Essays on Information Economics

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Abstract

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This dissertation is composed of three independent works in the field of Information Economics. Chapter 1 considers a Bayesian persuasion game in a new setting where a third party can endogenously choose to acquire outside information. I study how the precision of outside information or preferences of the signal-receivers can affect the timing to perform persuasion, which in turn affects the choice of optimal voting rule. Without incorporation of the new setting, unanimous rule is optimal while this result can break down under the new setting. Chapter 2 considers a collective decision problem in a setting where committee members have access to costly information of different precisions. I find that having all members who are all accessible to the most precise information does not result in the optimal design of the committee, and this chapter studies the conditions under which the optimal committee should include members who get access to less precise information. Chapter 3 studies Bayesian updating under uncertainty in the bias of public information. I find that more evidence supporting one opinion can possibly induce a Bayesian individual to be less convinced of that opinion. Also, when an information provider can improve the precision of his signals but this possibility is unknown to the information receivers, more precise information can result in a lower decision quality and a higher probability of polarization.

Chapter 1: I study a persuasion game between a sender and a group of voters under the presence of an advisor who can access outside information to affect the collective decision. In addition to strategically designing a public signal, the sender can also affect the voters' information environment by choosing the timing of persuasion: either before (early) or after (late) the advisor makes his information acquisition decision. I provide sufficient and necessary conditions for a sender to strictly prefer early-persuasion. The main insight is that early-persuasion is adopted when the benefit of avoiding the leak of outside information exceeds the loss of sending over-informative signal. I then examine the optimal voting rules with and without outside information, and find that unanimity is no longer always optimal with outside information.

Chapter 2: A group of members with identical preferences but heterogeneous abilities must make a collective decision under uncertainty about which decision is best. Before the decision is made, each member can acquire a costly and imperfect signal, and then members share with each other their signal results to reach an agreement on which alternative should be selected. I study how less accurate (weak) signals acquired by less efficient members (called inexperts) can contribute to the group's overall collection of information, and also derive the sufficient or necessary conditions for an optimal committee that should include positive number of inexperts. The insight behind the main results is that weak signals can help leverage the tradeoff between efficiently contributing to the group's overall collection of information and aggravating the free-rider problem among members. The main results are also robust to members having heterogeneous preferences, committee size being endogenous, or each signal becoming the provider's private information.

Chapter 3: This paper studies a model in which Bayesian agents observe signals that are informative about the truth but are uncertain about the informativeness (not the direction) of each signal. As they observe additional signals, Bayesian updating requires that they not only learn about the truth but also update how they interpret the signals. Upon observing the same information, even though they interpret every single signal in the same direction, their attitudes toward the strength of evidence each signal represents can differ, and I provide the conditions under which this difference can result in polarization. When reputation refers to agents' beliefs that information is unbiased, I find that, by inducing a higher reputation, an information provider's choice of less precise information can possibly lead to a higher chance that agents make correct decisions and a lower probability that they polarize. Under mild assumptions, the main results are robust to agents having heterogeneous and biased prior beliefs in the truth.

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Chapter 1

TIMING OF PERSUASION UNDER OUTSIDE INFORMATION

1.1 Introduction

A CEO would like to persuade her committee members to approve launching a new product of unknown quality. She can strategically design her test (e.g., selection of investigation methods, assessment of profits under favorable assumptions, etc) of the product so that there is a better chance of having a positive result and hence approval by committee members. In addition to the CEO's test result, a technical specialist can also choose whether to conduct an evaluation on which the approval decision is also based. To become a more successful persuader, which timing would the CEO choose to design her test (and reveal test result as well), before or after the specialist's possible involvement? Would different timings of persuasion impact committee members' welfare? If so, what is its implication on designing the optimal voting rule?

I investigate these questions in the broader setting of a sender (she) who can affect the decisions of a receiver (he) or a group of receivers by controlling his/their information environment as in Kamenica and Gentzkow (2011)[21] (KG henceforth). The receiver chooses the action (between two alternatives denoted as a and b) that maximizes his utility by seeking to match his action with an unknown state of the world. The sender wants to maximize the probability of choosing a by the receiver or voting result being a. I expand on the KG model by introducing an additional player called advisor (AD). The advisor can choose to acquire some outside information in order to increase the chance of making the correct decision by the receiver(s). Moreover, because AD and the receivers can have heterogenous preferences, to avoid the outcome that AD disagrees with the receivers' decision, AD may also decide not to acquire this information. This expansion enriches the KG model with the ability to study the timing of sender's persuasion: If she can only design her experiment once, then she can either persuade both AD and the receiver(s) by sending her signal before AD's action, or persuade only the receiver(s) by designing her test after AD's action is observed. In this paper, I study and derive the conditions that characterize the persuasion timing selected by the sender.

I first examine the model with a single receiver and focus on the sender's trade-off in persuading both AD and the receiver. Intuitively, persuading both players helps disincentivize AD's acquisition and hence the sender benefits from avoiding the risk of the information leaking from AD; however, the cost of sending a more informative signal might be required to generate enough disincentives. Therefore, when AD is relatively easy to persuade so that persuading both players is not costly, the sender would prefer to take her actions before AD's acquisition.

I then extend the model to a group of receivers with heterogenous preferences and examine how different voting rules impact welfare, defined as total receivers' payoffs. Under the KG model of multiple receivers as a benchmark, I show that unanimity is the optimal voting rule. This is because when the decisive voter becomes more difficult to persuade, a more informative signal is required for success in persuasion, indicating a higher level of welfare. By introducing additional source of information accessible by AD, I show that unanimous rule is not optimal under some parameter specifications where different voting rules induce different timings of persuasion. Intuitively, as voting rule approaches unanimity, AD and the decisive receiver's preferences are more heterogeneous, implying that disincentivizing AD's acquisition is less costly. When the sender are induced to switch persuading both AD and the receivers, the loss of AD's information explains why unanimity becomes non-optimal.

1.2 Literature

This paper is related to the Bayesian persuasion literature. In a one-sender-one-receiver model with common priors, KG develop the fundamental methodology to solve a broad class of strategic experimentation problems: they show that the sender's problem is simply a choice among distributions of the receiver's belief about the state subject to a condition called Bayes plausibility. Alonso and Câmara (2016)[2] generalizes to the case of heterogeneous priors and provide necessary and sufficient conditions for a sender to benefit from persuasion. Shimoji (2016)[29] maintains this heterogeneity in prior beliefs and investigates persuasion in a multi-receiver setting. Alonso and Câmara (2018)[4] consider an extension in which the sender has a chance of observing a private signal prior to her design of experimentation, and they study the conditions under which the sender can benefit from being more informed. The most related work to mine is Bizzotto, Rüdiger and Vigier (2017)[9]: they examine the KG model in a dynamic setting with constant inflow of outside public information. My paper differs from theirs in two major ways. First, the outside information is acquired endogenously rather than comes in exogenously. Second, the timings (of persuasion) are defined differently. Namely, the timing in their paper means the period in which the sender's signal induces the receiver to finalize his action rather than postpone it; while in my paper the timing refers to the two options regarding when the sender designs her test and reveals the result: before or after the outside information comes in.

A set of papers study persuasion in a multi-sender setting. The central question they revolve around is whether adding more senders leads to more information revelation. Gentzkow and Kamenica (2016)[15] show that the impact of competition among senders on information revelation is ambiguous in general and derive the condition under which competition is beneficial. Li and Norman (2018)[22] construct examples showing that adding senders can result in a loss of information under some generalizations (e.g., senders can play mixed strategies). Li and Norman (2018)[23] consider sequential persuasion and show that it cannot generate a more informative equilibrium than simultaneous persuasion. One extension of my model to multiple advisors fits in line with this strand of literature.

My paper also relates to the work on voting in a persuasion setting. Alonso and Câmara (2016)[3] study persuasion when there are more than two payoff-dependent states and examines whether persuasion benefits the voters. They show that, with a simple-majority rule, persuasion can make all voters strictly worse off because the sender would exploit voters' heterogeneity by designing an experiment with realizations targeting different winning coalitions. By imposing only two payoff-dependent states, Chan et al. (2016)[11] highlights another channel for taking advantage of the voters' heterogenous preferences by allowing private and correlated persuasion. They show that non-monotonicity is required for a voting rule to be optimal. Bardhi and Guo (2018)[7] investigate persuasion under unanimous rule when voters' preferences are correlated. My contribution to this strand of literature is providing a new channel that explains why a loss of information can be induced through different voting rules.

The rest of this chapter is organized as follows. The model with a single receiver is described in Section 1.3. I analyze the model and present the main results in Section 1.4. In Section 1.5, I extend the model to multiple receivers and examine the impact of voting rules on welfare. Section 1.6 concludes. Appendix A contains all proofs.

1.3 Model

A receiver (`he`) needs to decide between $x \in \{a, b\}$, and a sender (`she`) seeks to persuade him to choose action a. An outside information advisor (AD,`it`), as a third player newly introduced into the model, seeks the receiver to take the action that matches a binary true state of nature $\omega \in \Omega = \{A, B\}$. To allow preference heterogeneity between AD (A) and the receiver (R), following the utility function form in [11], their payoffs are given by

$$u_i(x, \omega) = \begin{cases} 1_{(\omega=A)} - (1_{(\omega=B)})l_i & \text{if } x = a \\ 0 & \text{if } x = b, \end{cases}$$

where $i \in \{R, A\}$ and $l_i > 0$ represents player i's units of loss for committing the error that $x = a$ but $\omega = B$ and hence characterizes their preferences. The sender prefers a regardless of the state and her payoff is hence given by $u_S(x, \omega) = 1_{(x=a)}$.

All players are uncertain about the state and share a common prior belief

$$\mu_0 = \Pr(\omega = A) = 0.5.$$

Informational learning takes place through outside information (whenever available) as well as the test which the sender selects. Throughout this paper, I call this outside information the news for illustration convenience. The news, following Bizzotto, Rüdiger and Vigier (2017)[9], is assumed to be a binary signal S_A with realization $s_A \in \{a, b\}$ and exogenous precision $p \in (\frac{1}{2}, 1)$ that is common knowledge to all players; which means the news has the following information structure

$$\Pr(S_A = a|\omega = A) = \Pr(S_A = b|\omega = B) = p.$$

In this paper, I further assume that this news is only accessible by AD with zero cost of acquisition. Once AD decides to acquire the news, the realization is publicly observed. While the news is free for AD, it may not always prefer to acquire it. This is because there could be a conflict in preferences between AD and the receiver (i.e., $l_R \neq l_A$). In particular, if they agree on the preferred decision $x \in \{a, b\}$ but the realized news could result in disagreement, then AD strictly prefers not to acquire the news.

For the information structure of the test which the sender selects, assume a binary one that is represented by a pair of probability distributions $\{\pi(\cdot|\omega)\}_{\omega\in\Omega}$, with $\pi(\cdot|\omega) \in \Delta(\{a, b\})$. The signal and its realization are denoted by S and s, respectively. In addition to π, the sender can also select the timing T of persuasion: either earlier or later (but not both) than AD makes its news acquisition decision, acquire or not. In this paper, the wordings early-persuasion or late-persuasion are used to explicitly indicate which timing is specified. Early-persuasion corresponds to persuading both the advisor and the receiver because the designed test would also aim for changing the advisor's belief so that its decision over whether to acquire the news is also affected. On the contrary, late-persuasion seeks to persuade the receiver only.

The game proceeds as follows. First, nature draws ω and its realization is unknown to all players. The sender then chooses the timing of persuasion, early-persuasion or late-persuasion. If she chooses early-persuasion, then her test π_0^- is designed and its result s is publicly observed before AD makes its action. If she chooses late-persuasion, then $\pi_{s_A}^+$ is

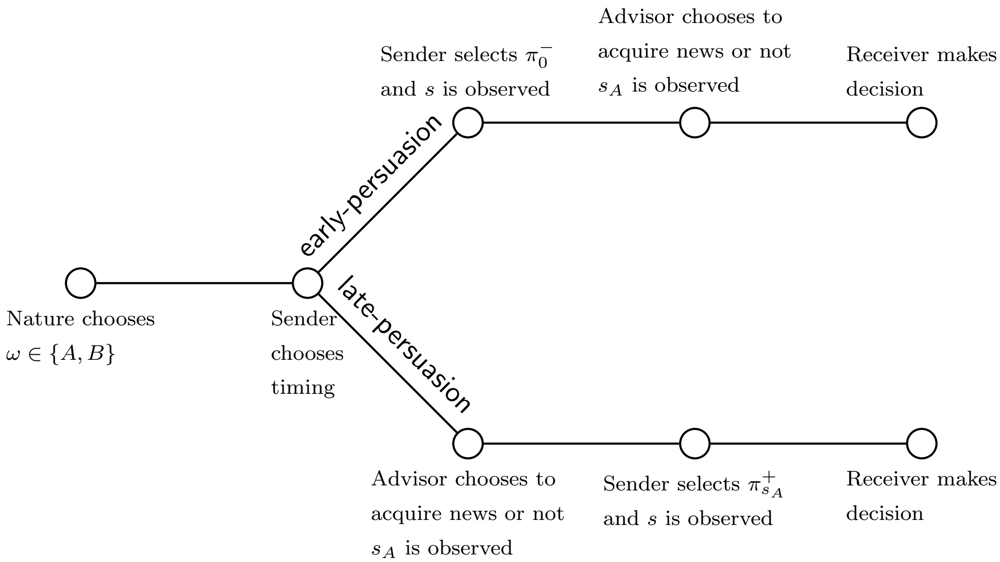

Figure 1.1

selected after s_A ($s_A = \emptyset$ if AD chooses not to acquire news) is observed by all players. After both $s_A \in \{\emptyset, a, b\}$ and $s \in \{a, b\}$ are realized and observed, the receiver then makes his decision. Figure 1.1 summarizes the timing. In this model, the state space, the receiver's and AD's action spaces, preference profiles (l_R, l_A), common prior μ_0, the news' precision p, and sender's choices $(T, \pi_0^-, \pi_\emptyset^+, \pi_a^+, \pi_b^+)$ together define a Bayesian game. In the next section, I first derive pure strategy equilibrium that highlights the sender's optimal choice of persuasion timing under the simplest case when $l_A = l_R$ as a baseline. By considering the possibility that the receiver and AD can have different preferences, I then study how this equilibrium timing, or persuasion target (both AD and the receiver or simply the receiver), can be affected when $l_A \neq l_R$.

1.4 Equilibrium Analysis

Before studying the equilibrium of this game, I first describe the sender's trade-off between different timings of persuasion. In early-persuasion, the sender loses the possibility to observe the news result but has the chance to eliminate the effect of (prospective) news on the success of persuasion by designing a test that persuades AD (not to acquire news) as well. AD prefers not to acquire news when the receiver and it agree on the preferred decision without the news but acquiring it would result in disagreement with positive probability and therefore yield a negative payoff. However, this chance of persuading both players might require a more informative test than persuading the receiver only does. This piece of extra information enhances the probability for the receiver to choose $x = b$ and hence results in a loss to the sender. For example, suppose a fully informative signal is required to induce both the receiver to prefer $x = a$ and the advisor not to acquire news, then the receiver makes the correct decision with probability 1, which is undesirable to the sender and hence is a case when early-persuasion becomes less likely to be chosen. In late-persuasion, if AD chooses to acquire news, then its result does affect the success of persuasion. Even though a chance of eliminating such effect is not provided, the sender can benefit from designing different tests based on different observed results of the news. For example, if $s_A = a$ is informative enough for the receiver to prefer $x = a$, then upon observing it the sender can succeed in persuasion with probability 1 by simply choosing to flip a fair coin (an uninformative test, $\pi_a^+ = \frac{1}{2}$ or $s = \emptyset$). In this case, only when observing $s_A = b$ would the sender face the risk of failure in persuasion.

From the sender's trade-off described above, she would prefer early-persuasion if persuading both AD and the receiver does not require a more informative test. The question of whether a more informative test is required depends on the difference between l_A and l_R. Intuitively, if l_A and l_R are very far apart (say, $l_A << l_R$), then at the belief where the receiver is indifferent between the two alternatives, AD is unwilling to acquire the news since $s_A = b$ can lead to the result such that AD prefers a-action but the receiver prefers b-action. So, in this case, a test informative enough to persuade only the receiver suffices to persuade both AD and the receiver. In other words, persuading these two players does not require a more informative test. Therefore, which timing is optimal for the sender depends on how different between l_A and l_R are.

Throughout the analysis, instead of solving the tests $(\pi_0^-, \pi_\emptyset^+, \pi_a^+, \pi_b^+)$ designed by the sender, I follow the approach introduced in KG that the optimal test corresponds to the Bayes plausible posteriors that solve the sender's corresponding optimization problem. Specifically, let μ_H and μ_L denote the posteriors, $\Pr(\omega = A|s)$, induced by observing $s = a$ or $s = b$, respectively. Then the sender's problem can be solved by finding the μ_H and μ_L such that $\Pr(x = a)$ is maximized subject to $\Pr(x = a)\mu_H + \Pr(x = b)\mu_L = \mu$ (Bayes plausibility condition), where μ is the belief (of true state being A) before observing s.

Assumption **1.** *We assume $p > \frac{1}{l_R+1}$ so that the news is precise enough for the receiver to choose b when observing $(s_A = b, s = \emptyset)$.*

Upon observing $I = (s_A = b, s = \emptyset)$, all players have posterior belief $\Pr(\omega = A|I)$ being $1 - p$, inducing the receiver to choose a iff $\mathbb{E}u_R(a, \omega) \geq 0$. This condition is equivalent to $p \leq \frac{1}{l_R+1}$, in which case an uninformative test can be chosen in any timing and for any news result so that the receiver chooses a with probability 1, making the game uninteresting.

1.4.1 The Baseline Case: $l_A = l_R$

When $l_A = l_R$, the receiver and AD have exactly the same payoff functions, reducing the game to a two-player one. Moreover, since the receiver is now responsible for deciding both x and whether to acquire the zero-cost news, acquiring news in this case becomes a (weakly) dominating strategy. So, throughout this subsection, the game is analyzed under the assumption that there is guaranteed arrival of the news.

With this assumption, the following argues that choosing late-persuasion is weakly preferred by the sender. For either observed news result in late-persuasion, if the sender chooses the test optimally designed in early-persuasion, then her expected payoff would be exactly the same as the one in early-persuasion. By choosing different tests upon observing different news results, she cannot become strictly worse off. In fact, with Assumption 1, the following proposition states that performing late-persuasion is ex-ante strictly better for the sender.

Proposition **1.** *Under Assumption 1, the sender's expected payoff generated by choosing late-persuasion is strictly larger than that generated by choosing early-persuasion.*

In Lemma 5, stated and proved in appendix A, I show that the sender's optimal design of test for early-persuasion requires choosing $\mu_H^* = \frac{l_R}{l_R+(1-p)/p}$, meaning that once $s = a$ is observed, the receiver would decide $x = a$ and ignore the news result. With this lemma, the sender's loss in early-persuasion becomes clear (and proposition 1 tells us that this loss is not small enough for early-persuasion to be profitable): to induce the receiver ignore the news result, she is required to design a signal strong enough such that $s_A = b$ does not turn success into failure. This signal, however, is "over-informative" for her when news result is $s_A = a$, in which case the receiver's payoff is strictly positive. On the other hand, upon observing $s_A = a$ in late-persuasion, she would optimally choose a less informative test such that success of persuasion makes the receiver indifferent between $x = a$ and $x = b$, yielding zero payoff for him. Therefore, we can also expect the receiver to prefer early-persuasion and the next proposition confirms this intuition.

Proposition **2.** *Under Assumption 1, although the equilibrium timing is late-persuasion, the receiver (weakly) prefers early-persuasion.*

1.4.2 General Case

In this subsection, I study how equilibrium timing (or persuasion target) is affected when l_A and l_R can be different. Specifically, to characterize the sender's choice of timing and the associated design of test, the sufficient and necessary conditions for her to strictly prefer early-persuasion are provided. From the special case when $l_A = l_R$, we learn that persuading AD (not to acquire news) is not profitable for the sender because when $s_A = a$ is observed, her designed test would be over-informative by yielding a strictly positive payoff for the receiver. So, for early-persuasion to be optimal, one necessary condition is that AD should be easier to persuade than the receiver (i.e., $l_A < l_R$). If $l_A \geq l_R$, then persuading AD is even more difficult than directly persuading the receiver, hence early-persuasion is too costly and the sender would prefer late-persuasion. To simplify the analysis under $l_A < l_R$, I assume that when AD feels indifferent between acquiring news and not, it would choose to acquire news. Also, we need the assumption below that allows the study of timing a meaningful question.

Assumption **2.** *Assume* $\mu_0 = \frac{1}{2} < \max\{\frac{l_A}{l_A+\frac{1-p}{p}}, \frac{l_R}{l_R+1}\}$.

Under $l_A < l_R$, assumption 2 acts like assumption 1 because if assumption 2 fails, then at prior belief μ_0, the receiver prefers $x = a$ and news result would not change the advisor's preferred decision. This further implies either news result would be ignored (if $\frac{1}{2} \geq \frac{l_R}{l_R+\frac{1-p}{p}}$) or news result is essential but the advisor prefers not to acquire it (if $\frac{1}{2} < \frac{l_R}{l_R+\frac{1-p}{p}}$ so that acquiring news yields strictly negative payoff by resulting in disagreement from agreement with positive probability). In either case, the study of timing is not meaningful because no persuasion is needed and the sender's payoff is simply 1. Note that under $l_A < l_R$, this assumption implies $p > \frac{1}{l_R+1}$, so we have a strict version of assumption 1.

If the game is played without introducing the public news, as the model in [11], then we already know that the optimally designed test would be choosing $(\mu_H^*, \mu_L^*) = (\frac{l_R}{l_R+1}, 0)$ (if persuasion is needed), suggesting that the receiver would be indifferent between $x = a$ and $x = b$ when persuasion succeeds. The associated payoff for the sender is $u_S^* = \left(\frac{1}{2} + \frac{1}{2l_R}\right)1_{\{l_R \geq 1\}} + 1_{\{l_R<1\}}$. Since the presence of the public news cannot strictly improve her success of persuasion, u_S^* is the maximum payoff she can attain (in either timing). From the proof of proposition 1, I found that, under assumption 1 implied by assumption 2, acquiring news can still yield a payoff of u_S^* for the sender in late-persuasion if $p \leq \frac{l_R}{l_R+1}$. In other words, the public news strictly hurts the sender only when it is informative enough, which makes intuitive sense. Therefore, for early-persuasion to be strictly preferred, we need $p > \frac{l_R}{l_R+1}$ listed below as our next necessary condition in addition to $l_A < l_R$ and assumption 2.

Assumption **3.** *Assume* $p > \frac{l_R}{l_R+1}$ *so that the presence of news strictly hurts the sender (in late-persuasion).*

With assumption 2 and 3, the next question is that l_A should be smaller than l_R by how much to induce early-persuasion. I start answering this question by first obtaining a bound for l_A under an extreme case when persuading the advisor results in zero loss. Then

I will derive the sender's optimal tests in early-persuasion so that this bound can be readily sharpen toward both sufficiency and necessity.

Suppose persuading the receiver to prefer $x = a$ is already enough to induce the advisor not to acquire news; that is, $\frac{l_R}{l_R+1} \geq \frac{l_A}{l_A+\frac{1-p}{p}}$, or equivalently $l_A \leq (\frac{1-p}{p})l_R$. Then by performing early-persuasion with $(\mu_H^*, \mu_L^*) = (\frac{l_R}{l_R+1}, 0)$, the advisor (strictly) prefers not to acquire news upon observing $s = a$ because $x = a$ is agreed preferred decision at μ_H^* but by acquiring news with result $s_A = b$, the receiver would prefer $x = b$ while the advisor disagrees. Note that in this extreme case, to persuade also the advisor does not require a more informative test and therefore results in zero loss, yielding the unhurt payoff, u_S^*, for the sender. While in late-persuasion, assumption 3 ensures that news strictly hurts the sender by attaining a payoff of $\frac{1}{2} + \frac{(1-p)}{2l_R}(l_R + 1)$ being strictly smaller than u_S^*. Therefore, assumption 2, 3, and $l_A \leq \frac{1-p}{p}l_R$ are sufficient to induce early-persuasion.

When $\frac{l_R}{l_R+1} < \frac{l_A}{l_A+\frac{1-p}{p}}$, persuading the advisor results in positive loss in sense that a more informative test is required with $\mu_H^* = \frac{l_A}{l_A+\frac{1-p}{p}}$, which is larger than the receiver's cutoff belief $\frac{l_R}{l_R+1}$ for choosing $x = a$. This result is formally stated below as a lemma.

Lemma 1. *Suppose* $\frac{l_R}{l_R+1} < \frac{l_A}{l_A+\frac{1-p}{p}}$ *and under* $l_A < l_R$*, assumption 2, 3, the sender's possible design of test for early-persuasion to be strictly preferred requires choosing* $\mu_H^* = \frac{l_A}{l_A+\frac{1-p}{p}}$*, and* $\mu_L^* \in \{0, \frac{l_R}{l_R+\frac{p}{1-p}}\}$.

Although it is not clear which design of test yields a higher payoff, an important common property for the two payoffs (provided in appendix A) is that they are both strictly decreasing in l_A. This is consistent with our intuition that as the advisor becomes more difficult to persuade, early-persuasion results in more loss. Therefore, for each payoff, there is a unique l_A such that the sender is indifferent between late-persuasion and early-persuasion. Specifically, the two l_A's are given by

$$\bar{l}_A^1 \triangleq \frac{l_R}{(l_R+1)p} \quad \text{and} \quad \bar{l}_A^2 \triangleq \frac{\frac{(1-p)^2+p^2}{p(1-p)} - (\frac{2p-1}{p})l_R}{l_R + (\frac{p}{1-p})\frac{1}{l_R}},$$

which means that for any $l_A < \max\{\bar{l}_A^1, \bar{l}_A^2\}$, the benefit of persuading the advisor exceeds its loss. Note that assumption 3 ensures that $\bar{l}_A^1 > (\frac{1-p}{p})l_R$, so that we strictly sharpen the bound for l_A to induce early-persuasion. In fact, $l_A < \max\{\bar{l}_A^1, \bar{l}_A^2\}$, $l_A < l_R$, assumption 2, and 3 form the sufficient and necessary conditions for the sender to strictly prefer early-persuasion.

PROPOSITION **3.** *The sender strictly prefers early-persuasion if and only if assumption 2, 3, $l_A < l_R$, and $l_A < \max\{\bar{l}_A^1, \bar{l}_A^2\}$ hold.*

1.5 Impact of Voting Rules

In this section, instead of a single receiver, consider to relax the model to a group of N receivers with heterogeneous preferences: $\{l_{R_i}\}_{i=1}^N$. With the tools and results established in the previous section, I study how the voting rule affects welfare (defined as total receivers' payoffs) and what the optimal voting rule would be across different parameter specifications. Without loss of generality, I assume $l_{R_1} \le l_{R_2} \le \cdots \le l_{R_N}$. Throughout this section, to simplify the analysis without losing the main insight into how the voting rule impacts welfare, I impose the following restriction on the set of tests that the sender can design (in any timing),

$$\pi(a|\omega = A) = 1 \text{ or equivalently } \mu_L^* = 0,$$

so that whenever $s = b$ is observed, all players know that $\Pr(\omega = B) = 1$.

The stage that the single receiver makes his decision $x \in \{a, b\}$ now becomes the voting stage. Suppose action $x = a$ is chosen if and only if there are at least k members (receivers) voting for action a, then with the following assumption, the k-th member becomes a decisive/pivotal voter and the voting rule is therefore fully characterized by $k \in \{1, ..., N\}$.

