Representations of Configurations and their Moduli

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A dissertation submitted
in partial fulfillment of the requirements
for the degree of

Doctor of Philosophy

University of Washington

2025

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Program Authorized to Offer Degree:
Mathematics

University of Washington

Abstract

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This thesis develops a scheme-theoretic framework for studying configurations—collections of points and blocks with incidence relations—and their representations in algebraic geometry. Generalizing classical notions, we define configurations as triples of schemes over a base and show that the moduli functor of representations into a geometric configuration is representable by a scheme. We construct fine moduli spaces for nondegenerate representations and realizations, and study their behavior under deformation. Applications include a modern perspective on Mnëv–Sturmfels universality and connections to classical projective theorems, highlighting new interactions between configuration theory, moduli spaces, and algebraic geometry.

Acknowledgments

I am deeply grateful to my advisor, Max Lieblich, for his mathematical insight, creative support, and above all, his patience, especially in letting me make my own (many) mistakes along the way. I thank Jarod Alper for introducing me to the foundations of algebraic geometry (and eventually the theory of stacks) during my first year. I also thank Sándor Kovács for reminding me that algebraic geometry, at its heart, is about geometry.

I'm grateful to my family for their constant support. I'm also thankful to Randy Campbell, Craig Cinderich, and Chris Harris, for showing me the discipline and playfulness it takes to make art.

Most of all, I thank Brittny Casian, my partner and my strongest supporter. We grew up with the odds against us, but we made it. Thanks, Birdy.

Contents

Notation

A **variety** over a scheme S is a geometrically integral, separated, finite-type, and flat S-scheme. For projective n-space $\mathbb{P}^n_S$ over a scheme S, we label the dual projective space as $\check{\mathbb{P}}^n_S$. For standard algebraic geometry terms we use [Stacks].

Chapter 1

Classical Configurations

1.1 Introduction

A **configuration** is a collection of elements we call "points" and "lines" that are related by incidence data. A **representation** of a configuration is a way of labeling points and lines in the plane with the points and lines from the configuration in a way that preserves incidence. By "plane", we mean the projective plane $\mathbb{P}^2_R$ over a ring R, which can be viewed as an extension of the affine plane of tuples (x, y) with $x, y \in R$ for which parallel lines intersect. For example, take the configuration described by Desargues' theorem (see figure 1.1).

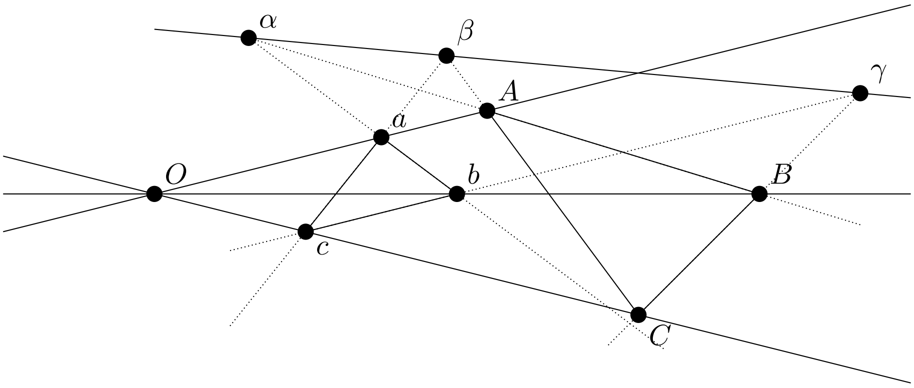

Figure 1.1: (Desargues' Theorem.) Consider two triangles Δabc and ΔABC in the projective plane over a field. If Δabc and ΔABC are perspective via point O (meaning $O \in \overline{aA} \cap \overline{bB} \cap \overline{cC}$), then $\alpha \in \overline{ab} \cap \overline{AB}$, $\beta \in \overline{ac} \cap \overline{AC}$, and $\gamma \in \overline{bc} \cap \overline{BC}$ are collinear (meaning that they all lie on a line).

In general, the arrangement produced by Desargues' theorem describes ten points and ten lines such that each line contains three of these points and each point is contained in three of these lines. This combinatorial data defines a configuration, and the labeled arrangement of ten lines and ten marked points in the plane defines a representation of this configuration.

The study of configurations began by studying redundancies of arrangements on the plane. Desargues' theorem was published in 1648 and is named after the French mathematician Girard Desargues (1591–1661), an early researcher of projective geometry who (independently from Johannes Kepler) developed the concept of a "point at infinity". Desargues' theorem has led to many interesting results in projective geometry. Model-theoretically, showing a projective plane over a ring R satisfies Desargues' theorem is equivalent to showing that R is a division ring (see [Wei07]). Such spaces are called *Desarguesian*. Thus Desargues' theorem can be viewed as a test to determine whether a ring is a division ring.

An analogue to Desargues' theorem is the Cayley-Bacharach theorem (see figure 1.2).

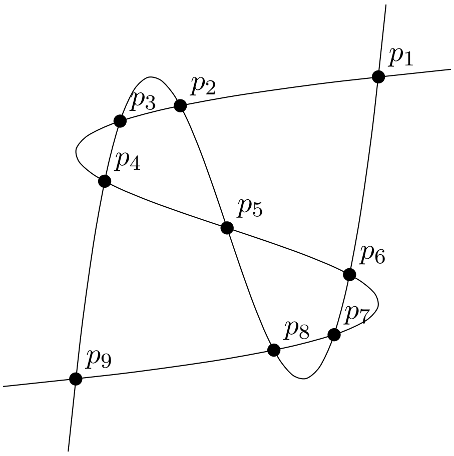

Figure 1.2: (Cayley-Bacharach theorem.) If two distinct cubic curves in $\mathbb{P}^2_k$ over field k intersect at $p_1, \cdots, p_9$ points, then any cubic curve containing $p_1, \cdots, p_8$ also contains p_9.

In the degenerate case when the cubics are unions of lines, the statement becomes Pappus' Hexagonal theorem (as shown in figure 1.3).

Just as Desargues' theorem determines whether a projective plane is defined over a division ring, Pappus' theorem determines whether a projective plane is defined over commutative ring. More generally, if a projective plane over a ring R satisfies the Cayley-Bacharach theorem, then the polynomial ring $R[x_0, x_1, x_2]$ is Gorenstein (see [DGO85]). It is interesting then that Desargues' theorem is not a special case of the Cayley-Bacharach theorem.

The theorems of Desargues, Cayley-Bacharach, and Pappus describe curves in the projective plane with marked singularity points. The first defi-

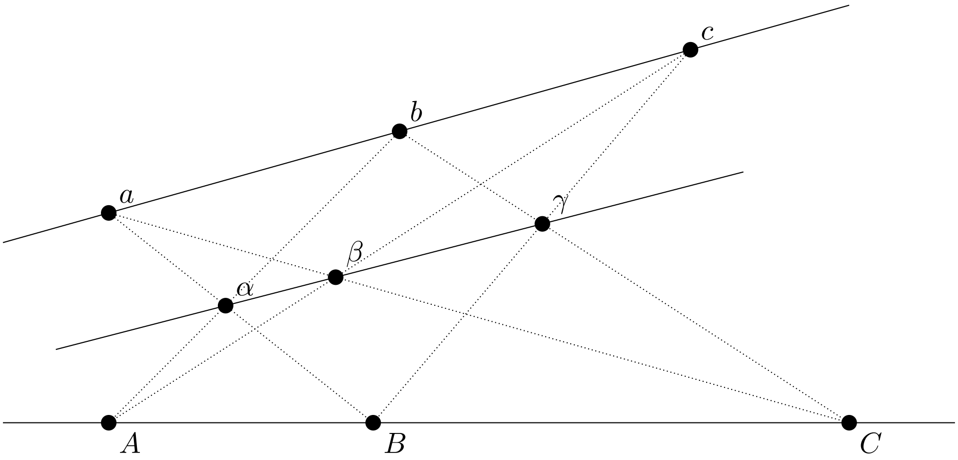

Figure 1.3: (Pappus' Theorem.) Consider two sets of collinear points $\{a, b, c\}$ and $\{A, B, C\}$ in $\mathbb{P}^2_k$ over a field k. Then $\alpha \in \overline{aB} \cap \overline{Ab}$, $\beta \in \overline{aC} \cap \overline{Ac}$, and $\gamma \in \overline{bC} \cap \overline{Bc}$ are collinear.

nition of a configuration was given by Reye in 1876 (albeit with a more rigid structure) in [Rey76].

1.2 Classical Definitions

In this section we review some classical definitions in the theory of configurations.

Definition 1. A **classical configuration** is a tuple $\mathcal{A} = (P, L, \Sigma)$ of finite sets with $\Sigma \subset P \times L$. We say $\mathcal{A}$ is **finite** if P and L are finite sets.

Notation 1. For a classical configuration $\mathcal{A} = (P, L, \Sigma)$, we call the elements of P **points** and the elements of L **lines** or **blocks**. We say p **lies in** l (or l **contains** p) and write "$p \in l$" if $(p, l) \in \Sigma$.

Example 1 (The Triangle)**.** Let $P = \{p_1, p_2, p_3\}$, $L = \{l_1, l_2, l_3\}$, and define $\Sigma \subset P \times L$ by declaring $p_i \in l_j$ for all $i \neq j$. Then the configuration $\mathcal{A} = (P, L, \Sigma)$ is called the **(standard) triangle configuration** (see figure 1.4).

Example 2 (The Complete Quadrangle)**.** The configuration $\mathcal{Q}$ of four points and six lines such that each line contains precisely two points and each point is contained in precisely two lines is called the **complete quadrangle configuration**, see figure 1.5.

Definition 2. A **representation** of $\mathcal{A}$ over a ring R is a set map $f : P \coprod L \to \mathbb{P}^2_R \coprod \check{\mathbb{P}}^2_R$ on R-points such that $f(P) \subset \mathbb{P}^2_R$, $f(L) \subset \check{\mathbb{P}}^2_R$, and

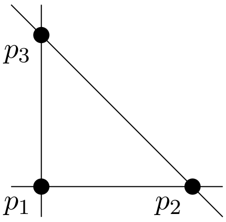

Figure 1.4: Three lines forming a triangle with vertices p_1, p_2, p_3.

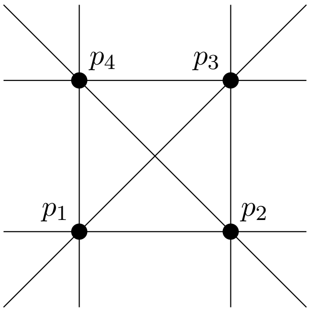

Figure 1.5: The **complete quadrangle**, containing four points and six lines.

$f(p) \in f(l)$ if $p \in l$. We say f is **nondegenerate** if f is an injective function. A nondegenerate representation f is a **realization** if in addition, $f(p) \in f(l)$ implies $p \in l$.

Notation 2. We write a representation as $f : \mathcal{A} \to \mathbb{P}^2_R$ for simplicity.

Example 3. The depiction of the Desargues configuration in figure 1.1 describes a nondegenerate representation. If the output line $\overline{\alpha\beta\gamma}$ contains O, then the representation is not a realization.

The history of representations of configurations of rich and outlined in [Gro94] and [Grü09], among other sources. We can also view classical configurations as bipartite graphs and their representations as morphisms of bipartite graphs, which is detailed in [PS10]. Lastly, there have been a tremendous amount of results using matroid theory, see [GMW22] for example.

1.3 Moduli of Representations and Classical Geometry

The following definition is a fundamental tool in the theory of classical representations. Let $\mathbb{P}^2_R$ have coordinates X, Y, Z and $\check{\mathbb{P}}^2_R$ have coordinates a, b, c. For nonnegative integers m and n, not both zero, we identify $(\mathbb{P}^2_R)^m \times_R (\check{\mathbb{P}}^2_R)^n$ with it's image in $\mathbb{P}^{3m+3n-1}_R$ under the Segre embedding.

Definition 3. Let $\mathcal{A}$ be a finite classical configuration with m points and n lines. We define the **moduli of representations** of $\mathcal{A}$ over R to be the closed subscheme of $\mathbb{P}_R^{3m+3n-1}$ given by the union of the zero sets

$$a_j X_i + b_j Y_i + c_j Z_i = 0,$$

whenever $p_i \in l_j$ in $\mathcal{A}$. We label this closed subscheme as $\mathrm{Hom}(\mathcal{A}, \mathbb{P}_R^2)$.

We will show in theorem 2.3.2 that $\mathrm{Hom}(\mathcal{A}, \mathbb{P}_R^2)$ is a fine moduli space parameterizing representations $f : \mathcal{A} \to \mathbb{P}_R^2$. Similarly, we will show in theorem 2.4.3 that there exists open subschemes $\mathrm{Real}_{\mathcal{A}}$ and $\mathrm{Re}_{\mathcal{A}}$ of $\mathrm{Hom}(\mathcal{A}, \mathbb{P}_R^2)$ parameterizing realization and nondegenerate representations of $\mathcal{A}$, respectively, such that the inclusions

$$\mathrm{Real}_{\mathcal{A}} \subset \mathrm{Re}_{\mathcal{A}} \subset \mathrm{Hom}(\mathcal{A}, \mathbb{P}_R^2)$$

are open immersions.

The moduli space of (nondegenerate) representations has been used to study various phenomena outside of linear geometry. For example, the moduli of representations can be used to determine which finitely-presented groups are fundamental groups of smooth complex algebraic varieties (see [KM98]). When $R = \mathbb{R}$, Kapovich and Millson in [KM02] show a relationship between linkages and representations of configurations.

A recent applications of design theory in algebraic geometry is Mnëv's theorem, which roughly states that every singularity type of a finite-type scheme can be realized in a representation space $\mathrm{Re}_{\mathcal{A}}$. Let $\mathcal{Q}$ denote the complete quadrangle configuration over a field k (see figure 1.5). Let $f_{std} : \mathcal{Q} \to \mathbb{P}^2$ the labeled arrangement in $\mathbb{P}_k^2$ representing $\mathcal{Q}$ by the assignment $p_i \mapsto e_i$ for each $i = 1, \cdots, 4$ (here $e_1, \cdots, e_4$ denotes the standard projective basis in $\mathbb{P}_k^2$).

We study the intrinsic geometry of $\mathcal{A}$ in $\mathbb{P}^2$ by studying the action of PGL_3 on $\mathrm{Re}_{\mathcal{A}}$, specifically by keeping track of a subcollection of points and lines in $\mathcal{A}$. When $\mathcal{Q}$ is a subconfiguration of $\mathcal{A}$ (meaning that the components of $\mathcal{Q}$ are subsets of the components of $\mathcal{A}$), there is a natural map $\mathrm{Re}_{\mathcal{A}} \to \mathrm{Re}_{\mathcal{Q}}$ by simply forgetting the other labelings outside of $\mathcal{Q}$. Define $BR_{\mathcal{A}} \subset \mathrm{Re}_{\mathcal{A}}$ as the fiber of $\mathrm{Re}_{\mathcal{A}} \to \mathrm{Re}_{\mathcal{Q}}$ over the standard complete quadrangle $f_{std} \in \mathrm{Re}_{\mathcal{Q}}$. It can be shown that $BR_{\mathcal{A}} = \mathrm{Re}_{\mathcal{A}} / PGL_3$ as a geometric quotient for sufficiently nice $\mathcal{A}$. In [LV12], both authors prove a version of Mnëv's theorem: every singularity type appearing on a finite-type scheme over $\mathrm{Spec}\,(\mathbb{Z})$ appears in a closed subscheme $BR_{\mathcal{A}}$ of $\mathrm{Re}_{\mathcal{A}}$ for some classical configuration $\mathcal{A}$. This is used to prove Murphy's law for singularities of schemes in [Vak06]. This is discussed further in section 5.2.

1.4 Existence and Variation of Representations

There are limitations to this theory. It would be interesting to generalize from points and lines in $\mathbb{P}^2$ to more exotic spaces, perhaps replacing lines with conics, or replacing the projective plane with another variety. Moreover, allowing for points and lines to collide in a configuration allows for a useful framework to study deformations of configurations. These concerns can be solved using the language of schemes.

Much of the study of configurations and their representations has been focused on answering the following question.

Question 1.4.1. *Given a configuration, does there exist a (nondegenerate) representation to a geometric incidence space? If so, how do these representations vary?*

I have attempted to answer question 1.4.1 using the language of schemes (definition 4). The question of existence in the projective plane has been successfully studied and outlined in [GMW22] using the language of matroids. Historically, the term "geometric incidence space" has been taken to mean the projective plane with the block-structure being lines (this is labeled as $\mathbb{P}^2_1$ in definition 5). The definition of a geometric configuration (definition 22) is an attempt to study representations of configurations in other spaces. The last question has deep roots in moduli theory, a theory in which one attempts to naturally parameterize a geometric phenomenon by a geometric space. Theorem 2.3.2 shows that there exists a scheme parameterizing representations of a finite configuration in a geometric configuration.

