Classification of Line Modules and Finite Dimensional Simple Modules over a Deformation of the Polynomial Ring in Three Variables

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Abstract

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Let $\Bbbk$ be a field and A the non-commutative $\Bbbk$-algebra generated by x_1, x_2, x_3 subject to the relations

$$qx_ix_j - q^{-1}x_jx_i \;=\; x_k$$

as (i, j, k) ranges over all cyclic permutations of $(1, 2, 3)$, where $q \in \Bbbk - \{0\}$. This thesis sets out to understand the representation theory of A. In particular, we classify all finite dimensional simple modules over A when q is not a root of unity. To this end, we introduce the notion of a linear module over a filtered $\Bbbk$-algebra, an analogue to the notion of a linear module defined for a connected graded $\Bbbk$-algebra.

Finite dimensional simple A-modules are closely related to certain linear modules for A of Gelfand-Kirillov dimension one, which we call line modules, in the sense that every finite dimensional simple A-module V appears in an exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow V \longrightarrow 0,$$

in which M and M' are line modules for A.

The main result shows that there are five non-isomorphic simple A-modules of each dimension.

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I am grateful to Professor S. Paul Smith for his advice, guidance and patience.

DEDICATION

to my family and my dear wife, Rizi Zheng

Chapter 1

INTRODUCTION

1.1 The goal

Let $\Bbbk$ be a field and $q \in \Bbbk - \{0\}$. Let $A(q)$ be the free algebra $\Bbbk\langle x_1, x_2, x_3\rangle$ modulo the three relations

$$\begin{aligned} qx_1x_2 - q^{-1}x_2x_1 &= x_3, \\ qx_2x_3 - q^{-1}x_3x_2 &= x_1, \\ qx_3x_1 - q^{-1}x_1x_3 &= x_2. \end{aligned}$$

The algebra $A(q)$ is a noetherian domain of Gelfand-Kirillov dimension 3. It has a Poincaré-Birkhoff-Witt basis of $\left\{x_1^i x_2^j x_3^k \mid (i,j,k) \in \mathbb{N}^3\right\}$.

Our goal is to understand the representation theory of $A(q)$. This includes the classification of all finite dimensional simple $A(q)$-modules (when q is not a root of unity). In addition, we introduce and classify certain infinite dimensional filtered $A(q)$-modules, which we call line modules for $A(q)$, with respect to a filtration of $A(q)$. Such line modules turn out to be closely related to the finite dimensional simple A-modules.

Note that the algebra $A(q)$ can be presented in various ways. For example, we shall show that $A(q)$ is isomorphic to the free algebra $\Bbbk\langle X_1, X_2, X_3\rangle$ modulo the relations

$$X_iX_j - pX_jX_i = (p-1)X_k,$$

where (i,j,k) runs over the cyclic permutations of $(1,2,3)$ and $p = q^{-2}$. In fact, the algebra $A(q)$ represents a very natural class of algebras with Poincaré-Birkhoff-Witt bases. We shall show that if R is the free algebra $\Bbbk\langle X, Y, Z\rangle$ modulo the relations

$$\begin{aligned} YX &= \alpha XY + aZ, \\ ZY &= \beta YZ + bX, \\ ZX &= \gamma XZ + cY, \end{aligned}$$

where $\alpha = q^2$, and has $\{X^iY^jZ^k \mid (i,j,k) \in \mathbb{N}^3\}$ as a basis, then $R \cong A(q)$.

Usually q will be fixed and we simply write A for $A(q)$.

1.2 The strategy and background

In order to classify the finite dimensional simple A-modules we first study certain infinite dimensional A-modules, namely its line modules.

The notion of a line module was first introduced by Artin-Tate-Van den Bergh in [3]. There they studied a class of graded $\Bbbk$-algebras S that are now called 3-dimensional Artin-Schelter regular algebras. As the name suggests this class of algebras was first introduced (and almost classified) by Artin and Schelter [1]. There are two basic classes of 3-dimensional Artin-Schelter regular algebras, those generated by two elements subject to two cubic relations and those generated by three elements subject to three quadratic relations. For simplicity we will only discuss the latter here.

The simplest such algebra is the polynomial ring, $R = \Bbbk[x, y, z]$, on three variables. The other ones should be viewed as non-commutative deformations of that polynomial ring. The polynomial ring on three variables is a homogeneous coordinate ring of the projective plane $\mathbb{P}^2$. The points in $\mathbb{P}^2$ are in natural bijection with the isomorphism classes of graded R-modules M such that $M = RM_0$ and $\dim_{\Bbbk}(M_i) = 1$ for all $i \geq 0$. Artin-Tate-Van den Bergh call such modules *point modules* and extend this definition to all 3-dimensional Artin-Schelter regular algebras. If S is as above (i.e., a 3-dimensional Artin-Schelter regular algebra generated by three elements subject to three quadratic relations), one views S as a "homogeneous coordinate ring" of a non-commutative analogue of $\mathbb{P}^2$. Point modules (up to isomorphism) for S are then considered as "points" in this non-commutative analogue of $\mathbb{P}^2$.

This idea can be formalized and made more precise in the following way. First, let $\mathsf{Gr}(S)$ denote the category of graded left S-modules. Let $\mathsf{Fdim}(S)$ denote the full subcategory of $\mathsf{Gr}(S)$ consisting of those modules M that are the sum of their finite dimensional submodules. One may then form the quotient category

$$\mathsf{QGr}(S) \;=\; \frac{\mathsf{Gr}(S)}{\mathsf{Fdim}(S)}.$$

If S *is* the polynomial ring $R = \Bbbk[x, y, z]$, then the category $\mathsf{QGr}(R)$ is equivalent to the category of quasi-coherent sheaves on $\mathbb{P}^2$. We therefore consider $\mathsf{QGr}(S)$ as a replacement for that and think of it as playing the role of the non-existent "category of quasi-coherent sheaves" on a non-commutative analogue of $\mathbb{P}^2$. The simple objects in $\mathsf{QGr}(R)$ are the skyscraper sheaves $\mathcal{O}_p$, one for each point $p \in \mathbb{P}^2$. Under the equivalence just mentioned the point modules, considered as objects in $\mathsf{QGr}(R)$, correspond to skyscraper sheaves. For all S, as above, the point modules for S become simple objects in $\mathsf{QGr}(S)$ so we think of them as the "skyscraper sheaves" at the "points" in this non-commutative analogue of $\mathbb{P}^2$.

Line modules are defined in a similar way: a graded S-module M is called a *line module* if $M = SM_0$ and $\dim_{\Bbbk}(M_i) = i + 1$ for all $i \geq 0$. The isomorphism classes of line modules for the polynomial ring $R = \Bbbk[x, y, z]$ are in natural bijection with the lines in $\mathbb{P}^2$. Under the equivalence of categories the line modules in $\mathsf{QGr}(R)$ correspond to the structure sheaves $\mathcal{O}_L$ of the lines $L \subseteq \mathbb{P}^2$. Line modules for S are therefore thought of as "structure sheaves" of "the lines" in this non-commutative analogue of $\mathbb{P}^2$.

The notions of point module and line module have since become ubiquitous in non-commutative algebraic geometry. Their definitions make sense for other graded rings. For example, let D be a 4-dimensional Artin-Schelter regular algebra whose degree-n component D_n has the same dimension as the degree-n component of the polynomial ring $\Bbbk[x_0, x_1, x_2, x_3]$. Isomorphism classes of point modules and line modules for $\Bbbk[x_0, x_1, x_2, x_3]$ are in bijection with the points and lines in the projective space $\mathbb{P}^3$ having $\Bbbk[x_0, x_1, x_2, x_3]$ as homogeneous coordinate ring. The category $\mathsf{QGr}(\Bbbk[x_0, x_1, x_2, x_3])$ is equivalent to the category of quasi-coherent sheaves on $\mathbb{P}^3$ and the point modules and line modules correspond to the structure sheaves of the points and lines in $\mathbb{P}^3$. Point modules and line modules for D are therefore thought of as "structure sheaves" of the "points" and "lines" in this non-commutative analogue of $\mathbb{P}^3$ "having D as homogeneous coordinate ring.

Despite the proven utility of point modules and line modules for graded algebras there has not been a parallel development of these (and related) notions for non-graded rings.

In Chapter 4 we introduce the notion of a *line module* for A and show amongst other things that if V is a finite dimensional simple A-module, then there is an exact sequence $0 \to L' \to L \to V \to 0$ in which L and L' are line modules for A. Such exact sequences will play a key role in our classification of finite dimensional simple A-modules.

Since $A = A(q)$ has a PBW basis $\{x^i y^j z^k \mid (i, j, k) \in \mathbb{N}^3\}$ it is a non-commutative analogue of the 3-dimensional polynomial ring so we think of it as a "coordinate ring" of a non-commutative analogue of affine 3-space $\mathbb{A}^3_{x,y,z}$ having coordinate functions x, y, z. Line modules for A then correspond to certain lines in $\mathbb{A}^3_{x,y,z}$.

We give two classifications of the line modules for A, a geometric one classifying the lines $\ell \subseteq \mathbb{A}^3_{x_1,x_2,x_3}$ for which $A/A\ell^\perp$ is a line module, and an algebraic one classifying the subspaces $\Bbbk a + \Bbbk b \subseteq \operatorname{span}\{1, x_1, x_2, x_3\}$ such that $A/Aa + Ab$ is a line module. The first is based on the close relation between line models over graded algebras and filtered modules. More specifically, in Chapter 5 we introduce a graded ring D with the following properties:

- D is a 4-dimensional Artin-Schelter regular algebra;
- D has a homogeneous central element $t \in D_1$ such that $D[t^{-1}]_0 \cong A$;
- the line modules for D have been "classified" by Le Bruyn-Smith-Van den Bergh in [6];
- line modules for A are in natural bijection with those line modules for D that have no t-torsion.

The line modules for A are thus obtained through the passage from $\mathsf{Gr}(D)$ to $\mathsf{Mod}(A)$. As expected, they correspond to certain lines in $\mathbb{A}^3$. Using geometric ideas as an organizing tool, we can almost says that $A/A\ell^{\perp}$ is a line module if and only if ℓ lies on one of a particular family of cubic surfaces.

An equivalent definition for line modules over A is that an A-module is a line module if and only if it is infinite dimensional and isomorphic to $A/(Aa + Ab)$ for some linearly independent elements $a, b \in \operatorname{span}\{1, x_1, x_2, x_3\}$. Based on this characteristic, a direct algebraic classification of the line modules for A is performed in Chapter 6, without using the results about the line modules for D. Among other reasons, such algebraic calculation reveals the algebraic intricacies that are somewhat obscured by the results about D, and provides an independent verification that the classification in Chapter 5 is correct.

In Chapter 7 we are devoted to the classification of the finite dimensional simple A-modules. As mentioned earlier, every finite dimensional simple A-module V appears in an exact sequence $0 \to L' \to L \to V \to 0$ in which L and L' are line modules for A. Guided by this fact, we shall determine all non-simple line modules for A and the conditions for one line module embedding into another. For the first task, it is not very difficult to narrow down to four types of line modules that are possibly non-simple. Using certain bases for line modules, we further filter two out of the four types of line modules whose quotients give all finite dimensional simple A-modules. While there are other types of line modules that could lead to the same results, we choose those that have simpler A-module structures (e.g., they have unique proper submodules). The final result shows that there are five non-isomorphic simple A-modules of each dimension.

1.3 Main results

Definition 1.1 (Line modules for A). *A left A-module M is a line module with respect to the filtration*

$$\Bbbk = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n \subseteq \cdots,$$

where $A_n = \operatorname{span}\left\{x_1^i x_2^j x_3^k \mid i+j+k \leq n\right\}$, *if there is an element* $m \in M$ *such that* $M = Am$ *and* $\dim_{\mathbb{k}}(A_n m) = n+1$ *for all* $n \geq 0$.

Denote by $\mathbb{A}^3_{x_1,x_2,x_3}$ the 3-dimensional affine space with x_1, x_2, x_3 as coordinate functions. For a line ℓ in $\mathbb{A}^3_{x_1,x_2,x_3}$, we denote by $\ell^\perp$ the set of elements in $\operatorname{span}\{1, x_1, x_2, x_3\}$ that vanish on ℓ. We call ℓ an A-line if $A/A\ell^\perp$ is a line module, which is denoted by $M(\ell)$.

The function $\boldsymbol{\lambda} : \mathbb{k} - \{0\} \to \mathbb{k}$ defined by

$$\boldsymbol{\lambda}(\alpha) \;=\; \frac{\alpha + \alpha^{-1}}{q^2 - q^{-2}}$$

will play a key role in formulating the A-lines.

Theorem 1.2 (See Theorem 6.3). *A line* ℓ *in* $\mathbb{A}^3_{x_1,x_2,x_3}$ *is an* A*-line if and only if*

$$\ell \;=\; \{x - \boldsymbol{\lambda}(q\alpha) \;=\; y - \alpha z - \beta \;=\; 0\}$$

for some cyclic permutation (x, y, z) *of* (x_1, x_2, x_3) *and some* $(\alpha, \beta) \in \mathbb{k}^2$ *such that* $\alpha \neq 0$ *and* $(\alpha^2 - q^{-4})\,\beta = 0$. *Furthermore, in that case,* $A = A\ell^\perp \oplus \mathbb{k}[z]$.

As a result, there are nine 1-parameter families of A-lines. These are, for each of the three cyclic permutations (i, j, k) of $(1, 2, 3)$, the lines

1. $\{x_i - \boldsymbol{\lambda}(q\alpha) \;=\; x_j - \alpha x_k \;=\; 0\}$, one for each $\alpha \in \mathbb{k} - \{\pm q^{-2}, 0\}$;

2. $\{x_i - \boldsymbol{\lambda}(q) \;=\; x_j - q^{-2} x_k - \beta \;=\; 0\}$, one for each $\beta \in \mathbb{k}$;

3. $\{x_i + \boldsymbol{\lambda}(q) \;=\; x_j + q^{-2} x_j - \beta \;=\; 0\}$, one for each $\beta \in \mathbb{k}$.

The following proposition serves as a guiding principle for our classification of finite dimensional simple A-modules.

Proposition 1.3 (See Proposition 6.13). *Every finite dimensional simple A-module V appears in an exact sequence*

$$0 \longrightarrow M(\ell') \longrightarrow M(\ell) \longrightarrow V \longrightarrow 0,$$

in which ℓ and ℓ' are A-lines.

We adopt the following notation: For $n \in \mathbb{Z}$, denote

$$[n] := \frac{q^{2n} - q^{-2n}}{q^2 - q^{-2}}.$$

For a set $S \subset \mathbb{Z}$, we denote $q^S := \{q^n \mid n \in S\}$ and $[S] := \{[n] \mid n \in S\}$. For two sets Λ and Γ, we denote by (Λ, Γ) the Cartesian product $\{(\lambda, \gamma) \mid \lambda \in \Lambda, \gamma \in \Gamma\}$.

The following three types [1] of line modules for A turn out to be of particular relevance to the finite dimensional simple A-modules.

Definition 1.4 (Types of A-lines). *Let (x, y, z) be a cyclic permutation of (x_1, x_2, x_3). We call the A-line*

$$\ell = \{x - \boldsymbol{\lambda}(q\alpha) = y - \alpha z - \beta = 0\}$$

and its respective line module $M(\ell)$ of

- *Type 1 if $(\alpha^2, \beta^2) \in \left(-q^{2\mathbb{N}-2}, 0\right)$;*
- *Type 2 if $(\alpha^2, \beta^2) \in (q^{-4}, q^{-2}[\mathbb{N}_+]^2)$;*
- *Type 3 if $(\alpha^2, \beta^2) \in \left(q^{4\mathbb{N}}, 0\right)$.*

Theorem 1.5 (See Theorem 7.21). *Every line module of Type 1, 2 or 3 has a finite dimensional simple quotient. All other line modules are simple.*

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[1] We actually defined four types of line modules in §7.4. The line modules of the fourth type, however, turn out to be simple.

Definition 1.6 (Shift and translation of an A-line)**.** *Let $n \in \mathbb{Z}$. For the A-line*

$$\ell \;=\; \{x - \boldsymbol{\lambda}(q\alpha) \;=\; y - \alpha z - \beta \;=\; 0\},$$

we define

$$\ell[n] \;:=\; \left\{x - \boldsymbol{\lambda}(q^{2n+1}\alpha) \;=\; y - q^{2n}\alpha z \;=\; 0\right\},$$

*called a **shift** of ℓ, and*

$$\ell + p \;:=\; \left\{(\mu,\nu,\eta) \in \mathbb{A}^3_{x,y,z} \mid (\mu,\nu,\eta) - p \;\in\; \ell\right\},$$

*called a **translation** of ℓ.*

There are 24 line modules that map onto r-dimensional simple A-modules. Let

$$\begin{aligned} \mathbb{L}_r \;&:=\; \left\{A\text{-lines mapping onto an } r\text{-dimensional simple module}\right\}, \text{and} \\ p_r \;&:=\; (0,0,q[r]-q) \in \mathbb{A}^3_{x,y,z}. \end{aligned}$$

for $r \geq 1$. Then $|\mathbb{L}_r| = 24$. It is an elementary exercise to find all A-lines in $\mathbb{L}_1$ (see §6.3.1). The following table shows every line in $\mathbb{L}_r$ is either a shift or translation of an line in $\mathbb{L}_1$.

$\mathbb{L}_1$	$\mathbb{L}_r$	Type
$\ell_1 \;=\; \left\{x = y - \mathbb{i}q^{-1}z = 0\right\}$	$\ell_1[\frac{1}{2}r - 1]$ if r is even $\ell_1[\frac{1}{2}(r-1)]$ if r is odd	1
$\ell_2 \;=\; \left\{x = y + \mathbb{i}q^{-1}z = 0\right\}$	$\ell_2[\frac{1}{2}r - 1]$ if r is even $\ell_2[\frac{1}{2}(r-1)]$ if r is odd	1
$\ell_3 \;=\; \left\{x + \boldsymbol{\lambda}(q) = y + q^{-2}z + q^{-1} = 0\right\}$	$\ell_3 - p_r$	2
$\ell_4 \;=\; \left\{x + \boldsymbol{\lambda}(q) = y + q^{-2}z - q^{-1} = 0\right\}$	$\ell_4 + p_r$	2
$\ell_5 \;=\; \left\{x - \boldsymbol{\lambda}(q) = y - q^{-2}z + q^{-1} = 0\right\}$	$\ell_5 - p_r$	2
$\ell_6 \;=\; \left\{x - \boldsymbol{\lambda}(q) = y - q^{-2}z - q^{-1} = 0\right\}$	$\ell_6 + p_r$	2
$\ell_7 \;=\; \{x - \boldsymbol{\lambda}(q) \;=\; y - z \;=\; 0\}$	$\ell_7[r-1]$	3
$\ell_8 \;=\; \{x + \boldsymbol{\lambda}(q) \;=\; y + z \;=\; 0\}$	$\ell_8[r-1]$	3

Let $\mathbb{i}$ be an element in $\Bbbk$ such that $\mathbb{i}^2 = -1$.

Theorem 1.7 (See Theorem 7.19)**.** *For every integer $r \geq 1$, there are five r-dimensional simple A-modules up to isomorphism. Each simple module S occurs in an exact sequence*

$$0 \longrightarrow M(\ell[-r]) \longrightarrow M(\ell) \longrightarrow S \longrightarrow 0$$

for some A-line $\ell \in \mathbb{A}^3_{x_1,x_2,x_3}$. The five A-lines, corresponding to the five simple modules, could be chosen to be

1. $\{x - \boldsymbol{\lambda}(\mathbb{i}q^{r-1}) \ = \ y - \mathbb{i}q^{r-2}z \ = \ 0\}$;
2. $\{x - \boldsymbol{\lambda}(q) \ = \ y - q^{-2}z - q^{-1}[r] \ = \ 0\}$;
3. $\{x + \boldsymbol{\lambda}(q) \ = \ y + q^{-2}z - q^{-1}[r] \ = \ 0\}$;
4. $\{x - \boldsymbol{\lambda}(q) \ = \ y - q^{-2}z + q^{-1}[r] \ = \ 0\}$;
5. $\{x + \boldsymbol{\lambda}(q) \ = \ y + q^{-2}z + q^{-1}[r] \ = \ 0\}$,

where (x, y, z) is a cyclic permutation of (x_1, x_2, x_3).

Chapter 2

PRELIMINARIES

2.1 Base field

Always, $\Bbbk$ is an algebraically closed field of characteristic zero and $q \in \Bbbk - \{0\}$. We assume that q is not a root of unity.

2.2 Algebras

All algebras are assumed to be $\Bbbk$-algebras. In principle, we use $\mathcal{A}$ for some arbitrary (usually filtered) $\Bbbk$-algebra, and A exclusively for the algebra defined in §3.1.

An element c in an algebra $\mathcal{A}$ is central if $ca = ac$ for all $a \in \mathcal{A}$. The center of $\mathcal{A}$, denoted by $Z(\mathcal{A})$, is the set of all central elements in $\mathcal{A}$.

2.2.1 Graded algebra

A $\mathbb{Z}$-graded algebra is an algebra $\mathcal{A}$ endowed with a family $\{\mathcal{A}_n\}_{n\in\mathbb{Z}}$ of subspaces such that

1. $\mathcal{A}_i\mathcal{A}_j \subset \mathcal{A}_{i+j}$ for all $i, j \in \mathbb{Z}$, and
2. $\mathcal{A} = \bigoplus_{n\in\mathbb{Z}} \mathcal{A}_n$.

If $\mathcal{A}_n = 0$ for all $n < 0$ then $\mathcal{A}$ is also called $\mathbb{N}$-graded.

An $\mathbb{N}$-graded algebra $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \cdots$ is connected if $\mathcal{A}_0 = \Bbbk$.

A $\mathbb{Z}$-graded algebra $\mathcal{A}$ is strongly graded if $\mathcal{A}_n\mathcal{A}_{-n} = \mathcal{A}_0$ for all $n \in \mathbb{Z}$. Clearly, a graded algebra having a unit of degree one is strongly graded.

The **Hilbert series** of a $\mathbb{Z}$-graded algebra $\mathcal{A}$ is the formal Laurent series

$$H_{\mathcal{A}}(t) \;:=\; \sum_{n\in\mathbb{Z}} \dim_{\Bbbk}\left(\mathcal{A}_n\right) t^n.$$

2.2.2 Filtered algebra

A filtered algebra is an algebra $\mathcal{A}$ endowed with an ascending sequence

$$\Bbbk \;=\; \mathcal{A}_0 \;\subseteq\; \mathcal{A}_1 \;\subseteq\; \cdots \;\subseteq\; \mathcal{A}_n \;\subseteq\; \cdots \;\subseteq\; \mathcal{A}$$

of subspaces such that

1. $\mathcal{A}_i\mathcal{A}_j \;\subset\; \mathcal{A}_{i+j}$ for all $i, j \in \mathbb{N}$, and
2. $\mathcal{A} \;=\; \bigcup_{n\in\mathbb{N}} \mathcal{A}_n$.

A filtered algebra is a generalization of the notion of a graded algebra.

Given a filtered algebra $\mathcal{A}$, its **associated graded algebra** is the graded algebra

$$\operatorname{gr}(\mathcal{A}) \;:=\; \bigoplus_{n=0}^{\infty} \mathcal{G}_n$$

where $\mathcal{G}_n = \mathcal{A}_{n+1}/\mathcal{A}_n$. The addition and multiplication in $\operatorname{gr}(\mathcal{A})$ are defined by

$$(a+\mathcal{G}_{n-1}) + (b+\mathcal{G}_{m-1}) \;=\; a+b+\mathcal{G}_{m+n-1} \quad\text{and}\quad (a+\mathcal{G}_{n-1})(b+\mathcal{G}_{m-1}) \;=\; ab+\mathcal{G}_{m+n-1},$$

where $a \in \mathcal{G}_n$ and $b \in \mathcal{G}_m$.

2.2.3 Gelfand-Kirillov dimension

Let $\mathcal{A}$ be a finitely generated $\Bbbk$-algebra and $V \subset \mathcal{A}$ a finite dimensional generating subspace containing $1_{\mathcal{A}}$. Then there is an ascending chain of subspaces

$$\Bbbk \ = \ V^0 \ \subset \ V \ \subset \ V^2 \ \subset \ \cdots \ \subset \ \bigcup_{n=0}^{\infty} V^n \ = \ \mathcal{A}.$$

The **Gelfand-Kirillov dimension** of $\mathcal{A}$, or GK-dimension of $\mathcal{A}$ for short, is defined as

$$\mathrm{GKdim}(\mathcal{A}) \ := \ \lim_{n \to \infty} \sup \log_n \left(\dim_{\Bbbk}(V^n) \right).$$

It is independent of the choice of V (See [11, page 14]).

It is well-known that if $R = \Bbbk[x_1, x_2, ..., x_n]$, the polynomial ring in n variables, then $\mathrm{GKdim}(R) = n$.

2.2.4 Artin-Schelter algebra

A connected graded algebra $\mathcal{A}$ is called **Artin-Schelter regular** of dimension d if

1. $\mathcal{A}$ has finite global dimension d;

2. $\mathcal{A}$ has finite GK-dimension (so the Hilbert function of $\mathcal{A}$ is bounded by a polynomial);

3. $\mathcal{A}$ is Gorenstein, i.e.,

$$\mathrm{Ext}^i_{\mathcal{A}}(\Bbbk, \mathcal{A}) \ = \ \begin{cases} 0 & i \neq d, \\ \Bbbk & i = d. \end{cases}$$

2.2.5 Quadratic algebra

If V is a finite dimensional vector space and R a subspace of $V^{\otimes 2}$ we call $T(V)/(R)$, the quotient of the tensor algebra by the ideal generated by R, a **quadratic algebra**. Its **quadratic** dual is the algebra $T(V^*)/(R^\perp)$ where $R^\perp$ is the subspace of $(V^*)^{\otimes 2}$ consisting of the linear forms that vanish on R.

2.3 Modules

All modules are left modules unless stated otherwise. We write $\mathsf{Mod}(\mathcal{A})$ for the category of left modules over an algebra $\mathcal{A}$.

A module is simple if its only submodules are 0 and itself, or equivalently, if every cyclic submodule generated by a non-zero element equals the module itself.

Given an $\mathcal{A}$-module M and $a \in \mathcal{A}$, by an a-eigenvector (resp. a-eigenvalue, a-eigenspace) we mean an eigenvector (resp. eigenvalue, eigenspace) for the action of a on M.

For a finite dimensional $\mathcal{A}$-module V and $a \in A$, we will denote by $\operatorname{tr}_V(a)$, or simply $\operatorname{tr}(a)$ if V is known, the trace for the action of a on V, which equals the sum of all a-eigenvalues (with multiplicities).

2.3.1 Linear modules over graded algebras

The notion of a linear module over a connected graded $\Bbbk$-algebra was introduced by Artin-Tate-Van den Bergh in [2].

Let $\mathcal{D}$ be a connected graded $\Bbbk$-algebra that is generated as a $\Bbbk$-algebra by its degree-one component, $\mathcal{D}_1$. Suppose that $\dim_\Bbbk(\mathcal{D}_1) < \infty$.

A graded left $\mathcal{D}$-module L is said to be d-linear if $L = \mathcal{D}L_0$ and

$$\dim_\Bbbk(L_n) \;=\; \binom{n+d}{n}$$

for all $n \geq 0$. For example, L is 0-linear if $L = \mathcal{D}L_0$ and $\dim_\Bbbk(L_n) = 1$ for all $n \geq 0$ and 1-linear if $L = \mathcal{D}L_0$ and $\dim_\Bbbk(L_n) = n+1$ for all $n \geq 0$. One calls 0-linear modules point modules and 1-linear modules line modules.

More details about linear modules are included in §4.1. We will extend the notion of a linear module over a graded algebra to that over a filtered algebra in §4.2.

2.3.2 Support of a module over a polynomial ring

Suppose $\Bbbk$ is algebraically closed and let $\Bbbk[t]$ denote the polynomial ring in one variable. A $\Bbbk[t]$-module is **torsion** if every element in it is annihilated by a non-zero element in $\Bbbk[t]$.

Let $\mathcal{A}$ be a $\Bbbk$-algebra and $a \in \mathcal{A}$. We regard a left $\mathcal{A}$-module M as a $\Bbbk[t]$-module with t acting as a does. We say M is $\Bbbk[a]$-torsion if it is $\Bbbk[t]$-torsion. For each $\beta \in \Bbbk$, let

$$M_\beta \; = \; \{m \in M \mid (t-\beta)^n m = 0 \text{ for } n \gg 0\}\,.$$

If M is $\Bbbk[a]$-torsion its a-**support** is defined as

$$\operatorname{Supp}_a(M) \; := \; \{\beta \in \Bbbk \mid M_\beta \neq 0\}\,.$$

By the general theory of modules over a PID, if M is $\Bbbk[a]$-torsion, then

$$M \; = \bigoplus_{\beta \in \operatorname{Supp}_a(M)} M_\beta.$$

2.3.3 Twisted modules

Let $\mathcal{A}$ be a $\Bbbk$-algebra. Given a left $\mathcal{A}$-module $(M, *)$ and an automorphism $\rho \in \operatorname{Aut}(\mathcal{A})$, the ρ-**twisted module** $(\rho^* M, \cdot)$ is the left $\mathcal{A}$-module with the same underlying vector space as M and the $\mathcal{A}$-module structure given by

$$a \cdot m \; = \; \rho(a) * m,$$

where $a \in \mathcal{A}$ and $m \in M$.

It is clear that ρ^* is an auto-equivalence of $\mathsf{Mod}(\mathcal{A})$, in fact an automorphism, with a quasi-inverse, in fact an inverse, given by $(\rho^{-1})^*$. We also have

$$(\rho_1\rho_2)^* \; = \; \rho_2^*\rho_1^*.$$

2.4 *Conventions and notation*

W adopt the following convention and notation:

- We will write $\mathbb{N}$ for the set of all non-negative integers, i.e., $\mathbb{N} = \{0, 1, 2, \cdots\}$. We will write $\mathbb{N}_+$ for $\mathbb{N} - \{0\}$.

- For $n \in \mathbb{Z}$, we denote
$$[n] \ := \ \frac{q^{2n} - q^{-2n}}{q^2 - q^{-2}}.$$
Clearly $[-n] = -[n]$. Also, if q is not a root of unity then $[n] = 0$ if and only if $n = 0$.

- Let $S \subset \mathbb{Z}$. By q^S we mean the set $\{q^n \mid n \in S\}$. Denote $[S] := \{[n] \mid n \in S\}$.

- For two sets S, T, denote by (S, T) the Cartesian product $\{(s, t) \mid s \in S, t \in T\}$.

- For $n \in \mathbb{N}_+$, we write $\mathbb{M}_n(\mathbb{k})$ for the $n \times n$ matrix algebra over $\mathbb{k}$.

- We frequently view x_1, x_2, x_3 as coordinate functions on $\mathbb{k}^3$ and denote $\mathbb{k}^3$ by $\mathbb{A}^3_{x_1,x_2,x_3}$ when we do this.

- To avoid subscripts, we often write (x, y, z) for any cyclic permutation of (x_1, x_2, x_3).

- We will fix $\mathbb{i}$ for an element in $\mathbb{k}$ such that $\mathbb{i}^2 = -1$.

- In principle, we use the letters a, b, c, d for elements in an algebra, the letters m, n, r, s, t for integers, the Greek letters $\alpha, \beta, \gamma, \delta, \lambda, \mu$ for scalars in the base field.

- The bold letter $\boldsymbol{\lambda}$ is reserved for the function defined in §5.3.2. For $n \in \mathbb{Z}$ and $\alpha \in \mathbb{k} - \{0\}$, denote $\lambda_n(\alpha) := \boldsymbol{\lambda}(q^{-2n+1}\alpha)$.

Other notation will be defined when they first appear.

Chapter 3

THE ALGEBRAS $A(q)$

Always, $\Bbbk$ is an algebraically closed field of characteristic zero and $q \in \Bbbk - \{0\}$.

3.1 The definition of $A(q)$

Let $A(q)$ be the free algebra $\Bbbk\langle x_1, x_2, x_3\rangle$ modulo the three relations

$$qx_ix_j - q^{-1}x_jx_i \;=\; x_k,$$

where (i,j,k) runs over all cyclic permutations of $(1,2,3)$. If (x,y,z) is a cyclic permutation of (x_1,x_2,x_3) then the relations for $A(q)$ are

$$\begin{aligned} qxy - q^{-1}yx &= z, && \text{(3-1)}\\ qyz - q^{-1}zy &= x, && \text{(3-2)}\\ qzx - q^{-1}xz &= y. && \text{(3-3)} \end{aligned}$$

Usually q will be fixed and we simply write A for $A(q)$.

A PBW basis for $A(q)$. The algebra A has a PBW (Poincaré-Birkhoff-Witt) basis.

Lemma 3.1. *If (x,y,z) is a cyclic permutation of (x_1,x_2,x_3), then*

$$\left\{x^iy^jz^k \;\middle|\; (i,j,k) \in \mathbb{N}^3\right\}$$

is a basis for A.

Proof. We use Bergman's Diamond Lemma [4] and the terminology in that paper. The replacements with respect to the lexicographic ordering induced by $x < y < z$ are

$$\begin{aligned} zy &= q^2 yz - qx, \\ yx &= q^2 xy - qz, \\ zx &= q^{-2} xz + q^{-1} y. \end{aligned}$$

The only ambiguity is zyx and this is resolvable because

$$\begin{aligned} (zy)x &= q^2 yzx - qx^2 \\ &= q^2 y \left(q^{-2} xz + q^{-1} y\right) - qx^2 \\ &= \left(q^2 xy - qz\right) z + qy^2 - qx^2 \\ &= q^2 xyz - q\left(z^2 - y^2 + x^2\right) \end{aligned}$$

and

$$\begin{aligned} z(yx) &= q^2 zxy - qz^2 \\ &= q^2 \left(q^{-2} xz + q^{-1} y\right) y - qz^2 \\ &= x\left(q^2 yz - qx\right) + qy^2 - qz^2 \\ &= q^2 xyz - q\left(x^2 - y^2 + z^2\right). \end{aligned}$$

Therefore, $\{x^i y^j z^k \mid i, j, k \geq 0\}$ is a basis for A. $\square$

The following result gives a different presentation of A. With this presentation, A appears in [15].

Proposition 3.2. *If $p = q^{-2}$, then $A(q)$ is isomorphic to the algebra $\Bbbk\langle X_1, X_2, X_3\rangle$ modulo the relations*

$$X_i X_j - p X_j X_i = (p - 1) X_k,$$

where (i,j,k) *runs over the cyclic permutations of* $(1,2,3)$*, via the map*

$$X_i \;\mapsto\; \left(q-q^{-1}\right) x_i, \qquad i=1,2,3.$$

Proof. This is can be verified directly; we leave it to the reader. $\square$

3.2 A natural class of PBW algebras

Although $A(q)$ may appear rather special, even artificial, at first glance, Proposition 3.3 shows it is not. That result shows that an apparently much larger, and very natural, family of algebras consists of precisely the algebras $A(q)$.[1]

Proposition 3.3. *Let* $\alpha,\beta,\gamma,a,b,c \in \Bbbk-\{0\}$ *and suppose that* $q^2=\alpha$*. Let* R *be the free algebra* $\Bbbk\langle x,y,z\rangle$ *modulo the relations*

$$\begin{cases} yx &=& \alpha xy + az, \\ zy &=& \beta yz + bx, \\ zx &=& \gamma xz + cy. \end{cases} \tag{3-4}$$

If $\{x^iy^jz^k \mid (i,j,k)\in\mathbb{N}^3\}$ *is a basis for* R*, then* $R\cong A(q)$*.*

Proof. Bergman's Diamond Lemma tells us that $\{x^iy^jz^k \mid i,j,k\geq 0\}$ is a basis for R if and only if the ambiguity zyx is resolvable (the reader should consult Bergman's paper [4] if this does not make immediate sense).

