COALESCENCE OF SYNCHRONOUS COUPLINGS

Krzysztof Burdzy and **Zhen-Qing Chen**

Abstract. We consider a pair of reflected Brownian motions in a Lipschitz planar domain starting from different points but driven by the same Brownian motion. First we construct such a pair of processes in a certain weak sense, since it is not known whether a strong solution to the Skorohod equation in Lipschitz domains exists. Then we prove that the distance between the two processes converges to zero with probability one if the domain has a polygonal boundary or it is a "lip domain", i.e., a domain between the graphs of two Lipschitz functions with Lipschitz constants strictly less than 1.

1. Introduction. We will consider a "synchronous coupling" of two Brownian motions reflected inside a planar domain D, i.e., a vector-valued process (X_t, Y_t) such that X_t and Y_t are reflected Brownian motions in D, and for every (random) interval (t_1, t_2) such that neither X_t nor Y_t hits the boundary of D when $t \in (t_1, t_2)$, we have $X_s - Y_s = X_u - Y_u$ for all $s, u \in (t_1, t_2)$. We will give a more precise definition of the synchronous coupling in the next section. Intuitively, the two processes X_t and Y_t move in unison as long as they both stay strictly inside the domain D. However, when one of them hits the boundary, that process experiences an additional "push," proportional to the local time on the boundary, which keeps the process inside the domain. We will consider only processes with the normal vector of reflection.

We will address the following question.

Question 1.1. *Do processes X_t and Y_t have to approach each other, i.e., is it true that $\lim_{t\to\infty} |X_t - Y_t| = 0$, a.s.?*

We will prove that the answer is "yes" for two classes of bounded domains.

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Recall that a domain D is called Lipschitz if for every point $x \in \partial D$, there is a neighborhood of x inside which D may be represented as the region above the graph of a Lipschitz function in some orthonormal coordinate system.

We will call D a *lip domain* if D is a Lipschitz domain and its boundary consists of the union of graphs of functions f_1 and f_2, i.e.,

$$D = \{(x_1, x_2) : x_2 = f_1(x_1), z_1 \le x_1 \le z_2\} \cup \{(x_1, x_2) : x_2 = f_2(x_1), z_1 \le x_1 \le z_2\},$$

such that $f_1(z_1) = f_2(z_1)$, $f_1(z_2) = f_2(z_2)$, $f_1(x_1) < f_2(x_1)$ for $z_1 < x_1 < z_2$, and functions f_1 and f_2 are Lipschitz with a constant $c_0 \in (0,1)$, i.e., for $k = 1, 2$,

$$|f_k(x_1) - f_k(\widetilde{x}_1)| \le c_0 |x_1 - \widetilde{x}_1|, \qquad \text{for all } z_1 \le x_1, \widetilde{x}_1 \le z_2. \tag{1.1}$$

Note that the assumption that D is a Lipschitz domain puts additional constraints on functions f_k besides the conditions (1.1). See (4.1)-(4.2) in Section 4 for a more formal treatment of this technical point. Figure 1 presents three examples of lip domains; D_1, D_2 and the interior of $\overline{D_1 \cup D_2}$ are all lip domains.

Figure 1.

Our main result is the following.

Theorem 1.1. *The distance between X_t and Y_t converges to* 0 *a.s. if $D \subset \mathbf{R}^2$ is a bounded domain satisfying one of the following conditions:*

(i) the boundary of D is a polygon or a finite union of disjoint polygons, or

(ii) D is a lip domain.

We will show in a forthcoming paper (Burdzy, Chen and Jones (2002)) that if D is a smooth domain then the answer to Question 1.1 may be positive or negative, depending on the geometry of the domain. The number of holes in D plays an important role for smooth domains, unlike in the case of polygonal domains.

One can construct reflected Brownian motion in a polygonal domain as a unique strong solution to a stochastic differential equation with a *given* "driving" Brownian motion. This easily shows that the synchronous coupling of reflected Brownian motions in a polygonal domain is unique in law. We do not know how to extend this claim to synchronous couplings in Lipschitz domains. Theorem 1.1(ii) holds for every synchronous coupling of reflected Brownian motions in D, as defined in Section 2.

Our article is inspired by some results on couplings in Burdzy and Kendall (2000) and Cranston and Le Jan (1989, 1990). If a domain D is convex then the answer to Question 1.1 is obviously "yes" because the vector of reflection is normal to the boundary and so its component parallel to the line passing through both particles is always pointing towards the other reflecting Brownian particle and never away from it. Cranston and Le Jan (1989, 1990) addressed a finer question: Do the processes meet in a finite time? They showed that the answer is "no", first for a disc D and then for a large class of planar convex domains. They also noted that the answer is "yes" if the boundary of the domain contains two perpendicular line segments.

Burdzy and Kendall (2000) studied the relationship between a coupling and the spectral gap for the Neumann problem in D. It had been known for some time that, informally speaking, if $E|X_t - Y_t| \approx \exp(-\mu t)$ then μ is a lower bound for the spectral gap. Burdzy and Kendall (2000) proved that μ is equal to the spectral gap for some couplings in some domains. Hence, it might be of interest to estimate the rate of convergence of $E|X_t - Y_t|$ to 0. The present paper is concerned with the related question of the pointwise convergence of $|X_t - Y_t|$ to 0 in the hope that the results and methods may shed some light on the quality of couplings as a technique for estimating the spectral gap.

The proof of Theorem 1.1(ii) is based on the techniques developed by Burdzy and Kendall (2000) and Bañuelos and Burdzy (1999). The first part of Theorem 1.1 is proved in a completely different way which is, in a sense, more elementary.

Section 2 presents some results on synchronous couplings in Lipschitz domains which have some independent interest, for example, they have been already applied by Atar and Burdzy (2002). In particular, we prove that the transition density $p(t, x, y)$ for reflected Brownian motion in a Lipschitz domain is jointly Hölder continuous in all three variables.

Couplings in polygonal domains are discussed in Section 3. Section 4 is devoted to lip domains. The core arguments proving both parts of Theorem 1.1 are rather short. The proof of part (i) is based on analysis of the coupling when both processes are close to a vertex of the polygon ∂D. Part (ii) is proved using a "monotonicity" property of synchronous couplings, first established in Burdzy and Kendall (2000) and Bañuelos and Burdzy (1999). The proof of part (i) needs some "self-evident" lemmas which require rather long, complicated and technical arguments. We have to show that the processes X_t and Y_t will not stay at a distance greater than some fixed $\varepsilon > 0$ forever, but will come close to each other, at least from time to time.

At times we will be informal with our notation—we will identify points in the plane with vectors; occasionally we will switch between vector and complex notation.

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2. Construction of synchronous couplings. Constructing a reflecting Brownian motion in an arbitrary Euclidean domain is a notoriously difficult task. The problem is mainly caused by the non-smoothness of the boundary. We will first recall a definition of reflected Brownian motion in a piecewise smooth domain. Then we will present a construction of a synchronous coupling of reflected Brownian motions in bounded Lipschitz domains in n-dimensional Euclidean space, though we will concentrate on planar Lipschitz domains in later sections.

Given a smooth domain $D \subset \mathbf{R}^n$, say C^3, and a standard Brownian motion B_t starting from 0, the origin, it is well known that there is a unique continuous conservative strong Markov process X_t adapted to the Brownian filtration and taking values in $\overline{D}$ such that

$$X_t = X_0 + B_t + \int_0^t \mathbf{n}(X_s) dL_s, \tag{2.1}$$

where $\mathbf{n}$ is the unit inward normal vector field on ∂D and L_t is the local time process of X on the boundary ∂D. The local time has the following properties,

$$L_t = \int_0^t 1_{\partial D}(X_s) dL_s \quad \text{and} \quad E\left[\int_0^t 1_{\partial D}(X_s) ds\right] = 0.$$

Process X_t is called a reflecting Brownian motion on $\overline{D}$. Formula (2.1) is called the Skorohod decomposition for X. In fact, for C^3-smooth domains, the following deterministic Skorohod problem can be solved. Given any continuous path $b(t)$ with $b(0) \in \overline{D}$, there are two unique continuous functions $x(t)$ and $l(t)$ such that $x(t) \in \overline{D}$ for all t, l is an increasing boundary function with $l(0) = 0$ that increases only when $x(t)$ is on the boundary ∂D, and

$$x(t) = b(t) + \int_0^t \mathbf{n}(x(s)) dl(s), \quad t \geq 0.$$

Hence, in sufficiently smooth domains we can identify the reflecting Brownian motion X in $\overline{D}$ with the solution x of the "deterministic" Skorohod problem where $b(t) = X_0 + B_t$.

It is still an open problem whether one can find a strong solution to the Skorohod equation (2.1) in any bounded Lipschitz domain D, given a Brownian motion B_t. The best known result so far gives an affirmative answer for bounded $C^{1,\alpha}$-domains—see Bass and Hsu (2000). However it is known that when D is a bounded Lipschitz domain one can always find a pair of continuous processes (X_t, W_t) so that (2.1) holds with W_t in place of B_t, where W_t is a Brownian motion with respect to the filtration generated by (X, W), X is a strong Markov process taking values in $\overline{D}$, and L is the local time of X on ∂D. The process X_t solving (2.1) in this weak sense is also called a reflecting Brownian motion in $\overline{D}$ with starting point X_0. It is known in this case that the distribution of X_t is unique. In fact the transition density $p(t, x, y)$ is the Neumann heat kernel in D and it is proved in Bass and Hsu (1991) that $p(t, x, y)$ is continuous on $\mathbf{R}_+ \times \overline{D} \times \overline{D}$.

If D is a bounded piecewise smooth Lipschitz domain, we will argue that (2.1) has a strong solution. Since D is a bounded Lipschitz domain, there exists a bounded extension operator that maps the Sobolev space $W^{1,2}(D)$ into $W^{1,2}(\mathbf{R}^n)$, where

$$W^{1,2}(D) \stackrel{\text{def}}{=} \{u \in L^2(D, dx) : \nabla u \in L^2(D, dx)\},$$

(cf. Chen (1993)). This implies (cf. Fukushima, Oshima and Takeda (1994)) that the capacity associated with reflecting Brownian motion is controlled by the capacity of Brownian motion in $\mathbf{R}^n$. So reflecting Brownian motion X does not visit any boundary set that is not hit by Brownian motion. In particular, reflecting Brownian motion on $\overline{D}$ will not visit the nonsmooth points of ∂D. The existence of a strong solution for (2.1) now follows from the results on the Skorohod problem in smooth domains and a standard localization argument.

Returning back to the case of bounded Lipschitz domains, we are going to strengthen a result of Bass and Hsu (1991) and show that $p(t, x, y)$ is Hölder continuous on $\mathbf{R}_+ \times \overline{D} \times \overline{D}$. In order to construct a synchronous coupling of reflected Brownian motions in bounded Lipschitz domains, we will establish uniform Hölder continuity of the Neumann heat kernels for an increasing sequence of Lipschitz domains.

Recall that a bounded domain D is said to be Lipschitz if there are constants $r_0 > 0$, $M > 0$ so that for every $z \in \partial D$ there is a ball $B(z, r_0)$ centered at z with radius r_0 such that $D \cap B(x, r_0)$ is the region above the graph of a Lipschitz function with Lipschitz constant no larger than M. We call constants (r_0, M) the Lipschitz characteristics of domain D. Now suppose that D is a bounded Lipschitz domain in $\mathbf{R}^n$. One can find an increasing sequence of piecewise smooth domains $\{D_k\}$ such that $\overline{D_k} \subset D_{k+1}$ with $\bigcup_{k\geq 1} D_k = D$

and such that they have the same Lipschitz characteristics (r_0, M). The existence of such $\{D_k\}_{k\geq 1}$ is especially evident when $n = 2$. For notational convenience, we denote D by D_∞ and let $p_k(t, x, y)$ be the Neumann heat kernel on $\overline{D}_k$, for $k = 1, 2, \cdots, \infty$.

Theorem 2.1 *For each fixed $t_0 > 0$, there are constants $c = c(t_0) > 0$ and $\gamma > 0$, independent of k, such that*

$$|p_k(t, x, y) - p_k(s, x_1, y_1)| \leq c \left(\sqrt{t - s} + |x - x_1| + |y - y_1|\right)^\gamma \tag{2.2}$$

for all $t_0 \leq s \leq t \leq 1$, all (x, y), $(x_1, y_1) \in \overline{D_k} \times \overline{D_k}$, and all $k = 1, 2, \ldots, \infty$.