The goal of this thesis is to illustrate the beautiful and historically rich theory of configurations and their representations via modern methods. Chapter 2 is a generalization of the classical theory of configurations using the language of schemes. Chapter 3 is a list of examples and properties of representations and their moduli. Chapter 4 is the study of deformations of these configurations. Chapter 5 is a revamp of topics through the lens of this new theory, along with a list of questions I have considered in regards to representations.

Chapter 2

Configurations

In this chapter we formally define configurations $\mathcal{A}$ over a scheme S, then study (nondegenerate) representations and realizations of representations $f : \mathcal{A} \to \mathcal{X}$. We show the moduli of such representations $\mathrm{Hom}(\mathcal{A}, \mathcal{X})$ exists as an S-scheme for sufficiently nice $\mathcal{A}$ and $\mathcal{X}$. We then show the nondegenerate representations and realization of $\mathcal{A}$ in $\mathcal{X}$ admit fine moduli spaces, labeled $\mathrm{Re}_{\mathcal{A}}^{\mathcal{X}}$ and $\mathrm{Real}_{\mathcal{A}}^{\mathcal{X}}$. Lastly, we show the inclusion morphisms

$$\mathrm{Real}_{\mathcal{A}}^{\mathcal{X}} \subset \mathrm{Re}_{\mathcal{A}}^{\mathcal{X}} \subset \mathrm{Hom}(\mathcal{A}, \mathcal{X})$$

are each open immmersions of S-schemes.

2.1 Basic Definitions

Definition 4. A **configuration** over a scheme S, or an S**-configuration**, is a triple $\mathcal{A} = (P, L, \Sigma)$ of S-schemes, where $\Sigma \subset P \times_S L$ is a closed subscheme. We refer to $P \to S$ as the space of **points**, $L \to S$ the space of **blocks** or **lines**, and $\Sigma \subset P \times_S L$ as the **incidence correspondence**.

Example 4 (Split Configurations)**.** Let $\mathcal{A}_0 = (P_0, L_0, \Sigma_0)$ be a classical configuration. Then for any scheme S, define

$$P = \coprod_{p \in P_0} S, \qquad L = \coprod_{l \in L_0} S, \qquad \Sigma = \coprod_{(p,l) \in \Sigma_0} S,$$

where $\Sigma \subset P \times_S L$ in the obvious way. This defines a configuration (P, L, Σ) over S, which we call the **split** S**-configuration associated** to $\mathcal{A}_0$. In general, a configuration of this form is called a **split** configuration. Lastly, a configuration associated to a finite classical configuration is called a **finite split configuration**.

Example 5 (The Configuration $\mathbb{P}^n_d$)**.** Consider the S-configuration

$$\mathbb{P}^n_{S,d} = (\mathbb{P}^n_S, |\mathcal{O}_{\mathbb{P}^2_S}(d)|, \Lambda),$$

where $|\mathcal{O}_{\mathbb{P}^2_S}(d)| = \mathbb{P}(f_*\mathcal{O}_{\mathbb{P}^n_S}(d))$ for structure morphism $f : \mathbb{P}^n_S \to S$ denotes the complete linear system of degree d hypersurfaces, and Λ is the restriction of the universal family $\Lambda_0 \subset X \times_S \mathrm{Hilb}_{X/S}$ to $X \times_S |\mathcal{O}_{\mathbb{P}^2_S}(d)|$. We write $\mathbb{P}^n_d$ when the base scheme is clear.

Definition 5. A morphism $\mathcal{A} \to \mathcal{A}'$ of configurations $\mathcal{A} = (P, L, \Sigma)$ over S and $\mathcal{A}' = (P', L', \Sigma')$ over S' is the data of the following commuting diagram of schemes

$$\begin{tikzcd}
\Sigma' \arrow[rr] \arrow[dr, "\gamma"] \arrow[dd] & & L' \arrow[dr, "\beta"] \arrow[dd] & \\
& \Sigma \arrow[rr] \arrow[dd] & & L \arrow[dd] \\
P' \arrow[rr] \arrow[dr, "\alpha"] & & S' \arrow[dr, "\psi"] & \\
& P \arrow[rr] & & S,
\end{tikzcd}$$

where the unlabeled morphisms from Σ and Σ' are the projection morphisms of $P \times_S L$ and $P' \times_{S'} L'$, respectively.

2.2 The Category of Configurations

Definition 6. The collection of configurations and morphisms of configurations form a category, which we label as Config.

Definition 7. Given a diagram of configurations $\{\varphi_i : \mathcal{A}_i \to \mathcal{A}'_i\}_{i\in I}$, we call $\{\alpha_i : P_i \to P'_i\}_{i\in I}$ the **point structure**, $\{\beta_i : L_i \to L'_i\}_{i\in I}$ the **block structure**, $\{\gamma_i : \Sigma_i \to \Sigma'_i\}_{i\in I}$ the **incidence structure**, and $\{\psi_i : S_i \to S'_i\}_{i\in I}$ the **base-scheme structure** of $\{\varphi_i\}_{i\in I}$.

Definition 8. Let $f : S' \to S$ be a morphism of schemes and $\mathcal{A} = (P, L, \Sigma)$ a configuration over S. Consider the S'-configuration $\mathcal{A}_{S'} = (P_{S'}, L_{S'}, \Sigma_{S'})$. The **pullback** of $\mathcal{A}$ by ψ is the morphism $\phi : \mathcal{A}_{S'} \to \mathcal{A}$ given by the canonical S-morphisms

$$P_{S'} \to P, \qquad L_{S'} \to L, \qquad \Sigma_{S'} \to \Sigma,$$

each defined by the standard pullbacks of schemes $\psi : S' \to S$.

Proposition 2.2.1. *Consider the forgetful functor* $\pi : \mathrm{Config} \to \mathrm{Sch}$ *given by assigning an S-configuration to it's base scheme S, and assigning a morphism of configurations $\mathcal{A}' \to \mathcal{A}$ to the morphism on base schemes $\psi : S' \to S$. The functor* $\pi : \mathrm{Config} \to \mathrm{Sch}$ *is a fibered category.*

Proof. Let $\psi : S' \to S$ be a morphism of schemes, $\mathcal{A}$ a configuration over S, and $\phi : \mathcal{A}_{S'} \to \mathcal{A}$ the pullback of $\mathcal{A}$ by ψ. For notational purposes, let $\mathcal{A}' =$

(P', L', Σ') denote $\mathcal{A}_{S'}$. We show that ϕ is cartesian in Config. Let $\mathcal{A}''$ be a configuration over S'' with a morphism $F : \mathcal{A}'' \to \mathcal{A}$ of configurations, then the above data can be represented by the following commutative diagrams:

$$\begin{tikzcd}
\Sigma' \arrow[rr] \arrow[dr, "\gamma"] \arrow[dd] & & L' \arrow[dr, "\beta"] \arrow[dd] & \\
& \Sigma \arrow[rr] \arrow[dd] & & L \arrow[dd] \\
P' \arrow[rr] \arrow[dr, "\alpha"] & & S' \arrow[dr, "\psi"] & \\
& P \arrow[rr] & & S
\end{tikzcd}
\qquad
\begin{tikzcd}
\Sigma'' \arrow[rr] \arrow[dr, "c"] \arrow[dd] & & L'' \arrow[dr, "b"] \arrow[dd] & \\
& \Sigma \arrow[rr] \arrow[dd] & & L \arrow[dd] \\
P'' \arrow[rr] \arrow[dr, "a"] & & S'' \arrow[dr, "f"] & \\
& P \arrow[rr] & & S.
\end{tikzcd}$$

Given a factorization of $f = \pi(F)$:

$$S'' \xrightarrow{f'} S' \xrightarrow{\psi} S,$$

we construct a morphism $F' : \mathcal{A}'' \to \mathcal{A}'$ over f' that uniquely factors F:

$$\begin{tikzcd}
\mathcal{A}'' \arrow[r, dashed, "\exists! F'"] & \mathcal{A}' \arrow[r, "\phi"] & \mathcal{A}.
\end{tikzcd}$$

A quick analysis of the point-structure of $\mathcal{A}' \to \mathcal{A} \leftarrow \mathcal{A}''$ yields a factorization of $a : P'' \to P$,

$$\begin{tikzcd}
P'' \arrow[r, dashed, "\exists! a'"] \arrow[d] & P' \arrow[r, "\alpha"] \arrow[d] \arrow[dr, phantom, "\square"] & P \arrow[d] \\
S'' \arrow[r, "f'"] & S' \arrow[r, "\psi"] & S,
\end{tikzcd}$$

where a' is given by the universal property of the right cartesian diagram. A similar analysis of the block and incidence structures yields unique factorizations $b' : L'' \to L'$ and $c' : \Sigma'' \to \Sigma'$ of b and c, respectively. An analysis of the following diagram shows that c is induced by $a' \times b'$:

$$\begin{tikzcd}
\Sigma'' \arrow[r, dashed, "\exists! c"] \arrow[d, hook] & \Sigma' \arrow[r, "\gamma"] \arrow[d, hook] \arrow[dr, phantom, "\square"] & \Sigma \arrow[d, hook] \\
P'' \times_{S''} L'' \arrow[r, "a' \times b'"] & P' \times_{S'} L' \arrow[r, "\alpha \times \beta"] & P \times_S L,
\end{tikzcd}$$

where the downward arrows are the inclusion morphisms.

□

For the remainder of the section, we embed Sch in Config fully-faithfully and show a useful result about fiber products in Config.

Lemma 2.2.2. *Consider the functor $\tau : \mathrm{Sch} \to \mathrm{Config}$ assigning a scheme T to the T-configuration (T, T, T), where $T \subset T \times_T T = T$ is equality, and a scheme morphism $f : T' \to T$ to $(T', T', T') \to (T, T, T)$, where each component morphism is f. Consider the forgetful functor $i : \mathrm{Config} \to \mathrm{Sch}$ assigning a configuration $\mathcal{A}$ to it's incidence structure Σ. Then $i \circ \tau = 1_{\mathrm{Sch}}$ as functors. Moreover, the functor-pair (τ, i) is adjoint.*

Proof. The first claim is clear. For the second claim, fix a scheme T and S-configuration $\mathcal{A} = (P, L, \Sigma)$. A morphism $\tau(T) \to \mathcal{A}$ is precisely the data of a commuting diagram

$$\begin{tikzcd}
T \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \Sigma \arrow[r] \arrow[d] & L \arrow[d] \\
& P \arrow[r] & S.
\end{tikzcd}$$

As a result, we see that

$$\mathrm{Hom}_{\mathrm{Config}}(\tau(T), \mathcal{A}) = \mathrm{Hom}_{Sch}(T, \Sigma) = \mathrm{Hom}_{Sch}(T, i(\mathcal{A})),$$

as desired.

□

Lemma 2.2.3. *The functor $\tau : \mathrm{Sch} \to \mathrm{Config}$ is fully-faithful and a section of π: i.e., $\pi \circ \tau = 1_{\mathrm{Sch}}$ as functors.*

Proof. The second statement is clear. To prove the first statement, let T and T' be schemes. Apply lemma 2.2.2 by setting $\mathcal{A} = \tau(T')$, then

$$\mathrm{Hom}_{\mathrm{Config}}(\tau(T), \tau(T')) = \mathrm{Hom}_{\mathrm{Sch}}(T, i \circ \tau(T')) = \mathrm{Hom}_{\mathrm{Sch}}(T, T'),$$

as desired. □

Notation 3. Because of lemma 2.2.3, we denote the S-configuration $\tau(S) = (S, S, S)$ simply as S.

Definition 9. Let $\mathcal{A}$ be a configuration over a scheme S. There exists a morphism of S-configurations $\mathcal{A} \to S$ given by the structure morphisms of P, L, and Σ as S-schemes, which we call the **structure morphism** of $\mathcal{A}$.

Proposition 2.2.4. *Let $\mathcal{A} = (P, L, \Sigma)$ be an S-configuration and $\mathcal{A}' = (P', L', \Sigma')$ be an S'-configuration. A morphism $\varphi = (a, b, c) : \mathcal{A} \to \mathcal{A}'$ induces a commutative diagram of configurations*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\varphi"] \arrow[d] & \mathcal{A}' \arrow[d] \\
S \arrow[r, "\pi(\varphi)"] & S',
\end{tikzcd} \tag{2.2.1}$$

where the downward arrows are the structure morphisms. This diagram is cartesian in Config *if and only if the point, block, and incidence structure form cartesian diagrams in* Sch*. That is, the following induced diagrams are cartesian:*

$$\begin{tikzcd}
P \arrow[r, "a"] \arrow[d] & P' \arrow[d] & L \arrow[r, "b"] \arrow[d] & L' \arrow[d] & \Sigma \arrow[r, "c"] \arrow[d] & \Sigma' \arrow[d] \\
S \arrow[r] & S', & S \arrow[r] & S', & S \arrow[r] & S'.
\end{tikzcd} \tag{2.2.2}$$

Proof. ($\Rightarrow$) Assume that φ induces a cartesian diagram of configurations. On the point structure, suppose there exists a scheme T fitting into the diagram

$$\begin{tikzcd}
T \arrow[rrd, bend left] \arrow[rdd, bend right] & & \\
& P \arrow[r] \arrow[d] & P' \arrow[d] \\
& S \arrow[r] & S'.
\end{tikzcd} \tag{2.2.3}$$

Define a configuration $(T, \emptyset, \emptyset)$ over T. This diagram induces a commutative diagram of configurations

$$\begin{tikzcd}
(T, \emptyset, \emptyset) \arrow[rrd, bend left] \arrow[rdd, bend right] & & \\
& \mathcal{A} \arrow[r] \arrow[d] & \mathcal{A}' \arrow[d] \\
& S \arrow[r] & S'.
\end{tikzcd}$$

By assumption, this induces a unique morphism of configurations $(T, \emptyset, \emptyset) \to \mathcal{A}$ filling in the above diagram. Thus there exists a morphism of schemes $T \to P$ filling diagram 2.2.3. Moreover, this morphism is unique as

$$\mathrm{Hom}_{\mathrm{Config}}((T, \emptyset, \emptyset), \mathcal{A}) = \mathrm{Hom}_{\mathrm{Sch}}(T, P)$$

as sets.

By a similar argument, one can show that the block structure of diagram 2.2.4 is cartesian, namely by choosing a T-configuration $(\emptyset, T, \emptyset)$. Again, use the same argument for the incidence structure and choose the T-configuration T (see notation 3). In this case we use that $\mathrm{Hom}_{\mathrm{Config}}(T, \mathcal{A}) = \mathrm{Hom}_{\mathrm{Sch}}(T, \Sigma)$ as sets.

($\Leftarrow$) Suppose that the collection of diagrams in 2.2.2 are cartesian. Let $\mathcal{A}''$ be an S''-configuration fitting into the following diagram

$$\begin{tikzcd}
\mathcal{A}'' \arrow[rrd, bend left] \arrow[rdd, bend right] & & \\
& \mathcal{A} \arrow[r] \arrow[d] & \mathcal{A}' \arrow[d] \\
& S \arrow[r] & S'.
\end{tikzcd} \tag{2.2.4}$$

Given our assumption, there exists morphisms $a' : P'' \to P$ and $b' : L'' \to L$. We then have a commuting diagram

$$\begin{tikzcd}
\Sigma'' \arrow[r, dashed, "\exists! c'"] \arrow[d, hook] \arrow[rr, bend left] & \Sigma \arrow[r] \arrow[d, hook] & \Sigma' \arrow[d, hook] \\
P'' \times_{S''} L'' \arrow[r, "a' \times b'"] \arrow[d] & P \times_S L \arrow[r] \arrow[d] & P' \times_{S'} L' \arrow[d] \\
S'' \arrow[r] & S \arrow[r] & S',
\end{tikzcd}$$

where $c' : \Sigma'' \to \Sigma$ is due to the outer right rectangle being cartesian. We then define a morphism of configurations $(a', b', c') : \mathcal{A}'' \to \mathcal{A}$ filling in diagram 2.2.4. Uniqueness is given by the uniqueness of a' and b' (which induce c').

$\square$

2.3 Representations of Configurations

Definition 10. A morphism $f : \mathcal{A} \to \mathcal{A}'$ in $\mathrm{Config}(S)$ is called a **representation** of $\mathcal{A}$ in $\mathcal{A}'$.

Example 6. Let $\mathcal{A}$ be a split configuration over a scheme S induced by $\mathcal{A}_0$. Then a representation $f : \mathcal{A} \to \mathbb{P}^2_1$ is a generalization of definition 2.

Definition 11. Let $\mathcal{A}$ and $\mathcal{A}'$ be configurations over a scheme S. The assignment $\mathrm{Hom}(\mathcal{A}, \mathcal{A}') : (\mathrm{Sch}/S)^{op} \to \mathrm{Sets}$ given by

$$\mathrm{Hom}(\mathcal{A}, \mathcal{A}')(T) := \{\text{Representations } \mathcal{A}_T \to \mathcal{A}'_T \text{ over } T\},$$

is functorial, which we call the **functor of representations** of $\mathcal{A}$ in $\mathcal{A}'$.

