CURVATURE OF THE CONVEX HULL OF PLANAR BROWNIAN MOTION NEAR ITS MINIMUM POINT

Krzysztof Burdzy[1]
and
Jaime San Martin[2]

Abstract. Let f be a (random) real-valued function whose graph represents the boundary of the convex hull of planar Brownian motion run until time 1 near its lowest point in a coordinate system so that f is non-negative and $f(0) = 0$. The ratio of $f(x)$ and $|x|/|\log |x||$ oscillates near 0 between 0 and infinity a.s.

1. Main results. Let $X = (X_1, X_2)$ be a 2-dimensional Brownian motion and let C denote the (closed) convex hull of $X([0,1])$. It is well known that a.s. there exists a unique $t_0 \in (0,1)$ such that $X_2(t_0) = \min\{X_2(t) : t \in [0,1]\}$. The boundary $\partial (C - X(t_0))$ of the translated convex hull $C - X(t_0)$ is a C^1-curve (Cranston et al. (1989)) so it is represented locally near 0 by the graph $\{(x, f(x)) : x \in \mathbb{R}\}$ of a random nonnegative C^1-function $f : \mathbb{R} \to \mathbb{R}$, such that $f(0) = 0$. Our main result is contained in the following

Theorem 1.1. *(i)*

$$\limsup_{x\to 0} \frac{f(x)}{|x|\,|\log|x||^{-1}} = \infty \quad a.s. \tag{1.1}$$

(ii)

$$\limsup_{x\to 0} \frac{f(x)}{|x|\,|\log|x||^{-1}\log|\log|x||} \le \pi \quad a.s. \tag{1.2}$$

For the sake of reference we state an obvious consequence of (1.2): for each $\varepsilon > 0$,

$$\lim_{x\to 0} \frac{f(x)}{|x|\,|\log|x||^{-1+\varepsilon}} = 0 \quad \text{a.s.} \tag{1.3}$$

The above statements give an idea about functions whose graphs stay locally (near 0) in $C - X(t_0)$. Cranston et al. (1989) investigated functions with graphs outside $C - X(t_0)$ in order to prove that ∂C is C^1-smooth. We will state one of their main results in a slightly changed form.

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[1]Department of Mathematics, University of Washington, Seattle, WA 98195.
[2]Department of Statistics, Purdue University, W. Lafayette, IN 47907. Permanent address: Universidad de Chile, Facultad de Ingenieria, Depto. de Mathemáticas, Beaucheff 850, Santiago, Chile.
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Theorem 1.2 (Cranston, Hsu and March (1989)). *Suppose that $g : \mathbb{R} \to \mathbb{R}$ is nonnegative, convex and $g(0) = 0$. Then*

$$\liminf_{x\to 0} \frac{f(x)}{g(x)} \begin{cases} = 0, \\ = \infty, \end{cases} \qquad a.s. \tag{1.4}$$

according as

$$\int_{-a}^{a} g(x)x^{-2}dx \begin{cases} = \infty & \text{for every } a > 0, \\ < \infty & \text{for some } a > 0. \end{cases} \tag{1.5}$$

We have adopted the convention that any positive number divided by zero is taken to be infinity.

We will give a proof of Theorem 1.2 as it will fit well into our paper and will not require much additional space. Our proof will differ from the original one only in its form, i.e. it will contain no new conceptual ideas.

As a consequence of Theorem 1.2, we have

$$\liminf_{x\to 0} \frac{f(x)}{|x|\,|\log|x||^{-1}} = 0 \quad \text{a.s.} \tag{1.6}$$

and

$$\lim_{x\to 0} \frac{f(x)}{|x|\,|\log|x||^{-1-\varepsilon}} = \infty \quad \text{a.s.} \tag{1.7}$$

for every $\varepsilon > 0$.

Notice the eye-pleasing symmetry between (1.1) and (1.6) and between (1.3) and (1.7). Unfortunately (from the aesthetic point of view) this symmetry breaks down in the case of (1.2) and its counterpart:

$$\liminf_{x\to 0} \frac{f(x)}{|x|\,|\log|x||^{-1}\log|\log|x||^{-1}} = 0 \quad \text{a.s.}$$

This statement also follows from Theorem 1.2.

Formulae (1.1) and (1.6) say that, in a vague sense, the curvature of ∂C at $X(t_0)$ is that of $|x|/|\log|x||$ at 0. More precisely, near 0 the ratio of $f(x)$ and $|x|/|\log|x||$ oscillates between 0 and infinity. It is not easy to visualize or illustrate such behavior while having in mind that both functions are C^1 and convex.

Probabilists are accustomed to comparing random functions to nonrandom ones as in the law of iterated logarithm for Brownian motion. One of the main reasons for doing so is the irregularity of the trajectories. There is no such excuse in the case of the function f representing $\partial(C - X(t_0))$. The next theorem about this function does not refer to any nonrandom functions for comparison.

Theorem 1.3. *(i) For every neighborhood U of 0 in $\mathbb{R}$ we have*

$$\int_U f(x)x^{-2}dx = \infty \quad a.s. \tag{1.8}$$

(ii)

$$\sum_{k=1}^{\infty} 2^{-kf\left(2^{-k}\right)/\left(\pi 2^{-k}\right)} = \infty \quad a.s. \tag{1.9}$$

The above conditions may be looked upon as a lower bound and an upper bound for f, respectively. Notice that (1.8) is a close cousin of (1.5). Based on Lemma 3.4 (see below), one can prove various statements that are slightly different from (1.9). For example one may replace 2 in (1.9) (in all three places) by a different constant greater than 1.