Since $\{x^iy^jz^k \mid i,j,k\geq 0\}$ is a basis for R. The ambiguity zyx is therefore resolvable. The relations for R imply that

$$(zy)x \;=\; \beta yzx + bx^2 \;=\; \beta\gamma yxz + \beta cy^2 + bx^2 \;=\; \beta\gamma\alpha xyz + \beta\gamma az^2 + \beta cy^2 + bx^2$$

---

[1] This result was stated, but not proved, in Note 1 on page 473 of the paper [9] by Havlicek and Posta.

and that

$$z(yx) \;=\; \alpha zxy + az^2 \;=\; \alpha\gamma xzy + \alpha cy^2 + az^2 \;=\; \alpha\gamma\beta xyz + \alpha\gamma bx^2 + \alpha cy^2 + az^2.$$

Since the ambiguity zyx is resolvable,

$$\beta\gamma az^2 + \beta cy^2 + bx^2 \;=\; \alpha\gamma bx^2 + \alpha cy^2 + az^2.$$

Since $\{x^2, y^2, z^2\}$ is a linearly independent subset of R,

$$\beta\gamma a \;=\; a, \qquad \beta c \;=\; \alpha c, \quad \text{and} \quad b \;=\; \alpha\gamma b.$$

Thus $\beta\gamma = 1$, $\beta = \alpha$, and $1 = \alpha\gamma$. It follows that the relations for R are

$$\begin{cases} yx \;=\; \alpha xy + az, \\ zy \;=\; \alpha yz + bx, \\ zx \;=\; \alpha^{-1}xz + cy. \end{cases} \tag{3-5}$$

Define X, Y, and Z, by

$$X \;=\; \frac{x}{\sqrt{-ac}}, \qquad Y \;=\; \frac{y}{-\sqrt{abq^{-2}}}, \qquad Z \;=\; \frac{z}{\sqrt{-bc}}\,.$$

It follows from (3-5) that

$$\begin{aligned} yx \;&=\; -\sqrt{-a^2bcq^{-2}}\,YX \;=\; -q^2\sqrt{-a^2bcq^{-2}}\,XY + a\sqrt{-bc}\,Z, \\ zy \;&=\; \sqrt{-ab^2cq^{-2}}\,ZY \;=\; -q^2\sqrt{-ab^2cq^{-2}}\,YZ + b\sqrt{-ac}\,X, \\ zx \;&=\; \sqrt{ac^2}\,ZX \;=\; q^2\sqrt{ac^2}\,XZ - c\sqrt{abq^{-2}}\,Y. \end{aligned}$$

Hence,

$$\begin{aligned} qXY - q^{-1}YX &= Z, \\ qYZ - q^{-1}ZY &= X, \\ qZX - q^{-1}XZ &= Y. \end{aligned}$$

More formally, if A is the algebra generated by X, Y, and Z, subject to these relations, then there is an algebra isomorphism $\Phi : R \to A$ given by

$$\Phi(x) = \sqrt{-ac}\, X, \qquad \Phi(y) = -\sqrt{abq^{-2}}\, Y, \qquad \Phi(z) = \sqrt{-bc}\, Z.$$

This completes the proof. $\square$

3.3 Remarks

We will now justify our earlier remark that the class of algebras in Proposition 3.3 is a very natural class to understand.

One of the algebras defined by the relations (3-4) is the enveloping algebra $U(\mathfrak{so}(3))$ of the Lie algebra $\mathfrak{so}(3)$; it occurs when $\alpha = \beta = \gamma = 1$ and $a = b = c = -1$. When the base field $\Bbbk$ is $\mathbb{C}$, $\mathfrak{so}(3)$ is isomorphic to the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ of traceless 2×2 matrices. A distinguishing feature of the enveloping algebra of a finite dimensional Lie algebra is that it has a PBW basis. When that Lie algebra has dimension 3, it has a basis $\{x, y, z\}$, and the statement that its enveloping algebra has a PBW basis is just the statement that $\{x^i y^j z^k \mid i, j, k \geq 0\}$ is a basis for its enveloping algebra.

The representations of $\mathfrak{sl}(2, \mathbb{C})$ are the same things as $U(\mathfrak{sl}(2, \mathbb{C}))$-modules. The representation theory of $U(\mathfrak{sl}(2, \mathbb{C}))$ has played a fundamental guiding role in the development of the representation theory of non-commutative noetherian rings since the 1960's. So too has the study of algebras having PBW bases. For these reasons it is important to have a complete understanding of the representation theory of the algebras having relations of the form (3-4) that have a PBW basis.

One of the most striking properties of $U(\mathfrak{sl}(2,\mathbb{C}))$ is that it has exactly one simple module of each dimension $n \geq 1$. We will show that when q is not a root of unity, the algebras in Proposition 3.3 have exactly *five* simple modules of each dimension $n \geq 1$. This came as quite a surprise to us and there does not appear to be any reason to expect such a result. At least, no good reason at first.

After proving this result we learned that it had already been proved around the year 2000 in the papers [8] and [9], and again, independently by N. Sasom in her 2005 Ph.D. thesis at the University of Sheffield. Sasom gives a geometric explanation for the appearance of five simple modules of each dimension. The algebras in Proposition 3.3 are deformation quantizations of the polynomial ring on three variables and the associated Poisson structure on that polynomial ring, more particularly on the affine space $\mathbb{A}^3$, has exactly 5 points that are symplectic leaves for that Poisson structure. Sasom shows how the simple modules correspond to those points in a natural way: the points (with multiplicities) are in a suitable sense the characteristic varieties of those simple modules. This is also explained in the paper [10].

3.4 The opposite algebra of $A(q)$

Proposition 3.4. *There is a* $\Bbbk$*-algebra isomorphism* $A(q) \cong A(q^{-1})^{\text{op}}$ *given by the map* $\Phi : A(q) \longrightarrow A(q^{-1})^{\text{op}}$ *defined by*

$$\Phi(x) \;=\; -x, \qquad \Phi(y) \;=\; -y, \qquad \Phi(z) \;=\; -z.$$

Proof. By definition, $A(q^{-1})$ is the free algebra $\Bbbk\langle X, Y, Z\rangle$ modulo the relations

$$q^{-1}XY - qYX \;=\; Z, \qquad q^{-1}YZ - qZY \;=\; X, \qquad q^{-1}ZX - qXZ \;=\; Y.$$

Write $*$ for the multiplication in $A(q^{-1})^{\mathrm{op}}$. Since

$$\begin{aligned} q\Phi(x)\Phi(y) - q^{-1}\Phi(y)\Phi(x) &= q(-X) * (-Y) - q^{-1}(-Y) * (-X) \\ &= qYX - q^{-1}XY \\ &= -Z \\ &= \Phi(z), \end{aligned}$$

Φ really does extend to a $\Bbbk$-algebra homomorphism. It is easy to see that this extension is bijective (by using PBW bases, for example), so Φ is an isomorphism. $\square$

Corollary 3.5. *The category of right $A(q)$-modules is equivalent to the category of left $A(q^{-1})$-modules.*

3.5 The group Alt_4 of automorphisms of $A(q)$

Let $\mathrm{Aut}(A)$ denote the group of $\Bbbk$-algebra automorphisms of A and write 1 for its identity. There are automorphisms $\sigma, \tau_1, \tau_2, \tau_3$ of A defined by

$$\begin{array}{lllllllll} \sigma(x_1) &=& x_2, & \sigma(x_2) &=& x_3, & \sigma(x_3) &=& x_1, \\ \tau_1(x_1) &=& x_1, & \tau_1(x_2) &=& -x_2, & \tau_1(x_3) &=& -x_3, \\ \tau_2(x_1) &=& -x_1, & \tau_2(x_2) &=& x_2, & \tau_2(x_3) &=& -x_3, \\ \tau_3(x_1) &=& -x_1, & \tau_3(x_2) &=& -x_2, & \tau_3(x_3) &=& x_3. \end{array}$$

If $\{i, j, k\} = \{1, 2, 3\}$, then $\tau_i(x_i) = x_i$, $\tau_i(x_j) = -x_j$, and $\tau_k(x_k) = -x_k$.

Proposition 3.6. *The subgroup $\langle \sigma, \tau_1, \tau_2, \tau_3 \rangle$ of $\mathrm{Aut}(A)$ is isomorphic to the alternating group Alt_4 via the map*

$$\sigma \mapsto (123), \quad \tau_1 \mapsto (12)(34), \quad \tau_2 \mapsto (13)(24), \quad \tau_3 \mapsto (23)(14).$$

Proof. It is obvious that $\sigma^3 = 1$ and it is easy to check that there is an isomorphism

$$\langle \sigma, \tau_1, \tau_2, \tau_3 \rangle \;=\; \langle \tau_1, \tau_2, \tau_3 \rangle \rtimes \langle \sigma \rangle \;\cong\; (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_3.$$

We leave the details to the reader. □

Whenever we speak of the group Alt_4 we mean this subgroup of $\mathrm{Aut}(A)$.

There are four subgroups of Alt_4 that are isomorphic to $\mathbb{Z}_3$, the cyclic group of order 3.

3.5.1 The action of Alt_4 *on the 1-dimensional* A*-modules*

We now determine the 1-dimensional A-modules and the action of the group Alt_4 of auto-equivalences on them. The methods and results are elementary but they provide a template that will appear later when we consider higher dimensional simple A-modules.

Denote

$$\delta \;:=\; \left(q - q^{-1}\right)^{-1}.$$

Proposition 3.7. *There are exactly five 1-dimensional* A*-modules, namely* $A/(x_1 - \mu, x_2 - \nu, x_3 - \eta)$ *as* (μ, ν, η) *runs over the points*

$$(0,0,0), \quad \delta(1,1,1), \quad \delta(1,-1,-1), \quad \delta(-1,1,-1), \quad \delta(-1,-1,1)$$

in $\mathbb{A}^3_{x_1,x_2,x_3}$.

Proof. The points (μ, ν, η) in $\Bbbk^3$ for which $A/(x_1 - \mu, x_2 - \nu, x_3 - \eta)$ has dimension 1 are the solutions to the system of equations

$$x_1x_2 \;=\; \delta x_3, \quad x_2x_3 \;=\; \delta x_1, \quad x_3x_1 \;=\; \delta x_2.$$

It is easy to verify that the only solutions are those in the statement of the proposition. □

Remark 3.8. If we homogenize the three equations in the previous proof by replacing x_i with $\delta x_i t^{-1}$, then we obtain three quadratic equations

$$x_1x_2 - x_3t \;=\; x_2x_3 - x_1t \;=\; x_3x_1 - x_2t \;=\; 0.$$

There are 8 solutions in $\mathbb{P}^3_{x_1,x_2,x_3,t}$ to this system of equations, namely,

$$(0,0,0,1), \quad (1,1,1,1), \quad (1,-1,-1,1), \quad (-1,1,-1,1), \quad (-1,-1,1,1),$$

$$(1,0,0,0), \quad (0,1,0,0), \quad (0,0,1,0).$$

We will write $S_{\mu,\nu,\eta}$ for $A/A(x_1-\delta\mu)+A(x_2-\delta\nu)+A(x_3-\delta\eta)$.

If $\rho \in \operatorname{Aut}(A)$ and S is a 1-dimensional A-module so is the ρ-twisted module ρ^*S.[2] The next lemma describes the action of Alt_4 on the set of isomorphism classes of 1-dimensional A-modules.

Corollary 3.9. *There are two orbits for the action of Alt_4 on the set of isomorphism classes of 1-dimensional A-modules, namely*

$$\begin{aligned} \operatorname{Orb}(S_{0,0,0}) &= \{S_{0,0,0}\} \quad \textit{and} \\ \operatorname{Orb}(S_{1,1,1}) &= \{S_{1,1,1}, S_{1,-1,-1}, S_{-1,-1,1}, S_{-1,1,-1}\}. \end{aligned}$$

Proof. It is straightforward to verify that

$$\sigma^*S_{0,0,0} \;\cong\; S_{0,0,0}, \qquad \tau_i^*S_{0,0,0} \;\cong\; S_{0,0,0} \;\text{ for } i=1,2,3,$$

and

$$\tau_1^*S_{1,1,1} \;\cong\; S_{1,-1,-1}, \quad \tau_2^*S_{1,1,1} \;\cong\; S_{-1,1,-1}, \quad \tau_3^*S_{1,1,1} \;\cong\; S_{-1,-1,1}.$$

Since $\text{Alt}_4 = \langle \sigma, \tau_1, \tau_2, \tau_3 \rangle$, the result follows. □

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[2] See §2.3.3 for the notion of a twisted module.

If M and N are A-modules such that $N \cong \rho^* M$ we will write $M \xrightarrow{\rho} N$ to indicate this relationship. Thus, the action of Alt_4 on the set of 1-dimensional A-modules can be depicted in the following commutative diagrams:

$$\begin{tikzcd}
S_{0,0,0} \arrow[loop above, "\tau_i"] \arrow[loop below, "\sigma"'] & S_{1,1,1} \arrow[r, "\tau_1"] & S_{1,-1,-1} \arrow[r, "\tau_2"] & S_{-1,-1,1} \arrow[r, "\tau_3"] & S_{-1,1,-1} \arrow[lll, bend left, "\tau_2"]
\end{tikzcd}$$

$$\begin{tikzcd}
 & S_{1,-1,-1} \arrow[dr, "\sigma"] & \\
S_{-1,-1,1} \arrow[ur, "\sigma"] & & S_{-1,1,-1} \arrow[ll, "\sigma"]
\end{tikzcd}$$

We will see in §7.10 that the orbits for the action of Alt_4 on the set of isomorphism classes of n-dimensional simple A-modules, where $n \geq 1$, have exactly the same behavior as it does on the 1-dimensional set.

3.6 A cubic central element in $A(q)$

Write $[-,-]$ for the commutator in A, that is, $[a,b] = ab - ba$ for $a, b \in A$.

Proposition 3.10. *If* (x, y, z) *is a cyclic permutation of* (x_1, x_2, x_3)*, then the element*

$$C \ := \ \left(q^{-2} - q^{2}\right) xyz + qx^2 + q^{-3}y^2 + qz^2$$

belongs to the center of $A(q)$*.*

Proof. In the free algebra $\Bbbk\langle x, y, z\rangle$, let $a = qxy - q^{-1}yx - z$, $b = qyz - q^{-1}zy - x$, and

$c = qzx - q^{-1}xz - y$. Calculations in $\Bbbk\langle x, y, z\rangle$ give

$$\begin{aligned} q[xyz, x] - x(y^2 - z^2) &= xyc - xaz, \\ q[xyz, y] - (z^2y - yx^2) &= azy - yxb, \\ q[xyz, z] - (x^2 - y^2)z &= xbz - cyz, \\ q^4\left(xy^2 - z^2x\right) - \left(y^2x - xz^2\right) &= q^3ay + q^{-3}ya - q^3zc - q^{-3}cz. \end{aligned}$$

Because $a = b = c = 0$ in $A(q)$, the following equalities hold in $A(q)$:

$$\begin{aligned} [xyz, x] &= q^{-1}x(y^2 - z^2), \\ [xyz, y] &= q^{-1}(z^2y - yx^2), \\ [xyz, z] &= q^{-1}(x^2 - y^2)z, \\ xy^2 - z^2x &= q^{-4}(y^2x - xz^2). \end{aligned}$$

Since $[xyz, x] = q^{-1}x(y^2 - z^2)$, it follows that

$$\begin{aligned} [C, x] &= \left(q^{-2} - q^2\right)[xyz, x] + q^{-3}[y^2, x] + q[z^2, x] \\ &= \left(q^{-3} - q\right)(xy^2 - xz^2) + q^{-3}(y^2x - xy^2) + q(z^2x - xz^2) \\ &= q(z^2x - xy^2) - q^{-3}(xz^2 - y^2x) \\ &= 0. \end{aligned}$$

A similar argument shows that $[C, y] = 0 = [C, z]$. Therefore, C is in the center of A. □

Remark 3.11. (1) Let σ be the automorphism of A defined by $\sigma(x) = y, \sigma(y) = z, \sigma(z) = x$. Then the element $C + \sigma(C) + \sigma^2(C)$, which equals

$$\left(q^{-2} - q^2\right)(x_1x_2x_3 + x_2x_3x_1 + x_3x_1x_2) + \left(2q + q^{-3}\right)\left(x_1^2 + x_2^2 + x_3^2\right),$$

also belongs to the center of A.

(2) Seemingly, there are three different cubic central elements in A:

$$C, \quad \sigma(C), \quad \sigma^2(C).$$

However, direct calculation in A shows that they are equal to each other. This is not a coincidence: in §5.6 we will show that the center of A equals $\Bbbk[C]$, the polynomial ring in C; thus $\sigma(C)$ is a scalar multiple of C. It is not difficult to see from the defining relations of A that

$$\begin{aligned}\sigma(C) &= \left(q^{-2}-q^{2}\right) yzx + qy^2 + q^{-3}z^2 + qx^2 \\ &= \left(q^{-2}-q^{2}\right) xyz + \text{lower degree monomials.}\end{aligned}$$

Since A has a PBW basis of $\left\{x^i y^j z^k \mid (i,j,k) \in \mathbb{N}^3\right\}$, it is necessary that $C = \sigma(C)$ and hence $\sigma^2(C) = \sigma(C) = C$.

Chapter 4

LINEAR MODULES

As the name suggests, linear modules are algebraic analogues of linear subspaces of affine and projective spaces.

The notion of a linear module over a connected graded $\Bbbk$-algebra was introduced by Artin-Tate-Van den Bergh in [2]. We recall their definition in §4.1. In §4.2 we introduce a new notion, that of a linear module over a filtered algebra. Linear modules over graded algebras are closely related to certain linear subspaces of projective spaces. Linear modules over filtered algebras are closely related to certain linear subspaces of affine spaces. These ideas will become clearer once we get to some examples.

We will be interested in linear modules over the algebra $A = A(q)$ introduced in §3.1. Of particular interest are those of GK-dimension one. We call them line modules. We will see that they correspond to certain lines in $\mathbb{A}^3$. Their importance for us is that every finite dimensional simple A-module is a quotient of a line module. If S is a finite dimensional simple A-module, there is an exact sequence $0 \to M(\ell') \to M(\ell) \to S \to 0$ in which $M(\ell')$ and $M(\ell)$ are the line modules corresponding to certain lines ℓ' and ℓ in $\mathbb{A}^3_{x,y,z}$. Thus, one way to hunt for such simple modules S is to classify the line modules (this is a relatively simple task) and then search for homomorphisms between line modules (this is a more difficult task). We will eventually do this. Because linear A-modules correspond to certain lines in $\mathbb{A}^3$ we are able to use geometric ideas as an organizing principle. For example, associated to the degree-3 central element $C \in A$ (degree is defined in terms of "the" filtration on A) there is a 1-parameter family of cubic surfaces $X_\mu \subseteq \mathbb{A}^3_{x,y,z}$, where $\mu \in \Bbbk$, such that every line module is annihilated by $C - \mu$ for some $\mu \in \Bbbk$ and the line in $\mathbb{A}^3_{x,y,z}$ corresponding to that line module lies on X_μ.

Linear modules over the enveloping algebra $U(\mathfrak{g})$ of a finite dimensional Lie algebra $\mathfrak{g}$, endowed with its standard filtration, are exactly the modules that are induced from 1-dimensional representations of subalgebras of $\mathfrak{g}$.[1] For example, if $\mathfrak{b}$ is a Borel subalgebra of a complex semisimple Lie algebra $\mathfrak{g}$, the Verma modules $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ are linear modules. Generalizations of Verma modules, often called Verma modules because of their similarity to the classical case just mentioned, have proved useful for algebras other than enveloping algebras. Most generalizations of Verma modules that we know of are linear modules.

Linear modules over graded algebras and filtered modules are closely related to one another. This relationship is like that between linear subspaces of the projective space $\mathbb{P}^r$ and the affine subspaces of the affine space $\mathbb{A}^r = \mathbb{P}^r - H$, the complement to a hyperplane $H \subseteq \mathbb{P}^r$. Given a filtered algebra $\mathcal{A}$ there is a graded algebra D having a degree-1 central regular element t such that $D[t^{-1}]_0 = \mathcal{A}$. The localization functor $\Psi^* : \mathsf{Gr}(D) \to \mathsf{Mod}(\mathcal{A})$ sends linear D-modules to linear $\mathcal{A}$-modules. We examine this relationship in §4.3.

4.1 Linear modules over graded algebras

Let D be a connected graded $\mathbb{k}$-algebra that is generated as a $\mathbb{k}$-algebra by its degree-one component, D_1. Suppose that $\dim_{\mathbb{k}}(D_1) < \infty$.

A graded left D-module L is said to be d-linear if $L = DL_0$ and

$$\dim_{\mathbb{k}}(L_n) \;=\; \binom{n+d}{n}$$

for all $n \geq 0$. For example, L is 0-linear if $L = DL_0$ and $\dim_{\mathbb{k}}(L_n) = 1$ for all $n \geq 0$ and 1-linear if $L = DL_0$ and $\dim_{\mathbb{k}}(L_n) = n + 1$ for all $n \geq 0$. One calls 0-linear modules point modules and 1-linear modules line modules.

The next result explains the terminology.

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[1] Prove this, cf., Le Bruyn-Van den Bergh [5] or Proposition 4.2.

The classical case. Let $D = \Bbbk[x_0, \dots, x_r]$ be the polynomial ring on $r+1$ variables. The standard grading on D is that given by the subspaces

$$D_n \;:=\; \operatorname{span}\{x_0^{i_0} \dots x_r^{i_r} \mid i_0 + \cdots + i_r = n\}.$$

The ring D is an r-linear D-module.

We view elements of D_1 as linear forms on the projective space

$$\mathbb{P}^r \;=\; \operatorname{Proj}(D) \;=\; \mathbb{P}(D_1^*)$$

whose points are the 1-dimensional subspaces of the vector space D_1^*. If V is an m-dimensional subspace of D_1 its vanishing locus is a linear subspace of $\mathbb{P}^r$ of co-dimension m. The quotient ring D/DV is then a polynomial ring on $r+1-m$ variables and $\operatorname{Proj}(D/DV)$ is that linear subspace of $\mathbb{P}^r$ (which has dimension $r-m$). The degree-n component of D/DV has dimension $\binom{n+r-m}{n}$ so as a graded D-module D/DV is an $(r-m)$-linear module.

Lemma 4.1. *Let $D = \Bbbk[x_0, \dots, x_r]$ be the polynomial ring on $r+1$ variables with its standard grading. The isomorphism classes of linear D-modules of dimension d are in bijection with the d-dimensional linear subspaces of $\mathbb{P}^r$. The d-linear module corresponding to a linear subspace $\ell \subseteq \mathbb{P}^r$ of dimension d is $D/D\ell^\perp$ where $\ell^\perp$ denotes the subspace of $D_1 = \Bbbk x_0 + \cdots + \Bbbk x_r$ vanishing on ℓ.*

Proof. Let L be a d-linear D-module. Thus $L = DL_0$ and $\dim_\Bbbk(L_n) = \binom{n+d}{n}$ for all $n \geq 0$. In particular, $\dim_\Bbbk(D_1L_0) = d+1$ so there is an $(r-d)$-dimensional subspace $V \subseteq D_1$ such that $VL_0 = 0$. Hence L is isomorphic to a quotient of D/DV. But D/DV is a polynomial ring on $d+1$ variables, say $D/DV \cong \Bbbk[z_0, \dots, z_d]$ where $\Bbbk z_0 + \cdots + \Bbbk z_d$ is the the image of D_1. The image of D_n in D/DV therefore has dimension $\binom{n+d}{n}$. Since this is the same as the dimension of L_n it follows that $L \cong D/DV$. The elements in D_1^* that vanish on V form a subspace, $V^\perp$, and the points in $\mathbb{P}^r = \mathbb{P}(D_1^*)$ that represent the 1-dimensional subspaces of $V^\perp$ form a linear subspace, ℓ say. Since $\ell^\perp = V$, $L \cong D/D\ell^\perp$. It is now easy to see there is a bijection as claimed. □

Thus, isomorphism classes of point modules are in natural bijection with the points in $\mathbb{P}^r$ and isomorphism classes of line modules are in natural bijection with the (projective) lines in $\mathbb{P}^r$, and so on. We often say, a little inaccurately, that point modules are in bijection with points in $\mathbb{P}^r$ and line modules are in bijection with the lines in $\mathbb{P}^r$. The bijection is simple and explicit. For example, if $p = (\lambda_0, \dots, \lambda_r)$ is a point in $\mathbb{P}^r$, the corresponding point module is

$$\frac{D}{(\lambda_j x_i - \lambda_i x_j \mid 0 \le i, j \le r)}.$$

If ℓ is the line $\{x - \alpha t = y - \beta z = 0\}$ in $\mathbb{P}^3 = \operatorname{Proj}(\Bbbk[x, y, z, t])$, the corresponding line module is

$$\frac{\Bbbk[x, y, z, t]}{(x - \alpha t, y - \beta z)}.$$

Suppose now that D is a quotient of the polynomial ring $\Bbbk[x_0, \dots, x_r]$ by an ideal generated by homogeneous elements and give D the inherited grading. It is not difficult to see that the point modules for D correspond to the points in $\operatorname{Proj}(D) \subseteq \mathbb{P}^r$ and that the line modules for D are in bijection with the lines in $\mathbb{P}^r$ that lie on $\operatorname{Proj}(D)$, and so on in higher dimensions. Thus, linear D-modules carry information about $\operatorname{Proj}(D)$. For example, if p and ℓ are, respectively, a point and a line on $\operatorname{Proj}(D)$, then $p \in \ell$ if and only if there is a surjective homomorphism $D/D\ell^\perp \to D/Dp^\perp$ between the corresponding linear modules. There is, of course, a corresponding map $\mathcal{O}_\ell \to \mathcal{O}_p$ between the corresponding $\mathcal{O}_{\operatorname{Proj}(D)}$-modules.

The non-commutative case Let D be a connected graded $\Bbbk$-algebra satisfying the assumptions at the beginning of §4.1. Suppose that $\dim_{\Bbbk}(D_1) = r + 1$, so $\mathbb{P}(D_1^*)$ is $\mathbb{P}^r$. Let L be a d-linear D-module. Since $\dim_{\Bbbk}(L_1) = d + 1$, L_0 is annihilated by an $(r - d)$-dimensional subspace, V say, of D_1. Since $L = DL_0$, L is isomorphic to a quotient of D/DV. In general, L will not be isomorphic to D/DV, though it is in many important situations, including the ones relevant to our investigation of A. Regardless of that, the vanishing locus of V is a linear subspace of $\mathbb{P}^r$. Thus, linear D-modules determine certain linear subspaces of $\mathbb{P}^r$.

In good cases, which include the ones relevant to our A, linear D-modules are in bijection with certain linear subspaces of $\mathbb{P}^r$. Only rarely will all subspaces of $\mathbb{P}^r$ occur in this bijection. In some sense, D only "sees" certain linear subspaces of $\mathbb{P}^r$. We will use the following terminology: if ℓ is a line in $\mathbb{P}^r$ such that $D/D\ell^{\perp}$ is a line module for D we call ℓ a D-line, and so on in other dimensions.

4.2 Linear modules over filtered algebras

For the rest of this chapter, $\mathcal{A}$ denotes a $\Bbbk$-algebra with a fixed filtration

$$\Bbbk \;=\; \mathcal{A}_0 \;\subseteq\; \mathcal{A}_1 \;\subseteq\; \cdots \;\subseteq\; \mathcal{A}_n \;\subseteq\; \cdots \;\subseteq\; \mathcal{A}$$

such that $\dim_{\Bbbk}(\mathcal{A}_1) < \infty$, $\mathcal{A}_n = (\mathcal{A}_1)^n$, and $\cup_{n=0}^{\infty}\mathcal{A}_n = \mathcal{A}$.

We call an $\mathcal{A}$-module M a a d-linear module, or a linear $\mathcal{A}$-module of dimension d, if there is an element $m \in M$ such that $M = \mathcal{A}m$ and

$$\dim_{\Bbbk}(\mathcal{A}_n m) \;=\; \binom{n+d}{n}$$

for all $n \geq 0$.

It is sometimes convenient to define $M_n = \mathcal{A}_n m$ and observe that these subspaces give M the structure of a filtered $\mathcal{A}$-module because $\mathcal{A}_i M_j \subseteq M_{i+j}$ for all $i, j \geq 0$.[2]

Thus, an $\mathcal{A}$-module M is a linear module of dimension 0 if and only if $\dim_{\Bbbk}(M) = 1$. An $\mathcal{A}$-module M is a linear module of dimension 1 if and only if there is an element $m \in M$ such that $M = \mathcal{A}m$ and $\dim_{\Bbbk}(\mathcal{A}_n m) = n + 1$ for all n. We call a linear module of dimension 1 a line module and a linear module of dimension 2 a plane module.

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[2] The definition of a linear $\mathcal{A}$-module depends on the dimensions of the subspaces in the ascending chain $M_0 \subseteq M_1 \subseteq \cdots$ whereas the definition of a linear D-module depends on the dimensions of the components in the decomposition $L_0 \oplus L_1 \oplus \cdots$. For example, the polynomial ring $\Bbbk[x, y]$ in two variables is a 1-linear module when viewed as a graded algebra and a 2-linear module when viewed as a filtered algebra. The numbers 1 and 2 correspond to the dimensions of the corresponding geometric objects, the projective line $\mathbb{P}^1$, and the affine plane $\mathbb{A}^2$, respectively.

Linear modules over the enveloping algebra $U(\mathfrak{g})$ of a finite dimensional Lie algebra $\mathfrak{g}$, endowed with its standard filtration, have been classified in [5]. We include its proof here for the reader's convenience.

Proposition 4.2 (Le Bruyn-Van den Bergh)**.** *[5, Proposition 2.3] Let $U = U(\mathfrak{g})$ be the enveloping algebra of a finite dimensional Lie algebra. Let $(U_i)_{i\geq 0}$ denote the standard filtration on $U(\mathfrak{g})$, namely $U_1 = \mathbb{C} + \mathfrak{g}$ and $U_i = (U_1)^i$ for $i \geq 1$. The d-linear $U(\mathfrak{g})$-modules are the modules $U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} \mathbb{C}_\lambda$ induced from the 1-dimensional representations of the co-dimension-d subalgebras $\mathfrak{h} \subseteq \mathfrak{g}$ such that $\lambda([\mathfrak{h}, \mathfrak{h}]) = 0$.*

Proof. First of all, if M is a d-linear $U(\mathfrak{g})$-module, then there exists some $m \in M$ such that $M = U.m$ and

$$\dim_{\mathbb{C}}(U_i.m) \;=\; \binom{i+d}{i} \quad \text{for } i \geq 0. \tag{4-1}$$

Denote $n := \dim_{\mathbb{C}}(\mathfrak{g})$.

Since $\dim_{\mathbb{C}}(U_1.m) = d+1$ and $\dim_{\mathbb{C}}(U_1) = n+1$, m is annihilated by an $(n-d)$-dimensional space, say $\mathfrak{h}'$, in U_1. In particular, M is a quotient of $U/U\mathfrak{h}'$.

Since U has a PBW-basis, it follows that $U_1\mathfrak{h}' \cap U_1 = [\mathfrak{h}', \mathfrak{h}'] + \mathfrak{h}'$. Furthermore, if V is the image of $U_1\mathfrak{h}'$ in U_2/U_1 then

$$\dim_{\mathbb{C}}(V) \;=\; \binom{n+1}{2} - \binom{d+1}{2}.$$

Hence,

$$\begin{aligned}\dim_{\mathbb{C}}(U_1\mathfrak{h}') &= \dim_{\mathbb{C}}(V) + \dim_{\mathbb{C}}(U_1\mathfrak{h}' \cap U_1) \\ &= \binom{n+1}{2} - \binom{d+1}{2} + \dim_{\mathbb{C}}([\mathfrak{h}', \mathfrak{h}'] + \mathfrak{h}').\end{aligned} \tag{4-2}$$

In addition, since $\dim_{\mathbb{C}}(U_2.m) = \binom{d+2}{2}$ and $U_1\mathfrak{h}'.M = 0$, we obtain

$$\dim_{\mathbb{C}}(U_1\mathfrak{h}') + \binom{d+2}{2} \;\leq\; \binom{n+2}{2}. \tag{4-3}$$

It then follows from (4-2) and (4-3) that

$$\begin{aligned}\dim_{\mathbb{C}}([\mathfrak{h}',\mathfrak{h}']+\mathfrak{h}') &\leq -\binom{n+1}{2}+\binom{d+1}{2}-\binom{d+2}{2}+\binom{n+2}{2}\\ &= n-d\\ &= \dim_{\mathbb{C}}(\mathfrak{h}').\end{aligned}$$

Hence, $[\mathfrak{h}',\mathfrak{h}'] \subset \mathfrak{h}'$ and $\mathfrak{h}'$ is a Lie subalgebra of U_1.

Let $\mathfrak{h}$ be the image of $\mathfrak{h}'$ in $\mathfrak{g}$ under the canonical decomposition $U_1 = \mathbb{C}\oplus\mathfrak{g}$. Since $1 \notin \mathfrak{h}'$, there will be a linear map $\lambda \in \mathfrak{h}^*$ such that

$$\mathfrak{h}' \;=\; \{u-\lambda(u) \mid u \in \mathfrak{h}\}.$$

Then $\lambda(\mathfrak{h}') = 0$. It follows that there is an isomorphism $U/U\mathfrak{h}' \;\cong\; U \otimes_{U(\mathfrak{h})} \mathbb{C}_\lambda$ and hence a surjective map:

$$U \otimes_{U(\mathfrak{h})} \mathbb{C}_\lambda \;\longrightarrow\; M.$$

By (4-1), this map must be a filtered isomorphism.

Lastly, if $u, v \in \mathfrak{h}$, then $[u,v] = [u-\lambda(u), v-\lambda(v)] \in \mathfrak{h}'$. Hence, $\lambda([u,v]) = 0$ and therefore $\lambda([\mathfrak{h},\mathfrak{h}]) = 0$. $\square$

4.3 The Rees ring construction

Let $\mathcal{A}[t]$ be the polynomial ring over $\mathcal{A}$ in a central indeterminate t.

The rings $\mathcal{A}[t] = \mathcal{A} \otimes_{\Bbbk} \Bbbk[t]$ and $\mathcal{A}[t^{\pm 1}] = \mathcal{A} \otimes_{\Bbbk} \Bbbk[t, t^{-1}]$ are made into graded rings by placing $\mathcal{A}$ in degree zero and t in degree one. The space

$$D \;:=\; \mathcal{A}_0 \oplus \mathcal{A}_1 t \oplus \mathcal{A}_2 t^2 \oplus \cdots$$

is a graded subalgebra of $\mathcal{A}[t]$ with homogeneous components

$$D_n \;=\; \mathcal{A}_n t^n.$$

It is called the **Rees ring** of $\mathcal{A}$ with respect to the filtration $\mathcal{A}_n$.

The basic properties of D are summarized in the following two well-known propositions.

Proposition 4.3. *The ring D is a connected graded $\Bbbk$-algebra generated as a $\Bbbk$-algebra by the finite dimensional subspace D_1. Furthermore, the element $t = 1t \in \mathcal{A}_1 t = D_1$ is a central regular element and $D[t^{-1}]_0 = \mathcal{A}$.*

Proposition 4.4. *Let D be a connected graded $\Bbbk$-algebra. Suppose further that D_1 has finite dimension, generates D as a $\Bbbk$-algebra and contains a central regular element t. Let $\mathcal{A} = D[t^{-1}]_0$ and define $\mathcal{A}_n = D_n t^{-n}$. Then $\mathcal{A}_0 = \Bbbk$, $\dim_{\Bbbk}(\mathcal{A}_1) < \infty$, $\mathcal{A}_n = (\mathcal{A}_1)^n$, $\mathcal{A} = \cup_{n=0}^{\infty} \mathcal{A}_n$, and D is the Rees ring of $\mathcal{A}$ with respect to the filtration $\mathcal{A}_n$.*

We also note the following well-known and important fact; we leave its proof to the reader.

Proposition 4.5. *There is an algebra isomorphism $\mathcal{A} \cong D/(t-1)$ and $D/(t)$ is isomorphic to the associated graded ring*

$$\mathrm{gr}(\mathcal{A}) \;=\; \bigoplus_{n=0}^{\infty} \frac{\mathcal{A}_{n+1}}{\mathcal{A}_n}.$$

The following result is also well-known. A proof can be found at [12, Section 3.5].

Proposition 4.6. *Let D be a positively graded $\Bbbk$-algebra and z a homogeneous central regular element of positive degree. Then D is left noetherian if and only $D/(z)$ is.*

Proposition 4.7. *The algebra $\mathcal{A}$ is left noetherian if and only if D is. Furthermore, $\mathcal{A}$ is a domain if and only if D is.*

Proof. Since D is a domain so is $D[t^{-1}]$. Hence $\mathcal{A}$ is a domain.

Since D is left and right noetherian so is its quotient ring $D/(t-1)$. But this quotient is isomorphic to $\mathcal{A}$, so $\mathcal{A}$ is a left and right noetherian. □

Corollary 4.8. *Suppose $\mathcal{A}$ has the properties stated at the beginning of §4.2. If the associated graded ring* $\mathrm{gr}(\mathcal{A})$ *is left noetherian so is $\mathcal{A}$.*

The algebra $A(q)$ defined in Section 3.1 is a filtered algebra endowed with the filtration $A_1 = \mathrm{span}\{1, x_1, x_2, x_3\}$ and $A_n = (A_1)^n$. Later in Section 5.1 we will prove the Rees ring D associated with this filtration is a noetherian domain. So is $A(q)$ by Proposition 4.7.

There is another way to prove that $A = A(q)$ is left noetherian but it depends on the fact that the categories $\mathsf{Gr}(D[t^{-1}])$ and $\mathsf{Mod}(A)$ are equivalent (we discuss this equivalence in §4.4 below). Under this equivalence the left modules $D[t^{-1}]$ and A correspond to one another. Since $D[t^{-1}]$ is a noetherian object in $\mathsf{Gr}(D[t^{-1}])$, A is a noetherian object in $\mathsf{Mod}(A)$; i.e., A is left noetherian. Since $A(q)$ is isomorphic to the opposite algebra of $A(q^{-1})$ (Proposition 3.4), it is also right noetherian.