Example 7. Let $\mathcal{A}$ be a finite split configuration over $S = \operatorname{Spec} R$, then $\mathrm{Hom}(\mathcal{A}, \mathbb{P}^2_1)$ is represented by the scheme described in definition 3.

We are interested in properties for which the functor $\operatorname{Hom}(\mathcal{A}, \mathcal{X})$ is representable by a scheme.

Definition 12. A configuration $\mathcal{A} = (P, L, \Sigma)$ over S is **finite** if P, L, and Σ are finite locally free over S.

Example 8 (Split Finite Configurations)**.** Let $\mathcal{A}_0$ be a finite classical configuration and $\mathcal{A}$ the split S-configuration generated by $\mathcal{A}_0$. Then $\mathcal{A}$ is a finite S-configuration. Thus a finite split configuration (definition 4) is exactly a finite configuration that is split.

Example 9 (Degeneration of a Triangle)**.** We construct a flat family of configurations over the affine line $\mathbb{A}^1_k = \operatorname{Spec} k[t]$ for field k of characteristic not 2, whose generic fiber is a triangle and whose special fiber degenerates in a controlled way. Let

$$P = \operatorname{Spec} k[t, x, y]/(x(x - t), y(y - t), xy),$$

and $L = \mathbb{A}^1 \coprod \mathbb{A}^1 \coprod \mathbb{A}^1$. Since $P \times_{\mathbb{A}^1} L = P \coprod P \coprod P$, we define closed subschemes $\Sigma_1, \Sigma_2, \Sigma_3 \subset P$ with $\Sigma = \Sigma_1 \coprod \Sigma_2 \coprod \Sigma_3$, where

$$\begin{aligned}
\Sigma_1 &= V(x) = \operatorname{Spec} k[t, y]/y(y - t),\\
\Sigma_2 &= V(y) = \operatorname{Spec} k[t, x]/x(x - t),\\
\Sigma_3 &= V(x + y - t) = \operatorname{Spec} k[t, x]/x(x - t).
\end{aligned}$$

The above defines a finite $\mathbb{A}^1$-configuration $\mathcal{A} = (P, L, \Sigma)$. When $t = a$ is nonzero, the configuration $\mathcal{A}_a$ is split over k, representing the standard triangle configuration. When $t = 0$, the points degenerate to

$$P_0 = \operatorname{Spec} k[x, y]/(x, y)^2,$$

the lines $L_0 = \coprod_3 \operatorname{Spec} k$ remain split, and the incidence structure degenerates to $\Sigma_0 = \coprod_3 \operatorname{Spec} k[\epsilon]$, where $k[\epsilon]$ are the dual numbers.

Lemma 2.3.1. *Let $\mathcal{A}$ be a finite configuration over S. Given a morphism of schemes $S' \to S$, then $\mathcal{A}_{S'}$ is a finite configuration over S'.*

Proof. The finite condition on $\mathcal{A}_{S'} \to S'$ is an immediate application of lemma 2.2.4 and the property that the finite locally free condition is stable under base-change.

□

Definition 13. We say that an S-configuration $\mathcal{X} = (X, H, \Lambda)$ has **quasi-projective components** if the point, line, and incidence schemes are quasi-projective over S. We define $\mathcal{A}$ having **projective components** similarly.

Remark 1. The configuration $\mathbb{P}^2_d$ over a field k has projective components. Moreover, for any projective variety X over k and closed subscheme $H \subset \mathrm{Hilb}_{X/k}$, the configuration $\mathcal{X} = (X, H, \Lambda)$, where $\Lambda \subset X \times_k H$ is the universal incidence scheme, has projective components. This is where the notation for $\mathcal{X}$ comes from. We will formally define this as a geometric configuration in definition 22.

Theorem 2.3.2. *Let $\mathcal{A}$ be a finite configuration and $\mathcal{X}$ is a configuration with quasi-projective components, both over a scheme S. Then the functor of representations* $\mathrm{Hom}(\mathcal{A}, \mathcal{X})$ *is representable by a scheme. If in addition S is noetherian, then* $\mathrm{Hom}(\mathcal{A}, \mathcal{X})$ *is represented by a quasi-projective scheme over S.*

Proof. It is not hard to show that $\mathrm{Hom}(\mathcal{A}, \mathcal{X})$ fits into the following cartesian diagram of functors

$$\begin{tikzcd}
\mathrm{Hom}(\mathcal{A}, \mathcal{X}) \arrow[r] \arrow[d] & \mathrm{Hom}_S(\Sigma, \Lambda) \arrow[d] \\
\mathrm{Hom}_S(P, X) \times_S \mathrm{Hom}_S(L, H) \arrow[r] & \mathrm{Hom}_S(\Sigma, X \times_S H).
\end{tikzcd} \tag{2.3.1}$$

By lemma A.1.2, we have that $\mathrm{Hom}(\mathcal{A}, \mathcal{X})$ is the fiber product of schemes, and is thus representable by an S-scheme.

For the second statement, if S is noetherian, then by lemma A.1.2, $\mathrm{Hom}(\mathcal{A}, \mathcal{X})$ is the fiber product of quasi-projective S-schemes and is thus quasi-projective over S. ☐

For the remainder of the section we show that the morphism

$$\mathrm{Hom}(\mathcal{A}, \mathcal{X}) \to \mathrm{Hom}_S(P, X) \times_S \mathrm{Hom}_S(L, H)$$

in diagram 2.3.1 is a closed immersion of schemes.

Definition 14. For a configuration $\mathcal{A} = (P, L, \Sigma)$, the **free configuration** $F(\mathcal{A})$ associated to $\mathcal{A}$ is the configuration $F(\mathcal{A}) = (P, L, \emptyset)$. In general, a configuration is **free** if it is of this form.

Remark 2. With this definition, it is easy to see that

$$\mathrm{Hom}(F(\mathcal{A}), \mathcal{X}) = \mathrm{Hom}(P, X) \times_S \mathrm{Hom}(L, H).$$

Thus we can rewrite figure 2.3.1 as the following cartesian diagram

$$\begin{tikzcd}
\mathrm{Hom}(\mathcal{A}, \mathcal{X}) \arrow[r] \arrow[d] & \mathrm{Hom}_S(\Sigma, \Lambda) \arrow[d] \\
\mathrm{Hom}(F(\mathcal{A}), \mathcal{X}) \arrow[r] & \mathrm{Hom}_S(\Sigma, X \times_S H).
\end{tikzcd}$$

Proposition 2.3.3. *The forgetful map* $\mathrm{Hom}_S(\mathcal{A},\mathcal{X}) \to \mathrm{Hom}_S(F(\mathcal{A}),\mathcal{X})$ *is a closed immersion.*

Proof. By figure 2.3.1 and base-change, it suffices to show the map

$$\mathrm{Hom}_S(\Sigma,\Lambda) \to \mathrm{Hom}_S(\Sigma, X \times_S H)$$

is a closed immersion. Since Σ is finite locally free over S and $\Lambda \subset X \times_S H$ is a closed subscheme, then this is a consequence of [BLR90, Proposition 7.6.2].

□

2.4 Nondegenerate Representations and Realizations

Throughout we assume $\mathcal{A} = (P, L, \Sigma)$ and $\mathcal{X} = (X, H, \Lambda)$ are configurations over a scheme S. By $\mathrm{Hom}^{cl}(P, X)$, we mean the subfunctor of $\mathrm{Hom}(P, L)$ of closed immersions (see appendix A.2).

Definition 15. A representation $f : \mathcal{A} = (P, L, \Sigma) \to \mathcal{A}' = (P', L', \Sigma')$ over S is **nondegenerate** if the component morphisms are closed immersions. In addition, we say f is a **realization** if the following diagram induced by f is cartesian:

$$\begin{tikzcd}
\Sigma \arrow[r, "c"] \arrow[d, hook] & \Sigma' \arrow[d, hook] \\
P \times_S L \arrow[r, "a\times b"] & P' \times_S L'.
\end{tikzcd}$$

Remark 3. The above notions are functorial, as both closed immersions and cartesian diagrams are stable under base change. This means that given a morphism of schemes $T \to S$, then if $f : \mathcal{A} \to \mathcal{A}'$ is a nondegenerate (resp. realization) morphism over S, then so is $f_T : \mathcal{A}_T \to \mathcal{A}'_T$.

Definition 16. Define the following functors $\mathrm{Real}^{\mathcal{X}}_{\mathcal{A}}, \mathrm{Re}^{\mathcal{X}}_{\mathcal{A}} : (\mathrm{Sch}/S)^{op} \to$ Sets given by

$$\mathrm{Re}^{\mathcal{X}}_{\mathcal{A}}(T) = \{\text{Nondegenerate representations } f : \mathcal{A}_T \to \mathcal{X}_T\},$$

$$\mathrm{Real}^{\mathcal{X}}_{\mathcal{A}}(T) = \{\text{Realizations } f : \mathcal{A}_T \to \mathcal{X}_T\}.$$

The goal of the remainder of the section is to prove a representability result about $\mathrm{Real}^{\mathcal{X}}_{\mathcal{A}}$ and $\mathrm{Re}^{\mathcal{X}}_{\mathcal{A}}$ and how they relate to $\mathrm{Hom}(\mathcal{A},\mathcal{X})$.

Lemma 2.4.1. *We have the following cartesian diagram of functors,*

$$\begin{tikzcd}
\mathrm{Re}^{\mathcal{X}}_{\mathcal{A}} \arrow[r] \arrow[d] & \mathrm{Hom}^{cl}(\Sigma, \Lambda) \arrow[d] \\
\mathrm{Re}^{\mathcal{X}}_{F(\mathcal{A})} \arrow[r] & \mathrm{Hom}^{cl}(\Sigma, P \times_S L).
\end{tikzcd}$$

Proof. It is not hard to see that

$$\mathrm{Re}^{\mathcal{X}}_{F(\mathcal{A})} = \mathrm{Hom}^{cl}(P, X) \times_S \mathrm{Hom}^{cl}(L, H).$$

The cartesian statement is by definition.

$\square$

Lemma 2.4.2. *The following diagram of forgetful functors is a commuting cube with cartesian sides (meaning all sides except the top and bottom are cartesian),*

$$\begin{tikzcd}
\mathrm{Re}^{\mathcal{X}}_{\mathcal{A}} \arrow[rr] \arrow[dr, "c"] \arrow[dd] & & \mathrm{Hom}^{cl}(\Sigma, \Lambda) \arrow[dr, "b"] \arrow[dd] & \\
& \mathrm{Hom}(\mathcal{A}, \mathcal{X}) \arrow[rr] \arrow[dd] & & \mathrm{Hom}(\Sigma, \Lambda) \arrow[dd] \\
\mathrm{Re}^{\mathcal{X}}_{F(\mathcal{A})} \arrow[rr] \arrow[dr, "a"] & & \mathrm{Hom}^{cl}(\Sigma, X \times_S H) \arrow[dr, "\psi"] & \\
& \mathrm{Hom}(F(\mathcal{A}), \mathcal{X}) \arrow[rr] & & \mathrm{Hom}(\Sigma, X \times_S H).
\end{tikzcd}$$

Proof. The front and back sides are cartesian (see remark 2). Consider the following diagram,

$$\begin{tikzcd}
\mathrm{Hom}^{cl}(\Sigma, \Lambda) \arrow[r] \arrow[d] & \mathrm{Hom}(\Sigma, \Lambda) \arrow[d] \\
\mathrm{Hom}^{cl}(\Sigma, X \times_S H) \arrow[r] & \mathrm{Hom}(\Sigma, X \times_S H).
\end{tikzcd}$$

This diagram is cartesian since a factoring c of closed immersions i, j,

$$\begin{tikzcd}
& \Lambda \arrow[d, hook, "i"] \\
\Sigma \arrow[ur, dashed, "c"] \arrow[r, hook, "j"'] & X \times_S H,
\end{tikzcd}$$

implies c is a closed immersion [Stacks, Tag 01QP].

Consider the following diagram

$$\begin{tikzcd}
\mathrm{Re}^{\mathcal{X}}_{\mathcal{A}} \arrow[r] \arrow[d] & \mathrm{Hom}(\mathcal{A}, \mathcal{X}) \arrow[d] \\
\mathrm{Re}^{\mathcal{X}}_{F(\mathcal{A})} \arrow[r] & \mathrm{Hom}(F(\mathcal{A}), X \times_S H).
\end{tikzcd}$$

This too is cartesian by a similar argument, as closed immersions $a : P \to X$ and $b : L \to H$ and a factorization

$$\begin{tikzcd}
& & \Lambda \arrow[d, hook] \\
\Sigma \arrow[r, hook] \arrow[urr, dashed, "c"] & P \times_S L \arrow[r, "a \times b"] & X \times_S H
\end{tikzcd}$$

imply c is a closed immersion (again by [Stacks, Tag 01QP]).

$\square$

Proposition 2.4.3. *Let $\mathcal{A}$ be a finite configuration and $\mathcal{X}$ a configuration with quasi-projective components, both over a scheme S. Then $\mathrm{Real}^{\mathcal{X}}_{\mathcal{A}}$ and $\mathrm{Re}^{\mathcal{X}}_{\mathcal{A}}$ are both representable. Moreover, the inclusion morphisms $\mathrm{Re}^{\mathcal{X}}_{\mathcal{A}} \subset \mathrm{Hom}(\mathcal{A}, \mathcal{X})$ and $\mathrm{Real}^{\mathcal{X}}_{\mathcal{A}} \subset \mathrm{Re}^{\mathcal{X}}_{\mathcal{A}}$ are open immersions.*

Proof. By definition, $\mathrm{Re}^{\mathcal{X}}_{\mathcal{A}}$ is the sublocus in $\mathrm{Hom}(\Sigma \subset P \times_S L, \Lambda \subset X \times_S H)$ where the induced map $P \times_S L \to X \times_S H$ is a closed immersion, and is thus representability by proposition 2.4.1. We show $\mathrm{Re}^{\mathcal{X}}_{\mathcal{A}} \subset \mathrm{Hom}(\mathcal{A}, \mathcal{X})$ is an open immersion. By lemma 2.4 and base-change, it suffices to show that $\mathrm{Re}^{\mathcal{X}}_{F(\mathcal{A})} \subset \mathrm{Hom}(F(\mathcal{A}), \mathcal{X})$ is an open immersion. Well $\mathrm{Re}^{\mathcal{X}}_{F(\mathcal{A})} = \mathrm{Hom}^{cl}(P, X) \times_S \mathrm{Hom}^{cl}(L, H)$ and $\mathrm{Hom}(F(\mathcal{A}), \mathcal{X}) = \mathrm{Hom}(P, X) \times_S \mathrm{Hom}(L, H)$. Because $\mathrm{Hom}^{cl}(P, X) \subset \mathrm{Hom}(P, X)$ and $\mathrm{Hom}^{cl}(L, H) \subset \mathrm{Hom}(L, H)$ are open subschemes (proposition A.2.1), then we are done.

By definition, $\mathrm{Real}^{\mathcal{X}}_{\mathcal{A}}$ is the locus in $\mathrm{Re}^{\mathcal{X}}_{\mathcal{A}}$ that induces a cartesian diagram; these are $H \subset \mathrm{Hom}(\Sigma \subset P \times_S L, \Lambda \subset X \times_S H)$ in proposition A.3.2. Thus the statement follows by the same proposition.

$\square$

Remark 4. We give another proof that $\mathrm{Real}^{\mathcal{X}}_{\mathcal{A}} \subset \mathrm{Re}^{\mathcal{X}}_{\mathcal{A}}$ is open. Consider an S-scheme T and nondegenerate representation $f : \mathcal{A}_T \to \mathcal{X}_T$ over T. The point and line structure of f induce a cartesian diagram of schemes

$$\begin{tikzcd}
\Sigma \arrow[dr, dashed, "\exists! \varphi"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \Sigma' \arrow[r] \arrow[d] & \Lambda \arrow[d] \\
& P_T \times_T L_T \arrow[r] & X_T \times_T H_T.
\end{tikzcd}$$

Each morphism is a closed immersion, thus φ is a closed immersion. With suitable identifications, the morphism φ is the inclusion of Σ in Σ' as closed subschemes of $P_T \times_T L_T$. Thus the condition that f is a realization is true if and only if $\Sigma = \Sigma'$ as closed subschemes of $P_T \times_T L_T$, which is an open condition.

The following is the main theorem of this thesis.

