On the Mod 2 General Linear Group Homology of Totally Real Number Rings

by

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Abstract

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We study the mod 2 homology of the general linear group of rings of integers in totally real number fields. In particular, for certain such rings R, we construct a space JKR and show that the mod 2 homology of JKR is a non-trivial quotient of the mod 2 homology of GLR. We explicitly calculate the mod 2 homology of JKR.

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DEDICATION

This dissertation is dedicated to my husband, Kyle Harris, for his encouragement, patience, and support, especially when I was discouraged or exhausted; to my father, Dwayne Nuzman, for sharing his love for mathematics with me; to my mother, Susan Nuzman, for her life-long encouragement and support; and lastly, to our first child, for motivating me to finish.

Chapter 1

INTRODUCTION

The purpose of this paper is to study the mod 2 homology of $GL\mathcal{O}_F[\frac{1}{2}]$ or, equivalently, the mod 2 homology of $BGL\mathcal{O}_F[\frac{1}{2}]^+$, where $\mathcal{O}_F$ is the ring of integers in a totally real number field.

Recall Quillen's ground-breaking calculation of the K-theory of finite fields [15]. Quillen identified the space $BGL\mathbb{F}_q^+$ with the homotopy fiber of the map, $\Psi^q - 1 : BU \to BU$. Here Ψ^q is the q-th Adams operation and $\mathbb{F}_q$ is the finite field with q elements. Let $F\Psi^q$ be the homotopy fiber of $\Psi^q - 1$. Quillen produced a map $BGL\mathbb{F}_q^+ \to F\Psi^q$, and showed that this map is in fact a homology isomorphism, by explicitly computing the homology of both spaces. Thus, $\Psi^q - 1$ must in fact be a homotopy isomorphism, and the calculation of $K_i(\mathbb{F}_q)$ then follows easily from the long exact sequence of a fibration.

Many results about the K-theory of rings of integers of number fields have been inspired by Quillen's methods. In order to make these calculations tractable, they are often carried out at a give prime l. For example, if R is the ring of integers in a number field F, Dwyer and Friedlander [6] have constructed a space Y_R and a map $BGLR^{+\wedge} \to Y_R$. Here, $BGLR^{+\wedge} = (BGLR^+)^{\wedge_l}$; that is, the l-adic completion of $BGLR^+$, where l is a prime. The space Y_R is an infinite loop space and is constructed out of familiar spaces which contain much K-theoretic information. Dwyer and Mitchell [7] have explicitly computed the homology and cohomology of Y_R at odd primes l.

The case $l = 2$ and $\sqrt{-1} \notin F$ must be studied separately. For the case $F = \mathbb{Q}$, $l = 2$, Mitchell [14] studied a space $JK\mathbb{Z}$, first constructed by Bökstedt [4]. The space has a particularly simple construction as the homotopy pull-back

$$\begin{array}{ccc} & & BO^{\wedge} \\ & & \downarrow \\ BGL\mathbb{F}_q^{+\wedge} & \longrightarrow & BU^{\wedge} \end{array}$$

which will be explained in more detail below. The space $JK\mathbb{Z}$ is in fact equivalent to the space Y_R, which Dwyer and Friedlander constructed by different methods, for $R = \mathbb{Z}[\frac{1}{2}]$. It is an infinite loop space and comes equipped with an infinite loop map

$$BGL\mathbb{Z}[\tfrac{1}{2}]^{+\wedge} \xrightarrow{f} JK\mathbb{Z}.$$

Mitchell explicitly calculated the cohomology and homology of $JK\mathbb{Z}$ and showed that f was a split epimorphism on homology [14]. Recent work of Voevodsky and others, verifying the Lichtenbaum-Quillen conjectures for $l = 2$, shows that f is in fact an isomorphism on homology. Hence, Mitchell's work gives an explicit description of the $\mathcal{A}$-Hopf algebra $H_*BGL\mathbb{Z}[\frac{1}{2}]$.

Suppose $R = \mathcal{O}[\frac{1}{2}]$, where $\mathcal{O}$ is the ring of integers in a totally real number field. We work at the prime 2, and we introduce a space JKR, analogous to $JK\mathbb{Z}$. The paper is organized as follows. We begin chapter 2 with some category-theoretic results. This enables us to define a space X_F, which will be a basic building block in the construction of JKR, and to construct a map $BGLR^{+\wedge} \to X_F$. In chapter 3, we discuss the spaces $JK_q\mathbb{Z}$ where $q = p^n$ for an odd prime p, and we classify these spaces up to homotopy. We are then able to choose a well-defined space $JK\mathbb{Z}$, which is in fact the space discussed above, and compute its homology and homotopy. In chapter 4, we construct a space JK_qR, which depends on the choice of a prime ideal in R. We also define a map $BGLR^{+\wedge} \to JK_qR$. Chapter 5 addresses the choice of prime ideal in R and contains some number theory results which allow us to work with a well-defined space JKR, independent of the choices made in its construction. We begin chapter 5 by restricting our study to certain totally real number fields, whose rings of integers possess several properties analogous to properties of the rational integers. In chapter 6, we study the space $BR^{\times}$ and maps from $BR^{\times}$ to BO and to $BGL\mathbb{F}_q^+$. Chapter 7 contains the main results. We explicitly compute the homology of JKR and show the map $H_*(BGLR^+;\mathbb{Z}/2) \to H_*(JKR;\mathbb{Z}/2)$ is an epimorphism of $\mathcal{A}$-Hopf algebras.

Notation and Terminology: All homology and cohomology groups have $\mathbb{Z}/2$ coefficients, unless explicitly stated otherwise. Give a space X, the space $X^\wedge$ is the 2-adic completion of X. F will be a totally real field of degree n over $\mathbb{Q}$, and $f_1, f_2, \ldots, f_n$ will be its n distinct real embeddings. The ring of integers in F will be denoted by $\mathcal{O}_F$ and $R = \mathcal{O}_F[\frac{1}{2}]$. $\mathcal{A}$ is the mod 2 Steenrod algebra.

Chapter 2

CONSTRUCTION OF X_F

The goal of this chapter is to construct a space X_F, along with maps $X_F \to BU$ and $BGLR^+ \to X_F$.We will then calculate the homology and homotopy of X_F. The space X_F will be used in chapter 3 to construct another space JKR and a map $BGLR^+ \to JKR$ which is an epimorphism on homology.

First we need some preliminary results. Let $\mathcal{C}$ be the category of compactly generated weak Hausdorff pointed spaces.

Proposition 2.1 *$\mathcal{C}$ has the structure of a proper closed model category, where*

- *Weak equivalences are weak homotopy equivalences.*
- *Fibrations are Serre fibrations.*
- *Cofibrations are maps which have the left lifting property with respect to acyclic fibrations.*

Proof: This model category structure is called the singular structure. Let $\mathcal{C}'$ be the category of compactly generated weak Hausdorff (unpointed) spaces. A proof that $\mathcal{C}'$ is a proper closed model category with the singular structure can be found in [19]. But $\mathcal{C}$ is just the "under category," $* \downarrow \mathcal{C}'$. That is, an object $X \in \mathcal{C}$ is an object $X \in \mathcal{C}'$, along with a map $* \to X$, which gives the basepoint of X. A morphism between X and Y is a morphism in $\mathcal{C}'$ so that $* \to X \to Y$ is equal to $* \to Y$; that is, a morphism which sends the basepoint of X to the basepoint of Y. Given any model category $\mathcal{D}$ and object A of $\mathcal{D}$, the under category $A \downarrow \mathcal{D}$ is also a model category, with the obvious structure inherited from $\mathcal{D}$ [8]. So, in fact, $\mathcal{C}$ is a proper closed model category. □

Fix a map $X \xrightarrow{f} Y$ in $\mathcal{C}$. Then we can factor this map as an acyclic cofibration followed by a fibration:

$$X \overset{\sim}{\rightarrowtail} X(f) \overset{\bar{f}}{\twoheadrightarrow} Y.$$

Lemma 2.2 *Given any two such compositions,*

$$X \overset{\sim}{\rightarrowtail} X(f) \overset{\bar{f}}{\twoheadrightarrow} Y$$

and

$$X \overset{\sim}{\rightarrowtail} X'(f) \overset{\bar{f}'}{\twoheadrightarrow} Y$$

then $X(f)$ is weakly equivalent to $X'(f)$ over Y.

Proof: Consider the commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\sim} & X(f) \\ {\scriptstyle\sim}\downarrow & \overset{L}{\nearrow} & \downarrow{\scriptstyle \bar{f}} \\ X'(f) & \xrightarrow{\bar{f}'} & Y. \end{array}$$

Since fibrations have the left lifting property with respect to acyclic cofibrations, there is a lift $L : X'(f) \rightarrow X(f)$. In fact L must be a weak equivalence (over Y) since pre-composition with $X \overset{\sim}{\rightarrowtail} X'(f)$ is a weak equivalence. Hence, the factorization above is unique, up to weak equivalence. Note that if f is already a fibration, then we may take $X(f) = X$. □

Lemma 2.3 *If f is (left) homotopic to g, then $X(f)$ is weakly equivalent to $X(g)$ over Y.*

Proof: Let IX be a cylinder object for X and let $F : IX \to Y$ be a homotopy from f to g. Then we have a commutative diagram:

$$\begin{array}{ccccc} X & \xrightarrow{\sim} & IX & \xleftarrow{\sim} & X \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle F} & \swarrow{\scriptstyle g} & \\ & & Y. & & \end{array}$$

But we also have commutative diagrams,

$$\begin{array}{ccc} X & \xrightarrow{\sim} & X(f) \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \bar{f}} \\ & & Y \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{\sim} & X(g) \\ & {\scriptstyle g}\searrow & \downarrow{\scriptstyle \bar{g}} \\ & & Y. \end{array}$$

Thus, $X(f) \sim X(g)$ over Y. □

Now, suppose we have a set of maps $f_i : X_i \to Y$ for $i = 1, 2, \ldots n$. Then define the *iterated pullback* of the (X_i, f_i) over Y as

$$P(f_1, f_2, \ldots, f_n) = X_1(f_1) \times_Y X_2(f_2) \times_Y \cdots \times_Y X_n(f_n).$$

Lemma 2.4 *The iterated pullback is associative; that is, the spaces $P(f_1, P(f_2, f_3))$ and $P(P(f_1, f_2), f_3)$ are weakly equivalent over Y. (Here, by abuse of notation, $P(f_i, f_j)$ denotes the obvious map from $P(f_i, f_j)$ to Y.)*

Proof: $P(f_1, (P(f_2, f_3))) = X_1(f_1) \times_Y P(P(f_2, f_3))$. But $P(f_2, f_3) \to Y$ is a fibration. For all of the maps in the pullback square

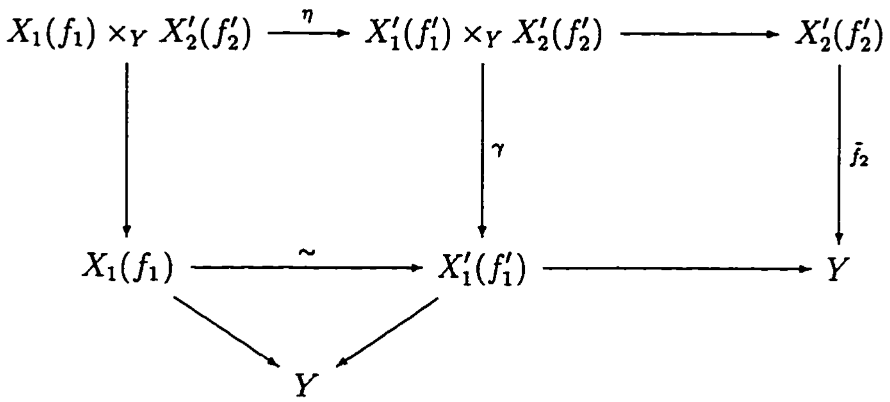
$$\begin{array}{ccc} P(f_2, f_3) & \longrightarrow & X(f_2) \\ \downarrow & & \downarrow{\scriptstyle \bar{f}_2} \\ X(f_3) & \xrightarrow{\bar{f}_3} & Y \end{array}$$

must be fibrations since $\bar{f}_2, \bar{f}_3$ are fibrations. But the composition of two fibrations is again a fibration. Hence, we can take $P(P(f_2, f_3)) = P(f_2, f_3)$. So, $P(f_1, (P(f_2, f_3))) = X(f_1) \times_Y (P(f_2, f_3)) = X(f_1) \times_Y X(f_2) \times_Y X(f_3) = P(P(f_1, f_2), f_3)$, over Y. □

For the next lemma, we need the following fact about pullback diagrams in model categories.

Lemma 2.5 *Suppose there are spaces X_i, X_i' and maps $f_i : X_i \to Y$, $f_i' : X_i' \to Y$ $(i = 1, 2, \ldots, n)$ such that each (X_i, f_i) is weakly equivalent over Y to (X_i', f_i'). Then $P(f_1, f_2, \ldots, f_n) \sim P(f_1', f_2', \ldots, f_n')$ over Y.*

Proof: By induction. The case of $n = 1$ is easy. For, $(X_1(f_1)) \sim X_1$ over Y and $X_1'(f_1) \sim X_1'$ over Y so $X_1(f_1) \sim X_1'(f_1')$ over Y. For $n = 2$, consider the diagram:



Note that both squares are pullbacks. Since $\tilde{f}_2$ is a fibration, so is γ. Since $\mathcal{C}$ is proper, this implies that η is a weak equivalence, and from the picture it is clear that the weak equivalence is in fact over Y. Similarly $X_1(f_1) \times_Y X_2'(f_2')$ is also weakly equivalent to $X_1(f_1) \times_Y X_2(f_2)$ over Y. This proves the case of $n = 2$. But by Lemma 2.4, this step is just the inductive step; thus, we are done. □

Lemma 2.6 *Suppose we have maps, $f_i : X \to Y$ and $f_i' : X \to Y$ for $i = 1, 2, \ldots, n$ such that each f_i is left-homotopic to f_i'. Then $P(f_1, f_2, \ldots f_n) \sim P(f_1', f_2', \ldots, f_n')$ over Y.*

Proof: This is immediate from Lemmas 2.3, 2.4, 2.5. □

We are now ready to construct X_F. We may regard F as a subfield of $\mathbb{R}$ via the embedding f_1. Then it is possible to choose automorphisms, say $\tau_i : \mathbb{C} \to \mathbb{C}$ for $1 \le i \le n$, such that $\tau_i f_i = f_1$ for all i. For simplicity, choose $\tau_1 = 1_{\mathbb{C}}$.

Each τ_i induces an infinite-loop map $\tau_i : BGL\mathbb{C}^{\delta+} \to BGL\mathbb{C}^{\delta+}$. (We write $\mathbb{C}^{\delta}$ to emphasize that here the complex numbers have the discrete topology.) The map $s : \mathbb{C}^{\delta} \to \mathbb{C}^{top}$ induces a map $s : BGL\mathbb{C}^{\delta+} \to BGL\mathbb{C}^{+}$ (here the latter $\mathbb{C}$ has the usual topology). The inclusion $U(n) \hookrightarrow GL(\mathbb{C}, n)$ is a group homomorphism and a homotopy equivalence. Hence, each inclusion induces a homotopy equivalence $BU(n) \to BGL_n\mathbb{C}$; these maps are clearly compatible and hence induce a map $\phi : BU \to BGL\mathbb{C}$, which is in fact a homotopy equivalence. Choose $\psi : BGL\mathbb{C} \to BU$ to be a homotopy inverse; clearly ψ factors through $BGL\mathbb{C}^{+}$. Hence, we have a map $\tilde{\psi} : BGL\mathbb{C}^{+} \to BU$. Define $h = \tilde{\psi} \circ s : BGL\mathbb{C}^{\delta+} \to BU$.

