Inverse Boundary-Value Problems on an Infinite Slab

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Abstract

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In this work, we study the stability aspect of two inverse boundary-value problems (IBVPs) on an infinite slab with partial data.

The uniqueness aspects of these IBVPs were considered and studied by Li and Uhlmann in [34] for the case of the Schrödinger equation as well as by Krupchyk, Lassas and Uhlmann in [32] for the case of the magnetic Schrödinger equation.

Here we quantify the method of uniqueness proposed by Li and Uhlmann and prove a log-log stability estimate for the IBVPs associated to the Schrödinger equation. The boundary measurements considered in these problems are modelled by partial knowledge of the Dirichlet-to-Neumann map; more precisely, we establish log-log stability estimates for each of the following two IBVPs:

- in the first inverse problem, the corresponding Dirichlet and Neumann data are known on different boundary hyperplanes of the slab;

- in the second inverse problem, they are known on the same boundary hyperplane of the slab.

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GLOSSARY

IBVP: an inverse boundary-value problem.

Σ: the domain[1] between two infinite parallel slabs in $\mathbb{R}^n$, $n \geq 3$;

$\partial\Sigma$: the boundary of Σ;

CGO: complex geometrical optics;

B, Ω: a smooth bounded subdomain of Σ.

[1]For brevity of expression, we shall occasionally refer to Σ as "the slab."

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DEDICATION

to my family and friends

Chapter 1

INTRODUCTION

1.1 Inverse problems: an overview

Inverse boundary-value problems (IBVPs) arise when one tries to use boundary measurements in order to determine internal parameters of a given medium. The physical phenomena are frequently described in a PDE framework. The foundational hypothesis in the statement of an IBVP is the so-called *measurement data*. (In many cases, the nature of the measurement data is mathematically encoded in a linear operator between appropriately chosen vector spaces.). The goal in the IBVP is to recover the coefficients of the PDE based on knowledge/measurements of a family of solutions along the boundary of the medium.

An inverse problem can be studied when full measurement data is available (corresponding to full knowledge of some linear operator) or when partial measurement data is available (corresponding to limited knowledge of some linear operator). After one has decided upon the measurement data (to be taken as a hypothesis in the analysis), there are three aspects of an inverse problem which can be studied: uniqueness, stability, and reconstruction.

1.2 The Calderón problem

One elliptic inverse problem which has attracted a lot of attention is Calderón's inverse problem. This inverse problem, also known as the "Calderón problem" and as the "inverse conductivity problem," was introduced by Calderón himself in [7]; broadly speaking, he considered the question of determining the internal electrical conductivity of a physical body by making voltage and current measurements along the exterior boundary of the body.

Before we go into more precise details, let us outline the physical phenomenon which underlies the Calderón problem: When an electrical potential is applied to the boundary of a physical object, an electrical current begins to flow through the interior of that object.

The Calderón problems starts by considering a physical body/object whose interior conductivity varies from point to point; we suppose further that we do not have direct access to the interior of the given physical object, but we can make direct measurements along its boundary. Next, based on the principle from the preceding paragraph, we can apply a family of electrical potentials to the boundary of our object and measure the corresponding electrical currents flowing out of the boundary. Then, the Calderón problem asks if this measurement data suffices to reconstruct the electrical conductivity at all interior points of our physical object.

Note that the inverse conductivity problem can be classified as a technique of non-invasive testing. This technique is referred to as electrical impedance tomography (EIT). When EIT is used for medical imaging, the physical object is a patient's body, one applies (sufficiently small) electrical potentials on the patient's skin and measures the outgoing currents on the skin, and finally one determines the electrical conductivity in the different tissues comprising the patient's body. Some applications of EIT include:

- early detection of breast cancer (because a malignant tumor contains a lot of blood and the electrical conductivity of blood is very high);

- oil prospection (this was one of Calderón's reasons for considering the inverse conductivity problem);

- monitoring the pulmonary function in the human body (cf. [13]).

We now turn to a mathematical formulation of the Calderón problem. Let the physical object (to be imaged by means of EIT) be modelled by some fixed open bounded domain $\Omega \subset \mathbb{R}^n$ such that $\partial\Omega$ is smooth. At a point x in the object Ω, we denote the electrical conductivity by $\gamma(x)$. As a result, the electrical conductivity of the object is modelled by a real-valued function defined on Ω; from now on, we assume this function to be strictly positive and finite-valued everywhere in Ω.

If there are no sources or sinks of current inside Ω, Ohm's law asserts that, when we apply a potential f to the boundary $\partial\Omega$, a voltage potential $u = u_f$ is induced in the interior of Ω, so that u solves the following Dirichlet problem:

$$\begin{cases} \operatorname{div}(\gamma \nabla u) &= 0 \text{ in } \Omega \\ u|_{\partial\Omega} &= f. \end{cases} \tag{1.2.1}$$

Provided that γ satisfies certain regularity hypotheses, the Dirichlet problem (1.2.1) for the conductivity equation admits a unique weak solution $u = u_f \in H^1(\Omega)$. This enables us to define the voltage-to-current map by

$$\Lambda_\gamma f := \gamma \frac{\partial u_f}{\partial \nu}\bigg|_{\partial\Omega}.$$

One can easily verify that Λ_γ is a bounded linear operator from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$. In practical terms, this operator encodes how the current flowing out of $\partial\Omega$ depends upon the voltage potential applied to $\partial\Omega$; in other words, the boundary-measurement operator in the inverse conductivity problem is Λ_γ.

Calderón studied the case of a physical body whose electrical conductivity is constant. (In this case, the conductivity equation reduces to the Laplace equation.) Next, by using the divergence theorem, he obtained

$$\langle \Lambda_\gamma f, g \rangle_{\partial\Omega} = \int_\Omega \gamma \nabla u \cdot \nabla v \, dx,$$

where $\langle \cdot, \cdot \rangle_{\partial\Omega}$ denotes the dual pairing between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$; u, v solve the conductivity equation in Ω with boundary values f, g, respectively. Calderón's strategy was to find a sufficiently large family $\mathcal{F}$ of solutions to the conductivity equation such that the dot products of the gradients of all functions in $\mathcal{F}$ comprise a dense subset of some appropriate function space. By taking $\mathcal{F}$ to be a collection of complex exponentials $u(x) := e^{\rho \cdot x}$, Calderón was able to establish unique identifiability for γ in the inverse conductivity problem; more precisely, the functions $x \mapsto e^{\rho \cdot x}$ in $\mathcal{F}$ were indexed by a complex vector $\rho \in \mathbb{C}^n$ with $\rho \cdot \rho = 0$ and with $|\rho|$ sufficiently large.

After Calderón's pioneering work, the inverse conductivity problem has been extensively researched. A landmark result in the literature on the Calderón problem is [43] by Sylvester and Uhlmann, where they established unique identifiability for C^2 conductivities in $\Omega \subset \mathbb{R}^n$, $n \geq 3$. Their result is as follows:

Theorem 1. *Let $n \geq 3$. Assume further that $\gamma_1, \gamma_2 \in C^2(\overline{\Omega})$ both satisfy*

$$M^{-1} \leq \gamma_j(x) \leq M, \quad x \in \Omega, \ j = 1, 2$$

for some finite constant $M > 0$. If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then $\gamma_1 = \gamma_2$ in $\overline{\Omega}$.

The first step in Sylvester and Uhlmann's proof of Theorem 1 was to reduce the problem to a question about unique identifiability of the coefficieint q for the Schrödinger operator $-\Delta + q$. Then, they were able to construct a family $\mathcal{F}'$ of solutions to the conductivity equation in Ω; each function in $\mathcal{F}'$ has the form $u(x) = e^{x \cdot \rho}(1 + r(x))$, where $\rho \in C^n$ is large enough in absolute value and it satisfies $\rho \cdot \rho = 0$; furthemore, r was shown to belong to $H^1(\Omega)$ with $\|r\|_{L^2(\Omega)} = \mathcal{O}\left(\frac{1}{|\rho|}\right)$ as $|\rho| \to \infty$. It was then shown that $\mathcal{F}'$ possesses a density property similar to $\mathcal{F}$. (The functions in $\mathcal{F}'$ are called *complex geometrical optics (CGO) solutions.*) Since Sylvester and Uhlmann's contribution, the construction of CGO solutions has been an essential part in the analysis of many inverse boundary-value problems associated with elliptic PDEs.

In recent years, inverse problems with partial data have been studied extensively. Bukhgeim and Uhlmann proved in [6] that, if the boundary measurements are given by Dirichlet data on the whole boundary but by Neumann data on (roughly) half of the boundary, then such measurements determine the potential in the Schrödinger equation uniquely in dimensions three or higher. Stability estimates in the same setting were established by Heck and Wang in [24].

In [33], Kenig, Sjöstrand and Uhlmann proved an improvement on the uniqueness result of [6] by showing that uniqueness continues to hold even when the Dirichltet data is given on an any (arbitrarily small) open subset of the boundary and the Neumann data is given on a slightly larger part of its complement. Nachman and Street in [39] analysed the reconstruction aspect of the inverse problem from [33].

1.3 Inverse boundary-value problems on an infinite slab

The geometry of an infinite slab is interesting and important, because it can be used to model a number of physical phenomena. First of all, the slab geometry is used in the modeling of wave propagation in shallow-ocean acoustics. For example, the authors of [45] considered the problem of imaging an obstacle which is embedded in a shallow-water waveguide, while the authors of [27] showed that one can uniquely identify the refraction index of a compactly supported inhomogeneity in an oceanic acoustic waveguide.

Another application of the slab geometry is that it provides a simple geometrical setting for medical imaging. It is pointed out in [5] that the slab framework can be used to probe biological tissues for breast tumors and brain hemorrhages; inverse problems of optical diffusion tomography were studied in [37] and [38]. Finally, the slab geometry also arises in some inverse problems having to do with the identification of an unknown embedded object; for instance, see [26] and [42].

This work is devoted to the study of IBVPs for the Schrödinger equation on an infinite slab. In broadest terms, the problem consists of recovering the electric potential q on the slab

$$\Sigma := \{x \in \mathbb{R}^3 : 0 < x_3 < L\},$$

from partial knowledge of the Dirichlet-to-Neumann map (DN map). Here, $L > 0$ is a constant, x_3 denotes the 3rd coordinate of x and q is compactly supported in

$$Q := \{(x', x_3) \in \mathbb{R}^3 : |x'| \leq R,\, 0 \leq x_3 \leq L\}$$

with $R > 0$ a constant. The DN map is roughly defined by

$$\Lambda_q : f \longmapsto \partial_\nu u|_{\partial\Sigma},$$

where $\partial\Sigma$ denotes the boundary of Σ, ν represents the outward-pointing unit normal vector along $\partial\Sigma$, $\partial_\nu = \nu \cdot \nabla$ and u solves the problem[1]

$$\begin{cases} (-\Delta - k^2 + q)u = 0 \,\text{in}\, \Sigma \\ u|_{\partial\Sigma} = f. \end{cases}$$

In [34], Li and Uhlmann proved two uniqueness results for the potential q; each result assumes a different kind of partial knowledge of the DN map. In order to precisely describe these uniqueness results, we need to introduce some notation. The boundary of Σ consists of the two hyperplanes

$$\Gamma_1 := \{x \in \mathbb{R}^3 : x_3 = L\}, \qquad \Gamma_2 := \{x \in \mathbb{R}^3 : x_3 = 0\}.$$

Choose $R' > 0$ with $R < R'$, and set

$$\Gamma_j^N := \{x \in \Gamma_j : |x'| < R'\}, \quad j = 1, 2.$$

Let Γ_1^D be a relatively open, precompact subset of Γ_1 such that

$$\overline{\Gamma_1^N} \subset \Gamma_1^D.$$

Let q_1 and q_2 be potentials from $L^\infty(\Sigma)$ such that both are (compactly) supported in Q, and let Λ_{q_1} and Λ_{q_2} denote their corresponding DN maps. Li and Uhlamann showed that if either

$$\Lambda_{q_1} f|_{\Gamma_1^N} = \Lambda_{q_2} f|_{\Gamma_1^N}$$

for all f supported in $\overline{\Gamma_1^D}$, or

$$\Lambda_{q_1} f|_{\Gamma_2^N} = \Lambda_{q_2} f|_{\Gamma_2^N}$$

for all f supported in $\overline{\Gamma_1^D}$, then

$$q_1 = q_2.$$

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[1]This problem is well-posed under certain conditions on f, k and q but, for the sake of simplicity, we omit details at this point.

These results were extended by Krupchyk, Lassas and Uhlmann in [32] to the case of the magnetic Scrödinger equation.

In the last fifteen years, IBVPs with partial data have attracted a lot of attention and nowadays there is a fairly long list of publications studying such problems. In [6], Bukhgeim and Uhlmann established, in dimension $n \geq 3$, uniqueness results for the IBVPs associated to the Schrödinger equation and the conductivity equation in the setting where the Dirichlet data is given on the whole boundary but the Neumann data is given only on (roughly speaking) half of the boundary. This result was improved by Kenig, Sjöstrand and Uhlmann in [33]. Stability estimates for these problems have been established in [24] for the Bukhgeim and Uhlmann's result and in [10] and [11] for the Kenig *et al*'s result. It is important to point out that, so far, the best known stability for these problems is of log-log type. A partial reconstruction procedure was proposed by Nachman and Street in [39]. Other related results are [18], [14], [44], [15], [17], [41] and [16]. Another important result with partial data is [30], where Isakov proved, in dimension $n = 3$, uniqueness for IBVPs associated to the Schrödinger equation and the conductivity equation with partial data. In his paper, Isakov assumed the boundary of the domain to be partially flat or spherical and the measurements to be taken on the complement of the flat or spherical part. Wang and Heck proved in [25] that Isakov's method provides the optimal stability for this inverse problem, that is, of log type (see [36] in connection with the optimality issue). Related results are [8], [9], [32] and [35]. Other interesting results for IBVPs with partial data are [3], [28] [23], [31], [2], [4] and [20].

The basic tools to deal with this kind of partial-data IBVPs are integration by parts to obtain Alessandrini formulas and the construction of appropriate complex geometric optics (CGOs). In [6], Bukhgeim and Uhlmann used a Carleman estimate with boundary terms to control the part of the boundary where no measurements were taken and then stated a type of Alessandrini formula. On the other hand, in [30], Isakov used a reflection argument across the flat part of the domain's boundary to construct CGOs vanishing on that flat part. In [34], Li and Uhlmann took advantage of the geometry of the slab to combine the ideas from

[6] and [30] to prove their uniqueness results.

This thesis is organized as follows. In Chapter 2, we formally introduce partial-data IBVPs on an infinite slab and sketch the proof of Li and Uhlmann's uniqueness result from [34]. In Chapter 3, we discuss the stability of two partial-data IBVPs on an infinite slab; the new results are stated in Section 3.2 and proved in the remainder of Chapter 3.

Chapter 2

UNIQUENESS

2.1 Chapter overview

To fix ideas, let $\Sigma, Q, \Gamma_1^N, \Gamma_2^N, \Gamma_1^D$ be chosen as in Section 1.3. In Section 2.2, we start by discussing the solvability of a direct boundary-value problem for $-\Delta + q$ on Σ; this will then enable us to introduce the DN map Λ_q. In Section 2.3, we state Li and Uhlmann's uniqueness results from [34]. Finally, in Section 2.4, we give an outline of the main points in Li and Uhlmann's proofs.

2.2 A direct boundary-value problem and the DN map

Our first order of business is to discuss the solvability of a direct boundary-value problem for $-\Delta + q$ on Σ, because this will enable us to introduce the DN map Λ_q. Furthermore, the DN map is of central importance, because it is the mathematical model for the boundary measurements in the main results of Chapters 2 and 3.