4.4 Strongly graded rings

A $\mathbb{Z}$-graded ring R is strongly graded if $R_iR_{-i} = R_0$ for all $i \in \mathbb{Z}$. Clearly, a graded ring having a unit of degree one is strongly graded. In particular, $D[t^{-1}]$ is strongly graded.

A fundamental theorem of E. Dade [7, Theorem 2.8] says that if R is strongly graded, then there is an equivalence of categories

$$\mathsf{Gr}(R) \;\equiv\; \mathsf{Mod}(R_0).$$

The equivalence sends a graded R module to its degree-0 component and its quasi-inverse sends an R_0-module N to $R \otimes_{R_0} N$ which is made into a graded R-module by declaring that its degree-i component is $R_i \otimes_{R_0} N$ and an element $r \in R_n$ acts on an element $a \otimes n$ by

$$r \cdot (a \otimes n) \;=\; ra \otimes n.$$

Since $D[t^{-1}]_0 = \mathcal{A}$ and $D[t^{-1}]$ is strongly graded, there is an equivalence of categories

$$\mathsf{Gr}(D[t^{-1}]) \;\equiv\; \mathsf{Mod}(D[t^{-1}]_0) \;=\; \mathsf{Mod}(\mathcal{A}).$$

The equivalence sends a graded $D[t^{-1}]$ module to its degree-0 component and its quasi-inverse sends a $D[t^{-1}]_0$-module N to $N \otimes_{\Bbbk} \Bbbk[t^{\pm 1}]$ which is made into a graded $D[t^{-1}]$-module by declaring that its degree-i component is $N \otimes t^i$ and an element $d \in D_n$ acts on an element $m \otimes t^i$ by

$$d \cdot (m \otimes t^i) \;=\; (dt^{-n})m \otimes t^{n+i}.$$

Composing this equivalence with the localization functor

$$\mathsf{Gr}(D) \to \mathsf{Gr}(D[t^{-1}]), \qquad M \rightsquigarrow M[t^{-1}],$$

gives a functor $\Psi^* : \mathsf{Gr}(D) \to \mathsf{Mod}(\mathcal{A})$.

4.5 The functor $\Psi^* : \mathsf{Gr}(D) \to \mathsf{Mod}(\mathcal{A})$

Define functors

$$\Psi^* : \mathsf{Gr}(D) \to \mathsf{Mod}(\mathcal{A}) \qquad \text{and} \qquad \Psi_* : \mathsf{Mod}(\mathcal{A}) \to \mathsf{Gr}(D)$$

by

$$\Psi^* M \;=\; M[t^{-1}]_0 \qquad \text{and} \qquad \Psi_* N \;=\; N \otimes_{\Bbbk} \Bbbk[t, t^{-1}],$$

where the grading on $\Psi_* N$ is given by $(\Psi_* N)_i = N \otimes t^i$. The actions of Ψ^* and Ψ_* on morphisms are the obvious ones.

It is clear that both Ψ^* and Ψ_* are exact functors.

It is clear that $\Psi^* \Psi_* \cong \mathsf{id}_{\mathsf{Mod}(\mathcal{A})}$.

Proposition 4.9. *The functor* Ψ^* *is left adjoint to* Ψ_*.

Proof. Fix $M \in \mathsf{Gr}(D)$ and $N \in \mathsf{Mod}(\mathcal{A})$. We will now define maps

$$\Phi : \mathrm{Hom}_{\mathcal{A}}(\Psi^* M, N) \longrightarrow \mathrm{Hom}_{\mathsf{Gr}(D)}(M, \Psi_* N)$$

and

$$\Phi' : \mathrm{Hom}_{\mathsf{Gr}(D)}(M, \Psi_* N) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(\Psi^* M, N).$$

Let $f : \Psi^* M \to N$ be an $\mathcal{A}$-module homomorphism. Let $m \in M_i$, whence $mt^{-i} \in M[t^{-1}]_0 = \Psi^* M$ and $f(mt^{-i}) \in N$. We define

$$\Phi(f)(m) \;=\; f(mt^{-i}) \otimes t^i.$$

Let $g : M \to \Psi_* N$ be a degree-preserving D-module homomorphism. Let $m \in M_i$, whence $mt^{-i} \in M[t^{-1}]_0 = \Psi^* M$ and $g(m) \in (\Psi_* N)_i = N \otimes t^i$. We define

$$\Phi'(g)(mt^{-i}) \;=\; g(m)t^{-i}.$$

The element $g(m)t^{-i}$ is in $N \otimes 1 \subseteq N \otimes_{\Bbbk} \Bbbk[t^{\pm 1}]$; we identify $N \otimes 1$ with N and so view $g(m)t^{-i}$ as an element in N.

To see that the map $\Phi'(g)$ is well-defined, suppose $mt^{-i} = m't^{-j}$ for some $m' \in M_j$. Since $mt^j - m't^i$ is in M and is zero in $M[t^{-1}]_0$, there is an equality $(mt^j - m't^i)t^k = 0$ in M for some $k \geq 0$. Hence $t^{j+k}g(m) = t^{i+k}g(m')$; this is an equality in $N \otimes_{\Bbbk} \Bbbk[t^{\pm 1}]$ on which t acts bijectively. Hence $g(m)t^{-i} = g(m')t^{-j}$; i.e., $\Phi(g')(mt^{-i}) = \Phi(g')(m't^{-j})$. Thus $\Phi'(g)$ is well-defined.

Keep the notation from the previous paragraph. We have

$$\begin{aligned}(\Phi'\Phi)(f)(mt^{-i}) &= \Phi(f)(m)t^{-i}\\ &= \big(f(mt^{-i})\otimes t^{i}\big)t^{-i}\\ &= f(mt^{-i})\otimes 1\\ &= f(mt^{-i}) \qquad \text{(using the identification } N\otimes 1 = N)\end{aligned}$$

whence $(\Phi'\Phi)(f) = f$. On the other hand, if $m \in M_i$, then

$$\begin{aligned}(\Phi\Phi')(g)(m) &= \Phi'(g)(mt^{-i})\otimes t^{i}\\ &= g(m)t^{-i}\otimes t^{i};\end{aligned}$$

since $g(m) \in N \otimes t^i$, it is equal to $n \otimes t^i$ for some $n \in N$; hence $g(m)t^{-i} = n \otimes 1$; under the identification of $N \otimes 1$ with N, $g(m)t^{-i}$ is identified with n; hence $g(m)t^{-i} \otimes t^i$ is equal to $n \otimes t^i$ which is $g(m)$; i.e., $g(m)t^{-i} \otimes t^i = g(m)$ whence $(\Phi\Phi')(g) = g$.

Since $\Phi'\Phi$ and $\Phi\Phi'$ are identity maps, Ψ^* is left adjoint to Ψ_*. □

Because Ψ^* is left adjoint to Ψ_*, there are natural transformations

$$\eta : \mathsf{id}_{\mathsf{Gr}(D)} \longrightarrow \Psi_*\Psi^*$$

and

$$\varepsilon : \Psi^*\Psi_* \longrightarrow \mathsf{id}_{\mathsf{Mod}(\mathcal{A})}.$$

We have already remarked that the co-unit ε is an isomorphism of functors. On a graded D-module M, $\eta_M : M \to M[t^{-1}]_0 \otimes_{\Bbbk} \Bbbk[t^{\pm 1}]$ is the map

$$\eta(m) = mt^{-i} \otimes t^i$$

for $m \in M_i$.

The effect of Ψ_* on finite dimensional modules. We want to make use of the fact that $\mathcal{A}$ and D are noetherian rings. However, the functor Ψ_* does not send noetherian $\mathcal{A}$-modules to noetherian D-modules: if N is a non-zero $\mathcal{A}$-module, then $\Psi_* N$ is not a finitely generated D-module. Nevertheless, there is a strong connection between finite dimensional $\mathcal{A}$-modules and noetherian D-modules thanks to the next result.

Lemma 4.10. *If N is a finite dimensional $\mathcal{A}$-module, then $(\Psi_* N)_{\geq n}$ is a finitely generated D-module for all $n \in \mathbb{Z}$.*

Proof. It is clear that $t(N \otimes t^i) = N \otimes t^{i+1}$. Since $t \in D_1$, $D_1 \cdot (N \otimes t^i) = N \otimes t^{i+1}$ for all i. Hence $(\Psi_* N)_{\geq n}$ is generated by $N \otimes t^n$. The result therefore follows from the assumption that N is finite dimensional. □

D-modules that are t-torsion-free. Let $\mathsf{Gr}_t(D)$ denote the full subcategory of $\mathsf{Gr}(D)$ consisting of the graded D-modules having no t-torsion. The functor $\Psi_* : \mathsf{Mod}(\mathcal{A}) \to \mathsf{Gr}(D)$ takes values in $\mathsf{Gr}_t(D)$ because t acts bijectively on $N \otimes_{\mathbb{k}} [t^{\pm 1}]$.

Because t acts injectively on $\Psi_* N$, if N is any $\mathcal{A}$-module then the only finite dimensional graded D-submodule of $\Psi_* N$ is the zero submodule.

If $M \in \mathsf{Gr}_t(D)$, then the map $\eta_M : M \to \Psi_* \Psi^* M$, $M_i \ni m \mapsto mt^{-i} \otimes t^i$ is injective.

4.6 Linear modules over $\mathcal{A}$ and D

We continue to assume that $\mathcal{A}$ is a $\mathbb{k}$-algebra with a fixed filtration $\mathbb{k} = \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \cdots$ such that $\dim_{\mathbb{k}}(\mathcal{A}_1) < \infty$, $\mathcal{A}_n = (\mathcal{A}_1)^n$, and $\cup_{n=0}^{\infty} \mathcal{A}_n = \mathcal{A}$. We also continue to assume that D is the Rees ring associated to $\mathcal{A}$ with respect to the filtration $\mathcal{A}_n$.

Proposition 4.11. *If L is a t-torsion-free d-linear D-module, then $M = \Psi^* L = L[t^{-1}]_0$ is a d-linear $\mathcal{A}$-module with associated filtration $M_n = L_n t^{-n}$.*

Proof. Let L be a d-linear D-module and let $M = \Psi^* L$. In particular, $M_0 = L_0$. Since $L_n = D_n L_0$, $M_n = D_n t^{-n} L_0 = \mathcal{A}_n L_0$. Since L has no t-torsion, the multiplication map

$L_n \mapsto L_n t^{-n}$, $v \mapsto vt^{-n}$ is injective. It is also surjective so $\dim_{\Bbbk}(\mathcal{A}_n M_0) = \dim_{\Bbbk}(L_n)$. Hence M is d-linear. □

If M is a left $\mathcal{A}$-module we write $M[t]$ for $M \otimes_{\Bbbk} \Bbbk[t]$ and view this as a graded $\mathcal{A}[t]$-module by declaring its degree n-component to be $M \otimes t^n$. Since D is a graded subalgebra of $\mathcal{A}[t]$, $M[t]$ is also a graded D-module. Whenever we talk about $M[t]$ as a D-module we always view it as a graded D-module with this grading. It is easy to see that

$$\bigoplus_{n=0}^{\infty} M_n t^n$$

is a graded D-submodule of $M[t]$.

Proposition 4.12. *Let M be a d-linear $\mathcal{A}$-module with respect to the filtration $(M_n)_{n\geq 0}$. Then*

$$L \;:=\; \bigoplus_{n=0}^{\infty} M_n t^n$$

is a t-torsion-free d-linear D-module and $\Psi^* L = M$.

Proof. Because $M_n = \mathcal{A}_n M_0$, $L_n = M_n t^n = \mathcal{A}_n t^n M_0 = D_n L_0$. Since $L_n = M_n t^n$, $\dim_{\Bbbk}(L_n) = \dim_{\Bbbk}(M_n)$. Since M is d-linear so is L.

Since $\Bbbk[t]$ is t-torsion-free so are $M[t]$ and its submodule L. Lastly, $L[t^{-1}]_0 = \sum_{n=0}^{\infty} L_n t^{-n} = \sum_{n=0}^{\infty} M_n = M$; i.e., $\Psi^* L = M$. □

Remark 4.13. If M is a linear module for $\mathcal{A}$, then $\Psi_* M$ is *not* a linear module for D because, for example, every homogeneous component of $\Psi_* M$ has infinite dimension. Nevertheless, if L is a linear D-module such that $M \cong \Psi^* L$, then $\Psi_* M \cong \Psi_* \Psi^* L$ so there is a graded D-module homomorphism

$$\eta_L : L \;\longrightarrow\; \Psi_* \Psi^* L \;\cong\; \Psi_* M.$$

4.7 Results on 4-dimensional regular algebras

In this section we assume that D is a 4-dimensional Artin-Schelter regular algebra that is generated by its degree one component and has the same Hilbert series as the polynomial ring on 4 variables with its standard grading.

A nonzero finitely generated $\mathbb{Z}$-graded D-module M is called $\mathsf{Cohen\text{-}Macaulay}$ if $\mathrm{pd}(M) = j(M)$, i.e., $\mathrm{Ext}^i_D(M,D) = 0$ if $i \neq j(M)$, where $\mathrm{pd}(M)$ is the projective dimension of M and

$$j(M) \ := \ \inf\{i \mid \mathrm{Ext}^i_D(M,D) \neq 0\}.$$

The following results appear in Levasseur and Smith's paper [13] on the 4-dimensional Sklyanin algebras.

Proposition 4.14. *[13, Proposition 2.12] If M is a Cohen-Macaulay module for D having GK-dimension 2 and multiplicity 1, then M is a shift of a line module.*

Proposition 4.15. *[13, Lemma 1.12] Let $0 \to M' \to M \to N \to 0$ be an exact sequence in $\mathsf{Gr}(D)$. Suppose that $j(N) = 1 + j(M)$.*

1. *If M and N are Cohen-Macaulay modules, so is M' and $j(M') = j(M)$.*

2. *If M and M' are Cohen-Macaulay modules, so is N.*

4.8 The passage from the line modules for D to those for $\mathcal{A}$

Let D be as in §4.7 having a degree-one central regular element t and let $\mathcal{A} = D[t^{-1}]_0$. Define $\mathcal{A}_n = D_n t^{-n}$. Because D is generated by D_1 as a $\Bbbk$-algebra, there is an ascending chain of finite dimensional subspaces $\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots$, whose union is $\mathcal{A}$ and $\mathcal{A}_i\mathcal{A}_j \subseteq \mathcal{A}_{i+j}$ for all $i, j \geq 0$.

Theorem 4.16. *An $\mathcal{A}$-module M is a line module if and only if there is a line module L for D such that $L[t^{-1}] \neq 0$ and $L[t^{-1}]_0 \cong M$.*

Proof. This follows from Proposition 4.11 and Proposition 4.12. □

Proposition 4.17. *Every non-zero submodule of a line module for $\mathcal{A}$ is a line module for $\mathcal{A}$.*

Proof. Let M be a line module for $\mathcal{A}$ and N a proper submodule of M. Let $V = M/N$. There is an exact sequence

$$0 \longrightarrow \Psi_* N \longrightarrow \Psi_* M \longrightarrow \Psi_* V \longrightarrow 0$$

in $\mathsf{Gr}(D)$. Let L be a line module for D such that $M \cong \Psi^* L$. Consider the diagram

$$\begin{tikzcd}
 & & L \arrow[d, "\eta"] & & \\
0 \arrow[r] & \Psi_* N \arrow[r] & \Psi_* M \arrow[r] & \Psi_* V \arrow[r] & 0
\end{tikzcd}$$

Let F denote the image and K the kernel of the composition $L \to \Psi_* M \to \Psi_* V$. There is an exact sequence

$$0 \longrightarrow K \longrightarrow L \longrightarrow F \longrightarrow 0.$$

Since F is a submodule of $\Psi_* V$ which has constant Hilbert series, $\mathrm{GKdim}(F) \leq 1$. Since $\Psi_* V$ is t-torsion-free so is F. Hence F has no finite dimensional submodule. Hence $\mathrm{GKdim}(F) = 1$ and, by [13, Proposition 2.1], F is a Cohen-Macaulay D-module of projective dimension 3. By Proposition 4.15, K is a Cohen-Macaulay D-module having GK-dimension 2. Because the multiplicity of L is 1, K also has multiplicity 1. By Proposition 4.14, K is a shift of a line module.

The vertical arrows in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d] & L \arrow[r] \arrow[d, "\eta"] & F \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Psi_* N \arrow[r] & \Psi_* M \arrow[r] & \Psi_* V \arrow[r] & 0
\end{tikzcd}$$

are injective so the vertical arrows in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Psi^*K \arrow[r] \arrow[d] & \Psi^*L \arrow[r] \arrow[d, "\eta^*"] & \Psi^*F \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Psi^*\Psi_*N \arrow[r] \arrow[d, equal] & \Psi^*\Psi_*M \arrow[r] \arrow[d, equal] & \Psi^*\Psi_*V \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r] & M \arrow[r] & V \arrow[r] & 0
\end{tikzcd}$$

are also injective. But the middle vertical map is an isomorphism so the left vertical map must be isomorphism by the Snake Lemma. Therefore, $N = \Psi^*K$ is a line module for $\mathcal{A}$. $\square$

4.9 Comparison with algebraic geometry

The functors Ψ^* and Ψ_* are "geometric".

Let $\Bbbk[x_0, \ldots, x_r]$ be the polynomial ring endowed with its standard grading and let D be a quotient of it by a homogeneous ideal. Let t denote the image of x_0 in D and assume that t is a regular element in D. Let

$$A \;=\; D[t^{-1}]_0.$$

Consider the projective scheme

$$X \;=\; \operatorname{Proj}(D) \;\subseteq\; \mathbb{P}^r$$

and the affine scheme

$$X^\circ \;=\; \operatorname{Spec}(A) \;=\; X - \{t = 0\} \;\subseteq \mathbb{A}^r \;=\; \mathbb{P}^r - \{x_0 = 0\}.$$

Thus X° is the open complement in X to the intersection of X with the hyperplane $\{t = 0\}$.

If ℓ is a line in $\mathbb{P}^r$ lying on X, then $\ell^\circ := \ell - \{t = 0\}$ is an affine line in $\mathbb{A}^r$ lying on X°. The closure in $\mathbb{P}^r$ of ℓ° is ℓ. On the other hand, if ℓ' is an affine line in X°, then the closure in $\mathbb{P}^r$ of ℓ' is a projective line whose intersection with X° is ℓ'. Thus, intersection and closure sets up a bijection between projective lines in X that do not lie on $\{t = 0\}$ and affine lines lying on X°. This bijection is not particular to lines: for every integer d, intersection and closure provide a bijection

$$\left\{\begin{matrix}\text{projective } d\text{-linear subspaces of } X \\ \text{that do not lie on } \{t = 0\}\end{matrix}\right\} \longleftrightarrow \left\{\text{affine } d\text{-linear subspaces of } X^\circ\right\}.$$

The inclusion $j : X^\circ \to X$ is a morphism of schemes. Associated to j is the inverse image functor j^* and the direct image functor j_*,

$$\begin{tikzcd} \mathrm{Qcoh}(X^\circ) \arrow[r, bend right, "j_*"'] & \mathrm{Qcoh}(X). \arrow[l, bend right, "j^*"'] \end{tikzcd}$$

The functor j^* is left adjoint to j_*. We write $j^* \dashv j_*$ to denote this fact.

The functors Ψ^* and Ψ_*, $\Psi^* \dashv \Psi_*$, which exist even when A and D are not commutative, are algebraic analogues of j^* and j_*. This becomes clearer when we write the categories of quasi-coherent sheaves in terms of module categories. When A and D are as above, i.e., commutative, the above diagram becomes

$$\begin{tikzcd} \mathrm{Qcoh}(X^\circ) \arrow[r, equal] & \mathsf{Mod}(A) \arrow[r, bend right, "j_*"'] & \mathsf{QGr}(D) \arrow[l, bend right, "j^*"'] \arrow[r, equal] & \mathrm{Qcoh}(X). \end{tikzcd}$$

By definition, $\mathsf{QGr}(D)$ is a quotient category of $\mathsf{Gr}(D)$ and the quotient functor $\pi^* : \mathsf{Gr}(D) \to \mathsf{QGr}(A)$ has a right adjoint π_*. Thus, there are diagrams

$$\begin{tikzcd} \mathsf{Mod}(A) \arrow[r, bend right, "j_*"'] & \mathsf{QGr}(D) \arrow[l, bend right, "j^*"'] \arrow[r, bend right, "\pi_*"'] & \mathsf{Gr}(D) \arrow[l, bend right, "\pi^*"'] \end{tikzcd} \tag{4-4}$$

and

$$\begin{tikzcd} \mathsf{Mod}(A) \arrow[r, bend right, "\pi_* j_*"'] & \mathsf{Gr}(D). \arrow[l, bend right, "j^*\pi^*"'] \end{tikzcd} \tag{4-5}$$

Of course, $j^*\pi^* \dashv \pi_* j_*$. In fact,

$$\Psi^* = j^*\pi^*$$

and

$$\Psi_* = \pi_* j_*.$$

Thus, even when D and A are not assumed to be commutative, but satisfy the hypotheses earlier in this chapter, and are related by the fact that D is the Rees ring for the filtration $(A_n)_{\geq n}$, the functors Ψ^* and Ψ_* provide an adjoint pair fitting into diagrams that are analogues of (4-5) and (4-4).

Chapter 5

THE GRADED ALGEBRAS $D(q)$

Let $q \in \Bbbk - \{0\}$. We now introduce, then study, a graded algebra $D(q)$ having a central regular element $t \in D(q)_1$ such that $A(q) \cong D(q)[t^{-1}]_0$. The algebra $D(q)$ is the Rees ring associated to the standard filtration on $A(q)$, i.e., the filtration whose n^{th} term is the linear span of the words in x_1, x_2, x_3 of length $\leq n$.

We classify the point modules and line modules for $D(q)$ using the methodology and results in [6]. By Theorem 4.16, there is a natural bijection between line modules for $A(q)$ and those for $D(q)$ that are t-torsion-free, based on which we then classify all line modules for $A(q)$.

The center of $D(q)$ is a polynomial ring in two central elements. Based on the center of $D(q)$, we calculate the center of $A(q)$.

From §5.5 onwards, we assume q is not a root of unity.

5.1 The definition of $D(q)$

Let $D(q)$ denote the free algebra $\Bbbk\langle x_1, x_2, x_3, t\rangle$ modulo the six relations

$$\begin{aligned} [t, x_1] \;=\; [t, x_2] \;=\; [t, x_3] \;&=\; 0, \\ q x_i x_j - q^{-1} x_j x_i \;&=\; \left(q - q^{-1}\right) x_k t, \end{aligned}$$

where (i, j, k) runs over the cyclic permutations of $(1, 2, 3)$.

Since q is fixed we will usually write D for $D(q)$.

Proposition 5.1. *Let* $\delta = (q - q^{-1})^{-1}$. *There is an algebra isomorphism*

$$\Phi : A(q) \longrightarrow D(q)[t^{-1}]_0, \qquad \Phi(x_i) \; = \; \delta x_i t^{-1} \quad \textit{for } i = 1, 2, 3.$$

Proof. The algebra $D[t^{-1}]_0$ is generated by $x_i t^{-1}$, $i = 1, 2, 3$. It follows from the relations for D that $D[t^{-1}]_0$ is generated by these three elements modulo the three relations

$$\left(q x_i x_j - q^{-1} x_j x_i\right) t^{-2} \; = \; \left(q - q^{-1}\right) x_k t t^{-2}$$

as (i, j, k) runs over the cyclic permutations of $(1, 2, 3)$. This relation can be rewritten as

$$q \delta x_i t^{-1} \delta x_j t^{-1} \; - \; q^{-1} \delta x_j t^{-1} \delta x_i t^{-1} \; = \; \delta x_k t^{-1},$$

i.e., as $q\Phi(x_i)\Phi(x_j) - q^{-1}\Phi(x_j)\Phi(x_i) = \Phi(x_k)$. It follows at once that Φ extends to an algebra isomorphism. □

We also note that $D/(t-1) \cong A$ although we will not use this fact in this chapter.

Proposition 5.2. *The algebra* $D(q)$ *is the Rees ring for* $A(q)$ *endowed with the filtration* $A_1 = \operatorname{span}\{1, x_1, x_2, x_3\}$ *and* $A_n = (A_1)^n$.

5.2 Quadratic algebras and central extensions

If V is a finite dimensional vector space and R a subspace of $V^{\otimes 2}$ we call $T(V)/(R)$, the quotient of the tensor algebra by the ideal generated by R, a quadratic algebra. Its quadratic dual is the algebra $T(V^*)/(R^{\perp})$ where $R^{\perp}$ is the subspace of $(V^*)^{\otimes 2}$ consisting of the linear forms that vanish on R.

Let $B = T(V)/(R)$ be a quadratic algebra and let $\Bbbk$ denote its *trivial module* $B/B_{\geq 1}$. There is a canonical $\Bbbk$-algebra homomorphism

$$B^! \; \longrightarrow \; \operatorname{Ext}^*_B(\Bbbk, \Bbbk),$$

where $B^!$ denotes the quadratic dual of B and the right-hand side is the Yoneda Ext algebra (Note that $B^{!!} = B$). If this homomorphism is an isomorphism we call B a **Koszul algebra**. See S. Priddy's paper [14]. In this case, $B^!$ is also a Koszul algebra and the Hilbert series of B and $B^!$ are related by the formula

$$H_B(t)H_{B^!}(-t) \;=\; 1.$$

Theorem 5.3. *[6, Theorem 2.6, Corollary 2.7] Let D be a finitely generated quadratic algebra with the following properties:*

1. *there is a 1-regular element $t \in D_1$ and $\phi \in \mathrm{Aut}(D)$ such that $td = \phi(d)t$ for all $d \in D$;*
2. *$D/(t)$ is a 3-dimensional Artin-Schelter regular with Hilbert series $H_{D/(t)}(t) = (1-t)^{-3}$;*
3. *$H_{D^!}(t) = (1+t)H_{(D/(t))^!}(t)$.*

Then D is a noetherian domain with Hilbert series $H_D(t) = (1-t)^{-4}$ and D is an Auslander-regular, Koszul algebra with the Cohen-Macaulay property.

A graded $\Bbbk$-algebra D having a central regular element t of degree 1 will be called a **central extension** of $D/(t)$ if $D/(t)$ is a 3-dimensional Artin-Schelter regular algebra. The terminology comes from the paper [6] by Le Bruyn-Smith-Van den Bergh.

Proposition 5.4. *The algebra D in §5.1 is a central extension of the 3-dimensional regular algebra*

$$S \;=\; \frac{\Bbbk\langle x_1, x_2, x_3\rangle}{(qx_ix_j - q^{-1}x_jx_i \mid (i,j) = (1,2),(2,3),(3,1))}$$

and is therefore a noetherian domain with Hilbert series $H_D(t) = (1-t)^{-4}$ and an Auslander-regular, Koszul algebra with the Cohen-Macaulay property.

Proof. We check that D satisfies the three properties in Theorem 5.3.

Certainly, the element t has property (1).

Clearly $S = D/(t)$. It is well-known that S is a 3-dimensional Artin-Schelter regular algebra with Hilbert series $(1-t)^{-3}$ and is therefore a Koszul algebra. In addition,

$$H_{S^!}(t) \;=\; H_S(-t)^{-1} \;=\; (1+t)^3.$$

Write $x_4 := t$. Then D is also a quadratic algebra $D = T(V)/(R)$ where $V = \oplus_{i=1}^{4} \Bbbk x_i$ and $\dim_{\Bbbk}(R) = 6$. It follows that $\dim_{\Bbbk}(R^{\perp}) = 4^2 - 6 = 10$ and a basis for $R^{\perp}$ is given by the following 10 elements

$$\hat{x_s} \otimes \hat{x_s}, \qquad q^{-1}\hat{x_i} \otimes \hat{x_j} + q\hat{x_j} \otimes \hat{x_i}, \qquad \left(q - q^{-1}\right) \hat{x_i} \otimes \hat{x_j} + q\hat{x_k} \otimes \hat{x_4}$$

where $\hat{x_s} \in V^*$ is the dual basis of x_s for $s = 1, \dots, 4$ and (i,j,k) runs over all cyclic permutations of $(1,2,3)$. A direct calculation shows $H_{D^!}(t) = 1 + 4t + 6t^2 + 4t^3 + t^4 = (1+t)^4$ and therefore $H_{D^!}(t) = (1+t)H_{S^!}(t)$. □

5.3 *Point modules and line modules for* D

We will use the methodology and results in [6] to classify the point modules and line modules for D.

5.3.1 Point modules

By [6, Theorem 4.2.2], the point modules for D are parametrized by the closed points of a scheme $\mathscr{P}_D$ called the point scheme. We refer the reader to [6] and [3] for further explanations.

The point module corresponding to a point $p \in \mathscr{P}_D$ is

$$M_p \;:=\; \frac{D}{Dp^{\perp}},$$

where $p^{\perp}$ is the subspace of D_1 consisting of the linear forms that vanish at p. If M_p is a point module so is $(M_p)_{\geq 1}(1)$. Keeping the notation in [6, Theorem 4.1.1] we write σ for the automorphism of $\mathscr{P}_D$ such that

$$M_{\sigma^{-1}(p)} = (M_p)_{\geq 1}(1). \tag{5-1}$$

Following the notation in [6, Section 3.1], we write

$$\mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} qx_2x_3 - q^{-1}x_3x_2 \\ qx_3x_1 - q^{-1}x_1x_3 \\ qx_1x_2 - q^{-1}x_2x_1 \end{bmatrix}, \qquad \mathbf{l} = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} = - \begin{bmatrix} (q - q^{-1})\, x_1 \\ (q - q^{-1})\, x_2 \\ (q - q^{-1})\, x_3 \end{bmatrix}$$

and

$$\begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \mathbf{f} + t\mathbf{l} = \begin{bmatrix} qx_2x_3 - q^{-1}x_3x_2 - (q - q^{-1})x_1t \\ qx_3x_1 - q^{-1}x_1x_3 - (q - q^{-1})x_2t \\ qx_1x_2 - q^{-1}x_2x_1 - (q - q^{-1})x_3t \end{bmatrix}. \tag{5-2}$$

Then the defining equations for the central extension D can be written as

$$\begin{aligned} tx_i - x_it &= 0, \\ g_j = f_j + tl_j &= 0, \end{aligned}$$

for $i, j = 1, 2, 3$, and the defining equations for $D/(t)$ are $f_1 = f_2 = f_3 = 0$. In addition, $\mathbf{f} = M\mathbf{x}$, where

$$M = \begin{bmatrix} 0 & -q^{-1}x_3 & qx_2 \\ qx_3 & 0 & -q^{-1}x_1 \\ -q^{-1}x_2 & qx_1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Note that f_1, f_2, f_3 are chosen in such way that there exists a 3×3 matrix Q satisfying

$\mathbf{x}^T M = (Q\mathbf{f})^T$. Indeed, since $\mathbf{x}^T M = \mathbf{f}^T$, Q is the identify matrix, that is,

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{5-3}$$

Proposition 5.5. *The equations for $\mathscr{P}_D$ are given by*

1. $g_1 = g_2 = g_3 = 0$ *on* $\mathscr{P}_D \cap \{t \neq 0\}$*, that is,*

$$x_2x_3 - x_1t \;=\; x_3x_1 - x_2t \;=\; x_1x_2 - x_3t \;=\; 0.$$

2. $tg_1 = tg_2 = tg_3 = h_i = 0$ *on* $\mathscr{P}_D \cap \{x_i \neq 0\}$*, where*

$$\begin{aligned} h_1 &= \left(q^3 - q^{-3}\right) x_1^2x_2x_3 - \left(q - q^{-1}\right)\left(x_1^3 + q^2x_1x_2^2 + q^{-2}x_1x_3^2\right)t, \\ h_2 &= \left(q^3 - q^{-3}\right) x_1x_2^2x_3 - \left(q - q^{-1}\right)\left(x_2^3 + q^2x_2x_3^2 + q^{-2}x_2x_1^2\right)t, \\ h_3 &= \left(q^3 - q^{-3}\right) x_1x_2x_3^2 - \left(q - q^{-1}\right)\left(x_3^3 + q^2x_3x_1^2 + q^{-2}x_3x_2^2\right)t. \end{aligned}$$

Proof. This is a direct application of [6, Lemma 4.2.1, Theorem 4.2.2]. It is worthy pointing out that the definitions for h_1, h_2, h_3 are given by [6, Lemma 4.2.1]. In our case, these definitions are reduced to

$$\begin{aligned} h_1 &= x_1 \det M + t \det[\mathbf{l}, M_2, M_3], \\ h_2 &= x_2 \det M + t \det[M_1, \mathbf{l}, M_3], \\ h_3 &= x_3 \det M + t \det[M_1, M_2, \mathbf{l}], \end{aligned}$$

where M_i is the i^{th} column vector of the matrix M. $\square$

Remark 5.6. It is by no means obvious, but the eventual result that A has exactly five non-isomorphic simple modules of dimension n for each $n \geq 1$ is intimately related to the observation in Part 1 of the next result that there are five points $p \in \mathscr{P}_D$ such that $\sigma(p) = p$.

Proposition 5.7. *The point scheme* $\mathscr{P}_D \subset \mathbb{P}^3_{x_1,x_2,x_3,t}$ *consists of*

- *the five points* $(0,0,0,1), (1,\pm 1,\pm 1,1)$, $(-1,\pm 1,\mp 1,1)$, *and*
- *the points on the lines* $\{x_i = t = 0\}$, *where* $i = 1,2,3$.

In addition,

1. *if* $p \in \{(0,0,0,1), (1,\pm 1,\pm 1,1), (-1,\pm 1,\mp 1,1)\}$, *then* $\sigma(p) = p$,
2. *if* $p = (0,\xi_2,\xi_3,0)$, *then* $\sigma(p) = (0, q\xi_2, q^{-1}\xi_3, 0)$,
3. *if* $p = (\xi_1, 0, \xi_3, 0)$, *then* $\sigma(p) = (q^{-1}\xi_1, 0, q\xi_3, 0)$,
4. *if* $p = (\xi_1,\xi_2,0,0)$, *then* $\sigma(p) = (q\xi_1, q^{-1}\xi_2, 0, 0)$.

Proof. By Proposition 5.5(1), the equations for $\mathscr{P}_D \cap \{t \neq 0\}$ are $g_i = 0$ for $i = 1,2,3$. If $p = (\xi_1,\xi_2,\xi_3,1) \in \mathbb{P}^3$ is a point in $\mathscr{P}_D \cap \{t \neq 0\}$, then

$$\xi_1\xi_2 - \xi_3 \;=\; \xi_2\xi_3 - \xi_1 \;=\; \xi_3\xi_1 - \xi_2 \;=\; 0, \tag{5-4}$$

If $\xi_i = 0$ for some i then by (5-4) $\xi_1 = \xi_2 = \xi_3 = 0$, so that $p = (0,0,0,1)$.