ASSUMPTION **4.** *At the voting stage, no member uses a weakly dominated strategy.*

Given this assumption, action $x = a$ is chosen if and only if the k-th member with preference l_{R_k} votes for action a. So, the model reduces to the previous persuasion game between a sender and a (pivotal) receiver, with the presence of an advisor.

Before deriving receivers' total payoffs, I briefly discuss the tradeoff induced by changing the voting rule k. Because a larger l_{R_k} indicates that the k-th member is more difficult to persuade, an increase in k suggests that a more informative test is required for persuasion and therefore generates a gain to the receivers. This gain persists regardless of which timing of persuasion is chosen. However, an increase in k also generates more incentives for the sender to perform early-persuasion, aiming to induce the advisor not to acquire news, which is clearly a loss to the receivers. Therefore, without introducing the public news that creates the issue of timing of persuasion, the previously mentioned loss would be missing and unanimous rule is expected to be optimal.

1.5.1 Without Public News

As a benchmark for what the optimal voting rule would be, we start with the persuasion game without introducing the public news and advisor. In this baseline model, the sender's designed test is represented by $(\mu_H^*, \mu_L^*) = (\frac{l_{R_k}}{l_{R_k}+1}, 0)$, suggesting that the member i's payoff is given by [1]

$$u_{R_i}(k) = \begin{cases} \Pr(S=a)\Bigg(\mu_H^* - (1-\mu_H^*)l_{R_i}\Bigg) = \frac{1}{2} - \frac{l_{R_i}}{2l_{R_k}} & \text{if } i \leq k \\ 0 & \text{if } i > k, \end{cases}$$

which is clearly increasing in l_{R_k}, and hence also increasing in k. Let welfare W be total members' payoffs; that is, $W(k) = \sum_{i=1}^{k} u_{R_i}(k)$. Then for any $k' > k$, we have

$$\begin{aligned} W(k') &= \sum_{i=1}^{k'} u_{R_i}(k') \\ &\geq \sum_{i=1}^{k} u_{R_i}(k') \quad \text{(each term is nonnegative)} \\ &\geq \sum_{i=1}^{k} u_{R_i}(k) \quad \text{(each term is increasing in } k\text{)} \\ &= W(k), \end{aligned}$$

which implies that unanimous rule is optimal since $\arg\max_k W(k) = N$.

[1] Here we assume persuasion is needed for all k; however, the result that welfare is increasing in k still holds even when this assumption fails because increasing k can only incentivize persuasion and persuasion provides information to the receivers.

1.5.2 With Public News

By introducing the public news whose acquisition decision being made by the advisor, an increase in k might cause a loss to the receivers through changes in the timing of persuasion. In the following, I divide the parameter regions into three and give a detailed discussion on what the optimal voting rule would be in each parameter region. In the first two regions, we have unanimous rule being optimal, and in the third region, I characterize the conditions when unanimous rule is not optimal.

Case 1: $\frac{l_{R_N}}{(l_{R_N}+1)p} \leq l_A$ (timing of persuasion does not change across voting rule k).

In this case, according to proposition 3, the sender would choose late-persuasion for all $k \in \{1, ..., N\}$. With the optimal test being $(\mu_H^*, \mu_L^*) = (\frac{l_{R_k}}{l_{R_k}+1}, 0)$ when $n = b$ is observed, the member i's payoff is given by [2]

$$\begin{aligned} u_{R_i}(k) &= \left[\Pr(S_A = a)(p - (1-p)l_{R_i}) + \Pr(S_A = b, S = a)\left(\mu_H^* - (1-\mu_H^*)l_{R_i}\right)\right] 1_{(i \leq k)} \\ &= \begin{cases} \frac{1}{2} - \frac{1}{2}(1-p)l_{R_i}\left(1 + \frac{1}{l_{R_k}}\right) & \text{if } i \leq k \\ 0 & \text{if } i > k, \end{cases} \end{aligned}$$

which is again increasing in l_{R_k}, and hence also increasing in k. Following similar argument as in the previous subsection, we have welfare $W(k)$ being increasing in k, so that unanimity is optimal.

Case 2: $l_{R_N}(\frac{1-p}{p}) < l_A < \frac{l_{R_N}}{(l_{R_N}+1)p}$ (early-persuasion and change of persuasion timing can happen but unanimity is still optimal).

In this case, there are three subcases: (1) the sender selects early-persuasion for all k, (2) the

[2] Here p is assumed large enough so that persuasion is needed for all k if and only if $n = b$ is observed. If persuasion is needed even when $n = a$ is observed, public news becomes irrelevant to the sender and the game reduces to the baseline case in the previous subsection. If persuasion is not needed even when $n = b$ is observed, then the game becomes uninteresting.

sender selects late-persuasion for all k, and (3) there exists a $k^* \in \{2, ..., N\}$ such that the sender performs late-persuasion for all $k < k^*$ and early-persuasion for all $k \geq k^*$. Since the first two subcases do not involve a change in both the timing of persuasion and the design of test across different values of k, we have each member's payoff being increasing in k and hence it can be easily shown that welfare is maximized at $k = N$, meaning that unanimous rule is optimal. We now focus on analyzing the impact of voting rule on welfare in the last subcase. Let $k_1 < k^*$ and $k_2 \geq k^*$, then from case 1, we have

$$W(k_1) = \sum_{i=1}^{k_1} u_{R_i}(k_1) = \sum_{i=1}^{k_1} \left(\frac{1}{2} - \frac{1}{2}(1-p) l_{R_i} (1 + \frac{1}{l_{R_{k_1}}}) \right).$$

Suppose assumption 2 and 3 hold for all k, then from proposition 3, early-persuasion is not selected under voting rule k_1 because either $l_A \geq l_{R_{k_1}}$, or $l_A \geq \frac{l_{R_{k_1}}}{(l_{R_{k_1}}+1)p}$.

From the previous section, we know that the assumption $l_{R_N}(\frac{1-p}{p}) < l_A$ implies that $(\mu_H^*, \mu_L^*) = (\frac{l_A}{l_A + \frac{1-p}{p}}, 0)$ being the optimal test in early-persuasion. Hence, member i's payoff is given by

$$u_{R_i}(k_2) = \Pr(S = a) \left(\mu_H^* - (1 - \mu_H^*) l_{R_i} \right) = \frac{1}{2} - \frac{1}{2} (\frac{1-p}{p}) \frac{l_{R_i}}{l_A}.$$

With all these expressions, it can be shown that $W(k_2) \geq W(k_1)$:

$$\begin{aligned}
W(k_2) &= \sum_{i=1}^{n} \left(\frac{1}{2} - \frac{1}{2} (\frac{1-p}{p}) \frac{l_{R_i}}{l_A} \right) \\
&\geq \sum_{i=1}^{k_1} \left(\frac{1}{2} - \frac{1}{2} (\frac{1-p}{p}) \frac{l_{R_i}}{l_A} \right) \quad \text{(each term is nonnegative)} \\
&\geq \sum_{i=1}^{k_1} \left(\frac{1}{2} - \frac{1}{2}(1-p) l_{R_i} (1 + \frac{1}{l_{R_{k_1}}}) \right) \quad \text{(either } l_A \geq l_{R_{k_1}} \text{ or } l_A \geq \tfrac{l_{R_{k_1}}}{(l_{R_{k_1}}+1)p}\text{)} \\
&= W(k_1),
\end{aligned}$$

which shows that unanimous rule is optimal.

Case 3: $l_A \leq l_{R_N}(\frac{1-p}{p})$ (early-persuasion and change of persuasion timing can happen and unanimity can be non-optimal).

In this case, there are two subcases: (1) $l_A \leq l_{R_1}(\frac{1-p}{p})$ (early-persuasion for all possible values of k) and (2) $l_A > l_{R_1}(\frac{1-p}{p})$. Since the first subcase does not involve a change in both the timing of persuasion and the design of test across different values of k, we again have unanimity as optimal voting rule. For the second subcase, we first define $k^* := \max\{i : l_A > l_{R_i}(\frac{1-p}{p})\}$. Note that $1 \in \{i : l_A > l_{R_i}(\frac{1-p}{p})\}$ so $k^* \in \{1, ..., N-1\}$ is well-defined. For any $k > k^*$, from the previous section, we know that a more informative test is not required for early-persuasion and hence the sender would perform early-persuasion with optimally designed test being $(\mu_H^*, \mu_L^*) = (\frac{l_{R_k}}{l_{R_k}+1}, 0)$. So, we can derive member i's payoff as

$$
\begin{aligned}
u_{R_i}(k) &= \Big[\Pr(S = a)\Big(\mu_H^* - (1-\mu_H^*)l_{R_i}\Big)\Big]1_{(i\leq k)} \\
&= \begin{cases} \frac{1}{2} - \frac{1}{2}\frac{l_{R_i}}{l_{R_k}} & \text{if } i \leq k \\ 0 & \text{if } i > k, \end{cases}
\end{aligned}
$$

implying that $\max_{k>k^*} W(k) = W(N)$. While from case 1 and case 2, we have $\max_{k\leq k^*} W(k) = W(k^*)$. Therefore, optimal voting rule is derived by comparing $W(N)$ and $W(k^*)$. If early-persuasion is performed under $k = k^*$, then

$$
\begin{aligned}
W(N) - W(k^*) &= \frac{1}{2l_A l_{R_N}} \sum_{i=1}^{N} \Big(\Big(\frac{1-p}{p}\Big)l_{R_N} - l_A\Big)l_{R_i} \\
&\geq 0,
\end{aligned}
$$

which shows that unanimous rule is optimal. While if late-persuasion is performed under $k = k^*$, then there can be a drop in member i's payoff for $i \leq k^*$:

$$
\begin{aligned}
u_{R_i}(k^*) &= \frac{1}{2} - \frac{1}{2}(1-p)l_{R_i}\Big(1 + \frac{1}{l_{R_{k^*}}}\Big) \\
&> \frac{1}{2} - \frac{1}{2}(\frac{l_{R_i}}{l_{R_N}}) \\
&= u_{R_i}(N),
\end{aligned}
$$

where the inequality holds if p is large enough. This payoff loss, which comes from the information missing from the public news while a more informative test is not required for early-persuasion, is where unanimous rule might not be optimal. Let $D \coloneqq \frac{1}{l_{R_N}} \sum_{i=1}^{N} l_{R_i} - (N - k^*)$, then $W(k^*) > W(N)$ if and only if

$$p > 1 - \frac{l_{R_{k^*}} D}{(l_{R_{k^*}} + 1) \sum_{i=1}^{k^*} l_{R_i}}.$$

If $D > 0$,[3] then there exists $p \in (\frac{1}{2}, 1)$ such that the above inequality holds, under which $\arg\max_k W(k) = k^* < N$ and unanimity is not optimal.

1.6 Conclusion

This paper studies the Bayesian persuasion of receivers (decision makers) by a signal sender (information designer) under the presence of an advisor who makes the decision over whether to acquire outside information (public news). Particularly, this setting raises the sender's trade-off between gaining control over the flow of public news and a loss due to the possibility of sending a signal more informative than necessary. This allows us to study the sender's timing of persuasion (before or after the advisor's news acquisition decision), and I derive the sufficient and necessary conditions for when the sender strictly prefers to perform early-persuasion.

With the observation that the equilibrium timing of persuasion depends on the receiver's preference, I also study the impact of a change in the voting rule on welfare (total receivers' payoffs). I find that, as the voting rule approaches unanimity, there can be a drop in welfare when the equilibrium timing changes, so unanimous rule can be non-optimal. This result is interesting because under the baseline model where the public news and the advisor are not introduced, we have unanimous rule being optimal.

[3] For example, if $k^* = N - 1$, then $D = \sum_{i=1}^{N-1} (l_{R_i}/l_{R_N}) > 0$.

Chapter 2

HETEROGENEOUS EXPERTISE AND COMMITTEE DESIGN

2.1 Introduction

Consider a setting in which a group of individuals with a common goal must make a collective decision between two alternatives, where the alternative that is best depends on an unknown state. To improve the quality of this collective decision, each group member can choose to invest in a costly signal that imperfectly correlates with the unknown state of interest, and then members share with each other their signal results to reach an agreement on which alternative should be selected. In some real world situations that fit to this setting, members' signals could have different precisions. For example, when a hiring committee is formed to decide whether to accept a job candidate, some members are more experienced and senior than others so that they are more able to correctly or accurately evaluate this candidate's qualifications. Regarding the quality of the group's decision, a not uncommon belief is that it is increasing in each member's efficiency to acquire her signal. That is, having all members accessible to highly accurate signals is regarded as a desirable characteristic for a committee. Therefore, an evaluation by members all having higher seniority and expertise is believed more reliable and hence convincing or unquestionable. In such situations, is it possible to further improve the decision's quality by considering to replace some efficient members with less efficient ones whose signals are less accurate?

Under certain conditions, the answer to this question is yes and this possibility is demonstrated in Example 1, which basically shows how a group of 5 less efficient members can yield a decision whose overall accuracy is higher than that of a decision made if all 5 members are efficient ones. In this paper, I investigate this question by first highlighting how less accurate information can contribute to the group's overall collection of information, then deriving the conditions under which further improvement could be possible or even is guaranteed to exist. In cases when further improvement exists, the optimal committee design includes positive number of less efficient members, a perhaps interesting result.

The intuition behind the main result is the following. In our setting, as information is a public good that is costly to obtain, an individual member's acquired signal contributes to the amount of information collected but incentivizes other members to free ride. When signals are inaccurate (weak), the probability that everyone is quite unsure about the qualities of the candidate, upon observing all acquired signals, is high and hence members will have a strong incentive to have more information. This suggests that weak signals can help leverage the tradeoff between efficiently contributing to the amount of information acquired and aggravating the free-rider problem among members. In certain situations this leverage can be profitable: when some members' incentive constraints fail but are close to binding, a small decrease in the precision of already acquired signals could be sufficient to induce these members to switch to acquire their information, resulting in a net increase in the total collection of information.

In this paper, the main focus is to analyze the case where committee size is given (so that designer cannot freely choose), members have homogeneous preference, and all the information acquired is publicly observable. In Section 2.5, I relax each of these assumptions and then explore some robustness of my main results. I find that any of these three directions of extending the model either unaffect the main results or even create more favorable conditions for them to carry over. The basic reason is that when settings do not entirely eradicate or can even aggravate the free-rider problem, there might be still room for weak signals to result in profitable leverage.

2.2 Literature

This paper is related to the literature on decision-making committees with endogenous information. Many of the literature assumes that committee members have homogeneous preferences. Gershkov and Szentes (2009)[19] and Li (2001)[24] study the incentivization for eliciting the collected information so as to maximize ex-ante social welfare. From a mechanism design perspective, Gershkov and Szentes (2009)[19] shows that it is optimal to consult these members with identical preferences sequentially, with each individual being unable to observe her position in the sequence and other members' reports. Instead of through how members are consulted, Li (2001)[24] finds that adopting a decision rule that is conservative and even overcautious can mitigate the free-rider problem in information acquisition and improve ex-ante efficiency. Persico (2004)[27] and Gerardi and Yariv (2008)[16] study homogeneous committee design, taking the size and decision rule of the committee as the choice variables. To produce the most informed decision, Persico (2004)[27] shows that the optimal voting rule cannot be close to being consensual unless the signals accessible by every member are precise enough. Gerardi and Yariv (2008)[16] consider more general decision rules that admits a mapping (by the designer) between members' reports and the group's decision, and they find that to provide strong incentives for information acquisition, the optimal device can be ex-post inefficient in sense that not all acquired information is utilized.

There are some other papers that assume heterogeneity in committee members' preferences. Most of these papers conclude that heterogeneous preferences can induce more information acquisition (for example, Chan et al. 2017; Zhao 2016; Cai 2009[10]). Chan et al. (2017)[12] consider a setting where public information keeps arriving over time until the committee collectively decides to stop it, and they find that greater heterogeneity in preferences can induce the pivotal agents to demand greater evidence prior to cast their decisions, leading to lengthier deliberation and hence more information acquisition on average. Zhao (2016)[31] and Gerardi and Yariv (2008)[17] consider similar settings where each member can choose the precision of her signal. They find that members with extreme preferences have strong incentive to persuade the pivotal voter and hence they acquire high-precision signals, meaning more amount of information. In my paper, the precision level of each member's signal is considered her innate ability (i.e., type) hence she cannot freely choose it.

Another set of papers assumes heterogeneity in some dimensions other than preferences.

For example, Che and Kartik (2009)[13], Hirsch (2016)[20], and Van den Steen (2010)[30] assume heterogeneity in prior beliefs between the decision maker and an information collector, and their findings are similar: when compared to the case of a common prior, a "persuasion effect" exists such that the information provider has additional incentive to acquire more information. Gersbach and Hahn (2012)[18] consider imposing heterogeneity on the dimension similar to that in my paper: members differ in their levels of efficiency to acquire information. However, there are two main setting differences that induce different research questions and goals. Firstly, in their paper, a less efficient member can still access to a high-precision signal by exerting effort more than efficient ones, while in my model efficient and less efficient members cannot access to signals of the same precision. Secondly, in their paper, members' efficiency levels are unobserved and hence cannot be freely chosen by the principal but can possibly be inferred from their individual decision at the voting stage, while in my model, a member's efficiency is public information to both the principal (committee designer) and other members, and the principal can freely choose which type of agent to include as member.

2.3 Model

Suppose a group of N (odd) members is assembled to make a collective decision d between A and B, seeking to match this decision with an unknown true state of nature $\omega \in \Omega = \{A, B\}$. In this paper, members' preferences are homogeneous and unbiased (i.e., their relative concerns over the two types of error are symmetric). Specifically, each member's payoff $u : \{A, B\} \times \Omega \to \mathbb{R}$ is given by $u(d, \omega) = 1_{(d=\omega)}$.

Members have a symmetric common prior over the state ω; $\Pr(\omega = A) = \Pr(\omega = B) = \frac{1}{2}$. To gain better information on the state, each member with type t can choose to acquire a (public) signal $s^t \in \{a^t, b^t\}$ that correlates with ω at her own cost $c > 0$, and signals from different members are conditionally independent on state ω. All members make their signal acquisition decisions simultaneously, so the signals are public in sense that their realizations are publicly observed prior to making the final decision. However, the precision of each signal depends on the member's type. The signal's precision is higher if she is an expert (H) and lower if she is not (L). That is, the conditional probabilities of signals are

$$\Pr(s^t = a^t | \omega = A) = \Pr(s^t = b^t | \omega = B) = p_t,$$

where type $t \in \{H, L\}$ and $p_H > p_L > \frac{1}{2}$. The costs c are the same across different types. Also, each member's type is known to the whole group.

Since all acquired signals are publicly observed by all members (prior to making the final decision) and preferences are assumed homogenous, there will be no disagreement over which decision is preferred. When all members are indifferent between A and B, I assume that everyone votes for B, referred to as the status quo. The impossibility of disagreement, along with this tie breaking rule, makes voting non-strategic and voting rule is hence irrelevant in this model. Therefore, the overall quality of the group decision only depends on how many strong and weak signals are acquired in equilibrium.

I define a mechanism as the following game:

1. **Designing Stage:** Given a fixed N, the mechanism designer picks n_H experts so that the committee is composed of n_H experts and $n_L = N - n_H$ inexperts.

2. **Information Acquisition Stage:** After observing n_H, each member chooses to either acquire her own signal, at a cost c, or not. These signal acquisition decisions are made simultaneously.

3. **Voting Stage** All the members update their beliefs on the state ω based on all acquired signals and the collective decision is assumed to be the one preferred by all members. This stage contains no strategic issue and is listed here simply for making the description of the game more complete.

Given any composition of the group, there might be multiple equilibria with different overall accuracy of acquired signals. Throughout this paper, I focus on the equilibrium that yields the highest overall accuracy given a composition. Also, I adopt the tie breaking rule that if a member is indifferent, she will acquire her signal. Let $\Sigma(n_H, n_L, c)$ be the set of all pairs of (x, y) such that n_H experts acquire x strong signals and n_L inexperts acquire y weak signals in equilibrium with cost c. Also, let $A(x, y)$ be the probability of making the correct decision (or equivalently, overall accuracy or a single member's expected payoff) based on x (y) strong (weak) signals. With these notations, the group's welfare is defined as $W(n_H, n_L, c) = \max_{(x,y)\in\Sigma(n_H,n_L,c)} A(x, y)$. Therefore, the choice of n_H^* that maximizes $W(n_H, n_L, c)$ becomes the designer's optimization problem.

Three interesting questions now arise: Firstly, what is the designer's incentive to consider having some inexperts in the group? Secondly, what are the sufficient and necessary conditions for an optimal committee design that includes positive number of inexperts? Thirdly, can we derive the optimal committee design by solving the designer's problem?

In this paper, I approach the calculations of incentive to acquire signals by first investigating the payoff under arbitrary amount of information. I then decompose this payoff in a way that facilitates the identification of the non-overlapping terms between the payoffs of acquiring and not acquiring additional information. This approach, yielding a convenient expression of the incentive, enables us to analyze how the incentive decays as more information comes in and acts as a central tool to study these three questions.

In the next section, I first study the group's welfare under the homogeneous type case (i.e., all experts or all inexperts) as a baseline that answers the first question. When extending to heterogeneous type case that starts to involve the designer's optimization problem, I then derive the necessary conditions, which are close to sufficiency, for an optimal committee design that includes positive number of inexperts. These conditions to some extent narrow down the search over for the optimal solution to the designer's problem, simplifying the complication of the third question.

2.4 Equilibrium Analysis

Before formally analyzing how weak signals can contribute to group's welfare in (the best) equilibrium, I first discuss some intuitions for it. As information is a public good that is costly to obtain, Zhao (2016)[31] mentions the free-rider problem in information acquisition: any individual's incentive to acquire information decreases if other members acquire information. This observation provides an insight into the value of weak signals: any individual's benefit of free-riding on other members' information grows smaller as their signals become less informative. Therefore, in equilibrium, more members can be incentivized to acquire signals, creating a possibility for improving overall accuracy. The following example confirms and supports this main insight.

Example **1.** *Consider a group with $N = 5$, $p_H = 0.9$, $p_L = 0.86$, and $c = 0.02$. If all members are experts, then we have*

$$A(3,0) - A(2,0) = 0.072 \geq c$$
$$A(4,0) - A(3,0) = 0 < c$$
$$A(5,0) - A(4,0) = 0.01944 < c,$$

implying that the maximum number of strong signals that can be sustained as an equilibrium is 3. The group's welfare in this case is $A(3,0) = 0.972$. If we replace all experts with inexperts, then

$$A(0,5) - A(0,4) = 0.03131171 \geq c,$$

implying that the maximum number of weak signals that can be sustained as an equilibrium is 5. The group's welfare improves by observing that $A(0,5) = 0.9779997 > 0.972 = A(3,0)$.

2.4.1 Value Added from Additional Information

This subsection, for pure technical purpose, is devoted to introducing the approach for yielding a convenient expression of the incentives of information acquisition. The reader who is primarily interested in the main results rather than the technical details is recommended to skip to the next subsection. There are two reasons for why it is worth creating one subsection simply for introducing this approach. Firstly, the contribution can stand on its own: this approach yields an expression that can be convenient for analysis and makes deriving incentives (or more generally, payoff differences) quite a simple task. Secondly, this approach is repeatedly used throughout almost every proof in this paper and hence its centrality should be highlighted.

Define a signal sequence of length n as a sequence $s_1 s_2 ... s_n$, where $s_i \in \{a^H, b^H, a^L, b^L\}$ for all i. Then let I be the set of all signal sequences of arbitrary length. To aggregate two pieces of information, define $\oplus : I \times I \to I$ as a binary operator that concatenates two signal sequences. For example, if $v_1 = s_1 s_2 s_3$ and $v_2 = s_4 s_5$, then $v_1 \oplus v_2 = s_1 s_2 s_3 s_4 s_5$. Finally, let $g : I \to \{A, B\}$ as the decision function that assigns to any signal sequence its rational decision. That is, $g(v) = A$ if and only if $\Pr(\omega = A|v) > \frac{1}{2}$.

To analyze how payoff can improve when additional information comes in, let V be the payoff based on the signal sequence v and V_+ be the payoff based on $v \oplus v_+$, the aggregation of given and new pieces of information. Then

$$V = \Pr(g(v) = \omega) = \sum_{u \in \Omega} \Pr(g(v) = \omega, g(v \oplus v_+) = u).$$

Similarly,

$$V_+ = \Pr(g(v \oplus v_+) = \omega) = \sum_{u \in \Omega} \Pr(g(v \oplus v_+) = \omega, g(v) = u).$$

Taking the difference and cancelling out the common terms yield

$$V_+ - V = \Pr(\omega = g(v \oplus v_+) \neq g(v)) - \Pr(\omega = g(v) \neq g(v \oplus v_+)).$$

This difference captures the gain and loss incurred by additional information. The first term, $\Pr(\omega = g(v \oplus v_+) \neq g(v))$, is the probability that additional information is pivotal and leads to a correct change in decision, and hence is interpreted as a gain. The second term, $\Pr(\omega = g(v) \neq g(v \oplus v_+))$, is the probability that additional information is pivotal but leads to an incorrect change in decision, and therefore is interpreted as a loss.

Although the derivation here is proceeded under definitions specific to this paper's setting, it is in fact compatible with more general model setup. One potential benefit is that this derivation might yield a possibly more convenient expression for analysis, creating chances to gain some further observations more easily.

2.4.2 The Baseline Case: Homogeneous Type

In this subsection, we focus on analyzing the case when all committee members are of the same type and then demonstrate the trade-off between weak and strong signals. This trade-off as a result can enhance our understanding that answers the question: what is the designer's incentive to consider having some inexperts in the group?

Let $D^H_m = A(m+1,0) - A(m,0)$ be the incentive to acquire one more strong signal given that other m experts acquire their signals, and $D^L_m = A(0,m+1) - A(0,m)$ is analogously defined. By plugging $v = s^t_1...s^t_m$ and $v_+ = s^t_{m+1}$ into the expression derived in the previous subsection, we can express the sequence $\{D^t_m\}^\infty_{m=0}$ explicitly in a form that highlights the value of weak signals. This first main result is presented in the following proposition.

Proposition 4. *Let $\rho_t = 4p_t(1-p_t)$, then a recursive formula of the sequence $\{D^t_m\}^\infty_{m=0}$ is given by*

$$D^t_0 = p_t - \frac{1}{2}, \quad D^t_1 = 0$$
$$D^t_{m+2} = \frac{m+1}{m+2}\rho_t D^t_m, \quad m \geq 0.$$

That is,

$$D^t_m = \begin{cases} 0 & \text{if } m \text{ is odd} \\ p_t - \frac{1}{2} & \text{if } m = 0 \\ (\frac{m-1}{m})(\frac{m-3}{m-2})\cdots(\frac{1}{2})\rho_t^{\frac{m}{2}} D^t_0 & \text{if } m (\geq 2) \text{ is even} \end{cases}$$

From the formula of the incentive sequence, there are two important observations. Firstly, $m(> 0)$ signals of type t can be sustained as an equilibrium if and only if m is odd and $D^t_{m-1} \geq c$. As we are interested in the maximum of such m, motivating us to define

$$N_t(c) = \begin{cases} \max\left\{n \in \{0,1,...,N\} : D^t_n \geq c\right\} + 1 & \text{if } p_t - \frac{1}{2} \geq c \\ 0 & \text{if } p_t - \frac{1}{2} < c. \end{cases}$$

When $N_t(c) > 0$, which is assumed throughout the paper, [1] it is odd and represents the maximum number of signals that can be sustained as an equilibrium in a group of all members being type t. Secondly, $\rho_t = 4p_t(1-p_t) < 1$ approximately captures how fast the nonzero part of $\{D^t_m\}$ is decreasing. And since $\rho_H = 4p_H(1-p_H) < 4p_L(1-p_L) = \rho_L$, $\{D^H_m\}$ is decreasing faster than $\{D^L_m\}$. Therefore, for a given $c > 0$, it is possible that $N_H(c) < N_L(c)$, as seen in example 1. This analytically demonstrates the value of weak signals: the incentive to free-ride on other members' information grows smaller and hence it is possible to induce more information acquisition that improves overall accuracy.