We begin with some preliminaries: Let K be an arbitrary compact subset of Γ_1, and define

$$H_K^{3/2}(\Gamma_1) := \{f \in H^{3/2}(\Gamma_1) : \operatorname{supp} f \subseteq K\}.$$

Fix a potential $q \in L^\infty(\Sigma)$ which is compactly supported in Q. For a certain frequency $k \geq 0$ that we call admissible for q, we know that, given a compactly supported $w \in L^2(\Sigma)$, there exists a unique $v \in H^2_{\text{loc}}(\overline{\Sigma})$ such that

$$\begin{cases} (-\Delta - k^2 + q)v &= w \text{ in } \Sigma, \\ v|_{\partial\Sigma} &= 0. \end{cases} \tag{2.2.1}$$

Moreover, for any bounded subset $\Omega \subset \Sigma$, we have the estimate

$$\|v\|_{H^2(\Omega)} \lesssim \|w\|_{L^2(\Sigma)},$$

where the implicit constant depends on k, Ω, and any upper bound on $\|q\|_{L^\infty(\Sigma)}$. For an account of this direct problem and a discussion of admissible frequencies, see [32].

The well-posedness of boundary value problem (2.2.1) implies that, given any $f \in H^{3/2}_K(\Gamma_1)$, there exists a unique admissible solution $u \in H^2_{\text{loc}}(\overline{\Sigma})$ to the following Dirichlet problem

$$\begin{cases} (-\Delta - k^2 + q) \quad u &= 0 \text{ in } \Sigma, \\ u|_{\Gamma_1} &= f, \\ u|_{\Gamma_2} &= 0. \end{cases} \tag{2.2.2}$$

Having discussed the well-posedness of the direct boundary-value problem [?] on Σ, we are now ready to define the following linear map:

$$\begin{aligned} \Lambda_q : H^{3/2}_K(\Gamma_1) &\rightarrow H^{1/2}_{\text{loc}}(\partial\Sigma). \\ f &\mapsto \partial_\nu u|_{\partial\Sigma} \end{aligned}$$

where u is the unique admissible solution to the problem (2.2.2). We refer to Λ_q as *the full boundary-measurement operator.*

Let Λ^1_q and Λ^2_q denote the maps defined by

$$\Lambda^1_q f := \Lambda_q f|_{\Gamma^N_1}, \quad \Lambda^2_q f := \Lambda_q f|_{\Gamma^N_2}, \qquad \forall\, f \in H^{3/2}_{\Gamma^D_1}(\Gamma_1). \tag{2.2.3}$$

We refer to Λ^1_q, Λ^2_q as *the partial boundary-measurement operators.*

2.3 Main uniqueness results

Li and Uhlmann's uniqueness results (for the operator $-\Delta + q$; from [34]) are as follows:

Theorem 2. *With $\Sigma, \Gamma^N_1, \Gamma^N_2, \Gamma^D_1$ fixed as before, let B be a smooth bounded subdomain of Σ such that*

$$\varnothing \neq \partial B \cap \Gamma_j \subseteq \Gamma^N_j \text{ for } j = 1, 2; \quad \partial B \cap \Gamma_1 \subseteq \Gamma^D_1.$$

Then,

$$\Lambda^2_{q_1} = \Lambda^2_{q_2} \quad \Rightarrow \quad q_1 = q_2.$$

Theorem 3. *With $\Sigma, \Gamma^N_1, \Gamma^N_2, \Gamma^D_1, B$ fixed just as in Theorem 2,*

$$\Lambda^1_{q_1} = \Lambda^1_{q_2} \quad \Rightarrow \quad q_1 = q_2.$$

2.4 Outline of proofs

The main steps in the proof of Theorem 2 can be summarized as follows:

- obtain an identity involving the quantity $\int_\Sigma (q_1 - q_2) v_1 u_2 \, dx$ for a large enough set of functions v_1 on Σ and u_2 on B
- use a Carleman estimate to obtain an integral inequality involving $\left| \int_\Sigma (q_1 - q_2) v_1 u_2 \, dx \right|$
- use a Runge-type approximation argument to extend the set of functions v_1 for which the integral inequality holds true
- construct CGO solutions to plug into the extended integral inequality
- take a limit to establish Theorems 2 and 3

The proof of Theorem 3 follows a similar scheme. The main difference is that, in the proof of Theorem 3, we construct different CGO solutions and, as a consequence, no Carleman estimate is required.

2.5 Schematic proof of Theorem 2

As we indicated at the end of the previous section, the proofs of Theorem 2 and 3 rely on similar ideas. For that reason and for the sake of brevity, we dedicate this section to a more detailed proof of Theorem 2 only.

2.5.1 Step 1: obtain an integral identity

Let $f \in H^{3/2}_{\Gamma_1^D}(\Gamma_1)$ be arbitrary. Then, let $v_j \in H^2_{\mathrm{loc}}(\overline{\Sigma})$ be the unique admissible solution of

$$\begin{aligned} (-\Delta + q_j - k^2)v_j &= 0 \quad \text{in } \Sigma \\ v_j|_{\Gamma_1} &= f \\ v_j|_{\Gamma_2} &= 0 \end{aligned}$$

Set $w := v_2 - v_1$.

Let $u_2 \in H^1(B)$ be any solution of $(-\Delta + q_2 - k^2)u_2 = 0$ in B.

The following facts about w follow from its very definition and from the hypothesis that $\Lambda^2_{q_1} = \Lambda^2_{q_2}$:

$$\begin{aligned} (-\Delta + q_2 - k^2)w &= (q_1 - q_2)v_1 \text{ in } \Sigma \\ w|_{\partial\Sigma} &= 0 \\ (\partial_\nu w)|_{\Gamma_2^N} &= 0 \end{aligned}$$

But we can get even more information about w by studying its properties on $\Sigma \setminus B$.

Before we do so, let us introduce the following notation:

$$l_j := \partial B \cap \Gamma_j, \quad j = 1, 2; \quad l_3 := \partial B \cap \Sigma.$$

As announced earlier, we now summarize the following properties of w on $\Sigma \setminus B$:

$$\begin{aligned} (-\Delta - k^2)w &= 0 \text{ in } \Sigma \setminus B \\ w|_{\Gamma_2^N \setminus l_2} &= 0 \\ (\partial_\nu w)|_{\Gamma_2^N \setminus l_2} &= 0 \end{aligned}$$

Now, by (qualitative) unique continuation for the Helmholtz operator $-\Delta - k^2$ on $\Sigma \setminus B$, the above properties of w imply that $w \equiv 0$ in $\Sigma \setminus B$.

To sum up the discussion so far, we now know that

$$w = 0 \text{ on } \partial B \tag{2.5.1}$$

$$\partial_\nu w = 0 \text{ on } l_2 \cup l_3 \tag{2.5.2}$$

We are now in a position to prove the desired integral inequality as follows:

$$\begin{aligned}\int_\Sigma (q_1-q_2)v_1u_2\,dx &= \int_B (q_1-q_2)v_1u_2\,dx\\ &= \int_B [(-\Delta+q_2-k^2)w]u_2\,dx\\ &= \int_B w[(-\Delta+q_2-k^2)u_2]\,dx - \int_{\partial B}(\partial_\nu w)u_2\,dS\\ &+ \int_{\partial B} w(\partial_\nu u_2)\,dS\end{aligned}$$

Utilizing properties of w and u_2, we obtain

$$\int_\Sigma (q_1-q_2)v_1u_2\,dx = -\int_{l_1}(\partial_\nu w)u_2\,dS. \tag{2.5.3}$$

2.5.2 Step 2: apply a Carleman estimate

Applying Bukhgeim and Uhlmann's Carleman estimate (cf. Proposition A.0.1) (with $Q=B$, $q=q_2-k^2$, $u=w$) to the integral identity (2.5.3), we arrive at

$$\begin{aligned}\left|\int_\Sigma (q_1-q_2)v_1u_2\ dx\right| &\le \left(\frac{1}{\tau(\eta\cdot e_n)}\right)^{1/2}\left(\int_B |e^{-\tau x\cdot\eta}(q_1-q_2)v_1|^2\ dx\right)^{1/2}\\ &\quad \left(\int_{l_1} |e^{\tau x\cdot\eta}u_2|^2\ dS\right)^{1/2}\end{aligned}$$

for all $\tau\ge\tau_0$ and for all $\eta\in\mathbb{R}^n$ with $|\eta|\ge a_0$, $\eta\cdot e_n>0$. (At this stage, we could take $a_0:=1/2$, but we don't have to.)

2.5.3 Step 3: obtain an extended integral inequality

Let us now combine the integral inequality from Step 2 with the Runge-type approximation result (Proposition B.0.2) to obtain the following *extended integral inequality*

Lemma 2.5.1. *We have*

$$\begin{aligned}\left|\int_\Sigma (q_1-q_2)u_1u_2\ dx\right| &\le \left(\frac{1}{\tau(\eta\cdot e_n)}\right)^{1/2}\left(\int_{l_1} |e^{\tau x\cdot\eta}u_2|^2\ dS\right)^{1/2}\\ &\quad \left(\int_B |e^{-\tau x\cdot\eta}(q_1-q_2)u_1|^2\ dx\right)^{1/2}\end{aligned}$$

whenever $\eta \in \mathbb{R}^n$ *satisfies* $|\eta| \geq a_0$, $\eta \cdot e_n > 0$, τ *satisfies* $\tau \geq \tau_0$, u_1 *and* u_2 *solve*

$$(-\Delta + q_j - k^2)u_j = 0 \text{ on } B \text{ for } j = 1, 2; \quad \text{and} \quad u_1|_{l_2} = 0.$$

2.5.4 Step 4: construct CGO solutions & concrete functions u_1, u_2

For any $\xi \in \mathbb{R}^3$ with $|\xi| \geq a_0$, $\xi_{1e} := \sqrt{\xi_1^2 + \xi_2^2} > 0$, we define

$$e(1) := \frac{1}{\xi_{1e}}(\xi_1, \xi_2, 0), \quad e(3) := (0, 0, 1),$$

and $e(2) \in \mathbb{R}^3$ as the unique unit vector such that $\{e(1), e(2), e(3)\}$ forms an oriented orthonormal basis of $\mathbb{R}^3$. Note that $\xi = (\xi_{1e}, 0, \xi_3)_e$. Define $\xi^\perp := (-\xi_3, 0, \xi_{1e})_e$.

In what follows, we shall apply the extended integral inequality (i.e. Lemma 2.5.4) with $\eta = \xi^\perp$ (which is the reason for requiring that $|\xi| \geq a_0$ as early as now).

For $\tau \geq \tau_0$, we choose

$$\begin{aligned} \rho_1 &:= \tau\xi^\perp + i\left(\frac{1}{2}\xi + \sqrt{\tau^2 - \frac{1}{4}}|\xi| e(2)\right), \\ \rho_2 &:= -\tau\xi^\perp + i\left(\frac{1}{2}\xi - \sqrt{\tau^2 - \frac{1}{4}}|\xi| e(2)\right). \end{aligned}$$

By a result of Sylvester-Uhlmann (i.e. Proposition C.0.3) and by increasing τ_0 (if necessary), we can choose

$$\begin{aligned} u_2(x) &:= e^{x\cdot\rho_2}(1 + \psi_2(x, \rho_2)), \\ u_1(x) &:= e^{x\cdot\rho_1}(1 + \psi_1(x, \rho_1)) - e^{x^*\cdot\rho_1}(1 + \psi_1^*(x, \rho_1)), \end{aligned}$$

where * denotes reflection in the line $\{x_3 = 0\}$ and the Sobolev norms of the ψ's satisfy certain decay estimates in τ.

Insert these concretely chosen u_1, u_2 into Lemma 2.5.4.

$$\begin{aligned} \left|\int_\Sigma (q_1 - q_2)u_1 u_2 \, dx\right| &\leq \left(\frac{1}{\tau(\eta \cdot e_n)}\right)^{1/2} \left(\int_B |e^{-\tau x\cdot\eta}(q_1 - q_2)u_1|^2 \, dx\right)^{1/2} \\ &\quad \left(\int_{\partial B \cap \Gamma_1} |e^{\tau x\cdot\eta} u_2|^2 \, d\sigma\right)^{1/2}; \end{aligned}$$

2.5.5 Step 5: take a limit

By letting $\tau \to \infty$ and using the decay estimates on ψ_1 and ψ_2 in the last estimate, one arrives at

$$(q_1 - q_2)\widehat{\ }(\xi) = 0$$

for all $\xi \in \mathbb{R}^3$ with $|\xi| \geq a_0$, $\xi \cdot e_3 > 0$.

From here, it follows (either by properties of the Fourier transform, or by the Payley-Wiener theorem) that

$$q_1 = q_2 \text{ in } \Sigma,$$

which establishes Theorem 2.

Chapter 3
STABILITY

3.1 Chapter overview

The new results in this thesis are Theorems 4, 5 and 6. The new results are quantitative versions of Li and Uhlmann's results (recorded in this thesis as Theorems 2 and 3). Theorems 5 and 6 consist of log-log-type stability estimates for two partial-data IBVPs on a slab. This is joint work with Pedro Caro (Institute of Mathematical Sciences, Spain).

We will now sketch the main points in our proof of Theorem 5, corresponding to the case where the Dirichlet and Neumann data are measured on different hyperplanes.

To fix ideas, let $\Sigma, Q, \Gamma_1^N, \Gamma_2^N, \Gamma_1^D$ be chosen as in Section 1.3. Let q_1 and q_2 denote two potentials with compact support in Q, and let $\Lambda_{q_1}^2$ and $\Lambda_{q_2}^2$ be defined by

$$\Lambda_{q_1}^2 f = \Lambda_{q_1} f|_{\Gamma_2^N}, \qquad \Lambda_{q_2}^2 f = \Lambda_{q_2} f|_{\Gamma_2^N},$$

for all f compactly supported in $\overline{\Gamma_1^D}$. The first step in our approach is to prove an integral estimate in which

$$\left| \int_\Sigma (q_1 - q_2) u_1 u_2 \, dx \right|$$

is bounded by $\|\Lambda_{q_1}^2 - \Lambda_{q_2}^2\|_*$ plus some controllable terms, for a large enough set of functions u_1 and u_2 solving the equations $(-\Delta - k^2 + q_1)u_1 = 0$ and $(-\Delta - k^2 + q_2)u_2 = 0$ in a bounded domain $\Omega \subset \Sigma$ satisfying

$$\{x \in \Sigma : |x'| \leq R\} \subset \Omega.$$

In order to obtain this estimate, we require u_1 to vanish along $\Gamma_2 \cap \partial\Omega$. The second step in our approach is to construct an appropriate family of solutions to extract information from the integral estimate. This will be a family of CGOs depending on a large parameter τ. In order to ensure that u_1 meets the requisite condition $u_1|_{\Gamma_2 \cap \partial\Omega} = 0$, we will use Isakov's reflection argument from [30]. The third step is to insert the CGOs into the integral estimate, which enables us to estimate (from above) the Fourier transform of $q_1 - q_2$ at frequencies from

$$\{\xi = (\xi', \xi_3) \in \mathbb{R}^3 : |\xi| < r, \, |\xi'| > 1\}$$

in terms of $\|\Lambda^2_{q_2} - \Lambda^2_{q_1}\|_*$ and the parameter τ. The forth step consists of extending the set of frequencies, at which the Fourier transform of $q_1 - q_2$ is controlled, to all of $\{\xi \in \mathbb{R}^3 : |\xi| < r\}$. To do so, we proceed as Liang did in [35]: we use that the Fourier transform of $q_1 - q_2$ is analytic and a result from [29]. Thus, we are able to control all the low frequencies in a ball of arbitrary radius. Finally, we follow the ideas proposed by Alessandrini in [1] to control first $\|q_1 - q_2\|_{H^{-1}(\mathbb{R}^3)}$ and then $\|q_1 - q_2\|_{L^\infty(\Sigma)}$.

The ingredients to achieve the first step are a Carleman estimate with boundary terms (proved and used in [6] by Bukhgeim and Uhlmann), a quantified unique continuation property from a proper boundary subset (due to Phung, see [40]), and a Runge-type approximation argument (performed by Li and Uhlmann in [34]). Furthermore, in order to be able to complete the proof of our first step (which consists of utilizing the Runge-type argument), we need to introduce a new operator norm $\| \centerdot \|_*$, which is used to establish the stability of the IBVPs under consideration.

The analytic unique continuation used in the fourth step does not produce any extra log since we are not enlarging the size of frequencies, we are just extending to low frequencies. This situation is different from [24], [10], [11] and [12].