Proof. Under our assumptions, there are positive constants c_1, c_2 independent of k such that

$$p_k(t, x, y) \leq c_1 t^{-n/2} \exp\left(-\frac{|x - y|^2}{c_2 t}\right) \quad \text{for } x, y \in \overline{D}_k \text{ and } t \leq 1 \tag{2.3}$$

(cf. Bass and Hsu (1991)). This implies in particular that for each $\varepsilon > 0$,

$$\sup_{\varepsilon \leq t \leq 1/\varepsilon} \sup_k \|p_k(t, \cdot, \cdot)\|_\infty < \infty.$$

By Nash's Hölder continuity result (cf. Stroock (1988)), estimate (2.2) is evident when x, y, x_1 and y_1 are inside D with distances to ∂D_k being larger than $\frac{r_0}{4M}$. As D_k is bounded, there exists a finite cover of $\{x \in \overline{D_k} : \text{dist}(x, \partial D_k) \leq \frac{r_0}{8M}\}$ consisting of balls $B(z_i, \frac{r_0}{4M})$, where $z_i \in \partial D_k$ and $1 \leq i \leq K$. It suffices to prove (2.2) in each $B(z_i, \frac{r_0}{4M}) \cap \overline{D_k}$. There is a coordinate system $(y_1, \cdots, y_{n-1}, y_n) := (\widetilde{y}, y_n)$ centered at point z_i and a Lipschitz function f with Lipschitz constant no larger than M such that $B(z_i, r_0) \cap \overline{D_k} = \{(\widetilde{y}, y_n) \in B(z_i, r_0) : y_n > f(y_1, \cdots, y_{n-1})\}$. Define a one-to-one map $\phi : \phi(\widetilde{y}, y_n) = (\widetilde{y}, y_n - f(\widetilde{y}))$. Assume without loss of generality that $M > 1$ and note that

$$\phi(B(z_i, r_0/(2M)) \cap \overline{D_k}) \subset \{(\widetilde{w}, w_n) : |\widetilde{w}| < r_0/(2M),\ 0 < w_n < r_0/2\} \subset \phi(B(z_i, r_0) \cap \overline{D_k}).$$

As f is Lipschitz, the Jacobians of ϕ and its inverse ϕ^{-1} are bounded, with the bound depending only on the Lipschitz constant M. If $u(t, y)$ is a solution of the heat equation in $[0, \infty) \times B(z_i, r_0) \cap \overline{D_k}$ with Neumann boundary conditions on $[0, \infty) \times \partial D_k$, then it is easy to check that $u(t, \phi^{-1}(x))$ solves a parabolic equation in $[0, \infty) \times \phi(B(z_i, r_0) \cap \overline{D_k})$. The equation has divergence form with bounded and uniformly elliptic coefficients. The bounds on the size of the coefficients and the ellipticity constant do not depend on k. Let $A_{k,i}$ be the "mirror" reflection of $\phi(B(z_i, r_0) \cap \overline{D_k})$ with respect to the hyperplane $\{(\widetilde{y}, y_n) : y_n = 0\}$. The corresponding reflection of the equation yields a solution to a parabolic equation of the same type in the set $[0,\infty) \times [A_{k,i} \cup \phi(B(z_i, r_0) \cap \overline{D_k})]$. This set includes $[0,\infty) \times \{(\widetilde{y}, y_n) : |\widetilde{y}| < r_0/(2M),\, |y_n| < r_0/2\}$. Applying this argument to the heat kernel $p_k(t,x,y)$ we see that $p_k(t,x,\phi^{-1}(w))$ satisfies a parabolic equation in $[0,\infty) \times \{(\widetilde{w}, w_n) : |\widetilde{w}| < r_0/(2M),\, 0 < w_n < r_0/2\}$ and so by Nash's Hölder continuity result,

$$|p_k(t,x,\phi^{-1}(w)) - p_k(s,x_1,\phi^{-1}(v)| \le c(t_0)\left(\sqrt{t-s} + |w-v|\right)^\gamma$$

for all $1 \ge t \ge s \ge t_0$ and $w, v \in \{(\widetilde{w}, w_n) : |\widetilde{w}| < r_0/(4M),\, 0 < w_n < r_0/4\}$. Since ϕ is Lipschitz, the function $(t,y) \to p_k(t,x,y)$ is Hölder continuous in $(t,y) \in (t_0,1] \times \left(B(z_i, r_0/4M) \cap \overline{D_k}\right)$. By symmetry, $(t,x) \to p_k(t,x,y)$ is also Hölder continuous. Combining Hölder continuity in x and y variables, we obtain (2.2) on $(t_0,1] \times (B(z_i, r_0/4M) \cap \overline{D_k}) \times (B(z_i, r_0/4M) \cap \overline{D_k})$. As indicated earlier in the proof, this implies Hölder continuity of $p_k(t,x,y)$ on $(t_0,1] \times \overline{D_k} \times \overline{D_k}$, uniform in k. □

Given a Brownian motion B_t, for $1 \le k < \infty$, let X^k_t be the unique reflecting Brownian motion on $\overline{D}_k$ satisfying

$$X^k_t = X^k_0 + B_t + \int_0^t \mathbf{n}_k(X^k_s)\, dL^{X^k}_s,$$

where L^{X^k} is the boundary local time of X^k on ∂D_k and $\mathbf{n}_k$ is the unit inward normal of ∂D_k. We use P^k_x to denote the law of X^k starting from x, that is with $X^k_0 = x$. For reflecting Brownian motion X^k with initial distribution ν, its law will be denoted by P^k_ν. The notation P_x and P_ν will be also used for Brownian motion in Lipschitz domains where we do not necessarily have strong solutions to the Skorohod equation but we do have uniqueness in law for the weak solutions.

The following two results extend the main result in Burdzy and Chen (1998). The approximation result proved in Burdzy and Chen (1998) is for reflecting Brownian motions in an increasing sequence of smooth domains approaching an arbitrary open domain D, with all processes in the sequence starting from a common fixed point in D. Here we assume D is Lipschitz.

Lemma 2.2. *For any sequence $x_k \in \overline{D}_k$ that converges to $x_\infty \in \overline{D}$, the finite dimensional distributions of $\{X^k_t,\, t \ge 0\}$ under $P^k_{x_k}$ converge to those of $\{X_t,\, t \ge 0\}$ under P_{x_∞}.*

Proof. It is proved in Theorem 3.6 in Chen (1993) that for each $T > 0$, there is a subset H of $[0,T]$ with Lebesgue measure T such that for $x \in D$, the finite dimensional distributions of $\{X^k_t,\, t \in H\}$ under P^k_x converge to those of $\{X_t,\, t \in H\}$ under P_x. Hence for any $t \in H$

and any bounded continuous function f on $\overline{D}$, $P_t^k f(x)$ converges to $P_t f(x)$ pointwise and in L^2. Thus for $0 < t_1 < t_2 < \cdots < t_k$ in H and bounded continuous functions $f_1, \cdots, f_k$ on $\overline{D}$,

$$E_{x_k}^k \left[\prod_{i=1}^k f_i(X_{t_i}^k) \right] = \int_{D_k} p_k(t_1, x_k, y) f_1(y) P_{t_2 - t_1}^k \left(f_2(P_{t_3-t_2}^k f_3 \cdots) \cdots \right) (y)\, dy$$

converges to

$$\int_D p(t_1, x_\infty, y) f_1(y) P_{t_2-t_1} \left(f_2(P_{t_3-t_2} f_3 \cdots) \cdots \right) (y)\, dy = E_{x_\infty} \left[\prod_{i=1}^k f_i(X_{t_i}) \right]$$

by the equi-continuity of $x \to p_k(t, x, y)$. By the equi-continuity of $t \to p_k(t, x, y)$, the above convergence holds for any $0 < t_1 < t_2 < \cdots < t_k$ and bounded continuous functions $f_1, \cdots, f_k$ on $\overline{D}$, and so the conclusion of the lemma follows. □

Theorem 2.3. *For every sequence of probability measures ν_k on $\overline{D_k}$, $\{P_{\nu_k}^k,\ k \geq 1\}$ is tight on $C([0,\, \infty), \mathbf{R}^n)$, the space of continuous $\mathbf{R}^n$-valued functions equipped with the local uniform topology. If ν_k converge weakly to a probability measure ν on $\overline{D}$ as $k \to \infty$, then $P_{\nu_k}^k$ converge weakly to P_ν on $C([0,\, \infty), \mathbf{R}^n)$.*

Proof. In view of Lemma 2.2 it suffices to show that the family $\{P_{\nu_k}^k\}$ is tight on $C([0,\, \infty), \mathbf{R}^n)$.

Fix a small $r > 0$ and define $\tau_k = \inf\{t > 0 : |X_t^k - X_0^k| \geq r\}$. It follows from Theorem 3.2 of Bass and Hsu (1991) that there is a constant $c > 0$ such that for each $k \in [k_0,\, \infty]$,

$$P_x^k(t > \tau_k) \leq c \exp\left(-\frac{r}{c\,t} \right) \quad \text{for all } t > 0 \text{ and } x \in \overline{D_k}. \tag{2.4}$$

Let $a > 0$. For each $T > 0$ and $\varepsilon > 0$, by (2.4) and the strong Markov property of X^k,

$$\begin{aligned} P_{\nu_k}^k \left(\sup_{\substack{0 \leq s,t \leq T \\ |t-s| \leq \delta}} \left| X_t^k - X_s^k \right| > \varepsilon \right) &\leq P_{\nu_k}^k \left(a \geq \tau_k \right) + P_{\nu_k}^k \left(\sup_{\substack{a \leq s,t \leq T \\ |t-s| \leq \delta}} \left| X_t^k - X_s^k \right| > \varepsilon,\ a < \tau_k \right) \\ &\leq c \exp\left(-\frac{r}{c\,a} \right) + P_{\mu_k}^k \left(\sup_{\substack{0 \leq s,t \leq T \\ |t-s| \leq \delta}} \left| X_t^k - X_s^k \right| > \varepsilon \right), \end{aligned} \tag{2.5}$$

where μ_k is the sub-probability distribution at time a of reflecting Brownian motion X^k in $\overline{D_k}$ killed upon leaving $B(X_0^k, r) \cap \overline{D_k}$. Let m_k denote the Lebesgue measure on D_k. Note that $\mu_k(dy) \le \left(\int_{\overline{D}_k} p_k(a,x,y)\nu_k(dx)\right) dy \le C m_k(dy)$, where $C := \sup_k \|p_k(a,\cdot,\cdot)\|_\infty < \infty$. We will use an idea of Takeda (Theorem 3.1 of Takeda (1989)) to show that

$$\lim_{\delta\to 0} \sup_{k\ge 1} P^k_{\mu_k}\left(\sup_{\substack{0\le s,t\le T\\ |t-s|\le\delta}} \left|X^k_t - X^k_s\right| > \varepsilon\right) = 0. \tag{2.6}$$

By Lyons-Zheng's forward and backward martingale decomposition (see Theorem 5.7.1 of Fukushima, Oshima and Takeda (1994)),

$$X^k_t - X^k_0 = \frac{1}{2}W^k_t - \frac{1}{2}\left(W^k_T\circ r^k_T - W^k_{T-t}\circ r^k_T\right) \quad \text{for all } 0\le t\le T,\ P^k_{m_k}\text{-a.s.},$$

where W^k is a martingale additive functional of X^k which is an n-dimensional Brownian motion, and r^k_T is the time reversal operator of X^k at time T, i.e., $X^k_t(r^k_T(\omega)) = X^k_{T-t}(\omega)$ for each $0\le t\le T$. Since X^k is symmetric under $P^k_{m_k}$, for $k\ge k_0$,

$$\begin{aligned}
P^k_{\mu_k}\left(\sup_{\substack{0\le s,t\le T\\ |t-s|\le\delta}} \left|X^k_t - X^k_s\right| > \varepsilon\right) &\le C\, P^k_{m_k}\left(\sup_{\substack{0\le s,t\le T\\ |t-s|\le\delta}} \left|X^k_t - X^k_s\right| > \varepsilon\right)\\
&\le C\, P^k_{m_k}\left(\sup_{\substack{0\le s,t\le T\\ |t-s|\le\delta}} \left|W^k_t - W^k_s\right| > \varepsilon\right)\\
&\quad + C P^k_{m_k}\left(\sup_{\substack{0\le s,t\le T\\ |t-s|\le\delta}} \left|W^k_{T-t}\circ r^k_T - W^k_{T-s}\circ r^k_T\right| > \varepsilon\right)\\
&= 2C\, P^k_{m_k}\left(\sup_{\substack{0\le s,t\le T\\ |t-s|\le\delta}} \left|W^k_t - W^k_s\right| > \varepsilon\right)\\
&= 2C\,|D_k|\, P\left(\sup_{\substack{0\le s,t\le T\\ |t-s|\le\delta}} |B_t - B_s| > \varepsilon\right),
\end{aligned}$$

where B is the standard n-dimensional Brownian motion and $|D_k|$ is the volume of D_k. Claim (2.6) follows because

$$\lim_{\delta\to 0} P\left(\sup_{\substack{0\le s,t\le T\\ |t-s|\le\delta}} |B_t - B_s| > \varepsilon\right) = 0.$$