Assume $\xi_1\xi_2\xi_3 \neq 0$. Since $\xi_i\xi_j = \xi_k$ and $\xi_j\xi_k = \xi_i$, it follows that $(\xi_j^2 - 1)\xi_k = 0$; hence $\xi_j^2 = 1$ for $j = 1,2,3$. If $\xi_1 = 1$, then by (5-4) $\xi_2 - \xi_3 = 0$. Hence, $p = (1,\pm 1,\pm 1,1)$. Similarly, if $\xi_1 = -1$ then $p = (-1,\pm 1,\mp 1,1)$. Therefore,

$$\mathscr{P}_D \cap \{t \neq 0\} \;=\; \{(0,0,0,1), (1,\pm 1,\pm 1,1), (-1,\pm 1,\mp 1,1)\}.$$

Next, we determine $\mathscr{P}_D \cap \{t = 0\}$. First note that

$$\mathscr{P}_D \cap \{t = 0\} \;=\; \bigcup_{i=1}^{3} \mathscr{P}_D \cap \{t = 0\} \cap \{x_i \neq 0\}.$$

By Proposition 5.5 (2), $\mathscr{P}_D \cap \{t=0\} \cap \{x_i \neq 0\} = \{x_j x_k = t = 0\}$. It follows that

$$\mathscr{P}_D \cap \{t=0\} \; = \; \bigcup_{i=1}^{3} \{x_i = t = 0\}.$$

Let $p \in \mathscr{P}_D$. Suppose $p = (\xi_1, \xi_2, \xi_3, \xi_4)$ and $\sigma(p) = (\xi_1', \xi_2', \xi_3', \xi_4')$. If $M_{\sigma(p)}$ is the point module corresponding to $\sigma(p)$ with a homogeneous basis $\{e_0, e_1, \ldots, e_n, \ldots,\}$ where $\deg(e_n) = n$, then it follows from (5-1) that

$$x_i e_0 \; = \; \xi_i' e_1, \quad t e_0 \; = \; \xi_4' e_1 \qquad \text{and} \qquad x_i e_1 \; = \; \xi_i e_2, \quad t e_1 \; = \; \xi_4 e_2$$

for $i = 1, 2, 3$. It follows that

$$\begin{aligned} 0 \; &= \; \left(q x_i x_j - q^{-1} x_j x_i - \delta^{-1} x_k t\right) e_0 \; = \; \left(q \xi_i \xi_j' - q^{-1} \xi_j \xi_i' - \delta^{-1} \xi_k \xi_4'\right) e_2, \\ 0 \; &= \; (x_i t - t x_i) e_0 \; = \; \left(\xi_i \xi_4' - \xi_4 \xi_i'\right) e_2, \end{aligned}$$

and hence

$$q \xi_i \xi_j' - q^{-1} \xi_j \xi_i' - (q - q^{-1}) \xi_k \xi_4' \; = \; 0 \; = \; \xi_i \xi_4' - \xi_4 \xi_i',$$

where (i, j, k) runs over all cyclic permutations of $(1, 2, 3)$. These six equations can be written as

$$\begin{bmatrix} -q^{-1}\xi_2 & q\xi_1 & 0 & -(q-q^{-1})\xi_3 \\ 0 & -q^{-1}\xi_3 & q\xi_2 & -(q-q^{-1})\xi_1 \\ q\xi_3 & 0 & -q^{-1}\xi_1 & -(q-q^{-1})\xi_2 \\ -\xi_4 & 0 & 0 & \xi_1 \\ 0 & -\xi_4 & 0 & \xi_2 \\ 0 & 0 & \xi_4 & \xi_3 \end{bmatrix} \begin{bmatrix} \xi_1' \\ \xi_2' \\ \xi_3' \\ \xi_4' \end{bmatrix} \; = \; \mathbf{0}.$$

Now, for each point p in Parts (1) - (4), one can verify that $\sigma(p)$ satisfies the linear system above. $\square$

5.3.2 An important function: $\boldsymbol{\lambda}(\alpha)$

The function $\boldsymbol{\lambda} : \Bbbk - \{0\} \to \Bbbk$ defined by

$$\boldsymbol{\lambda}(\alpha) \;=\; \frac{\alpha + \alpha^{-1}}{q^2 - q^{-2}}$$

will play a key role in all that follows. We note that

$$\boldsymbol{\lambda}(-\alpha) \;=\; -\boldsymbol{\lambda}(\alpha) \qquad \text{and} \qquad \boldsymbol{\lambda}\left(\alpha^{-1}\right) \;=\; \boldsymbol{\lambda}(\alpha).$$

5.3.3 Line modules

We will use the results in [6, Section 5] to classify the line modules for D.

By [13, Theorem 4.5], if M is a line module for D, then there is a unique line $\ell \subseteq \mathbb{P}(D_1^*) = \mathbb{P}^3$ such that

$$M \;\cong\; \frac{D}{D\ell^{\perp}},$$

where $\ell^{\perp}$ is the set of points in D_1 that vanish on ℓ. Let

$$\begin{aligned} \mathscr{L}_D \;&=\; \{\ell \subseteq \mathbb{P}^3 \mid D/D\ell^{\perp} \text{ is a line module}\} \;\; \text{and} \\ \mathbb{L}_p \;&=\; \{\ell \in \mathscr{L}_D \mid p \in \ell \text{ and } p \in \mathscr{P}_S\}, \end{aligned}$$

where

$$S \;=\; \frac{D}{(t)} \;=\; \frac{\Bbbk\langle x_1, x_2, x_3\rangle}{(qx_ix_j - q^{-1}x_jx_i \mid (i,j) = (1,2),(2,3),(3,1))}.$$

It is well known that the point scheme $\mathscr{P}_S$ is the “triangle” $x_1x_2x_3 = 0$ on $\mathbb{P}^2 = \{t = 0\} \subseteq \mathbb{P}^3$. As a subvariety of $\mathbb{P}^3_{x_1,x_2,x_3,t}$

$$\mathscr{P}_S \;=\; \{(\alpha_1,1,0,0),\; (0,\alpha_2,1,0),\; (1,0,\alpha_3,0) \mid \alpha_i \in \Bbbk\} \;\subset\; \mathscr{P}_D.$$

According to the discussion in [6, page 204],

$$\mathscr{L}_D \;=\; \{\text{lines on } t=0\} \cup \bigcup_{p \in \mathscr{P}_S} \mathbb{L}_p. \tag{5-5}$$

Furthermore, if $p \in \mathscr{P}_S$ there is a quadric Q_p containing p such that

$$\mathbb{L}_p \;=\; \{\text{the lines } \ell \subseteq \mathscr{V}(t) \mid p \in \ell\} \cup \{\text{the lines } \ell \subseteq Q_p \mid p \in \ell\}.$$

By [6, Proposition 5.1.7], if $\sigma(p) = (\xi_1, \xi_2, \xi_3, 0)$ then Q_p is given by the equation $\boldsymbol{\xi}^T Q \mathbf{g} = 0$, where $\boldsymbol{\xi} = [\xi_1, \xi_2, \xi_3]^T$ and $\mathbf{g}, Q$ are given by (5-2), (5-3), respectively. Since Q is the identity matrix, Q_p is given by the equation $\boldsymbol{\xi}^T \mathbf{g} = 0$, that is,

$$\begin{bmatrix} \xi_1, \xi_2, \xi_3 \end{bmatrix} \begin{bmatrix} x_2x_3 - x_1t \\ x_3x_1 - x_2t \\ x_1x_2 - x_3t \end{bmatrix} \;=\; 0. \tag{5-6}$$

Proposition 5.8. *There are exactly 6 points $p \in \mathscr{P}_S$ for which the quadric Q_p is singular, namely*

1. $p = (1, \pm q^2, 0, 0)$ *in which case* $Q_p = \{x_1 \pm x_2 = 0\} \cup \{x_3 \mp t = 0\}$*;*

2. $p = (0, 1, \pm q^2, 0)$ *in which case* $Q_p = \{x_2 \pm x_3 = 0\} \cup \{x_1 \pm t = 0\}$*;*

3. $p = (\pm q^2, 0, 1, 0)$ *in which case* $Q_p = \{x_3 \pm x_1 = 0\} \cup \{x_2 \pm t = 0\}$*.*

Proof. Let $p \in \mathscr{P}_S$. We can, and do, choose a cyclic permutation (i, j, k) of $(1, 2, 3)$ and a scalar α such that $p = (1, \alpha, 0, 0)$ with respect to the ordered coordinates (x_i, x_j, x_k, t). By Proposition 5.7, $\sigma(p) = (q, q^{-1}\alpha, 0, 0)$. By (5-6), the equation for Q_p is

$$q(x_jx_k - x_it) + q^{-1}\alpha(x_kx_i - x_jt) \;=\; 0.$$

The partial derivatives for x_i, x_j, x_k and t are $q^{-1}\alpha x_k - qt$, $qx_k - q^{-1}\alpha t$, $q^{-1}\alpha x_i + qx_j$, and $-qx_i - q^{-1}\alpha x_j$, respectively. If $a \neq \pm q^2$ then there is no point in $\mathbb{P}^3$ where all four of these partial derivatives vanish. Hence Q_p is smooth if $\alpha \neq \pm q^2$. On the other hand, if $\alpha = \pm q^2$, then the equation defining Q_p factors as $(x_i \pm x_j)(x_k \mp t)$. □

The kind of a point. The following terminology is adopted in [6]. If $p \in \mathscr{P}_S$, then there exist $u, v, a, b \in S_1$ such that $p = \mathscr{V}(u, v, t)$ and $a \otimes v - b \otimes u \in R_S$. Following [6, Definition 5.1.9], the point p is said to be

- of the first kind if $a = v$ and u, v, b are linearly independent;
- of the second kind if $a = u$ and u, v, b are linearly independent;
- of the third kind if $p = \mathscr{V}(a, b, t)$.

The next result determines $\mathbb{L}_p$ for each $p \in \mathscr{P}_S$ and therefore determines all line modules for D, according to (5-5).

Recall that $\delta = (q - q^{-1})^{-1}$.

Proposition 5.9. *Let p be a point in $\mathscr{P}_S$. Then*

1. *p is of the second kind;*
2. *if $p = (0, \alpha, 1, 0) \in \mathbb{P}^3_{x_i,x_j,x_k,t}$, where (i, j, k) is a cyclic permutation of $(1, 2, 3)$ and $\alpha \in \Bbbk$, then*

$$\mathbb{L}_p = \begin{cases} \mathscr{V}\left(\delta x_i - \boldsymbol{\lambda}(q\alpha)t, x_j - \alpha x_k\right) \cup \mathscr{V}(x_i, t) & \alpha^2 \notin \{q^{-4}, 0\}, \\ \mathscr{V}(x_i, t) \cup \mathscr{V}(x_j, t) & \alpha = 0, \\ \mathscr{V}(x_i, t) \cup \mathscr{V}\left(\delta x_i - \boldsymbol{\lambda}(q\alpha)t, x_j - \alpha x_k - \beta t\right) & \alpha^2 = q^{-4}; \end{cases}$$

3. *$\mathscr{L}_D$ consists of*

- *the lines on the plane* $\mathscr{V}(t)$, *and*

- *the lines* $\mathscr{V}\left(\delta x-\boldsymbol{\lambda}(q\alpha)t, y-\alpha z-\beta t\right)$, *where* (x,y,z) *runs over all cyclic permutations of* (x_1,x_2,x_3) *and* $(\alpha,\beta)\in\Bbbk^2$ *is such that* $\alpha\neq 0$ *and* $(\alpha^2-q^{-4})\beta=0$.

Proof. (1) Let $p\in\mathscr{P}_S$. There is a cyclic permutation (i,j,k) of $(1,2,3)$ such that p is the point $\{x_i=x_j-\alpha x_k=t=0\}$ for some $\alpha\in\Bbbk$. Write $u=x_i$, $v=x_j-\alpha x_k$, and $w=q^{-2}\alpha x_j-q^2x_k$. Then the element

$$u\otimes v-w\otimes u \;=\; q^{-1}(qx_i\otimes x_j-q^{-1}x_j\otimes x_i)+q\alpha(qx_k\otimes x_i-q^{-1}x_i\otimes x_k)$$

is a relation in $S=D/(t)$, so p is of the second kind.

(2) By Proposition 5.7, $\sigma(p)=(0,q\alpha,q^{-1},0)\in\mathbb{P}^3_{x_i,x_j,x_k,t}$. Now by (5-6), the quadric Q_p is given by the equation $q\alpha(x_kx_i-x_jt)+q^{-1}(x_ix_j-x_kt)=0$, or equivalently,

$$x_i(q^{-1}x_j+q\alpha x_k)-t(q\alpha x_j+q^{-1}x_k)\;=\;0. \tag{5-7}$$

Clearly the line $\mathscr{V}(x_i,t)$ is in $\mathbb{L}_p$.

Let ℓ be a line on Q_p that contains p and suppose $\ell\neq\mathscr{V}(x_i,t)$. Then ℓ is the line

$$\lambda x_i-\mu t\;=\;0\;=\;\mu(q^{-1}x_j+q\alpha x_k)-\lambda(q\alpha x_j+q^{-1}x_k)$$

for some $(\lambda,\mu)\in\mathbb{P}^1$. Since $p\in\ell$, $\mu\left(q^{-1}\alpha+q\alpha\right)-\lambda\left(q\alpha^2+q^{-1}\right)=0$; hence $(\lambda,\mu)=(q^{-1}\alpha+q\alpha, q\alpha^2+q^{-1})$. Therefore, ℓ is contained in the intersection of the planes

$$\begin{aligned}(q+q^{-1})\alpha x_i-\left(q\alpha^2+q^{-1}\right)t &= 0 \quad\text{and}\\ \left(\alpha^2-q^{-4}\right)(x_j-\alpha x_k) &= 0.\end{aligned}$$

The three cases in Part 2 of the proposition are discussed below.

(i) Suppose $\alpha^2 \notin \{q^{-4}, 0\}$. The two planes are different so ℓ is the line

$$\delta x_i - \boldsymbol{\lambda}(q\alpha)t \;=\; x_j - \alpha x_k \;=\; 0.$$

Hence, $\mathbb{L}_p$ consists of the lines ℓ and $\mathscr{V}(x_i, t)$.

(ii) If $\alpha = 0$, then $\ell = \mathscr{V}(x_j, t)$ so $\mathbb{L}_p$ consists of the lines $\mathscr{V}(x_i, t)$ and $\mathscr{V}(x_j, t)$.

(iii) If $\alpha^2 = q^{-4}$, then ℓ lies on the plane $\delta x_i - \boldsymbol{\lambda}(q\alpha)t = 0$ so $\mathbb{L}_p$ is the pencil of lines on this plane that passes through p.

Write $\lambda = \boldsymbol{\lambda}(q\alpha)$ for short. Suppose that

$$\ell \;=\; \{ax_i + bx_j + cx_k + dt \;=\; \delta x_i - \lambda t \;=\; 0\},$$

where $a, b, c, d \in \Bbbk$. If $b = c = 0$, then $\ell = \{ax_i + dt = \delta x_i - \lambda t = 0\} = \{x_i = t = 0\}$, contradicting our assumption that $\ell \neq \mathscr{V}(x_i, t)$. So $(b, c) \neq (0, 0)$.

Furthermore, since $p \in \ell$, $p = (0, \alpha, 1, 0)$ lies on the plane $ax_i + bx_j + cx_k + dt = 0$, whence $\alpha b + c = 0$. Since $(b, c) \neq (0, 0)$, it is necessary that $b \neq 0$.

It follows that

$$\begin{aligned} \ell \;&=\; \left\{bx_j - \alpha b x_k + \left(\delta^{-1} a\lambda + d\right) t = \delta x_i - \lambda t = 0\right\} \\ &=\; \{\delta x_i - \lambda t = x_j - \alpha x_k - \beta t = 0\}, \end{aligned}$$

where $\beta = -\left(\delta^{-1} a\lambda + d\right) b^{-1}$.

Therefore, $\mathbb{L}_p = \mathscr{V}\left(\delta x_i - \boldsymbol{\lambda}(q\alpha)t, x_j - \alpha x_k - \beta t\right) \cup \mathscr{V}(x_i, t)$.

(3) This follows from (5-5) and Part (2) immediately. $\square$

5.4 Line modules for $A(q)$

Based on the passage from $\mathsf{Gr}(D)$ to $\mathsf{Mod}(A)$ established in Chapter 4, we are now able to classify all line modules for A.

To distinguish the generators for A and those for D, *for the rest of this chapter* we assume A to be the free algebra $\Bbbk\langle X_1, X_2, X_3\rangle$ modulo the relations $qX_iX_j - q^{-1}X_jX_i = X_k$ where (i, j, k) runs over all cyclic permutations of $(1, 2, 3)$.

For a line ℓ in $\mathbb{A}^3_{X_1,X_2X_3}$, we write $\ell^\perp$ for the set of elements in $\operatorname{span}\{1, X_1, X_2, X_3\}$ that vanish on ℓ.

Proposition 5.10. *Every line module for A is isomorphic to $A/A\ell^\perp$, where*

$$\ell \;=\; \{X - \boldsymbol{\lambda}(q\alpha) \;=\; Y - \alpha Z - \beta \;=\; 0\}$$

for some cyclic permutation (X, Y, Z) of (X_1, X_2, X_3) and some $(\alpha, \beta) \in \Bbbk^2$ such that $\alpha \neq 0$ and $(\alpha^2 - q^{-4})\,\beta = 0$.

Proof. By Proposition 5.1, $A \cong D[t^{-1}]_0$ given by $X_i \mapsto \delta x_i t^{-1}$ for $i = 1, 2, 3$, where $\delta = (q - q^{-1})^{-1}$.

By Theorem 4.16, an A-module M is a line module if and only if there exists a line module L for D such that L is t-torsion free and $L[t^{-1}]_0 \cong M$.

By Proposition 5.9(3), there exist a cyclic permutation (x, y, z) of (x_1, x_2, x_3) and $(\alpha, \beta) \in \Bbbk^2$ with $\alpha \neq 0$ and $(\alpha^2 - q^{-4})\,\beta = 0$ such that

$$L \;\cong\; \frac{D}{D\,(\delta x - \boldsymbol{\lambda}(q\alpha)t) + D\,(y - \alpha z - \beta t)}.$$

It follows that

$$\begin{aligned} L[t^{-1}]_0 \;&\cong\; \frac{D[t^{-1}]_0}{D[t^{-1}]_0\,(\delta t^{-1}x - \boldsymbol{\lambda}(q\alpha)) + D[t^{-1}]_0\,(\delta t^{-1}y - \alpha\delta t^{-1}z - \delta\beta)} \\ &\cong\; \frac{A}{A(X - \boldsymbol{\lambda}(q\alpha)) + A(Y - \alpha Z - \beta')}, \end{aligned}$$

where $\beta' = \delta\beta$. This completes the proof. $\square$

5.5 *The center of* $D(q)$

We determine the center of $D(q)$ in this section.

Proposition 5.11. *Let* (x, y, z) *be a cyclic permutation of* (x_1, x_2, x_3).

1. *The element*

$$\Omega \ := \ -\left(q+q^{-1}\right)xyz + qx^2t + q^{-3}y^2t + qz^2t.$$

   is central in $D(q)$.

2. *The center of* $D(q)$ *is* $\Bbbk[\Omega, t]$.

Proof. (1) Write $\gamma := (q - q^{-1})$. In the free algebra $\Bbbk\langle x, y, z, t\rangle$, let

$$a \ = \ qxy - q^{-1}yx - \gamma zt, \quad b \ = \ qyz - q^{-1}zy - \gamma xt, \quad c \ = \ qzx - q^{-1}xz - \gamma yt.$$

Calculations in $\Bbbk\langle x, y, z, t\rangle$ give

$$\begin{aligned} q[xyz, x] - \gamma x(y^2 - z^2)t &= xyc - xaz, \\ q[xyz, y] - \gamma(z^2y - yx^2)t &= azy - yxb, \\ q[xyz, z] - \gamma(x^2 - y^2)zt &= xbz - cyz, \\ q^4\left(xy^2 - z^2x\right) - \left(y^2x - xz^2\right)t &= q^3ay + q^{-3}ya - q^3zc - q^{-3}cz. \end{aligned}$$

Because $a = b = c = 0$ in $D(q)$, the following equalities hold in $D(q)$:

$$\begin{aligned} [xyz, x] &= \left(1-q^{-2}\right)x(y^2 - z^2)t, \\ [xyz, y] &= \left(1-q^{-2}\right)(z^2y - yx^2)t, \\ [xyz, z] &= \left(1-q^{-2}\right)(x^2 - y^2)zt, \\ xy^2 - z^2x &= q^{-4}\left(y^2x - xz^2\right). \end{aligned}$$

Since $[xyz, x] = (1 - q^{-2})x(y^2 - z^2)t$, it follows that

$$\begin{aligned}[\Omega, x] &= -\left(q + q^{-1}\right)[xyz, x] + q^{-3}[y^2t, x] + q[z^2t, x] \\ &= \left(q^{-3} - q\right)(xy^2 - xz^2)t + q^{-3}(y^2x - xy^2)t + q(z^2x - xz^2)t \\ &= q(z^2x - xy^2)t - q^{-3}(xz^2 - y^2x)t \\ &= 0.\end{aligned}$$

A similar argument shows that $[\Omega, y] = 0 = [\Omega, z]$. Therefore, Ω is central.

(2) We first show that the center of $R := D/(t)$ is $\mathbb{k}[xyz]$, the polynomial ring in the variable xyz. Write $Z = Z(R)$.

Since R is the free algebra $\mathbb{k}\langle x, y, z\rangle$ modulo the relations

$$xy = q^{-2}yx, \quad yz = q^{-2}zy, \quad zx = q^{-2}xz, \tag{5-8}$$

$R = \bigoplus_{n\geq 0} R_n$ is $\mathbb{N}$-graded, where $R_n = \bigoplus_{i+j+k=n} \mathbb{k}x^iy^jz^k$ for $n \geq 0$. Hence, the center of R is also $\mathbb{N}$-graded, i.e.,

$$Z = \bigoplus_{n\geq 0} (Z \cap R_n).$$

Let $c = \sum_{i+j+k=n} \alpha_{i,j,k}x^iy^jz^k \in Z \cap R_n$, where $n \geq 0$ and $\alpha_{i,j,k} \in \mathbb{k}$. Since $xc = cx$,

$$\sum_{i+j+k=n} \alpha_{i,j,k}x^{i+1}y^jz^k = \left(\sum_{i+j+k=n} \alpha_{i,j,k}x^iy^jz^k\right)x \overset{(5\text{-}8)}{=} \sum_{i+j+k=n} q^{2j-2k}\alpha_{i,j,k}x^{i+1}y^jz^k.$$

Since q is not a root of unity, if $\alpha_{i,j,k} \neq 0$ then $j = k$. By a similar argument, we show $i = j$ if $\alpha_{i,j,k} \neq 0$. It follows that $c \in \sum_{i\geq 0} \mathbb{k}x^iy^iz^i = \sum_{i\geq 0} \mathbb{k}(xyz)^i$. So $Z(R) \subset \mathbb{k}[xyz]$.

On the other hand, it is not difficult to verify that $xyz \in Z(R)$. Therefore, $Z(R) = \mathbb{k}[xyz]$.

Now, we show that $Z(D) = \Bbbk[\Omega, t]$. Let

$$\phi : D \ \rightarrow \ D/(t)$$

be the canonical surjection. Then $\phi(\Omega) = -\left(q + q^{-1}\right) xyz$. Since D is $\mathbb{N}$-graded, it suffices to show

$$Z(D)_n \ := \ Z(D) \cap D_n \ = \ \Bbbk[\Omega, t]_n \tag{5-9}$$

for all $n \geq 0$. We argue this by induction on n. Clearly this is true when $n = 0$.

Suppose (5-9) is true for some $n \geq 0$. Let $d \in Z(D)_{n+1}$. Since $Z(D/(t)) = \Bbbk[xyz]$, there exists a polynomial g in one variable over $\Bbbk$ such that

$$\phi(d) \ = \ g\left(xyz\right) \ = \ g\left(\phi(\Omega)\right) \ = \ \phi\left(g(\Omega)\right),$$

Hence, $d = g(\Omega) + d't$ for some $d' \in D$. Since $d \in Z(D)_{n+1}$, we may, and do assume that $g(\Omega) \in D_{n+1}$ and $d' \in D_n$ without loss of generality.

Since both d and $g(\Omega)$ are central, so is $d't$. Since D is domain and t is central, d' is central in D and hence belongs to $Z(D) \cap D_n$. By the inductive hypothesis, $d' \in \Bbbk[\Omega, t]_n$. Therefore, $d \in (\Bbbk[\Omega, t])_{n+1}$. The proof is now complete. □

Remark 5.12. The central element Ω is independent of the choice of the cyclic permutation of $(x_1.x_2.x_3)$. More formally, let

$$\Omega \ := \ -\left(q + q^{-1}\right) x_1x_2x_3 + qx_1^2t + q^{-3}x_2^2t + qx_3^2t,$$

and let σ be the automorphism of D defined by $\sigma(x_1) = x_2, \sigma(x_2) = x_3, \sigma(x_3) = x_1$ and $\sigma(t) = t$. We claim $\Omega = \sigma(\Omega) = \sigma^2(\Omega)$.

Since D has a PBW basis of $\left\{x_1^i x_2^j x_3^k t^l \ \mid (i, j, k, l) \in \mathbb{N}^4\right\}$ and $Z(D) = \Bbbk[\Omega, t]$, $\sigma(\Omega) =$

$\alpha\Omega + \beta t^3$ for some $\alpha, \beta \in \Bbbk$. Furthermore, a direct calculation in D shows

$$\sigma(\Omega) \;=\; -(q+q^{-1})x_1x_2x_3 + \text{other degree-three monomials in } x_1, x_2, x_3 \text{ and } t \text{ (except } t^3\text{)}.$$

It is necessary that $\alpha = 1$ and $\beta = 0$. Hence, $\sigma(\Omega) = \Omega$ and therefore $\sigma^2(\Omega) = \sigma(\Omega) = \Omega$.

Corollary 5.13. *Let (x, y, z) be a cyclic permutation of (x_1, x_2, x_3) and ℓ the D-line*

$$\ell \;=\; \{\delta x - \boldsymbol{\lambda}(q\alpha)t = y - \alpha z - \beta t = 0\}.$$

Then $M(\ell)$ is annihilated by $\Omega - \delta^{-2}\mu t^3$, where $\delta = (q - q^{-1})^{-1}$ and

$$\mu \;=\; q^{-1}\boldsymbol{\lambda}^2(q^2\alpha) + q\beta^2 + \frac{q}{(1+q^2)^2}.$$

Proof. Write $\lambda = \boldsymbol{\lambda}(q\alpha)$. By Proposition 5.11(1),

$$\Omega \;=\; -\left(q+q^{-1}\right)yzx + qy^2t + q^{-3}z^2t + qx^2t$$

is central in D. The result follows from the fact that

$$\delta^2\Omega - \mu t^3 \;=\; a(\delta x - \lambda t) \,+\, b(y - \alpha z - \beta t) + q\alpha^{-1}(\alpha^2 - q^{-4})\beta z t^2,$$

where $a = -\left(q+q^{-1}\right)yz + q\delta^{-1}xt + \left(q\delta^{-2}\lambda - q^{-2}\delta^{-1}\alpha^{-1}\right)t^2$ and $b = qyz + q\beta t^2$. Since $(\alpha^2 - q^{-4})\beta = 0$, the result holds. $\square$

5.6 *The center of* $A(q)$

We continue writing A to be the free algebra $\Bbbk\langle X_1, X_2, X_3\rangle$ modulo the relations $qX_iX_j - q^{-1}X_jX_i = X_k$ where (i, j, k) runs over all cyclic permutations of $(1, 2, 3)$.

Recall from Proposition 3.10 that the element

$$C \;=\; -\left(q^{2}-q^{-2}\right) X_1X_2X_3 + qX_1^2 + q^{-3}X_2^2 + qX_3^2$$

belongs to the center of $A(q)$.

Proposition 5.14. *The center of $A(q)$ is the polynomial ring $\Bbbk[C]$.*

Proof. By Proposition 5.11(2), the center of D is $\Bbbk[\Omega, t]$, where

$$\Omega \;=\; -\left(q+q^{-1}\right) x_1x_2x_3 + qx_1^2t + q^{-3}x_2^2t + qx_3^2t.$$

Since $A \cong D[t^{-1}]_0$ via $X_i \mapsto \delta x_i t^{-1}$ for $i = 1, 2, 3$, it suffices to prove $Z(D[t^{-1}]_0) = \Bbbk\,[\Omega t^{-3}]$. Write $R = D[t^{-1}]_0$ for short.

Let $c \in Z(R)$. Then $ct^n \in D$ for some $n \geq 0$. Since t is central, $ct^n \in Z(D) = \Bbbk[\Omega, t]$. It follows that $Z(R) \subset \Bbbk[\Omega, t^{\pm 1}]_0$.

On the other hand, it is easy to see that $\Bbbk[\Omega, t^{\pm 1}]_0 \subset Z(R)$. Hence, $Z(R) = \Bbbk[\Omega, t^{\pm 1}]_0$. But $\Bbbk[\Omega, t^{\pm 1}]_0 = \Bbbk[\Omega t^{-3}]$. $\square$

Chapter 6

LINE MODULES FOR A

We begin this chapter by classifying, in Theorem 6.3, the line modules for A *without using the results about line modules for D.* There are several reasons for doing this. First, the more direct classification shows the algebraic intricacies that are somewhat obscured by the results about D. It also illustrates the broad applicability of the results in [6]. Second, the results in this chapter provide an independent check that the classification of the line modules for A in §5.4 is correct, as the calculations in this chapter and the next are somewhat error prone. Third, the methods used in this chapter to classify the line modules for A are in the same spirit as the methods N. Sasom uses in her thesis [15] to classify what she calls Verma modules. The name of Verma module is quite appropriate because, although line modules are not induced from 1-dimensional representations of subalgebras, they behave like Verma modules. This last point deserves emphasis: line modules (and point modules, and plane modules) for many algebras play the role that Verma modules play in the representation theory of semisimple Lie algebras. They also have similar homological properties though we do not emphasize that point here.

Sasom's Verma modules are our line modules.[1] However, our line modules form a larger family of A-modules, which later on will make a difference in the classification of finite

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[1] Sasom defines the algebra $T = T_q$ to be $\Bbbk\langle x, y, z\rangle$ modulo the relations

$$xy - qyx \;=\; z, \quad yz - qzy \;=\; x, \quad zx - qxz \;=\; y.$$

By Proposition 3.3 Sasom's family of algebras T_q is the same as our family of algebras $A(q)$ (Note our q is not equal to her q). In [15, Remark 3.4.2.] Sasom defines the left ideals $I_\eta = T(x - \lambda) + T(y - \eta z)$ where $\lambda, \eta \in \Bbbk$, and when $\lambda \;=\; \frac{\eta^2+q}{(1-q^2)\eta}$, she also defines the Verma modules

$$V(\eta) \;=\; T/I_\eta.$$

simple dimensional simple A-modules. We will explain this in more detail in Remark 6.12 and Chapter 7.

6.1 Definition and notation for line modules

In this chapter we view $A = A(q)$ as the quotient of $\Bbbk\langle x_1, x_2, x_3\rangle$ modulo the three relations $qx_ix_j - q^{-1}x_jx_i = x_k$ obtained by allowing (i, j, k) to run over the cyclic permutations of $(1, 2, 3)$. Frequently, (x, y, z) will denote a cyclic permutation of (x_1, x_2, x_3).

Lemma 6.1. *Let* $\alpha, \beta, \delta, \lambda \in \Bbbk$. *Let* $I = A(x - \delta z - \lambda) + A(y - \alpha z - \beta)$ *and let* $V_n = \operatorname{span}\{1, z, \ldots, z^n\}$. *Then*

1. $xz^n - q^{2n}\delta z^{n+1}$ *and* $yz^n - q^{-2n}\alpha z^{n+1}$ *belong to* $V_n + I$ *for all* $n \geq 0$;
2. $x^iy^jz^k \in V_{i+j+k} + I$ *for all* $(i, j, k) \in \mathbb{N}^3$;
3. $A = I + \Bbbk[z]$.

Proof. (1) We argue by induction on n. Clearly, this is true when $n = 0$.

Suppose $xz^n - q^{2n}\delta z^{n+1} \in V_n + I$ and $yz^n - q^{-2n}\alpha z^{n+1} \in V_n + I$. Then

$$\begin{aligned} xz^{n+1} &= (q^2zx - qy)z^n \\ &\in q^2z\left(q^{2n}\delta z^{n+1} + V_n + I\right) - q\left(q^{-2n}\alpha z^{n+1} + V_n + I\right) \\ &\subset q^{2n+2}\delta z^{n+2} + V_{n+1} + I. \end{aligned}$$

Hence, $xz^{n+1} - q^{2n+2}\delta z^{n+2} \in V_{n+1} + I$.

A similar argument shows that $yz^{n+1} - q^{-2n-2}\alpha z^{n+2} \in V_{n+1} + I$.

(2) By Part (1), $xz^n, yz^n \in V_{n+1} + I$ for $n \geq 0$. Inductively we can show $x^mz^n, y^mz^n \in V_{m+n} + I$ for $m \geq 0$. It follows that

$$x^iy^jz^k \in x^iV_{j+k} + I \subset V_{i+j+k} + I.$$

(3) By Lemma 3.1, the set $\{z^k x^i y^j \mid (k,i,j) \in \mathbb{N}^3\}$ is a basis for A. It thus suffices to show that $z^k x^i y^j \in \Bbbk[z] + I$ for all i, j, k.

By Part (1), both xz^n and yz^n are contained in $\Bbbk[z] + I$ for all $n \geq 0$. Hence, $x\Bbbk[z] + y\Bbbk[z] \subseteq \Bbbk[z] + I$.

If $x^{i-1}\Bbbk[z] \subseteq \Bbbk[z] + I$ for some $i \geq 1$, then

$$x^i\Bbbk[z] \;=\; x\left(x^{i-1}\Bbbk[z]\right) \;\subseteq\; x\left(\Bbbk[z] + I\right) \;\subseteq\; \Bbbk[z] + I.$$

Hence, $x^i\Bbbk[z] \subseteq \Bbbk[z] + I$ for all $i \geq 0$. A similar argument shows that $y^j\Bbbk[z] \subset \Bbbk[z] + I$ for all $j \geq 0$. It follows that

$$z^k x^i y^j \;\in\; z^k x^i \Bbbk[z] + I \;\subseteq\; z^k \Bbbk[z] + I \;\subseteq\; \Bbbk[z] + I$$

for all $i, j, k \geq 0$. □

Definition of line modules for A. Line modules for A are always defined with respect to the filtration

$$A_0 \;\subseteq\; A_1 \;\subseteq\; \cdots \;\subseteq\; A_n \subseteq \cdots,$$

where $A_n = \operatorname{span}\left\{x_1^i x_2^j x_3^k \mid i + j + k \leq n\right\}$ for $n \geq 0$. Thus, a left A-module M is a line module if there is an element $m \in M$ such that $M = Am$ and $\dim_{\Bbbk}(A_n m) = n + 1$ for all $n \geq 0$.

We frequently view x_1, x_2, x_3 as coordinate functions on $\Bbbk^3$ and denote $\Bbbk^3$ by $\mathbb{A}^3_{x_1,x_2,x_3}$ when we do this. In this way, subspaces $\Bbbk a + \Bbbk b$ of A spanned by two linearly independent elements a and b in $\operatorname{span}\{1, x_1, x_2, x_3\}$ such that $1 \notin \Bbbk a + \Bbbk b$ are in natural bijection with lines in $\Bbbk^3$; the line corresponding to $\Bbbk a + \Bbbk b$ is the line cut out by the equations $a = b = 0$. Note that every line in $\Bbbk^3$ is given by equations of the form

$$x - \delta z - \gamma \;=\; y - \alpha z - \beta \;=\; 0$$

for some cyclic permutation (x, y, z) of (x_1, x_2, x_3) and some $\alpha, \beta, \gamma, \delta \in \Bbbk$.[2]

The next result shows that every line module for A corresponds to an affine line in $\mathbb{A}^3_{x_1,x_2,x_3}$.

Proposition 6.2. *A left A-module is a line module if and only if it is infinite dimensional and isomorphic to*

$$\frac{A}{Aa + Ab}$$

for some linearly independent elements $a, b \in \operatorname{span}\{1, x_1, x_2, x_3\}$.

Proof. ($\Rightarrow$) Suppose $M = Am$ is a line module. Since $\dim_\Bbbk(A_1) = 4$ and $\dim_\Bbbk(A_1 m) = 2$, m is annihilated by a 2-dimensional subspace, $\Bbbk a + \Bbbk b$ say, of A_1. In particular, M is a quotient of $A/Aa + Ab$. There is a cyclic permutation (x, y, z) of (x_1, x_2, x_3) and scalars $\alpha, \beta, \gamma, \delta$ such that

$$a \;=\; x - \delta z - \gamma \qquad \text{and} \qquad b \;=\; y - \alpha z - \beta.$$

By Lemma 6.1(3), $(Aa + Ab) + \Bbbk[z] = A$. Since M is a quotient of $A/Aa + Ab$ it is also a $\Bbbk[z]$-module quotient of $\Bbbk[z]$. But $\dim_\Bbbk(M) = \infty$ so the composition $\Bbbk[z] \to A/Aa+Ab \to M$ is an isomorphism and $M \cong A/Aa + Ab$.

($\Leftarrow$) Let $\Bbbk a + \Bbbk b$ be a 2-dimensional subspace of A_1 such that $A/Aa + Ab$ is infinite dimensional. There is a cyclic permutation (x, y, z) of (x_1, x_2, x_3) and scalars $\alpha, \beta, \gamma, \delta$ such that

$$a \;=\; x - \delta z - \gamma \qquad \text{and} \qquad b \;=\; y - \alpha z - \beta.$$

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[2] The proof of this is a simple exercise in linear algebra. If details are needed here they are. Let L be a line in $\Bbbk^3$. Then L is the set of solutions to a system of linear equations of the form

$$\begin{cases} \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 \;=\; \alpha_4 \\ \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 \;=\; \beta_4 \end{cases}$$

Because L is a line,

$$\operatorname{rank}\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{pmatrix} \;=\; 2.$$

After replacing this matrix by a row equivalent matrix in row reduced echelon form and replacing (x_1, x_2, x_3) by a suitable cyclic permutation (x, y, z) we see that L is cut out by equations of the form as claimed.

Write $I = Aa + Ab$.