The approach used in the case where the Dirichlet and Neumann data are measured on the same hyperplane is quite similar to this one. In that case, we use CGOs to construct u_1 and u_2 in a such a way that both of them vanish on $\Gamma_2 \cap \partial\Omega$; as a consequence, no Carleman estimate is required, so the proof of the integal estimate turns out to be simpler. However, the rest of the argument requires a quantification of the Riemann-Lebesgue lemma (cf. the proof of Theorem 8.22(f) from [21] for functions in $C^\infty_c(\mathbb{R}^n)$).

In Section 3.2, we state the new results in this thesis. In Section 3.3, we prove the integral estimates for the two IBVPs under consideration. In Section 3.4, we prove the stability of the problem when the Dirichlet and Neumann data are measured on different hyperplanes. Section 3.5 is dedicated to the case where measurements are made on the same hyperplane.

3.2 Main stability results

Our stability estimates for these two IBVPs are contained in Theorems 5 and 6; Theorem 4 enables us to quantify the difference between different boundary measurements. At this point, it is useful to recall the definition of our (partial) boundary-measurement operators $\Lambda^j_{q_1}, \Lambda^j_{q_2}$ from (2.2.3).

Theorem 4. *Let $k \geq 0$ be an admissible frequency for the zero potential. Then, there exists a norm $\vert\vert\vert \cdot \vert\vert\vert$ on $H^{3/2}_{\Gamma_1^D}(\Gamma_1)$, which depends on k, such that, if $q \in L^\infty(\Sigma)$ with $\operatorname{supp} q \subseteq Q$ and if k is admissible for q, then Λ^l_q is a bounded operator from $\left(H^{3/2}_{\Gamma_1^D}(\Gamma_1), \vert\vert\vert \cdot \vert\vert\vert\right)$ to $H^{-3/2}(\Gamma_l^N)$.*

Let $\| \cdot \|_*$ denote the operator norm of bounded linear operators from $\left(H^{3/2}_{\Gamma_1^D}(\Gamma_1), \vert\vert\vert \cdot \vert\vert\vert\right)$ to $H^{-3/2}(\Gamma_l^N)$.

Theorem 5. *Consider $s > 3/2$, and let q_1, q_2 belong to $H^s(\Sigma)$ and have their supports contained in Q. Consider $k \geq 0$ to be admissible for q_1, q_2 and the zero potential. Let M denote an upper bound on $\|q_j\|_{H^s(\Sigma)} \leq M$. Then, there exists $\delta = \delta(L, R, k) > 0$ such that, if $\|\Lambda^2_{q_2} - \Lambda^2_{q_1}\|_* < 1/\delta$, then*

$$\|q_1 - q_2\|_{L^\infty(\Sigma)} \lesssim \left(\log[1 + |\log(\delta\|\Lambda^2_{q_2} - \Lambda^2_{q_1}\|_*)|]\right)^{-\theta\frac{s-3/2}{s+1}}$$

with $0 < \theta < 1/10$. The implicit constant[1] only depends on L, R, k, M, s and δ.

Theorem 6. *Consider $s > 3/2$, and let q_1, q_2 belong to $H^s(\mathbb{R}^3)$ and have their supports contained in Q. Consider k to be admissible for q_1, q_2 and the zero potential. Let M denote an upper bound on $\|q_j\|_{H^s(\mathbb{R}^3)} \leq M$. Then, there exists $\delta = \delta(L, R, k) > 0$ such that, if*

---

[1]Throughout the paper, we will write $a \lesssim b$ whenever a and b are non-negative quantities that satisfy $a \leq Cb$ for a certain constant $C > 0$. A constant $C > 0$ satisfying the previous inequality will be called an *implicit constant* and it will only depend on unimportant quantities such as L, R, k, M, s and δ.

$\|\Lambda^1_{q_2} - \Lambda^1_{q_1}\|_* < 1/\delta$, *then*

$$\|q_1 - q_2\|_{L^\infty(\Sigma)} \lesssim \left(\log[1 + |\log(\delta\|\Lambda^1_{q_2} - \Lambda^1_{q_1}\|_*)|]\right)^{-\theta\frac{s-3/2}{s+1}}.$$

with $0 < \theta < 1/5$. *The implicit constant in the last inequality depends on the same parameters as the implicit constant from the inequality in Theorem 5.*

Our results hold in dimension $n = 3$. We have only considered the three dimensional case for the sake of simplicity but we believe that these results also hold for $n > 3$ following similar arguments.

3.3 Integral estimates

The main goal of this section is to prove the integral estimates that we announced in the introduction. Before stating these estimates, we will introduce a norm for $H^{3/2}_{\Gamma_1^D}(\Gamma_1)$ and we will prove Theorem 4.

Let $k \geq 0$ be an admissible frequency for the zero potential in Σ; we define, for each $f \in H^{3/2}_{\Gamma_1^D}(\Gamma_1)$, the norm

$$|||f||| := \|v_f\|_{L^2(\Omega)}, \tag{3.3.1}$$

where $v_f \in H^2_{\text{loc}}(\overline{\Sigma})$ is the unique solution to

$$\begin{cases} -(\Delta + k^2)v_f &= 0 \text{ in } \Sigma, \\ v_f|_{\Gamma_1} &= f, \\ v_f|_{\Gamma_2} &= 0; \end{cases} \tag{3.3.2}$$

and Ω is a bounded open subset of Σ which satisfies

$$\{x \in \Sigma : |x'| \leq R'\} \subset \Omega$$

and has a smooth boundary $\partial\Omega$ such that

$$\partial\Omega \cap \Gamma_1 \subseteq \Gamma_1^D, \qquad \overline{\Gamma_j^N} \subseteq \text{int}_{\Gamma_j}(\partial\Omega \cap \Gamma_j)$$

for $j = 1, 2$. Since we want Γ_1^D and Γ_j^N to be as small as possible, we now assume $R' < 2R$; at this moment, we fix Ω satisfying all of the above conditions together with

$$\Omega \subset \{x \in \Sigma : |x'| \leq 2R\}.$$

The norm $|||\,\boldsymbol{.}\,|||$ obviously depends on Ω and k but these dependences are harmless for our problems. The well-posedness of the problem (3.3.2), together with the fact that

$$|||f||| = 0 \Rightarrow f = 0, \tag{3.3.3}$$

guarantee that $|||\,\boldsymbol{.}\,|||$ is a norm on $H^{3/2}_{\Gamma_1^D}(\Gamma_1)$. The property (3.3.3) follows from the weak unique continuation property for the equation $-(\Delta + k^2)v_f = 0$ in Σ.

With this new norm on $H^{3/2}_{\Gamma_1^D}(\Gamma_1)$, we will show that Λ_q^j is a bounded operator.

Lemma 3.3.1. *The following inequality holds*

$$\|\Lambda_q^j f\|_{H^{-3/2}(\Gamma_j')} \lesssim |||f|||,$$

for every $f \in H^{3/2}_{\Gamma_1^D}(\Gamma_1)$, *where*

$$\|\Lambda_q^j f\|_{H^{-3/2}(\Gamma_j^N)} := \sup_{g \in H^{3/2}_{\Gamma_j^N}(\Gamma_j) \setminus \{0\}} \frac{|\int_{\Gamma_j} \Lambda_q f\, g\, dx'|}{\|g\|_{H^{3/2}(\Gamma_j)}}. \tag{3.3.4}$$

The implicit constant here depends on k, *any upper bound on* $\|q\|_{L^\infty(\Sigma)}$ *and* Ω.

Note that Theorem 4 is an immediate consequence of this lemma. Moreover, Lemma 3.3.1 still holds if each occurrence of $\overline{\Gamma_1^D}$ in its statement is replaced by any compact subset K of Γ_1. In particular, the intersection between $\overline{\Omega}$ and K is even allowed to be empty.

Proof. Fix $f \in H^{3/2}_{\Gamma_1^D}(\Gamma_1)$. For any $g \in H^{3/2}_{\Gamma_j^N}(\Gamma_j)$, we have that

$$\int_{\Gamma_j} \Lambda_q f\, g\, dx' = \int_{\Gamma_j} \partial_\nu u\, g\, dx'$$

with u solving (2.2.2). By the trace theorem for Ω, there exists $v \in H^2(\Omega)$ such that $v(x) = g(x)$ for almost every $x \in \Gamma_j^N$, $v(x) = 0$ for almost every $x \in \partial\Omega \setminus \Gamma_j^N$, $\partial_\eta v|_{\partial\Omega} = 0$ and

$$\|v\|_{H^2(\Omega)} \lesssim \|g\|_{H^{3/2}(\Gamma_j)}. \tag{3.3.5}$$

Here η denotes the outward-pointing unit normal vector along $\partial\Omega$, and the implicit constant depends on Ω. Then, using Green's formula, we get that

$$\int_{\Gamma_j} \Lambda_q f \, g \, dx' = \int_\Omega \Delta u v - u \Delta v \, dx$$

which, by (3.3.5), implies

$$\left| \int_{\Gamma_j} \Lambda_q f \, g \, dx' \right| \lesssim \big(\|u\|_{L^2(\Omega)} + \|\Delta u\|_{L^2(\Omega)}\big) \|g\|_{H^{3/2}(\Gamma_j)}.$$

Since u is solution to (2.2.2), we have

$$\|\Delta u\|_{L^2(\Omega)} \leq (k^2 + \|q\|_{L^\infty(\Sigma)}) \|u\|_{L^2(\Sigma)}$$

and therefore, by (3.3.4),

$$\|\Lambda_q^j f\|_{H^{-3/2}(\Gamma_j')} \lesssim \|u\|_{L^2(\Omega)},$$

where the implicit constant depends on k, any upper bound on $\|q\|_{L^\infty(\Sigma)}$ and Ω.

Let w be defined by $w := u - v_f$ with v_f as in (3.3.2). Then, $u = w + v_f$ with w being the unique solution to the Dirichlet problem

$$\begin{cases} (-\Delta - k^2 + q)w &= -q v_f \text{ in } \Sigma, \\ w|_{\partial\Sigma} &= 0. \end{cases}$$

By the triangle inequality and the well-posedness of this problem, we deduce

$$\|\Lambda_q^j f\|_{H^{-3/2}(\Gamma_j')} \lesssim \|v_f\|_{L^2(\Omega)},$$

which is nothing but the claimed inequality. □

Next, we turn our attention to the integral estimates, which can be stated as follows.

Proposition 3.3.1. *Fix potentials $q_1, q_2 \in L^\infty(\Sigma)$ both of which are compactly supported in Q, and let $M > 0$ denote an upper bound on $\|q_j\|_{L^\infty(\Sigma)} \leq M$ for $j = 1, 2$. Consider $k \geq 0$ to be admissible for q_1, q_2 and the zero potential. Assume that u_1 and u_2 belong to $H^2(\Omega)$ and are solutions to*

$$\begin{aligned}(-\Delta - k^2 + q_1)u_1 &= 0 \textit{ in } \Omega, \\ u_1|_{\Gamma_2 \cap \partial\Omega} &= 0\end{aligned}$$

and

$$(-\Delta - k^2 + q_2)u_2 = 0 \textit{ in } \Omega,$$

respectively.

(a) *If $u_2|_{\Gamma_2 \cap \partial\Omega} = 0$, then there exists a constant $\delta = \delta(L, R, k) > 0$ such that, if $\|\Lambda^1_{q_2} - \Lambda^1_{q_1}\|_* < 1/\delta$, then*

$$\left| \int_\Omega (q_1 - q_2)u_1 u_2 \, dx \right| \lesssim \frac{\|u_1\|_{L^2(\Omega)} \|u_2\|_{H^2(\Omega)}}{\left[1 + \left| \log\left(\delta \|\Lambda^1_{q_2} - \Lambda^1_{q_1}\|_*\right) \right|\right]^{1/2}}.$$

(b) *There exist constants $C = C(L, R) > 0$ and $\delta = \delta(L, R, k) > 0$ such that, if $\|\Lambda^2_{q_2} - \Lambda^2_{q_1}\|_* < 1/\delta$, then*

$$\begin{aligned}\left| \int_\Omega (q_1 - q_2)u_1 u_2 \, dx \right| \lesssim & e^{c\tau|\zeta|} \frac{\|u_1\|_{L^2(\Omega)} \|u_2\|_{H^2(\Omega)}}{\left[1 + \left| \log\left(\delta \|\Lambda^2_{q_2} - \Lambda^2_{q_1}\|_*\right) \right|\right]^{1/2}} \\ & + \frac{1}{\tau^{1/2}} \|e^{\tau x \cdot \zeta} u_2\|_{H^1(\Omega)} \|e^{-\tau x \cdot \zeta} u_1\|_{L^2(\Omega)}\end{aligned}$$

for all $\tau \geq \tau_0 := C(k^2 + M)$ and $\zeta \in \mathbb{R}^3$ with $\zeta \cdot \eta|_{\Gamma_1^N} \geq 1$; here, $c > 2(2R + L)$.

Proof. Let $v_1 \in H^2_{\mathrm{loc}}(\overline{\Sigma})$ be a solution to $(-\Delta - k^2 + q_1)v_1 = 0$ in Σ with $\mathrm{supp}(v_1|_{\partial\Sigma}) \subseteq \overline{\Gamma_1^D}$. Writing $f := v_1|_{\Gamma_1}$, we know that there exists a unique $v_2 \in H^2_{\mathrm{loc}}(\overline{\Sigma})$ such that

$$\begin{aligned}(-\Delta - k^2 + q_2)v_2 &= 0 \text{ in } \Sigma, \\ v_2|_{\Gamma_1} &= f, \\ v_2|_{\Gamma_2} &= 0.\end{aligned}$$

Then, $w := v_2 - v_1$ belongs to $H^2_{\mathrm{loc}}(\overline{\Sigma})$, and it is the unique admissible solution of

$$\begin{aligned} (-\Delta - k^2 + q_2)w &= (q_1 - q_2)v_1 \text{ in } \Sigma, \\ w|_{\partial\Sigma} &= 0. \end{aligned} \tag{3.3.6}$$

Obviously,

$$\begin{aligned} &\int_\Omega (q_1 - q_2)u_1 u_2 \, dx \\ &\qquad = \int_\Omega (q_1 - q_2)v_1 \chi u_2 \, dx + \int_\Omega (q_1 - q_2)(u_1 - v_1)u_2 \, dx \end{aligned} \tag{3.3.7}$$

where χ is a bump function in $\mathbb{R}^2$ which satisfies $\chi(x') = 1$ for $|x'| \le R + \epsilon$ and $\operatorname{supp}\chi \subset \{|x'| \le R' - \epsilon\}$ for a small enough $\epsilon > 0$. Using the equation solved by w, applying Green's formula in Ω, utilizing the equation satisfied by u_2 together with $w|_{\partial\Sigma} = 0$ and taking advantage of $\chi = 0$ in a neighbourhood of $\partial\Omega \cap \Sigma$, we get

$$\begin{aligned} &\int_\Omega (q_1 - q_2)v_1 \chi u_2 \, dx \\ &\qquad = -\int_\Omega w(\Delta\chi u_2 + 2\nabla\chi \cdot \nabla u_2)\, dx - \int_{\Gamma_1^N \cup \Gamma_2^N} \chi u_2 \partial_\nu w \, dx'. \end{aligned} \tag{3.3.8}$$

Using (3.3.7), (3.3.8) and that $\operatorname{supp} q_j \subseteq Q$ for $j = 1, 2$, we immediately see that

$$\begin{aligned} \left| \int_\Omega (q_1 - q_2)u_1 u_2 \, dx \right| &\lesssim \|\chi(u_1 - v_1)\|_{L^2(\Omega)} \|\chi u_2\|_{L^2(\Omega)} \\ &+ \left| \int_\Omega w(\Delta\chi u_2 + 2\nabla\chi \cdot \nabla u_2)\, dx \right| + \left| \int_{\Gamma_1^N \cup \Gamma_2^N} \chi u_2 \partial_\nu w \, dx' \right|. \end{aligned} \tag{3.3.9}$$

We next have to obtain an upper bound on each term in the previous inequality. The method for estimating each of the two boundary integrals depends on whether the domain of integration does or does not coincide with the part of $\partial\Sigma$ on which the Neumann data is measured. The method for estimating the interior integral on the right-hand side of (3.3.9) relies on a quantified unique continuation property for w.

