The Debye Mechanism in Superconductors

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Abstract

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This thesis will study a new mechanism of charge transport in various types of superconductors, the Debye mechanism. This mechanism gives rise to a conductivity that is controlled by the *inelastic* relaxation time, τ_{in}. Conventionally, the conductivity is controlled by the *elastic* relaxation time, τ_{el}, which is typically much smaller than the inelastic relaxation time. Thus, the Debye mechanism can provide the dominant contribution to the conductivity. This thesis will introduce the Debye mechanism and how it contributes to the linear and non-linear conductivities in bulk S-wave superconductors, in both the clean and dirty limits and at high and low temperatures, the linear conductivity of bulk D-wave superconductors with and without impurity scattering, and at high and low temperatures, the flux flow conductivity in type-II superconductors, and the linear conductivity of non-centrosymmetric superconductors.

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Chapter 1

INTRODUCTION

The Debye mechanism was first studied by Peter Debye in 1928 in the context of dilute molecular gasses [11]. In this thesis I will explain the Debye mechanism and show how it arises to give a new contribution to the conductivity in a variety of superconductors. First we will start by giving a brief introduction to Debye's theory of dilute molecular gasses and then give a brief introduction on how this mechanism arises in several different types of superconductors.

1.1 Debye's Theory of Dilute Molecular Gasses

Debye wrote down his theory for dilute molecular gasses in 1928 [11]. The Debye mechanism arises in a dilute molecular gas under application of a monochromatic electric field $\boldsymbol{E}$ with frequency ω such that $\omega \ll \Delta E$, where ΔE is the level spacing between the ground state and the first excited state of a single molecule. In this case, the energy levels change as a function of time t due to the oscillating electric field, but there are no electric-field-induced transitons between the energy levels. Thus, in the absence of inelastic relaxation, the molecules occupying these energy levels move in energy space, creating a non-equilibrium distribution δn (See Fig. 1.1). This non-equilibrium distribution can only relax via *inelastic* relaxation.

We can see that the conductivity is proportional to the *inelastic* relaxation time τ_{in} by a simple consideration. Let us write down the entropy production due to this process, which can be related to the conductivity via the heat generated due to Joule heating:

$$T\dot{S} = \sigma E^2. \tag{1.1}$$

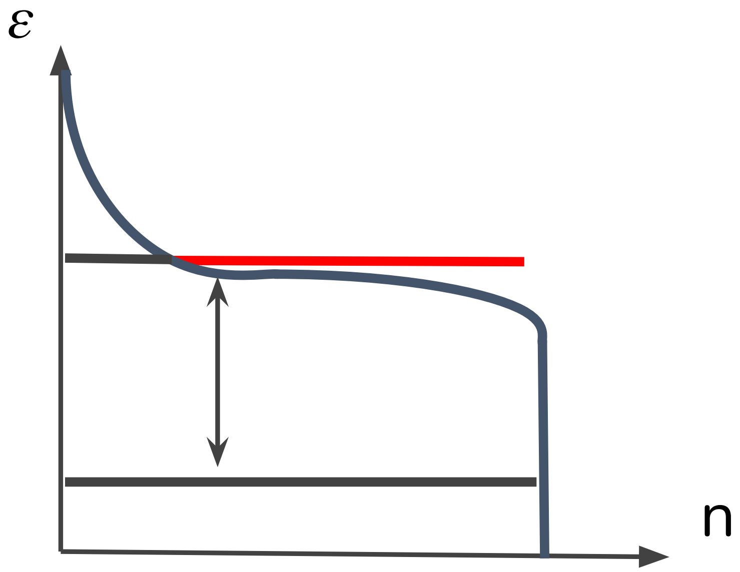

Figure 1.1: Schematic plot of a single occupied energy level moving in energy space, generating a non-equilibrium distribution (red line), whcih can only relax via inelastic scattering.

The entropy production can be estimated as

$$T\dot{S} \sim T\frac{(\delta n)^2}{\tau_{in}} \tag{1.2}$$

and as $\delta n \sim \tau_{in}$ we have

$$\sigma_{DB} \sim \tau_{in} \tag{1.3}$$

This is significant, as the conventional contribution to the conductivity in many systems, the Drude conductivity, is controlled not by the inelastic relaxation time τ_{in}, but by the elastic relaxation time τ_{el}, which is typically much smaller than the inelastic relaxation time $\tau_{in} \gg \tau_{el}$. As such the ratio between the Debye conductivity σ_{DB} and the Drude conductivity can be very large:

$$\frac{\sigma_{DB}}{\sigma_D} \sim \frac{\tau_{in}}{\tau_{el}} \gg 1 \tag{1.4}$$

Thus, the Debye conductivity can give the dominant contribution to the conductivity, and is important to take into account when considering dissipative proccesses.

1.2 How it Arises in s- and d- wave supercondutors with inversion symmetry

Now let us consider the theory of microwave absorption in superconductors. In linear response to the microwave field $\boldsymbol{E}(t) = \boldsymbol{E}_\omega \cos(\omega t)$, and in the limit of low frequencies ω, the current density in a superconductor may be written as

$$\mathbf{j} = \frac{e}{m} N_\mathrm{s}\, \boldsymbol{p}_\mathrm{s} + \sigma \mathbf{E}. \tag{1.5}$$

Here N_s is the superfluid density, e and m are, respectively, the charge and the mass of the electron, and the superfluid momentum is defined by $\boldsymbol{p}_s = \frac{\hbar}{2} \boldsymbol{\nabla} \chi - \frac{e}{c} \boldsymbol{A}$, with χ being the phase of the order parameter, and $\boldsymbol{A}$ the vector potential. The second term in Eq. (1.5), characterized by the conductivity σ, represents the dissipative part of the current.

The microwave absorption coefficient is controlled by the conductivity σ. The value of σ is determined by the quasiparticle scattering processes in the superconductor, which are generally characterized by two relaxation times: elastic, τ_el, and inelastic, τ_in, ones. In a typical situation, which we assume below, $\tau_\mathrm{in} \gg \tau_\mathrm{el}$. The theory of transport phenomena in conventional superconductors was developed long ago, see for example [51, 36, 3, 45, 41]. The conventional result is that the conductivity, and consequently the microwave absorption coefficient, are proportional to the elastic relaxation time τ_el. For example, at temperatures T near the critical temperature T_c the conductivity of a superconductor σ is [3, 45]

$$\sigma = \sigma_\mathrm{D} \left(1 + \frac{\Delta}{2T} \ln \frac{\Delta}{\omega} \right), \tag{1.6}$$

where σ_D is Drude conductivity of a normal metal [1]. Measurements of the microwave absorption in s-wave superconductors in the absence of an applied *dc* supercurrent generally agree with the aforementioned theory [24, 27]. However, there is a decisive lack of experimental literature on measurements of the microwave absorption in the presence of an applied *dc*

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[1] Equation (1.6) applies at sufficiently high frequencies

supercurrent. The recent paper by Santavicca et. al. [49] shows that their dependence of the microwave absorption in an s-wave superconductor on an applied *dc* supercurrent is not described by the conventional theory, as their measurements are several orders of magnitude larger than the conventional theory would suggest. We believe this dependence of the microwave absorption coefficient on an applied *dc* supercurrent could be described by our new contribution discussed below.

In this article we discuss another contribution to the conductivity, σ_{DB}, that is proportional to the inelastic relaxation time τ_{in}. As a result it may exceed the conventional contribution by orders of magnitude. This contribution to the linear conductivity exists only in the presence of a *dc* supercurrent. Furthermore, this contribution is strongly anisotropic and depends on the relative orientation between $\boldsymbol{E}_\omega$ and the supercurrent. Even in situations where this contribution is small in comparison to the conventional result, it determines the dependence of the conductivity on both the magnitude and direction of the *dc* supercurrent. This enables determination of τ_{in}, which is difficult to measure by other methods.

In the case of the nonlinear conductivity, a contribution proportional to τ_{in} exists even at zero *dc* supercurrent. Consequently, the nonlinear threshold for microwave absorption turns out to be anomalously low.

The physical mechanism of this contribution to the conductivity is similar to the Debye mechanism of microwave absorption in gases [11], Mandelstam-Leontovich mechanism of the second viscosity in liquids [31], and Pollak-Geballe mechanism of microwave absorption in the hopping conductivity regime [47]. It can be qualitatively understood as follows. Let us separate the superfluid momentum $\boldsymbol{p}_{\mathrm{s}}(t) = \bar{\boldsymbol{p}}_{\mathrm{s}} + \delta\boldsymbol{p}_{\mathrm{s}}(t)$ into a *dc* part $\bar{\boldsymbol{p}}_{\mathrm{s}}$ and the *ac* part $\delta\boldsymbol{p}_{\mathrm{s}}(t)$ whose time evolution is determined by the microwave field

$$\delta\dot{\boldsymbol{p}}_s(t) = e\mathbf{E}(t). \tag{1.7}$$

Below we assume $\hbar\omega \ll \Delta$, where Δ is the pairing gap. In this regime the quasiparticles may be characterized by the instantaneous energy spectrum. Furthermore, for $\omega \ll \tau_{\mathrm{el}}^{-1}$, the quasiparticle distribution function n depends only on the energy ϵ. Importantly, the density of states per unit energy, $\nu(\epsilon, p_{\mathrm{s}})$, depends on the instantaneous value of the superfluid momentum p_{s}. In other words, as the value of $\mathbf{p}_s$ changes individual quasiparticle levels move in energy space. At finite temperature the quasiparticles occupying these levels travel in energy space as well. This motion creates a non-equilibrium quasiparticle distribution. The relaxation of the latter due to inelastic scattering causes entropy production and energy dissipation. The corresponding contribution to the conductivity is proportional to τ_{in}. The reason why the Debye contribution to the linear conductivity exists only at $\bar{\boldsymbol{p}}_{\mathrm{s}} \neq 0$ is the following. The density of states is invariant under time reversal and thus can depend only on the magnitude of the condensate momentum $p_{\mathrm{s}} = |\boldsymbol{p}_{\mathrm{s}}|$. As a result, in the linear in $\boldsymbol{E}$ approximation $\nu(\epsilon)$ changes in time proportionally to $\delta\boldsymbol{p}_{\mathrm{s}}(t) \cdot \bar{\boldsymbol{p}}_{\mathrm{s}}$.

1.3 How it arises in type-II s-wave superconductors

When a type-II superconductor is subject to a magnetic field H in the mixed state interval, $H_{\mathrm{c1}} < H < H_{\mathrm{c2}}$, the magnetic field penetrates into the sample in the form of vortices [1]. Here $H = n_{\mathrm{v}}\Phi_0$ is the average magnetic field, with n_{v} being the flux line density and $\Phi_0 = \pi\hbar c/e$ - the flux quantum. Typically, defects and intrinsic disorder of the underlying crystalline lattice induce inhomogeneities in the superconducting order parameter. As a result, the vortex lattice becomes pinned to the crystalline lattice. For current densities j below some critical value j_{c} the vortices remain pinned, and the current in this metastable state is dissipationless. However, at $j > j_{\mathrm{c}}$, or if the flux lattice is melted by thermal fluctuations, the vortices begin to move, generating dissipation, and the system acquires a finite conductivity σ. This phenomenon has been extensively studied both experimentally and theoretically (see, for example, Refs. [5, 44, 19, 33, 20, 35, 37, 8, 56, 7, 22, 6], and references therein.).

Near the critical current density j_{c} this motion proceeds by creep [2], but as the current density is increased the system enters the flux flow regime, in which the vortices move with a macroscopic velocity $\mathbf{V}$. The latter is related to the macroscopic electric field $\mathbf{E}$ by the

Josephson relation [25]

$$\mathbf{E} = -\frac{1}{c}[\mathbf{V} \times \mathbf{H}], \tag{1.8}$$

which implies that in the reference frame moving with the vortex lattice the electric field vanishes. The nonlinear conductivity σ in the flux flow regime can be expressed in terms of the energy dissipation rate as

$$\sigma E^2 = n_{\mathrm{v}} W, \tag{1.9}$$

where W is the energy dissipation rate per unit length of the vortex.

At relatively weak magnetic fields, $H \ll H_{\mathrm{c2}}$, the dependence of the conductivity on the magnetic field can be established from rather general considerations. The energy dissipation in this case occurs in the vortex cores. In the ohmic regime the dissipation rate in each vortex is quadratic in $\mathbf{V}$. From here, using Eqs. (1.8) and (1.9) one arrives at the conclusion that the conductivity is inversely proportional to the magnetic field, $\sigma = C/H$. Evaluation of the coefficient C requires a microscopic theory.

The problem of flux flow conductivity in superconductors has been studied for a long time. It is generally accepted that in the regime where temperature is not too close to the critical temperature T_{c} and the magnetic field is not too close to H_{c2}, the longitudinal conductivity in the flux flow regime is given by the Bardeen-Stephen relation [5] (see also reviews [20, 37]):

$$\sigma_{\mathrm{BS}} = \zeta \sigma_{\mathrm{n}} \frac{H_{\mathrm{c2}}}{H}, \quad H_{\mathrm{c2}} = \frac{\Phi_0}{2\pi\xi^2}. \tag{1.10}$$

Here, ζ is a number of order unity, ξ is the superconducting coherence length, and $\sigma_{\mathrm{n}} = e^2 \nu_{\mathrm{n}} D_{\mathrm{n}}$ is the conductivity of normal metal, with ν_{n} being the density of states at the Fermi energy, and D_n - the electron diffusion coefficient. The latter can be expressed in terms of the Fermi velocity v_F and the elastic momentum relaxation time, τ_{el}, as $D_{\mathrm{n}} = v_{\mathrm{F}}^2 \tau_{\mathrm{el}}/3$. Equation (1.10) reflects the fact that the core region of a vortex (of area $\pi\xi^2$) may be considered, with respect to its electronic properties, as a normal metal. It is important that the Bardeen-Stephen expression for the conductivity is proportional to the elastic relaxation time τ_{el}, and is independent of the energy relaxation time. This means that at $T \ll T_{\mathrm{c}}$ the flux flow conductivity Eq. (1.10) is temperature-independent.

In the dirty limit, $T_c\tau_{\rm el} \ll 1$, the Bardeen-Stephen relation (1.10) was confirmed by microscopic calculations in Refs. [19, 35, 33, 37] in the approximation neglecting pinning of vortices, which is valid at the current density $j \gg j_c$. It was also found [35] that up to a factor of order unity, the same formulas describe the flux flow conductivity of superconductors in the clean limit, $T_c\tau_{\rm el} \gg 1$.

In this thesis we take into account random spacial fluctuations of the system parameters, which were neglected in Refs. [19, 35, 33, 20, 37], and show that they lead to an additional [to the Bardeen-Stephen value of Eq. (1.10)] contribution to the conductivity, which is proportional to the inelastic relaxation time $\tau_{\rm in}$. Since typically $\tau_{\rm in}$ is orders of magnitude larger than the elastic relaxation time [2], this contribution can significantly exceed the one given by Eq. (1.10). At low temperatures this contribution is strongly temperature dependent. The physical mechanism that gives rise to this new contribution is similar to the Debye mechanism of microwave absorption in gases [11], superconductors [53, 52], and the Mandelstam-Leontovich mechanism of second viscosity in liquids [31]. In this paper we will assume the inequality $\tau_{\rm el} \ll k_F\xi/\Delta$ to be satisfied and not consider the opposite, so-called superclean limit, which was discussed in a number of papers starting from [29], since in this case special attention should be paid to a large Hall angle.

1.4 How it arises in non-centrosymmetric superconductors

The microscopic mechanism that gives rise to this new contribution to the magnetoconductivity in non-centrosymmetric superconductors can be understood from the following consideration. At fast momentum relaxation the quasiparticle distribution function depends only on the energy ϵ. The quasiparticle energy levels depend on the condensate momentum $\boldsymbol{p}_s = \frac{1}{2}\left(\boldsymbol{\nabla}\chi - \frac{2e}{c}\boldsymbol{A}\right)$, where χ is the phase of the superconducting order parameter and $\boldsymbol{A}$ is the vector potential. In the presence of a low frequency in-plane electric field,

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[2] Depending on parameters of the system and temperature the ratio $\tau_{\rm in}/\tau_{\rm el}$ can be is big as 10^{10}. See for example [18]

$\boldsymbol{E}(t) = \boldsymbol{E}_\omega \cos\omega t$, the superfluid momentum changes in time in accordance with the condensate acceleration equation $\dot{\mathbf{p}}_{\rm s} = e\mathbf{E}$ and can be written as

$$\mathbf{p}_{\rm s}(t) = \mathbf{p}_{\rm s}^{(0)} + \frac{e}{\omega}\mathbf{E}_\omega \sin(\omega t). \tag{1.11}$$

The time dependence of $\mathbf{p}_{\rm s}$ leads to the motion of quasiparticle energy levels. As a result, an initially equilibrium quasiparticle distribution acquires a nonequilibrium component. The ensuing energy relaxation results in dissipation that is proportional to the inelastic relaxation time $\tau_{\rm in}$. This dissipation mechanism is similar to the Debye mechanism in centrosymmetric superconductors [53, 52].

The dissipation due to the Debye mechanism is completely described by the dependence of the quasiparticle density of states in a superconductor, $\nu(\epsilon)$, on the superfluid momentum. However, being a scalar quantity, the density of states $\nu(\epsilon)$ can depend only on the square of the condensate momentum $\mathbf{p}_{\rm s}(t)$ or a scalar product of $\mathbf{p}_{\rm s}(t)$ with another vector. As a result, the Debye mechanism contributes to the linear conductivity only in the presence of a steady state supercurrent current, $\mathbf{p}_{\rm s}^{(0)} \neq 0$. In centrosymmetric materials, this *dc*-supercurrent must be applied to the sample. However, in non-centrosymmetric materials, the application of an in-plane magnetic field generates an inhomogenous superconducting order parameter which gives a finite value of $\mathbf{p}_{\rm s}^{(\mathbf{0})}$. Thus, the Debye mechanism will contribute to the linear magnetoconductivity $\sigma(H)$ in non-centrosymmetric superconductors.

Chapter 2

S-WAVE SUPERCONDUCTORS

2.1 Derivation of general formula

To describe the motion of energy levels we note that the number of levels in the system is conserved. Therefore the density of states $\nu(\epsilon, p_{\rm s}(t))$ is subject to the continuity equation in energy space $\partial_t \nu(\epsilon, p_s) + \partial_\epsilon [v_\nu(\epsilon) \nu(\epsilon, p_s)]$=0, where $v_\nu(\epsilon, p_s)$ is the level "velocity" in energy space. Using the condensate acceleration equationẽqrefeq:acceleration. we can express the latter in the form $v_\nu(\epsilon, p_s) = e\boldsymbol{E} \cdot \boldsymbol{V}(\epsilon, \boldsymbol{p}_{\rm s})$, where

$$\boldsymbol{V}(\epsilon, \boldsymbol{p}_{\rm s}) = -\frac{1}{\nu(\epsilon, p_{\rm s})} \int_0^\epsilon d\tilde{\epsilon} \frac{\partial \nu(\tilde{\epsilon}, p_{\rm s})}{\partial \boldsymbol{p}_{\rm s}} \tag{2.1}$$

characterizes the sensitivity of the energy levels to changes of $\mathbf{p}_s$. The quasiparticle distribution function $n(\epsilon, t)$ describes the occupancy of energy levels. In the absence of inelastic scattering its time evolution due to the spectral flow is described by the continuity equation $\partial_t(\nu n) + \partial_\epsilon(v_\nu \nu n) = 0$. Combining this with the continuity equation for $\nu(\epsilon, p_s)$ and allowing for inelastic collisions we obtain the kinetic equation

$$\partial_t n(\epsilon, t) + e\boldsymbol{E}(t) \cdot \boldsymbol{V}(\epsilon, \boldsymbol{p}_{\rm s})\, \partial_\epsilon n(\epsilon, t) = I\{n\}, \tag{2.2}$$

where $I\{n\}$ is the collision integral describing inelastic scattering of quasiparticles.

The power W of microwave radiation absorbed per unit volume of the superconductor may be obtained by evaluating the rate of work performed by the microwave field on the quasiparticles, which is given by

$$W = \int_0^\infty d\epsilon \big\langle \nu(\epsilon, p_s(t)) n(\epsilon, t) e\boldsymbol{E}(t) \cdot \boldsymbol{V}(\epsilon, \boldsymbol{p}_{\rm s}(t)) \big\rangle. \tag{2.3}$$

Here $\langle \ldots \rangle$ denotes time averaging. Below we characterize the absorption power by the dissipative part of the conductivity $\sigma_{\rm DB}$ defined by

$$\frac{\sigma_{\rm DB}}{2}\,E_\omega^2 = W. \tag{2.4}$$

Linear regime.—For an equilibrium distribution the integrand in Eq. (2.3) is a total derivative and $W = 0$. At small microwave fields we can linearize the kinetic equation (2.2) in $\mathbf{E}(t)$ and the deviation of the quasiparticle distribution function from equilibrium, $\delta n(\epsilon,t) = n(\epsilon,t) - n_F(\epsilon) \ll 1$ (here $n_F(\epsilon) = [\exp(\epsilon/T)+1]^{-1}$ is the Fermi function). Below we assume that the temperature is near the critical temperature, $|T - T_c| \ll T_c$. In this case the density of states is affected by the condensate momentum in a narrow energy window $|\epsilon - \Delta| \ll T$. Since the energy transfer in a typical inelastic collision is of order T the inelastic collision integral in Eq. (2.2) may be written in the relaxation time approximation,

$$I\{n\} = -\frac{\delta n(\epsilon,t)}{\tau_{\rm in}}, \tag{2.5}$$

where the inelastic relaxation time $\tau_{\rm in}(T)$ depends only on the temperature T.

For an isotropic spectrum, which we assume below for simplicity, the vector $\boldsymbol{V}(\epsilon,\boldsymbol{p}_{\rm s})$ is parallel to $\boldsymbol{p}_{\rm s}$. In this case only the longitudinal conductivity, which corresponds to $\boldsymbol{E}_\omega \parallel \bar{\boldsymbol{p}}_{\rm s}$, is affected by inelastic relaxation.

For a monochromatic electric field, the solution of the linearized kinetic equation (4.8), with the collision integral in the form Eq. (2.5), is given by

$$\delta n_\omega(\epsilon) = -\partial_\epsilon n_F(\epsilon)\frac{e\boldsymbol{E}_\omega\cdot\boldsymbol{V}(\epsilon,\bar{p}_{\rm s})}{-i\omega+\tau_{\rm in}^{-1}}. \tag{2.6}$$

Next we substitute this expression into Eq. (2.3) for the rate of energy dissipation and note that in the relevant energy interval where the density of states depends appreciably on $p_{\rm s}$ we may approximate $-\partial_\epsilon n_F(\epsilon) \approx \frac{1}{4T}$. Doing so and using Eq. (2.4) we obtain

$$\frac{\sigma_{\rm DB}}{\sigma_{\rm D}} = \frac{3\tau_{\rm in}}{4\tau_{\rm el}}\frac{1}{\left[1+(\omega\tau_{\rm in})^2\right]}\int_0^\infty \frac{d\epsilon}{T}\frac{\nu(\epsilon,\bar{p}_{\rm s})V^2(\epsilon,\bar{p}_{\rm s})}{\nu_n v_{\rm F}^2}. \tag{2.7}$$

Here ν_n is the normal state density of states at the Fermi level, $v_{\rm F}$ is the Fermi velocity, and $\sigma_{\rm D} = e^2\nu_n D$, with $D = v_F^2\tau_{\rm el}/3$ being the diffusion coefficient. Equation (2.7) expresses

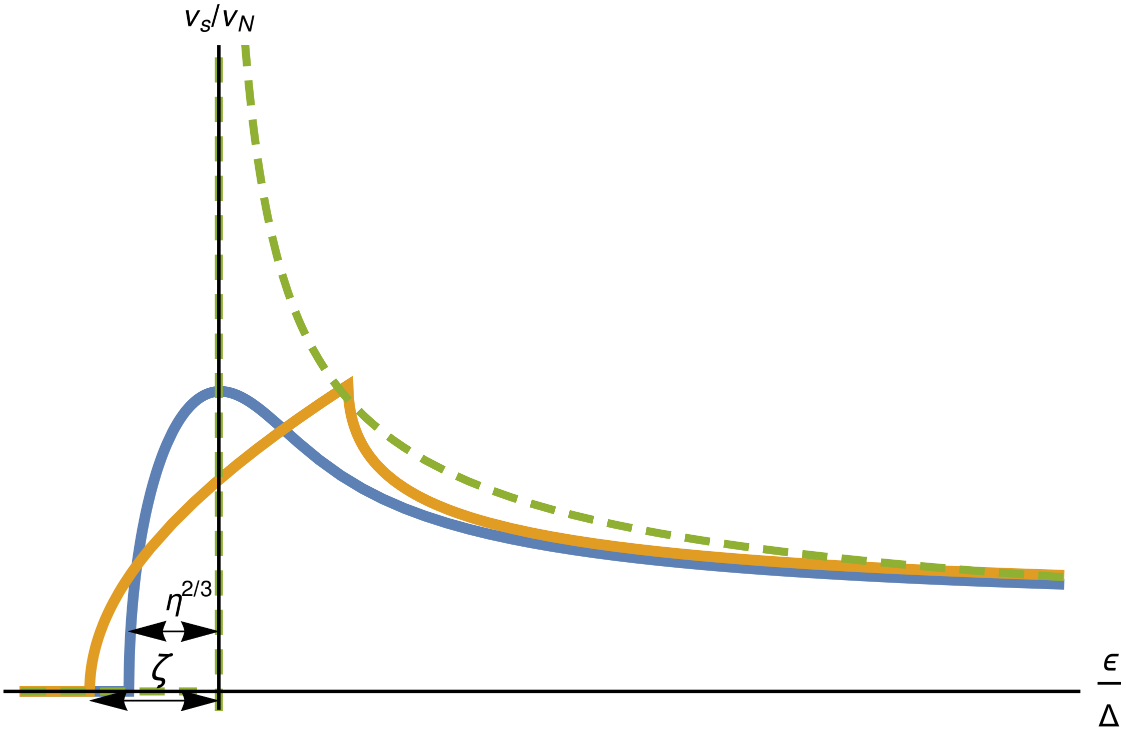

Figure 2.1: Schematic plots of $\nu(\epsilon, p_\mathrm{s})$ at: $p_\mathrm{s} = 0$ - dashed green line, in the diffusive regime $\Delta v_F p_\mathrm{s} \tau_\mathrm{el}^2 \ll 1$ - blue line, and in the ballistic regime $\Delta v_F p_\mathrm{s} \tau_\mathrm{el}^2 \gg 1$ – orange line.

the Debye contribution to the conductivity in terms of the density of states in a current-carrying superconductor. It applies to superconductors with arbitrary symmetry of the order parameter.