The incentive sequence $\{D^t_m\}$ can also help us study the gain and loss of having weak signals. To see this, observe that we can express $A(m,0)$ as a telescoping sum. That is,

$$A(m,0) = \sum_{k=1}^{m}\Big(A(k,0) - A(k-1,0)\Big) + A(0,0) = \Big(\sum_{k=1}^{m} D^H_{k-1}\Big) + \frac{1}{2}.$$

Therefore, suppose $N_H(c) < N_L(c)$ for a given c, the loss of having (all) weak signals is

$$A(N_H(c),0) - A(0,N_H(c)) = \sum_{k=1}^{N_H(c)} D^H_{k-1} - \sum_{k=1}^{N_H(c)} D^L_{k-1},$$

while the gain comes from those signals that are additionally incentivized:

$$A(0,N_L(c)) - A(0,N_H(c)) = \sum_{k=N_H(c)+1}^{N_L(c)} D^L_{k-1},$$

where the incentive constraints are satisfied: $D^H_{N_H(c)+1} < c \leq D^L_{N_L(c)-1}$. Observe that the loss depends on how close p_H and p_L are, while the magnitude of the gain does not. So, when p_H and p_L are close, the loss can be small, in which case the contribution of additionally acquired signals can be essential and hence the group's welfare can be improved by replacing all experts by ones that are mildly less professional.

To discuss some comparative statics, regard p_L as a variable rather than a fixed parameter. Then the following lemma facilitates our analysis of the impact of a small increase in p_L on the (equilibrium) welfare.

[1] The $N_t(c) = 0$ case makes the game uninteresting.

Lemma **2.** *For any $m \geq 0$, we have the following integral expression for the series $\sum_{i=0}^{m} D_i^t$:*

$$\sum_{i=0}^{m} D_i^t = \int_{\frac{1}{2}}^{p_t} b_{\lfloor \frac{m}{2} \rfloor} x^{\lfloor \frac{m}{2} \rfloor} (1-x)^{\lfloor \frac{m}{2} \rfloor} dx,$$

where the sequence $\{b_k\}$ satisfies the recurrence relation $b_k = (k+1)a_{k+1}$ for any $k \geq 0$, and the sequence $\{a_k\}$ satisfies $a_1 = 1$ and $a_{k+1} = \frac{2k+1}{2k+2}(4a_k)$ for any $k \geq 1$.

Suppose, for simplicity, that $N = 5$ and $N_L(c) = 5$. With this lemma, we have that 5 weak signals can be sustained as an equilibrium with welfare given by

$$A(0,5) = \frac{1}{2} + \sum_{i=0}^{4} D_i^L = \frac{1}{2} + \int_{\frac{1}{2}}^{p_L} 30x^2(1-x)^2 dx.$$

When p_L increases to $p_{L'} = p_L + \Delta p_L$, then there are two cases: $D_4^{L'} \geq c$ or $D_4^{L'} < c$. That is, 5 signals might become unsustainable under $p_{L'}$, in which case $N_{L'}(c) = 3$ by proposition 4. [2] In the first case, 5 signals are still sustainable and therefore welfare is for sure improved, and the improvement is approximately given by

$$\left(\frac{\partial}{\partial p_L} A(0,5)\right) \Delta p_L = 30 p_L^2 (1-p_L)^2 \Delta p_L.$$

In the second case, only 3 signals can be sustained and the change in welfare is approximately given by

$$\left(\frac{\partial}{\partial p_L} A(0,3)\right) \Delta p_L - D_4^L = 6 p_L (1-p_L) \Delta p_L - 3 p_L^2 (1-p_L)^2 (2p_L - 1),$$

where the first term approximates the gain from better quality of the 3 acquired signals, while the second term represents exactly the loss from the two signals no longer being acquired. When Δp_L is small so that loss dominates gain, making members more professional can decrease group's welfare. If p_L increases by a larger amount, then the gain grows continuously at the expense of more possible failures in incentive constraints, resulting in a discontinuous increase in loss.

[2] Here $\Delta p_L > 0$ is assumed small enough such that when $N_L(c)$ decreases, it decreases by the smallest possible units, which are 2.

2.4.3 The General Case

Under the general case, where committee members' types can be heterogeneous, what are the sufficient or necessary conditions for an optimal committee design that includes positive number of inexperts? This subsection is devoted to investigating this question.

Intuitively, if the inexperts are highly inefficient, then it becomes less likely to benefit from including any inexperts. The following assumption imposes a mild restriction on how different experts and inexperts can be.

ASSUMPTION **5.** *Assume* $(\frac{p_L}{1-p_L})^2 \geq \frac{p_H}{1-p_H}$. [3] *That is, when observing a sequence such as* $v = a^L a^L b^H$, *then posterior belief is in favor of* $\omega = A$, *or equivalently,* $\Pr(\omega = A|v) \geq \frac{1}{2}$.

With assumption 5, the second main result, which describes the necessary conditions for an optimal committee composition to include positive number of inexperts, is summarized in the following proposition.

PROPOSITION **5.** *Under assumption 5, if* $n_L^*(> 0)$ *is the optimal choice of number of inexperts in the committee associated with* x *strong signals and* $y > 0$ *weak signals acquired in the best equilibrium, then the following properties hold.*

(1) $x + y \geq N_H(c) + 2$.

(2) $x < N_H(c)$.

(3) $3 \leq y(\leq n_L^*)$.

(4) $N_H(c) + 2 \leq N_L(c)$.

The intuition for why these four properties hold is now described. For (1), note that $N_H(c)$ is odd so that adding one more strong signal does not strictly improve the overall accuracy induced by $N_H(c)$ strong signals. Therefore, at least $N_H(c)+2$ signals are required

[3] This assumption is equivalent to $\varepsilon \leq \dfrac{1}{p_L^2 + (1-p_L)^2} p_L(1-p_L)(2p_L-1)$, where $\varepsilon = p_H - p_L$.

so that it is possible to have $A(x,y) > A(N_H(c),0)$. For (2), it means that to achieve a higher welfare by inducing more signals, we must sacrifice some strong signals. Property (3), which is simply a corollary after having (1) and (2), means that once weak signals can be helpful in improving welfare, we need at least three inexperts, otherwise (strict) improvement in welfare and incentive constraints to sustain equilibrium cannot be both satisfied simultaneously. Property (4) indicates that to sustain at least $N_H(c)+2$ signals, the least requirement is that $N_H(c)+2$ inexperts can acquire signals in equilibrium. That is, to sustain x strong and y weak signals such that $x+y \geq N_H(c)+2$, an inexpert's incentive to acquire her signal is largest when $(x,y) = (0, N_H(c)+2)$.

These conditions can be helpful in the search for the optimal design given model parameters, but they are not sufficient to ensure that inexperts are needed to strictly improve welfare, as demonstrated by the following example.

Example **2.** *Consider a group with $N = 5$, $p_H = 0.9$, $p_L = 0.8$, and $c = 0.02$. As example 1 shows, $N_H(c) = 3$ and therefore, if all members are experts, the associated welfare is* $A(3,0) = 0.972$. *Since Assumption 5 is satisfied, Proposition 5 indicates that the only three possible better equilibria are x strong and y weak signals with $(x,y) \in \{(0,5),(1,4),(2,3)\}$. However, none of them is welfare improving:*

$$A(0,5) = 0.94208 \qquad A(1,4) = 0.95744 \qquad A(2,3) = 0.97128$$

Therefore, the optimal design does not require inexperts and $n_H^* = 5$ associated with 3 strong signals acquired in the best equilibrium.

The only difference between Example 1 and 2 is how relatively imprecise the weak signals are. When inexperts are not efficient enough, it is intuitive that signals more than $N_H(c)+2$ are required for welfare improving. The following assumption, which ensures Assumption 5, imposes more restriction on the difference between p_H and p_L.

Assumption **6.** *Let $\varepsilon = p_H - p_L$, assume that $\varepsilon \leq \dfrac{2}{N_H(c)+1} p_L(1-p_L)(2p_L-1)$.* [4]

[4] For any $N_H(c) > 1$, $\frac{2}{N_H(c)+1} \leq 1$, meaning that Assumption 5 is implied.

Although this assumption seems technical, its implication, whose proof provided in appendix B, is simple to understand: the overall accuracy based on $N_H(c)+2$ weak signals is strictly higher than that based on $N_H(c)$ strong signals (i.e., $A(N_H(c),0) < A(0,N_H(c)+2)$). Therefore, as seen in Example 1, if $N_H(c)+2$ weak signals can be sustained as an equilibrium and the associated overall accuracy strictly improves over that of $N_H(c)$ strong signals, then the optimal design should include positive number of inexperts. This third main result is formally described in the proposition below.

Proposition **6.** *The following two conditions are sufficient to ensure that the optimal committee design should include positive number of inexperts (i.e., $n_L^* > 0$).*

(1) Assumption 6: $\varepsilon \leq \dfrac{2}{N_H(c)+1} p_L(1-p_L)(2p_L-1)$.

(2) $N_H(c)+2 \leq N_L(c)$.

This proposition indicates that when inexperts are efficient enough but not so efficient as to disincentivise more signal acquisitions, including inexperts can for sure improve group welfare. As efficient inexperts (large p_L) would make it less probable to induce more signals, some readers might doubt whether these two conditions would contradict to each other. and this is indeed the case when expert's incentive constraints (to sustain $N_H(c)+2$ strong signals) are far away from binding. If expert's incentive constraints are close to binding, then being a bit less inefficient would be enough to induce more signal acquisitions, making the two conditions able to hold simultaneously, as was the case for Example 1:

$$N_H(c)+2 = 5 = N_L(c) \quad \text{and} \quad \varepsilon = 0.04 < 0.043 = \frac{2}{N_H(c)+1} p_L(1-p_L)(2p_L-1).$$

Obviously, for inclusion of inexperts to be profitable, Assumption 6 is too strong. However, Proposition 6 and the discussion so far at least suggests that to derive more conclusive results regarding whether inexperts are needed, more detail in how inexperts are relatively inefficient is required. I conclude this subsection by revisiting Example 1, showing that it can be possible for the optimal design to include both experts and inexperts.

Example **3.** *(Revisit of Example 1): Recall that* $N_H(c) = 3$ *and the group's welfare is* $A(3,0) = 0.972$ *if all members are experts. Now consider a configuration of* 1 *experts and* 4 *inexperts and observe that an inexpert's incentive constraint holds:*

$$A(1,4) - A(1,3) = 0.02034 \geq c,$$

implying that an expert's incentive constraint is satisfied as well. The group's welfare also improves by having $A(1,4) = 0.9814$*. From Proposition 5, the only possible better configuration is* 2 *experts and* 3 *inexperts. However,* 2 *strong and* 3 *weak signals cannot be sustained as an equilibrium and hence* $(n_H^*, n_L^*) = (1,4)$ *is the optimal design.*

2.4.4 Committee Design

In this subsection, we discuss how the tool and results established previously can be exploited to solve the designer's problem. Specifically, similar to our analysis (of incentives) in the homogeneous type case, I attempt to generalize the incentive sequences and study how they behave. Their behaviors, along with the necessary conditions in the previous subsection, might further narrow down the search over for the optimal design.

To begin, let $D^{H,L}_{n,m} = A(n, m+1) - A(n,m)$ be an inexpert's incentive to acquire one more (weak) signal given that other n strong and m weak signals are acquired. We are interested in analyzing the sequence in m for each fixed n because of the observation that once inexperts are willing to acquire their signals, experts would be willing as well. The behavior of $\{D^{H,L}_{n,m}\}_{m\geq 0}$ for each n highly depends on how close p_H and p_L are. If they are pretty close such as Assumption 6, then we expect Proposition 4 serves as a good approximation and hence $D^{H,L}_{n,m} \approx D^H_{n+m}$. Therefore, we proceed by assuming that they are mildly close (that makes Assumption 5 close to fail): $\left(\frac{p_L}{1-p_L}\right)^2 = \frac{p_H}{1-p_H}$. With this assumption, the minimum weak signals that are required for improving welfare now changes from 3 to 5, and expressions of the incentive sequences are tractable. Both results are described in the following proposition.

Proposition **7.** *Suppose* $\left(\dfrac{p_L}{1-p_L}\right)^2 = \dfrac{p_H}{1-p_H}$ *and let* $k^{(m)}_{a,t}$ *be the number of excess signals of* a^t *out of* m *signals. Then once inexperts can be included to improve welfare, at least* 5 *(rather than 3) are required. Next, for any $n \in \{0, 1, ..., N_H(c)\}$, the sequence $\{D^{H,L}_{n,m}\}_{m\geq 4}$ is given by* [5]

$$D^{H,L}_{n,m} = \begin{cases} 0, & \textit{if } m \textit{ is odd,} \\ \Pr\left(2k^{(n)}_{a,H} + k^{(m)}_{a,L} = 0\right) D^L_0, & \textit{if } m (\geq 4) \textit{ is even.} \end{cases}$$

This result suggests that n strong and $m(\geq 5)$ weak signals can be sustained as an equilibrium if and only if m is odd and $D^{H,L}_{n,m-1} \geq c$. Since it is clear that $\Pr\left(2k^{(n)}_{a,H} + k^{(m)}_{a,L} = 0\right) \leq \Pr\left(k^{(m+n)}_{a,H} = 0\right)$, there exists an m^*_n such that for any $m > m^*_n$, n strong and m weak signals cannot be sustained. Therefore, we can define

$$N^{(n)}_L = \begin{cases} \max\left\{m \geq 4 : D^{H,L}_{n,m} \geq c\right\} + 1 & \text{if } \left\{m \geq 4 : D^{H,L}_{n,m} \geq c\right\} \neq \emptyset \\ 0 & \text{if } \left\{m \geq 4 : D^{H,L}_{n,m} \geq c\right\} = \emptyset. \end{cases}$$

Regarding this definition, it is worth noting that when $n = 0$, we have $D^{H,L}_{0,m} = D^L_m$ and hence $N^{(0)}_L = N_L(c)$. Also, the reasons for letting $N^{(n)}_L = 0$ in the undefined case are two. Firstly, when $n = N_H(c)$, 0 is the only choice that makes $N^{(n)}_L$ consistent with (2) of Proposition 5. Secondly, if there exists some $n_0 \neq N_H(c)$ such that $\left\{m \geq 4 : D^{H,L}_{n_0,m} \geq c\right\}$ is empty, then the optimal number of experts n^*_H cannot be n_0 and hence defining $N^{(n_0)}_L$ as 0 could ensure that $\max_n\{A(n, N^{(n)}_L)\}^{N_H(c)}_{n=0} \neq A(n_0, 0)$.

When Assumption 5 holds but $\left(\frac{p_L}{1-p_L}\right)^2 \neq \frac{p_H}{1-p_H}$, we can expect that $N^{(n)}_L$ can still be defined similarly. However, there is one essential difference that would highly complicate equilibrium analysis: the incentive sequence $D^{H,L}_{n,m}$ is not necessarily equal to 0 for odd m and is proportional to $p_H - p_L$. Therefore, to draw some conclusions when the magnitude of $p_H - p_L$ is fairly far away from Assumption 5 and 6, separate analysis is required and can be considered as future work.

Under the assumption $\left(\frac{p_L}{1-p_L}\right)^2 = \frac{p_H}{1-p_H}$, our current results can help solve the designer's optimization problem as follows:

[5] We need only pay attention to those m's such that $m \geq 4$ for welfare-improving.

- if $N_L(c) < N_H(c) + 4$, then no inexperts are needed and $n_H^* = N$. [6]

- if $N_L(c) \geq N_H(c) + 4$, then

$$n_H^* = \underset{n=0,1,\ldots,N_H(c)}{\arg\max} \{A(n, N_L^{(n)})\}_{n=0}^{N_H(c)}.$$

2.5 Discussions

2.5.1 Heterogeneous Preferences

In this subsection, I extend the discussion to the case in which members' preferences can be heterogeneous. Specifically, if some of the members lean more/bias toward one decision, say A, over the other (i.e., B), then does this relaxation of model assumption create more favorable conditions for the main results to carry over?

In some settings when members with relatively more extreme preference have more incentive to acquire information (for example, Zhao 2016[31]), they might become information providers so that other members could free-ride. This is where the value of weak signals can come into play: if those extremely biased members are inexperts, then they could still have enough incentives to acquire weak signals, and at the same time by mitigating the free-riding problem, also incentivize other experts to acquire their strong signals. The example below demonstrates how this idea works.

Example **4.** *(Revisit of Example 3) Recall that a* 5*-member group with* $p_H = 0.9$*,* $p_L = 0.86$*, and* $c = 0.02$*, Example 3 mentions that when members' preferences are homogeneous and unbiased,* 2 *strong and* 3 *weak signals cannot be sustained as an equilibrium. The reason for this unsustainability is that inexperts' incentive constraints are not satisfied, while experts have enough incentives to acquire their signals:*

$$\textit{expert's incentive:} \qquad A(2,3) - A(1,3) = 0.0234 \geq c$$

$$\textit{inexpert's incentive:} \qquad A(2,3) - A(2,2) = 0.0197 < c.$$

[6] In this case, any n from $N_H(c)$ to N can be a choice of n_H^*.

Now consider that there are two types of preferences: a member could be either extremely biased toward decision A with payoff $u(d,\omega) = 1_{(d=A)}$ or unbiased as previously assumed. To induce information acquisition, the designer must include at least one unbiased member as the pivotal voter in the group. Then, if all members are experts (regardless of their preference types), then we still have 3 strong signals acquired in the best equilibrium. However, allowing the inclusion of inexperts can now improve welfare even further by sustaining 2 strong and 3 weak signals. The designer can attain this goal by choosing 2 unbiased experts and 3 biased inexperts. The experts' incentive constraints are still the same and hence satisfied, while any of the inexperts also has enough incentive to acquire her signal:

$$\Pr\left(g(s_1^H s_2^H s_3^L s_4^L s_5^L) = A\right) - \Pr\left(g(s_1^H s_2^H s_3^L s_4^L) = A\right) = 0.5 - 0.478328 = 0.021672 \geq c.$$

The reason of why biased members' incentive to acquire their signals is larger compared to if they were unbiased comes from the setting where the tie breaking rule is in favor of decision B. Therefore, when the group has less efficient members with higher incentive to acquire information, it can be more possible that they would be willing to provide information and at the same time prevent other members from free-riding, and hence inducing more information acquisition.

2.5.2 Committee Size

We now relax the assumption that committee size is given. Instead, the designer can freely choose N to maximize welfare. Can this relaxation create more chances for the inexperts to be included? The answer to this question is yes based on the argument below.

To begin, instead of directly considering the designer's optimal N, I proceed my argument by first assuming that N is given, and then discuss possible resulting changes when the designer changes N. For an arbitrarily given N, there are two possible cases: inexperts can either strictly improve welfare or not.

In the first case, since inexperts are included, we have $N_H(c) < N$ and the group's welfare is strictly larger than $A(N_H(c), 0)$. However, Proposition 4 implies that when all members are experts, the maximum number of strong signals (acquired in equilibrium) cannot exceed $N_H(c)$ regardless of what N is. Therefore, even if there exists another $N' \neq N$ such that welfare can be further improved, inexperts are still needed. Otherwise, group's welfare cannot exceed $A(N_H(c), 0)$, which is a contradiction.

In the second case, if there exists any other $N' \neq N$ such that inexperts would be included under N', then we are jumping into the first case, whose result indicates that inexperts would still be included under optimal choice of committee size. The existence of such N' is not an impossibility: revisit Example 1 but consider $N = 3$. Under this N, inexperts are not needed because $N_H(c) = 3$, but when committee size changes to $N' = 5$, we have already seen that inexperts should be included.

2.5.3 Private Signals

In the model, we assume all acquired signals are public information before the group's decision is made. To introduce some private information, we now discuss the case when each member's acquired signal is her private information (to other members but not to the designer) but its realization and precision can be credibly revealed via hard evidence. With this introduction, in the information acquisition stage, members simultaneously decide whether to acquire and then whether to reveal their signals. In the voting stage, since disagreement becomes possible, voting rule is hence relevant and we proceed our discussion by assuming voting rule is not unanimous so that every single member cannot be pivotal given that other members agree on which decision is preferred.

Firstly, once each member can choose to conceal her signal, then the observability of the action of acquiring information is irrelevant. This is because if a member decides to acquire her signal but chooses to conceal it, then the observability of this acquisition decision provides no information regarding what the realization might be. Therefore, the option to hide signal realizations is enough to describe each member's signal being private.

Secondly, in this private information case with non-unanimous rule, I argue that there is an equilibrium in which every member makes the same information acquisition decision as in the public information case and always reveals her acquired signal. Given that all other acquired signals would be revealed, a member's incentive to acquire her signal is the same as in the public information case. Hence, incentive constraints for signal acquisition does not change. Regarding the incentive constraints for signal revelation, note that after a member's private observation of either realization (say a) of her signal, her incentive to reveal it is represented by her incentive to acquire signal with each probability in the expression being replaced by conditional probability on observing a. Since c is a sunk cost for revelation decision, the incentive constraint for signal revelation is therefore the conditional probability difference being nonnegative, which clearly holds because of the unbiased-preference assumption.

Intuitively, given any certain pair of the numbers of strong and weak signals acquired in equilibrium, the committee cannot be worse off when every acquired signal is revealed. Therefore, the largest possible overall accuracy that the group can enjoy is still completely determined by how many strong and weak signals can be acquired in equilibrium. This suggests that there is no need to redefine the group's welfare and our discussion so far indicates that all results established in previous sections carry over to this private information case.

2.6 Conclusion

In collective decision-making problems, members of the decision-making committees can collect information to improve the quality of their joint decision before voting for their final choices. The amount of information acquired by the members can be affected by decision-making environments such as committee composition, size, and the decision/voting rule. In this paper, by assuming that members of a committee can have different levels of expertise in acquiring information, I study how an informational source that is less precise can contribute to the incentives for the members to acquire information, and provide the sufficient or necessary conditions for when the replacement of some efficient members by less efficient ones can be included to induce more overall collection of information. The idea behind the main results is that mildly less accurate signals can help leverage the tradeoff between efficiently contributing to the amount of information acquired and aggravating the free-rider problem among members.

To explore the robustness of my main results, I also discuss how the relaxation of some assumptions impacts the model. Firstly, members can also have different and biased preferences over the decision, and the committee designer's problem becomes how to choose each member's type in both dimensions: preference and level of expertise. Secondly, instead of having a committee of fixed size, the designer can freely choose the size of the committee (to maximize group's welfare). Thirdly, I assume that each member's acquired signal is her private information (to other members but not to the designer) but its realization and precision can be credibly revealed via hard evidence. The discussion section concludes that any of these three directions of extending the model either unaffect the main results or can even create more favorable conditions for them to carry over.

Chapter 3

UNCERTAINTY IN INTERPRETING EVIDENCE: NONMONOTONIC BELIEFS, POLARIZATION, AND REPUTATION OF INFORMATION SOURCES

3.1 Introduction

In many real world situations when being provided information on an issue, individuals have to decide how to interpret the information to update their opinions. Their interpretations of each piece of evidence, however, can often be data-dependent and hence yield seemingly inconsistent results. For example, when a shopper reads online reviews to form an evaluation for a product, suppose observing three 5-star reviews along with one 2-star review would induce her to believe that the product is fairly okay, then it is possible that observing four 5-star reviews along with one 2-star review would switch her opinion. A similar phenomenon can arise in the context of an election. Some laboratory experiment indicates that people with higher opinions (which can be possibly formed by a series of positive news) of a political candidate tend to be more amenable to negative information on him/her (Mattes 2008[25]). Are these phenomena evidence that people process information in a biased manner?

In this paper, I argue that, on the contrary, these phenomena are perfectly consistent with Bayesian updating. That is, the mere fact that people's interpretation of a certain piece of news depends on what other information they observe is not evidence of biased reasoning. The key insight is that when people are unsure about whether the available information is biased, Bayesian updating requires that, in response to the observed information, they update their opinions of interest and at the same time adjust their beliefs in its biasedness. Consider the shopper example above, suppose after reading three 5-star reviews and a 2-star one, she is close to buying the product but decides to search for more information, say one more review, before the purchase decision. Then another 5-star review can affect her opinions in two ways. Firstly, she forms a higher opinion of the product because of the additional positive news. Secondly, this new review strengthen her belief that this review site is, to some degree, biased toward the product. Consequently, her positive interpretation of the three originally observed 5-star reviews would be adjusted downward. In fact, it is possible that this second effect dominates the first one, in which case she becomes less convinced of the product being good enough.

This insight of a model with multidimensional uncertainty also provides an explanation for why some people's opinions can move further apart following common information, or polarize. Prior to reading any reviews, a shopper who strongly believes that the site is biased (toward the product) by occasionally screening negative feedback or comments would rationally treat a 2-star review as strong evidence against the product. In contrast, another shopper who holds a different opinion on the site would be more amenable to positive reviews. As a result, exposing the two shoppers to the same mixed evidence may cause their evaluations (on the product) move in different directions, even though they interpret every single review in the same direction. In section 3.4, one of the main results states the conditions under which individuals in the model polarize.

Another insight that yields interesting results of this paper starts from the observation that information which seems conclusive is consistent with it being biased. Consequently, to develop a solid reputation for unbiased news, the information provider needs to release his information with some level of ambiguity. That is, the information is not definitive evidence in one direction or another. This insight has an interesting implication: precise information can negatively impact the provider's reputation and hence his released information can be less efficient in achieving consensus or inducing correct decisions. In subsection 3.5.2, I consider a dynamic extension of the model to exemplify how high-precision signals, compared to low-precision ones, can yield a higher probability that signal receivers polarize but a lower chance that they make correct decisions.

Lastly, to explore some robustness of my main results, I relax the assumption that prior to observing any signals, signal receivers' opinions on the unknown truth of interest are unbiased and identical. Under biased and heterogeneous prior beliefs, all the insights above still hold with one exception: information that seems conclusive can be no longer treated (by receivers) as evidence of biasedness and hence does necessarily hurt the provider's reputation. This is because receivers anticipate both unbiased and biased information to generate evidence in the direction they expect. In subsection 3.5.1, for each main result of this paper, by assuming that data is fairly large to avoid the receivers' priors dominating their posteriors, I either extend it or obtain a result similar to it.

3.2 Literature

This paper is related to the literature on polarization. Rabin and Schrag (1999)[28] explains polarization via confirmation bias. In their model, agents update beliefs using Bayes' rule but would misinterpret signals that are against their current beliefs with an exogenous probability. Given this misinterpretation mechanism, agents with different prior opinions would interpret some commonly observed signals differently and hence polarization can be possible. Fryer, Harms and Jackson (2018)[14] consider the possibility that some signals are ambiguous and an agent would choose to interpret them based on her current belief. Hence, the confirmation bias seems to arise from an individual's inclination to store conclusive rather than ambiguous information. Baliga, Hanany and Klibanoff (2013)[6] provide a different approach to explaining polarization. In their model, agents have hedging motives: they are prone to make predictions so as to limit the variation in the loss across different possible states of the world. This hedging effect can lead agents with different priors to update in different directions based on the same information. Nimark and Sundaresan (2019)[26] consider a setting where agents' information is endogenous. Their model explains polarization by showing the optimal information choice displays a confirmation effect: agents choose information that is more likely to reinforce their prior beliefs.