Theorem 2.7 *(Suslin, [17]) This map $h : BGL\mathbb{C}^{\delta+} \to BU$ is an infinite loop map and a homotopy equivalence after completing at a given prime p. (Similarly, there is an infinite loop map $g : BGL\mathbb{R}^{\delta+} \to BO$ which is a homotopy equivalence after completing at a prime p.)*

It is clear that it is necessary to complete to get such an equivalence. For, the fundamental group of $BGL\mathbb{C}^{\delta+}$ is $\mathbb{C}^{\times}$, while BU is simply connected. For our purposes, we will be completing at the prime 2.

After completing at the prime 2, this gives us a map $h\tau_i h^{-1} : BU^{\wedge} \to BU^{\wedge}$ for each τ_i. For simplicity, we will often denote this map merely τ_i. Let $\hat{c} : BO^{\wedge} \to BU^{\wedge}$ denote the map induced by complexification.

Definition: *We define*

$$X_F = P(\tau_1\hat{c}, \tau_2\hat{c}, \ldots \tau_n\hat{c}).$$

We get a canonical map $BGLR^{+\wedge} \to X_F$. Let f_i also denote the induced map $BGLR^{+\wedge} \to BGL\mathbb{R}^{+\wedge}$. Then we have the following commutative diagram for each

i:

$$\begin{array}{ccc} BGLR^{+\wedge} & & \\ \Big\downarrow{\scriptstyle gf_i} & & \\ BO^{\wedge} & \xrightarrow{\phi_i} & BO^{\wedge}(\tau_i\hat{c}) \\ \Big\downarrow{\scriptstyle (h\tau_i h^{-1})\hat{c}} & \swarrow{\scriptstyle \overline{\tau_i\hat{c}}} & \\ BU^{\wedge}. & & \end{array}$$

Note that if $i : \mathbb{R} \to \mathbb{C}$ is the usual inclusion, then the following diagram commutes:

$$\begin{array}{ccc} BGL\mathbb{R}^{+\wedge} & \xrightarrow{Bi} & BGL\mathbb{C}^{+\wedge} \\ \Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle h} \\ BO^{\wedge} & \xrightarrow{\hat{c}} & BU^{\wedge} \end{array}$$

So in fact for each j we have $h\tau_j h^{-1}\hat{c}gf_j = h\tau_j h^{-1}(hBig^{-1})gf_j = h\tau_j Bif_j$. But by choice of the τ_i, $h\tau_j Bif_j = h\tau_k Bif_k$ for all j, k. So the following diagram commutes:

$$\begin{array}{ccc} BGLR^{+\wedge} & \xrightarrow{\phi_j g f_j} & BO^{\wedge}(\tau_j\hat{c}) \\ \Big\downarrow{\scriptstyle \phi_k g f_k} & & \Big\downarrow{\scriptstyle \overline{\tau_j\hat{c}}} \\ BO^{\wedge}(\tau_k\hat{c}) & \xrightarrow[\overline{\tau_k\hat{c}}]{} & BU^{\wedge} \end{array}$$

for each j, k. So in fact we have a map $h_0 = (\phi_1 gf_1, \phi_2 gf_2, \ldots, \phi_n gf_n) : BGLR^{+\wedge} \to P(\tau_1\hat{c}, \tau_2\hat{c}, \ldots \tau_n\hat{c})$, as desired.

Proposition 2.8 *$X_F \sim BO^{\wedge}(\hat{c}, \hat{c}, \ldots, \hat{c})$ over $BU^{\wedge}$. In particular, X_F is independent of the choice of the τ_i.*

Proof: First we show that τ_i is homotopy equivalent to Ψ_i^{α}, a "2-adic Adams operation", defined below. Here we follow the exposition in [12].

First, we must define what is meant by a "2-adic" Adams operation. These ideas go back to Atiyah, Tall, and Sullivan [3] [16]. If Y is a finite CW-complex, then we define $K^\wedge(Y)$ to be 2-adic completion of the finitely generated group $K^0(Y)$. That is,

$$K^\wedge(Y) = (K^0(Y))^\wedge = \varprojlim \frac{K(Y)}{2^n}.$$

Given an integer k relatively prime to 2, we have a well-defined operation Ψ^k : $K(-)/2^n \rightarrow K(-)/2^n$ on finite complexes for each n. These operations are compatible, so in fact we have a well-defined operation $K^\wedge(-) \rightarrow K^\wedge(-)$ on finite complexes. In fact, it is possible to define 2-adic Adams operations Ψ^α on $K^\wedge(Y)$, for any $\alpha \in \mathbb{Z}_2^\times$. Given such an α, write $\alpha = \{[\alpha_n]\}$, where each α_n is a non-negative integer less than 2^n. Each Ψ^{α_n} is then a well-defined operation on $K(-)/2^n$ on finite complexes. These operations are compatible; hence we get a well-defined operation $K^\wedge(-) \rightarrow K^\wedge(-)$ on finite complexes.

Fix $\alpha \in \mathbb{Z}_2^\times$. Write $BU^\wedge$ as a direct limit of finite complexes, say $BU^\wedge = \varinjlim X_n$. It is possible to choose X_n to have only even-dimensional cells. Since $K^\wedge Y = [Y, BU^\wedge]$, and since these Adams operations are in fact a natural transformations on the functor $K^\wedge$, we get a compatible sequence of maps $\Psi^\alpha : X_n \rightarrow BU^\wedge$, which in fact induces a unique map $\Psi^\alpha : BU^\wedge \rightarrow BU^\wedge$. To see this, note that because X_n has only even-dimensional cells, $[X_n, U] = 0$ for all n. So in fact $\varinjlim^1 [X_n, U] = 0$, and Milnor's exact sequence becomes

$$0 \rightarrow [BU, BU] \rightarrow \varinjlim [X_n, BU] \rightarrow 0.$$

In fact, there is an injection $\mathbb{Z}_2[[\mathbb{Z}_2^\times]] \rightarrow [BU^\wedge, BU^\wedge]$ [12]. This map comes from sending α of $\mathbb{Z}_2^\times$ to the corresponding map Ψ^α, discussed in the last chapter. Roughly speaking, $\mathbb{Z}_2[[\mathbb{Z}_2^\times]]$ can be thought of as power series in elements of $\mathbb{Z}_2^\times$ with coefficients in $\mathbb{Z}_2$.

Given an automorphism, σ, of $\mathbb{C}$ over $\mathbb{Q}$, note that σ restricts to an automorphism of $\mathbb{Z}_{2^\infty}$, the 2-powered roots of unity. But $\mathrm{Aut}(\mathbb{Z}_{2^\infty})$ is $\mathbb{Z}_2$. Hence we have a map $\mathrm{Aut}(\mathbb{C}) \rightarrow \mathbb{Z}_2$. Let α_i be the image of τ_i under this map. We wish to show that $\Psi^{\alpha_i} \sim \tau_i : BU^\wedge \rightarrow BU^\wedge$.

Lemma 2.9 *(Adams [1]): Suppose $f, g : BU^\wedge \to BU^\wedge$ are H-maps such that $f_* = g_* : \pi_* BU^\wedge \to \pi_* BU^\wedge$. Then f and g are homotopic.*

Hence, we need only to compute Ψ^{α_i} and τ_i on π_* and show that they are equal. To do so, we will use the fact that $\pi_* BU^\wedge$ is a ring; in fact, $\pi_* BU^\wedge = \mathbb{Z}_2[\beta]$, where β is a generator of $\pi_2(BU^\wedge)$. Further, Ψ^{α_i} and τ_i induce ring homomorphisms on π_*. This makes the computation of Ψ^{α_i} immediate. Recall that for $x \in K(S^{2n})$, $\Psi^q(x) = q^n x$; hence, for $x \in \pi_{2n} BU^\wedge$, $\Psi^{\alpha_i}(x) = \alpha_i^n x$.

Consider now τ_i. We claim that for $x \in \pi_2 BGL\mathbb{C}^{+\wedge}, \tau_i(x) = \alpha_i x$. To see this, recall that we have a Universal Coefficient Theorem for completion that gives us a short exact sequence:

$$0 \to Ext(\mathbb{Z}/2^\infty, \pi_2 B) \to \pi_2 B^\wedge \to Hom(\mathbb{Z}/2^\infty, \pi_1 B) \to 0$$

where $B = BGL\mathbb{C}^{\delta+}$. But recall that a theorem of Matsumoto [11] tells us that

$$K_2\mathbb{C} = (\mathbb{C}^* \otimes_{\mathbb{Z}} \mathbb{C}^*)/I$$

where I is the subgroup generated by all $a \otimes (1-a)$. $\mathbb{C}^*$ is divisible, hence so is $K_2\mathbb{C}$. Thus, the Ext term above is actually zero. But

$$\mathrm{Hom}(\mathbb{Z}/2^\infty, \pi_1 B) = \mathrm{Hom}(\varinjlim \mathbb{Z}/2^n, \mathbb{C}^\times)$$

where $\tau_i : \mathbb{C}^\times \to \mathbb{C}^\times$ induces a map from $\mathrm{Hom}(\varinjlim \mathbb{Z}/2^n, \mathbb{C}^\times)$ to itself by post-composition. But in fact,

$$\begin{aligned} \mathrm{Hom}(\mathbb{Z}/2^\infty, \pi_1 B) &= \varprojlim \mathrm{Hom}(\mathbb{Z}/2^n, \mathbb{C}^\times) \\ &= \varprojlim \mathrm{Hom}(\mathbb{Z}/2^n, \mu_{2^n}) \\ &= \varprojlim \mathbb{Z}/2^n \end{aligned}$$

where μ_{2^n} is the 2^n-th roots of unity. On the first and second lines, τ_i acts by post-composition; on the third line, τ_i acts by multiplication by α_i.

Suslin's equivalence, shows that $\pi_* BGL\mathbb{C}^{+\wedge} = \mathbb{Z}_2[\beta]$ where β is in π_2. Hence, the calculation above suffices to prove that the two maps are equal on π_*. □

We now have two homotopic maps $BU^\wedge \to BU^\wedge$, namely $h\tau_i h^{-1}$ and Ψ^{α_i}. Lemma 2.6 then implies that

$$X_F = BO^\wedge(\tau_1\hat{c}, \tau_2\hat{c}, \ldots, \tau_n\hat{c}) \sim BO^\wedge(\Psi^{\alpha_1}\hat{c}, \ldots, \Psi^{\alpha_n}\hat{c}) \text{ over } BU^\wedge.$$

Lemma 2.10 *Let $\Psi_{\mathbf{R}}^k$ denote the k-th real Adams operations, for k a non-negative integer. Then $\Psi^k c \sim c\Psi_{\mathbf{R}}^k$.*

Proof: This is easy to verify by working on the level of vector bundles and is proved, for instance, in [9]. The key point is that the Adams operations are sums of products of exterior powers, and sums, products, and exterior powers all commute with complexification. □

We may extend Lemma 2.10 to the p-adic case, as above. Hence, again by Lemma 2.6, we have

$$X_F \sim BO^\wedge(\hat{c}\Psi_{\mathbf{R}}^{\alpha_1}, \ldots, \hat{c}\Psi_{\mathbf{R}}^{\alpha_n}) \text{ over } BU^\wedge.$$

Since each α_i is a 2-adic unit, $\Psi_{\mathbf{R}}^{\alpha_i}$ has an inverse; namely $\Psi_{\mathbf{R}}^{\beta_i}$, where $\beta_i = \alpha_i^{-1}$. In particular, each $\Psi_{\mathbf{R}}^{\alpha_i} : BO^\wedge \to BO^\wedge$ is a homotopy equivalence. So we have a commutative diagram:

$$\begin{array}{ccc} BO^\wedge & \xrightarrow[\Psi_{\mathbf{R}}^{\alpha_i}]{\sim} & BO^\wedge \\ & {\scriptstyle \hat{c}\Psi_{\mathbf{R}}^{\alpha_i}} \searrow \quad \swarrow {\scriptstyle \hat{c}} & \\ & BU^\wedge. & \end{array}$$

Then Lemma 2.5 implies that

$$X_f \sim BO^\wedge(\hat{c}, \hat{c}, \ldots, \hat{c}) \text{ over } BU^\wedge$$

as desired. In particular this shows that X_F is independent of the choice of the τ_i. We also have the following.

Proposition 2.11

$$X_F \sim BO^\wedge \times ((U/O)^\wedge)^{n-1}$$

Proof: Let $X_k = BO^\wedge(\hat{c}, \hat{c}, \ldots, \hat{c})$ where the pullback is over k copies $\hat{c} : BO^\wedge \to BU^\wedge$. Inductively, we'll show that $X_k \sim BO^\wedge \times ((U/O)^\wedge)^{n-1}$. The case of $n = 1$ is clear, and the fiber of $\hat{c} : BO^\wedge \to BU^\wedge$ is $(U/O)^\wedge$. For the inductive step, consider the pullback diagram:

$$\begin{array}{ccc}
((U/O)^\wedge)^{k-1} & \xrightarrow{=} & ((U/O)^\wedge)^{k-1} \\
\downarrow{\scriptstyle i} & & \downarrow \\
X_k & \longrightarrow & X_{k-1} \\
\downarrow{\scriptstyle f} & & \downarrow \\
BO^\wedge & \xrightarrow{\hat{c}} & BU^\wedge.
\end{array}$$

Inductively, we know that the fiber of $X_{k-1} \to BU^\wedge$ is $((U/O)^\wedge)^{k-1}$. Also, we have a diagonal map $\Delta : BO^\wedge \to X_k$; note that $f \circ \Delta = 1_{BO^\wedge}$. Since all the spaces in the diagram are infinite loop spaces, we have a split long exact sequence:

$$\cdots \to \pi_i((U/O)^\wedge)^{k-1} \to \pi_i X_k \to \pi_i BO^\wedge \to \pi_{i-1}((U/O)^\wedge)^{k-1} \to \cdots$$

Since $BO^\wedge$ is an H-space and $\hat{c}$ is an H-space map, X_k inherits an H-space multiplication from $BO^\wedge$ in an obvious way. Call this multiplication m. Then we can deduce from the long exact sequence above that the map $m(\Delta \times i) : BO^\wedge \times ((U/O)^\wedge)^{k-1} \to X_k$ in fact is a weak equivalence. Hence, by induction, we are done.

Corollary 2.12

$$\begin{aligned}
H^*X_F &\cong H^*(BO) \otimes (H^*(U/O))^{n-1} \\
&\cong \mathbb{Z}/2[w_1, w_2, w_3, \ldots] \otimes (\mathbb{Z}/2\langle u_1, u_2, u_3 \ldots \rangle)^{n-1} \\
H_*X_F &\cong H_*(BO) \otimes (H_*(U/O))^{n-1} \\
&\cong \mathbb{Z}/2[b_1, b_2, b_3, \ldots] \otimes (\mathbb{Z}/2[s_1, s_2, s_3, \ldots])^{n-1} \\
\pi_*X_F &\cong \pi_*(BO^\wedge) \times (\pi_*((U/O)^\wedge))^{n-1}
\end{aligned}$$

Here, $w_i \in H^i(BO)$, $u_i \in H^i(U/O)$, $b_i \in H_i(BO)$, *and* $s_i \in H_{2i-1}(U/O)$.