From (2.5) and (2.6),

$$\limsup_{\delta\to 0} \sup_{k\ge 1} P^k_{\nu_k}\left(\sup_{\substack{0\le s,t\le T\\ |t-s|\le\delta}} \left|X^k_t - X^k_s\right| > \varepsilon\right) \le c\exp\left(-\frac{r}{c\,a}\right).$$

Letting $a \downarrow 0$, we have that for each $T > 0$ and $\varepsilon > 0$,

$$\lim_{\delta \to 0} \sup_{k \geq 1} P^k_{\nu_k} \left(\sup_{\substack{0 \leq s,t \leq T \\ |t-s| \leq \delta}} \left| X^k_t - X^k_s \right| > \varepsilon \right) = 0.$$

Thus the family $\{P^k_{\nu_k},\, k \geq 1\}$ is tight on $C([0,\, \infty), \mathbf{R}^n)$.

Now assume that ν_k converge weakly to ν. Since each of weak limit distributions of $P^k_{\nu_k}$'s must be equal to P_ν by Lemmas 2.1 and 2.2, we conclude that $P^k_{\nu_k}$'s converge weakly to P_ν on $C([0,\, \infty), \mathbf{R}^n)$ as $k \to \infty$. □

Let B_t be a standard Brownian motion starting from 0; it will serve the role of the "driving Brownian motion" for reflecting Brownian motions. As ∂D_k is piecewise smooth, given two points $x_k, y_k \in \overline{D_k}$, there exist unique reflecting Brownian motions X^k and Y^k driven by the same Brownian motion B_t:

$$X^k_t = x_k + B_t + \int_0^t \mathbf{n}_k(X^k_s) dL^{X^k}_s$$
$$Y^k_t = y_k + B_t + \int_0^t \mathbf{n}_k(Y^k_s) dL^{Y^k}_s.$$

We will call (X^k_t, Y^k_t) a synchronous coupling of reflected Brownian motions in D_k. The above construction does not apply in a Lipschitz domain because we do not know if the stochastic differential equations have strong solutions in such a case. The following is the main result of this section. It presents a "weak" construction of a synchronous coupling of reflected Brownian motions in a bounded Lipschitz domain D.

Theorem 2.4. *Consider a sequence of synchronous couplings (X^k_t, Y^k_t) of reflecting Brownian motion in $\overline{D_k}$ with $(X^k_0, Y^k_0) = (x_k, y_k)$ for some $x_k, y_k \in \overline{D_k}$.*

(i) The sequence of distributions of (X^k, Y^k) is tight on $C([0,\, \infty), \mathbf{R}^n \times \mathbf{R}^n)$.

(ii) Suppose that $x_k \to x_\infty$ and $y_k \to y_\infty$. Let (X, Y) be any subsequential limit of (X^k, Y^k). Its components X and Y are reflecting Brownian motion in $\overline{D}$ starting from x_∞ and y_∞. Moreover there is a continuous process W which is a Brownian motion with respect to the filtration generated by (W, X, Y) such that (X, Y) admits the following Skorohod representation:

$$X_t = x_\infty + W_t + \int_0^t \mathbf{n}(X_s) dL^X_s \quad \text{and} \quad Y_t = y_\infty + W_t + \int_0^t \mathbf{n}(Y_s) dL^Y_s,$$

where L^X and L^Y are boundary local times of X and Y, respectively. In particular, $P(X_t \in \partial D) = 0$, for every fixed $t > 0$.

Proof. Let (X_t^k, Y_t^k) be the synchronous coupling of reflecting Brownian motion in $\overline{D_k}$ with $(X_0^k, Y_0^k) = (x_k, y_k)$, driven by a Brownian motion B^k. By Theorem 2.3, we see that the sequence of distributions of (B^k, X^k, Y^k) is tight on the space $C([0,\infty), (\mathbf{R}^n)^3)$. Consider any subsequential limit (W, X, Y) of distributions of (B^k, X^k, Y^k). Using the Skorohod lemma, we may assume, by changing the underlying probability spaces if necessary, that processes (B^k, X^k, Y^k) converge to (W, X, Y) almost surely. We will prove that W is a Brownian motion with respect to the filtration generated by (W, X, Y). To see this, note that for any $t > s > s_m > \cdots > s_1$, bounded continuous functions $f_1, \cdots, f_m$ on $\mathbf{R}^{3n}$ and a bounded continuous function ϕ on $\mathbf{R}^n$,

$$
\begin{aligned}
&E\left[\phi(W_t - W_s)\prod_{i=1}^m f_i(W_{s_i}, X_{s_i}, Y_{s_i})\right] \\
&\quad = \lim_{k\to\infty} E\left[\phi(B_t^k - B_s^k)\prod_{i=1}^m f_i(B_{s_i}^k, X_{s_i}^k, Y_{s_i}^k)\right] \\
&\quad = \lim_{k\to\infty} E\left[\phi(B_t^k - B_s^k)\right] \, E\left[\prod_{i=1}^m f_i(B_{s_i}^k, X_{s_i}^k, Y_{s_i}^k)\right] \\
&\quad = E\left[\phi(B_t - B_s)\right] E\left[\prod_{i=1}^m f_i(W_{s_i}, X_{s_i}, Y_{s_i})\right],
\end{aligned}
$$

which shows that $W_t - W_s$ has a Gaussian distribution with zero mean and variance $t - s$ which is independent of the filtration $\sigma\{(W_r, X_r, Y_r) : r \le s\}$. Similarly it can be shown that W is a standard Brownian motion with respect to the filtration generated by (W, X) as well as with respect to the filtration generated by (W, Y). It follows from Theorem 2.3 that X and Y are reflecting Brownian motions in $\overline{D}$ starting from x_∞ and y_∞, respectively. So $P(X_t \in \partial D) = \int_{\partial D} p(t, x_\infty, y)dy = 0$ for any $t > 0$. Note that by (2.3),

$$
\sup_k E_{x_k}^k \left[L_t^{X^k}\right] = \sup_k \int_0^t \int_{\partial D_k} p_k(s, x, y)\sigma_k(dy)ds \le c\sqrt{t},
$$

where σ_k is the surface measure of ∂D_k. Since $\int_0^t \mathbf{n}_k(X_s^k)dL_s^{X^k} = X_t^k - x_k - B_t^k$ converges weakly to $A_t := X_t - x_\infty - W_t$, we have for any partition $\{t_i,\, 0 \le i \le l\}$ of interval $[0, t]$,

$$
E\left[\sum_{i=1}^l \left|A_{t_i} - A_{t_{i-1}}\right|\right] = \lim_{k\to\infty} E_{x_k}^k \left[\sum_{i=1}^l \left|\int_{t_{i-1}}^{t_i} \mathbf{n}_k(X_s^k)dL_s^{X^k}\right|\right] \le \limsup_{k\to\infty} E_{x_k}^k \left[L_t^{X^k}\right] \le c\sqrt{t}.
$$

Hence A_t is a continuous process of finite variation and therefore $X_t = x_\infty + W_t + A_t$ is a semimartingale. Similarly it can be shown that $C_t := Y_t - y_\infty - W_t$ is a continuous process of finite variation and so $Y_t = y_\infty + W_t + C_t$ is a semimartingale. As A_t, C_t are measurable with respect to the filtrations generated by (W, X) and (W, Y) respectively, W is the unique continuous martingale part in the Doob-Meyer decomposition of the continuous semimartingales X and Y with respect to the filtrations generated by (W, X) and (W, Y), respectively. Processes X and Y have the desired Skorohod decompositions since they are reflecting Brownian motions in D starting from x_∞ and y_∞, respectively. $\square$

Remarks 2.5 (i) In view of Theorem 2.4, we are justified to call any subsequential limit (X, Y) from any subsequential limit (X, Y) of (X^k, Y^k) a synchronous coupling of reflecting Brownian motion in $\overline{D}$.

(ii) Though the weak convergence in Theorem 2.4 is stated and proved for synchronous reflecting Brownian motions having deterministic starting points, the theorem holds for those with random starting points as well. The same proof shows that if the initial distributions for synchronous couplings of reflecting Brownian motions (X^k, Y^k) converge weakly to a probability distribution ν on $\overline{D} \times \overline{D}$, then the sequence of the distributions of (X^k, Y^k) is tight on $C([0, \infty) \times \mathbf{R}^n \times \mathbf{R}^n)$, and any of its subsequential limits is a synchronous coupling of reflecting Brownian motions in $\overline{D}$ with initial distribution ν. Hence every synchronous coupling of reflecting Brownian motions (X, Y) in $\overline{D}$ has Markov property in the following sense: for any $s > 0$, the process $t \to (X_{s+t}, Y_{s+t})$ is a synchronous coupling of reflecting Brownian motions in $\overline{D}$ with the initial distribution equal to that of (X_s, Y_s).

3. Polygonal domains. The proof of Theorem 1.1 (i) consists of two major steps. First we will prove that the distance between X_t and Y_t has to be small from time to time. Then we will show that if the processes start not far from each other then there is good a chance that the distance between them will get smaller and smaller in time. The first step will require a series of lemmas.

Let us recall the Skorohod construction of a reflected path generated from a "free" path. The construction is real analytic in nature and so it applies path by path to processes with continuous trajectories. First we will discuss the case of normal reflection on a straight line.

Suppose that $\Gamma : [s_1, s_2) \to \mathbf{R}^2$ is a continuous function; we will often take $[s_1, s_2) = [0, \infty)$. Let $\Gamma(t) = (\Gamma_1(t), \Gamma_2(t))$, assume that $\Gamma_2(s_1) \geq 0$ and let $\Gamma_2^*(t) = \Gamma_2(t) - 0 \wedge \inf_{s_1 \leq s \leq t} \Gamma_2(s)$. We will call the function $\Gamma^*(t) = (\Gamma_1(t), \Gamma_2^*(t))$ the Skorohod transform of Γ relative to the halfspace $H = \{(x_1, x_2) : x_2 \geq 0\}$. In other words, Γ^* is Γ reflected on the x_1-axis (horizontal axis).

A more familiar form of the Skorohod transform is the following. Let $L^\Gamma_t = 0 \wedge \inf_{s_1 \le s \le t} \Gamma_2(s)$ and let $\mathbf{n} = (0,1)$. Then

$$\Gamma^*_t = \Gamma(t) + \int_{s_1}^t \mathbf{n} dL^\Gamma_s, \tag{3.1}$$

for $t \in [s_1, s_2)$.

We will record for future reference several simple properties of the Skorohod mapping.

(SM1) If $[s_3, s_4) \subset [s_1, s_2)$ and $\{\Gamma^*(t), t \in [s_1, s_2)\}$ is a normal reflection of $\{\Gamma(t), t \in [s_1, s_2)\}$ in H then $\{\Gamma^*(t), t \in [s_3, s_4)\}$ is a normal reflection of $\{\Gamma(t) + \Gamma^*(s_3) - \Gamma(s_3), t \in [s_3, s_4)\}$ in H.