2.2 Linear Debye Conductivity in S-Wave Superconductors

Below we focus on s-wave superconductors with an isotropic spectrum and assume $v_F \bar{p}_\mathrm{s} \ll \Delta$. In this case the density of states is most strongly affected by the supercurrent at energies near the gap Δ. Namely at $\bar{p}_\mathrm{s} \neq 0$ the peak in the BCS density of states, $\nu(\epsilon, 0) \to \nu_n \sqrt{\frac{\Delta}{2(\epsilon - \Delta)}}$ at $\epsilon \to \Delta$, is broadened. The width and the shape of the broadening depends on the magnitude of the condensate momentum $\bar{p}_s$ and the strength of disorder.

2.2.1 Ballistic Regime

In the ballistic regime, $v_{\rm F}\bar{p}_{\rm s}\tau_{\rm el}^2\Delta \gg 1$, (which can be realized only in clean superconductors, $\Delta\tau_{\rm el} \gg 1$) the density of states can be found from a simple consideration. In this case one can use the standard expression for the quasiparticle spectrum [51, 56, 17], $\epsilon(\boldsymbol{k}) = \sqrt{\xi_{\boldsymbol{k}}^2 + |\Delta|^2} + \boldsymbol{v}_{\boldsymbol{k}} \cdot \bar{\boldsymbol{p}}_{\rm s}$, where $\boldsymbol{k}$ is the quasiparticle momentum, $\xi_{\boldsymbol{k}}$ is the electron energy measured from the Fermi level, and $\boldsymbol{v}_{\boldsymbol{k}} = d\xi_{\boldsymbol{k}}/d\boldsymbol{k}$ is the electron velocity. The density of states at $|\epsilon - \Delta| \ll \Delta$ is given by

$$\frac{\nu(\epsilon, p_s)}{\nu_n} = \sqrt{\frac{\Delta}{2v_{\rm F}p_{\rm s}}}\left[\theta(z+1)\sqrt{z+1} - \theta(z-1)\sqrt{z-1}\right], \tag{2.8}$$

where $z = (\epsilon-\Delta)/v_F p_{\rm s}$, and $\theta(z)$ is the Heavyside step-function. The width of the broadening of the BCS peak is $\delta\epsilon \sim v_F\bar{p}_{\rm s}$. The shape of the broadening is illustrated in Fig. 2.1. Using Eq. (2.1) and Eq. (2.7) we obtain for the Debye contribution to the conductivity in the ballistic regime

$$\frac{\sigma_{\rm DB}}{\sigma_{\rm D}} = I_{\rm b}\frac{\tau_{\rm in}}{\tau_{\rm el}\left[1 + (\omega\tau_{\rm in})^2\right]}\frac{\Delta}{T}\sqrt{\frac{v_{\rm F}\bar{p}_{\rm s}}{\Delta}} \tag{2.9}$$

where $I_{\rm b} = \frac{8}{45}$ is a dimensionless integral defined in Eq. (A.16) of Appendix A. The powerlaw dependence of $\sigma_{\rm DB}$ on the condensate momentum $\bar{p}_{\rm s}$ follows from the simple scaling form of the density of states in Eq. (2.8). The exponent of this power law dependence, $\sigma_{\rm DB} \propto \sqrt{\bar{p}_{\rm s}}$, can be understood from the following consideration. The quasiparticle states whose energies are affected by the supercurrent lie in a narrow energy window of width $\delta\epsilon \sim v_{\rm F}\bar{p}_{\rm s}$ near $\epsilon = \Delta$. The number of such states per unit volume may be estimated as $\nu_n\sqrt{\Delta v_{\rm F} p_{\rm s}}$. Since the characteristic level displacement in the microwave field is given by $v_{\rm F}\delta p_{\rm s} \sim v_{\rm F} e E_0/\omega$ one obtains an estimate for the absorption power consistent with Eq. (2.9).

The above consideration of the density of states is valid as long the condition of ballistic motion $v_{\rm F}\bar{p}_{\rm s}\tau_{\rm el}(\epsilon) \gg 1$ is satisfied for most of the quasiparticles in the relevant energy interval $|\epsilon - \Delta| \lesssim v_{\rm F}\bar{p}_{\rm s}$. Here $\tau_{\rm el}(\epsilon)$ is the energy-dependent quasiparticle mean free time, which for $|\epsilon - \Delta| \ll \Delta$ is given by the standard expression $\tau_{\rm el}^{-1}(\epsilon) \approx \tau_{\rm el}^{-1}\sqrt{\frac{2(\epsilon-\Delta)}{\Delta}}$ (see for example Ref. [42]). Therefore the regime of ballistic motion of quasiparticles participating in the Debye mechanism of microwave absorption is realized at relatively large supercurrent densities, where $v_{\mathrm{F}}\bar{p}_{\mathrm{s}}\tau_{\mathrm{el}}^2\Delta \gg 1$.

2.2.2 *Diffusive Regime*

To study the Debye contribution to the conductivity outside the ballistic regime we express the density of states in terms of the disorder-averaged Green's functions. This enables us to utilize the standard theoretical methods developed in the theory of disordered superconductors [36, 39]. We show in Appendix A that the density of states can be expressed as

$$\frac{\nu(\epsilon, p_{\mathrm{s}})}{\nu_n} = \frac{1}{\sqrt{2}}\Im x^{-1}, \tag{2.10}$$

where the variable x satisfies the quintic equation (S.9a). In the ballistic regime, $v_{\mathrm{F}}p_{\mathrm{s}}\tau_{\mathrm{el}}^2\Delta \gg 1$, the latter reduces to the biquadratic equation (S.11) whose solution, when substituted into Eq. (2.10), reproduces Eq. (2.8). In the opposite regime $v_{\mathrm{F}}p_{\mathrm{s}}\tau_{\mathrm{el}}^2\Delta \ll 1$, which corresponds to diffusive motion of quasiparticles in the relevant energy interval, the quintic equation (S.9a) simplifies to

$$x\left(x^2 + w\right) + \frac{\sqrt{2}\zeta^2}{3\gamma} = 0. \tag{2.11}$$

Here $\zeta = v_{\mathrm{F}}p_{\mathrm{s}}/\Delta$, $\gamma = (\tau_{\mathrm{el}}\Delta)^{-1}$, and $w = (\epsilon - \Delta)/\Delta$. The solutions of this equation can be written in the scaling form $x = \frac{\zeta^{2/3}}{\gamma^{1/3}}\,\tilde{x}\left(\frac{w\gamma^{2/3}}{\zeta^{4/3}}\right)$, where the explicit form of $\tilde{x}\left(\frac{w\gamma^{2/3}}{\zeta^{4/3}}\right)$ is given by the Cardano formula, Eq. (S.17). Substituting this form into Eq. (2.10) (the corresponding $\nu(\epsilon, p_{\mathrm{s}})$ is plotted in Fig. 2.1), and using Eqs. (2.1) and (2.7), we obtain

$$\frac{\sigma_{\mathrm{DB}}}{\sigma_{\mathrm{D}}} = I_{\mathrm{d}}\frac{\tau_{\mathrm{in}}}{\tau_{\mathrm{el}}}\frac{\Delta}{T}\frac{\tau_{\mathrm{el}}\left(\Delta D^2\bar{p}_{\mathrm{s}}^4\right)^{1/3}}{1+(\omega\tau_{\mathrm{in}})^2}, \tag{2.12}$$

where $I_{\mathrm{d}} \approx 0.0549$ is defined in Eq. (D.13). This expression is consistent with the result obtained in Ref. [45] by a different method.

The exponent of the powerlaw dependence $\sigma_{DB} \propto \bar{p}_{\mathrm{s}}^{4/3}$ in Eq. (2.12) and its order of magnitude can be obtained by noting that the width of the broadening of the BCS singularity in the diffusive regime is $\delta\epsilon \sim (\Delta D^2\bar{p}_{\mathrm{s}}^4)^{1/3}$ and the number per unit volume of levels that participate in microwave absorption is $\sim \nu_n\,(\Delta^2 D\bar{p}_{\mathrm{s}}^2)^{1/3}$.

It is worth noting that the diffusive regime can be realized in both clean, $\Delta\tau_{\mathrm{el}} \ll 1$, and dirty $\Delta\tau_{\mathrm{el}} \gg 1$, superconductors. Accordingly Eqs. (2.10) and (2.11) for the density of states and the resulting conductivity (2.12) can be obtained using either the Gorkov equations or the Usadel equation, see Appendix A

2.3 Low Temperature Linear Conductivity in S-Wave Superconductors

2.3.1 Derivation of Low Temperature Debye Conductivity

Low temperature quasiparticle kinetics in *s*- and *d*-wave superconductors have common features. In both cases the low energy density of states is suppressed. Therefore, in both cases the quasiparticle concentration decreases with temperature more rapidly than in normal metals. Consequently the electron-electron scattering rate is suppressed and the quasiparticle energy relaxation is controlled by electron-phonon scattering.

Furthermore, one needs to distinguish between two different types of inelastic scattering processes in superconductors. The quasiparticle-phonon relaxation processes that conserve the number of quasiparticles are characterized by the rate $1/\tau_{\mathrm{in}}^{(st)}(T)$, which is independent of quasiparticle concentration. The second type of inelastic relaxation processes corresponds to recombination, which changes the total number of quasiparticles. The rate $1/\tau_{\mathrm{r}}(T)$ of such processes is proportional to the quasiparticle concentration $x(T)$. Therefore at low temperatures it becomes much smaller than $1/\tau_{\mathrm{in}}^{(st)}(T)$;

$$\tau_{\mathrm{r}}(T) \propto \frac{\tau_{\mathrm{r}}^{(0)}(T)}{x(T)} \gg \tau_{\mathrm{in}}^{(st)}(T). \tag{2.13}$$

The Debye contribution to the dissipative kinetic coefficients is proportional to the longest relaxation time in a system (see for example [31]), which in our case is $\tau_{\mathrm{r}}(T)$. On the other hand σ_{DB} is also proportional to the density of thermal quasiparticles. We show below that, as a consequence, the Debye contribution to the conductivity becomes independent of the quasiparticle concentration $x(T)$. As a result, its magnitude in the low temperature regime is roughly speaking of the same order as that near T_{c}.

In order to obtain an estimate for $\sigma_{\rm DB}$ in this regime we note that since the recombination time is the longest time scale in the problem, $\tau_{\rm r} \gg \tau_{\rm in}^{(st)}$, at relatively short time scales of order of $\tau_{\rm in}^{(st)}$ the number of quasiparticles is approximately conserved. As a result, at such time scales the system of quasiparticles reaches a quasi-equilibrium form which is characterized by a nonzero chemical potential,

$$n(\epsilon) = \frac{1}{1+\exp(\frac{\epsilon-\mu}{T})}, \tag{2.14}$$

while in thermal equilibrium $\mu = 0$. To find the value of μ in the presence of microwave radiation one has to integrate Eq. (2.2) over ϵ bearing in mind that the relaxation processes conserve the number of quasiparticles, $\int I_{\rm st} d\epsilon = 0$. Doing so, we get the following estimate for the chemical potential

$$\mu \sim \frac{\tau_{\rm r}}{n_{\rm F}(\epsilon^*)} \int e\boldsymbol{E}(t)\cdot\boldsymbol{V}(\epsilon,\boldsymbol{p}_{\rm s})\,\frac{dn_{\rm F}(\epsilon)}{d\epsilon}d\epsilon. \tag{2.15}$$

Here $\epsilon^* = \Delta$ in the case of s-wave superconductors, and $\epsilon^* = 0$ for the case of d-wave superconductors. To get $\sigma_{\rm DB}$ one should substitute $\delta n(\epsilon) \sim \mu dn_{\rm F}(\epsilon)/d\epsilon$ into Eqs. (2.3) and (2.4). Since in this regime the relaxation time approximation for the recombination collision integral is only applicable to accuracy within a factor of order unity, both Eq. (2.15) and subsequent estimates for $\sigma_{\rm DB}$ are valid only with the same accuracy.

2.3.2 s-wave superconductors

In s-wave superconductors the dimensionless quasiparticle concentration $x_{\rm s}(T)$ defined by

$$x_{\rm s}(T) = (\nu_n\Delta)^{-1}\int_0^\infty d\epsilon\nu(\epsilon)n_{\rm F}(\epsilon) \sim \sqrt{\frac{T}{\Delta}}\exp(-\Delta/T) \tag{2.16}$$

is exponentially small. Consequently, the conventional contribution to the microwave absorption coefficient is exponentially small as well. On the other hand, since the recombination rate in Eq. (2.13) is inversely proportional to the quasiparticle concentration,[1] in the low

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[1] The parameter $\tau_{\rm r}^{(0)}$ in Eq. (2.13) may be estimated as $1/\tau_{\rm r}^{(0)} \sim \Delta^3/\theta_{\rm D}^2$, where $\Theta_{\rm D}$ is the Debye temperature.

frequency limit, $\omega\tau_{\mathrm{r}} \ll 1$, the exponentially small factor $\exp(-\Delta/T)$ is canceled from the expression for the conductivity. Below we illustrate this fact in the diffusive regime, and at $T \ll \delta\epsilon(\bar{p}_{\mathrm{s}}) \ll \Delta$. In this case the magnitude of the level sensitivity in the energy interval $|\epsilon - \Delta| \lesssim T$ is $V \sim \frac{1}{\bar{p}_{\mathrm{s}}}\delta\epsilon \sim \left(\Delta D^2 \bar{p}_{\mathrm{s}}\right)^{1/3}$. Thus, we get

$$\frac{\sigma_{\mathrm{DB}}}{\sigma_{\mathrm{D}}} \sim \frac{\tau_{\mathrm{r}}^{(0)}}{\tau_{\mathrm{el}}} \sqrt{\frac{\Delta}{T}} \tau_{\mathrm{el}} \left(\Delta D^2 \bar{p}_{\mathrm{s}}^4\right)^{1/3}. \tag{2.17}$$

We note that the value of the conductivity at zero superfluid momentum may be estimated as $\sigma(\bar{p}_s = 0) \sim x_{\mathrm{s}}(T)\sigma_{\mathrm{D}}$, and is exponentially small at $T \ll \Delta$. Thus, in this regime the Debye contribution to the conductivity becomes exponentially enhanced at low temperatures in comparison to the conventional contribution.

2.4 Non-Linear Conductivity

Let us now consider the situation in which the *dc* supercurrent is absent, $\bar{p}_{\mathrm{s}} = 0$. In the presence of the microwave field the oscillation amplitude of the condensate momentum is given by $\delta p_{\mathrm{s}} = eE_{\omega}/\omega$. Since the Debye contribution to the nonlinear conductivity defined by Eq. (2.4) is proportional to τ_{in} the nonlinear threshold for the microwave absorption is anomalously low. To evaluate microwave absorption in the nonlinear regime it is convenient to introduce the integrated over energy density of states

$$N(\epsilon, t) = \int_0^{\epsilon} d\varepsilon \ \nu(\varepsilon, t) \tag{2.18}$$

and consider the quasiparticle distribution function not as a function of energy ϵ and time t but rather as a function of N and t. The change of variables $n(\epsilon, t) \to n(N, t)$ is equivalent to the transformation from Eulerian to Lagrangian variables in fluid mechanics [31]. In this representation the kinetic equation (2.2) acquires a very simple form,

$$\partial_t n(N, t) = I\{n\} = -\frac{n(N, t) - n_F(\epsilon(N, t))}{\tau_{\mathrm{in}}}. \tag{2.19}$$

Note that the electric field enters this equation only via the time-dependence of the quasiparticle energy level $\epsilon(N, t)$. In the presence of the microwave field the latter undergoes nonlinear oscillations $\epsilon(N,t) = \epsilon_0(N) + \delta\epsilon(N,t)$ whose form is determined by Eq. (2.18). Note that the linearization of the collision integral is justified in the nonlinear regime as long as the amplitude of $\delta\epsilon(N,t)$ is small as compared to T. The solution of Eq. (2.19) can be written as $n(N,t) = n_F(\epsilon_0(N)) + \frac{dn_F(\epsilon_0(N))}{d\epsilon_0(N)} \int_0^\infty \frac{d\tau}{\tau_{\rm in}} e^{-\frac{\tau}{\tau_{\rm in}}} \delta\epsilon(N, t-\tau)$. The absorption power per unit volume in this representation is given by $W = \int_0^\infty dN\, \langle n(N,t)\partial_t\epsilon(N,t)\rangle$. Substituting the solution for $n(N,t)$ into this expression we get

$$W = \int_0^\infty dN \int_0^\infty \frac{d\tau}{4T} e^{-\frac{\tau}{\tau_{\rm in}}} \langle \partial_t\epsilon(N,t)\partial_t\epsilon(N,t-\tau)\rangle. \tag{2.20}$$

Here the level velocity is given by $\partial_t\epsilon(N,t) = v_\nu(\epsilon,t) = e\boldsymbol{E}\cdot\boldsymbol{V}(\epsilon,\boldsymbol{p}_{\rm s})$, with $\boldsymbol{V}(\epsilon,\boldsymbol{p}_{\rm s})$ defined in Eq. (2.1). This can be shown by taking the time derivative of (2.18).

Equation (2.20) expresses the power of nonlinear microwave absorption in terms of the correlation function of level velocities $\partial_t\epsilon(N,t)$, which are defined by Eq. (2.18). Similarly to the linear regime, the results depend of the degree of disorder, the amplitude of the microwave field E_ω and the frequency of radiation ω. They simplify in the ballistic regime $eE_\omega v_{\rm F}\Delta\tau_{\rm el}^2 \gg \omega$ and in the diffusive regime $eE_\omega v_{\rm F}\Delta\tau_{\rm el}^2 \ll \omega$ where the nonlinear conductivity has a simple powerlaw dependence on the amplitude of the microwave field E_ω. In the ballistic regime we obtain

$$\frac{\sigma_{\rm DB}^{\rm nl}}{\sigma_{\rm D}} = \frac{\tau_{\rm in}}{\tau_{\rm el}}\frac{\Delta}{T}\sqrt{\frac{v_{\rm F}eE_\omega}{\omega\Delta}}\, F_{\rm b}(\omega\tau_{\rm in}), \tag{2.21}$$

while in the diffusive regime we find

$$\frac{\sigma_{\rm DB}^{\rm nl}}{\sigma_{\rm D}} = \frac{\tau_{\rm in}}{\tau_{\rm el}}\frac{\Delta}{T}\frac{\Delta^{1/3}D^{5/3}|eE_\omega|^{4/3}}{v_F^2\omega^{4/3}}\, F_{\rm d}(\omega\tau_{\rm in}). \tag{2.22}$$

The functions $F_{\rm b}(\omega\tau_{\rm in})$ and $F_{\rm d}(\omega\tau_{\rm in})$ that describe the frequency dependence of nonlinear microwave conductivity are given Eqs. (B.5) and (B.11) in Appendix B Although, in contrast to Eqs. (2.9) and (2.12), they do not have a simple Lorentzian form, their high- and low-frequency asymptotic behavior is similar; at low frequency $F_{\rm b}(0) \approx 0.10848$ and $F_{\rm d}(0) \approx 0.10909$ while at high frequencies they behave as $1/(\omega\tau_{\rm in})^2$.

Chapter 3

D-WAVE SUPERCONDUCTORS

In clean superconductors where the mean free path exceeds the superconducting coherence length the nonequilibrium state of the superconductor may be described by the quasiparticle distribution function $n_{\boldsymbol{p}}$. In this case the current density is expressed in terms of the quasiparticle distribution function as

$$\boldsymbol{j} = eN\frac{\boldsymbol{p}_{\mathrm{s}}}{m} + 2e\int \frac{d^3p}{(2\pi)^3}\boldsymbol{v}\, n_{\boldsymbol{p}}. \tag{3.1}$$

Here N is the electron density and $\boldsymbol{v} = \boldsymbol{p}/m$ is the band velocity of the electron with quasimomentum $\boldsymbol{p}$.

The time evolution of the distribution function is described by the Boltzmann kinetic equation, which in the spatially uniform case takes a simple form

$$\partial_t n_{\boldsymbol{p}} = I_{\mathrm{el}} + I_{\mathrm{in}}. \tag{3.2}$$

Here I_{el} and I_{in} are the collision integrals describing, correspondingly, the elastic and inelastic scattering processes.

The reason the conductivity is affected by the inelastic collisions is that in the presence of supercurrent the quasiparticle energy spectrum,

$$\tilde{\epsilon}_{\boldsymbol{p}}(\boldsymbol{p}_{\mathrm{s}}) = \sqrt{|\Delta(\boldsymbol{p})|^2 + \xi_{\boldsymbol{p}}^2} + \boldsymbol{p}_{\mathrm{s}} \cdot \boldsymbol{v}, \tag{3.3}$$

contains an odd-in-momentum part described by the second term above. Since we are interested in the regime $\tau_{\mathrm{el}} \ll \tau_{\mathrm{in}}$, $\omega\tau_{\mathrm{el}} \ll 1$ the quasiparticle distribution function depends only on the quasiparticle energy $n_{\boldsymbol{p}} = n\left(\tilde{\epsilon}_{\boldsymbol{p}}(\boldsymbol{p}_{\mathrm{s}}), t\right)$. Substituting this form into Eq. (3.1), noting that $\boldsymbol{v} = \frac{d}{d\boldsymbol{p}_{\mathrm{s}}}\tilde{\epsilon}_{\boldsymbol{p}}(\boldsymbol{p}_{\mathrm{s}})$, and using the resolution of identity $1 = \int_0^\infty d\epsilon\delta[\epsilon - \tilde{\epsilon}_{\boldsymbol{p}}(\boldsymbol{p}_{\mathrm{s}})]$ we can express the current density as

$$\boldsymbol{j} = eN\frac{\boldsymbol{p}_{\rm s}}{m} + e\int_0^\infty d\epsilon\, n(\epsilon,t)\nu(\epsilon,\boldsymbol{p}_{\rm s})\boldsymbol{V}(\epsilon,\boldsymbol{p}_{\rm s}), \tag{3.4}$$

where

$$\nu(\epsilon,\boldsymbol{p}_{\rm s}) = 2\int\frac{d^3p}{(2\pi\hbar)^3}\delta[\epsilon-\tilde{\epsilon}_{\boldsymbol{p}}(\boldsymbol{p}_{\rm s})]. \tag{3.5}$$

is the density of states, and

$$\boldsymbol{V}(\epsilon,\boldsymbol{p}_{\rm s}) = \frac{1}{\nu[\epsilon,\boldsymbol{p}_{\rm s}]}\int\frac{d^3p}{(2\pi\hbar)^3}\delta[\epsilon-\tilde{\epsilon}_{\boldsymbol{p}}(\boldsymbol{p}_{\rm s})]\,\frac{d}{d\boldsymbol{p}_{\rm s}}\tilde{\epsilon}_{\boldsymbol{p}}[\boldsymbol{p}_{\rm s}]. \tag{3.6}$$

Writing the δ-function in the integrand above as a derivative of the step-function, and integrating by parts it is easy to show that Eq. (3.6) reduces to Eq. (2.1). Thus Eq. (3.4) expresses the current density in terms of the energy-dependent distribution function $n(\epsilon,t)$ and $p_{\rm s}$-dependence of the density of states.