By Lemma 6.1(3), $I + \Bbbk[z] = A$. Hence, $I \oplus \Bbbk[z] = A$. Let V_n be the subspace of A/I spanned by the images of $\{1, z, \ldots, z^n\}$ for $n \geq 0$ and let e be the natural image of 1_A in A/I. By Lemma 6.1(2), $x^i y^j z^k.e \in V_n$ if $i + j + k = n$. It follows that

$$A_n e \;=\; \operatorname{span}\left\{x^i y^j z^k.e \mid i + j + k \leq n\right\} \;=\; V_n;$$

hence $\dim_{\Bbbk}(A_n e) = \dim_{\Bbbk}(V_n) = n + 1$, and A/I is a line module. □

Notation and terminology. Let ℓ be a line in $\mathbb{A}^3_{x,y,z}$. We will write

$$\begin{aligned} \ell^{\perp} &:= \{a \in \operatorname{span}\{1, x, y, z\} \mid a \text{ vanishes on } \ell\}, \\ M(\ell) &:= A/A\ell^{\perp}, \\ \mathbb{L}_A &:= \left\{\text{the lines } \ell \subseteq \mathbb{A}^3_{x,y,z} \text{ such that } A/A\ell^{\perp} \text{ is a line module}\right\}. \end{aligned}$$

A line ℓ in $\mathbb{A}^3_{x,y,z}$ will be called an A-line if $A/A\ell^{\perp}$ is a line module. We will use the notation $M(\ell)$ *only* when $M(\ell)$ is a line module.

6.2 *Classification of line modules for* A

We give two classifications, an algebraic one classifying the subspaces $\Bbbk a + \Bbbk b \subset A_1$ such that $A/Aa + Ab$ is a line module, and a geometric one classifying the lines $\ell \subseteq \mathbb{A}^3_{x_1,x_2,x_3}$ for which $A/A\ell^{\perp}$ is a line module. The second almost says that $A/A\ell^{\perp}$ is a line module if and only if ℓ lies on one of a particular family of cubic surfaces.

Theorem 6.3. *A line ℓ in $\mathbb{A}^3_{x_1,x_2,x_3}$ is an A-line if and only if*

$$\ell \;=\; \{x - \boldsymbol{\lambda}(q\alpha) \;=\; y - \alpha z - \beta \;=\; 0\} \tag{6-1}$$

for some cyclic permutation (x, y, z) of (x_1, x_2, x_3) and some $(\alpha, \beta) \in \Bbbk^2$ such that $\alpha \neq 0$ and

$(\alpha^2 - q^{-4})\,\beta = 0$. *Furthermore, in that case,* $A = A\ell^\perp \oplus \Bbbk[z]$.

Proof. By Proposition 6.2, every A-line ℓ must have the form

$$\ell \;=\; \{x - \delta z - \gamma \;=\; y - \alpha z - \beta \;=\; 0\}$$

for some cyclic permutation (x, y, z) of (x_1, x_2, x_3) and some $\alpha, \beta, \gamma, \delta \in \Bbbk$.

For the rest of the proof we write

$$I \;:=\; A\ell^\perp \;=\; A(x - \delta z - \gamma) + A(y - \alpha z - \beta).$$

We will prove that $\dim_\Bbbk(A/I) = \infty$ if and only if either

Case 1 $\alpha = 0$ and $\delta \neq 0$ and $\beta = \boldsymbol{\lambda}(q^{-1}\delta)$ and $\gamma(\delta^2 - q^4) = 0$, or

Case 2 $\delta = 0$ and $\alpha \neq 0$ and $\gamma = \boldsymbol{\lambda}(q\alpha)$ and $\beta(\alpha^2 - q^{-4}) = 0$.

Before embarking on the proof we make an important observation. If Case 2 holds then the theorem is obviously true. If Case 1 holds the truth of the theorem is less obvious. Nevertheless, suppose Case 1 holds. Since $\boldsymbol{\lambda}(q^{-1}\delta) = \boldsymbol{\lambda}(q\delta^{-1})$, $I = A(x-\delta z-\gamma)+A\left(y - \boldsymbol{\lambda}(q\delta^{-1})\right)$. Replace (x, y, z) in this description of I by the cyclic permutation (z, x, y), to get $I = A(z - \delta y - \gamma) + A\left(x - \boldsymbol{\lambda}(q\delta^{-1})\right)$. In other words, $I = A(x - \boldsymbol{\lambda}(q\delta^{-1})) + A(y - \delta^{-1}z - \delta^{-1}\gamma)$. Now replacing δ^{-1} by α and $\delta^{-1}\gamma$ by β we obtain $I = A(x - \boldsymbol{\lambda}(q\alpha)) + A(y - \alpha z - \beta)$. In addition, $\alpha \neq 0$ and $\beta(\alpha^2 - q^{-4}) = 0$. This shows that the theorem is true when Case 1 holds. Thus, to prove the theorem is suffices to prove the statement in previous paragraph, as shown below.

By Lemma 6.1(3), $\dim_\Bbbk(A/I) = \infty$ if and only if $I \cap \Bbbk[z] = 0$.

In the next computation we use the following notation: for $a, b \in A$, we write $a \equiv b \pmod{I}$ if $a - b \in I$.

We have

$$\begin{aligned}
0 &= qxy - q^{-1}yx - z \\
&\equiv qx(\alpha z + \beta) - q^{-1}y(\delta z + \gamma) - z \\
&\equiv q^2\alpha(qzx - y) - q^{-2}\delta(q^{-1}zy + x) + q\beta x - q^{-1}\gamma y - z \\
&\equiv \left(q^3\alpha z + q\beta - q^{-2}\delta\right) x - \left(q^{-3}\delta z + q^{-1}\gamma + q^2\alpha\right) y - z \\
&\equiv \left(q^3\alpha z + q\beta - q^{-2}\delta\right)(\delta z + \gamma) - \left(q^{-3}\delta z + q^{-1}\gamma + q^2\alpha\right)(\alpha z + \beta) - z \\
&\equiv \left(q^3 - q^{-3}\right)\alpha\delta z^2 + \left[(q^3 - q^{-1})\alpha\gamma + (q - q^{-3})\beta\delta - q^2\alpha^2 - q^{-2}\delta^2 - 1\right] z \\
&\quad + \left(q - q^{-1}\right)\beta\gamma - q^2\alpha\beta - q^{-2}\gamma\delta \pmod{I}.
\end{aligned}$$

Hence, if $\dim_{\Bbbk}(A/I) = \infty$ then $\alpha\delta = 0$, in which case,

$$\begin{aligned}
&\left[(q^3 - q^{-1})\alpha\gamma + (q - q^{-3})\beta\delta - q^2\alpha^2 - q^{-2}\delta^2 - 1\right] z \\
&+ \left(q - q^{-1}\right)\beta\gamma - q^2\alpha\beta - q^{-2}\gamma\delta \in I.
\end{aligned} \tag{6-2}$$

On the other hand, if $\alpha = 0 = \delta$, then $I = A(x-\gamma) + A(y-\beta)$ so $z = qxy - q^{-1}yx \in \Bbbk + I$, whence $\dim_{\Bbbk}(A/I) < \infty$.

Thus, to determine necessary and sufficient conditions on $(\alpha, \beta, \gamma, \delta)$ for $\dim_{\Bbbk}(A/I) = \infty$, it suffices to consider two cases:

1. $\alpha = 0$ and $\delta \neq 0$;

2. $\alpha \neq 0$ and $\delta = 0$,

as discussed below.

(1) Assume $\alpha = 0$ and $\delta \neq 0$. It follows from (6-2) that the element

$$\left[(q - q^{-3})\beta\delta - q^{-2}\delta^2 - 1\right] z + \gamma\left[(q - q^{-1})\beta - q^{-2}\delta\right]$$

is contained in $I \cap \Bbbk[z]$. It follows that $I \cap \Bbbk[z]$ is the ideal of $\Bbbk[z]$ generated by this element.

Thus, $\dim_{\Bbbk}(A/I) = \infty$ if and only if

$$\left(q - q^{-3}\right)\beta\delta - q^{-2}\delta^2 - 1 \;=\; \left(q - q^{-1}\right)\beta\gamma - q^{-2}\gamma\delta \;=\; 0.$$

The first equality is equivalent to the condition that

$$\beta \;=\; \left(q\delta^{-1} + q^{-1}\delta\right) / \left(q^2 - q^{-2}\right),$$

i.e., that $\beta = \boldsymbol{\lambda}(q\delta^{-1}) = \boldsymbol{\lambda}(q^{-1}\delta)$. If the first equality holds, then the second can be replaced by the condition that $\gamma(\delta^2 - q^4) = 0$. Thus, $\dim_{\Bbbk}(A/I) = \infty$ if and only if $\beta = \boldsymbol{\lambda}(q^{-1}\delta)$ and $\gamma(\delta^2 - q^4) = 0$. This completes the proof of Case 1.

(2) Assume $\alpha \neq 0$ and $\delta = 0$. An argument like that in the previous paragraph shows that $\dim_{\Bbbk}(A/I) = \infty$ if and only if $\gamma = \boldsymbol{\lambda}(q\alpha)$ and $\beta(\alpha^2 - q^{-4}) = 0$.

The proof is now complete. $\square$

Recall that $\mathbb{L}_A$ denotes the set of A-lines in $\mathbb{A}^3_{x_1,x_2,x_3}$. Let

$$\mathcal{S} \;=\; \{(1,2,3), (2,3,1), (3,1,2)\}$$

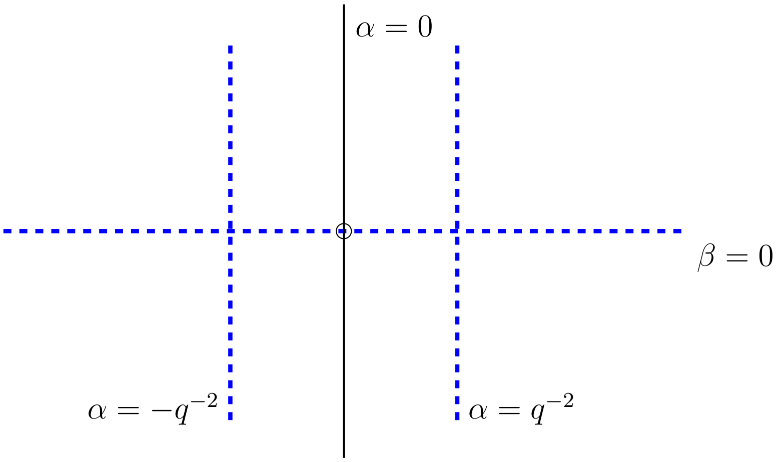
be the set of all cyclic permutations of $(1,2,3)$, and let

$$\mathcal{T} \;=\; \left\{(\alpha,\beta) \in \Bbbk^2 \mid \alpha \neq 0 \;\text{ and }\; \left(\alpha^2 - q^{-4}\right)\beta = 0\right\}.$$

Theorem 6.3 says that there is a bijection

$$\begin{aligned} \mathcal{S} \times \mathcal{T} \;&\longrightarrow\; \mathbb{L}_A, \\ (i,j,k) \times (\alpha,\beta) \;&\mapsto\; \{x_i - \boldsymbol{\lambda}(q\alpha) \;=\; x_j - \alpha x_k - \beta \;=\; 0\}. \end{aligned}$$

The set $\mathcal{T}$ consists of the points on the three dashed lines in $\mathbb{A}^2_{\alpha,\beta}$, with the origin excluded, in the following picture:



Corollary 6.4. *Every proper quotient of a line module is finite dimensional.*

Proof. Let M be a line module over A and N a non-zero submodule.

By Theorem 6.3, there is a left ideal I of A such that $M \cong A/I$ and $I + \Bbbk[z] = A$ for some $z \in \{x_1, x_2, x_3\}$. Hence M is isomorphic to $\Bbbk[z]$ as a left $\Bbbk[z]$-module. But every quotient of $\Bbbk[z]$ by a non-zero ideal has finite dimension, so M/N has finite dimension. □

Conversely, if V is a finite dimensional A-module there may not be a line module that maps onto it. However, if V is a finite dimensional simple A-module then it is always a quotient of a line module. We will prove this in Proposition 6.10.

Lemma 6.5. *If* $\ell = \{x - \boldsymbol{\lambda}(q\alpha) = y - \alpha z - \beta\}$ *is an* A*-line, then there is no* y*-eigenvector or* z*-eigenvector in* $M(\ell)$*.*

Proof. By Theorem 6.3, $A = A\ell^{\perp} \oplus \Bbbk[z]$. Hence, the image of $\{z^n \mid n \geq 0\}$ in $M(\ell)$ is a basis of $M(\ell)$. For a nonzero element in $M(\ell)$ we define its degree to be the standard degree when it is viewed as an element in $\Bbbk[z]$. More formally, let $v = \sum_{n \geq 0} \alpha_n z^n \in M(\ell) - \{0\}$ with $\alpha_n \in \Bbbk$, we define

$$\deg(v) \;:=\; \max\{n \geq 0 \mid \alpha_n \neq 0\}.$$

By Lemma 6.1(1), $\deg(yv) = \deg(zv) = \deg(v) + 1$ when $v \neq 0$.

If v is a y-eigenvector in $M(\ell)$, then $yv \in \Bbbk v$, hence $\deg(yv) \leq \deg(v)$. But, $\deg(yv) = \deg(v) + 1$, a contradiction. Therefore $M(\ell)$ has no y-eigenvector.

By a similar argument we show that there is no z-eigenvector in $M(\ell)$. $\square$

Proposition 6.6. *The isomorphism classes of line modules for A are in bijection with the A-lines via $\ell \longleftrightarrow A/A\ell^{\perp}$.*

Proof. Let ℓ and ℓ' be the A-lines

$$\begin{aligned} \ell &= \{x - \boldsymbol{\lambda}(q\alpha) = y - \alpha z - \beta = 0\}, \\ \ell' &= \{x' - \boldsymbol{\lambda}(q\alpha') = y' - \alpha' z' - \beta' = 0\}, \end{aligned}$$

where (x, y, z) and (x', y', z') are cyclic permutations of (x_1, x_2, x_3). To show that distinct A-lines give non-isomorphic line modules, it suffices to show that if there is an A-module isomorphism

$$\phi : A/A\ell'^{\perp} \xrightarrow{\sim} A/A\ell^{\perp},$$

then $\ell' = \ell$.

Let e denote the image of 1_A in $A/A\ell'^{\perp}$. Then $x'e = \lambda' e$ where $\lambda' = \boldsymbol{\lambda}(q\alpha')$, and $(y - \alpha' z - \beta')e = 0$.

To show $\ell' = \ell$, it suffices to show (1) $x' = x$, and (2) $(\alpha', \beta') = (\alpha, \beta)$, as follows.

(1) Since $x'e = \lambda' e$, it follows that

$$x'.\phi(e) = \phi(x'.e) = \lambda' \phi(e).$$

So $\phi(e)$ is an x'-eigenvector in $M(\ell')$. In view of Lemma 6.5, it is necessary that $x = x'$ and hence $(y', z') = (y, z)$.

(2) Write $\phi(e) = \sum_{i=0}^{n} \gamma_i z^i$, where $n \geq 0, \gamma_i \in \Bbbk$ and $\gamma_n \neq 0$. By Lemma 6.1(1),

$$\begin{aligned}
(y - \alpha' z - \beta').\phi(e) &= (y - \alpha' z - \beta') \left(\sum_{i=0}^{n} \gamma_i z^i \right) \\
&= \gamma_n \left(y - \alpha' z - \beta' \right) z^n + (y - \alpha' z - \beta') \left(\sum_{i=0}^{n-1} \gamma_i z^i \right) \\
&= \gamma z^{n+1} + \text{ lower degree monomials in } \Bbbk[z],
\end{aligned}$$

where $\gamma = \gamma_n(q^{-2n}\alpha - \alpha')$. On the other hand,

$$(y - \alpha' z - \beta').\phi(e) \; = \; \phi\left((y - \alpha' z - \beta').e\right) \; = \; \phi(0) \; = \; 0. \tag{6-3}$$

Hence, $\gamma = 0$. Since $\gamma_n \neq 0$, it follows that $\alpha' = q^{-2n}\alpha$.

An analogous argument shows that $\alpha = q^{-2m}\alpha'$ where $m \geq 0$. It follows that

$$\alpha \; = \; q^{-2m}\alpha' \; = \; q^{-2(m+n)}\alpha;$$

hence $q^{-2(m+n)} = 1$. Since q is not a root of unity, $m + n = 0$. But $m, n \geq 0$, so $m = n = 0$. Hence, $\alpha = \alpha'$.

Furthermore, since $\phi(e)$ is a scalar element in $M(\ell)$, it is annihilated by $y - \alpha z - \beta$. In view of (6-3), $\phi(e)$ is also annihilated by $y - \alpha' z - \beta'$. Since $\alpha = \alpha'$, $(\beta - \beta')\phi(e) = 0$ and hence $\beta = \beta'$.

The proof is now complete. □

Corollary 6.7. *There are nine 1-parameter families of A-lines. These are, for each of the three cyclic permutations* (i, j, k) *of* $(1, 2, 3)$*, the lines*

1. $\{x_i - \boldsymbol{\lambda}(q\alpha) \; = \; x_j - \alpha x_k \; = \; 0\}$*, one for each* $\alpha \in \Bbbk - \{\pm q^{-2}, 0\}$*;*

2. $\{x_i - \boldsymbol{\lambda}(q) \; = \; x_j - q^{-2}x_k - \beta \; = \; 0\}$*, one for each* $\beta \in \Bbbk$*;*

3. $\{x_i + \boldsymbol{\lambda}(q) = x_j + q^{-2}x_j - \beta = 0\}$, *one for each* $\beta \in \Bbbk$.

Proof. This is an immediate consequence of Theorem 6.3 and Proposition 6.6. $\square$

6.2.1 Shifting and translation of an A-line

Let $n \in \mathbb{Z}$ and p be a point on $\{x = 0\}$. For the A-line

$$\ell \;=\; \{x - \boldsymbol{\lambda}(q\alpha) \;=\; y - \alpha z - \beta \;=\; 0\}\,,$$

we define

$$\ell[n] \;:=\; \left\{x - \boldsymbol{\lambda}(q^{2n+1}\alpha) \;=\; y - q^{2n}\alpha z \;=\; 0\right\},$$

called a shift of ℓ, and

$$\ell + p \;:=\; \left\{(\mu,\nu,\eta) \in \mathbb{A}^3_{x,y,z} \,|\, (\mu,\nu,\eta) - p \;\in\; \ell\right\},$$

called a translation of ℓ.

Clearly, $\ell[n]$ is an A-line. If $\alpha^2 = q^{-4}$ then $\ell + p$ is also an A-line. In addition, we note that

$$\begin{aligned}\ell[m][n] &= \ell[m+n] \quad \text{and}\\ \ell + p + o &= \ell + (p + o).\end{aligned}$$

for all $m, n \in \mathbb{Z}$ and all $p, o \in \{x = 0\}$.

6.2.2 The action of Alt_4 *on A-lines and line modules*

If $\rho \in \mathrm{Alt}_4$ and ℓ is the A-line

$$\ell \;=\; \{x - \boldsymbol{\lambda}(q\alpha) \;=\; y - \alpha z - \beta \;=\; 0\}\,,$$

we define

$$\rho(\ell) \;:=\; \{\rho(x) - \boldsymbol{\lambda}(q\alpha) \;=\; \rho(y) - \alpha\rho(z) - \beta \;=\; 0\}\,.$$

Recall the equivalence $\rho^* : \mathsf{Mod}(A) \to \mathsf{Mod}(A)$ induced by an automorphism ρ of A defined in §2.3.3.

Proposition 6.8. *Let ℓ be an A-line and $\rho \in Alt_4$. Then*

1. *$\rho(\ell)$ is an A-line, and*
2. *$\rho^* M$ is a line module and isomorphic to $M(\rho(\ell))$.*

Proof. Suppose $\ell = \{x - \boldsymbol{\lambda}(q\alpha) = y - \alpha z - \beta = 0\}$. Write $M = M(\ell)$.

Since $M = A/A\ell^\perp$,

$$\rho^* M \;\cong\; \frac{A}{A\left(\rho(x) - \boldsymbol{\lambda}(q\alpha)\right) + A\left(\rho(y) - \alpha\rho(z) - \beta\right)} \;=\; \frac{A}{A\rho(\ell)^\perp},$$

where $\rho(x), \rho(y), \rho(z) \in \operatorname{span}\{1, x_1, x_2, x_3\}$.

Since M is infinite dimensional, so is $\rho^* M$. By Proposition 6.2, $\rho(\ell)$ is an A-line and $\rho^* M$ is a line module isomorphic to $M(\rho(\ell))$. □

6.3 Finite dimensional simples and line modules

In this section we reveal an important relationship between finite dimensional simple A-modules and line modules for A, that is, every finite dimensional simple A-module is a quotient of a line module.

First we find factorizations in A of the form

$$(x - \lambda)(ay + bz + \gamma) \;=\; (cy + dz + \delta)(x - \nu),$$

where $a, b, c, d, \gamma, \delta, \lambda, \nu \in \Bbbk$. The reason for doing this is as follows. Let V be a finite dimensional simple A-module. There will be some $v \in V$ that is an x-eigenvector—say

$(x-\nu)v = 0$. In trying to examine V it might be useful to determine whether there will be other x-eigenvectors in it. One is therefore led to ask whether there are elements f in $\Bbbk y + \Bbbk z + \Bbbk$ having the property that fv is an x-eigenvector. The next result addresses this question.

Lemma 6.9. *If* (x, y, z) *is a cyclic permutation of* (x_1, x_2, x_3) *and* $(\alpha, \beta) \in \Bbbk^2$ *is such that* $\alpha \neq 0$ *and* $(\alpha^2 - 1)\beta = 0$*, then*

$$(x - \boldsymbol{\lambda}(q\alpha))\left(q^2 y - \alpha z - \beta\right) \;=\; \left(y - q^2\alpha z - \beta\right)\left(x - \boldsymbol{\lambda}(q^{-1}\alpha)\right)$$

in A.

Proof. First of all, note that $(y - q^2\alpha z)(x - \boldsymbol{\lambda}(q^{-1}\alpha))$ is equal to

$$\begin{aligned}
&\left(y - q^2\alpha z\right)\left(x - \frac{q^{-1}\alpha + q\alpha^{-1}}{q^2 - q^{-2}}\right) \\
=\;& q^2 xy - qz - \alpha xz - q\alpha y - \frac{q^{-1}\alpha + q\alpha^{-1}}{q^2 - q^{-2}} y + \frac{q\alpha^2 + q^3}{q^2 - q^{-2}} z \\
=\;& x\left(q^2 y - \alpha z\right) - \frac{q\alpha + q^{-1}\alpha^{-1}}{q^2 - q^{-2}}\left(q^2 y - \alpha z\right) \\
=\;& \left(x - \frac{q\alpha + q^{-1}\alpha^{-1}}{q^2 - q^{-2}}\right)\left(q^2 y - \alpha z\right),
\end{aligned}$$

so

$$\left(y - q^2\alpha z\right)\left(x - \boldsymbol{\lambda}(q^{-1}\alpha)\right) \;=\; (x - \boldsymbol{\lambda}(q\alpha))\left(q^2 y - \alpha z\right). \tag{6-4}$$

If $\alpha^2 = 1$, then $\boldsymbol{\lambda}(q\alpha) = \boldsymbol{\lambda}(q^{-1}\alpha)$. Thus, the equality in the lemma is obtained by adding $-\beta\,(x - \boldsymbol{\lambda}(q\alpha))$ to both sides of (6-4). □

Proposition 6.10. *Every finite dimensional simple* A*-module is a quotient of a line module.*

Proof. Let V be a finite dimensional simple A-module and let (x, y, z) be a cyclic permutation of (x_1, x_2, x_3). Let

$$E \;:=\; \{\text{all eigenvalues for the action of } x \text{ on } V\}.$$

Then E is non-empty and finite.

Note that $\boldsymbol{\lambda}(\Bbbk - \{0\}) = \Bbbk$. In addition, if $\lambda \in \Bbbk$, there are at most two $\alpha \in \Bbbk - \{0\}$ such that $\boldsymbol{\lambda}(\alpha) = \lambda$. Hence, there exists some $\alpha \in \Bbbk$ such that $\boldsymbol{\lambda}(q^{-1}\alpha) \in E$ but $\boldsymbol{\lambda}(q\alpha) \notin E$.

Let v be an x-eigenvector in V with eigenvalue $\boldsymbol{\lambda}(q^{-1}\alpha)$. It follows from Lemma 6.9 that

$$(x - \boldsymbol{\lambda}(q\alpha))\left(q^2y - \alpha z\right)v \;=\; \left(y - q^2\alpha z\right)\left(x - \boldsymbol{\lambda}(q^{-1}\alpha)\right)v \;=\; 0.$$

Since $\boldsymbol{\lambda}(q\alpha) \notin E$, it is necessary that $(q^2y - \alpha z)v = 0$. Thus Av, which is equal to V, is a quotient of the A-module

$$\frac{A}{A(x - \boldsymbol{\lambda}(q^{-1}\alpha)) + A(q^2y - \alpha z)} \;=\; \frac{A}{A(x - \boldsymbol{\lambda}(q\alpha')) + A(y - \alpha' z)},$$

where $\alpha' = q^{-2}\alpha$. Since the latter is a line module, the proof is complete. □

In view of the proof of Proposition 6.10, we actually proved a stronger result, as stated below. It is useful in the classification of finite dimensional simple A-modules, since we can thus fix a cyclic permutation of (x_1, x_2, x_3) and only consider a subset of the A-lines.

Proposition 6.11. *Given a cyclic permutation (x, y, z) of (x_1, x_2, x_3), every finite dimensional simple A-module is a quotient of a line module $M(\ell)$, where*

$$\ell \;=\; \{x - \boldsymbol{\lambda}(q\alpha) \;=\; y - \alpha z \;=\; 0\}$$

for some $\alpha \in \Bbbk - \{0\}$.

Remark 6.12. (1) The A-lines in Proposition 6.11 are those in Part 1 of Corollary 6.7. Nonetheless, later on we will see that there are finite dimensional simple A-modules as quotients of line modules corresponding to the A-lines in Part 2 and Part 3 of Corollary 6.7.

(2) N. Sasom's line modules defined in her Ph.D. thesis [15, Remark 3.4.2.], which she calls Verma modules, correspond precisely to the A-lines in Part 1 of Corollary 6.7. While these A-lines are sufficient to discover all finite dimensional simple A-modules based on Proposition

6.11, it is more efficient to define some finite dimensional simple A-modules using the A-lines in Part 2 and 3 of Proposition 6.11. We will explain this in more detail in Chapter 7.

The following result serves as a guiding principle for our classification of finite dimensional simple A-modules in Chapter 7.

Proposition 6.13. *Given a cyclic permutation (x, y, z) of (x_1, x_2, x_3), every finite dimensional simple A-module V appears in an exact sequence*

$$0 \longrightarrow M(\ell') \longrightarrow M(\ell) \longrightarrow V \longrightarrow 0$$

for some A-lines

$$\begin{aligned} \ell \;&=\; \{x - \boldsymbol{\lambda}(q\alpha) \;=\; y - \alpha z - \beta \;=\; 0\} \;\; \textit{and} \\ \ell' \;&=\; \{x - \boldsymbol{\lambda}(q\alpha') \;=\; y - \alpha' z - \beta' \;=\; 0\}\,. \end{aligned}$$

Proof. By Proposition 6.10, there is a surjective homomorphism $M(\ell) \to V$ for some A-line ℓ of the form

$$\ell \;=\; \{x - \boldsymbol{\lambda}(q\alpha) \;=\; y - \alpha z - \beta \;=\; 0\}\,.$$

By Proposition 4.17, the kernel of this homomorphism is isomorphic to $M(\ell')$ for some A-line ℓ' of the form

$$\ell' \;=\; \{x' - \boldsymbol{\lambda}(q\alpha') \;=\; y' - \alpha' z' - \beta' \;=\; 0\}\,.$$

It remains to show $(x, y, z) = (x', y', z')$.

Since $M(\ell')$ is isomorphic to a submodule of $M(\ell)$, there exists a nonzero element $v \in M(\ell)$ such that $(x' - \lambda')v = 0$, where $\lambda' = \boldsymbol{\lambda}(q\alpha')$.

If x' equals y (resp. z), then v is a y-eigenvector (resp. z-eigenvector) in M. This contradicts Lemma 6.5. Hence, $x' = x$ and therefore $(x', y', z') = (x, y, z)$. $\square$

6.3.1 1-dimensional quotients of line modules

This subsection determines which line modules have 1-dimensional quotients. The methods here are elementary and independent of the general methods in Chapter 7 that determine for each finite dimensional simple module the line modules that map onto it. Nevertheless, these simple cases provide a template for what happens for higher dimensional simple modules; we say more about this in §6.3.2.

We continue to use the notation

$$\delta := \left(q - q^{-1}\right)^{-1}.$$

Also recall the notions of a shift and a translation of an A-line defined in §6.2.1.

In §3.5.1 we showed there are five 1-dimensional A-modules up to isomorphism, namely

$$S_{\mu,\nu,\eta} = \frac{A}{A(x_1 - \delta\mu) + A(x_2 - \delta\nu) + A(x_3 - \delta\eta)}$$

where (μ, ν, η) runs over the points

$$(0,0,0), \quad (1,1,1), \quad (1,-1,-1), \quad (-1,1,-1), \quad (-1,-1,1)$$

in $\mathbb{A}^3_{x_1,x_2,x_3}$.

Proposition 6.14. *Let $\alpha = \mathbb{i}q^{-1}$. The simple module $S_{0,0,0}$ is a quotient of exactly 6 line modules, namely the $M(\ell)$'s for which ℓ is one of the following lines:*

$$\begin{aligned} &\{x_1 = x_2 - \alpha x_3 = 0\}, \qquad &&\{x_1 = x_2 + \alpha x_3 = 0\}, \\ &\{x_2 = x_3 - \alpha x_1 = 0\}, \qquad &&\{x_2 = x_3 + \alpha x_1 = 0\}, \\ &\{x_3 = x_1 - \alpha x_2 = 0\}, \qquad &&\{x_3 = x_1 + \alpha x_2 = 0\}. \end{aligned}$$

Furthermore, if ℓ is one of these lines there is an exact sequence

$$0 \;\to\; M(\ell[-1]) \;\to\; M(\ell) \;\to\; S_{0,0,0} \;\to\; 0.$$

Proof. By Theorem 6.3, the lines in the statement of the proposition are A-lines. They all pass through $(0,0,0)$ so if ℓ is any one of them there is a surjective homomorphism

$$\phi : M(\ell) \;\twoheadrightarrow\; S_{0,0,0}.$$

Conversely, if $S_{0,0,0}$ is a quotient of a line module $M(\ell)$, then ℓ must pass through $(0,0,0)$. According to Corollary 6.7, the only A-lines passing through $(0,0,0)$ are those in the statement of the proposition.

We now determine the kernel of ϕ. For each of the six A-lines, we can write $\ell = \{x = y - \alpha z = 0\}$ for a suitable cyclic permutation (x, y, z) of (x_1, x_2, x_3) and an α such that $\alpha^2 + q^{-2} = 0$ (i.e., $\boldsymbol{\lambda}(q\alpha) = 0$). Let $\alpha' = q^{-2}\alpha$.

Since the kernel of ϕ is generated by the image of z as a $\Bbbk[z]$-module it is also generated by the image of z as an A-module. Since

$$(x + q\alpha)\, z \;=\; q^2 zx - q(y - \alpha z)$$

and

$$\left(y - q^{-2}\alpha z\right) z \;=\; q^{-2} z(y - \alpha z) + q^{-1} x,$$

the left ideal $A(x + q\alpha) + A\left(y - q^{-2}\alpha z\right)$ annihilates the image of z in $A/A\ell^{\perp}$. Hence there is a surjective homomorphism $A/A\ell'^{\perp} \to \ker(\phi)$ where

$$\ell' \;=\; \left\{x + q\alpha = y - q^{-2}\alpha z = 0\right\} \;=\; \{x - \boldsymbol{\lambda}(q\alpha') = y - \alpha' z = 0\} \;=\; \ell[-1].$$

Since both $M(\ell)$ and $M(\ell')$ are line modules they are critical of GK-dimension 1, whence $\ker(\phi) \cong M(\ell')$.

Therefore, there is an exact sequence $0 \to M(\ell[-1]) \to M(\ell) \to S_{0,0,0} \to 0$ as claimed. $\square$

In a similar manner, we can identify the line modules that have $S_{1,1,1}$ as a quotient. The proof is similar to that of Proposition 6.14.

Proposition 6.15. *The simple module $S_{1,1,1}$ is a quotient of exactly 6 line modules, namely the $M(\ell)$'s for which ℓ is one of the following lines:*

$$\begin{array}{ll} \{x_1 - \delta \ = \ x_2 - x_3 \ = \ 0\}, & \{x_1 - \delta \ = \ x_2 - q^{-2}x_3 - q^{-1} \ = \ 0\}, \\ \{x_2 - \delta \ = \ x_3 - x_1 \ = \ 0\}, & \{x_2 - \delta \ = \ x_3 - q^{-2}x_1 - q^{-1} \ = \ 0\}, \\ \{x_3 - \delta \ = \ x_1 - x_2 \ = \ 0\}, & \{x_3 - \delta \ = \ x_1 - q^{-2}x_2 - q^{-1} \ = \ 0\}. \end{array}$$

Furthermore, if ℓ is one of these lines there is an exact sequence

$$0 \to M(\ell[-1] + p) \ \to \ M(\ell) \ \to \ S_{1,1,1} \ \to \ 0,$$

where p equals $(0, q^{-1}, 0)$ if ℓ is one of three A-lines on the left, and equals $(0,0,0)$ otherwise.

Proof. Arguments like those in the proof of Proposition 6.14 show $S_{1,1,1}$ is a quotient of exactly the line modules associated to these six lines. We leave the details to the reader.

We now determine the kernel of the surjection $M(\ell) \twoheadrightarrow S_{1,1,1}$, where $\ell = \{x - \delta = y - z = 0\}$. Since the kernel is the submodule generated by the image of $z - \delta$ in $M(\ell)$, the kernel is isomorphic to A/J, where $J = \{a \in A \mid a\,(z-\delta) \in A\,(x-\delta) + A(y-z)\}$.

Since

$$\big(y - q^{-2}z - q^{-1}\big)\,(z-\delta) \ = \ \big(q^{-2}z - \delta\big)\,(y-z) + q^{-1}(x-\delta)$$

and

$$(x-\delta)\,(z-\delta) \ = \ -q(y-z) + \big(q^2 z - \delta\big)\,(x-\delta)\,,$$

$J \ = \ A\,(x-\delta) + A\,(y - q^{-2}z + q^{-1}) \ = \ A\,(x - \boldsymbol{\lambda}(q\alpha)) + A\,(y - \alpha z + q^{-1})$, where $\alpha \ = \ q^{-2}$. Hence, $A/J \cong M(\ell')$, where $\ell' = \{x - \delta = y - q^{-2}z + q^{-1} = 0\} = \ell[-1] + p$. It follows that there is a short exact sequence of the type claimed.

Similarly, if $\ell = \{x - \delta = y - q^{-2}z - q^{-1} = 0\}$, then $\ell' = \{x - \delta - \delta^{-1} = y - q^{-4}z = 0\} = \ell[-1]$. $\square$

The line modules that map onto the other three 1-dimensional A-modules can be obtained using the action of the automorphism group Alt_4 on the line modules. Recall the action of Alt_4 on the A-lines defined in §6.2.1. Let $\rho \in \mathrm{Alt}_4$ and S a 1-dimensional A-module. If there is a surjection $M(\ell) \twoheadrightarrow S$, it induces a surjection

$$\rho^* M(\ell) \ \twoheadrightarrow \ \rho^* S,$$

where $\rho^* M(\ell) \cong M(\rho(\ell))$ by Proposition 6.8(2). In addition, by Proposition 6.6, $M(\rho(\ell))$ and $M(\ell)$ are not isomorphic unless ρ is the identity map.

In Corollary 3.9, we observed that the orbit of $S_{1,1,1}$ under the action of Alt_4 is

$$\{S_{1,1,1},\ S_{1,-1,-1},\ S_{-1,-1,1},\ S_{-1,1,-1}\}.$$

Using the action of Alt_4, we have the following facts.

- The simple module $S_{1,-1,-1}$ is a quotient of $M(\ell)$ if and only if ℓ is either

  $$\{x - \delta = y - z = 0\} \quad \text{or} \quad \left\{x - \delta = y - q^{-2}z + q^{-1} = 0\right\}$$

  for some cyclic permutation (x, y, z) of (x_1, x_2, x_3).

- The simple module $S_{-1,1,-1}$ is a quotient of the line module $M(\ell)$ if and only if ℓ is either

  $$\{x + \delta = y + z = 0\} \quad \text{or} \quad \left\{x + \delta = y + q^{-2}z - q^{-1} = 0\right\}$$

  for some cyclic permutation (x, y, z) of (x_1, x_2, x_3).

- The simple module $S_{-1,-1,1}$ is a quotient of the line module $M(\ell)$ if and only if ℓ is either

$$\{x+\delta = y+z = 0\} \quad \text{or} \quad \left\{x+\delta = y+q^{-2}z+q^{-1} = 0\right\}$$

for some cyclic permutation (x, y, z) of (x_1, x_2, x_3).