Before stating the next result, we will give some more definitions. Let X^h be the h-process in the upper half-plane $\{(x, y) \in \mathbb{R}^2 : y > 0\}$ starting at $(0, 1)$ and converging to $(0, 0)$. In other words, X^h is a 2-dimensional Brownian motion starting from $(0, 1)$ and conditioned to hit the horizontal axis at $(0, 0)$. Let $\widetilde{C}$ denote the convex hull of $X^h([0, R))$ where R is the (random) lifetime of X^h. We will identify $\mathbb{R}^2$ and $\mathbb{C}$.

The proof of Theorem 1.1 hinges on the following estimate.

Lemma 1.1. *There exist $c_1 > 0$ and $c_2 < \infty$ such that for $r \in (0, 1/10)$ and $\alpha \in (0, 1/10)$ we have*

$$c_1 \left(\frac{r}{|\log r|}\right)^{\alpha/(1-\alpha)} \leq P\left(re^{i\alpha\pi} \notin \widetilde{C}\right) \leq c_2 r^{\alpha/(1-\alpha)}. \tag{1.10}$$

The lemma estimates the chance that a fixed point is in the convex hull $\widetilde{C}$. This and convexity will be used to compute the chance that a polygonal line is inside $\widetilde{C}$; and polygonal lines will be used to approximate smooth curves.

Now we will present some multidimensional results.

Suppose that $X = (X_1, X_2, \ldots, X_n)$ is an n-dimensional Brownian motion for some $n \geq 2$, and $t_0 \in (0, 1)$ is the unique point satisfying $X_n(t_0) = \min\{X_n(t) : t \in [0, 1]\}$. Let C be the convex hull of $X([0, 1])$ and denote by f the function mapping $\mathbb{R}^{n-1}$ into $\mathbb{R}$ whose graph represents $\partial(C - X(t_0))$ near 0.

In the next theorem, x will denote a member of $\mathbb{R}^{n-1}$ and dx will denote $(n-1)$-dimensional Lebesgue measure.

Theorem 1.4. *(i) Suppose that $g : \mathbb{R}^{n-1} \to \mathbb{R}$ is a nonnegative convex function such that $g(0) = 0$. Then*

$$\liminf_{x \to 0} \frac{f(x)}{g(x)} \begin{cases} = 0, \\ = \infty, \end{cases} \quad a.s. \tag{1.11}$$

according as

$$\int_{\{|x|<a\}} g(x)|x|^{-n} dx \begin{cases} = \infty & \textit{for every } a > 0, \\ < \infty & \textit{for some } a > 0. \end{cases} \tag{1.12}$$

(ii)

$$\limsup_{x \to 0} \frac{f(x)}{|x| \, |\log |x||^{-1}} = \infty \quad a.s.$$

(iii) For every neighborhood U of 0 in $\mathbb{R}^{n-1}$ we have

$$\int_U f(x)|x|^{-n}dx = \infty \quad a.s.$$

The results presented here are not as complete as one would like them to be. Here is a list of open problems.

(i) Find estimates of $P\left(re^{i\alpha\pi} \notin \widetilde{C}\right)$ more precise than those in Lemma 1.1.
(ii) Does (1.2) remain valid with π replaced with a smaller constant, for example 0?
(iii) What is a multi-dimensional analogue of (1.2)?
(iv) Is it possible to find a simple necessary and sufficient condition (for example an integral test analogous to (1.5)) for a function so that its graph stays locally inside $C - X(t_0)$?
(v) Does there exist an LIL-type theorem for the internal side of $\partial(C - X(t_0))$? Notice that there is no such LIL-type result for the external side of $\partial(C - X(t_0))$ due to the particular form of the test (1.5).

Our proofs will use the conformal invariance of Brownian motion, the theory of h-processes and elements of exit systems. In order to save space, we will not review these notions here. Information on these subjects is available in Burdzy (1987), Doob (1984), Durrett (1984) and Williams (1979).

The convex hull of planar Brownian motion was studied by Lévy (1948), El Bachir (1983) and Cranston et al. (1989), among others. In particular, the C^1-smoothness of ∂C was either conjectured or proved in each of these publications.

We would like to say that our research was inspired by the paper of Cranston et al. (1989) as indicated by the title of the present article. We would like to thank the Referee for a very detailed report.

2. Preliminaries. In this section, we will introduce notation and present a lemma which is fundamental to our study.

We will identify $\mathbb{R}^2$ and $\mathbb{C}$. The imaginary unit, the real and imaginary parts of z will be denoted i, $\Re z$ and $\Im z$, respectively. By the convex hull of A we will mean the smallest convex and closed set containing A.

We will use a probability space $(\Omega, \mathcal{F})$ where Ω is the family of all paths $\omega : [0,\infty) \to \mathbb{R}^n \cup \{\delta\}$ which are continuous on $[0, R)$ and equal to δ afterwards. The lifetime R of ω may be infinite. The canonical process will be denoted X, i.e. $X_t(\omega) = \omega(t)$ for all ω and t (sometimes we will use different symbols to denote a process). The σ-field generated by $\{X_t,\ t \geq 0\}$ will be denoted $\mathcal{F}$.

We will use several measures on $(\Omega, \mathcal{F})$. First of all, P^x will denote the measure which makes X a standard Brownian motion in $\mathbb{R}^n$ starting from $x \in \mathbb{R}^n$. The symbol P^x_D will denote the distribution of Brownian motion in D (i.e. Brownian motion killed at the time of first hitting of $\mathbb{R}^n \backslash D$) and P^x_h will stand for the distribution of the h-process starting from x (i.e. conditioned Brownian motion). See Doob (1984) for the definitions and properties of P^x, P^x_D and P^x_h.