To fix ideas, let the Neumann data be measured on Γ_l^N. Start by estimating the boundary integral along Γ_l^N from (3.3.9). Using $\partial_\nu w|_{\partial\Sigma} = (\Lambda_{q_2} - \Lambda_{q_1})f$, $\operatorname{supp}\chi \subseteq \{|x'| \le R'\}$, and (3.3.4), we get

$$\left| \int_{\Gamma_l^N} \chi u_2 \partial_\nu w \, dx' \right| \le \|\chi u_2\|_{H^{3/2}(\Gamma_l)} \|(\Lambda_{q_1}^l - \Lambda_{q_2}^l) f\|_{H^{-3/2}(\Gamma_l^N)}.$$

The last term on the right-hand side can be estimated using the definition of the operator norm and (3.3.1) as follows:

$$\|(\Lambda^l_{q_1} - \Lambda^l_{q_2})f\|_{H^{-3/2}(\Gamma^N_l)} \leq \|\Lambda^l_{q_1} - \Lambda^l_{q_2}\|_*(\|v_f - v_1\|_{L^2(\Omega)} + \|v_1\|_{L^2(\Omega)}), \tag{3.3.10}$$

where v_f satisfies (3.3.2). Note that $v_f - v_1$ satisfies

$$\begin{cases} (-\Delta - k^2)(v_f - v_1) &= q_1 v_1 \text{ in } \Sigma, \\ (v_f - v_1)|_{\partial\Sigma} &= 0. \end{cases}$$

By the well-posedness of this problem, we have

$$\|v_f - v_1\|_{L^2(\Omega)} \lesssim \|\chi v_1\|_{L^2(\Sigma)}. \tag{3.3.11}$$

Thus, using (3.3.10), (3.3.11) and the boundedness of the trace operator associated with Ω, the boundary term under consideration is bounded in the following way:

$$\left| \int_{\Gamma^N_l} \chi u_2 \partial_\nu w \, dx' \right| \lesssim \|\Lambda^l_{q_1} - \Lambda^l_{q_2}\|_* \|u_2\|_{H^2(\Omega)} \|v_1\|_{L^2(\Omega)}. \tag{3.3.12}$$

Under the assumptions in (a), the inequalities (3.3.12) and (3.3.9) imply

$$\begin{aligned} \left| \int_\Omega (q_1 - q_2) u_1 u_2 \, dx \right| &\lesssim \|u_1 - v_1\|_{L^2(\Omega)} \|u_2\|_{L^2(\Omega)} \\ &\quad + \|w\|_{L^2(Q')} \|u_2\|_{H^1(\Omega)} + \|\Lambda^1_{q_1} - \Lambda^1_{q_2}\|_* \|u_2\|_{H^2(\Omega)} \|v_1\|_{L^2(\Omega)}, \end{aligned} \tag{3.3.13}$$

where $Q' := \{x \in \Sigma : R + \epsilon < |x'| < R' - \epsilon\}$. In order to get the estimate in (a) from (3.3.13), we have to control w in Q' and $u_1 - v_1$ in Ω. We postpone this for a while; instead, we now focus on estimating the other boundary term in (3.3.9), which only appears under the assumptions in (b). More concretely, we focus on estimating the term

$$\left| \int_{\Gamma^N_1} \chi u_2 \partial_\nu w \, dx' \right| \tag{3.3.14}$$

in terms of $\|\Lambda^2_{q_2} - \Lambda^2_{q_1}\|_*$ and a sufficiently large parameter τ.

To fix ideas, let $\zeta \in \mathbb{R}^3$ be arbitrarily chosen with $\zeta \cdot e_3 \geq 1$. In order to control (3.3.14), we use a Carleman inequality proven by Bukhgeim and Uhlmann in [6] (see Corollary 2.3).

Since $|\zeta| \geq |\zeta \cdot e_3| \geq 1$, the Carleman inequality can be applied to our situation as follows: For any $q \in L^\infty(\Omega)$ with $\|q\|_{L^\infty(\Omega)} \leq M$, there exists a constant $C = C(L, R) > 0$ such that

$$\begin{aligned} \tau^2 \int_\Omega |e^{-\tau x\cdot\zeta} u|^2 \, dx + \tau \int_{\partial\Omega} (\zeta \cdot \eta) |e^{-\tau x\cdot\zeta} \partial_\eta u|^2 \, dS \\ \lesssim \int_\Omega |e^{-\tau x\cdot\zeta} (-\Delta - k^2 + q) u|^2 \, dx \end{aligned} \tag{3.3.15}$$

for all $u \in H^2(\Omega)$ with $u|_{\partial\Omega} = 0$, $\tau \geq C(k^2 + M)$; the implicit constant in (3.3.15) depends on R and L.

Start by noting that

$$\left| \int_{\Gamma_1^N} \chi u_2 \partial_\nu w \, dx' \right| \leq \|e^{\tau x\cdot\zeta} u_2\|_{L^2(\Gamma_1^N)} \|e^{-\tau x\cdot\zeta} \partial_\eta (\chi w)\|_{L^2(\Gamma_1^N)} \tag{3.3.16}$$

since $\eta|_{\Gamma_1^N} = \nu|_{\Gamma_1^N}$ is a constant multiple of e_3 and since $\partial_{x_3}\chi = 0$. Here e_3 denotes the vector satisfying $x_3 = e_3 \cdot x$. The first term on the right-hand side can be bounded as follows

$$\|e^{\tau x\cdot\zeta} u_2\|_{L^2(\Gamma_1^N)} \lesssim \|e^{\tau x\cdot\zeta} u_2\|_{H^1(\Omega)} \tag{3.3.17}$$

using the boundedness of the trace operator associated with Ω, where the implicit constant depends on Ω. We estimate the second term on the right-hand side of (3.3.16) as

$$\|e^{-\tau x\cdot\zeta} \partial_\eta (\chi w)\|^2_{L^2(\Gamma_1^N)} \leq \int_{\Gamma_1^N} \zeta \cdot \eta |e^{-\tau x\cdot\zeta} \partial_\eta (\chi w)|^2 \, dx'. \tag{3.3.18}$$

Since $\chi w \in H^2(\Omega)$ vanishes on $\partial\Omega$, an application of (3.3.15) with u replaced by χw and q replaced by q_2 shows that the right-hand side of (3.3.18) can be bounded by

$$\frac{1}{\tau} \int_\Omega |e^{-\tau x\cdot\zeta} (-\Delta - k^2 + q_2)(\chi w)|^2 \, dx + |\zeta| e^{2c\tau|\zeta|} \|\chi \partial_\nu w\|^2_{L^2(\Gamma_2)}, \tag{3.3.19}$$

where $c := 2R + L$ is not the c from the statement of Proposition 3.3.1 (b). Furthermore, since w solves (3.3.6), we have

$$\begin{aligned} \int_\Omega |e^{-\tau x\cdot\zeta} (-\Delta - k^2 + q_2)(\chi w)|^2 \, dx &\lesssim e^{2c\tau|\zeta|} \|v_1 - u_1\|^2_{L^2(\Omega)} \\ &\quad + \|e^{-\tau x\cdot\zeta} u_1\|^2_{L^2(\Omega)} + \int_\Omega |e^{-\tau x\cdot\zeta} (\Delta\chi w + 2\nabla\chi \cdot \nabla w)|^2 \, dx \\ &\lesssim e^{2c\tau|\zeta|} \|w\|^2_{H^1(Q')} + \left(e^{2c\tau|\zeta|} \|v_1 - u_1\|^2_{L^2(\Omega)} + \|e^{-\tau x\cdot\zeta} u_1\|^2_{L^2(\Omega)}\right). \end{aligned}$$

These computations are meant to bound the first term in (3.3.19). We now take care of the second one. By interpolation and using that $\partial_\nu w|_{\partial\Sigma} = (\Lambda_{q_2} - \Lambda_{q_1})f$, we get

$$\|\chi\partial_\nu w\|_{L^2(\Gamma_2)} \leq \|\chi(\Lambda_{q_2} - \Lambda_{q_1})f\|^{1/4}_{H^{-3/2}(\Gamma_2)} \|\chi\partial_\nu w\|^{3/4}_{H^{1/2}(\Gamma_2)}.$$

It is a simple computation to show that

$$\|\chi(\Lambda_{q_2} - \Lambda_{q_1})f\|_{H^{-3/2}(\Gamma_2)} \lesssim \|(\Lambda^2_{q_2} - \Lambda^2_{q_1})f\|_{H^{-3/2}(\Gamma_2^N)}$$

with the implicit constant depending on R. Following (3.3.10) and (3.3.11), we get

$$\|\chi\partial_\nu w\|_{L^2(\Gamma_2)} \lesssim \|\Lambda^2_{q_2} - \Lambda^2_{q_1}\|^{1/4}_{*} \|v_1\|^{1/4}_{L^2(\Omega)} \|\chi\partial_\nu w\|^{3/4}_{H^{1/2}(\Gamma_2)}.$$

In order to estimate the last factor on the right-hand side, we are going to use the boundedness of the trace operator in Ω and the well-posedness of (3.3.6) to get control on $\|w\|_{H^2(\Omega)}$. Thus, we get

$$\|\chi\partial_\nu w\|_{H^{1/2}(\Gamma_2)} \lesssim \|w\|_{H^2(\Omega)} \lesssim \|v_1\|_{L^2(\Omega)},$$

which implies

$$\|\chi\partial_\nu w\|_{L^2(\Gamma_2)} \lesssim \|\Lambda^2_{q_2} - \Lambda^2_{q_1}\|^{1/4}_{*} \|v_1\|_{L^2(\Omega)}. \tag{3.3.20}$$

Finally, gathering (3.3.16), (3.3.17), (3.3.18) and the computations to estimate each term on (3.3.19), we can state that

$$\begin{aligned}\left|\int_{\Gamma_1^N} \chi u_2 \partial_\nu w \, dx'\right| \lesssim \|e^{\tau x\cdot\zeta} u_2\|_{H^1(\Omega)} \Bigg[&\frac{1}{\tau^{1/2}} \|e^{-\tau x\cdot\zeta} u_1\|_{L^2(\Omega)} \\ &+ \frac{e^{c\tau|\zeta|}}{\tau^{1/2}} \big(\|w\|_{H^1(Q')} + \|v_1 - u_1\|_{L^2(\Omega)}\big) \\ &+ |\zeta|^{1/2} e^{c\tau|\zeta|} \|\Lambda^2_{q_2} - \Lambda^2_{q_1}\|^{1/4}_{*} \|v_1\|_{L^2(\Omega)} \Bigg].\end{aligned} \tag{3.3.21}$$

Before proceeding with the proof of the claimed integral estimates, let us write down what the estimate, under the assumptions in (b), looks like at this stage: by (3.3.9), (3.3.12) and (3.3.21), we obtain

$$\begin{aligned} \left| \int_\Omega (q_1 - q_2) u_1 u_2 \, dx \right| &\lesssim \|u_1 - v_1\|_{L^2(\Omega)} \|u_2\|_{L^2(\Omega)} \\ &+ \|w\|_{L^2(Q')} \|u_2\|_{H^1(\Omega)} + \|\Lambda^2_{q_1} - \Lambda^2_{q_2}\|_* \|u_2\|_{H^2(\Omega)} \|v_1\|_{L^2(\Omega)} \\ &+ \|e^{\tau x \cdot \zeta} u_2\|_{H^1(\Omega)} \Bigg[\frac{1}{\tau^{1/2}} \|e^{-\tau x \cdot \zeta} u_1\|_{L^2(\Omega)} \\ &+ \frac{e^{c\tau|\zeta|}}{\tau^{1/2}} \big(\|w\|_{H^1(Q')} + \|v_1 - u_1\|_{L^2(\Omega)} \big) \\ &+ |\zeta|^{1/2} e^{c\tau|\zeta|} \|\Lambda^2_{q_2} - \Lambda^2_{q_1}\|_*^{1/4} \|v_1\|_{L^2(\Omega)} \Bigg] \end{aligned} \tag{3.3.22}$$

for all $\tau \geq C(k^2 + M)$ and $\zeta \in \mathbb{R}^3$ with $\zeta \cdot e_3 \geq 1$.

In the next step, we will control w in Q' by using quantified unique continuation from the boundary. This will be applied to (3.3.13) and (3.3.22) to obtain the estimates in (a) and (b), respectively.

Proceed with the control of w in Q'. We may assume w not to vanish identically in Q', otherwise we do not have anything to control. In order to estimate a non-identically-vanishing w, we will apply an estimate due to Phung (see Théorème 1.1 in [40]) which reads as follows in our particular case: Let U be a smooth open subset of Ω containing Q' with $U \cap Q = \emptyset$. Then, there exists a $d > 0$, which depends on U, Γ and k, such that, if

$$\frac{\|w\|_{H^2(U)}}{\|\partial_\nu w\|_{L^2(\Gamma)}} \geq \frac{1}{d}, \tag{3.3.23}$$

with $\Gamma := \{x \in \Gamma_l^N : R + \epsilon < |x'| < R' - \epsilon\}$ for the ϵ already chosen, then

$$\|w\|_{H^1(U)} \lesssim \frac{\|w\|_{H^2(U)}}{\left[\log\left(e \frac{d\|w\|_{H^2(U)}}{\|\partial_\nu w\|_{L^2(\Gamma)}}\right)\right]^{1/2}}. \tag{3.3.24}$$

Obviously, $\Gamma \subseteq \partial U \cap \Gamma_l$. Note that, by $w|_{\partial\Sigma} = 0$ and by unique continuation from the boundary, we can ensure that $\|\partial_\nu w\|_{L^2(\Gamma)} > 0$.

On the one hand, by the well-posedness of the problem satisfied by w, we know that

$$\|w\|_{H^2(\Omega)} \lesssim \|v_1 - u_1\|_{L^2(\Omega)} + \|u_1\|_{L^2(\Omega)}$$

with the implicit constant depending on Ω, M and k. On the other hand, considering another bump function χ' in $\mathbb{R}^2$ such that $\chi(x') = 1$ for $|x'| \leq R' - \epsilon$ and $\chi'(x') = 0$ for $|x'| > R' - \epsilon/2$, we have, by the same argument that we used to get (3.3.20) with χ' instead of χ, that

$$\begin{aligned}\|\partial_\nu w\|_{L^2(\Gamma)} &\leq \|\chi' \partial_\nu w\|_{L^2(\Gamma_l)} \\ &\lesssim \|\Lambda^l_{q_2} - \Lambda^l_{q_1}\|_*^{1/4} \big(\|u_1\|_{L^2(\Omega)} + \|v_1 - u_1\|_{L^2(\Omega)}\big).\end{aligned}$$

Obviously, the implicit constants in the previous inequalities can be chosen to be the same. Thus, since the function

$$t \mapsto \frac{t}{(\log t)^{1/2}}$$

is increasing on (e, ∞) and since the right-hand side of (3.3.24) can be written as

$$\frac{\|\partial_\nu w\|_{L^2(\Gamma)}}{ed} \frac{e^{\frac{d\|w\|_{H^2(U)}}{\|\partial_\nu w\|_{L^2(\Gamma)}}}}{\left[\log\left(e^{\frac{d\|w\|_{H^2(U)}}{\|\partial_\nu w\|_{L^2(\Gamma)}}}\right)\right]^{1/2}},$$

the last two inequalities can be combined with (3.3.23) and (3.3.24) to deduce the following: if $\|\Lambda^l_{q_2} - \Lambda^l_{q_1}\|_* < d^4$, we have

$$\|w\|_{H^1(U)} \lesssim \frac{\|v_1 - u_1\|_{L^2(\Omega)} + \|u_1\|_{L^2(\Omega)}}{\left[1 + \left|\log\left(d^{-4}\|\Lambda^l_{q_2} - \Lambda^l_{q_1}\|_*\right)\right|\right]^{1/2}}. \tag{3.3.25}$$

From now until the end of the proof, we shall write $\delta := d^{-4}$ and we shall assume $\|\Lambda^l_{q_2} - \Lambda^l_{q_1}\|_* < \delta^{-1}$ (so that we do not have to state this condition explicitly every time).