This is immediate from Proposition 2.11.

Chapter 3

THE SPACES JK_Q

Recall, F is a number field, $\mathcal{O}_F$ is the ring of integers in F and $R = \mathcal{O}_F[\frac{1}{2}]$. In what follows, we complete the space the space $BGLR^+$ at the prime 2, and work with mod 2 homology. In order to study $H_*BGLR^{+\wedge}$, we will construct a space JKR from well-studied spaces and show that JKR contains much homology information about $BGLR^+$ at the prime 2. In particular, we will construct a map $BGLR^{+\wedge} \to JKR$ which is an epimorphism on homology. Since we will be able to compute the homology of JKR, this will allow us to find a non-trivial quotient of the homology of $BGLR^{+\wedge}$.

The case of $F = \mathbb{Q}$ has already been studied via this program by Mitchell [14], following the work of Bökstedt, Dwyer, and Friedlander. In this chapter, we consider this case.

First we need some preliminaries. We begin by recalling the construction of the Brauer lift $\theta : BGL\mathbb{F}_q^+ \to BU$, where $q = p^m$ for some prime p and integer $m \geq 1$. First, we construct compatible maps $BGL_n\mathbb{F}_q \to BU$. To do this, let G be any finite group, and let $\mathcal{R}_{\mathbb{C}}G$ denote the representation ring of G over $\mathbb{C}$. This ring is generated as an abelian group by the isomorphism classes of (irreducible) representations of G over $\mathbb{C}$. Thus an element of $\mathcal{R}_{\mathbb{C}}G$ is a formal sum of representations; we often call such an element a virtual representation. Notice that a representation V yields a complex vector bundle over G; namely, $EG \times_G V$. But each complex n-bundle is classified by a map $BG \to BU$. In fact, this induces a map $\mathcal{R}_{\mathbb{C}}G \to K^0BG = [BG, BU \times \mathbb{Z}]$ which is in fact a ring homomorphism.

For each n, there is an obvious representation of $GL_n\mathbb{F}_q$ over $\mathbb{F}_q$ with representation space $\mathbb{F}_q^n$; these representations are clearly compatible with the inclusions $GL_n\mathbb{F}_q \hookrightarrow GL_{n+1}\mathbb{F}_q$ and $\mathbb{F}_q^n \hookrightarrow \mathbb{F}_q^{n+1}$. We wish to "lift" these representations to representations of $\mathbb{C}$. Let $\overline{\mathbb{F}}_q$ denote the algebraic closure of $\mathbb{F}_q$. Then it is possible to choose an injective homomorphism $\overline{\mathbb{F}}_q^\times \hookrightarrow \mathbb{C}^\times$. Choose one such map, i. For what follows, we will keep this homomorphism fixed. Later, we will see that the construction of JK_q is independent of the choice of i.

Define the so-called Brauer character of the representation of $GL_n\mathbb{F}_q$ by

$$\chi_n(M) = \sum_{\alpha \in E_n(M)} i(\alpha)$$

where $E_n(M)$ is the set of eigenvalues (with multiplicity) of M for $M \in GL_n\mathbb{F}_q$.

Theorem 3.1 *(Green) The Brauer character χ_n is an element of $\mathcal{R}_\mathbb{C}(GL_n\mathbb{F}_q)$; that is, it is a virtual complex character of $GL_n\mathbb{F}_q$.*

Clearly, $\chi_n|_{GL_{n-1}} = \chi_{n-1}$; hence we have a set of compatible elements of $\mathcal{R}_\mathbb{C}GL_n\mathbb{F}_q$ and thus compatible elements of $[BGL_n\mathbb{F}_q, BU]$ and hence in fact an element of $\varprojlim[BGL_n\mathbb{F}_q, BU]$.

Lemma 3.2

$$[BGL\mathbb{F}_q^+, BU] = [BGL\mathbb{F}_q, BU] = \varprojlim[BGL_n\mathbb{F}_q, BU].$$

Proof: The first equality is clear because $\pi_1 BU = 0$. Hence, by the universal property of the plus construction [10], any map $BGLR \to BU$ factors uniquely through $BGLR^+$.

The second equality comes from Milnor's exact sequence:

$$0 \to \lim{}^1[BGL_n\mathbb{F}_q, U] \to [BGL\mathbb{F}_q, BU] \to \lim[BG_n\mathbb{F}_q, BU] \to 0$$

By a theorem of Atiyah [3], we know that for a finite group G, $K^1BG = 0$. Hence the $\lim^1$ term above vanishes, as desired.

Let $\theta : (BGL\mathbb{F}_q^+)^\wedge \to BU^\wedge$ be the map corresponding via Lemma 3.2 to the family of virtual characters χ_n, after completing at the prime 2. In what follows, everything will be completed at the prime 2 and we will take $BU^\wedge$ to be our fixed base space for the iterated pull-back construction, as in Chapter 2. Recall that the map $\hat{c} : BO^\wedge \to BU^\wedge$ is the complexification map.

Definition:

$$JK_q = P(\theta, \hat{c})$$

In other words, JK_q is the homotopy pull-back:

$$\begin{array}{ccc} JK_q & \longrightarrow & BO^\wedge \\ \big\downarrow & & \big\downarrow \hat{c} \\ BGL\mathbb{F}_q^{+\wedge} & \xrightarrow{\theta} & BU^\wedge. \end{array}$$

Claim 3.3 *JK_q is independent of the choice of embedding $i : \overline{\mathbb{F}}_q^\times \to \mathbb{C}$.*

Proof: Suppose i' is another such embedding and θ' the induced Brauer lift. Since i and i' are injective maps with a common image (roots of unity of order prime to p), then it is possible to define an automorphism $\phi : \overline{\mathbb{F}}_q^\times \to \overline{\mathbb{F}}_q^\times$ such that $i\phi = i'$. But we can restrict ϕ to an automorphism $\phi : \mathbb{F}_q^\times \to \mathbb{F}_q^\times$, which induces a weak equivalence $\Phi : BGL\mathbb{F}_q^\wedge \to BGL\mathbb{F}_q^\wedge$. Then clearly $\theta\Phi = \theta'$ and $P(\theta', \hat{c}) = P(\theta\Phi, \hat{c}) = P(\theta, \hat{c})$, as proved in Chapter 2. □

For brevity, we write $\Lambda = \mathbb{Z}_2^\times$. Let $\langle a_1, a_2, ..., a_n\rangle$ denote the closed subgroup generated by $a_1, a_2, \ldots, a_n$.

Proposition 3.4 *$JK_q \sim JK_{q'}$ over $BU^\wedge$ if and only if $\langle q, -1\rangle$ is equal to $\langle q', -1\rangle$.*

Lemma 3.5 *$(BGL\mathbb{F}_q^+)^\wedge$ is weakly equivalent to $(BGL\mathbb{F}_{q'}^+)^\wedge$ over $BU^\wedge$ if and only if $\langle q\rangle = \langle q'\rangle$.*

Proof: Suppose first of all that q and q' generate the same subgroup of $\mathbb{Z}_2^\times$. Then there must exist a sequence of integers k_n such that $\lim\limits_{n\to\infty} q^{k_n} = q'$. Now consider the subgroup generated by $q-1$ in Λ, a profinite, and hence compact, topological ring. Notice $(q-1)\Lambda$ is the continuous image of a compact group and hence compact and therefore closed in $\mathbb{Z}_2[[\Lambda]]$. Clearly $q^{k_n} - 1 \in (q-1)\Lambda$ for each n. Therefore, $q'-1 \in (q-1)\Lambda$. But the same argument shows that $q-1 \in (q'-1)\Lambda$. So there exist a unit u in Λ such that $q-1 = u(q'-1)$. Hence we have a commutative diagram:

$$\begin{array}{ccccc}
(BGL\mathbb{F}_{q'}^+)^\wedge & \longrightarrow & BU^\wedge & \xrightarrow{\Psi^{q'}-1} & BU^\wedge \\
\vdots & & \Big\| & & \Big\downarrow u \\
(BGL\mathbb{F}_q^+)^\wedge & \longrightarrow & BU^\wedge & \xrightarrow{\Psi^{q}-1} & BU^\wedge.
\end{array}$$

Since each row is a fiber sequence, the dotted arrow exists. In fact, this induced map is unique, for $[BGL\mathbb{F}_q^+, U] = [BGL\mathbb{F}_q, U] = \varprojlim [BGL_n\mathbb{F}_q, U] = 0$. As before, the first equality comes from the universal property for the plus construction. The second and third equalities come from Milnor's $\lim^1$ sequence and the fact that $[BG, U] = 0$ for all finite groups G. But in fact this means that u must be a homotopy equivalence. For, the map $(BGL\mathbb{F}_q^+)^\wedge \to (BGL\mathbb{F}_q^+)^\wedge$ induced by u^{-1} is also unique and the composition of the two must be the identity. Hence $(BGL\mathbb{F}_q^+)^\wedge$ is weakly equivalent to $(BGL\mathbb{F}_{q'}^+)^\wedge$ over BU, as desired.

To prove the converse, we need another lemma:

Lemma 3.6 *Suppose q and q' are odd integers, $|q| \neq 1, |q'| \neq 1$. Then $\langle q\rangle = \langle q'\rangle$ if and only if either*

1. *$q = q' = 1 \bmod 4$ and $\nu_2(q-1) = \nu_2(q'-1)$, or*

2. *if $q = q' = 3 \bmod 4$ and $\nu_2(q+1) = \nu_2(q'+1)$.*

(Here $\nu_2(m)$ is the highest power of 2 dividing m.)

Proof: Suppose first that $q = 1+2^k s$ and $q' = 1+2^k s'$ where $k \geq 2$ and s, s' are odd. For any integer m, $q^m \equiv 1 \bmod 2^k$. Fix $n \geq 2k$. For $m, m' \in \{1, 2, \ldots, 2^{n-k}\}, m \neq m'$, $q^m \neq q^{m'} \bmod 2^n$. But there are exactly 2^{n-k} elements of $\mathbb{Z}/2^n$ which are congruent to 1 mod 2^k namely, $\{1 + 2^k t : t = 0, 1, \ldots, 2^{n-k} - 1\}$. Hence

$$\{q^m : m = 1, 2, \ldots, 2^{n-k}\} = \{1 + 2^k t : t = 0, 1, \ldots, 2^{n-k} - 1\}$$

in $\mathbb{Z}/2^n$. But that means that there exists an integer m such that $q^m \equiv q' \bmod 2^n$. Hence q' is an element of the closed subgroup generated by q. Similarly, q is an element of the closed subgroup generated by q'.

On the other hand, suppose $q = -1 + 2^k s$ and $q' = -1 + 2^k s'$ where $k \geq 2$ and s, s' are odd. For a given integer m, $q^{2m} \equiv 1 \bmod 2^k$ and $q^{2m+1} \equiv -1 \bmod 2^k$. Fix $n \geq 2k$. For $m, m' \in \{1, 2, \ldots, 2^{n-k}\}, m \neq m'$, $q^m \neq q^{m'} \bmod 2^n$. So there are exactly 2^{n-k} distinct elements in the set $\{q^m : m = 1, 2, \ldots 2^{n-k}\}$. There are of course $2(2^{n-k})$ elements of $\mathbb{Z}/2^n$ which are congruent to $\pm 1 \bmod 2^k$. However, $q^m = (-1)^m + ms(2^k) + S_m$, where S_m is divisible by 2^{2k} and hence is an even multiple of 2^k. So q^m must lie in the set $\{1+(2j)2^k : j = 0, 1, \ldots 2^{n-k-1}\} \cup \{-1+(2j+1)2^k : j = 0, 1, \ldots, 2^{n-k-1}\}$ (considered as elements of $\mathbb{Z}/2^n$). So, as before the two sets coincide. But q' is in the latter set, so there exists an integer m such that $q^m \equiv q' \bmod 2^n$. Hence q' is an element of the closed subgroup generated by q. Similarly, q is an element of the closed subgroup generated by q'.

Suppose conversely that q and q' generate the same subgroup of $\mathbb{Z}_2$. They must then be equivalent mod 4. For if $q \equiv 1 \bmod 4$, then $q^m \equiv 1 \bmod 4$; hence, we must have $q' \equiv 1 \bmod 4$. Similarly, if $q' \equiv 1 \bmod 4$, then $q \equiv 1 \bmod 4$.

Now suppose $q \equiv q' \equiv 1 \bmod 4$. Let $q = 1 + 2^k s$, $q' = 1 + 2^{k'} s'$ for s, s' odd. If, for instance, $k > k'$, then $q^m \equiv 1 \bmod 2^k$ hence, $q^m \not\equiv q' \bmod 2^k$ for any m. Hence, in fact $k = k'$. Similarly, if $q \equiv q' \equiv 3 \bmod 4$, then $\nu_2(q+1) = \nu_2(q'+1)$. Hence Lemma 3.6 is proved. □

To prove the converse of Lemma 3.5, note that if $(BGL\mathbb{F}_q^+)^\wedge$ is weakly equivalent to $(BGL\mathbb{F}_{q'}^{+})^\wedge$ over $BU^\wedge$, then their fundamental groups must be equal, so $(\frac{\mathbf{Z}}{q-1})^\wedge = (\frac{\mathbf{Z}}{q'-1})^\wedge$. This means $q-1 = 2^k s$ and $q'-1 = 2^k s'$ with s, s' odd integers. If $k \geq 2$, then by Lemma 3.6, q and q' generate the same subgroup. On the other hand, if $k = 1$, then $q \equiv q' \equiv 3 \bmod 4$. Now look at π_3. This tells us that $(\frac{\mathbf{Z}}{q^2-1})^\wedge = (\frac{\mathbf{Z}}{q'^2-1})^\wedge$, which implies that $\nu_2(q+1) = \nu_2(q'+1)$, as desired. □

Lemma 3.7 *The subgroup* $\langle q, -1\rangle = \langle q', -1\rangle$ *if and only if* $\nu_2(q^2-1) = \nu_2(q'^2-1)$.

Proof: Suppose $\nu_2(q^2-1) = \nu_2(q'^2-1)$. Clearly $-1 \in \langle -1, q\rangle$. We may assume that $q \equiv q' \equiv 1 \bmod 4$, by replacing q or q' by $-q$ or $-q'$, if necessary. Write $q = 1+4r, q' = 1+4r'$. Then $q^2 - 1 = 8r(1+2r)$ and $q'^2-1 = 8r'(1+2r')$. Hence, $\nu_2(8r(1+2r)) = \nu_2(8r'(1+2r'))$, which implies that $\nu_2(r) = \nu_2(r')$. So $\nu_2(q-1) = \nu_2(q'-1)$ and $\langle q\rangle = \langle q'\rangle$, as desired.

