a_{11}a_{22} + \frac{1}{2}a_{12}a_{21}$.

(2) $-\frac{1}{2}a_{12}^2 = 0$.

(3) $-\frac{1}{2}a_{13}^2 = 0$.

(4) $-\frac{1}{2}a_{11}a_{13} = -pa_{13}a_{21} + pa_{11}a_{23}$.

(5) $pa_{22}a_{32} = 0$.

(6) $pa_{21}a_{32} + pa_{22}a_{31} = 0$.

Notice that (2) and (3) imply that $a_{12} = a_{13} = 0$ and $a_{11} \neq 0$. The remaining simplifications are: (1) implies that $a_{11} = a_{22}$, (4) implies that $a_{23} = 0$, (5) implies that $a_{32} = 0$, and (6) implies that $a_{31} = 0$. In conclusion,

$$\mathrm{PAut}_{\mathrm{gr}}(P) = \left\{ \begin{bmatrix} a & 0 & 0 \\ b & a & 0 \\ 0 & 0 & c \end{bmatrix} : a, c \neq 0 \right\}.$$

Non-unimodular 7. $\{x_1, x_2\} = px_1^2 + px_2^2$, $\{x_2, x_3\} = qx_2x_3 + (2p+r)x_1x_3$, $\{x_3, x_1\} = (2p+r)x_2x_3 - qx_1x_3$.

Restrictions on coefficients: $p, q \neq 0$, $2p + r \neq 0$, $(2p+r)^2 + q^2 \neq 0$.

A straightforward calculation that solves the equation $(ax_1+bx_2+cx_3)(d_ix_1+e_ix_2+f_ix_3) = \{ax_1 + bx_2 + cx_3, x_i\}$, $1 \leq i \leq 3$, shows that the linear Poisson normal elements of P are scalar multiples of $\pm ix_1 + x_2$ and x_3. Let $\phi \in \mathrm{PAut}_{\mathrm{gr}}(P)$. By Lemma 3.1.2, there are two scenarios:

- If $\phi(x_3)$ is a scalar multiple of $\pm ix_1 + x_2$, then $a_{31} = \pm ia_{32} \neq 0$ and $a_{33} = 0$.

- If $\phi(x_3)$ is a scalar multiple of x_3, then $a_{31}, a_{32} = 0$ and $a_{33} \neq 0$.

By Lemma 3.1.1, ϕ satisfies the following system of equations, with redundant equations omitted:

(1) $a_{13}^2 + a_{23}^2 = 0$.

(2) $(2p+r)a_{23}a_{33} - qa_{13}a_{33} = 0$.

(3) $qa_{21}a_{33} + qa_{23}a_{31} + (2p+r)a_{11}a_{33} + (2p+r)a_{13}a_{31} = (2p+r)(a_{22}a_{33} - a_{23}a_{32}) - q(a_{23}a_{31} - a_{21}a_{33})$.

(4) $qa_{22}a_{33} + qa_{23}a_{32} + (2p+r)a_{12}a_{33} + (2p+r)a_{13}a_{32} = q(a_{22}a_{33} - a_{23}a_{32}) + (2p+r)(a_{23}a_{31} - a_{21}a_{33})$.

(5) $(2p+r)a_{22}a_{33} + (2p+r)a_{23}a_{32} - qa_{12}a_{33} - qa_{13}a_{32} = q(a_{13}a_{32} - a_{12}a_{33}) + (2p+r)(a_{11}a_{33} - a_{13}a_{31})$.

The simplification process is analogous to that of *Non-unimodular 2*. Suppose that $a_{31} = \pm ia_{32} \neq 0$ and $a_{33} = 0$. Equation (5) can be translated as $a_{13}^2 + a_{23}^2 = \left(\frac{4q^2}{(2p+r)^2} \pm \frac{4q}{2p+r}i\right)a_{13}^2$. Notice that $\frac{4q^2}{(2p+r)^2} \pm \frac{4q}{2p+r}i \neq 0$ as we are assuming $(2p+r)^2 + q^2 \neq 0$. However, (1) requires

$a_{13}^2 + a_{23}^2 = 0$, a contradiction unless $a_{13} = a_{23} = 0$. In that case, $a_{13} = a_{23} = a_{33} = 0$ contradicts the invertibility of ϕ.

Suppose that $a_{31} = a_{32} = 0$ and $a_{33} \neq 0$. A combination of (1) and (2) necessitates that $a_{13} = a_{23} = 0$. Simplify the remaining equations. Equation (3) and (4) imply $a_{11} = a_{22}$ and $a_{12} = -a_{21}$. In conclusion,

$$\mathrm{PAut}_{\mathrm{gr}}(P) = \left\{ \begin{bmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & c \end{bmatrix} : a, b, c \neq 0 \right\}.$$

Non-unimodular 8. $\{x_1, x_2\} = \frac{p+q}{2}x_1^2 + \frac{p+q}{2}x_2^2 \pm x_3^2$, $\{x_2, x_3\} = px_1x_3$, $\{x_3, x_1\} = px_2x_3$.

Restrictions on coefficients: $p \neq 0$, $p + q \neq 0$.

A straightforward calculation that solves the equation $(ax_1+bx_2+cx_3)(d_ix_1+e_ix_2+f_ix_3) = \{ax_1 + bx_2 + cx_3, x_i\}$, $1 \leq i \leq 3$, shows that the linear Poisson normal elements of P are scalar multiples of x_3. Let $\phi \in \mathrm{PAut}_{\mathrm{gr}}(P)$. By Lemma 3.1.2, $a_{31} = a_{32} = 0$ and $a_{33} \neq 0$. By Lemma 3.1.1, ϕ satisfies the following system of equations, with redundant equations omitted:

(1) $\frac{p+q}{2}a_{13}^2 + \frac{p+q}{2}a_{23}^2 \pm a_{33}^2 = \pm a_{11}a_{22} \mp a_{12}a_{21}$.

(2) $pa_{13}a_{33} = \pm a_{21}a_{32} \mp a_{22}a_{31}$.

(3) $pa_{11}a_{33} + pa_{13}a_{31} = pa_{22}a_{33} - pa_{23}a_{32}$.

(4) $pa_{12}a_{33} + pa_{13}a_{32} = pa_{23}a_{31} - pa_{21}a_{33}$.

(5) $pa_{23}a_{33} = \pm a_{12}a_{31} \mp a_{11}a_{32}$.

Equations (2), (3), (4), and (5) impose the following constraints: $a_{13} = 0$, $a_{11} = a_{22}$, $a_{12} = -a_{21}$, $a_{23} = 0$, respectively. Finally, equation (1) asserts that $a_{12} = \pm\sqrt{a_{33}^2 - a_{11}a_{22}}$.

In conclusion,

$$\mathrm{PAut}_{\mathrm{gr}}(P) = \left\{ \begin{bmatrix} a & \pm\sqrt{b^2-a^2} & 0 \\ \mp\sqrt{b^2-a^2} & a & 0 \\ 0 & 0 & b \end{bmatrix} : b \neq 0 \right\}.$$

Non-unimodular 9. $\{x_1, x_2\} = -\frac{1}{3}x_1^2$, $\{x_2, x_3\} = px_1^2 - \frac{1}{3}x_2^2 + \frac{1}{3}x_1x_3$, $\{x_3, x_1\} = \frac{1}{3}x_1x_2$.

Restrictions on coefficients: $p \neq 0$.

A straightforward calculation that solves the equation $(ax_1+bx_2+cx_3)(d_ix_1+e_ix_2+f_ix_3) = \{ax_1 + bx_2 + cx_3, x_i\}$, $1 \leq i \leq 3$, shows that the linear Poisson normal elements of P are scalar multiples of x_1. By Lemma 3.1.2, $a_{12} = a_{12} = 0$ and $a_{11} \neq 0$. By Lemma 3.1.1, ϕ satisfies the following system of equations, with redundant equations omitted:

(1) $-\frac{1}{3}a_{11}^2 = -\frac{1}{3}a_{11}a_{22} + \frac{1}{3}a_{12}a_{21} + pa_{12}a_{23} - pa_{13}a_{22}$.

(2) $-\frac{2}{3}a_{11}a_{12} = \frac{1}{3}a_{13}a_{21} - \frac{1}{3}a_{11}a_{23}$.

(3) $\frac{1}{3}a_{11}a_{21} = -\frac{1}{3}a_{12}a_{31} + \frac{1}{3}a_{11}a_{32} + pa_{13}a_{32} - pa_{12}a_{33}$.

(4) $\frac{1}{3}a_{11}a_{22} + \frac{1}{3}a_{12}a_{21} = \frac{1}{3}a_{11}a_{33} - \frac{1}{3}a_{13}a_{31}$.

Immediately, it follows that $a_{23} = 0$, $a_{11} = a_{22}$, $a_{21} = a_{32}$, $a_{22} = a_{33}$ in the corresponding order of (2), (1), (3), (4). In conclusion,

$$\mathrm{PAut}_{\mathrm{gr}}(P) = \left\{ \begin{bmatrix} a & 0 & 0 \\ b & a & 0 \\ c & b & a \end{bmatrix} : a \neq 0 \right\}.$$

Non-unimodular 10. $\{x_1, x_2\} = -(2p+1)x_1^2$, $\{x_2, x_3\} = px_2^2 - (4p+1)x_1x_3$, $\{x_3, x_1\} = (2p+1)x_1x_2$.

Restrictions on coefficients: $p \neq 0, -\frac{1}{4}, -\frac{1}{3}, -\frac{1}{2}$.

A straightforward calculation that solves the equation $(ax_1+bx_2+cx_3)(d_ix_1+e_ix_2+f_ix_3) = \{ax_1 + bx_2 + cx_3, x_i\}$, $1 \leq i \leq 3$, shows that the linear Poisson normal elements of P are scalar multiples of x_1. By Lemma 3.1.2, $a_{12} = a_{12} = 0$ and $a_{11} \neq 0$. By Lemma 3.1.1, ϕ satisfies the following system of equations, with redundant equations omitted:

(1) $-(2p+1)a_{11}^2 = -(2p+1)(a_{11}a_{22} - a_{12}a_{21})$.

(2) $-2(2p+1)a_{11}a_{12} = (1+2p)(a_{13}a_{21} - a_{11}a_{23})$.

(3) $pa_{21}^2 - (1+4p)a_{11}a_{31} = -(2p+1)(a_{21}a_{32} - a_{22}a_{31})$.

(4) $pa_{22}^2 - (1+4p)a_{12}a_{32} = p(a_{22}a_{33} - a_{23}a_{32})$.

(5) $(1+2p)a_{11}a_{21} = -(2p+1)(a_{12}a_{31} - a_{11}a_{32})$.

Immediately, it follows that $a_{11} = a_{22}$, $a_{23} = 0$, $a_{22} = a_{33}$, $a_{21} = a_{32}$, $a_{31} = \frac{a_{21}^2}{2a_{11}}$ in the corresponding order of (1), (2), (4), (5), (3). In conclusion,

$$\mathrm{PAut}_{\mathrm{gr}}(P) = \left\{ \begin{bmatrix} a & 0 & 0 \\ b & a & 0 \\ \frac{b^2}{2a} & b & a \end{bmatrix} : a \neq 0 \right\}.$$

Non-unimodular 11. $\{x_1, x_2\} = px_1^2 + qx_3^2$, $\{x_2, x_3\} = (2p+1)x_1x_3$, $\{x_3, x_1\} = 0$.

Restriction on coefficients: $p \neq 0, -\frac{1}{2}, -\frac{1}{4}$, $q \neq 0$.

A straightforward calculation that solves the equation $(ax_1+bx_2+cx_3)(d_ix_1+e_ix_2+f_ix_3) = \{ax_1 + bx_2 + cx_3, x_i\}$, $1 \leq i \leq 3$, shows that the linear Poisson normal elements of P are scalar multiples of $\pm i\sqrt{\frac{3p+1}{q}}x_1 + x_3$ and x_3. Let $\phi \in \mathrm{PAut}_{\mathrm{gr}}(P)$. By Lemma 3.1.2, there are two scenarios:

- If $\phi(x_3)$ is a scalar multiple of $\pm i\sqrt{\frac{3p+1}{q}}x_1 + x_3$, then $a_{31} = \pm i\sqrt{\frac{3p+1}{q}}a_{33} \neq 0$ and $a_{32} = 0$.

- If $\phi(x_3)$ is a scalar multiple of x_3, then $a_{31}, a_{32} = 0$ and $a_{33} \neq 0$.

By Lemma 3.1.1, ϕ satisfies the following system of equations, with redundant equations omitted:

(1) $pa_{11}^2 + qa_{31}^2 = pa_{11}a_{22} - pa_{12}a_{21}$.

(2) $pa_{12}^2 + qa_{32}^2 = 0$.

(3) $pa_{13}^2 + qa_{33}^2 = qa_{11}a_{22} - qa_{12}a_{21}$.

(4) $2pa_{11}a_{13} + 2qa_{31}a_{33} = (2p+1)a_{12}a_{23} - (2p+1)a_{13}a_{22}$.

(5) $(2p+1)a_{11}a_{31} = pa_{21}a_{32} - pa_{22}a_{31}$.

(6) $(2p+1)a_{13}a_{33} = qa_{21}a_{32} - qa_{22}a_{31}$.

Suppose that $a_{31} = \pm i\sqrt{\frac{3p+1}{q}}a_{33} \neq 0$ and $a_{32} = 0$. By (2), $a_{12} = 0$ and $a_{22} \neq 0$. Since $a_{31} \neq 0$, we may rewrite (5) as $a_{22} = -\frac{2p+1}{p}a_{11}$. Constrained by (6), we deduce that $a_{13} = \mp i\frac{\sqrt{(3p+1)q}}{2p+1}a_{22} = \pm i\frac{\sqrt{(3p+1)q}}{p}a_{11}$. Upon substituting this equality into (3): $a_{33} = \pm a_{11}$. However, a simplification of (4) shows that $4p+1 = 0$, a contradiction.

Suppose that $a_{31} = a_{32} = 0$. It follows from (2) and (6) that $a_{12} = a_{13} = 0$ and $a_{11} \neq 0$, and subsequently from (1) and (3) that $a_{11} = a_{22}$ and $a_{33} = \pm a_{11}$. In conclusion,

$$\text{PAut}_{\text{gr}}(P) = \left\{ \begin{bmatrix} a & 0 & 0 \\ b & a & c \\ 0 & 0 & \pm a \end{bmatrix} : a \neq 0 \right\}.$$

Non-unimodular 12. $\{x_1, x_2\} = px_1^2 + qx_3^2$, $\{x_2, x_3\} = x_1^2 + (1+2p)x_1x_3$, $\{x_3, x_1\} = 0$.

Restriction on coefficients: $p \neq 0, -\frac{1}{2}, -\frac{1}{3}$, $q = 0$.

Let $\phi \in \text{Aut}_{\text{gr}}(P)$. By Lemma 3.1.1, we have the following system of equations, with redundant equations omitted:

(1) $pa_{11}^2 = pa_{11}a_{22} - a_{13}a_{22}$.

(2) $pa_{12}^2 = 0$.

(3) $pa_{13}^2 = 0$.

(4) $a_{11}^2 + (1+2p)a_{11}a_{31} = pa_{21}a_{32} - pa_{22}a_{31} + a_{22}a_{33} - a_{23}a_{32}$.

(5) $2a_{11}a_{12} + (1+2p)a_{11}a_{32} + (1+2p)a_{12}a_{31} = 0$.

At once, based on (2) and (3), $a_{12} = a_{13} = 0$ and $a_{11} \neq 0$. Subsequently, it can be deduced from (1) and (5) that $a_{11} = a_{22}$ and $a_{32} = 0$. Finally, (4) dedicates that $a_{31} = \frac{a_{33}-a_{11}}{1+3p}$. In conclusion,

$$\mathrm{PAut}_{\mathrm{gr}}(P) = \left\{ \begin{bmatrix} a & 0 & 0 \\ b & a & c \\ \frac{d-a}{1+3p} & 0 & d \end{bmatrix} : a, d \neq 0 \right\}.$$

Non-unimodular 13. $\{x_1, x_2\} = px_1^2 + qx_3^2 + 2x_1x_3$, $\{x_2, x_3\} = rx_1^2 + x_3^2 + (2p+1)x_1x_3$, $\{x_3, x_1\} = 0$.

Restriction on coefficients: $p \neq 0, -\frac{1}{2}$, $q = 0$, $r = 0$.

A straightforward calculation that solves the equation $(ax_1+bx_2+cx_3)(d_ix_1+e_ix_2+f_ix_3) = \{ax_1 + bx_2 + cx_3, x_i\}$, $1 \leq i \leq 3$, shows that the linear Poisson normal elements of P are scalar multiples of x_1, x_3, and $(p+\frac{1}{3})x_1 + x_3$. Let $\phi \in \mathrm{PAut}_{\mathrm{gr}}(P)$. By Lemma 3.1.2, $\phi\big|_{P_1}$ takes one of the following forms:

$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & 0 & a_{33} \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & 0 & a_{33} \end{bmatrix}, \quad \begin{bmatrix} (p+\frac{1}{3})a_{13} & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} (p+\frac{1}{3})a_{13} & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}.$$

By Lemma 3.1.1, ϕ satisfies the following system of equations:

(1) $pa_{11}^2 + 2a_{11}a_{31} = pa_{11}a_{22} - pa_{12}a_{21}$.

(2) $pa_{13}^2 + 2a_{13}a_{33} = a_{12}a_{23} - a_{13}a_{22}$.

(3) $2pa_{11}a_{13} + 2a_{11}a_{33} + 2a_{13}a_{31} = 2a_{11}a_{22} - 2a_{12}a_{21} + (2p+1)a_{12}a_{23} - (2p+1)a_{13}a_{22}$.

(4) $2pa_{12}a_{13} + 2a_{12}a_{33} + 2a_{13}a_{32} = 0$.

(5) $a_{31}^2 + (2p+1)a_{11}a_{31} = pa_{21}a_{32} - pa_{22}a_{31}$.

(6) $a_{33}^2 + (2p+1)a_{13}a_{33} = a_{22}a_{33} - a_{23}a_{32}$.

(7) $2a_{31}a_{33}+(2p+1)a_{11}a_{33}+(2p+1)a_{13}a_{31} = 2a_{21}a_{32}-2a_{22}a_{31}+(2p+1)a_{22}a_{33}-(2p+1)a_{23}a_{32}$.

Suppose that $\phi\big|_{P_1}$ takes the form of the first matrix. Based on (1) and (5), we can derive that $a_{31} = \frac{p}{2}(a_{22} - a_{11})$ and $a_{22} = a_{33}$. Substituting these values into (11), we ascertain that $a_{11} = a_{22}$ and $a_{31} = 0$.

Suppose that $\phi\big|_{P_1}$ takes the form of the second matrix. Referring to (6), it is clear that $a_{32} = 0$. Additionally, by examining (5) and (7), it can be deduced that $a_{22} = 0$, contradicting to the invertibility of ϕ. As a consequence, no graded Poisson automorphism can take this form.

Suppose that $\phi\big|_{P_1}$ takes the form of the third matrix. Upon simplifying (3) and (11), it follows that $a_{11} = a_{13} = a_{22} = 0$, contradicting to the invertibility of ϕ. As a consequence, no graded Poisson automorphism can take this form.

Suppose that $\phi\big|_{P_1}$ takes the form of the fourth matrix. By combining (3) and (9), it is deduced that $a_{22} = (p + \frac{2}{3})a_{13}$. However, (1) simplifies to $a_{11} = a_{22} = (p + \frac{1}{3})a_{13}$, implying that $a_{11} = a_{13} = a_{22} = 0$, contradicting to the invertibility of ϕ. As a consequence, no graded Poisson automorphism can take this form.

In conclusion,

$$\mathrm{PAut}_{\mathrm{gr}}(P) = \left\{ \begin{bmatrix} a & 0 & 0 \\ b & a & c \\ 0 & 0 & a \end{bmatrix} : a \neq 0 \right\}.$$

3.4 Classification for Deformation Quantizations

In this section, our objective is to provide a classification of graded automorphisms for the standard deformation quantizations of quadratic Poisson structures on $\Bbbk[x_1, x_2, x_3]$.

Let $P = \Bbbk[x_1, x_2, x_3]$ be a quadratic Poisson algebra and $P_\hbar$ be its standard deformation quantization. Suppose that $\phi \in \mathrm{Aut}_{\mathrm{gr}}(P_\hbar)$. The graded automorphism ϕ can be uniquely represented by a 3×3 invertible bmatrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ for some $a_{ij} \in \Bbbk$, $1 \le i, j \le 3$, satisfying $\phi(y_i) = \sum_{j=1}^{3} a_{ij} y_j$, $1 \le i \le 3$ and the following compatibility conditions on a_{ij}'s:

$$\frac{\hbar}{2} \sum_{k,l} c_{i,j}^{k,l} \phi(y_k y_l + y_l y_k) = \phi([y_i, y_j]), 1 \le i, j \le 3.$$

Expand the RHS:

$$\begin{aligned}
\frac{\hbar}{2} \sum_{k,l} c_{i,j}^{k,l} \phi(y_k y_l + y_l y_k) =&(a_{i1}a_{j2} - a_{i2}a_{j1})[y_1, y_2] \\
&+(a_{i2}a_{j3} - a_{i3}a_{j2})[y_2, y_3] \\
&+(a_{i3}a_{j1} - a_{i1}a_{j3})[y_3, y_1] \\
=&(a_{i1}a_{j2} - a_{i2}a_{j1}) \frac{\hbar}{2} \sum_{k',l'} c_{1,2}^{k',l'} (y_{k'} y_{l'} + y_{l'} y_{k'}) \\
&+(a_{i2}a_{j3} - a_{i3}a_{j2}) \frac{\hbar}{2} \sum_{k',l'} c_{2,3}^{k',l'} (y_{k'} y_{l'} + y_{l'} y_{k'}) \\
&+(a_{i3}a_{j1} - a_{i1}a_{j3}) \frac{\hbar}{2} \sum_{k',l'} c_{3,1}^{k',l'} (y_{k'} y_{l'} + y_{l'} y_{k'}).
\end{aligned}$$

Given that the coefficients of $y_k y_l$ and $y_l y_k$ on the LHS coincide, and that the coefficients of $y_{k'}y_{l'}$ and $y_{l'}y_{k'}$ on the RHS also coincide, we may reduce the equation to:

$$\begin{aligned}\sum_{k,l} c_{i,j}^{k,l}\phi(y_k y_l) =&(a_{i1}a_{j2} - a_{i2}a_{j1})\sum_{k',l'} c_{1,2}^{k',l'} y_{k'}y_{l'}\\ &+(a_{i2}a_{j3} - a_{i3}a_{j2})\sum_{k',l'} c_{2,3}^{k',l'} y_{k'}y_{l'}\\ &+(a_{i3}a_{j1} - a_{i1}a_{j3})\sum_{k',l'} c_{3,1}^{k',l'} y_{k'}y_{l'}.\end{aligned}$$

In the following lemma, we state the equivalency of classifying the graded automorphisms of $P_\hbar$ and classifying the graded Poisson automorphisms of P.