Finally, in order to obtain the time evolution equation for $n(\epsilon,t)$ we substitute the distribution function in the form $n_{\boldsymbol{p}} = n\left(\tilde{\epsilon}_{\boldsymbol{p}}(\boldsymbol{p}_{\rm s}),t\right)$ into Eq. (3.2), multiply it by $\delta[\epsilon-\tilde{\epsilon}_{\boldsymbol{p}}(\boldsymbol{p}_{\rm s})]$ and integrate over $\frac{d^3p}{(2\pi)^3}$. Then, using the fact that $\partial_t n_{\boldsymbol{p}} = \partial_t n\left(\tilde{\epsilon}_{\boldsymbol{p}},t\right) + \boldsymbol{v}\cdot\dot{\boldsymbol{p}}_{\rm s}\,\partial_{\tilde{\epsilon}_{\boldsymbol{p}}} n\left(\tilde{\epsilon}_{\boldsymbol{p}},t\right)$, and noting that the elastic collision integral is nullified by an arbitrary distribution function that depends only on $\tilde{\epsilon}_{\boldsymbol{p}}(\boldsymbol{p}_{\rm s})$ we reproduce Eq. (2.2). Linearizing it and substituting the result for δn into Eq. (3.1) we get Eq. (2.7).

3.1 *Clean Regime, High T*

Let us now apply the general expression (2.7) to study the Debye contribution to the conductivity of d-wave superconductors. The order parameter in d-wave superconductors $\Delta(\mathbf{p})$ changes its sign upon rotation of the momentum by $\pi/2$ in the xy plane, and can be modeled by the form

$$\Delta(\boldsymbol{p}) = \Delta_0(\sin^2 p_x a - \cos^2 p_y a), \tag{3.7}$$

where $\Delta_0(T,\tau_{\rm el})$ is the gap maximum at the antinode, which generally depends on temperature and $\tau_{\rm el}$. In this article we focus on the limit $\Delta_0\tau_{\rm el} \gg 1$. In this case the density of states in the presence of supercurrent may be evaluated with the aid of Eqs. (A.2) and (A.3). The integral in Eq. (2.7) for the Debye contribution to the conductivity is dominated by a narrow energy interval $|\epsilon - \Delta_0| \ll \Delta_0$, which corresponds to quasiparticles with momenta near the antinodes.

Let us begin with the clean limit, $\tau_{\mathrm{el}} \to \infty$. In this case the density of states may be evaluated using Eq. (3.5). For $|\epsilon - \Delta_0| \ll \Delta_0$ we obtain

$$\nu(\epsilon, p_{\mathrm{s}}) = \frac{\nu_n}{\pi} \sum_i \ln \frac{\Delta_0}{|(\epsilon - \Delta_0) + v_{\mathrm{F}}(\boldsymbol{n}_i \cdot \boldsymbol{p}_{\mathrm{s}})|}, \tag{3.8}$$

where the summation is performed over all antinodal lines and $\boldsymbol{n}_i$ is the unit vector in the direction of the i-th antinodal line.

The energy level sensitivity $\boldsymbol{V}(\epsilon)$ in the clean limit may be determined from Eq. (3.6), and is given by

$$\boldsymbol{V}(\epsilon) = v_{\mathrm{F}} \frac{\sum_i (\boldsymbol{n}_i \cdot \hat{\boldsymbol{p}}_{\mathrm{s}}) \boldsymbol{n}_i \ln \left(\frac{\Delta_0}{|(\epsilon - \Delta_0) + v_{\mathrm{F}}(\boldsymbol{n}_i \cdot \boldsymbol{p}_{\mathrm{s}})|} \right)}{\sum_i \ln \frac{\Delta_0}{|(\epsilon - \Delta_0) + v_{\mathrm{F}}(\boldsymbol{n}_i \cdot \boldsymbol{p}_{\mathrm{s}})|}}. \tag{3.9}$$

Substituting Eqs. (3.8), and (3.9) into Eq. (2.7) and assuming $T \gg v_{\mathrm{F}} \bar{p}_{\mathrm{s}}$ within logarithmic accuracy we obtain the following expression for the Debye contribution to the conductivity,

$$\frac{\sigma_{\mathrm{DB}}}{\sigma_{\mathrm{D}}} = \frac{3}{\pi} \frac{\tau_{\mathrm{in}}}{\tau_{\mathrm{el}}} \frac{1}{[1 + (\omega \tau_{\mathrm{in}})^2]} \left(\frac{v_{\mathrm{F}} \bar{p}_{\mathrm{s}}}{T} \right) \ln \left(\frac{\Delta_0}{v_{\mathrm{F}} \bar{p}_{\mathrm{s}}} \right). \tag{3.10}$$

To derive this result we neglected the contributions of quasiparticles near the nodal lines to σ_{DB} because they are small in the ratio $v_F \bar{p}_{\mathrm{s}} / \Delta$ as compared to that in Eq. (3.10).

Equations (3.8) and (3.10) are valid provided $v_{\mathrm{F}} \bar{p}_{\mathrm{s}} > \tau_{\mathrm{el}}^{-1}$. In the presence of disorder the non-analyticity of the density of states as a function of ϵ, Eq. (3.8), is smeared in the interval of energies of order τ_{el}^{-1}. In the limit of small supercurrent, $v_{\mathrm{F}} \bar{p}_{\mathrm{s}} \ll \tau_{\mathrm{el}}^{-1}$, the Debye contribution to the conductivity is expected to be analytic in $\bar{p}_s$, namely $\sigma_{\mathrm{DB}} \sim a \bar{p}_{\mathrm{s}}^2$. The value of the coefficient a can be estimated by matching this expression with Eq. (3.10) at $v_{\mathrm{F}} \bar{p}_{\mathrm{s}} \sim \tau_{\mathrm{el}}^{-1}$. This yields

$$\frac{\sigma_{\mathrm{DB}}}{\sigma_{\mathrm{D}}} \sim \frac{\tau_{\mathrm{in}}}{T} \frac{(v_{\mathrm{F}} \bar{p}_{\mathrm{s}})^2}{[1 + (\omega \tau_{\mathrm{in}})^2]}. \tag{3.11}$$

In the Born approximation this result can be obtained from Eqs.Eqs. (A.2) and (A.3).

3.2 *Dirty regime*

The low energy density of states in d-wave superconductors is dominated by momenta in the vicinity of the nodal lines, and in the clean limit $\tau_{\mathrm{el}} \to \infty$ is given by [60]

$$\nu(\epsilon, p_{\mathrm{s}}) = \nu_n \sum_i \frac{|\epsilon + v_{\mathrm{F}}(\boldsymbol{m}_i \cdot \boldsymbol{p}_{\mathrm{s}})|}{\Delta_0}, \tag{3.12}$$

where $\boldsymbol{m}_i$ denotes the unit vector pointing in the direction of the i-th nodal line. Using Eq. (2.1) we find that at $\epsilon \ll \Delta_0$ the level sensitivity $\boldsymbol{V}(\epsilon)$ is given by

$$\boldsymbol{V}(\epsilon) = v_{\mathrm{F}} \frac{\sum_i \boldsymbol{m}_i(\boldsymbol{m}_i \cdot \hat{\boldsymbol{p}}_{\mathrm{s}})|\epsilon + v_{\mathrm{F}}(\boldsymbol{m}_i \cdot \boldsymbol{p}_{\mathrm{s}})|}{\sum_i |\epsilon + v_{\mathrm{F}}(\boldsymbol{m}_i \cdot \boldsymbol{p}_{\mathrm{s}})|}. \tag{3.13}$$

Substituting Eqs. (3.12) and (3.13) into Eq. (2.7) we find

$$\frac{\sigma_{\mathrm{DB}}}{\sigma_{\mathrm{D}}} \sim \frac{\tau_{\mathrm{r}}(T)}{\tau_{\mathrm{el}}} \frac{1}{\left[1 + (\omega\tau_{\mathrm{r}}(T))^2\right]} \begin{cases} \left(\frac{v_{\mathrm{F}}\bar{p}_{\mathrm{s}}}{\Delta_0}\right)^2 \frac{\Delta_0}{T} \ln\left(\frac{T}{v_{\mathrm{F}}\bar{p}_{\mathrm{s}}}\right) & \text{for } T \gg v_{\mathrm{F}}\bar{p}_{\mathrm{s}}, \\ \frac{T^2}{v_{\mathrm{F}}\bar{p}_{\mathrm{s}}\Delta_0} & \text{for } T \ll v_{\mathrm{F}}\bar{p}_{\mathrm{s}}. \end{cases} \tag{3.14}$$

The recombination time here may be estimated using Eq. (2.13) by noting that in d-wave superconductors the dimensionless quasiparticle concentration decreases only as a power law in T

$$x_{\mathrm{d}}(T) = (\nu_n T)^{-1} \int_0^\infty d\epsilon \nu(\epsilon) n_{\mathrm{F}}(\epsilon) \sim \frac{T}{\Delta_0}, \tag{3.15}$$

while $\tau_{\mathrm{r}}^{(0)}$ in Eq. (2.13) may be estimated as $\tau_{\mathrm{r}}^{(0)} \sim \tau_{\mathrm{st}}$.

In Eqs. (3.12), (3.13), and (3.14) we neglected impurity scattering, which broadens the quasiparticle energy levels. Consequently the result (3.14) is valid provided $v_{\mathrm{F}}\bar{p}_{\mathrm{s}}, T \gg \Gamma_{\mathrm{el}}$, where Γ_{el} is the characteristic broadening scale of low energy quasiparticle levels. The value of Γ_{el} is not universal, and depends on the details of the scattering potential. For example, for weak impurities $\Gamma_{\mathrm{el}} \sim \Delta_0^2 \tau_{\mathrm{el}} \exp(-\Delta_0 \tau_{\mathrm{el}})$ [38, 42], while in the case of strong impurities whose scattering cross-section is close to the unitary limit $\Gamma_{\mathrm{el}} \sim \Delta_0/\sqrt{\Delta_0 \tau_{\mathrm{el}}}$, see Refs. [23, 46, 50]. In order to estimate σ_{DB} in the presence of disorder we may evaluate the density of states using Eqs. (A.2) and (A.3) by setting $\tilde{\epsilon} \to \epsilon + i\Gamma_{\mathrm{el}}$. At relatively large energies, $\Gamma_{el} < \epsilon < \Delta_0$, the density of states is practically unaffected by disorder and superfluid momentum,

$$\nu(\epsilon > \Gamma_{\mathrm{el}}, p_{\mathrm{s}}) \sim \nu_n \frac{\epsilon}{\Delta_0}. \tag{3.16}$$

At lower energies, $\epsilon \lesssim \Gamma_{\rm el}$ it becomes independent of the energy. In the absence of superfluid current it may be estimated as

$$\nu(\epsilon < \Gamma_{\rm el}, p_s = 0) \sim \nu_n \frac{\Gamma_{\rm el}}{\Delta_0}, \tag{3.17}$$

while the correction to due to the presence of supercurrent, $\delta\nu(\epsilon, p_{\rm s}) = \nu(\epsilon, p_s) - \nu(\epsilon, 0)$ may be estimated at $v_F p_{\rm s} \ll \Gamma_{\rm el}$ as

$$\frac{\delta\nu(\epsilon < \Gamma_{\rm el}, p_{\rm s})}{\nu(\epsilon < \Gamma_{\rm el}, p_{\rm s} = 0)} \sim \left(\frac{v_{\rm F} p_{\rm s}}{\Gamma_{\rm el}}\right)^2. \tag{3.18}$$

Using Eqs. (3.16), (3.17), and (3.18) we can estimate the level sensitivity $V(\epsilon, p_{\rm s})$ in Eq. (2.1) as

$$V(\epsilon, \boldsymbol{p}_{\rm s}) \sim v_{\rm F} \begin{cases} \epsilon \frac{v_{\rm F} p_{\rm s}}{\Gamma_{\rm el}^2}, & \text{for } \epsilon < \Gamma_{\rm el} \\ \frac{v_{\rm F} p_{\rm s}}{\epsilon}, & \text{for } \epsilon > \Gamma_{\rm el}. \end{cases} \tag{3.19}$$

Using these estimates, in the temperature interval $v_{\rm F} \bar{p}_{\rm s} < \Gamma_{\rm el} < T$ we get

$$\frac{\sigma_{\rm DB}}{\sigma_{\rm D}} \sim \frac{\tau_{\rm r}}{\tau_{\rm el}} \frac{1}{[1 + (\omega\tau_{\rm r})^2]} \frac{\Delta_0}{T} \left(\frac{v_{\rm F} \bar{p}_{\rm s}}{\Delta_0}\right)^2 \ln\left(\frac{T}{\Gamma_{\rm el}}\right). \tag{3.20}$$

We note that at $\Delta_0 \gg T \gg \Gamma_{\rm el}$, the conductivity at zero superfluid momentum, $\sigma(\bar{p}_{\rm s} = 0) \sim \sigma_{\rm D}$, is of order the Drude conductivity. [38, 55].

Finally, in the regime $T, v_{\rm F} \bar{p}_{\rm s} \ll \Gamma_{\rm el}$ using Eqs. (3.17) and (3.18) we get

$$\frac{\sigma_{\rm DB}}{\sigma_{\rm D}} \sim \frac{\tau_{\rm r}}{\tau_{\rm el}} \frac{1}{[1 + (\omega\tau_{\rm r})^2]} \frac{\Gamma_{\rm el}}{\Delta_0} \left(\frac{T}{\Gamma_{\rm el}}\right)^2 \left(\frac{v_{\rm F} \bar{p}_{\rm s}}{\Gamma_{\rm el}}\right)^2 \tag{3.21}$$

We note that in this temperature interval $\sigma(\bar{p}_{\rm s} = 0) \sim \sigma_{\rm D}/\Delta_0\tau_{\rm el} \ll \sigma_{\rm D}$ [16, 38].

3.3 Summary

We have shown that supercurrent dependence of the microwave conductivity of superconductors is proportional to the inelastic relaxation time. Therefore in the presence of supercurrent the absorption coefficient can be larger than the conventional contribution, which determines the conductivity at $p_{\rm s} = 0$ and is generally proportional to the elastic mean free time. We note that such mechanism should exist even in the absence of *dc* supercurrent in superconductors with broken time-reversal symmetry. For example in topological superconductors with $p_x + ip_y$ structure of the order parameter where breaking of time reversal symmetry leads to the existence of edge quasiparticle states [40, 54, 26]. In time-reversal symmetric superconductors in the absence of *dc* supercurrent, $\bar{p}_{\rm s} = 0$, the Debye mechanism of microwave absorption manifests itself in the anomalously strong non-linear microwave absorption.

The situation with a spatially uniform supercurrent density and electric field, which was considered above, can be realized in sufficiently thin superconducting films. In bulk superconductors in the presence of a magnetic field $H < H_{c1}$ that is parallel to the surface $\bar{p}_{\rm s}$ is nonzero only within the London penetration depth $\lambda_{\rm H}$ near the surface. In this case the situation is different for *s*- and *d*-wave superconductors.

In bulk samples of gapless *d*-wave superconductors in the presence of a magnetic field parallel to the surface the situation is different. The reason is that the quasiparticles in the relevant energy interval can diffuse into the bulk. Therefore in this case the inelastic relaxation time in corresponding formulas for d-wave superconductors should be substituted by the minimum between the inelastic relaxation time and the time of diffusion from the surface layer of thickness $\lambda_{\rm H}$.

Finally we would like to note that the considered above mechanism of the microwave absorption is closely related to the mechanism of *ac* conductivity of SNS junctions discussed in Refs. [4, 63, 62].

Chapter 4

DEBYE CONDUCTIVITY IN THE FLUX FLOW REGIME OF TYPE-II SUPERCONDUCTORS

When a type-II superconductor is subject to a magnetic field H in the mixed state interval, $H_{\rm c1} < H < H_{\rm c2}$, the magnetic field penetrates into the sample in the form of vortices [1]. Here $H = n_{\rm v}\Phi_0$ is the average magnetic field, with $n_{\rm v}$ being the flux line density and $\Phi_0 = \pi\hbar c/e$ - the flux quantum. Typically, defects and intrinsic disorder of the underlying crystalline lattice induce inhomogeneities in the superconducting order parameter. As a result, the vortex lattice becomes pinned to the crystalline lattice. For current densities j below some critical value $j_{\rm c}$ the vortices remain pinned, and the current in this metastable state is dissipationless. However, at $j > j_{\rm c}$, or if the flux lattice is melted by thermal fluctuations, the vortices begin to move, generating dissipation, and the system acquires a finite conductivity σ. This phenomenon has been extensively studied both experimentally and theoretically (see, for example, Refs. [5, 44, 19, 33, 20, 35, 37, 8, 56, 7, 22, 6], and references therein.).

Near the critical current density $j_{\rm c}$ this motion proceeds by creep [2], but as the current density is increased the system enters the flux flow regime, in which the vortices move with a macroscopic velocity $\mathbf{V}$. The latter is related to the macroscopic electric field $\mathbf{E}$ by the Josephson relation [25]

$$\mathbf{E} = -\frac{1}{c}[\mathbf{V} \times \mathbf{H}], \tag{4.1}$$

which implies that in the reference frame moving with the vortex lattice the electric field vanishes. The nonlinear conductivity σ in the flux flow regime can be expressed in terms of the energy dissipation rate as

$$\sigma E^2 = n_{\rm v} W, \tag{4.2}$$

where W is the energy dissipation rate per unit length of the vortex.

At relatively weak magnetic fields, $H \ll H_{\rm c2}$, the dependence of the conductivity on the magnetic field can be established from rather general considerations. The energy dissipation in this case occurs in the vortex cores. In the ohmic regime the dissipation rate in each vortex is quadratic in $\mathbf{V}$. From here, using Eqs. (4.1) and (4.2) one arrives at the conclusion that the conductivity is inversely proportional to the magnetic field, $\sigma = C/H$. Evaluation of the coefficient C requires a microscopic theory.

The problem of flux flow conductivity in superconductors has been studied for a long time. It is generally accepted that in the regime where temperature is not too close to the critical temperature $T_{\rm c}$ and the magnetic field is not too close to $H_{\rm c2}$, the longitudinal conductivity in the flux flow regime is given by the Bardeen-Stephen relation [5] (see also reviews [20, 37]):

$$\sigma_{\rm BS} = \zeta \sigma_{\rm n} \frac{H_{\rm c2}}{H}, \quad H_{\rm c2} = \frac{\Phi_0}{2\pi\xi^2}. \tag{4.3}$$

Here, ζ is a number of order unity, ξ is the superconducting coherence length, and $\sigma_{\rm n} = e^2 \nu_{\rm n} D_{\rm n}$ is the conductivity of normal metal, with $\nu_{\rm n}$ being the density of states at the Fermi energy, and D_n - the electron diffusion coefficient. The latter can be expressed in terms of the Fermi velocity v_F and the elastic momentum relaxation time, $\tau_{\rm el}$, as $D_{\rm n} = v_{\rm F}^2 \tau_{\rm el}/3$. Equation (4.3) reflects the fact that the core region of a vortex (of area $\pi\xi^2$) may be considered, with respect to its electronic properties, as a normal metal. It is important that the Bardeen-Stephen expression for the conductivity is proportional to the elastic relaxation time $\tau_{\rm el}$, and is independent of the energy relaxation time. This means that at $T \ll T_{\rm c}$ the flux flow conductivity Eq. (4.3) is temperature-independent.

In the dirty limit, $T_{\rm c}\tau_{\rm el} \ll 1$, the Bardeen-Stephen relation (4.3) was confirmed by microscopic calculations in Refs. [19, 35, 33, 37] in the approximation neglecting pinning of vortices, which is valid at the current density $j \gg j_{\rm c}$. It was also found [35] that up to a factor of order unity, the same formulas describe the flux flow conductivity of superconductors in the clean limit, $T_{\rm c}\tau_{\rm el} \gg 1$.

In this Rapid Communication we take into account random spacial fluctuations of the system parameters, which were neglected in Refs. [19, 35, 33, 20, 37], and show that they lead to an additional [to the Bardeen-Stephen value of Eq. (4.3)] contribution to the conductivity, which is proportional to the inelastic relaxation time τ_{in}. Since typically τ_{in} is orders of magnitude larger than the elastic relaxation time [1], this contribution can significantly exceed the one given by Eq. (4.3). At low temperatures this contribution is strongly temperature dependent. The physical mechanism that gives rise to this new contribution is similar to the Debye mechanism of microwave absorption in gases [11], superconductors [53, 52], and the Mandelstam-Leontovich mechanism of second viscosity in liquids [31]. In this paper we will assume the inequality $\tau_{\text{el}} \ll k_{\text{F}}\xi/\Delta$ to be satisfied and not consider the opposite, so-called superclean limit, which was discussed in a number of papers starting from [29], since in this case special attention should be paid to a large Hall angle.

4.1 2D

Below we will adopt a model where, in the absence of a magnetic field, both the modulus of the order parameter $\Delta(\mathbf{r}) = \bar{\Delta} + \delta\Delta(\mathbf{r})$ and the diffusion coefficient $D_{\text{n}}(\mathbf{r}) = \bar{D}_{\text{n}} + \delta D_{\text{n}}(\mathbf{r})$ exhibit random spatial variations. For brevity we introduce a parameter $\alpha(\mathbf{r}) \equiv (\Delta(\mathbf{r}), D_{\text{n}}(\mathbf{r}))$ which denotes both the above parameters. We assume that the spatial variations are small, $\delta\alpha \ll \bar{\alpha}$, and denote their correlation function by

$$\langle \delta\alpha(\mathbf{r})\delta\alpha(\mathbf{r}')\rangle = \left\langle (\delta\alpha)^2 \right\rangle g\left(\frac{|\mathbf{r}-\mathbf{r}'|}{L_{\text{c}}}\right), \tag{4.4}$$

where $\langle \ldots \rangle$ denotes averaging over random realizations of $\alpha(\mathbf{r})$. For simplicity we assume the correlation radius to be large, $L_{\text{c}} > \xi$.

We begin with the simplest case of a thin film of s-wave superconductor at $H \ll H_{\text{c2}}$, where the distance between vortices exceeds the coherence length ξ, while film thickness $d \leq \xi$. In this case the modulus of the order parameter changes from zero at the center of a vortex, to its maximal value Δ_0 at $|\mathbf{r}|$ of order of the inter-vortex distance. Below

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[1] Depending on parameters of the system and temperature the ratio $\tau_{\text{in}}/\tau_{\text{el}}$ can be is big as 10^{10}. See for example [18]

we assume that the temperature exceeds the mean level spacing in the core. We therefore neglect discreteness of the quasiparticle energy spectrum, and introduce the density of states $\nu(\epsilon)$ per vortex at $\epsilon < \Delta_0$. At low energies, $\epsilon \ll \Delta_0$, the density of states is $\nu(\epsilon) \sim \nu_{\mathrm{n}}\xi^2 d$. It changes by a factor of order unity at $\epsilon \sim \Delta_0$, and dramatically increases as $\epsilon \to \Delta_0$.

In the flux flow regime the vortices pass through sample regions with different values of $\alpha(\mathbf{r})$, which changes the spatial profile and amplitude of the order parameter $\Delta(\mathbf{r})$ near the vortex cores. As a result the density of states in the vortex core, $\nu(\epsilon, \alpha)$, changes in time. Since the number of energy levels is conserved, the time evolution of the density of states is described by the continuity equation:

$$\frac{\partial \nu(\epsilon, \alpha)}{\partial t} + \frac{\partial [v_\nu(\epsilon, \alpha)\nu(\epsilon, \alpha)]}{\partial \epsilon} = 0, \tag{4.5}$$

where $v_\nu(\epsilon, \alpha)$ is the level "velocity" in energy space. Integrating this equation over energy and bearing in mind that the spectral flow vanishes at $\epsilon = 0$ we can express $v_\nu(\epsilon, \alpha)$ in the form $v_\nu(\epsilon, \alpha) = -\frac{\dot{\alpha}}{\nu(\epsilon, \alpha(t))} \int_0^\epsilon d\tilde{\epsilon} \partial_\alpha \nu(\tilde{\epsilon}, \alpha)$, where $\dot{\alpha}$ denotes the time derivative of α along the trajectory of the vortex motion. To leading order in inhomogeneity we have

$$v_\nu(\epsilon, t) = A(\epsilon)\, \dot{\alpha}, \tag{4.6}$$

where

$$A(\epsilon) = -\frac{1}{\nu(\epsilon, \bar{\alpha})} \int_0^\epsilon d\tilde{\epsilon}\; \partial_\alpha \nu(\tilde{\epsilon}, \alpha)\big|_{\alpha = \bar{\alpha}}\,. \tag{4.7}$$

characterizes the sensitivity of the density of states in the vortex cores to local variations of α. The level velocities $v_\nu(\epsilon, t)$ oscillate in time as the vortices move. The typical frequency of these oscillations is $\omega_{\mathrm{E}} \sim cE/HL_{\mathrm{c}}$.