Theorem 2.4.4. *Let* $\mathcal{A}$ *be a finite configuration and* $\mathcal{X}$ *a configuration with quasi-projective components over* S*. The functors*

$$\mathrm{Real}_{\mathcal{A}}^{\mathcal{X}} \subset \mathrm{Re}_{\mathcal{A}}^{\mathcal{X}} \subset \mathrm{Hom}(\mathcal{A}, \mathcal{X}),$$

are representable by schemes. Moreover, each inclusion is an open immersion. If in addition S *is a noetherian scheme, then* $\mathrm{Real}_{\mathcal{A}}^{\mathcal{X}}$*,* $\mathrm{Re}_{\mathcal{A}}^{\mathcal{X}}$*, and* $\mathrm{Hom}(\mathcal{A}, \mathcal{X})$ *are quasi-projective* S*-schemes.*

Proof. This is a clear consequence of proposition 2.4.3 and theorem 2.3.2. □

Corollary 2.4.5. *Given finite configurations* $\mathcal{A}' = (P', L', \Sigma')$ *and* $\mathcal{A} = (P, L, \Sigma)$ *over* S*, then a nondegenerate representation* $\iota : \mathcal{A}' \to \mathcal{A}$ *induces morphisms*

$$\mathrm{Re}_{\mathcal{A}}^{\mathcal{X}} \to \mathrm{Re}_{\mathcal{A}'}^{\mathcal{X}} \qquad \text{and} \qquad \mathrm{Hom}(\mathcal{A}, \mathcal{X}) \to \mathrm{Hom}(\mathcal{A}', \mathcal{X}).$$

If in addition $\iota : \mathcal{A}' \to \mathcal{A}$ *is a realization, then* ι *induces a morphism of* S*-schemes at the level of realizations,*

$$\mathrm{Real}_{\mathcal{A}}^{\mathcal{X}} \to \mathrm{Real}_{\mathcal{A}'}^{\mathcal{X}}.$$

Corollary 2.4.6. *Given* S*-configurations* $\mathcal{X}' = (X', H', \Lambda')$ *and* $\mathcal{A} = (X, H, \Lambda)$ *with quasi-projective components over* S*. A nondegenerate representation* $\iota : \mathcal{X} \to \mathcal{X}'$ *induces morphisms*

$$\mathrm{Re}_{\mathcal{A}}^{\mathcal{X}} \to \mathrm{Re}_{\mathcal{A}}^{\mathcal{X}'} \qquad \text{and} \qquad \mathrm{Hom}(\mathcal{A}, \mathcal{X}) \to \mathrm{Hom}(\mathcal{A}, \mathcal{X}').$$

If in addition $\iota : \mathcal{X} \to \mathcal{X}'$ *is a realization, then* ι *induces a morphism of* S*-schemes at the level of realizations,*

$$\mathrm{Real}_{\mathcal{A}}^{\mathcal{X}} \to \mathrm{Real}_{\mathcal{A}}^{\mathcal{X}'}.$$

Chapter 3

Properties of Configurations

3.1 Dual and symmetric configurations

Definition 17. Given an S-configuration $\mathcal{A} = (P, L, \Sigma)$, we define the **dual configuration** of $\mathcal{A}$ as the S-configuration $\mathcal{A}^* := (L, P, \Sigma)$, where we identify Σ with it's image under the involution $P \times_S L \to L \times_S P$.

Lemma 3.1.1. *The functor* $* : \mathrm{Config} \to \mathrm{Config}$ *assigning a configuration* $\mathcal{A}$ *to it's dual* $\mathcal{A}^*$ *is an equivalence of categories. This equivalence descends to an equivalence of categories* $\mathrm{Config}(S) \to \mathrm{Config}(S)$ *for any scheme* S.

Proof. The result follows by verifying the functor $*$ is it's own inverse. The second statement follows since $*$ acts trivially on the base scheme structure. $\square$

Definition 18. An S-configuration $\mathcal{A} = (P, L, \Sigma)$ is **symmetric** or **self-dual** if there exists an isomorphism of S-configurations $\tau : \mathcal{A} \to \mathcal{A}^*$, which we call a **duality**.

Lemma 3.1.2. *Then* S*-configuration* $\mathbb{P}^n_1$ *is symmetric.*

Proof. By base-change, we can assume $S = \operatorname{Spec} \mathbb{Z}$. Since there exists an isomorphism $\sigma : |\mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1)| \to \mathbb{P}^n_{\mathbb{Z}}$, then the morphism of configurations $\tau = (\sigma^{-1}, \sigma, \sigma^{-1} \times \sigma|_\Lambda)$ is a duality. $\square$

Lemma 3.1.3. *Suppose* $\mathcal{A}$ *is a symmetric* S*-configuration with duality* τ*. Then the morphism of functors given by pre-composition by* τ

$$\mathrm{Hom}(\mathcal{A}^*, \mathcal{B}) \to \mathrm{Hom}(\mathcal{A}, \mathcal{B}),$$

is an isomorphism. Moreover, this isomorphism descends to isomorphisms $\mathrm{Re}^{\mathcal{B}}_{\mathcal{A}^*} \to \mathrm{Re}^{\mathcal{B}}_{\mathcal{A}}$ *and* $\mathrm{Real}^{\mathcal{B}}_{\mathcal{A}^*} \to \mathrm{Real}^{\mathcal{B}}_{\mathcal{A}}$*. Similarly, the morphism of functors given by post-composition by* τ

$$\mathrm{Hom}(\mathcal{B}, \mathcal{A}) \to \mathrm{Hom}(\mathcal{B}, \mathcal{A}^*)$$

is an isomorphism, which descends to isomorphisms $\mathrm{Re}_{\mathcal{B}}^{\mathcal{A}} \to \mathrm{Re}_{\mathcal{B}}^{\mathcal{A}^*}$ *and* $\mathrm{Real}_{\mathcal{B}}^{\mathcal{A}} \to \mathrm{Real}_{\mathcal{B}}^{\mathcal{A}^*}$.

Proof. This result is given by functoriality and lemma 3.1.1. ☐

3.2 Separated schemes and configurations

Definition 19. For a separated S-scheme T, identify T with it's image under the closed immersion $\Delta : T \to T \times_S T$. The S-configuration (T, T, T) is called the **induced configuration of** $T \to S$.

Remark 5. For separated S-scheme T, the induced S-configuration T is symmetric with a duality $\mathcal{A} \to \mathcal{A}^*$ being the identity morphism.

Lemma 3.2.1. *For a separated S-scheme T, the forgetful transformation to the incidence structure*

$$\mathrm{Hom}((T, T, T), \mathcal{A}) \to \mathrm{Hom}(T, \Sigma)$$

is a natural isomorphism.

Proof. Consider the assignment $\mathrm{Hom}(T, \Sigma) \to \mathrm{Hom}((T, T, T), \mathcal{A})$ given by assigning a morphism $\varphi : T \to \Sigma$ to $(T, T, T) \to \mathcal{A}$, given by the diagram

$$\begin{tikzcd}
(T,T,T): \arrow[d] & T \arrow[r, equal] \arrow[d, "p_1\circ\varphi"] & T \arrow[r, equal] \arrow[d, "\varphi"] & T \arrow[d, "p_2\circ\varphi"] \\
\mathcal{A}: & P & \Sigma \arrow[l, "p_1"] \arrow[r, "p_2"] & L,
\end{tikzcd}$$

where p_i is the i-th projection map. This assignment is clearly functorial, and is the inverse of the forgetful transformation above. ☐

Proposition 3.2.2. *The assignment* $(\mathrm{Sch}/S)^{sep} \to \mathrm{Config}/S$ *given in definition 19 is functorial, meaning an S-morphism $T \to T'$ induces a morphism of configurations $(T, T, T) \to (T', T', T')$. Moreover, this functor is fully-faithful.*

Proof. Functoriality is obvious. The fully-faithful condition is given by lemma 3.2.1. ☐

Remark 6. The awkward condition for T to be separated over S can be seen as a reflection of our requirement for $\Sigma \subset P \times_S L$ to be a closed subscheme. Since the diagonal morphism $\Delta : T \to T \times_S T$ is a locally closed immersion for any scheme T, one can define a generalized version of a configuration as a tuple (P, L, Σ) where $\Sigma \subset P \times_S L$ is a locally closed subscheme. If we let Config' denote the category enlarging Config, then we can construct a functor $\mathrm{Sch}/S \to \mathrm{Config}'/S$ defined as in proposition 3.2.2. By the same argument, this functor is fully-faithful.

Corollary 3.2.3. *Let $\mathcal{A} = (P, L, \Sigma)$ be a configuration over S.*

(i) *As an S-configuration, S is final in* $\mathrm{Config}(S)$ *with* $\mathrm{Hom}(S, \mathcal{A}) = \Sigma$.

(ii) *Consider $\emptyset$ as a separated (vacuously) S-scheme. Then the induced configuration $\emptyset$ is initial in* $\mathrm{Config}(S)$ *with* $\mathrm{Hom}(\emptyset, \mathcal{A}) = S$.

Proof. This is an immediate application of lemma 3.2.1. ☐

Theorem 3.2.4. *Let $\mathcal{A}$ be a finite configuration and $\mathcal{X}$ a configuration with quasi-projective components, both over S. Then the natural morphisms of configurations*

$$\emptyset \to \mathcal{A} \to S$$

induce morphisms of schemes

$$\Lambda \to \mathrm{Hom}(\mathcal{A}, \mathcal{X}) \to S,$$

where $\mathrm{Hom}(\mathcal{A}, \mathcal{X}) \to S$ *is the structure morphism. Moreover, if the structure morphism $\mathcal{A} \to S$ admits a section $\Lambda \to$* $\mathrm{Hom}(\mathcal{A}, \mathcal{X})$ *is a closed immersion.*

Proof. The first few statements are given by functoriality and corollary 3.2.3. To prove $\Lambda \to \mathrm{Hom}(\mathcal{A}, \mathcal{X})$ is a closed immersion, by proposition 2.3.3, consider Λ and $\mathrm{Hom}(\mathcal{A}, \mathcal{X})$ as closed subschemes of $\mathrm{Hom}(F(\mathcal{A}), \mathcal{X})$. The morphism $\Lambda \to \mathrm{Hom}(\mathcal{A}, \mathcal{X})$ is a factorization

$$\Lambda \to \mathrm{Hom}(\mathcal{A}, \mathcal{X}) \subset \mathrm{Hom}(F(\mathcal{A}), \mathcal{X})$$

of $\Lambda \subset \mathrm{Hom}(F(\mathcal{A}), \mathcal{X})$, and is thus a closed immersion by [Stacks, Tag 01QP]. ☐

Remark 7. Let $\mathcal{A}$ be an S-configuration whose components are separated over S, then the structure morphism $\mathcal{A} \to S$ admits a section if and only if there exists a nondegenerate representation $S \to \mathcal{A}$. This is because a section of $\mathcal{A} \to S$ is a representation $S \to \mathcal{A}$. Then the result follows since sections of a separated morphism of schemes is a closed immersion (see [Stacks, Tag 01QP]).

Construction 1. Fix a configuration $\mathcal{X} = (X, H, \Lambda)$ with quasi-projective components over S and finite locally free S-schemes P and L. Then there is a map of sets

$$\varphi : \left\{ \begin{array}{l} \text{Finite } S\text{-configurations} \\ \mathcal{A} \text{ with } F(\mathcal{A}) = (P, L, \emptyset) \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{Closed subschemes of} \\ \mathrm{Hom}(P, X) \times_S \mathrm{Hom}(L, H) \\ \text{containing } \Lambda \end{array} \right\},$$

defined by $\varphi(\mathcal{A}) = \mathrm{Hom}(\mathcal{A}, \mathcal{X})$.

Question 3.2.1 (Properties of φ)**.** *Fix a configuration* $\mathcal{X} = (X, H, \Lambda)$ *with quasi-projective components over* S*, and let* P, L *be finite locally free schemes over* S*. Classify closed subschemes* $Z \subset \operatorname{Hom}((P, L, \emptyset), \mathcal{X})$ *containing* Λ *of the form* $\operatorname{Hom}(\mathcal{A}, \mathcal{X})$*; i.e., such that there exists a closed subscheme* $\Sigma \subset P \times_S L$ *such that* $\operatorname{Hom}((P, L, \Sigma), \mathcal{X}) = Z$ *as closed subschemes of* $\operatorname{Hom}((P, L, \emptyset), \mathcal{X})$*.*

3.3 Points and blocks of a configuration

Definition 20. The S-configurations $\mathcal{P}(\mathcal{A}) = (P, \emptyset, \emptyset)$ and $\mathcal{L}(\mathcal{A}) = (\emptyset, L, \emptyset)$ are called the **point configuration** and **block/line configuration** of $\mathcal{A}$ over S, respectively.

Lemma 3.3.1. *Given a configuration* $\mathcal{X} = (X, H, \Lambda)$ *over* S*, the inclusion* $\operatorname{Real}^{\mathcal{X}}_{\mathcal{P}(\mathcal{A})} \subset \operatorname{Re}^{\mathcal{X}}_{\mathcal{P}(\mathcal{A})}$ *from Theorem 2.3.2 is an equality. Moreover, the forgetful map to the point-structure*

$$\operatorname{Hom}(\mathcal{P}(\mathcal{A}), \mathcal{X}) \to \operatorname{Hom}_{\mathrm{Sch}/S}(P, X)$$

is an isomorphism. Similarly, we have an equality $\operatorname{Real}^{\mathcal{X}}_{\mathcal{L}(\mathcal{A})} = \operatorname{Re}^{\mathcal{X}}_{\mathcal{L}(\mathcal{A})}$ *with an isomorphism*

$$\operatorname{Hom}(\mathcal{L}(\mathcal{A}), \mathcal{X}) \to \operatorname{Hom}_{\mathrm{Sch}/S}(L, H).$$

Proof. The inclusion $\operatorname{Real}^{\mathcal{X}}_{\mathcal{P}(\mathcal{A})} \subset \operatorname{Re}^{\mathcal{X}}_{\mathcal{P}(\mathcal{A})}$ is an equality vacuously. A representation $\mathcal{P}(\mathcal{A}) \to \mathcal{X}$ is the data of a morphism $P \to X$, and thus $\operatorname{Hom}(\mathcal{P}(\mathcal{A}), \mathcal{X}) \to \operatorname{Hom}_{\mathrm{Sch}/S}(P, X)$ is an isomorphism, trivially. The rest of the statements are given using $\mathcal{P}(\mathcal{A}^*) = \mathcal{L}(\mathcal{A})$. $\square$

This is beneficial as the non-reduced structure of P can give us information on X. For instance, we can reinterpret a theorem of Mustaţă, summarized in [KK13], as follows.

Corollary 3.3.2 (Mustaţă)**.** *Let* X *be a normal, quasi-projective, local complete intersection variety over a field* k *of characteristic 0. Consider the* S*-configurations* $\mathcal{A}_m = (\mathit{Spec}\,(k[\epsilon]/\epsilon^m), \emptyset, \emptyset)$ *and* $\mathcal{X} = (X, H, \Lambda)$ *with quasi-projective components.*

(i) $\operatorname{Real}^{\mathcal{X}}_{\mathcal{A}_m}$ *has pure dimension* $(m+1)\dim X$ *for every* m *if and only if* X *is log canonical.*

(ii) $\operatorname{Real}^{\mathcal{X}}_{\mathcal{A}_m}$ *is irreducible for every* m *if and only if* X *is canonical.*

(iii) $\operatorname{Real}^{\mathcal{X}}_{\mathcal{A}_m}$ *is normal for every* m *if and only if* X *is terminal.*

3.4 Closed Subconfiguration

Definition 21. A **closed subconfiguration** of an S-configuration $\mathcal{A}$ is an S-configuration $\mathcal{A}' = (P', L', \Sigma')$, where $P' \subset P$, $L' \subset L$, and $\Sigma' \subset \Sigma$ are closed subschemes.

Remark 8. The inclusion morphism of a closed subconfiguration $\mathcal{A}' \subset \mathcal{A}$ is a nondegenerate representation. Moreover, every nondegenerate representation can be viewed as an inclusion morphism after suitable identifications. Sometimes we will conflate the two.

Example 10. The unique morphism $\emptyset \to \mathcal{A}$ in $\mathrm{Config}(S)$ realizes $\emptyset$ as a closed subconfiguration of $\mathcal{A}$. Similarly, the forgetful morphism $F(\mathcal{A}) \to \mathcal{A}$ realizes $F(\mathcal{A})$ as a subconfiguration of $\mathcal{A}$. Thus every configuration $\mathcal{A}$ satisfies the chain of subconfigurations

$$\emptyset \subset F(\mathcal{A}) \subset \mathcal{A}.$$

Proposition 3.4.1. *Let $\mathcal{C}$ be a closed subconfiguration of $\mathcal{A}$ and $\mathcal{B}$ as S-configurations. Then the coproduct $\mathcal{A} \cup_{\mathcal{C}} \mathcal{B}$ exists in* $\mathrm{Config}(S)$*, and thus the following diagram is cocartesian:*

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] & \mathcal{B} \arrow[d] \\
\mathcal{A} \arrow[r] & \mathcal{A} \cup_{\mathcal{C}} \mathcal{B}.
\end{tikzcd}$$

Moreover, the induced maps $\mathcal{A} \to \mathcal{A} \cup_{\mathcal{C}} \mathcal{B}$ and $\mathcal{B} \to \mathcal{A} \cup_{\mathcal{C}} \mathcal{B}$ are nondegenerate representations, making $\mathcal{A}$ and $\mathcal{B}$ closed subconfigurations of $\mathcal{A} \cup_{\mathcal{C}} \mathcal{B}$.