On the other hand, suppose $\langle q, -1\rangle = \langle q', -1\rangle$. As before, we may assume that $q, q' \equiv 1 \bmod 4$. Since $q' \in \langle q, -1\rangle$, for each $k > 1$, there exist whole numbers n_k, m_k such that $q' \equiv -1^{m_k} q^{n_k} \bmod 2^k$. In fact, since $q, q' \equiv 1 \bmod 4$, m_k must be even for each k. Hence, we have $q' \in \langle q\rangle$. Hence, by a previous lemma, we may write $q = 1+2^k s$ and $q' = 1+2^k s'$ for s, s' odd and $k \geq 2$. But then $q^2-1 = 2^{k+1}s(1+2^{k-1}s)$ and $q'^2-1 = 2^{k+1}s'(1+2^{k-1}s')$. So, $\nu_2(q^2-1) = \nu_2(q'^2-1) = 2^{k+1}$. Hence, Lemma 3.7 is proved. □

Recall, we are trying to prove that $JK_q \sim JK_{q'}$ if and only if $\langle q, -1\rangle = \langle q', -1\rangle$.

Proof of Proposition 3.4: Suppose first of all that $\langle q, -1\rangle = \langle q', -1\rangle$. Then consider the diagram

$$\begin{array}{ccccc} JK_q & \longrightarrow & BO^\wedge & \xrightarrow{(\Psi^q-1)\hat{c}} & BU^\wedge \\ & & \Big\downarrow = & & \vdots \\ JK_{q'} & \longrightarrow & BO^\wedge & \xrightarrow{(\Psi^{q'}-1)\hat{c}} & BU^\wedge \end{array}$$

We wish to fill in the dotted arrow. Suppose first of all that $q \equiv q' \equiv 1 \bmod 4$. Then, as in the proof of Lemma 3.7, we in fact have that $\langle q\rangle = \langle q'\rangle$. But in this case we have a unit $u \in \mathbb{Z}_2^\times$ such that $u(\Psi^q - 1) = \Psi^{q'} - 1$, so we can take the dotted arrow to be u. But then we have in fact the following diagram:

$$\begin{array}{ccccc} JK_q & \longrightarrow & BO^\wedge & \xrightarrow{(\Psi^q-1)\hat{c}} & BU^\wedge \\ \vdots\, i & & \Big\downarrow = & & \Big\downarrow u \\ JK_{q'} & \longrightarrow & BO^\wedge & \xrightarrow{(\Psi^{q'}-1)\hat{c}} & BU \end{array}$$

Here, i exists because the bottom row is a fibration. In fact, both rows are fibrations and give rise to long exact sequences on homotopy; since the two vertical maps on the right are equivalences, i must also be an equivalence.

Now consider the case when either $q \not\equiv 1 \bmod 4$ or $q' \not\equiv 1 \bmod 4$. In fact, we can replace q with $-q$ or q' with $-q'$ as necessary. Clearly $\langle q, -1\rangle = \langle -q, -1\rangle$. It is also true that map $(\Psi^q - 1)\hat{c} = (\Psi^{-q} - 1)\hat{c}$, for any q. To see this, recall that in Chapter 2 we proved that given $\sigma \in$ Aut $\mathbb{C}$, then σ induces the map $\Psi^{\alpha(\sigma)} : BU^\wedge \to BU^\wedge$. Here, $\alpha :$ Aut $\mathbb{C} \to \mathbb{Z}_2^\times$ was defined in Chapter 2, and it is clear that α send complex conjugation to -1. Hence, $\Psi^{-1}\hat{c} = \hat{c}$, since complex conjugation is the identity when restricted to the real numbers embedded in the complex numbers in the usual way. Hence, we have proved the first half of the claim.

On the other hand, suppose that $JK_q \sim JK_{q'}$. Then $\pi_3(JK_q) = \pi_3(JK_{q'})$ and hence $\nu_2(q^2 - 1) = \nu_2(q'^2 - 1)$, as desired. □

As an example, consider the spaces JK_3 and JK_5. Note that $\langle 3\rangle \neq \langle 5\rangle$ because $5 \not\equiv 3 \bmod 4$. So $BGLF_3^{+\wedge} \not\cong BGLF_5^{+\wedge}$. On the other hand, $\langle 3, -1\rangle = \langle 5, -1\rangle$ since $\nu_2(3^2 - 1) = \nu_2(5^2 - 1)$. So, $JK_3 \cong JK_5$.

Now that we have classified JK_q by the subgroup $\langle q, -1\rangle$, we follow [14] in computing the homology of JK_q. By construction, JK_q fits into the fiber sequence:

$$U^\wedge \to JK_q \to BO^\wedge \stackrel{(\Psi^q-1)\hat{c}}{\to} BU^\wedge$$

Note that the sequence $U^\wedge \to JK_q \to BO^\wedge$ is pulled back from the sequence $U^\wedge \to BGL\mathbb{F}_q^{+\wedge} \to BU^\wedge$; since $\pi_1 BU^\wedge = 0$, the local coefficient systems of both fiber sequences are trivial. Thus, the Serre spectral sequence for the fiber sequence $U^\wedge \to JK_q \to BO^\wedge$ has

$$E^2_{p,q} = H_p(BO;\mathbb{Z}/2) \otimes H_q(U;\mathbb{Z}/2) = \mathbb{Z}/2[b_1, b_2, \ldots] \otimes \mathbb{Z}/2\langle x_1, x_2, \ldots\rangle$$

where $|b_i| = i$ and $|x_i| = 2i-1$. In fact, the Serre spectral sequence collapses. To see this, map $j : \mathbb{R}P^{\infty\wedge} \to BO^\wedge$ as the usual inclusion. Then b_i is in the image of H_*j for each i. But $(\Psi^q - 1)\hat{c}j$ is null homotopic, so j lifts to JK_q. Hence, each b_i is in the image of $JK_q \to BO^\wedge$, so the edge homorphism theorem implies that $d_n(b_n) = 0$ and thus $d_n \equiv 0$ and the spectral sequence collapses. So we have

$$E^\infty = \mathbb{Z}/2[b_1, b_2, \ldots] \otimes \mathbb{Z}/2\langle x_1, x_2, \ldots\rangle.$$

But in fact there is no extension problem, since the each exterior generator x_i resides in $E^\infty_{0,2i-1} \subseteq H_*(JK_q)$.

To compute the cohomology of JK_q, we can use the Eilenberg-Moore spectral sequence of the pullback square

$$\begin{array}{ccc} JK_q & \longrightarrow & BO^\wedge \\ \downarrow & & \downarrow \\ BGL\mathbb{F}_q^{+\wedge} & \longrightarrow & BU^\wedge. \end{array}$$

The Eilenberg-Moore spectral sequence collapses, since $BU^\wedge$ is simply connected and all of the spaces in the pullback square have finite type. In addition, $H^*BO^\wedge$ is a free module over $H^*BU^\wedge$, so in fact

$$H^*JK_q = H^*BGL\mathbb{F}_q^{+\wedge} \otimes_{H^*BU^\wedge} H^*BO^\wedge.$$

Dually, the homology Eilenberg-Moore spectral sequence shows that H_*JK_q is the cotensor product of $H_*BGL\mathbb{F}_q^{+\wedge}$ and $H_*BO^\wedge$ over $H_*BU^\wedge$. This implies that the map $H_*JK_q \to H_*BO^\wedge \otimes H_*BGL\mathbb{F}_q^{+\wedge}$ is injective.

To make computations about the homotopy of JK_q, we can use the Mayer-Vietoris sequence of the pull-back square defining JK_q, as well as the long exact sequence in homotopy derived from the fibrations $JK_q \to BO^\wedge$ and $JK_q \to BGL\mathbb{F}_q^{+\wedge}$. For example, such computations show that

$$\operatorname{rank}(\pi_n JK_q) = \begin{cases} 0 & n \equiv 0, 2, 3 \bmod 4 \\ 1 & n \equiv 1 \bmod 4. \end{cases}$$

Chapter 4

THE SPACES JK_QR

In chapter 3, we constructed JK_q, for q a power of an odd prime. If we fix $q \equiv \pm 3$ mod 8, then JK_q is the space $JK\mathbb{Z}$ mentioned in Chapter 1. Mitchell showed that the infinite loop map, $BGL\mathbb{Z}[\frac{1}{2}] \to JK\mathbb{Z}$, first constructed by Bökstedt, induces a split epimorphism of $\mathcal{A}$-Hopf algebras on $\mathbb{Z}/2$-homology.

In this chapter, we will construct the analogous spaces JK_qR as well as a map $BGLR^+ \to JK_qR$. In Chapter 6, we will show that for a suitable choice of q this map induces on homology an epimorphism of $\mathcal{A}$-Hopf algebras.

Choose an odd prime ideal P in $\mathcal{O}_F$. Let $\hat{R}_P$ be the P-adic completion of R; that is $\hat{R}_P = \varprojlim(R/P^n)$. Then $\hat{R}_P/P = R/P = \mathbb{F}_q$ where $q = p^r$ for some prime p. Let $\theta : BGL\mathbb{F}_q^{+\wedge} \to BU^\wedge$ be the Brauer map, discussed in chapter 3.

Given such a prime ideal P, we construct JK_qR by extending the ideas of the previous chapter. In fact, we will show in this chapter that the space JK_qR depends only on q and not on P. In Chapter 5, we will discuss the choice of P. We might first guess that a suitable definition of JK_qR would be the space we have already defined as JK_q; that is, the homotopy pull-back of the diagram:

$$\begin{array}{ccc} & & BO^\wedge \\ & & \Big\downarrow{\scriptstyle \hat{c}} \\ BGL\mathbb{F}_q^{+\wedge} & \xrightarrow[\theta]{} & BU^\wedge \end{array}$$

However, recall from Chapter 3 that the rank of $\pi_k(JK_q)$ is one for $k \equiv 1$ mod 4. From a theorem of Borel [5], however, we know that the rank of $\pi_k(BGLR^{+\wedge})$ is equal to n for $k \equiv 1$ mod 4. We would like JKR to share this property. Hence, we define

JKR as follows. Recall we have maps $\tau_i\hat{c} : BO^\wedge \to BU^\wedge$ for each $i = 1, 2, \ldots, n$ such that $\tau_i\hat{c} = \hat{c}$. Define

$$JK_qR = P(\theta, \tau_1\hat{c}, \tau_2\hat{c}, \ldots \tau_n\hat{c}) = P(\theta, \hat{c}, \hat{c}, \ldots, \hat{c}).$$

Notice that this is the homotopy pullback

$$\begin{array}{ccc} JK_qR & \longrightarrow & X_F \\ \downarrow & & \downarrow \\ BGL\mathbb{F}_q^{+\wedge} & \xrightarrow[\theta]{} & BU^\wedge. \end{array}$$

In fact,

$$JK_qR = P(\theta, \hat{c}) \times_{BU^\wedge} P(\underbrace{\hat{c}, \hat{c}, \ldots, \hat{c}}_{n-1}).$$

So we have the following pull-back diagram:

$$\begin{array}{ccc} ((U/O)^\wedge)^{n-1} & \cdots\cdots\to & ((U/O)^\wedge)^{n-1} \\ \downarrow & & \downarrow \\ JK_qR & \longrightarrow & P(\underbrace{\hat{c}, \hat{c}, \ldots, \hat{c}}_{n-1}) \\ g\downarrow & & \downarrow \\ P(\theta, \hat{c}) & \longrightarrow & BU^\wedge. \end{array}$$

Note that in fact g has a section s; namely, $s(x, y) = (x, y, y, \ldots, y)$. The composite $gs = 1$ and the spaces in the fibration

$$((U/O)^\wedge)^{n-1} \to JK_qR \to P(\theta, \hat{c})$$

are all H-spaces. Hence, in fact the long exact sequence of the fibration is split and

$$\begin{aligned} JK_qR &\sim ((U/O)^\wedge)^{n-1} \times P(\theta, \hat{c}) \\ &\sim ((U/O)^\wedge)^{n-1} \times JK_q. \end{aligned}$$

Recall that the Eilenberg-Moore spectral sequence of the pull-back square defining JK_q collapses, which implies that the map $H_*JK_q \to H_*BO^\wedge \otimes H_*BGL\mathbb{F}_q^{+\wedge}$ is injective. Using the same argument inductively, it is clear that the natural map

$$H_*JK_qR \to \underbrace{H_*BO^\wedge \otimes \cdots \otimes H_*BO^\wedge}_{n} \otimes H_*BGL\mathbb{F}_q^{+\wedge}$$

is injective.

We now wish to construct a map $BGLR^{+\wedge} \to JK_qR$. In Chapter 3, we constructed a map $BGLR^{+\wedge} \to X_F$. Now we consider the problem of mapping $BGLR^{+\wedge}$ to $BGL\mathbb{F}_q^+$ in such a way as to commute with the map $BGLR^{+\wedge} \to X_F$ over $BU^\wedge$. The reduction map $R \to \mathbb{F}_q$ induces a map $r : BGLR^{+\wedge} \to BGL\mathbb{F}_q^{+\wedge}$. Recall that $\hat{R}_P$ is the P-adic completion of R. Suslin [18] showed that the map $\hat{R}_P \to \mathbb{F}_q$ induces a homotopy equivalence $s : BGL\hat{R}_P^{+\wedge} \to BGL\mathbb{F}_q^{+\wedge}$. It is possible to choose an injective homomorphism $\iota : \hat{R}_P \hookrightarrow \mathbb{C}$ such that the diagram

$$\begin{array}{ccc} R & \longrightarrow & \mathbb{R} \\ \downarrow & & \downarrow \\ \hat{R}_P & \xrightarrow{\iota} & \mathbb{C} \end{array}$$

commutes. This induces a map $\iota : BGL\hat{R}_P^{+\wedge} \to BU^\wedge$, along with a commutative diagram:

$$\begin{array}{ccc} BGLR^{+\wedge} & \longrightarrow & BO^\wedge \\ \downarrow & & \downarrow \\ BGL\hat{R}_P^{+\wedge} & \xrightarrow{\iota} & BU^\wedge. \end{array}$$

Lemma 4.1 *The following diagram:*

$$\begin{array}{ccc} BGL\hat{R}_P^{+\wedge} & \xrightarrow{s} & BGL\mathbb{F}_q^{+\wedge} \\ & \searrow^{\iota} & \downarrow \theta \\ & & BU^\wedge \end{array}$$

commutes.

See [14] [13].

The lemma implies that the diagram

$$\begin{array}{ccc} BGLR^{+\wedge} & \xrightarrow{f_1} & BO^{\wedge} \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle c} \\ BGL\mathbb{F}_q^{+\wedge} & \xrightarrow{\theta} & BU^{\wedge} \end{array}$$

commutes. Let $\overline{\theta}$ and ψ be the maps defined by the diagram:

$$\begin{array}{ccc} BGL\mathbb{F}_q^{+\wedge} & \xrightarrow{\psi} & BGL\mathbb{F}_q^{+\wedge}(\theta) \\ {\scriptstyle \theta}\downarrow & \swarrow{\scriptstyle \overline{\theta}} & \\ BU^{\wedge} & & \end{array}$$

where ψ and $\overline{\theta}$ factor θ as an acyclic cofibration followed by a fibration. Then it follows from the above commutative diagrams that

$$\begin{array}{ccc} BGLR^{+\wedge} & \xrightarrow{\phi_k g f_k} & BO^{\wedge}(\tau_k \hat{c}) \\ {\scriptstyle \psi r}\downarrow & & \downarrow \\ BGL\mathbb{F}_q^{+\wedge}(\theta) & \longrightarrow & BU^{\wedge} \end{array}$$

commutes for each $k = 1, 2, \ldots, n$. Hence, we have a map $\Psi : BGLR^{+\wedge} \to JK_qR$, as desired. Note that Ψ is a (commutative) H-space map, so it induces a map of $\mathcal{A}$-Hopf algebras on homology.