At $T > 0$ the quasiparticle states in the vortex cores are populated. As a result, the time-dependence of the density of states $\nu(\epsilon, t)$ caused by the vortex motion creates a non-equilibrium quasiparticle distribution. At low vortex velocities V, the quasiparticle distribution function $n(\epsilon, t)$ depends only on the energy ϵ. In the absence of inelastic scattering its time evolution due to the spectral flow is described by the continuity equation $\partial_{\mathrm{t}}(\nu n) + \partial_\epsilon(v_\nu \nu n) = 0$. Combining this equation with the continuity equation (4.5) for

$\nu(\epsilon, t)$, allowing for inelastic collisions, and working to lowest order in inhomogeneity, we obtain the following kinetic equation

$$\partial_{\mathrm{t}}\delta n(\epsilon, t) + v_\nu(\epsilon, t)\frac{dn_{\mathrm{F}}(\epsilon)}{d\epsilon} = I_{\mathrm{in}}\{n\}. \tag{4.8}$$

Here $n_{\mathrm{F}}(\epsilon) = (e^{\epsilon/T} + 1)^{-1}$ is the Fermi function, $\delta n(\epsilon) = n(\epsilon) - n_{\mathrm{F}}(\epsilon)$ is the nonequilibrium part of the distribution functiton, and $I_{\mathrm{in}}\{n\}$ is the linearized inelastic collision integral, which we write in the relaxation time approximation, $I_{\mathrm{in}}\{n\} = -\delta n(\epsilon, t)/\tau_{\mathrm{in}}$.

The rate of energy absorption per unit length due to the quasiparticles in the vortex core in Eq. (4.2) is given by [53, 52] $W = \frac{1}{d}\int_0^\infty d\epsilon\, \overline{\nu(\epsilon, \alpha(t))n(\epsilon, t)v_\nu(\epsilon, t)}$, where $\overline{\cdots}$ denotes time averaging along the vortex trajectory. If one replaces the quasiparticle distribution function here by the equilibrium distribution $n_{\mathrm{F}}(\epsilon)$, the energy dissipation rate vanishes as the integrand becomes a total derivative. Therefore, to lowest order in inhomogeneity we have

$$W = \frac{1}{d}\int_0^\infty d\epsilon\, \nu(\epsilon, \bar{\alpha})\, \overline{\delta n(\epsilon, t)v_\nu(\epsilon, t)}. \tag{4.9}$$

Substituting here the solution of the linearized kinetic equation (4.8), and using Eqs. (4.6), (4.7) we get

$$W \;=\; \frac{1}{d}\int_0^\infty d\epsilon \left(-\frac{dn_{\mathrm{F}}(\epsilon)}{d\epsilon}\right) \nu(\epsilon, \bar{\alpha})A^2(\epsilon)\mathcal{C}(E), \tag{4.10}$$

where the dependence on the electric field is described by the quantity $\mathcal{C}(E)$ defined as

$$\mathcal{C}(E) = \int_0^\infty e^{-\frac{\tau}{\tau_{\mathrm{in}}}}d\tau\, \overline{\dot{\alpha}(t)\dot{\alpha}(t-\tau)}. \tag{4.11}$$

The correlator of $\dot{\alpha}$ in the integrand must be averaged over the trajectories of the vortex motion at a given electric field E. Substituting Eq. (4.10) into (4.2) we obtain for the Debye contribution to the nonlinear conductivity

$$\sigma = \frac{n_{\mathrm{v}}}{d}\frac{\mathcal{C}(E)}{E^2}\int_0^\infty \frac{d\epsilon}{4T}\frac{\nu(\epsilon, \bar{\alpha})A^2(\epsilon)}{\cosh^2\left(\frac{\epsilon}{2T}\right)}. \tag{4.12}$$

This expression, with $\mathcal{C}(E)$ in the form (4.11), explicitly depends on the inelastic relaxation time τ_{in}. However, the correlator in the integrand of Eq. (4.11) depends on the statistical properties of vortex trajectories in the presence of disorder. As a result, its dependence on τ_{in} and the electric field E is difficult to establish in the general case.

The situation simplifies dramatically in the flux flow regime. In this case the vortices move with the velocity $\mathbf{V} = c\,[\mathbf{E}\times\mathbf{H}]/H^2$ along straight lines, and thus $\alpha(t) = \alpha(\mathbf{r}_0 + \mathbf{V}t)$, where $\mathbf{r}_0$ is the initial position of the vortex. As a result, $\mathcal{C}(E)$ in Eq. (4.11) can be expressed in terms of the disorder correlation function in Eq. (4.4). Passing to the Fourier representation (see Supplementary Material for a detailed derivation) we obtain

$$\mathcal{C}\,(E) = \frac{\langle(\delta\alpha)^2\rangle}{\tau_{\text{in}}}\int\frac{d\tilde{\omega}}{2\pi}\,\frac{\tilde{\omega}^2\,\tilde{g}\,(\tilde{\omega})}{\left(\frac{E^*}{E}\right)^2+\tilde{\omega}^2},\qquad E^* = \frac{HL_{\text{c}}}{c\tau_{\text{in}}}. \tag{4.13}$$

Here $\tilde{g}(\tilde{\omega}) = \int dx g(x)e^{i\tilde{\omega}x}$ denotes the Fourier transform of the function $g(x)$ in Eq. (4.4), and E^* is the characteristic electric field of the onset of nonlinearity for the Debye contribution to the conductivity.

At small electric fields, $E < E^*$, which corresponds to low flow velocities, $V\tau_{\text{in}} < L_{\text{c}}$, $\mathcal{C}(E)$ in Eq. (4.13) may be estimated as $\mathcal{C}(E) \sim (cE/H)^2\tau_{\text{in}}\langle(\nabla\alpha)^2\rangle$. Substituting this into Eq. (4.12) we obtain the following estimate for the Debye contribution to the linear flux flow conductivity,

$$\sigma_{\text{DB}} \sim \frac{1}{d}\frac{e^2}{\hbar^2}\tau_{\text{in}}\frac{H_{\text{c2}}}{H}\langle(\nabla\alpha)^2\rangle\xi^2\int_0^\infty\frac{d\epsilon}{T}\frac{\nu(\epsilon,\bar{\alpha})A^2(\epsilon)}{\cosh^2\left(\frac{\epsilon}{2T}\right)}. \tag{4.14}$$

This expression applies at an arbitrary value of the parameter $T_{\text{c}}\tau_{\text{el}}$. In the clean ($T_{\text{c}}\tau_{\text{el}} \gg 1$) and dirty ($T_{\text{c}}\tau_{\text{el}} \ll 1$) limits the coherence length ξ here is given by, respectively, $\xi = \hbar v_{\text{F}}/\pi\Delta$ and $\xi = \sqrt{\hbar D_{\text{n}}/2\Delta}$.

At low temperatures, $T \ll \Delta$, the integral in Eq. (4.14) is dominated by energies $\epsilon \sim T$. In this energy range $A(\epsilon)$ in Eq. (4.7) may be estimated as $A(\epsilon \sim T) \simeq T/\bar{\alpha}$. Taking into account that $\nabla\alpha \sim \delta\alpha/L_{\text{c}}$ and $\nu(\epsilon,\bar{\alpha}) \sim \nu_{\text{n}}\xi^2 d$, we find the Debye-type contribution to the flux flow conductivity:

$$\sigma_{\text{DB}} \sim e^2\nu_{\text{n}}\tau_{\text{in}}\frac{H_{\text{c2}}}{H}\frac{\langle(\delta\alpha)^2\rangle}{\bar{\alpha}^2}\frac{\xi^2}{L_{\text{c}}^2}\left(\frac{\xi T}{\hbar}\right)^2,\quad T \ll T_{\text{c}}. \tag{4.15}$$

The ratio between the Debye contribution to the conductivity, Eq. (4.15), and the Bardeen-

Stephen expression in Eq. (4.3) is of the order of

$$\frac{\sigma_{\mathrm{DB}}}{\sigma_{\mathrm{BS}}} \sim \frac{\tau_{\mathrm{in}}}{\tau_{\mathrm{el}}} \frac{\langle(\delta\alpha)^2\rangle}{\bar{\alpha}^2} \frac{\xi^2}{L_{\mathrm{c}}^2} \left(\frac{T\xi}{\hbar v_{\mathrm{F}}}\right)^2, \quad T \ll T_{\mathrm{c}}. \tag{4.16}$$

This ratio is proportional to a product of a very large factor $(\tau_{\mathrm{in}}/\tau_{\mathrm{el}}) \gg 1$ and other factors which are moderately small. Since $\tau_{\mathrm{in}}/\tau_{\mathrm{el}}$ may reach many orders of magnitude at low temperatures (some estimates are provided below), the whole ratio (4.16) may become large. Then the Debye contribution to the conductivity (4.14) is the dominant one. In this case the flux flow conductivity will exhibit strong temperature dependence.

The estimates (4.14)-(4.16) are obtained under the condition $\omega_{\mathrm{E}}\tau_{\mathrm{in}} \leq 1$, which corresponds to low electric fields $E < E^*$. The maximal current density attainable in the linear regime, $j_{\max} \sim \sigma_{\mathrm{DB}}E^*$ is independent of τ_{in},

$$j_{\max} \sim e^2\nu_{\mathrm{n}} \frac{\Phi_0}{cL_{\mathrm{c}}} \frac{\langle(\delta\alpha)^2\rangle}{\bar{\alpha}^2} \left(\frac{\xi T}{\hbar}\right)^2. \tag{4.17}$$

The linear regime in the current-voltage characteristic (CVC) that is dominated by the Debye conductivity (4.15) exists provided $j_{\max}$ exceeds the critical current density $j_{\mathrm{c}} \ll j_{\max}$, which is determined by the strength of vortex pinning.

If $E \gtrsim E^*$ the CVC becomes non-linear. From Eqs. (4.13), (4.10) and (4.2) it follows that at $E \gg E^*$ the Debye contribution to the current density is $j(E) \propto \sigma_{\mathrm{DB}}\,(E^*)^2/E$. At arbitrary electric fields the current density can be described by an interpolation formula

$$j_{\mathrm{DB}}(E) = \frac{\sigma_{\mathrm{DB}}\,E}{1 + a(E/E^*)^2}, \tag{4.18}$$

where a is a number of order unity. The denominator in Eq. (4.18) can be rewritten in the form $(1 + (\omega_{\mathrm{E}}\tau_{\mathrm{in}})^2)$, which is characteristic of the Debye absorption mechanism.

Since at $E > E^*$ the current density is a decreasing function of the electric field, in this regime spatially uniform flow becomes unstable. A similar scenario based on a thermal instability of the Bardeen-Stephen flux flow was proposed in Ref. [34], with the characteristic electric field $E_{\mathrm{LO}} \sim \frac{H}{c}\sqrt{D_{\mathrm{n}}/\tau_{\mathrm{in}}}$. The ratio $E^*/E_{\mathrm{LO}} = L_{\mathrm{c}}/\sqrt{D_{\mathrm{n}}\tau_{\mathrm{in}}}$ is typically small due to the large value of τ_{in}.

If $j_{\max} < j_{\rm c}$, then upon depinning at $j > j_{\rm c}$ the system would jump into the unstable branch of the CVC with the negative differential conductance, $-dj/dE \propto 1/E^2$. However, the depinning electric field may exceed the field E_1 at which the Debye contribution becomes of order $\sigma_{\rm BS}$; $\sigma_{\rm DB}/[1+(E_1/E^*)^2] \sim \sigma_{\rm BS}$. In this case the instability develops at $E \sim E_{\rm LO}$. The interval $E_1 < E < E_{\rm LO}$ exists if

$$E_1/E_{\rm LO} \sim \frac{\langle(\delta\alpha)^2\rangle}{\bar{\alpha}^2}\frac{\xi^2}{(v_{\rm F}\tau_{\rm el})^2}\left(\frac{T\xi}{\hbar v_{\rm F}}\right)^2 \ll 1. \tag{4.19}$$

Consideration of the nonlinear regime is beyond the scope of our article.

The inelastic relaxation rate $1/\tau_{\rm in}$, which controls the value of the Debye contribution to the conductivity Eq. (4.14), may be dominated by electron-electron or electron-phonon scattering. In bulk metals the rate of electron-electron collisions typically exceeds the rate of electron-phonon scattering at temperatures below a few Kelvin. We are not aware of systematic studies of these rate for quasiparticles in the vortex cores. At low temperatures, where the wavelength $\lambda_{\rm ph}$ of thermal phonons exceeds the core size ξ the electron-phonon contribution to the inelastic relaxation rate is expected to be smaller than that in bulk metals in the parameter $\xi/\lambda_{\rm ph}$. In this temperature interval the energy relaxation rate is dominated by electron-electron scattering, $1/\tau_{\rm in} = 1/\tau_{\rm (ee)}$. At $T \sim \Delta$ this rate is roughly the same as the electron-electron scattering rate in normal metals.

At $T \ll \Delta$ the electron-electron relaxation processes are characterized by two relaxation times. The shorter time, $\tau_{\rm ee}$, corresponds to relaxation processes involving only quasiparticles with typical thermal energies. Such relaxation processes conserve the total energy of quasiparticles in the vortex core and lead to the establishment of a local electron temperature in the vortex core. Subsequent relaxation to equilibrium characterized by a global electron temperature requires energy exchange between different cores and must involve quasiparticles with energies $\epsilon > \Delta_0$, which can propagate between different vortices. As a result, the relaxation time associated with such processes is much longer, $\tau_{\rm ee1} > \tau_{\rm ee}$. The Debye contribution to the linear kinetic coefficient is proportional to the longest relaxation time in the system [31]. Therefore, at $T \ll \Delta$ we must set $\tau_{\rm in} \sim \tau_{\rm ee1}$ in Eq. (4.14). We also note that at $T \ll \Delta$ there are two nonlinear electric field thresholds corresponding to the two relaxation times. The above estimates of relaxation times assumed that quasiparticles with energies $\epsilon < \Delta$ are confined to the vortex cores. However, in disordered superconductors the density of states in this energy range can be nonzero even outside the vortex cores. In this case the value of τ_{in} in Eqs. (4.14), (4.15) will be decreased.

4.2 3D

The above results apply to the case of thin films where the quasiparticles with $\epsilon < \Delta$ are confined in the cores of the pancake vortices. In bulk superconductors non-equilibrium quasiparticles can diffuse along vortex lines, which effectively shortens the energy relaxation time. To account for this effect we allow for the dependence of the quasiparticle distribution function on the coordinate z along the vortex, $\delta n(\epsilon, z, t)$, and modify the kinetic equation Eq. (4.8) as follows

$$\left[\partial_{\text{t}} - D_{\text{v}}\partial_{\text{z}}^2 + \frac{1}{\tau_{\text{in}}}\right]\delta n(\epsilon, z, t) = -\frac{dn_{\text{F}}(\epsilon)}{d\epsilon}v_\nu(\epsilon, \alpha, z), \tag{4.20}$$

where $D_{\text{v}}(\epsilon)$ is the diffusion coefficient of quasiparticles inside the vortex core. In this case the z-dependent level velocity $v_\nu(\epsilon, \alpha, z)$ is still described by Eqs. (4.6) and (4.7), but $\nu(\epsilon)$ should be understood as the density of states per unit length of the vortex. Finally, Eq. (4.9) for the energy absorption rate should be modified as follows, $W = \frac{1}{L}\int dz \int_0^\infty d\epsilon\, \nu(\epsilon, \bar{\alpha})\, \overline{\delta n(\epsilon, z, t) v_\nu(\epsilon, z, t)}$, where L is the length of the vortex line. Using Eq. (4.20) and following the arguments that lead to Eq. (4.13) we obtain (see Supplemental Material for the details):

$$W = \text{Re}\int\frac{dq d\omega}{(2\pi)^2}\int_0^\infty \frac{d\epsilon \nu(\epsilon, \bar{\alpha}) A^2(\epsilon)}{4T\cosh^2\left(\frac{\epsilon}{2T}\right)} \frac{\tau_{\text{in}}\langle(\delta\alpha)^2\rangle\omega^2\tilde{g}(q, \omega)}{1 + D_{\text{v}}q^2\tau_{\text{in}} - i\omega\tau_{\text{in}}}, \tag{4.21}$$

where $\tilde{g}(q, \omega) = \int dz dt e^{i\omega t - iqz} g\left(\frac{\sqrt{z^2+V^2t^2}}{L_{\text{c}}}\right)$.

If $D_{\text{v}}\tau_{\text{in}} < L_{\text{c}}^2$ diffusion along the vortex is irrelevant, and the energy dissipation per unit length, and thus the conductivity are the same as those for thin films, which are given by Eqs. (4.12), and (4.13).

In the opposite limit, $\sqrt{D_\mathrm{v}\tau_\mathrm{in}} \gg L_\mathrm{c}$, one finds for the Debye contribution to the conductivity

$$\sigma_\mathrm{DB}^{(3D)} \sim e^2\nu_\mathrm{n}\sqrt{\frac{\tau_\mathrm{in}}{D_\mathrm{v}}}L_\mathrm{c}\frac{H_\mathrm{c2}}{H}\frac{\langle(\delta\alpha)^2\rangle}{\bar{\alpha}^2}\frac{\xi^2}{L_\mathrm{c}^2}\left(\frac{\xi T}{\hbar}\right)^2 \tag{4.22}$$

which is smaller than the 2D result in Eq. (4.15) by a factor of order $L_\mathrm{c}/\sqrt{D_\mathrm{v}\tau_\mathrm{in}} \ll 1$. The physical reason for this is that the fluctuations $\delta\alpha(x)$ are effectively averaged over a segment of the vortex with length $\sim \sqrt{D_\mathrm{v}\tau_\mathrm{in}} \gg L_\mathrm{c}$. In this case the Debye contribution may still exceed the Bardeen-Stephen result, $\sigma_\mathrm{DB} > \sigma_\mathrm{BS}$. However, since $j_\mathrm{max}^{(3D)} \sim 1/\sqrt{\tau_\mathrm{in}}$ the range of current densities corresponding to the stable branch of the CVC ($j_\mathrm{max}^{(3D)} > j_\mathrm{c}$) turns out to be much smaller than in the 2D case.

The value of the diffusion coefficient D_v depends on the value of the parameter $\Delta\tau_\mathrm{el}$. In isotropic dirty superconductors, $\Delta\tau_\mathrm{el} \ll 1$, it can be shown [9] with the aid of the Usadel equation that $D_\mathrm{v} \approx D_\mathrm{n}$. In clean superconductors the value of D_v can be significantly smaller. In this case quasiparticle states inside a vortex are described by the Caroli-deGennes-Matricon (CdGM) solution [10] with energy dispersion $\epsilon_\mu(p_z) \approx \mu\omega^*/\sqrt{1-p_z^2/p_F^2}$, where $\mu + 1/2$ is an integer, and $\omega^* = \Delta/(k_\mathrm{F}\xi_0)$. At small energies, $\epsilon \ll \Delta$, the quasiparticle velocities along the vortex are greatly reduced in comparison to the Fermi velocity, and may be estimated as $v_\mathrm{v} \sim v_\mathrm{F}\frac{\epsilon}{\Delta}(k_\mathrm{F}\xi_0)^{-1}$, where $\epsilon = \mu\omega^*$. Determination of the elastic relaxation time in the core, τ_el^v, requires a careful consideration of quasiparticle wave functions in the core and is beyond the scope of the present paper. Assuming no delicate cancellation of the scattering amplitude for electron- and hole-components of the quasiparticle wave functions occurs, τ_el^v may be estimated using the density of states in the core as $\tau_\mathrm{el}^v \sim \tau_\mathrm{el}$. The corresponding diffusion coefficient, $D_\mathrm{v} \sim \frac{D_\mathrm{n}}{k_\mathrm{F}^2\xi_0^2}\frac{\epsilon^2}{\Delta_0^2} \sim \frac{D_\mathrm{n}}{k_\mathrm{F}^2\xi_0^2}\frac{T^2}{\Delta_0^2}$, may be several orders of magnitude smaller than that in the normal state. In such a situation diffusion of quasiparticles along the vortex line is inefficient and the 2D regime of inelastic relaxation is realized.

4.3 Summary

Finally, we mention a related effect. Microwave absorption in type-II superconductors in a mixed state may be greatly enhanced due to the Debye mechanism even without depinning of vortices by a strong transport current. The microwave field will exert a time-dependent Magnus force on the vortices, which in turn cause them to oscillate about their equilibrium positions. Because of the inhomogeneity of $\alpha(\mathbf{r})$ the density of quasiparticle states in the vortex cores will vary in time. Relaxation of quasiparticles to equilibrium will produce a contribution to microwave absorption which is proportional to the inelastic relaxation time τ_{in} at low frequencies. Thus microwave absorption measurements in the mixed state could be used to extract τ_{in} for quasiparticles in vortex cores. The present mechanism relies on the inhomogeneity of the sample parameters $\alpha(\mathbf{r})$ and produces a contribution to microwave absorption proportional to τ_{in} even in the absence of macroscopic supercurrent through the sample. In contrast, in the absence of inhomogeneity of $\alpha(\mathbf{r})$ the linear microwave absorption coefficient depends on τ_{in} only in the presence of a macrosopic supercurrent [45, 53, 52].

We developed a theory of the Debye dissipation mechanism in the flux flow regime of type-II superconductors. The energy dissipation rate due to this mechanism is controlled by the inelastic relaxation time τ_{in}, and becomes nonlinear at rather weak electric fields $E \sim E^* \sim 1/\tau_{\mathrm{in}}$, see Eq. (4.13). At weak fields, $E \lesssim E^*$, the Debye contribution to the conductivity, Eqs. (4.15), (4.22), increases as τ_{in} increases, and greatly exceeds the Bardeen-Stephen result, the enhancement being especially pronounced at low temperatures, $T \ll T_c$. In such a case the flux-flow resistivity $\rho_{\mathrm{xx}}(T) \propto 1/\tau_{\mathrm{in}}(T)$ is expected to be strongly temperature-dependent; the accompanying Hall resistance ρ_{xy} is small and scales as $\rho_{\mathrm{xy}}(T) \propto \rho_{\mathrm{xx}}^2(T)$ for the reasons outlined in Ref. [59]. Currently, we are not aware of experimental results indicating significant enhancement of the conductivity compared to the Bardeen-Stephen value. We expect however that the proposed mechanism may be observable at low temperatures in clean two-dimensional or layered materials (such as $\mathrm{NbSe_2}$ and $\mathrm{MoS_2}$), and under magnetic fields $H \ll H_{c2}$ perpendicular to the layers. It is important to work under weak pinning conditions, where the critical depinning current density j_{c} is much smaller than the pair-breaking current density j_0. This condition can be satisfied for $H \ll H_{\mathrm{c2}}$ in clean superconductors in the regime of weak collective pinning [37, 8], where j_{c} is proportional to a high power of the disorder parameter $\langle \delta\alpha^2 \rangle$, while the maximal dissipative current, Eq. (4.17) is proportional to $\langle \delta\alpha^2 \rangle$. We expect that in such materials the crossover to the unstable branch of the CVC should occur at very weak electric fields $E^* \sim 1/\tau_{\mathrm{in}}$, see Eq. (4.13). In contrast, in dirty superconductors (e.g. [43, 28, 48]), which exhibit the Bardeen-Stephen flux flow resistance (4.3) the instability occurs at a much higher field, $E_{\mathrm{LO}} \gg E^*$, predicted by Larkin and Ovchinnikov [34, 37].

Chapter 5

NON-CENTROSYMMETRIC S-WAVE

The existence or absence of many phenomena in crystalline solids may be established by analyzing their spatial symmetries. In particular, non-centrosymmetric materials display several physical effects that are forbidden by symmetry in centrosymmetric materials. Examples include the circular photogalvanic effect [12], magneto-electric effect [13], magnetochiral effect [61], and non-reciprocal magnon transport [57].