Proof. The point, block, and incidence structure of the diagram

$$\mathcal{A} \leftarrow \mathcal{C} \rightarrow \mathcal{B},$$

have coproducts making the relative diagram cocartesian, see [Sch05, corollary 3.9]. In other words, we can glue along closed subschemes. It is not hard to show that this process of gluing yields an S-configuration $\mathcal{A} \cup_{\mathcal{C}} \mathcal{B}$. The cartesian statement can be seen by inspecting the point, line, and incidence structures. ☐

Corollary 3.4.2. *Let $\mathcal{C}$ be a closed subconfiguration of $\mathcal{A}$ and $\mathcal{B}$. Then*

$$\mathrm{Hom}(\mathcal{A} \cup_{\mathcal{C}} \mathcal{B}, \mathcal{X}) = \mathrm{Hom}(\mathcal{A}, \mathcal{X}) \times_{\mathrm{Hom}(\mathcal{C}, \mathcal{X})} \mathrm{Hom}(\mathcal{B}, \mathcal{X}).$$

Remark 9. The importance of corollary 3.4.2 is substantial. With this, we can study representations of configurations $f : \mathcal{A}' \to \mathcal{X}$ by decomposing the source configuration $\mathcal{A}'$ by subconfigurations $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and studying their representations in $\mathcal{X}$. This also allows us to build more intricate configurations.

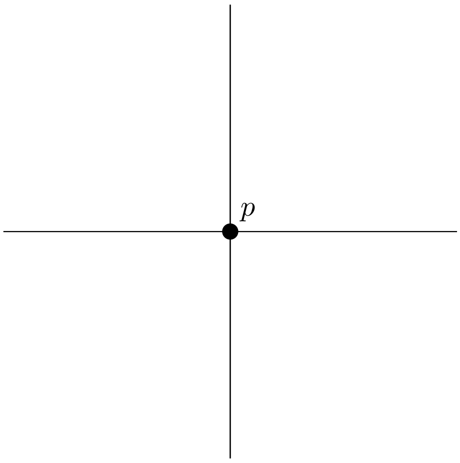

Figure 3.1: The **cross configuration**.

Example 11. Consider S as an S-configuration and let $\mathcal{A} = \mathcal{B} = S$. Consider $\mathcal{C} = \mathcal{P}(S) = (S, \emptyset, \emptyset)$ as a subconfiguration of both $\mathcal{A}$ and $\mathcal{B}$. Then $\mathcal{A} \cup_{\mathcal{C}} \mathcal{B}$ is the cross configuration (see figure 3.1).

Moreover, by corollary 3.4.2 and lemma 3.2.1, we have that

$$\operatorname{Hom}(\mathcal{A} \cup_{\mathcal{C}} \mathcal{B}, \mathcal{X}) = \Lambda \times_X \Lambda.$$

Example 12. We generalize the previous example. For positive integer n, let $\mathcal{A}_n$ be the split S-configuration induced by the n**-star configuration** (see figure 3.2). By the previous example, the n-star configuration can be build recursively, and so it is not hard to show that

$$\operatorname{Hom}(\mathcal{A}_n, \mathcal{X}) = \Lambda \times_X \cdots \times_X \Lambda,$$

where there are n factors of Λ.

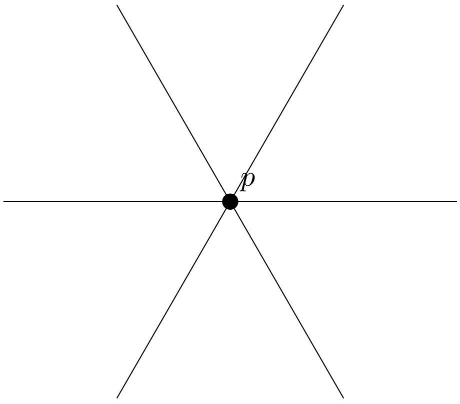

Figure 3.2: The **star configuration** of $n = 3$ lines incident at a single point p.

3.5 Geometric Configurations

Throughout we assume S is a noetherian scheme. Thus, if X is a projective variety over S, then the Hilbert functor $\mathrm{Hilb}_{X/S}$ is representable by an S-scheme.

Definition 22. A **geometric configuration** $\mathcal{X} = (X, H, \Lambda)$ over a scheme S is the data of a projective variety X over S, a closed subscheme $H \subset \mathrm{Hilb}_{X/S}$ of the Hilbert scheme of X, and Λ is the restriction to the universal family $\Lambda_0 \subset X \times_S \mathrm{Hilb}_{X/S}$.

Remark 10. Since the universal family $\Lambda_0 \subset X \times_S \mathrm{Hilb}_{X/S}$ is a closed subscheme for a projective variety X, then by base-change, so is $\Lambda = \Lambda_0|_H \subset X \times_S H$. Thus the notion of a geometric configuration is well-defined.

Lemma 3.5.1. *Let $\mathcal{X} = (X, H, \Lambda)$ be a geometric configuration over S, then H and Λ are projective over S. Thus $\mathcal{X}$ has projective components over S.*

Proof. It is a classical result that X projective over S implies $\mathrm{Hilb}_{X/S}$ is represented by a projective scheme over S. Thus the closed subscheme $H \subset \mathrm{Hilb}_{X/S}$ is projective over S. Since X and H are projective S-schemes, the closed subscheme $\Lambda \subset X \times_S H$ is also projective over S. □

Example 13. The S-configuration $\mathbb{P}^n_d$ is a geometric configuration. To generalize, let X be a projective S-variety and $\mathcal{O}_X(1)$ a relatively very ample line bundle of $X \to S$. Then $(X, |\mathcal{O}_X(1)|, \Lambda)$, where $\Lambda \subset X \times_S |\mathcal{O}_X(1)|$ is the universal incidence scheme, is a geometric S-configuration.

Definition 23. Let X be a projective variety over a scheme S. Consider the S-configuration

$$\mathrm{Geo}(X) = (X, \mathrm{Hilb}_{X/S}, \Lambda_0),$$

where $\Lambda_0 \subset X \times_S Hilb_{X/S}$ is the universal family of $Hilb_{X/S}$. We call $\mathrm{Geo}(X)$ the **geometric configuration induced by X.**

Lemma 3.5.2. *Let X be a projective variety over a scheme S, and T an S-scheme. Then $\mathrm{Geo}(X)_T = \mathrm{Geo}(X_T)$ up to canonical isomorphism.*

Proof. The configuration $\mathrm{Geo}(X)_T$ is the tuple $(X_T, Hilb_{X_T/T}, \Lambda_T)$ (definition 8). Then the result follows from proposition 2.2.2 and properties of the Hilbert scheme. □

Lemma 3.5.3. *Let $\mathcal{A} = (P, L, \Sigma)$ be an S-configuration with P an S-variety. Then $\mathcal{A}$ is geometric if and only if there exists a realization $\mathcal{A} \to \mathrm{Geo}(X)$ over S, for some projective S-variety X.*

Proof. An S-configuration $\mathcal{A} = (P, L, \Sigma)$ being a geometric configuration is the data of a projective S-variety P and a closed subscheme $\iota : L \subset Hilb_{X/S}$ such that the induced diagram

$$\begin{tikzcd}
\Sigma \arrow[r] \arrow[d, phantom, "\cap"] & \Lambda_0 \arrow[d, phantom, "\cap"] \\
P \times_S L \arrow[r, "\mathrm{id}_P \times \iota"] & P \times_S Hilb_{P/S}
\end{tikzcd}$$

is cartesian. This is precisely the data of a realization $\mathcal{A} \to \mathrm{Geo}(P)$ induced by (id_P, ι). Lastly, the closed immersion $P \to X$ implies P is projective over S.

□

Lemma 3.5.4. *Let $\mathcal{A} = (P, L, \Sigma)$ be an S-configuration with P a projective variety over S, and suppose $\mathrm{Real}_{\mathcal{A}}^{Geo(P)}$ is representable by a S-scheme. Then $\mathcal{A}$ is geometric if and only if the structure morphism $\mathrm{Real}_{\mathcal{A}}^{\mathrm{Geo}(P)} \to S$ admits a section.*

Proof. This is an immediate consequence of lemma 3.5.3. □

Proposition 3.5.5. *Let $\mathcal{X} = (X, H, \Lambda)$ be a geometric configuration over S and T an S-scheme. Then $\mathcal{X}_T$ is a geometric configuration over T.*

Proof. By lemma 3.5.3, let $\varphi : \mathcal{X} \to \mathrm{Geo}(X)$ be the realization defining $\mathcal{X}$ as a geometric S-configuration. By functoriality, we have a realization $\varphi' : \mathcal{X}_T \to \mathrm{Geo}(X)_T$ as T-configurations. Since $\mathrm{Geo}(X)_T = \mathrm{Geo}(X_T)$ by lemma 3.5.2, this gives a realization $\varphi' : \mathcal{X}_T \to \mathrm{Geo}(X_T)$, implying $\mathcal{X}_T$ is geometric over T. □

Example 14. Let X be a projective variety with polarization $\mathcal{O}_X(1)$. Let $\Lambda \subset X \times_S |\mathcal{O}(1)|$ denote the incidence subscheme, then $\mathcal{X} = (X, |\mathcal{O}_X(1)|, \Lambda)$ is a geometric configuration over S. Moreover, for $n = \dim |\mathcal{O}_X(1)|$, the inclusion $X \to \mathbb{P}^n_S$ by $\mathcal{O}_X(1)$ induces a realization $\mathcal{X} \to \mathbb{P}^n_1$. This induces $\mathrm{Hom}(\mathcal{A}, \mathcal{X}) \to \mathrm{Hom}(\mathcal{A}, \mathbb{P}^n_1)$, which is a closed immersion by hom properties. Thus $\mathrm{Re}_{\mathcal{A}}^{\mathcal{X}} \to \mathrm{Re}_{\mathcal{A}}^{\mathbb{P}^n_1}$ and $\mathrm{Real}_{\mathcal{A}}^{\mathcal{X}} \to \mathrm{Real}_{\mathcal{A}}^{\mathbb{P}^n_1}$ are closed immersions. This says that many representation problems can be understood by studying representations to projective space.

We conclude this section by showing representability of $\mathrm{Hom}(\mathcal{X}, \mathcal{X}')$ of geometric configurations with the domain having flat components.

Theorem 3.5.6. *Let S be a noetherian scheme. Suppose $\mathcal{X} = (X, H, \Lambda)$ is a geometric configuration with flat components and $\mathcal{X}' = (X', H', \Lambda')$ is a configuration with quasi-projective components, both over S. Then $\mathrm{Hom}(\mathcal{X}, \mathcal{X}')$ is representable by an S-scheme.*

Proof. By A.1.1 , then $\mathrm{Hom}(\Lambda, \Lambda')$, $\mathrm{Hom}(\Lambda, X' \times_S H')$, and $\mathrm{Hom}(F(\mathcal{X}), \mathcal{X}') = \mathrm{Hom}(X, X') \times_S \mathrm{Hom}(H, H')$ are representable by schemes. The result follows since $\mathrm{Hom}(\mathcal{X}, \mathcal{X}')$ fits into the following cartesian diagram of schemes,

$$\begin{tikzcd}
\mathrm{Hom}(\mathcal{X}, \mathcal{X}') \arrow[r] \arrow[d] & \mathrm{Hom}(\Lambda, \Lambda') \arrow[d] \\
\mathrm{Hom}(F(\mathcal{X}), \mathcal{X}') \arrow[r] & \mathrm{Hom}(\Lambda, X' \times_S H').
\end{tikzcd}$$

□

Corollary 3.5.7. *In the scenario of theorem 3.5.6,* $\mathrm{Real}_{\mathcal{X}}^{\mathcal{X}'}$ *and* $\mathrm{Re}_{\mathcal{X}}^{\mathcal{X}'}$ *are representable by* S*-schemes. Moreover, the inclusion morphisms*

$$\mathrm{Real}_{\mathcal{X}}^{\mathcal{X}'} \subset \mathrm{Re}_{\mathcal{X}}^{\mathcal{X}'} \subset \mathrm{Hom}(\mathcal{X}, \mathcal{X}')$$

are open immersions.

3.6 Flat Moduli

Throughout, let S be a noetherian scheme.

Proposition 3.6.1. *Let* $\mathcal{A}$ *be a finite* S*-configuration whose components are étale over* S*, and* $\mathcal{X}$ *a geometric* S*-configuration whose components are flat over* S*, then* $\mathrm{Hom}(\mathcal{A}, \mathcal{X})$ *is flat over* S*.*

Proof. The conditions on $\mathcal{A}$ imply that $\Sigma = \coprod_{i=1}^{N} S_i$ for some integer N and S_i étale over S for each i. The natural morphism $\mathrm{Hom}(\Sigma, \Lambda) \to \mathrm{Hom}(\Sigma, X \times_S H)$ then becomes

$$\prod_{i=1}^{N} \Lambda(S_i) \to \prod_{i=1}^{N} X \times_S H(S_i),$$

which is flat by [Ji+22, Proposition 2.9] By the same proposition, $\mathrm{Hom}(P, X)$ and $\mathrm{Hom}(L, H)$ are flat over S, thus so is $\mathrm{Hom}(P, X) \times_S \mathrm{Hom}(L, H)$. From diagram 2.3.1 and base change, we have that the natural map $\mathrm{Hom}(\mathcal{A}, \mathcal{X}) \to \mathrm{Hom}(F(\mathcal{A}), \mathcal{X}) = \mathrm{Hom}(P, X) \times_S \mathrm{Hom}(L, H)$ is flat. The result then follows since compositions of flat morphisms are flat.

□

Corollary 3.6.2. *Let* $\mathcal{A}$ *be a finite split configuration over a scheme* S*, then* $\mathrm{Hom}(\mathcal{A}, \mathbb{P}_1^2)$ *is flat over* S*.*

3.7 Finite Split Configurations over Fields

Definition 24. Let $\mathcal{A} = (P, L, \Sigma)$ be a finite split configuration over S, meaning P, L, and Σ are finite disjoint unions of S. The **number of points** of $\mathcal{A}$ is the degree of P over S, the **number of lines/blocks** of $\mathcal{A}$ is the degree of L over S, and the **number of incidences** of $\mathcal{A}$ is the degree of Σ over S.

Throughout, let $\mathcal{A} = (P, L, \Sigma)$ be a finite split configuration with m points, n lines, and N incidences, and $\mathcal{X} = (X, H, \Lambda)$ geometric configuration, both over a field k.

Proposition 3.7.1. *The scheme* $\operatorname{Hom}(\mathcal{A}, \mathcal{X})$ *is projective over* S*.*

Proof. The natural map $\operatorname{Hom}(\mathcal{A}, \mathcal{X}) \to \operatorname{Hom}(F(\mathcal{A}), \mathcal{X})$ is a closed immersion (proposition 2.3.3) and $\operatorname{Hom}(F(\mathcal{A}), \mathcal{X}) = X^m \times_S H^n$ is projective over S, as desired. ☐

Lemma 3.7.2. *Suppose X and H are smooth varieties over k. If Λ is a smooth variety over k, then*

$$\dim \operatorname{Hom}(\mathcal{A}, \mathcal{X}) \geq m \dim X + n \dim H - N.$$

Moreover, $\operatorname{Real}_{\mathcal{A}}^{\mathcal{X}}$ *and* $\operatorname{Re}_{\mathcal{A}}^{\mathcal{X}}$ *are either empty or of dimension at least* $m \dim X + n \dim H - N$*.*

Proof. Because $\operatorname{Hom}(P, X) = X^m$, $\operatorname{Hom}(L, H) = H^n$, and $\operatorname{Hom}(\Sigma, X \times H) = (X \times_k H)^N$, the natural map $\operatorname{Hom}(P, X) \times \operatorname{Hom}(L, H) \to \operatorname{Hom}(\Sigma, X \times H)$ from diagram 2.3.1 is a closed immersion. Thus, after suitable identifications, $X^m \times_k H^n \subset (X \times_k H)^N$ is a closed subvariety of a smooth k-variety. By [Stacks, Tag 0AZP], each irreducible component of $\operatorname{Hom}(\mathcal{A}, \mathcal{X})$ is of dimension at least $m \dim X + n \dim H - N$. The last statement is due to $\operatorname{Real}_{\mathcal{A}}^{\mathcal{X}}$ and $\operatorname{Re}_{\mathcal{A}}^{\mathcal{X}}$ being open subschemes of $\operatorname{Hom}(\mathcal{A}, \mathcal{X})$ (theorem 2.4.4). ☐

Corollary 3.7.3. *Let $\mathcal{A}$ be a split configuration over a field k with m points, n lines, and N incidences. If* $\operatorname{Hom}(\mathcal{A}, \mathbb{P}_1^2)$ *is not the empty scheme, then*

$$\dim \operatorname{Hom}(\mathcal{A}, \mathbb{P}_1^2) \geq 2(m + n) - N.$$

Moreover, $\operatorname{Real}_{\mathcal{A}}^{\mathbb{P}_1^2}$ *and* $\operatorname{Re}_{\mathcal{A}}^{\mathbb{P}_1^2}$ *are either empty or of dimension at least* $2(m + n) - N$*.*

Chapter 4

Deformation Theory

In this chapter we introduce deformations of configurations and their representations. The goal is to connect these topics with deformations of schemes.