Chapter 5

NUMBER THEORY RESULTS

The construction of JK_qR depends on the choice of a prime ideal P. In this chapter, we discuss this choice of prime ideal. First, we restrict our study to totally real fields F which satisfy three conditions, listed below. Given such a field F, we discuss the choice of P and discuss the natural maps $BR^\times \to BGL\mathbb{F}_q^+$ and $BR^\times \to BO$.

Suppose u is a unit in $\mathcal{O}_F^\times$. Call u *totally positive* if each $f_i(u)$ is positive. For the rest of this dissertation, we assume our fixed totally real field F satisfies the following three conditions:

(C1) There is a unique prime ideal, β, dividing two in $\mathcal{O}_F$, and this prime ideal is principal.

(C2) The only totally positive units in $\mathcal{O}_F^\times$ are the squares.

(C3) There is no 2-torsion in the class group of R.

Note that in fact conditions (C1) and (C3) imply:

(C3') There is no 2-torsion in the class group of $\mathcal{O}_F$.

To see this, consider the map $\phi : Cl(\mathcal{O}_F) \to Cl(R)$, where $Cl(-)$ denotes the class group. Let $[I]$ denote the equivalence class of the ideal I. Given $[J]$ in $Cl(R)$, $\phi([J \cap \mathcal{O}_F]) = [J]$; that is, ϕ is onto. On the other hand, suppose $\phi([J]) = 0$. Then $JR = (d)$ for some $d \in R$. In fact, we may choose $d \in \mathcal{O}_F$. Since $d = rj$ for some

$r \in R$ and $j \in J$, there is some non-negative integer k such that $2^k d \in J$, or $2^k d = JI$ for some ideal I in $\mathcal{O}_F$. So $[I] = [J]^{-1}$ and $IR = 2^k R = R$. Hence, $1 \in IR$ and in fact $2^j \in I$ for some $j \geq 0$. But then $I|(2^j)$; or, I is a product of primes which lie over 2. Hence $[J] = [I]^{-1} = [P_1^{a_1} \cdots P_i^{a_i}]$ where the P_i are primes over 2 and $a_i \geq 0$. So, in fact the kernel of ϕ is generated by the primes over 2. But by (C1), the only prime over 2 is principal; hence in fact ϕ is injective and therefore an isomorphism.

Although these conditions are fairly restrictive, there are interesting examples of such fields. For example, if ζ_{2^k} is a primitive root of unity, then the totally real field $E = \mathbb{Q}(\zeta_{2^k} + \zeta_{2^k}^{-1})$ is the maximal real subfield of $\mathbb{Q}(\zeta_{2^k})$. E satisfies conditions (C1) and (C3) [20]. It is also known that E satisfies (C2).

For instance, the maximal real subfield of $\mathbb{Q}(\zeta_8)$ is $\mathbb{Q}(\sqrt{2})$. In this case, $(\sqrt{2})$ is the unique prime ideal dividing 2, the class number is 1, and the fundamental unit is $1 + \sqrt{2}$.

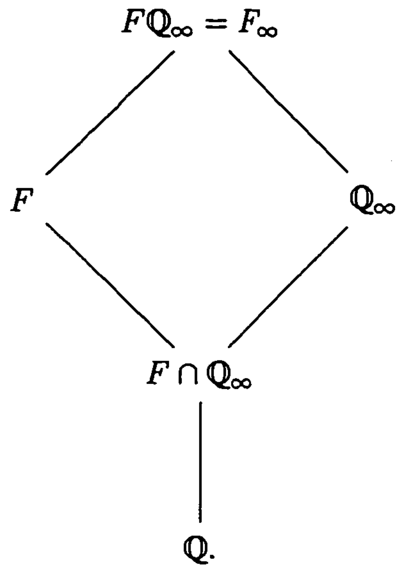
The embeddings $f_i : \mathcal{O}_F^\times \to \mathbb{R}$ yield a natural map

$$f : \mathcal{O}_F^\times/(\mathcal{O}_F^\times)^2 \to \Pi^n(\mathbb{R}^\times/(\mathbb{R}^\times)^2) = (\mathbb{Z}/2)^n.$$

Note that condition (C2) implies that f is injective. But $\mathcal{O}_F^\times \cong \mathbb{Z}/2 \times (\mathbb{Z})^{n-1}$, so $\mathcal{O}_F^\times/(\mathcal{O}_F^\times)^2 \cong (\mathbb{Z}/2)^n$. So f is an injective map of $\mathbb{Z}/2$ vector spaces of dimension n; hence, f is in fact an isomorphism.

Let b' be generator of β, the unique prime ideal dividing 2. By the surjectivity of f, it is possible to choose a unit u such that $f_i(u) < 0$ if and only if $f_i(b') < 0$ for $i = 1, 2, \ldots, n$. Then $b = b'u$ is a totally positive generator for β. For example, $\frac{\sqrt{2}}{1+\sqrt{2}}$ is a totally positive generator for $(\sqrt{2})$ in $\mathbb{Q}[\sqrt{2}]$.

We want to focus on prime ideals P in R such that b generates the 2-torsion in R/P. As a first step, we consider what we call "2-good" primes P. Let F_∞ be the field generated by F and the set $\{\zeta_{2^k}\}_{k>1}$. Similarly, define $\mathbb{Q}_\infty$ as the field that contains $\mathbb{Q}$ and every 2-powered root of unity. Consider the diagram



Since $Gal(\mathbb{Q}_\infty, \mathbb{Q}) = \mathbb{Z}_2^\times$, $Gal(F_\infty, F) = Gal(\mathbb{Q}_\infty, F \cap \mathbb{Q}_\infty)$ is a subgroup of $\mathbb{Z}_2^\times$. Let $\Gamma = Gal(F_\infty, F)$.

For every unramified prime ideal $P \subset \mathcal{O}_F$, it is possible to define the Frobenius element of Γ, σ_P. We work first at the finite level. Let $F \subset F_0 = F(\sqrt{-1}) \subset F_1 \subset \ldots \subset F_n \ldots \subset F_\infty$, where each $[F_n : F_{n-1}] = 2$ and $F_\infty = \cup F_n$. For a fixed n, let α_n be a prime ideal in $\mathcal{O}_{F_n}$ lying over P. Then let $\mathbb{F}_P = \mathcal{O}_F/P$ and similarly define $\mathbb{F}_{\alpha_n} = \mathcal{O}_{F_n}/\alpha_n$. Let $\Gamma_{P,n} = \{\sigma \in Gal(F_n, F) | \sigma(\alpha_n) = \alpha_n\}$. (Recall that for an abelian Galois group, this so-called "decomposition group" does not depend on the choice of α_n over P, so this notation is appropriate.) Then $\Gamma_{P,n} \cong Gal(\mathbb{F}_{\alpha_n}, \mathbb{F}_P)$ via the natural map $\eta : \Gamma_{P,n} \to Gal(\mathbb{F}_{\alpha_n}, \mathbb{F}_P)$. This maps sends a given $\sigma \in \Gamma_{P,n}$ to $\bar{\sigma}$, where $\bar{\sigma}([u]) = [\sigma(u)]$, for $u \in \mathcal{O}_{F_n}$. The Frobenius element of $Gal(F_n, F)$, $\sigma_{P,n}$ is defined to be $\eta^{-1}(\sigma)$, where σ is the Frobenius in $Gal(\mathbb{F}_{\alpha_n}, \mathbb{F}_P)$. Again, it it true that because $Gal(F_n, F)$ is abelian, the Frobenius element is independent of the choice of α_n.

Clearly we can map each $Gal(F_n, F)$ to $Gal(F_{n-1}, F)$ by sending τ to $\tau|_{F_{n-1}}$. In fact the Frobenius elements $\sigma_{P,n}$ are compatible under these mappings, and so give an element of $\Gamma = \varprojlim Gal(F_n, F)$; namely, $\sigma_P = \{\sigma_{P,n}\}$.

If $a_1, a_2, \ldots, a_n$ are elements of Γ, let $\langle a_1, a_2, \ldots, a_n \rangle$ be the closed subgroup of Γ generated by $a_1, a_2 \ldots a_n$.

Definition: *A prime ideal P of $\mathcal{O}_F$ is* 2-good *if P is unramified in F_∞/F and $\langle \sigma_P, c \rangle = \Gamma$, where c is complex conjugation.*

For example, a prime ideal (p) of $\mathbb{Z}$ is 2-good exactly when $p \cong \pm 3 \bmod 8$. We show this in two steps. First, we show that (3) is 2-good, which is equivalent to showing that each $(\mathbb{Z}/2^k)^\times$ is generated by -1 and 3. If $k = 1$ or $k = 2$, this is easy to verify.

If $k \geq 3$, consider the homomorphism $\mathbb{Z}/2 \times \mathbb{Z}/2^{k-2} \to (\mathbb{Z}/2^k)^\times$ given by sending $(1,0)$ to -1 and $(0,1)$ to 3. Since these are both finite groups, we need only show this map is injective to show that $(\mathbb{Z}/2^k)^\times$ is generated by -1 and 3. Suppose $(-1)^a 3^b \equiv 1 \bmod 2^k$ where $a \in \{0,1\}$ and $b \in \{0, 1, 2, \ldots, 2^{k-2} - 1\}$. Assume first that $a = 0$. If $b \neq 0$, $3^b - 1 = (3-1)(1 + 3 + 3^2 + \cdots 3^{b-1}) \equiv 0 \bmod 2^k$. One can show inductively that

$$\nu_2(1 + 3 + 3^2 + \cdots + 3^r) \begin{cases} < k+1 & \text{if } r < 2^k - 1 \\ = k+1 & \text{if } r = 2^k - 1 \end{cases}$$

But this forces $b > 2^{k-2} - 1$. Hence, in fact we must have $b = 0$. On the other hand, suppose $a = 1$. Then $3^b \equiv -1 \bmod 2^k$. But this means $3^{2b} \equiv 1 \bmod 2^k$. The calculation above show that in fact $3^{2^{k-2}} = 1 \bmod 2^k$. So $3^{2b'} \equiv 1 \bmod 2^k$, where $b' = b$ if $2b < 2^{k-2}$ and $b' = b - 2^{k-2}$ otherwise. Then $0 \leq 2b' \leq 2^{k-2} - 1$. But this means $b' = 0$; hence, in fact $b = 0$. But clearly, $3^0 \not\equiv -1 \bmod 2^k$; hence in fact, $a \neq 1$. This proves that -1 and 3 do indeed generate $(\mathbb{Z}/2^k)^\times$, as claimed.

Secondly, note that $\nu_2(3^2 - 1) = 3$, so by a theorem proved in chapter 3, (p) is 2-good if and only if $\nu_2(p^2 - 1) = 3$. Let $p = k + 8s$, where $k \in \{-3, -1, 1, 3\}, s \in \mathbb{Z}$. Then $p^2 - 1 = (k^2 - 1) + 16s(k + 4s)$. If $k = \pm 3$, then $p^2 - 1 = 8(1 + 2s(k + 4s))$, so

$\nu_2(p^2-1)=3$. On the other hand, if $k=\pm 1$, $p^2-1=16s(k+4s)$, so $\nu_2(p^2-1)\geq 3$. So, (p) is 2-good if and only if $p\equiv \pm 3 \bmod 8$.

Let $\mathbf{E} = (R/P)^\times_{(2)}$ denote the localization of $(R/P)^\times$ at the prime 2. Since $(R/P) = (\mathcal{O}_F)/P$, then $\mathbf{E} = (\mathcal{O}_F/P)^\times_{(2)}$.

Theorem 5.1 *If P is a 2-good ideal, then b generates $(R/P)^\times_{(2)}$.*

Theorem 5.2 *There are infinitely many 2-good primes P.*

For example, consider the case of $F=\mathbb{Q}$. In this case, $b=2$. If p is a prime, then 2 generates the 2-torsion in $\mathbb{F}_p^\times$ if and only if 2 is not a square in $\mathbb{F}_p^\times$. But it is a well-known number theory result that 2 is not a square mod p if and only if $p\equiv\pm 3$ mod 8. Hence, for $F=\mathbb{Q}$, Theorems 5.1 and 5.2 clearly hold.

Before we prove Theorem 5.1, we prove Theorem 5.2. Theorem 5.2 is a consequence of the following:

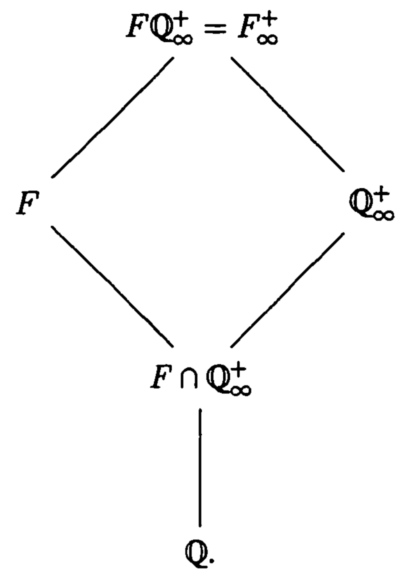
Cebotarev Density Theorem: Let E/E_0 be an abelian Galois extension with group G. Let $g\in G$. Then there are infinitely prime ideals P in $\mathcal{O}_{E_0}$ such that $g=\sigma_P$.

In fact, Cebotarev's result is stronger. He explicitly calculated the "density" of such primes and proved the result for non-abelian extensions as well, by replacing g and σ_P with their conjugacy classes. But the above version is sufficient for our purposes.

Proof that Theorem 5.2 follows from Cebotarev:

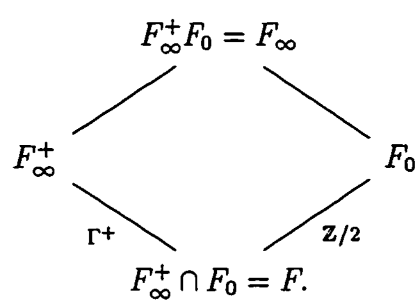
We have already shown that $\Gamma\subset\mathbb{Z}_2^\times$. For a given subfield K of $\mathbb{C}$, let $K^+ = K\cap\mathbb{R}$.

Let $\Gamma^+ = Gal(F_\infty^+, F)$. Then we have the diagram:

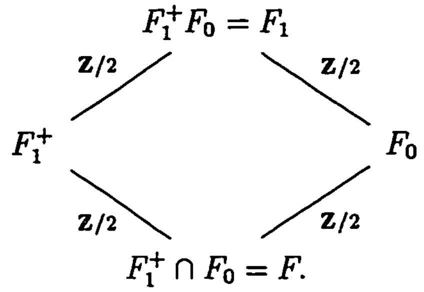


Then a similar argument to the one above show $\Gamma^+ \subseteq Gal(\mathbb{Q}_\infty^+, \mathbb{Q}) = \mathbb{Z}_2$. (Here we mean the additive subgroup $\mathbb{Z}_2$.) In fact, it is clear that Γ^+ is a non-trivial subgroup, because $[F \cap \mathbb{Q}_\infty^+, \mathbb{Q}]$ is finite. Thus, $\Gamma^+ \cong \mathbb{Z}_2$.