At this stage, the proofs of both parts of Proposition 3.3.1 are almost complete. What remains for us to do is, firstly, to apply (3.3.25) to each inequality of (3.3.13), (3.3.22) thus obtaining two new inequalities and, secondly, to apply the announced Runge-type approximation to the two new inequalities. We now go on to finish the proof of Proposition 3.3.1, whereby we shall omit all lengthy but straightforward calculations.

The Runge-type approximation can be stated as follows: For all u_1 as in the statement of Proposition 3.3.1 and $\varepsilon > 0$, there exists a $v_1 \in H^2_{\mathrm{loc}}(\overline{\Sigma})$ solving $(-\Delta - k^2 + q_1)v_1 = 0$ in Σ with $\operatorname{supp}(v_1|_{\partial\Sigma}) \subseteq \overline{\Gamma^D_1}$ such that

$$\|v_1 - u_1\|_{L^2(\Omega)} < \varepsilon.$$

With regard to part (a) of Proposition 3.3.1: by applying (3.3.25) to (3.3.13) and then by applying the approximation result to the resulting inequality, we obtain

$$\left|\int_\Omega (q_1 - q_2)u_1u_2\,dx\right| \lesssim \frac{\|u_1\|_{L^2(\Omega)}\|u_2\|_{H^1(\Omega)}}{\left[1+\left|\log\left(\delta\|\Lambda^1_{q_2} - \Lambda^1_{q_1}\|_*\right)\right|\right]^{1/2}} + \|\Lambda^1_{q_1} - \Lambda^1_{q_2}\|_*\|u_1\|_{L^2(\Omega)}\|u_2\|_{H^2(\Omega)}. \tag{3.3.26}$$

With regard to part (b) of Proposition 3.3.1: we argue analogously by firstly applying (3.3.25) to (3.3.22) and secondly by applying the approximation result to obtain

$$\begin{aligned}\left|\int_\Omega (q_1 - q_2)u_1u_2\,dx\right| &\lesssim \frac{\|u_1\|_{L^2(\Omega)}\|u_2\|_{H^1(\Omega)}}{\left[1+\left|\log\left(\delta\|\Lambda^2_{q_2} - \Lambda^2_{q_1}\|_*\right)\right|\right]^{1/2}} \\ &+ \|\Lambda^2_{q_1} - \Lambda^2_{q_2}\|_*\|u_1\|_{L^2(\Omega)}\|u_2\|_{H^2(\Omega)} \\ &+ \|e^{\tau x\cdot\zeta}u_2\|_{H^1(\Omega)}\left(\frac{e^{c\tau|\zeta|}}{\tau^{1/2}}\frac{\|u_1\|_{L^2(\Omega)}}{\left[1+\left|\log\left(\delta\|\Lambda^2_{q_2} - \Lambda^2_{q_1}\|_*\right)\right|\right]^{1/2}}\right. \\ &\left.+ \frac{1}{\tau^{1/2}}\|e^{-\tau x\cdot\zeta}u_1\|_{L^2(\Omega)} + |\zeta|^{1/2}e^{c\tau|\zeta|}\|\Lambda^2_{q_2} - \Lambda^2_{q_1}\|_*^{1/4}\|u_1\|_{L^2(\Omega)}\right).\end{aligned} \tag{3.3.27}$$

By dropping higher-order terms from the right-hand side of (3.3.26) and (3.3.27) (possibly at the cost of increasing the implicit constants in each of these inequalities), we arrive at the estimate claimed in (a) and (b). □

3.4 Proof of Theorem 5 (Dirichlet and Neumann data on different slabs)

In this section, we prove Theorem 5. To achieve this task, we will construct appropriate CGOs, use those CGOs to construct the functions u_1 and u_2 appearing in the integral estimate of Proposition 3.3.1 (b), and eventually obtain an upper bound on $(\widehat{q_1} - \widehat{q_2})(\xi)$ at each frequency ξ from

$$\{\xi = (\xi', \xi_3) \in \mathbb{R}^3 : 1 \leq |\xi'| < r, |\xi_3| < r\};$$

then, we will extend our control on $\widehat{q_1} - \widehat{q_2}$ to the ball

$$\{\xi \in \mathbb{R}^3 : |\xi| < r\}.$$

After this, we will carry out a classical argument due to Alessandrini (see [1]) in order to obtain the stability estimate.

From now until the end of this section, we abuse notation by letting q_j stand both for the potential from the statement of Theorem 5 (which is only defined on Σ) and for its trivial extension to all of $\mathbb{R}^3$. The meaning will be clear from the context; for example, $\widehat{q_j}$ refers to the Fourier transform of the trivial extension of q_j to all of $\mathbb{R}^3$.

Start by stating the CGOs used to prove Theorem 5. We perform the reflection argument originating from the work of Isakov in [30]. Let $r > 2$, which will be specified later on in this section.

Let $\xi \in \mathbb{R}^3$ with

$$1 \leq \xi_{1e} := \sqrt{\xi_1^2 + \xi_2^2} < r \quad \text{and} \quad |\xi_3| < r, \tag{3.4.1}$$

be arbitrarily chosen. We define

$$\begin{aligned} e(1) &:= \frac{1}{\xi_{1e}}(\xi_1, \xi_2, 0), \\ e(3) &:= (0, 0, 1), \\ e(2) &:= e(3) \times e(1) = \frac{1}{\xi_{1e}}(-\xi_2, \xi_1, 0). \end{aligned}$$

We set $x^* := (x_1, x_2, -x_3)$ for any $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $f^*(x) := f(x^*)$ for any function f, and $G^* = \{x^* : x \in G\}$ for any domain G. The coordinates of any $x \in \mathbb{R}^3$ with respect to the orthonormal basis $\{e(j)\}_{j=1}^3$ shall be denoted by $x = (x_{1e}, x_{2e}, x_{3e})_e$. Note $\xi = (\xi_{1e}, 0, \xi_3)_e$. We also write $\xi^\perp := (-\xi_3, 0, \xi_{1e})_e$.

As preparation for the reflection argument, we now fix a smooth bounded domain $B \subseteq \mathbb{R}^3$ such that

$$\overline{\Omega \cup \Omega^*} \subseteq B, \quad B^* = B.$$

Let $Q_1 \in L^\infty(B)$ be the even extension of q_1 about the coordinate variable x_3 and $Q_2 \in L^\infty(B)$ be the trivial extension of q_2 to all of B; explicitly, we define

$$\begin{aligned} Q_1(x) &:= q_1(x)\chi_\Sigma(x) + q_1(x^*)\chi_{\Sigma^*}(x), \\ Q_2(x) &:= q_2(x)\chi_\Sigma(x), \end{aligned}$$

for a.e. $x \in \mathbb{R}^3$.

As in [34], we introduce

$$\begin{aligned} \rho_1 &:= \left(-\tau\xi_3 + \frac{i}{2}\xi_{1e}, i|\xi|(\tau^2 - 1/4)^{1/2}, \tau\xi_{1e} + \frac{i}{2}\xi_3\right)_e \\ &= \tau\xi^\perp + i\left(\frac{1}{2}\xi + |\xi|(\tau^2 - 1/4)^{1/2}e(2)\right), \\ \rho_2 &:= \left(\tau\xi_3 + \frac{i}{2}\xi_{1e}, -i|\xi|(\tau^2 - 1/4)^{1/2}, -\tau\xi_{1e} + \frac{i}{2}\xi_3\right)_e \\ &= -\tau\xi^\perp + i\left(\frac{1}{2}\xi - |\xi|(\tau^2 - 1/4)^{1/2}e(2)\right). \end{aligned}$$

One immediately computes that

$$\rho_m \cdot \rho_m = 0, \quad |\rho_m| = \sqrt{2}\tau|\xi|, \quad m = 1, 2. \tag{3.4.2}$$

The ρ_1 and ρ_2 will be the candidates to construct the family of CGOs. It is a well-known fact that there exists a function $V_m \in H^2(B)$ solving

$$(-\Delta + Q_m - k^2)V_m = 0 \text{ in } B \tag{3.4.3}$$

and having the form $V_m = e^{x\cdot\rho_m}(1 + \psi_m)$, where the remainder ψ_m obeys

$$\|\psi_m\|_{H^k(B)} \lesssim \frac{1}{\tau^{1-k}}, \quad k = 0, 1, 2, \tag{3.4.4}$$

for all $\tau \geq \tau_1 := \max(C_0(M + k^2), 1)$, with $C_0 \geq 1$ depending on B. The implicit constant in (3.4.4) depends on B, M and k.

Recall that Proposition 3.3.1 (b) requires for u_1 to satisfy $u_1|_{\partial\Omega\cap\Gamma_2} = 0$; this boundary condition can be arranged to hold via Isakov's reflection argument from [30]. Employing the same idea as in [34], we set

$$u_1(x) \;:=\; e^{x\cdot\rho_1}(1+\psi_1(x)) - e^{x^*\cdot\rho_1}(1+\psi_1^*(x)), \tag{3.4.5}$$

$$u_2(x) \;:=\; e^{x\cdot\rho_2}(1+\psi_2(x)). \tag{3.4.6}$$

The construction of ψ_1, ψ_2, u_1, u_2 ensures that $u_1|_\Omega$ and $u_2|_\Omega$ satisfy the hypotheses of Proposition 3.3.1 (b).

Let us compute that

$$\int_\Sigma (q_1 - q_2)u_1 u_2\,dx = \int_\Sigma e^{ix\cdot\xi}(1+\psi_1)(1+\psi_2)(q_1-q_2)\,dx - \int_\Sigma (q_1-q_2)e^{ix_{1e}\xi_{1e}}e^{-2\tau x_3\xi_{1e}}(1+\psi_1^*)(1+\psi_2)\,dx.$$

As a consequence, we obtain

$$\left|\int_\Sigma e^{ix\cdot\xi}(q_1-q_2)\,dx\right| \le \left|\int_\Sigma (q_1-q_2)u_1u_2\,dx\right| + \left|\int_\Sigma e^{ix\cdot\xi}(q_1-q_2)(\psi_1+\psi_2+\psi_1\psi_2)\,dx\right| + \left|\int_\Sigma (q_1-q_2)e^{ix_{1e}\xi_{1e}}e^{-2\tau x_3\xi_{1e}}(1+\psi_1^*)(1+\psi_2)\,dx\right|. \tag{3.4.7}$$

By applying the triangle inequality, using that $\mathrm{supp}(q_m) \subseteq Q$, and using $\|q_m\|_{L^\infty(\Sigma)} \lesssim 1$, we verify that

$$\left|\int_\Sigma (q_1-q_2)e^{ix_{1e}\xi_{1e}}e^{-2\tau x_3\xi_{1e}}\,dx\right| \lesssim \frac{1}{\tau}. \tag{3.4.8}$$

Let us now apply (3.4.4) and (3.4.8) to (3.4.7) to obtain

$$\left|\int_\Sigma e^{ix\cdot\xi}(q_1-q_2)\,dx\right| \lesssim \left|\int_\Sigma (q_1-q_2)u_1u_2\,dx\right| + \frac{1}{\tau}$$

for $\tau \ge \tau_1$, and u_1, u_2 defined by (3.4.5) and (3.4.6). As noted earlier, the functions $u_1|_\Omega$ and $u_2|_\Omega$ satisfy the hypotheses of Proposition 3.3.1, so we may apply Proposition 3.3.1 (b) with $\zeta = \xi^\perp$ to deduce that, if $\|\Lambda^2_{q_1} - \Lambda^2_{q_2}\|_* < 1/\delta$, then

$$\left|\int_\Sigma (q_1-q_2)e^{ix\cdot\xi}\,dx\right| \lesssim \frac{1}{\tau} + e^{c\tau|\xi|}\frac{\|u_1\|_{L^2(\Omega)}\|u_2\|_{H^2(\Omega)}}{\left[1+\left|\log\left(\delta\|\Lambda^2_{q_2}-\Lambda^2_{q_1}\|_*\right)\right|\right]^{1/2}} + \frac{1}{\tau^{1/2}}\|e^{\tau x\cdot\xi^\perp}u_2\|_{H^1(\Omega)}\|e^{-\tau x\cdot\xi^\perp}u_1\|_{L^2(\Omega)}$$

for all $\tau \geq \max(\tau_0, \tau_1)$.

The choices of ρ_m and u_m can be combined with (3.4.4) to deduce that

$$\left| \int_\Sigma (q_1 - q_2) e^{ix\cdot\xi}\, dx \right| \lesssim \frac{e^{c\tau|\xi|}}{\left[1 + \left| \log\left(\delta \|\Lambda^2_{q_2} - \Lambda^2_{q_1}\|_*\right) \right|\right]^{1/2}} + \frac{1}{\tau^{1/2}}$$

for all $\tau \geq \max(\tau_0, \tau_1)$, with $c > 4(2R + L)$. Thus, we obtain the uniform estimate

$$\left|\widehat{q_1}(\xi) - \widehat{q_2}(\xi)\right| \lesssim \frac{e^{c\tau r}}{\left[1 + \left| \log\left(\delta \|\Lambda^2_{q_2} - \Lambda^2_{q_1}\|_*\right) \right|\right]^{1/2}} + \frac{1}{\tau^{1/2}}. \tag{3.4.9}$$

for all $\tau \geq \max(\tau_0, \tau_1)$ and all $\xi \in \mathbb{R}^3$ with $1 \leq |\xi_{1e} < r$, $|\xi_3| < r$. Now, we are going to use analytic continuation in order to extend the set of frequencies, at which we control the difference $\widehat{q_1} - \widehat{q_2}$, to all of $\{|\xi| < r\}$.

Let $\xi \in \mathbb{R}^3$ with $0 < \xi_{1e} < 1$, $|\xi_3| < r$ be arbitrarily chosen; define $e(1), e(2), e(3)$ as we did earlier. By the Payley-Wiener theorem, $\widehat{q_1} - \widehat{q_2}$ is the restriction to $\mathbb{R}^3$ of an entire function on $\mathbb{C}^3$. Therefore, the function f defined by

$$\begin{aligned} f : \mathbb{C} &\rightarrow \mathbb{C} \\ z &\mapsto (\widehat{q_1} - \widehat{q_2})\left((z, 0, \xi_3)_e\right) \end{aligned}$$

is entire. If we define

$$G := \{s + it \in \mathbb{C} : |s| < 2, |t| < 2\},$$

$$\gamma := \{s + it \in \mathbb{C} : 0 < s < 1, t = 0\},$$

$$\Gamma_0 := \{s + it \in \mathbb{C} : 1 < s < 2, t = 0\},$$

then Corollary 1.2.2 (b) from [29] implies that there exist constants $C_0 > 0$ and $\lambda \in (0, 1)$, both of which depend on γ, such that

$$\sup_\gamma |f(s)| \leq C_0 (\sup_G |f(s + it)|)^{1-\lambda} (\sup_{\Gamma_0} |f(s)|)^\lambda.$$

Since

$$\sup_G |f(s + it)| \lesssim 1,$$

and since $\sup_{\Gamma_0} |f(s)|$ can be bounded by means of (3.4.9), we can conclude

$$\left|\widehat{q_1}(\xi) - \widehat{q_2}(\xi)\right| \lesssim \frac{e^{c\lambda\tau r}}{\left[1 + \left|\log\left(\delta\|\Lambda^2_{q_2} - \Lambda^2_{q_1}\|_*\right)\right|\right]^{\lambda/2}} + \frac{1}{\tau^{\lambda/2}} \tag{3.4.10}$$

for all $\tau \geq \max(\tau_0, \tau_1)$ and $\xi \in \mathbb{R}^3$ with $0 < \xi_{1e} < 1$, $|\xi_3| < r$.