For each set A in $\mathbb{R}^n$, let

$$T(A) = \inf\left\{t > 0 : \lim_{s\uparrow\uparrow t} X(s) \in A\right\}.$$

If the hitting time $\inf\{t > 0 : X(t) \in A\}$ of A is less than infinity then it is equal to $T(A)$, by the continuity of paths.

We will write C for the convex hull of $X([0,1])$ and $\widetilde{C}$ for the convex hull of $X([0,R))$. The first symbol will be usually used in conjunction with P^x and the second one with P^x_h.

For a convex closed subset C_1 of $\mathbb{R}^n$, let $M = (M_1, M_2, \ldots, M_n) = M(C_1)$ denote its lowest point, i.e.

$$M_n = \min\{z_n : (z_1, z_2, \ldots, z_n) \in C_1\}$$

provided such a point M exists. If $\partial(C - M(C))$ (resp. $\partial(\widetilde{C} - M(\widetilde{C}))$) may be represented locally near 0 by the graph of a function defined on $\mathbb{R}^{n-1}$ then this function will be denoted f (resp. $\widetilde{f}$). This representation is not unique but this is irrelevant to our study. If the graphs of functions f_1 and f_2 represent $\partial(C - M(C))$ locally near 0 then these functions are equal on some neighborhood U of 0 in $\mathbb{R}^{n-1}$ but they are not necessarily identical on the whole of $\mathbb{R}^{n-1}$.

A property A of a convex closed subset of $\mathbb{R}^n$ will be called local if the following holds. Suppose that C_1 and C_2 are convex and closed sets and for some neighborhood U of 0 we have

$$U \cap (C_1 - M(C_1)) = U \cap (C_2 - M(C_2));$$

then both sets C_1 and C_2 have the property A or both sets do not have A. We will say that a property A is preserved under unions if for every pair of convex and closed sets which have A, the convex hull of their union also has A.

Let h be the Poisson kernel in $D \stackrel{\text{df}}{=} \{z \in \mathbb{C} : \Im z > 0\}$ corresponding to $0 \in \partial D$. Notice that with this choice of h, the process X under P^i_h is the Brownian motion starting from i and conditioned to exit D at 0.

Lemma 2.1. *Suppose that A is a local property of convex sets which is preserved under unions. If $\widetilde{C}$ has the property A P^i_h-a.s. then C has A P^z-a.s. for each $z \in \mathbb{C}$.*

Proof. Suppose that $\widetilde{C}$ has the property A P^i_h-a.s. Notice that $\lim_{t\uparrow R} X(t) = 0$, X is continuous and $X([0,R)) \subset D$, P^i_h-a.s. Since for each $\alpha \in (0,1)$, the half-line $\{z \in \mathbb{C} : z = re^{i\alpha\pi}, r > 0\}$ is not minimal thin in D at 0 (Example 1 XII 12 (b) of Doob (1984)), Theorem 3 III 3 of Doob (1984) implies that $\widetilde{C}$ does not stay in any cone with vertex 0 and angle less that π, P^i_h-a.s. Let C_ε denote the convex hull of $X((R-\varepsilon, R))$. It is easy to see that with P^i_h-probability 1, for every $\varepsilon > 0$, we can find a neighborhood U of 0 such that $U \cap \widetilde{C} = U \cap C_\varepsilon$. It follows that there exists an event A_1 in the tail σ-field of the P^i_h-process such that P^i_h-a.s., the event A_1 holds if and only if $\widetilde{C}$ has the property A.

Let $G_D(\cdot\,,\cdot)$ be the Green function in D and $h_1(x) \stackrel{\text{df}}{=} G_D(x,i)$. If X has the distribution P^i_h then $Y(t) \stackrel{\text{df}}{=} X(R-t)$ has the distribution $P^0_{h_1}$ (see Doob (1984) 3 III 1). Let A_2 be the time-reversed version of A_1, i.e. the event A_1 holds for X if and only if A_2 holds for Y. Then A_2 belongs to the σ-field $\mathcal{F}_{0+}$ defined relative to Y and holds $P^0_{h_1}$-a.s.

Let H^0 denote the standard excursion law of Brownian motion in D starting at 0 (see Burdzy (1987) Chapter 3). Since A_2 holds $P^0_{h_1}$-a.s., the 0-0 law (see

Proposition 3.3 and its proof in Burdzy (1987)) implies that $H^0(A_2^c) = 0$. By the translation and rotation invariance of Brownian motion, the analogous result is true for all standard excursion laws H^x in D and $D_1 = \{z \in \mathbb{C} : \Im z < 0\}$, i.e. $H^x (A_2^c$ holds for $X - x) = 0$ for all $x \in \partial D$.

For s such that $X_s \in \partial D$, the excursion e_s of X in $D \cup D_1$ is defined by

$$e_s(t) = \begin{cases} X(s+t) & \text{if } s+t < \inf\{u > s : X_s \in \partial D\}, \\ \delta & \text{otherwise.} \end{cases}$$

If $e_s \equiv \delta$ then this excursion is called trivial.

The exit system formula (3.3) of Burdzy (1987) and the above argument concerning excursion laws imply that P^0-a.s., for every non-trivial excursion e_s of X in $D \cup D_1$, $e_s - e_s(0)$ has the property A_2.