Lemma 3.4.1. Let $P = \Bbbk[x_1, x_2, x_3]$ be a quadratic Poisson algebra and let $P_\hbar$ be its standard deformation quantization. Then $\mathrm{Aut}_{\mathrm{gr}}(P_\hbar) \cong \mathrm{PAut}_{\mathrm{gr}}(P)$ as groups.

Proof. Let $\phi_\hbar \in \mathrm{Aut}_{\mathrm{gr}}(P_\hbar)$, written as $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ for some $a_{ij} \in \Bbbk$, $1 \le i, j \le 3$. Define $\phi : P \to P$ as follows: $\phi(x_i) = \sum_{j=1}^{3} a_{ij}x_j$, $1 \le i \le 3$. It is clear that invertibility of $\phi_\hbar$ is equivalent to the invertibility of ϕ and it remains to show that ϕ preserves the Poisson bracket. For $1 \le i < j \le 3$,

$$\phi(\{x_i, x_j\}) =(a_{i1}a_{j2} - a_{i2}a_{j1})\{x_1, x_2\} + (a_{i2}a_{j3} - a_{i3}a_{j2})\{x_2, x_3\} + (a_{i3}a_{j1} - a_{i1}a_{j3})\{x_3, x_1\}.$$

Since $\{x_i, x_j\} = \sum_{k,l} c_{i,j}^{k,l} x_k x_l$, the LHS can be expanded into:

$$\begin{aligned}\frac{1}{2}\sum_{k',l'} c_{i,j}^{k',l'}\phi(x_{k'}x_{l'}) =&(a_{i1}a_{j2} - a_{i2}a_{j1})\{x_1, x_2\}\\ &+(a_{i2}a_{j3} - a_{i3}a_{j2})\{x_2, x_3\}\\ &+(a_{i3}a_{j1} - a_{i1}a_{j3})\{x_3, x_1\}\end{aligned}$$

$$
\begin{aligned}
=&(a_{i1}a_{j2}-a_{i2}a_{j1})\sum_{k',l'}c_{1,2}^{k',l'}x_{k'}x_{l'}\\
+&(a_{i2}a_{j3}-a_{i3}a_{j2})\sum_{k',l'}c_{2,3}^{k',l'}x_{k'}x_{l'}\\
+&(a_{i3}a_{j1}-a_{i1}a_{j3})\sum_{k',l'}c_{3,1}^{k',l'}x_{k'}x_{l'}.
\end{aligned}
$$

Notice that this condition is precisely what makes $\phi_\hbar$ a graded homomorphism of $P_\hbar$ (by a factor of $\frac{\hbar}{2}$). Up to this point, we have defined a mapping $\mathrm{Aut}_{\mathrm{gr}}(P_\hbar) \to \mathrm{PAut}_{\mathrm{gr}}(P) : \phi_\hbar \mapsto \phi$ that is apparently a group homomorphism. Its inverse can be constructed in a similar manner. □

Lemma 3.4.1 states that the classification of $\mathrm{Aut}_{\mathrm{gr}}(P_\hbar)$ is equivalent to the classification of $\mathrm{PAut}_{\mathrm{gr}}(P)$. In practices, the latter is preferred due to its simplicity (smaller basis), and a comprehensive classification has already been provided in Section 3.2 and 3.3.

Proposition 3.4.2. Let $P = \Bbbk[x_1, x_2, x_3]$ be one of the quadratic Poisson algebras and let $P_\hbar$ be its standard deformation quantization. Then $\mathrm{PAut}_{\mathrm{gr}}(P) \cong \mathrm{Aut}_{\mathrm{gr}}(P_\hbar)$ contains the following automorphisms:

Case	$\mathbf{PAut_{gr}(P) \cong Aut_{gr}(\boldsymbol{P_\hbar})}$
14-1	$\begin{bmatrix} \pm\sqrt{bf-ce} & 0 & 0 \\ a & b & c \\ d & e & f \end{bmatrix}$
14-2	$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ c & d & a \end{bmatrix}$

Case	$\mathbf{PAut_{gr}(P)} \cong \mathbf{Aut_{gr}}(\boldsymbol{P_\hbar})$
14-3	$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}, \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & a \\ b & 0 & 0 \\ 0 & c & 0 \end{bmatrix}$
14-4	$\begin{bmatrix} 0 & a & 0 \\ -a & -a & 0 \\ b & c & a \end{bmatrix}, \begin{bmatrix} 0 & -a & 0 \\ -a & 0 & 0 \\ b & c & a \end{bmatrix}, \begin{bmatrix} -a & 0 & 0 \\ a & a & 0 \\ b & c & a \end{bmatrix}, \begin{bmatrix} -a & -a & 0 \\ a & 0 & 0 \\ b & c & a \end{bmatrix}, \begin{bmatrix} a & a & 0 \\ 0 & -a & 0 \\ b & c & a \end{bmatrix}, \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ b & c & a \end{bmatrix}$
14-5	$\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$
14-6	$\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}, \begin{bmatrix} -\frac{1}{2}a & \pm\frac{\sqrt{3}}{2}a & 0 \\ \mp\frac{\sqrt{3}}{2}a & -\frac{1}{2}a & 0 \\ \frac{9}{8}a & \mp\frac{3\sqrt{3}}{8}a & a \end{bmatrix}$
14-7	$\left(\langle \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} : a \neq 0 \rangle \times \langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b^2 \end{bmatrix} : b = \xi_3, \xi_3^2 \rangle \right) \rtimes \langle \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \rangle,$
14-8	$\begin{bmatrix} a & 0 & 0 \\ 0 & \frac{a^2}{b} & 0 \\ b-a & 0 & b \end{bmatrix}$

Case	$\mathbf{PAut_{gr}(P)} \cong \mathbf{Aut_{gr}}(\boldsymbol{P}_\hbar)$
14-9	$\begin{bmatrix} a & 0 & 0 \\ b & a & 0 \\ -\frac{b^2}{a} & -b & a \end{bmatrix}$

Case	$\mathbf{PAut_{gr}(P)} \cong \mathbf{Aut_{gr}}(\boldsymbol{P}_\hbar)$	Restrictions
1	$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$	$p, q, r \neq 0.$
2	$\begin{bmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & c \end{bmatrix}$	$p, q \neq 0$, $p \neq q$, $4p^2 + q^2 \neq 0.$
3	$\begin{bmatrix} a & 0 & 0 \\ b & a & 0 \\ 0 & 0 & c \end{bmatrix}$	$p \neq 0.$
4	$\begin{bmatrix} \pm\sqrt{ab} & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}$	$p, q \neq 0.$
5	$\begin{bmatrix} a & 0 & 0 \\ b & a & 0 \\ 0 & 0 & c \end{bmatrix}$	$p \neq 0, \frac{1}{2}.$

Case	$\mathbf{PAut_{gr}}(\boldsymbol{P})$	**Restrictions**
6	$\begin{bmatrix} a & 0 & 0 \\ b & a & 0 \\ 0 & 0 & c \end{bmatrix}$	$p \neq 0.$
7	$\begin{bmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & c \end{bmatrix}$	$p, q \neq 0,\ 2p + r \neq 0,$ $(2p+r)^2 + q^2 \neq 0.$
8	$\begin{bmatrix} a & \pm\sqrt{b^2 - a^2} & 0 \\ \mp\sqrt{b^2 - a^2} & a & 0 \\ 0 & 0 & b \end{bmatrix}$	$p \neq 0,\ p + q \neq 0.$
9	$\begin{bmatrix} a & 0 & 0 \\ b & a & 0 \\ c & b & a \end{bmatrix}$	$p \neq 0.$
10	$\begin{bmatrix} a & 0 & 0 \\ b & a & 0 \\ \frac{b^2}{2a} & b & a \end{bmatrix}$	$p \neq 0, -\frac{1}{4}, -\frac{1}{3}, -\frac{1}{2}.$
11	$\begin{bmatrix} a & 0 & 0 \\ b & a & c \\ 0 & 0 & \pm a \end{bmatrix}$	$p \neq 0, -\frac{1}{2}, -\frac{1}{4},\ q \neq 0.$
12	$\begin{bmatrix} a & 0 & 0 \\ b & a & c \\ \frac{d-a}{1+3p} & 0 & d \end{bmatrix}$	$p \neq 0, -\frac{1}{2},\ q = 0..$

Case	$\mathbf{PAut_{gr}}(\boldsymbol{P})$	**Restrictions**
13	$\begin{bmatrix} a & 0 & 0 \\ b & a & c \\ 0 & 0 & a \end{bmatrix}$	$p \neq 0, -\frac{1}{2}$, $q = 0$, $r = 0$.

Recall that Lemma 3.1.4 [HTWZ23, Lemma 3.3] states that a graded Poisson automorphism of a $\mathbb{Z}$-graded unimodular Poisson structure on $\Bbbk[x_1, x_2, x_3]$ can be characterized as a graded automorphism of the underlying $\Bbbk$-algebra that preserves the superpotential up to a scalar factor. It is desirable to establish a corresponding lemma for non-unimodular Poisson structures on $\Bbbk[x_1, x_2, x_3]$; however, a challenge arises due to the absence of the notion of twisted superpotentials for non-unimodular Poisson structures (in contrast to the well-established notion of twisted superpotentials for skew Calabi-Yau algebras [MS16, Definition 2.5]). Nonetheless, Lemma 3.4.1 allows us to circumvent this problem by transferring the lemma to standard deformation quantizations:

Lemma 3.4.3. Let $P = \Bbbk[x_1, x_2, x_3]$ be a non-unimodular quadratic Poisson algebra. There exists a polynomial $\omega \in \Bbbk\langle y_1, y_2, y_3\rangle$ such that

$$\mathrm{PAut_{gr}}(P) = \{\phi \in \mathrm{GL}_n(\Bbbk) : \phi(\omega) = \lambda_\phi \omega\},$$

for some $\lambda_\phi \in \Bbbk^\times$.

Proof. Since the standard deformation quantization $P_\hbar$ is Artin-Schelter regular, its minimal free resolution of $\Bbbk$ takes the form:

$$0 \to P_\hbar(-3) \longrightarrow P_\hbar(-2)^{\oplus 3} \longrightarrow P_\hbar(-1)^{\oplus 3} \longrightarrow P_\hbar \to \Bbbk \to 0.$$

In particular, every boundary matrix is over $(P_\hbar)_1$, and $P_\hbar$ is 2-Koszul. Dubois-Violette has demonstrated that any m-Koszul Artin-Schelter regular algebra is necessarily isomorphic to a derivation quotient algebra $D(\omega, 1)$ for some twisted superpotential $\omega \in \Bbbk\langle y_1, y_2, y_3\rangle$ [DV07, Theorem 11]. Subsequently, Mori and Smith has established that the graded automorphism group of $P_\hbar$ has the following property: $\mathrm{Aut}_{\mathrm{gr}}(P_\hbar) = \{\phi \in \mathrm{GL}_n(\Bbbk) : \phi(\omega) = \lambda_\phi \omega\}$ for some $\lambda_\phi \in \Bbbk^\times$ [MS16, Theorem 3.2], and if $\phi \in \mathrm{Aut}_{\mathrm{gr}}(P_\hbar)$, the value λ_ϕ is equal to $\mathrm{hdet}(\phi)$ [MS16, Theorem 3.3]. Using the isomorphism $\mathrm{PAut}_{\mathrm{gr}}(P) \cong \mathrm{Aut}_{\mathrm{gr}}(P_\hbar)$ in Lemma 3.4.1, we obtain the desired result. □

According to Lemma 2.4.4, the standard deformation quantization $P_\hbar$ of P is Artin-Schelter regular; consequently, $P_\hbar$ is skew Calabi-Yau, as these two concepts are equivalent for connected graded algebras, as established in [RRZ14, Lemma 1.2]. It should be noted that the ω in Lemma 3.4.3 is associated with the skew Calabi-Yau property of $P_\hbar$ as in [Pym15, Theorem 2.6].

As present, our knowledge regarding the ω in Lemma 3.4.3 is limited to its existence. An explicit formula of ω would be helpful. My fellow graduate student Jiahao (Eric) Zhang is currently researching on this topic. Those interested in further developments are encouraged to follow his work.

Chapter 4

CLASSIFICATIONS OF POISSON REFLECTIONS OF $P = \Bbbk[X_1, X_2, X_3]$

In this chapter, we delve into the classification of Poisson reflections for all quadratic Poisson structures on $\Bbbk[x_1, x_2, x_3]$, as well as the classification of quasi-reflections for their standard deformation quantizations.

4.1 Classification for Unimodular Poisson Algebras

In this section, our attention is centered on the classification of Poisson reflections for unimodular Poisson structures on $\Bbbk[x_1, x_2, x_3]$.

Unimodular 5, Unimodular 9.

It is clear that there are no Poisson reflections for these Poisson structures, as any graded Poisson automorphism has three repeated eigenvalues.

Unimodular 1.

A graded Poisson automorphism ϕ has the form $\phi\big|_{P_1} = \begin{bmatrix} \pm\sqrt{bf-ce} & 0 & 0 \\ a & b & c \\ d & e & f \end{bmatrix}$. Its eigenvalues are $\lambda_1 = \pm\sqrt{bf-ce}$, $\lambda_2, \lambda_3 = \dfrac{b+f \pm \sqrt{(b-f)^2+4ce}}{2}$. Notice that $\lambda_2\lambda_3 = \lambda_1^2$. If ϕ is a Poisson reflection, $\{\lambda_1, \lambda_2, \lambda_3\} = \{1, 1, \xi\}$ for some primitive root of unity ξ. If $\lambda_1 = 1$, then $\lambda_2\lambda_3 = 1$, contradicting to $\{\lambda_2, \lambda_3\} = \{1, \xi\}$. If $\lambda_1 = \xi$, then $\lambda_2\lambda_3 = \xi^2$, contradicting to $\{\lambda_2, \lambda_3\} = \{1, 1\}$ unless $\xi = -1$. In that case, $\lambda_1 = \pm\sqrt{bf-ce} = -1$, $\lambda_2 + \lambda_3 = b + f = 2$, and $\phi\big|_{P_1}$ takes the form $\begin{bmatrix} -1 & 0 & 0 \\ a & b & c \\ d & e & 2-b \end{bmatrix}$ subject to the constraint $b(2-b) - ce = 1$. Upon computation, it is found that the (2,3)-entry of the matrix, when raised to the nth power, is nc. If ϕ has finite order, then $c = 0$. Given that $b(2-b) = 1$, it follows that $b = 1$ and $\phi\big|_{P_1}$ takes a simpler form $\begin{bmatrix} -1 & 0 & 0 \\ a & 1 & 0 \\ d & e & 1 \end{bmatrix}$. Again, upon computation, it is found that the (3,2)-entry of the matrix, when raised to the nth power, is ne. If ϕ has finite order, then $e = 0$. Upon substituting this value, the resulting matrix has finite order 2. In conclusion,

$$\mathrm{PR}(P) = \left\{ \begin{bmatrix} -1 & 0 & 0 \\ a & 1 & 0 \\ d & 0 & 1 \end{bmatrix} \right\}.$$

Unimodular 2.

A graded Poisson automorphism ϕ has the form $\phi\big|_{P_1} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ c & d & a \end{bmatrix}$. Its eigenvalues are a, a, b. If ϕ is a Poisson reflection, $a = 1$, $b = \xi$ for some primitive root of unity ξ, and $\phi\big|_{P_1}$ takes the form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \xi & 0 \\ c & d & 1 \end{bmatrix}$. When raised to the nth power, the (3,1)-entry of the matrix $(\phi\big|_{P_1})^n$ is equal to nc, implying that $c = 0$. In the meantime, the (2,2)-entry of the matrix is equal to ξ^n and the (3,2)-entry of the matrix is equal to $d\dfrac{(\xi^n - 1)}{\xi - 1}$. If n is a multiple of the order of ξ, the matrix $(\phi\big|_{P_1})^n$ is equal to the identity matrix. In conclusion,

$$\mathrm{PR}(P) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & \xi & 0 \\ 0 & d & 1 \end{bmatrix} \right\}.$$

Unimodular 3.

A graded Poisson automorphism ϕ has three possible forms: $\phi\big|_{P_1} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}, \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{bmatrix},$ $\begin{bmatrix} 0 & 0 & a \\ b & 0 & 0 \\ 0 & c & 0 \end{bmatrix}$. It is apparent that the first matrix encompasses the following Poisson reflections: $\begin{bmatrix} \xi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \xi \end{bmatrix}$ for some primitive root of unity ξ. For the second and third matrices, their eigenvalues are $\lambda_1 = \sqrt[3]{abc}, \lambda_2 = \sqrt[3]{abc}\xi_3, \lambda_3 = \sqrt[3]{abc}\xi_3^2$, where ξ_3 is a primitive 3rd root of unity. If ϕ is a Poisson reflection, then $\{\lambda_1, \lambda_2, \lambda_3\} = \{1, 1, \xi\}$ for some primitive root of unity ξ. If $\lambda_1 = 1$, then $\lambda_2\lambda_3 = \xi$, contradicting to $\lambda_2\lambda_3 = \lambda_1^2$. If $\lambda_1 = \xi$, then $\dfrac{\lambda_3}{\lambda_2} = \dfrac{1}{1} = 1$, contradicting to $\dfrac{\lambda_3}{\lambda_2} = \dfrac{\sqrt[3]{abc}\xi_3^2}{\sqrt[3]{abc}\xi_3} = \xi_3$. Consequently, the second and third matrices cannot be Poisson reflections. In conclusion,

$$\mathrm{PR}(P) = \left\{ \begin{bmatrix} \xi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \xi \end{bmatrix} \right\}.$$

Unimodular 4.

A graded Poisson automorphism ϕ has six possible forms: $\phi\big|_{P_1} = \begin{bmatrix} 0 & a & 0 \\ -a & -a & 0 \\ b & c & a \end{bmatrix},$ $\begin{bmatrix} 0 & -a & 0 \\ -a & 0 & 0 \\ b & c & a \end{bmatrix}, \begin{bmatrix} -a & 0 & 0 \\ a & a & 0 \\ b & c & a \end{bmatrix}, \begin{bmatrix} -a & -a & 0 \\ a & 0 & 0 \\ b & c & a \end{bmatrix}, \begin{bmatrix} a & a & 0 \\ 0 & -a & 0 \\ b & c & a \end{bmatrix}, \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ b & c & a \end{bmatrix}$. The eigenvalues of the first and fourth matrices are $\lambda_1 = a, \lambda_2, \lambda_3 = \dfrac{(-1 \pm i\sqrt{3})a}{2}$. If ϕ is a Poisson reflection, then $\{\lambda_1, \lambda_2, \lambda_3\} = \{1, 1, \xi\}$ for some primitive root of unity ξ. In any case, $\lambda_2 + \lambda_3 \neq -\lambda_1$, a contradiction. The eigenvalues of the sixth matrix are a, a, a; consequently, it is impossible for such a matrix to be a Poisson reflection. The eigenvalues of the second, third, and fifth matrices are $\lambda_1 = -a, \lambda_2, \lambda_3 = a$. If ϕ is a Poisson reflection, then $a = 1$ and there exists three candidates for ϕ: $\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ b & c & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ b & c & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ b & c & 1 \end{bmatrix}$. By a straightforward calculation,

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ b & c & 1 \end{bmatrix}^n = \begin{cases} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{n(b-c)}{2} & \frac{n(c-b)}{2} & 1 \end{bmatrix} & n \text{ is even,} \\ \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ \frac{n+1}{2}b - \frac{n-1}{2}c & \frac{n+1}{2}c - \frac{n-1}{2}b & 1 \end{bmatrix} & n \text{ is odd.} \end{cases}$$

In this case, a finite order Poisson reflection ϕ satisfies $b = c$.

$$\begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ b & c & 1 \end{bmatrix}^n = \begin{cases} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{n}{2}c & nc & 1 \end{bmatrix} & n \text{ is even,} \\ \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ b + \frac{n-1}{2}c & nc & 1 \end{bmatrix} & n \text{ is odd.} \end{cases}$$

In this case, a finite order Poisson reflection ϕ satisfies $c = 0$.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ b & c & 1 \end{bmatrix}^n = \begin{cases} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ nb & \frac{n}{2}b & 1 \end{bmatrix} & n \text{ is even,} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ nb & \frac{n-1}{2}b + c & 1 \end{bmatrix} & n \text{ is odd.} \end{cases}$$

In this case, a finite order Poisson reflection ϕ satisfies $b = 0$.

In conclusion,

$$\text{PR}(P) = \left\{ \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ b & b & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ b & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & c & 1 \end{bmatrix} \right\}.$$

Unimodular 6.

A graded Poisson automorphism ϕ has two possible forms: $\phi\big|_{P_1} = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$, $\begin{bmatrix} -\frac{1}{2}a & \pm\frac{\sqrt{3}}{2}a & 0 \\ \mp\frac{\sqrt{3}}{2}a & -\frac{1}{2}a & 0 \\ \frac{9}{8}a & \mp\frac{3\sqrt{3}}{8}a & a \end{bmatrix}$. The eigenvalues of the former matrix and the latter matrix are $\lambda_1, \lambda_2, \lambda_3 = a, a, a$ and $\lambda_1 = a, \lambda_2, \lambda_3 = \dfrac{(-1 \pm i\sqrt{3})a}{2}$, respectively. Neither can be Poisson reflections, as discussed in prior instances.