4.1 Deformation of Configurations

Throughout, let $\mathcal{A}_0 = (P_0, L_0, \Sigma_0)$ be a configuration over a field k.

Definition 25. A **deformation** of $\mathcal{A}_0$ consists of a configuration $\mathcal{A} = (P, L, \Sigma)$ over a scheme S, together with a morphism of configurations $\varphi : \mathcal{A}_0 \to \mathcal{A}$ fitting into a cartesian diagram

$$\begin{tikzcd}
\mathcal{A}_0 \arrow[r, "\varphi"] \arrow[d] & \mathcal{A} \arrow[d] \\
\operatorname{Spec} k \arrow[r] & S
\end{tikzcd}$$

where the vertical arrows are the respective structure morphisms, and where P, L, and Σ are flat over S.

We say that a deformation φ is **infinitesimal** (resp., **first-order**) if $S = \operatorname{Spec} A$ for some Artinian local k-algebra A (resp., for $A = k[\epsilon]$).

Example 15. Let $\mathcal{A}$ be the $\mathbb{A}^1_k$-configuration described in example 9. Then $\mathcal{A}$ is a deformation of the standard triangle configuration.

Definition 26. A configuration $\mathcal{A}$ is **rigid** if all first-order deformations are trivial.

We see from example 15 that the standard triangle is not rigid.

Definition 27. Two deformations $\varphi : \mathcal{A}_0 \to \mathcal{A}$ and $\varphi' : \mathcal{A}_0 \to \mathcal{A}'$ of $\mathcal{A}_0$ over a scheme S are **isomorphic** if there exists an automorphism $\psi : \mathcal{A}_0 \to \mathcal{A}_0$ over S such that $\varphi' = \varphi \circ \psi$.

Definition 28. Define $\mathrm{Def}_{\mathcal{A}_0} : \mathrm{Art}_k \to \mathit{Sets}$ given by

$\mathrm{Def}_{\mathcal{A}_0}(A) = \{$Isomorphism classes of infinitesimal deformations of $\mathcal{A}_0$ over $A\}$.

Lemma 4.1.1. *Let A be an Artin local k-algebra for field k and $\mathcal{A} = (P, L, \Sigma)$ a configuration over A. A morphism $\varphi : \mathcal{A}_0 \to \mathcal{A}$ of configurations is a deformation of $\mathcal{A}_0 = (P_0, L_0, \Sigma_0)$ over A if and only if the induced morphisms $P_0 \to P$, $L_0, \to L$, and $\Sigma_0 \to \Sigma$ are infinitesimal deformations of their respective domain over A.*

Proof. This is an immediate application of proposition 2.2.4.

□

For the rest of the section we describe $\mathrm{Def}_{\mathcal{A}_0}$ as a fiber product of deformation functors of schemes.

Definition 29. Let $\mathrm{Def}_{\Sigma_0 \subset P_0 \times L_0} : \mathrm{Art}_k \to \mathrm{Sets}$ denote the functor where an element of $\mathrm{Def}_{\Sigma_0 \subset P_0 \times L_0}(A)$ is the data of the equivalence class of the following commuting diagram

$$\begin{tikzcd}
\Sigma_0 \arrow[r] \arrow[d, phantom, "\cap"] & \Sigma \arrow[d, phantom, "\cap"] \\
P_0 \times L_0 \arrow[r] \arrow[d] & D \arrow[d] \\
\mathrm{Spec}\, k \arrow[r] & \mathrm{Spec}\, A,
\end{tikzcd}$$

where each square is cartesian, $\Sigma \subset D$ is a closed subscheme, and Σ and D are flat over $\mathrm{Spec}\, A$.

Lemma 4.1.2. *We have the following cartesian diagram of functors*

$$\begin{tikzcd}
\mathrm{Def}_{\mathcal{A}_0} \arrow[r] \arrow[d] & \mathrm{Def}_{\Sigma_0 \subset P_0 \times L_0} \arrow[d] \\
\mathrm{Def}_{P_0} \times \mathrm{Def}_{L_0} \arrow[r] & \mathrm{Def}_{P_0 \times L_0} .
\end{tikzcd}$$

Proof. The result is immediate, as a deformation $(a, b, c) : \mathcal{A}_0 \to \mathcal{A} = (P, L, \Sigma)$ over A is the data of the following commuting diagram of schemes,

$$\begin{tikzcd}
\Sigma_0 \arrow[r, "c"] \arrow[d, phantom, "\cap"] & \Sigma \arrow[d, phantom, "\cap"] \\
P_0 \times_A L_0 \arrow[r, "a \times b"] \arrow[d] & P \times_A L \arrow[d] \\
\mathrm{Spec}\, k \arrow[r] & \mathrm{Spec}\, A,
\end{tikzcd}$$

where each square is cartesian. □

Proposition 4.1.3. *Let $\mathcal{A}_0$ be a configuration over k with projective components. Then the functor $\mathrm{Def}_{\mathcal{A}_0}$ satisfies the following properties,*

(i) $\mathrm{Def}_{\mathcal{A}_0}(k)$ *is a singleton,*

(ii) α *is surjective when $A'' \to A$ is a small extension,*

(iii) α *is bijective when $A = k$ and $A'' = k[\epsilon]$.*

Proof. Following [Ser06, Theorem 2.3.2], we fix notation. Let

$$A' \to A \leftarrow A''$$

be a diagram in Art_k, and $R = A' \times_A A''$. Let

$$\alpha : \mathrm{Def}_{\mathcal{A}_0}(R) \to \mathrm{Def}_{\mathcal{A}_0}(A') \times_{\mathrm{Def}_{\mathcal{A}_0}(A)} \mathrm{Def}_{\mathcal{A}_0}(A'')$$

be the natural map.

(i) It is clear that $\mathrm{Def}_{\mathcal{A}_0}(k)$ is a singleton.

(ii) Let $\mathcal{A} = (P, L, \Sigma)$ be a deformation of $\mathcal{A}_0$ over A, and let $\mathcal{A}' = (P', L', \Sigma')$ and $\mathcal{A}'' = (P'', L'', \Sigma'')$ be deformations of $\mathcal{A}_0$ over A' and A'', respectively, whose restrictions to A is $\mathcal{A}$. Because the deformation functor of a scheme is prorepresentable, there exists schemes P''' and L''' over R that extend $P' \leftarrow P \to P''$ and $L' \leftarrow L \to L''$, respectively. Thus $X = P''' \times_R L'''$ is an extension of $X|_k = P_0 \times_k L_0$ over R. Because the relative hilbert functor $\mathrm{Hilb}^{X/R}_{\Sigma_0}$ is prorepresentable by [Ser06, Theorem 3.2.12], there exists a closed subscheme $\Sigma''' \subset X$ over R that extends $\Sigma_0 \subset X|_k = P_0 \times_k L_0$. We can then define the configuration $\mathcal{A}''' = (P''', L''', \Sigma''')$ over R that has the desired conditions.

(iii) Again by prorepresentabilty of Def_{P_0} and Def_{L_0}, there exists unique (up to R-isomorphism) schemes P'' and L'' over R that extend $P' \leftarrow P_0 \to P$ and $L' \leftarrow L_0 \to L$, respectively. Thus $X = P'' \times_R L''$ is an extension of $X|_k = P_0 \times_k L_0$ over R. Because the relative hilbert functor $\mathrm{Hilb}^{X/R}_{\Sigma_0}$ is prorepresentable by [Ser06, Theorem 3.2.12], there exists a unique closed subscheme $\Sigma'' \subset X$ over R that extends $\Sigma_0 \subset X|_k = P_0 \times_k L_0$. The configuration $\mathcal{A}'' = (P'', L'', \Sigma'')$ over R is the desired configuration. It is unique since it's components are unique. This defines an inverse of α, showing bijectivity.

□

Remark 11. It is false that $\dim_k \mathrm{Def}_{\mathcal{A}_0}(k[\epsilon]) < \infty$ in general (see [Ser06, Remark 3.4.18]).

4.2 Rigidity of $\mathbb{P}^n_1$

Definition 30. An infinitesimal deformation $\varphi : \mathcal{A}_0 \to \mathcal{A}$ with $\mathcal{A}$ an Spec A-configuration is **geometric** if $\mathcal{A}$ is geometric over Spec A.

Definition 31. A configuration $\mathcal{A}$ is **geometrizable** if it is isomorphic to a geometric configuration.

Lemma 4.2.1. *Let $D \subset \mathbb{P}^n_{k[\epsilon]} \times |\mathcal{O}_{\mathbb{P}^n_{k[\epsilon]}}(1)|$ be a first-order deformation of the universal incidence divisor $\Lambda_0 \subset \mathbb{P}^n_k \times |\mathcal{O}_{\mathbb{P}^n_k}(1)|$. Then there exists an infinitesimal automorphism of $\mathbb{P}^n_k$ such that the induced automorphism on $\mathbb{P}^n_{k[\epsilon]} \times |\mathcal{O}_{\mathbb{P}^n_{k[\epsilon]}}(1)|$ maps D to the universal incidence divisor $\Lambda_0 \times_k A \subset \mathbb{P}^n_{k[\epsilon]} \times |\mathcal{O}_{\mathbb{P}^n_{k[\epsilon]}}(1)|$.*

Proof. Let $D \subset \mathbb{P}^n_{k[\epsilon]} \times |\mathcal{O}_{\mathbb{P}^n_{k[\epsilon]}}(1)|$ as above, then D is an infinitesimal deformation of a $(1,1)$-divisor, and is thus a $(1,1)$-divisor as well. Thus D can be described by a linear equation

$$\begin{bmatrix} a_0 & \cdots & a_n \end{bmatrix} M_D \begin{bmatrix} x_0 \\ \vdots \\ x_n \end{bmatrix} = 0,$$

for some $M_D \in PGL_{n+1}(k[\epsilon])$. Because D is an infinitesimal deformation of Λ_0, we have that $M_D = I + \epsilon M$ for I the identity matrix and some square matrix $M \in M_{n+1}(k)$. To show the desired statement, it suffices to show there exists a matrix $N \in PGL_{n+1}(k[\epsilon])$ such that $M_D N = I$. Set $N = I - \epsilon M$, then

$$M_D N = (I + \epsilon M)(I - \epsilon M) = I,$$

as desired.

□

Proposition 4.2.2. *The configuration $\mathbb{P}^n_1$ over k is rigid.*

Proof. Let $\varphi : \mathbb{P}^n_1 \to \mathcal{A} = (P, L, \Sigma)$ be a first-order deformation. By rigidity of projective space, there exists isomorphisms $\alpha : P \to \mathbb{P}^n_{k[\epsilon]}$ and $\beta : L \to |\mathcal{O}_{k[\epsilon]}(1)| \simeq \mathbb{P}^n_{k[\epsilon]}$. Let $D \subset \mathbb{P}^n_{k[\epsilon]} \times |\mathcal{O}_{\mathbb{P}^n_{k[\epsilon]}}(1)|$ be the image of Σ under $\alpha \times \beta$. We can define $\mathcal{X} = (\mathbb{P}^n_{k[\epsilon]}, |\mathcal{O}_{\mathbb{P}^n_{k[\epsilon]}}(1)|, D)$ and an isomorphism of $k[\epsilon]$-configurations $\varphi : \mathcal{A} \to \mathcal{X}$ given by α and β. By lemma 4.2.1, we can define an isomorphism $\mathcal{X} \to \mathbb{P}^n_{1,k[\epsilon]}$, as desired.

□

4.3 Deformations of Representations

Throughout, let $\mathcal{A}_0 = (P_0, L_0, \Sigma_0)$ be a finite configuration and $\mathcal{X}_0 = (X_0, H_0, \Lambda_0)$ a configuration with quasi-projective point and line structures, both over a field k. Fix a representation $f_0 : \mathcal{A}_0 \to \mathcal{X}_0$.

Definition 32. An **infinitesimal deformation** of f_0 is a morphism of configurations $f : \mathcal{A} \to \mathcal{A}'$ over an artin local k-algebra A fitting into a commuting diagram of configurations

$$\begin{tikzcd}
\mathcal{A}_0 \arrow[r] \arrow[d, "f_0"] & \mathcal{A} \arrow[d, "f"] \\
\mathcal{X}_0 \arrow[r] \arrow[d] & \mathcal{A}' \arrow[d] \\
\operatorname{Spec} k \arrow[r] & \operatorname{Spec} A,
\end{tikzcd}$$

where the bottom downward arrows are the structure morphisms, both squares are cartesian, and the point, block, and incidence schemes of both $\mathcal{A}$ and $\mathcal{A}'$ are flat over Spec A.

4.3.1 Deformations of Representations with fixed domain and target

Definition 33. An infinitesimal deformation $f : \mathcal{A} \to \mathcal{A}'$ of f_0 over A has **fixed domain and target** if $\mathcal{A} = \mathcal{A}_0 \times A$ and $\mathcal{X} = \mathcal{X}_0 \times A$. This defines a functor $\operatorname{Def}_{\mathcal{A}_0/f/\mathcal{X}_0} : Art_k \to \text{Sets}$ given by

$$\operatorname{Def}_{\mathcal{A}_0/f_0/\mathcal{X}_0}(A) = \{\text{Deformations of } f_0 \text{ over } A \text{ with fixed domain and target}\}.$$

Proposition 4.3.1. *Consider a representation of configurations $f_0 : \mathcal{A}_0 \to \mathcal{A}_0'$ over a field k. Then the following is a cartesian diagram of functors*

$$\begin{tikzcd}
\operatorname{Def}_{\mathcal{A}_0/f_0/\mathcal{A}_0'} \arrow[r] \arrow[d] & \operatorname{Def}_{\Sigma_0/c_0/\Sigma_0'} \arrow[d] \\
\operatorname{Def}_{P_0/a_0/P_0'} \times \operatorname{Def}_{L_0/b_0/L_0'} \arrow[r] & \operatorname{Def}_{P_0 \times L_0/c_0'/P_0' \times L_0'},
\end{tikzcd}$$

where c_0' denotes the composition $\Sigma_0 \xrightarrow{c_0} \Sigma_0' \subset P_0' \times L_0'$.

Proof. A deformation $f : \mathcal{A} \to \mathcal{A}'$ of $f_0 : \mathcal{A}_0 \to \mathcal{A}_0'$ is the data of deformations $a : P \to X$ of a_0, $b : L \to H$ of b_0, and $c : \Sigma \to \Sigma'$ of c_0 such that the following diagram commutes,

$$\begin{tikzcd}
\Sigma \arrow[r, "c"] & \Sigma' \\
\cap & \cap \\
P \times_A L \arrow[r, "a\times b"] & P' \times_A L'.
\end{tikzcd}$$

Thus the cartesian statement is given by construction. ☐

Proposition 4.3.2. *Suppose $\mathcal{A}_0$ is finite split over k, and $\mathcal{A}'_0$ with non-singular components over k (such as $\mathbb{P}^n_1$). Then $\mathrm{Def}_{\mathcal{A}_0/f_0/\mathcal{A}'_0}$ fits into the following cartesian diagram of functors,*

$$\begin{tikzcd}
\mathrm{Def}_{\mathcal{A}_0/f_0/\mathcal{A}'_0} \arrow[r] \arrow[d] & \mathrm{Hilb}_{\Gamma_c/\Sigma_0\times\Sigma'_0} \arrow[d] \\
\mathrm{Hilb}_{\Gamma_{a_0}/P_0\times P'_0} \times \mathrm{Hilb}_{\Gamma_{b_0}/L_0\times L'_0} \arrow[r] & \mathrm{Hilb}_{\Gamma_{c'}/\Sigma_0\times P'_0\times L'_0}.
\end{tikzcd}$$

Moreover, evaluating at $k[\epsilon]$ gives the following cartesian diagram of k-vector spaces.

$$\begin{tikzcd}
\mathrm{Def}_{\mathcal{A}_0/f_0/\mathcal{A}'_0}(k[\epsilon]) \arrow[r] \arrow[d] & \displaystyle\bigoplus_{(i,j)\in\Sigma_0}^{m} k^{\dim P'_0} \arrow[d] \\
\left(\displaystyle\bigoplus_{i=1}^{m} k^{\dim P'_0}\right) \times \left(\displaystyle\bigoplus_{j=1}^{n} k^{\dim L'_0}\right) \arrow[r] & \displaystyle\bigoplus_{(i,j)\in\Sigma_0} k^{\dim P'_0+\dim L'_0}.
\end{tikzcd}$$

Proof. This is an immediate application of proposition 4.3.2 and [Ser06, Proposition 3.4.2] ☐

4.4 Deformations of Closed Subconfigurations

Throughout, we will assume $\mathcal{A}_0 = (P_0, L_0, \Sigma_0)$ is a closed subconfiguration of a geometric configuration $\mathcal{X}_0 = (X_0, H_0, \Lambda_0)$ over a field k.