(SM2) The normal reflection does not depend on the parameterization of a curve which is reflected, i.e., if $\psi : [s_1, s_2) \to [s_3, s_4)$ is continuous, strictly increasing and one-to-one, and $\Lambda(t) = \Gamma(\psi^{-1}(t))$ then if $\{\Gamma^*(t), t \in [s_1, s_2)\}$ is the normal reflection of $\{\Gamma(t), t \in [s_1, s_2)\}$ in H and $\{\Lambda^*(t), t \in [s_3, s_4)\}$ is the normal reflection of $\{\Lambda(t), t \in [s_3, s_4)\}$ in H then $\Lambda^*(t) = \Gamma^*(\psi^{-1}(t))$.

(SM3) If $\Gamma(t)$ is a piecewise linear function and $\Gamma^*(t)$ is its normal reflection in a halfplane then $|\frac{d}{dt}\Gamma^*(t)| \le |\frac{d}{dt}\Gamma(t)|$ for all t where the derivatives are well defined.

The above remarks (SM1)-(SM3), suitably modified, apply to the normal reflection in polygonal domains.

Next we discuss the case of normal reflection in polygonal domains. We start with convex domains. We refer the reader to Dupuis and Ishii (1991) for the proofs of the following results. If D is a convex polygonal domain and $\{\Gamma(t), t \in [s_1, s_2)\}$ is a piecewise linear function with $\Gamma(s_1) \in \overline{D}$ then it has a unique normal reflection $\{\Gamma^*(t), t \in [s_1, s_2)\}$ in D. Moreover, the Skorohod mapping is continuous in convex polygonal domains, i.e., there exists $\alpha = \alpha(D) < \infty$ such that if $\{\Gamma(t), t \in [s_1, s_2)\}$ and $\{\Lambda(t), t \in [s_1, s_2)\}$ are piecewise linear then

$$\sup_{t \in [s_1, s_2)} |\Gamma^*(t) - \Lambda^*(t)| \le \alpha \sup_{t \in [s_1, s_2)} |\Gamma(t) - \Lambda(t)|.$$

In this paper, we do not want to limit ourselves to convex domains. The lack of convexity presents some new challenges. The reflection might not be uniquely defined at vertices where the domain is not locally convex. We will tackle the problem of non-uniqueness by perturbing the original function $\{\Gamma(t), t \in [s_1, s_2)\}$ so that its normal reflection avoids the non-convex vertices. The details are presented in the next lemma.

Suppose D is a polygonal domain. We will call a vertex x of ∂D convex if for some neighborhood U of x, the set $U \cap D$ is convex. A vertex which is not convex will be called concave.

Consider a polygonal, not necessarily convex domain D and a piecewise linear function $\{\Gamma(t), t \in [s_1, s_2)\}$ with $\Gamma(s_1) \in \overline{D}$. Suppose that $\{\Gamma^*(t), t \in [s_1, s_2)\}$ and $\{\Gamma^{**}(t), t \in [s_1, s_2)\}$ are normal reflections of $\Gamma(t)$ in D in the following sense. If U is an open polygonal domain such that $U \cap D$ is convex and $\{\Gamma^*(t), t \in [s_3, s_4)\} \subset U$ then $\{\Gamma^*(t), t \in [s_3, s_4)\}$ is a normal reflection of $\{\Gamma(t) + \Gamma^*(s_3) - \Gamma(s_3), t \in [s_3, s_4)\}$ in $U \cap D$. Suppose the same applies to $\Gamma^{**}(t)$. Moreover, suppose that Γ^* and Γ^{**} do not visit concave vertices of ∂D. Then $\Gamma^* \equiv \Gamma^{**}$. This claim follows from the uniqueness of the Skorohod mapping in convex polygonal domains via a localization argument. We will call Γ^* a normal reflection of Γ in D, provided it satisfies the conditions listed above.

Lemma 3.1. *Suppose D is a polygonal domain and $\{\Gamma(t), t \in [s_1, s_2]\}$ is a piecewise linear function with $\Gamma(s_1) \in \overline{D}$ whose range does not contain any concave vertices of ∂D. Fix arbitrarily small $\delta > 0$. Then there exists a piecewise linear function $\{\widetilde{\Gamma}(t), t \in [s_1, s_2]\}$ with $\widetilde{\Gamma}(s_1) = \Gamma(s_1)$, satisfying the following properties. There exists a normal reflection $\{\widetilde{\Gamma}^*(t), t \in [s_1, s_2]\}$ of $\widetilde{\Gamma}(t)$ in $\overline{D}$ which does not hit any concave vertices of ∂D. Moreover, $|\widetilde{\Gamma}(t) - \Gamma(t)| < \delta$ for $t \in [s_1, s_2]$ and $|\frac{d}{dt}\widetilde{\Gamma}(t)| \leq (1 + \delta)|\frac{d}{dt}\Gamma(t)|$ for every t where the derivatives exist.*

Proof. We will construct a finite sequence of curves Γ^j whose last element will satisfy the conditions of the lemma. Let $\Gamma^1 = \Gamma$ and let s_3 be the supremum of s such that $\{\Gamma^1(t), t \in [s_1, s)\}$ has a normal reflection in D which does not visit any concave vertices. If $s_3 = s_2$ then we are done. Suppose that $s_3 < s_2$. Then $\Gamma^{1,*}(s_3-)$ is a concave vertex y. We may choose the coordinate system so that $y = (0, 0)$ and $\Gamma^1(t) = \Gamma^1(s_3) - (t - s_3)(a, 0)$ for some $a \geq 0$, $s_4 > s_3$, and all $t \in [s_3, s_4]$.

There exists some $s_5 < s_3$ such that the functions Γ^1 and $\Gamma^{1,*}$ are linear on $[s_5, s_3]$. It is easy to see that we can find small $\varepsilon_1, \varepsilon_2 > 0$ with the following properties. The distance from $\Gamma^{1,*}(s_5)$ to $(0, 0)$ is greater than ε_1. Let $s_6 \in (s_5, s_3)$ be the smallest s with $|\Gamma^{1,*}(s)| = \varepsilon_1$. Since $(0, 0)$ is a concave vertex, at least one of the points $(0, \varepsilon_2)$ or $(0, -\varepsilon_2)$ belongs to D and at least one of the line segments connecting $\Gamma^{1,*}(s_6)$ to $(0, \varepsilon_2)$ and $(0, -\varepsilon_2)$ lies in D, except possibly for the endpoint $\Gamma^{1,*}(s_6)$. Assume that this property holds for $(0,\varepsilon_2)$. Then, if ε_1 and ε_2 are small and properly chosen,

$$|\Gamma^{1,*}(s_6) - (0,\varepsilon_2)| \le (1+\delta)(s_3 - s_6)\left|\frac{d}{dt}\Gamma^{1,*}(s_6)\right|. \tag{3.2}$$

Let

$$\mathbf{v} = \frac{(0,\varepsilon_2) - \Gamma^{1,*}(s_6)}{s_3 - s_6}.$$

We now define a new piecewise linear function $\Gamma^2(t)$. We let $\Gamma^2(t) = \Gamma^1(t)$ for $t \le s_6$. For $t \in (s_6, s_3]$ we let $\Gamma^2(t) = \Gamma^1(s_6) + (t - s_6)\mathbf{v}$. Finally,

$$\Gamma^2(t) = \Gamma^1(t) - \Gamma^1(s_3) + \Gamma^1(s_6) + (0,\varepsilon_2) - \Gamma^{1,*}(s_6),$$

for $t \in (s_3, s_2)$.

It is routine to check that Γ^2 has the following properties. Let s_7 be the supremum of s such that $\{\Gamma^2(t), t \in [s_1, s)\}$ has a normal reflection in D which does not visit any concave vertices. Then $s_7 > s_3$. We have $\Gamma^{2,*}(t) = \Gamma^{1,*}(t)$ for $t \le s_6$, $\Gamma^{2,*}(s_3) = (0,\varepsilon_2)$, and $\Gamma^{2,*}$ is linear on $[s_6, s_3]$.

From property (SM3) of Skorohod mappings, $|\frac{d}{dt}\Gamma^{1,*}(t)| \le |\frac{d}{dt}\Gamma^1(t)|$ for $t \in [s_6, s_3)$. The definition of Γ^2 and (3.2) imply that $|\frac{d}{dt}\Gamma^2(t)| \le (1+\delta)|\frac{d}{dt}\Gamma^{1,*}(t)|$, for $t \in [s_6, s_3)$. Hence, $|\frac{d}{dt}\Gamma^2(t)| \le (1+\delta)|\frac{d}{dt}\Gamma^1(t)|$, for $t \in [s_6, s_3)$ and in fact for all $t \in [s_1, s_2]$ except possibly for finitely many points.

Consider arbitrarily small $\delta_1 > 0$. We can make the values of ε_1 and ε_2 smaller, if necessary, so that $|\Gamma^2(t) - \Gamma^1(t)| < \delta_1$, for all $t \in [s_1, s_2]$.

We continue the construction by induction. Given Γ^j, we construct Γ^{j+1} from Γ^j in a way analogous to the way Γ^2 was defined relative to Γ^1.

We will argue that the inductive procedure will terminate after a finite number of steps with a function satisfying the conditions of the lemma. Recall that $\Gamma^1(t) = \Gamma^1(s_3) - (t - s_3)(a, 0)$ for $t \in [s_3, s_4]$. Assume that s_4 is the largest s with the property that $\frac{d}{dt}\Gamma^1$ is constant on (s_3, s_4). Note that if $s_7 \le s_4$ then the vertex approached by $\Gamma^{2,*}$ at s_7 lies to the left of $(0,0)$. More generally, all concave vertices approached by $\Gamma^{2,*}$ at times $s \in (s_3, s_4)$ must lie to the left of $(0,0)$. It follows that the number of inductive steps needed to construct a function on (s_3, s_4) avoiding concave vertices is not larger than the number N_1 of all vertices of ∂D. Since Γ^1 is piecewise linear, its slope is constant on a finite number N_2 of intervals whose lengths sum up to $s_2 - s_1$. Our inductive procedure will require no more than $N_1 N_2$ steps.

If we let $\delta_1 = \delta/(N_1 N_2)$, then the function $\widetilde{\Gamma} = \Gamma^{N_1 N_2 + 1}$ will satisfy all the properties stated in the lemma. $\square$

Recall that a Brownian path reflected in a polygonal domain D does not hit any vertices of ∂D, a.s. The problem of non-uniqueness of the reflected paths does not arise in the case of Brownian trajectories.

Lemma 3.2. *Suppose that D is a polygonal domain and $\{\Gamma(t), t \in [0,v]\}$ is a piecewise linear function such that its normal reflection $\{\Gamma^*(t), t \in [0,v]\}$ in D does not visit any concave vertices of ∂D. There exist $\varepsilon > 0$ and $\beta < \infty$ such that the following holds. Assume that $\{f(t), t \in [0,v]\}$ is a continuous function with $\sup_{t\in[0,v]} |f(t) - \Gamma(t)| = a \le \varepsilon$, such that its normal reflection $\{f^*(t), t \in [0,v]\}$ in D does not hit any vertices of ∂D. Then $\sup_{t\in[0,v]} |f^*(t) - \Gamma^*(t)| \le a\beta$.*

Proof. The set of $t \in [0,v]$ such that $\Gamma^*(t) \in \partial D$ consists of a finite number of disjoint closed intervals $A_1, A_2, \dots, A_k$, ordered in this way. Some intervals A_j may be degenerate, i.e., they may consist of a single point. Find s_j and s'_j, $j = 1, \dots, k$, with the following properties. Every interval A_j is inside (s_j, s'_j), except when A_1 contains 0 then $s_1 = 0$ and $A_1 \subset [s_1, s'_1)$. Similarly, if $v \in A_k$ then $s'_k = v$ and $A_k \subset (s_k, s'_k]$. We require that $s'_j < s_{j+1}$ for $j = 1, \dots, k-1$. Let M_j denote the union of those edges of ∂D which are visited by $\Gamma^*(t)$ for $t \in A_j$. Choose $\delta > 0$ so small that

(i) if some continuous curve $\{f(t), t \in [0,v]\}$ satisfies $|f(t) - \Gamma^*(t)| \le \delta$ for $t \in [0,s]$ then $f(t)$ does not hit $\partial D \setminus M_j$ for any $t \in [s_j, s'_j] \cap [0,s]$, and

(ii) if $|f(t) - \Gamma^*(t)| \le \delta$ for $t \in [0,v]$ then $f(t)$ does not hit ∂D for $t \notin \bigcup_j [s_j, s'_j]$.