We go on to combine (3.4.9) and (3.4.10), then drop higher-order terms (possibly at the cost of increasing the implicit constant), and thus conclude the following:

$$\left|\widehat{q_1}(\xi) - \widehat{q_2}(\xi)\right| \lesssim \frac{e^{c\tau r}}{\left[1 + \left|\log\left(\delta\|\Lambda^2_{q_2} - \Lambda^2_{q_1}\|_*\right)\right|\right]^{\lambda/2}} + \frac{1}{\tau^{\lambda/2}} \tag{3.4.11}$$

for all $\tau \geq \max(\tau_0, \tau_1)$ and $\xi \in \mathbb{R}^3$ with $|\xi| < r$.

Next, we finish the proof of Theorem 5 by performing the classical argument due to Alessandrini [1]. If we put $\varepsilon := \frac{s - \frac{3}{2}}{2}$ (so that $s = \frac{3}{2} + 2\varepsilon$), we may apply the Sobolev embedding theorem and interpolation together with the a-priori bounds on q_1, q_2 to obtain

$$\begin{aligned} \|q_1 - q_2\|_{L^\infty(\Sigma)} &= \|q_1 - q_2\|_{L^\infty(\Omega)} \lesssim \|q_1 - q_2\|_{H^{\frac{3}{2}+\varepsilon}(\Omega)} \\ &\leq \|q_1 - q_2\|^{\frac{\varepsilon}{s+1}}_{H^{-1}(\Omega)} \|q_1 - q_2\|^{\frac{s-\varepsilon+1}{s+1}}_{H^s(\Omega)} \\ &\lesssim \|q_1 - q_2\|^{\frac{\varepsilon}{s+1}}_{H^{-1}(\Omega)} \leq \|q_1 - q_2\|^{\frac{\varepsilon}{s+1}}_{H^{-1}(\mathbb{R}^3)}. \end{aligned} \tag{3.4.12}$$

On the other hand, by using the definition of $\|\centerdot\|_{H^{-1}(\mathbb{R}^3)}$ in terms of the Fourier transform, then splitting the integral into high and low frequencies, and lastly using Plancharel's theorem, we get

$$\|q_1 - q_2\|^2_{H^{-1}(\mathbb{R}^3)} \lesssim r^3 \sup_{\{|\xi|<r\}} |\widehat{q_1}(\xi) - \widehat{q_2}(\xi)|^2 + r^{-2}.$$

Applying (3.4.11) to the last estimate, utilizing $\tau \geq 1$, and for $c > 4(2R + L) + 1$, we get

$$\|q_1 - q_2\|_{H^{-1}(\mathbb{R}^3)} \lesssim \frac{e^{c\tau r}}{\left[1 + \left|\log\left(\delta\|\Lambda^2_{q_2} - \Lambda^2_{q_1}\|_*\right)\right|\right]^{\lambda/2}} + r^{3/2}\tau^{-\lambda/2} + r^{-1}.$$

Upon selecting τ so that $r^{-1} = r^{3/2}\tau^{-\lambda/2}$ or, equivalently, as $\tau := r^{5/\lambda}$, the preceding estimate implies

$$\|q_1 - q_2\|_{H^{-1}(\mathbb{R}^3)} \lesssim \frac{e^{cr^{\frac{\lambda+5}{\lambda}}}}{\left[1 + \left|\log\left(\delta\|\Lambda^2_{q_2} - \Lambda^2_{q_1}\|_*\right)\right|\right]^{\lambda/2}} + r^{-1}.$$

Choose $r > 0$ so that

$$r^{\frac{\lambda+5}{\lambda}} = c^{-1} \log\left\{ \left[1 + \left|\log(\delta \|\Lambda^2_{q_2} - \Lambda^2_{q_1}\|_*)\right|\right]^{\lambda/4} \right\}$$

in the last inequality and combine it with (3.4.12); in the resulting inequality, we drop higher-order terms (possibly at the cost of increasing the implicit constant), and thus derive the stability estimate of Theorem 5 with $\theta := \frac{\lambda}{2(\lambda+5)}$.

3.5 Proof of Theorem 6 (Dirichlet and Neumann data on the same slab)

In this section, we prove Theorem 6. In doing so, we imitate the arguments from Section 3.4; broadly speaking, the main difference is that occurrences of $(\widehat{q_1} - \widehat{q_2})$ from Section 3.4 will now be replaced by occurrences of $(Q_1^{\text{even}} - Q_2^{\text{even}})\widehat{\ }$, where Q_j^{even} stands for the even extension of $q_j|_{\{x_3 \geq 0\}}$ to $\mathbb{R}^3$ about the coordinate variable x_3.

As in Section 3.4, we begin by constructing appropriate CGOs by means of Isakov's reflection argument from [30]. Consider $r > 2$, which will be specified later on in this section.

Let $\xi \in \mathbb{R}^3$ with

$$1 \leq \xi_{1e} < r \quad \text{and} \quad |\xi_3| < r,$$

be arbitrarily chosen. Define $e(1), e(2), e(3)$ as in Section 3.4.

From now until the end of this section, we let Q_j^{even} stand for the even extension of q_j about the coordinate variable x_3; explicitly, we set

$$Q_j^{\text{even}}(x) := q_j(x) + q_j(x^*) \qquad \text{for a.e. } x \in \mathbb{R}^3.$$

Thanks to the regularity hypotheses on q_j, we have that Q_1^{even} and Q_2^{even} belong to $H^s(\mathbb{R}^3)$ and have their supports contained in $Q \cup Q^*$.

Fix B as in Section 3.4. Following the idea from Section 4 in [34], we will construct u_1

and u_2 via Isakov's reflection argument. Firstly, we define

$$\rho_1 := \left(i\left(\frac{\xi_{1e}}{2} - \alpha\xi_3\right), -(\alpha^2+1/4)^{1/2}|\xi|, i\left(\frac{\xi_3}{2} + \alpha\xi_{1e}\right)\right)_e, \tag{3.5.1}$$

$$\rho_2 := \left(i\left(\frac{\xi_{1e}}{2} + \alpha\xi_3\right), (\alpha^2+1/4)^{1/2}|\xi|, i\left(\frac{\xi_3}{2} - \alpha\xi_{1e}\right)\right)_e, \tag{3.5.2}$$

where $\alpha > 0$ is a parameter. One readily verifies that

$$\rho_m \cdot \rho_m = 0, \quad |\rho_m| = \sqrt{2}|\xi|(\alpha^2+1/4)^{1/2}, \quad m = 1,2.$$

It is a well-known fact that there exists a constant $C_0 = C_0(B, M, k) \geq 1$ such that, for each $\alpha \geq \alpha_2 := \max(C_0(M+k^2), 1)$, there exists a function $\psi_m \in H^2(B)$ satisfying

$$\|\psi_m\|_{H^k(B)} \lesssim \frac{1}{\left[(\alpha^2+1/4)^{1/2}|\xi|\right]^{1-k}}, \quad k = 0,1,2, \quad m = 1,2 \tag{3.5.3}$$

such that $V_m(x) := e^{x\cdot\rho_m}(1+\psi_m)$ belongs to $H^2(B)$ and satisfies

$$(-\Delta + Q_m^{\text{even}} - k^2)V_m = 0 \text{ in } B.$$

The implicit constant in (3.5.3) depends on B, M, k. Employing the same idea as in [34], we set

$$u_m(x) := e^{x\cdot\rho_m}(1+\psi_m) - e^{x^*\cdot\rho_m}(1+\psi_m^*); \tag{3.5.4}$$

it then follows that $u_1|_\Omega$ and $u_2|_\Omega$ satisfy the hypotheses of Proposition 3.3.1 (a). On the one hand, a routine computation utilizing the decay estimates from (3.5.3) shows that

$$\|u_1\|_{L^2(\Omega)} \lesssim e^{cr(\alpha^2+1/4)^{1/2}}, \tag{3.5.5}$$

$$\|u_2\|_{H^2(\Omega)} \lesssim e^{cr(\alpha^2+1/4)^{1/2}}, \tag{3.5.6}$$

with the implicit constants depending on B, n, M, k; at this stage, we have increased c if necessary. On the other hand, a direct calculation shows that

$$\begin{aligned}
&\int_\Sigma (q_1 - q_2) u_1 u_2 \, dx = \\
&\quad \int_\Sigma (q_1 - q_2) e^{ix\cdot\xi} \, dx + \int_\Sigma (q_1 - q_2) e^{ix\cdot\xi} (\psi_1 + \psi_2 + \psi_1 \psi_2) \, dx \\
&\quad - \int_\Sigma (q_1 - q_2) e^{ix\cdot(\xi_{1e}, 0, 2\alpha\xi_{1e})_e} \, dx \\
&\quad - \int_\Sigma (q_1 - q_2) e^{ix\cdot(\xi_{1e}, 0, 2\alpha\xi_{1e})_e} (\psi_1 + \psi_2^* + \psi_1 \psi_2^*) \, dx \\
&\quad - \int_\Sigma (q_1 - q_2) e^{ix\cdot(\xi_{1e}, 0, -2\alpha\xi_{1e})_e} \, dx \\
&\quad - \int_\Sigma (q_1 - q_2) e^{ix\cdot(\xi_{1e}, 0, -2\alpha\xi_{1e})_e} (\psi_1^* + \psi_2 + \psi_1^* \psi_2) \, dx \\
&\quad + \int_\Sigma (q_1 - q_2) e^{ix^*\cdot\xi} \, dx + \int_\Sigma (q_1 - q_2) e^{ix^*\cdot\xi} (\psi_1^* + \psi_2^* + \psi_1^* \psi_2^*) \, dx;
\end{aligned}$$

combining this with the hypotheses on $q_1 - q_2$, (3.5.3), and $|\xi| \geq \xi_{1e} \geq 1$ establishes

$$\begin{aligned}
&\left| \int_\Sigma (q_1 - q_2) e^{ix\cdot\xi} \, dx + \int_\Sigma (q_1 - q_2) e^{ix^*\cdot\xi} \, dx \right| \lesssim \left| \int_\Sigma (q_1 - q_2) u_1 u_2 \, dx \right| \\
&+ \left| (q_1 - q_2)\widehat{\ } \left((-\xi_{1e}, 0, -2\alpha\xi_{1e})_e \right) \right| + \left| (q_1 - q_2)\widehat{\ } \left((-\xi_{1e}, 0, 2\alpha\xi_{1e})_e \right) \right| \\
&+ \frac{1}{(\alpha^2 + 1/4)^{1/2}},
\end{aligned}$$

where the implicit constant depends on B, M, k. For technical reasons, let us replace ξ by $-\xi$. Now, we apply the quantified Riemann-Lebesgue lemma to $f := q_1 - q_2$ (in order to handle the Fourier transforms on the right-hand side in the last inequality), Proposition 3.3.1 (a), (3.5.5), and (3.5.6) to obtain

$$|(Q_1^{\text{even}} - Q_2^{\text{even}})\widehat{\ }(\xi)| \lesssim \frac{e^{cr(\alpha^2 + 1/4)^{1/2}}}{\left[1 + |\log(\delta \|\Lambda_{q_1}^1 - \Lambda_{q_2}^1\|_*)|\right]^{1/2}} + \frac{1}{(\alpha^2 + 1/4)^{1/2}}$$

whenever $\alpha \geq \alpha_2$. Here we have increased c.

For the sake of brevity and the ease of comparison with the arguments from the previous section, let us introduce a new parameter $\tau := (\alpha^2 + 1/4)^{1/2}$. Using this new parameter, we have obtained the following inequality: there exists a constant $T_2 := (\alpha_2^2 + 1/4)^{1/2}$ such that

$$|(Q_1^{\text{even}} - Q_2^{\text{even}})\widehat{\ }(\xi)| \lesssim \frac{e^{cr\tau}}{\left[1 + |\log(\delta \|\Lambda_{q_1}^1 - \Lambda_{q_2}^1\|_*)|\right]^{1/2}} + \frac{1}{\tau} \tag{3.5.7}$$

for all $\tau \geq T_2$ and all $\xi \in \mathbb{R}^3$ with $1 \leq \xi_{1e} < r$, $|\xi_3| < r$.

Now, we are going to use analytic continuation in order to extend the set of frequencies, at which we control the difference $(Q_1^{\text{even}} - Q_2^{\text{even}})\hat{}$, to all of $\{|\xi| < r\}$.

Let $\xi \in \mathbb{R}^3$ with $0 < \xi_{1e} < 1$, $|\xi_3| < r$ be arbitrarily chosen; define $e(1), e(2), e(3)$ as we did earlier. By the Payley-Wiener theorem, $(Q_1^{\text{even}} - Q_2^{\text{even}})\hat{}$ is the restriction to $\mathbb{R}^3$ of an entire function on $\mathbb{C}^3$. Therefore, the function g defined by

$$\begin{aligned} g : \mathbb{C} &\rightarrow \mathbb{C} \\ z &\mapsto (Q_1^{\text{even}} - Q_2^{\text{even}})\hat{}((z, 0, \xi_3)_e) \end{aligned}$$

is entire. If G, γ, Γ_0 stand for the same sets as in Section 3.4, then Corollary 1.2.2 (b) from [29] implies that there exist constants $C_0 > 0$ and $\lambda \in (0,1)$, both of which depend on γ, such that

$$\sup_{\gamma} |f(s)| \leq C_0 (\sup_{G} |f(s+it)|)^{1-\lambda} (\sup_{\Gamma_0} |f(s)|)^{\lambda}.$$

Again, as in Section 3.4, we verify that $\sup_G |g(s)| \lesssim 1$ while $\sup_{\Gamma_0} |g(s)|$ can be bounded by means of (3.5.7), enabling us to conclude

$$\left|(Q_1^{\text{even}} - Q_2^{\text{even}})\hat{}(\xi)\right| \lesssim \frac{e^{c\lambda\tau r}}{\left[1 + \left|\log\left(\delta\|\Lambda^1_{q_2} - \Lambda^1_{q_1}\|_*\right)\right|\right]^{\lambda/2}} + \frac{1}{\tau^{\lambda}} \qquad (3.5.8)$$

for all $\tau \geq T_2$ and $\xi \in \mathbb{R}^3$ with $0 < \xi_{1e} < 1$, $|\xi_3| < r$.

We go on to combine (3.5.7) and (3.5.8), then drop higher-order terms (possibly at the cost of increasing the implicit constant), and thus conclude the following:

$$\left|(Q_1^{\text{even}} - Q_2^{\text{even}})\hat{}(\xi)\right| \lesssim \frac{e^{c\tau r}}{\left[1 + \left|\log\left(\delta\|\Lambda^1_{q_2} - \Lambda^1_{q_1}\|_*\right)\right|\right]^{\lambda/2}} + \frac{1}{\tau^{\lambda}} \qquad (3.5.9)$$

for all $\tau \geq T_2$ and $\xi \in \mathbb{R}^3$ with $|\xi| < r$.

Next, we finish the proof of Theorem 6 by performing the classical argument due to Alessandrini [1]. If we put $\varepsilon := \frac{s-\frac{3}{2}}{2}$ (so that $s = \frac{3}{2} + 2\varepsilon$), we may apply the Sobolev embedding theorem and interpolation together with the a-priori bounds on q_1, q_2 to obtain

$$\begin{aligned} \|q_1 - q_2\|_{L^\infty(\Sigma)} &= \|Q_1^{\text{even}} - Q_2^{\text{even}}\|_{L^\infty(\Omega)} \\ &\lesssim \|Q_1^{\text{even}} - Q_2^{\text{even}}\|_{H^{\frac{3}{2}+\varepsilon}(\Omega)} \\ &\leq \|Q_1^{\text{even}} - Q_2^{\text{even}}\|_{H^{-1}(\Omega)}^{\frac{\varepsilon}{s+1}} \|Q_1^{\text{even}} - Q_2^{\text{even}}\|_{H^s(\Omega)}^{\frac{s-\varepsilon+1}{s+1}} \\ &\lesssim \|Q_1^{\text{even}} - Q_2^{\text{even}}\|_{H^{-1}(\Omega)}^{\frac{\varepsilon}{s+1}} \leq \|Q_1^{\text{even}} - Q_2^{\text{even}}\|_{H^{-1}(\mathbb{R}^3)}^{\frac{\varepsilon}{s+1}}. \end{aligned} \tag{3.5.10}$$

Again, as in Section 3.4, by using the definition of $\|\,\boldsymbol{\cdot}\,\|_{H^{-1}(\mathbb{R}^3)}$ in terms of the Fourier transform, then splitting the integral into high and low frequencies, and lastly using Plancharel's theorem, we get

$$\|Q_1^{\text{even}} - Q_2^{\text{even}}\|_{H^{-1}(\mathbb{R}^3)}^2 \lesssim r^3 \sup_{\{|\xi|<r\}} |(Q_1^{\text{even}} - Q_2^{\text{even}})\widehat{\ }(\xi)|^2 + r^{-2}.$$

We proceed by imitating the argument from Section 3.4: apply (3.5.9) to the last inequality, insert the resulting inequaliting into (3.5.10), then select $\tau := r^{5/(2\lambda)}$, next select r such that

$$r^{\frac{2\lambda+5}{2\lambda}} = c^{-1} \log\left\{\left[1 + \left|\log(\delta\|\Lambda_{q_2}^1 - \Lambda_{q_1}^1\|_*)\right|\right]^{\lambda/4}\right\},$$

drop higher-order terms (possibly at the cost of increasing the implicit constant), and thus derive the stability estimate of Theorem 6 with $\theta := \frac{\lambda}{2\lambda+5}$.