Let $Z(t) = \Re X(t) + i|\Im X(t)|$ and let $s_0 = \max\{t < 1 : Z(t) \in \partial D\}$. If X has the distribution P^0 then Z is a reflected Brownian motion in D and the above result about excursions and the obvious symmetry show that $\{Z(s_0+t) - Z(s_0),\ t \in [0, 1-s_0]\}$ has the property A_2 P^0-a.s.

By a theorem of Lévy (see Williams (1979) Theorem II 61) the processes $|\Im X(t)|$ and $\Im X(t) - \min_{s\le t} \Im X(s)$ have the same distribution under P^0. Since $\Re X$ and $\Im X$ are independent under P^0, the processes $Z(t) = \Re X(t) + i|\Im X(t)|$ and

$$V(t) = \Re X(t) + i\left(\Im X(t) - \min_{s\le t} \Im X(s)\right)$$

also have identical distributions. Let t_0 be the unique time such that $X(t_0) = M(C)$ P^0-a.s. Then the distributions of $\{Z(t),\ t \in [s_0, 1]\}$ and $\{V(t),\ t \in [t_0, 1]\}$ under P^0 are the same. It follows that

$$\{V(t_0+t) - V(t_0),\ t \in [0, 1-t_0]\}$$

and, therefore,

$$\{X(t_0+t) - X(t_0),\ t \in [0, 1-t_0]\}$$

have the property A_2 P^0-a.s. This implies that the convex hull C_1 of $\{X(t) - X(t_0)$, $t \in [t_0, 1]\}$ has the property A P^0-a.s.

Observe that $\{X(1-t) - X(1),\ t \in [0,1]\}$ has the same distribution as $\{X(t),\ t \in [0,1]\}$ under P^0. Hence, the convex hull C_2 of $\{X(t) - X(t_0),\ t \in [0, t_0]\}$ has the property A P^0-a.s.

Notice that $C - M(C)$ is the convex hull of $C_1 \cup C_2$ so it has the property A P^0-a.s. as well, since we assumed that A is preserved under unions. This also holds P^x-a.s. for every $x \in \mathbb{C}$, by the translation invariance of Brownian motion. □

Remarks 2.1. (i) We will use a multidimensional version of the above lemma. It can be proved in a completely analogous way.

(ii) Theorems 1.1-1.4 deal with properties of C which satisfy our definition of the local property which is preserved under unions.

3. Proofs. Let

$$\begin{aligned} D &= \{z \in \mathbb{C} : \Im z > 0\}, \\ L(\alpha, z) &= \left\{v \in \mathbb{C} : v = z + re^{i\alpha\pi},\ r > 0\right\}, \\ T(\alpha) &= T(L(\alpha, 0)). \end{aligned}$$

It is well known that the Brownian hitting distribution on a line is a Cauchy distribution, in particular we have for $z = x + iy \in D$, $(r, 0) \in \partial D$,

$$P^z \left(X(T(\partial D)) \in dr\right) = \frac{y}{\pi\left((r-x)^2 + y^2\right)} dr. \tag{3.1}$$

First we will derive a formula for the hitting density of the side of a wedge.

Lemma 3.1. *Suppose that* $\alpha \in (0, 1/2)$ *and let* $\gamma = 1/(1-\alpha)$ *and* $\left(ie^{-i\alpha\pi}\right)^\gamma = x_0 + iy_0$. *Then*

$$P^i_D(|X(T(\alpha))| \in dr) = \frac{y_0 \gamma r^{\gamma-1}}{\pi\left(\left(r^\gamma - x_0\right)^2 + y_0^2\right)} dr, \quad r > 0. \tag{3.2}$$

Proof. Let D_1 be the connected component of $D \setminus L(\alpha, 0)$ which contains i. Notice that

$$P^i_D(|X(T(\alpha))| \in dr) = P^i_{D_1}(|X(T(\alpha))| \in dr). \tag{3.3}$$

Let $f(z) = \left(ze^{-i\alpha\pi}\right)^\gamma$. The function f maps D_1 onto D, is analytic in D_1 and one-to-one. The conformal invariance of Brownian motion (see Durrett (1984)) implies that for a subset A of $L(\alpha, 0)$ we have

$$P^i_{D_1}(X(T(\alpha)) \in A) = P^{f(i)}_D(X(T(\partial D)) \in f(A)).$$

This formula, (3.1) and (3.3), together with some elementary calculations yield (3.2). □

Next we will find a formula for the hitting probability of a subset of $L(\alpha, 0)$ by the conditioned Brownian motion.

Let h be the Poisson kernel in D with the pole at 0, i.e.

$$h\left(re^{i\alpha\pi}\right) = \frac{\sin \alpha\pi}{r} \quad \text{for } r > 0,\ \alpha \in (0, 1). \tag{3.4}$$

The function h is positive and harmonic in D and every h-process converges to 0 a.s.

Lemma 3.2. *For* $0 < r < 1$ *and* $0 < \alpha < 1/4$ *we have*

$$P^i_h(|X(T(\alpha))| > r) \geq 1 - 2r^{\alpha/(1-\alpha)}. \tag{3.5}$$

Proof. First we will derive a formula for the P^i_h-hitting density of $T(\alpha)$. By formula (2.1), page 672 of Doob (1984) we have

$$P^i_h(|X(T(\alpha))| \in dr) = P^i_D(|X(T(\alpha))| \in dr) \frac{h\left(re^{i\alpha\pi}\right)}{h(i)}.$$