My paper also relates to the work on Bayesian updating or learning in a setting of multidimensional uncertainty. Andreoni and Mylovanov (2012)[5] consider a model where each agent's private and public signals convey information of different dimensions, both of which are important for identifying the state of nature. Hence, public information can resolve uncertainty in one dimension but can exaggerate the impact of private signals about the other dimension on the posterior beliefs, explaining why agents' opinions can diverge. Benoît and Dubra (2014)[8] and Acemoglu, Chernozhukov and Yildiz (2016)[1] both study how Bayesian agents can update differently or polarize when there they are also unsure about the interpretation of commonly observed signals.

The most related works to mine are Benoît and Dubra (2014)[8] and Acemoglu, Chernozhukov and Yildiz (2016)[1]. My paper differs from theirs in two major ways. First, when agents update their interpretations, both papers allow the possibility that they interpret positive news negatively; while in my paper, the model generates polarization when a piece of positive (negative) news is always treated as positive (negative). Second, both papers focus on analyzing agents' beliefs in the major dimension of interest since they are interested in whether they learn (the truth) in the long-run or polarize; while my paper is also interested in how agents' beliefs in the other dimension behave as their behaviors have interesting implications on how the model can be extended.

The rest of this chapter is organized as follows. In Section 3.3, I describe a model of two-dimensional uncertainty. I present and discuss the main results, and introduce an information provider into the model in Section 3.4. In Section 3.5, I extend the model to the case of asymmetric and heterogeneous priors, and explore some robustness of my main results. Section 3.6 concludes. Appendix C contains all proofs, purely technical lemmas, and computation details.

3.3 Model

There are two possible unknown states of nature: $\omega \in \{A, B\}$. Two agents, labeled 1 and 2, are assumed Bayesians and share a symmetric common prior belief over the state ω:

$$\mathbb{P}_k(\omega = A) = \mathbb{P}_k(A) = 1 - \mathbb{P}_k(\omega = B) = \frac{1}{2},$$

where $k = 1, 2$ and $\mathbb{P}_k(A)$ is simply a notational abbreviation. Moreover, to gain better information on the state, they observe a common set of signals $\{s_t\}_{t=1}^{n}$ that imperfectly correlate with ω.

In addition to the unknown state of interest ω, following Benoît and Dubra (2014)[8], I introduce another dimension of uncertainty $\pi \in \Pi = \{\pi_U, \pi_A, \pi_B\}$ that also correlates with ω. However, in my model, this ancillary dimension has a specific meaning: it represents the situations where agents are also uncertain about how signals correlate with ω and its value indicates that the observations are unbiased, or biased toward either A or B. Therefore, the ancillary dimension π would affect the interpretation of signals. In this model, I assume that the two dimensions of uncertainty, ω and π, are independent. Also, two agents have different prior beliefs over π:

$$\mathbb{P}_k(\pi = \pi_U) = m_k, \qquad \mathbb{P}_k(\pi = \pi_A) = \mathbb{P}_k(\pi = \pi_B) = \left(\frac{1 - m_k}{2}\right),$$

where $k = 1, 2$. Without loss of generality, I assume that $m_1 > m_2$.

Agents believe that signals are conditionally independent on ω and π. In addition, the following likelihood function, which represents the agents' beliefs about the conditional probability distributions of each signal $s_t \in \{a, b\}$, is given by

$$\begin{aligned} &\mathbb{P}(s_t = a|\pi = \pi_U, \omega = A) = p, && \mathbb{P}(s_t = b|\pi = \pi_U, \omega = B) = p, \\ &\mathbb{P}(s_t = a|\pi = \pi_A, \omega = A) = \bar{p}, && \mathbb{P}(s_t = a|\pi = \pi_A, \omega = B) = p, \\ &\mathbb{P}(s_t = b|\pi = \pi_B, \omega = A) = p, && \mathbb{P}(s_t = b|\pi = \pi_B, \omega = B) = \bar{p}, \end{aligned}$$

where $\bar{p} > p \in \left(\frac{1}{2}, 1\right)$. When $\pi = \pi_U$, the resulting observations are called **unbiased**, which means that signal realizations a and b are equally strong evidence of opposite directions. When $\pi = \pi_A(\pi_B)$, the resulting observations are called **biased toward state A**(**B**), which means that, regardless of the true state ω, $a(b)$ is more likely to be observed.

The assumption that $\bar{p} > p$ has two implications on agents' posterior beliefs when observations are biased. Firstly, $\bar{p} \neq p$ suggests that biased signals are still informative and hence agents could still learn ω. Secondly, $\bar{p} > p$ implies that biasedness does not change the direction of interpretation of a single signal. In other words, regardless of π, a single observation of $a(b)$ suggests that agents believe even more that true state ω is $A(B)$. However, the biasedness does affect each signal's strength in determining the agents' posterior beliefs. For example, if the observations are biased toward A (i.e., $\pi = \pi_A$), then agents would account for the bias and hence treat a single signal of a as relatively weak evidence supporting state A while b as relatively strong evidence against it. Therefore, an observation of $aaabb$ could result in agents believing B even more, which is the intuition for why this introduced biasedness could result in a change in the direction of interpretation of a set of signals.

To study why the two agents' beliefs can differ upon observing signals $\{s_t\}_{t=1}^n$, a formal definition of polarization and an example of polarized beliefs are given below.

Definition **1.** *Agents* 1 *and* 2 ***polarize*** *if*

$$\mathbb{P}_1(A|\{s_t\}_1^n) > \frac{1}{2} > \mathbb{P}_2(A|\{s_t\}_1^n) \quad \textit{or} \quad \mathbb{P}_2(A|\{s_t\}_1^n) > \frac{1}{2} > \mathbb{P}_1(A|\{s_t\}_1^n).$$

That is, they interpret the signals $\{s_t\}_{t=1}^n$ *in different directions.*

Example **5.** *Suppose* $n = 5$*,* $p = 0.75$*,* $\bar{p} = 0.9$*, and* $(m_1, m_2) = (0.4, 0.1)$*. Then upon observing signals* $\{s_t\}_1^5 = aabbb$*, the two agents' posterior beliefs are given by*

$$\mathbb{P}_1(A|aabbb) = \frac{\sum_{k\in\Pi} \mathbb{P}_1(\omega = A, \pi = k, \{s_t\}_1^5 = aabbb)}{\sum_{i\in\{A,B\}} \sum_{k\in\Pi} \mathbb{P}_1(\omega = i, \pi = k, \{s_t\}_1^5 = aabbb)} = 0.4315,$$
$$\mathbb{P}_2(A|aabbb) = \frac{\sum_{k\in\Pi} \mathbb{P}_2(\omega = A, \pi = k, \{s_t\}_1^5 = aabbb)}{\sum_{i\in\{A,B\}} \sum_{k\in\Pi} \mathbb{P}_2(\omega = i, \pi = k, \{s_t\}_1^5 = aabbb)} = 0.5704,$$

where two agents polarize since $\mathbb{P}_1(A|aabbb) < \frac{1}{2}$ *and* $\mathbb{P}_2(A|aabbb) > \frac{1}{2}$*.*

The intuition here for why the two agents update in different directions is that for agent 1, signals $aabbb$ are relatively more consistent with unbiased observations, while for agent 2, the signals are relatively more consistent with biased observations toward state B. So, agent 2 treats each signal of b as weaker evidence (against A) compared to agent 1, resulting in their polarized posterior beliefs.

3.4 Main Results

In this section, the three main results of this paper will be presented and discussed. It seems intuitive that any agent's posterior belief of true state being A is increasing in the number of occurrence of a in the signal set $\{s_t\}_{t=1}^n$. In my model, however, this is not always true and the first main result states the condition for when posterior beliefs can be non-monotonic. The second main result specifies the conditions for the two agents to polarize. Perhaps interestingly, the third main result basically means that more precise information on average yields a lower belief (by either agent) that the underlying information source is unbiased.

3.4.1 The Non-monotonic Behavior of Posterior Beliefs

To formalize the exposition and discussion of the first result, let x_a be the number of occurrences of a in the signal set $\{s_t\}_{t=1}^n$ and express an agent's posterior belief of true state being A (i.e., $\mathbb{P}_k(A|\{s_t\}_{t=1}^n)$, $k \in \{1, 2\}$) as a function of x_a. Then since, as discussed in the previous section, a single observation of $a(b)$ is evidence supporting state $A(B)$ regardless of π, it seems to make perfect sense that $\mathbb{P}_k(A|\{s_t\}_{t=1}^n)$ is at least nondecreasing in x_a. However, this is not always true and the following counterexample serves as a disproof.

EXAMPLE **6.** *(Revisit of Example 5): Consider agent 2's posterior beliefs upon observing $\{s_t\}_1^5 = aabbb$ versus $\{s_t\}_1^5 = aaabb$:*

$$\mathbb{P}_2(A|aabbb) = 0.5704 > 0.4295 = \mathbb{P}_2(A|aaabb),$$

which shows that more signals of a could induce a strictly lower belief that true state is A.

The intuition here is that for agent 2 who largely believes the observations are biased, signals of $aabbb$ are relatively more consistent with $\pi = \pi_B$ (biased toward state B) while signals of $aaabb$ are relatively more consistent with $\pi = \pi_A$ (biased toward state A). So, each signal of b in $aabbb$ counts for weak evidence against state A while each signal of b in $aaabb$ is fairly strong evidence supporting state B. The proposition below states the condition for when an agent's posterior belief might exhibit this non-monotonic behavior.

Proposition 8. *Define x_a as the number of occurrences of a in the signal set $\{s_t\}_{t=1}^n$, then agent k's posterior belief in A, $\mathbb{P}_k(A|\{s_t\}_{t=1}^n)$, is nondecreasing in x_a if $m_k \geq \frac{1}{3}$. If $m_k < \frac{1}{3}$ and $|\bar{p} - p|$ is large enough, then $\mathbb{P}_k(A|\{s_t\}_{t=1}^n)$ is non-monotonic in x_a.*

A large m_k means that agent k is treating all possible observations as unbiased with enough weights when updating her belief, implying that the relative strength of evidence supporting state A for each signal of a remains fairly constant across different observations. Therefore, in the case of m_k being large enough, agent k's posterior belief in A, $\mathbb{P}_k(A|\{s_t\}_{t=1}^n)$, is expected to be increasing in the number of signals of a.

3.4.2 Belief Polarization

Unlike the model with one dimension of uncertainty only, being also unsure about how signals correlate with ω makes the interpretation of each signal realization depend on the actual observations. In other words, as we have seen previously, when an agent updates her belief, each signal of a's relative strength of evidence supporting state A can be data-dependent. To provide a more clear illustration, consider signals of $s_1s_2s_3 = aba$ to be observed sequentially: agent k first observes $s_1s_2 = ab$ and then $s_3 = a$. Upon observing s_1s_2, since signals of $s_1s_2 = ab$ are equally likely to be generated by $\pi = \pi_A$ and $\pi = \pi_B$, each signal of a and b are equally strong evidence supporting opposite states of ω, inducing an unchanged belief: $\mathbb{P}_k(A|ab) = \mathbb{P}_k(A)$. When the final signal of $s_3 = a$ comes in, observations (i.e., aba) become more consistent with $\pi = \pi_A$ than $\pi = \pi_B$, implying that each signal of a's relative strength of evidence supporting state A is being adjusted downward. Therefore, how strong evidence each signal represents is data-dependent.

When interpretation of each signal is data-dependent, the impact of observing additional signals of a on an agent's comes from two sources. The first source is the change in **quantity** of evidence: there are more pieces of evidence supporting state A. The second source is the change in (subjective) **quality** of each piece of evidence: each signal of a's relative strength of evidence supporting state A, as illustrated above, is being adjusted downward. These two sources can be readily distinguished via the following expression of an agent's posterior belief on ω:

$$\mathbb{P}_k(A|\{s_t\}_1^n) = \sum_{\pi\in\Pi} \underbrace{\mathbb{P}_k(A|\pi,\{s_t\}_1^n)}_{\text{source of quantity effect}} \times \underbrace{\mathbb{P}_k(\pi|\{s_t\}_1^n)}_{\text{source of quality effect}}$$

The aggregation of all the effects (induced by these two sources) determines the direction in which she updates her belief. If the quantity (quality) induced effects dominate, the agent would believe more (less) that true state is A. As we have seen in example 6, it is possible that the quality induced effects dominate. In fact, it is also possible that when the number of signals of a is larger than that of b, an agent holds an opinion in favor of $\omega = A$ (i.e., $\mathbb{P}_k(A|\{s_t\}_{t=1}^n) > \mathbb{P}_k(A)$), as shown by the following example.

Example **7.** *Suppose* $n = 9$, $p = 0.75$, $\bar{p} = 0.9$, *and* $m_k = 0.1$. *Let* x_a *be the number of occurrence of* a *in* $\{s_t\}_1^9$, *then all possible posterior beliefs are*

$$\begin{aligned}&\left(\mathbb{P}_k(A|\{s_t\}_1^9)\big|_{x_a=0}, ..., \mathbb{P}_k(A|\{s_t\}_1^9)\big|_{x_a=9}\right)\\ =&\big(0.15, 0.33, 0.55, 0.69, 0.63, 0.36, 0.30, 0.44, 0.66, 0.84\big).\end{aligned}$$

So, after observing $x_a \in \{2, 3, 4, 8, 9\}$, *this agent believes even more (i.e.,* $\mathbb{P}_k(A|\{s_t\}_1^9) > \frac{1}{2}$*) that true state is* A.

When the number of signals of a is only a bit larger than that of b, the quality induced effects dominate, while when most of the signals are a, the quantity induced effects dominate. A generalization of Example 7 under odd n, which would be useful in deriving the conditions for when the two agents polarize, is stated as a proposition below.

Proposition **9.** *Assume that* n *is odd and let* x_a*, as previously defined, be the number of occurrences of* a *in the signal set* $\{s_t\}_{t=1}^n$. *Also, let* $H(y) = \frac{p^y(1-p)^{n-y} - p^{n-y}(1-p)^y}{\bar{p}^y(1-\bar{p})^{n-y} - \bar{p}^{n-y}(1-\bar{p})^y}$, $y \in [0, n]$, $y \neq \frac{n}{2}$. *Then the following result describes the direction that agent* k *updates her belief.*

(***Update downward***) $\mathbb{P}_k(A|\{s_t\}_1^n) < \mathbb{P}_k(A) = \frac{1}{2}$ *if* $0 \leq x_a < \underline{x}_k$ *or* $\frac{n+1}{2} \leq x_a < \overline{x}_k$,

(***Update upward***) $\mathbb{P}_k(A|\{s_t\}_1^n) > \mathbb{P}_k(A) = \frac{1}{2}$ *if* $\underline{x}_k < x_a \leq \frac{n-1}{2}$ *or* $\overline{x}_k < x_a \leq n$,

$\big($***No belief change***$\big)$ $\mathbb{P}_k(A|\{s_t\}_1^n) = \mathbb{P}_k(A) = \frac{1}{2}$ *if* $x_a \in \{\underline{x}_k, \overline{x}_k\}$,

where $\underline{x}_k = \min\left(H^{-1}\left(\frac{1-m_k}{1-3m_k}\right) \cup \{\frac{n}{2}\}\right)$ *and* $\overline{x}_k = \max\left(H^{-1}\left(\frac{1-m_k}{1-3m_k}\right) \cup \{\frac{n}{2}\}\right)$.

In other words, proposition 9 specifies the thresholds that describe how large or fairly small the number of occurrence of a should be for an agent to believe even more that true state is A. Note that when agent k has a high (prior) belief that the observations are unbiased (i.e., large m_k), $H(y) = \frac{1-m_k}{1-3m_k}$ has no solutions, implying $\underline{x}_k = \overline{x}_k = \frac{n}{2}$ and this proposition degenerates to a result that is intuitive: $\mathbb{P}_k(A|\{s_t\}_1^n) > \mathbb{P}_k(A)$ if and only if $x_a > \frac{n}{2}$. Lemma 10, stated in appendix C, provides the lower bound for m_k that goes to the degenerate case.

Proposition 9 is useful in deriving, understanding, and interpreting the conditions for when the two agents polarize. These conditions are summarized in the following proposition that serves as the second main result of this paper.

Proposition **10.** *With the notations previously defined, the two agents polarize if and only if the observation* x_a *satisfies one of the two following conditions:*

(1) $\underline{x}_2 < x_a < \underline{x}_1$ $\big($*resulting beliefs:* $\mathbb{P}_1(A|\{s_t\}_1^n) < \frac{1}{2} < \mathbb{P}_2(A|\{s_t\}_1^n)\big)$.

(2) $\overline{x}_1 < x_a < \overline{x}_2$ $\big($*resulting beliefs:* $\mathbb{P}_2(A|\{s_t\}_1^n) < \frac{1}{2} < \mathbb{P}_1(A|\{s_t\}_1^n)\big)$.

Moreover, when $m_k \geq \frac{1}{3}$ *for both agents, they do not polarize for any observation of* x_a.

In words, proposition 10 states that polarization takes place when the numbers of signals of a and b do not differ a lot, the two agents' different (prior) beliefs in the relative strength of evidence of each signal can be essential, making the resulting overall interpretations in opposite directions. When the numbers of signals of a and b do differ enough (i.e., strong evidence supporting either state) or when both agents treat the observations as relatively unbiased (i.e., large m's), it makes intuitive sense that resulting overall interpretations would be in the same direction.

3.4.3 Reputation of Information Provider

To present the third main result, an incorporation of the role of an information provider into the model is needed. The signal provider (SP) provides the signals $\{s_t\}_{t=1}^n$ (data size n is exogenous) that are unbiased with endogenous precision. To be more specific, the SP can choose between the following two conditional distributions that each s_t follows:

$$\mathbb{P}(s_t = a|\omega = A) = \mathbb{P}(s_t = b|\omega = B) = \widetilde{p}_j \quad \text{with} \quad j \in \{H, L\},$$

where $\widetilde{p}_H > \widetilde{p}_L > \frac{1}{2}$. The agents do not know the set of distributions that SP faces and hence his choice either. In other words, SP's information technology is unobservable (or unverifiable) to the agents. Therefore, the agents still have two dimensions of uncertainty with priors as described in the previous section.

This asymmetric setting that SP has the option to improve the precision of his information while the agents do not account for this possibility creates a misunderstanding between SP and the agents. By assuming this setting, one goal of this subsection and section 3.5.2 is to explore the consequences of this misunderstanding.

Instead of specifying SP's utility or objective function that characterizes how he makes his decision, I am interested in understanding and deriving the consequences under each choice of precision. From the previous main results, we know that agents do not polarize when observations are extreme, therefore high-precision signals imply a lower probability that polarization takes place. Another consequence of interest can be agent k's posterior belief that observations are unbiased, $\mathbb{P}_k(\pi = \pi_U|\{s_t\}_1^n)$, to which I refer as the **SP's reputation rated by agent** k. So, the central question in this subsection becomes which choice of precision yields a higher expected reputation rated by agent k. That is, we are interested in solving the following optimization problems:

$$\max_{\widetilde{p} \in \{\widetilde{p}_L, \widetilde{p}_H\}} \mathbb{E}^{\widetilde{p}}\bigg(\mathbb{P}_k(\pi = \pi_U|\{s_t\}_1^n)\bigg), \quad k = 1, 2.$$

The meaning of $\mathbb{P}(\omega = A)$ that is involved in the calculation of expected reputation (rated by agent k) determines how we interpret this expectation. $\mathbb{P}(\omega = A)$ can be referred to as

SP's prior belief or the true probability distribution of ω. If the former meaning is adopted, together with an assumption that how agents update their beliefs upon observing the signals is part of SP's knowledge, then the expectation is SP's expected reputation. If the latter one is adopted, then the value represents the true expected reputation of SP. However, either interpretation leaves the discussion and analysis hereafter unaffected, and yields the same result: low-precision signals solve the optimization problems formulated above.

The key insight here is that when the numbers of signals of a and b do differ a lot, whose likelihood is larger under high-precision than under low precision, the observations are more consistent with draws from $\pi \in \{\pi_A, \pi_B\}$ than $\pi = \pi_U$ for the agents. That is, a large enough (or small enough) x_a is evidence of observations being biased. This result is formally stated in the lemma below.

LEMMA **3.** *Assume that $n \geq 2$ and $m_k \in (0,1)$, then there exists $x_L, x_U \in (0,n)$ such that*

$\big($***Reputation Improves***$\big)$ $\mathbb{P}_k(\pi = \pi_U | \{s_t\}_1^n) > m_k$ *when* $x_L < x_a < x_U$.

$\big($***Reputation Wanes***$\big)$ $\mathbb{P}_k(\pi = \pi_U | \{s_t\}_1^n) < m_k$ *when* $x_a > x_U$ *or* $x_a < x_L$,

Moreover, x_L and x_U are independent of k, so the two agents share the same two thresholds.

From lemma 3, extreme observations, whose likelihood is larger under high precision, are detrimental to the SP's reputation rated by either agent. Therefore, it is natural to conjecture that low-precision signals solve the optimization problems. The following proposition, which serves as the third main result, confirms this conjecture.

PROPOSITION **11.** *For any $\mathbb{P}(\omega = A) \in [0,1]$ and any $m_k \in [0,1]$, low-precision signals solve the optimization problem. That is,*

$$\max_{\widetilde{p} \in \{\widetilde{p}_L, \widetilde{p}_H\}} \mathbb{E}^{\widetilde{p}}\Big(\mathbb{P}_k(\pi = \pi_U | \{s_t\}_1^n)\Big) = \mathbb{E}^{\widetilde{p}_L}\Big(\mathbb{P}_k(\pi = \pi_U | \{s_t\}_1^n)\Big).$$

This proposition indicates that choosing high-precision signals (for whatever reason) on average hurts SP's reputation rated by every agent. A detailed discussion of the implication of this third main result is left to the next section since it involves some specific possible extensions of the model.

3.5 Extensions and Discussions

3.5.1 Heterogeneous Priors

In this subsection, I consider an extension of the model to the case in which the two agents' priors on ω can be asymmetric and different. That is, for an agent (labeled k), assume $\gamma_k = \mathbb{P}_k(A) \in [0,1]$. However, $\gamma_1 = \gamma_2$ is not assumed. Under this relaxation of model assumption, I study whether the three main results carry over.

To see whether the first main result carries over, I examine the idea that drives proposition 8: when the number of signals of a increases, the observations would become more consistent with $\pi = \pi_A$ relative to $\pi = \pi_B$. Since the idea that it is more likely to observe more signals of a under π_A than under π_B is true for any $\omega \in \{A, B\}$, my conjecture is that proposition 8 holds under asymmetric prior on ω. The following proposition confirms this conjecture.

Proposition **12.** *For any $\gamma_k \in (0,1)$, agent k's posterior belief in A, $\mathbb{P}_k(A|\{s_t\}_{t=1}^n)$, is nondecreasing in x_a if $m_k \geq \frac{1}{3}$. If $m_k < \frac{1}{3}$, then $\mathbb{P}_k(A|\{s_t\}_{t=1}^n)$ can be non-monotonic in x_a.*

Under heterogeneous priors on ω, when both agents update their beliefs in different directions, it does not necessarily imply that their disagreement is aggravated. In such case, this is not consistent with our understanding of what polarization means. Therefore, to study whether the second main result carries over requires the following redefinition of polarization.

Definition **2.** *Agents 1 and 2* ***polarize*** *if*

$$\mathbb{P}_1(A|\{s_t\}_1^n) > \gamma_1 \geq \gamma_2 > \mathbb{P}_2(A|\{s_t\}_1^n) \quad \text{or} \quad \mathbb{P}_2(A|\{s_t\}_1^n) > \gamma_2 \geq \gamma_1 > \mathbb{P}_1(A|\{s_t\}_1^n).$$

That is, they interpret the signals $\{s_t\}_{t=1}^n$ in different directions and their disagreement over ω is aggravated.

In fact, when agents are Bayesians and the two dimensions of uncertainty, ω and π, are independent, the directions in which their beliefs (on ω) are updated are unaffected by their priors on ω. This seems counter-intuitive at first glance because to convince an agent with a higher prior on $\omega = A$ to believe more that true state is A, stronger evidence (supporting state A) should be expected. However, at the same time, she would also treat the observations as being more consistent with $\omega = A$ than if she were with a lower prior on $\omega = A$. Under the independence between ω and π, this requirement of stronger evidence is no more or less met by a more extreme posterior induced by the same observations.

The fact that an agent's prior on ω is irrelevant to the direction in which she updates her belief explains why proposition 9 and the second main result (proposition 10) can extend to the case of asymmetric and heterogeneous priors on ω. The following proposition summarizes these extension results.

PROPOSITION **13.** *Proposition 9 holds for any $\gamma_k \in (0,1)$ with all the definitions in it being unaffected. Moreover, the two agents polarize if and only if the observation x_a satisfies the following case-dependent condition:*

$$\underline{x}_2 < x_a < \underline{x}_1 \quad \text{if} \quad \gamma_2 \geq \gamma_1 \qquad \text{or} \qquad \overline{x}_1 < x_a < \overline{x}_2 \quad \text{if} \quad \gamma_1 \geq \gamma_2.$$

Again, when $m_k \geq \frac{1}{3}$ for both agents, they do not polarize for any observation of x_a.

To carry over the third main result, I impose the assumption that $\mathbb{P}(\omega)$ is symmetric. The failure of this assumption can make the solution to the optimization problem agent-dependent. That is, the solution depends on which agent is targeted. For example, suppose $\gamma_1 = 1$, $\gamma_2 = 0$, and $\mathbb{P}(\omega = A) = 1$. Then, when high-precision signals are chosen, it is expected that most of the signals are a, which is consistent with agent 1's prior belief and hence her rating on SP can be high. On the other hand, when most of the signals are a, agent 2 would treat the observations as highly biased toward state A, inducing her rating on SP to be pretty low. Therefore, assuming that $\mathbb{P}(\omega)$ is symmetric can be helpful in excluding this agent-dependency issue.

ASSUMPTION **7.** $\mathbb{P}(\omega)$ *is symmetric. That is,* $\mathbb{P}(\omega = A) = \frac{1}{2}$.

Under asymmetric priors on ω, extreme observations can also be fairly consistent with $\pi = \pi_U$ and hence do not necessarily induce a low reputation. In this case, the insight behind the third main result no longer holds. However, when the number of signals is large enough, it makes statistical sense that the data would dominate the agents' posterior beliefs. The following lemma, which states that agents' tail beliefs on $\pi = \pi_U$ can be taken arbitrarily small by choosing n large enough, confirms this statistical intuition.

Lemma **4.** *For each $\gamma_k \in [0,1]$, $x \in \mathbb{N} \cup \{0\}$ and $\delta > 0$, there exists an $n^* \in \mathbb{N}$ such that $\mathbb{P}_k(\pi = \pi_U | \{s_t\}_1^n)\big|_{x_a = x} \leq \delta$ and $\mathbb{P}_k(\pi = \pi_U | \{s_t\}_1^n)\big|_{x_a = n-x} \leq \delta$ for all $n \geq n^*$.* [1]

When data size n is large enough, this lemma ensures the existence of an $\ell \in \mathbb{N} \cup \{0\}$ such that $\mathbb{P}_k(\pi = \pi_U | \{s_t\}_1^n)\big|_{x_a = x} + \mathbb{P}_k(\pi = \pi_U | \{s_t\}_1^n)\big|_{x_a = n-x} \leq m_k$ for all $x \in \{0, 1, ..., \ell\}$ and $k \in \{1, 2\}$. That is, if agents face enough data, extreme observations would hurt the SP's reputation. This assumption is formally stated below.