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Appendices

Appendix A

A CARLEMAN ESTIMATE

In Sections 2.5 and 3.4, we shall have occasion to use Bukhgeim and Uhlmann's Carleman estimate from [6]. In Section 3.4, we shall actually need a mild generalization of the Carleman estimate from [6]; we dedicate this appendix to stating and proving the needed generalization.

The Poincaré inequality from [19] can be generalized in a straightforward way as follows:

Lemma A.0.1. *Fix $\eta \in \mathbb{R}^n \setminus \{0\}$. Let $Q \subseteq \mathbb{R}^n$ be open for which there exist $a, b \in \mathbb{R}$ with $a < b$ such that*

$$Q \subseteq \{x \in \mathbb{R}^n : a < x \cdot \eta < b\}.$$

Then,

$$\forall u \in H_0^1(Q): \quad \|u\|_{L^2(Q)} \leq \frac{1}{|\eta|^2} \frac{b-a}{\sqrt{2}} \|\eta \cdot \nabla u\|_{L^2(Q)}.$$

Proof. The proof is broken down into two cases.

Suppose first that $|\eta| = 1$. Since $C_c^\infty(Q)$ is dense in $H_0^1(Q)$, it suffices to prove the inequality for functions in $C_c^\infty(Q)$. Let $u \in C_c^\infty(Q)$ be arbitrary. If P denotes the hyperplane in $\mathbb{R}^n$ orthogonal to η, then

$$\begin{aligned} \forall x \in P, \forall t \in [a,b]: \quad |u(x+t\eta)| &= \left| \int_a^t \eta \cdot \nabla u(x+s\eta)\, ds \right| \\ &\leq \left(\int_a^t 1\, ds \right)^{1/2} \left(\int_a^t |\eta \cdot \nabla u(x+s\eta)|^2\, ds \right)^{1/2} \\ &= (t-a)^{1/2} \left(\int_a^t |\eta \cdot \nabla u(x+s\eta)|^2\, ds \right)^{1/2}, \end{aligned}$$

from which it trivially follows that

$$|u(x+t\eta)|^2 \leq (t-a) \left(\int_a^b |\eta \cdot \nabla u(x+s\eta)|^2\, ds \right).$$

Then, we integrate the last inequality with respect to t over $[a,b]$ and with respect to $d\mathcal{H}^{n-1}$ over P to obtain

$$\begin{aligned}
\int_P \int_a^b |u(x+t\eta)|^2 \, dt \, d\mathcal{H}^{n-1} &\leq \int_P \int_a^b \left[(t-a)\left(\int_a^b |\eta \cdot \nabla u(x+s\eta)|^2 \, ds\right)\right] dt \, d\mathcal{H}^{n-1} \\
&= \left(\int_a^b (t-a)\, dt\right)\left(\int_P \int_a^b |\eta \cdot \nabla u(x+s\eta)|^2 \, d\mathcal{H}^{n-1} \, ds\right) \\
&= \frac{(b-a)^2}{2}\left(\int_P \int_a^b |\eta \cdot \nabla u(x+s\eta)|^2 \, d\mathcal{H}^{n-1} \, ds\right),
\end{aligned}$$

which is equivalent to $\|u\|_{L^2(Q)} \leq \frac{b-a}{\sqrt{2}} \|\eta \cdot \nabla u\|_{L^2(Q)}$; this completes the proof of the case when $|\eta| = 1$.

Consider now the second case, i.e. when $\eta \in \mathbb{R}^n \setminus \{0\}$. Write

$$\widehat{\eta} := \frac{\eta}{|\eta|} \in \mathbb{S}^{n-1}.$$

It is clear that, the condition $\Omega \subseteq \{x \in \mathbb{R}^n \mid a < x \cdot \eta < b\}$ is equivalent to

$$\Omega \subseteq \left\{ x \in \mathbb{R}^n \;\middle|\; \frac{a}{|\eta|} < x \cdot \widehat{\eta} < \frac{b}{|\eta|} \right\};$$

by appealing to the case of Lemma A.0.1 which has just been established, we conclude that

$$\|u\|_{L^2(Q)} \leq \frac{\frac{b}{|\eta|} - \frac{a}{|\eta|}}{\sqrt{2}} \|\widehat{\eta} \cdot \nabla u\|_{L^2(Q)} = \frac{1}{|\eta|^2} \frac{b-a}{\sqrt{2}} \|\eta \cdot \nabla u\|_{L^2(Q)}.$$

□

In [6], the authors derived a Carleman estimate (Corollary 2.3) for a bounded domain Q. By using Lemma A.0.1, the Carleman estimate of [6] can be generalized as follows:

Lemma A.0.2. *Let $Q \subseteq \mathbb{R}^n$ be a bounded domain with C^2 boundary. Let $r_Q > 0$ be such that $Q \subseteq B(0, r_Q)$. Then, for all $\eta \in \mathbb{R}^n \setminus \{0\}$, $\tau > 0$, and $u \in C^2(Q;\mathbb{C})$ with $u|_{\partial Q} = 0$, we have*

$$\int_Q e^{-2\tau x \cdot \eta} |\Delta u|^2 \, dx \geq \frac{8\tau^2 |\eta|^4}{(2r_Q|\eta|)^2} \int_Q e^{-2\tau x \cdot \eta} |u|^2 \, dx + 2\tau \int_{\partial Q} (\nu \cdot \eta) e^{-2\tau x \cdot \eta} |\partial_\nu u|^2 \, d\sigma,$$

where ν is the outward-pointing normal along $\partial\Omega$.

Proof. Fix $\eta \in \mathbb{R}^n \setminus \{0\}$, $\tau > 0$, and $u \in C^2(Q;\mathbb{C})$ with $u|_{\partial Q} = 0$. Abbreviate

$$\partial_\eta := \eta \cdot \nabla, \quad v_k := \partial_k v, \quad v_{kj} := \partial_{kj} v.$$

Put

$$I := \int_Q e^{-2\tau x\cdot\eta} |\Delta u|^2 \, dx,$$

and we shall estimate I from below. Upon setting $v := e^{-\tau x\cdot\eta} u$, we have

$$\begin{aligned}
I &= \int_Q \left| e^{-\tau x\cdot\eta} \Delta \left(e^{\tau x\cdot\eta} v \right) \right|^2 \, dx \\
&= \int_Q \left| e^{-\tau x\cdot\eta} \nabla \cdot \left(\tau\eta e^{\tau x\cdot\eta} v + e^{\tau x\cdot\eta} \nabla v \right) \right|^2 \, dx \\
&= \int_Q \left| e^{-\tau x\cdot\eta} \left(\tau^2 |\eta|^2 e^{\tau x\cdot\eta} v + \tau e^{\tau x\cdot\eta} (\partial_\eta v) + \tau (\partial_\eta v) e^{\tau x\cdot\eta} + e^{\tau x\cdot\eta} \Delta v \right) \right|^2 \, dx \\
&= \int_Q \left| \underbrace{(\Delta + |\eta|^2\tau^2) v}_{P_+ v :=} + \underbrace{(2\tau\partial_\eta) v}_{P_- v :=} \right|^2 \, dx \\
&= \int_Q \left(|P_+ v|^2 + |P_- v|^2 + 2 \,\mathrm{Re}(P_+ v \overline{P_- v}) \right) dx.
\end{aligned}$$

Let us also calculate that

$$\begin{aligned}
2 \,\mathrm{Re}(P_+ v \overline{P_- v}) &= 2 \,\mathrm{Re} \left[\left((\Delta + |\eta|^2\tau^2) v \right) \left((2\tau\partial_\eta) \overline{v} \right) \right] \\
&= 2\tau \left\{ 2 \,\mathrm{Re} \left[(\partial_\eta \overline{v})(\Delta v) \right] \right\} + 4\tau^3 |\eta|^2 \mathrm{Re} \left[v (\partial_\eta \overline{v}) \right] \\
&= 2\tau \left\{ 2 \,\mathrm{Re} \left[(\partial_\eta \overline{v})(\Delta v) \right] \right\} + 2\tau^3 |\eta|^2 \partial_\eta \left(|v|^2 \right),
\end{aligned}$$

and

$$\begin{aligned}
2\tau\bigl\{2\ \mathrm{Re}\,[(\partial_\eta\overline{v})(\Delta v)]\bigr\} &= 2\tau\ \left\{2\ \mathrm{Re}\left[\sum_{k,j} v_{kk}\eta_j\overline{v}_j\right]\right\}\\
&= 2\tau\ \left\{\sum_{k,j}[2\ \mathrm{Re}(\partial_k(v_k\overline{v}_j)) - 2\ \mathrm{Re}(v_k\overline{v}_{kj})]\,\eta_j\right\}\\
&= 2\tau\ \left\{\sum_{k,j}\left[2\ \mathrm{Re}(\partial_k(v_k\overline{v}_j)) - \partial_j\left(|v_k|^2\right)\right]\eta_j\right\}\\
&= 2\tau\ \left\{\sum_{k} 2\ \partial_k\left[\mathrm{Re}\left(\sum_j v_k\overline{v}_j\eta_j\right)\right] - \sum_j\sum_k \partial_j\left(|v_k|^2\right)\eta_j\right\}\\
&= 2\tau\ \left\{\sum_k 2\ \partial_k\left[\mathrm{Re}\left(v_k(\partial_\eta\overline{v})\right)\right] - \sum_j \partial_j\left(|\nabla v|^2\right)\eta_j\right\}\\
&= 2\tau\ \left\{2\ \mathrm{div}\bigl[\mathrm{Re}\bigl((\partial_\eta\overline{v})\nabla v\bigr)\bigr] - \mathrm{div}\left(|\nabla v|^2\eta\right)\right\},
\end{aligned}$$

from which it follows that

$$2\ \mathrm{Re}(P_+v\overline{P_-v}) = \mathrm{div}\bigl[4\tau\ \mathrm{Re}\bigl((\partial_\eta\overline{v})\nabla v\bigr) - 2\tau|\nabla v|^2\eta + 2\tau^3|\eta|^2|v|^2\eta\bigr].$$

In view of the last displayed relation, we may compute the following integral by the divergence theorem to get

$$\int_Q 2\ \mathrm{Re}(P_+v\overline{P_-v})\,dx = \int_{\partial Q}\bigl[4\tau\ \mathrm{Re}\bigl((\partial_\eta\overline{v})(\partial_\nu v)\bigr) - 2\tau(\nu\cdot\eta)|\nabla v|^2 + 2\tau^3|\eta|^2|v|^2(\nu\cdot\eta)\bigr]\ d\sigma;$$

by $u|_{\partial Q} = 0$, it follows that $v|_{\partial Q} = 0$, so

$$(\nabla v)|_{\partial Q} = (\nu\cdot\nabla v)\nu + \underbrace{(\nabla v - (\nu\cdot\nabla v)\nu)}_{=0\text{ as the tang. comp. of }\nabla v} = (\partial_\nu v)\nu,$$

i.e.

$$v|_{\partial Q} = 0;\quad |\nabla v|^2 = |\partial_\nu v|^2 \text{ on } \partial Q;\quad \partial_\eta v = (\eta\cdot\nu)(\partial_\nu v) \text{ on } \partial Q,$$

so by inserting these three relations into the identity we obtained via the divergence theorem, we arrive at

$$\begin{aligned}
\int_Q 2\,\mathrm{Re}(P_+ v\overline{P_- v})\,dx &= \int_{\partial Q}\left[4\tau\ \mathrm{Re}\big((\eta\cdot\nu)|\partial_\nu v|^2\big) - 2\tau(\nu\cdot\eta)|\partial_\nu v|^2\right]\,d\sigma \\
&= 2\tau\int_{\partial Q}\mathrm{Re}\big((\nu\cdot\eta)|\partial_\nu v|^2\big)\,d\sigma \\
&= 2\tau\int_{\partial Q}(\nu\cdot\eta)|\partial_\nu v|^2\,d\sigma \\
&= 2\tau\int_{\partial Q}(\nu\cdot\eta)\left|\partial_\nu\left(e^{-\tau x\cdot\eta}u\right)\right|^2\,d\sigma \\
&= 2\tau\int_{\partial Q}(\nu\cdot\eta)\left|\nu\cdot\left(-\tau\eta e^{-\tau x\cdot\eta}u + e^{-\tau x\cdot\eta}\nabla u\right)\right|^2\,d\sigma \\
&= 2\tau\int_{\partial Q}(\nu\cdot\eta)e^{-2\tau x\cdot\eta}\left|\partial_\nu u\right|^2\,d\sigma,
\end{aligned}$$

where the first term vanishes because $u|_{\partial Q} = 0$.

Let us summarize our computations so far:

$$\begin{aligned}
I &\geq \int_Q |P_- v|^2\,dx + \int_Q 2\,\mathrm{Re}\left(P_+ v\overline{P_- v}\right)\,dx \\
&= 4\tau^2\|\partial_\eta v\|^2_{L^2(Q)} + 2\tau\int_{\partial Q}(\nu\cdot\eta)e^{-2\tau x\cdot\eta}\left|\partial_\nu u\right|^2\,d\sigma. \qquad \text{(A.0.1)}
\end{aligned}$$

By $Q \subseteq B(0, r_Q)$ and the Cauchy-Schwarz inequality, we have

$$Q \subseteq \{x \in \mathbb{R}^n : -r_Q|\eta| < x\cdot\eta < r_Q|\eta|\},$$

so Lemma A.0.1 implies that

$$\|\partial_\eta v\|_{L^2(Q)} \geq \frac{\sqrt{2}|\eta|^2}{r_Q|\eta| - (-r_Q|\eta|)}\|v\|_{L^2(Q)};$$

plug this into (A.0.1) to get

$$\int_Q e^{-2\tau x\cdot\eta}|\Delta u|^2\,dx \geq \frac{8\tau^2|\eta|^4}{(2r_Q|\eta|)^2}\|v\|^2_{L^2(Q)} + 2\tau\int_{\partial Q}(\nu\cdot\eta)e^{-2\tau x\cdot\eta}\left|\partial_\nu u\right|^2\,d\sigma,$$

as desired. □

We now obtain a Carleman estimate for a certain class of Schrödinger operators $\Delta - q$, which shall be needed in the study of the inverse problem.