Definition 34. An **infinitesimal deformation** of $\mathcal{A}_0 \subset \mathcal{X}_0$ over artin local k-algebra A is given by the following cartesian diagram of configurations:

$$\begin{tikzcd}
\mathcal{A}_0 \arrow[r] \arrow[d] & \mathcal{A} \arrow[d] \\
\mathcal{X}_0 \arrow[r] & (\mathcal{X}_0)_A,
\end{tikzcd}$$

where $\mathcal{A}$ is flat over A. This defines a functor of artin rings $\mathrm{Hilb}_{\mathcal{A}_0/\mathcal{X}_0} : \mathrm{Art}_k \to \mathrm{Sets}$, which we call the **local hilbert functor** of $\mathcal{A}_0 \subset \mathcal{X}_0$.

Definition 35. A closed subconfiguration $\mathcal{A}_0 \subset \mathcal{X}_0$ yields a diagram of closed immersions of schemes

$$\begin{tikzcd}
\Sigma_0 \arrow[r] \arrow[d] & \Lambda_0 \arrow[d] \\
P_0 \times L_0 \arrow[r] & X_0 \times H_0,
\end{tikzcd}$$

which we call the incidence diagram of $\mathcal{A}_0 \subset \mathcal{X}_0$. A **deformation of the incidence diagram** over an artin local k-algebra A is the following cartesian diagram of schemes:

$$\begin{tikzcd}
\Sigma_0 \arrow[rr] \arrow[dr] \arrow[dd] & & \Lambda_0 \arrow[dr] \arrow[dd] & \\
& \Sigma \arrow[rr, crossing over] & & (\Lambda_0)_A \arrow[dd] \\
P_0 \times L_0 \arrow[rr] \arrow[dr] & & X_0 \times H_0 \arrow[dr] & \\
& D \arrow[rr] \arrow[from=uu, crossing over] & & (X_0 \times H_0)_A.
\end{tikzcd}$$

This defines a deformation functor $\mathrm{Def}_\bullet$.

Proposition 4.4.1. *The functor* $\mathrm{Def}_\bullet$ *of* $\mathcal{A}_0 \subset \mathcal{X}_0$ *fits into the following cartesian diagram of deformation functors,*

$$\begin{tikzcd}
\mathrm{Def}_\bullet \arrow[r] \arrow[d] & \mathrm{Hilb}_{\Sigma_0/\Lambda_0} \arrow[d] \\
FHilb_{\Sigma_0/P_0\times L_0/X_0\times H_0} \arrow[r] & \mathrm{Hilb}_{\Sigma_0/X_0\times H_0},
\end{tikzcd}$$

where $FHilb_{\Sigma_0/P_0\times L_0/X_0\times H_0}$ *denoted the flagged hilbert functor of the diagram of closed subschemes* $\Sigma_0 \subset P_0 \times L_0 \subset X_0 \times H_0$.

Proof. This is by the definition of $\mathrm{Def}_\bullet$.

□

Proposition 4.4.2. *The functor* $\mathrm{Hilb}_{\mathcal{A}_0/\mathcal{X}_0}$ *fits into the following cartesian diagram of deformation functors.*

$$\begin{tikzcd}
\mathrm{Hilb}_{\mathcal{A}_0/\mathcal{X}_0} \arrow[r] \arrow[d] & \mathrm{Def}_\bullet \arrow[d] \\
\mathrm{Hilb}_{P_0/X_0} \times Hilb_{L_0/H_0} \arrow[r] & \mathrm{Hilb}_{P_0\times L_0/X_0\times H_0}
\end{tikzcd}$$

Proof. This is an application of propositions 4.3.1 and 4.4.1. ☐

Proposition 4.4.3. *Consider a closed subconfiguration* $\mathcal{A} \subset \mathcal{X}$. *If* $\Sigma \subset P \times L$ *is étale (such as when* $\mathcal{A}$ *is split), then the projection map* $\mathrm{Def}_\bullet \to \mathrm{Hilb}_{\Sigma/\Lambda}$ *is an isomorphism.*

Proof. Since $\Sigma \subset P \times L$ is étale, we have that the natural morphism

$$\mathrm{Hilb}_{P/X} \times \mathrm{Hilb}_{L/H} \to Hilb_{P\times L/X\times H}$$

is an isomorphism. Thus the result is true by base-change. ☐

Corollary 4.4.4. *Consider a closed subconfiguration* $\mathcal{A} = (P, L, \Sigma) \subset \mathcal{X}$ *such that* $\Sigma \subset P \times L$ *is étale. Then* $\mathrm{Hilb}_{\mathcal{A}/\mathcal{X}}$ *fits into the following cartesian diagram,*

$$\begin{tikzcd}
\mathrm{Hilb}_{\mathcal{A}/\mathcal{X}} \arrow[r] \arrow[d] & \mathrm{Hilb}_{\Sigma/\Lambda} \arrow[d] \\
\mathrm{Hilb}_{P/X} \times \mathrm{Hilb}_{L/H} \arrow[r] & \mathrm{Hilb}_{P\times L/X\times H}
\end{tikzcd}$$

Proof. This is an immediate application of proposition 4.4.2 and 4.4.3. ☐

Chapter 5

Topics in Configurations

5.1 Interpolation

We work over field k. Let P be n points, and $\mathcal{A}$ be the split configuration of one block containing n points. Then the inclusion $P \subset \mathcal{A}$ induces a morphism of schemes $\mathrm{Re}_{\mathcal{A}}^{\mathcal{X}} \to \mathrm{Re}_{P}^{\mathcal{X}}$, where $\mathcal{X}$ is a geometric configuration. It is clear that $\mathrm{Re}_{P}^{\mathcal{X}} = X^n \setminus \Delta$, which is a quasi-projective variety.

Definition 36. We say $\mathcal{X}$ (resp. **generically**) **interpolates** n**-points** if the induced map

$$\mathrm{Re}_{\mathcal{A}}^{\mathcal{X}} \to \mathrm{Re}_{P}^{\mathcal{X}} = X^n \setminus \Delta$$

is (resp. generically) surjective.

Question 5.1.1. *For which n does a geometric space $\mathcal{X} = (X, H, \Lambda)$ generically interpolate n-points?*

Proposition 5.1.1. *The configuration $\mathbb{P}_1^2$ interpolates n-points if and only if $n = 1, 2$.*

Proof. This is a classical result. ☐

Proposition 5.1.2 (Atanasov, Larson, Yang)**.** *Let $\mathcal{H}_{d,g,r}$ denote the hilbert scheme of degree d curves of genus g in $\mathbb{P}^r$ (with $d \geq g + r$). The geometric configuration $\mathcal{X} = (\mathbb{P}^r, \mathcal{H}_{d,g,r}, \Lambda)$ generically interpolates n-points if and only if*

$$\begin{cases} (r-1)n \leq (r+1)d - (r-3)(g-1), & \textit{if } (d,g,r) \notin \{(5,2,3),(7,2,5)\} \\ n \leq 9, & \textit{if } (d,g,r) \in \{(5,2,3),(7,2,5)\}. \end{cases}$$

Proof. This is [ALY19, corollary 1.4].

☐

5.2 Mnëv-Sturmfels Universality

Let $\mathcal{Q}$ complete quadrangle configuration (see figure 1.5). A **$\mathcal{Q}$-based configuration** is the data of a configuration $\mathcal{X}$ over a scheme S and a closed subconfiguration $f : \mathcal{Q} \subset \mathcal{X}$. If $S = \operatorname{Spec} \mathbb{Q}$, then the natural action of $\operatorname{Aut} \mathbb{P}^2_1 \simeq PGL_3$ on $\operatorname{Re}^{\mathbb{P}^2_1}_{\mathcal{Q}}$ is fully-faithful, and thus $\operatorname{Re}^{\mathbb{P}^2_1}_{\mathcal{Q}}$ is a trivial left PGL_3-torsor over $\mathbb{Q}$.

As described in section 1.3, we can realize $\mathbb{P}^2_1$ as a $\mathcal{Q}$-based configuration by considering the standard complete quadrangle representation $f_{std} : \mathcal{Q} \subset \mathbb{P}^2_1$ over $\mathbb{Z}$. For a finite split configuration $\mathcal{A}$, it can be shown that the action $\operatorname{Aut} \mathbb{P}^2$ on $\operatorname{Re}^{\mathbb{P}^2_1}_{\mathcal{A}}$ has finite reduced stabilizer groups, and thus $[\operatorname{Re}^{\mathbb{P}^2_1}_{\mathcal{A}} / \operatorname{Aut} \mathbb{P}^2_1]$ is a Deligne-Mumford stack. Suppose $\mathcal{A}$ is also $\mathcal{Q}$-based, then by corollary 2.4.5 there is a morphism of schemes $\operatorname{Re}^{\mathcal{X}}_{\mathcal{A}} \to \operatorname{Re}^{\mathcal{X}}_{\mathcal{Q}}$. For sufficiently nice configurations $\mathcal{A}$, the moduli stack has coarse moduli space $BR^{\mathbb{P}^2_1}_{\mathcal{A}}$ fitting into a cartesian diagram

$$\begin{tikzcd}
BR^{\mathbb{P}^2_1}_{\mathcal{A}} \arrow[r] \arrow[d] & \operatorname{Re}^{\mathbb{P}^2_1}_{\mathcal{A}} \arrow[d] \\
\operatorname{Spec} \mathbb{Z} \arrow[r, "f_{std}"] & \operatorname{Re}^{\mathbb{P}^2_1}_{\mathcal{Q}} .
\end{tikzcd}$$

The morphism $BR^{\mathbb{P}^2_1}_{\mathcal{A}} \to \operatorname{Re}^{\mathbb{P}^2_1}_{\mathcal{A}}$ is a closed immersion as $f_{std} : \operatorname{Spec} \mathbb{Z} \to \operatorname{Re}^{\mathbb{P}^2_1}_{\mathcal{Q}}$ is a closed immersion. By [LV12], we have that every singularity type appearing on a finite-type scheme over $\operatorname{Spec} \mathbb{Z}$ appears in $BR^{\mathbb{P}^2_1}_{\mathcal{A}}$ for some $\mathcal{Q}$-based finite configuration $\mathcal{A}$.

We can generalize the above in order to determine whether Mnëv's theorem holds for other geometric configurations.

Definition 37. Let $\mathcal{B}$ be a configuration over S. A **$\mathcal{B}$-based configuration** is the data of a configuration $\mathcal{X}$ over S and a closed subconfiguration $f : \mathcal{B} \subset \mathcal{X}$.

Consider a finite configuration $\mathcal{A}$ and geometric configuration $\mathcal{X}$, both over S and both $\mathcal{B}$-based. Label $f : \mathcal{B} \subset \mathcal{X}$ as the given morphism of configurations. Construct $BR^{\mathcal{X}}_{\mathcal{A}}$ as the scheme-theoretic fiber of $\operatorname{Re}^{\mathcal{X}}_{\mathcal{A}} \to \operatorname{Re}^{\mathcal{X}}_{\mathcal{Q}}$ over $f \in \operatorname{Re}^{\mathcal{X}}_{\mathcal{Q}}$. A pair $(\mathcal{B}, \mathcal{X})$ of a geometric $\mathcal{B}$-based configuration $\mathcal{X}$ over $\operatorname{Spec} \mathbb{Z}$, is said to **satisfy Mnëv's theorem** if every singularity type appearing on a finite-type scheme over $\operatorname{Spec} \mathbb{Z}$ appears in $BR^{\mathcal{X}}_{\mathcal{A}}$ for some based finite configuration $\mathcal{A}$. The above discussion shows that the standard complete quadrangle defines a closed subconfiguration $\mathcal{Q} \subset \mathbb{P}^2_1$ for which the pair $(\mathcal{Q}, \mathbb{P}^2_1)$ satisfies Mnëv's theorem.

Question 5.2.1. *Which pairs $(\mathcal{B}, \mathcal{X})$ over $\operatorname{Spec} \mathbb{Z}$ satisfy Mnëv's theorem?*

5.3 The Image of a Nondegenerate Representation

Let $f : \mathcal{A} \to \mathcal{X}$ be a nondegenerate representation between a finite configuration $\mathcal{A} = (P, L, \Sigma)$ and geometric configuration $\mathcal{X} = (X, H, \Lambda)$, both over a scheme S. The closed immersion $a : P \to X$ determines a closed subscheme $Z_a \subset X$ trivially. The closed immersion $b : L \to H$ induces a closed subscheme $Z_b \subset X$ by the following construction. Let $\Lambda_L \subset X \times_S L$ be the restriction of the universal family $\Lambda \subset X \times_S H$, and the composition $p : \Lambda_L \subset X \times_S L \to X$ be the first projection map. The morphism $X \times_S L \to X$ is finite by base-change, and thus proper. Hence $p : \Lambda_L \to X$ is proper and as a consequence has closed image. Let $Z_b \subset X$ denote the scheme-theoretic image of $p : \Lambda_L \to X$.

Definition 38. The **induced closed subscheme** (or **image**) of a nondegenerate representation $f = (a, b, c) : \mathcal{A} \to \mathcal{X}$ is the closed subscheme $Z_f \subset X$ defined as the scheme-theoretic union,

$$Z_f = Z_a \cup Z_b,$$

where Z_a and Z_b are constructed above.

5.4 Intrinsic Geometry of Representations

Let $\mathcal{X} = (X, H, \Lambda)$ be a geometric configuration over S such that $H \subset \mathrm{Hilb}_{X/S}$ is closed under the action $\mathrm{Aut}\, X$ on $\mathrm{Hilb}_{X/S}$.

5.4.1 Aut $\mathcal{X}$-action on $\mathrm{Hom}(\mathcal{A}, \mathcal{X})$

Let $\mathrm{Aut}\, \mathcal{X}$ denote the automorphism group of $\mathcal{X}$ in $\mathrm{Config}(S)$. An automorphism σ of X induces an automorphism σ' of the Hilbert scheme $\mathrm{Hilb}_{X/S}$. Since H is closed under the automorphism group action by assumptions, the restriction of σ' to $H \subset \mathrm{Hilb}_{X/S}$, which we also label as σ', is well-defined. The pair σ and σ' induce an automorphism of $\mathcal{X}$ in $\mathrm{Config}(S)$. Moreover, this assignment defines a group homomorphism $\psi : \mathrm{Aut}\, X \to \mathrm{Aut}\, \mathcal{X}$.

Lemma 5.4.1. *Let $\mathcal{X} = (X, H, \Lambda)$ be a geometric configuration. Then the forgetful map to the point-structure*

$$\varphi : Aut\mathcal{X} \to Aut X$$

is an isomorphism of groups with inverse $\psi : Aut X \to Aut\mathcal{X}$, assigning

$$\sigma \mapsto (\sigma, \sigma', \sigma \times \sigma'|_{\Lambda}).$$

Proof. It is clear that ψ is a section of φ. To show $\psi \circ \varphi$ is the identity on $\mathcal{X}$, it suffices to show that for an automorphism (Id_X, b, c) on $\mathcal{X}$, that b is induced by Id_X (as described by ψ). If $H = \emptyset$ then we are done. Suppose not, we show $b([Z]) = [Z]$ for all $[Z] \in H$. For $x \in Z$, we have that $(x, [Z]) \in \Lambda$, so $c(x, [Z]) = (x, b([Z])) \in \Lambda$. Thus $x \in b([Z])$, and hence Z is a closed subscheme of $b([Z])$. By a symmetric argument, we have that $b([Z])$ is a closed subscheme of Z, hence $b([Z]) = [Z]$, as desired.

□

We can thus define an action of $\operatorname{Aut} X$ on $\operatorname{Hom}(\mathcal{A}, \mathcal{X})$, $\mathrm{Re}_{\mathcal{A}}^{\mathcal{X}}$, and $\mathrm{Real}_{\mathcal{A}}^{\mathcal{X}}$ by post-composition. Hence, we can form the quotient stacks

$$[\operatorname{Hom}(\mathcal{A}, \mathcal{X})/\operatorname{Aut} X], \qquad [\mathrm{Re}_{\mathcal{A}}^{\mathcal{X}}/\operatorname{Aut} X], \qquad [\mathrm{Real}_{\mathcal{A}}^{\mathcal{X}}/\operatorname{Aut} X].$$

These quotient stacks parameterize representations, nondegenerate representations, and realizations up to the automorphisms of $\mathcal{X}$.