It follows from Lemma 3.1 and (3.4) that

$$P_h^i(|X(T(\alpha))| \in dr) = \frac{y_0 \gamma r^{\gamma-1}}{\pi\left((r^\gamma - x_0)^2 + y_0^2\right)} \frac{\sin \alpha\pi}{r}\, dr, \tag{3.6}$$

where $\gamma = 1/(1-\alpha)$ and $x_0 + iy_0 = \left(ie^{-i\alpha\pi}\right)^\gamma$. Notice that $y_0 > 1/2$ for $\alpha < 1/4$. Hence,

$$\begin{aligned} P_h^i(|X(T(\alpha))| < u) &= \frac{y_0 \gamma \sin \alpha\pi}{\pi} \int_0^u \frac{r^{\gamma-2}}{(r^\gamma - x_0)^2 + y_0^2}\, dr \\ &\le \frac{y_0 \gamma \sin \alpha\pi}{\pi} \int_0^u \frac{r^{\gamma-2}}{y_0^2}\, dr \\ &= \frac{1}{y_0} \frac{\sin \alpha\pi}{\pi} \gamma \frac{1}{\gamma - 1} u^{\gamma-1} \\ &= \frac{1}{y_0} \frac{\sin \alpha\pi}{\alpha\pi} u^{\alpha/(1-\alpha)} \\ &\le \frac{1}{y_0} u^{\alpha/(1-\alpha)} \\ &\le 2u^{\alpha/(1-\alpha)}. \end{aligned} \tag{3.7}$$

The set $L(\alpha, 0)$ is not minimal thin in D at 0 (Example 1 XII 12 (b) of Doob (1984)). Then Theorem 3 III 3 of Doob (1984) implies

$$P_h^i(T(\alpha) < \infty) = 1. \tag{3.8}$$

The lemma now follows from (3.7) and (3.8). □

Now we will estimate the chance that P_h^i-process does not hit a line $L(\alpha, z)$. We will write $T(\alpha, z) = T(L(\alpha, z))$.

Lemma 3.3. *There exists $c_1 > 0$ such that for $z = (x, 0) : 0 < x < 1/2$ and $0 < \alpha < 1/4$ we have*

$$P_h^i(T(\alpha, z) = \infty) \ge c_1 x^{\alpha/(1-\alpha)}.$$

Proof. Let D_1, γ, x_0 and y_0 be as in Lemma 3.1. The formula (3.2) is equivalent to

$$P_{D_1}^i(|X(T(\alpha))| \in dr) = \frac{y_0 \gamma r^{\gamma-1}}{\pi\left((r^\gamma - x_0)^2 + y_0^2\right)} dr. \tag{3.9}$$

Let $v = e^{i(\alpha+1/2)\pi}$, i.e. v is the point symmetric to i with respect to the axis of symmetry of D_1. Then (3.9) and this symmetry imply

$$P_{D_1}^v(|X(T(\partial D))| \in dr) = \frac{y_0 \gamma r^{\gamma-1}}{\pi\left((r^\gamma - x_0)^2 + y_0^2\right)} dr. \tag{3.10}$$

Let D_2 be the component of $D \setminus L(\alpha, z)$ which contains i. By the translation invariance, we obtain from (3.10),

$$P_{D_2}^{v+z}(|X(T(\partial D)) - z| \in dr) = \frac{y_0 \gamma r^{\gamma-1}}{\pi\left((r^\gamma - x_0)^2 + y_0^2\right)} dr. \tag{3.11}$$

The points i and $v+z$ are contained in the set

$$A \stackrel{\text{df}}{=} \{w \in \mathbb{C} : |w| < 2,\ \Im w > \sqrt{2}/4,\ \Im w > \Re w + 1/4\},$$

assuming that $0 < x < 1/2$ and $0 < \alpha < 1/4$. Denote

$$B = \{w \in \mathbb{C} : |w| < 3,\ \Im w > \sqrt{2}/8,\ \Im w > \Re w + 1/8\}.$$

The Harnack principle applied in the sets A and B shows that there is a constant $c_2 > 0$ such that $g(z_1) \geq c_2 g(z_2)$ for every choice of $z_1,\ z_2 \in A$ and every function g which is positive and harmonic in B. The function

$$w \to P^w_{D_2}(|X(T(\partial D)) - z| \in dr)$$

is positive and harmonic in B so (3.11) and the Harnack principle imply that

$$P^i_{D_2}(|X(T(\partial D)) - z| \in dr) \geq c_2 \frac{y_0 \gamma r^{\gamma-1}}{\pi\left((r^\gamma - x_0)^2 + y_0^2\right)} dr. \tag{3.12}$$

Recall that the P^i_h-process may be interpreted as the P^i-process conditioned to hit ∂D at 0. Hence,

$$P^i_h(T(\alpha, z) = \infty) = \frac{P^i_{D_2}(|X(T(\partial D)) - z| \in dr)\,|_{r=x}}{P^i(X(T(\partial D)) \in du)\,|_{u=0}}.$$

In view of (3.1) and (3.12), the last quantity is greater or equal to

$$c_2 \ \frac{y_0 \gamma z^{\gamma-1}}{\left((z^\gamma - x_0)^2 + y_0^2\right)}.$$

There is $c_1 > 0$ such that for all $z \in (0, 1/2)$ and $\alpha \in (0, 1/4)$ we have

$$c_2 \ \frac{y_0 \gamma}{\left((z^\gamma - x_0)^2 + y_0^2\right)} > c_1.$$

Thus

$$P^i_h(T(\alpha, z) = \infty) \geq c_1 x^{\gamma-1} = c_1 x^{\alpha/(1-\alpha)}.$$

□

Proof of Lemma 1.1. Recall that $\widetilde{C}$ denotes the convex hull of $X([0, R))$.