Assumption **8.** *The number of signals n is large enough such that there exists an $\ell \in \mathbb{N} \cup \{0\}$ satisfying $\mathbb{P}_k(\pi = \pi_U | \{s_t\}_1^n)\big|_{x_a = x} + \mathbb{P}_k(\pi = \pi_U | \{s_t\}_1^n)\big|_{x_a = n-x} \leq m_k$ for all $x \in \{0, 1, ..., \ell\}$ and $k \in \{1, 2\}$.*

This assumption is fairly mild because we require the amount of data to be enough to influence agents' beliefs in $\pi = \pi_U$ significantly only upon observing all a's or no a's. With assumption 7 and 8, the insight behind the third main result carries over to the case of heterogeneous and asymmetric priors (on ω), hence we have a result similar to proposition 11.

Proposition **14.** *Under assumption 7 and 8, then for any agent k with $m_k \in [0,1]$ and $\gamma_k \in [0,1]$, there exists a $\widetilde{p}_0 \in \left(\frac{1}{2}, 1\right)$ such that for any $\widetilde{p}_H > \widetilde{p}_L \geq \widetilde{p}_0$, low-precision signals solve the optimization problem. That is,*

$$\max_{\widetilde{p} \in \{\widetilde{p}_L, \widetilde{p}_H\}} \mathbb{E}^{\widetilde{p}}\Big(\mathbb{P}_k(\pi = \pi_U | \{s_t\}_1^n)\Big) = \mathbb{E}^{\widetilde{p}_L}\Big(\mathbb{P}_k(\pi = \pi_U | \{s_t\}_1^n)\Big).$$

[1] Following notational conventions in mathematics, $\mathbb{N}$ is the set of natural numbers.

3.5.2 A Simple Two-Period Example

In this subsection, I extend the model to two-period setting so that why reputation is of interest becomes more clear. This extension is motivated by the intuition that when the SP develops a solid reputation for unbiased news, the information he releases is influential in two aspects. Firstly, agents will interpret the released information less differently and hence consensus can be achieved more likely. This is the influence in reducing disagreement and is in fact what we have in proposition 10: when reputations are high enough, agents do not polarize for any possible observations. Secondly, if SP is truly giving unbiased news, then agents will interpret the information correctly as demonstrated through a high probability that they decide between A and B correctly. This is the influence in improving learning quality. Therefore, if the SP is seeking his information in period 2 to be "influential", then he has incentive to improve his reputation in period 1.

This extended model is described as follows.

In period 1:

(1) Nature chooses $\omega \in \{A, B\}$ with equal probability.

(2) SP (privately) chooses precision $\widetilde{p} \in \{\widetilde{p}_L, \widetilde{p}_H\}$ for the signals $\{s_t^{(1)}\}_{t=1}^n$.

(3) Upon observing $\{s_t^{(1)}\}_{t=1}^n$, agents update beliefs and the game enters period 2.

In period 2:

(1) Nature resets the value for ω (for distinction purpose, let ω_2 denote the value after reset) with equal probability.

(2) All players reset their priors to $\frac{1}{2}$: $\mathbb{P}_k(\omega_2 = A) = \frac{1}{2}, \quad k \in \{1, 2, SP\}$.

(3) SP provides the 2nd-period signals $\{s_t^{(2)}\}_{t=1}^n$ with precision chosen in period 1.

(4) Upon observing $\{s_t^{(2)}\}_{t=1}^n$, each agent updates her belief and decides between A and B that matches ω_2.

Regarding the decision rule for the agents, it is natural to assume that agent k chooses $d_k \in \{A, B\}$ if $\mathbb{P}_k(\omega_2 = d_k | \{s_t^{(1)}\}_{t=1}^n, \{s_t^{(2)}\}_{t=1}^n) > \frac{1}{2}$. In case of indifference, I suppose she flips a fair coin and chooses A if and only if a head appears. For SP, as in the previous section, I again do not specify how his decision regarding the precision is made. Instead, I intend to compare the consequences under the two choices and see if any interesting result can be found.

For the dynamic aspect of this game, it is worth mentioning that because both SP has to commit to the same choice of precision in period 2 and this is part of agents' knowledge, each agent's posterior over π in period 1 becomes her prior belief in period 2. Therefore, in period 2, each agent's prior beliefs on $\pi = \pi_A$ and $\pi = \pi_B$ would not in general be symmetric, enhancing the intractability of this two-period model. However, it is possible to derive an example whose result indicates that high-precision signals can induce lower probabilities of correct decisions (by both agents) and a higher probability that polarization takes place. The intuition here is that low precision signals on average induce a higher reputation, making the SP's signals in period 2 more effective uniformly across both agents and hence the result. The example is summarized below and the derivation details are provided in appendix C.

Example **8.** *Suppose $n = 99$, $p = 0.75$, $\bar{p} = 0.9$, $(m_1, m_2) = (0.5, 0.4)$, and $(\widetilde{p}_H, \widetilde{p}_L) = (0.80, 0.75)$. If the SP chooses low-precision signals, then the probabilities of correct decisions are given by*

$$\mathbb{P}^{\widetilde{p}_L}(d_1 = \omega_2) = 0.9884 \quad \text{and} \quad \mathbb{P}^{\widetilde{p}_L}(d_2 = \omega_2) = 0.9799,$$

and the probability that polarization takes place is given by

$$\mathbb{P}^{\widetilde{p}_L}\bigg(\Big(\mathbb{P}_1\big(\omega_2 = A|\{s_t^{(1)}, s_t^{(2)}\}_1^n\big) - \frac{1}{2}\Big)\Big(\mathbb{P}_2\big(\omega_2 = A|\{s_t^{(1)}, s_t^{(2)}\}_1^n\big) - \frac{1}{2}\Big) < 0\bigg) = 0.0084.$$

On the other hand, if the SP chooses high-precision signals, then the probabilities of correct decisions both decrease:

$$\mathbb{P}^{\widetilde{p}_H}(d_1 = \omega_2) = 0.9210 \quad \text{and} \quad \mathbb{P}^{\widetilde{p}_H}(d_2 = \omega_2) = 0.8957,$$

and the probability that polarization takes place increases:

$$\mathbb{P}^{\widetilde{p}_H}\left(\left(\mathbb{P}_1\big(\omega_2 = A|\{s_t^{(1)}, s_t^{(2)}\}_1^n\big) - \frac{1}{2}\right)\left(\mathbb{P}_2\big(\omega_2 = A|\{s_t^{(1)}, s_t^{(2)}\}_1^n\big) - \frac{1}{2}\right) < 0\right) = 0.0253.$$

To conclude this subsection, an important remark has to be made: we do not need a dynamic model to generate these results. In other words, if the agents are assumed to decide between A and B at the end of period 1, then, with the same intuition, it is also possible that high-precision signals can induce lower probabilities of correct decisions and a higher probability that polarization takes place. However, a dynamic setting creates a more favorable environment for these results to happen. To be more specific, to generate the results in a one-period example, we require both agents to start with small m_k's since the main insight here is coming from low reputation. When there is an exogenous shock so that agents reset their beliefs in ω, example 8 tells us that the results can happen not just under small m_k's.

3.6 Conclusion

This paper studies how Bayesian individuals update their beliefs on an issue when they are unsure about the underlying information environment and are aware of the possibility that the observed information can be biased. In such setting, the impact of observing additional information on an individual's belief over the state of interest has two sources: one is the change in quantity of evidence and the other is the change in quality of each piece of evidence. The quality-induced effects, which can appear only in models of multidimensional uncertainty, provide one explanation for why it is fully rational for an individual's interpretation of each piece of evidence to be data-dependent. The model in this paper also gives us an answer to some interesting questions. These questions are: (1) Why can more evidence supporting one opinion induce individuals to be less convinced of that opinion? (2) Why does polarization take place? and (3) What is one possible incentive for a news source to provide less precise information?

The main results of this paper carry over to the case of asymmetric and heterogeneous priors. In addition to this extension, a setting in which the model is extended to two-period is considered. Under this dynamic setting, I provide a simple example that demonstrates how more precise information can lead to higher probabilities of incorrect decisions and polarization. A generalization of that example is worth exploring and can be left as future work.

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Appendix A

LEMMAS AND PROOFS OF CHAPTER 1

Lemma 5. *Under Assumption 1, the sender's optimal design of test for early-persuasion requires choosing* $\mu_H^* = \frac{l_R}{l_R+(1-p)/p}$, *and* $\mu_L^* \in \{0, \frac{l_R}{l_R+p/(1-p)}\}$.

Proof of Lemma 5: In early-persuasion, all possible choices of (μ_L, μ_H) can be classified into the following four cases.

1. the receiver decides $x = a$ if and only if $(s = a, s_A = a)$.

2. the receiver decides $x = a$ if and only if $(s = a, s_A \in \{a, b\})$.

3. the receiver decides $x = a$ if and only if $(s \in \{a, b\}, s_A = a)$.

4. the receiver decides $x = a$ if and only if $(s \in \{a, b\}, s_A = a)$, or $(s = a, s_A = b)$.

We now derive the sender's optimal choice of (μ_L, μ_H) for each case and calculate her associated expected payoff.

[Case 1] the receiver decides $x = a$ if and only if $(s = a, s_A = a)$. In this case, the sender's constrained optimization problem is

$$\begin{aligned} &\max_{\mu_H,\mu_L} \phi(\mu_H p + (1-\mu_H)(1-p)) \\ \text{s.t.} \quad &\mu_H\phi + \mu_L(1-\phi) = \mu_0 = \frac{1}{2} \\ &\mu_L < \frac{l_R}{l_R + \frac{p}{1-p}} \\ &\mu_H \in [\frac{l_R}{l_R + \frac{p}{1-p}}, \frac{l_R}{l_R + \frac{1-p}{p}}). \end{aligned}$$

Without solving μ_H, $\mu_L^* = 0$ is a necessary condition for the solution, which simplifies the sender's expected payoff to $\phi^*(\mu_H^* p + (1-\mu_H^*)(1-p)) = (p-\frac{1}{2}) + \phi^*(1-p) \leq (p-\frac{1}{2}) + (1-p) = \frac{1}{2}$.

[Case 2] the receiver decides $x = a$ if and only if $(s = a, s_A \in \{a, b\})$. In this case, the sender's constrained optimization problem becomes

$$\begin{aligned} &\max_{\mu_H, \mu_L} \phi \\ \text{s.t.} \quad &\mu_H \phi + \mu_L(1-\phi) = \mu_0 = \frac{1}{2} \\ &\mu_H \geq \frac{l_R}{l_R + \frac{1-p}{p}} > \frac{1}{2}, \end{aligned}$$

where $\frac{l_R}{l_R + \frac{1-p}{p}} > \frac{1}{2}$ holds because of Assumption 1. $(\mu_L^*, \mu_H^*) = (0, \frac{l_R}{l_R + \frac{1-p}{p}})$ solves this problem and the sender's expected payoff is

$$\phi^* = \frac{1}{2\mu_H^*} = \frac{1}{2} + \frac{1-p}{2 l_R p} > \frac{1}{2}.$$

[Case 3] the receiver decides $x = a$ if and only if $(s \in \{a, b\}, s_A = a)$. In this case, the sender's constrained optimization problem becomes

$$\begin{aligned} &\max_{\mu_H, \mu_L} \phi[\mu_H p + (1-\mu_H)(1-p)] + (1-\phi)[\mu_L p + (1-\mu_L)(1-p)] \\ \text{s.t.} \quad &\mu_H \phi + \mu_L(1-\phi) = \mu_0 = \frac{1}{2} \\ &\mu_H \geq \mu_L \geq \frac{l_R}{l_R + \frac{p}{1-p}}. \end{aligned}$$

There is no need to solve this problem because the Bayes plausibility constraint already simplifies the objective function to $\frac{1}{2}$, which means that the sender's expected payoff is $\frac{1}{2}$ for any feasible (μ_H, μ_L).

[Case 4] the receiver decides $x = a$ if and only if $(s \in \{a, b\}, s_A = a)$, or $(s = a, s_A = b)$.

In this case, the sender's constrained optimization problem becomes

$$\begin{aligned} &\max_{\mu_H,\mu_L} \phi + (1-\phi)[\mu_L p + (1-\mu_L)(1-p)] \\ \text{s.t.}\quad &\mu_H\phi + \mu_L(1-\phi) = \mu_0 = \frac{1}{2} \\ &\mu_H \geq \frac{l_R}{l_R + \frac{1-p}{p}} > \frac{1}{2} \\ &\mu_L \geq \frac{l_R}{l_R + \frac{p}{1-p}}. \end{aligned}$$

Choosing $(\mu_H^*, \mu_L^*) = (\frac{l_R}{l_R+\frac{1-p}{p}}, \frac{l_R}{l_R+\frac{p}{1-p}})$ solves this problem and the sender's expected payoff is

$$\phi^* + (1-\phi^*)(\mu_L^* p + (1-\mu_L^*)(1-p)) = \frac{1}{2} + \frac{1}{2}(1-p)\left(p + \frac{p^2 - p(1-p)l_R^2}{l_R(2p-1)}\right).$$

Note that the payoff under case 4 can be strictly greater than $\frac{1}{2}$, so the optimal choice of (μ_H^*, μ_L^*) for early-persuasion is characterized by either case 2 or case 4. ∎

Proof of Proposition 1: We first calculate sender's expected payoff when choosing late-persuasion. Let $\mu \in \{p, 1-p\}$ be the posterior belief formed upon observing public news n. If $\mu \geq \frac{l_R}{l_R+1}$, then the sender chooses to send an uninformative signal and the receiver would choose action a. If $\mu < \frac{l_R}{l_R+1}$, then the sender's optimization problem becomes

$$\begin{aligned} &\max_{\mu_H,\mu_L} \phi \\ \text{s.t.}\quad &\mu_H\phi + \mu_L(1-\phi) = \mu \\ &\mu_L \leq \mu < \frac{l_R}{l_R+1} \leq \mu_H. \end{aligned}$$

By choosing $\mu_H^* = \frac{l_R}{l_R+1}$ and $\mu_L^* = 0$, the maximized probability of successful persuasion is $\phi^* = \frac{\mu}{\mu_H^*} = \mu\frac{l_R+1}{l_R}$. Therefore, the sender's expected payoff in late-persuasion is either

$$u_S^1(+) = \Pr(S_A = a) + \Pr(S_A = b)(1-p)\frac{l_R+1}{l_R} = \frac{1}{2} + \frac{1}{2}(1-p)\frac{l_R+1}{l_R},$$

if $p \geq \frac{l_R}{l_R+1}$, or

$$u_S^2(+) = \Pr(S_A = a)(p)\frac{l_R+1}{l_R} + \Pr(S_A = b)(1-p)\frac{l_R+1}{l_R} = \frac{1}{2} + \frac{1}{2l_R},$$

if $p < \frac{l_R}{l_R+1}$. From the proof of Lemma 5, we know that the sender's expected payoff in early persuasion is

$$u_S(-) = \max\left\{\frac{1}{2} + \frac{1-p}{2l_R p}, \frac{1}{2} + \frac{1-p}{2}\left(p + \frac{p^2 - p(1-p)l_R^2}{l_R(2p-1)}\right)\right\}.$$

We complete the proof by showing

(i) $\frac{1}{2} + \frac{1-p}{2l_R p} < u_S^1(+)$.

(ii) $\frac{1}{2} + \frac{1-p}{2l_R p} < u_S^2(+)$.

(iii) $\frac{1}{2} + \frac{1-p}{2}\left(p + \frac{p^2-p(1-p)l_R^2}{l_R(2p-1)}\right) < u_S^1(+)$.

(iv) $\frac{1}{2} + \frac{1-p}{2}\left(p + \frac{p^2-p(1-p)l_R^2}{l_R(2p-1)}\right) < u_S^2(+)$ under $p < \frac{l_R}{l_R+1}$.

Inequality (i) is equivalent to $p > \frac{1}{l_R+1}$, which is assumed in Assumption 1. Inequality (ii) is equivalent to $\frac{1-p}{p} < 1$, which holds because $p > \frac{1}{2}$. With some rearrangement, inequality (iii) can be simplified to $(pl_R - (1-p))(l_R+1) > 0$, which clearly holds because $l_R > \frac{1-p}{p}$ from Assumption 1 and $l_R > 0$. For inequality (iv), let $f(l_R) = (1-p)p\big(-(1-p)l_R^2 + (2p-1)l_R + p\big)$, then (iv) can be rearranged into $2p-1 > f(l_R)$. Now observe that $f(l_R)$ is quadratic in l_R that opens downward with critical point $l_R^* = \frac{p-\frac{1}{2}}{1-p}$, so the restriction $p < \frac{l_R}{l_R+1}$, which is equivalent to $l_R > \frac{p}{1-p} > l_R^*$, implies that

$$\sup_{l_R > \frac{p}{1-p}} f(l_R) = f(\frac{p}{1-p}) = 0,$$

which shows $2p-1 > f(l_R)$ because $p > \frac{1}{2}$. ∎

Proof of Proposition 2: We compare the payoffs under two cases: $p < \frac{l_R}{l_R+1}$ and $p \geq \frac{l_R}{l_R+1}$.

[Case 1] $p < \frac{l_R}{l_R+1}$, or equivalently $l_R > \frac{p}{1-p}$. In this case, the precision of the public news is relatively weak so that simply observing $s_A = a$ is not enough for the receiver to decide $x = a$. Therefore, in late-persuasion, the receiver's payoff is zero since the sender would design her test such that he is indifferent between a and b when $s = a$ is observed. Now we argue that the receiver's payoff under early-persuasion in this case is positive. Note that the optimization problem of Case 4 in the proof of Lemma 5 implies that $l_R < \frac{p}{1-p}$, which contradicts to the assumption of this case. Therefore, the optimal information structure for early-persuasion is $\pi_0(a|A) = 1$ and $\pi_0(b|B) = \frac{l_R - \frac{1-p}{p}}{l_R}$ that is induced by $(\mu_L^*, \mu_H^*) = (0, \frac{l_R}{l_R+(1-p)/p})$. This suggests that the receiver's payoff is given by

$$\begin{aligned} u_R(-) &= [\Pr(\omega = A|s = a) - \Pr(\omega = B|s = a)l_R] \Pr(S = a) \\ &= \left[\mu_H^* - (1-\mu_H^*)l_R\right]\left(\frac{1}{2}\pi_0(a|A) + \frac{1}{2}\pi_0(a|B)\right) \\ &= \frac{2p-1}{2p} > 0. \end{aligned}$$

[Case 2] $p \geq \frac{l_R}{l_R+1}$, or equivalently $l_R \leq \frac{p}{1-p}$. In this case, the posterior belief is in favor of a once $n = a$ is observed and if late-persuasion is performed, the sender would simply design an uninformative signal. If $s_A = b$ is (first) observed, then the receiver would again be indifferent between a and b once $s = a$. Hence, the receiver's payoff in late-persuasion is given by

$$u_R(+) = \Pr(S_A = a)\big(p - (1-p)l_R\big) = \frac{1}{2}\big(p - (1-p)l_R\big).$$

The receiver's payoff in early-persuasion and $(\mu_L^*, \mu_H^*) = (0, \frac{l_R}{l_R+(1-p)/p})$ is derived in Case 1 as $u_R(-; \mu_L^* = 0) = \frac{2p-1}{2p}$. With some rearrangement, $u_R(-; \mu_L^* = 0) > u_R(+)$ is equivalent to Assumption 1, $p > \frac{1}{l_R+1}$. Now consider his payoff in early-persuasion and $(\mu_L^*, \mu_H^*) = (\frac{l_R}{l_R+p/(1-p)}, \frac{l_R}{l_R+(1-p)/p})$. Under such (μ_L^*, μ_H^*), the induced design of test is $\pi_0(a|A) = \frac{p^2-p(1-p)l_R}{2p-1}$ and $\pi_0(b|B) = \frac{p^2 l_R - p(1-p)}{(2p-1)l_R}$ and the only possible realization of (s, s_A) that makes him decide $x = a$ and not indifferent between a and b is $(s = a, s_A = \emptyset)$ (i.e., public news is ignored). Hence, his payoff is given by

$$\begin{aligned} u_R(-; \mu_L^* > 0) &= [\Pr(\omega = A|s = a) - \Pr(\omega = B|s = a)l_R] \Pr(S = a) \\ &= \left[\mu_H^* - (1-\mu_H^*)l_R\right]\left(\frac{1}{2}\pi_0(a|A) + \frac{1}{2}\pi_0(a|B)\right) \\ &= \frac{1}{2}\big(p - (1-p)l_R\big) \\ &\geq u_R(+), \end{aligned}$$

so the receiver at least weakly prefers early-persuasion. ∎

Proof of Lemma 1: $\frac{l_R}{l_R+1} < \frac{l_A}{l_A+\frac{1-p}{p}}$ and assumption 2, 3 jointly imply that the prior belief $\mu_0 = \frac{1}{2} \in (\frac{l_R}{l_R+\frac{p}{1-p}}, \frac{l_A}{l_A+\frac{1-p}{p}})$. To induce the advisor not to acquire news, any choice of μ_H that is larger than $\frac{l_A}{l_A+\frac{1-p}{p}}$ is not optimal since the sender's payoff can be strictly improved by choosing a smaller one (without changing μ_L). Moreover, for any μ_H that cannot persuade the advisor, the associated optimal payoff is independent of l_A and would not beat the one in late-persuasion, as proved in proposition 1. Since we are interested in the values of l_A such that early-persuasion yields a higher payoff and this happens when $\frac{l_R}{l_R+1} = \frac{l_A}{l_A+\frac{1-p}{p}}$ (or equivalently, $l_A = \frac{1-p}{p} l_R$) with $\mu_H^* = \frac{l_A}{l_A+\frac{1-p}{p}}$, it is necessary that the optimally designed test in early-persuasion requires $\mu_H^* = \frac{l_A}{l_A+\frac{1-p}{p}}$.

Regarding the optimal choice of μ_L, only two possibilities need to be considered: upon observing $s = b$, the advisor either acquire news (and it is still possible to change the receiver's mind) or news result is ignored. In either case, the sender's payoff can be improved through decreasing μ_L. For the news result to be relevant (to the receiver), $\mu_L \geq \frac{l_R}{l_R+\frac{p}{1-p}}$ is needed and this belief can also induce the advisor to acquire news. Therefore, one optimal candidate for choosing μ_L is $\frac{l_R}{l_R+\frac{p}{1-p}}$. Similarly, for the news result to be irrelevant, we have $\mu_L \in [0, \frac{l_R}{l_R+\frac{p}{1-p}})$, implying the optimal μ_L in this case is simply 0.

Which one of the two payoffs is larger depends on the precision p of the news. The derivation of the threshold that answers this question is tedious and having such threshold does not help much to our subsequent analysis of obtaining sufficient and necessary condition to induce early-persuasion. Therefore, I choose not to compare the two payoffs and simply list them below. The payoffs associated with $(\mu_H^*, \mu_L^*) = (\frac{l_A}{l_A+\frac{1-p}{p}}, 0)$ and $(\mu_H^*, \mu_L^*) = (\frac{l_A}{l_A+\frac{1-p}{p}}, \frac{l_R}{l_R+\frac{p}{1-p}})$ are respectively given by

$$\frac{1}{2} + \frac{1-p}{2l_A p},$$

and

$$\frac{1}{2} + \frac{\frac{p}{1-p} - l_R}{2\left(\frac{p}{1-p} l_A - \frac{1-p}{p} l_R\right)}(1-p)(l_A+1).$$

▮

Proof of Proposition 3: We first prove the sufficiency part. The sender's payoffs in different timings of persuasion are compared under two cases: $\frac{l_R}{l_R+1} > \frac{l_A}{l_A+\frac{1-p}{p}}$ and $\frac{l_R}{l_R+1} \leq \frac{l_A}{l_A+\frac{1-p}{p}}$. In either case, her payoff in late-persuasion would remain the same. Specifically, her design of test would be $(\mu_H^*, \mu_L^*) = (\frac{l_R}{l_R+1}, 0)$ upon observing $s_A = b$ (persuasion is needed because $\frac{1}{2} < \frac{l_R}{l_R+\frac{1-p}{p}}$, implied by assumption 2 and $l_A < l_R$) and no persuasion is needed upon observing $s_A = a$ because of assumption 3. Under $l_A < l_R$, note that this design of test ensures that he advisor and receiver would for sure agree on their preferred decision at the end of the game, regardless of whether news is acquired or not. So, the advisor cannot be strictly worse off by acquiring news and it would choose to do it. The sender's payoff is hence given by

$$\Pr(S_A = a) + \Pr(S_A = b)\frac{(1-p)}{l_R}(l_R+1) = \frac{1}{2} + \frac{(1-p)}{2l_R}(l_R+1).$$

[Case 1] $\frac{l_R}{l_R+1} > \frac{l_A}{l_A+\frac{1-p}{p}}$.

Now consider early-persuasion with $(\mu_H^*, \mu_L^*) = (\frac{l_R}{l_R+1}, 0)$. If $s = b$ is observed, then belief $\mu = 0$ and the game ends with the receiver choosing $x = b$. If $s = a$ is observed, then at μ_H^*, both the advisor and receiver prefer $x = a$, but acquiring news would result in disagreement upon observing $s_A = b$, implying that the advisor strictly prefers not to acquire it and the game ends with choosing $x = a$. Therefore, the sender's payoff in this case is given by

$$\Pr(S = a) = \frac{1}{2} + \frac{1}{2l_R},$$

which is strictly larger than her payoff in late-persuasion because of assumption 3. So, the sender strictly prefers early-persuasion in this case.

[Case 2] $\frac{l_R}{l_R+1} \leq \frac{l_A}{l_A+\frac{1-p}{p}}$.

Now consider early-persuasion with $\mu_H^* = \frac{l_A}{l_A+\frac{1-p}{p}}$ and $\mu_L^* \in \{0, \frac{l_R}{l_R+\frac{p}{1-p}}\}$. If $s = a$ is observed, then as in the previous case, the advisor strictly prefers not to acquire news and the game ends with choosing $x = a$. If $s = b$ is observed under $\mu_L^* = 0$, the game ends with choosing $x = b$. If $s = b$ is observed under $\mu_L^* = \frac{l_R}{l_R+\frac{p}{1-p}}$, then $l_A < l_R$ ensures that acquiring news has no chance of resulting in disagreement; so, the advisor would choose to acquire news and news result determines the receiver's decision. Therefore, the sender's payoff in this case is given by

$$\Pr(S=a)=\frac{1}{2}+\frac{1-p}{2l_A p},$$

if $\mu_L^*=0$ and

$$\Pr(S=a)+\Pr(S=b)\Pr(S_A=a)=\frac{1}{2}+\frac{\frac{p}{1-p}-l_R}{2\left(\frac{p}{1-p}l_A-\frac{1-p}{p}l_R\right)}(1-p)(l_A+1),$$

if $\mu_L^*=\frac{l_R}{l_R+\frac{p}{1-p}}$. Since both payoffs are strictly decreasing in l_A, the assumption $l_A<\max\{\bar{l}_A^1,\bar{l}_A^2\}$ means that at least one of the two payoffs is strictly larger than the one in late-persuasion because $\bar{l}_A^i$ is defined as the unique solution for l_A such that early persuasion under i-th design of test and late persuasion yields the same payoff. So, the sender also strictly prefers early-persuasion in this case.