Question 5.4.1. *Describe the* $\operatorname{Aut} X$*-invariant properties of* $\mathrm{Real}_{\mathcal{A}}^{\mathcal{X}}, \mathrm{Re}_{\mathcal{A}}^{\mathcal{X}}$, *and* $\operatorname{Hom}(\mathcal{A}, \mathcal{X})$.

Proposition 5.4.2. *Let* $\mathcal{A}$ *be a finite configuration over a field* k. *Then the following are algebraic stacks:*

$$[\operatorname{Hom}(\mathcal{A}, \mathbb{P}_1^n)/Aut\mathbb{P}_1^n], \qquad [\mathrm{Re}_{\mathcal{A}}^{\mathbb{P}_1^n}/Aut\mathbb{P}_1^n], \qquad [\mathrm{Real}_{\mathcal{A}}^{\mathbb{P}_1^n}/Aut\mathbb{P}_1^n].$$

Proof. Because $\operatorname{Aut} \mathbb{P}_1^n = PGL_{n+1}$ is a smooth affine group scheme over k, the result follows from [Alp25, Theorem 3.1.10]. □

5.4.2 Nondegenerate Representation on $\mathbb{P}_1^2$

We work over a field k of characteristic 0.

Definition 39. An n_b**-configuration** (or a b**-configuration**) is a split configuration $\mathcal{A}$ over a scheme S such that there are precisely n points and n lines with the conditions that every point is contained in precisely b lines and each line contains precisely b points.

Example 16. A 3_2-configuration is the standard triangle configuration (example 1). In general, if $b = 2$, then there is only one isomorphism class of n_2-configurations.

Definition 40. Let $\mathcal{A}$ and $\mathcal{X}$ be configurations over a scheme S. We say $\mathcal{A}$ is **rigid** in $\mathcal{X}$ if its realizations form a single class under the natural $\operatorname{Aut} \mathcal{X}$-action.

Conjecture 5.4.3 (Grünbaum)**.** *Let* k *be a field of characteristic 0. There are no rigid* b*-configurations in* $\mathbb{P}_{1,k}^2$ *for* $b \geq 3$.

Lemma 5.4.4. *Let k a field, possibly of positive characteristic. Let I be the inertia group scheme of the natural action of PGL_3 on U, and $F : I \to U$ the morphism. If $N > 4$, then $F : I \to U$ is unramified over a dense open $V \subset U$. If F is ramified, then the irreducible components of the ramification locus $Z = U \setminus V$ of F are of codimension at least 1 in U.*

Proof. The open subscheme $U \subset (\mathbb{P}^2)^{(N)}$ is a smooth variety over k, as it is the quotient of a smooth quasi-projective variety by a free finite group action. Consider the natural action map

$$f : PGL_3 \times U \to U \times U$$

defined by $(g, u) \mapsto (g \cdot u, u)$. Let I be the inertia group scheme of this action; that is, I is the pullback of the diagonal map $\Delta : U \to U \times U$ along f,

$$\begin{tikzcd}
I \arrow[r, "F"] \arrow[d] & U \arrow[d, "\Delta"] \\
PGL_3 \times U \arrow[r, "f"] & U \times U
\end{tikzcd}$$

By construction, I is the subscheme of $PGL_3 \times U$ parameterizing pairs (A, u) for which A stabilizes u. It is easy to see that $f : PGL_3 \times U \to U \times U$ is a quasi-finite morphism between smooth varieties, and the branch locus of F in U parameterizes N-points in $\mathbb{P}^2_k$ in general position that are permuted by a nontrivial automorphism. Let Z denote this branch locus, we show $V = U \setminus Z$ is dense in U. By base change, we see that F is a finite-type morphism between noetherian schemes, thus by [Stacks, Tag 039N] the non-branch locus $V \subset U$ is open. It is nonempty since there exists $u \in U$ with trivial stabilizer. Since U is a variety (in particular, irreducible), we have that $V \subset U$ is dense.

Suppose F is ramified, we prove the branch locus $Z = U \setminus V$ of F is of codimension 1 in U. The morphism f is affine by [Stacks, Tag 0ECD], and thus by [Gro67, Corollary 21.12.7] the non-branch locus $V' \subset U \times U$ of f is affine. Since U is separated over k, then Δ is a closed immersion, and thus $\Delta^{-1}(V') \subset U$ is an affine scheme. Since f is flat and locally of finite presentation, by [Stacks, Tag 02V4] we have that $\Delta^{-1}(V') = V$. Using [Gro67, Corollary 21.12.7] again on $V \subset U$ implies every irreducible component of $Z = U \setminus V$ is codimension 1 in U, as desired. ☐

Lemma 5.4.5. *Let k be a field of characteristic zero. If $1 \leq N \leq 3$, then $F : I \to U$ is ramified everywhere. If $N \geq 4$, then $F : I \to U$ is unramified. Moreover, if $N > 4$, then there exists a dense open $V \subset U$ for which F has trivial fibers.*

Proof. The statement for $1 \leq N \leq 3$ is clear. Let $N \geq 4$. An automorphism of $\mathbb{P}^2$ is completely determined by where it maps 4 points of general position, thus each fiber of F can be embedded in $S_{N'}$, for $N' = \binom{n}{4}$, such that the inclusion morphism is an open and closed immersion. Thus, the fibers of F are finite disjoint unions of $\operatorname{Spec} k$ and are thus unramified, implying F is unramified (see [Mil17, Proposition 3.2]). The last statement holds, as U is a variety over k and $V \subset U$ is a nonempty open, and thus dense. $\square$

Theorem 5.4.6. *Let $\mathcal{A}$ be an n_b-configuration over k for which there exists a nondegenerate representation $f : \mathcal{A} \to \mathbb{P}^2_1$. Let N be the maximum number of points in the image of f that are in general position. If $N \geq 4$, then $[\mathrm{Real}_{\mathcal{A}}^{\mathbb{P}^2_1}/PGL_3]$ is a Deligne-Mumford stack over k.*

Proof. The first statement follows from lemma 5.4.5 and [Alp25, Theorem 3.6.8].

$\square$

Corollary 5.4.7 (Grünbaum's Conjecture for $b = 3$)**.** *Let $\mathcal{A}$ be an 3-configuration over $\mathbb{R}$. If there exists a nondegenerate representation of $\mathcal{A}$ in $\mathbb{P}^2_1$, then $[\mathrm{Real}_{\mathcal{A}}^{\mathbb{P}^2_1}/Aut\mathbb{P}^2_1]$ is a Deligne-Mumford stack of dimension at least 1.*

Proof. By [Grü09], we know that m and n are at least 9. Apply theorem 5.4.6 and theorem 3.7.3 with $m = n = 9$ and $N = 3m$. $\square$

5.4.3 Desargues Configurations

Let D denote the Desargues configuration over field k of characteristic 0 (see figure 1.1). We describe a coarse moduli space for $[\mathrm{Re}_D^{\mathbb{P}^2_1}/PGL_3(k)]$. We are given a labeling for the points of D:

$$\mathcal{O}, a, b, c, A, B, C, \alpha, \beta, \gamma.$$

Since $\mathbb{P}^2$ is four-transitive, assume $\mathcal{O} = [1,1,1]$, $a = [1,0,0]$, $b = [0,1,0]$, and $c = [0,0,1]$, then $A = [r,1,1]$, $B = [1,s,1]$, and $C = [1,1,t]$ for some $r,s,t \in k$. With these coordinates, then $\alpha = [r-1, 1-s, 0]$, $\beta = [r-1, 0, 1-t]$, and $\gamma = [0, s-1, 1-t]$. The output line is given by the equation

$$\overline{\alpha\beta\gamma} : (s-1)(t-1)X + (r-1)(t-1)Y + (r-1)(s-1)Z = 0.$$

We now describe the conditions for $r,s,t \in \mathbb{A}^3$. Since each triple point is distinct, then $r,s,t \neq 1$. We also require that A,B,C are not collinear. Three points in $\mathbb{P}^2$ are collinear if and only if their corresponding pencil of lines are concurrent in $\check{\mathbb{P}}^2$. Thus A,B,C are collinear if and only if

$$f_1(r,s,t) = det \begin{pmatrix} r & 1 & 1 \\ 1 & s & 1 \\ 1 & 1 & t \end{pmatrix} = rst - (r+s+t) + 2 = 0.$$

From this computation we require

$$f_1(r,s,t) = rst - (r+s+t) + 2 \neq 0.$$

The last condition we require is that $\mathcal{O} \notin \overline{\alpha\beta\gamma}$, which occurs if and only if

$$f_2(r,s,t) = (s-1)(t-1) + (r-1)(t-1) + (r-1)(s-1) \neq 0.$$

In summary, the locus of (nondegenerate) Desargues configurations is given by the quotient complement of the surface in $\mathbb{A}^3$ whose defining equation is given by

$$f(r,s,t) = (r-1)(s-1)(t-1)f_1(r,s,t)f_2(r,s,t) = 0.$$

It is a degree 6 reducible surface.

Now we describe another coarse moduli space of $[\mathrm{Re}_{D}^{\mathbb{P}^2_1}/PGL_3]$. In 1883, Cyparissos Stephanos gave a deep relationship between Desargues configurations in $\mathbb{P}^2_{\mathbb{C}}$ and binary stable sextics. It was shown in [AL02] that the coarse moduli space M_D of $[\mathrm{Re}_{\mathcal{D}}^{\mathbb{P}^2_1}/PGL_3]$ admits an injective birational morphism to the space of binary stable sextics M_6^b, which we call the **Stephanos map**. Recall that M_6^b is the coarse moduli space of the stack $\mathcal{M}_2$ of genus two curves. We can ask whether the Stephanos map lifts to a morphism of stacks.

Question 5.4.2. *Is there a morphism of stacks* $[\mathrm{Re}_{\mathcal{D}}^{\mathbb{P}^2_1}/PGL_3] \to \mathcal{M}_2$ *that descends to the Stephanos map* $M_D \to M_6^b$ *as described in [AL02]?*

A clear obstruction is that the natural map $\mathcal{M}_2 \to M_6^b$ is a $\mathbb{Z}/2$-gerbe generated by the hyperelliptic involution.

Appendix A

Hom Scheme

A.1 Hom functor

Definition 41. Let X, Y be schemes over a scheme S. Define the functor $\mathrm{Hom}_S(X, Y)$ assigning an S-scheme T to the set

$$\mathrm{Hom}_S(X, Y)(T) = \mathrm{Hom}_T(X_T, Y_T).$$

We sometimes write $\mathrm{Hom}(X, Y)$ when the base scheme is clear.

Proposition A.1.1 (Grothendieck)**.** *Let S be a noetherian scheme. If X is projective and flat over S, and Y is quasi-projective over S, then* $\mathrm{Hom}(X, Y)$ *is representable by an open scheme of* $\mathrm{Hilb}_{X \times_S Y/S}$.

Proof. See [Fan+05, Theorem 5.23]. □

In this document, we are usually in the case when the source scheme is finite locally free over the base scheme.

Proposition A.1.2. *Let S' be a finite locally free scheme over a scheme S. If X is quasi-projective over S, then* $\mathrm{Hom}_S(S', X)$ *is representable by a scheme. If in addition the base scheme S is noetherian, then* $\mathrm{Hom}_S(S', X)$ *is quasi-projective over S.*

Proof. See [Ji+22, Theorem 1.3 and 1.10].

□

Proposition A.1.3. *Let S' be a finite locally free scheme over a scheme S, and let X and Y be quasi-projective schemes over S. Let P be one of the following properties*

(i) *monomorphism,*

(ii) *open immersion,*

(iii) *closed immersion,*

(iv) *separated,*

(v) *smooth,*

(vi) *étale,*

(vii) *locally of finite type,*

(viii) *locally of finite presentation,*

(ix) *finite presentation.*

If $f : X \to Y$ is a morphism of S-schemes with property P, then the induced map $\mathrm{Hom}_S(S', X) \to \mathrm{Hom}_S(S', Y)$ of S-schemes also has the property P.

Proof. See [Ji+22, Proposition 2.6]. □

A.2 Functor of closed immersions

Definition 42. Let X, Y be schemes over a scheme S. Define the subfunctor $\mathrm{Hom}^{cl}_S(X, Y) \subset \mathrm{Hom}_S(X, Y)$ assigning an S-scheme T to the set

$$\mathrm{Hom}^{cl}_S(X, Y)(T) = \{\varphi : X_T \to Y_T \mid \ \varphi \text{ is a closed immersion}\}.$$

We write $\mathrm{Hom}^{cl}(X, Y)$ when the base scheme is clear.

Proposition A.2.1. *Let P be a finite locally free scheme over S. If X is quasi-projective over S, then $\mathrm{Hom}^{cl}_S(P, X)$ is representable by an open subscheme of $\mathrm{Hom}_S(P, X)$.*

Proof. By proposition A.1.2, $\mathrm{Hom}(P, X)$ is representable by an S-scheme. Then this induces a universal morphism

$$\varphi : P \times_S \mathrm{Hom}(P, X) \to X \times_S \mathrm{Hom}(P, X).$$

Then $\mathrm{Hom}^{cl}(P, X) \subset \mathrm{Hom}(P, X)$ is precisely the locus where φ is a closed immersion, which is an open condition. □

Proposition A.2.2. *In the scenario of proposition A.1.3, if $f : X \to Y$ is a morphism of S-schemes with property P from A.1.3, then the induced map $\mathrm{Hom}^{cl}(S', X) \to \mathrm{Hom}^{cl}(S', Y)$ also has the property P.*

Proof. It is not hard to show that the natural inclusian diagram

$$\begin{tikzcd}
\mathrm{Hom}^{cl}(S', X) \arrow[r] \arrow[d] & \mathrm{Hom}^{cl}(S', Y) \arrow[d] \\
\mathrm{Hom}(S', X) \arrow[r] & \mathrm{Hom}(S', Y),
\end{tikzcd}$$

is a cartesian diagram of schemes. Thus the result follows from proposition A.1.3 and that property P is stable under base-change. □

A.3 Functor of cartesian diagrams

Definition 43. For a scheme S, let $Z \subset X$ and $Z' \subset X'$ be closed subschemes of S-schemes X and X', respectively. Define a functor

$$\mathrm{Hom}_S(Z \subset X, Z' \subset X') : (\mathrm{Sch}/S)^{op} \to Sets$$

assigning an S-scheme T to the set of morphisms of S-schemes $f : Z_T \to Z'_T$ and $g : X_T \to X'_T$ making the following diagram commute

$$\begin{tikzcd}
Z_T \arrow[r, "f"] \arrow[d, phantom, "\cap"] & Z'_T \arrow[d, phantom, "\cap"] \\
X_T \arrow[r, "g"] & X'_T.
\end{tikzcd} \tag{A.3.1}$$

We write $\mathrm{Hom}(Z \subset X, Z' \subset X')$ when the base scheme is clear.

Proposition A.3.1. *Let Z and X be finite locally free schemes over a scheme S, and Z' and X' both quasi-projective and flat over S. Then* $\mathrm{Hom}(Z \subset X, Z' \subset X')$ *is representable by an S-scheme.*

Proof. It is not hard to show that the following is cartesian diagram of functors,

$$\begin{tikzcd}
\mathrm{Hom}(Z \subset X, Z' \subset X') \arrow[r] \arrow[d] & \mathrm{Hom}(Z, Z') \arrow[d] \\
\mathrm{Hom}(X, X') \arrow[r] & \mathrm{Hom}(Z, X').
\end{tikzcd}$$

By proposition A.1.2, we have that $\mathrm{Hom}(Z \subset X, Z' \subset X')$ is the fiber product of schemes, as desired. □

Proposition A.3.2. *Suppose we satisfy the hypothesis of proposition A.3.1. Let H denote the locus in* $\mathrm{Hom}(Z \subset X, Z' \subset X')$ *such that diagram A.3.1 is cartesian. Then H is an open subscheme of* $\mathrm{Hom}(Z \subset X, Z' \subset X')$.

Proof. Let $\mathcal{U}$ denote the universal family of $\operatorname{Hom}(Z \subset X, Z' \subset X')$, then the natural $\mathcal{U} \to \operatorname{Hom}(Z \subset X, Z' \subset X')$ is the data of a commutative diagram of schemes

$$\begin{tikzcd}
Z_{\mathcal{U}} \arrow[r, "f"] \arrow[d, phantom, "\cap"] & Z'_{\mathcal{U}} \arrow[d, phantom, "\cap"] \\
X_{\mathcal{U}} \arrow[r, "g"] & X'_{\mathcal{U}}.
\end{tikzcd}$$

Define the scheme

$$P = X_{\mathcal{U}} \times_{X'_{\mathcal{U}}} Z'_{\mathcal{U}},$$

then there exists a unique morphism $\varphi : Z_{\mathcal{U}} \to P$ respecting the above diagram. The locus $H \subset \operatorname{Hom}(Z \subset X, Z' \subset X')$ is precisely the locus in $\mathcal{U}$ where φ is an isomorphism, which is an open condition. ☐

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