Non-centrosymmetric superconductors display a number of additional effects. The absence of inversion symmetry allows for the coexistance of singlet and triplet Cooper pairs, which is typically forbidden by symmetry. This singlet-triplet mixing allows for novel magneto-electric effects. For example, a superconducting film with Rashba-type spin-orbit coupling subjected to a magnetic field parallel to the plane develops an inhomogeneous order parameter in the absence of supercurrent [14, 15]. Non-centrosymmetric superconductors can also display an anisotropic upper critical field H_{c2} that greatly exceeds the Pauli paramagnetic limiting field.

If a particular effect is allowed by symmetry, there may be several microscopic mechanisms responsible. In this case, some mechanisms can only be realized in systems where a particular symmetry is broken. If the symmetry-sensitive mechanism is the dominant one, the effect turns out to be much stronger in systems with this broken symmetry. In the present article we consider an example of this type. We develop a microscopic theory for the linear response *ac* conductivity, $\sigma(H)$, of superconducting films in the presence of an external in-plane magnetic field H, and show that the dominant microscopic mechanism responsible for the dependence of the conductivity on H is qualitatively different from that in centrosymmetric materials.

The conductivity determines absorption of the *ac* electromagnetic field, and can be measured in microwave absorption experiments. In centrosymmetric materials and at frequencies below the superconducting gap the conductivity is controlled by elastic scattering of quasiparticles in the superconductor. As a result, $\sigma(H)$ turns out to be proportional to the elastic relaxation time τ_{el}, and independent of the inelastic relaxation time τ_{in}. In particular, near the critical temperature, the conductivity of a superconductor is of the order of the Drude conductivity of a normal metal. Below we show that in the case of non-centrosymmetric materials the magnetoconductivity is controlled by inelastic scattering of quasiparticles. As a result, the magnetoconductivity in films of non-centrosymmetric superconductors turn out to be proportional to τ_{in}. Since in typical superconductors τ_{in} exceeds τ_{el} by several orders of magnitude, the magnetoconductivity of superconducting films is significantly enhanced by inversion symmetry breaking.

The microscopic mechanism that gives rise to this new contribution to the magnetoconductivity in non-centrosymmetric superconductors can be understood from the following consideration. At fast momentum relaxation the quasiparticle distribution function depends only on the energy ϵ. The quasiparticle energy levels depend on the condensate momentum $\boldsymbol{p}_{\mathrm{s}} = \frac{1}{2}\left(\boldsymbol{\nabla}\chi - \frac{2e}{c}\boldsymbol{A}\right)$, where χ is the phase of the superconducting order parameter and $\boldsymbol{A}$ is the vector potential. In the presence of a low frequency in-plane electric field, $\boldsymbol{E}(t) = \boldsymbol{E}_{\omega}\cos\omega t$, the superfluid momentum changes in time in accordance with the condensate acceleration equation $\dot{\mathbf{p}}_{\mathrm{s}} = e\mathbf{E}$ and can be written as

$$\mathbf{p}_{\mathrm{s}}(t) = \mathbf{p}_{\mathrm{s}}^{(0)} + \frac{e}{\omega}\mathbf{E}_{\omega}\sin(\omega t). \tag{5.1}$$

The time dependence of $\mathbf{p}_{\mathrm{s}}$ leads to the motion of quasiparticle energy levels. As a result, an initially equilibrium quasiparticle distribution acquires a nonequilibrium component. The ensuing energy relaxation results in dissipation that is proportional to the inelastic relaxation time τ_{in}. This dissipation mechanism is similar to the Debye mechanism in centrosymmetric superconductors [53, 52, **?**].

The dissipation due to the Debye mechanism is completely described by the dependence of the quasiparticle density of states in a superconductor, $\nu(\epsilon)$, on the superfluid momentum.

However, being a scalar quantity, the density of states $\nu(\epsilon)$ can depend only on the square of the condensate momentum $\mathbf{p}_\mathrm{s}(t)$ or a scalar product of $\mathbf{p}_\mathrm{s}(t)$ with another vector. As a result, the Debye mechanism contributes to the linear conductivity only in the presence of a steady state supercurrent current, $\mathbf{p}_\mathrm{s}^{(0)} \neq 0$. In centrosymmetric materials, this *dc*-supercurrent must be applied to the sample. However, in non-centrosymmetric materials, the application of an in-plane magnetic field generates an inhomogenous superconducting order parameter which gives a finite value of $\mathbf{p}_\mathrm{s}^{(\mathbf{0})}$. Thus, the Debye mechanism will contribute to the linear magnetoconductivity $\sigma(H)$ in non-centrosymmetric superconductors.

Below, we develop a microscopic theory of the Debye contribution to the linear magnetoconductivity in non-centrosymmetric thin film superconductors. This theory might be useful in describing the magnetoconductivity in the interface of $\mathrm{LaAlO_3/SrTiO_3}$ [**?**] and $\mathrm{LaTiO_3/SrTiO_3}$[**?**] systems as well as two-dimensional Kondo lattices [**?**, **?**]. We hope this article serves as further motivation for experimental studies on thin film non-centrosymmetric superconducting systems with strong in-plane spin-orbit coupling.

5.1 General Formula

In this section we derive an expression for the Debye conductivity two ways. We consider the clean limit, making use of the quasiclassical kinetic equation, and write the current induced by our new Debye-type mechanism. This is only strictly valid in the limit $\xi \ll l$, where ξ is the superconducting coherence length and l is the quasiparticle mean free path [3].

We start by writing down the Hamiltonian of a normal metal,

$$H_n = \xi_\mathbf{p} + \boldsymbol{b}(\boldsymbol{p}) \cdot \boldsymbol{\sigma} - \boldsymbol{\mu} \cdot \boldsymbol{\sigma} \tag{5.2}$$

where $\xi_\mathbf{p} = E(\boldsymbol{p}) - \zeta$ is the quasiparticle energy relative to the chemical potential ζ, $\boldsymbol{\mu} = g\mu_\mathrm{B}\boldsymbol{H}/2$, where g is the electron g-factor, μ_B is the Bohr magneton, and $\boldsymbol{H}$ is the applied Zeeman field, and $\boldsymbol{b}(\boldsymbol{p})$ describes our spin-orbit coupling. For a superconductor with spin-orbit coupling, we write our Hamiltonian in particle hole space:

$$\begin{aligned} H &= \begin{pmatrix} H_n(\boldsymbol{p}) & \hat{\Delta} \\ \hat{\Delta} & -H_n^T(-\boldsymbol{p}) \end{pmatrix} \\ &= \left(\xi_{\mathbf{p}} + \boldsymbol{b}(\boldsymbol{p}) \cdot \boldsymbol{\sigma}\right) \tau_3 + \hat{\Delta}\tau_1 - \boldsymbol{\mu} \cdot \boldsymbol{\sigma} \end{aligned} \tag{5.3}$$

Here $\boldsymbol{\tau}$ are the Pauli matrices in particle-hole space, and we have rotated our basis vectors so that $\hat{\Delta} = \Delta \hat{I}$ is proportional to the identity in spin space and describes spin-singlet pairing [39]. We neglect the triplet component of the superconducting order parameter that arises in non-centrosymmetric superconductors with spin-orbit coupling as it is subleading compared to the singlet component [21], and its inclusion does not alter the physics below.

We can account for the motion of the condensate by making a gauge transformation $\boldsymbol{p} \rightarrow \boldsymbol{p} - \boldsymbol{p}_s \tau_3$ [30]. Thus, up to first order in the superfluid momentum, our energy eigenvalues are given by

$$\tilde{\epsilon}_{\mathbf{p}\sigma} = \epsilon_{\mathbf{p}\sigma} - \boldsymbol{v}_{\mathbf{p}\sigma} \cdot \boldsymbol{p}_s. \tag{5.4}$$

Here $\epsilon_{\mathbf{p}\sigma}$ are the eigenvalues of Eq. (5.3), and $\boldsymbol{v}_{\mathbf{p}\sigma}$ are the diagonal matrix elements of the velocity operator, $\hat{\boldsymbol{v}}_{\mathbf{p}} = \frac{\partial}{\partial \boldsymbol{p}} \left(\xi_{\boldsymbol{p}} + \boldsymbol{b}(\boldsymbol{p}) \cdot \boldsymbol{\sigma}\right)$. In clean superconductors, when the elastic mean free path is larger than the superconducting coherence length, the quasiparticle distribution function can be described by $\hat{f}_{\mathbf{p}}$, which is an operator in spin-space and is diagonal, with diagonal elements $f_{\mathbf{p}\sigma}$. The current density can be written [3]

$$\boldsymbol{j} = e\frac{N}{m}\boldsymbol{p}_{\mathrm{s}} + e\sum_{p} \mathrm{Tr}\left[\hat{\boldsymbol{v}}_{\mathbf{p}} \hat{f}_{\mathbf{p}}\right] \tag{5.5}$$

where N is the total electron density. Thus, we need to find the distribution function $\hat{f}_{\mathbf{p}}$. The kinetic equation for the distribution function matrix elements in spin space, $f_{\mathbf{p}\sigma}$, considering only elastic relaxation first,

$$\frac{\partial f_{\mathbf{p}\sigma}}{\partial t} + \boldsymbol{v}_{\mathbf{p}\sigma} \cdot \frac{\partial f_{\mathbf{p}\sigma}}{\partial \mathbf{p}} - \frac{\partial \tilde{\epsilon}_{\mathbf{p}\sigma}}{\partial \mathbf{r}} \cdot \frac{\partial f_{\mathbf{p}\sigma}}{\partial \mathbf{p}} = I_{\mathrm{el}}\left(\{f_{\mathbf{p}\sigma}\}\right). \tag{5.6}$$

Here the elastic scattering integral $I_{\mathrm{el}}\left(\{f_{\mathbf{p}\sigma}\}\right)$ takes into account inter- and intra-band (σ) hopping. In the situation under consideration the second and third terms on the left hand side are zero, and so Eq. (5.6) reduces to

$$\frac{\partial f_{\mathbf{p}\sigma}}{\partial t} = I_{\text{el}}\left(\{f_{\mathbf{p}\sigma}\}\right). \tag{5.7}$$

We multiply (5.7) by $\delta\left(\epsilon - \tilde{\epsilon}_{\mathbf{p}\sigma}\right)$ and sum over $\boldsymbol{p}$ and σ. It is convenient to define

$$\sum_{\mathbf{p}\sigma} \delta\left(\epsilon - \tilde{\epsilon}_{\mathbf{p}\sigma}\right) f_{\mathbf{p}\sigma} = n(\epsilon)\nu(\epsilon) \tag{5.8a}$$

$$\sum_{\mathbf{p}\sigma} \delta\left(\epsilon - \tilde{\epsilon}_{\mathbf{p}\sigma}\right) \partial_t\left(\tilde{\epsilon}_{\mathbf{p}\sigma}\right) f_{\mathbf{p}\sigma} = v_\nu(\epsilon)\nu(\epsilon)n(\epsilon) \tag{5.8b}$$

where $\nu(\epsilon)$ describes the quasiparticle occupancy of state ϵ and $v_\nu(\epsilon)$ is the level 'sensitivity', which can be written

$$v_\nu(\epsilon) = e\boldsymbol{E}\cdot\boldsymbol{V}(\epsilon) = -\frac{e\boldsymbol{E}}{\nu(\epsilon)}\sum_{\mathbf{p}\sigma}\delta(\epsilon - \tilde{\epsilon}_{\mathbf{p}\sigma})\frac{d\tilde{\epsilon}_{\mathbf{p}\sigma}}{d\boldsymbol{p}_s} \tag{5.9}$$

where $\boldsymbol{V}(\epsilon)$ is the level 'sensitivity'. v_ν can be obtained by solving the quasiparticle energy level continuity equation $\partial_t\nu(\epsilon) + \partial_\epsilon\left(v_\nu(\epsilon)\nu(\epsilon)\right) = 0$, which can be derived by taking a time derivative of the quasiparticle density of states.

The elastic scattering integral is nullified by an arbitrary function of energy ϵ, and therefore our kinetic equation in the absence of inelastic relaxation is

$$\frac{\partial\left(\nu(\epsilon)n(\epsilon)\right)}{\partial t} + \frac{\partial\left(v_\nu(\epsilon)\nu(\epsilon)n(\epsilon)\right)}{\partial\epsilon} = 0. \tag{5.10}$$

Including inelastic relaxation as well as using the continuity of levels, our kinetic equation becomes

$$\frac{\partial n(\epsilon)}{\partial t} + v_\nu(\epsilon)\frac{\partial n(\epsilon)}{\partial\epsilon} = I_{\text{in}}\{n\} \tag{5.11}$$

We use the relaxation time approximation, $I_{\text{in}} = -\frac{\delta n(\epsilon)}{\tau_{\text{in}}}$, which is valid as long as $\delta\epsilon \ll T$, where $\delta\epsilon$ is the relevant energy interval for the Debye mechanism, and $\delta n = n(\epsilon) - n_{\text{F}}(\epsilon)$, where $n_{\text{F}}(\epsilon) = \left[e^{-\epsilon/T}+1\right]^{-1}$ is the Fermi distribution. Fourier transforming in time and linearizing with respect to the applied electric field $\boldsymbol{E}$, we solve for the non-equilibrium distribution function

$$\delta n_\omega(\epsilon) = \tau_{\text{in}}\frac{e\boldsymbol{E}\cdot\boldsymbol{V}(\epsilon)}{1 - i\omega\tau_{\text{in}}}\left[-\partial_\epsilon n_{\text{F}}\right]. \tag{5.12}$$

The dissipative component of the current can be written

$$\boldsymbol{j}_{\mathrm{D}} = e\sum_{\mathbf{p}\sigma} \boldsymbol{v}_{\mathbf{p}\sigma} f_{\mathbf{p}\sigma} = e\int d\epsilon\ \nu(\epsilon)\boldsymbol{V}(\epsilon)n(\epsilon) \tag{5.13}$$

which is zero when $n(\epsilon)$ is the equilibrium distribution. Using Eq. (5.12) we get for the dissipative part of the current

$$\boldsymbol{j}_{\mathrm{D}} = e^2\frac{\tau_{\mathrm{in}}}{1+(\omega\tau_{\mathrm{in}})^2}\int d\epsilon\ \nu(\epsilon)\boldsymbol{V}(\epsilon)\left(\boldsymbol{V}(\epsilon)\cdot\boldsymbol{E}\right)\left[-\partial_\epsilon n_{\mathrm{F}}\right]. \tag{5.14}$$

assuming the electric field is parallel to the level velocity $\boldsymbol{V}(\epsilon)$, and making the approximation that $-\partial_\epsilon n_{\mathrm{F}} = 1/4T$, which is justified when $\delta\epsilon \ll T$, we arrive at the expression for the Debye conductivity for systems with strong spin-orbit coupling

$$\frac{\sigma_{\mathrm{DB}}}{\sigma_{\mathrm{D}}} = \frac{3}{4}\frac{\tau_{\mathrm{in}}}{\tau_{\mathrm{el}}}\int\frac{d\epsilon}{T}\frac{\nu(\epsilon)}{\nu_{\mathrm{n}}}\frac{V^2(\epsilon)}{v_{\mathrm{F}}^2}. \tag{5.15}$$

One of the most important distinctions between the derivation above and the case in the absence of spin-orbit coupling [52, 53] is that by including inter-band relaxation in $I_{el}\left(\{f_{p\sigma}\}\right)$ we are also including the spin-orbit relaxation time τ_{so}. Thus, the distribution function $f_{\mathbf{p}\sigma}$ will be a function of only $\tilde{\epsilon}_{p\sigma}$ for times much longer than τ_{el} and τ_{so}.

5.2 S-wave clean

In the case of ballistic motion of quasiparticles ($\mu\tau_{\mathrm{el}}^2\Delta \ll 1$) we can write down the density of states using the single particle excitation spectrum. In the presence of an in-plane Zeeman field and a supercurrent, the single particle excitation spectrum takes the form (up to first order in μ and $\bar{p}_{\mathrm{s}}$)

$$\tilde{\epsilon}_{\pm} = \sqrt{(\xi_{\mathbf{p}} \pm |\boldsymbol{b}(\boldsymbol{p})|)^2 + \Delta^2} - \tilde{\boldsymbol{v}}_{\pm}\cdot\bar{\boldsymbol{p}}_{\mathrm{s}} \mp \boldsymbol{\mu}\cdot\hat{\boldsymbol{b}} \tag{5.16}$$

Here $\bar{\boldsymbol{p}}_{\mathrm{s}}$ is the supercurent generated via the magneto-electric effect due to the presence of the in-plane Zeeman field, $\hat{\boldsymbol{b}} = \boldsymbol{b}(\boldsymbol{p})/|\boldsymbol{b}(\boldsymbol{p})|$, and

$$\tilde{v}_{\pm}^{\alpha} = v^{\alpha} \pm \frac{d\boldsymbol{b}(\boldsymbol{p})}{dp^{\alpha}}\cdot\hat{\boldsymbol{b}}. \tag{5.17}$$

The density of states in two spatial dimensions is

$$\nu(\epsilon) = \frac{\nu_n}{2} \sum_{\pm} \int \frac{d\phi}{2\pi} f_{\pm}(\phi) \frac{\epsilon + \tilde{\boldsymbol{v}}_{\mathrm{F}\pm} \cdot \bar{\boldsymbol{p}}_{\mathrm{s}} \pm \boldsymbol{\mu} \cdot \hat{\boldsymbol{b}}}{\sqrt{(\epsilon + \tilde{\boldsymbol{v}}_{\mathrm{F}\pm} \cdot \bar{\boldsymbol{p}}_{\mathrm{s}} \pm \boldsymbol{\mu} \cdot \hat{\boldsymbol{b}})^2 - \Delta^2}} \tag{5.18}$$

where $\tilde{\boldsymbol{v}}_{\mathrm{F}\pm}$ is the band velocity Eq. (5.17) evaluated on the respective Fermi circle ±. For linear spin-orbit coupling, this reduces to $\boldsymbol{v}_F$, to linear order in $|\boldsymbol{b}|/E_F$. The function $f_{\pm}(\phi)$ characterizes the Fermi circle's deviation from being perfectly circular. In the case of a perfectly circular Fermi circle, $f_{\pm}(\phi) = 1$, which we will assume below.

It is important to note that the above perturbation theory fails when $\boldsymbol{b}(\boldsymbol{p}) = 0$. In this case, however, the angles that contribute to the density of states where perturbation theory is valid are much larger than the angles where perturbation theory fails (which go as $\sim \mu/|\boldsymbol{b}|_{\max} \ll 1$, where $|\boldsymbol{b}|_{\max}$ is the spin-orbit coupling evaluated when $\boldsymbol{\mu} \parallel \boldsymbol{b}$), and thus the above treatment is still applicable to materials with point-like nodes of $\boldsymbol{b}(\boldsymbol{p})$ on their Fermi surfaces.

The angular integral in Eq. (5.19) depends on the relative angle between $\boldsymbol{\mu}$ and $\boldsymbol{b}$. We assume the supercurrent $\bar{p}_{\mathrm{s}}$, which is generated due to the magnetization μ, is in the same direction as $\boldsymbol{\mu}$ for Dresselhaus spin-orbit coupling, $\boldsymbol{b}_{\mathrm{D}} = \beta \boldsymbol{p}$, and is perpendicular to $\boldsymbol{\mu}$ for Rashba spin-orbit coupling, $\boldsymbol{b}_{\mathrm{R}} = \alpha \boldsymbol{p} \times \hat{\boldsymbol{n}}$, where $\hat{\boldsymbol{n}}$ is one of the two non-equivalent normals to the sample. For energies near Δ the density of states is given by

$$\nu(\epsilon) = \frac{\nu_{\mathrm{n}}}{2\pi} \sum_{\pm} \theta\left(w - \gamma_{\pm}\right) \frac{K\left(\frac{w+\gamma_{\pm}}{2\gamma_{\pm}}\right)}{\sqrt{\gamma_{\pm}}} \tag{5.19}$$

where $w = \frac{\epsilon - \Delta}{\Delta}$, $\gamma_{\pm} = \frac{v_{\mathrm{F}} \bar{p}_{\mathrm{s}} \pm \mu}{\Delta}$, $\nu_{\mathrm{n}} = \frac{m}{\pi}$ is the density of states in 2D, and $K(m) = \int_0^{\pi/2} d\phi (1 - m \sin^2 \phi)^{-1/2}$ is the complete elliptic integral of the first kind.

The most important energies for microwave absorption are those near the gap, Δ. It should be noted that there is an energy interval below Δ where only one Fermi surface (+) contributes to the density of states. This energy interval will give the dominant contribution to the level velocity as at higher energies, when both Fermi surfaces contribute to the density of states, there is a near cancellation of the level velocity

$$\boldsymbol{V}(\epsilon) = -\frac{1}{\nu(\epsilon)} \int_0^{\epsilon} d\tilde{\epsilon} \frac{\partial \nu(\epsilon)}{\partial \boldsymbol{p}_{\mathrm{s}}}. \tag{5.20}$$

Where it is clear that in the energy interval $-\gamma_+ \leq w \leq -|\gamma_-|$ only the + Fermi surface gives a non-zero contribution. Thus the level sensitivity in this interval is independent of energy:

$$V(\epsilon) = -v_{\mathrm{F}}. \tag{5.21}$$

For higher energies the level sensitivity is smaller by a factor $|\boldsymbol{b}(p_{\mathrm{F}})|/E_{\mathrm{F}}$. Using Eqs. (5.19) and (5.21) in Eq. (5.15) we obtain

$$\frac{\sigma_{\mathrm{DB}}}{\sigma_{\mathrm{D}}} = \frac{3}{8} \frac{\tau_{\mathrm{in}}}{\tau_{\mathrm{el}}} \frac{\Delta}{T} \frac{v_{\mathrm{F}} \bar{p}_{\mathrm{s}}}{\sqrt{\mu \Delta}}. \tag{5.22}$$

The supercurrent in the presence of an in-plane Zeeman field in a ballistic superconductor is given by $v_F \bar{p}_s \sim \frac{|\boldsymbol{b}(p_{\mathrm{F}})|}{E_{\mathrm{F}}} \mu$, giving us

$$\frac{\sigma_{\mathrm{DB}}}{\sigma_{\mathrm{D}}} \sim \frac{\tau_{\mathrm{in}}}{\tau_{\mathrm{el}}} \frac{\Delta}{T} \sqrt{\frac{\mu}{\Delta}} \frac{|\boldsymbol{b}(p_{\mathrm{F}})|}{E_{\mathrm{F}}}. \tag{5.23}$$

While $|\boldsymbol{b}(p_{\mathrm{F}})|/E_{\mathrm{F}} \ll 1$ the factor μ/Δ can be as large as order 1.

5.3 S-wave dirty

While the above derivation of Eq. (5.15) effectively illustrates the microscopic mechanism behind the Debye contribution to the magnetoconductivity, it is only strictly valid in the clean limit, when the kinetic equation for superconductors is applicable. For arbitary disorder, one can relate the conductivity to the work per unit volume:

$$W = \frac{1}{2} \sigma E^2, \tag{5.24}$$

where the work is given by

$$W = \int d\epsilon \, \overline{\nu(\epsilon) n(\epsilon, t) v_{\nu}(\epsilon)}, \tag{5.25}$$

and the bar denotes a time average. The work will be zero when using the equilibrium distribution function for $n(\epsilon, t)$, as the integrand reduces to a total derivative. Plugging in $v_\nu = e\boldsymbol{E} \cdot \boldsymbol{V}(\epsilon)$, Eq. (5.12) for δn, and using Eq. (5.24) we arrive at equation (5.15).