Unimodular 7.

Gaddis, Veerapen, and Wang provided a comprehensive analysis of the linear Poisson normal elements of this particular instance [GVW23, Lemma 4.3]. Given that $\lambda^3 \neq 1$, it follows that there are no linear Poisson normal elements, thus excluding any Poisson reflections [GVW23, Lemma 2.2].

Unimodular 8.

A graded Poisson automorphism ϕ has the form $\phi\big|_{P_1} = \begin{bmatrix} a & 0 & 0 \\ 0 & \frac{a^2}{b} & 0 \\ b-a & 0 & b \end{bmatrix}$. Its eigenvalues are $\lambda_1 = a, \lambda_2 = \dfrac{a^2}{b}, \lambda_3 = b$. If ϕ is a Poisson reflection, then $\{\lambda_1, \lambda_2, \lambda_3\} = \{1, 1, \xi\}$ for some primitive root of unity ξ. If $\lambda_1 = 1$, then $\lambda_2\lambda_3 = \xi \neq \lambda_1^2$, a contradiction. If $\lambda_1 = \xi$, then the constraint $\lambda_2\lambda_3 = \lambda_1^2$ implies that $\xi = -1$ and $\phi\big|_{P_1}$ takes the form $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$. Such ϕ has finite order 2. In conclusion,

$$\mathrm{PR}(P) = \left\{ \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \right\}.$$

4.2 Classification for Non-unimodular Poisson Algebras

In this section, our attention is centered on the classification of Poisson reflections for unimodular Poisson structures on $\Bbbk[x_1, x_2, x_3]$.

Non-unimodular 9, *Non-unimodular 10*, *Non-unimodular 13.*

It is clear that there are no Poisson reflections for these Poisson structures, as any graded Poisson automorphism has three repeated eigenvalues.

Non-unimodular 1.

A graded Poisson automorphism ϕ has the form $\phi\big|_{P_1} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$. If ϕ is a Poisson reflection, then two of the variables are equal to 1 and the remaining variable is a primitive root of unity ξ. In conclusion,

$$\mathrm{PR}(P) = \left\{ \begin{bmatrix} \xi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \xi \end{bmatrix} \right\}.$$

Non-unimodular 2, Non-unimodular 7.

A graded Poisson automorphism ϕ has the form $\phi\big|_{P_1} = \begin{bmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & c \end{bmatrix}$. Its eigenvalues are $a \pm ib, c$. If ϕ is a Poisson reflection, either one of the following occurs:

- $a = \dfrac{1+\xi}{2}, b = \pm\dfrac{1-\xi}{2i}, c = 1,$
- $a = 1, b = 0, c = \xi,$

for some primitive root of unity. Both matrices have finite order. In conclusion,

$$\mathrm{PR}(P) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \xi \end{bmatrix}, \begin{bmatrix} \frac{1+\xi}{2} & \pm\frac{1-\xi}{2i} & 0 \\ \mp\frac{1-\xi}{2i} & \frac{1+\xi}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

Non-unimodular 3, Non-unimodular 5, Non-unimodular 6.

A graded Poisson automorphism ϕ has the form $\phi\big|_{P_1} = \begin{bmatrix} a & 0 & 0 \\ b & a & 0 \\ 0 & 0 & c \end{bmatrix}$. If ϕ is a Poisson reflection, then $a = 1$ and $c = \xi$ for some primitive root of unity. Such matrices have finite order if and only if $b = 0$. In conclusion,

$$\mathrm{PR}(P) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \xi \end{bmatrix} \right\}.$$

Non-unimodular 4.

A graded Poisson automorphism ϕ has the form $\phi\big|_{P_1} = \begin{bmatrix} \pm\sqrt{ab} & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}$. Its eigenvalues are $\pm\sqrt{ab}, a, b$. If ϕ is a Poisson, then $\{\pm\sqrt{ab}, a, b\} = \{1, 1, \xi\}$ for some primitive root of unity ξ. If $\{a, b\} = \{1, 1\}$, then choosing $-\sqrt{ab}$ results in a non-trivial finite order matrix; otherwise, if $\{a, b\} = \{1, \xi\}$, then $\pm\sqrt{ab}$ cannot be equal to 1, a contradiction. In conclusion,

$$\mathrm{PR}(P) = \left\{ \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}.$$

Non-unimodular 8.

A graded Poisson automorphism ϕ has the form $\phi\big|_{P_1} = \begin{bmatrix} a & \pm\sqrt{b^2 - a^2} & 0 \\ \mp\sqrt{b^2 - a^2} & a & 0 \\ 0 & 0 & b \end{bmatrix}$. Its eigenvalues are $a \pm \sqrt{a^2 - b^2}, b$. If ϕ is a Poisson reflection, then $\{a \pm \sqrt{a^2 - b^2}, b\} = \{1, 1, \xi\}$ for some primitive root of unity ξ. If $\{a \pm \sqrt{a^2 - b^2}\} = \{1, 1\}$, then choosing $b = -1$ results in a non-trivial finite order matrix; otherwise, if $\{a \pm \sqrt{a^2 - b^2}\} = \{1, \xi\}$, or equivalently, $b = 1$, then $a = \dfrac{1 + \xi}{2}$ and either one of the following occurs:

$$\begin{aligned} a + \sqrt{a^2 - 1} &= 1 \;\Rightarrow\; \sqrt{a^2 - 1} = 1 - a \;\Rightarrow\; a^2 - 1 = (1 - a)^2 \;\Rightarrow\; a = 1, \\ a - \sqrt{a^2 - 1} &= 1 \;\Rightarrow\; -\sqrt{a^2 - 1} = 1 - a \;\Rightarrow\; a^2 - 1 = (1 - a)^2 \;\Rightarrow\; a = 1. \end{aligned}$$

In either case, we arrive at a contradiction $\xi = 1$. In conclusion,

$$\mathrm{PR}(P) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}.$$

Non-unimodular 11.

A graded Poisson automorphism ϕ has the form $\phi\big|_{P_1} = \begin{bmatrix} a & 0 & 0 \\ b & a & c \\ 0 & 0 & \pm a \end{bmatrix}$. Its eigenvalues are $a, a, \pm a$, and it is clear that selecting $a = 1$ leads to a Poisson reflection. In conclusion,

$$\mathrm{PR}(P) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}.$$

Non-unimodular 12.

A graded Poisson automorphism ϕ has the form $\phi\big|_{P_1} = \begin{bmatrix} a & 0 & 0 \\ b & a & c \\ \frac{d-a}{3p+1} & 0 & d \end{bmatrix}$. Its eigenvalues are a, a, d. Set $a = 1$ and $d = \xi$ for some primitive root of unity ξ. By a straightforward calculation,

$$(\phi\big|_{P_1})^n = \begin{bmatrix} 1 & 0 & 0 \\ nb + \dfrac{(\sum_{i=0}^{n-1} \xi^i)c - nc}{3p+1} & 1 & (\sum_{i=0}^{n-1} \xi^i)c \\ \dfrac{\xi^n - 1}{3p+1} & 0 & \xi^n \end{bmatrix}$$

In order to have finite order, $b = \dfrac{c}{3p+1}$. In conclusion,

$$\mathrm{PR}(P) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ \frac{c}{3p+1} & 1 & c \\ \frac{\xi-1}{3p+1} & 0 & \xi \end{bmatrix} \right\}.$$

It is time to record all Poisson reflections of non-unimodular quadratic Poisson structures on $\Bbbk[x_1, x_2, x_3]$ into a table for future reference.

Proposition 4.2.1. Let $P = \Bbbk[x_1, x_2, x_3]$ be one of the non-unimodular quadratic Poisson algebras. Then PR(P) contains the following automorphisms:

Case	**PR(P)**	**Conditions**
1	$\begin{bmatrix} \xi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \xi \end{bmatrix}$	$p, q, r \neq 0.$
2	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \xi \end{bmatrix}, \begin{bmatrix} \frac{1+\xi}{2} & \pm\frac{1-\xi}{2i} & 0 \\ \mp\frac{1-\xi}{2i} & \frac{1+\xi}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$p, q \neq 0$, $p \neq q$, $4p^2 + q^2 \neq 0.$
3	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \xi \end{bmatrix}$	$p \neq 0.$
4	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$p, q \neq 0.$
5	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \xi \end{bmatrix}$	$p \neq 0, \frac{1}{2}.$
6	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \xi \end{bmatrix}$	$p \neq 0.$

Case	PR(*P*)	Conditions
7	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&\xi\end{bmatrix}, \begin{bmatrix}\frac{1+\xi}{2}&\pm\frac{1-\xi}{2i}&0\\\mp\frac{1-\xi}{2i}&\frac{1+\xi}{2}&0\\0&0&1\end{bmatrix}$	$p, q \neq 0,\ 2p+r \neq 0,$ $(2p+r)^2+q^2 \neq 0.$
8	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}$	$p \neq 0,\ p+q \neq 0.$
9	$\emptyset$	$p \neq 0.$
10	$\emptyset$	$p \neq 0, -\frac{1}{4}, -\frac{1}{3}, -\frac{1}{2}.$
11	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}$	$p \neq 0, -\frac{1}{2}, -\frac{1}{4},\ q \neq 0.$
12	$\begin{bmatrix}1&0&0\\\frac{c}{3p+1}&1&c\\\frac{\xi-1}{3p+1}&0&\xi\end{bmatrix}$	$p \neq 0, -\frac{1}{2}, -\frac{1}{3},\ q \neq 0.$
13	$\emptyset$	$p \neq 0, -\frac{1}{2},\ q = 0,\ r = 0.$

4.3 Classification for Deformation Quantizations

In this section, our attention is centered on the classification of quasi-reflections for the standard deformation quantizations of unimodular Poisson structures on $\Bbbk[x_1, x_2, x_3]$.

We begin by noting the following:

Lemma 4.3.1. Let $P = \Bbbk[x_1, x_2, x_3]$ be a unimodular quadratic Poisson algebra and let $P_\hbar$ be its standard deformation quantization. Suppose that ϕ is a graded automorphism of $P_\hbar$. If ϕ satisfies the following conditions:

(1) ϕ has the form $\phi\big|_{P_1} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$,

(2) the coefficients of y_1y_3, y_2y_3, y_3^2 are equal to 0 in the commutator bracket $[y_1, y_2]$,

(3) the coefficient of y_3^2 are equal to 0 in the commutator brackets $[y_2, y_3]$ and $[y_3, y_1]$,

and either of the following conditions:

(4) the commutator bracket $[y_1, y_2]$ is equal to 0,

(4') the entries a_{12}, a_{21} are equal to 0,

(4") the entry a_{12} is equal to 0 and the coefficient of y_2^2 in the commutator bracket $[y_1, y_2]$ is equal to 0,

then ϕ is a quasi-reflection if and only if ϕ is a classical reflection.

Proof. Fix a degree $d \geq 0$. Take an arbitrary basis element $y_1^i y_2^j y_3^k$ from $(P_\hbar)_d$: $i + j + k = d$ [DML98, page 254]. Apply ϕ to $y_1^i y_2^j y_3^k$:

$$\phi(y_1^i y_2^j y_3^k) = (a_{11}y_1 + a_{12}y_2)^i (a_{21}y_1 + a_{22}y_2)^j (a_{31}y_1 + a_{32}y_2 + a_{33}y_3)^k.$$

To compute the trace series $\mathrm{Tr}_{P_\hbar}(\phi, t)$, we determine the coefficient of the term $y_1^i y_2^j y_3^k$ after rearranging the terms of $\phi(y_1^i y_2^j y_3^k)$ in lexicographical order.

Given that flipping y_2 and y_1 does not result in the generation of y_3, the exclusive source of y_3 is the product $(a_{31}y_1 + a_{32}y_2 + a_{33}y_3)^k$. In order to generate y_3^k, the only choice is the product of k copies of $a_{33}y_3$. This is due to the observation that choosing $a_{31}y_1$ or $a_{32}y_2$ in the product $(a_{31}y_1 + a_{32}y_2 + a_{33}y_3)^k$ results in a reduction by one in the number of copies of y_3, based on the constraints (2) and (3).

It remains to generate $y_1^i y_2^j$. If either condition (4), (4'), or (4") is met, it follows that the coefficient of the term $y_1^i y_2^j$ is identical to the coefficient of the term $y_1^i y_2^j$ when we rearrange the terms lexicographically subjecting to the relations $y_1y_2 = y_2y_1$. In conjunction with the above argument regarding y_3^k, the coefficient of the term $y_1^i y_2^j y_3^k$ is identical to the coefficient of the term $y_1^i y_2^j y_3^k$ when we rearrange the terms lexicographically in a polynomial ring, that is, subjecting to the relations $y_1y_2 = y_2y_1$, $y_2y_3 = y_3y_2$, $y_3y_1 = y_1y_3$. Consequently,

$$\mathrm{Tr}_{P_\hbar}(\phi, t) = \frac{1}{\det(I_n - t\phi\big|_{(P_\hbar)_1})} = \frac{1}{(1-\lambda_1 t)\cdots(1-\lambda_n t)},$$

where $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of the matrix $\phi\big|_{(P_\hbar)_1}$.

According to [KKZ09, Theorem 3.1], a quasi-reflection of a quantum polynomial ring, to which $P_\hbar$ belongs as established by Lemma 2.4.4, must be either a classical reflection or a mystic reflection. However, in the latter case, $\mathrm{Tr}_{P_\hbar}(\phi, t) = \dfrac{1}{(1-t)(1+t^2)} \neq \dfrac{1}{(1-t)^2 f(t)}$ for some polynomial $f(t)$ satisfying $f(1) \neq 0$. This rules out the possibility of ϕ being a mystic reflection, leading to the conclusion that ϕ is a quasi-reflection if and only if ϕ is a classical reflection. □

Lemma 4.3.1 resolves 70% of the instances of classifying quasi-reflections for unimodular quadratic Poisson structures on $\Bbbk[x_1, x_2, x_3]$. However, there are certain cases that remain unaddressed, and these will be discussed in the subsequent lemma.

Lemma 4.3.2. Let $P = \Bbbk[x_1, x_2, x_3]$ be a unimodular quadratic Poisson algebra derived from the following superpotentials:

$$\Omega_1 : x_1^3, \quad \Omega_3 : 2x_1x_2x_3, \quad \Omega_6 : x_1^3 + x_1^2x_3 + x_2^2x_3, \quad \Omega_7 : \frac{1}{3}(x_1^3 + x_2^3 + x_3^3) - \lambda x_1x_2x_3, \lambda^3 \neq 1.$$

If ϕ is a graded automorphism of the standard deformation quantization $P_\hbar$ of P, then ϕ is a quasi-reflection if and only if ϕ is a classical reflection.

Proof. For *Unimodular 1* and *Unimodular 3*, our objective is to establish that for any graded automorphism of the standard deformation quantization $P_\hbar$, the trace series

$$\mathrm{Tr}_{P_\hbar}(\phi, t) = \frac{1}{(1-\lambda_1 t)\cdots(1-\lambda_n t)},$$

where $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of $\phi\big|_{(P_\hbar)_1}$. As illustrate in Lemma 4.3.1, this equality enables us to conclude that a classical reflection is necessarily a quasi-reflection, and vice versa.

Unimodular 1. $\Omega_1 = x_1^3$.

A graded automorphism ϕ of $P_\hbar$ has the form $\phi\big|_{(P_\hbar)_1} = \begin{bmatrix} \pm\sqrt{bf-ce} & 0 & 0 \\ a & b & c \\ d & e & f \end{bmatrix}$. Fix a degree $d \geq 0$. Take an arbitrary basis element $y_1^i y_2^j y_3^k$ from $(P_\hbar)_d$: $i+j+k=d$ [DML98, page 254]. Apply ϕ to $y_1^i y_2^j y_3^k$:

$$\phi(y_1^i y_2^j y_3^k) = \pm\sqrt{bf-ce}\, y_1^i (ay_1 + by_2 + cy_3)^j (dy_1 + ey_2 + fy_3)^k.$$

To compute the trace series $\mathrm{Tr}_{P_\hbar}(\phi, t)$, we determine the coefficient of the term $y_1^i y_2^j y_3^k$ after rearranging the terms of $\phi(y_1^i y_2^j y_3^k)$ in lexicographical order.

As $\pm\sqrt{bf-ce}\, y_1^i$ already supplies y_1^i, and flipping the order of any of the variables will not result in an increase of y_2 and y_3, our attention is solely on $(by_2 + cy_3)^j (ey_2 + fy_3)^k$ within the last two multiplicands. Using the commutator relation $[y_2, y_3] = 3\hbar y_1^2$, the coefficient of the term $y_1^i y_2^j y_3^k$ is identical to the coefficient of the term $y_1^i y_2^j y_3^k$ when we rearrange the terms lexicographically in a polynomial ring. Using the same reasoning as in Lemma 4.3.1, $\mathrm{Tr}_{P_\hbar}(\phi, t) = \dfrac{1}{(1-\lambda_1 t)\cdots(1-\lambda_n t)}$, where $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of $\phi\big|_{(P_\hbar)_1}$, and we conclude that ϕ cannot be a mystic reflection.

Unimodular 3. $\Omega_3 = 2x_1x_2x_3$.

Applying Lemma 4.3.1, it is sufficient to compute the trace series $\mathrm{Tr}_{P_\hbar}(\phi, t)$ of ϕ of the forms $\phi_1\big|_{(P_\hbar)_1} = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{bmatrix}, \phi_2\big|_{(P_\hbar)_1} = \begin{bmatrix} 0 & 0 & a \\ b & 0 & 0 \\ 0 & c & 0 \end{bmatrix}$. Again, fix a degree $d \geq 0$ and take an arbitrary basis element $y_1^i y_2^j y_3^k$ from $(P_\hbar)_d$: $i + j + k = d$ [DML98, page 254]. Apply ϕ_1 and ϕ_2:

$$\phi_1(y_1^i y_2^j y_3^k) = a^i b^j c^k y_2^i y_3^j y_1^k = a^i b^j c^k (\frac{1-\hbar}{1+\hbar})^{(i-j)k} y_1^k y_2^i y_3^j,$$

$$\phi_2(y_1^i y_2^j y_3^k) = a^i b^j c^k y_3^i y_1^j y_2^k = a^i b^j c^k (\frac{1-\hbar}{1+\hbar})^{(k-j)i} y_1^j y_2^k y_3^i.$$

In both instances, the existence of the term $y_1^i y_2^j y_3^k$ within $\phi(y_1^i y_2^j y_3^k)$ is equivalent to the condition that $i = j = k$, and when this specific condition is satisfied, the coefficient of $y_1^i y_2^j y_3^k$ is identical to the coefficient of $y_1^i y_2^j y_3^k$ when we rearrange the terms lexicographically in a polynomial ring. Using the same reasoning as in Lemma 4.3.1, $\mathrm{Tr}_{P_\hbar}(\phi, t) = \dfrac{1}{(1-\lambda_1 t)\cdots(1-\lambda_n t)}$, where $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of $\phi\big|_{(P_\hbar)_1}$, and we conclude that ϕ cannot be a mystic reflection.

For *Unimodular 6*, we take an alternative approach.

Unimodular 6. $\Omega_6 : x_1^3 + x_1^2 x_3 + x_2^2 x_3$.

Our analysis in Section 3.2 shows that there is no classical reflection. As established in Lemma 2.4.4, the standard deformation quantization $P_\hbar$ is a quantum polynomial ring. Therefore, our objective boils down to demonstrating that a graded automorphism ϕ of the form $\phi\big|_{(P_\hbar)_1} = \begin{bmatrix} -\frac{1}{2}a & \pm\frac{\sqrt{3}}{2}a & 0 \\ \mp\frac{\sqrt{3}}{2}a & -\frac{1}{2}a & 0 \\ \frac{9}{8}a & \mp\frac{3\sqrt{3}}{8}a & a \end{bmatrix}$ cannot be a mystic reflection by a combination of Lemma 4.3.1 and [KKZ09, Theorem 3.1]. By calculation,

$$\begin{bmatrix} -\frac{1}{2}a & \pm\frac{\sqrt{3}}{2}a & 0 \\ \mp\frac{\sqrt{3}}{2}a & -\frac{1}{2}a & 0 \\ \frac{9}{8}a & \mp\frac{3\sqrt{3}}{8}a & a \end{bmatrix}^4 = \begin{bmatrix} -\frac{1}{2}a^4 & \pm\frac{\sqrt{3}}{2}a^4 & 0 \\ \mp\frac{\sqrt{3}}{2}a^4 & -\frac{1}{2}a^4 & 0 \\ \frac{9}{8}a^4 & \mp\frac{3\sqrt{3}}{8}a^4 & a^4 \end{bmatrix},$$

is not the identity matrix, and, therefore, is not a mystic reflection.

For *Unimodular 7*, [KKZ09, Theorem 6.2, Corollary 6.3] has provided a comprehensive analysis of its standard deformation quantization $P_\hbar$, proving that it has no quasi-reflections. As $P_\hbar$ is a quantum polynomial ring, it is tautological to state that ϕ is a quasi-reflection if and only if ϕ is a classical reflection. □

It is time to record all Poisson reflections (resp. quasi-reflections) of unimodular quadratic Poisson structures on $\Bbbk[x_1, x_2, x_3]$ (resp. standard deformation quantization of unimodular quadratic Poisson structures on $\Bbbk[x_1, x_2, x_3]$) into a table for future reference.

Proposition 4.3.3. Let $P = \Bbbk[x_1, x_2, x_3]$ be one of the unimodular quadratic Poisson algebras and let $P_\hbar$ be its standard deformation quantization. Then $\mathrm{PR}(P) = \mathrm{QR}(P_\hbar)$ contains the following automorphisms:

Case	$\mathbf{PR}\ (\boldsymbol{P}) = \mathbf{QR}(\boldsymbol{P}_\hbar)$
14-1	$\begin{bmatrix} -1 & 0 & 0 \\ a & 1 & 0 \\ d & 0 & 1 \end{bmatrix}$
14-2	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \xi & 0 \\ 0 & d & 1 \end{bmatrix}$
14-3	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \xi \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \xi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Case	$\mathbf{PR}\ (\boldsymbol{P}) = \mathbf{QR}(\boldsymbol{P}_\hbar)$
14-4	$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ b & b & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ b & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & c & 1 \end{bmatrix}$
14-5	$\emptyset$
14-6	$\emptyset$
14-7	$\emptyset$
14-8	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$
14-9	$\emptyset$

Chapter 5

VARIANTS OF THE SHEPHARD-TODD-CHEVALLEY THEOREM OF $P = \Bbbk[X_1, X_2, X_3]$

In this chapter, we prove the majority of the main results of this dissertation: variants of the Shephard-Todd-Chevalley Theorem for quadratic Poisson structures on $\Bbbk[x_1, x_2, x_3]$ and their associated algebraic structures.