We can see by the following diagram, that $\Gamma = \Gamma^+ \times \mathbb{Z}/2$:



Clearly, complex conjugation, c, generates $Gal(F_0, F)$. Let g be the generator of $Gal(F_1, F_0)$. The following diagram makes it clear that $Gal(F_1, F) = \mathbb{Z}/2 \times \mathbb{Z}/2$ and is generated by c and g:



By Cebotarev's theorem, there are infinitely many P such that $\sigma_{P,1} = g$. Fix one such P. Then σ_P and c topologically generate Γ. Indeed, given any two elements, x, y in $\mathbb{Z}_2 \times \mathbb{Z}/2$, then x, y are topological generators, if and only if x, y generate $\mathbb{Z}_2/2 \times \mathbb{Z}/2$. This proves Theorem 5.2. □

Similarly, Cebotarev's Theorem implies that the analogue of Theorem 5.2 holds for l-good primes, where l is odd. In this case, we define P to be l-good if it is unramified and if σ_P topologically generates $Gal(F_\infty, F)$. In the case of l odd, Theorem 5.2 holds not just for totally real number fields, but for arbitrary number fields.

Proof of Theorem 5.1: Consider the extension F_1 of F. (Recall $[F_1 : F] = 4$.) Let K be the maximal real subfield of this extension, so $[K : F] = 2$ and $Gal(F_1, K)$ is generated by c. Thus, we can write $K = F(\sqrt{d})$ for some $d \in F^\times$.

Claim 5.3 *In fact we can take $d = b$.*

Proof: Write $(d) = \Pi_\lambda \lambda^{a_\lambda}$ for prime ideals λ in $\mathcal{O}_F$ and $a_\lambda \in \mathbb{Z}$. Notice that if $\lambda \nmid 2$, then λ is unramified in F_1/F and hence unramified in K/F. But in this case, a_λ must be even. To see this, fix one such λ. Let D be the ring of integers $\mathcal{O}_F$, completed with respect to the λ-valuation ν_λ. Let L be the field of fractions of D. D is a discrete valuation ring, and the unique maximal ideal (also the unique prime ideal), λ, is principal; say, $\lambda = (\pi)$. The field extension $L(\sqrt{d})/L$ is unramified, and hence $(\pi)\mathcal{O}_{L(\sqrt{d})}$ is maximal. But $(\sqrt{d}) = (\pi)^k$ and hence $d = (\pi)^{2k}$ in $\mathcal{O}_{L(\sqrt{d})}$. One the other hand, $d = (\pi)^{a_\lambda}$, so in fact $a_\lambda = 2k$, as desired.

Since the only prime ideal dividing (2) is (b), we may in fact write $(d) = (b)^e I^2$, where I is an ideal of $\mathcal{O}_F$. In fact, I^2 must be principal, since (d) and (b) are. But since there is no 2-torsion in the class group of $\mathcal{O}_F$, then I is principal. So, $(d) = (b)^e(\gamma)^2$ for some $\gamma \in \mathcal{O}_F^\times$. So in fact $d = b^e\gamma^2\eta$, where $\eta \in \mathcal{O}_F^\times$.

Note that in fact K is a totally real field. Hence d must be totally positive. But b is also totally positive; therefore, η must be. But that means $\eta = \varepsilon^2$ for some $\varepsilon \in \mathcal{O}_F^\times$. So, in fact we can take $d = b^e$. Clearly, e must be odd, and we can take $d = b$, as claimed. □

Now consider the tower

$$F \hookrightarrow F(\sqrt{b}) \hookrightarrow F_1.$$

Notice, by an argument given previously, we need only show that σ_P, c generate $Gal(F_1, F)$; then σ_P, c topologically generate Γ. But clearly σ_P, c generate $Gal(F_1, F)$ if and only if σ_P generates $Gal(F(\sqrt{b}), F)$, since c generates $Gal(F_1, F(\sqrt{b}))$.

Note that an unramified prime P is inert in $F(\sqrt{b})$ if and only if σ_P generates $Gal(F(\sqrt{b}), F) = \mathbb{Z}/2$. For in a quadratic extension, if P splits, it splits completely, in which case σ_P is trivial. Otherwise, σ_P must in fact be non-trivial, and hence generate $Gal(F(\sqrt{b}), F)$.

On the other hand, b generates $(\mathcal{O}_F/P)^\times_{(2)}$ if and only if b is not a square in $(\mathcal{O}_F/P)^\times$. For, $(\mathcal{O}_F/P)^\times \cong \mathbb{Z}/2^k \times \mathbb{Z}/r$ where $k \geq 1$ and r is odd. But b is a non-square in the units $\mathcal{O}_F/P$ if and only if its image in $\mathbb{Z}/2^k \times \mathbb{Z}/r$ is not divisible by 2. But since all elements in $\mathbb{Z}/r$ are divisible by 2, b is a non-square if and only if its image in $\mathbb{Z}/2^k$ is not divisible by 2 if and only if its image generates $\mathbb{Z}/2^k = (\mathcal{O}_F/P)^\times_{(2)}$.

So we will be done if we can show that P is inert in $F(\sqrt{b})$ if and only if $b \notin$

$((\mathcal{O}_F/P)^\times)^2$. Hence, we are done if we prove the following proposition:

Proposition 5.4 *Let P be a prime ideal of $\mathcal{O}_F$ which is unramified in $\mathcal{O}_K$, $K = F(\sqrt{b})$. P is inert if and only if $b \notin ((\mathcal{O}_F/P)^\times)^2$.*

Lemma 5.5 *The ring of algebraic integers in K, $\mathcal{O}_K$, is equal to $\mathcal{O}_F[\sqrt{b}]$.*

We will first prove the proposition, assuming the lemma. Then we prove the lemma.

Proof of Proposition 5.4: Let P in $\mathcal{O}_F$ be an unramified prime ideal; $P \neq (b)$, since b is clearly ramified. Let Q be a prime ideal in $\mathcal{O}_K$ containing P. Since P is unramified, then either P splits completely and $[\mathcal{O}_K/Q : \mathcal{O}_F/P] = 1$ or P is inert, $Q = P$, and $[\mathcal{O}_K/P : \mathcal{O}_F/P] = 2$. The inclusion map $\mathcal{O}_F \hookrightarrow \mathcal{O}_K$ induces a map $\eta : (\mathcal{O}_F/P)^\times \to (\mathcal{O}_K/Q)^\times$. Since $P \neq (b)$, $b \in (\mathcal{O}_F/P)^\times$. If P splits, then η must in fact be an isomorphism. But that means that $(\eta^{-1}(\sqrt{b}))^2 = b$ in $(\mathcal{O}_F/P)^\times$, so b is a square. Hence, if b is not a square, then P must be inert. On the other hand, suppose P is inert. Then η is not onto, which means that $\sqrt{b} + P$ is not in the image of η. Suppose by way of contradiction, that b is a square in $(\mathcal{O}_F/P)^\times$. Then there exists $r \in \mathcal{O}_F$ such that $r^2 - b \in P$. So $(r - \sqrt{b})(r + \sqrt{b}) \subseteq P$ in $\mathcal{O}_k$. Hence, either $r - \sqrt{b} \in P$ or $r + \sqrt{b} \in P$. In the first case $\eta(r + P) = \sqrt{b} + P$; in the second $\eta(-r + P) = \sqrt{b} + P$. Hence if P is inert, b must in fact be a non-square, as desired. □

Proof of Lemma 5.5: Let $N = N_{K/F}$ be the norm in K/F and $Tr = Tr_{K/F}$ be the trace in K/F. Given $\gamma \in K$, γ satisfies the polynomial $p(x) = x^2 - Tr(\gamma)x + N(\gamma)$. Thus, $\gamma \in \mathcal{O}_K$ if and only if $Tr(\gamma), N(\gamma) \in \mathcal{O}_F$. We can write $\gamma = r + s\sqrt{b}$ where $r, s \in F$. Assume $\gamma \in \mathcal{O}_K$. We will show that in fact this means $r, s \in \mathcal{O}_F$. First, note that $N(\gamma) = r^2 - bs^2 \in \mathcal{O}_F$ and $Tr(\gamma) = 2r \in \mathcal{O}_F$. Thus, $4(r^2 - bs^2) \in \mathcal{O}_F$, which implies that $m = b(2s)^2 \in \mathcal{O}_F$. But (b) is a prime ideal and hence $(2s)^2$ must be in $\mathcal{O}_F$, since b could not "cancel" any factors in the denominator of s^2. So in fact, $2s \in \mathcal{O}_F$.

Write $r = \frac{c}{2}$ and $s = \frac{d}{2}$ for $c, d \in \mathcal{O}_F$. Then $c^2 - bd^2 = 4g$, for some $g \in \mathcal{O}_F$. But $b|4$, so $b|c^2$, which implies that in fact $b|c$.

Write $c = bc'$. Then $b^2c'^2 - bd^2 = 4g = b^{2k}g'$ for some $g' \in \mathcal{O}_F$ and some k such that $(b)^k = (2)$. So indeed, b must divide d^2 and hence, $b|d$. If $k = 1$, then clearly $r, s \in \mathcal{O}_F$. Otherwise, we write $d = bd'$. Then we have $b^2c'^2 - b^3d'^2 = b^{2k}g'$, or $c'^2 - bd'^2 = b^{2k-2}g'$. Repeating the above procedure as necessary, we eventually see that in fact $b^k|c$ and $b^k|d$. Hence, $r, s \in \mathcal{O}_F$, as desired. $\square$

For the rest of the paper, we fix a 2-good prime ideal P with $R/P = \mathbb{F}_q$. Then we may write JKR for JK_qR, since Proposition 3.4 of Chapter 3 shows if P and P' are 2-good prime ideals, then the associated spaces JK_qR and $JK_{q'}R$ are weakly equivalent over $BU^\wedge$.

Chapter 6

THE SPACE $BR^{\times}$

In this chapter we will consider the natural map $BR^{\times} \to BGL\mathbb{F}_q^{+\wedge}$ and the maps $(BR^{\times})^{\wedge} \hookrightarrow BGLR^{+\wedge} \overset{gf_i}{\to} BO^{\wedge}$. In particular, we will compute the corresponding maps on homology. We will also consider some primitive elements in $S(H_*(BR^{\times})^{\wedge})$. These primitive elements will play an important role in proving that Ψ, defined in chapter 2, is a homology epimorphism. For the rest of the this chapter we will usually drop the $(-)^{\wedge}$ notation, since for the spaces X we consider, $H_*(X) = H_*(X;\mathbb{Z}/2) \cong H_*(X^{\wedge}) = H_*(X^{\wedge};\mathbb{Z}/2)$.

Fix now a 2-good prime ideal P, and let q be such that $R/P = \mathbb{F}_q$. Recall that F satisfies the conditions (C1), (C2), and (C3), and that $f_1, f_2, \ldots, f_n$ are the real embeddings of F. Recall as well that β is the unique prime ideal dividing 2 and that b is a totally positive generator for β.

Maps from $BR^{\times}$ to $BGL\mathbb{F}_q^{+}$:

For brevity, let $U = \mathcal{O}_F^{\times}$. We wish to consider the image of U under each f_i, but only up to sign. Recall that we have defined a group homomorphism, in fact an isomorphism, $f : U/U^2 \to \Pi^n \mathbb{R}^{\times}/(\mathbb{R}^{\times})^2 = (\mathbb{Z}/2)^n$. For the present, use additive notation for $\mathbb{Z}/2$; that is, the i-th component of $f([u])$ is 0 if $f_i(u) > 0$ and 1 if $f_i(u) < 0$, for each $u \in U$. This will be convenient for what follows.

By a theorem of Dirichlet, we know we can choose a set of fundamental units for U, i.e. we can choose $\tilde{\varepsilon}_1, \ldots \tilde{\varepsilon}_{n-1}$ such that every $u \in U$ can be written uniquely in the form $u = (-1)^{\lambda}\tilde{\varepsilon}_1^{i_1} \cdots \tilde{\varepsilon}_{n-1}^{i_{n-1}}$, for $\lambda \in \mathbb{Z}/2$ and $i_k \in Z$. Clearly U/U^2 is an n-dimensional $\mathbb{Z}/2$-vector space and each $\bar{u} \in U/U^2$ can be written uniquely as $\bar{u} = (-1)^{i_0}\bar{\varepsilon}_1^{i_1} \cdots \bar{\varepsilon}_{n-1}^{i_{n-1}}$, for $i_k \in \mathbb{Z}/2$. Thus we can represent each $\bar{u}$ as a vector

$(i_0, i_1, \ldots, i_{n-1})$ with respect to the basis $\{-1, \tilde{\varepsilon}_1, \cdots, \tilde{\varepsilon}_{n-1}\}$. Let $\bar{f}_i$ be the i-th component of f. The map f is an isomorphism and can be represented, with respect to this basis for U/U^2 and the standard basis for $(\mathbb{Z}/2)^n$, by the $n \times n$ matrix M:

$$\begin{pmatrix} 1 & \bar{f}_1(\tilde{\varepsilon}_1) & \bar{f}_1(\tilde{\varepsilon}_2) & \ldots & \bar{f}_1(\tilde{\varepsilon}_{n-1}) \\ 1 & \bar{f}_2(\tilde{\varepsilon}_1) & \bar{f}_2(\tilde{\varepsilon}_2) & \ldots & \bar{f}_2(\tilde{\varepsilon}_{n-1}) \\ 1 & \bar{f}_3(\tilde{\varepsilon}_1) & \bar{f}_3(\tilde{\varepsilon}_2) & \ldots & \bar{f}_3(\tilde{\varepsilon}_{n-1}) \\ & & \cdot & & \\ & & \cdot & & \\ & & \cdot & & \\ 1 & \bar{f}_n(\tilde{\varepsilon}_1) & \bar{f}_n(\tilde{\varepsilon}_2) & \ldots & \bar{f}_n(\tilde{\varepsilon}_{n-1}) \end{pmatrix}$$

Since f is an isomorphism, this matrix is invertible. In particular, we can make a change of basis for U/U^2 in such a way that we can represent f by another invertible matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 1 & 0 & \ldots & 0 \\ \cdot & & \cdot & & & \\ \cdot & & & \cdot & & \\ \cdot & & & & \cdot & \\ 1 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}$$

Call this new basis $(-1, \varepsilon_1, \ldots, \varepsilon_{n-1})$. (Note that in fact we keep -1 as our first basis element, since we did not change the first column of the matrix.) Note that each ε_i is positive. Further note that $f_i(\varepsilon_k) > 0$ for $i \neq k+1$ and that $f_{k+1}(\varepsilon_k) < 0$ for $1 \leq k \leq n-1$.

Let $q - 1 = 2^k r$, where 2 is relatively prime to r, and consider the composite $s : R^\times \to \mathbb{F}_q^\times = \mathbb{Z}/(q-1) = \mathbb{Z}/(2^k) \times \mathbb{Z}/r \xrightarrow{\pi} \mathbb{Z}/(2^k)$, where the first arrow is reduction mod P and π is projection onto the first factor. Since b generates the

2-torsion in $(\mathbb{F}_q^\times)_{(2)}$, $s(\varepsilon_i) = s(b)^{n_i}$ for some n_i, for $i = 1, 2, \ldots, n-1$. If we replace each ε_i with $\varepsilon_i b^{-n_i}$, then $s(\varepsilon_i) = 1$. Since b is totally positive, it still holds that $f_i(\varepsilon_k) > 0$ for $i \neq k+1$ and that $f_{k+1}(\varepsilon_k) < 0$ for $1 \leq k \leq n-1$.