By Theorem 2.2 of Dupuis and Ishii (1991), for every convex polygonal domain $\widetilde{D}$ there exists a constant $\alpha = \alpha(\widetilde{D}) < \infty$ such that if $\{f_1(t), t \in [0,v]\}$ and $\{f_2(t), t \in [0,v]\}$ are continuous functions with $\sup_{t\in[0,v]} |f_1(t) - f_2(t)| \le a$ and $\{f_1^*(t), t \in [0,v]\}$ and $\{f_2^*(t), t \in [0,v]\}$ denote the normal reflections of f_1 and f_2 in $\widetilde{D}$ then $\sup_{t\in[0,v]} |f_1^*(t) - f_2^*(t)| \le a\alpha$.

Since $\Gamma^*(t)$ can visit only convex vertices, every set M_j is either a single edge of ∂D or it is the union of several adjacent edges with convex vertices between them. The obvious localization argument allows us to adapt Dupuis and Ishii's result on the Lipschitz continuity of the Skorohod map as follows. For every j there exists a constant $\alpha_j \in (1, \infty)$ with the following property. Suppose that $\{f(t), t \in [s_j, s'_j]\}$ is a continuous function with $\sup_{t\in[s_j,s'_j]} |f(t) - \Gamma(t)| \le a$, such that its normal reflection $\{f^*(t), t \in [0,v]\}$ in D does not hit $\partial D \setminus M_j$. Then $\sup_{t\in[s_j,s'_j]} |f^*(t) - \Gamma^*(t)| \le a\alpha_j$.

Let $\alpha = \max_j \alpha_j$ and $\varepsilon = \delta 2^{-k-2}(1+\alpha)^{-k}$.

Assume that $\{f(t), t \in [0,v]\}$ is a continuous function which satisfies

$$\sup_{t\in[0,v]} |f(t) - \Gamma(t)| \leq \delta_1 2^{-k}(1+\alpha)^{-k}, \tag{3.3}$$

where $\delta_1 \leq \delta$. We will show that $\sup_{t\in[0,v]} |f^*(t) - \Gamma^*(t)| \leq \delta_1$. Suppose otherwise and let $v_1 = \inf\{t : |f^*(t) - \Gamma^*(t)| \geq \delta_1\}$. The function $f^*(t)$ cannot hit ∂D at any $t \in [0, v_1] \setminus \bigcup_j [s_j, s'_j]$ in view of the definition of δ and the assumption that $\delta_1 \leq \delta$. We have chosen s_j's and s'_j's so that $\Gamma^*(t) \notin \partial D$ for $t \in [0, v_1] \setminus \bigcup_j [s_j, s'_j]$ so the distance between $f^*(t)$ and $\Gamma^*(t)$ remains fixed on every interval in $[0, v_1] \setminus \bigcup_j [s_j, s'_j]$. It follows that $v_1 \in [s_j, s'_j]$ for some j, say, for $j = j_0$. Suppose for the moment that $j_0 \geq 3$.

By (SM1), the functions $\{f^*(t), t \in [s_1, s'_1]\}$ and $\{\Gamma^*(t), t \in [s_1, s'_1]\}$ can be viewed as normal reflections in D of $\{f(t)+f^*(s_1)-f(s_1), t \in [s_1, s'_1]\}$ and $\{\Gamma(t)+\Gamma^*(s_1)-\Gamma(s_1), t \in [s_1, s'_1]\}$. The norm of the difference of the last two functions is bounded by $\delta_1 2^{-k}(1+\alpha)^{-k}$ in view of (3.3) because $f^*(s_1) = f(s_1)$ and $\Gamma^*(s_1) = \Gamma(s_1)$, The Lipschitz continuity of the Skorohod map implies that $|f^*(s'_1) - \Gamma^*(s'_1)| \leq \delta_1 2^{-k}(1+\alpha)^{-k}\alpha$. The same estimate holds at time s_2.

We iterate the estimate. We look at the functions $\{f^*(t), t \in [s_2, s'_2]\}$ and $\{\Gamma^*(t), t \in [s_2, s'_2]\}$ as normal reflections in D of $\{f(t) + f^*(s_2) - f(s_2), t \in [s_2, s'_2]\}$ and $\{\Gamma(t) + \Gamma^*(s_2) - \Gamma(s_2), t \in [s_2, s'_2]\}$. We have

$$\begin{aligned}
&|f(t) + f^*(s_2) - f(s_2) - (\Gamma(t) + \Gamma^*(s_2) - \Gamma(s_2))| \\
&\quad \leq |f(t) - \Gamma(t)| + |f(s_2) - \Gamma(s_2)| + |f^*(s_2) - \Gamma^*(s_2)| \\
&\quad \leq 2 \cdot \delta_1 2^{-k}(1+\alpha)^{-k} + \delta_1 2^{-k}(1+\alpha)^{-k}\alpha \\
&\quad \leq \delta_1 2^{-k+1}(1+\alpha)^{-k+1},
\end{aligned}$$

for $t \in [s_2, s'_2]$. By the Lipschitz continuity, $|f^*(s'_2) - \Gamma^*(s'_2)| \leq \delta_1 2^{-k+1}(1+\alpha)^{-k+1}\alpha$, and the same is true at time s_3. Proceeding by induction, we obtain

$$|f^*(s'_{j_0-1}) - \Gamma^*(s'_{j_0-1})| \leq \delta 2^{-k+j_0-2}(1+\alpha)^{-k+j_0-2}\alpha.$$

The same inequality is valid at time s_{j_0}. We now drop the assumption that $j_0 \geq 3$; it was imposed only to enable us to illustrate the method with a few non-trivial steps.

We repeat our earlier argument involving the translation of the paths, this time for the functions on the interval $[s_{j_0}, s'_{j_0}]$. We may view $\{f^*(t), t \in [s_{j_0}, s'_{j_0}]\}$ and $\{\Gamma^*(t), t \in [s_{j_0}, s'_{j_0}]\}$ as normal reflections in D of $\{f(t) + f^*(s_{j_0}) - f(s_{j_0}), t \in [s_{j_0}, s'_{j_0}]\}$ and $\{\Gamma(t) +$

$\Gamma^*(s_{j_0}) - \Gamma(s_{j_0}), t \in [s_{j_0}, s'_{j_0}]\}$. We have for $t \in [s_{j_0}, s'_{j_0}]$,

$$\begin{aligned}
&|f(t) + f^*(s_{j_0}) - f(s_{j_0}) - (\Gamma(t) + \Gamma^*(s_{j_0}) - \Gamma(s_{j_0}))| \\
&\leq |f(t) - \Gamma(t)| + |f(s_{j_0}) - \Gamma(s_{j_0})| + |f^*(s_{j_0}) - \Gamma^*(s_{j_0})| \\
&\leq 2 \cdot \delta_1 2^{-k}(1+\alpha)^{-k} + \delta_1 2^{-k+j_0-2}(1+\alpha)^{-k+j_0-2}\alpha \\
&\leq \delta_1 2^{-k+j_0-1}(1+\alpha)^{-k+j_0-1} \leq \delta_1/(2\alpha).
\end{aligned}$$

By the Lipschitz property of the Skorohod map, we must have $|f^*(t) - \Gamma^*(t)| \leq \delta_1/2$ on the interval $[s_{j_0}, s'_{j_0}]$. This contradicts the definition of j_0 and proves that $\sup_{t\in[0,v]} |f^*(t) - \Gamma^*(t)| \leq \delta_1$.

The lemma holds with ε defined earlier in the proof and $\beta = 2^k(1+\alpha)^k$. □

Lemma 3.3. *Suppose that (X_t, Y_t) is a synchronous coupling of reflected Brownian motions in D, starting from (x, y). For some $\varepsilon_1 > 0$, all $\varepsilon_2 > 0$, and all $x, y \in \overline{D}$, with probability 1, there exists $t < \infty$ such that $|X_t - Y_t| \leq \varepsilon_2$, $\operatorname{dist}(X_t, \partial D) \geq \varepsilon_1$ and $\operatorname{dist}(Y_t, \partial D) \geq \varepsilon_1$.*

Proof. *Step 1.* In this part of the proof, we will find points whose neighborhoods are visited infinitely often by the synchronous coupling. Let D_1 be a non-empty open subset of D, such that $\overline{D}_1 \subset D$. The stationary distribution of X_t has the uniform density in D. This clearly implies that $X_n \in D_1$ for infinitely many n, a.s. For every integer $k \geq 1$, choose a finite sequence of open balls $\mathcal{B}(z^k_j, 1/k)$ whose union covers $\overline{D}$. Let A_{j_1,j_2} be the event that $(X_n, Y_n) \in (\mathcal{B}(z^k_{j_1}, 1/k) \cap D_1) \times \mathcal{B}(z^k_{j_2}, 1/k)$ for infinitely many integers $n \geq 0$. Since $X_n \in D_1$ for infinitely many n, and the family of balls $\mathcal{B}(z^k_{j_2}, 1/k)$ is finite, the union of events A_{j_1,j_2} has probability one. Every event A_{j_1,j_2} belongs to the tail σ-field so its probability is 0 or 1. There are only finitely many events A_{j_1,j_2} so not all of them can have probability 0. Let j^k_1 and j^k_2 be such that $P(A_{j^k_1, j^k_2}) = 1$. By compactness, a subsequence of $(z^k_{j^k_1}, z^k_{j^k_1})$ converges to a pair $(z_1, z_2) \in \overline{D}_1 \times \overline{D}$.

Step 2. This step is devoted to the construction of a piecewise linear path $\{\Gamma(t), t \geq 0\}$ inside D with $\Gamma(0) = z_1$ which chases the normal reflection of $\{\Pi(t), t \geq 0\} \stackrel{\text{def}}{=} \{\Gamma(t) - z_1 + z_3, t \geq 0\}$, where $z_3 \in D$ is a point very close to z_2.

Fix some $\varepsilon_3 > 0$. We start by defining an initial piece of Γ. We choose $t_1 > 0$ and $\{\Gamma^1(t), 0 \leq t \leq t_1\}$ so that the range of this function is the shortest polygonal line inside $\overline{D}$ connecting $\Gamma^1(0) = z_1$ and $\Gamma^1(t_1) = z_2$. Moreover, we require that the derivative of $\Gamma^1(t)$ has the unit length for all t where it is defined. Next we slightly perturb Γ^1 so that it fits inside D. More precisely, we find a piecewise linear function $\{\Gamma^2(t), 0 \le t \le t_1\}$ such that $\Gamma^2(t) \notin \partial D$ for $t \in [0, t_1]$, $\Gamma^2(0) = \Gamma^1(0) = z_1$, $|\Gamma^2(t_1) - \Gamma^1(t_1)| \le \varepsilon_3$, and $|\frac{d}{dt}\Gamma^2(t)| = |\frac{d}{dt}\Gamma^1(t)| = 1$ for almost all $t \in [0, t_1]$. The existence of such a function Γ^2 is not hard to prove. Denote $\Gamma^2(t_1)$ by z_3 and note that $|z_3 - z_2| \le \varepsilon_3$, by the choice of Γ^2.

Using Lemma 3.1 we can modify $\{\Gamma^2(t) - z_1 + z_3,\ 0 \le t \le t_1\}$ to obtain a piecewise linear function $\{\Pi(t), 0 \le t \le t_1\}$ such that $\Pi(0) = z_3$, $\sup_{0 \le t \le t_1} |\Pi(t) - (\Gamma^2(t) - z_1 + z_3))|$ is so small that $\{\Pi(t) - z_3 + z_1,\ 0 \le t \le t_1\} \subset D$, and the normal reflection of $\{\Pi(t), 0 \le t \le t_1\}$ in D does not visit any concave vertices of ∂D. Moreover, $|\frac{d}{dt}\Pi(t)| \le (1 + 2^{-1})|\frac{d}{dt}\Gamma^2(t)| = (1 + 2^{-1})$ for all $t \in [0, t_1]$ where the derivatives are defined. Then we let $\Gamma(t) = \Pi(t) - z_3 + z_1$ for $t \in [0, t_1]$; note that this curve lies inside D.