First we will prove the right hand side of (1.10). Let $z = re^{i\alpha\pi},\ r \in (0,1),\ \alpha \in (0, 1/10)$. The points 0 and $X(T(\alpha))$ belong to $\widetilde{C}$ P^i_h-a.s. If $|X(T(\alpha))| > r$ then z belongs to the line segment joining 0 and $X(T(\alpha))$ and, therefore, is contained in $\widetilde{C}$. This and Lemma 3.2 imply that

$$\begin{aligned} P^i_h\left(re^{i\alpha\pi} \notin \widetilde{C}\right) &\leq 1 - P^i_h(|X(T(\alpha))| > r) \\ &\leq 2r^{\alpha/(1-\alpha)}. \end{aligned}$$

Now we will prove the left hand side of (1.10). Let $z = x + iy = re^{i\alpha\pi}$ and assume that $r \in \left(0,\ e^{-2}\right),\ \alpha \in (0, 1/10)$. Denote $v = (x/\,|\log x|, 0)$ and choose β so that $z \in L(\beta, v)$. Note that $\beta \in (0, 1/4)$. Observe that

$$P^i_h(z \notin \widetilde{C}) \geq P^i_h(T(\beta, v) = \infty)$$

and, by Lemma 3.3,

$$P^i_h(z \notin \widetilde{C}) \geq c_1 \left| \frac{x}{\log x} \right|^{\beta/(1-\beta)}.$$

Let $a = 1/\,|\log x|$. By the convexity of the tan function on $(0, \pi/2)$, we have

$$\begin{aligned} \tan((1-a)\beta\pi) &= \tan((1-a)\beta\pi + a \cdot 0) \\ &\leq (1-a)\tan\beta\pi. \end{aligned}$$

Elementary geometry shows that

$$(1-a)\tan(\beta\pi) = \tan(\alpha\pi),$$

so

$$\tan\left((1-a)\beta\pi\right) \leq \tan(\alpha\pi).$$

This and the fact that tan is increasing yield $(1-a)\beta\pi \leq \alpha\pi$ and $\beta \leq \alpha/\,(1-a)$. The function $\beta \to \beta/\,(1-\beta)$ is increasing, so

$$\begin{aligned} \frac{\beta}{1-\beta} &\leq \frac{\alpha/\,(1-a)}{1-\alpha/\,(1-a)} \\ &= \frac{\alpha}{1-\alpha} + \frac{a\alpha}{(1-\alpha)(1-a-\alpha)} \\ &\overset{\text{df}}{=} \frac{\alpha}{1-\alpha} + b. \end{aligned}$$

Since $\left|\dfrac{x}{\log x}\right| < 1$,

$$\left| \frac{x}{\log x} \right|^{\beta/(1-\beta)} \geq \left| \frac{x}{\log x} \right|^{\alpha/(1-\alpha)} \left| \frac{x}{\log x} \right|^{b}.$$

For $x \in \left(0, e^{-2}\right)$ and $\alpha \in (0, 1/10)$ we have

$$\begin{aligned} \log\left(\left| \frac{x}{\log x} \right|^{b} \right) &= b \log \left| \frac{x}{\log x} \right| \\ &= \frac{a\alpha}{(1-\alpha)(1-a-\alpha)} (\log x - \log|\log x|) \\ &= \frac{\alpha/\,|\log x|}{(1-\alpha)(1-\alpha-1/\,|\log x|)} (\log x - \log|\log x|) \\ &= \frac{\alpha}{(1-\alpha)(1-\alpha-1/\,|\log x|)} \left(-1 - \frac{\log|\log x|}{|\log x|} \right) \\ &\geq \frac{1/10}{(1-1/10)(1-1/10-1/2)} \left(-1 - \frac{1}{e} \right) \\ &> -1 \end{aligned}$$

and, therefore,

$$\left|\frac{x}{\log x}\right|^{b} \geq e^{-1}.$$

It follows that

$$\left|\frac{x}{\log x}\right|^{\beta/(1-\beta)} \geq e^{-1}\left|\frac{x}{\log x}\right|^{\alpha/(1-\alpha)}$$

and

$$P_h^i(z \notin \widetilde{C}) \geq c_2\left|\frac{x}{\log x}\right|^{\alpha/(1-\alpha)}. \tag{3.13}$$

Later we will use (3.13) rather than the left hand side of (1.10). Nevertheless let us complete the proof. Observe that for $r \in \left(0, e^{-2}\right)$ and $\alpha \in (0, 1/10)$ we have $r \leq 2x$ and $|\log r| > |\log x|/2$. Thus,

$$\begin{aligned}
P_h^i(z \notin \widetilde{C}) &\geq c_2\left|\frac{x}{\log x}\right|^{\alpha/(1-\alpha)} \\
&\geq c_2\left|\frac{r}{\log r}\right|^{\alpha/(1-\alpha)}\left(\frac{1}{4}\right)^{\alpha/(1-\alpha)} \\
&\geq c_2\left|\frac{r}{\log r}\right|^{\alpha/(1-\alpha)}\left(\frac{1}{4}\right)^{(1/4)/(1-1/4)} \\
&\geq c_3\left|\frac{r}{\log r}\right|^{\alpha/(1-\alpha)}.
\end{aligned}$$