For the necessity part. Suppose assumption 2 does not hold; that is, $\mu_0=\frac{1}{2}\geq\max\{\frac{l_A}{l_A+\frac{1-p}{p}},\frac{l_R}{l_R+1}\}$. Then if $\frac{1}{2}\geq\frac{l_R}{l_R+\frac{1-p}{p}}$, news result is irrelevant and the receiver would choose $x=a$; if $\frac{1}{2}<\frac{l_R}{l_R+\frac{1-p}{p}}$, $s_A=b$ would result in disagreement while the advisor and receiver agrees on $x=a$ being preferred at $\mu_0=\frac{1}{2}$, suggesting that news would not be acquired. So, in either case, the sender's payoff is simply 1 and no persuasion is needed. Suppose $p\leq\frac{l_R}{l_R+1}$ (assumption 3 fails) while assumption 2 holds (so that persuasion is needed), then in late-persuasion, the sender's payoff would be $\frac{1}{2}+\frac{1}{2l_R}$ regardless of whether news is acquired. Since news in this case does not strictly hurt the sender (by late-persuasion), early-persuasion cannot be strictly preferred even when it is costless (i.e., $\frac{l_R}{l_R+1}>\frac{l_A}{l_A+\frac{1-p}{p}}$ so that persuading the receiver to prefer $x=a$ is enough to induce the advisor not to acquire news). Now suppose $l_A\geq l_R$ while assumption 2 holds. If $\frac{1}{2}\geq\frac{l_R}{l_R+\frac{1-p}{p}}$, then no persuasion is needed. If $\frac{1}{2}<\frac{l_R}{l_R+\frac{1-p}{p}}$ (or equivalently, assumption 1), then regardless of whether or not to persuade the advisor (in early-persuasion), the corresponding optimal designs of test all require choosing $\mu_H^*=\frac{l_R}{l_R+\frac{1-p}{p}}$ (see lemma 5). However, proposition 1 tells us that early-persuasion in this case cannot be strictly preferred. Finally, suppose $l_A\geq\max\{\bar{l}_A^1,\bar{l}_A^2\}$ while assumption 2 holds, then we need only consider whether early-persuasion is strictly preferred under $l_A < l_R$ and assumption 3. Observe that $l_A \geq \bar{l}_A^1 \triangleq \frac{1}{(l_R+1)p} l_R > (\frac{1-p}{p}) l_R$ with the latter strict inequality implied by assumption 3. So, with $l_A > (\frac{1-p}{p}) l_R$, which is equivalent to $\frac{l_R}{l_R+1} < \frac{l_A}{l_A+\frac{1-p}{p}}$, then lemma 1 suggests that the two possible payoffs in early-persuasion required consideration are

$$\frac{1}{2} + \frac{1-p}{2l_A p},$$

and

$$\frac{1}{2} + \frac{\frac{p}{1-p} - l_R}{2\left(\frac{p}{1-p} l_A - \frac{1-p}{p} l_R\right)}(1-p)(l_A+1),$$

where $l_A \geq \max\{\bar{l}_A^1, \bar{l}_A^2\}$ implies that neither payoff is strictly larger than the payoff in late-persuasion, which is $\frac{1}{2} + \frac{(1-p)}{2l_R}(l_R+1)$. Therefore, early persuasion is therefore not strictly preferred by the sender. ∎

Appendix B

LEMMAS AND PROOFS OF CHAPTER 2

Proof of Proposition 4: It is easy to see that $D_0^t = p_t - \frac{1}{2}$. Let $k_{a,t}^{(m)}$ be the number of excess signals of a^t out of m signals. I split the proof into two cases: m is odd and m is even.

[Case 1] Suppose m is odd. By plugging $v = s_1^t ... s_m^t$ and $v_+ = s_{m+1}^t$ into the expression of $V_+ - V$, we have

$$\begin{aligned}
D_m^t &= \Pr(\omega = g(v \oplus v_+) \neq g(v)) - \Pr(\omega = g(v) \neq g(v \oplus v_+)) \\
&= \Pr(g(v) = B, g(v \oplus v_+) = A, \omega = A) - \Pr(g(v) = B, g(v \oplus v_+) = A, \omega = B) \\
&+ \Pr(g(v) = A, g(v \oplus v_+) = B, \omega = B) - \Pr(g(v) = A, g(v \oplus v_+) = B, \omega = A) \\
&= \Pr(k_{a,t}^{(m)} = 1, s^t = b^t, \omega = B) - \Pr(k_{a,t}^{(m)} = 1, s^t = b^t, \omega = A) \\
&= \frac{1}{2}\binom{m}{\frac{m-1}{2}}\Big[p_t(1-p_t)\Big]^{\frac{m-1}{2}}(1-p_t)p_t - \frac{1}{2}\binom{m}{\frac{m-1}{2}}\Big[p_t(1-p_t)\Big]^{\frac{m-1}{2}}p_t(1-p_t) \\
&= 0.
\end{aligned}$$

[Case 2] Suppose m is even. Then

$$\begin{aligned}
D_m^t &= \Pr(\omega = g(v \oplus v_+) \neq g(v)) - \Pr(\omega = g(v) \neq g(v \oplus v_+)) \\
&= \Pr(g(v) = B, g(v \oplus v_+) = A, \omega = A) - \Pr(g(v) = B, g(v \oplus v_+) = A, \omega = B) \\
&+ \Pr(g(v) = A, g(v \oplus v_+) = B, \omega = B) - \Pr(g(v) = A, g(v \oplus v_+) = B, \omega = A) \\
&= \Pr(k_{a,t}^{(m)} = 0, s^t = a^t, \omega = A) - \Pr(k_{a,t}^{(m)} = 0, s^t = a^t, \omega = B) \\
&= \binom{m}{\frac{m}{2}}\Big[p_t(1-p_t)\Big]^{\frac{m}{2}}(p_t - \frac{1}{2}).
\end{aligned}$$

Replace m by $m+2$ in the above expression, we have

$$\begin{aligned}
D^t_{m+2} &= \binom{m+2}{\frac{m}{2}+1}\Big[p_t(1-p_t)\Big]^{\frac{m}{2}+1}(p_t-\frac{1}{2}) \\
&= \frac{(m+2)(m+1)}{(\frac{m}{2}+1)(\frac{m}{2}+1)}p_t(1-p_t)\binom{m}{\frac{m}{2}}\Big[p_t(1-p_t)\Big]^{\frac{m}{2}}(p_t-\frac{1}{2}) \\
&= \frac{m+1}{m+2}\Big[4p_t(1-p_t)\Big]D^t_m \\
&= \frac{m+1}{m+2}\rho_t D^t_m.
\end{aligned}$$

∎

Proof of Lemma 2: Define $a_0 = \frac{1}{2}$, and I first claim that $D^t_m = a_{\frac{m}{2}}\Big[p_t(1-p_t)\Big]^{\frac{m}{2}}(2p_t-1)$ for any even m. From proposition 4, $D^t_0 = p_t - \frac{1}{2} = a_0\Big[p_t(1-p_t)\Big]^0(2p_t-1)$ and $D^t_2 = \Big[p_t(1-p_t)\Big](2p_t-1) = a_1\Big[p_t(1-p_t)\Big](2p_t-1)$. Now suppose the relation hold for a even $m \geq 2$. Then consider D^t_{m+2}.

$$\begin{aligned}
D^t_{m+2} &= \frac{m+1}{m+2}\rho_t D^t_m && \text{(proposition 4)} \\
&= \frac{m+1}{m+2}\Big[p_t(1-p_t)\Big]4a_{\frac{m}{2}}\Big[p_t(1-p_t)\Big]^{\frac{m}{2}}(2p_t-1) && \text{(induction assumption)} \\
&= \frac{m+1}{m+2}4a_{\frac{m}{2}}\Big[p_t(1-p_t)\Big]^{\frac{m}{2}+1}(2p_t-1) \\
&= a_{\frac{m}{2}+1}\Big[p_t(1-p_t)\Big]^{\frac{m}{2}+1}(2p_t-1).
\end{aligned}$$

So, the claim is proved by induction. With this result, we are ready to prove the lemma. For the base case when $m = 0$, then

$$D^t_0 = p_t - \frac{1}{2} = \int_{\frac{1}{2}}^{p_t} 1dx = \int_{\frac{1}{2}}^{p_t} b_0\Big[x(1-x)\Big]^0 dx.$$

Now suppose the lemma hold for some $m \geq 0$. If m is even, then

$$\sum_{i=0}^{m+1} D^t_i = \sum_{i=0}^{m} D^t_i + 0 \qquad \text{(proposition 4)}$$

$$= \int_{\frac{1}{2}}^{p_t} b_{\lfloor \frac{m}{2} \rfloor} x^{\lfloor \frac{m}{2} \rfloor} (1-x)^{\lfloor \frac{m}{2} \rfloor} dx + 0 \qquad \text{(induction assumption)}$$
$$= \int_{\frac{1}{2}}^{p_t} b_{\frac{m}{2}} x^{\frac{m}{2}} (1-x)^{\frac{m}{2}} dx \qquad (m \text{ is even})$$
$$= \int_{\frac{1}{2}}^{p_t} b_{\lfloor \frac{m+1}{2} \rfloor} x^{\lfloor \frac{m+1}{2} \rfloor} (1-x)^{\lfloor \frac{m+1}{2} \rfloor} dx, \qquad (m \text{ is even})$$

as desired. If m is odd, let $\rho(x) = x(1-x)$, then

$$D^t_{m+1} = a_{\frac{m+1}{2}} \Big[p_t(1-p_t) \Big]^{\frac{m+1}{2}} (2p_t - 1) \qquad \text{(proved claim)}$$
$$= \int_{\frac{1}{2}}^{p_t} a_{\frac{m+1}{2}} \rho(x)^{\frac{m-1}{2}} \Big(-\frac{m+1}{2}(2x-1)^2 + 2\rho(x) \Big) dx$$
$$= \int_{\frac{1}{2}}^{p_t} \Big((2m+4) a_{\frac{m+1}{2}} \rho(x)^{\frac{m+1}{2}} - b_{\frac{m-1}{2}} \rho(x)^{\frac{m-1}{2}} \Big) dx$$
$$= \int_{\frac{1}{2}}^{p_t} \Big((2m+4) a_{\frac{m+1}{2}} \rho(x)^{\frac{m+1}{2}} - b_{\lfloor \frac{m}{2} \rfloor} \rho(x)^{\lfloor \frac{m}{2} \rfloor} \Big) dx \qquad (m \text{ is odd})$$
$$= \int_{\frac{1}{2}}^{p_t} (2m+4) a_{\frac{m+1}{2}} \rho(x)^{\frac{m+1}{2}} dx - \sum_{i=0}^{m} D^t_i. \qquad \text{(induction assumption)}$$

Therefore,

$$\sum_{i=0}^{m+1} D^t_i = (2m+4) \int_{\frac{1}{2}}^{p_t} a_{\frac{m+1}{2}} \Big[x(1-x) \Big]^{\frac{m+1}{2}} dx$$
$$= \frac{m+3}{2} \int_{\frac{1}{2}}^{p_t} a_{\frac{m+3}{2}} \Big[x(1-x) \Big]^{\frac{m+1}{2}} dx$$
$$= \int_{\frac{1}{2}}^{p_t} b_{\frac{m+1}{2}} \Big[x(1-x) \Big]^{\frac{m+1}{2}} dx$$
$$= \int_{\frac{1}{2}}^{p_t} b_{\lfloor \frac{m+1}{2} \rfloor} \Big[x(1-x) \Big]^{\lfloor \frac{m+1}{2} \rfloor} dx, \qquad (m \text{ is odd})$$

as desired. By induction, the proof is complete. ∎

LEMMA **6.** *Suppose m strong signals (with 0 weak signals) can be sustained as an equilibrium. Let x and y be the numbers of strong and weak signals, respectively. If (x, y) improves over $(m, 0)$ (i.e., $A(x, y) > A(m, 0)$), then $x + y \geq m + 2$.*

Proof of Lemma 6: It is trivial that $x+y \geq m+1$. If we can show that $x+y \neq m+1$ holds, then the proof is complete. First note that for any $k \in \{0,1,...,m+1\}$, $A(m+1-k,k)$ is (weakly) decreasing in k. Then

$$A(m,0) = A(m+1,0) \geq A(m+1-k,k),$$

where $k \in \{0,1,...,m+1\}$. This means that we need at least $m+2$ signals to achieve a strictly higher overall accuracy over $A(m,0)$. ∎

LEMMA **7.** *Let x and y be the numbers of strong and weak signals, respectively. Under Assumption 5 and the restriction that $x+y=m$, where m is odd and $y \geq 1$, an individual inexpert's incentive to acquire her signal is largest when $(x,y)=(0,m)$.*

Proof of Lemma 7: An inexpert's incentive to acquire her signal when $(x,y)=(0,m)$ is $D^L_{m-1} = QD^L_0$, where $Q = \binom{m-1}{\frac{m-1}{2}}\left[p_L(1-p_L)\right]^{\frac{m-1}{2}}$. Let $D(k)$ be an inexpert's incentive to acquire her signal when $(x,y)=(k,m-k)$. Then the proof is done if we can show that $D(k) \leq D^L_{m-1}$ for any $k \in \{1,...,m-1\}$.

Let $v_k = s^H_1...s^H_k s^L_{k+1}...s^L_{m-1}$ and C_k as the set that includes all possible realizations of v_k. Then define $\underline{C}_k = \{v \in C_k : \Pr(\omega = A|v) \in (1-p_L, \frac{1}{2}]\}$ and $\overline{C}_k = \{v \in C_k : \Pr(\omega = A|v) \in (\frac{1}{2}, p_L]\}$. With these definitions, $D(k)$ can be written as

$$\begin{aligned}
D(k) &= \sum_{v \in \underline{C}_k}\left[\Pr(v, s^L_m = a^L, \omega = A) - \Pr(v, s^L_m = a^L, \omega = B)\right] \\
&+ \sum_{v \in \overline{C}_k}\left[\Pr(v, s^L_m = b^L, \omega = B) - \Pr(v, s^L_m = b^L, \omega = A)\right] \\
&= \sum_{v \in \underline{C}_k} \Pr(v)\left[p_L \Pr(\omega = A|v) - (1-p_L)\Pr(\omega = B|v)\right] \\
&+ \sum_{v \in \overline{C}_k} \Pr(v)\left[p_L \Pr(\omega = B|v) - (1-p_L)\Pr(\omega = A|v)\right] \\
&\leq \Pr\Big(\Pr(\omega = A|v_k) \in (1-p_L, p_L]\Big) D^L_0,
\end{aligned}$$

where $\Pr\Bigg(\Pr(\omega = A|v_k) \in (1-p_L, p_L]\Bigg)$ is weakly decreasing in k. So, it remains to show that $\Pr\Bigg(\Pr(\omega = A|v_1) \in (1-p_L, p_L]\Bigg) \leq Q$.

To establish this desired inequality, let $G = \Pr\Bigg(\Pr(\omega = A|v_1) \in (1-p_L, p_L]\Bigg)$ and derive the exact expression for G:

$$\begin{aligned} G &= 2\Pr\Bigg((k_{a,L}^{(m-2)}, s_1^H) \in \{(1,b),(-1,a)\}, \omega = A\Bigg) \qquad \text{(Assumption 5)} \\ &= \binom{m-2}{\frac{m-3}{2}}\Bigg[p_L(1-p_L)\Bigg]^{\frac{m-1}{2}}\Bigg(\frac{1-p_H}{1-p_L} + \frac{p_H}{p_L}\Bigg) \\ &= \Bigg(\frac{1}{2}\Bigg)Q\Bigg(2 + \frac{(1-2p_L)}{p_L(1-p_L)}\Delta\Bigg) \\ &\leq Q, \end{aligned}$$

where $\Delta = p_H - p_L$. ∎

Lemma 8. *Let x and y be the numbers of strong and weak signals, respectively. Under Assumption 5 and the restriction that $x + y = m + 1$, where m is odd and $y \geq 1$, an individual inexpert's incentive to acquire her signal is smaller than D_{m-1}^L.*

Proof of Lemma 8: Let $D(k)$ be an inexpert's incentive to acquire her signal when $(x, y) = (k, m+1-k)$. Then the proof is done if the following statements are true.

- There exists another sequence $\{\overline{D}(k)\}_{k=1}^m$ such that it is decreasing in k and $D(k) \leq \overline{D}(k)$ for all $k \in \{1, ..., m\}$.
- $\overline{D}(1) \leq D_{m-1}^L$.

Note that the first statement is true from the proof of Lemma 7. To prove the second statement, we again derive the exact expression for $\overline{D}(1)$, and then establish the desired inequality. Let $Q = \binom{m-1}{\frac{m-1}{2}}\Bigg[p_L(1-p_L)\Bigg]^{\frac{m-1}{2}}$ and $v = s_1^L ... s_{m-1}^L s_m^H$, we have

$$\overline{D}(1) = \Pr\Bigg(\Pr(\omega = A|v) \in (1-p_L, p_L]\Bigg)D_0^L$$

$$
\begin{aligned}
&= 2\Pr\left((k_{a,L}^{(m-1)}, s_1^H) \in \{(2,b),(-2,a)\}, \omega = A\right) D_0^L && \text{(Assumption 5)}\\
&= \left(\frac{m-1}{m+1}\right) Q\left(\frac{p_L}{1-p_L}(1-p_H) + \frac{1-p_L}{p_L} p_H\right) D_0^L\\
&\leq Q\left(1 + \frac{(1-2p_L)}{p_L(1-p_L)}\Delta\right) D_0^L\\
&\leq D_{m-1}^L,
\end{aligned}
$$

where $\Delta = p_H - p_L$. ∎

Proof of (1) of Proposition 5: This is a corollary of Lemma 6 by letting $m = N_H(c)$. ∎

Proof of (3) of Proposition 5: We assume $y = 1$ or $y = 2$ and try to derive contradictions. Firstly, assume $y = 1$, then $x \neq 0$ since $D_0^H > D_0^L$. If x is odd, then this single weak signal is not pivotal in changing the decision induced by all realizations of these x strong signals, meaning that the incentive is 0 and hence x strong and y weak signals cannot be sustained. If $x(\geq 2)$ is even, then this single inexpert's incentive is $K\rho_H^{\frac{m}{2}} D_0^L \geq c$, where $K = (\frac{m-1}{m})(\frac{m-3}{m-2})\cdots(\frac{1}{2})$. Observe that if this single inexpert is replaced by an expert, then every individual expert's (among the $x+1$ experts) incentive constraint is still satisfied: $K\rho_H^{\frac{m}{2}} D_0^H > K\rho_H^{\frac{m}{2}} D_0^L \geq c$. So, there exists another design that has a better equilibrium, yielding a contradiction.

Secondly, assume $y = 2$, then again split the proof into two cases (x is odd or x is even). If $x(\geq 0)$ is even, then any individual inexpert's incentive is $0 \ngeq c$, yielding a contradiction. If x is odd, then it can be easily shown that an individual inexpert's incentive is strictly smaller than D_{x+1}^H, implying that if we replace the two inexperts by experts, then $x+2$ strong signals (with 0 weak signals) can be sustained as an equilibrium. So, there exists another design that has a better equilibrium, yielding a contradiction. ∎

Proof of (2) of Proposition 5: Let $m(\geq N_H(c))$ be odd, $D(k)$ be an inexpert's incentive to acquire signals (necessary to sustain m strong signals and k weak signals). From (3) of Proposition 5, we can assume $k \geq 3$. The following statements capture the structure of the proof.

- $D(3) \leq D^H_{m+1}$ and $D(4) \leq D^H_{m+1}$ (Note: by definition, $D^H_{m+1} < c$).

- $D(k+2) \leq \max\{D(k), D(k+1)\}$ for all $k \geq 3$.

The second statement is an observation from Proposition 4, and the proof is done if we repeat proving these two under a even m, which will be skipped because the proof is almost the same as the odd m case. To prove the first statement, we first derive the exact expression for $D(3)$ and $D(4)$, then establish the desired inequalities. Let $\Delta = p_H - p_L$, the expression for $D(3)$ is given by

$$\begin{aligned} D(3) = 2\Big[&\Pr(k^{(m)}_{a,H} = 1, k^{(2)}_{a,L} = -2, s^L_{\text{new}} = a, \omega = A) \\ &- \Pr(k^{(m)}_{a,H} = 1, k^{(2)}_{a,L} = -2, s^L_{\text{new}} = a, \omega = B)\Big] \\ &= \binom{m}{\frac{m-1}{2}} \left[p_H(1-p_H)\right]^{\frac{m-1}{2}} p_L(1-p_L)\Delta, \end{aligned}$$

which is smaller than D^H_{m+1} if and only if $p_L(1-p_L)\Delta \leq p_H(1-p_H)(2p_H-1)$. Observe that Assumption 5 implies that $\Delta \leq \frac{p_L(1-p_L)}{p_L^2+(1-p_L)^2}(2p_L-1)$, which is equivalent to $\frac{(1-p_L)^2}{p_L^2+(1-p_L)^2} \leq (1-p_H)$. With this result, we have

$$\begin{aligned} p_L(1-p_L)\Delta &\leq p_L^2 \frac{(1-p_L)^2}{p_L^2+(1-p_L)^2}(2p_L-1) \\ &\leq p_H(1-p_H)(2p_H-1), \end{aligned}$$

as desired. The expression for $D(4)$ is given by

$$\begin{aligned} D(4) = 2\Big[&\Pr(k^{(m)}_{a,H} = -1, k^{(3)}_{a,L} = 1, s^L_{\text{new}} = a, \omega = A) \\ &- \Pr(k^{(m)}_{a,H} = -1, k^{(3)}_{a,L} = 1, s^L_{\text{new}} = a, \omega = B)\Big] \\ &= \binom{m}{\frac{m-1}{2}} \left[p_H(1-p_H)\right]^{\frac{m-1}{2}} \Big(3p_L(1-p_L)\Big)\Big((1-p_H)p_L^2 - p_H(1-p_L)^2\Big) \\ &\leq \binom{m}{\frac{m-1}{2}} \left[p_H(1-p_H)\right]^{\frac{m+1}{2}} \Big(2p_H-1\Big) \\ &= D^H_{m+1}, \end{aligned}$$

where the first equality holds under an further assumption that $\left(\frac{p_L}{1-p_L}\right)^4 \leq \left(\frac{p_H}{1-p_H}\right)^3$. However, even if p_L and p_H are close enough so that this further assumption fails, $D(4) \leq D^H_{m+1}$ still holds. The only resulting difference in the derivation is that the expression for $D(4)$ would be dominated by the third line expression by replacing the $\big(3p_L(1-p_L)\big)$ by $\big(4p_L(1-p_L)\big)$, which clearly does not affect the last (desired) inequality. ▮

Proof of (4) of Proposition 5: Let $m = N_H(c)+2$, and by definition we know that $m-2$ strong signals can be sustained as an equilibrium (if the group has at least $m-2$ experts). So, to sustain x strong and $y(\geq 1)$ weak signals such that $A(x,y) > A(m-2,0)$, we must have $x+y \geq m-1$. From Lemma 6, $x+y \neq m-1$, and the restriction becomes $x+y \geq m$.

From Lemma 7, Lemma 8, and Proposition 4, an inexpert's incentive to acquire her signal is largest when $x=0$ and $y=m$ among all pairs of (x,y) such that $x+y \geq m$ and $y \geq 1$. So, if $N_H(c)+2$ weak signals cannot be sustained as an equilibrium (if all members are inexperts), then no pairs of (x,y) can be both sustainable and overall-accuracy improving, meaning that the optimal design's best equilibrium corresponds to $N_H(c)$ strong signals with 0 weak signals and hence including any inexperts is not needed. ▮

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Proof of Proposition 6: It suffices to only show that $A(N_H(c),0) < A(0,N_H(c)+2)$ under

Assumption 6. Let $m = N_H(c)$, then from Lemma 2 and its proof, we have

$$\begin{aligned} A(m,0) - A(0,m+2) &= \left(\sum_{i=0}^{m-1} D_i^H - \sum_{i=0}^{m-1} D_i^L\right) - D_{m+1}^L \\ &= \int_{p_L}^{p_H} b_{\frac{m-1}{2}} \Big[x(1-x)\Big]^{\frac{m-1}{2}} dx - a_{\frac{m+1}{2}} \Big[p_L(1-p_L)\Big]^{\frac{m+1}{2}} (2p_L - 1) \\ &< b_{\frac{m-1}{2}} \Big[p_L(1-p_L)\Big]^{\frac{m-1}{2}} \varepsilon - a_{\frac{m+1}{2}} \Big[p_L(1-p_L)\Big]^{\frac{m+1}{2}} (2p_L - 1) \\ &= a_{\frac{m+1}{2}} \Big[p_L(1-p_L)\Big]^{\frac{m-1}{2}} \left(\frac{m+1}{2}\varepsilon - p_L(1-p_L)(2p_L-1)\right) \\ &\leq 0. \end{aligned}$$

∎

Proof of part 1 of Proposition 7: I claim that $A(N_H(c)-1,4) < A(N_H(c),0)$, which implies that once inclusion of inexperts is welfare improving, at least 5 are required. Let $m = N_H(c)$ and $Q = \binom{m-1}{\frac{m-1}{2}} \Big[p_L(1-p_L)\Big]^{\frac{m-1}{2}}$. Then

$$A(m,0) - A(m-1,0) = Q(2p_H - 1) = Q(2p_L - 1 + 2\varepsilon),$$

where $\varepsilon = p_H - p_L$. And

$$A(m-1,4) - A(m-1,0) = Q\big(1 + 2p_L(1-p_L)\big)(2p_L - 1).$$

Therefore, $A(m-1,4) < A(m,0)$ if and only if $\varepsilon > p_L(1-p_L)(2p_L-1)$, which is true because the assumption $\big(\frac{p_L}{1-p_L}\big)^2 = \frac{p_H}{1-p_H}$ implies that

$$\varepsilon = \frac{1}{p_L^2 + (1-p_L)^2} p_L(1-p_L)(2p_L-1) > p_L(1-p_L)(2p_L-1).$$

∎

Proof of part 2 of Proposition 7: This part is devoted to deriving the expression of the incentive sequences. Let $K = 2k_{a,H}^{(n)} + k_{a,L}^{(m)}$ and $v = s_1^H...s_n^H s_{n+1}^L...s_{n+m}^L$. Observe that

$g(v) = B$ if and only if $K \leq 0$. By plugging v and $v_+ = s^L_{n+m+1}$ into the expression of $V_+ - V$, we have

$$\begin{aligned} D^{H,L}_{n,m} &= \Pr(K \in (-1,0], v_+ = a^L, \omega = A) - \Pr(K \in (-1,0], v_+ = a^L, \omega = B) \\ &\quad + \Pr(K \in (0,1], v_+ = b^L, \omega = B) - \Pr(K \in (0,1], v_+ = b^L, \omega = A). \end{aligned}$$

Therefore, when m is odd, then K is odd, implying that

$$\begin{aligned} D^{H,L}_{n,m} &= \Pr(K = 1, v_+ = b^L, \omega = B) - \Pr(K = 1, v_+ = b^L, \omega = A) \\ &= \Pr(K = 1)\Big(\Pr(v_+ = b^L, \omega = B | K = 1) - \Pr(v_+ = b^L, \omega = A | K = 1)\Big) \\ &= \Pr(K = 1)\Big(p_L(1 - p_L) - (1 - p_L)p_L\Big) \\ &= 0. \end{aligned}$$

While when m is even, then K is even, implying that

$$\begin{aligned} D^{H,L}_{n,m} &= \Pr(K = 0, v_+ = a^L, \omega = A) - \Pr(K = 1, v_+ = a^L, \omega = B) \\ &= \Pr(K = 0)\Big(\Pr(v_+ = a^L, \omega = A | K = 0) - \Pr(v_+ = a^L, \omega = B | K = 0)\Big) \\ &= \Pr(K = 0) D^L_0. \end{aligned}$$

∎

Appendix C

LEMMAS, PROOFS, AND COMPUTATION DETAILS OF CHAPTER 3

Proof of Proposition 8: Let x be the number of occurrences of a in $\{s_t\}_{t=1}^n$. Then to express $\mathbb{P}_k(A|\{s_t\}_{t=1}^n)$ as a function of x, we define the following,

$$T_1(x) = m_k p^x q^{n-x} + \left(\frac{1-m_k}{2}\right)\bar{p}^x\bar{q}^{n-x} + \left(\frac{1-m_k}{2}\right)p^{n-x}q^x,$$
$$T_2(x) = m_k p^{n-x} q^x + \left(\frac{1-m_k}{2}\right)p^x q^{n-x} + \left(\frac{1-m_k}{2}\right)\bar{p}^{n-x}\bar{q}^x,$$

where $q = 1-p$ and $\bar{q} = 1-\bar{p}$. With these definitions, we have $\mathbb{P}_k(A|\{s_t\}_{t=1}^n) = \frac{T_1(x)}{T_1(x)+T_2(x)}$. So, $\mathbb{P}_k(A|\{s_t\}_{t=1}^n)$ is nondecreasing in x if and only if

$$\frac{T_1(x+1)}{T_1(x+1)+T_2(x+1)} \geq \frac{T_1(x)}{T_1(x)+T_2(x)}.$$

With some rearrangement and cancellation of terms, this inequality is equivalent to

$$\begin{aligned}\frac{(1-3m_k)(1+m_k)}{4}\big(pq\big)^{n'}\big(2p-1\big) \leq{}& \frac{(1-m_k)^2}{4}\big(\bar{p}\bar{q}\big)^{n'}(2\bar{p}-1)\\ &+\frac{(1-m_k)^2}{4}(\bar{p}-p)\left(\left(\frac{q\bar{q}}{p\bar{p}}\right)^x(p\bar{p})^{n'} + \left(\frac{p\bar{p}}{q\bar{q}}\right)^x\big(q\bar{q}\big)^{n'}\right)\\ &+\frac{m_k(1-m_k)}{2}(p-\bar{q})\left(\left(\frac{p\bar{q}}{\bar{p}q}\right)^x(\bar{p}q)^{n'} + \left(\frac{\bar{p}q}{p\bar{q}}\right)^x\big(p\bar{q}\big)^{n'}\right),\end{aligned}$$

where $n' = n-1$. Since the right-hand side cannot be negative, this inequality holds for all possible parameter specifications when $m_k \geq \frac{1}{3}$. When $m_k < \frac{1}{3}$, example 6 shows that it is possible the inequality does not hold. Particularly, when $x \neq 0$ and $n'-x \neq 0$, then by taking $\bar{p} \longrightarrow 1$, the above inequality does not hold:

$$\frac{(1-3m_k)(1+m_k)}{4}\big(pq\big)^{n'}\big(2p-1\big) \not\leq 0,$$

meaning that $\mathbb{P}_k(A|\{s_t\}_{t=1}^n)|_{x_a=x+1} < \mathbb{P}_k(A|\{s_t\}_{t=1}^n)|_{x_a=x}$.