We will use induction to extend $\{\Gamma(t), 0 \le t \le t_1\}$ to a piecewise linear path $\{\Gamma(t), t \ge 0\}$ inside D such that the normal reflection Π^* in D of the path $\Pi(t) \stackrel{\text{def}}{=} \Gamma(t) - z_1 + z_3$ does not visit any concave vertices of ∂D, $|\Gamma(s) - \Pi^*(s - t_1)| \le 2^{-n}$ for $s \in [nt_1, (n+1)t_1)]$ and $n \ge 1$, and

$$\left|\frac{d}{ds}\Gamma(s)\right| \le (1 + 2^{-n}) \left|\frac{d}{ds}\Pi^*(s - t_1)\right|$$

for almost every $s \in [nt_1, (n+1)t_2]$.

Suppose that $\{\Gamma(t), nt_1 \le t \le (n+1)t_1\}$ has been defined. Let $\Pi(t) = \Gamma(t) - z_1 + z_3$ for $t \in [nt_1, (n+1)t_1]$; an equivalent formula for $\Pi(t)$ is $\Gamma(t) - \Gamma(nt_1) + \Pi(nt_1)$. Let $\{\Pi^*(t), nt_1 \le t \le (n+1)t_1\}$ be the normal reflection of $\{\Pi(t) - \Pi(nt_1) + \Pi^*(nt_1), nt_1 \le t \le (n+1)t_1\}$ in D. The latter is the same as $\{\Gamma(t) - \Gamma(nt_1) + \Pi^*(nt_1),\ nt_1 \le t \le (n+1)t_1\}$. We set $\Gamma^1(t) = \Pi^*(t - t_1)$ for $t \in ((n+1)t_1, (n+2)t_1]$. We then modify $\{\Gamma^1(t), (n+1)t_1 \le t \le (n+2)t_1\}$ in several steps, just as in the case of $\{\Gamma^1(t), 0 \le t \le t_1\}$. First we find a piecewise linear function $\{\Gamma^2(t), (n+1)t_1 \le t \le (n+2)t_1\}$ lying strictly inside D such that $\Gamma^2((n+1)t_1) = \Gamma^1((n+1)t_1)$,

$$|\Gamma^2(t) - \Gamma^1(t)| \le 2^{-(n+2)},$$

and $|\frac{d}{dt}\Gamma^2(t)| = |\frac{d}{dt}\Gamma^1(t)|$ for almost all $t \in [(n+1)t_1, (n+2)t_1]$.

Next we use Lemma 3.1 to modify

$$\{\Gamma^2(t) + \Gamma^2((n+2)t_1) - \Gamma^2((n+1)t_1),\ (n+1)t_1 \le t \le (n+2)t_1\}$$

to obtain a piecewise linear function $\{\Pi(t),\ (n+1)t_1 \le t \le (n+2)t_1\}$ such that $\Pi((n+1)t_1) = \Gamma^2((n+2)t_1)$,

$$\sup_{(n+1)t_1 \le t \le (n+2)t_1} |\Pi(t) - (\Gamma^2(t) + \Gamma^2((n+2)t_1) - \Gamma^2((n+1)t_1))|$$

is smaller than $2^{-(n+2)}$ for $t \in [(n+1)t_1, (n+2)t_2]$ and so small that

$$\{\Pi(t) - \Pi((n+1)t_1) + \Gamma^2((n+1)t_1),\ (n+1)t_1 \le t \le (n+2)t_1\} \subset D.$$

Moreover, the normal reflection of $\{\Pi(t)-\Pi((n+1)t_1)+\Pi^*((n+1)t_1),\ (n+1)t_1 \le t \le (n+2)t_1\}$ in D does not visit any concave vertices of ∂D and $|\frac{d}{dt}\Pi(t)| \le (1+2^{-(n+1)})|\frac{d}{dt}\Gamma^2(t)|$ for all $t \in [(n+1)t_1, (n+2)t_1]$ where the derivatives are defined. Then we let $\Gamma(t) = \Pi(t) - \Pi((n+1)t_1) + \Gamma^2((n+1)t_1)$ for $t \in [(n+1)t_1, (n+2)t_1]$; this path lies inside D.

Proceeding by induction, we define $\Gamma(t)$ on $[0,\infty)$. Let $\Pi(t) = \Gamma(t) - z_1 + z_3$, which is a parallel translation of the path Γ. Denote by Π^* the normal reflection of Π in D. Note that $|\Gamma((n+2)t_1) - \Pi^*((n+1)t_1)| \le 2^{-(n+1)}$ for any $n \ge 0$.

Step 3. We will show that $\liminf_{n\to\infty} |\Gamma(nt_1) - \Pi^*(nt_1)| = 0$. Suppose that this is not the case. Then $|\Gamma(nt_1) - \Pi^*(nt_1)| \ge b$ for some $b > 0$ and large n. Since $|\Gamma((n+2)t_1) - \Pi^*((n+1)t_1)| \le 2^{-(n+1)}$, we must have $|\Gamma(nt_1) - \Gamma((n+1)t_1)| \ge b/2$ for large n. We will reparametrize Γ using a continuous strictly monotone one-to-one function $\psi : [0,\infty) \to [0,\infty)$ so that $\Lambda(t) = \Gamma(\psi^{-1}(t))$ is parameterized according to its length, i.e., $|\frac{d}{dt}\Lambda(t)| = 1$ for all t where the derivative is well defined. Let $s_n = \psi(nt_1)$ so that $\Lambda(s_n) = \Gamma(nt_1)$. Let $\Phi(t) = \Pi(\psi^{-1}(t))$ and note that the normal reflection of Φ in D is $\Phi^*(t) = \Pi^*(\psi^{-1}(t))$.

Recall that $|\Gamma((n+1)t_1) - \Pi^*(nt_1)| \le 2^{-n}$. It follows that

$$\Lambda(s_{n+1}) - \Lambda(s_n) = \Phi^*(s_n) - \Phi^*(s_{n-1}) + \mathbf{v}_n,$$

for some $\mathbf{v}_n$ satisfying $|\mathbf{v}_n| \le 2^{-n+2}$. From the "local time" representation (3.1) of reflected paths we obtain

$$\Lambda(s_{n+1}) - \Lambda(s_n) = \mathbf{v}_n + \Phi^*(s_n) - \Phi^*(s_{n-1}) = \mathbf{v}_n + \Lambda(s_n) - \Lambda(s_{n-1}) + \int_{s_{n-1}}^{s_n} \mathbf{n}_{\Phi^*(s)} dL_s^\Phi,$$

so, using induction,

$$\Lambda(s_{n+k+1}) - \Lambda(s_{n+k}) = \sum_{m=n}^{n+k} \mathbf{v}_m + \Lambda(s_n) - \Lambda(s_{n-1}) + \int_{s_{n-1}}^{s_{n+k}} \mathbf{n}_{\Phi^*(s)} dL_s^\Phi,$$

for integer $k \ge 0$. From this we obtain,

$$\begin{aligned}
\Lambda(s_{n+j+1}) - \Lambda(s_n) &= \sum_{k=0}^{j} \Lambda(s_{n+k+1}) - \Lambda(s_{n+k}) \\
&= \sum_{k=0}^{j} \left(\sum_{m=n}^{n+k} \mathbf{v}_m + \Lambda(s_n) - \Lambda(s_{n-1}) + \int_{s_{n-1}}^{s_{n+k}} \mathbf{n}_{\Phi^*(s)} dL_s^\Phi \right) \\
&= \sum_{k=0}^{j} \sum_{m=n}^{n+k} \mathbf{v}_m + (j+1)(\Lambda(s_n) - \Lambda(s_{n-1})) + \sum_{k=0}^{j} \left(\int_{s_{n-1}}^{s_{n+k}} \mathbf{n}_{\Phi^*(s)} dL_s^\Phi \right).
\end{aligned}$$

Let b_2 be the diameter of D and let j be 1 plus the integer part of $4b_2/b$. Since $|\Lambda(s_n) - \Lambda(s_{n-1}))| \geq b/2$ for large n,

$$|(j+1)(\Lambda(s_n) - \Lambda(s_{n-1}))| \geq 2b_2.$$

Consider n sufficiently large so that

$$\left| \sum_{k=0}^{j} \sum_{m=n}^{n+k} \mathbf{v}_m \right| \leq (j+1)2^{-n+4} \leq 2^{-n+4} 8b_2/b \leq b_2/2.$$

Since both points $\Lambda(s_{n+j+1})$ and $\Lambda(s_n)$ lie in D, we must have for some $k = k(n) \leq j$,

$$\left| \int_{s_{n-1}}^{s_{n+k}} \mathbf{n}_{\Phi^*(s)} dL_s^{\Phi} \right| \geq b_2/(2(j+1)) \geq b/8.$$

So for this $k = k(n)$, $L^{\Phi}(s_{n+k}) - L^{\Phi}(s_{n-1}) \geq b/8$. This shows that there is a constant $a \in (0,1)$ depending only on b and j (in fact, a can be taken to be $\sqrt{1 - b^2 16^{-2}(j+1)^{-2}}$) such that $|\frac{d}{ds}\Phi^*(s)| \leq a$ for s in a union of intervals in $[s_{n-1}, s_{n+k}]$ of total length no less than $b/16$. Hence, the length of the curve $\{\Phi^*(s), s \in [s_{n-1}, s_{n+k(n)}]\}$ is smaller than that of $\{\Phi(s), s \in [s_{n-1}, s_{n+k(n)}]\}$, and therefore of $\{\Lambda(s), s \in [s_{n-1}, s_{n+k(n)}]\}$, by $(1-a)b/16$.

Since $|\frac{d}{dt}\Pi^*(t)| \leq |\frac{d}{dt}\Pi(t)| = |\frac{d}{dt}\Gamma(t)|$ and $|\frac{d}{dt}\Gamma(t+t_1)| \leq (1 + 2^{-n-1})|\frac{d}{dt}\Pi^*(t)|$, for $t \in (nt_1, (n+1)t_1]$, we see that the length of $\{\Gamma(t), (n+1)t_1 \leq t \leq (n+2)t_1\}$ is not greater than $1 + 2^{-n-1}$ times the length of $\{\Pi^*(t), nt_1 \leq t \leq (n+1)t_1\}$; it is therefore bounded by $1 + 2^{-n-1}$ times the length of $\{\Gamma(t), nt_1 \leq t \leq (n+1)t_1\}$. Hence, for any $\delta > 0$, $m \geq 1$ and all sufficiently large n, the length of $\{\Gamma(t), nt_1 \leq t \leq (n+k(n)+1)t_1\}$ is not greater than $1 + \delta$ times the length of $\{\Gamma(t), (n-1)t_1 \leq t \leq (n+k(n))t_1\}$ and the length of $\{\Gamma(t), (n+k(n)+1)t_1 \leq t \leq (j+1)t_1\}$ is bounded by $1+\delta$ times the length of $\{\Gamma(t), (n+k(n))t_1 \leq t \leq jt_1\}$. Translated to the language of Λ, this means that for large n, the length of the curve $\{\Lambda(s), s \in [s_n, s_{n+k+1}]\}$ is smaller than that of $\{\Lambda(s), s \in [s_{n-1}, s_{n+k}]\}$ by $(1-a)b/32$, and the length of $\{\Lambda(s), s \in [s_{n+k+1}, s_{n+j+1}]\}$ is not greater than the length of $\{\Lambda(s), s \in [s_{n+k}, s_{n+j}]\}$ times $1+\delta$. Combining these estimates, we obtain for large n that the length of $\{\Lambda(s), s \in [s_n, s_{n+j+1}]\}$ is smaller than that of $\{\Lambda(s), s \in [s_{n-1}, s_{n+j}]\}$ by $(1-a)b/64$. As $(1-a)b/64$ is independent of n, an induction argument now implies that for sufficiently large n, the length of $\{\Lambda(s), s \in [s_n, s_{n+j+1}]\}$ is negative, which is impossible. This completes the proof of the claim that $\liminf_{n\to\infty} |\Gamma(nt_1) - \Pi^*(nt_1)| = 0$.