5.1 Overview

Let $P = \Bbbk[x_1, \cdots, x_n]$ be a Poisson algebra and let G be a finite subgroup of the graded Poisson automorphism group of P. The invariant subalgebra of P under the action of G

$$P^G = \{f \in P : \phi(f) = f \text{ for all } \phi \in G\}$$

is a Poisson algebra under the Poisson structure inherited from P. Establishing P and P^G are non-isomorphic Poisson algebras, by demonstrating the absence of suitable mappings, can be a challenging task. In this dissertation, we are primarily relying on the following four lemmas to distinguish Poisson algebras from their invariant subalgebras.

The first two lemmas assert that unimodularity can distinguish Poisson algebras. Furthermore, when unimodularity is not provided, the Poisson twistings are invariant under Poisson isomorphisms, provided that appropriate conditions are met.

Lemma 5.1.1. Let $P = \Bbbk[x_1, \cdots, x_n]$, $Q = \Bbbk[y_1, \cdots, y_n]$ be Poisson algebras. If P is unimodular and Q is non-unimodular, then P is not isomorphic to Q as Poisson algebras.

Proof. Let $\underline{m}_P, \underline{m}_Q$ be the modular derivations of P, Q, respectively. Suppose that P is isomorphic to Q as Poisson algebras through an isomorphism ϕ. Given that the modular derivation $\underline{m}_Q$ is independent of the generators, we may compute $\underline{m}_Q$ with respect to the generators $\{\phi(x_1), \cdots, \phi(x_n)\}$, instead of the conventional generators $\{y_1, \cdots, y_n\}$. For all $f \in Q$,

$$\underline{m}_Q(f) = \sum_{i=1}^{n} \frac{\partial\{\phi(x_i), \phi(g)\}}{\partial \phi(x_i)} = \sum_{i=1}^{n} \frac{\partial \phi\{x_i, g\}}{\partial \phi(x_i)} = \phi\left(\sum_{i=1}^{n} \frac{\partial\{x_i, g\}}{\partial x_i}\right) = \phi(m_P(g)) = 0,$$

for some $g \in P$. By definition, the Poisson algebra Q is unimodular, a contradiction. □

It should be remarked that to prove the modular derivation of Poisson structure on $\Bbbk[x_1, \cdots, x_n]$ is independent of the choice of generators, or more precisely, the choice of volume forms, requires a more general definition than the one given earlier. For a Poisson algebra P admitting a volume form ν, the divergence of a derivation δ with respect to ν can be defined as the unique value that satisfies the equation $\mathcal{L}_\nu(\delta) = \underline{\quad}\,\nu$, and the modular derivation $\underline{m}$ can be defined as the mapping that assigns $f \in P$ to the negative of the divergence of the Hamiltonian derivation $\{f, -\}$. Using this set of definitions, following a computation analogous to [TWZ22, page 6], we can establish that when P is a polynomial Poisson algebra, its modular derivation is independent of the choice of generators.

Lemma 5.1.2. Let $P = \Bbbk[x_1, \cdots, x_n]$, $Q = \Bbbk[y_1, \cdots, y_n]$ be Poisson algebras under the standard grading, with their modular derivations written as $\underline{m}_P, \underline{m}_Q$, respectively. If P and Q are isomorphic as Poisson algebras, then their Poisson twistings $P^{-\frac{1}{n}\underline{m}_P}$ and $Q^{-\frac{1}{n}\underline{m}_Q}$ are isomorphic as Poisson algebras.

Proof. Suppose that P is isomorphic to Q as Poisson algebras through an isomorphism ϕ. Given that P, Q are connected $\mathbb{N}$-graded Poisson algebras that are finitely generated in degree 1, we may assume ϕ is graded [GW20, Theorem 4.2]. First, we claim the following diagram is commutative:

$$\begin{array}{ccc} P & \xrightarrow{\phi} & Q \\ {\scriptstyle \underline{m}_P}\downarrow & & \downarrow{\scriptstyle \underline{m}_Q} \\ P & \xrightarrow[\phi]{} & Q \end{array}$$

For all $f \in P$,

$$\phi(\underline{m}_P(f)) = \phi\left(\sum_{i=1}^{n} \frac{\partial\{x_i, f\}}{\partial x_i}\right) = \sum_{i=1}^{n} \frac{\partial\phi(\{x_i, f\})}{\partial\phi(x_i)} = \sum_{i=1}^{n} \frac{\partial\{\phi(x_i), \phi(f)\}}{\partial\phi(x_i)} = \underline{m}_Q(\phi(f)),$$

where the second equality follows from the commutativity of ϕ and differential operators, and the fourth equality follows from the independence of choice of generators of $\underline{m}_Q$.

Given that $P^{-\frac{1}{n}\underline{m}_P} = P$ and $Q^{-\frac{1}{n}\underline{m}_Q} = Q$ as $\mathbb{k}$-algebras, we still have a $\mathbb{k}$-algebra isomorphism ϕ between the Poisson twistings, and our task is to demonstrate that ϕ preserves the Poisson structures. For all $1 \le i, j \le n$,

$$\begin{aligned} \phi(\langle x_i, x_j\rangle)_P =& \phi(\{x_i, x_j\}_P - \frac{1}{n} \cdot 1 \cdot x_i\underline{m}_P(x_j) + \frac{1}{n} \cdot 1 \cdot x_j\underline{m}_P(x_i)) \\ =& \phi(\{x_i, x_j\}_P) - \frac{1}{n}\phi(x_i)\phi(\underline{m}_P(x_j)) + \frac{1}{n}\phi(x_j)\phi(\underline{m}_P(x_i)) \\ =& \{\phi(x_i), \phi(x_j)\}_Q - \frac{1}{n}\phi(x_i)\underline{m}_Q(\phi(x_j)) + \frac{1}{n}\phi(x_j)\underline{m}_Q(\phi(x_i)) \\ =& \langle\phi(x_i), \phi(x_j)\rangle_Q, \end{aligned}$$

where the third equality follows from the commutativity of the above diagram. □

There are two additional lemmas that, although less general than the former two, are nonetheless highly useful in this dissertation.

Lemma 5.1.3. Let $P = \mathbb{k}[x_1, x_2, x_3]$ be a quadratic Poisson algebra. If $Q = \mathbb{k}[y_1, y_2, y_3]$ is a Poisson algebra satisfying the following conditions:

(1) there exists a pair $(i, j) \in \{1, 2, 3\}^{\oplus 2}$ such that $\{y_i, y_j\} = f(y_1, y_2) - ry_3$ for some

homogeneous f with respect to the grading $\deg(y_1) = \deg(y_2) = 1$, and some $r \in \Bbbk^\times$,

(2) the bracket $\{y_k, y_l\}$ is a scalar multiple of a single monomial in y_1, y_2, y_3, for all $(k,l) \in \{1,2,3\}^{\oplus 2} \backslash \{(i,j)\}$,

then P is not isomorphic to Q as Poisson algebras.

Proof. Suppose, for the sake of contradiction, that P and Q are isomorphic as Poisson algebras. Observe that such a Poisson isomorphism passes to a $\Bbbk$-algebra isomorphism:

$$P/(\{P,P\}) \xrightarrow{\sim} Q/(\{Q,Q\})$$

For the $\Bbbk$-algebra P, the ideal $(\{P,P\})$ is generated by $\{x_i, x_j\}$, where $1 \leq i,j \leq 3$, because:

$$\{g,h\} = \sum_{1 \leq i,j \leq 3} \frac{\partial g}{\partial x_i} \frac{\partial h}{\partial x_j} \{x_i, x_j\},$$

for all $g, h \in P$. The quotient algebra $P/(\{P,P\})$ is a connected $\mathbb{N}$-graded algebra that is finitely generated in degree 1, as the Poisson algebra P is quadratic and consequently the ideal $(\{P,P\})$ admits a set of homogeneous generators of degree 2.

For the $\Bbbk$-algebra Q, by the same reasoning, the ideal $\{Q,Q\}$ is generated by $\{y_i, y_j\}$, where $1 \leq i,j \leq 3$. Based on the assumptions, we can write

$$\{y_a, y_b\}_Q = f(y_1,y_2) - ry_1, \quad \{y_b, y_c\}_Q = sy_1^{\alpha_1} y_2^{\alpha_2} y_3^{\alpha_3}, \quad \{y_c, y_a\}_Q = ty_1^{\beta_1} y_2^{\beta_2} y_3^{\beta_3},$$

for some $\{a,b,c\}$ that is a relabeling of $\{1,2,3\}$, $s,t \in \Bbbk$, and $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \geq 0$. The quotient algebra

$$\begin{aligned} Q/(\{Q,Q\}) =& \Bbbk[y_1,y_2,y_3]/(f - ry_1, sy_1^{\alpha_1} y_2^{\alpha_2} y_3^{\alpha_3}, ty_1^{\beta_1} y_2^{\beta_2} y_3^{\beta_3}) \\ \cong& \Bbbk[y_1,y_2]/(\frac{s}{r^{\alpha_3}} y_1^{\alpha_1} y_2^{\alpha_2} f^{\alpha_3}, \frac{t}{r^{\beta_3}} y_1^{\beta_1} y_2^{\beta_2} f^{\beta_3}) \end{aligned}$$

is a connected $\mathbb{N}$-graded algebra that is finitely generated in degree 1, as f is a homogeneous polynomial with respect to the grading $\deg(y_1) = \deg(y_2) = 1$.

Based on the preceding argument, [GW20, Lemma 4.1] applies. However,

$$\dim_{\Bbbk}\big(P/(\{P,P\})\big) = 3 \neq 2 \geq \dim_{\Bbbk}\big(Q/(\{Q,Q\})\big),$$

a contradiction. □

Lemma 5.1.4. Let $P = \Bbbk[x_1, x_2, x_3]$, $Q = \Bbbk[y_1, y_2, y_3]$ be the following Poisson algebras:

$$\begin{aligned} \{x_1,x_2\} =& rx_1^2, \quad \{x_2,x_3\} = \ s_1x_1x_3 + s_2x_2x_3, \quad \{x_3,x_1\} = tx_1x_3, \\ \{y_1,y_2\} =& ry_1^2, \quad \{y_2,y_3\} = ns_1y_1y_3 + ns_2y_2y_3, \quad \{y_3,y_1\} = nty_1y_3, \end{aligned}$$

for some $r, t \in \Bbbk^{\times}$, $s_1, s_2 \in \Bbbk$, and $n \neq 0, 1$. Then P is not isomorphic to Q as Poisson algebras.

Proof. The Poisson algebras P and Q are connected $\mathbb{N}$-graded Poisson algebras that are finitely generated in degree 1. If P and Q are isomorphic as Poisson algebras through a Poisson isomorphism ϕ, then we can assume that ϕ is graded [GW20, Theorem 4.2] and record it as $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ for some $a_{ij} \in \Bbbk$, $1 \leq i, j \leq 3$, satisfying $\phi(x_i) = \sum_{j=1}^{3} a_{ij} y_j$, $1 \leq i \leq 3$. Given that ϕ is a Poisson homomorphism, ϕ preserves the Poisson structures:

$$\begin{aligned} \phi(\{x_1,x_2\}_P) =& \{\phi(x_1), \phi(x_2)\}_Q, \\ \phi(\{x_3,x_1\}_P) =& \{\phi(x_3), \phi(x_1)\}_Q. \end{aligned}$$

By examining the coefficients of the terms x_2^2, x_3^2, x_1x_3 in the first equality, and the coefficient of the term x_1x_3 in the second equality, we establish the following constraints on $\phi|_{P_1}$:

(1) $ra_{12}^2 = 0$,

(2) $ra_{13}^2 = 0$,

(3) $2ra_{11}a_{13} = ns_1(a_{12}a_{23} - a_{13}a_{22}) + nt(a_{13}a_{21} - a_{11}a_{23})$,

(4) $t(a_{11}a_{33} + a_{13}a_{31}) = ns_1(a_{13}a_{32} - a_{12}a_{33}) + nt(a_{11}a_{33} - a_{13}a_{31})$

Solving the four equalities results in $a_{13} = a_{23} = a_{33} = 0$, contradicting to the invertibility of ϕ. Therefore, no such Poisson isomorphism ϕ exists in the first place. $\square$

Finally, we introduce an additional approach for distinguishing Poisson algebras: Poisson cohomology (homology). While our dissertation does not require any computation of Poisson cohomology (homology) due to the availability of simpler arguments, we include it because it is a valuable tool for differentiating Poisson algebras from one another in broader contexts. Those interested in this subject are encouraged to read [LGPV13], [LWZ17], [TWZ22], [HTWZ23]

Definition 5.1.5. Let P be a Poisson algebra.

- A *derivation* of P is a $\Bbbk$-linear map $\delta : P \to P$ such that $\delta(fg) = \delta(f)g + f\delta(g)$ for all $f, g \in P$.

- A *Hamiltonian derivation* of P is a derivation $\delta : P \to P$ such that δ coincides with $\{-, f\}$ for some $f \in P$. The $\Bbbk$-vector space of Hamiltonian derivations of P will be denoted as $\mathrm{Ham}(P)$.

- A *Poisson derivation* of P is a derivation $\delta : P \to P$ such that $\delta(\{f, g\}) = \{\delta(f), g\} + \{f, \delta(g)\}$ for all $f, g \in P$. The $\Bbbk$-vector space of Poisson derivations of P will be denoted as $\mathrm{PDer}(P)$.

- A *skew-symmetric p-derivation* of P is a skew-symmetric $\Bbbk$-linear map $\delta : \bigwedge^p P \to P$ that is a derivation in all coordinates. The $\Bbbk$-vector space of all skew-symmetric p-derivation of P will be denoted as $\mathscr{X}^p(P)$.

Definition-Lemma 5.1.6. Let P be a Poisson algebra, with its Poisson structure written as π. Define a sequence of boundary maps

$$d_\pi^\bullet : \mathscr{X}^\bullet(P) \to \mathscr{X}^{\bullet+1}(P)$$

as follows: for all $Q \in \mathscr{X}^{p+1}(P)$,

$$\begin{aligned} d_\pi^p(Q)[f_0, \cdots, f_p] = &\sum_{i=0}^{p} (-1)^i \pi(f_i, Q[f_0, \cdots, \widehat{f_i}, \cdots, f_p]) \\ &+ \sum_{0 \le i,j \le n} Q[\pi(f_i, f_j), f_0, \cdots, \widehat{f_i}, \cdots, \widehat{f_j}, \cdots, f_k]. \end{aligned}$$

The boundary maps $d_\pi^\bullet$ can be realized as a particular instance of the *Schouten bracket* $[\pi, -]_S$, allowing us to demonstrate that $d_\pi^\bullet$ satisfies the condition $d_\pi^2 = 0$ [LGPV13, page 95]. Consequently, the pair $(\mathscr{X}^\bullet(P), d_\pi^\bullet)$ forms a complex, with the pth cohomology group defined as the *pth Poisson cohomology*:

$$PH^p(P) = \ker d_\pi^p / \mathrm{im} d_\pi^{p-1}.$$

As one may have discovered, Poisson cohomology can be realized as the Poisson counterpart of Hochschild cohomology of associative algebras. The parallels can be observed in several ways. For instance, the 0th cohomology PH^0 corresponds to the Poisson center $Z_\pi(P)$, the 1st cohomology PH^1 represents the quotient of the Poisson derivations and the Hamiltonian derivations $\mathrm{PDer}(P)/\mathrm{Ham}(P)$, the 2nd cohomology PH^2 is related to the classification of infinitesimal deformations, and the 3rd cohomology PH^3 is associated with the obstructions of lifting infinitesimal deformations to formal deformations. Furthermore, the Poisson complex $\mathscr{X}^\bullet(P) = \bigoplus_{p\ge 0} \mathscr{X}^p(P)$ is a Gerstenhabor algebra under the following operations $\wedge$ and $[-,-]_S$: for all $Q \in \mathscr{X}^q(P)$ and $R \in \mathscr{X}^r(P)$,

$$(Q \wedge R)[f_1, \cdots, f_{q+r}] = \sum_{\sigma \in S_{q,r}} \operatorname{sgn}(\sigma) Q[f_{\sigma(1)}, \cdots, f_{\sigma(q)}] R[f_{\sigma(q+1)}, \cdots, f_{\sigma(q+r)}],$$

$$\begin{aligned}[Q, R]_S[f_1, \cdots, f_{q+r-1}] = &\sum_{\sigma \in S_{q,r-1}} \operatorname{sgn}(\sigma) Q[R[f_{\sigma(1)}, \cdots f_{\sigma(r)}], f_{\sigma(r+1)}, \cdots, f_{\sigma(q+r-1)}] \\ &- \sum_{\sigma \in S_{r,q-1}} \operatorname{sgn}(\sigma) R[Q[f_{\sigma(1)}, \cdots f_{\sigma(q)}], f_{\sigma(r+1)}, \cdots, f_{\sigma(q+r-1)}],\end{aligned}$$

where $S_{q,r} = \{\sigma \in S_{q+r} : \sigma(1) < \cdots < \sigma(q), \sigma(r+1) < \cdots < \sigma(q+r)\}$.

Example 5.1.7. Let $P = \Bbbk[x, y, z]/(xy, y^2, yz)$ under the Poisson structure $\pi = y\partial y \wedge \partial z$. The Poisson algebra P admits a $\Bbbk$-vector space decomposition $P = \Bbbk[x, z]_{\geq 1} \oplus \Bbbk \oplus \Bbbk y$, and the Poisson complex $\mathscr{X}^\bullet(P)$ can be computed based on the relations of P:

$\mathscr{X}^1(P) = (\Bbbk[x, z]_{\geq 1} \oplus \Bbbk y)\partial x + (\Bbbk y)\partial y + (\Bbbk[x, z]_{\geq 1} \oplus \Bbbk y)\partial z.$

$\mathscr{X}^2(P) = (\Bbbk y)\partial x \wedge \partial y + (\Bbbk y)\partial y \wedge \partial z + (\Bbbk[x, z]_{\geq 1} \oplus \Bbbk y)\partial x \wedge \partial z.$

$\mathscr{X}^3(P) = (\Bbbk y)\partial x \wedge \partial y \wedge \partial z.$

$$0 \longrightarrow P \xrightarrow{d_\pi^0} \mathscr{X}^1(P) \xrightarrow{d_\pi^1} \mathscr{X}^2(P) \xrightarrow{d_\pi^2} \mathscr{X}^3(P) \longrightarrow 0.$$

After an extensive computation, $PH^i(P) \cong \begin{cases} \Bbbk[x, xz, z^2, z^3] & i = 0 \\ (\Bbbk[x, xz, z^2, z^3]_{\geq 1})^{\oplus 2} & i = 1 \\ \Bbbk[x, xz, z^2, z^3]_{\geq 1} & i = 2 \\ 0 & i \geq 3 \end{cases}.$

When the Poisson algebra P has a limited number of relations, the computation of the Poisson cohomology of P becomes a formidable task. As an alternative approach, one can introduce a grading for the Poisson algebra P and use the Hilbert series to describe its Poisson cohomology. This approach is exemplified in [TWZ22] and [HTWZ23].

In a dual manner, we can define Poisson homology of a Poisson algebra P using the Kähler differentials $\Omega^1(P)$ and $\Omega^p(P) = \bigwedge^p \Omega^1(P)$. In cases where the Poisson algebra P is affine and its Kähler differentials $\Omega^1(P)$ admits a free P-basis $\Omega^1(P) = \langle \{dx_1, \cdots, dx_l\} \rangle$ for some dx_i satisfying $\mathrm{tr}(dx_i) = 0$, there is a duality between Poisson cohomology and Poisson homology:

$$PH_n(P) \cong PH^{l-n}(P)$$

for all $n \geq 0$ [LWZ20].

5.2 A Variant for Unimodular Poisson Algebras

In this section, we prove a variant of the Shephard-Todd-Chevalley Theorem for unimodular Poisson structures on $\Bbbk[x_1, x_2, x_3]$, stated as follows:

Theorem 5.2.1. (Shephard-Todd-Chevalley Theorem) Let $P = \Bbbk[x_1, x_2, x_3]$ be a unimodular quadratic Poisson algebra and let $G \subseteq \mathrm{PAut}_{\mathrm{gr}}(P)$ be a finite subgroup. Then the invariant subalgebra P^G is isomorphic to P as Poisson algebras if and only if G is trivial.

Proof. It suffices to prove $\Rightarrow$. The classical Shephard-Todd-Chevalley Theorem states that if $P^G \cong P$ as algebras, then G is generated by reflections. Since $G \subseteq \mathrm{PAut}_{\mathrm{gr}}(P)$, G is generated by Poisson reflections. By Proposition 4.3.3, *Unimodular 5*, *Unimodular 6*, *Unimodular 7*, *Unimodular 9* have no Poisson reflections, hence the statement is trivially true. For *Unimodular 1*, *Unimodular 2*, *Unimodular 3*, *Unimodular 4*, *Unimodular 8*, we proceed with a detailed analysis of each instance.

Unimodular 1. $\{x_1, x_2\} = 0$, $\{x_2, x_3\} = 3x_1^2$, $\{x_3, x_1\} = 0$.