The diffusive regime is realized when $\mu\tau_{\mathrm{el}}^2\Delta \ll 1$ and can be realized in both clean ($\Delta\tau_{\mathrm{el}} \gg 1$) and dirty ($\Delta\tau_{\mathrm{el}} \ll 1$) superconductors. We can arrive at the crossover parameter between the ballistic and diffusive regimes via a simple consideration: for the relevant quasiparticles to move ballistically the energy-dependent elastic scattering rate $\tau_{\mathrm{el}}(\epsilon) \sim \sqrt{\frac{\epsilon-\Delta}{\Delta}}$ [42] must be smaller than the splitting of the BCS singularity $\delta\epsilon \sim \mu$. We arrive at $\mu\tau_{\mathrm{el}}^2\Delta$ controlling the crossover between the relevant quasiparticles moving ballistically or diffusively. To account for impurity scattering we consider the self energy due to impurity scattering in the Born approximation. It can be shown (See Appendix **??** for details) that in the diffusive limit the density of states can be written

$$\nu(\epsilon) = \frac{\nu_{\mathrm{n}}}{\sqrt{2}}\Re y \tag{5.26}$$

where y is given by the proper root ($\Re y \geq 0$) of the cubic equation

$$\frac{i\gamma^2}{2\sqrt{2}\beta}y^3 - y^2 w + 1 = 0 \tag{5.27}$$

with $\beta = \frac{1}{\tau_{\mathrm{el}}\Delta}$ and $\gamma^2 = \frac{\mu^2 + v_{\mathrm{F}}^2\bar{p}_{\mathrm{s}}^2}{\Delta^2}$. The density of states takes the form

$$\nu(\epsilon) = \theta\left(\tilde{w} + \frac{3}{2^{5/3}}\right)\frac{\nu_{\mathrm{n}}}{\sqrt{3}}\frac{\beta^{1/3}}{\gamma^{2/3}}\left[\frac{\tilde{\alpha}(\tilde{w})}{2^{4/3}} - \frac{2^{4/3}\tilde{w}^2}{\tilde{\alpha}(\tilde{w})}\right], \tag{5.28a}$$

$$\tilde{\alpha}(\tilde{w}) = \left(16\tilde{w}^3 + 27 + 3\sqrt{3}\sqrt{32\tilde{w}^3 + 27}\right)^{1/3}. \tag{5.28b}$$

Here $\tilde{w} = w\left(\frac{\beta}{\gamma^2}\right)^{2/3}$. Note that, unlike the ballistic regime where we could separate the contributions to the density of states coming from different Fermi surfaces, in the diffusive regime we can no longer distinguish between contributions from individual Fermi surfaces due to $\tau_{\mathrm{el}}^{-1} \gg |\boldsymbol{b}(p_{\mathrm{F}})|$. From here we use Eqs. (5.20) and (5.15) and arrive at

$$\frac{\sigma_{\mathrm{DB}}}{\sigma_{\mathrm{D}}} = I_{\mathrm{D}}\frac{\tau_{\mathrm{in}}}{\tau_{\mathrm{el}}}\frac{\Delta}{T}\frac{\tau_{\mathrm{el}}^{5/3}\Delta^{1/3}(v_{\mathrm{F}}\bar{p}_{\mathrm{s}})^2}{\mu^{2/3}} \tag{5.29}$$

where $I_{\mathrm{D}} = 0.38727$ is a definite integral defined in Eq. (D.13) of the Appendix. Using $v_F \bar{p}_s \sim \frac{|\boldsymbol{b}(p_{\mathrm{F}})|}{E_{\mathrm{F}}} \mu$ for the ground state superfluid momentum

$$\frac{\sigma_{\mathrm{DB}}}{\sigma_{\mathrm{D}}} \sim \frac{\tau_{\mathrm{in}}}{\tau_{\mathrm{el}}} \frac{\Delta}{T} \left(\frac{|\boldsymbol{b}(p_{\mathrm{F}})|}{E_{\mathrm{F}}} \right)^2 \tau_{\mathrm{el}} \left(\Delta \tau_{\mathrm{el}}^2 \mu^4 \right)^{1/3} . \tag{5.30}$$

Chapter 6

SUMMARY

As demonstrated above, the Debye mechanism appears in a wide variety of superconducting systems and can contribute to the linear and non-linear conductivities. The Debye mechanism only requires a time and energy dependent density of states [11]. As such, on top of an AC electric field, one needs only apply an DC supercurrent to give rise to the Debye mechanism in bulk S- and D-wave superconductors, or an in-plane magnetic field in the case of non-centrosymmetric superconductors. It also arises due to long-range disorder in the flux-flow regime of Type-II superconductors. Previous work has shown this effect also arises naturally in the conductivity of SNS junctions discussed in Refs. [4, 63, 62]. There are still many avenues of research on the Debye mechanism in superconductors, namely further work in SNS junctions. Of particular experimental relevance is in SNS junctions with strong spin-orbit coupling and an in-plane magnetic field, where there should be an anisotropic critical current in the ground state, and anisotropic conductivity upon the application of a voltage difference across the junction.

Appendix A

DERIVATION OF THE DEBYE CONTRIBUTION TO THE LINEAR CONDUCTIVITY IN S-WAVE SUPERCONDUCTORS

In order to evaluate the linear conductivity in the presence of supercurrent we express the density of states in a superconductor in terms of the dimensionless disorder-averaged retarded Green's function $g(\epsilon)$ at coinciding points

$$\frac{\nu(\epsilon,p_{\rm s})}{\nu_n} = -\frac{2}{\pi}{\rm Im}\, g(\epsilon). \tag{A.1}$$

The latter can be expressed as [1, 39]

$$g(\epsilon) \;=\; \left\langle \frac{\tilde{\epsilon}-\boldsymbol{v}\cdot\boldsymbol{p}_{\rm s}}{\sqrt{(\tilde{\epsilon}-\boldsymbol{v}\cdot\boldsymbol{p}_{\rm s})^2-|\tilde{\Delta}|^2}} \right\rangle, \tag{A.2}$$

where $\langle\ldots\rangle$ denotes averaging over the Fermi surface and the disorder-renormalized energy $\tilde{\epsilon}$ and order parameter $\tilde{\Delta}$ are given by

$$\tilde{\epsilon} \;=\; \epsilon+\frac{i}{2\tau_{\rm el}}\left\langle \frac{\tilde{\epsilon}-\boldsymbol{v}\cdot\boldsymbol{p}_{\rm s}}{\sqrt{(\tilde{\epsilon}-\boldsymbol{v}\cdot\boldsymbol{p}_{\rm s})^2-|\tilde{\Delta}|^2}} \right\rangle, \tag{A.3a}$$

$$\tilde{\Delta} \;=\; \Delta+\frac{i}{2\tau_{\rm el}}\left\langle \frac{\tilde{\Delta}}{\sqrt{(\tilde{\epsilon}-\boldsymbol{v}\cdot\boldsymbol{p}_{\rm s})^2-|\tilde{\Delta}|^2}} \right\rangle, \tag{A.3b}$$

where we set $\hbar=1$. For simplicity we assume the Fermi surface to be spherical. Performing the angular averaging and introducing the dimensionless variables $\delta=\tilde{\Delta}/\Delta$, $u=\tilde{\epsilon}/\tilde{\Delta}$ we can write the disorder-renormalization equations for $\tilde{\epsilon}$ and $\tilde{\Delta}$ in the following form

$$u\delta \;=\; \frac{\epsilon}{\Delta}+\frac{i\delta\gamma}{4\zeta}\left(\sqrt{u_+^2-1}-\sqrt{u_-^2-1}\right), \tag{A.4a}$$

$$\delta \;=\; 1+\frac{i\delta\gamma}{4\zeta}\ln\left(\frac{u_- - \sqrt{u_-^2-1}}{u_+ - \sqrt{u_+^2-1}}\right). \tag{A.4b}$$

Here $\gamma = 1/(\tau_{\rm el}\Delta)$ characterizes the disorder strength, $\zeta = v_{\rm F}p_{\rm s}/\Delta$ is a dimensionless measure of the condensate momentum, and we introduced $u_\pm = u \pm \zeta/\delta$. The Green's function $g(\epsilon)$ in Eqs. (A.1) and (A.2) is expressed in terms of these variables as

$$g = \frac{2i}{\gamma}\left(\frac{\epsilon}{\Delta} - u\delta\right). \tag{A.5}$$

The system of Eqs. (A.4), (A.5) and (A.1) describes the energy dependence of the density of states in the presence of supercurrent.

In the limit $\zeta \to 0$ the solution of Eqs. (A.4) is given by $u_0 = \epsilon/\Delta$, and $\delta_0 = 1 + \frac{i\gamma}{2\sqrt{u^2-1}}$. When substituted into Eqs. (A.1) and (A.5) this yields the conventional result for the density of states, $\nu(\epsilon)/\nu_n = \epsilon/\sqrt{\epsilon^2 - \Delta^2}$.

At $\zeta = v_{\rm F}p_{\rm s}/\Delta \ll 1$ the density of sates $\nu(\epsilon)$ is significantly affected by the condensate momentum $p_{\rm s} \neq 0$ only for energies near the gap, $|\epsilon - \Delta| \ll |\Delta|$, see Eq. (2.8). This interval of energies corresponds to $|u_\pm - 1| \ll 1$. The $p_{\rm s}$ dependence of $\nu(\epsilon, p_{\rm s})$ in this interval is nonanalytic, c.f. Eq. (2.8), and the expansion of Eqs. (A.4) in powers of $u_+ - u_- = 2\zeta/\delta$ fails. Formally, the expansion fails because at $\zeta = 0$ the solution of Eq. (A.4) approaches the branching point, $u = 1$, of the radicals in that equation. To circumvent this difficulty we introduce new variables x and y via

$$u = 1 - x^2 - y^2, \quad \delta = -\frac{\zeta}{2xy} \tag{A.6}$$

and rewrite Eqs. (A.4) in the form

$$-\frac{\zeta(1-x^2-y^2)}{2xy} = 1 + w - \frac{\gamma}{4\zeta d}\left[(x+y)\sqrt{2-(x+y)^2} - (x-y)\sqrt{2-(x-y)^2}\right] \tag{A.7a}$$

$$-\frac{\zeta}{2xy} = 1 + \frac{i\gamma}{4\zeta d}\ln\left[\frac{1-(x-y)^2 - i(x-y)\sqrt{2-(x-y)^2}}{1-(x+y)^2 - i(x+y)\sqrt{2-(x+y)^2}}\right]. \tag{A.7b}$$

Here we have introduced the notation $w = (\epsilon - \Delta)/\Delta$.

The advantage of writing the disorder renormalization equations in terms of the variables x and y is that the branching points of the radicals in Eqs. (A.4), which are located at

$u_\pm = 1 - (x \pm y)^2 = 1$, are resolved in terms of rational functions of x and y. As a result the expressions in the right hand side in Eqs. (A.7) become amenable to a series expansion in x and y.

Although x and y enter Eqs. (A.6) and (A.7) on equal footing we choose y to be odd in ζ so that in the absence of supercurrent $y = 0$. At small supercurrent, $\zeta \ll 1$, the variable y is also small. In this regime, the density of states in the relevant energy interval near the spectrum edge may be determined by expanding the expressions in the right hand side of Eqs. (A.7) to third order in y:

$$\frac{\zeta}{\gamma}\left(1 - x^2 - y^2 + \frac{2xy(1+w)}{\zeta}\right) = \frac{y}{\sqrt{2-x^2}}\left[x^2 - 1 + \frac{y^2}{(2-x^2)^2}\right], \tag{A.8a}$$

$$-\frac{\zeta}{\gamma}\left(\frac{2xy}{\zeta} + 1\right) = \frac{y}{\sqrt{2-x^2}}\left[1 + \frac{1}{3}\frac{(1+x^2)y^2}{(2-x^2)^2}\right]. \tag{A.8b}$$

The relevant energy interval, $|w| \ll 1$, corresponds to $|x| \ll 1$. In this case Eqs. (A.8) can be simplified to

$$x\big(x^2 + w\big)\left(2x + \frac{\gamma}{\sqrt{2}}\right)^2 = -\zeta^2\left(x + \frac{\gamma}{3\sqrt{2}}\right), \tag{A.9a}$$

$$y = -\frac{\sqrt{2}\zeta}{2\sqrt{2}x + \gamma}. \tag{A.9b}$$

The density of states is obtained by substituting Eqs. (A.6) into (A.5) and expanding in small y then x. Within the accuracy of our approximation $\nu(\epsilon, p_\mathrm{s})$ in the relevant energy interval, $|w| \ll 1$, is given by

$$\nu(\epsilon, p_\mathrm{s}) = \frac{\nu_n}{\sqrt{2}}\Im x^{-1}, \tag{A.10}$$

and may be determined by solving Eq. (A.9a).

The quintic equation (A.9a) for the variable x has five roots. The complex solutions come in pairs of complex conjugate numbers corresponding to retarded and advanced Green's functions. The retarded solutions correspond to $\Im x^{-1} \geq 0$, c.f. Eq. (A.10). The roots of a general quintic equation can be expressed via the Jacobi theta functions, however Eq. (A.9a) simplifies significantly in the limiting regimes $\zeta/\gamma^2 = v_\mathrm{F}\bar{p}_\mathrm{s}\tau_\mathrm{el}^2\Delta \gg 1$, and $\zeta/\gamma^2 = v_\mathrm{F}\bar{p}_\mathrm{s}\tau_\mathrm{el}^2\Delta \ll$

1, which correspond to, respectively, ballistic and diffusive motion of quasiparticles that participate in microwave absorption.

A.1 Ballistic Regime

At relatively large supercurrent densities, $\zeta/\gamma^2 \gg 1$, the relevant root of Eq. (A.9a) satisfies the condition $|x| \gg \gamma$. In this case we may neglect γ in Eq. (A.9a) to get a quadratic equation for x^2:

$$x^4 + wx^2 + \frac{\zeta^2}{4} = 0, \tag{A.11}$$

which yields the solution

$$x^2 = \frac{\zeta^2}{\left(\sqrt{-w-\zeta} - \sqrt{-w+\zeta}\right)^2}. \tag{A.12}$$

Substituting this into Eq. (A.10) we obtain Eq. (2.8);

$$\frac{\nu(\epsilon, p_\mathrm{s})}{\nu_n} = \frac{1}{\sqrt{2\zeta}} \left[\eta_b\left(z+1\right) - \eta_b\left(z-1\right)\right]. \tag{A.13}$$

Here the dimensionless energy variable z and the function $\eta_b(x)$ are defined by

$$z = \frac{w}{\zeta}, \quad \eta_b(x) = \theta(x)\sqrt{x}. \tag{A.14}$$

Substituting Eq. (A.13) into Eq. (2.1) we obtain,

$$\begin{aligned} \frac{V(\epsilon, p_s)}{v_F} &= \frac{\eta_b(z+1) + \eta_b(z-1)}{\eta_b(z+1) - \eta_b(z-1)} \\ &- \frac{2}{3}\frac{\eta_b^3\left(z+1\right) - \eta_b^3\left(z-1\right)}{\eta_b\left(z+1\right) - \eta_b\left(z-1\right)}. \end{aligned} \tag{A.15}$$

Substituting Eq. (A.13) into Eq. (5.15) and using the dimensionless variable z we obtain Eq. (5.23) for the longitudinal conductivity with I_b given by

$$I_b = \frac{3}{4}\int_{-1}^{\infty} dz \frac{\left[\eta_b\left(z+1\right) - \eta_b\left(z-1\right)\right]}{\sqrt{2}} \frac{V^2(\epsilon, p_s)}{v_F^2}. \tag{A.16}$$

Substituting Eq. (A.15) into Eq. (A.16) we obtain

$$I_b = \frac{8}{45}. \tag{A.17}$$

A.2 Diffusive Regime

At small supercurrent densities, $\zeta/\gamma^2 \ll 1$, the relevant root of Eq. (A.9a) satisfies the condition $|x| \ll \gamma$. In this regime, which corresponds to diffusive motion of quasiparticles participating in microwave absorption, Eq. (A.9a) simplifies to the cubic equation

$$x\left(x^2+w\right)+\frac{\sqrt{2}\zeta^2}{3\gamma}=0. \tag{A.18}$$

Using the Cardano formula and substituting the root with $\Im x^{-1} \geq 0$ into Eq. (A.10) we can express the density of states in terms of rescaled variables $\eta \equiv \frac{2\zeta^2}{3\gamma}$, and $\tilde{w} = w\eta^{-2/3}$ in the form

$$\nu(\epsilon, p_s) = \nu_n \frac{\tilde{\nu}_d(\tilde{w})}{\eta^{1/3}}, \tag{A.19}$$

$$\tilde{\nu}_d(\tilde{w}) = \frac{1}{2\sqrt{3}}\theta\left(\tilde{w}+\frac{3}{2}\right)\left[\frac{\tilde{\alpha}(\tilde{w})}{2^{2/3}} - \frac{2^{2/3}w^2}{\tilde{\alpha}(\tilde{w})}\right], \tag{A.20}$$

where the function $\tilde{\alpha}(\tilde{w})$ is defined by

$$\tilde{\alpha}(\tilde{w}) = \left(4\tilde{w}^3+27+\sqrt{27}\sqrt{8\tilde{w}^3+27}\right)^{1/3}. \tag{A.21}$$

Substituting this form into Eq. (4) we obtain

$$V(\epsilon, p_s) = \frac{1}{\tilde{\nu}_d(\tilde{w})}\frac{Dp_s}{\eta^{1/3}} I(\tilde{w}), \tag{A.22}$$

where D is the diffusion coefficient and $I(\tilde{w})$ is given by

$$\begin{aligned} I(\tilde{w}) = \int_{-\frac{3}{2}}^{\tilde{w}} \frac{d\tilde{x}}{2^{5/3}\sqrt{3}} \Bigg[& \frac{2^{4/3}\tilde{x}^2}{\tilde{\alpha}(\tilde{x})} - \tilde{\alpha}(\tilde{x}) + \\ & \left(\frac{1}{\tilde{\alpha}^2(\tilde{x})}+\frac{2^{4/3}\tilde{x}^2}{\tilde{\alpha}^4(\tilde{x})}\right)\left(18+\frac{\sqrt{3}}{2}\left[\frac{16\tilde{x}^3+108}{\sqrt{8\tilde{x}^3+27}}\right]\right)\Bigg]. \end{aligned}$$

Using Eqs. (A.22) and (A.19) in Eq. (5.15) we get Eq. (5.30) for the longitudinal Debye conductivity where the definite integral I_d is given by

$$\begin{aligned} I_d &= \frac{1}{2^{5/3}}\int_{-3/2}^{\infty} d\tilde{x} \frac{2\sqrt{3}}{\left(\frac{\tilde{\alpha}(\tilde{x})}{2^{2/3}} - \frac{2^{2/3}\tilde{x}^2}{\tilde{\alpha}(\tilde{x})}\right)} I^2(\tilde{x}) \\ &\approx 0.054886. \end{aligned} \tag{A.23}$$

A.2.1 Usadel Equation

In dirty superconductors, when $\Delta\tau_{\text{el}} \ll 1$, the density of states may be evaluated using the Usadel equation [58], which has the following form for the retarded Green's function

$$[\hat{\tau}_3\epsilon + \hat{\Delta}, \hat{g}_s^R] = iD\boldsymbol{\nabla}[\hat{g}_s^R\boldsymbol{\nabla}\hat{g}_s^R]. \tag{A.24}$$

Here [,] denotes the commutator and the hat indicates 2×2 matrices in Gor'kov-Nambu space, which are given by

$$\hat{\tau}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{\Delta} = \begin{pmatrix} 0 & \Delta \\ \Delta^* & 0 \end{pmatrix}, \tag{A.25}$$

$$\hat{g}_s = \begin{pmatrix} g_s^R & F_s^R \\ F_s^{R*} & g_s^{R*} \end{pmatrix}. \tag{A.26}$$

The Green's function satisfies the nonlinear constraint $\hat{g}_s^R \cdot \hat{g}_s^R = 1$, and can be expressed in terms of the angles θ and χ in the form $g_s^R = \cos\theta$, $F_s^R = e^{i\chi}\sin\theta$. The density of states is given by

$$\nu(\epsilon) = \nu_n \Re\cos\theta(E) \tag{A.27}$$

and Eq. (A.24) reduces to

$$\frac{D}{2}\left(p_s^2\sin\theta\cos\theta - \nabla^2\theta\right) = i\epsilon\sin\theta + \Delta\cos\theta, \tag{A.28a}$$

$$D\boldsymbol{\nabla}\left(\boldsymbol{p}_s\sin^2\theta\right) = 0. \tag{A.28b}$$

Here $\boldsymbol{p}_s = \boldsymbol{\nabla}\chi$. As we are considering a thin film we assume $\boldsymbol{p}_s$ to be spatially uniform. Then Eq. (A.28b) yields $\boldsymbol{\nabla}\theta = 0$, while Eq. (A.28a) reduces to

$$\epsilon + i\Gamma\cos\theta = i\Delta\cot\theta \tag{A.29}$$

with $\Gamma = Dp_s^2/2$.

Defining $\xi \equiv e^{i\theta}$ we can write Eq. (A.29) as a polynomial in ξ

$$2 - \xi^{-2}w - i\frac{\eta}{2}\xi^{-3} = 0 \tag{A.30}$$

where $w \equiv (\epsilon - \Delta)/\Delta$ as before, $\eta \equiv \Gamma/\Delta$, and we have used the fact that we are concerned only with the energy range $|w| \ll 1$, which corresponds to $|\xi| \ll 1$. The density of states in this approximation becomes

$$\nu(\epsilon) = \frac{\nu_n}{2} \Re \xi^{-1}. \tag{A.31}$$

One can easily see that the substitution $\xi = ix/\sqrt{2}$ and $\eta = \frac{2\zeta^2}{3\gamma}$ renders Eqs. (A.30) and (A.31) identical to Eqs. (A.18) and (A.10).

Appendix B

DERIVATION OF THE NONLINEAR CONDUCTIVITY IN THE ABSENCE OF *DC* SUPERCURRENT

The microwave power absorbed per unit volume of the sample is expressed in Eq. (2.20) in terms of the time-dependence of the energy level $\epsilon(N,t)$. The latter is determined by (c.f. Eq. (2.18))

$$N = \int_0^{\epsilon(N,t)} d\varepsilon\, \nu(\varepsilon, t). \tag{B.1}$$

Although the time dependence of the condensate momentum in the presence of the microwave field is very simple, $p_{\mathrm{s}}(t) = \frac{eE_\omega}{\omega}\sin(\omega t)$, the density of states $\nu(\varepsilon,t)$, being a nonlinear function of $p_{\mathrm{s}}(t)$, is a complicated function of time in the presence of a microwave field. As a result $\epsilon(N,t)$ given by the solution of Eq. (B.1) has a complicated dependence not only on time but also on the amplitude of the microwave field, E_ω. Because of this the absorbed power in Eq. (2.20) and the nonlinear conductivity are complicated functions of E_ω and ω. The situation simplifies dramatically in the limiting regimes $v_F e E_\omega \Delta\tau_{el}^2/\omega \gg 1$, and $v_F e E_\omega \Delta\tau_{el}^2/\omega \ll 1$. In these regimes the nonlinear conductivity has a simple power law dependence on E_ω and ω.

B.1 Ballisic Regime

For $v_F e E_\omega \Delta\tau_{el}^2/\omega \gg 1$ the quasiparticles contributing to the Debye conductivity are in the ballistic regime and the density of states may be described by Eq. (A.13). In this case the time-dependent width of the energy window in which the density of states is affected by microwave radiation is $\delta\epsilon(t) \sim v_F p_{\mathrm{s}}(t)$. The characteristic density of quasiparticle levels within the energy window where $\nu(\varepsilon,t)$ is affected by microwave radiation may be estimated as $\delta N \sim \nu_n \sqrt{v_F e E_\omega \Delta/\omega}$. Therefore it is convenient to introduce a rescaled level density

$$\mathcal{N}_b = \frac{N}{\nu_n \Delta} \sqrt{\frac{\omega \Delta}{v_F e E_\omega}} \tag{B.2}$$

and express the time-dependent energy $\epsilon(N,t)$ in terms of a new function $z_{\mathcal{N}_b}(\omega t)$ as

$$\epsilon(N,t) = \Delta + v_F |p_s(t)| z_{\mathcal{N}_b}(\omega t). \tag{B.3}$$

Substituting the change of variables (B.2), (B.3) into Eq. (B.1) and using the form of the density of states in the ballistic regime, Eqs. (A.13) and (A.14), we find that the dependence of $z_{\mathcal{N}}(\phi)$ on the phase of the microwave field $\phi = \omega t$ is determined by the equation

$$\begin{aligned} \frac{\mathcal{N}_b}{\sqrt{|\sin(\phi)|}} &= \frac{\sqrt{2}}{3} \Big[\theta(z_{\mathcal{N}_b}(\phi) + 1) \left(z_{\mathcal{N}_b}(\phi) + 1\right)^{3/2} \\ &- \theta(z_{\mathcal{N}_b}(\phi) - 1) \left(z_{\mathcal{N}_b}(\phi) - 1\right)^{3/2} \Big], \end{aligned} \tag{B.4}$$

which does not contain the amplitude E_ω. Substituting the change of variables (B.3) into Eq. (2.20) we find that the nonlinear conductivity has a simple power law dependence on E_ω given by Eq. (2.21) with the function $F_b(x)$ given by

$$F_b(x) = 3 \int_0^\infty d\mathcal{N}_b \int_0^\pi \frac{d\phi}{2\pi} \int_0^\infty d\bar{\tau}\ e^{-\bar{\tau}} f_b(\mathcal{N}_b, \phi) f_b(\mathcal{N}_b, \phi - x\bar{\tau}). \tag{B.5}$$

Here $f_b(\mathcal{N}_b, \phi)$ denotes the function

$$f_b(\mathcal{N}_b, \phi) = \cos(\phi) z_{\mathcal{N}_b}(\phi) + \sin(\phi) \partial_\phi z_{\mathcal{N}_b}(\phi). \tag{B.6}$$

In the low frequency limit, $\omega\tau_{in} \ll 1$, $F_b(\omega\tau_{\rm in})$ in Eq. (B.5) can be easily evaluated using Eqs. (A.13) and (A.15),

$$F_b(0) \;=\; 2\langle \sin^{1/2}(\omega t)\cos^2(\omega t)\rangle I_b \approx 0.108. \tag{B.7}$$

Here $\langle \ldots \rangle$ denotes averaging over half the oscillation period and I_b was defined in Eq. (A.16).