Step 4. Fix some $x, y \in \overline{D}$ and suppose that (X_t, Y_t) is a synchronous coupling of reflected Brownian motions in D, starting from (x, y). Recall z_1 and z_2 from Step 1 of the proof. Find $v < \infty$ such that $|\Gamma(v) - \Pi^*(v)| \leq \varepsilon_2/4$. Let δ_0 be the distance of $\{\Gamma(t), t \in [0, v]\}$ from ∂D and recall that $\delta_0 > 0$. Brownian paths reflected in a polygonal domain do not hit any vertices, a.s., so Lemma 3.2 can be applied to a Brownian path in place of the function f. Lemma 3.2 implies that we can find $\delta_1 \in (0, \varepsilon_2/4 \wedge \delta_0/2)$ so small that if $|f(t) - \Pi(t))| \leq \delta_1$ for all $t \in [0, v]$ then $|f^*(t) - \Pi^*(t)| \leq \varepsilon_2/4$ for $t \in [0, v]$. Let $\mathcal{B}_j = \mathcal{B}(z_j, \delta_1/3)$, for $j = 1, 2$. We may and will assume that $|z_3 - z_2| \leq \delta_1/3$. Let T_1 be the first time t when $X_t \in \mathcal{B}_1$ and $Y_t \in \mathcal{B}_2$. This stopping time is finite a.s., by the definition of z_1 and z_2. Recall that B_t is the Brownian motion "driving" X_t and Y_t. Let A be the event that $|B_{T_1+t} - B_{T_1} - \Gamma(t) + z_1| \leq \delta_1/3$ for all $t \in [0, v]$. By the support theorem, the probability of A is greater than some $p_1 > 0$. The event A implies that $X_t = B_t + X_{T_1} - B_{T_1}$ for $t \in [T_1, T_1 + v]$ because the process $t \to B_{T_1+t} + X_{T_1} - B_{T_1}$ will not hit the boundary on the interval $[0, v]$, by the choice of δ_0 and δ_1. Hence,

$$
\begin{aligned}
|X_{T_1+v} - \Gamma(v)| &= |B_{T_1+v} - \Gamma(v) + X_{T_1} - B_{T_1}| \\
&\leq |B_{T_1+v} - B_{T_1} - \Gamma(v) + z_1| + |X_{T_1} - z_1| \\
&\leq \delta_1/3 + \delta_1/3 < \delta_1 < \varepsilon_2/4.
\end{aligned}
$$

Recall that $\Pi(t) = \Gamma(t) - z_1 + z_3$. If A occurs then

$$
\begin{aligned}
&|B_{T_1+t} - B_{T_1} + Y_{T_1} - \Pi(t)| \\
&\qquad \leq |B_{T_1+t} - B_{T_1} - \Gamma(t) + z_1| + |Y_{T_1} - z_2| + |z_2 - z_3| \\
&\qquad \leq \delta_1/3 + \delta_1/3 + \delta_1/3 = \delta_1.
\end{aligned}
$$

for all $t \in [0, v]$. Since $\{Y_t, t \in [T_1, T_1 + v]\}$ is a normal reflection of $\{B_{T_1+t} - B_{T_1} + Y_{T_1}, t \in [T_1, T_1 + v]\}$ in D, the definition of δ_1 implies that $|Y(T_1 + t) - \Pi^*(t)| \leq \varepsilon_2/4$ for $t \in [0, v]$. Then

$$
\begin{aligned}
|X_{T_1+v} - Y_{T_1+v}| &\leq |X_{T_1+v} - \Gamma(v)| + |Y_{T_1+v} - \Pi^*(v)| + |\Pi^*(v) - \Gamma(v)| \\
&\leq \varepsilon_2/4 + \varepsilon_2/4 + \varepsilon_2/4 < 3\varepsilon_2/4.
\end{aligned}
$$

We see that $|X_{T_1+v} - Y_{T_1+v}| \leq 3\varepsilon_2/4$ with probability greater than p_1.

For $k \geq 2$, let T_k be the first time $t \geq T_{k-1} + v$, such that $X_t \in \mathcal{B}_1$ and $Y_t \in \mathcal{B}_2$. All stopping times T_k are finite a.s. The strong Markov property applied at time T_k implies that given the events $\{|X_{T_j+v} - Y_{T_j+v}| \leq 3\varepsilon_2/4\}$ did not occur for any $j < k$, we have $|X_{T_k+v} - Y_{T_k+v}| \leq 3\varepsilon_2/4$ with probability greater than p_1, by the same argument as above. This easily implies that $|X_{T_k+v} - Y_{T_k+v}| \leq 3\varepsilon_2/4$ for some k with probability 1.

It is easy to see that if $\varepsilon_1 > 0$ is sufficiently small then there exists $p_2 > 0$, such that for any $x, y \in \overline{D}$ with $|x - y| = 3\varepsilon_2/4$, if $(X_0, Y_0) = (x, y)$ then both X_t and Y_t will move at least ε_1 units away from the boundary before moving away from each other to the distance greater than ε_2, in 1 time unit, with probability greater than p_2. The proof of this fact is quite elementary and is left to the reader. We note that Lemma 4.1 tackles similar estimates in the considerably harder case of domains with Lipschitz boundaries.

Let S_1 be the first time t such that $|X_t - Y_t| \leq 3\varepsilon_2/4$, and let $S_k = \inf\{t > S_{k-1} + 1 : |X_t - Y_t| \leq 3\varepsilon_2/4\}$. The main part of the proof shows that all stopping times S_k are finite. By the strong Markov property applied at S_k, there exists $t \in [S_k, S_k + 1]$ such that $|X_t - Y_t| \leq \varepsilon_2$, $\text{dist}(X_t, \partial D) \geq \varepsilon_1$ and $\text{dist}(Y_t, \partial D) \geq \varepsilon_1$ with probability greater than p_2, assuming that there is no t with such properties in any of the intervals $[S_j, S_j + 1]$, $j = 1, 2, \dots, k-1$. A routine argument now shows that with probability 1, there exists a time t such that $|X_t - Y_t| \leq \varepsilon_2$, $\text{dist}(X_t, \partial D) \geq \varepsilon_1$ and $\text{dist}(Y_t, \partial D) \geq \varepsilon_1$. □

Let $K = \{re^{i\theta} \in \mathbf{C} : r > 0, \theta \in (0, \theta_0)\}$, a wedge with angle $\theta_0 \in (0, 2\pi)$. In Lemma 3.4 we will consider a process (X_t, Y_t) involving a sequence of synchronous couplings in K. Later on, we will apply the lemma to a synchronous coupling of reflected Brownian motions in a polygonal domain. Such a pair of processes behaves like a synchronous coupling of reflected Brownian motions in a wedge K whenever both processes are inside a neighborhood of a vertex of ∂D.

Processes (X_t, Y_t) in the statement of Lemma 3.4 are supposed to have the following properties. Fix some integer n, not necessarily positive. Assume there exist sequences of stopping times $S_k, k \geq 1$ and $U_k, k \geq 1$, such that

(i) $S_{k+1} > U_k > S_k$ for all k,

(ii) $|X_{S_k}| = 2^n$ for every k,

(iii) $\{(X_t, Y_t), t \in [S_k, U_k]\}$ is a synchronous coupling of reflected Brownian motions in K, independent of $\{(X_t, Y_t), t \in [0, S_k)\}$ given (X_{S_k}, Y_{S_k}),

(iv) $U_k = \inf\{t > S_k : |X_t| = 2^{n+1}\}$,

(v) $|X_{S_1} - Y_{S_1}| \leq |X_0 - Y_0|$,

(vi) $|X_{S_k} - Y_{S_k}| \leq |X_{U_{k-1}} - Y_{U_{k-1}}|$ if $|X_{U_{k-1}} - Y_{U_{k-1}}|/|X_{U_{k-1}}| \leq 1/2$.

Let $I = \bigcup_{j \geq 1} [S_j, U_j]$.

Lemma 3.4. *If* $|X_{S_1} - Y_{S_1}| \leq |X_{S_1}|/2^m$ *then* $P\left(\sup_{t \in I} |X_t - Y_t|/|X_t| \leq 1/2\right) > q_m$, *for some* q_m*'s such that* $q_m \to 1$ *as* $m \to \infty$.

Proof. Let $T_1 = S_1$ and for $k \geq 1$,

$$T_k = \inf\{t \in (T_{k-1}, \infty) \cap I : |X_t| = 2|X_{T_{k-1}}| \text{ or } |X_t| = |X_{T_{k-1}}|/2\}.$$

The modulus $R_t = |X_t|$ of X_t is a 2-dimensional Bessel process in random intervals I because the normal reflection on ∂K does not affect the radial part of the driving Brownian motion B_t (see the discussion of "mirror" couplings in Section 3 of Burdzy and Kendall (2000)). It follows that $N_k = \log_2 R_{T_k}$ is the symmetric random walk on $(-\infty, n+1] \cap \mathbf{Z}$, with reflection at $n+1$.

We will define events A_k in terms of $\{X_t, t \in [T_k, T_{k+1}]\}$ and $\{B_t, t \in [T_k, T_{k+1}]\}$. First, we declare that $A_k \subset \{|X_{T_k}| \neq 2^{n+1}\}$. Suppose that $|X_{T_k}| = 2^m \neq 2^{n+1}$. Let

$$T_k^1 = \inf\{t > T_k : |X_t| = (3/2)2^m \text{ or } |X_t| = (3/4)2^m\},$$
$$T_k^2 = \inf\{t > T_k : \arg X_t = 0\}.$$

Note that $T_k^1 < T_{k+1}$ and let $A_k^1 = \{T_k^2 < T_k^1\}$. Write $B_t = (B_t^1, B_t^2)$. Let

$$T_k^3 = \inf\{t > T_k^2 : B_t^2 \leq B_{T_k^2}^2 - 2^m\}$$

and

$$A_k^2 = \Big\{T_k^3 < \inf\{t > T_k^2 : |B_t^1 - B_{T_k^2}^1| \geq 1/16 \cdot 2^m\} \wedge \inf\{t > T_k^2 : B_t^2 - \inf_{s \in [T_k^2, t]} B_s^2 \geq 1/16 \cdot 2^m\}\Big\}.$$

If $A_k^1 \cap A_k^2$ holds then $|X_{T_k^3}| \in [(5/8)2^m, (13/8)2^m]$. Next let

$$T_k^4 = \inf\{t > T_k^3 : |X_t| = (7/4)2^m \text{ or } |X_t| = (9/16)2^m\},$$
$$T_k^5 = \inf\{t > T_k^3 : \arg X_t = \theta_0\},$$

and $A_k^3 = \{T_k^5 < T_k^4\}$. Let $\widetilde{B}_t^1$ and $\widetilde{B}_t^2$ be the real and imaginary parts of $B_t e^{-i\theta_0}$. Let

$$T_k^6 = \inf\{t > T_k^5 : \widetilde{B}_t^2 \geq \widetilde{B}_{T_k^5}^2 + 2^m\}$$

and

$$A_k^4 = \Big\{T_k^6 < \inf\{t > T_k^5 : |\widetilde{B}_t^1 - \widetilde{B}_{T_k^5}^1| \geq 1/64 \cdot 2^m\} \wedge \inf\{t > T_k^5 : -\widetilde{B}_t^2 + \sup_{s \in [T_k^5, t]} \widetilde{B}_s^2 \geq 1/64 \cdot 2^m\}\Big\}.$$

We let $A_k = A_k^1 \cap A_k^2 \cap A_k^3 \cap A_k^4$. Note that if A_k holds then $|X_{T_k^6}| \in [(17/32)2^m, (57/32)2^m]$.

We will now argue that $P(A_k \mid B_{T_k}, X_{T_k}, |X_{T_k}| \neq 2^{n+1}) > p_1$ for some $p_1 > 0$, independent of k. First, the event A_k^1 occurs if the driving Brownian motion B_t, $t \geq T_k$, makes a clockwise loop within the annulus with center $B_{T_k} - X_{T_k}$ and radii $(3/2)2^m$ and $(3/4)2^m$; this event has a positive probability not depending on m, by scaling. Assuming that A_k^1 occurred, the event A_k^2 can occur with a positive probability, by the support theorem; this probability is independent of m, again by the scaling property of Brownian motion. Similar arguments apply to A_k^3 and A_k^4. This proves our claim about the probability of A_k.