A Poisson reflection ϕ has the form $\phi\big|_{P_1} = \begin{bmatrix} -1 & 0 & 0 \\ a & 1 & 0 \\ d & 0 & 1 \end{bmatrix}$. If G contains two distinct Poisson reflections: $\phi_1\big|_{P_1} = \begin{bmatrix} -1 & 0 & 0 \\ a_1 & 1 & 0 \\ d_1 & 0 & 1 \end{bmatrix}$, $\phi_2\big|_{P_1} = \begin{bmatrix} -1 & 0 & 0 \\ a_2 & 1 & 0 \\ d_2 & 0 & 1 \end{bmatrix}$. However, the product $(\phi_1\phi_2)\big|_{P_1} = \begin{bmatrix} 1 & 0 & 0 \\ a_2 - a_1 & 1 & 0 \\ d_2 - d_1 & 0 & 1 \end{bmatrix}$, a matrix of infinite order unless $a_2 = a_1$ and $d_2 = d_1$. Consequently, we may assume that G is cyclic generated: $G = \begin{bmatrix} -1 & 0 & 0 \\ a & 1 & 0 \\ d & 0 & 1 \end{bmatrix} \cong \mathbb{Z}_2$. To compute the invariant subalgebra, we start by observing that the polynomials $y_1 = x_1^2$, $y_2 = \dfrac{a}{2}x_1 + x_2$, $y_3 = \dfrac{d}{2}x_1 + x_3$ are algebraically independent and remain invariant under the action of G. Embed the $\mathbb{k}$-algebra generated by y_1, y_2, y_3 into the invariant subalgebra P^G. Using Molien's Theorem to compute $h_{P^G}(t)$, we compare the Hilbert series of these two $\mathbb{k}$-algebras, and conclude that $P^G = \mathbb{k}[y_1, y_2, y_3]$. The Poisson structure on the invariant subalgebra P^G is:

$$\{y_1, y_2\} = 0, \quad \{y_2, y_3\} = 3y_1, \quad \{y_3, y_1\} = 0.$$

By invoking Lemma 5.1.3, it becomes evident that the Poisson algebras P and Q are non-isomorphic Poisson algebras.

Unimodular 2. $\{x_1, x_2\} = 0$, $\{x_2, x_3\} = 2x_1x_2$, $\{x_3, x_1\} = x_1^2$.

A Poisson reflection ϕ has the form $\phi\big|_{P_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \xi & 0 \\ 0 & d & 1 \end{bmatrix}$. If G contains two non-commuting Poisson reflections: $\phi_1\big|_{P_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \xi_{n_1} & 0 \\ 0 & d_1 & 1 \end{bmatrix}$ and $\phi_2\big|_{P_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \xi_{n_2} & 0 \\ 0 & d_2 & 1 \end{bmatrix}$, where ξ_{n_1} (resp. ξ_{n_2}) is a primitive n_1th (resp. n_2th) root of unity. Since $\phi_1\phi_2 \neq \phi_2\phi_1$, the product $(\phi_1\phi_2\phi_1^{n_1-1}\phi_2^{n_2-1})\big|_{P_1} =$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & d & 1 \end{bmatrix}$ for some $d \neq 0$; however, such a matrix has infinite order, a contradiction. Therefore, G is a finite abelian group. Keeping ϕ_1 and ϕ_2 as above, the commutativity $\phi_1\phi_2 = \phi_2\phi_1$ is equivalent to $d_2 = \dfrac{\xi_{n_2} - 1}{\xi_{n_1} - 1} d_1$.

Decompose $G \cong \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_m}$ for some $n_i \in \mathbb{N}$ satisfying $n_i | n_{i+1}$. According to the commutativity condition, we may take the cyclic generator of $\mathbb{Z}_{n_i}$ to be $\phi_i\big|_{P_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \xi_{n_i} & 0 \\ 0 & \frac{\xi_{n_i}-1}{\xi_{n_1}-1} d_1 & 1 \end{bmatrix}$ for some primitive n_ith root of unity ξ_{n_i}, for all $1 \leq i \leq m$. Define $S = \{\alpha \in \mathbb{N}^m : 1 \leq \alpha_i \leq n_i\}$. For $\alpha = (\alpha_i)_{1 \leq i \leq m} \in S$, we define $\xi^\alpha = \xi_{n_1}^{\alpha_1} \cdots \xi_{n_m}^{\alpha_m}$. According to the Molien's Theorem,

$$h_{P^G}(t) = \frac{1}{n_1 \cdots n_m} \sum_{\alpha \in S} \frac{1}{(1-t)^2(1-\xi^\alpha t)} = \frac{1}{(1-t)^2} \left(\frac{1}{n_1 \cdots n_m} \sum_{\alpha \in S} \frac{1}{(1-\xi^\alpha t)} \right).$$

Before computing $\sum_{\alpha \in S} \dfrac{1}{(1-\xi^\alpha t)}$, we claim that $\{\xi^\alpha : \alpha \in S\}$ is precisely $n_1 \cdots n_{m-1}$ copies of $\{\xi_{n_m}^{\alpha_m} : 1 \leq \alpha_m \leq n_m\}$. To prove the claim, we proceed by induction. When $m = 2$, the set $\{\xi^\alpha : \alpha \in S\}$ contains the following elements:

$$\begin{array}{ccccc} 1 & \xi_{n_2} & \xi_{n_2}^2 & \cdots & \xi_{n_2}^{n_2-1} \\ 1 & \xi_{n_1}\xi_{n_2} & \xi_{n_1}\xi_{n_2}^2 & \cdots & \xi_{n_1}\xi_{n_2}^{n_2-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \xi_{n_1}^{n_1-1}\xi_{n_2} & \xi_{n_1}^{n_1-1}\xi_{n_2}^2 & \cdots & \xi_{n_1}^{n_1-1}\xi_{n_2}^{n_2-1} \end{array}$$

Each line can be realized as the image of a left multiplication map $l_{\xi_{n_1}^{\alpha_1}} : \langle \xi_{n_2} \rangle \to \langle \xi_{n_2} \rangle$, for all $0 \leq \alpha_1 \leq n_1 - 1$. Since $n_1 | n_2$, and consequently, $\xi_{n_1}^{\alpha_1}$ is an element of $\langle \xi_{n_2} \rangle$, each line is a permutation of the first line. This proves the base case: $\{\xi_{n_1}^{\alpha_1}\xi_{n_2}^{\alpha_2} | 1 \leq \alpha_1 \leq n_1, 1 \leq \alpha_2 \leq n_2\}$ is precisely n_1 copies of $\{\xi_{n_2}^{\alpha_2} | 1 \leq \alpha_2 \leq n_2\}$. Inductively, when m is arbitrary, the set $\{\xi^\alpha | \alpha \in S\}$

contains the following elements:

$$1 \quad E_{m-1} \quad E_{m-1}\xi_{n_m} \quad \cdots \quad E_{m-1}\xi_{n_m}^{n_m-1},$$

where $E_{m-1} = \{\xi_{n_1}^{\alpha_1} \cdots \xi_{n_{m-1}}^{\alpha_{n_{m-1}}} : 1 \leq \alpha_i \leq n_i\}$. By the induction hypothesis, the set $\{\xi_{n_1}^{\alpha_1} \cdots \xi_{n_{m-1}}^{\alpha_{n_{m-1}}} : 1 \leq \alpha_i \leq n_i\}$ is $n_1 n_2 \cdots n_{m-2}$ copies of $\{\xi_{n_{m-1}}^{\alpha_{m-1}} : 1 \leq \alpha_{m-1} \leq n_{m-1}\}$. Consequently, we can view the preceding lines of elements as $n_1 n_2 \cdots n_{m-2}$ layers of the subsequent lines of elements:

$$\begin{array}{cccccc} 1 & \xi_{n_m} & \xi_{n_m}^2 & \cdots & \xi_{n_m}^{n_m-1} \\ 1 & \xi_{n_{m-1}}\xi_{n_m} & \xi_{n_{m-1}}\xi_{n_m}^2 & \cdots & \xi_{n_{m-1}}\xi_{n_m}^{n_m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \xi_{n_{m-1}}^{n_{m-1}-1}\xi_{n_m} & \xi_{n_{m-1}}^{n_{m-1}-1}\xi_{n_m}^2 & \cdots & \xi_{n_{m-1}}^{n_{m-1}-1}\xi_{n_m}^{n_m-1} \end{array}$$

Again, from the induction hypothesis, the above lines of elements are $n_1 n_2 \cdots n_{m-1}$ copies $\{\xi_{n_m}^{\alpha_m} | 1 \leq \alpha_m \leq n_m\}$ as claimed. We can compute the Hilbert series of the invariant subalgebra P^G by using the claim:

$$\begin{aligned} h_{P^G}(t) =& \frac{1}{(1-t)^2}\left(\frac{n_1 \cdots n_{m-1}}{n_1 \cdots n_m} \sum_{1 \leq \alpha_m \leq n_m} \frac{1}{(1-\xi_{n_m}^{\alpha_m} t)}\right) \\ =& \frac{1}{(1-t)^2}\left(\frac{1}{n_m} \sum_{1 \leq \alpha_m \leq n_m} \frac{1}{(1-\xi_{n_m}^{\alpha_m} t)}\right) \\ =& \frac{1}{(1-t)^2(1-t^{n_m})}. \end{aligned}$$

Set $l = n_m$. Furthermore, set $y_1 = x_1$, $y_2 = d_1 x_2 + (1-\xi_{n_1})x_3$, $y_3 = x_2^l$. The elements y_1, y_2, y_3 are three algebraically independent polynomials that are invariant under the action of G. Consequently, we may embed $\Bbbk[y_1, y_2, y_3]$ into the invariant subalgebra P^G. It is evident that the Hilbert series of $\Bbbk[y_1, y_2, y_3]$ is $\dfrac{1}{(1-t)^2(1-t^l)}$, and therefore, the embedding $\Bbbk[y_1, y_2, y_3] \hookrightarrow P^G$ is surjective because the cokernel has Hilbert series 0. Accordingly, we conclude that the invariant subalgebra $P^G = \Bbbk[y_1, y_2, y_3]$ and has the following Poisson structure:

$$\{y_1, y_2\} = (\xi_{n_1} - 1)y_1^2, \quad \{y_2, y_3\} = 2l(\xi_{n_1} - 1)y_1y_3, \quad \{y_3, y_1\} = 0.$$

Suppose that $P^G \cong P$ as Poisson algebras. Specifically, by applying (the contrapositive of) Lemma 5.1.1, the invariant subalgebra is unimodular. This implies that we can find a superpotential $\Omega \in \Bbbk[y_1, y_2, y_3]$ satisfying:

$$\frac{\partial \Omega}{\partial y_3} = (\xi_{n_1} - 1)y_1^2, \quad \frac{\partial \Omega}{\partial y_1} = 2l(\xi_{n_1} - 1)y_1y_3, \quad \frac{\partial \Omega}{\partial y_2} = 0.$$

It is straightforward to verify no such Ω exists unless $l = 1$, or equivalently, unless G is trivial.

Unimodular 3. $\{x_1, x_2\} = 2x_1x_2$, $\{x_2, x_3\} = 2x_2x_3, \{x_3, x_1\} = 2x_1x_3$.

This case is addressed in [GVW23, Theorem 4.5]. The conclusion is as follows: $P^G \cong P$ as Poisson algebras if and only if G is trivial.

Unimodular 4. $\{x_1, x_2\} = 0$, $\{x_2, x_3\} = 2x_1x_2 + x_2^2$, $\{x_3, x_1\} = x_1^2 + 2x_1x_2$.

A Poisson reflection ϕ has three possible forms: $\phi\big|_{P_1} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ b & b & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ b & 0 & 1 \end{bmatrix},$ $\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & c & 1 \end{bmatrix}$. First, notice that the finite automorphism group G cannot contain two Poisson reflections of the same form.

- Suppose that G contains two distinct Poisson reflections of the first form: $\phi_1\big|_{P_1} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ b_1 & b_1 & 1 \end{bmatrix}$, $\phi_2\big|_{P_1} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ b_2 & b_2 & 1 \end{bmatrix}$. The product $(\phi_1\phi_2)\big|_{P_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b_2 - b_1 & b_2 - b_1 & 1 \end{bmatrix}$

is a matrix of infinite order, contradicting to the finiteness of G.

- Suppose that G contains two distinct Poisson reflections of the second form: $\phi_1\big|_{P_1} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ b_1 & 0 & 1 \end{bmatrix}$, $\phi_2\big|_{P_1} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ b_2 & 0 & 1 \end{bmatrix}$. The product $(\phi_1\phi_2)\big|_{P_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b_2 - b_1 & 0 & 1 \end{bmatrix}$ is a matrix of infinite order, contradicting to the finiteness of G.

- Suppose that G contains two distinct Poisson reflections of the third form: $\phi_1\big|_{P_1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & c_1 & 1 \end{bmatrix}$, $\phi_2\big|_{P_1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & c_2 & 1 \end{bmatrix}$. The product $(\phi_1\phi_2)\big|_{P_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & b_2 - b_1 & 1 \end{bmatrix}$ is a matrix of infinite order, contradicting to the finiteness of G.

Consequently, the finite automorphism group G falls into one of the following three categories:

(1) The group G is generated by three Poisson reflections of distinct types.

(2) The group G is generated by two Poisson reflections of distinct types.

(3) The group G is generated by a single Poisson reflection.

In Case (1), $G = \langle \phi_1\big|_{P_1} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ a & a & 1 \end{bmatrix}, \phi_2\big|_{P_1} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ b & 0 & 1 \end{bmatrix}, \phi_3\big|_{P_1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & c & 1 \end{bmatrix} \rangle$.
By calculation, the product $(\phi_1\phi_2\phi_3)\big|_{P_1}^n$ is not equal to the identity matrix I_3 when n is odd, and it equals the following matrix:

$$(\phi_1\phi_2\phi_3)\big|_{P_1}^n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{n}{2}(b+c-a) & n(b+c-a) & 1 \end{bmatrix}$$

when n is even. As a result, the finiteness of G necessitates the condition: $a = b + c$. Given this equality, through further calculations, we observe that:

$$(\phi_1\phi_2\phi_1)\big|_{P_1} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ b+c & b+c & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ b & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ b+c & b+c & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & c & 1 \end{bmatrix} = \phi_3\big|_{P_1},$$

$$(\phi_2\phi_3\phi_2)\big|_{P_1} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ b & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & c & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ b & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ b+c & b+c & 1 \end{bmatrix} = \phi_1\big|_{P_1},$$

$$(\phi_3\phi_1\phi_3)\big|_{P_1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & c & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ b+c & b+c & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & c & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ b & 0 & 1 \end{bmatrix} = \phi_2\big|_{P_1}.$$

In conclusion, we have established that $G = \langle \phi_1, \phi_2, \phi_3 \rangle = \langle \phi_1, \phi_2 \rangle = \langle \phi_2, \phi_3 \rangle = \langle \phi_3, \phi_1 \rangle$, and it is sufficient to consider the case when $G = \langle \phi_1, \phi_2 \rangle$ for both Case (1) and Case (2).

In Case (2), $G = \langle \phi_1\big|_{P_1} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ a & a & 1 \end{bmatrix}, \phi_2\big|_{P_1} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ b & 0 & 1 \end{bmatrix} \rangle$. By calculation,

$$\phi_1^2\big|_{P_1} = \phi_2^2\big|_{P_1} = (\phi_1\phi_2)^3\big|_{P_1} = I_3, \quad \phi_1\phi_2 \neq \phi_2\phi_1.$$

The first equality implies that G is isomorphic to a quotient of the symmetric group S_3. Further, the second inequality implies that G is isomorphic to S_3, as S_3 is the smallest finite non-abelian group. Apply the Molien's Theorem:

$$h_{P^G}(t) = \frac{1}{6}\left(\frac{1}{(1-t)^3} + \frac{3}{(1-t)^2(1+t)} + \frac{2}{(1-t)(1+t+t^2)}\right) = \frac{1}{(1-t)(1-t^2)(1-t^3)}.$$

After calculating $\int f = \frac{1}{|G|}\sum_{\phi \in G} \phi(f)$ for all monomials $f \in \Bbbk[x_1, x_2, x_3]$ with degree less or equal to 6, we discover that the elements $y_1 = \frac{1}{3}(a+b)x_1 + \frac{1}{3}(2a-b)x_2 + x_3$, $y_2 = x_1^2 + x_2^2 + x_1x_2$, $y_3 = 2x_1^3 + 3x_1^2x_2 - 3x_1x_2^2 - 2x_2^3$ are three algebraically independent polynomials that are invariant under the action of G. Embed $\Bbbk[y_1, y_2, y_3]$ into the invariant subalgebra P^G. Since the domain and codomain share the same Hilbert series, the cokernel is trivial. In other words, the invariant subalgebra $P^G = \Bbbk[y_1, y_2, y_3]$ and has the following Poisson structure:

$$\{y_1, y_2\} = y_3, \quad \{y_2, y_3\} = 0, \quad \{y_3, y_1\} = -6y_2^2.$$

By applying Lemma 5.1.3, P^G is not isomorphic to P as Poisson algebras.

In Case (3), we will concurrently discuss three subcases: $G_1 = \langle \phi_1|_{P_1} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ a & a & 1 \end{bmatrix} \rangle$, $G_2 = \langle \phi_2|_{P_1} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ b & 0 & 1 \end{bmatrix} \rangle$, and $G_3 = \langle \phi_3|_{P_1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & c & 1 \end{bmatrix} \rangle$. Notice that the finite automorphism group G_i is isomorphic to $\mathbb{Z}_2$ for all $1 \leq i \leq 3$, and the Hilbert series of the invariant subalgebra is

$$h_{P^G}(t) = \frac{1}{2}\left(\frac{1}{(1-t)^3} + \frac{1}{(1-t)^2(1+t)}\right) = \frac{1}{(1-t)^2(1-t^2)}$$

by applying Molien's Theorem. As in our previous analysis, we can find three algebraically independent polynomials that are invariant under the action of G_i:

	$\boldsymbol{y_1}$	$\boldsymbol{y_2}$	$\boldsymbol{y_3}$
G_1	x_1x_2	$-x_1 + x_2$	$ax_1 + x_3$
G_2	x_1^2	$\frac{b}{2}x_1 + x_3$	$\frac{1}{2}x_1 + x_2$
G_3	x_2^2	$-cx_1 + x_3$	$2x_1 + x_2$

By comparing the Hilbert series of the domain and the codomain of the embedding

$\Bbbk[y_1, y_2, y_3] \hookrightarrow P^G$, we ascertain that the invariant subalgebra $P^G = \Bbbk[y_1, y_2, y_3]$ and has the following Poisson structures:

	$\{y_1, y_2\}$	$\{y_2, y_3\}$	$\{y_3, y_1\}$
G_1	0	$6y_1 + y_2^2$	$y_1 y_2$
G_2	$-4y_1y_3$	$\frac{3}{4}y_1 - y_3^2$	0
G_3	$2y_1y_3$	$-\frac{3}{2}y_1 + \frac{1}{2}y_3^2$	0

Suppose that $P^G \cong P$ as Poisson algebras. Specifically, by applying (the contrapositive of) Lemma 5.1.1, the invariant subalgebra is unimodular. This implies that we can find a superpotential $\Omega \in \Bbbk[y_1, y_2, y_3]$ satisfying:

$$\frac{\partial \Omega}{\partial y_3} = \{y_1, y_2\}, \quad \frac{\partial \Omega}{\partial y_1} = \{y_2, y_3\}, \quad \frac{\partial \Omega}{\partial y_2} = \{y_3, y_1\}.$$

It is straightforward to verify no such Ω exists for G_1, G_2, and G_3, and therefore, P^G is not unimodular, a contradiction.

In summary, if G is generated by Poisson reflections, then P^G is not isomorphic to P as Poisson algebras. Consequently, the condition $P^G \cong P$ as Poisson algebras implies that G is necessarily trivial.

Unimodular 8. $\{x_1, x_2\} = x_1x_2$, $\{x_2, x_3\} = 3x_1^2 + 2x_1x_2 + x_2x_3$, $\{x_3, x_1\} = x_1^2 + x_1x_3$.

A Poisson reflection ϕ has the form $\phi\big|_{P_1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$. Suppose that $G = \langle \phi \rangle \cong \mathbb{Z}_2$. By the Molien's Theorem,

$$h_{P^G}(t) = \frac{1}{2}\left(\frac{1}{(1-t)^3} + \frac{1}{(1-t)^2(1+t)}\right) = \frac{1}{(1-t)^2(1-t^2)}.$$

It is not difficult to show that the elements $y_1 = x_2$, $y_2 = x_1 + x_3$, $y_3 = x_1^2$ are algebraically independent and are invariant under the action of G. Consider the natural inclusion $\Bbbk[y_1, y_2, y_3] \hookrightarrow P^G$. Extend the natural inclusion into a short exact sequence and apply Hilbert series. Given that the Hilbert series of a short exact sequence sums up to 0, we deduce that the invariant subalgebra $P^G = \Bbbk[y_1, y_2, y_3]$. The Poisson structure on the invariant subalgebra is:

$$\{y_1, y_2\} = y_1 y_2 + 3y_3, \quad \{y_2, y_3\} = 2y_2 y_3, \quad \{y_3, y_1\} = 2y_1 y_3.$$

To prove that P^G is not isomorphic to P as Poisson algebras, one can either invoke Lemma 5.1.1 or Lemma 5.1.3. Both approaches are equally straightforward. □

5.3 A Variant for Non-unimodular Poisson Algebras

In this section, we prove a variant of the Shephard-Todd-Chevalley Theorem for non-unimodular Poisson structures on $\Bbbk[x_1, x_2, x_3]$, stated as follows:

Theorem 5.3.1. (Shephard-Todd-Chevalley Theorem) Let $P = \Bbbk[x_1, x_2, x_3]$ be a non-unimodular quadratic Poisson algebra and let $G \subseteq \mathrm{PAut}_{\mathrm{gr}}(P)$ be a finite subgroup. If $\{-, -\}$ satisfies the following conditions on its coefficients:

- *Non-unimodular 1.* $p, q, r \neq 0$.
- *Non-unimodular 2.* $p, q \neq 0$, $p \neq q$, $4p^2 + q^2 \neq 0$
- *Non-unimodular 3.* $p \neq 0$.
- *Non-unimodular 4.* $p, q \neq 0$.
- *Non-unimodular 5.* $p \neq 0, \frac{1}{2}$.
- *Non-unimodular 6.* $p \neq 0$.
- *Non-unimodular 7.* $p, q \neq 0$, $2p + r \neq 0$, $(2p + r)^2 + q^2 \neq 0$
- *Non-unimodular 8.* $p \neq 0$, $p + q \neq 0$.
- *Non-unimodular 9.* $p \neq 0$.
- *Non-unimodular 10.* $p \neq 0, -\frac{1}{4}, -\frac{1}{3}, -\frac{1}{2}$.
- *Non-unimodular 11.* $p \neq 0, -\frac{1}{2}, -\frac{1}{4}$, $q \neq 0$.