In the high frequency regime $\omega\tau_{in} \gg 1$ we find that $F_b(\omega\tau_{in}) \sim \frac{1}{\omega^2\tau_{in}^2}$, which can most readily be seen by Fourier transforming Eq. (2.20).

B.2 Diffusive Regime

At $v_F e E_\omega \Delta \tau_{el}^2/\omega \ll 1$ the quasiparticles contributing to the Debye conductivity are in the diffusive regime, and the density of states may be described by Eq. (A.19). In this case the time dependent broadening of the singularity in the quasiparticle density of states is $\delta\epsilon(t) \sim (\Delta D^2 p_s^4(t))^{1/3}$. The characteristic density of quasiparticle levels that participate in microwave absorption can then be estimated as $\delta N \sim \nu_n \left(\frac{\Delta^2 D e^2 E_\omega^2}{\omega^2}\right)^{1/3}$. Introducing a rescaled level density

$$\mathcal{N}_d = \frac{1}{\nu_n \Delta}\left(\frac{2\Delta\omega^2}{e^2 D E_\omega^2}\right)^{1/3} N \tag{B.8}$$

we express the time-dependent energy $\epsilon(N,t)$ in terms of a new function $\tilde{w}_{\mathcal{N}_b}(\omega t)$ as

$$\epsilon(N,t) = \Delta + \left(\frac{\Delta D^2 p_s^4(t)}{2}\right)^{1/3} \tilde{w}_{\mathcal{N}_d}(\omega t), \tag{B.9}$$

Substituting Eqs. (B.8) and (B.9) into (B.1) and using the form of the density of states in the diffusive regime, Eq. (A.19), we find the dependence of $\tilde{w}_{\mathcal{N}_d}(\phi)$ on the phase $\phi = \omega t$ is determined by

$$\frac{\mathcal{N}_d}{\sin^{2/3}(\phi)} = \int_0^{\tilde{w}_{\mathcal{N}_d}(\phi)} d\tilde{w}' \tilde{\nu}_d(\tilde{w}'), \tag{B.10}$$

where $\tilde{\nu}_d(\tilde{w}')$ is defined by Eq. (A.20). Note that (B.10) is independent of the amplitude of the applied field E_ω. Substituting Eqs. (B.8) and (B.9) into Eq. (2.20) we obtain Eq. (2.22) with the function $F_d(x)$ given by

$$F_d(x) = \frac{1}{2^{2/3}\pi}\int_0^\infty d\mathcal{N}_d \int_0^\pi d\phi \int_0^\infty d\bar{\tau}\; e^{-\bar{\tau}} f_d(\mathcal{N}_d,\phi) f_d(\mathcal{N}_d,\phi - x\bar{\tau}) \tag{B.11}$$

Here the function $f_d(\mathcal{N}_d,\phi)$ is defined by

$$\begin{aligned} f_d(\mathcal{N}_d,\phi) &= \frac{2}{3}\sin^{1/3}(\phi)\cos(\phi)\tilde{w}_{\mathcal{N}_d}(\phi) \\ &\quad + \frac{1}{2}\sin^{4/3}(\phi)\partial_\phi \tilde{w}_{\mathcal{N}_d}(\phi). \end{aligned} \tag{B.12}$$

In the low frequency regime, $\omega\tau_{in} \ll 1$, $F_d(\omega\tau_{\rm in})$ can be easily evaluated. Using Eqs. (A.19) and (A.22) we obtain

$$F_d(0) = 2 < \sin^{4/3}(\omega t)\cos^2(\omega t) > I_d \approx 0.109, \tag{B.13}$$

where $\langle \ldots \rangle$ denotes averaging over half the oscillation period, and I_d is given by Eq. (D.13). In the high frequency limit, $\omega\tau_{in} \gg 1$, $F_d(\omega\tau_{in}) \sim \frac{1}{\omega^2\tau_{in}^2}$ which is most easily seen by Fourier transforming Eq. (2.20).

Appendix C

DEBYE MECHANISM IN THE FLUX FLOW REGIME

C.0.1 Derivation of expressions for the energy dissipation rate in thin films

Here we provide a detailed derivation of Eqs. (4.9), (4.10) and (4.13) for the energy absorption rate in thin films.

According to Ehrenfest's theorem [32], the energy absorption rate is given by $\frac{d}{dt}\langle\hat{H}\rangle = \left\langle\frac{\partial\hat{H}(t)}{\partial t}\right\rangle$, where $\hat{H}$ is the system Hamiltonian, and $\langle\ldots\rangle$ denotes statistical averaging. Adapting this expression to quasiparticles in the vortex core we write the energy absorption rate per unit length of the vortex in the form

$$W = \frac{1}{d}\int_0^\infty d\epsilon\, \overline{\nu(\epsilon,\alpha(t))n(\epsilon,t)v_\nu(\epsilon,t)}, \tag{C.1}$$

where $\overline{\cdots}$ denotes time averaging over the trajectory of the vortex motion. For the equilibrium quasiparticle distribution, $n(\epsilon,t) = n_{\mathrm{F}}(\epsilon)$ the integrand above is a total derivative, and the energy dissipation rate vanishes. Therefore, to lowest order in inhomogeneity we may replace $n(\epsilon,t) \to \delta n(\epsilon,t)$ in the above equation. This yields Eq. (4.9)

Writing the solution of the linearized kinetic equation (4.8) in the form

$$\delta n(\epsilon,t) = \left(-\frac{dn_{\mathrm{F}}(\epsilon)}{d\epsilon}\right) A(\epsilon)\int_0^\infty d\tau e^{-\frac{\tau}{\tau_{\mathrm{in}}}}\dot{\alpha}(t-\tau),$$

and using Eqs. (4.6) and (4.7) we arrive at Eqs. (4.10) and Eq. (4.11)

Next, using the fact that in the flux flow regime the vortex trajectories are given by $\alpha(t) = \alpha(\mathbf{r}_0 + \mathbf{V}t)$, where $\mathbf{r}_0$ is the initial position of the vortex, and $\mathbf{V} = c\,[\mathbf{E}\times\mathbf{H}]/H^2$ is the drift velocity of the lattice, we can convert time averaging into spatial averaging over inhomogeneity in Eq. (4.4). We thus express the quantity $\mathcal{C}(E)$ in Eq. (4.11) in the form

$$\mathcal{C}\left(E\right) \;=\; -\langle(\delta\alpha)^2\rangle\int_0^\infty d\tilde{t}e^{-\tilde{t}/\tau_{\mathrm{in}}}\frac{d^2}{d\tilde{t}^2}g\left(\frac{cE|\tilde{t}|}{HL_{\mathrm{c}}}\right). \tag{C.2}$$

Introducing the Fourier transform $\tilde{g}(\tilde{\omega}) \equiv \int dx e^{i\tilde{\omega}x} g(x)$ of the function $g(x)$ in Eq. (4.4) we obtain

$$\mathcal{C}(E) \;=\; \langle(\delta\alpha)^2\rangle \int \frac{d\omega}{2\pi} \frac{L_{\rm c}H}{cE} \frac{\tau_{\rm in}\omega^2}{1+\omega^2\tau_{\rm in}^2} \tilde{g}\left(\frac{\omega L_{\rm c} H}{cE}\right). \tag{C.3}$$

Finally, introducing the dimensionless frequency $\tilde{\omega} = \frac{\omega L_{\rm c} H}{cE}$ and the characteristic electric field $E^* = H\frac{L_{\rm c}}{c\tau_{\rm in}}$ we arrive at Eq. (4.13)

C.0.2 Derivation of expressions for the energy dissipation rate in bulk superconductors

The generalization Eq. (4.9) for the energy dissipation rate per unit length of the vortex to bulk superconductors is

$$W = \frac{1}{L}\int_0^\infty d\epsilon \int dz\, \nu(\epsilon,\bar{\alpha}) \overline{\delta n(\epsilon,t,z) v_\nu(\epsilon,t,z)}, \tag{C.4}$$

where L is the length of the vortex line, $\nu(\epsilon,\bar{\alpha})$ is the density of states per unit length of the vortex, and $v_\nu(\epsilon,t,z)$ is given by the obvious generalization of Eq. (4.6),

$$v_\nu(\epsilon,t,z) = A(\epsilon)\,\dot{\alpha}(t,z), \tag{C.5}$$

where $\nu(\epsilon,\bar{\alpha})$ in expression (4.7) should be understood as the density of states per unit length of the vortex.

Writing the solution of the kinetic equation (4.20) in the Fourier representation and using the fact that in the flux flow regime $\alpha(t,z) = \alpha\left(\mathbf{r}_0 + \mathbf{V}t\right)$ we obtain Eq. (4.21):

$$W = \int_0^\infty \frac{d\epsilon \nu(\epsilon,\bar{\alpha}) A^2(\epsilon)}{4T\cosh^2\left(\frac{\epsilon}{2T}\right)} \mathcal{C}_3(E) \tag{C.6}$$

$$\mathcal{C}_3(E) = \int \frac{dq d\omega}{(2\pi)^2} \frac{\tau_{\rm in}\langle(\delta\alpha)^2\rangle\,\omega^2\left(1+D_{\rm v}q^2\tau_{\rm in}\right)\tilde{g}(q,\omega)}{\left(1+D_{\rm v}q^2\tau_{\rm in}\right)^2+\omega^2\tau_{\rm in}^2}. \tag{C.7}$$

$$\tilde{g}(q,\omega) = \int dz dt e^{i\omega t - iqz} g\left(\frac{\sqrt{z^2+V^2t^2}}{L_{\rm c}}\right). \tag{C.8}$$

Assuming the disorder correlation function is isotropic we have

$$\begin{aligned}\tilde{g}(q,\omega) &\propto \frac{L_{\rm c}^2}{V}\int_0^\infty x dx J_0\left(xL_{\rm c}\sqrt{q^2+(\omega/V)^2}\right)g(x)\\ &\equiv \frac{L_{\rm c}^2}{V}G\left(L_{\rm c}\sqrt{q^2+(\omega/V)^2}\right). \end{aligned} \tag{C.9}$$

Substituting this into (C.7) we get

$$\mathcal{C}_3(E)=\frac{V^2}{L_{\rm c}^2}\int\frac{d\tilde{q}d\tilde{\omega}}{(2\pi)^2}\frac{\tau_{\rm in}\langle(\delta\alpha)^2\rangle\,\tilde{\omega}^2\left(1+\tilde{q}^2\frac{D_{\rm v}\tau_{\rm in}}{L_{\rm c}^2}\right)\tilde{G}\left(\sqrt{\tilde{q}^2+\tilde{\omega}^2}\right)}{\left(1+\tilde{q}^2\frac{D_{\rm v}\tau_{\rm in}}{L_{\rm c}^2}\right)^2+\tilde{\omega}^2\frac{V^2\tau_{\rm in}^2}{L_{\rm c}^2}}. \tag{C.10}$$

At the smallest electric fields we have $\mathcal{C}_3(E)\sim\frac{L_{\rm c}V^2}{\sqrt{D_{\rm v}\tau_{\rm in}}}\tau_{\rm in}\frac{\langle(\delta\alpha)^2\rangle}{L_{\rm c}^2}$. However, since the integral is dominated by wavevectors q for which the first term in the denominator is of order unity, the characteristic electric field for the crossover into the nonlinear regime is unaffected by diffusion, i.e. remains the same as in 2D, $V^*\sim L_{\rm c}/\tau_{\rm in}$.

Appendix D

NON-CENTROSYMMETRIC SUPERCONDUCTORS

D.1 Derivation of the Debye contribution to the linear conductivity in non-centrosymmetric superconductors including impurity scattering

The self energy in the Born approximation is

$$\hat{\Sigma} = \frac{-1}{2\pi\nu_{\rm n}\tau_{\rm el}} \int \frac{d^2p}{(2\pi)^2}\hat{G} \tag{D.1}$$

where $\hat{G}$ is the Green's function in particle-hole and spin space. It is given by

$$\hat{G} = \frac{1}{\epsilon - H}. \tag{D.2}$$

Then, using Eq. (5.16) we can shift the poles up to first order in $\bar{\boldsymbol{p}}_{\rm s}$ and $\boldsymbol{\mu}$ and obtain the renormalization equations for the the energy $\tilde{\epsilon}$ and gap $\tilde{\Delta}$:

$$\begin{aligned}
\tilde{\epsilon} &= \epsilon + \frac{i}{4\tau_{\rm el}}\sum_{\pm}\int \frac{d\phi}{2\pi}\frac{u - d\gamma_{\pm}\cos\phi}{\sqrt{(u-d\gamma_{\pm}\cos\phi)^2-1}} && \text{(D.3)}\\
\tilde{\Delta} &= \Delta + \frac{i}{4\tau_{\rm el}}\sum_{\pm}\int \frac{d\phi}{2\pi}\frac{1}{\sqrt{(u-d\gamma_{\pm}\cos\phi)^2-1}}. && \text{(D.4)}
\end{aligned}$$

Here $u = \tilde{\epsilon}/\tilde{\Delta}$ and $d = \Delta/\tilde{\Delta}$. Writing $u = 1 + z$ these equations can be reduced to

$$\begin{aligned}
\frac{z}{d} &= w + \frac{i\beta}{4\sqrt{2}\pi}\left[\sqrt{z-d\gamma_+}E\left(\frac{-2d\gamma_+}{z-d\gamma_+}\right) + \sqrt{z+d\gamma_+}E\left(\frac{2d\gamma_+}{z+d\gamma_+}\right) + \sqrt{z-d\gamma_-}E\left(\frac{-2d\gamma_-}{z-d\gamma_-}\right)\right.\\
&+ \left.\sqrt{z+d\gamma_-}E\left(\frac{2d\gamma_-}{z+d\gamma_-}\right)\right] && \text{(D.5)}\\
\frac{1}{d} &= 1 + \frac{i\beta}{4\sqrt{2}\pi}\left[\frac{K\left(\frac{-2d\gamma_+}{z-d\gamma_+}\right)}{\sqrt{z-d\gamma_+}} + \frac{K\left(\frac{2d\gamma_+}{z+d\gamma_+}\right)}{\sqrt{z+d\gamma_+}} + \frac{K\left(\frac{-2d\gamma_-}{z-d\gamma_-}\right)}{\sqrt{z-d\gamma_-}} + \frac{K\left(\frac{2d\gamma_-}{z+d\gamma_-}\right)}{\sqrt{z+d\gamma_-}}\right]. && \text{(D.6)}
\end{aligned}$$

As before, $\beta = \frac{1}{\tau_{\mathrm{el}}\Delta}$. In the energy interval of interest, $|w| \ll 1$, and in the diffusive regime, $z \ll d\gamma_\pm$, we can expand these equations in the small parameter $\frac{z}{d\gamma_\pm}$. Doing so, and keeping terms only up to γ^2, we arrive at

$$z = w - \frac{i\gamma^2}{2\sqrt{2}\beta}\frac{1}{\sqrt{z}}. \tag{D.7}$$

Making the subsitution $y = z^{-1/2}$ we arrive at Eq. (5.27). The density of states can be written

$$\nu(\epsilon) = -\frac{1}{\pi}\Im\int\frac{d^2p}{(2\pi)^2}\,\mathrm{Tr}_{sp}\,G^R(\boldsymbol{p},\epsilon) \tag{D.8}$$

where Tr_{sp} denotes a trace over the spin degrees of freedom and G^R is the retarded quasi-particle Green's function. Using. Eq. (D.2) and Eq. (5.16)

$$\nu(\epsilon) = \frac{\nu_n}{2}\sum_\pm \Re\int_0^{2\pi}\frac{d\phi}{2\pi}\frac{1+z-d\gamma_\pm\cos\phi}{\sqrt{2(z-d\gamma_\pm\cos\phi)}}. \tag{D.9}$$

Making the same approximations as above, we arrive at Eq. (5.26).

We introduce the dimensionless density of states

$$\begin{aligned}\tilde\nu(\epsilon) &= \theta\left(\tilde w + \frac{3}{2^{5/3}}\right)\frac{1}{\sqrt{3}}\left[\frac{\tilde\alpha(\tilde w)}{2^{4/3}} - \frac{2^{4/3}\tilde w^2}{\tilde\alpha(\tilde w)}\right] &\text{(D.10a)}\\ \tilde\alpha(\tilde w) &= \left(16\tilde w^3 + 27 + 3\sqrt{3}\sqrt{32\tilde w^3+27}\right)^{1/3} &\text{(D.10b)}\end{aligned}$$

Using Eq. (5.28a) in Eq. (5.20) the level sensitivity can be written

$$V(\epsilon) = -v_{\mathrm{F}}\frac{v_{\mathrm{F}}\bar p_{\mathrm{s}}}{\Delta}\frac{\gamma^{2/3}}{\beta^{2/3}}\tilde V(\tilde w) \tag{D.11}$$

where the dimensionless level sensitivity $\tilde V$ is given by

$$\tilde V(\tilde w) = \frac{2}{\tilde\nu(\tilde w)}\int_{-3/2^{5/3}}^{\tilde w} d\tilde w'\frac{1}{\sqrt{3}}\left[\frac{2^{4/3}\tilde w^2}{\tilde\alpha(\tilde w)} - \frac{\tilde\alpha(\tilde w)}{2^{4/3}} + \left(\frac{1}{2^{4/3}} + \frac{2^{4/3}\tilde w^2}{\tilde\alpha^2(\tilde w)}\right)\frac{2}{3\tilde\alpha^2(\tilde w)}\left[27+\sqrt{27}\frac{16\tilde w^3+27}{\sqrt{32\tilde w^3+27}}\right]\right]. \tag{D.12}$$

Plugging Eq. (5.28a) and (D.11) into Eq. (5.15) we get Eq. (5.29) with I_{d} given by

$$I_{\mathrm{d}} = \frac{3}{4}\int_{-3/2^{5/3}}^{\infty} d\tilde w\;\tilde\nu(\tilde w)\;\tilde V^2(\tilde w) \tag{D.13}$$

D.2 Derivation of the Supercurrent in the Ground State of a Non-Centrosymmetric Superconductor

In this section we derive a general expression for the supercurrent in the ground state of a NCS thin film superconductor in the presence of an in-plane magnetic field for arbitrary in-plane spin-orbit coupling.

The current response to an in-plane magnetic field is given by

$$\boldsymbol{j} = eT\sum_n \int \frac{d^2p}{(2\pi)^2}\,\mathrm{Tr}\left\{\hat{\boldsymbol{v}}(\boldsymbol{p})\underline{G}_{11}(i\epsilon_n, \boldsymbol{p})\right\} \tag{D.14}$$

where e is the electric charge, T is the temperature, ϵ_n is a fermionic Matsubara frequency, $\hat{\boldsymbol{v}}$ is the velocity operator, $\underline{G}_{11}$ is the 11 component of the Green's function in particle-hole space, and Tr denotes a trace over spin space. The Green's function in Nambu spin space is

$$\underline{G} = \frac{1}{i\epsilon_n - \hat{H}} \tag{D.15}$$

where our Hamiltonian in Nambu spin space is

$$\hat{H} = \hat{H}_0 + \hat{H}_1 \tag{D.16a}$$

$$\hat{H}_0 = (\xi + \boldsymbol{b}(\boldsymbol{p})\cdot\boldsymbol{\sigma})\tau_3 + \hat{\Delta} \tag{D.16b}$$

$$\hat{H}_1 = \boldsymbol{\mu}\cdot\boldsymbol{\sigma} + \hat{\boldsymbol{v}}(\boldsymbol{p})\cdot\boldsymbol{p}_{\mathrm{s}}. \tag{D.16c}$$

Here ξ is our spectrum relative to the chemical potential, which for an isotropic spectrum takes the form $\xi = p^2/2m - E_F$, $\boldsymbol{b}(\boldsymbol{p})$ is our spin-orbit coupling, σ_i denotes matrices in spin space, τ_i denotes matrices in particle-hole space, $\boldsymbol{\mu} = \mu_B g\boldsymbol{B}/2$ with μ_B the Bohr magenton, g the electronic g factor, and $\boldsymbol{B}$ the applied in-plane magnetic field. Further $\boldsymbol{p}_{\mathrm{s}}$ is the superfluid momentum and $\underline{\Delta} = \Delta\tau_1$, with Δ the superconducting order parameter. We have performed a rotation in spin space to remove the conventional factor of $i\sigma_2$. Below we will treat $\hat{H}_1$ as a perturbation, and expand $\underline{G}$ in $\hat{H}_1$. As we are concerned with temperatures near T_c, we will also expand in $\underline{\Delta}$. The resulting expression for the current is

$$\boldsymbol{j} = eT\sum_n \int \frac{d^2p}{(2\pi)^2}\,\mathrm{Tr}\left\{\hat{\boldsymbol{v}}\left(\underline{G}_n\underline{\Delta}\underline{G}_n\hat{H}_1\underline{G}_n\underline{\Delta}\underline{G}_n + \underline{G}_n\hat{H}_1\underline{G}_n\underline{\Delta}\underline{G}_n\underline{\Delta}\underline{G}_n + \underline{G}_n\underline{\Delta}\underline{G}_n\underline{\Delta}\underline{G}_n\hat{H}_1\underline{G}_n\right)\right\} \tag{D.17}$$

where $\underline{G}_n$ is given by

$$\underline{G}_n = \begin{pmatrix} \hat{G}(i\epsilon_n, \boldsymbol{p}) & 0 \\ 0 & -\hat{G}^\dagger(i\epsilon_n, \boldsymbol{p}) \end{pmatrix} \tag{D.18a}$$

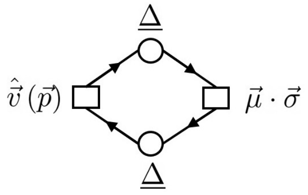
$$\hat{G} = (i\epsilon_n - \xi - \boldsymbol{b}\cdot\boldsymbol{\sigma})^{-1} = \sum_{\nu=\pm}\hat{\Pi}^\nu G_\nu(i\epsilon_n, \boldsymbol{p}) \tag{D.18b}$$

with

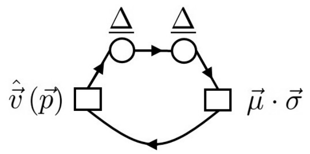
$$\hat{\Pi}^\nu = \frac{1}{2}(1 + \nu\hat{b}(\boldsymbol{p})\cdot\boldsymbol{\sigma}) \tag{D.19a}$$

$$G_\nu(i\epsilon_n, \boldsymbol{p}) = (i\epsilon_n - \xi - \nu b(\boldsymbol{p}))\,. \tag{D.19b}$$

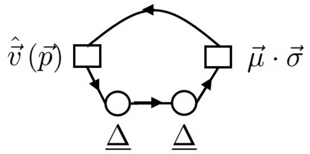
D.2.1 Clean



(a) Figure 1a



(b) Figure 1b



(c) Figure 1c

Figure D.1: The diagrams contributing to the current due to an in-plane magnetic field in the clean limit $\tau_{\mathrm{el}}T_c \gg 1$. The black lines with arrows denote the Green's function $\underline{G}_n$. We take the 11 component in particle-hole space of each diagram.