□

Proof of (1.1). Consider a function $g(x) = c_4|x|/|\log|x||$ for some $c_4 \in (0, \infty)$. Let $z = re^{i\alpha\pi}$ belong to the graph of g, i.e. $z = x + ig(x)$. Assume that $\alpha \in (0, 1/8)$. Then we have

$$\alpha\pi \leq \tan(\alpha\pi) = c_4/|\log|x||$$

and, therefore,

$$\frac{\alpha}{1-\alpha} \leq 2\alpha \leq \frac{2c_4}{\pi|\log|x||}.$$

Now suppose that $x \in \left(0, e^{-2}\right)$ so that $|\log x| \geq 2$ and $x/|\log x| < 1$. It follows that

$$\left|\frac{x}{\log x}\right|^{\alpha/(1-\alpha)} \geq \left|\frac{x}{\log x}\right|^{2c_4/(\pi|\log x|)}$$

and

$$\begin{aligned}
&\log\left(\left|\frac{x}{\log x}\right|^{2c_4/(\pi|\log x|)}\right) \\
&\quad = \frac{2c_4}{\pi|\log x|}(\log x - \log|\log x|) \\
&\quad = \frac{2c_4}{\pi}\left(-1 - \frac{\log|\log x|}{|\log x|}\right) \\
&\quad \geq \frac{2c_4}{\pi}\left(-1 - \frac{1}{e}\right) \\
&\quad > -c_4.
\end{aligned}$$

Hence,

$$\left|\frac{x}{\log x}\right|^{\alpha/(1-\alpha)} \geq e^{-c_4}.$$

This and (3.13) imply that for all points z on the graph of g which are close to 0 we have

$$P_h^i(z \notin \widetilde{C}) \geq c_5 > 0. \tag{3.14}$$

Let $\{z_k\}$ be a sequence of points on the graph of g which converges to 0. Then, by (3.14),

$$P_h^i\left(\bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} \left\{z_k \notin \widetilde{C}\right\}\right) \geq c_5. \tag{3.15}$$

The event in this formula belongs to the tail σ-field so (3.15) and the 0-1 law (see Doob (1984) 2 X 12 (c1)) imply that the probability in (3.15) is in fact equal to 1.

Recall from Section 2 that the graph of $\widetilde{f}$ represents $\partial\widetilde{C}$ near 0. We obtain from (3.15)

$$\limsup_{x\to 0} \frac{\widetilde{f}(x)}{c_4|x|\,|\log|x||^{-1}} \geq 1 \quad P_h^i\text{-a.s.}$$

for all rational $c_4 \in (0,\infty)$ simultaneously. It follows that

$$\limsup_{x\to 0} \frac{\widetilde{f}(x)}{|x|\,|\log|x||^{-1}} = \infty \quad P_h^i\text{-a.s.}$$

which implies (1.1) in view of Lemma 2.1. □

Lemma 3.4. *Suppose that* $g:\mathbb{R}\to\mathbb{R}$, $g\geq 0$, $g(0)=0$ *and* $g(x)/x \to 0$ *as* $x\to 0$. *Let* $z_k = x_k + ig(x_k) = r_k e^{i\alpha_k\pi}$ *and assume that* $x_k > 0$ *and* $x_k \to 0$ *as* $k\to\infty$. *If*

$$\sum_{k=1}^{\infty} x_k^{g(x_k)/(\pi x_k)} < \infty$$

then

$$P_h^i\left(\exists k_0 : z_k \in \widetilde{C} \text{ for all } k > k_0\right) = 1.$$

Proof. For large k we have $r_k < 2x_k$ and $\alpha_k/(1-\alpha_k) < 1/2$ so

$$r_k^{\alpha_k/(1-\alpha_k)} \leq x_k^{\alpha_k/(1-\alpha_k)}\sqrt{2}.$$

For small α we have

$$\alpha/(1-\alpha) > \tan(\alpha\pi)/\pi.$$

Notice that $\tan(\alpha_k\pi) = g(x_k)/x_k$. Thus, for large k, so that α_k and x_k are small, we have

$$\begin{aligned} r_k^{\alpha_k/(1-\alpha_k)} &\leq c_1 x_k^{\alpha_k/(1-\alpha_k)} \\ &\leq c_1 x_k^{\tan(\alpha_k\pi)/\pi} \\ &\leq c_1 x_k^{g(x_k)/(\pi x_k)}. \end{aligned}$$

Now it follows from Lemma 1.1 that for suitably large k_0,

$$\sum_{k=k_0}^{\infty} P_h^i\left(z_k \notin \widetilde{C}\right) \le \sum_{k=k_0}^{\infty} c_2 r_k^{\alpha_k/(1-\alpha_k)} \le \sum_{k=k_0}^{\infty} c_2 c_1 x_k^{g(x_k)/(\pi x_k)} < \infty.$$

The conclusion of our lemma follows from this inequality, by the Borel-Cantelli lemma. □

Proof of (1.2). First we will apply Lemma 3.4 with

$$g(x) = g_\varepsilon(x) = (\pi+\varepsilon)|x|\,|\log|x||^{-1}\log|\log|x||,$$

where $\varepsilon > 0$ and we will take $x_k = e^{-k}$. We have

$$\begin{aligned}\log\left(x_k^{g(x_k)/(\pi x_k)}\right) &= \frac{g(x_k)}{\pi x_k}\log x_k \\ &= (\pi+\varepsilon)e^{-k}\left|\log e^{-k}\right|^{-1}\log\left|\log e^{-k}\right|\log e^{-k}/\left(\pi e^{-k}\right) \\ &= -(1+\varepsilon/\pi)\log k.\end{aligned}$$

Thus

$$x_k^{g(x_k)/(\pi x_k)} = k^{-1-\varepsilon/\pi}$$

and, therefore,

$$\sum_{k=1}^{\infty} x_k^{g(x_k)/(\pi x_k)} < \infty.$$

Let $z_k = x_k + ig(x_k)$. Lemma 3.4 implies that for each fixed $\varepsilon > 0$, the sequence $\{z_k\}$ stays eventually in $\widetilde{C}$ P_h^i-a.s.