- *Non-unimodular 12.* $p \neq 0, -\frac{1}{2}, -\frac{1}{3}$, $q = 0$.
- *Non-unimodular 13.* $p \neq 0, -\frac{1}{2}$, $q = 0$, $r \neq 0$.

Then the invariant subalgebra P^G is isomorphic to P as Poisson algebras if and only if G is trivial.

Proof. In a manner analogous to Theorem 5.2.1, it is sufficient to consider finite automorphism groups that are generated by Poisson reflections. By Proposition 4.2.1, *Non-unimodular 9*, *Non-unimodular 10* have no Poisson reflections, hence the statement is trivially true. For the remaining Poisson structures, we proceed with a detailed analysis of each instance.

Non-unimodular 1.

A Poisson reflection ϕ is one of the following commuting matrices: $(\phi_1)\big|_{P_1} = \begin{bmatrix} \xi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $(\phi_2)\big|_{P_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $(\phi_3)\big|_{P_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \xi \end{bmatrix}$. Similarly to the analysis in [GVW23, Proposition 2.6] and [GVW23, Theorem 4.5], it is clear that a finite Poisson reflection group admits a decomposition $G \cong G_{x_1} \times G_{x_2} \times G_{x_3}$, where the subgroups G_{x_1}, G_{x_2}, G_{x_3} acts on the subalgebras $\Bbbk[x_1]$, $\Bbbk[x_2]$, $\Bbbk[x_3]$, respectively. Set $n = \exp(G_{x_1})$, $y_1 = x_1^n$, $m = \exp(G_{x_2})$, $y_2 = x_2^m$, $l = \exp(G_{x_3})$, $y_3 = x_3^l$. The invariant subalgebra $P^G = \Bbbk[x_1^n, x_2^m, x_3^l]$ has the following Poisson structures:

$$\{y_1, y_2\} = pnmy_1y_2, \quad \{y_2, y_3\} = qmly_2y_3, \quad \{y_3, y_1\} = rnly_1y_3.$$

According to [GW20, Theorem 4.6], a Poisson isomorphism $P^G \cong P$, or more explicitly, a Poisson isomorphism between two Poisson structures that are defined by two skew-symmetric matrix $M_P = \begin{bmatrix} 0 & p & r \\ -p & 0 & q \\ -r & -q & 0 \end{bmatrix}$ and $M_{PG} = \begin{bmatrix} 0 & pnm & rnl \\ -pnm & 0 & qml \\ -rnl & -qml & 0 \end{bmatrix}$, necessitates that $M_P[i,j] = M_{PG}[\sigma(i), \sigma(j)]$ for some $\sigma \in S_3$, for all $1 \leq i,j \leq 3$. Upon scrutiny of the permutations in S_3, it is observed that each odd permutation results in a contradiction, and each even permutation results in the equality $n = m = l = 1$, or equivalently, G is trivial.

Non-unimodular 2, Non-unimodular 7.

For these two Poisson structures, a Poisson reflection ϕ has two possible forms: $\phi_1\big|_{P_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \xi \end{bmatrix}$, $\phi_2\big|_{P_1} = \begin{bmatrix} \frac{1+\xi}{2} & \pm\frac{1-\xi}{2i} & 0 \\ \mp\frac{1-\xi}{2i} & \frac{1+\xi}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Since $\phi_1\big|_{P_1}$ and $\phi_2\big|_{P_1}$ are commuting matrices, a finite Poisson reflection group G is necessarily abelian. Suppose that the finite Poisson reflection group G is generated by $\phi_{11}, \cdots, \phi_{1c_1}$, $\phi_{21}, \cdots, \phi_{2c_2}$, $\phi_{31}, \cdots, \phi_{3c_3}$, where $\phi_{1\_}$ are Poisson reflections of the form $(\phi_{1\_})\big|_{P_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \xi \end{bmatrix}$, $\phi_{2\_}$ are Poisson reflections of the form $(\phi_{2\_})\big|_{P_1} = \begin{bmatrix} \frac{1+\xi}{2} & \frac{1-\xi}{2i} & 0 \\ -\frac{1-\xi}{2i} & \frac{1+\xi}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and $\phi_{3\_}$ are Poisson reflections of the form $(\phi_{3\_})\big|_{P_1} = \begin{bmatrix} \frac{1+\xi}{2} & -\frac{1-\xi}{2i} & 0 \\ \frac{1-\xi}{2i} & \frac{1+\xi}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Through extensive (and routine) computations, the Poisson reflection group G admits a decomposition:

$$G \cong \langle \phi_{11}, \cdots, \phi_{1c_1} \rangle \times \langle \phi_{21}, \cdots, \phi_{2c_2} \rangle \times \langle \phi_{31}, \cdots, \phi_{3c_3} \rangle,$$

and each of the three components admits a further decomposition into invariant factors:

$$\langle \phi_{11}, \cdots, \phi_{1c_1} \rangle \cong \langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \xi_{l_1} \end{bmatrix}, \cdots, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \xi_{l_{d_1}} \end{bmatrix} \rangle,$$

$$\langle \phi_{21}, \cdots, \phi_{2c_2} \rangle \cong \langle \begin{bmatrix} \frac{1+\xi_{m_1}}{2} & \frac{1-\xi_{m_1}}{2i} & 0 \\ -\frac{1-\xi_{m_1}}{2i} & \frac{1+\xi_{m_1}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \cdots, \begin{bmatrix} \frac{1+\xi_{m_{d_2}}}{2} & \frac{1-\xi_{m_{d_2}}}{2i} & 0 \\ -\frac{1-\xi_{m_{d_2}}}{2i} & \frac{1+\xi_{m_{d_2}}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \rangle,$$

$$\langle \phi_{31}, \cdots, \phi_{3c_3} \rangle \cong \langle \begin{bmatrix} \frac{1+\xi_{n_1}}{2} & -\frac{1-\xi_{n_1}}{2i} & 0 \\ \frac{1-\xi_{n_1}}{2i} & \frac{1+\xi_{n_1}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \cdots, \begin{bmatrix} \frac{1+\xi_{n_{d_3}}}{2} & -\frac{1-\xi_{n_{d_3}}}{2i} & 0 \\ \frac{1-\xi_{n_{d_3}}}{2i} & \frac{1+\xi_{n_{d_3}}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \rangle.$$

for some $n_i | n_{i+1}$, $m_j | m_{j+1}$, $l_k | l_{k+1}$, for all $1 \leq i \leq d_1$, $1 \leq j \leq d_2$, $1 \leq k \leq d_3$. Define

$$S_1 = \{\alpha \in \mathbb{N}^{d_1} : 1 \leq \alpha_i \leq l_i\},\ S_2 = \{\beta \in \mathbb{N}^{d_2} : 1 \leq \alpha_i \leq m_i\},\ S_3 = \{\gamma \in \mathbb{N}^{d_3} : 1 \leq \alpha_i \leq n_i\}.$$

For $\alpha = (\alpha_i)_{1 \leq i \leq d_1} \in S_1$, $\beta = (\beta_i)_{1 \leq i \leq d_2} \in S_2$, $\gamma = (\gamma_i)_{1 \leq i \leq d_3} \in S_3$, we will write

$$\xi^\alpha = \xi_{l_1}^{\alpha_1} \cdots \xi_{l_{d_1}}^{\alpha_{d_1}}, \quad \xi^\beta = \xi_{m_1}^{\beta_1} \cdots \xi_{m_{d_2}}^{\beta_{d_2}}, \quad \xi^\gamma = \xi_{n_1}^{\gamma_1} \cdots \xi_{n_{d_3}}^{\gamma_{d_3}}.$$

According to the Molien's Theorem and the computation in Theorem 5.2.1 *Unimodular 2*,

$$\begin{aligned} h_{P^G}(t) =& \frac{1}{(l1 \cdots l_{d_1} m_1 \cdots m_{d_2} \cdots n_1 \cdots n_{d_3})} \sum_{\gamma \in S_3} \sum_{\beta \in S_2} \sum_{\alpha \in S_1} \frac{1}{(1-\xi^\alpha t)(1-\xi^\beta t)(1-\xi^\gamma t)} \\ =& \frac{1}{l1 \cdots l_{d_1}} \sum_{\gamma \in S_3} \left(\frac{1}{m_1 \cdots m_{d_2}} \sum_{\beta \in S_2} \left(\frac{1}{n_1 \cdots n_{d_3}} \sum_{\alpha \in S_1} \frac{1}{1-\xi^\alpha t} \right) \frac{1}{1-\xi^\beta t} \right) \frac{1}{1-\xi^\gamma t} \\ =& \frac{1}{(1-t^{l_{d_1}})(1-t^{m_{d_2}})(1-t^{n_{d_3}})}. \end{aligned}$$

Set $l = l_{d_1}$, $m = m_{d_2}$, $n = n_{d_3}$. Identifying three algebraically independent polynomials that are invariant under the action of G is straightforward: $y_1 = (-ix_1 + x_2)^n$, $y_2 = (ix_1 +$

$x_2)^m$, $y_3 = x_3^l$. The $\Bbbk$-algebra $\Bbbk[y_1, y_2, y_3]$ can be embedded into the invariant subalgebra P^G. After comparing the Hilbert series of $\Bbbk[y_1, y_2, y_3]$ and P^G, it becomes evident that $P^G = \Bbbk[y_1, y_2, y_3]$. The Poisson structures on P^G are:

Non-unimodular	$\{y_1, y_2\}$	$\{y_2, y_3\}$	$\{y_3, y_1\}$
2	$-2imnpy_1y_2$	$-ml(2ip+q)y_2y_3$	$-nl(2ip-q)y_1y_3$
7	$-2imnpy_1y_2$	$-ml(i(2p+r)-q)y_2y_3$	$-nl(i(2p+r)+q)y_1y_3$

Suppose that P^G is isomorphic to P as Poisson algebras via some Poisson isomorphism ϕ. The Poisson normal elements of P^G are scalar multiples of y_1, y_2, y_3, while those of P are scalar multiples of $\pm ix_1 + x_2$, x_3. Then Lemma 3.1.2 states that $\phi : P^G \to P$ necessarily takes one of the following forms on P_1^G:

$$\begin{bmatrix} -ia & a & 0 \\ ib & b & 0 \\ 0 & 0 & c \end{bmatrix}, \begin{bmatrix} -ia & a & 0 \\ 0 & 0 & c \\ ib & b & 0 \end{bmatrix}, \begin{bmatrix} ib & b & 0 \\ -ia & a & 0 \\ 0 & 0 & c \end{bmatrix}, \begin{bmatrix} ib & b & 0 \\ 0 & 0 & c \\ -ia & a & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & c \\ -ia & a & 0 \\ ib & b & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & c \\ ib & b & 0 \\ -ia & a & 0 \end{bmatrix},$$

for some $a, b, c \in \Bbbk^\times$.

For *Unimodular 2*, if $\phi|_{P_{P_1^G}}$ takes the form of the first matrix,

$$\begin{aligned} \phi(\{y_1, y_2\}) &= -2imnpab(x_1^2 + x_2^2), & \{\phi(y_1, y_2)\} &= -2ipab(x_1^2 + x_2^2), \\ \phi(\{y_2, y_3\}) &= -ml(2ip+q)bc(ix_1 + x_2)x_3, & \{\phi(y_1, y_2)\} &= -(2ip+q)bcx_2x_3. \end{aligned}$$

The condition that ϕ preserves the Poisson structures implies that $m = n = l = 1$, or equivalently, G is trivial. Similarly, one can verify that if $\phi|_{P_{P_1^G}}$ takes the form of the remaining five matrices, then G is trivial, although the argument may be less straightforward than taking the form of first matrix.

For *Unimodular 7*, the argument can be carried out in a similar manner.

It should be noted that Lemma 5.1.2 enables us to simplify the computations, at the cost of missing a set of coefficients with measure 0 in $(\Bbbk^\times)^3$. In both instances, we can compute the superpotential associated with the Poisson twistings of the Poisson algebra and its invariant subalgebra. In the table below, we will label the *Non-unimodular 2* (resp. *Non-unimodular 7*) Poisson algebra as P_2 (resp. P_7), and its invariant subalgebra as P_2^G (resp. P_7^G).

	Superpotential of The Poisson Twistings
P_2	$p(x_1^2+x_2^2)x_3$
P_2^G	$\frac{1}{3}(-2i(mn+ml+nl)p+(n-m)lq)y_1y_2y_3$
P_7	$(p+\frac{1}{3}r)(x_1^2+x_2^2)x_3$
P_7^G	$\frac{1}{3}(-2i(mn+ml+nl)p-i(m+n)lr+(m-n)lq)y_1y_2y_3$

First, observe that the invariant subalgebra P^G admits the standard grading: $\deg(y_1) = \deg(y_2) = \deg(y_3) = 1$. According to the analysis in Lemma 5.1.2, the Poisson isomorphism $P^G \xrightarrow{\phi} P$ passes to another Poisson isomorphism $(P^G)^{-\frac{1}{3}\underline{m}_{P^G}} \to P^{-\frac{1}{3}\underline{m}_P}$. Since $(P^G)^{-\frac{1}{3}\underline{m}_{P^G}} = P^G$ and $P^{-\frac{1}{3}\underline{m}_P} = P$ as $\Bbbk$-algebras, we will, for the sake of simplicity, continue to denote this Poisson isomorphism on the Poisson twistings as ϕ. The Poisson normal elements of $(P^G)^{-\frac{1}{3}\underline{m}_{P^G}}$ are scalar multiples of y_1, y_2, y_3, while those of $P^{-\frac{1}{3}\underline{m}_P}$ are scalar multiples of $\pm ix_1 + x_2$, x_3. Again, Lemma 3.1.2 states that $\phi : P^G \to P$ necessarily takes the following forms on $\left((P^G)^{-\frac{1}{3}\underline{m}_{P^G}}\right)_1$:

$$\begin{bmatrix} -ia & a & 0 \\ ib & b & 0 \\ 0 & 0 & c \end{bmatrix},$$

for some $a, b, c \in \Bbbk^\times$, up to some re-labelling of y_1, y_2, y_3. This simplifies our computation to a single case. Now, [HTWZ23, Lemma 3.3] comes into play, establishing that $\phi(\Omega_{P^G}) = \Omega_P$. Comparing the coefficients of $\phi(\Omega_{P^G})$ and Ω_P, it can be established that $\phi(\Omega_{P^G}) \neq \Omega_P$, except in the case where $m = n = l = 1$ (G is trivial), or under the following circumstances:

- *Non-unimodular 2.* $2(mn + ml + nl - 3)p + i(n - m)lq \neq 0$ for all positive integers

m, n, l.

- *Non-unimodular 7.* $2(mn+ml+nl-3)p+((m+n)l-2)r+i(m-n)lq \neq 0$ for all positive integers m, n, l.

This missing set of coefficients has measure 0 in $(\Bbbk^\times)^3$.

Non-unimodular 3, *Non-unimodular 5*, *Non-unimodular 6*.

For these three Poisson structures, a Poisson reflection ϕ has the form: $\phi\big|_{P_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \xi \end{bmatrix}$.
It is clear that a finite Poisson reflection group G is abelian, and therefore G admits a decomposition

$$G \cong \langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \xi_{n_1} \end{bmatrix} \rangle \times \cdots \times \langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \xi_{n_m} \end{bmatrix} \rangle,$$

for some primitive root of unity ξ_{n_i} satisfying $n_i | n_{i+1}$ for all $1 \leq i \leq m$. In these three cases, the Poisson structures on $\Bbbk[x_1, x_2, x_3]$ satisfies the following properties:

- $\{x_1, x_2\} \in \Bbbk[x_1, x_2]$,

- $\{x_2, x_3\} = f_{23}x_3$ and $\{x_3, x_1\} = f_{31}x_3$ for some polynomials $f_{23}, f_{31} \in \Bbbk[x_1, x_2]$.

Set $n = n_m$, $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_3^n$, and φ be the correspondence $x_1 \leftrightarrow y_1$ and $x_2 \leftrightarrow y_2$. The invariant subalgebra $P^G = \Bbbk[y_1, y_2, y_3]$ together with the following Poisson structure:

$$\{y_1, y_2\}_{P^G} = \varphi(\{x_1, x_2\}_P), \quad \{y_2, y_3\}_{P^G} = n\varphi(f_{23})y_3, \quad \{y_3, y_1\}_{P^G} = n\varphi(f_{31})y_3.$$

According to Lemma 5.1.4, P^G is not isomorphic to P unless $n = 1$, or equivalently, G is trivial.

Non-unimodular 4, *Non-unimodular 8*, *Non-unimodular 11*.

For these three Poisson structures, a Poisson reflection ϕ has the form: $\phi\big|_{P_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. Suppose that $G = \langle \phi \rangle$. Due to the simplicity of G, the computations of the invariant subalgebras are straightforward. Set $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_3^2$. The invariant subalgebra $P^G = \Bbbk[y_1, y_2, y_3]$ and has the following Poisson structures:

Non-unimodular	$\{y_1, y_2\}$	$\{y_2, y_3\}$	$\{y_3, y_1\}$
4	$2py_1y_2$	$y_1 + qy_2y_3$	$2py_1y_3$
8	$\frac{p+q}{2}y_1^2 + \frac{p+q}{2} \pm y_3^2$	$2py_1y_3$	$2py_2y_3$
11	$py_1^2 + qy_3$	$2(2p+1)y_1y_3$	0

In each instance, it is possible to apply Lemma 5.1.3 to demonstrate that P^G is not isomorphic to P as Poisson algebras.

Non-unimodular 12.

A Poisson reflection ϕ has the form: $\phi\big|_{P_1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{c}{3p+1} & 1 & c \\ \frac{\xi-1}{3p+1} & 0 & \xi \end{bmatrix}$. Suppose that a finite Poisson reflection group G has at least two generators: $\phi_1\big|_{P_1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{a}{3p+1} & 1 & a \\ \frac{\xi_n-1}{3p+1} & 0 & \xi_n \end{bmatrix}$, $\phi_2\big|_{P_1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{b}{3p+1} & 1 & b \\ \frac{\xi_m-1}{3p+1} & 0 & \xi_m \end{bmatrix}$, for some primitive root of unity ξ_n, ξ_m. Through an exceedingly complicated computation,

$$(\phi_1\phi_2\phi_1^{n-1}\phi_2^{m-1})\big|_{P_1} = \begin{bmatrix} 1 & 0 & 0 \\ \dfrac{\xi_n^{n-1}(\xi_m^{m-1}-1)(a\xi_m+b)}{3p+1} & 1 & \xi_n^{n-1}\xi_m^{m-1}(a\xi_m+b) \\ 0 & 0 & 1 \end{bmatrix},$$

$$(\phi_2\phi_1\phi_2^{n-1}\phi_1^{m-1})\big|_{P_1} = \begin{bmatrix} 1 & 0 & 0 \\ \dfrac{\xi_n^{n-1}(\xi_m^{m-1}-1)(a+b\xi_n)}{3p+1} & 1 & \xi_n^{n-1}\xi_m^{m-1}(a+b\xi_n) \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that both matrices have infinite order unless $a\xi_m = -b$ and $b\xi_n = -a$. In combination, we have $b = -a\xi_n^{n-1}$, $\xi_m = \xi_n^{n-1}$, and

$$\phi_2\big|_{P_1} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{a\xi_n^{n-1}}{3p+1} & 1 & -a\xi_n^{n-1} \\ \frac{\xi_n^{n-1}-1}{3p+1} & 0 & \xi_n^{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{a}{3p+1} & 1 & a \\ \frac{\xi_n-1}{3p+1} & 0 & \xi_n \end{bmatrix}^{n-1} = (\phi_1)^{n-1}\big|_{P_1}.$$

In conclusion, a Poisson reflection group G is necessarily isomorphic to $\mathbb{Z}_n = \langle \begin{bmatrix} 1 & 0 & 0 \\ \frac{c}{3p+1} & 1 & c \\ \frac{\xi-1}{3p+1} & 0 & \xi \end{bmatrix} \rangle$.

Once this is established, the computation of the invariant subalgebra becomes straightforward. Let $y_1 = x_1,\ y_2 = x_2 + \dfrac{c}{1-\xi_n}x_3,\ y_3 = (\dfrac{1}{3p+1}x_1 + x_3)^n$. The elements y_1, y_2, y_3 are three algebraically independent polynomials that are invariant under the action of G. The Hilbert series of $\Bbbk[y_1, y_2, y_3]$ is $\dfrac{1}{(1-t)^2(1-t^n)}$, coinciding with the Hilbert series of the invariant subalgebra P^G, which can be obtained using the Molien's Theorem. As a consequence, $P^G = \Bbbk[y_1, y_2, y_3]$ has the following Poisson structure:

$$\{y_1, y_2\} = py_1^2, \quad \{y_2, y_3\} = (2p+1)ny_1y_3, \quad \{y_3, y_1\} = 0.$$

Apply Lemma 5.1.4, P^G is not isomorphic to P unless G is trivial. □

It may be helpful to document the proof of the Shephard-Todd-Chevalley Theorem in a table for more efficient reference.