The total current is equal to zero as there is no current in equilibrium. We thus separate the contributions in Eq. (D.17) corresponding to the in-plane magnetic field and the superfluid momentum, and focus on the former. There are 3 distinct terms in Eq. (D.17), represented diagramatically in Fig. D.1. Their contributions to the current are

$$\boldsymbol{j}^c_\mu = e\Delta^2 T \sum_n \int d\xi \int \frac{dS}{(2\pi)^2 v}\left[\eta_1 \frac{db^\alpha}{d\boldsymbol{p}}\mu^\alpha + \eta_2 \frac{db^\alpha}{d\boldsymbol{p}} b^\alpha \boldsymbol{\mu}\cdot\boldsymbol{b} + \eta_3 \boldsymbol{v}\boldsymbol{\mu}\cdot\boldsymbol{b}\right]. \tag{D.20}$$

where

$$\eta_1 = \sum_{\nu\mu} \frac{1}{2}(1-\mu\nu) I_c^{\nu\mu} \tag{D.21a}$$

$$\eta_2 = \sum_{\nu\mu} \mu\nu I_c^{\nu\mu} \tag{D.21b}$$

$$\eta_3 = \sum_{\nu\mu} \frac{1}{2}(\mu+\nu) I_c^{\nu\mu} \tag{D.21c}$$

and $I_c^{\nu\mu}$ is

$$I_c^{\nu\mu} = |G_\nu|^2 |G_\mu|^2 - G_\nu |G_\mu|^2 G_\nu - |G_\nu|^2 G_\nu G_\mu. \tag{D.22}$$

We have defined $v = \partial\xi/\partial p$ and dS is an infintesimal line on the Fermi surface (which is 1 dimensional in this case). If we assume a spherical Fermi surface, then we may evaluate the dS, and to lowest order in b/E_F obtain

$$\boldsymbol{j}^c_\mu = 2e\Delta^2 T\pi \sum_n \frac{\nu_0}{|\epsilon_n|(\epsilon_n^2+b_0^2)}\left\langle \frac{db_0^\alpha}{d\boldsymbol{p}}\mu^\alpha \right\rangle + \frac{\nu_0}{|\epsilon_n|^3(\epsilon_n^2+b_0^2)}\left\langle \frac{db_0^\alpha}{d\boldsymbol{p}} b_0^\alpha \boldsymbol{\mu}\cdot\boldsymbol{b}_0 \right\rangle - \frac{1}{|\epsilon_n|^3}\langle \nu_0\left(\boldsymbol{v}_0\boldsymbol{b}_1\cdot\boldsymbol{\mu} + \boldsymbol{v}_1\boldsymbol{b}_0\cdot\boldsymbol{\mu}\right) + \tag{D.23}$$

Here $\boldsymbol{v}_0 = \boldsymbol{v}|_{\xi=0}$ is the Fermi velocity, $\boldsymbol{v}_1 = d\boldsymbol{v}/d\xi|_{\xi=0}$, $\boldsymbol{b}_0 = \boldsymbol{b}|_{\xi=0}$ is the spin-orbit coupling on the Fermi surface, $\boldsymbol{b}_1 = d\boldsymbol{b}/d\xi|_{\xi=0}$, $\nu_0 = m/2\pi$ is the density of states to lowest order in b_0, and $\nu_1 = d\nu(\xi)/d\xi|_{\xi=0}$. Further $\langle\ldots\rangle = \int_0^{2\pi}\frac{d\phi}{2\pi}\ldots$ denotes an average over the Fermi surface. It is important to note that in the case of an isotropic spectrum, particle-hole symmetry, and

Rashba spin-orbit coupling $\boldsymbol{b}^R = \alpha \boldsymbol{p} \times \boldsymbol{c} \cdot \boldsymbol{\sigma}$, where $\boldsymbol{c}$ is one of the non-equivalent normals, our result is in agreement with Edelsten [15], and at small values of the spin-orbit coupling ($b_0 \ll T$) there is a cancellation of the linear in b_0 term:

$$\boldsymbol{j}_\mu^{c(R)} = -\frac{b_0}{2E_F} e\Delta^2 p_F (\boldsymbol{\mu} \times \boldsymbol{c}) T \sum_n \frac{b_0^2}{2|\epsilon_n|^3(\epsilon_n^2 + b_0^2)}. \tag{D.24}$$

However, it is important to note that removing one of the conditions listed above; a spherical Fermi surface, an isotropic spectrum, linear in $\boldsymbol{p}$ spin-orbit coupling, or particle-hole symmetry will lead to the linear in b term to be non-zero, and thus in any realistic system, there will be a linear in b term as demonstrated by Eq. (D.23).

To obtain the supercurrent, we use the fact that $\boldsymbol{J}_s = -\boldsymbol{j}_\mu$.

D.2.2 Dirty

We must now account for impurity scattering in the regime $\tau_{\text{el}} T_{\text{c}} \ll 1$. There are now several diagrams as show in Figs. D.3 and D.5 . First, we must renormalize the energy ϵ_n. In the presence of point-like impurities the quasiparticle energy $\epsilon_n \to \tilde{\epsilon}_n = \epsilon_n + \frac{1}{2\tau_{\text{el}}} \operatorname{sgn} \epsilon_n$, where $\tau_{\text{el}}^{-1} = m n_i u^2$ is the elastic scattering rate, n_i is the impurity concentration, m is the electron mass, and u is the amplitude of the impurity potential in momentum space.

Renormalization of Order Parameter

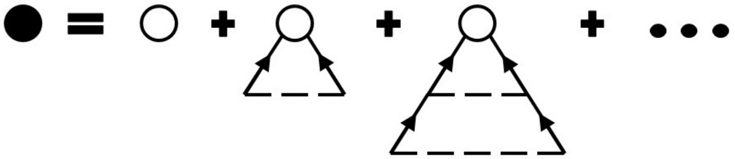


Figure D.2: The renormalization of the order parameter due to impurity scattering

The renormalization of the order parameter is show diagramatically in Fig. D.2 . It is given by

$$\tilde{\hat{\Delta}} = \hat{\Delta} + n_i \underline{u} \int \frac{d^2p}{(2\pi)^2} \underline{G}_n \tilde{\hat{\Delta}} \underline{G}_n \underline{u} \tag{D.25}$$

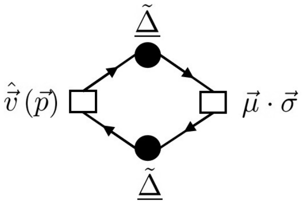
where $\underline{u} = u\tau_3$. Thus, the renormalization for the order parameter reduces to

$$\tilde{\Delta} = \Delta + n_i u^2 \tilde{\Delta} \int \frac{d^2p}{(2\pi)^2} \hat{G}(i\tilde{\epsilon}_n, \boldsymbol{p}) \hat{G}^\dagger(i\tilde{\epsilon}_n, \boldsymbol{p}) \tag{D.26}$$

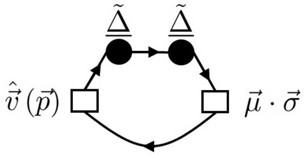
$$\frac{\tilde{\Delta}}{\Delta} = \frac{\tilde{\epsilon}_n}{\epsilon_n} = 1 + \frac{1}{2\tau_{\text{el}}|\epsilon_n|} \tag{D.27}$$

as is required by Anderson's theorem.

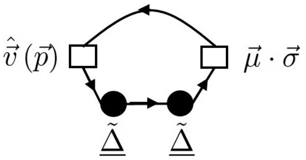
Subleading Contribution



(a) Figure 3a



(b) Figure 3b



(c) Figure 3c

Figure D.3: The three diagrams that give subleading contributions to the current due to an in-plane magnetic field in the dirty limit $\tau_{\text{el}} T_c \ll 1$. As before, black lines with arrows are the Green's function $\underline{G}_n$ and we take the 11 component of the diagram in particle-hole space.

The sub-leading diagrams, which do not contain a Cooperon line, are shown in Fig. D.3 . These can be obtained by taking the results of the clean case and renormalizing the energy and order parameter in Eq. (D.20), $\epsilon_n \to \tilde{\epsilon}_n$, $\Delta \to \tilde{\Delta}$. As will be shown below, the diagrams with a Cooperon line are the dominant contribution in the dirty ($\tau_{\text{el}} T_c \ll 1$) regime.

Cooperon Ladder

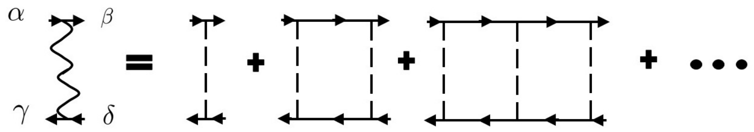


Figure D.4: The cooperon ladder that appears in the diagrams Fig. D.5

The cooperon is given diagrammatically in Fig D.4 and its analytical expression is

$$\begin{aligned} T^{\alpha\beta}_{\gamma\delta} &= -n_i u^2 \delta^{\alpha\beta} \delta^{\gamma\delta} \\ &+ n_i u^2 \int \frac{d^2 p}{(2\pi)^2} \underline{T}^{\alpha\alpha_1}_{\gamma\gamma_1} \hat{G}_{\alpha_1\beta}(i\tilde{\epsilon}_n, \boldsymbol{p}) \hat{G}^{\dagger}_{\delta\gamma_1}(i\tilde{\epsilon}_n, \boldsymbol{p}) \end{aligned} \tag{D.28}$$

We assume $\left\langle \hat{b}^a \hat{b}^b \right\rangle = \delta_2^{ab}/2$, where δ_2^{ab} is the two-dimensional kronicker delta. Schwarz decomposition gives the resulting T:

$$T^{\alpha\beta}_{\gamma\delta} = \sum_{i=0}^{3} A_i \sigma_i^{\alpha\gamma} \sigma_i^{\delta\beta} \tag{D.29}$$

$$\begin{aligned} &= -\frac{1}{4\pi\tau_{\text{el}}\nu_0} \left[\frac{\delta^{\alpha\gamma}\delta^{\delta\beta}}{\tilde{Z}_1(\tilde{\epsilon}_n) + \tilde{Z}_2(\tilde{\epsilon}_n)} + \frac{\sigma_3^{\alpha\gamma}\sigma_3^{\delta\beta}}{\tilde{Z}_1(\tilde{\epsilon}_n) - \tilde{Z}_2(\tilde{\epsilon}_n)} \right. \\ &\left. + \frac{\sigma_1^{\alpha\gamma}\sigma_1^{\delta\beta}}{\tilde{Z}_1(\tilde{\epsilon}_n)} + \frac{\sigma_2^{\alpha\gamma}\sigma_2^{\delta\beta}}{\tilde{Z}_1(\tilde{\epsilon}_n)} \right] \end{aligned} \tag{D.30}$$

where

$$\tilde{Z}_1(\tilde{\epsilon}_n) = 1 - \frac{1}{4\tau_{\text{el}}\tilde{\epsilon}_n|} \left(1 + \frac{\tilde{\epsilon}_n^2}{\tilde{\epsilon}_n^2 + b^2} \right) \tag{D.31a}$$

$$\tilde{Z}_2(\tilde{\epsilon}_n) = -\frac{1}{4\tau_{\text{el}}|\tilde{\epsilon}_n|} \frac{b^2}{\tilde{\epsilon}_n^2 + b^2} \tag{D.31b}$$

which is in agreement with Edelstein for Rashba spin-orbit coupling [15]

Leading Contribution

(a) Figure 5a

(b) Figure 5b

(c) Figure 5c

(d) Figure 5d

Figure D.5: The diagrams giving the leading contribution to the current due to an in-plane magnetic field in the dirty limit. The black line with an arrow is the Green's function $\underline{G}_n$ and we take the 11 component in particle-hole space.

The diagrams in Fig D.5 give the following contributions to $\boldsymbol{j}_\mu$:

$$\boldsymbol{j}_\mu^{5a} = eT\sum_n \int \frac{d^2p}{(2\pi)^2}\int \frac{d^2p'}{(2\pi)^2} U_2^{\gamma\delta}(\boldsymbol{p}) T^{\delta\alpha}_{\gamma\beta} U_1^{\alpha\beta}(\boldsymbol{p}') \tag{D.32a}$$

$$\boldsymbol{j}_\mu^{5b} = eT\sum_n \int \frac{d^2p}{(2\pi)^2}\int \frac{d^2p'}{(2\pi)^2} U_4^{\gamma\delta}(\boldsymbol{p}) T^{\delta\alpha}_{\gamma\beta} U_3^{\alpha\beta}(\boldsymbol{p}') \tag{D.32b}$$

$$\boldsymbol{j}_\mu^{5c} = eT\sum_n \int \frac{d^2p}{(2\pi)^2}\int \frac{d^2p'}{(2\pi)^2} U_4^{\gamma\delta}(\boldsymbol{p}) T^{\delta\alpha}_{\gamma\beta} U_5^{\alpha\beta}(\boldsymbol{p}') \tag{D.32c}$$

$$\boldsymbol{j}_\mu^{5d} = eT\sum_n \int \frac{d^2p}{(2\pi)^2}\int \frac{d^2p'}{(2\pi)^2} U_2^{\gamma\delta}(\boldsymbol{p}) T^{\delta\alpha}_{\gamma\beta} U_6^{\alpha\beta}(\boldsymbol{p}') \tag{D.32d}$$

where

$$U_1(\boldsymbol{p}) = \hat{G}(i\tilde{\epsilon}_n, \boldsymbol{p})\tilde{\Delta}\hat{G}^\dagger(i\tilde{\epsilon}_n, \boldsymbol{p})\boldsymbol{\mu}\cdot\boldsymbol{\sigma}\hat{G}^\dagger(i\tilde{\epsilon}_n, \boldsymbol{p}) \tag{D.33a}$$

$$U_2(\boldsymbol{p}) = -\hat{G}^\dagger(i\tilde{\epsilon}_n, \boldsymbol{p})\tilde{\Delta}\hat{G}(i\tilde{\epsilon}_n, \boldsymbol{p})\hat{\boldsymbol{v}}(\boldsymbol{p})\hat{G}(i\tilde{\epsilon}_n, \boldsymbol{p}) \tag{D.33b}$$

$$U_3(\boldsymbol{p}) = \hat{G}^\dagger(i\tilde{\epsilon}_n, \boldsymbol{p})\boldsymbol{\mu}\cdot\boldsymbol{\sigma}\hat{G}^\dagger(i\tilde{\epsilon}_n, \boldsymbol{p})\tilde{\Delta}\hat{G}(i\tilde{\epsilon}_n, \boldsymbol{p}) \tag{D.33c}$$

$$U_4(\boldsymbol{p}) = -\hat{G}(i\tilde{\epsilon}_n, \boldsymbol{p})\hat{\boldsymbol{v}}(\boldsymbol{p})\hat{G}(i\tilde{\epsilon}_n, \boldsymbol{p})\tilde{\Delta}\hat{G}^\dagger(i\tilde{\epsilon}_n, \boldsymbol{p}) \tag{D.33d}$$

$$U_5(\boldsymbol{p}) = -\hat{G}^\dagger(i\tilde{\epsilon}_n, \boldsymbol{p})\tilde{\Delta}\hat{G}(i\tilde{\epsilon}_n, \boldsymbol{p})\boldsymbol{\mu}\cdot\boldsymbol{\sigma}\hat{G}(i\tilde{\epsilon}_n, \boldsymbol{p}) \tag{D.33e}$$

$$U_6(\boldsymbol{p}) = -\hat{G}(i\tilde{\epsilon}_n, \boldsymbol{p})\boldsymbol{\mu}\cdot\boldsymbol{\sigma}\hat{G}(i\tilde{\epsilon}_n, \boldsymbol{p})\tilde{\Delta}\hat{G}^\dagger(i\tilde{\epsilon}_n, \boldsymbol{p}). \tag{D.33f}$$

Using Eqs. (D.18b) and (D.29) we have for the dominant contribution to the current

$$\begin{aligned}\boldsymbol{j}^l_\mu = -eT\sum_n \tilde{\Delta}^2 \int d\xi d\xi' \int \frac{dS}{(2\pi)^2 v}\frac{dS'}{(2\pi)^2 v'} \Big[A_1 \Big(& \alpha_1 \boldsymbol{v}(\boldsymbol{p})\hat{\boldsymbol{b}}(\boldsymbol{p})\cdot\boldsymbol{\mu} + \alpha_2 \hat{\boldsymbol{b}}(\boldsymbol{p})\cdot\boldsymbol{b}'(\boldsymbol{p})\hat{\boldsymbol{b}}(\boldsymbol{p})\cdot\boldsymbol{\mu} + \alpha_3 \boldsymbol{b}'(\boldsymbol{p})\cdot\boldsymbol{\mu} \\ & + \alpha_4 \boldsymbol{v}(\boldsymbol{p})\hat{\boldsymbol{b}}(\boldsymbol{p}')\cdot\boldsymbol{\mu}\hat{\boldsymbol{b}}(\boldsymbol{p}')\cdot\hat{\boldsymbol{b}}(\boldsymbol{p}) + \alpha_5 \hat{\boldsymbol{b}}(\boldsymbol{p})\cdot\boldsymbol{b}'(\boldsymbol{p})\hat{\boldsymbol{b}}(\boldsymbol{p})\cdot\hat{\boldsymbol{b}}(\boldsymbol{p}')\boldsymbol{\mu}\cdot\hat{\boldsymbol{b}}(\boldsymbol{p}') + \alpha_6 \boldsymbol{b}'(\boldsymbol{p})\cdot\hat{\boldsymbol{b}}(\boldsymbol{p}')\hat{\boldsymbol{b}}(\boldsymbol{p}')\cdot\boldsymbol{\mu}\Big) \\ & + A_0 \Big(\alpha_7 \boldsymbol{v}(\boldsymbol{p})\boldsymbol{\mu}\cdot\hat{\boldsymbol{b}}(\boldsymbol{p}') + \alpha_8 \hat{\boldsymbol{b}}(\boldsymbol{p})\boldsymbol{\mu}\cdot\hat{\boldsymbol{b}}(\boldsymbol{p}')\Big) + A_3\alpha_9 \boldsymbol{b}'(\boldsymbol{p})\times\hat{\boldsymbol{b}}(\boldsymbol{p})\cdot\boldsymbol{\mu}\times\hat{\boldsymbol{b}}(\boldsymbol{p}')\Big]\end{aligned} \tag{D.34}$$

where

$$\alpha_1 = \sum_{\nu\mu\nu'\mu'} \frac{1}{2}\nu(1-\nu'\mu')\delta^{\mu\nu} I^{\nu\mu}_{\nu'\mu'} \tag{D.35a}$$

$$\alpha_2 = \sum_{\nu\mu\nu'\mu'} \frac{1}{2}\mu\nu(1-\mu'\nu') I^{\nu\mu}_{\nu'\mu'} \tag{D.35b}$$

$$\alpha_3 = \sum_{\nu\mu\nu'\mu'} \frac{1}{4}(1-\mu\nu)(1-\mu'\nu') I^{\nu\mu}_{\nu'\mu'} \tag{D.35c}$$

$$\alpha_4 = \sum_{\nu\mu\nu'\mu'} \nu\mu'\nu'\delta^{\mu\nu} I^{\nu\mu}_{\nu'\mu'} \tag{D.35d}$$

$$\alpha_5 = \sum_{\nu\mu\nu'\mu'} \mu\nu\mu'\nu' I^{\nu\mu}_{\nu'\mu'} \tag{D.35e}$$

$$\alpha_6 = \sum_{\nu\mu\nu'\mu'} \frac{1}{2}(1-\mu\nu)\mu'\nu' I^{\nu\mu}_{\nu'\mu'} \tag{D.35f}$$

$$\alpha_7 = \sum_{\nu\mu\nu'\mu'} \nu'\delta^{\mu\nu}\delta^{\mu'\nu'} I^{\nu\mu}_{\nu'\mu'} \tag{D.35g}$$

$$\alpha_8 = \sum_{\nu\mu\nu'\mu'} \nu\nu'\delta^{\mu\nu}\delta^{\mu'\nu'} I^{\nu\mu}_{\nu'\mu'} \tag{D.35h}$$

$$\alpha_9 = -\sum_{\nu\mu\nu'\mu'} \frac{1}{4}(\mu-\nu)(\mu'-\nu') I^{\nu\mu}_{\nu'\mu'} \tag{D.35i}$$

and

$$\begin{aligned} I^{\nu\mu}_{\nu'\mu'} &= |G_\nu(i\tilde{\epsilon}_n,\boldsymbol{p})|^2 G_\mu(i\tilde{\epsilon}_n,\boldsymbol{p})|G_{\nu'}(i\tilde{\epsilon}_n,\boldsymbol{p}')|^2 G^\dagger_{\mu'}(i\tilde{\epsilon}_n,\boldsymbol{p}') \\ &+ G_\nu(i\tilde{\epsilon}_n,\boldsymbol{p})|G_\mu(i\tilde{\epsilon}_n,\boldsymbol{p})|^2 G^\dagger_{\nu'}(i\tilde{\epsilon}_n,\boldsymbol{p}')|G_{\mu'}(i\tilde{\epsilon}_n,\boldsymbol{p}')|^2 \\ &- G_\nu(i\tilde{\epsilon}_n,\boldsymbol{p})|G_\mu(i\tilde{\epsilon}_n,\boldsymbol{p})|^2|G_{\nu'}(i\tilde{\epsilon}_n,\boldsymbol{p}')|^2 G_{\mu'}(i\tilde{\epsilon}_n,\boldsymbol{p}') \\ &- |G_\nu(i\tilde{\epsilon}_n,\boldsymbol{p})|^2 G_\mu(i\tilde{\epsilon}_n,\boldsymbol{p}) G_{\nu'}(i\tilde{\epsilon}_n,\boldsymbol{p}')|G_{\mu'}(i\tilde{\epsilon}_n,\boldsymbol{p}'). \end{aligned} \tag{D.36}$$

Assuming a spherical Fermi surface, our answer simplifies dramatically:

$$
\begin{aligned}
\boldsymbol{j}^l_\mu = -4eT \sum_n \tilde{\Delta}^2 A_1 \nu_0 \frac{\pi^2}{\tilde{\epsilon}_n^2 + b^2} \Bigg[& \frac{-1}{2\tilde{\epsilon}_n^2} \left\langle \nu_0 \left(\boldsymbol{v}_0(p) \boldsymbol{b}_1(p) \cdot \boldsymbol{\mu} + \boldsymbol{v}_1(p) \boldsymbol{b}_0(p) \cdot \boldsymbol{\mu} \right) + \nu_1 \boldsymbol{v}_0(p) \boldsymbol{b}_0(p) \cdot \boldsymbol{\mu} \right\rangle \\
& + \frac{\nu_0}{2\tilde{\epsilon}_n^4 (\tilde{\epsilon}_n^2 + b^2)} \left\langle \boldsymbol{b}_0'(p) \cdot \boldsymbol{b}_0(p) \boldsymbol{\mu} \cdot \boldsymbol{b}_0(p) \right\rangle + \frac{\nu_0}{\tilde{\epsilon}_n^2 + b^2} \left\langle \boldsymbol{\mu} \cdot \boldsymbol{b}_0'(p) \right\rangle \\
& - \frac{1}{\tilde{\epsilon}_n^4} \left\langle \boldsymbol{\mu} \cdot \boldsymbol{b}_0(p') \left[\nu_0 \left(\boldsymbol{v}_0(p) \boldsymbol{b}_1(p) \cdot \boldsymbol{b}_0(p') + \boldsymbol{v}_1(p) \boldsymbol{b}_0(p) \cdot \boldsymbol{b}_0(p') \right) + \nu_1 \boldsymbol{v}_0 \boldsymbol{b}_0(p) \cdot \boldsymbol{b}_0(p') \right] \right\rangle \\
& + \frac{\nu_0}{\tilde{\epsilon}_n^4 (\tilde{\epsilon}_n^2 + b^2)} \left\langle \boldsymbol{b}_0'(p) \cdot \boldsymbol{b}_0(p) \boldsymbol{b}_0(p') \cdot \boldsymbol{b}(p) \boldsymbol{\mu} \cdot \boldsymbol{b}(p') \right\rangle + \frac{\nu_0}{\tilde{\epsilon}_n^2 (\tilde{\epsilon}_n^2 + b^2)} \left\langle \boldsymbol{\mu} \cdot \boldsymbol{b}_0(p') \boldsymbol{b}_0(p') \cdot \boldsymbol{b}_0'(p) \right\rangle \Bigg]
\end{aligned}
\tag{D.37}
$$

where $\langle ... \rangle = \int_0^{2\pi} \frac{d\phi}{2\pi} ...$ denotes an average over the Fermi surface. If we further assume Rashba spin-orbit coupling and particle-hole symmetry, our answer reduces to that of Edelstein

$$
\boldsymbol{j}^{l(R)}_\mu = -\frac{1}{2} eT\pi\nu_n \beta (\boldsymbol{\mu} \times \boldsymbol{c}) \sum_n \frac{\tilde{\Delta}^2}{\tau_{\mathrm{el}} \tilde{Z}_1} \frac{b_0^2}{2\tilde{\epsilon}_n^4 (\tilde{\epsilon}_n^2 + b_0^2)}
\tag{D.38}
$$

where again we see the cancellation of the b_0 term. However, as in the clean case, in a more reasilistic system, which may have anisotropic Fermi surfaces, particle-hole asymmetries, or a more general form of spin-orbit coupling, the b_0 term will survive.

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