In particular, both $S_{1,1,1}$ and $S_{1,-1,-1}$ are quotients of the line module $M(\ell)$ with $\ell = \{x-\delta = y-z = 0\}$. Both $S_{-1,1,-1}$ and $S_{-1.-1.1}$ are quotients of the line module $M(\ell')$ with $\ell' = \{x+\delta = y+z = 0\}$.

In summary, every 1-dimensional simple A-module is a quotient of exactly six line modules. In total, there are 24 line modules that map onto 1-dimensional A-modules.

6.3.2 Looking ahead

One reason for working out in detail what happens for the 1-dimensional A-modules is that the higher dimensional simple A-modules behave in a remarkably similar way, as shown in Chapter 7. For example, for each $n \geq 1$

1. there are five n-dimensional simple A-modules up to isomorphism;
2. every n-dimensional simple A-module S appears in an exact sequence $0 \to M(\ell[-n]) \to M(\ell) \to S \to 0$ for some A-line ℓ;
3. for every finite dimensional simple A-module there are exactly six line modules that map onto it;
4. there are exactly twenty-four A-lines for which the corresponding line modules have quotients of n-dimensional A-modules;
5. there are two orbits for the action of Alt_4 on the set of the isomorphism classes of n-dimensional simple A-modules, one of length 1 and one of length 4.

Such similarities provide a good template to test our work on higher dimensional simple A-modules in Chapter 7. The proofs for these properties, however, are not as elementary as those for the 1-dimensional A-modules.

6.4 Geometric classification of the A-lines

In this section we give a geometric description for the A-lines. The geometric classification complements the algebraic classification in Corollary 6.7.

Lemma 6.16.

1. *In $\mathbb{A}^3_{x,y,z}$, the lines*

$$\{x - \boldsymbol{\lambda}(q\alpha) \;=\; y - \alpha z - \beta \;=\; 0\}$$

and

$$\left\{x - \boldsymbol{\lambda}(q^{-1}\alpha^{-1}) \;=\; y - q^{-2}\alpha^{-1} z - \beta \;=\; 0\right\}$$

lie on the surface

$$\left(q^2 - q^{-2}\right) x(y-\beta)z - q(y-\beta)^2 - q^{-1}z^2 \;=\; 0.$$

Furthermore, $\boldsymbol{\lambda}(q\alpha) = \boldsymbol{\lambda}(q^{-1}\alpha^{-1})$ so these two lines are the intersection of this surface with the plane $x = \boldsymbol{\lambda}(q\alpha)$.

2. *The A-line $\{x - \boldsymbol{\lambda}(q\alpha) = y - \alpha z = 0\}$ lies on the surface*

$$\left(q^2 - q^{-2}\right) xyz - qy^2 - q^{-1}z^2 \;=\; 0.$$

Proof. This is because $(q^2 - q^{-2})\, x(y-\beta)z - q(y-\beta)^2 - q^{-1}z^2$ equals

$$\left(q^2 - q^{-2}\right)\left(x - \boldsymbol{\lambda}(q\alpha)\right)(y-\beta)z \;-\; q(y - \alpha z - \beta)\left(y - q^{-2}\alpha^{-1}z - \beta\right).$$

The second statement is a special case of the first. □

Lemma 6.17. *Let S be the surface in $\mathbb{A}^3 = \operatorname{Spec}(\Bbbk[X, Y, z])$ given by $qY^2+XYz+q^{-1}z^2 = 0$. The lines on S are*

1. *$\{Y = z = 0\}$, which is the singular locus of S, and*
2. *the lines $\{X + q\alpha + q^{-1}\alpha^{-1} = Y - \alpha z = 0\}$ for all $\alpha \in \Bbbk - \{0\}$.*

Proof. Let $f = qY^2 + XYz + q^{-1}z^2$. It is clear that the line $\{Y = z = 0\}$ lies on S and that the partial derivatives f_X, f_Y, and f_z, vanish at all points on $\{Y = z = 0\}$.

The intersection of S with the open subset $\{z \neq 0\} = \operatorname{Spec}(\Bbbk[X, Y, z^{\pm 1}])$ is the zero locus of $q(Yz^{-1})^2 + X(Yz^{-1}) + q^{-1} = 0$. This is non-singular because $q \neq 0$. Hence $\{Y = z = 0\}$ is the singular locus of S.

Let $\alpha \in \Bbbk - \{0\}$. The line $\{X + q\alpha + q^{-1}\alpha^{-1} = Y - \alpha z = 0\}$ lies on S because

$$qY^2 + XYz + q^{-1}z^2 \;=\; \left(qY - q^{-1}\alpha^{-1}z\right)(Y - \alpha z) \;+\; \left(X + q\alpha + q^{-1}\alpha^{-1}\right)Yz.$$

On the other hand, a line ℓ on S can be represented by

$$sY - tz \;=\; 0 \;=\; t(qY + xz) + q^{-1}sz \tag{6-5}$$

for some $(s, t) \in \Bbbk^2$. If $st = 0$ then $\ell = \{Y = z = 0\}$. Assume now that $\ell \neq \{Y = z = 0\}$. Thus, $st \neq 0$.

It follows from (6-5) that $(X + qs^{-1}t + q^{-1}st^{-1})\,Y = Y - s^{-1}tz = 0$. If $\ell \subset \{Y = 0\}$ then $\ell = \{Y = z = 0\}$, contradicting to our assumption that $\ell \neq \{Y = z = 0\}$. Hence,

$$\ell \;=\; \left\{X + pst^{-1} + s^{-1}t = Ys^{-1}tz = 0\right\}.$$

Write $\alpha = s^{-1}t \in \Bbbk - \{0\}$. Thus, $\ell = \{X + q\alpha + q\alpha^{-1} = Y - \alpha z = 0\}$. □

Proposition 6.18. *Let $q \in \Bbbk - \{0\}$ and assume that q is not a root of unity. Let $\beta \in \Bbbk$ and let $S \subseteq \mathbb{A}^3_{x,y,z}$ be the surface*

$$\left(q^2 - q^{-2}\right) x(y-\beta)z - q(y-\beta)^2 - q^{-1}z^2 \;=\; 0.$$

The lines on S consist of

1. *the line $\{y - \beta = z = 0\}$, which is the singular locus of S, and*
2. *the lines $\{x - \boldsymbol{\lambda}(q\alpha) = y - \alpha z - \beta = 0\}$ for all $\alpha \in \Bbbk - \{0\}$.*

The line in (1) is not an A-line. The line in (2) is an A-line if and only if $\left(\alpha^2 - q^{-4}\right)\beta = 0$.

Proof. Let $Y = y - \beta$ and $X = -\left(q^2 - q^{-2}\right)x$. The surface S is now given by the equation $qY^2 + XYz + q^{-1}z^2 = 0$. By Lemma 6.17(1), the singular locus of S is the line $\{y - \beta = z = 0\}$, which is not in $\mathbb{L}_A$ because $A/A(y-\beta) + Az = A/Ax + Ay + Az$ and this is certainly not a line module.

By Lemma 6.17(2), the other lines on S are the zero loci of the equations

$$-\left(q^2 - q^{-2}\right)x + q\alpha + q^{-1}\alpha^{-1} \;=\; y - \alpha z - \beta \;=\; 0,$$

where $\alpha \in \Bbbk - \{0\}$ and $\beta \in \Bbbk$. Since $\boldsymbol{\lambda}(\alpha) = (\alpha + \alpha^{-1})/\left(q^2 - q^{-2}\right)$, these are the lines $\{x - \boldsymbol{\lambda}(q\alpha) = y - \alpha z - \beta = 0\}$.

By Theorem 6.3, the line $\{x - \boldsymbol{\lambda}(q\alpha) = y - \alpha z - \beta = 0\}$ belongs to $\mathbb{L}_A$ if $\left(\alpha^2 - q^{-4}\right)\beta = 0$. This finishes the proof. $\square$

The observation in Proposition 6.18 that the A-lines lie on a 1-parameter family of cubic surfaces has a nice algebraic analogue: the line modules are annihilated by a 1-parameter family of cubic central elements. To state it we use the central element

$$C \;=\; -\left(q^2 - q^{-2}\right)x_1x_2x_3 + qx_1^2 + q^{-3}x_2^2 + qx_3^2$$

identified in Proposition 3.10.

Proposition 6.19. *Let (x, y, z) be a cyclic permutation of (x_1, x_2, x_3) and*

$$\ell \;=\; \{x - \boldsymbol{\lambda}(q\alpha) = y - \alpha z - \beta = 0\} \;\in\; \mathbb{L}_A,$$

where $(\alpha^2 - q^{-4})\,\beta = 0$. Let $\mu \in \Bbbk$. The following three statements are equivalent:

1. *$M(\ell)$ is annihilated by $C - \mu$;*
2. *ℓ lies on the surface*

$$q^3x(y-\beta)^2 + q^{-3}xz^2 + \left(q^3 - q^{-1}\right)(\mu - q\beta^2)(y-\beta)z \;=\; 0;$$

3. *$\mu \;=\; q^{-1}\boldsymbol{\lambda}^2(q^2\alpha) + q\beta^2 + q\,(1+q^2)^{-2}$.*

Proof. Write $\lambda := \boldsymbol{\lambda}(q^2\alpha)$ for short.

(3) ⇔ (1) By Remark 3.11(2), the element C equals $-(q^2 - q^{-2})yzx + qy^2 + q^{-3}z^2 + qx^2$. A straightforward calculation using the defining relations for A shows that

$$C \;\in\; q\alpha^{-1}\left(\alpha^2 - q^{-4}\right)\beta z + q^{-1}\lambda^2 + q\beta^2 + q(1+q^2)^{-2} + A\ell^{\perp}.$$

Since $(\alpha^2 - q^{-4})\,\beta = 0$, the result follows.

(3) ⇒ (2) If $\mu = q^{-1}\lambda^2 + q\beta^2 + q(1+q^2)^{-2}$, then it is straightforward to verify that the element

$$q^3x(y-\beta)^2 + q^{-3}xz^2 + \left(q^3 - q^{-1}\right)\left(\mu - q\beta^2\right)(y-\beta)z$$

is equal to

$$(x-\lambda)\left[q^3(y+\beta)^2 + q^{-3}z^2\right] + \lambda\left(q^3y - q^{-3}\alpha^{-1}z - q^3\beta\right)(y - \alpha z - \beta).$$

(2) ⇒ (3) If ℓ lies on the surface in (2), then the relations $x = \boldsymbol{\lambda}(q\alpha)$ and $y = \alpha z + \beta$ satisfy the equation of the surface. Hence,

$$\begin{aligned}
0 &= x\left[q^3(y-\beta)^2 + q^{-3}z^2\right] + \left(q^3 - q^{-1}\right)(\mu - q\beta^2)(y-\beta)z \\
&= \boldsymbol{\lambda}(q\alpha)\left(q^3\alpha^2z^2 + q^{-3}z^2\right) + \left(q^3 - q^{-1}\right)(\mu - q\beta^2)\alpha z^2 \\
&= \left[\boldsymbol{\lambda}(q\alpha)\left(q^3\alpha^2 + q^{-3}\right) + \left(q^3 - q^{-1}\right)(\mu - q\beta^2)\alpha\right]z^2.
\end{aligned}$$

It follows that $\boldsymbol{\lambda}(q\alpha)\left(q^3\alpha + q^{-3}\alpha^{-1}\right) + \left(q^3 - q^{-1}\right)(\mu - q\beta^2) = 0$, which implies (3). □

We conclude this chapter by a useful application of Proposition 6.19. Let M and M' be line modules for A. By Proposition 6.19 there are $\mu, \mu' \in \Bbbk$ such that $(C - \mu)M = (C - \mu')M' = 0$. If M' can be embedded in M, then it is necessary that $\mu' = \mu$. This observation leads to the next result, which describes the relations between M and M'. We will revisit this result in Proposition 7.11.

Corollary 6.20. *Let ℓ and ℓ' be the A-lines*

$$\begin{aligned}
\ell &= \{x - \boldsymbol{\lambda}(q\alpha) \,=\, y - \alpha z - \beta \,=\, 0\}, \\
\ell' &= \{x - \boldsymbol{\lambda}(q\alpha') \,=\, y - \alpha' z - \beta' \,=\, 0\},
\end{aligned}$$

where (x, y, z) is a cyclic permutation of (x_1, x_2, x_3). Suppose $M(\ell')$ is isomorphic to a submodule of $M(\ell)$.

1. *If $\beta = \beta' = 0$, then $\alpha' \in \{\pm\alpha, \pm q^{-4}\alpha^{-1}\}$;*

2. *If $\alpha'^2 = q^{-4}$ and $\beta = 0$, then*

$$\beta'^2 = \frac{1}{q^2}\left(\frac{q^2\alpha - q^{-2}\alpha^{-1}}{q^2 - q^{-2}}\right)^2;$$

3. If $\alpha^2 = q^{-4}$ *and* $\beta' = 0$*, then*

$$\beta^2 \;=\; \frac{1}{q^2}\left(\frac{q^2\alpha' - q^{-2}\alpha'^{-1}}{q^2 - q^{-2}}\right)^2.$$

Proof. Write $M = M(\ell)$ and $M' = (\ell')$ for short.

By Proposition 6.19, M and M' are annihilated by $C - \mu$ and $C - \mu'$, respectively, where

$$\mu \;=\; q^{-1}\boldsymbol{\lambda}^2(q^2\alpha) + q\beta^2 + \frac{q}{(1+q^2)^2} \quad \text{and} \quad \mu' \;=\; q^{-1}\boldsymbol{\lambda}^2(q^2\alpha') + q\beta'^2 + \frac{q}{(1+q^2)^2}.$$

Since $M' \subset M$, $\mu = \mu'$. It follows that

$$q^{-1}\boldsymbol{\lambda}^2(q^2\alpha) + q\beta^2 \;=\; q^{-1}\boldsymbol{\lambda}^2(q^2\alpha') + q\beta'^2.$$

The three cases in the statement of the corollary are derived from the equality above:

(1) If $\beta = \beta' = 0$, then $\boldsymbol{\lambda}^2(q^2\alpha) = \boldsymbol{\lambda}^2(q^2\alpha')$ and hence $\boldsymbol{\lambda}(q^2\alpha) = \pm\boldsymbol{\lambda}(q^2\alpha)$. It follows that $\alpha' = \pm\alpha$ or $\alpha' = \pm q^{-4}\alpha^{-1}$.

(2) If $\alpha'^2 = q^{-4}$ and $\beta = 0$, then $q^{-1}\boldsymbol{\lambda}^2(q^2\alpha) = q^{-1}\boldsymbol{\lambda}^2(1) + q\beta'^2$. It follows that

$$\beta'^2 \;=\; q^{-2}\left(\boldsymbol{\lambda}^2(q^2\alpha) - \boldsymbol{\lambda}^2(1)\right) \;=\; q^{-2}\left(\frac{q^2\alpha - q^{-2}\alpha^{-1}}{q^2 - q^{-2}}\right)^2.$$

(3) This can be proved by a similar argument in Part 2. □

Chapter 7

FINITE DIMENSIONAL SIMPLE A-MODULES

In this chapter, we will classify all finite dimensional simple modules over A. The main result is that there are five non-isomorphic simple A-modules of each dimension.

Proposition 6.13 shows that every finite dimensional simple A-module S appears in an exact sequence

$$0 \to M(\ell') \to M(\ell) \to S \to 0$$

in which ℓ and ℓ' are A-lines. Inspired by this fact, our approach is to determine all non-simple line modules for A and when one line module can be embedded into another. This approach is in the same spirit as the method N. Sasom used in her Ph.D. thesis [15]. Notwithstanding, compared with her method ours has several improvements: First, we show that all but four types of line modules for A are simple using the action of the central element in A identified in §3.6 on line modules, while Sasom achieved a similar result mainly relying on certain basis elements for line modules. From a technical standpoint, our method is simpler and cleaner.

Secondly, as mentioned in Remark 6.12(2), Sasom's line modules, which she called Verma modules, form a proper subset of our line modules — those corresponding to the A-lines

$$\{x - \boldsymbol{\lambda}(q\alpha) \;=\; y - \alpha z \;=\; 0\} \tag{7-1}$$

as (x, y, z) runs over all the cyclic permutations of (x_1, x_2, x_3) and $\alpha \in \Bbbk - \{0\}$. Although this subset of line modules is sufficient for classifying finite dimensional simple modules, we additionally consider the line modules corresponding to the A-lines

$$\left\{x - \boldsymbol{\lambda}(q) \;=\; y - q^{-2}z - \beta \;=\; 0\right\} \qquad \text{and} \qquad \left\{x + \boldsymbol{\lambda}(q) \;=\; y + q^{-2}z - \beta \;=\; 0\right\},$$

where $\beta \in \Bbbk$. For certain non-zero β, such line modules have unique proper submodules, modulo which their quotients lead to four simple A-modules of each dimension. If, instead, we restrict ourselves to A-lines in the form of (7-1), we would have to use more complex line modules to discover the same simple A-modules. Thus, our version of line modules provides a more efficient way to identify finite dimensional simple A-modules.

Thirdly, both Sasom and the author construct certain bases of line modules for A consisting of eigenvectors (or generalized eigenvectors) for the action of some $x \in \{x_1, x_2, x_3\}$. However, Sasom's construction ([15, Section 3.6]) is somewhat implicit and descriptive. In contrast, we explicitly define the bases in an inductive manner using the elements f_n introduced in §7.2. Thus, it is revealed that the x-eigenvectors are related with each other in a very nice way: when f_n's act on a line module they send eigenvectors to eigenvectors, and as they do the corresponding eigenvalues form a $\boldsymbol{\lambda}$-sequence.

We begin this chapter by introducing the $\boldsymbol{\lambda}$-sequence. As its name suggested, it is defined based on the $\boldsymbol{\lambda}$-function introduced in §5.3.2.

7.1 $\boldsymbol{\lambda}$-sequences

Let $\alpha \in \Bbbk - \{0\}$. For each $n \in \mathbb{Z}$ we define

$$\lambda_n(\alpha) \ := \ \boldsymbol{\lambda}\left(q^{-2n+1}\alpha\right) \ = \ \frac{q^{-2n+1}\alpha + q^{2n-1}\alpha^{-1}}{q^2 - q^{-2}}$$

and call the sequence $(\lambda_n(\alpha))_{n\in\mathbb{Z}}$, denoted by $\Lambda(\alpha)$, a $\boldsymbol{\lambda}$-sequence with parameter α, or simply $\boldsymbol{\lambda}$-sequence. When α is fixed, we may simply write λ_n for $\lambda_n(\alpha)$.

Lemma 7.1. *Given a $\boldsymbol{\lambda}$-sequence $\Lambda(\alpha) = (\lambda_n)_{n\in\mathbb{Z}}$,*

1. *the scalars in $\Lambda(\alpha)$ are mutually distinct if and only if $\alpha^2 \notin q^{2\mathbb{Z}}$;*

2. *if $\alpha^2 \in q^{4\mathbb{Z}}$, say $\alpha^2 = q^{4r}$, then $\lambda_n = \lambda_{2r+1-n}$ for all $n \in \mathbb{Z}$ and $\lambda_n \neq \lambda_m$ for all $m > n > r$;*

3. *if* $\alpha^2 \in q^{4\mathbb{Z}-2}$*, say* $\alpha^2 = q^{4r-2}$*, then* $\lambda_n = \lambda_{2r-n}$ *for all* $n \in \mathbb{Z}$ *and* $\lambda_n \neq \lambda_m$ *for all* $m > n \geq r$*.*

Proof. (1) Let $s, t \in \mathbb{Z}$. It is easy to verify that

$$q^{-2s+1}\alpha + q^{2s-1}\alpha^{-1} \;=\; q^{-2t+1}\alpha + q^{2t-1}\alpha^{-1}$$

if and only if $\alpha = \pm q^{s+t-1}$. Thus, $\lambda_s = \lambda_t$ for some $s \neq t$ if and only if $\alpha \in \pm q^{\mathbb{Z}}$, or equivalently, $\alpha^2 \in q^{2\mathbb{Z}}$.

(2) Since $\alpha^{-1} = q^{-4r}\alpha$, it follows that for all $n \in \mathbb{Z}$

$$\lambda_{2r+1-n} \;=\; \boldsymbol{\lambda}\left(q^{2n-4r-1}\alpha\right) \;=\; \boldsymbol{\lambda}\left(q^{-2n+4r+1}\alpha^{-1}\right) \;=\; \boldsymbol{\lambda}(q^{-2n+1}\alpha) \;=\; \lambda_n.$$

If $\lambda_m = \lambda_n$ for some $m > n > r$, then $q^{2(m+n-1)} = \alpha^2 = q^{4r}$, so that $m + n = 2r + 2$ as q is not a root of unity. But this is impossible.

(3) This can be proved by a similar argument in Part 2. $\square$

One observation from Lemma 7.1 is that no three elements in a $\boldsymbol{\lambda}$-sequence are equal. This property will play a crucial role in the structure of a line module for A as a module over a polynomial ring, as discussed in §7.3.

Write $\Lambda_{\geq 0}(\alpha) = (\lambda_n(\alpha) \mid n \geq 0)$ for $\alpha \in \Bbbk - \{0\}$.

Lemma 7.2. *Let* $\alpha, \alpha' \in \Bbbk - \{0\}$*. If* $\Lambda_{\geq 0}(\alpha') \subsetneq \Lambda_{\geq 0}(\alpha)$*, then* $\alpha' \in q^{-2\mathbb{N}-2}\alpha$*.*

Proof. Above all, since $\Lambda_{\geq 0}(\alpha') \neq \Lambda_{\geq 0}(\alpha)$, $\alpha' \neq \alpha$.

Since $\Lambda_{\geq 0}(\alpha') \subset \Lambda_{\geq 0}(\alpha)$, there exists some $n_0 \geq 0$ such that $\lambda_0(\alpha') = \lambda_{n_0}(\alpha)$, that is,

$$q\alpha' + q^{-1}\alpha'^{-1} \;=\; q^{-2n_0+1}\alpha + q^{2n_0-1}\alpha^{-1},$$

which implies that either $\alpha' = q^{-2n_0}\alpha$ or $\alpha' = q^{2n_0-2}\alpha^{-1}$. If the former occurs, then $\alpha' \in q^{-2\mathbb{N}}\alpha$. But $\alpha' \neq \alpha$, so $\alpha' \in q^{-2\mathbb{N}-2}\alpha$. We are done.

Now suppose $\alpha' = q^{2n_0-2}\alpha^{-1}$. We further consider three cases: (1) $\alpha^2 \notin q^{2\mathbb{Z}}$, (2) $\alpha^2 \in q^{4\mathbb{Z}}$, and (3) $\alpha^2 \in q^{4\mathbb{Z}-2}$, as discussed below.

(1) Suppose $\alpha^2 \notin q^{2\mathbb{Z}}$. For $n \geq 0$

$$\lambda_n(\alpha') \;=\; \boldsymbol{\lambda}\left(q^{-2n+2n_0-1}\alpha^{-1}\right) \;=\; \boldsymbol{\lambda}(q^{2n-2n_0+1}\alpha) \;=\; \lambda_{-n+n_0}(\alpha).$$

By Lemma 7.1(1), the elements in $\Lambda(\alpha)$ are mutually distinct. Hence, $\lambda_n(\alpha') \notin \Lambda_{\geq 0}(\alpha)$ when $n > n_0$. This is a contradiction. So, $\alpha^2 \in q^{2\mathbb{Z}}$, or equivalently, either $\alpha^2 \in q^{4\mathbb{Z}}$ or $\alpha^2 \in q^{4\mathbb{Z}-2}$.

(2) Suppose $\alpha^2 \in q^{4\mathbb{Z}}$, say $\alpha^2 = q^{4r}$ where $r \in \mathbb{Z}$. Then $\alpha' = q^{2n_0-2}\alpha^{-1} = q^{2(n_0-2r-1)}\alpha$. Write $t = n_0 - 2r - 1$. It follows that

$$\lambda_{n+t}(\alpha') \;=\; \lambda_{n+t}\left(q^{2t}\alpha\right) \;=\; \lambda_n(\alpha)$$

for all $n \in \mathbb{Z}$.

If $t \geq 0$, then $\lambda_n(\alpha) \in \Lambda_{\geq 0}(\alpha')$ for all $n \geq 0$, so that $\Lambda_{\geq 0}(\alpha) \subset \Lambda_{\geq 0}(\alpha')$ and hence $\Lambda_{\geq 0}(\alpha) = \Lambda_{\geq 0}(\alpha')$. This is a contradiction. Hence, $t < 0$ and thus $\alpha' = q^{2t}\alpha \in q^{-2\mathbb{N}-2}\alpha$.

(3) When $\alpha^2 \in q^{4\mathbb{Z}-2}$, by a similar argument in Part (2) we show $\alpha' \in q^{-2\mathbb{N}-2}\alpha$. □

7.2 *The elements f_n and f_n^- in A*

Given a cyclic permutation (x, y, z) of (x_1, x_2, x_3) and $\alpha \in \Bbbk$, we define

$$f_n(\alpha) \;:=\; q^{-2n+2}\alpha y - z \qquad \text{and} \qquad f_n^-(\alpha) \;:=\; q^{2n-2}\alpha^{-1}y - q^{-2}z$$

for $n \in \mathbb{Z}$. Note that $f_n^-(\alpha) = q^{-2}f_{-n+1}(\alpha^{-1})$.

When α is fixed, we may simply write f_n (resp. f_n^-) for $f_n(\alpha)$ (resp. $f_n^-(\alpha)$).

We understand the elements $f_n(\alpha)$ and $f_n^-(\alpha)$ are dependent on some presupposed cyclic permutation (x, y, z), which is not reflected in our notation. To avoid ambiguity, $f_n(\alpha)$ and $f_n^-(\alpha)$ are always associated with the cyclic permutation (x, y, z) used in the context, unless specified otherwise.

Recall our notation: For $n \in \mathbb{N}$, we denote

$$[n] \ := \ \frac{q^{2n} - q^{-2n}}{q^2 - q^{-2}}.$$

Lemma 7.3. *Let $m, n \in \mathbb{Z}$. Suppose $\alpha^2 \neq q^{4n-2}$ and write*

$$\delta \ := \ \left(q^{-2n+1}\alpha - q^{2n-1}\alpha^{-1}\right)^{-1}.$$

The following relations are true in A:

1. $y \ = \ (q^{-1} f_n - q f_n^-)\, \delta;$

2. $z \ = \ (q^{2n-1}\alpha^{-1} f_n - q^{-2n+3}\alpha f_n^-)\, \delta;$

3. $f_m \ = \ [(q^{-2m+1}\alpha - q^{2n-1}\alpha^{-1})\, f_n + (q^{-2n+3} - q^{-2m+3})\, \alpha f_n^-]\, \delta;$

4. $f_m^- \ = \ [(q^{2m-3} - q^{2n-3})\, \alpha^{-1} f_n + (q^{-2n+1}\alpha - q^{2m-1}\alpha^{-1})\, f_n^-]\, \delta.$

Proof. This is linear algebra; we leave it to the reader. □

Lemma 7.4. *Given a $\boldsymbol{\lambda}$-sequence $(\lambda_n(\alpha))_{n \in \mathbb{Z}}$, the following equalities hold in A:*

1. $(x - \lambda_{n+1}) f_n \ = \ q^2 f_{n+2} (x - \lambda_n);$

2. $(x - \lambda_{n-1})\, f_n^- \ = \ q^2 f_{n-2}^- \,(x - \lambda_n);$

3. $f_{n+1}^- f_n - f_{n-1} f_n^- \ = \ (q^{-2n+1}\alpha - q^{2n-1}\alpha^{-1})\, x.$

Proof. By Lemma 6.9, there is an identity in A:

$$(x - \boldsymbol{\lambda}(q\alpha'))\left(q^2 y - \alpha' z\right) \ = \ \left(y - q^2 \alpha' z\right)\left(x - \boldsymbol{\lambda}(q^{-1}\alpha')\right), \tag{7-2}$$

where $\alpha' \in \Bbbk - \{0\}$.

(1) This is obtained from (7-2) by replacing α' with $q^{2n}\alpha^{-1}$.

(2) This is obtained from (7-2) by replacing α' with $q^{-2n+2}\alpha$.

(3) Write $\gamma = q^{-2n+2}\alpha$. Then $f_n = \gamma y - z$ and $f_n^- = \gamma^{-1}y - q^{-2}z$. So $z = q^2\gamma^{-1}y - q^2 f_n^-$. It follows that

$$\begin{aligned} f_{n+1}^- f_n &= \left(q^2\gamma^{-1}y - q^{-2}z\right)(\gamma y - z) \\ &= q^2y^2 - q^2\gamma^{-1}yz - q^{-2}\gamma zy + q^{-2}z^2 \\ &= q^2y^2 - q^2\gamma^{-1}yz - \gamma\left(yz - q^{-1}x\right) + q^{-2}z\left(q^2\gamma^{-1}y - q^2 f_n^-\right) \\ &= q^2y^2 - (q^2\gamma^{-1} + \gamma)yz + q^{-1}\gamma x - \gamma^{-1}\left(q^2yz - qx\right) - zf_n^- \\ &= q^2\gamma y\left(\gamma^{-1}y - q^{-2}z\right) - zf_n^- + \left(q^{-1}\gamma - q\gamma\right)x \\ &= f_{n-1}f_n^- + \left(q^{-1}\gamma - q\gamma\right)x. \end{aligned}$$

This proves the desired equality. □

The relevance of the elements f_n and f_n^- becomes apparent in the next result: roughly speaking, when acting on an A-module they send x-eigenvectors to x-eigenvectors and, as they do this, the new eigenvalues fit into a $\boldsymbol{\lambda}$-sequence (or more precisely, the $\boldsymbol{\lambda}$-sequence with positive indices).

Proposition 7.5. *Let v_0 be an x-eigenvector in an A-module with eigenvalue $\boldsymbol{\lambda}(q\alpha)$, where $\alpha \in \mathbb{k} - \{0\}$. Let $\beta \in \mathbb{k}$. Define elements*

$$v_n := f_{n-1}(\alpha).v_{n-1} \quad \text{for } n \geq 1.$$

Then

1. $x.v_n = \lambda_n(\alpha)\, v_n$ *for all* $n \geq 0$, *and*

2. $f_n^-(\alpha).v_n = \mu_n v_{n-1}$ *for all* $n \geq 1$, *if, in addition,* $f_0^-(\alpha).v_0 = q^2\alpha\beta v_0$ *and* $(\alpha^2 - q^{-4})\,\beta =$

0, *where*

$$\mu_n \ = \ \frac{q^{-2n+4}\alpha^2 - q^{2n-4}\alpha^{-2}}{q^2 - q^{-2}}[n] + q^2\beta^2.$$

Proof. (1) We argue by induction on n. Obviously, this is true when $n = 0$ by our hypothesis on v_0.

Suppose $xv_n = \lambda_n v_n$ for some $n \geq 0$. It follows from Lemma 7.4(1) that

$$(x - \lambda_{n+1})v_{n+1} \ = \ (x - \lambda_{n+1})(f_n v_n) \ = \ q^2 f_{n+1}(x - \lambda_n)v_n \ = \ 0;$$

hence $xv_{n+1} = \lambda_{n+1}v_{n+1}$. Thus Part 1 is proved.

(2) First, we show that

$$\left(f_{-1}f_0^-\right).v_0 \ = \ q^2\beta^2 v_0. \tag{7-3}$$

Indeed, if $\alpha^2 \neq q^{-4}$, then $\beta = 0$ and hence $f_{-1}\left(f_0^- v_0\right) = q^2\alpha\beta f_{-1}v_0 = 0 = q^2\beta^2 v_0$. Suppose $\alpha^2 = q^{-4}$. Then $f_{-1} = q^4\alpha y - z = q^2(q^{-2}\alpha^{-1}y - q^{-2}z) = q^2 f_0^-$. Since $f_0^-.v_0 = q^2\alpha\beta v_0$, it follows that

$$f_{-1}f_0^-.v_0 \ = \ q^2\alpha\beta f_{-1}.v_0 \ = \ q^4\alpha\beta f_0^-.v_0 \ = \ q^6\alpha^2\beta^2 v_0 \ = \ q^2\beta^2 v_0.$$

We, again, argue by induction on n. Since $v_1 = f_0 v_0$, it follows from Lemma 7.4(3) with $n = 0$ that

$$\begin{aligned}
f_1^-.v_1 \ &= \ f_{-1}f_0^- v_0 + \left(q\alpha - q^{-1}\alpha^{-1}\right)xv_0 \\
&\overset{(7\text{-}3)}{=} \ \left[q^2\beta^2 + \left(q\alpha - q^{-1}\alpha^{-1}\right)\lambda_0\right]v_0, \\
&= \ \left[q^2\beta^2 + \left(q\alpha - q^{-1}\alpha^{-1}\right)\frac{q\alpha + q^{-1}\alpha^{-1}}{q^2 - q^{-2}}\right]v_0 \\
&= \ \left(\frac{\alpha^2 q^2 - \alpha^{-2}q^{-2}}{q^2 - q^{-2}} + q^2\beta^2\right)v_0 \\
&= \ \mu_1 v_0.
\end{aligned}$$

Thus, Part 2 is true when $n = 1$.

Suppose $f_n^- v_n = \mu_n v_{n-1}$ for some $n \geq 1$. Since $f_{n-1}(f_n^- v_n) = \mu_n f_{n-1} v_{n-1} \;=\; \mu_n v_n$, it then follows from Lemma 7.4(3) that

$$\begin{aligned} f_{n+1}^- . v_{n+1} &= (f_{n+1}^- f_n) . v_n \\ &= f_{n-1} f_n^- v_n + \left(q^{-2n+1}\alpha - q^{2n-1}\alpha^{-1}\right) x v_n \\ &= \left[\mu_n + \left(q^{-2n+1}\alpha - q^{2n-1}\alpha^{-1}\right)\lambda_n\right] v_n \\ &= \mu_{n+1} v_n, \end{aligned}$$

where the last equality holds because $\mu_n + (q^{-2n+1}\alpha - q^{2n-1}\alpha^{-1})\,\lambda_n$ equals

$$\begin{aligned} & \frac{q^{-2n+4}\alpha^2 - q^{2n-4}\alpha^2}{q^2 - q^{-2}}[n] + q^2\beta^2 + \frac{q^{-4n+2}\alpha^2 - q^{-4n-2}\alpha^{-2}}{q^2 - q^{-2}} \\ = \;& \frac{(q^{-2n+4}\alpha^2 - q^{2n-4}\alpha^2)\,[n] + (q^{-4n+2}\alpha^2 - q^{-4n-2}\alpha^{-2})}{q^2 - q^{-2}} + q^2\beta^2 \\ = \;& \frac{q^{-2n+2}\alpha^2 - q^{2n-2}\alpha^{-2}}{q^2 - q^{-2}}[n+1] + q^2\beta^2 \\ = \;& \mu_{n+1}. \end{aligned}$$

Therefore, Part 2 is true for all $n \geq 1$. □

7.3 The $\Bbbk[x]$-module structure of a line module

Let $\Bbbk[x]$ be the polynomial ring in the variable x. In this section we determine the structure of the line module $M = M(\ell)$ as a $\Bbbk[x]$-module, where $\ell = \{x - \boldsymbol{\lambda}(q\alpha) = y - \alpha z - \beta = 0\}$. To this end, we construct a certain basis for the line module consisting of x-eigenvectors or generalized x-eigenvectors.

Roughly speaking, M is a direct sum of indecomposable $\Bbbk[x]$-submodules of dimensions 1 or 2, and the x-support[1] of M is the set $\{\lambda_n(\alpha) \mid n \geq 0\}$. The fact that no indecomposable $\Bbbk[x]$-submodule has dimension ≥ 3 is a consequence of the fact that no three elements in a

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[1] See §2.3.2 for the definition of x-support.

$\boldsymbol{\lambda}$-sequence are equal.

Throughout §7.3, we will fix a cyclic permutation (x, y, z) of (x_1, x_2, x_3), and write

$$\ell_{\alpha,\beta} \;:=\; \{x - \boldsymbol{\lambda}(q\alpha) \;=\; y - \alpha z - \beta \;=\; 0\},$$

where $(\alpha, \beta) \in \Bbbk^2$ *always* satisfies $\alpha \neq 0$ and $(\alpha^2 - q^{-4})\,\beta = 0$.

Unless specified otherwise, the elements $f_n(\alpha)$ and $f_n^-(\alpha)$ are always associated with the fixed cyclic permutation (x, y, z) for $n \in \mathbb{Z}$.