Readers might question that $\frac{T_1(x+1)}{T_1(x+1)+T_2(x+1)} < \frac{T_1(x)}{T_1(x)+T_2(x)}$ for some x does not ensure that $\mathbb{P}_k(A|\{s_t\}_{t=1}^n)$ is non-monotonic. However, it is clear that an agent's belief that $\omega = A$ is higher upon observing all signals being a than all signals being b. That is, we must have $\frac{T_1(n)}{T_1(n)+T_2(n)} > \frac{T_1(0)}{T_1(0)+T_2(0)}$. Therefore, once $\frac{T_1(x+1)}{T_1(x+1)+T_2(x+1)} < \frac{T_1(x)}{T_1(x)+T_2(x)}$ for some x, $\mathbb{P}_k(A|\{s_t\}_{t=1}^n)$ is for sure non-monotonic. ∎

Lemma **9.** *Assume that n is odd and let $H(x) = \frac{g(x|p)}{g(x|\bar{p})}$, where $g(x|p) = p^x(1-p)^{n-x} - p^{n-x}(1-p)^x$. Then $H(x)$ is strictly increasing on $[0, \frac{n}{2})$ and strictly decreasing on $(\frac{n}{2}, n]$.*

Proof of Lemma 9: Let $u = \frac{p}{1-p}$ and define $\theta = \log\left(\frac{\bar{p}}{1-\bar{p}}\right) / \log\left(\frac{p}{1-p}\right)$. It would be useful to also define $A_1 = (1-p)^n u^x$, $A_2 = p^n u^{-x}$, $B_1 = (1-\bar{p})^n u^{\theta x}$, and $B_2 = \bar{p}^n u^{-\theta x}$ so that $H(x) = \frac{A_1 - A_2}{B_1 - B_2}$. With some algebra and the fact that $u > 1$, we have $H'(x) > 0$ if and only if

$$(A_1 + A_2)(B_1 - B_2) - \theta(A_1 - A_2)(B_1 + B_2) > 0.$$

Observe that $A_2 = u^{n-2x} A_1$ and $B_2 = \left(u^{n-2x}\right)^\theta B_1$, this inequality can be further simplified to

$$(v+1)(1-v^\theta) - \theta(1-v)(1+v^\theta) > 0,$$

where $v = u^{n-2x}$. Now I claim that this inequality holds for all $x \in (0, \frac{n}{2})$. To prove this, let $h(v) = (v+1)(1-v^\theta) - \theta(1-v)(1+v^\theta)$, and then notice the following three results:

- $h''(v) = (\theta+1)\theta(\theta-1)v^{\theta-2}(v-1) > 0$ for all $x \in (0, \frac{n}{2})$ since $v > 1$ and $\theta > 1$.
- $h'(1) = 0$.
- $h(1) = 0$.

By applying mean-value theorem on the first two results, we have $h'(v) > 0$ for all $v > 1$. Now applying mean-value theorem again on this and the third result yields $h(v) > 0$ for all $v > 1$, which proves the claim since $x \in (0, \frac{n}{2})$ implies that $v > 1$. Therefore, $H'(x) > 0$ for any $x \in (0, \frac{n}{2})$ and hence is strictly increasing on $[0, \frac{n}{2})$.

The proof for showing that $H(x)$ is strictly decreasing on $(\frac{n}{2}, n]$ is quite analogous and hence is omitted. ▮

Lemma **10.** *Assume that n is odd and let x_a, as previously defined, be the number of occurrence of a in the signal sequence $\{s_t\}_{t=1}^n$. Also, define $H : [0, \frac{n}{2}) \cup (\frac{n}{2}, n] \to \mathbb{R}$ by*

$$H(y) = \frac{p^y(1-p)^{n-y} - p^{n-y}(1-p)^y}{\bar{p}^y(1-\bar{p})^{n-y} - \bar{p}^{n-y}(1-\bar{p})^y}.$$

Moreover, let $A_L^{(k)} = \{y \in [0, \frac{n}{2}) : H(y) = \frac{1-m_k}{1-3m_k}\} \cup \{\frac{n}{2}\}$ and $A_U^{(k)} = \{y \in (\frac{n}{2}, n] : H(y) = \frac{1-m_k}{1-3m_k}\} \cup \{\frac{n}{2}\}$. With these notations, we have the following results regarding the direction that agent k updates her belief (recall that agent k's prior on ω, $\mathbb{P}_k(A)$, is $\frac{1}{2}$).

$\big($***Case 1:*** $x_a \leq \frac{n-1}{2}$ *and* $m_k \geq \frac{1}{3}\big)$ $\mathbb{P}_k(A|\{s_t\}_1^n) < \frac{1}{2}$.

$\big($***Case 2:*** $x_a \geq \frac{n+1}{2}$ *and* $m_k \geq \frac{1}{3}\big)$ $\mathbb{P}_k(A|\{s_t\}_1^n) > \frac{1}{2}$.

$\big($***Case 3:*** $x_a \leq \frac{n-1}{2}$ *and* $m_k < \frac{1}{3}\big)$ *Let* $\underline{x}_k = \min A_L^{(k)} = \min\big(H^{-1}\big(\frac{1-m_k}{1-3m_k}\big) \cup \{\frac{n}{2}\}\big)$, *then*

$$\mathbb{P}_k(A|\{s_t\}_1^n) < \frac{1}{2} \quad \textit{if} \quad 0 \leq x_a < \underline{x}_k, \ \textit{and} \ \mathbb{P}_k(A|\{s_t\}_1^n) > \frac{1}{2} \quad \textit{if} \quad \underline{x}_k < x_a \leq \frac{n-1}{2}.$$

$\big($***Case 4:*** $x_a \geq \frac{n+1}{2}$ *and* $m_k < \frac{1}{3}\big)$ *Let* $\overline{x}_k = \max A_U^{(k)} = \max\big(H^{-1}\big(\frac{1-m_k}{1-3m_k}\big) \cup \{\frac{n}{2}\}\big)$, *then*

$$\mathbb{P}_k(A|\{s_t\}_1^n) < \frac{1}{2} \quad \textit{if} \quad \frac{n+1}{2} \leq x_a < \overline{x}_k, \ \textit{and} \ \mathbb{P}_k(A|\{s_t\}_1^n) > \frac{1}{2} \quad \textit{if} \quad \overline{x}_k < x_a \leq n.$$

Proof of Lemma 10: Again let x be the number of occurrences of a in $\{s_t\}_{t=1}^n$. Then with some algebra, we have $\mathbb{P}_k(A|\{s_t\}_1^n) > \mathbb{P}_k(A) = \frac{1}{2}$ if and only if

$$(3m_k - 1)\Big(p^x(1-p)^{n-x} - p^{n-x}(1-p)^x\Big) > (1-m_k)\Big(\bar{p}^{n-x}(1-\bar{p})^x - \bar{p}^x(1-\bar{p})^{n-x}\Big),$$

where the exact expression for $\mathbb{P}_k(A|\{s_t\}_1^n)$ is provided in the proof of proposition 8. In case 2, the left-hand side is positive while the right-hand side is negative, so the inequality holds. In case 1, this strict inequality flips, which implies that $\mathbb{P}_k(A|\{s_t\}_1^n) < \frac{1}{2}$. The proof for the first two cases is therefore complete.

In case 3, by following the definition that $H(y) = \frac{g(y|p)}{g(y|\bar{p})}$, where $g(y|p) = p^y(1-p)^{n-y} - p^{n-y}(1-p)^y$, we can further rearrange the above inequality so that it is equivalent to $H(x) > \frac{1-m_k}{1-3m_k}$. To see when this inequality holds, consider the following two subcases:

$$\lim_{y \to \frac{n}{2}} H(y) \leq \frac{1-m_k}{1-3m_k} \qquad \text{or} \qquad \lim_{y \to \frac{n}{2}} H(y) > \frac{1-m_k}{1-3m_k}.$$

Recall that $A_L^{(k)} = \{y \in [0, \frac{n}{2}) : H(y) = \frac{1-m_k}{1-3m_k}\} \cup \{\frac{n}{2}\}$. In the first subcase, lemma 9 implies that $H(y) < \frac{1-m_k}{1-3m_k}$ for all $y \in [0, \frac{n}{2})$ and $\underline{x}_k = \min A_L = \frac{n}{2}$. Therefore, $\mathbb{P}_k(A|\{s_t\}_1^n) < \frac{1}{2}$ for all possible $x \in \{0, 1, ..., \frac{n-1}{2}\}$, which is equivalent to saying that $\mathbb{P}_k(A|\{s_t\}_1^n) < \frac{1}{2}$ if $0 \leq x < \underline{x}_k$ and notice that no x would satisfy $\underline{x}_k < x \leq \frac{n-1}{2}$. In the second subcase, from the fact that $H(0) < 1 \leq \frac{1-m_k}{1-3m_k}$ and lemma 9, intermediate value theorem ensures the unique existence of $x_0 \in (0, \frac{n}{2})$ such that $H(x_0) = \frac{1-m_k}{1-3m_k}$. Therefore, we have $\underline{x}_k = \min A_L = x_0$, $H(y) < \frac{1-m_k}{1-3m_k}$ on $[0, \underline{x}_k)$, and $H(y) > \frac{1-m_k}{1-3m_k}$ on $(\underline{x}_k, \frac{n}{2})$. This means that $\mathbb{P}_k(A|\{s_t\}_1^n) < \frac{1}{2}$ if $0 \leq x < \underline{x}_k$, and $\mathbb{P}_k(A|\{s_t\}_1^n) > \frac{1}{2}$ if $\underline{x}_k < x \leq \frac{n-1}{2}$.

The proof for case 4 is quite analogous and hence is omitted. ∎

Proof of Proposition 9: From lemma 10, we know that $\mathbb{P}_k(A|\{s_t\}_1^n) > \mathbb{P}_k(A) = \frac{1}{2}$ if and only if one of the following cases (in lemma 10) holds:

- Case 2.
- Case 3 with $\underline{x}_k < x_a \leq \frac{n-1}{2}$.
- Case 4 with $\overline{x}_k < x_a \leq n$.

When $\underline{x}_k < x_a \leq \frac{n-1}{2}$, we have $\underline{x}_k \neq \frac{n}{2}$, implying that $H^{-1}\left(\frac{1-m_k}{1-3m_k}\right) \neq \emptyset$ and hence $m_k < \frac{1}{3}$. Therefore, this case falls into the Case 3 with $\underline{x}_k < x_a \leq \frac{n-1}{2}$. When $\overline{x}_k < x_a \leq n$ with $m_k \geq \frac{1}{3}$, we have $H^{-1}\left(\frac{1-m_k}{1-3m_k}\right) = \emptyset$ and hence $\overline{x}_k = \frac{n}{2}$, implying that $x_a \in \{\frac{n+1}{2}, ..., n\}$ and hence this case falls into the Case 2. When $\overline{x}_k < x_a \leq n$ with $m_k < \frac{1}{3}$, obviously this case is the Case 4 with $\overline{x}_k < x_a \leq n$.

The proof for update downward or no belief change is quite analogous and hence is omitted. ∎

Proof of Proposition 10: Suppose the first condition (i.e., $\underline{x}_2 < x_a < \underline{x}_1$) holds. Since $\underline{x}_1 \leq \frac{n}{2}$ and x_a is an integer, we have $x_a \leq \frac{n-1}{2}$. Therefore, x_a satisfies both inequalities: $0 \leq x_a < \underline{x}_1$ and $\underline{x}_2 < x_a \leq \frac{n-1}{2}$, which implies $\mathbb{P}_1(A|\{s_t\}_1^n) < \frac{1}{2} < \mathbb{P}_2(A|\{s_t\}_1^n)$ from proposition 9. Similarly, again from proposition 9, the second condition (i.e., $\overline{x}_1 < x_a < \overline{x}_2$) implies that $\mathbb{P}_2(A|\{s_t\}_1^n) < \frac{1}{2} < \mathbb{P}_1(A|\{s_t\}_1^n)$.

Conversely, suppose two agents polarize with $\mathbb{P}_1(A|\{s_t\}_1^n) < \frac{1}{2} < \mathbb{P}_2(A|\{s_t\}_1^n)$. Then consider the only two possible cases: $x_a \leq \frac{n-1}{2}$ or $x_a \geq \frac{n+1}{2}$. In the first case, we have $0 \leq x_a < \underline{x}_1$ and $\underline{x}_2 < x_a \leq \frac{n-1}{2}$ from proposition 9. These two conditions jointly imply that $\underline{x}_2 < x_a < \underline{x}_1$ once $\underline{x}_2 < \underline{x}_1$ can be established. To prove this, we need the behavior of $H(\cdot)$. As shown in lemma 9 that $H(\cdot)$ is strictly increasing on $[0, \frac{n-1}{2}]$, $m_1 > m_2$ ensures that $\underline{x}_2 \leq \underline{x}_1$. However, equality does not hold since the existence of x_a such that agent 2 updates upward implies that $H(\underline{x}_2) = \frac{1-m_2}{1-3m_2} < H(\underline{x}_1)$. Therefore, we establish that $\underline{x}_2 < \underline{x}_1$. In the second case, we have $\frac{n+1}{2} \leq x_a < \overline{x}_1$ and $\overline{x}_2 < x_a \leq n$ again from proposition 9. However, with lemma 9 that ensures $\overline{x}_1 \leq \overline{x}_2$, combining all these conditions yields $x_a < \overline{x}_1 \leq \overline{x}_2 < x_a$, which is a contradiction. Therefore, only the first case is possible and hence we can conclude $\underline{x}_2 < x_a < \underline{x}_1$. Similarly, $\mathbb{P}_2(A|\{s_t\}_1^n) < \frac{1}{2} < \mathbb{P}_1(A|\{s_t\}_1^n)$ implies that $\overline{x}_1 < x_a < \overline{x}_2$. ▮

Proof of Lemma 3: Let x be the number of occurrences of a in $\{s_t\}_{t=1}^n$. Then to express $\mathbb{P}_k(\pi = \pi_U|\{s_t\}_{t=1}^n)$ as a function of x, we define the following,

$$\begin{aligned} T_1(x) &= m_k p^x (1-p)^{n-x} + m_k p^{n-x}(1-p)^x, \\ T_2(x) &= \left(\frac{1-m_k}{2}\right)\bar{p}^x(1-\bar{p})^{n-x} + \left(\frac{1-m_k}{2}\right)p^x(1-p)^{n-x}, \\ T_3(x) &= \left(\frac{1-m_k}{2}\right)p^{n-x}(1-p)^x + \left(\frac{1-m_k}{2}\right)\bar{p}^{n-x}(1-\bar{p})^x. \end{aligned}$$

With these definitions, $\mathbb{P}_k(\pi = \pi_U|\{s_t\}_{t=1}^n) = \frac{T_1(x)}{T_1(x)+T_2(x)+T_3(x)}$. With some rearrangements and algebra, we have $\mathbb{P}_k(\pi = \pi_U|\{s_t\}_{t=1}^n) < \mathbb{P}_k(\pi = \pi_U) = m_k$ if and only if $m_k \in (0,1)$ and $p^x(1-p)^{n-x} + p^{n-x}(1-p)^x < \bar{p}^x(1-\bar{p})^{n-x} + \bar{p}^{n-x}(1-\bar{p})^x$. Note that whether this inequality holds does not depend on the value of m_k once $m_k \notin \{0,1\}$.

Under $m_k \in (0,1)$ and let $g(s;y,n) = s^y(1-s)^{n-y} + s^{n-y}(1-s)^y$. Hence $\mathbb{P}_k(\pi =$

$\pi_U|\{s_t\}_{t=1}^n) < \mathbb{P}_k(\pi = \pi_U) = m_k$ if and only if $g(p; x, n) < g(\bar{p}; x, n)$. To derive the conditions under which this inequality holds, we analyze how the function g behaves. With some algebra, we have

$$\frac{\partial g}{\partial s} = s^{y-1}(1-s)^{n-y-1}\big(y - ns\big) + s^{n-y-1}(1-s)^{y-1}\big(n(1-s) - y\big),$$

which implies that $\frac{\partial g}{\partial s} < 0$ on $s \in [p, \bar{p}]$ for all $y \in [n(1-p), np]$. Therefore, when $x \in [n(1-p), np]$, mean value theorem implies that $g(p; x, n) > g(\bar{p}; x, n)$ and hence $\mathbb{P}_k(\pi = \pi_U|\{s_t\}_{t=1}^n) > m_k$.

Now consider $x \geq np$ and observe that $g(\bar{p}; n, n) = \bar{p}^n + (1-\bar{p})^n > p^n + (1-p)^n = g(p; n, n)$ (from $\frac{\partial g}{\partial s}\big|_{x=n} > 0$ on $s \in [p, \bar{p}]$). Since $g(\cdot)$ is continuous in y, intermediate value theorem indicates that there exists an $x_U \in (np, n)$ such that $g(\bar{p}; x_U, n) = g(p; x_U, n)$. To argue that such x_U is unique, observe the following facts:

- $\frac{\partial g}{\partial y}(s; x, n) = \log\left(\frac{s}{1-s}\right)\left(s^x(1-s)^{n-x} - s^{n-x}(1-s)^x\right)$.
- when $x > np$, both of the terms above are increasing in s, which can be easily verified using calculus. Therefore, we have
  - $\log\left(\frac{\bar{p}}{1-\bar{p}}\right) > \log\left(\frac{p}{1-p}\right)$.
  - $\bar{p}^x(1-\bar{p})^{n-x} - \bar{p}^{n-x}(1-\bar{p})^x > p^x(1-p)^{n-x} - p^{n-x}(1-p)^x$.

The facts above jointly imply that when $g(\bar{p}; x, n)$ and $g(p; x, n)$ meet at some $x = x^*$, then $g(\bar{p}; x, n) > g(p; x, n)$ on $(x^*, n]$ and hence there is no chance for the two functions to meet at any $x > x^*$. Therefore, x_U is the only point larger than np such that the two functions meet and

$$g(p; x, n) < g(\bar{p}; x, n) \quad \text{if} \quad x \in (x_U, n] \quad \text{and} \quad g(p; x, n) > g(\bar{p}; x, n) \quad \text{if} \quad x \in [np, x_U)$$

implying that $\mathbb{P}_k(\pi = \pi_U|\{s_t\}_{t=1}^n) < m_k$ if $x \in (x_U, n]$ and $\mathbb{P}_k(\pi = \pi_U|\{s_t\}_{t=1}^n) > m_k$ if $x \in [np, x_U)$. Similar analysis would show that there exists a unique $x_L \in \big(0, n(1-p)\big)$ such that $\mathbb{P}_k(\pi = \pi_U|\{s_t\}_{t=1}^n) < m_k$ if $x \in [0, x_L)$ and $\mathbb{P}_k(\pi = \pi_U|\{s_t\}_{t=1}^n) > m_k$ if $x \in (x_L, n(1-p)]$. ∎

Lemma **11.** *Under $m_k \in [0,1]$, the agent k's posterior belief on the observations being unbiased, $\mathbb{P}_k(\pi = \pi_U|\{s_t\}_{t=1}^n)$, is inverse U-shaped and symmetric around $\frac{n}{2}$. That is, $\mathbb{P}_k(\pi = \pi_U|\{s_t\}_{t=1}^n)$ is (weakly) increasing if $x_a < n - x_a$ and (weakly) decreasing if $x_a > n - x_a$.*

Proof of Lemma 11: Let x be the number of occurrences of a in $\{s_t\}_{t=1}^n$, and $p_k(x) = \mathbb{P}_k(\pi = \pi_U|\{s_t\}_{t=1}^n)\big|_{x_a=x}$. Since $p_k(x) = p_k(n-x)$, $\mathbb{P}_k(\pi = \pi_U|\{s_t\}_{t=1}^n)$ is symmetric around $\frac{n}{2}$. Therefore, once we show that $p_k(x+1) \geq p_k(x)$ for all $x \in \mathbb{Z}^*$ such that $x+1 \leq \frac{n}{2}$, the proof would be complete ($\mathbb{Z}^*$ is the set of nonnegative integers).

If $m_k \in \{0,1\}$, then $p_k(x) = m_k$ for all x, in which case $p_k(x+1) \geq p_k(x)$ for all $x \leq \frac{n}{2} - 1$. If $m_k \in (0,1)$, then with some tedious derivations, we have $p_k(x+1) \geq p_k(x)$ if and only if

$$\Big(A(x) - A(n'-x)\Big)\Big(\bar{p} + p - 1\Big) + \Big(B(x) - B(n'-x)\Big)\Big(\bar{p} - p\Big) \geq 0, \qquad (*)$$

where $A(x) = \big(p(1-\bar{p})\big)^x \big((1-p)\bar{p}\big)^{n'-x}$, $B(x) = \big((1-p)(1-\bar{p})\big)^x \big(p\bar{p}\big)^{n'-x}$, and $n' = n-1$. Then when $x \in \mathbb{Z}^*$ such that $x+1 \leq \frac{n}{2}$, we have $x \leq n' - x$, which implies that

$$A(x) \geq A(n'-x) \quad \text{and} \quad B(x) \geq B(n'-x).$$

So, the inequality $(*)$ holds and hence $p_k(x+1) \geq p_k(x)$. ∎

Proof of Proposition 11: Let x_a as previously defined and $\gamma = \mathbb{P}(\omega = A) \in [0,1]$. For any $\widetilde{p} \in (\frac{1}{2}, 1)$, from the symmetry of our model, we have

$$\mathbb{E}^{\widetilde{p}}\Big(\mathbb{P}_k(\pi = \pi_U|\{s_t\}_1^n)\big|\omega = A\Big) = \mathbb{E}^{\widetilde{p}}\Big(\mathbb{P}_k(\pi = \pi_U|\{s_t\}_1^n)\big|\omega = B\Big),$$

implying that

$$\mathbb{E}^{\widetilde{p}}\Big(\mathbb{P}_k(\pi = \pi_U|\{s_t\}_1^n)\Big) = \mathbb{E}^{\widetilde{p}}\Big(\mathbb{P}_k(\pi = \pi_U|\{s_t\}_1^n)\big|\omega = A\Big), \quad \text{for any } \gamma \in [0,1].$$

So, we can assume $\gamma = \frac{1}{2}$ without loss of generality.

Let $w_j(x) = \mathbb{P}(x_a = x|\widetilde{p}_j) = \frac{1}{2}\binom{n}{x}\Big[(\widetilde{p}_j)^x(1-\widetilde{p}_j)^{n-x} + (\widetilde{p}_j)^{n-x}(1-\widetilde{p}_j)^x\Big]$, where $j \in \{H, L\}$. Also, let $p_k(x) = \mathbb{P}_k(\pi = \pi_U|\{s_t\}_{t=1}^n)\big|_{x_a=x}$, $W_H = \{y \in \{0,...,n\} : w_H(y) \geq w_L(y)\}$, and $W_L = \{y \in \{0,...,n\} : w_H(y) < w_L(y)\}$. With these notations, we have

$$\mathbb{E}^{\widetilde{p}_H}\Big(\mathbb{P}_k(\pi = \pi_U|\{s_t\}_1^n)\Big) = \sum_{x=0}^n p_k(x) w_H(x)$$

$$= \sum_{x=0}^{n} p_k(x) w_L(x) + \sum_{x=0}^{n} p_k(x)\Big(w_H(x) - w_L(x)\Big)$$
$$= \mathbb{E}^{\widetilde{p}_L}\Big(\mathbb{P}_k(\pi = \pi_U | \{s_t\}_1^n)\Big) + \sum_{x=0}^{n} p_k(x)\Big(w_H(x) - w_L(x)\Big).$$

If we can show that $\sum_{x=0}^{n} p_k(x)\Big(w_H(x) - w_L(x)\Big) \le 0$ for any $m_k \in [0,1]$, then the proof is complete. From the proof of lemma 3, we know that $w_H(x) \ge w_L(x)$ on the two tails. That is, there exists $x_1 \in [0, \frac{n}{2}]$ and $x_2 \in [\frac{n}{2}, n]$ such that $W_H = \{0, 1, ..., x_1\} \cup \{x_2, x_2+1, ..., n\}$. Also, from lemma 11, we have shown that $p_k(x)$ is inverse U-shaped and symmetric around $x = \frac{n}{2}$ for any $m_k \in [0,1]$. With these two results, we have $\max_{x \in W_H} p_k(x) \le \min_{x \in W_L} p_k(x)$ for all $m_k \in [0,1]$, which yields the following, as desired:

$$\begin{aligned}
&\sum_{x=0}^{n} p_k(x)\Big(w_H(x) - w_L(x)\Big) \\
=&\sum_{x \in W_H} p_k(x)\Big(w_H(x) - w_L(x)\Big) + \sum_{x \in W_L} p_k(x)\Big(w_H(x) - w_L(x)\Big) \\
\le&\Big(\max_{x \in W_H} p_k(x)\Big) \sum_{x \in W_H} \Big(w_H(x) - w_L(x)\Big) + \Big(\min_{x \in W_L} p_k(x)\Big) \sum_{x \in W_L} \Big(w_H(x) - w_L(x)\Big) \\
\le&\Big(\min_{x \in W_L} p_k(x)\Big) \sum_{x \in W_H} \Big(w_H(x) - w_L(x)\Big) + \Big(\min_{x \in W_L} p_k(x)\Big) \sum_{x \in W_L} \Big(w_H(x) - w_L(x)\Big) \\
=&\Big(\min_{x \in W_L} p_k(x)\Big)\Big(\sum_{x=0}^{n} w_H(x) - \sum_{x=0}^{n} w_L(x)\Big) \\
=&0,
\end{aligned}$$

where the last equality holds because $\sum_{x=0}^{n} w_H(x) = \sum_{x=0}^{n} w_L(x) = 1$. ∎

Proof of Proposition 12: Following all the notations in the proof of proposition 8, we have $\mathbb{P}_k(A|\{s_t\}_1^n) = \frac{\gamma_k T_1(x)}{\gamma_k T_1(x) + (1-\gamma_k) T_2(x)}$. Then $\mathbb{P}_k(A|\{s_t\}_1^n)$ is nondecreasing in x if and only if

$$\frac{\gamma_k T_1(x+1)}{\gamma_k T_1(x+1) + (1-\gamma_k) T_2(x+1)} \ge \frac{\gamma_k T_1(x)}{\gamma_k T_1(x) + (1-\gamma_k) T_2(x)},$$

which is equivalent to $\gamma_k(1-\gamma_k) T_1(x+1) T_2(x) \ge \gamma_k(1-\gamma_k) T_1(x) T_2(x+1)$. Since $\gamma_k(1-\gamma_k) \ne 0$, this inequality collapses to

$$T_1(x+1) T_2(x) \ge T_1(x) T_2(x+1),$$

implying that whether $\mathbb{P}_k(A|\{s_t\}_1^n)$ is nondecreasing in x does not depend on $\gamma_k \in (0,1)$. Therefore, the rest of the proof follows from the proof of proposition 8. ▮

Proof of Proposition 13: Let $q = 1-p$ and $\bar{q} = 1-\bar{p}$. Then for any $\gamma_k \in (0,1)$, we have $\mathbb{P}_k(\omega = A|\{s_t\}_{t=1}^n) \gtreqless \gamma_k$ if and only if

$$m_k\Big(p^{x_a}q^{n-x_a} - p^{n-x_a}q^{x_a}\Big) \gtreqless \Big(\frac{1-m_k}{2}\Big)\Big(\bar{p}^{n-x_a}\bar{q}^{x_a} - \bar{p}^{x_a}\bar{q}^{n-x_a} + p^{x_a}q^{n-x_a} - p^{n-x_a}q^{x_a}\Big),$$

which is independent of γ_k. Therefore, the rest of the proof follows from the proof of proposition 9 and 10. ▮

Proof of Lemma 4: Let $q = 1-p$, $\bar{q} = 1-\bar{p}$, $c_k = \frac{1-m_k}{2m_k}$, and $p_k(x,n|\gamma_k) = \mathbb{P}_k(\pi = \pi_U|\{s_t\}_1^n)\Big|_{x_a=x}$. Once we can show that $p_k(x,n|\gamma_k) \to 0$ and $p_k(n-x,n|\gamma_k) \to 0$ as $n \to \infty$, the proof would be complete.