Clearly, any unit in $R^\times$ can be written uniquely as $u = (-1)^i b^{m_0} \varepsilon_1^{m_1} \varepsilon_2^{m_2} \cdots \varepsilon_{n-1}^{m_{n-1}}$, where $i = 0, 1$ and $m_i \epsilon \mathbb{Z}$. Hence, $R^\times = \{\pm 1\} \times \mathbb{Z}b \times \mathbb{Z}\varepsilon_1 \times \ldots \times \mathbb{Z}\varepsilon_{n-1}$ and $H_*BR^\times = H_*\mathbb{R}P^\infty \otimes H_*S^1 \otimes H_*S^1 \otimes \cdots \otimes H_*S^1$ (where there are n H_*S^1 terms). Let b_i be the non-zero element of $H_i\mathbb{R}P^\infty$, for $i \geq 0$ ($b_0 = 1$). Let e_i be the exterior generator of the i-th H_*S^1 term.

Let $\varphi : BR^\times \to BGL\mathbb{F}_q^+$ be the map induced by

$$R^\times \to \mathbb{F}_q^\times = GL_1\mathbb{F}_q \hookrightarrow GL\mathbb{F}_q.$$

Notice that the map $\varphi_* : H_*(BR^\times) \to H_*BGL\mathbb{F}_q^+$ factors as $H_*BR^\times \xrightarrow{\varphi'} H_*B\mathbb{Z}/2^k \to H_*BGL\mathbb{F}_q^+$. Here, as above, $q - 1 = 2^k r$ where r is odd; then $H_*B\mathbb{F}_q^\times \cong H_*B\mathbb{Z}/2^k \otimes H_*B\mathbb{Z}/r \cong H_*B\mathbb{Z}/2^k$ since $\tilde{H}_*B\mathbb{Z}/r \cong \tilde{H}_*\mathbb{Z}/r = 0$. In fact, by [15], the map $S'(H_*B\mathbb{Z}/2^k) \to H_*BGL\mathbb{F}_q^+$ is an isomorphism. Here $S'(V)$ is the strict symmetric algebra on the graded vector space V. That is, $S'(V) = S(V)/I$ where $I \subset S(V)$ is the ideal generated by all a^2 such that the degree of a is odd.

Let $d_i \epsilon H_i B(\mathbb{Z}/2^k)$ be the non-zero element in each dimension. We consider two cases. **Case 1** ($k = 1$): In this case, $S'(\varphi_*)$ restricted to the $\mathbb{R}P^\infty$ factor is clearly an isomorphism, so $\varphi_*(b_i) = d_i$ for all i. On the other hand, we know that b generates the 2-torsion in $\mathbb{F}_q^\times$, so $\pi_1(\varphi)(b) \neq 0$. Thus, $H_1(\varphi)(B\mathbb{Z}b) \neq 0$ which forces $\varphi_*(e_1) = d_1$, the unique non-zero element in $H_1B\mathbb{Z}/2^k$. Recall that we chose ε_i such that $s(\varepsilon_i) = 1$; hence, $\varphi_*(e_i) = 0$ for $2 \leq i \leq n$. **Case 2** ($k > 1$): It is well known that $\mathbb{Z}/2 \to \mathbb{Z}/2^k$ induces on homology a map which is an isomorphism in even degrees and the zero map in odd degrees. So $\varphi_*(b_{2i}) = d_{2i}$ and $\varphi_*(b_{2i+1}) = 0$. On the other hand, the same argument as above shows that $\varphi_*(e_1) = d_1$ and $\varphi_*(e_i) = 0$ for $2 \leq i \leq n$.

Maps from $BR^\times$ to BO:

Recall the maps $gf_i : BGLR^+ \to BO$ defined in Chapter 2. Let gf_i also denote the pre-composition of this map with the inclusion $BR^\times \hookrightarrow BGLR^+$. Now we are ready to compute the map induced by each gf_i on homology. Since $\mathbb{Z}/2 \hookrightarrow R^\times \to O(1)$ is the unique isomorphism which induces $gf_j|_{B\mathbb{Z}/2}$ for each j, we may use b_i to denote both the non-zero element in $H_i\mathbb{R}P^\infty$ and its image under gf_{k*} in $H_*BO^\wedge = \mathbb{Z}/2[b_1, b_2, \ldots.]$. To compute $gf_{k*}(e_i)$, first note that we can extend the map $f : U \to (\mathbb{Z}/2)^n$ to $R^\times$. That is, if $i : U \to R^\times$ is the usual inclusion, then there exists an $\tilde{f} : R^\times \to (\mathbb{Z}/2)^n$ such that $i\tilde{f} = f$. In fact, $\tilde{f}$ is given by the same rule as f; namely, the i-th component of $f(u)$ is 0 if $f_i(u) > 0$ and 1 if $f_i(u) < 0$, for each $u \in U$. Then for $1 \le j \le n$, $\pi_1(gf_j)(u) = \tilde{f}_j(u)$, the j-th component of $\tilde{f}(u)$. Hence, $\pi_1 gf_j(b) = 0$ for each j and $\pi_1 gf_j(\varepsilon_i) = 0$ for $i \neq j-1$ while $\pi_1 gf_{i+1}(\varepsilon_i) \neq 0$.

Since $\pi_1(BR^\times)$ is abelian, $\pi_1(gf_j) = H_1(gf_j;\mathbb{Z})$. The Universal Coefficients Theorem now implies that the following diagram commutes:

$$\begin{array}{ccc} H_1(BR^\times;\mathbb{Z})\otimes\mathbb{Z}/2 & \xrightarrow{H_1(gf_j;\mathbb{Z})\otimes 1_{\mathbb{Z}/2}} & H_1(BO;\mathbb{Z}/2) \\ \downarrow \cong & & \downarrow \cong \\ H_1(BR^\times;\mathbb{Z}/2) & \xrightarrow{H_1(gf_j;\mathbb{Z}/2)} & H_1(BO;\mathbb{Z})\otimes\mathbb{Z}/2 \end{array}$$

Let $1 \le j \le n$. Then, $gf_{1*}(e_l) = gf_{j*}(e_l) = 0$ for $j \neq l$ while $gf_{j*}(e_j) \neq 0$, for $j \neq 1$. Hence, $gf_{j*}(e_j) = b_1$ for $j > 1$.

Primitives in $S(H_*BR^\times)$:

Recall that $\tilde{H}_*(BR^\times)$ is a bicommutative Hopf algebra since $BR^\times$ is a homotopy commutative H-space. We can extend the coproduct on $\tilde{H}_*(BR^\times)$ to $S(H_*BR^\times)$, and in fact the latter is a Hopf algebra. We will use $*$ to denote the product in the former space and juxtaposition to denote the product in the latter.

Let x_0 be the basepoint of S^1; then define

$$\begin{array}{rcl} \mathbb{R}P^\infty \times S^1 \times \cdots \times S^1 & \xrightarrow{\pi_0} & \mathbb{R}P^\infty \times S^1 \times \cdots \times S^1 \\ (a_0, a_1, \ldots a_n) & \mapsto & (a_0, x_0, \ldots x_0) \\ \mathbb{R}P^\infty \times S^1 \times \cdots \times S^1 & \xrightarrow{\pi_l} & \mathbb{R}P^\infty \times S^1 \times \cdots \times S^1 \\ (a_0, a_1, \ldots a_n) & \mapsto & (a_0, x_0, x_0, \ldots, x_0, a_l, x_0 \ldots x_0) \end{array}$$

for $1 \le l \le n$. By abuse of notation, let $\pi_0 = S(\pi_0)_*$, $\pi_l = S(\pi_l)_*$. Recall that e_i is the generator of the i-th H_*S^1 factor. Let $p_{m,l} = (\pi_l \boxplus \chi\pi_0)(b_m * e_l)$. (Here $\boxplus$ and χ refer to the Hopf sum and inverse, respectively.)

It is easy to show that $\chi\pi_0(b_l)$ can be found inductively from the formulas $\chi\pi_0(b_0) = b_0 = 1$ and $\chi\pi_0(b_l) = b_l + b_{l-1}(\chi\pi_0(b_1)) + \ldots + b_1(\chi\pi_0(b_{l-1}))$ and that

$$p_{m,t} = \sum_{i+j=m} (b_i * e_t)(\chi\pi_0 b_j).$$

We wish to replace this recursive formula with one that is more straightforward; to do so we need some notation.

Let $I_j = \{i_1, i_2, \ldots, i_n\}$ be a partition of j; that is, $j = i_1 + i_2 + \cdots i_n$ and each i_k is a positive integer. Let $p(j)$ be the set of partitions of j. Define $\alpha(I_j)$ to be the number of ways to order the set $I_j = \{i_1, i_2, \ldots, i_n\}$. For example, if $j = 4$ and $I_4 = \{1, 1, 2\}$ then $\alpha(I_4) = 3$ (namely, $(1,1,2), (1,2,1), (2,1,1)$). Finally, let $b_{I_k} = b_{i_1} b_{i_2} \cdots b_{i_n}$.

Claim 6.1

$$\chi\pi_0 b_k = \sum_{I_k \epsilon p(k)} \alpha(I_k) b_{I_k}, \quad k \ge 1.$$

Proof: The proof is by induction on k. The case $k = 1$ is clear, for in this case the equation in question becomes $b_1 = b_1$. Suppose the induction hypothesis holds for $1, 2, \ldots, k$. Let $A = \sum_{I_{k+1} \in p(k+1)} \alpha(I_{k+1}) b_{I_{k+1}}$ and $B = \chi\pi_0 b_{k+1}$. By the induction hypothesis,

$$B = b_{k+1} + b_k\Big(\sum_{I_1 \in p(1)} \alpha(I_1) b_{I_1}\Big) + \cdots + b_1\Big(\sum_{I_k \in p(k)} \alpha(I_k) b_{I_k}\Big).$$

Note first that b_{k+1} occurs exactly once in A and exactly once in B. Now fix $I_{k+1} = \{i_1, i_2, \ldots, i_n\}$, a partition of $k+1$ such that $n > 1$. Note that the term $b_{I_{k+1}}$ occurs with multiplicity $\alpha(I_{k+1})$ times in A. On the other hand, note that

$$\{i_1, i_2, \ldots, i_{j-1}, i_{j+1}, \ldots, i_n\} \in p(k+1-i_j)$$

for $j = 1, 2, \ldots, n$. For a given $i_j \in I_{K+1}$,

$$b_{i_j}(b_{\{i_1,\ldots i_{j-1}, i_{j+1},\ldots i_n\}})$$

occurs in B

$$\alpha(I_{\{i_1,\ldots i_{j-1}, i_{j+1},\ldots i_n\}})$$

times. Hence, being careful to count each occurrence just once, $b_{I_{k+1}}$ occurs with multiplicity

$$\sum_{i_j \notin \{i_1,\ldots,i_{j-1}\}} \alpha(\{i_1, i_2, \ldots, i_{j-1}, i_{j+1}, \ldots, i_n\})$$

in B. But observe that for each i_j, there are exactly $\alpha(\{i_1, i_2, \ldots, i_{j-1}, i_{j+1}, \ldots, i_n\})$ possible ways to order the set I_{k+1} if we require that i_j be in the first slot. So, again being careful to count each possibility only once,

$$\alpha(I_{k+1}) = \sum_{i_j \notin \{i_1,\ldots,i_{j-1}\}} \alpha(\{i_1, i_2, \ldots, i_{j-1}, i_{j+1}, \ldots, i_n\}).$$

Hence, $b_{I_{k+1}}$ occurs in A and B with the same multiplicity. Further, it is clear that we have accounted for every term on both sides of the equality above. □

Finally, we wish to show that the $p_{m,t}$ are indeed primitives. Fix t, $1 \leq t \leq n$ and m a non-negative integer. Then $p_{m,t} = (\pi_t \boxplus \chi\pi_0)(b_m * e_t)$. Let

$$C = (\mathbb{Z}/2)(b_0) \oplus \bigoplus_j (\mathbb{Z}/2)(b_j * e_t)$$

be the the coalgebra with one non-zero element in each degree ($b_j * e_t$ in degree $j+1$ and b_0 in degree 0) and the trivial coalgebra structure; i.e., each $b_j * e_t$ is primitive. Let e_t be the generator of H_*S^1. Then the obvious projection map $p : H_*(\mathbb{R}P^\infty \times S^1) \to C$, is, in fact, a map of coalgebras. For

$$\begin{aligned} p \otimes p(\Delta(b_j * e_t)) &= p \otimes p(\sum_{r+s=j} ((b_r * e_t) \otimes b_s) + (b_r \otimes (b_s * e_t))) \\ &= (b_j * e_t) \otimes 1 + 1 \otimes (b_j * e_t) \\ &= \Delta(p(b_j * e_t)) \end{aligned}$$

In addition, $p \otimes p(\Delta b_j) = 0$, for $j \neq 0$.

Also note that $\pi_t(b_i) = \pi_0(b_i)$, so $(\pi_t \boxplus \chi\pi_0)(b_i) = 0$ while $(\pi_t \boxplus \chi\pi_0)(b_i * e_t) \equiv b_i * e_t$ mod decomposables. So, in fact, the map $\pi_t \boxplus \chi\pi_0$ factors through C:

$$\begin{array}{ccc} H_*\mathbb{R}P^\infty \times S^1 & \xrightarrow{\pi_t \boxplus \chi\pi_0} & S(H_*BR^\times) \\ {\scriptstyle p}\downarrow & \nearrow & \\ C & & \end{array}$$

Hence, in fact $p_{m,t} = (\pi_t \boxplus \chi\pi_0)(b_m * e_t)$ is primitive. Furthermore, for a fixed non-negative integer m, there is a unique non-zero primitive which is a polynomial in the b_i of degree m. Let p_m denote this primitive element of $S(H_*BR^\times)$.

Chapter 7

THE HOMOLOGY OF JKR AND $BGLR^+$

In this chapter, we compute the homology of JKR and prove that $H_*(\Psi)$ is an epimorphism of $\mathcal{A}$-Hopf algebras. To do so, we will work with the primitives in $S(H_*BR^\times)$ which we identified in chapter 6, and we will consider their images in H_*JKR.

Images of the Primitives in $H_*BO \otimes \cdots \otimes H_*BO \otimes H_*BGL\mathbb{F}_q^+$:

Recall from the definition of $p_{m,l}$ for $l = 1, \ldots, n$, that $p_{m,l}$ is congruent to $b_m * e_l$ mod decomposables. In Chapter 6, we saw that $(gf_j)_*(e_l) = 0$ for $j = 1$ or $j \neq l$. Since $(gf_j)_*$ is in fact a Hopf algebra map with respect to the $*$ multiplication, then

$$S(gf_{j*})(p_{2m,l}) = 0$$

for $j = 1$ or $j \neq l$. On the other hand, for $j \neq 1$,

$$S(gf_{j*})(p_{2m,j}) \equiv b_{2m} * b_1 = b_{2m+1}$$

mod decomposables.

Similarly, $\varphi(p_{2m,1}) \equiv d_{2m+1}$ mod decomposables and $\varphi(p_{2m,l}) = 0$ for $l \neq 1$.

Define

$$\tilde{\Psi} = (gf_1, \ldots gf_n, \varphi) : BR^\times \to \underbrace{BO \times BO \times \cdots \times BO}_{n} \times BGL\mathbb{F}_q^+.$$

Then the map $S(\tilde{\Psi}_*)$ factors as

$$S(H_*BR^\times) \xrightarrow{i_*} H_*BGLR^+ \xrightarrow{\Psi_*} H_*JKR \to H_*BO \otimes \cdots \otimes H_*BO \otimes H_*BGL\mathbb{F}_q^+.$$

Recall from Chapter 4 that the last map is known to be injective.