Case	**Poisson Reflection Group**	**Solved By**
14-1	$\mathbb{Z}_2$	Lemma 5.1.3
14-2	$\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_m}$, $n_i \vert n_{i+1}$	Lemma 5.1.1
14-3	$(\mathbb{Z}_{l_1} \times \cdots \times \mathbb{Z}_{l_{d_1}})\times$ $(\mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_{d_2}})\times$ $(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_{d_3}})$ $l_i \vert l_{i+1}$, $m_i \vert m_{i+1}$, $n_i \vert n_{i+1}$	[GVW23, Theorem 4.5]
14-4	S_3, $\mathbb{Z}_2$	Lemma 5.1.1, Lemma 5.1.3
14-5	$\emptyset$	Classical STC
14-6	$\emptyset$	Classical STC
14-7	$\emptyset$	Classical STC
14-8	$\mathbb{Z}_2$	Lemma 5.1.1
14-9	$\emptyset$	Classical STC
1	$(\mathbb{Z}_{l_1} \times \cdots \times \mathbb{Z}_{l_{d_1}})\times$ $(\mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_{d_2}})\times$ $(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_{d_3}})$ $l_i \vert l_{i+1}$, $m_i \vert m_{i+1}$, $n_i \vert n_{i+1}$	[GW20, Theorem 4.6]

Case	**Poisson Reflection Group**	**Solved By**
2	$(\mathbb{Z}_{l_1} \times \cdots \times \mathbb{Z}_{l_{d_1}})\times$ $(\mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_{d_2}})\times$ $(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_{d_3}})$ $l_i \mid l_{i+1},\ m_i \mid m_{i+1},\ n_i \mid n_{i+1}$	Lemma 5.1.2
3	$\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_m},\ n_i \mid n_{i+1}$	Lemma 5.1.4
4	$\mathbb{Z}_2$	Lemma 5.1.3
5	$\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_m},\ n_i \mid n_{i+1}$	Lemma 5.1.4
6	$\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_m},\ n_i \mid n_{i+1}$	Lemma 5.1.4
7	$(\mathbb{Z}_{l_1} \times \cdots \times \mathbb{Z}_{l_{d_1}})\times$ $(\mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_{d_2}})\times$ $(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_{d_3}})$ $l_i \mid l_{i+1},\ m_i \mid m_{i+1},\ n_i \mid n_{i+1}$	Lemma 5.1.2
8	$\mathbb{Z}_2$	Lemma 5.1.3
9	$\emptyset$	Classical STC
10	$\emptyset$	Classical STC
11	$\mathbb{Z}_2$	Lemma 5.1.3
12	$\mathbb{Z}_n$	Lemma 5.1.4
13	$\emptyset$	Classical STC

5.4 A Variant for Poisson Enveloping Algebras

In this section, we prove a variant of the Shephard-Todd-Chevalley Theorem for Poisson enveloping algebras of quadratic Poisson structures on $\Bbbk[x_1, \cdots, x_n]$ under the induced action $\mathrm{PAut}_{\mathrm{gr}}(P) \to \mathrm{Aut}_{\mathrm{gr}}(U(P))$. This variant of the Shephard-Todd-Chevalley Theorem is encapsulated in the following lemma.

Lemma 5.4.1. Let $P = \Bbbk[x_1, \cdots, x_n]$ be a quadratic Poisson algebra. Let ϕ be a graded Poisson automorphism of P and let $\widetilde{\phi}$ be the corresponding graded automorphism of $U(P)$. Suppose that $\phi\big|_{P_1}$ has eigenvalues $\lambda_1, \cdots, \lambda_m$, with multiplicity $c_1, \cdots, c_m$, respectively. Then $\widetilde{\phi}\big|_{U(P)_1}$ has eigenvalues $\lambda_1, \cdots, \lambda_m$, with multiplicity $2c_1, \cdots, 2c_m$, respectively.

Proof. The Poisson enveloping algebra $U(P)$ is a quadratic $\Bbbk$-algebra generated by $x_1, \cdots, x_n$, $y_1, \cdots, y_n$. Recall that in Lemma 2.3.5, $\widetilde{\phi}$ is constructed as follows:

$$\widetilde{\phi}(x_i) = \phi(x_i), \quad \widetilde{\phi}(y_i) = \sum_{j=1}^{n} \frac{\partial \phi(x_i)}{\partial x_j} y_j,$$

for all $1 \le i \le n$.

Fix $1 \le i \le m$. Let $\{v_{i,1}, \cdots, v_{i,c_i}\}$ be a basis for the eigenspace of λ_i in P_1. By calculation,

$$\widetilde{\phi}(v_{i,j}) = \phi(v_{i,j}) = \lambda_i v_{i,j},$$

$$\begin{aligned}\widetilde{\phi}(\sum_{k=1}^{n} \frac{\partial \phi(v_{i,j})}{\partial x_k} y_k) = \sum_{k=1}^{n} \phi\left(\frac{\partial(\lambda_i v_{i,j})}{\partial x_k}\right)\widetilde{\phi}(y_k) =& \lambda_i \sum_{k=1}^{n} \phi(\frac{\partial v_{i,j}}{\partial x_k})\widetilde{\phi}(y_k) \\ =& \lambda_i \sum_{k=1}^{n}\sum_{l=1}^{n} \frac{\partial \phi(v_{i,j})}{\partial \phi(x_k)} \frac{\partial \phi(x_k)}{\partial x_l} y_l = \lambda_i \sum_{l=1}^{n} \frac{\partial \phi(v_{i,j})}{\partial x_l} y_l.\end{aligned}$$

for all $1 \le j \le c_i$. Given that $\widetilde{\phi}$ is a graded automorphism, the vectors $\widetilde{\phi}(v_{i,1}), \cdots, \widetilde{\phi}(v_{i,c_i})$, $\widetilde{\phi}(\sum_{k=1}^{n} \frac{\partial \phi(v_{i,j})}{\partial x_k} y_k), \cdots, \widetilde{\phi}(\sum_{k=1}^{n} \frac{\partial \phi(v_{c_i})}{\partial x_k} y_k)$ are linearly independent in $U(P)_1$. Since $\sum_{i=1}^{m} c_i = n$,

$\sum_{i=1}^{m} 2c_i = 2n$. Because $\sum_{i=1}^{m} c_i = n$, it follows that $\sum_{i=1}^{m} 2c_i = 2n$. This assertion implies that the set

$$\left\{\widetilde{\phi}(v_{i,1}), \cdots, \widetilde{\phi}(v_{i,c_i}), \widetilde{\phi}(\sum_{k=1}^{n} \frac{\partial \phi(v_{i,j})}{\partial x_k} y_k), \cdots, \widetilde{\phi}(\sum_{k=1}^{n} \frac{\partial \phi(v_{c_i})}{\partial x_k} y_k)\right\}_{1 \le i \le m}$$

forms an eigenbasis for $U(P)_1$. In particular, the multiplicity of λ_i in $U(P)_1$ equals to twice of multiplicity of λ_i in P_1. □

Now, we are prepared to establish a variant of the Shephard-Todd-Chevalley Theorem for the Poisson enveloping algebras of quadratic Poisson structures on $\Bbbk[x_1, \cdots, x_n]$ under the induced action $\mathrm{PAut}_{\mathrm{gr}}(P) \to \mathrm{Aut}_{\mathrm{gr}}(U(P))$.

Theorem 5.4.2. Let $P = \Bbbk[x_1, \cdots, x_n]$ be a quadratic Poisson algebra and $U(P)$ be its Poisson enveloping algebra. Let G be a finite subgroup of the graded Poisson automorphism group of P and let $\widetilde{G} = \{\widetilde{\phi} : \phi \in G\}$ be the corresponding finite subgroup of the graded automorphism group of $U(P)$. The invariant subalgebra $U(P)^{\widetilde{G}}$ is Artin-Schelter regular if and only if G is trivial.

Proof. It suffices to assume the graded Poisson automorphism group G is nontrivial. The Poisson enveloping algebra $U(P)$, according to [GVW23, Lemma 5.4], is a quantum polynomial ring. Therefore its quasi-reflections, as discussed in [KKZ09, Theorem 3.1], is either a classical reflection or a mystic reflection:

- The eigenvalues of $\widetilde{\phi}$ are $\underbrace{1, \cdots, 1}_{2n-1}, \xi$, for some primitive root of unity ξ.
- The order of $\widetilde{\phi}$ is 4 and the eigenvalues of $\widetilde{\phi}$ are $\underbrace{1, \cdots, 1}_{2n-2}$, i, $-i$.

Comparing to the eigenvalues listed in Lemma 5.4.1, the induced graded automorphism group $\widetilde{G}$ contains no quasi-reflections, which, as indicated in [KKZ09, Lemma 6.1], results in the invariant subalgebra $U(P)^{\widetilde{G}}$ having infinite global dimension. As a consequence, the invariant subalgebra $U(P)^{\widetilde{G}}$ is not Artin-Schelter regular. □

Example 5.4.3. Let $P = \Bbbk[x_1, x_2]$ be the Poisson algebra $\{f, g\} = \left(\frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_2} - \frac{\partial g}{\partial x_1}\frac{\partial f}{\partial x_2}\right)qx_1x_2$, for some $q \neq 0$, for all $f, g \in P$. Let ξ_n be a primitive nth root of unity. Consider the graded Poisson automorphism group $G = \langle \phi = [x_1 \mapsto \xi_n x_1, x_2 \mapsto x_2] \rangle$ of P, and its induced graded automorphism group $\widetilde{G} = \langle \widetilde{\phi} = [x_1 \mapsto \xi_n x_1, x_2 \mapsto x_2, y_1 \mapsto \xi_n y_1, y_2 \mapsto y_2] \rangle$ of $U(P)$. The invariant subalgebra $U(P)^{\widetilde{G}}$ is isomorphic to the graded $\Bbbk$-algebra generated by $a_1, \cdots, a_{n+1}$, b, c subjecting to the following relations:

- $[a_i, a_j]$,
- $a_i a_j = a_k a_{i+j-k}$,
- $a_{i+1}b - \sum_{j=0}^{i} \binom{i}{j} q^j b a_{i+1-j}$,
- $a_{i+1}c - \sum_{j=0}^{i} \binom{i}{j} q^j (c + npb) a_{i+1-j}$.

Its Hilbert series is

$$H_{U(P)^{\widetilde{G}}}(t) = \frac{1}{n} \sum_{\widetilde{\phi} \in \widetilde{G}} \mathrm{Tr}_{U(P)}(\widetilde{\phi}, t) = \frac{1}{n} \sum_{i=0}^{n-1} \frac{1}{(1-t)^2(1-\xi^i t)^2} = \frac{(n-1)t^n + 1}{(1-t)^2(1-t^n)^2}.$$

However, according to [JZ97, Theorem 2.3], if $U(P)^{\widetilde{G}}$ has finite global dimension, then its Hilbert series has the form $\frac{1}{p(t)}$ for some $p(t) \in \Bbbk[t]$ with $p(0) = 1$. This is clearly not the case.

5.5 A Variant for Deformation Quantizations

In this section, we prove a variant of the Shephard-Todd-Chevalley Theorem for the standard deformation quantizations of unimodular Poisson structures on $\Bbbk[x_1, x_2, x_3]$, stated as follows:

Theorem 5.5.1. (Shephard-Todd-Chevalley Theorem) Let $P = \Bbbk[x_1, x_2, x_3]$ be a unimodular quadratic Poisson algebra. Let G be a finite subgroup of the graded Poisson automorphism group of P, and $G_\hbar$ be the corresponding finite subgroup of the graded automorphism group of $P_\hbar$ under the isomorphism $\mathrm{PAut}_{\mathrm{gr}}(P) \cong \mathrm{Aut}_{\mathrm{gr}}(P_\hbar)$. The following are equivalent:

(1) G is generated by Poisson reflections.

(2) $G_\hbar$ is generated by quasi-reflections.

(3) $P_\hbar^{G_\hbar}$ is Artin-Schelter regular.

Proof. (1) $\Leftrightarrow$ (2) is established by Proposition 4.3.3. For (2) $\Leftrightarrow$ (3), we can apply a combination of [KKZ09, Lemma 1.10(c)] and [KKZ10, Proposition 2.5] to simplify the proof, reducing it to proving $P_\hbar^{G_\hbar}$ has finite global dimension for every finite $G_\hbar \subseteq \mathrm{Aut}_{\mathrm{gr}}(P_\hbar)$ generated by quasi-reflections. In order to prove this statement, we invoke [KKZ10, Theorem 5.3] and conclude that for every finite abelian $G_\hbar \subseteq \mathrm{Aut}_{\mathrm{gr}}(P_\hbar)$ generated by quasi-reflections, $P_\hbar^{G_\hbar}$ has finite global dimension. This immediately resolves all cases, except when P is determined by the superpotential $\Omega_4 = x_1^2x_2 + x_1x_2^2$ and $G_\hbar$ is generated by $\left\{ \phi_1\big|_{P_1} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ a & a & 1 \end{bmatrix}, \right.$ $\left. \phi_2\big|_{P_1} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ b & 0 & 1 \end{bmatrix} \right\}.$

To address this particular case, define a $\Bbbk$-algebra homomorphism: $\varphi : \Bbbk\langle z_1, z_2, z_3 \rangle \to P_\hbar$, by mapping

$$z_1 \mapsto \frac{1}{3}(a+b)y_1 + \frac{1}{3}(a-2b)y_2 + y_3, \quad z_2 \mapsto y_1^2 + y_2^2 + y_1y_2, \quad z_3 \mapsto 2y_1^3 + 3y_1^2y_2 - 3y_1y_2^2 - 2y_2^3.$$

Straightforward computation shows that $[z_1, z_2] = \hbar z_3$, $[z_2, z_3] = 0$, $[z_3, z_1] = -6\hbar z_2^2$ are annihilated by φ. As such, φ passes through the following quotient:

$$\widetilde{\varphi}: \Bbbk\langle z_1, z_2, z_3\rangle/([z_1, z_2] = \hbar z_3, [z_2, z_3] = 0, [z_3, z_1] = -6\hbar z_2^2) \to P_\hbar.$$

The domain has a $\Bbbk$-linear basis $\{z_1^i z_2^j z_3^k : i, j, k \geq 0\}$. In the following argument, we will assume every polynomial in the domain is written with respect to this $\Bbbk$-linear basis. Suppose that $f(z_1, z_2, z_3) \in \ker \widetilde{\varphi}$. Decompose f into two parts: a part consisting of all the terms containing z_1, called g, and another part consisting of all the remaining terms, called h. Because $f \in \ker \widetilde{\varphi}$, we have $\widetilde{\varphi}(g) = -\widetilde{\varphi}(h)$. Rewrite $g = z_1 g'$ for some $g' \in \Bbbk\langle z_1, z_2, z_3\rangle$. In the equation $\widetilde{\varphi}(g) = -\widetilde{\varphi}(h)$,

$$\text{LHS} = \frac{1}{3}(a+b)y_1\widetilde{\varphi}(g') + \frac{1}{3}(a-2b)y_2\widetilde{\varphi}(g') + y_3\widetilde{\varphi}(g').$$

In particular, LHS must contain a term containing y_3, as flipping any variable does not produce any additional y_3 in $P_\hbar$. However, RHS does not contain any y_3, as $\varphi(z_2), \varphi(z_3) \in \Bbbk[y_1, y_2] \subseteq P_\hbar$. This is a contradiction unless $g = 0$.

Now, the domain of $\widetilde{\varphi}$ can be realized as an Ore-extension $\Bbbk[z_2, z_3][z_1, \mathrm{id}, \delta]$, where δ is the derivation satisfying $\delta(z_2) = \hbar z_3$ and $\delta(z_3) = 6\hbar z_2^2$. Since $\ker \widetilde{\varphi} \subseteq \Bbbk[z_2, z_3] \subseteq \Bbbk[z_2, z_3][z_1, \mathrm{id}, \delta]$, we may restrict the domain to $\Bbbk[z_2, z_3]$ and subsequently restrict the codomain to $\Bbbk[y_1, y_2]$:

$$\underbrace{\Bbbk[z_2, z_3] \xrightarrow{\phi} \Bbbk[y_1, y_2] \subseteq P_\hbar.}_{\widetilde{\varphi}\big|_{\Bbbk[z_2, z_3]}}$$

The $\Bbbk$-algebra homomorphism $\widetilde{\varphi}$ if and only if the $\Bbbk$-algebra homomorphism ϕ is injective, and if and only if $\phi(z_2)$, $\phi(z_3)$ are algebraically independent. To assert algebraic independence, it is sufficient to compute the following determinant as a result of [ER93, Theorem

2.4]:

$$\det\begin{bmatrix}\dfrac{\partial\phi(z_2)}{\partial y_1} & \dfrac{\partial\phi(z_2)}{\partial y_2}\\ \dfrac{\partial\phi(z_3)}{\partial y_1} & \dfrac{\partial\phi(z_3)}{\partial y_2}\end{bmatrix} = \begin{vmatrix} 2y_1+y_2 & y_1+2y_2 \\ 6y_1^2+6y_1y_2-3y_2^2 & 3y_1^2-6y_1y_2-6y_2^2\end{vmatrix} = 27(y_1^2y_2+y_1y_2^2)\neq 0.$$

Consequently, $\widetilde{\varphi}$ is injective as desired. In addition, it should be noted that the elements z_1, z_2, z_3 along with their commutator relations, remain invariant under the action of the group $G_\hbar$. The detailed calculation required to confirm this assertion is routine and, for the sake of brevity, will be omitted. This, in conjunction with the preceding argument, demonstrates that $\mathrm{Dom}(\widetilde{\varphi})$ can be embedded into $P_\hbar^{G_\hbar}$. To establish an isomorphism, we complete the embedding into a short exact sequence:

$$0 \to \mathrm{Dom}(\widetilde{\varphi}) \to P_\hbar^{G_\hbar} \to P_\hbar^{G_\hbar}/\mathrm{Dom}(\widetilde{\varphi}) \to 0.$$

First, since $\mathrm{Dom}(\widetilde{\varphi})$ has a $\mathbb{k}$-linear basis $\{z_1^i z_2^j z_3^k : i,j,k \geq 0\}$, its Hilbert series $h_{\mathrm{Dom}(\widetilde{\varphi})}(t) = \dfrac{1}{(1-t)(1-t^2)(1-t^3)}$. On a different note, we can calculate the Hilbert series of $P_\hbar^{G_\hbar}$ using the Molien's Theorem:

$$\begin{aligned} h_{P_\hbar{}^{G_\hbar}}(t) =& \frac{1}{6}\left(\frac{1}{(1-t)^3}+\frac{3}{(1-t)^2(1+t)}+\frac{2}{(1-t)(1+t+t^2)}\right)\\ =&\frac{1}{(1-t)(1-t^2)(1-t^3)}. \end{aligned}$$

By the additivity of Hilbert series in a short exact sequence, the embedding $\mathrm{Dom}\widetilde{\varphi} \hookrightarrow P_\hbar^{G_\hbar}$ translates into $\mathrm{Dom}\widetilde{\varphi} \cong P_\hbar^{G_\hbar}$. From the isomorphism, we can realize $P_\hbar^{G_\hbar}$ as an iterated Ore-extension of an Artin-Schelter regular algebra, and therefore is Artin-Schelter regular (and has finite global dimension). $\square$

5.6 Commutativity of Invariants and Deformation Quantizations

In this section, we prove taking invariant subalgebras under finite Poisson reflection groups and taking standard deformation quantizations are two commutative processes for unimodular quadratic Poisson structures on $\Bbbk[x_1, x_2, x_3]$.

Let $P = \Bbbk[x_1, x_2, x_3]$ be a unimodular quadratic Poisson algebra. Let G be a finite subgroup of the graded Poisson automorphism group of P and let $G_\hbar$ be the corresponding finite subgroup of the graded automorphism group of $P_\hbar$ under the isomorphism $\mathrm{PAut_{gr}}(P) \cong \mathrm{Aut_{gr}}(P_\hbar)$. After a thorough examination of the proof of Theorem 5.2.1, there are eight possible combinations of Poisson algebras P and finite Poisson reflection groups G.

Case	**Superpotential**	$\boldsymbol{G_\hbar}$
1	Ω_1	$\mathbb{Z}_2$
2	Ω_2	$\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_m}$, $n_i \mid n_{i+1}$
3	Ω_3	$(\mathbb{Z}_{l_1} \times \cdots \times \mathbb{Z}_{l_{d_1}})\times$ $(\mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_{d_2}})\times$ $(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_{d_3}})$ $l_i \mid l_{i+1}$, $m_i \mid m_{i+1}$, $n_i \mid n_{i+1}$
4, 5, 6	Ω_4	$\mathbb{Z}_2$
7	Ω_4	S_3
8	Ω_8	$\mathbb{Z}_2$

Consider the following polynomials and commutator relations in the standard deformation quantization $P_\hbar$:

Case	w_1	w_2	w_3
1	y_1^2	$\frac{a}{2}y_1 + y_2$	$\frac{d}{2}y_1 + y_3$
2	y_1	$d_1y_2 + (1-\xi_{n_1})y_3$	y_2^l
3	y_1^m	y_2^n	y_3^l
4	y_1y_2	$-y_1 + y_2$	$ay_1 + y_3$
5	y_1^2	$\frac{b}{2}y_1 + y_3$	$\frac{1}{2}y_1 + y_2$
6	y_2^2	$-cy_1 + y_3$	$2y_1 + y_2$
7	$\frac{1}{3}(a+b)y_1+\frac{1}{3}(2a-b)y_2+y_3$	$y_1^2 + y_2^2 + y_1y_2$	$2y_1^3 + 3y_1^2y_2 - 3y_1y_2^2 - 2y_2^3$
8	y_2	$y_1 + y_3$	y_1^2

Case	$[w_1, w_2]$	$[w_2, w_3]$	$[w_3, w_1]$
1	0	$3\hbar w_1$	0
2	$(\xi_{n_1} - 1)\hbar w_1^2$	$2l(\xi_{n_1} - 1)\hbar w_1w_3$	0
3	$\hbar\frac{2mn+higher\ degree\ terms}{(1+\hbar)^{mn}}w_1w_2$	$\hbar\frac{2nl+higher\ degree\ terms}{(1+\hbar)^{mn}}w_2w_3$	$\hbar\frac{2ml+higher\ degree\ terms}{(1-\hbar)^{mn}}w_1w_3$
4	0	$\hbar(6w_1 + w_2^2)$	$\hbar w_1w_2$
5	$-4\hbar w_1w_3$	$\hbar(\frac{3}{4}w_1 - w_3^2)$	0
6	$2\hbar w_1w_3$	$-\frac{3}{2}w_1 + \frac{1}{2}w_3^2$	0
7	$\hbar w_3$	0	$-6\hbar w_2^2$
8	$\hbar(\frac{2}{2+\hbar}w_1w_2 + \frac{6}{2+\hbar}w_3)$	$\hbar(\frac{8}{(2+\hbar)^2}w_2w_3)$	$\hbar(\frac{8}{(2-\hbar)^2}w_1w_3)$

Lemma 5.6.1. The polynomials w_i are invariant under the action of $G_\hbar$, and the commutator relations $[w_i, w_j]$ are preserved in $P_\hbar$.