Fix an arbitrary $\varepsilon > 0$. With P_h^i-probability 1, for some random k_0 and all $k > k_0$, the points

$$e^{-k} + i(\pi+\varepsilon)e^{-k}(\log k)/k$$

and

$$e^{-k} + i(\pi+\varepsilon/2)e^{-k}(\log k)/k$$

belong to $\widetilde{C}$. It follows that

$$e^{-k} + i(\pi+\varepsilon)e^{-k}(\log(k+1))/(k+1)$$

also belongs to $\widetilde{C}$ since this point is contained in the line segment joining the previous two points, at least for large k. Notice that the points 0,

$$e^{-k-1} + i(\pi+\varepsilon)e^{-k-1}(\log(k+1))/(k+1)$$

and

$$e^{-k} + i(\pi+\varepsilon)e^{-k}(\log(k+1))/(k+1)$$

belong to a straight line K and also belong to $\widetilde{C}$, for large k. The first two points and only these two points of K belong to the graph of g_ε since this function is strictly convex. It follows that the part of the graph of $g_\varepsilon(x)$ between $x = e^{-k-1}$ and $x = e^{-k}$ stays above K and, therefore, inside $\widetilde{C}$. This is true for all large k, P^i_h-a.s., so the graph of g_ε stays in $\widetilde{C}$ in some random interval $(0, \eta)$ and, by symmetry, in $(-\eta_1, \eta_1)$, $\eta_1 > 0$. This may be expressed by saying that for each $\varepsilon > 0$,

$$\limsup_{x\to 0} \frac{\widetilde{f}(x)}{(\pi+\varepsilon)|x|\,|\log|x||^{-1}\log|\log|x||} \le 1 \quad P^i_h\text{-a.s.}$$

Since $\varepsilon > 0$ is arbitrary, the inequality holds even with $\varepsilon = 0$. This and Lemma 2.1 imply (1.2). □

Proof of Theorem 1.2 and Theorem 1.4 (i). Let $\mathbf{1} = (0, 0, \dots, 0, 1) \in \mathbb{R}^n$ and let h be the Poisson kernel in $\{x \in \mathbb{R}^n : x_n > 0\}$ corresponding to 0. Suppose that $g : \mathbb{R}^{n-1} \to \mathbb{R}$ is convex, nonnegative and $g(0) = 0$. Then g is Lipschitz in a neighborhood of 0 and the set A below the graph of g is minimal thin in $\{x \in \mathbb{R}^n : x_n > 0\}$ at 0 if and only if (Burdzy (1987) Theorem 8.2)

$$\int_{\{|x|<a\}} g(x)|x|^{-n}dx < \infty, \quad \text{for some } a > 0. \tag{3.16}$$

Minimal thinness of A is equivalent in this case to the fact that the convex hull $\widetilde{C}$ of $X([0, R))$ stays above A locally near 0 $P^{\mathbf{1}}_h$-a.s. (Doob (1984) 3 III 3). This we may write as

$$\liminf_{x\to 0} \frac{\widetilde{f}(x)}{g(x)} \ge 1 \quad P^{\mathbf{1}}_h\text{-a.s.} \tag{3.17}$$

If (3.16) holds for some g then it holds for every function cg, for every rational $c \in (0, \infty)$, and the same may be said, consequently, about (3.17). In such a case we have

$$\liminf_{x\to 0} \frac{\widetilde{f}(x)}{g(x)} = \infty \quad P^{\mathbf{1}}_h\text{-a.s.}$$

As in the previous proofs, it remains to invoke Lemma 2.1 to complete the proof of Theorem 1.2 and Theorem 1.4 (i). □

Proof of Theorem 1.3 and Theorem 1.4 (iii). First we will prove (1.8). In view of Lemma 2.1 it will suffice to prove (1.8) with $\widetilde{f}$ in place of f.

Observe that the event $\left\{\int \widetilde{f}(x)x^{-2}dx = \infty\right\}$ belongs to the tail σ-field of the P^i_h-process, so its probability is either 0 or 1 (Doob (1984) 2 X 12 (c1)).

Suppose that the P^i_h-probability of this event is zero. We will show that this assumption leads to a contradiction.

Consider two independent processes X and Y, each having the distribution P^i_h. Let functions $\widetilde{f}_X$ and $\widetilde{f}_Y$ represent locally the boundaries of the convex hulls of $X([0, R))$ and $Y([0, R))$ near 0, respectively.

Since, by assumption, $\int \widetilde{f}_Y(x)x^{-2}dx < \infty$ a.s., it follows that

$$\liminf_{x\to 0} \widetilde{f}_X(x)/\widetilde{f}_Y(x) = \infty \quad \text{a.s.,} \tag{3.18}$$

by Theorem 1.2. By symmetry,

$$\liminf_{x\to 0} \widetilde{f}_Y(x)/\widetilde{f}_X(x) = \infty \quad \text{a.s.}$$

which contradicts (3.18). This shows that

$$P_h^i\left(\int \widetilde{f}(x)x^{-2}dx = \infty\right) = 1.$$

The proofs of (1.9) and Theorem 1.4 (iii) are completely analogous and therefore are omitted. The proof of (1.9) uses Lemma 3.4 with $x_k = 2^{-k}$. □

Proof of Theorem 1.4 (ii). Notice that (X_{n-1}, X_n) is a 2-dimensional Brownian motion and Theorem 1.1 (i) may be applied to this process. Then Theorem 1.4 (ii) is an obvious consequence of (1.1). □

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