For any $\gamma_k \in [0,1)$, we have

$$p_k(x,n|\gamma_k) = \frac{A+1}{A+B+C+1+\big(\frac{\gamma_k}{1-\gamma_k}\big)\big(\frac{1-m_k}{2m_k}\big)},$$

where $A = \frac{\gamma_k}{1-\gamma_k}\big(\frac{p}{q}\big)^{2x-n}$, $B = c_k\Big(\frac{\gamma_k}{1-\gamma_k}\big(\frac{\bar{p}}{q}\big)^{x}\big(\frac{\bar{q}}{p}\big)^{n-x} + \big(\frac{p}{q}\big)^{2x-n}\Big)$, and $C = c_k\big(\frac{\bar{p}}{p}\big)^{n-x}\big(\frac{\bar{q}}{q}\big)^{x}$. Since $A \to 0$, $B \to 0$, and $C \to \infty$ as $n \to \infty$, we get $p_k(x,n|\gamma_k) \to 0$ under $\gamma_k \in [0,1)$. If $\gamma_k = 1$, then

$$p_k(x,n|\gamma_k = 1) = \frac{1}{1+\frac{1-m_k}{2m_k}\big(\frac{\bar{p}}{p}\big)^{x}\big(\frac{\bar{q}}{q}\big)^{n-x} + \frac{1-m_k}{2m_k}\big(\frac{p}{q}\big)^{n-2x}} \to 0,$$

where the limit holds from $(p/q)^{n-2x} \to 0$ as $n \to \infty$.

$p_k(n-x,n|\gamma_k) \to 0$ as $n \to \infty$ follows from the observation that $p_k(n-x,n|\gamma_k) = p_k(x,n|1-\gamma_k)$. ▮

Lemma **12.** *Let $p_k(x|\gamma_k) = \mathbb{P}_k(\pi = \pi_U|\{s_t\}_{t=1}^n)\big|_{x_a=x}$ and $\gamma_k > \frac{1}{2}$ be agent k's prior on ω. Also, define $Q : [0, \frac{n}{2}] \to \mathbb{R}$ by*

$$Q(y|\gamma_k) = \begin{cases} p_k(y|\gamma_k) + p_k(n-y|\gamma_k) & \text{if } \ y \in [0, \frac{n}{2}); \\ \\ p_k(y|\gamma_k) & \text{if } \ y = \frac{n}{2}. \end{cases}$$

Then there exists an $x^* \in (0, \frac{n}{2})$ such that the function $Q(\cdot)$ has the following properties:

(i) $Q(x|\gamma_k) \geq m_k$ *for any* $x \in [x^*, \frac{n}{2}] \cap \mathbb{Z}^*$*, where* $\mathbb{Z}^*$ *is the set of nonnegative integers.*

(ii) $Q(y|\gamma_k)$ *is increasing on* $[0, x^*]$*.*

Proof of Lemma 12: To show the first property, let $x^* = n \log \frac{\bar{p}}{p} \Big/ \log \frac{\bar{p}(1-p)}{p(1-\bar{p})}$ and we make the follow observations:

- $\bar{p}(1-\bar{p}) < p(1-p)$, which implies that $x^* < \frac{n}{2}$.
- x^* solves the equation: $\left(\frac{\bar{p}}{p}\right)^{n-y} = \left(\frac{1-p}{1-\bar{p}}\right)^{y}$.
- The left-hand side (right-hand side) of the equation is decreasing (increasing) in $y \in [0, \frac{n}{2}]$, implying that $\left(\frac{\bar{p}}{p}\right)^{n-y} < \left(\frac{1-p}{1-\bar{p}}\right)^{y}$ when $y \in (x^*, \frac{n}{2}]$, and $\left(\frac{\bar{p}}{p}\right)^{n-y} > \left(\frac{1-p}{1-\bar{p}}\right)^{y}$ when $y \in [0, x^*)$.
- With some rearrangement, we have $\bar{p}^{n-y}(1-\bar{p})^y > p^{n-y}(1-p)^y$ for all $y \in [0, x^*)$ and $\bar{p}^{n-y}(1-\bar{p})^y < p^{n-y}(1-p)^y$ for all $y \in (x^*, \frac{n}{2}]$, with the two terms being equal at $y = x^*$.
- For any $y \in [0, \frac{n}{2}]$, we have $\bar{p}^y(1-\bar{p})^{n-y} < p^y(1-p)^{n-y}$.
- Some rearrangement of $p_k(\cdot)$ yields that $p_k(x|\gamma_k) \geq m_k$ if and only if

$$\begin{aligned}&\big(2\gamma_k - 1\big)\Big[p^x(1-p)^{n-x} - p^{n-x}(1-p)^x\Big] + \gamma_k p^x(1-p)^{n-x} + (1-\gamma_k)p^{n-x}(1-p)^x \\ &\geq \gamma_k \bar{p}^x(1-\bar{p})^{n-x} + (1-\gamma_k)\bar{p}^{n-x}(1-\bar{p})^x.\end{aligned}$$

When $x \in [x^*, \frac{n}{2}] \cap \mathbb{Z}^*$, the following inequality holds for any $\gamma_k \in [0, 1]$:

$$\gamma_k p^x(1-p)^{n-x} + (1-\gamma_k)p^{n-x}(1-p)^x \geq \gamma_k \bar{p}^x(1-\bar{p})^{n-x} + (1-\gamma_k)\bar{p}^{n-x}(1-\bar{p})^x. \qquad (*)$$

Therefore, if $p_k(x|\gamma_k) < m_k$, then it must because the term $\big(2\gamma_k - 1\big)\Big[p^x(1-p)^{n-x} - p^{n-x}(1-p)^x\Big]$ is negative. Now note that $p_k(n-x|\gamma_k) = p_k(x|1-\gamma_k)$ and replacing γ_k in $p_k(x|\gamma_k) \geq m_k$

by $1-\gamma_k$ would reverse the sign of the term but leaves the inequality $(*)$ unaffected, implying that $p_k(n-x|\gamma_k) = p_k(x|1-\gamma_k) \geq m_k$ and therefore $Q(x|\gamma_k) \geq m_k$. Clearly, if $p_k(x|\gamma_k) \geq m_k$, $Q(x|\gamma_k) \geq m_k$ holds trivially. So, $Q(x|\gamma_k) \geq m_k$ for any $x \in [x^*, \frac{n}{2}] \cap \mathbb{Z}^*$ and the proof for the first property is complete.

To establish the second property, I first define some notations for convenience. Let $q = 1-p$, $\bar{q} = 1-\bar{p}$, $K = m_k\big(\frac{1-m_k}{2}\big) > 0$, $C_p = \log\frac{p}{1-p}$, $C_{\bar{p}} = \log\frac{\bar{p}}{1-\bar{p}}$, and the following functions:

- $A_1(y) = \gamma_k m_k p^y q^{n-y}$, $A_2(y) = \big(1-\gamma_k\big) m_k p^{n-y} q^y$,
- $A_3(y) = \gamma_k\big(\frac{1-m_k}{2}\big)\bar{p}^y\bar{q}^{n-y}$, $A_4(y) = \big(1-\gamma_k\big)\big(\frac{1-m_k}{2}\big)p^y q^{n-y}$,
- $A_5(y) = \gamma_k\big(\frac{1-m_k}{2}\big)p^{n-y}q^y$, $A_6(y) = \big(1-\gamma_k\big)\big(\frac{1-m_k}{2}\big)\bar{p}^{n-y}\bar{q}^y$.
- $B_i(y) = A_i(n-y)$, $i \in \{1, ..., 6\}$.

With these notations, we have $Q(y|\gamma_k) = \frac{A_1(y)+A_2(y)}{\sum_i A_i(y)} + \frac{B_1(y)+B_2(y)}{\sum_i B_i(y)}$. Now note that when $y \in [0, x^*]$, the observations in the proof of the first property indicate that

$$\bar{p}^{n-y} - \bar{q}^y \geq p^{n-y}q^y \quad \text{and} \quad -\bar{p}^y\bar{q}^{n-y} \geq -p^y(1-p)^{n-y},$$

implying that $\sum_i B_i(y) \geq \sum_i A_i(y)$. Some (tedious) calculations yield that $Q'(y|\gamma_k) \geq 0$ if and only if

$$\frac{(2\gamma_k-1)G + (1-\gamma_k)^2L - \gamma_k^2S + H}{\Big(\sum_i A_i(y)\Big)^2} + \frac{(1-2\gamma_k)G + \gamma_k^2L - (1-\gamma_k)^2S + H}{\Big(\sum_i B_i(y)\Big)^2} \geq 0,$$

where $G = 2KC_p(pq)^n$, $L = \big(C_{\bar{p}} - C_p\big)K(p\bar{p})^{n-y}(q\bar{q})^y$, $S = \big(C_{\bar{p}} - C_p\big)K(p\bar{p})^y(q\bar{q})^{n-y}$, and $H = \big(C_{\bar{p}} + C_p\big)K\big(\gamma_k(1-\gamma_k)\big)\Big((\bar{p}q)^{n-y}(p\bar{q})^y - (\bar{p}q)^y(p\bar{q})^{n-y}\Big)$. Under $y \in [0, x^*)$ and $\gamma_k > \frac{1}{2}$, the fact that $G \leq 0$, $H \geq 0$, $L \geq S$, and $\sum_i B_i(y) \geq \sum_i A_i(y)$ indicates that when adding up the two fractions, many terms are positive (nonnegative) and the only term whose sign seems indefinite is $\Big(\sum_i A_i(y)\Big)^2 L - \Big(\sum_i B_i(y)\Big)^2 S$. Once we can show that this term is also nonnegative, the proof for the second property would be complete.

The term $\Big(\sum_i A_i(y)\Big)^2 L - \Big(\sum_i B_i(y)\Big)^2 S$ can be expressed as a sum of the following three terms:

(T1) $L\big(A_1 + A_3 + A_5\big)\big(A_2 + A_4 + A_6\big) - S\big(B_1 + B_3 + B_5\big)\big(B_2 + B_4 + B_6\big)$.

(T2) $L\big(A_1 + A_3 + A_5\big)^2 - S\big(B_1 + B_3 + B_5\big)^2$.

(T3) $L\big(A_2 + A_4 + A_6\big)^2 - S\big(B_2 + B_4 + B_6\big)^2$.

I claim that all these three terms are nonnegative. For the term in (T1), observe that $A_1 = \frac{\gamma_k}{1-\gamma_k} B_2$, $A_2 = \frac{1-\gamma_k}{\gamma_k} B_1$, $A_3 = \frac{\gamma_k}{1-\gamma_k} B_6$, $A_4 = \frac{1-\gamma_k}{\gamma_k} B_5$, $A_5 = \frac{\gamma_k}{1-\gamma_k} B_4$, and $A_6 = \frac{1-\gamma_k}{\gamma_k} B_3$. With these relationships, we have

$$\begin{aligned} L\big(A_1 + A_3 + A_5\big)\big(A_2 + A_4 + A_6\big) &= L\big(B_2 + B_6 + B_4\big)\big(B_1 + B_5 + B_3\big) \\ &\geq S\big(B_1 + B_3 + B_5\big)\big(B_2 + B_4 + B_6\big), \end{aligned}$$

where the inequality holds because $L \geq S$. For the term in (T2), by positive scaling, it suffices to consider $B_1 B_3 \big(A_1 + A_3 + A_5\big)^2 - A_1 A_3 \big(B_1 + B_3 + B_5\big)^2$. Notice that $A_1 A_5 = B_1 B_5$ and hence expansion of this term yields

$$\begin{aligned} & B_1 B_3 \sum_{i \in \{1,3,5\}} A_i^2 - A_1 A_3 \sum_{i \in \{1,3,5\}} B_i^2 + 2A_1 A_5 \big(B_1 B_3 - A_1 A_3\big) + 2A_3 B_3 \big(A_5 B_1 - A_1 B_5\big) \\ \geq\ & B_1 B_3 \sum_{i \in \{1,3,5\}} A_i^2 - A_1 A_3 \sum_{i \in \{1,3,5\}} B_i^2, \end{aligned}$$

where the inequality holds because $B_1 B_3 \geq A_1 A_3$ (from $L \geq S$) and $\frac{A_5 B_1}{A_1 B_5} = \big(\frac{p}{1-p}\big)^{2n-4y} \geq 1$. Now since $y \leq x^* < \frac{n}{2}$ implies that $A_5 \geq B_5$ and

$$\begin{aligned} B_1 B_3 - A_1 A_3 &= \gamma_k^2 K \Big((p\bar{p})^{n-y} (q\bar{q})^y - (p\bar{p})^y (q\bar{q})^{n-y}\Big) \\ &\geq \gamma_k^2 K \Big((\bar{p}q)^{n-y} (p\bar{q})^y - (\bar{p}q)^y (p\bar{q})^{n-y}\Big) \\ &= A_1 B_3 - A_3 B_1 \\ &\geq 0, \end{aligned}$$

we get to the following conclusion:

$$\sum_{i\in\{1,3,5\}} \left(B_1B_3A_i^2 - A_1A_3B_i^2\right) = (A_1B_3 - A_3B_1)(A_1B_1 - A_3B_3) + B_1B_3A_5^2 - A_1A_3B_5^2$$
$$\geq (A_1B_3 - A_3B_1)(A_1B_1 - A_3B_3) + A_5B_5(B_1B_3 - A_1A_3)$$
$$\geq (A_1B_3 - A_3B_1)(A_1B_1 - A_3B_3) + A_5B_5(A_1B_3 - A_3B_1)$$
$$= (A_1B_3 - A_3B_1)(A_1B_1 - A_3B_3 + A_5B_5)$$
$$\geq 0,$$

where the last inequality holds because $A_5B_5 = \gamma_k^2\left(\frac{1-m_k}{2}\right)^2(pq)^n > \gamma_k^2\left(\frac{1-m_k}{2}\right)^2(\bar{p}\bar{q})^n = A_3B_3$. Therefore, the term in (T2) is nonnegative. The proof for showing that the term in (T3) is nonnegative is quite analogous and hence is omitted. ▮

Proof of Proposition 14: The proof is trivial if $m_k \in \{0,1\}$, so we assume $m_k \in (0,1)$. Let $p_k(\cdot|\gamma_k)$ as previously defined in the proof of lemma 12 and $\widetilde{p}$ is the precision of the SP's signals, then for any $\widetilde{p} > \frac{1}{2}$, observe the following derivations:

$$\sum_{x=0}^{n} p_k(x|\gamma_k)\mathbb{P}(x_a = x|\widetilde{p}) = \sum_{x=0}^{n} p_k(n-x|1-\gamma_k)\mathbb{P}(x_a = x|\widetilde{p})$$
$$= \sum_{x=0}^{n} p_k(n-x|1-\gamma_k)\mathbb{P}(x_a = n-x|\widetilde{p})$$
$$= \sum_{y=0}^{n} p_k(y|1-\gamma_k)\mathbb{P}(x_a = y|\widetilde{p}),$$

where the equalities hold because $p_k(x|\gamma_k) = p_k(n-x|1-\gamma_k)$ and assumption 7 implies that $\mathbb{P}(x_a = x|\widetilde{p}) = \mathbb{P}(x_a = n-x|\widetilde{p})$. Therefore, for the rest of the proof, we can assume $\gamma_k > \frac{1}{2}$ without loss of generality.

Let $\ell^* = \min\{\lfloor x^* \rfloor, \ell\}$, where $x^* = n\log(\bar{p}/p)\big/\log\left(\bar{p}(1-p)/p(1-\bar{p})\right)$ and ℓ is defined in assumption 8. Then consider the only two possible cases for ℓ^* in assumption 8: $\ell^* > 0$ or $\ell^* = 0$. In the first case, there exists a $\widetilde{p}_0 \in \left(\frac{1}{2}, 1\right)$ such that $n(1-\widetilde{p}) < \ell^*$ for any $\widetilde{p} \geq \widetilde{p}_0$. In the second case, there exists a $\widetilde{p}_0 \in \left(\frac{1}{2}, 1\right)$ such that $n(1-\widetilde{p}) < 1$ for any $\widetilde{p} \geq \widetilde{p}_0$. In either case, according to the proof of lemma 3, for any $\widetilde{p}_H > \widetilde{p}_L \geq \widetilde{p}_0$

there exists a nonnegative integer $x_H \leq \ell^*$ such that $\big\{y \in \{0,1,...,\lfloor \frac{n}{2} \rfloor\} : \mathbb{P}(x_a = y|\widetilde{p}_H) \geq \mathbb{P}(x_a = y|\widetilde{p}_L)\big\} = \{0,1,...,x_H\}$. Now following the (notational) definition for $Q(\cdot|\gamma_k)$ in lemma 12, and let $w_j(x) = \mathbb{P}(x_a = x|\widetilde{p}_j)$, $W_H = \big\{y \in \{0,1,...,\lfloor \frac{n}{2} \rfloor\} : w_H(y) \geq w_L(y)\big\}$, and $W_L = \big\{y \in \{0,1,...,\lfloor \frac{n}{2} \rfloor\} : w_H(y) < w_L(y)\big\}$. Then to establish the desired inequality, we make the following observations:

- $W_H = \{0,1,...,x_H\}$ with $x_H \leq \ell^* \leq x^*$, so $\max_{x \in W_H} Q(x|\gamma_k) \leq Q(\max W_H|\gamma_k) = Q(x_H|\gamma_k)$ from lemma 12.
- $W_H = \{0,1,...,x_H\} \subseteq \{0,1,...,\ell\}$ implies that $\max_{x \in W_H} Q(x|\gamma_k) = Q(x_H|\gamma_k) \leq m_k$ from definition of ℓ in assumption 8.
- If $x_H + 1 \leq x^*$, then $\min_{x \in W_L} Q(x|\gamma_k) \geq \min\{Q(x_H + 1|\gamma_k), m_k\}$ from lemma 12.
- If $x_H + 1 > x^*$, then $\min_{x \in W_L} Q(x|\gamma_k) \geq m_k$ from lemma 12.
- If $x_H + 1 \leq x^*$, then $Q(x_H|\gamma_k) \leq Q(x_H + 1|\gamma_k)$ from lemma 12 and $Q(x_H|\gamma_k) \leq m_k$ from above. So, $\max_{x \in W_H} Q(x|\gamma_k) \leq \min_{x \in W_L} Q(x|\gamma_k)$ in this case.
- If $x_H + 1 > x^*$, then again $\max_{x \in W_H} Q(x|\gamma_k) \leq m_k \leq \min_{x \in W_L} Q(x|\gamma_k)$ in this case.

The goal of these observations is to conclude that $\max_{x \in W_H} Q(x|\gamma_k) \leq \min_{x \in W_L} Q(x|\gamma_k)$. With this inequality, we have

$$\begin{aligned}
&\sum_{x=0}^{n} p_k(x|\gamma_k)\Big(w_H(x) - w_L(x)\Big) \\
=&\sum_{x=0}^{\lfloor n/2 \rfloor} Q(x|\gamma_k)\Big(w_H(x) - w_L(x)\Big) \\
=&\sum_{x \in W_H} Q(x|\gamma_k)\Big(w_H(x) - w_L(x)\Big) + \sum_{x \in W_L} Q(x|\gamma_k)\Big(w_H(x) - w_L(x)\Big) \\
\leq&\Big(\max_{x \in W_H} Q(x|\gamma_k)\Big) \sum_{x \in W_H} \Big(w_H(x) - w_L(x)\Big) + \Big(\min_{x \in W_L} Q(x|\gamma_k)\Big) \sum_{x \in W_L} \Big(w_H(x) - w_L(x)\Big)
\end{aligned}$$

$$\le \Big(\min_{x\in W_L} Q(x|\gamma_k)\Big) \sum_{x\in W_H} \Big(w_H(x) - w_L(x)\Big) + \Big(\min_{x\in W_L} Q(x|\gamma_k)\Big) \sum_{x\in W_L} \Big(w_H(x) - w_L(x)\Big)$$

$$= \Big(\min_{x\in W_L} Q(x|\gamma_k)\Big) \Big(\sum_{x=0}^{\lfloor n/2 \rfloor} w_H(x) - \sum_{x=0}^{\lfloor n/2 \rfloor} w_L(x)\Big)$$

$$\le 0,$$

where the first equality holds because assumption 7 implies that $\sum_{x=0}^{n} p_k(x|\gamma_k) w_j(x) = \sum_{x=0}^{\lfloor \frac{n}{2} \rfloor} Q(x|\gamma_k) w_j(x)$ for all $j \in \{H, L\}$, and the last inequality holds because

$$\sum_{x=0}^{\lfloor n/2 \rfloor} w_H(x) - \sum_{x=0}^{\lfloor n/2 \rfloor} w_L(x) = \begin{cases} 0 & \text{if } n \text{ is odd;} \\ \frac{1}{2}\Big(w_H(n/2) - w_L(n/2)\Big) < 0 & \text{if } n \text{ is even.} \end{cases}$$

Therefore, we establish the desired result:

$$\mathbb{E}^{\widetilde{p}_H} \mathbb{P}_k(\pi = \pi_U | \{s_t\}_1^n) - \mathbb{E}^{\widetilde{p}_L} \mathbb{P}_k(\pi = \pi_U | \{s_t\}_1^n) = \sum_{x=0}^{n} p_k(x|\gamma_k)\Big(w_H(x) - w_L(x)\Big) \le 0.$$

∎

Derivation Details of Example 8: Let x_a and y_a be the number of signals of a in period 1 and 2, respectively. Also, let $G_k(x) = (g_k^U(x), g_k^A(x), g_k^B(x))$, where $g_k^\ell(x) = \mathbb{P}_k(\pi = \pi_\ell | \{s_t^{(1)}\}_1^n)$, $\ell \in \{U, A, B\}$, $k \in \{1, 2\}$. Moreover, let $\mathbb{P}_k(\omega_2 = A | c_U, c_A, c_B, \{s_t^{(2)}\}_1^n)$ be agent k's posterior belief that $\omega_2 = A$ when her prior (in period 2) on $\pi = \pi_U$, $\pi = \pi_A$, $\pi = \pi_B$ are given by c_U, c_A, and c_B, respectively. With these notations, we have

$$\begin{aligned}\mathbb{P}^{\widetilde{p}}(d_k = \omega_2) &= \sum_{x=0}^{n} \mathbb{P}^{\widetilde{p}}(x_a = x) \mathbb{P}^{\widetilde{p}}\Big(d_k = \omega_2 | G_k(x)\Big) \\ &= \sum_{x=0}^{n} \mathbb{P}^{\widetilde{p}}(x_a = x) \Big(\frac{1}{2} \sum_{w\in\{A,B\}} \mathbb{P}^{\widetilde{p}}(d_k = w | G_k(x), \omega_2 = w)\Big),\end{aligned}$$

where

$$\begin{aligned}\mathbb{P}^{\widetilde{p}}(d_k = w | G_k(x), \omega_2 = w) = &\sum_{y=0}^{n} \mathbb{P}^{\widetilde{p}}(y_a = y | \omega_2 = w) \mathbb{I}_{\left(\mathbb{P}_k(\omega_2 = w | G_k(x), \{s_t^{(2)}\}_1^n)\big|_{y_a = y} > \frac{1}{2}\right)} \\ &+ \frac{1}{2} \sum_{y=0}^{n} \mathbb{P}^{\widetilde{p}}(y_a = y | \omega_2 = w) \mathbb{I}_{\left(\mathbb{P}_k(\omega_2 = w | G_k(x), \{s_t^{(2)}\}_1^n)\big|_{y_a = y} = \frac{1}{2}\right)},\end{aligned}$$

for any $w \in \{A, B\}$ and

$$\mathbb{P}^{\widetilde{p}}(x_a = x) = \frac{1}{2} \sum_{w \in \{A,B\}} \mathbb{P}^{\widetilde{p}}(y_a = x | \omega_2 = w) = \frac{1}{2} \binom{n}{x} \Big[(\widetilde{p})^x (1 - \widetilde{p})^{n-x} + (\widetilde{p})^{n-x} (1 - \widetilde{p})^x \Big].$$

Now let $\mu_k(x_a, y_a) = \mathbb{P}_k(\omega_2 = A | G_k(x_a), \{s_t^{(2)}\}_1^n) - \frac{1}{2}$, then the probability that polarization takes place is given by

$$\sum_{x=0}^{n} \mathbb{P}^{\widetilde{p}}(x_a = x) \mathbb{P}^{\widetilde{p}}(\mu_1(x, y_a) \mu_2(x, y_a) < 0),$$

where $\mathbb{P}^{\widetilde{p}}(\mu_1(x, y_a)\mu_2(x, y_a) < 0) = \sum_{y=0}^{n} \mathbb{P}^{\widetilde{p}}(y_a = y) \mathbb{I}(\mu_1(x, y)\mu_2(x, y) < 0)$. Here all the expressions that we currently have are simple enough to code up numerically. ∎