7.3.1 The elements e_n

Let $\Bbbk[z]$ be the polynomial ring in the variable z. By Theorem 6.3, there is a vector space decomposition $A = A\ell^{\perp}_{\alpha,\beta} \oplus \Bbbk[z]$, which induces a natural left $\Bbbk[z]$-module isomorphism

$$\theta : M(\ell_{\alpha,\beta}) \;\xrightarrow{\sim}\; \Bbbk[z].$$

Lemma 7.6. *Let e_0 be the natural image of 1_A in $M(\ell_{\alpha,\beta})$. Define elements*

$$e_n \;:=\; f_{n-1}(\alpha).e_{n-1} \quad \textit{for } n \geq 1.$$

Then

1. $x.e_n \;=\; \lambda_n(\alpha)e_n$*;*

2. $f_n^-(\alpha).e_n \;=\; \mu_n e_{n-1}$*, where* $e_{-1} := e_0$ *and*

$$\mu_n \;=\; \begin{cases} q^2\alpha\beta & n = 0, \\ \frac{q^{-2n+4}\alpha^2 - q^{2n-4}\alpha^{-2}}{q^2 - q^{-2}}[n] + q^2\beta^2 & n > 0; \end{cases}$$

3. $\theta(e_n) \;=\; \beta_n z^n +$ *lower degree monomials in* $\Bbbk[z]$ *for* $n \geq 0$*, where*

$$\beta_n \;=\; \begin{cases} 1 & n = 0, \\ \Pi_{i=0}^{n-1}\left(q^{2-4i}\alpha^2 - 1\right) & n > 0. \end{cases} \tag{7-4}$$

Proof. Part 1 and 2 are an application of Proposition 7.5(1)(2) with $v_0 = e_0$. We prove Part 3 below.

Write $M = M(\ell_{\alpha,\beta})$. We still write x, y and z for their natural images in M, respectively. Let V_n be the subspace in M spanned by $\{1, z, \ldots, z^n\}$ for $n \geq 0$.

We argue by induction on n. Clearly, this is true when $n = 0$, as $\theta(e_0) = 1_{\Bbbk[z]}$.

Suppose $e_n = \beta_n z^n + v$ for some $n \geq 0$, where $v \in V_{n-1}$. By Lemma 6.1(1), the following are true in M:

$$yz^n - q^{-2n}\alpha z^{n+1} \;\in\; V_n \qquad \text{and} \qquad yV_{n-1} \;\subset\; V_n. \tag{7-5}$$

Since $e_{n+1} = f_n(\alpha).e_n$, it follows that

$$\begin{aligned} e_{n+1} &= \left(q^{-2n+2}\alpha y - z\right)\left(\beta_n z^n + v\right) \\ &= q^{-2n+2}\alpha\beta_n yz^n - \beta_n z^{n+1} + q^{-2n+2}\alpha yv - zv \\ &\overset{(7\text{-}5)}{\in} q^{-4n+2}\alpha^2\beta_n z^{n+1} - \beta_n z^{n+1} + V_n \\ &= \left(q^{-4n+2}\alpha^2 - 1\right)\beta_n z^{n+1} + V_n \\ &= \beta_{n+1} z^{n+1} + V_n, \end{aligned}$$

where the last equality holds because $\beta_{n+1} = \left(q^{-4n+2}\alpha^2 - 1\right)\beta_n$. This proves Part 3. □

We thus obtain a sequence of x-eigenvectors (including possible zero elements) in $M(\ell_{\alpha,\beta})$. Note that the coefficient β_n in Lemma 7.6(3) is nonzero for all $n \in \mathbb{N}$ if and only if $\alpha^2 \neq q^{4\mathbb{N}-2}$. Whether this is true differentiates the $\Bbbk[x]$-module structure of $M(\ell_{\alpha,\beta})$, as discussed in the next two subsections.

7.3.2 The case when $\alpha^2 \notin q^{4\mathbb{N}-2}$

Proposition 7.7. *Suppose $\alpha^2 \notin q^{4\mathbb{N}-2}$. Write $M = M(\ell_{\alpha,\beta})$.*

1. *The elements $\{e_n \mid n \geq 0\}$ defined in Lemma 7.6 is a basis of M;*

2. *There is a left $\Bbbk[x]$-module isomorphism*

$$M \;\cong\; \bigoplus_{n\geq 0} \frac{\Bbbk[x]}{(x-\lambda_n(\alpha))}.$$

If, in particular, $\alpha^2 \in q^{4\mathbb{N}}$, say $\alpha^2 = q^{4r}$, then

$$M \;\cong\; \bigoplus_{n=r+1}^{2r+1} \left(\frac{\Bbbk[x]}{(x-\lambda_n(\alpha))}\right)^{\oplus 2} \;\oplus\; \bigoplus_{n\geq 2r+2} \frac{\Bbbk[x]}{(x-\lambda_n(\alpha))}.$$

Proof. (1) By Lemma 7.6(3), $e_n = \beta_n z^n +$ lower degree monomials in $\Bbbk[z]$, where $\beta_n \in \Bbbk$. Because $\alpha^2 \notin q^{4\mathbb{N}-2}$, the coefficient β_n is nonzero for all $n \geq 0$. Since $\{z^n \mid n \geq 0\}$ is a basis in M, so is $\{e_n \mid n \geq 0\}$.

(2)The first isomorphism is an immediate consequence of Lemma 7.6(1) and Part 1.

By Lemma 7.1(2), if $\alpha^2 = q^{4r}$ then $\lambda_n(\alpha) = \lambda_{2r+1-n}(\alpha)$ for $n \in \mathbb{Z}$. This leads to the second isomorphism. □

7.3.3 The case when $\alpha^2 \in q^{4\mathbb{N}-2}$

In this case, we construct a similar basis for $M(\ell_{\alpha,0})$ as for the previous case, i.e., a basis consisting of generalized x-eigenvectors. Its construction and the action of A on this basis, however, is more complicated.

Lemma 7.8. *Suppose $\alpha^2 = q^{4r-2}$ where $r \geq 0$. There is a basis $\{b_n \mid n \geq 0\}$ of the line module $M(\ell_{\alpha,0})$ such that*

1. $$(x-\lambda_n(\alpha)).b_n \;=\; \begin{cases} 0 & n=0,\dots,r \;\;\textit{or}\;\; n>2r, \\ [n-r]b_{2r-n} & n=r+1,\dots,2r; \end{cases}$$

2. $$f_n(\alpha).b_n \;=\; \begin{cases} q^2\mu_{n+1}b_{n+1} & n=0,\dots,r-1, \\ b_{n+1}+\rho_n b_{2r-n+1} & n=r+1,\dots,2r; \end{cases}$$

3. $$y.b_r \;=\; q^{-2r}\alpha b_{r-1}-q^{-1}b_{r+1};$$

4. $$z.b_r \;=\; -q^{1-2r}\alpha b_{r+1};$$

5. $$f_n^-(\alpha).b_n \;=\; \begin{cases} q^{-2}b_{n-1} & n=0,\dots,r, \\ \alpha_n b_{2r-n+1}+\delta_n b_{2r-n-1} & n=r+1, \\ \mu_n b_{n-1}+\alpha_n b_{2r-n+1}+\delta_n b_{2r-n-1} & n=r+1,\dots,2r, \end{cases}$$

where $b_{-1}:=0$*,* $\alpha_n\in\Bbbk$ *and*

$$\rho_n \;=\; \frac{\alpha[n][2r-n+1]}{q^{4n-2r-3}-q^{2r-3}}, \qquad \mu_n \;=\; [n][2r-n+1], \qquad \delta_n \;=\; \frac{\alpha}{q^{2r+1}-q^{6r-4n+1}}.$$

Proof. Write $M=M(\ell_{\alpha,0})$. Let $\{e_n\in M \mid n\geq 0\}$ be the elements defined in Lemma 7.6.

First of all, we prove that if $\alpha^2=q^{4r-2}$ then

(a) $e_n \;=\; \gamma_n e_{2r-n}$ for $n=r,\dots,2r$, where

$$\gamma_n \;=\; \begin{cases} 1 & n=r, \\ q^{2n-2r}\prod_{i=2r-n+1}^{n}[i] & n=r+1,\dots,2r, \end{cases}$$

and

(b) $e_n \;=\; 0$ for all $n>2r$,

as follows.

(a) We argue by induction on n. This is obviously true when $n = r$. Suppose $e_n = \gamma_n e_{2r-n}$ for some $n \geq r$. Since $\alpha^2 = q^{4r-2}$, it follows from Lemma 7.6(2) that

$$f_n^-.e_n \;=\; [2r-n+1][n]e_{n-1} \quad \text{for } n \geq 1. \tag{7-6}$$

In addition,

$$f_n \;=\; q^{-2n+2}\alpha y - z \;=\; q^2\left(q^{4r-2n-2}\alpha^{-1}y - q^{-2}z\right) \;=\; q^2 f_{2r-n}^- \tag{7-7}$$

for all $n \in \mathbb{Z}$. It follows that

$$e_{n+1} \;=\; f_n e_n \overset{\text{(7-7)}}{=} q^2\gamma_n f_{2r-n}^- e_{2r-n} \overset{\text{(7-6)}}{=} q^2[2r-n][n+1]\gamma_n e_{2r-n-1} \;=\; \gamma_{n+1}e_{2r-n-1},$$

where the last equality holds because $\gamma_{n+1} = q^2[2r-n][n+1]\gamma_n$. This proves (a).

(b) By Part (a), $e_{2r} = \gamma_{2r}e_0$. Since $f_0^-.e_0 = 0$, it follows that

$$e_{2r+1} \;=\; f_{2r}e_{2r} \overset{\text{(7-7)}}{=} q^2\gamma_{2r}f_0^- e_0 \;=\; 0.$$

Therefore, $e_n = 0$ for all $n > 2r$ due to the definition of e_n. This proves (b).

Now, we define

$$b_n \;:=\; \begin{cases} e_{2r-n} & n = -1, 0, \ldots, r, \\ -q^{-2r+1}\alpha z.b_r & n = r+1, \\ f_{n-1}.b_{n-1} - \rho_{n-1}e_{n-2} & n > r+1. \end{cases} \tag{7-8}$$

We first show $\{b_n \mid n \geq 0\}$ is a basis of M.

We adopt the following notation: by V_n we denote the subspace of M spanned by $\{1, z, \ldots, z^n\}$ for $n \in \mathbb{N}$. For $v, w \in M$, we write $v \equiv w \pmod{V_n}$ if $v - w \in V_n$.

By Lemma 7.6(3), $e_n \equiv \beta_n z^n \pmod{V_{n-1}}$ for $n \geq 0$, where $\beta_0 = 1$ and

$$\beta_n \;=\; \prod_{i=0}^{n-1} \left(q^{2-4i}\alpha^2 - 1\right) \;=\; \prod_{i=0}^{n-1} \left(q^{4r-4i} - 1\right) \text{ for } n \geq 1.$$

Thus, $\beta_n \neq 0$ when $0 \leq n \leq r$. In addition, by (a) above,

$$b_n \;=\; e_{2r-n} \;=\; \gamma_{2r-n} e_n \;\equiv\; \beta_n \gamma_{2r-n} z^n \pmod{V_{n-1}} \quad \text{for } n = 0, \ldots, r, \tag{7-9}$$

where $\gamma_r = 1$ and $\gamma_{2r-n} = q^{2r-2n} \prod_{i=n+1}^{2r-n} [i]$ for $n = 0, \ldots, r-1$. Note that $\beta_n \gamma_{2r-n} \neq 0$.

In particular, it follows from (7-9) that $b_{r+1} = -q^{-2r+1}\alpha z b_r \equiv -q^{-2r+1}\alpha \beta_r z^{r+1} \pmod{V_r}$. By a similar argument in Lemma 7.6(3), we can inductively show that

$$b_n \;\equiv\; \beta'_n z^n \pmod{V_{n-1}} \qquad \text{for } n \geq r+1, \tag{7-10}$$

where

$$\beta'_n \;=\; -q^{-2r+1}\alpha \prod_{\substack{0 \leq i \leq n-1 \\ i \neq r}} \left(q^{4r-4i} - 1\right) \;\neq\; 0.$$

Since $\{z^n \mid n \geq 0\}$ is a basis of M, it follows from (7-9) and (7-10) that $\{b_n \mid n \geq 0\}$ is also a basis of M.

Next, we prove this basis satisfies Properties (1) - (5) listed in the statement of the lemma, as follows.

(1) First suppose $n \leq r$. Since $b_n = e_{2r-n}$ and $\lambda_n = \lambda_{2r-n}$, it follows from Lemma 7.6(1) that

$$(x - \lambda_n(\alpha)).b_n \;=\; (x - \lambda_{2r-n}(\alpha)).e_{2r-n} \;=\; 0.$$

When $n > r$, we argue by induction on n. Since $e_{r+1} = f_r.e_r = (q^{-2r+2}\alpha y - z).e_r$,

$$q^{-2r+2}\alpha y.e_r \;=\; z.e_r + e_{r+1}. \tag{7-11}$$

Since $b_{r+1} = -q^{-2r+1}\alpha z.e_r$, we have

$$\begin{aligned}
(x-\lambda_{r+1})\,b_{r+1} &= -q^{-2r+1}\alpha\,(xz-\lambda_{r+1}z)\,e_r \\
&= \alpha\left(-q^{-2r+3}zx + q^{-2r+2}y + q^{-2r+1}\lambda_{r+1}z\right)e_r \\
&\overset{(7\text{-}11)}{=} -q^{-2r+3}\alpha\lambda_r z e_r + z e_r + e_{r+1} + q^{-2r+1}\alpha\lambda_{r+1} z e_r \\
&= \left(-q^{-2r+3}\alpha\lambda_r + 1 + q^{-2r+1}\alpha\lambda_{r+1}\right) z e_r + e_{r+1} \\
&= b_{r-1},
\end{aligned}$$

where the last equality holds because $b_{r-1} = e_{r+1}$ and

$$\begin{aligned}
&\left(q^2-q^{-2}\right)\left(-q^{-2r+3}\alpha\lambda_r + 1 + q^{-2r+1}\alpha\lambda_{r+1}\right) \\
=\;& -q^{-4r+4}\alpha^2 - q^2 + q^2 - q^{-2} + q^{-4r}\alpha^2 + q^2 \\
=\;& 0.
\end{aligned}$$

Thus, (1) is true when $n = r+1$.

Suppose $(x-\lambda_n)b_n = [n-r]b_{2r-n}$ for some $n \geq r+1$. By Lemma 7.3(3),

$$q^2[n-r]f_{n+2} \;=\; [n+1-r]f_n - q^{2r-2n+2}f_n^- . \tag{7-12}$$

It follows from Lemma 7.4(1) that

$$\begin{aligned}
(x-\lambda_{n+1})\,f_n b_n &= q^2 f_{n+2}(x-\lambda_n)b_n \\
&= q^2[n-r]f_{n+2}e_n \\
&\overset{(7\text{-}12)}{=} \left([n+1-r]f_n - q^{2r-2n+2}f_n^-\right)e_n \\
&\overset{(7\text{-}6)}{=} [n+1-r]e_{n+1} - q^{2r-2n+2}[n][2r-n+1]e_{n-1}.
\end{aligned}$$

This equality will be used in the next calculation. Since $b_{n+1} = f_n b_n - \rho_n e_{n-1}$, it follows that

$$\begin{aligned} (x-\lambda_{n+1}).b_{n+1} &= (x-\lambda_{n+1})\left(f_n b_n - \rho_n e_{n-1}\right) \\ &= (x-\lambda_{n+1}) f_n b_n - \rho_n (x-\lambda_{n+1}) e_{n-1} \\ &= [n+1-r] e_{n+1} - q^{2r-2n+2}[n][2r-n+1] e_{n-1} - \rho_n(\lambda_{n-1}-\lambda_{n+1}) e_{n-1} \\ &= [n+1-r] b_{2r-n-1}, \end{aligned}$$

where the last equality holds because $e_{n+1} = b_{2r-n-1}$ and

$$\rho_n\left(\lambda_{n-1}-\lambda_{n+1}\right) = -q^{2r-2n+2}[n][2r-n+1].$$

This proves Property (1).

(2) When $n = 0, \ldots, r$,

$$f_n.b_n \overset{(7\text{-}7)}{=} q^2 f^-_{2r-n}.e_{2r-n} \overset{(7\text{-}6)}{=} q^2[2r-n][n+1] e_{2r-n-1} = q^2 \mu_{n+1} b_{n+1}.$$

When $n > r$, this is clearly true due to the definition (7-8) for b_n.

(3) This follows from (7-11).

(4) This is the definition (7-8) for b_{r+1}.

(5) When $n = 0, \ldots, r$,

$$f^-_n.b_n \overset{(7\text{-}7)}{=} q^{-2} f_{2r-n}.e_{2r-n} = q^{-2} e_{2r-n+1} \overset{(7\text{-}8)}{=} q^{-2} b_{n-1},$$

Note $b_{-1} = e_{2r+1} = 0$ by (b).

Now, suppose $r+1 \le n \le 2r$. Since $\alpha^2 = q^{4r-2} \ne q^{4n-2}$, it follows from Lemma 7.3(4) that

$$q^2[n-r] f^-_{n-2} = [n-r-1] f^-_n + q^{2n-2r-2} f_n. \tag{7-13}$$

Since $x.e_{n+1} = \lambda_{n+1} e_{n+1}$, we have

$$\begin{aligned}
\delta_n(x - \lambda_{n-1}).e_{n+1} &= \delta_n(\lambda_{n+1} - \lambda_{n-1})e_{n+1} \\
&= \frac{\alpha\left(q^{-2n-1}\alpha + q^{2n+1}\alpha^{-1} - q^{-2n+3}\alpha^2 - q^{2n-3}\alpha^{-1}\right)}{\left(q^2 - q^{-2}\right)\left(q^{2r+1} - q^{-4n+6r+1}\right)} e_{n+1} \\
&= \frac{q^{4r-2n-3} + q^{2n+1} - q^{4r-2n+1} - q^{2n-3}}{\left(q^2 - q^{-2}\right)\left(q^{2r+1} - q^{-4n+6r+1}\right)} e_{n+1} \\
&= q^{2n-2r-2} e_{n+1} \\
&= q^{2n-2r-2} b_{2r-n-1}. \qquad (7\text{-}14)
\end{aligned}$$

It follows from Lemma 7.4(2) that

$$\begin{aligned}
(x - \lambda_{n-1})\, f_n^- .b_n &= q^2 f_{n-2}^- \,(x - \lambda_n)\, b_n \\
&\overset{(1)}{=} q^2 [n-r] f_{n-2}^- e_n \\
&\overset{(7\text{-}13)}{=} \left([n-r-1] f_n^- + q^{2n-2r-2} f_n\right) e_n \\
&\overset{(7\text{-}6)}{=} [n-r-1][2r-n+1][n] e_{n-1} + q^{2n-2r-2} e_{n+1} \\
&\overset{(7\text{-}14)}{=} [n-r-1][2r-n+1][n] e_{n-1} + \delta_n (x - \lambda_{n-1}) b_{2r-n-1} \\
&\overset{(1)}{=} \begin{cases} \delta_n (x - \lambda_{n-1}) b_{2r-n-1} & n = r+1, \\ \mu_n (x - \lambda_{n-1}) b_{n-1} + \delta_n\,(x - \lambda_{n-1})\, b_{2r-n-1} & n = r+2, \dots, 2r. \end{cases}
\end{aligned}$$

Thus, $(x - \lambda_{n-1})\,.b'_{n-1} = 0$, where

$$b'_{n-1} := \begin{cases} f_n^- b_n - \delta_n e_{n+1} & n = r+1, \\ f_n^- b_n - \mu_n b_{n-1} - \delta_n e_{n+1} & n = r+2, \dots, 2r. \end{cases} \qquad (7\text{-}15)$$

Furthermore, by Lemma 7.1(3), $\lambda_n = \lambda_{2r-n}$ and the scalars in $\{\lambda_n \mid n \geq r\}$ are distinct. Since $\{b_n \mid n \geq 0\}$ is a basis, it then follows from Property (1) that the x-eigenspace in M with eigenvalue λ_{n-1} is the 1-dimensional space $\Bbbk b_{2r-n+1}$ for $n = r+1, \dots, 2r$. Hence,

$b'_{n-1} = \alpha_n b_{2r-n+1}$ for some $\alpha_n \in \Bbbk$. It then follows from (7-15) that

$$f_n^-.b_n \;=\; \begin{cases} \alpha_n b_{2r-n+1} + \delta_n b_{2r-n-1} & n = r+1, \\ \mu_n b_{n-1} + \alpha_n b_{2r-n+1} + \delta_n b_{2r-n-1} & n = r+2, \ldots, 2r. \end{cases}$$

This proves Property (5) and the proof is complete. □

Proposition 7.9. *If $\alpha^2 \in q^{4\mathbb{N}-2}$, say $\alpha^2 = q^{4r-2}$ with $r \geq 0$, then as left $\Bbbk[x]$-modules*

$$M(\ell_{\alpha,0}) \;\cong\; \frac{\Bbbk[x]}{(x-\lambda_r(\alpha))} \;\oplus\; \bigoplus_{n=r+1}^{2r} \frac{\Bbbk[x]}{(x-\lambda_n(\alpha))^2} \;\oplus\; \bigoplus_{n>2r} \frac{\Bbbk[x]}{(x-\lambda_n(\alpha))}.$$

If, in particular, $\alpha^2 = q^{-2}$ (i.e., $r = 0$) then

$$M(\ell_{\alpha,0}) \;\cong\; \bigoplus_{n\geq 0} \frac{\Bbbk[x]}{(x-\lambda_n(\alpha))}.$$

Proof. By Lemma 7.1(3), $\lambda_n = \lambda_{2r-n}$ and the scalars in $\{\lambda_n \mid n \geq r\}$ are distinct. The first isomorphism is then an immediate consequence of Lemma 7.8(1). The second is a special case of the first. □

7.3.4 Summary

Recall from §2.3.2 that the x-support of an A-module M is the set of all $\lambda \in \Bbbk$ such that

$$\{m \in M \mid (x-\lambda)^n m = 0 \text{ for } n \gg 0\} \;\neq\; 0.$$

Corollary 7.10. *Let $M = M(\ell_{\alpha,\beta})$ and $M' = M(\ell_{\alpha',\beta'})$ be two line modules for A.*

1. *The line module M is a direct sum of*
   - *1-dimensional $\Bbbk[x]$-modules when $\alpha^2 \notin q^{4\mathbb{N}+2}$, and*

  - *1- and 2-dimensional indecomposable* $\Bbbk[x]$*-modules when* $\alpha^2 \in q^{4\mathbb{N}+2}$.

  In addition, the x*-support of* M *is* $\{\lambda_n(\alpha) \mid n \geq 0\}$.

2. *If* M' *is isomorphic to a proper submodule of* M*, then* $\alpha' \in q^{-2\mathbb{N}-2}\alpha$.

Proof. (1) This follows from Proposition 7.7 and Proposition 7.9.

(2) Write $M = M(\ell_{\alpha,\beta})$ and $M' = M(\ell_{\alpha',\beta'})$.

By Part 1, the x-support of M' is the set $\{\lambda_n(\alpha') \mid n \geq 0\}$. Since $M' \subset M$, the x-support for M' is a subset of that for M, i.e., $\{\lambda_n(\alpha')\}_{n\geq 0} \subset \{\lambda_n(\alpha)\}_{n\geq 0}$.

Since M is generated by the 1-dimensional x-eigenspace with eigenvalue $\lambda_0(\alpha)$, it is necessary that $\lambda_0(\alpha) \notin \{\lambda_n(\alpha')\}_{n\geq 0}$; otherwise $M = M'$. Therefore,

$$\{\lambda_n(\alpha')\}_{n\geq 0} \ \subsetneq \ \{\lambda_n(\alpha)\}_{n\geq 0}.$$

By Lemma 7.2, $\alpha' \in q^{-2\mathbb{N}-2}\alpha$. ☐

7.4 Types of *A*-lines

To give more explicit description of the A-lines in relation to finite dimensional simple A-modules, we introduce the following terminology. Fix a cyclic permutation (x, y, z) of (x_1, x_2, x_3). We call the A-line

$$\ell \ = \ \{x - \boldsymbol{\lambda}(q\alpha) \ = \ y - \alpha z - \beta \ = \ 0\}$$

and its respective line module $M(\ell)$ of

- Type 1 if $(\alpha^2, \beta^2) \in \big(-q^{2\mathbb{N}-2}, 0\big)$;
- Type 2 if $(\alpha^2, \beta^2) \in (q^{-4}, q^{-2}[\mathbb{N}_+]^2)$;
- Type 3 if $(\alpha^2, \beta^2) \in \big(q^{4\mathbb{N}}, 0\big)$;

- Type 4 if $(\alpha^2, \beta^2) \in \left(q^{4\mathbb{N}-2}, 0\right)$.

The relevance of these types of line modules to finite dimensional simple A-modules is indicated by the next result, which improves Corollary 6.20.

Proposition 7.11. *Let ℓ and ℓ' be the A-lines*

$$\begin{aligned} \ell &= \{x - \boldsymbol{\lambda}(q\alpha) = y - \alpha z - \beta = 0\}, \\ \ell' &= \{x - \boldsymbol{\lambda}(q\alpha') = y - \alpha' z - \beta' = 0\}, \end{aligned}$$

where (x, y, z) is a cyclic permutations of (x_1, x_2, x_3). If $M(\ell')$ is isomorphic to a proper submodule of $M(\ell)$, then one of the following occurs:

1. $(\alpha^2, \beta^2) \in \left(-q^{2\mathbb{N}-2}, 0\right)$ *and* $(\alpha', \beta') = (-q^{-4}\alpha^{-1}, 0)$;
2. $(\alpha^2, \beta^2) = (q^{-4}, q^{-2}[m]^2)$ *and* $(\alpha', \beta') = (q^{-2m}\alpha, 0)$ *for some* $m \geq 1$;
3. $(\alpha^2, \beta^2) = (q^{4n-4}, 0)$ *and*

   $$(\alpha', \beta') \in \left(q^{-2n}\alpha, q^{-1}[n]\right) \cup \left(q^{-2n}\alpha, -q^{-1}[n]\right) \cup \left(q^{-4n-4}\alpha, 0\right)$$

   for some $n \geq 1$;
4. $(\alpha^2, \beta^2) \in \left(q^{4\mathbb{N}-2}, 0\right)$ *and* $(\alpha', \beta') = (q^{-4}\alpha^{-1}, 0)$.

The A-lines in Cases 1 - 4 are of Types 1 - 4, respectively.

Proof. Write $M = M(\ell)$ and $M' = M(\ell')$.

Since $M' \subsetneq M$, it follows from Corollary 7.10(2) that

$$\alpha' \in q^{-2\mathbb{N}-2}\alpha. \tag{7-16}$$

Suppose $\beta\beta' \neq 0$. Since $(\alpha^2 - q^{-4})\beta = (\alpha'^2 - q^{-4})\beta' = 0$, $\alpha^2 = \alpha'^2 = q^{-4}$. But this is impossible due to (7-16). Therefore $\beta\beta' = 0$, which thus leaves us with three possibilities:

1. $\beta = \beta' = 0$,
2. $\alpha'^2 = q^{-4}$ and $\beta' \neq 0$, and
3. $\alpha^2 = q^{-4}$ and $\beta \neq 0$.

We discuss each of them below.

(1) Since $\beta = \beta' = 0$, it follows from Corollary 6.20(1) that $\alpha' \in \{\pm\alpha, \pm q^{-4}\alpha^{-1}\}$. Clearly, $\alpha \neq \alpha'$.

If $\alpha' = -\alpha$, then $-\alpha \in q^{-2\mathbb{N}-2}\alpha$. But this is impossible, as q is not a root of unity.

If $\alpha' = -q^{-4}a^{-1}$, then $\alpha^2 \in -q^{2\mathbb{N}-2}$. This gives us **Case 1** in the statement of the lemma.

If $\alpha' = q^{-4}a^{-1}$, then $q^{-4}\alpha^{-1} \in q^{-2\mathbb{N}-2}\alpha$, so that

$$\alpha^2 \ \in \ q^{2\mathbb{N}-2} \ = \ q^{4\mathbb{N}-2} \cup q^{4\mathbb{N}}.$$

In particular, when $\alpha^2 \in q^{4\mathbb{N}-2}$, it leads to **Case 4** in the lemma. The other possibility is

$$\big(\alpha^2, \beta^2\big) \ \in \ \big(q^{4\mathbb{N}}, 0\big) \qquad \text{and} \qquad (\alpha', \beta') \ = \ \big(q^{-4}\alpha^{-1}, 0\big) \tag{7-17}$$

(2) By (7-16), $\alpha = q^{2n}\alpha'$ for some $n \geq 1$. Since $\alpha'^2 = q^{-4}$, $\alpha^2 \in q^{4n}\alpha'^2 = q^{4n-4}$. By Corollary 6.20(2),

$$\beta'^2 \ = \ q^{-2}\left(\frac{q^2\alpha - q^{-2}\alpha^{-1}}{q^2 - q^{-2}}\right)^2 \ = \ q^{-2}\left(\frac{q^{2n} - q^{-2n}}{q^2 - q^{-2}}\right)^2 \ = \ q^{-2}[n]^2.$$

Thus, $(\alpha^2, \beta^2) = (q^{4n-4}, 0)$ and $(\alpha', \beta') = (q^{-2n}\alpha, \pm q^{-1}[n])$. This, together with (7-17), gives **Case 3**.

(3) Suppose $\alpha^2 = q^{-4}$ and write $\alpha' = q^{-2m}\alpha$ where $m \geq 1$. By a similar argument in Part (2), we can show $\beta^2 \in q^{-2}[m]^2$. This is **Case 2** in the statement of the lemma. $\square$

Proposition 7.11 suggests line modules of Types 1 - 4 *may* be non-simple. The next result finds proper submodules of the line modules of Types 1-3 using the basis elements given by

Proposition 7.7.

Lemma 7.12. *Let $\{e_n \,|\, n \geq 0\}$ be the elements in a line module M as defined in Lemma 7.6.*

1. *If M is of Type 1, 2 or 3, then there exists $r \geq 1$ such that Ae_r is a nonzero proper submodule.*

2. *If M is of Type 1 or 2, then it has a unique nonzero proper submodule.*

Proof. Assume $M = M(\ell)$, where $\ell = \{x - \boldsymbol{\lambda}(q\alpha) = y - \alpha z - \beta = 0\}$.

(1) Since $\alpha^2 \notin q^{4\mathbb{N}-2}$ when ℓ is of Type 1, 2 or 3, it follows from Proposition 7.7(1) that $\{e_n \,|\, n \geq 0\}$ is a basis of M.

First suppose ℓ is of Type 1, i.e., $(\alpha^2, \beta^2) = (-q^{2r-4}, 0)$ for some $r \geq 1$. By Lemma 7.6(2),

$$f_r^-(\alpha).e_r = \left(\frac{q^{-2r+4}\alpha^2 - q^{2r-4}\alpha^{-2}}{q^2 - q^{-2}}[r] + q^2\beta^2\right) e_{r-1} = \frac{-1+1}{q^2 - q^{-2}}[r]e_{r-1} = 0.$$

Hence, $\left(y - q^{-2r}\alpha z\right).e_r = q^{-2r+2}\alpha f_r^-(\alpha).e_r = 0$.

Write $\alpha_0 = q^{-2r}\alpha$. Since $\left(x - \boldsymbol{\lambda}(q\alpha_0)\right)e_r = \left(y - \alpha_0 z\right)e_r = 0$, there is a surjection

$$\phi : M(\ell_0) \twoheadrightarrow Ae_r,$$

where $\ell_0 = \{x - \boldsymbol{\lambda}(q\alpha_0) = y - \alpha_0 z = 0\} = \ell[-r]$.

If $\ker\phi \neq 0$, then $\dim_{\Bbbk}(Ae_r) < \infty$ by Corollary 6.4. But, Ae_r is infinite dimensional as $e_n \in Ae_r$ for all $n \geq r$. Hence, $\ker\phi = 0$ and ϕ is an isomorphism. Therefore, $Ae_r \cong M(\ell[-r])$ and it is a proper submodule.

By similar arguments we show that Part (1) holds for Type 2 and Type 3 line modules as well.

(2) We, again, only prove the case when ℓ is of Type 1, i.e., $(\alpha^2, \beta^2) = (-q^{2r-4}, 0)$. The case when ℓ is of Type 2 can be proved similarly.

Suppose N is a nonzero proper submodule of M. Then $N \cong M(\ell')$ for some A-line

$$\ell' = \{x - \boldsymbol{\lambda}(q\alpha') = y - \alpha' z - \beta' = 0\}.$$

By Proposition 7.11(1), $\alpha' = -q^{-4}\alpha^{-1} = q^{-2r}\alpha$ and $\beta' = 0$. Hence, N is generated by an x-eigenvector in M with eigenvalue $\lambda_0(\alpha') = \lambda_r(\alpha)$. However, by Proposition 7.7(2), the x-eigenspace in M with eigenvalue $\lambda_r(\alpha)$ is 1-dimensional. Therefore, there is a unique nonzero proper submodule in M. □

7.4.1 The traces for the actions of x, y and z

We calculate the traces for the actions of x, y and z on proper quotients of line modules of Types 1, 2 and 4. It will be useful in distinguishing non-isomorphic finite dimensional simple A-modules from each other.

Lemma 7.13. *Let (x, y, z) be a cyclic permutation of (x_1, x_2, x_3) and $r \geq 1$. Let V be an r-dimensional quotient of a line module $M(\ell)$, where*

$$\ell = \{x - \boldsymbol{\lambda}(q\alpha) = y - \alpha z - \beta = 0\}.$$

If ℓ is of Type 1, 2 or 4, then the traces for the actions of x, y and z on V are

$$\begin{aligned} \mathrm{tr}_V(x) &= \boldsymbol{\lambda}(q^{-r+2}\alpha)\left(q^r - q^{-r}\right) / \left(q - q^{-1}\right), \\ \mathrm{tr}_V(y) &= \beta/(1 - q^{-2}), \\ \mathrm{tr}_V(z) &= \alpha\beta/(q^{-2} - q^{-4}). \end{aligned}$$

Proof. First of all, by Proposition 6.13, there is an exact sequence

$$0 \longrightarrow M(\ell') \longrightarrow M(\ell) \longrightarrow V \longrightarrow 0,$$

where $\ell' = \{x - \boldsymbol{\lambda}(q\alpha') = y - \alpha' z - \beta' = 0\}$. Write $M = M(\ell)$ and $M' = M(\ell')$ for short. We will view M' as a submodule of M.

We prove the result when (1) ℓ is of Type 1 or 2, and when (2) ℓ of Type 4, separately.

(1) Suppose ℓ is of Type 1 or 2. By Lemma 7.1(1)(2), $\lambda_m(\alpha) \neq \lambda_n(\alpha)$ when $m > n \geq 0$. By Corollary 7.10(2), $\alpha' = q^{-2t}\alpha$ for some $t \geq 1$.

Since $\alpha^2 \notin q^{4\mathbb{N}-2}$, it follows from Proposition 7.7(1) that there is a basis $\{e_n \mid n \geq 0\}$ for M such that for $n \geq 0$

$$x.e_n = \lambda_n(\alpha)e_n \qquad f_n.e_n = e_{n+1}, \qquad f_n^-.e_n = \mu_n e_{n-1} \tag{7-18}$$

where $e_{-1} := e_0$, $\mu_n \in \Bbbk$ and $\mu_0 = q^2\alpha\beta$.

Since $\alpha'^2 = q^{-4t}\alpha^2 \notin q^{4\mathbb{N}-2}$, it follows from Proposition 7.7(2) that M' is a direct sum of 1-dimensional x-eigenspaces with eigenvalues $\{\lambda_n(\alpha') \mid n \geq 0\} = \{\lambda_n(\alpha) \mid n \geq t\}$. Hence, $\{e_n \mid n \geq t\}$ is a basis for M'. It follows that V has a basis of $\{v_n \mid n = 0, \ldots, t-1\}$, where v_n is the natural image of e_n under the surjection $M \twoheadrightarrow V$. Since $\dim_{\Bbbk}(V) = r$, $t = r$.

By (7-18), $\mathrm{tr}_V(x)$ equals

$$\begin{aligned}
\sum_{n=0}^{r-1} \boldsymbol{\lambda}\left(q^{-2n+1}\alpha\right) &= \sum_{n=0}^{r-1} \frac{q^{-2n+1}\alpha + q^{2n-1}\alpha^{-1}}{q^2 - q^{-2}} \\
&= \frac{\left(\alpha q^{-r+2} + \alpha^{-1}q^{r-2}\right)\left(q^r - q^{-r}\right)}{\left(q - q^{-1}\right)\left(q^2 - q^{-2}\right)} \\
&= \frac{\boldsymbol{\lambda}(q^{-r+2}\alpha)\left(q^r - q^{-r}\right)}{q - q^{-1}}.
\end{aligned}$$

Furthermore, by Lemma 7.3(1), for all $n \geq 0$

$$y = \left(q^{-1}f_n - qf_n^-\right)\delta_n,$$

where $\delta_n = (q^{-2n+1}\alpha - q^{2n-1}\alpha^{-1})^{-1}$. It follows that

$$\begin{aligned} y.v_n &= \left(q^{-1}f_n.v_n - qf_n^-.v_n\right)\delta_n \\ &\overset{(7\text{-}18)}{=} \left(-q\mu_n v_{n-1} + q^{-1}v_{n+1}\right)\delta_n \\ &= \begin{cases} \left(-q\mu_0 v_0 + q^{-1}v_1\right)\delta_0 & n = 0, \\ \left(-q\mu_n v_{n-1} + q^{-1}v_{n+1}\right)\delta_n & n = 1, \ldots, r-1, \end{cases} \end{aligned}$$

where $v_r := 0$. Thus, the action of y on V can be presented by a trigonal matrix with diagonal

$$\left(\frac{-q^3\alpha\beta}{q\alpha - q^{-1}\alpha^{-1}},\ 0, \ldots, 0\right).$$

Hence,

$$\mathrm{tr}_V(y) = \frac{-q^3\alpha\beta}{q\alpha - q^{-1}\alpha^{-1}} = \frac{-q^4\alpha^2\beta}{q^2\alpha^2 - 1} = \frac{\beta}{1 - q^{-2}}.$$

where the last equality holds because $(\alpha^2 - q^{-4})\beta = 0$.