Proposition A.0.1 (from [6]). *Let $Q \subseteq \mathbb{R}^n$ be a bounded domain with C^2 boundary. Let $r_Q > 0$ be such that $Q \subseteq B(0, r_Q)$. Fix $m > 0$ and $a_0 > 0$. Then, there exist constants $\tau_0 = \tau_0(r_Q, m, a_0) > 0$ and $C = C(r_Q, m, a_0) > 0$ such that, for all $q \in L^\infty(Q)$ with $\|q\|_{L^\infty(Q)} \leq m$, $u \in C^2(\overline{Q}; \mathbb{C})$ with $u|_{\partial Q} = 0$, $\eta \in \mathbb{R}^n$ with $|\eta| \geq a_0$, and $\tau \geq \tau_0$, we have:*

$$C\tau^2 \|e^{-\tau x\cdot\eta} u\|^2_{L^2(Q)} + \tau \int_{\partial Q} (\eta \cdot \nu) |e^{-\tau x\cdot\eta} \partial_\nu u|^2 \, dS \leq \|e^{-\tau x\cdot\eta}(\Delta - q)u\|^2_{L^2(Q)}.$$

Proof. Let us estimate that

$$\begin{aligned}\int_Q e^{-2\tau x\cdot\eta} |\Delta u|^2 \, dx &\leq \int_Q e^{-2\tau x\cdot\eta} \left(|(\Delta - q)u| + |qu|\right)^2 \, dx \\ &\leq \int_Q e^{-2\tau x\cdot\eta} \left(2|(\Delta - q)u|^2 + 2|qu|^2\right) \, dx \\ &\leq 2\int_Q e^{-2\tau x\cdot\eta} |(\Delta - q)u|^2 \, dx + 2\|q\|^2_{L^\infty(Q)} \int_Q e^{-2\tau x\cdot\eta} |u|^2 \, dx,\end{aligned}$$

at which point we may apply Lemma A.0.2 to the left-hand side and thus arrive at

$$\begin{aligned}2\int_Q e^{-2\tau x\cdot\eta} |(\Delta - q)u|^2 \, dx &\geq \left(\frac{8\tau^2 |\eta|^4}{(2r_Q|\eta|)^2} - 2\|q\|^2_{L^\infty(Q)}\right) \int_Q e^{-2\tau x\cdot\eta} |u|^2 \, dx \\ &+ 2\tau \int_{\partial Q} (\nu \cdot \eta) e^{-2\tau x\cdot\eta} |\partial_\nu u|^2 \, d\sigma.\end{aligned}$$

Therefore, we get

$$\begin{aligned}2\int_Q e^{-2\tau x\cdot\eta} |(\Delta - q)u|^2 \, dx &\geq \left(\frac{2a_0^2}{r_Q^2}\tau^2 - 2m^2\right) \int_Q e^{-2\tau x\cdot\eta} |u|^2 \, dx \\ &+ 2\tau \int_{\partial Q} (\nu \cdot \eta) e^{-2\tau x\cdot\eta} |\partial_\nu u|^2 \, d\sigma.\end{aligned}$$

Since $\frac{\frac{2a_0^2}{r_Q^2}\tau^2 - 2m^2}{\tau^2} \to \frac{2a_0^2}{r_Q^2}$ as $\tau \to \infty$, there exists some $\tau_0 = \tau_0(r_Q, m, a_0) > 0$ such that

$$\forall \tau \geq \tau_0 : \quad \frac{\frac{2\tau^2 a_0^2}{r_Q^2} - 2m^2}{\tau^2} \geq \frac{1}{17} \cdot \frac{2a_0^2}{r_Q^2}.$$

Combining the last two inequalities and dividing throughout by a factor of 2 completes the proof. □

Appendix B

A RUNGE-TYPE APPROXIMATION RESULT

Let us emphasize that Lemma 3.3 from [34] is an important step in Li and Uhlmann's uniqueness results. Additionally, when we study the stability aspect of our IBVPs in the slab Σ, we shall need a simple generalization of Lemma 3.3 from [34]. For the sake of completeness, we state and prove the generalization here. If K is an arbitrary compact subset of Γ_1, we define

$$\begin{aligned} \mathcal{W}_j(\Sigma, K) &:= \{v \in H^2_{\text{loc}}(\overline{\Sigma}) : (-\Delta + q_j - k^2)v = 0 \text{ in } \Sigma, v|_{\Gamma_2} = 0, \\ &\quad\ \text{supp}(v|_{\Gamma_1}) \subseteq K, v \text{ is admissible in the sense of [KLU]}\} \\ \mathcal{W}_j(\Omega) &:= \{u \in H^2(\Omega) : (-\Delta + q_j - k^2)u = 0 \text{ in } \Omega, u|_{\Gamma_2 \cap \partial\Omega} = 0\} \end{aligned}$$

Proposition B.0.2 (Lemma 3.3 from [34])**.** *Let K be a compact subset of Γ_1 such that the relative interior of K in Γ_1 contains $\partial\Omega \cap \Gamma_1$. Then, the space $\{u|_\Omega : u \in W_j(\Sigma, K)\}$ is dense in $W_j(\Omega)$ with respect to $\|\centerdot\|_{L^2(\Omega)}$ for $j = 1, 2$.*

Let the relative interior of K in Γ_1 be denoted by γ_1. It is noteworthy that the hypotheses on K are necessary for ensuring that $\gamma_1 \setminus (\partial\Omega \cap \Gamma_1) \neq \varnothing$, which in turn is a critical step in the proof of Proposition B.0.2.

Proof. Fix $j \in \{1, 2\}$. Abbreviate

$$l_1 := \partial\Omega \cap \Gamma_1,\ l_2 := \partial\Omega \cap \Gamma_2,\ l_3 := \partial\Omega \cap \Sigma.$$

Clearly, $\{u|_\Omega : u \in W_j(\Sigma)\} \subseteq W_j(\Omega)$, so

$$\overline{\{u|_\Omega : u \in W_j(\Sigma)\}}^{\|\cdot\|_{L^2(\Omega)}} \subseteq \left(\overline{W_j(\Omega)}^{\|\cdot\|_{L^2(\Omega)}}, (\cdot, \cdot)_{L^2(\Omega)}\right) =: \mathcal{H}. \tag{B.0.1}$$

We point out that $\mathcal{H}$ is a closed subspace of $L^2(\Omega)$, so $\mathcal{H}$ is itself a Hilbert space. Assume (for the sake of contradiction) that the containment in (B.0.1) is proper, i.e. that there exists some[1]

$$g \in \left(\overline{\{u|_\Omega : u \in W_j(\Sigma)\}}^{\|\cdot\|_{L^2(\Omega)}} \right)^\perp \subseteq \mathcal{H},$$

with

$$\|g\|_{L^2(\Omega)} > 0. \tag{B.0.2}$$

The fact that g belongs to the orthogonal complement of $\overline{\{u|_\Omega : u \in W_j(\Sigma)\}}^{\|\cdot\|_{L^2(\Omega)}}$ in the Hilbert space $\mathcal{H}$ guarantees that

$$\forall u \in W_j(\Sigma) : \quad \int_\Omega ug\,dx = 0. \tag{B.0.3}$$

By the very definition of $\mathcal{H}$ and $g \in \mathcal{H}$, we deduce that

$$\exists \{u_k\}_{k=1}^\infty \subseteq W_j(\Omega) : \quad \lim_{k\to\infty} \|u_k - g\|_{L^2(\Omega)} = 0. \tag{B.0.4}$$

Define $\widetilde{g} \in L^2(\Sigma) \cap \mathcal{E}'(\overline{\Sigma})$ by

$$\widetilde{g}(x) := \begin{cases} g(x) & \text{for } x \in \Omega \\ 0 & \text{for } x \in \Sigma \setminus \Omega \end{cases}$$

By the solvability of the direct problem established in [32], there exists a unique admissible $U \in H^2_{\text{loc}}(\overline{\Sigma})$ such that

$$(-\Delta + q_j - k^2)U = \widetilde{g} \quad \text{in } \Sigma, \tag{B.0.5}$$

$$U|_{\partial\Sigma} = 0. \tag{B.0.6}$$

By (B.0.3), the definition of $\widetilde{g}$, and (B.0.5), we get

$$\forall u \in W_j(\Sigma) : \quad 0 = \int_\Sigma u \left[(-\Delta + q_j - k^2)U\right] \, dx.$$

[1]Since $\mathcal{H} \subseteq L^2(\Omega)$, it follows that $g \in L^2(\Omega)$.

Apply Green's formula in Σ to get:

$$\begin{aligned}\forall u \in W_j(\Sigma): \quad 0 &= \int_{\Sigma} \left[(-\Delta + q_j - k^2)u\right] U \, dx \\ &- \int_{\partial\Sigma} (\partial_\nu U) u \, d\sigma + \int_{\partial\Sigma} U(\partial_\nu u)\, d\sigma.\end{aligned}$$

By $u \in W_j(\Sigma)$ and by (B.0.6), the preceding relation reduces to

$$\forall u \in W_j(\Sigma): \quad 0 = -\int_{\Gamma_1} (\partial_\nu U) u \, d\sigma;$$

since $u|_{\Gamma_1}$ can be an arbitrary element of $C_c^\infty(\gamma_1)$, the above relation leads us to conclude that

$$(\partial_\nu U)|_{\gamma_1} = 0. \tag{B.0.7}$$

Combine (B.0.5), (B.0.6), the definition of $\widetilde{g}$, $\operatorname{supp}(q_j) \subseteq \{x \in \overline{\Sigma} : |x'| < R\} \subseteq \Omega$, and (B.0.7) to deduce that

$$(-\Delta - k^2)U = 0 \quad \text{in } \Sigma \setminus \Omega, \tag{B.0.8}$$

$$U|_{\Gamma_1} = 0, \tag{B.0.9}$$

$$(\partial_\nu U)|_{\gamma_1} = 0. \tag{B.0.10}$$

By (B.0.8)–(B.0.10), the function U solves the Helmholtz equation in $\Sigma \setminus \Omega$ and U has zero Cauchy data along $\gamma_1 \setminus l_1 \neq \varnothing$ so, by (the qualitative version of) unique continuation for the Helmholtz equation in $\Sigma \setminus \Omega$, it follows that $U = 0$ in $\Sigma \setminus \Omega$, so

$$U|_{l_3} = 0, \quad (\partial_\nu U)|_{l_3} = 0. \tag{B.0.11}$$

By (B.0.2) and (B.0.4), we have

$$\lim_{k\to\infty} \int_\Omega u_k g \, dx = \|g\|^2_{L^2(\Omega)} > 0,$$

so there exists some $k_0 \in \mathbb{N}$ such that

$$0 < \int_\Omega u_{k_0} g \, dx. \tag{B.0.12}$$

Apply first (B.0.5) and then Green's formula in Ω to get

$$\begin{aligned}\int_\Omega u_{k_0} g\,dx &= \int_\Omega u_{k_0}\left[(-\Delta+q_j-k^2)U\right]\,dx\\ &= \int_\Omega \left[(-\Delta+q_j-k^2)u_{k_0}\right]U\,dx\\ &- \int_{\partial\Omega}(\partial_\nu U)u_{k_0}\,d\sigma+\int_{\partial\Omega}U(\partial_\nu u_{k_0})\,d\sigma \end{aligned} \tag{B.0.13}$$

By $u_{k_0}\in W_j(\Omega)$, we see

$$\int_\Omega \left[(-\Delta+q_j-k^2)u_{k_0}\right]U\,dx=0. \tag{B.0.14}$$

By $u_{k_0}\in W_j(\Omega)$, (B.0.10), (B.0.11), we obtain

$$\int_{\partial\Omega}(\partial_\nu U)u_{k_0}\,d\sigma=0. \tag{B.0.15}$$

By (B.0.6) and (B.0.11), we deduce

$$\int_{\partial\Omega}U(\partial_\nu u_{k_0})\,d\sigma=0. \tag{B.0.16}$$

The combination of (B.0.13)–(B.0.16) implies that

$$\int_\Omega u_{k_0} g\,dx=0,$$

which produces a contradiction with (B.0.12). □

Appendix C
OTHER IMPORTANT RESULTS

The main tool for constructing CGO solutions which we shall be needing was first introduced by Sylvester and Uhlmann in [43]. In each of Sections 2.5, 3.4, and 3.5, we shall be using the following version of Sylvester and Uhlmann's result:

Proposition C.0.3 (from [22]). *Let $Q \subseteq \mathbb{R}^n$ be a bounded open set, and let $q \in L^\infty(Q)$. There exists a constant $C_0 = C_0(Q,n) > 0$ such that, for any $\rho \in C^n$ with*

$$\rho \cdot \rho = 0, |\rho| \geq \max(C_0\|q\|_{L^\infty(Q)}, 1),$$

and for any function $a \in H^2(Q)$ with

$$\rho \cdot \nabla a = 0 \text{ in } Q,$$

there exists a $\psi \in H^2(Q)$ with

$$\|\psi\|_{H^k(Q)} \leq \frac{C_0\|(-\Delta + q)a\|_{L^2(Q)}}{|\rho|^{1-k}}, \quad k = 0, 1, 2,$$

such that $u(x) := e^{\rho\cdot x}(a + \psi)$ belongs to $H^2(Q)$ and solves

$$(-\Delta + q)u = 0 \text{ in } Q.$$

For the analysis in Section 3.5, we shall need a quantified version of the Riemann-Lebesgue lemma, so we state and prove such a quantification below.

Lemma C.0.3 (A quantified Riemann-Lebesgue lemma). *Let $s > n/2 + 1$, $f \in H^s(\mathbb{R}^n) \cap \mathcal{E}'(\mathbb{R}^n)$. If $F \subseteq \mathbb{R}^n$ denotes the support of f, then:*

$$|\hat{f}(\xi)| \lesssim \mathcal{L}^n(F)\frac{\|f\|_{H^s(\mathbb{R}^n)}}{|\xi|}, \quad \xi \in \mathbb{R}^n.$$

The implicit constant depends on the operator norm of the Sobolev embedding $H^s(\mathbb{R}^n) \hookrightarrow C^1_0(\mathbb{R}^n)$. (Here, $C^1_0(\mathbb{R}^n)$ denotes the space of once-differentiable functions which vanish at infinity.)

Remark C.0.4. For this proof, we shall use the definition of the Fourier transform from [21]:

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} f(x)\,dx, \quad f \in L^1(\mathbb{R}^n), \quad \xi \in \mathbb{R}^n.$$

Proof. Fix $f \in H^s(\mathbb{R}^n) \cap \mathcal{E}'(\mathbb{R}^n)$. Since $s > n/2+1$, the Sobolev Embedding theorem implies

$$f \in C^1_c(\mathbb{R}^n), \quad \|f\|_{C^1(\mathbb{R}^n)} \leq C\|f\|_{H^s(\mathbb{R}^n)}.$$

It now follows that $f \in C^1(\mathbb{R}^n)$, all partials of f of order less or equal than 1 are in $L^1(\mathbb{R}^n)$, and $f \in C_0(\mathbb{R}^n)$; by Theorem 8.22(e) in [21], we get

$$(\partial^\alpha f)\widehat{\ } = (2\pi i\xi)^\alpha \widehat{f}(\xi), \quad |\alpha| \leq 1, \quad \xi \in \mathbb{R}^n.$$

Next, we calculate

$$\begin{aligned}
|\xi\widehat{f}(\xi)| &= \left(\sum_{j=1}^n \xi_j^2\right)^{1/2} |\widehat{f}(\xi)| \leq \left(\sum_{j=1}^n |\xi_j|\right)|\widehat{f}(\xi)| \\
&= \sum_{j=1}^n |\xi_j\widehat{f}(\xi)| = \frac{1}{2\pi}\sum_{j=1}^n |(2\pi i\xi)^{e_j}\widehat{f}(\xi)| \\
&= \frac{1}{2\pi}\sum_{j=1}^n |(2\pi i\xi)^{e_j}\widehat{f}(\xi)| = \frac{1}{2\pi}\sum_{j=1}^n |(\partial_j f)\widehat{\ }(\xi)| \\
&\leq \frac{1}{2\pi}\sum_{j=1}^n \|(\partial_j f)\widehat{\ }\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{2\pi}\sum_{j=1}^n \|\partial_j f\|_{L^1(\mathbb{R}^n)} \\
&= \frac{1}{2\pi}\sum_{j=1}^n \int_{\mathbb{R}^n} |\partial_j f(x)|\,dx \leq \frac{1}{2\pi}\left(\sum_{j=1}^n \|\partial_j f\|_{L^\infty(\mathbb{R}^n)}\mathcal{L}^n(F)\right) \\
&\leq \frac{1}{2\pi}\mathcal{L}^n(F)\|f\|_{C^1(\mathbb{R}^n)} \leq \frac{C}{2\pi}\mathcal{L}^n(F)\|f\|_{H^s(\mathbb{R}^n)}.
\end{aligned}$$

□