We shall introduce one notation before presenting the proof for Lemma 5.6.1. In this section, for a pair of variables $v = (v_1, v_2, v_3), w = (w_1, w_2, w_3)$, we will use φ_{vw} to denote the $\Bbbk$-vector space map:

$$\varphi_{vw} : \bigoplus_{d\geq 0} \Bbbk \left\{ v_1^i v_2^j v_3^k : i+j+k=d \right\} \to \bigoplus_{d\geq 0} \Bbbk \left\{ w_1^i w_2^j w_3^k : i+j+k=d \right\}$$
$$v_1^i v_2^j v_3^k \mapsto w_1^i w_2^j w_3^k.$$

Proof. Choose an arbitrary graded Poisson automorphism $\phi \in G$. In each case, the polynomial w_i is either linear, or it satisfies $w_i = y_j^n$ and $\phi(x_j) = \lambda x_k$, or it is contained in some $\Bbbk\langle \Bbbk y_j \oplus \Bbbk y_k \rangle$ such that $[y_j, y_k] = 0$ in the standard deformation quantization $P_\hbar$. As a consequence of the construction of $\widetilde{\phi}$, the polynomial $\varphi_{xy}(w_i)$ is invariant under the action of $\widetilde{\phi}$.

Regarding the commutator relations, it is challenging to prove that these relations are preserved in $P_\hbar$ systematically because we are working with a variant of the classic deformation quantization, and there is no natural $\Bbbk$-algebra homomorphism from $P_\hbar$ to P. Alternatively, we can conduct a manual verification of each relation in $P_\hbar$ since there are a limited number of cases. For the sake of conciseness, we will omit the calculations and assumes the statement for free. □

Based on Lemma 5.6.1, for each case, we can define a $\Bbbk$-algebra homomorphism:

$$\Phi : \Bbbk\langle z_1, z_2, z_3 \rangle / ([z_i, z_j] = \varphi_{wz}([w_i, w_j]))_{1\leq i,j\leq 3} \to P_\hbar^{G_\hbar} \subseteq P_\hbar,$$
$$z_1, z_2, z_3 \mapsto w_1, w_2, w_3.$$

In the subsequent three lemmas, we will establish either injectivity or surjectivity for Φ in all cases.

Lemma 5.6.2. For Case 1, Case 2, Case 4, Case 5, Case 6, Case 7, the $\Bbbk$-algebra homomorphism Φ is injective.

Proof. A common characteristic shared by Case 1, Case 2, Case 4, Case 5, Case 6, and Case 7 is the existence of a pair of indices $(i, j) \in \{1, 2, 3\}^{\oplus 2}$ satisfying:

(1) the variable y_i is contained in w_j,

(2) the variable y_i is not contained in w_k for all $k \in \{1, 2, 3\} \backslash \{j\}$,

(3) the variables y_k, y_l commute for all $k, l \in \{1, 2, 3\} \backslash \{i\}$,

(4) the commutator bracket $[y_i, y_k]$ does not contain a term containing y_i for all $k \in \{1, 2, 3\}$,

(5) the variables w_k, w_l commute for all $k, l \in \{1, 2, 3\} \backslash \{j\}$.

Suppose that $f \in \ker \Phi$. Decompose f into two parts: a part consisting of all the terms containing z_j, called g, and another part consisting of all the remaining terms, called h. Because $f \in \ker \Phi$, we have $\Phi(g) = -\Phi(h)$. In the equation,

$$\text{LHS} = y_i g' + \textit{remaining terms},$$

for some polynomial g'. In particular, LHS contain at least one term containing y_i, as flipping any variable does not produce an additional y_i. However, RHS cannot contain any term containing y_i, as y_i is not contained in w_k for all $k \in \{1, 2, 3\} \backslash \{j\}$. This is a contradiction unless $g = 0$.

Let $\{a, b\} = \{1, 2, 3\} \backslash \{j\}$ and $\{c, d\} = \{1, 2, 3\} \backslash \{i\}$. The domain of Φ can be realized as an Ore-extension $\Bbbk[z_a, z_b][z_j, \mathrm{id}, \delta]$, for some appropriate choice of δ. Since $\ker \Phi \subseteq \Bbbk[z_a, z_b] \subseteq \Bbbk[z_a, z_b][z_j, \mathrm{id}, \delta]$, we may restrict the domain to $\Bbbk[z_a, z_b]$ and subsequently restrict the codomain to $\Bbbk[y_c, y_d]$:

$$\underbrace{\Bbbk[z_a, z_b] \xrightarrow{\Phi'} \Bbbk[y_c, y_d] \subseteq P_\hbar.}_{\Phi\big|_{\Bbbk[z_a, z_b]}}$$

The $\Bbbk$-algebra homomorphism Φ is injective if and only if the $\Bbbk$-algebra homomorphism Φ' is injective, and if and only if $\Phi'(z_a)$, $\Phi'(z_b)$ are algebraically independent. To assert algebraic independence, it is sufficient to compute the following determinant as a result of [ER93, Theorem 2.4]:

$$\det \begin{bmatrix} \frac{\partial \Phi'(z_a)}{\partial y_c} & \frac{\partial \Phi'(z_a)}{\partial y_d} \\ \frac{\partial \Phi'(z_b)}{\partial y_c} & \frac{\partial \Phi'(z_b)}{\partial y_d} \end{bmatrix},$$

which, in the above cases, does not vanish. Consequently, Φ is injective as desired. □

Lemma 5.6.3. For Case 3, the $\Bbbk$-algebra homomorphism Φ is injective.

Proof. This lemma becomes immediate if we repeat the former half of the proof of Lemma 5.2 (LHS, RHS argument) three times, because in this case, $z_1 \mapsto y_1^m$, $z_2 \mapsto y_2^n$, and $z_3 \mapsto y_3^l$. □

Lemma 5.6.4. For Case 8, the $\Bbbk$-algebra homomorphism Φ is surjective.

Proof. According to Theorem 4.1 and [Ste96, Proposition 1.1], the invariant subalgebra $P_\hbar^{G_\hbar}$ has either two generators subjecting to two homogeneous relations or three generators subjecting to three homogeneous relations. We fall into the latter category because the elements w_1, w_2, w_3 cannot be minimally generated by two elements in $P_\hbar$, and thus, not in the invariant subalgebra $P_\hbar^{G_\hbar}$ either. Furthermore, the elements w_1, w_2, w_3 form a minimal set of generators of $P_\hbar^{G_\hbar}$ because:

- the elements w_1, w_2, w_3 are contained in $P_\hbar^{G_\hbar}$,
- the linear polynomials w_1, w_2 are linearly independent,
- no linear elements apart from linear combinations of w_1 and w_2 is contained in $P_\hbar^{G_\hbar}$,

- the quadratic polynomial w_3 cannot be generated by w_1 and w_2.

Given that w_1, w_2, w_3 lie in the image of Φ, the $\Bbbk$-algebra homomorphism Φ is surjective.

$\square$

We now turn to the proof of Theorem 0.2:

Theorem 5.6.5. Let $P = \Bbbk[x_1, x_2, x_3]$ be a unimodular quadratic Poisson algebra. Let G be a finite Poisson reflection group of P and let $G_\hbar$ be the corresponding finite quasi-reflection group of $P_\hbar$ under the isomorphism $\mathrm{PAut}_{\mathrm{gr}}(P) \cong \mathrm{Aut}_{\mathrm{gr}}(P_\hbar)$. Define $Q_\hbar := P_\hbar^{G_\hbar}$, with $\hbar$ viewed as a formal parameter (as opposed to a scalar value). Then

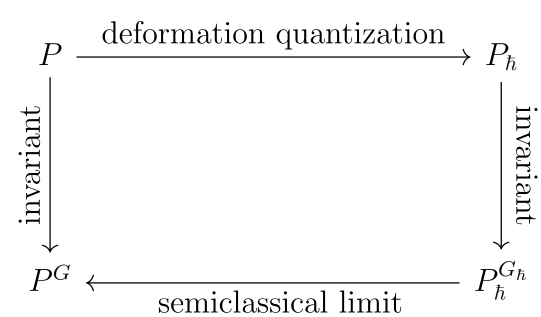
(1) $Q_\hbar/(\hbar) \cong P^G$ as $\Bbbk$-algebras.

(2) For all $f, g \in Q_\hbar$, $fg - gf = \hbar\pi_1(f, g)$ for some $\pi_1(f, g) \in Q_\hbar$.

(3) $Q_\hbar/(\hbar)$ together with the following Poisson bracket:

$$\{\overline{f}, \overline{g}\} = \overline{\pi_1(f, g)},$$

where $\overline{(\ )}$ denotes the image under the natural projection $Q_\hbar \to Q_\hbar/(\hbar)$, is isomorphic to P^G as Poisson algebras.



Proof. By Lemma 5.6.2, Lemma 5.6.3, Lemma 5.6.4, the $\Bbbk$-algebra homomorphism Φ is either injective or surjective. In either case, complete Φ to a short exact sequence by adding a cokernel (injective) or a kernel (surjective). Assign a degree to z_i such that $\deg(z_i) = \deg(w_i)$, for $1 \leq i \leq 3$. Observe that the commutator relations are homogeneous and the $\Bbbk$-algebra homomorphism Φ is graded. For Case 1 - Case 8, the domain of Φ admits a $\Bbbk$-linear basis $\{z_1^i z_2^j z_3^k : i, j, k \geq 0\}$, leading to a Hilbert series of $\dfrac{1}{(1-t^{\deg(w_1)})(1-t^{\deg(w_2)})(1-t^{\deg(w_3)})}$. By applying Molien's Theorem, it can be deduced that the Hilbert series of the codomain of Φ, namely $P_\hbar^{G_\hbar}$, is also given by $\dfrac{1}{(1-t^{\deg(w_1)})(1-t^{\deg(w_2)})(1-t^{\deg(w_3)})}$. By addivity of the Hilbert series, Φ is a $\Bbbk$-algebra isomorphism.

Let $Q_\hbar := P_\hbar^{G_\hbar}$, with $\hbar$ viewed as a formal parameter instead of a scalar value. One can scrutinize the conditions outlined in Theorem 5.1. It is clear that (1) and (2) hold. For (3),

Case	$[\boldsymbol{w_1}, \boldsymbol{w_2}]$	$[\boldsymbol{w_2}, \boldsymbol{w_3}]$	$[\boldsymbol{w_3}, \boldsymbol{w_1}]$
1	0	$3w_1$	0
2	$\frac{\xi_{n_1}-1}{a_1} z_1^2$	$\frac{2l(\xi_{n_1}-1)}{a_1} z_1 z_3$	0
3	$2mnz_1z_2$	$2nlz_2z_3$	$2mlz_1z_3$
4	$-2z_1^2 - 12z_3$	$-2z_1z_3$	0
5	$z_1^2 - 3z_3$	$4z_1z_3$	0
6	$-z_1^2 + 3z_3$	$-4z_1z_3$	0
7	z_3	0	$-6z_2^2$
8	$z_1z_2 + 3z_3$	$2z_2z_3$	$2z_1z_3$

Comparing with Theorem 5.2.1, the Poisson algebra $Q_\hbar$ is isomorphic to P^G as Poisson algebras, after suitable re-labelings of z_1, z_2, z_3. $\square$

Chapter 6

A VARIANT OF THE WATANABE THEOREM OF $P = \Bbbk[X_1, \cdots, X_N]$

Let $P = \Bbbk[x_1, \cdots, x_n]$ be a quadratic Poisson algebra and let $U(P)$ be its Poisson enveloping algebra. Let ϕ be a graded Poisson automorphism of P and let $\widetilde{\phi}$ be the corresponding graded automorphism of $U(P)$. In contrast to Theorem 5.4.2, $U(P)^{\widetilde{G}}$ may be Artin–Schelter Gorenstein in certain instances. In practice, it is challenging to verify this through homological approaches because, in general, there is no systematic way to describe the generators and relations of $U(P)^{\widetilde{G}}$. Instead, we shift our attention to [JZ00, Theorem 3.3], the "noncommutative Watanabe Theorem":

Theorem 6.0.1. [JZ00, Theorem 3.3] Let A be a Noetherian Artin-Schelter Gorenstein $\Bbbk$-algebra and let H be a finite subgroup of the graded automorphism group of A. If $\mathrm{hdet} h = 1$ for all $h \in H$, then A^H is Artin-Schelter Gorenstein.

To apply Theorem 6.0.1, we require a formula for computing the homological determinant of each induced graded automorphism $\widetilde{\phi}$ of $U(P)$, which will be the primary goal of this chapter.

6.1 A Formula for Calculating Homological Determinant

The subsequent lemma, as an generalization of [GVW23, Theorem 5.6], plays an important role in deriving such a formula.

Lemma 6.1.1. Let $P = \Bbbk[x_1, \cdots, x_n]$ be a quadratic Poisson algebra and let ϕ be a finite-order graded Poisson automorphism of P. Suppose that $\phi\Big|_{P_1}$ has eigenvalues $\lambda_1, \cdots, \lambda_m$, with multiplicity $c_1, \cdots, c_m$, respectively. Then

$$\mathrm{Tr}_{U(P)}(\widetilde{\phi}, t) = \frac{1}{(1-\lambda_1 t)^{2c_1} \cdots (1-\lambda_m t)^{2c_m}}.$$

Proof. Let $1 + \sum_{i\geq 1} a_i t^i$ be the Taylor expansion of $\mathrm{Tr}_P(\phi, t)$, where $a_i \in \Bbbk$ for all $i \geq 1$. It is sufficient to prove that $(1 + \sum_{i\geq 1} a_i t^i)^2 = \mathrm{Tr}_{U(P)}(\widetilde{\phi}, t)$. Fix $d \geq 1$. By [OPS06, Theorem 3.7], the degree d component of the Poisson enveloping algebra $U(P)_d$ admits a $\Bbbk$-linear basis $\{x_1^{p_1} \cdots x_n^{p_n} y_1^{q_1} \cdots y_n^{q_n} : \sum_{j=1}^{n} (p_j + q_j) = d\}$. Let $r_1, \cdots, r_n \geq 0$, and $b_{r_1,\cdots,r_n}$ be the coefficient of the term $x_1^{r_1} \cdots x_n^{r_n}$ in $\phi(x_1^{r_1} \cdots x_n^{r_n})$. Consider the coefficient of the term $x_1^{p_1} \cdots x_n^{p_n} y_1^{q_1} \cdots y_n^{q_n}$ in $\widetilde{\phi}(x_1^{p_1} \cdots x_n^{p_n} y_1^{q_1} \cdots y_n^{q_n})$. There are three observations:

(1) Given that $[x_i, x_j] = 0$ in $U(P)$, the coefficient of the term $x_1^{p_1} \cdots x_n^{p_n}$ in $\widetilde{\phi}(x_1^{p_1} \cdots x_n^{p_n})$ is $b_{p_1,\cdots,p_n}$.

(2) Given that $[y_i, y_j] = \sum_{k=1}^{n} \frac{\partial \{x_i, x_j\}}{\partial x_k} y_k$ and $[y_i, x_j] = \{x_i, x_j\}$ in $U(P)$, the coefficient of the term $y_1^{q_1} \cdots y_n^{q_n}$ in $\widetilde{\phi}(y_1^{q_1} \cdots y_n^{q_n})$ is $b_{q_1,\cdots,q_n}$.

(3) Given that $\widetilde{\phi}(x_j) \subseteq \bigoplus_{k=1}^{n} \Bbbk x_k$, the coefficient of $x_1^{p_1} \cdots x_n^{p_n} y_1^{q_1} \cdots y_n^{q_n}$ in $\widetilde{\phi}(x_1^{p_1} \cdots x_n^{p_n} y_1^{q_1} \cdots y_n^{q_n})$ is $b_{p_1,\cdots,p_n} b_{q_1,\cdots,q_n}$.

Let $1 + \sum_{i\geq 1} c_i t^i$ be the Taylor expansion of $\mathrm{Tr}_{U(P)}(\widetilde{\phi}, t)$, where $c_i \in \Bbbk$ for all $i \geq 1$. According to the definition of the trace series and the above argument, the coefficient relating to the dimension of the degree d component c_d equals to the summation of all $b_{p_1,\cdots,p_n} b_{q_1,\cdots,q_n}$ ranging over $p_1 + \cdots + p_n + q_1 + \cdots + q_n = d$. Equivalently, $c_d = \sum_{i+j=d} a_i a_j$, and therefore, $(1 + \sum_{i\geq 1} a_i t^i)^2 = 1 + \sum_{i\geq 1} c_i t^i$.

Finally, since P is a commutative polynomial ring, $\mathrm{Tr}_P(g, t) = \frac{1}{(1-\lambda_1 t)^{c_1} \cdots (1-\lambda_m t)^{c_m}}$, and consequently,

$$\mathrm{Tr}_{U(P)}(\widetilde{\phi}, t) = \frac{1}{(1-\lambda_1 t)^{2c_1} \cdots (1-\lambda_m t)^{2c_m}},$$

as desired. □

The following result provides a simple formula for calculating the homological determinant of an induced graded automorphism ϕ:

Theorem 6.1.2. Let $P = \Bbbk[x_1, \cdots, x_n]$ be a quadratic Poisson algebra and let $U(P)$ be its Poisson enveloping algebra. Let ϕ be a finite-order graded Poisson automorphism of P and let $\widetilde{\phi}$ be the induced graded automorphism of $U(P)$. Then

$$\mathrm{hdet}\widetilde{\phi} = (\det \phi\big|_{P_1})^2.$$

Proof. By [JZ00, Proposition 4.2], the graded automorphism $\widetilde{\phi}$ is rational over $\Bbbk$. Apply [JZ00, Lemma 2.6], the trace series

$$\mathrm{Tr}_{U(P)}(\widetilde{\phi}, t) = (\mathrm{hdet}\widetilde{\phi})^{-1} t^{-2n} + \textit{lower terms},$$

when written as a Laurent series in t^{-1}. By Lemma 6.1.1,

$$\begin{aligned}\mathrm{Tr}_{U(P)}(\widetilde{\phi}, t) =& \frac{1}{(1-\lambda_1 t)^{2c_1} \cdots (1-\lambda_m t)^{2c_m}} \\ =& \frac{1}{\left(\det(\phi\big|_{P_1})\right)^2 t^{2n} + \textit{lower terms}} \\ =& \left(\det(\phi\big|_{P_1})^2\right)^{-1} t^{-2n} + \textit{lower terms.}\end{aligned}$$

Comparing the leading coefficient, $\mathrm{hdet}\widetilde{\phi} = (\det \phi\big|_{P_1})^2$. □

With this formula in mind, the answer to the Artin-Schelter Gorensteinness question becomes a straightforward consequence of Theorem 6.0.1 [JZ00, Theorem 3.3].

Theorem 6.1.3. Let $P = \Bbbk[x_1, \cdots, x_n]$ be a quadratic Poisson algebra and let $U(P)$ be its Poisson enveloping algebra. Let G be a finite subgroup of the graded Poisson automorphism group of P and let $\widetilde{G} = \{\widetilde{\phi} : \phi \in G\}$ be the corresponding finite subgroup of the graded automorphism group of $U(P)$. If G is generated by graded Poisson automorphisms $\phi_1, \cdots, \phi_m$ such that $\det(\phi_i|_{P_1}) = \pm 1$ for all $1 \leq i \leq m$, then $U(P)^{\widetilde{G}}$ is Artin-Schelter Gorenstein.

Proof. This is an application of [JZ00, Theorem 3.3] when we substitute the value of the homological determinant in the formula provided in Theorem 6.1.2. □

6.2 Examples

In this section, we present two examples to illustrate the effectiveness of Theorem 6.1.3.

Example 6.2.1. Let $P = \Bbbk[x_1, x_2]$ be the Poisson algebra

$$\{f, g\} = \left(\frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_2} - \frac{\partial g}{\partial x_1}\frac{\partial f}{\partial x_2}\right) q x_1 x_2,$$

for some $q \neq 0$, for all $f, g \in P$. Let $G \cong \mathbb{Z}_n \times \mathbb{Z}_m$ be the graded Poisson automorphism group generated by:

$$\phi_1|_{P_1} = \begin{bmatrix} \xi_n & 0 \\ 0 & 1 \end{bmatrix}, \phi_2|_{P_1} = \begin{bmatrix} 1 & 0 \\ 0 & \xi_m \end{bmatrix},$$

where ξ_n, ξ_m are primitive nth, mth root of unity, respectively. It is straightforward to compute that $\det(\phi_1|_{P_1}) = \xi_n$ and $\det(\phi_2|_{P_1}) = \xi_m$. As a consequence, $U(P)^{\widetilde{G}}$ is Artin-Schelter Gorenstein if and only if $n, m \in \{1, 2\}$.

Example 6.2.2. Let $P = \Bbbk[x_1, x_2, x_3]$ be the Poisson algebra

$$\{f, g\} = \left(\frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_3} - \frac{\partial g}{\partial x_2}\frac{\partial f}{\partial x_3}\right)(2x_1x_2 + x_2^2) + \left(\frac{\partial f}{\partial x_3}\frac{\partial g}{\partial x_1} - \frac{\partial g}{\partial x_3}\frac{\partial f}{\partial x_1}\right)(x_1^2 + 2x_1x_2),$$

for all $f, g \in P$. Let $G \cong S_3$ be the graded Poisson automorphism group generated by:

$$\phi_1\big|_{P_1} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ a & a & 1 \end{bmatrix}, \phi_2\big|_{P_1} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ b & 0 & 1 \end{bmatrix},$$

for some $a, b \in \Bbbk$. Again, it is straightforward to compute that $\det(\phi_1\big|_{P_1}) = -1$ and $\det(\phi_2\big|_{P_1}) = -1$. As a consequence, the invariant subalgebra $U(P)^{\widetilde{G}}$ is Artin-Schelter Gorenstein.

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