To simplify notation, let $b_{i,l}$ denote $1 \otimes \cdots \otimes 1 \otimes b_i \otimes 1 \cdots \otimes 1$, where the b_i falls in the l-th slot. For simplicity, abuse notation and let d_i denote $1 \otimes \cdots \otimes 1 \otimes d_i$.

From the calculations above, we see that

$$S(\tilde{\Psi}_*)(p_{2m,1}) \equiv d_{2m+1}$$

and

$$S(\tilde{\Psi}_*)(p_{2m,l}) \equiv b_{2m+1,l}$$

mod decomposables for $2 \leq l \leq n$. Recall that p_n denotes the n-th primitive in $H_*BO = S(H_*\mathbb{R}P^\infty)$. Recall that $p_{2m+1} \equiv b_{2m+1}$ mod decomposables. If $\mathbb{F}_q^\times = \mathbb{Z}/2 \times \mathbb{Z}/r$ for r odd, then

$$S(\tilde{\Psi}_*)(p_{2m+1}) \equiv b_{2m+1,1} + b_{2m+1,2} + \cdots + b_{2m+1,n} + d_{2m+1}$$

mod decomposables. Otherwise,

$$S(\tilde{\Psi}_*)(p_{2m+1}) \equiv b_{2m+1,1} + b_{2m+1,2} + \cdots + b_{2m+1,n}$$

mod decomposables.

Let $\mathcal{B}$ be the subalgebra in $S(H_*BR^\times)$ generated by all the b_m and all the $p_{2m,l}$, for $m \geq 0$ and $l = 1, 2, \ldots n$. Thus

$$\mathcal{B} = \mathbb{Z}/2[b_m; m \geq 1] \otimes \mathbb{Z}/2[p_{2m,1}; m \geq 0] \otimes \cdots \otimes \mathbb{Z}/2[p_{2m,n}; m \geq 0].$$

Let $\tilde{\mathcal{B}} = \mathcal{B}/(p_{2m,1}^2)$ and let $\pi : \mathcal{B} \to \tilde{\mathcal{B}}$ be the obvious projection map. Then

$$\tilde{\mathcal{B}} = \mathbb{Z}/2[b_m; m \geq 1] \otimes \mathbb{Z}/2\langle p_{2m,1}; m \geq 0\rangle \otimes \mathbb{Z}/2[p_{2m,2}; m \geq 0] \otimes \cdots \otimes \mathbb{Z}/2[p_{2m,n}; m \geq 0].$$

Now we are ready to state the main theorem.

Theorem 7.1 *Let F be a totally real field which satisfies conditions (C1), (C2), and (C3) of chapter 5. Let $\Psi : BGLR^{+\wedge} \to JKR$ be the map defined in chapter 4. Then $H_*\Psi$ is an epimorphism.*

Note that $H_*\Psi$ is in fact an $\mathcal{A}$-Hopf-algebra map. To prove the theorem, we will show that $H_*JKR \cong \tilde{\mathcal{B}}$.

Note that in fact, $(\tilde{\Psi}(p_{2m,1}))^2 = 0$. To see, this note that

$$S(\tilde{\Psi}_*)(p_{2m,1}) = S(\varphi_*)(p_{2m,1}) = S(\varphi_*)(\sum_{i+j=m}(b_i * e_1)(\chi\pi_\emptyset b_j)).$$

But each $\varphi_*(b_i * e_1)$ clearly has square zero. Hence, $S(\tilde{\Psi}_*)$ factors through $\tilde{\mathcal{B}}$; say $S(\tilde{\Psi}_*) = \psi\pi$.

Claim 7.2 *The map ψ is injective.*

Proof: Recall that a map on Hopf algebras is injective if and only if it is injective on the primitives. We first identify the primitives in $\tilde{\mathcal{B}}$ and then show ψ is injective on the primitives.

In order to identify the primitives, it is helpful to break down $\tilde{\mathcal{B}}$ into smaller pieces. In fact, the following is true:

Claim 7.3 *Let H_1 and H_2 be connected graded algebras of finite type over a field K. Then*

$$P(H_1 \otimes H_2) \cong P(H_1) \oplus P(H_2)$$

where $P(H)$ denotes the submodule of primitives in a given algebra H.

Proof of Claim 7.3: We prove the dual statement: $Q(H_1^*) \oplus Q(H_2^*) \cong Q((H_1 \otimes H_2)^*)$ where $Q(C)$ denotes the module of indecomposable elements of a connected graded co-algebra C. To simplify notation, let $A = H_1^*$ and $B = H_2^*$. For a graded co-algebra C, let $\overline{C} = \oplus_{n>0} C_n$. Then $Q(C) = \overline{C}/\overline{C}^2$. Consider the map

$$F : \frac{\overline{A}}{\overline{A}^2} \oplus \frac{\overline{B}}{\overline{B}^2} \to \frac{\overline{A \otimes B}}{\overline{A \otimes B}^2}$$

$$(a, b) \mapsto a \otimes 1 + 1 \otimes b$$

We wish to show that F is an isomorphism. (We are using the fact that because of the finite type hypothesis, $(H_1 \otimes H_2)^* \cong H_1^* \otimes H_2^*$.)

Onto: Let $\sum_i a_i \otimes b_i \in \overline{A \otimes B}$. We may assume that each a_i and b_i is homogeneous. Suppose for a given k that $|a_k||b_k| > 0$. Then $a_k \otimes b_k = (a_k \otimes 1)(1 \otimes b_k) \in \overline{A \otimes B}^2$. So $\sum_i a_i \otimes b_i = \sum_j a_j \otimes k_j + \sum_l k_l \otimes b_l \mod \overline{A \otimes B}^2$, where each $k_i \in K$. So $\sum_i a_i \otimes b_i = F(\sum_j (a_j k_j), \sum_l (k_l b_l))$. Hence, the map is onto.

One-to-one: Consider the composition

$$\overline{A \otimes B} \hookrightarrow A \otimes B \to \frac{A}{\overline{A}^2} \otimes \frac{B}{\overline{B}} \cong \frac{A}{\overline{A}^2}$$

and similarly

$$\overline{A \otimes B} \hookrightarrow A \otimes B \to \frac{A}{\overline{A}} \otimes \frac{B}{\overline{B}^2} \cong \frac{B}{\overline{B}^2}.$$

Note that $\overline{A \otimes B}^2$ is in the kernel of each of these maps. To see this, consider the first map $\overline{A \otimes B} \to A/\overline{A}^2$. Note that a typical element of $\overline{A \otimes B}^2$ is of the form $(\sum_i a_i' \otimes b_i')(\sum_j a_j'' \otimes b_j'') = \sum_{i,j} (-1)^{|a_j''||b_i'|} a_i' a_j'' \otimes b_i' b_j''$. We may assume that the a_i', a_j'' are homogeneous elements of A and similarly for b_i', b_j''. Either $|a_i'| > 0$ or $|b_i'| > 0$ for each i. Similarly, either $|a_j''| > 0$ or $|b_j''| > 0$ for each j. Fix i, j. If either $|b_i'| > 0$ or $|b_j''| > 0$, then $b_i' b_j'' \in \overline{B}$, and this term goes to zero under the above map. Otherwise, both $|a_i'|$ and $|a_j''|$ must be greater than zero, so $a_i' a_j'' \in \overline{A}^2$, and the term still goes to zero. Thus we see that $\overline{A \otimes B}^2$ is in the kernel of the first map. The proof that it is also in the kernel of the second map is almost identical.

Hence, we have a map

$$\frac{\overline{A \otimes B}}{\overline{A \otimes B}^2} \to \frac{A}{\overline{A}^2} \oplus \frac{B}{\overline{B}^2}$$

such that the composition

$$\frac{\overline{A}}{\overline{A}^2} \oplus \frac{\overline{B}}{\overline{B}^2} \xrightarrow{F} \frac{\overline{A \otimes B}}{\overline{A \otimes B}^2} \to \frac{A}{\overline{A}^2} \oplus \frac{B}{\overline{B}^2}$$

sends $([a], [b]) \to ([a], [b])$; hence, F is one-to-one, as desired. □

Hence it suffices to identify the primitives in each component of $\tilde{\mathcal{B}}$. In fact, the primitives in the first component, $H_* BO$, are known to be the $\mathbb{Z}/2$ vector space generated by $\{p_{2m+1}^{2^i} | m, i \geq 0\}$. In the second component, the primitives consist of the the $\mathbb{Z}/2$ vector space generated by $\{p_{2m,1} | m \geq 0\}$. In the last $n-2$ components, the primitives are precisely the $\mathbb{Z}/2$ vector space generated by $\{p_{2m,k}^{2^i} | m, i \geq 0, k = 2, \ldots n\}$.

Now we show that ψ is injective on the primitives. Consider first a non-zero primitive of odd degree, say degree $2l+1$. If p is a primitive of odd degree, $p = \lambda p_{2l+1} + p_{2l,i_1} + p_{2l,i_2} + \cdots + p_{2l,i_t}$ for $1 \leq i_1 < i_2 < \cdots < i_t \leq n$ and $\lambda = 0$ or $\lambda = 1$. The image of p must be primitive; that is,

$$\begin{aligned} \psi(p) &\in P(H_*BO \otimes \cdots \otimes H_*BO \otimes H_*BGL\mathbb{F}_q^+) \\ &= P(H_*BO) \oplus \cdots \oplus P(H_*BO) \oplus P(H_*BGL\mathbb{F}_q^+). \end{aligned}$$

Let

$$\rho : P(H_*BO) \oplus \cdots \oplus P(H_*BO) \oplus P(H_*BGL\mathbb{F}_q^+) \to P(H_*BO) \oplus \cdots \oplus P(H_*BO)$$

be the obvious projection map. Then $\rho\psi(p)$, mod decomposables, is given by

$$\rho\psi(p) \equiv \begin{cases} b_{2l+1,i_1} + \cdots + b_{2l+1,i_t} & i_1 > 1, \lambda = 0 \\ b_{2l+1,i_2} + \cdots + b_{2l+1,i_t} & i_1 = 1, \lambda = 0 \\ \sum_{i \in N(p)} b_{2l+1,i} & \lambda = 1 \end{cases}$$

where $N(p) = \{1, 2, \ldots, n\} - (\{2, 3, \ldots, n\} \cap \{i_1, i_2, \ldots, i_t\})$. Then if $p \neq p_{2l,1}$ is a primitive of degree $2l+1$, $\rho\psi(p) \neq 0$; hence, $\psi(p) \neq 0$. But we have already calculated that $\psi(p_{2l,1}) \equiv d_{2l+1} \neq 0$ mod decomposables. On the other hand, suppose p is a non-zero primitive of degree $2^k(2l+1)$ for $k > 0$. Then $p = \lambda p_{2l+1}^{2^k} + p_{2l,i_1}^{2^k} + p_{2l,i_2}^{2^k} + \cdots + p_{2l,i_t}^{2^k}$ for $2 \leq i_1 < i_2 < \cdots < i_t \leq n, \lambda = 0$, or $\lambda = 1$. But then $p = (\lambda p_{2l+1} + p_{2l,i_1} + p_{2l,i_2} + \cdots + p_{2l,i_t})^{2^k}$, so $\psi(p) \neq 0$. Hence, ψ is injective on the primitives, as claimed. But this proves Claim 7.2. □

Isomorphism of Vector Spaces:

Note that $\psi(\tilde{\mathcal{B}}) \cong \tilde{\Psi}\mathcal{B}$.

Claim 7.4 $\psi(\tilde{\mathcal{B}}) \cong H_*JKR$.

Proof: First, note that $(\psi(\tilde{\mathcal{B}})) \subseteq H_*JKR$. But note also that *as vector spaces over* $\mathbb{Z}/2$:

$$\begin{aligned} H_*JKR &\cong H^*JKR \cong \underbrace{H^*BO \otimes_{H^*BU} \cdots \otimes_{H^*BU} H^*BO}_{n} \otimes_{H^*BU} H^*BGL\mathbb{F}_q^+ \\ &\cong \underbrace{\mathbb{Z}/2[w_i] \otimes_{\mathbb{Z}/2[c_i]} \cdots \otimes_{\mathbb{Z}/2[c_i]} \mathbb{Z}/2[w_i]}_{n} \otimes_{\mathbb{Z}/2[c_i]} (\mathbb{Z}/2[c_i] \otimes \mathbb{Z}/2\langle z_{2i+1}\rangle) \\ &\cong \mathbb{Z}/2[w_i] \otimes \underbrace{\mathbb{Z}/2\langle w_i\rangle \otimes \cdots \otimes \mathbb{Z}/2\langle w_i\rangle}_{n-1} \otimes \mathbb{Z}/2\langle z_{2i+1}\rangle \end{aligned}$$

where $|w_i| = |w_i'| = i$; $|z_{2i+1}| = 2i+1$. But $\mathbb{Z}/2\langle w_1, w_2, \ldots\rangle$ is isomorphic to $\mathbb{Z}/2[s_1, s_3, s_5, \ldots]$, $|s_{2i-1}| = 2i-1$, as a vector space. To see this, consider a monomial of the form

$$s_{j_1}^{n_1} s_{j_2}^{n_2} \cdots s_{j_k}^{n_k}.$$

Note that any positive integer n can be written uniquely as $n = 2^{l_1} + 2^{l_2} + \cdots + 2^{l_t}$. Hence, we can write this monomial in a unique way in the form

$$s_{j_1}^{2^{l_{11}}} s_{j_1}^{2^{l_{12}}} \cdots s_{j_1}^{2^{l_{1t}}} \cdots s_{j_k}^{2^{l_{k1}}} s_{j_k}^{2^{l_{k2}}} \cdots s_{j_k}^{2^{l_{kt'}}}.$$

Now this corresponds to a monomial in the exterior algebra via $w_{2^k i} \leftrightarrow s_i^{2^k}$; that is, the above monomial in $s_{j_1}, s_{j_2}, \ldots, s_{j_k}$ is sent to

$$w_{j_1 2^{l_{11}}} w_{j_1 2^{l_{12}}} \cdots w_{j_1 2^{l_{1t}}} \cdots w_{j_k 2^{l_{k1}}} \cdots w_{j_k 2^{l_{kt'}}}.$$

This is clearly a nonzero element of the exterior algebra. To see this, note that any positive integer j can be written uniquely in the form $j = (2^k)i$, where $k \geq 0$ and i is an odd positive integer. Hence, each w_n in the above expression is distinct from the others: each j_i, $i = 1, \ldots, k$ is distinct and for a fixed j_i, each $l_{j_i m}$ is distinct. It is clear that this correspondence is injective and surjective, and thus induces an isomorphism of graded vector spaces, as desired.

So now, we clearly have

$$H_*JKR \cong \tilde{\mathcal{B}} \cong \psi(\tilde{\mathcal{B}})$$

as vector spaces. But the map ψ is in fact a Hopf-algebra map. This proves Claim 7.4.

But $\psi(\tilde{\mathcal{B}}) \cong \tilde{\Psi}\mathcal{B} \cong (\Psi \circ i)\mathcal{B}$, since H_*JKR injects into $H_*BO \otimes \cdots \otimes H_*BO \otimes H_*BGL\mathbb{F}_q^+$. Hence, $\Psi \circ i$ must be onto; therefore, Ψ must be a surjective map, proving Theorem 7.1. □

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