In a similar manner we show that $\mathrm{tr}_V(z) = \alpha\beta/\left(q^{-2} - q^{-4}\right)$.

(2) Suppose ℓ is of Type 4, i.e., $\alpha^2 = q^{4s-2}$ and $\beta = 0$ for some $s \geq 0$. The treatment is similar but the calculation is more complicated.

Let $\{b_n \mid n \geq 0\}$ be the basis of M given by Lemma 7.8. Then the x-eigenspace in M with eigenvalue $\lambda_n(\alpha)$ is the 1-dimensional space $\Bbbk b_n$ for $n \geq 2s+1$.

By Proposition 7.11(4), $\alpha' = q^{-4}\alpha^{-1} = q^{-4s-2}\alpha$; whence $\alpha'^2 = q^{-8}\alpha^{-2} \in q^{-6-4s} \notin q^{4\mathbb{N}+2}$. It then follows from Proposition 7.7(1) that a basis of M' is given by a set of x-eigenvectors with eigenvalues

$$\{\lambda_0(\alpha'), \lambda_1(\alpha'), \cdots\} = \{\lambda_{2s+1}(\alpha), \lambda_{2s+2}(\alpha), \cdots\}.$$

Hence, the set $\{b_n \mid n \geq 2s+1\}$ is a basis of M'.

Since $V \cong M/M'$, it has a basis $\{w_n \in V \mid n = 0, \ldots, 2s\}$, where w_n is the image of b_n

under the surjection $M \twoheadrightarrow V$, such that

$$x.w_n \;=\; \begin{cases} \lambda_n(\alpha) w_n & n = 0, \ldots, s, \\ \lambda_n(\alpha) w_n + [n-s] w_{2s-n} & n = s+1, \ldots, 2s; \end{cases}$$

$$f_n.w_n \;\in\; \begin{cases} \Bbbk w_{n+1} & n = 0, \ldots, s-1, \\ \Bbbk w_{n+1} + \Bbbk w_{2s-n+1} & n = s+1, \ldots, 2s; \end{cases}$$

$$f_n^-.w_n \;\in\; \begin{cases} \Bbbk w_{n-1} & n = 0, \ldots, s-1, \\ \Bbbk w_{n-1} + \Bbbk w_{2s-n+1} + \Bbbk w_{2s-n-1} & n = s+1, \ldots, 2s. \end{cases}$$

In addition, $y.w_s = q^{-2s}\alpha w_{s-1} - q^{-1} w_{s+1}$ and $z.w_s = -q^{1-2s}\alpha w_{s+1}$. Since $\dim_{\Bbbk}(V) = r$, $r = 2s$.

Similar to Part (1), $\operatorname{tr}_V(x) = \sum_{n=0}^{r} \lambda_n(\alpha) = \boldsymbol{\lambda}(q^{-r+2}\alpha)\,(q^r - q^{-r}) \,/\, (q - q^{-1})$.

Note $\beta = 0$ in this case. It thus remains to show $\operatorname{tr}_V(y) = \operatorname{tr}_V(z) = 0$. By Lemma 7.3(1), $y = (q^{-1} f_n.v_n - q f_n^-.v_n)\,\delta_n$ when $\alpha^2 = q^{4s-2} \neq q^{4n-2}$, i.e., when $n \neq s$. It follows that

$$\begin{aligned} y.w_n \;&=\; \begin{cases} q^{-2s}\alpha w_{s-1} - q^{-1} w_{s+1} & n = s, \\ (q^{-1} f_n.w_n - q f_n^-.w_n)\,\delta_n & n \neq s, \end{cases} \\ &\in\; \begin{cases} \Bbbk w_{n-1} + \Bbbk w_{n+1} & n = 0, \ldots, s, \\ \Bbbk v_{2s-n-1} + \Bbbk w_{2s-n+1} + \Bbbk w_{n-1} + \Bbbk w_{n+1} & n = s+1, \ldots, 2s, \end{cases} \end{aligned}$$

where $w_{-1} = w_{2r+1} := 0$. So, the action of y on V can be presented by a matrix with zero diagonal elements. So, $\operatorname{tr}_V(y) = 0$. By a similar argument, we show $\operatorname{tr}_V(z) = 0$. $\square$

7.4.2 The simple module $S(\ell)$

Let ℓ be an A-line of Type 1 or 2. Assured by Lemma 7.12(2), the line module $M(\ell)$ has a unique simple quotient module. We denote it by $S(\ell)$.

We will call $S(\ell)$ of Type 1 (resp. Type 2) if ℓ is of Type 1 (resp. Type 2).

In the subsequent subsections §7.5 - §7.8, we will further examine the simple quotients of line modules of Type 1-4. In particular, we will show every finite dimensional simple A-module is isomorphic to $S(\ell)$ for some A-line ℓ of Type 1 or 2.

Notation From §7.5 to §7.6, we fix a cyclic permutation (x, y, z) of (x_1, x_2, x_3) and denote

$$\ell_{\alpha,\beta} \;=\; \{x - \boldsymbol{\lambda}(q\alpha) \;=\; y - \alpha z - \beta \;=\; 0\}\,,$$

where $(\alpha, \beta) \in \Bbbk^2$ is such that $\alpha \neq 0$ and $(\alpha^2 - q^{-4})\,\beta = 0$.

7.5 *Type 1:* $\alpha^2 \in -q^{2\mathbb{N}-2}$ *and* $\beta = 0$

The following results shows that exactly one of the five r-dimensional simples is a quotient of a Type 1 line module.

Lemma 7.14. *If* $\alpha^2 \in -q^{2r-4}$ *with* $r \geq 1$, *then*

1. *there is an exact sequence* $0 \to M(\ell_{\alpha,0}[-r]) \to M(\ell_{\alpha,0}) \to S(\ell_{\alpha,0}) \to 0$,
2. $\dim_{\Bbbk}(S(\ell_{\alpha,0})) = r$, *and*
3. *there is an A-module isomorphism* $S(\ell_{\alpha,0}) \cong S(\ell_{-\alpha,0})$.

Proof. Write $\ell = \ell_{\alpha,0}$ and $S = S(\ell)$ for short.

(1) There is an exact sequence $0 \to M(\ell') \to M(\ell) \to S \to 0$, where

$$\ell' \;=\; \{x - \boldsymbol{\lambda}(q\alpha') = y - \alpha' z - \beta' = 0\}\,.$$

By Proposition 7.11(1), $\alpha' = -q^{-4}\alpha^{-1} = q^{-2r}\alpha$ and $\beta' = 0$. So, $\ell' = \ell[-r]$.

(2) By Proposition 7.7(2), $M(\ell)$ is a direct sum of 1-dimensional x-eigenspaces with eigenvalues $\{\lambda_n(\alpha) \mid n \geq 0\}$. Similarly, $M(\ell[-r])$ is a direct sum of 1-dimensional x-eigenspaces

with eigenvalues $\{\lambda_n(\alpha') \mid n \geq 0\} = \{\lambda_n(\alpha) \mid n \geq r\}$. Since $S \cong M(\ell)/M(\ell[-r])$, it follows that $\dim_{\Bbbk}(S) = r$.

(3) First, it is direct to verify that $\lambda_n(\alpha) = \lambda_{r-n-1}(-\alpha)$ for all $n \in \mathbb{Z}$ when $\alpha^2 \in -q^{2r-4}$. In addition, by Lemma 7.1(1), the scalars in $\{\lambda(\alpha) \mid n \geq 0\}$ are distinct.

In view of the proof in Part 2, the eigenvalues for the action of x on S are the distinct scalers $\lambda_0(\alpha), \ldots, \lambda_{r-1}(\alpha)$. Hence, there exists an x-eigenvector, say v, with eigenvalue $\lambda_{r-1}(\alpha) = \lambda_0(-\alpha)$, that is,

$$(x - \lambda_0(-\alpha)).v \;=\; 0.$$

It follows from Lemma 7.4(2) with $n = 0$ that

$$(x - \lambda_{-1}(-\alpha))\, f_0^-(-\alpha).v \;=\; q^2 f_{-2}^-(-\alpha)\,(x - \lambda_0(-\alpha))\,.v \;=\; 0.$$

But, $\lambda_{-1}(-\alpha) = \lambda_r(\alpha)$ is not an x-eigenvalue for S, so $f_0^-(-\alpha).v = 0$. Hence,

$$(y + \alpha z)v \;=\; -q^2 \alpha f_0^-(-\alpha)v \;=\; 0.$$

It follows that there is a surjection $M(\ell_{-\alpha,0}) \twoheadrightarrow Av = S$. By Lemma 7.12(2), $M(\ell_{-\alpha,0})$ has a unique simple quotient, which is $S(\ell_{-\alpha,0})$. Therefore, $S(\ell_{\alpha,0}) \cong S(\ell_{-\alpha,0})$. □

7.6 Type 2: $\alpha^2 = q^{-4}$ and $\beta^2 \in q^{-2}[\mathbb{N}_+]^2$

The following shows that exactly four of the r-dimensional simples are quotients of Type 2 line modules.

Lemma 7.15. *If* $\alpha^2 = q^{-4}$ *and* $\beta^2 \in q^{-2}[r]^2$ *where* $r \geq 1$*, then*

1. *there is an exact sequence* $0 \to M(\ell_{\alpha,\beta}[-r]) \to M(\ell_{\alpha,\beta}) \to S(\ell_{\alpha,\beta}) \to 0$*;*
2. $\dim_{\Bbbk}(S(\ell_{\alpha,\beta})) = r$*;*

3. *the four simple A-modules*

$$S\left(\ell_{\alpha,\beta}\right), \quad S\left(\ell_{-\alpha,\beta}\right), \quad S\left(\ell_{\alpha,-\beta}\right), \quad S\left(\ell_{-\alpha,-\beta}\right)$$

are non-isomorphic;

4. *none of the four simple modules in Part 3 is isomorphic to the r-dimensional simple A-module in Lemma 7.14.*

Proof. (1) The proof is similar to that of Lemma 7.14(1).

(2) The proof is similar to that of Lemma 7.14(2).

(3) Let $S_i = S(\ell_{\alpha_i,\beta_i})$, where $\alpha_i^2 = q^{-4}$ and $\beta_i^2 = q^{-2}[r]^2$ for $i = 1, 2$. It suffices to show that if $S_1 \cong S_2$ then $(\alpha_1, \beta_1) = (\alpha_2, \beta_2)$.

By Lemma 7.13,

$$(\mathrm{tr}_{S_i}(x), \mathrm{tr}_{S_i}(y)) \;=\; \left(\boldsymbol{\lambda}(q^{-r+2}\alpha_i)\frac{q^r - q^{-r}}{q - q^{-1}}, \frac{\beta_i}{1 - q^{-2}}\right) \;=\; \left(\frac{q^2\alpha_i[r]}{q - q^{-1}}, \frac{\beta_i}{1 - q^{-2}}\right).$$

If $S_1 \cong S_2$, then $(\mathrm{tr}_{S_1}(x), \mathrm{tr}_{S_1}(y)) = (\mathrm{tr}_{S_2}(x), \mathrm{tr}_{S_2}(y))$, that is,

$$\left(\frac{q^2\alpha_1[r]}{q - q^{-1}}, \frac{\beta_1}{1 - q^{-2}}\right) \;=\; \left(\frac{q^2\alpha_2[r]}{q - q^{-1}}, \frac{\beta_2}{1 - q^{-2}}\right),$$

which implies $(\alpha_1, \beta_1) = (\alpha_2, \beta_2)$. This proves Part 3.

(4) Let V be the r-dimensional simple A-module in Lemma 7.14. By Lemma 7.13, $\mathrm{tr}_V(x) = 0$. However, the trace for the action of x on $S(\ell_{\alpha,\beta})$ is nonzero, as shown in Part 3. So V must not be isomorphic to $S(\ell_{\alpha,\beta})$. ☐

7.7 *Type 3:* $\alpha^2 \in q^{4\mathbb{N}}$ *and* $\beta = 0$

The following result shows that Type 3 line modules provide no more simple modules (up to isomorphism) by their quotients than Type 2 line modules do.

Lemma 7.16. *Every simple quotient of a Type 3 line module is also a quotient of a Type 2 line module.*

Proof. Let $M = M(\ell_{\alpha,0})$ be a Type 3 line module, where $\alpha^2 = q^{4r-4}$ with $r \geq 1$. Let V be a simple quotient of M.

Write $\alpha_0 := q^{-2r}\alpha$. Note that $\alpha_0^2 = q^{-4}$ and $\lambda_{-1}(\alpha_0) = \lambda_0(\alpha_0)$.

By Proposition 7.7(2), the x-eigenspace in M with eigenvalue $\lambda_r(\alpha) = \lambda_0(\alpha_0)$ is 2-dimensional.

By Proposition 6.11, there is an exact sequence $0 \to M(\ell_{\alpha',\beta'}) \to M(\ell_{\alpha,0}) \to V \to 0$. Write $M' = M(\ell_{\alpha',\beta'})$. By Proposition 7.11(3), $\alpha' = \alpha_0$ or $q^{-4r-4}\alpha$, so $\alpha'^2 = q^{-4}$ or q^{-4r}. In either case, the x-eigenspace in M' with eigenvalue $\lambda_0(\alpha_0)$ has dimension no greater than 2, according to Proposition 7.7(2).

Since $V \cong M/M'$, it follows that the x-eigenspace in V with eigenvalue $\lambda_0(\alpha_0)$, denoted by W, is finite dimensional and nonzero. Applying Lemma 7.4(2) with $n = 0$, we see that

$$(x - \lambda_{-1}(\alpha_0))f_0^-(\alpha_0).W \;=\; q^2 f_{-2}^-(\alpha_0)(x - \lambda_0(\alpha_0)).W \;=\; 0.$$

Since $\lambda_{-1}(\alpha_0) = \lambda_0(\alpha_0)$, $f_0^-(\alpha_0).W \subset W$. Since $\Bbbk$ is algebraically closed, there is a $q^2\alpha_0 f_0^-(\alpha_0)$-eigenvector in W, say v_0, with eigenvalue β_0 for some $\beta_0 \in \Bbbk$, that is,

$$\left(q^2\alpha_0 f_0^-(\alpha_0) - \beta_0\right).v_0 \;=\; (y - \alpha_0 z - \beta_0)\,.v_0 \;=\; 0.$$

Since $v_0 \in W$, we also have $(x - \lambda_0(\alpha_0)).v_0 = 0$. Hence, there is a surjective homomorphism

$$M\left(\ell_{\alpha_0,\beta_0}\right) \twoheadrightarrow Av_0 \;=\; V.$$

Lastly, since $M\left(\ell_{\alpha_0,\beta_0}\right)$ has a finite dimensional quotient, it must not be simple. By Proposition 7.11(2), $\beta_0^2 \in q^{-2}[\mathbb{N}_+]^2$. Therefore the line module $M(\ell_{\alpha_0,\beta_0})$ is of Type 2. $\square$

While it is not needed for the classification of finite dimensional simple A-modules, the next result shows the structure of a Type 3 line module.

Proposition 7.17. *Every Type 3 line module has three proper submodules, which are line modules. Two submodules are Type 2 and one is simple.*

Proof. Let $M = M(\ell_{\alpha,0})$ be a line module of Type 3, where $\alpha^2 = q^{4r-4}$ with $r \geq 1$. Then $\lambda_n(\alpha) = \lambda_{2r-1-n}(\alpha)$ and $f_n(\alpha) = q^2 f^-_{2r-1-n}(\alpha)$ for all $n \in \mathbb{Z}$.

Let $\{e_n | n \geq 0\}$ be the basis of M given by Proposition 7.7(1). By Lemma 7.6, $f_n e_n = e_{n+1}$ and

$$f^-_n(\alpha).e_n \;=\; \frac{q^{-2n+4}\alpha^2 - q^{2n-4}\alpha^{-2}}{q^2 - q^{-2}}[n]e_{n-1} \;=\; [2r-n][n]e_{n-1}. \tag{7-19}$$

Let $v = e_{r-1} + q^{-1}[r]^{-1}e_r$. Write $\alpha_0 = q^{-2r}\alpha$ and $\beta_0 = q^{-2r+1}\alpha[r]$. Then $\alpha_0^2 = q^{-4}$ and $\beta_0^2 = q^{-2}[r]^2$. It follows that

$$\begin{aligned} f_{r-1}(\alpha).v &= f_{r-1}.\left(e_{r-1} + q^{-1}[r]^{-1}e_r\right) \\ &= e_r + q[r]^{-1}f^-_r e_r \\ &\overset{7-19}{=} e_r + q[r]e_{r-1} \\ &= q[r]v. \end{aligned}$$

Since $f_{r-1}(\alpha) = q^{-2r+4}\alpha y - z = q^4\alpha_0 y - z$, it follows that $(y - \alpha_0 z - \beta_0).v = 0$. In a similar manner, we show that if $v' = e_{r-1} - q^{-1}[r]^{-1}e_r$ then $(y - \alpha_0 z + \beta_0).v' = 0$.

Since $\lambda_r(\alpha) = \lambda_{r-1}(\alpha) = \lambda_0(\alpha_0)$, $(x - \lambda_0(\alpha_0))v = (x - \lambda_0(\alpha_0))v' = 0$. Hence, there are surjective homomorphisms

$$M(\ell_{\alpha_0,\beta_0}) \twoheadrightarrow Av \qquad \text{and} \qquad M(\ell_{\alpha_0,-\beta_0}) \twoheadrightarrow Av'.$$

By Proposition 4.17, both Av and Av' are line modules and hence infinite dimensional. As a result, both of the above surjections are isomorphisms. Note that $M(\ell_{\alpha_0,\pm\beta_0})$ are of Type 2.

By Lemma 7.14 and Lemma 7.15, both Av and Av' have a proper submodule, which is simple and isomorphic to

$$\bigoplus_{n\geq 0} \frac{\Bbbk[x]}{(x-\lambda_n(\alpha_0))} = \bigoplus_{n\geq 2r} \frac{\Bbbk[x]}{(x-\lambda_n(\alpha))}$$

as $\Bbbk[x]$-modules. However, by Proposition 7.7(2), as a $\Bbbk[x]$-module M has one and only one such submodule. This proves the result. □

7.8 Type 4: $\alpha^2 \in q^{4\mathbb{N}-2}$ *and* $\beta = 0$

Lemma 7.18. *Every Type 4 line module is simple.*

Proof. Let $M = M(\ell_{a,0})$ be a Type 4 line module, where $\alpha^2 = q^{4r-2}$ with $r \geq 0$. If M is not simple then there is a finite dimensional simple quotient V of M.

By Lemma 7.13,

$$\mathrm{tr}_V(x) = \frac{\boldsymbol{\lambda}(q^{-r+2}\alpha)\,(q^r - q^{-r})}{q-q^{-1}} = \frac{(q^r + q^{-r})\,(q^{r+1} - q^{-r-1})}{q^{2r} - q^{2r-2}} \neq 0,$$

and $\mathrm{tr}_V(y) = \mathrm{tr}_V(z) = 0$.

By Proposition 6.11, with respect to the cyclic permutation (y, z, x), there is a surjection $M(\ell') \twoheadrightarrow S$, for some A-line

$$\ell' = \{y - \boldsymbol{\lambda}(q\alpha') = z - \alpha' x = 0\}\,.$$

By Proposition 7.11, ℓ' is of Type 1, 3 or 4. According to Lemma 7.13,

- if ℓ' is of type 1, then $\mathrm{tr}_V(x_i) = 0$ for $i = 1, 2, 3$, a contradiction;
- if ℓ' is of type 3, then V is isomorphic to a Type 2 simple module by Lemma 7.16. It follows that $\mathrm{tr}_V(x_i) \neq 0$ for $i = 1, 2, 3$, a contradiction;
- if ℓ' is of type 4, then $\mathrm{tr}_V(y) \neq 0$ and $\mathrm{tr}_V(z) = \mathrm{tr}_V(x) = 0$, a contradiction again.

Therefore, such simple quotient V does not exist and M must be simple. $\square$

7.9 Main results

Now we complete the classification of finite dimensional simple A-modules and summarize the results below. Recall the notion of shift and translation of an A-line defined in §6.2.1.

Theorem 7.19. *For every integer $r \geq 1$, there are five r-dimensional simple A-modules up to isomorphism. Each simple module S occurs in an exact sequence*

$$0 \longrightarrow M(\ell[-r]) \longrightarrow M(\ell) \longrightarrow S \longrightarrow 0$$

for some A-line $\ell \in \mathbb{A}^3_{x_1,x_2,x_3}$. The five A-lines, corresponding to the five simple modules, could be chosen to be

1. $\{x - \boldsymbol{\lambda}(\mathbb{i}q^{r-1}) \;=\; y - \mathbb{i}q^{r-2}z \;=\; 0\}$;
2. $\{x - \boldsymbol{\lambda}(q) \;=\; y - q^{-2}z - q^{-1}[r] \;=\; 0\}$;
3. $\{x + \boldsymbol{\lambda}(q) \;=\; y + q^{-2}z - q^{-1}[r] \;=\; 0\}$;
4. $\{x - \boldsymbol{\lambda}(q) \;=\; y - q^{-2}z + q^{-1}[r] \;=\; 0\}$;
5. $\{x + \boldsymbol{\lambda}(q) \;=\; y + q^{-2}z + q^{-1}[r] \;=\; 0\}$,

where (x, y, z) is a cyclic permutation of (x_1, x_2, x_3).

Proof. By Proposition 6.13, every r-dimensional simple A-module V appears in an exact sequence $0 \to M(\ell') \to M(\ell) \to V \to 0$ for some A-lines

$$\begin{aligned} \ell &= \{x - \boldsymbol{\lambda}(q\alpha) \;=\; y - \alpha z - \beta \;=\; 0\} \text{ and} \\ \ell' &= \{x - \boldsymbol{\lambda}(q\alpha') \;=\; y - \alpha' z - \beta' \;=\; 0\}\,. \end{aligned}$$

By Proposition 7.11, ℓ is of Type 1, 2, 3 or 4.

Thanks to Lemma 7.16 and 7.18 we may (and will) further assume ℓ to be of Type 1 or 2, without loss of generality.

Lemma 7.14 and 7.15 classify all r-dimensional simple quotients of line modules of Type 1 and 2, respectively. It is shown that there are five non-isomorphic r-dimensional simple A-modules, for each of which $\ell' = \ell[-r]$. In addition,

- if ℓ is of Type 1, then $(\alpha, \beta) = (\mathbb{i}q^{r-1}, 0)$, and
- if ℓ is of Type 2, then (α, β) is one of the following

$$\left(q^{-2}, q^{-1}[r]\right), \quad \left(-q^{-2}, q^{-1}[r]\right), \quad \left(q^{-2}, -q^{-1}[r]\right), \quad \left(-q^{-2}, -q^{-1}[r]\right).$$

The proof is now complete. □

Notation Let $r \geq 1$ and fix a cyclic permutation (x, y, z) of (x_1, x_2, x_3). Denote the five A-lines listed in the statement of Theorem 7.19 by

$$\begin{aligned} \ell_{0,0} &:= \left\{x - \boldsymbol{\lambda}(\mathbb{i}q^{r-1}) \;=\; y - \mathbb{i}q^{r-2}z \;=\; 0\right\} \text{ and} \\ \ell_{i,j} &:= \left\{x - i\boldsymbol{\lambda}(q) \;=\; y - q^{-2}iz - q^{-1}[r]j \;=\; 0\right\} \text{ for } i, j \in \{1, -1\}. \end{aligned}$$

Thus, the five r-dimensional simple A-modules are [2]

$$S_{i,j}(r) \;:=\; \frac{M(\ell_{i,j})}{M(\ell_{i,j}[-r])},$$

where $(i, j) \in \{(0,0), (1,1), (1,-1), (-1,1), (-1,-1)\}$.

The next corollary provides a useful tool to distinguish one finite dimensional simple A-module from another.

---

[2] More accurately, here $M(\ell_{i,j}[-r])$ should be replaced by its isomorphic image in $M(\ell_{i,j})$.

Corollary 7.20. *Let $S_{i,j}(r)$ be as defined above.*

1. *The traces for the actions of x, y and z on $S_{i,j}(r)$ are*

$$(\mathrm{tr}(x), \mathrm{tr}(y), \mathrm{tr}(z)) \;=\; \frac{[r]}{q-q^{-1}}\,(i, j, ij)\,,$$

where $(i,j) \in \{(0,0), (1,1), (1,-1), (-1,1), (-1,-1)\}$.

2. *Two finite dimensional simple A-modules V and W are isomorphic if and only if*

$$(\mathrm{tr}_V(x), \mathrm{tr}_V(y)) \;=\; (\mathrm{tr}_W(x), \mathrm{tr}_W(y))\,.$$

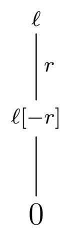
Proof. Part 1 follows from Lemma 7.13. Part 2 is an immediate consequence of Part 1. □

As a closely related topic, all simple line modules for A are identified. Simply speaking, a line module for A is simple if and only if it is anything but of Type 1, 2, or 3.

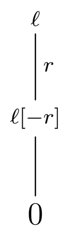
Theorem 7.21. *Every line module of Type 1, 2 or 3 has a finite dimensional simple quotient. All other line modules are simple.*

Proof. This follows from Proposition 7.11, Lemma 7.12 and Lemma 7.18. □

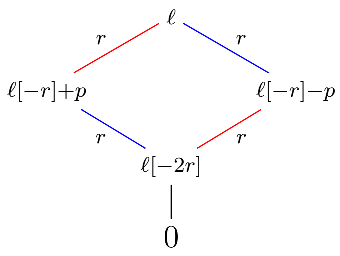
In addition, the structure of the non-simple line modules are illustrated in the following diagrams. Write $p = (0,0,q[r]) \in \mathbb{A}^3_{x,y,z}$.



Type 1



Type 2



Type 3

There are 24 line modules that map onto r-dimensional simple A-modules. Let

$$\mathbb{L}_r := \{A\text{-lines mapping onto an } r\text{-dimensional simple}\}$$

for $r \geq 1$. Then $|\mathbb{L}_r| = 24$. Recall that we have identified all lines in $\mathbb{L}_1$ in §6.3.1. The following table shows every line in $\mathbb{L}_r$ is either a shift or translation of an line in $\mathbb{L}_1$.

Write $p_r := (0,0,q[r]-q) \in \mathbb{A}^3_{x,y,z}$ for $r \geq 1$.

$\mathbb{L}_1$	$\mathbb{L}_r$	Type
$\ell_1 = \{x = y - \mathbb{i}q^{-1}z = 0\}$	$\ell_1[\frac{1}{2}r-1]$ if r is even $\ell_1[\frac{1}{2}(r-1)]$ if r is odd	1
$\ell_2 = \{x = y + \mathbb{i}q^{-1}z = 0\}$	$\ell_2[\frac{1}{2}r-1]$ if r is even $\ell_2[\frac{1}{2}(r-1)]$ if r is odd	1
$\ell_3 = \{x + \boldsymbol{\lambda}(q) = y + q^{-2}z + q^{-1} = 0\}$	$\ell_3 - p_r$	2
$\ell_4 = \{x + \boldsymbol{\lambda}(q) = y + q^{-2}z - q^{-1} = 0\}$	$\ell_4 + p_r$	2
$\ell_5 = \{x - \boldsymbol{\lambda}(q) = y - q^{-2}z + q^{-1} = 0\}$	$\ell_5 - p_r$	2
$\ell_6 = \{x - \boldsymbol{\lambda}(q) = y - q^{-2}z - q^{-1} = 0\}$	$\ell_6 + p_r$	2
$\ell_7 = \{x - \boldsymbol{\lambda}(q) = y - z = 0\}$	$\ell_7[r-1]$	3
$\ell_8 = \{x + \boldsymbol{\lambda}(q) = y + z = 0\}$	$\ell_8[r-1]$	3

The following example gives the five 2-dimensional simple A-modules.

Example 7.22 (2-dimensional simple A-modules). Write $\gamma = (q+q^{-1})^{-1}$ and $\delta = (q-q^{-1})^{-1}$. The five non-isomorphic 2-dimensional simple A-modules are given by

$$\frac{A}{Aa + Ab + Ac},$$

where $(a,b,c) \in A^{\oplus 3}$ equals one of the following:

1. $(x_1 - \mathbb{i}\gamma, x_2 - \mathbb{i}x_3, x_3^2 + \gamma^2)$;

2. $(x_1 - \delta, x_2 - q^{-2}x_3 - q - q^{-3}, x_3^2 - (2\delta + \delta^{-1})x_3 + \delta^2 + 1)$;

3. $(x_1 + \delta, x_2 + q^{-2}x_3 - q - q^{-3}, x_3^2 + (2\delta + \delta^{-1})x_3 + \delta^2 + 1)$;

4. $(x_1 - \delta, x_2 - q^{-2}x_3 + q + q^{-3}, x_3^2 + (2\delta + \delta^{-1})x_3 + \delta^2 + 1)$;

5. $(x_1 + \delta, x_2 + q^{-2}x_3 + q + q^{-3}, x_3^2 - (2\delta + \delta^{-1})x_3 + \delta^2 + 1)$.

The matrix presentation $A \to \mathbb{M}_2(\mathbb{k})$ for each of the five simple modules is determined by the images of x_1, x_2 and x_3, as given below.

1. $\begin{bmatrix} \mathbb{i}\gamma & 0 \\ 0 & -\mathbb{i}\gamma \end{bmatrix}, \qquad \begin{bmatrix} 0 & -\mathbb{i}q^{-1}\gamma \\ -\mathbb{i}q\gamma & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & -q^{-1}\gamma \\ q\gamma & 0 \end{bmatrix},$
2. $\begin{bmatrix} \delta & 0 \\ 0 & \delta + \delta^{-1} \end{bmatrix}, \qquad \begin{bmatrix} 2\delta + \delta^{-1} & -q^{-1}\delta \\ q(\delta + \delta^{-1}) & 0 \end{bmatrix}, \qquad \begin{bmatrix} 2\delta + \delta^{-1} & -q\delta \\ q^{-1}(\delta + \delta^{-1}) & 0 \end{bmatrix},$
3. $\begin{bmatrix} -\delta & 0 \\ 0 & -\delta - \delta^{-1} \end{bmatrix}, \qquad \begin{bmatrix} 2\delta + \delta^{-1} & -q^{-1}\delta \\ -q(\delta + \delta^{-1}) & 0 \end{bmatrix}, \qquad \begin{bmatrix} -2\delta - \delta^{-1} & -q\delta \\ q^{-1}(\delta + \delta^{-1}) & 0 \end{bmatrix},$
4. $\begin{bmatrix} \delta & 0 \\ 0 & \delta + \delta^{-1} \end{bmatrix}, \qquad \begin{bmatrix} -2\delta - \delta^{-1} & -q^{-1}\delta \\ q(\delta + \delta^{-1}) & 0 \end{bmatrix}, \qquad \begin{bmatrix} -2\delta - \delta^{-1} & -q\delta \\ q^{-1}(\delta + \delta^{-1}) & 0 \end{bmatrix},$
5. $\begin{bmatrix} -\delta & 0 \\ 0 & -\delta - \delta^{-1} \end{bmatrix}, \qquad \begin{bmatrix} -2\delta - \delta^{-1} & q^{-1}\delta \\ -q(\delta + \delta^{-1}) & 0 \end{bmatrix}, \qquad \begin{bmatrix} 2\delta + \delta^{-1} & -q\delta \\ q^{-1}(\delta + \delta^{-1}) & 0 \end{bmatrix}.$

7.10 The action of Alt_4 *on the simple* A*-modules*

Recall from §3.5 that the alternating group $\mathrm{Alt}_4 = \langle \sigma, \tau_1, \tau_2, \tau_3 \rangle$ acts as automorphisms of A, given by

$$\begin{array}{lllllllll}
\sigma(x_1) & = & x_2, & \sigma(x_2) & = & x_3, & \sigma(x_3) & = & x_1, \\
\tau_1(x_1) & = & x_1, & \tau_1(x_2) & = & -x_2, & \tau_1(x_3) & = & -x_3, \\
\tau_2(x_1) & = & -x_1, & \tau_2(x_2) & = & x_2, & \tau_2(x_3) & = & -x_3, \\
\tau_3(x_1) & = & -x_1, & \tau_3(x_2) & = & -x_2, & \tau_3(x_3) & = & x_3.
\end{array}$$

There is an auto-equivalence $\rho^* : \mathsf{Mod}(A) \to \mathsf{Mod}(A)$ associated to each $\rho \in \mathrm{Alt}_4$. It is clear that $\rho^* M$ is a finite dimensional simple A-module if M is. The dimensions of $\rho^* M$ and M are the same. We now examine the action of Alt_4 on the five isomorphism classes of simple A-modules of each dimension, as a generalization of Corollary 3.9.

Specifically, we will see that the Type 1 simple modules are fixed by the action of Alt_4 and the four Type 2 simple modules of dimension r form a single orbit. Since that orbit has size 4 the stabilizer of each of them is a cyclic subgroup of Alt_4 having size 3. There are four such subgroups of Alt_4. Thus, these four simple modules are naturally labelled by the four order-3 subgroups of Alt_4.

We adopt the following notation. If M and N are A-modules such that $N \cong \rho^* M$ we will write

$$M \xrightarrow{\rho} N$$

to indicate this relationship.

Proposition 7.23. *Let* $r \geq 1$ *and write* $S_{i,j} = S_{i,j}(r)$.

1. *The action of Alt*$_4$ *on the isomorphism classes of* r*-dimensional simple* A*-modules is*

given by the following commutative diagrams:

$$\begin{tikzcd}
S_{0,0} \arrow[loop left, "\sigma"] \arrow[loop right, "\tau_i"]
\end{tikzcd}
\qquad
\begin{tikzcd}
S_{1,1} \arrow[loop left, "\sigma"]
\end{tikzcd}
\qquad
\begin{tikzcd}
S_{-1,1} \arrow[r, shift left, "\tau_2"] \arrow[d, shift left, "\tau_1"] \arrow[dr, shift left, "\tau_3"] & S_{1,1} \arrow[l, shift left, "\tau_2"] \arrow[d, shift left, "\tau_1"] \\
S_{-1,-1} \arrow[u, shift left, "\tau_1"] \arrow[r, shift left, "\tau_2"] & S_{1,-1} \arrow[l, shift left, "\tau_2"] \arrow[u, shift left, "\tau_1"] \arrow[ul, shift left, "\tau_3"]
\end{tikzcd}
\qquad
\begin{tikzcd}
& S_{-1,1} \arrow[dl, "\sigma"'] & \\
S_{-1,-1} \arrow[rr, "\sigma"'] & & S_{1,-1} \arrow[ul, "\sigma"']
\end{tikzcd}$$

2. *There are two orbits for the action of* Alt_4 *on the set of isomorphism classes of* r*-dimensional* A*-modules, namely*

$$\begin{aligned}
\mathrm{Orb}(S_{0,0}) &= \{S_{0,0}\} \quad \textit{and} \\
\mathrm{Orb}(S_{1,1}) &= \{S_{1,1}, S_{1,-1}, S_{-1,1}, S_{-1,-1}\}\,.
\end{aligned}$$

Proof. (1) Write $(x, y, z) = (x_1, x_2, x_3)$. The result can be verified using Corollary 7.20 and the fact that if V is a finite dimensional A-module and $\rho \in \mathrm{Alt}_4$, then

$$(\mathrm{tr}_{\rho^*V}(x), \mathrm{tr}_{\rho^*V}(y), \mathrm{tr}_{\rho^*V}(z)) \;=\; (\mathrm{tr}_V(\rho(x)), \mathrm{tr}_V(\rho(y)), \mathrm{tr}_V(\rho(z)))\,.$$

For example, by Corollary 7.20(1),

$$\big(\mathrm{tr}_{\tau_1^* S_{1,-1}}(x), \mathrm{tr}_{\tau_1^* S_{1,-1}}(y)\big) \;=\; \big(\mathrm{tr}_{S_{1,-1}}(x), -\,\mathrm{tr}_{S_{1,-1}}(y)\big) \;=\; \frac{[r]}{q-q^{-1}}\,(1,1)$$

and

$$\big(\mathrm{tr}_{S_{1,1}}(x), \mathrm{tr}_{S_{1,1}}(y)\big) \;=\; \frac{[r]}{q-q^{-1}}\,(1,1)\,.$$

Hence, $\tau_1^* S_{1,-1} \cong S_{1,1}$ by Corollary 7.20(2). We leave the rest of the verification to the reader.

(2) This is an immediate consequence of Part 1. □

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