Let M_j be the number of $k \leq j$ such that $|X_{T_k}| \neq 2^{n+1}$. Let V_j be the number of $k \leq j$ such that A_k holds. Recall that A_k is defined in terms of $\{X_t, t \in [T_k, T_{k+1}]\}$ and $\{B_t, t \in [T_k, T_{k+1}]\}$. Hence, by the strong Markov property, assuming $|X_{T_k}| \neq 2^{n+1}$, the conditional probability of A_k given the σ-field generated by $\{A_j, j < k\}$ is greater than p_1. The strong law of large numbers implies that $\lim_{j\to\infty} V_j/M_j > p_1$. However, standard results for the reflected random walk N_k show that $\lim_{j\to\infty} M_j/j = 1$ so $\lim_{j\to\infty} V_j/j > p_1$, a.s.

We note that if $|X_t - Y_t|/|X_t| \leq 1$ for $t \leq s$ then the distance between X_t and Y_t cannot increase during the time interval $[0, s] \cap I$—we will refer to this observation as (*) later in the proof. This is because for all $t \in I$ not larger than s, if one of the processes reflects on a side of the wedge K, the other process stays on the same side of the straight line containing this side. The claim now follows easily from the Skorohod representation of reflecting Brownian motion in K and assumption (vi).

If $|X_{T_k} - Y_{T_k}|/|X_{T_k}| \leq 1/2$ and the event A_k occurs then $|X_t - Y_t|$ is reduced by at least a factor of $|\cos\theta_0|$ on the interval $[T_k, T_{k+1}]$. To see this, first note that $|X_t - Y_t|/|X_t| \leq 1$ for $t \in [T_k, T_{k+1}]$. Hence, by (*), the function $t \to |X_t - Y_t|$ cannot increase during the time interval $[T_k, T_{k+1}]$. The occurrence of A_k^2 implies that at some time $s_k^1 \in [T_k, T_k^3]$, both processes X_t and Y_t lie on the horizontal axis. Later, at some time $s_k^2 \in [T_k^3, T_k^6]$, in view of the occurrence of A_k^4, the processes will lie on the side of the wedge K which is inclined at the angle θ_0. Elementary geometry shows that $|X_{s_k^2} - Y_{s_k^2}| \leq |\cos\theta_0||X_{s_k^1} - Y_{s_k^1}|$.

From now on we will assume that $\theta_0 \neq \pi/2, \pi, 3\pi/2$ so that $|\cos\theta_0| \in (0,1)$. The argument would have to be modified if we allowed θ_0 to take one of the values $\pi/2, \pi$ or $3\pi/2$ but the proof would be in fact easier than the one given below; we leave the analysis of those special three cases to the reader. Recall that the process $N_k = \log_2 |X_{T_k}|$ is a symmetric random walk on $(-\infty, n+1] \cap \mathbf{Z}$ reflected at $n+1$. Fix some $q \in (0,1)$, arbitrarily close to 1 and note that $N_1 = n$. The law of iterated logarithm implies that we can find

m so large that the probability that $N_k \le n - m/2 - k\big|\log_2|\cos\theta_0|\big|p_1/2$ for some $k \ge 0$ is less than $(1-q)/2$. Since $\lim_{k\to\infty} V_k/k > p_1$, a.s., we can increase m, if necessary, so that $\big|\log_2|\cos\theta_0|\big|V_k \ge -m/2 + k\big|\log_2|\cos\theta_0|\big|p_1/2$ for all $k \ge 0$ with probability greater than $(1-q)/2$. It follows that for some large m, $N_k \ge n - m - \big|\log_2|\cos\theta_0|\big|V_k$ for all $k \ge 0$ with probability greater than q.

Assume that $|X_{S_1} - Y_{S_1}|/|X_{S_1}| \le 2^{-m-3}$. We will argue that if $N_k \ge n - m - \big|\log_2|\cos\theta_0|\big|V_k$ for all $k \ge 0$ then $|X_t - Y_t|/|X_t| \le 1/2$ for all $t \ge S_1$. Suppose otherwise. Let j be the smallest integer such that $|X_t - Y_t|/|X_t| > 1/2$ for some $t \in [T_j, T_{j+1}]$. This implies that the function $t \to |X_t - Y_t|$ is non-decreasing on the interval $[0, T_j] \cap I$, in view of (*) and assumption (vi). Since $V_{j-1} \ge -(N_{j-1} - n + m)/\big|\log_2|\cos\theta_0|\big|$, the distance $|X_{T_j} - Y_{T_j}|$ is less than or equal to $|\cos\theta_0|^{-(N_{j-1}-n+m)/|\log_2|\cos\theta_0||}|X_{S_1} - Y_{S_1}|$. Thus

$$\log_2|X_{T_j} - Y_{T_j}| \le (N_{j-1} - n + m) + \log_2|X_{S_1} - Y_{S_1}| \le N_j + 1 - n + m + \log_2|X_{S_1} - Y_{S_1}|,$$

so

$$\log_2|X_{T_j} - Y_{T_j}| - N_j \le 1 - \log_2|X_{S_1}| + m + \log_2|X_{S_1} - Y_{S_1}|,$$

$$\log_2\left(|X_{T_j} - Y_{T_j}|/|X_{T_j}|\right) \le 1 + m + \log_2\left(|X_{S_1} - Y_{S_1}|/|X_{S_1}|\right).$$

Recall that $|X_{S_1} - Y_{S_1}|/|X_{S_1}| \le 2^{-m-3}$ to see that $\log_2|X_{T_j} - Y_{T_j}|/|X_{T_j}| \le -2$ and so $|X_{T_j} - Y_{T_j}|/|X_{T_j}| \le 1/4$. This and (*) imply that $|X_t - Y_t|/|X_t| \le 1/2$ for $t \in [T_j, T_{j+1}]$, because $t \to |X_t - Y_t|$ is non-increasing for $t \in I$ as long as $|X_t - Y_t|/|X_t| \le 1$, and there is no t for which this condition fails before the first time $s > T_j$ when $|X_s|$ is equal to $|X_{T_j}|/2$. This contradicts the definition of j.

We have shown that for any $q < 1$, there exists m such that if $|X_{S_1} - Y_{S_1}|/|X_{S_1}| \le 2^{-m-3}$ then $|X_t - Y_t|/|X_t| \le 1/2$ for all $t \in [S_1, \infty) \cap I$ with probability greater than or equal to q. The lemma follows from this statement in a straightforward way. □

Proof of Theorem 1.1 (i). Let $\{x_1, x_2, \dots, x_{k_0}\}$ be the set of all vertices of ∂D. Choose n so negative that the discs $\mathcal{B}(x_j, 2^{n+2})$ are disjoint. For every $j = 1, 2, \dots, k_0$, let

$$\begin{aligned} S_1^j &= \inf\{t \ge 0 : |X_t - x_j| = 2^n\}, \\ U_k^j &= \inf\{t > S_k^j : |X_t - x_j| = 2^{n+1}\}, \\ S_{k+1}^j &= \inf\{t > U_k^j : |X_t - x_j| = 2^n\}. \end{aligned}$$

For a point $x \in \partial D$ which is not a vertex, let $K(x)$ be a straight line containing the edge of ∂D to which x belongs. Let $H(x)$ be a halfplane whose boundary is $K(x)$, and which intersects D in every neighborhood of x. Let $\varepsilon > 0$ be so small that if $|X_t - Y_t| \le \varepsilon$, X_t is at a distance greater than 2^{n-1} from all vertices, and $X_t \in \partial D$ then $Y_t \in \overline{H(X_t)}$. This property is symmetric in X_t and Y_t, of course. As long as $|X_t - Y_t| \le \varepsilon$, the function $t \to |X_t - Y_t|$ is non-increasing outside $\bigcup_{j,k}[S^j_k, U^j_k]$.

Choose arbitrarily small $\delta > 0$ and let m be so large that q_m defined in Lemma 3.4 satisfies $k_0(1 - q_m) < \delta$. Recall ε_1 and ε_2 from Lemma 3.3. Assume without loss of generality that ε_2 is smaller than ε chosen in the previous paragraph. Decrease n, if necessary, so that $2^{n+2} < \varepsilon_1$. Decrease ε_2 so that $|X_t - Y_t| \le |X_t - x_j|/2^m$ for all j if $|X_t - Y_t| \le \varepsilon_2$ and $\operatorname{dist}(X_t, \partial D) \ge \varepsilon_1$. Let T_1 be the first time t when $|X_t - Y_t| \le \varepsilon_2$, and for all j we have $|X_t - Y_t| \le |X_t - x_j|/2^m$ and $|X_t - x_j| \ge 2^{n+2}$. The stopping time T_1 is finite a.s. because there exists $t < \infty$ with $|X_t - Y_t| \le \varepsilon_2$ and $\operatorname{dist}(X_t, \partial D) \ge \varepsilon_1$, by Lemma 3.3.

The process $t \to |X_t - Y_t|$ will be non-increasing at least until the first time s when $|X_s - Y_s|/|X_s - x_j| \ge 1/2$ for some j. This follows from observation (*) made in the proof of Lemma 3.4. For every j, Lemma 3.4 can be applied to the process (X_t, Y_t) relative to wedge K_j for which $K_j \cap V_j = D \cap V_j$ and some neighborhood V_j of x_j. Lemma 3.4 shows that the probability that there exists $s \ge T_1$ such that $|X_s - Y_s|/|X_s - x_j| \ge 1/2$ for some j is less than $k_0(1 - q_m) < \delta$. If there is no such s, the process $t \to |X_t - Y_t|$ is non-increasing for all $t \ge T_1$.

If there is $s \ge T_1$ such that $|X_s - Y_s|/|X_s - x_j| \ge 1/2$ for some j, we let T_2 be the first time $t > s$ when $|X_t - Y_t| \le \varepsilon_2$, and for all j we have $|X_t - Y_t| \le |X_t - x_j|/2^m$ and $|X_t - x_j| \ge 2^{n+2}$. Lemma 3.3 implies that $T_2 < \infty$ a.s. It should be now clear how to use an induction argument to define a sequence of stopping times T_k and to show that with probability 1, for some k, we will have $|X_t - Y_t|/|X_t - x_j| \le 1/2$ for all j and all $t \ge T_k$. Remark (*) in the proof of Lemma 3.4 implies that with probability 1, the distance between X_t and Y_t will be non-increasing after such a time T_k. In view of Lemma 3.3, the distance will converge to 0. □

4. Lip domains. We will prove Theorem 1.1 (ii) in this section.

Let D be a lip domain. We will need a sequence of Lipschitz domains D_n with piecewise smooth boundaries, increasing to D. It is easy to see that we can find D_n's with the following properties. The boundary of D_n is the union of graphs of C^2 functions f^n_1 and f^n_2, i.e.,

$$D_n = \{(x_1, x_2) : x_2 = f^n_1(x_1), z^n_1 \le x_1 \le z^n_2\} \cup \{(x_1, x_2) : x_2 = f^n_2(x_1), z^n_1 \le x_1 \le z^n_2\},$$

such that $f_1^n(z_1^n) = f_2(z_1^n)$, $f_1^n(z_2^n) = f_2^n(z_2^n)$, $f_1^n(x_1^n) < f_2^n(x_1^n)$ for $z_1^n < x_1 < z_2^n$, and functions f_1^n and f_2^n are Lipschitz with the constant c_0 (same as in (1.1)), i.e., for $k = 1, 2$,

$$|f_k^n(x_1) - f_k^n(\widetilde{x}_1)| \le c_0 |x_1 - \widetilde{x}_1|, \qquad \text{for all } z_1^n \le x_1, \widetilde{x}_1 \le z_2^n.$$

Moreover, $D_n \subset D$ for every $n \ge 1$ and $D = \bigcup_n D_n$.

We will assume that the boundary of D near its left and right endpoints can be represented as a Lipschitz function in the coordinate system rotated by $\pi/2$ relative to the standard one. The general case does not present any fundamentally new technical challenges. Because of this assumption, there exist $c_1 \in (0, c_0]$ and $r_1 > 0$ such that,

$$\begin{aligned} f_1(x_1) - f_1(x_2) &\ge c_1(x_2 - x_1), \\ f_2(x_1) - f_2(x_2) &\le -c_1(x_2 - x_1), \end{aligned} \tag{4.1}$$

for $z_1 \le x_1 < x_2 \le z_1 + r_1$, and

$$\begin{aligned} f_1(x_1) - f_1(x_2) &\le -c_1(x_2 - x_1), \\ f_2(x_1) - f_2(x_2) &\ge c_1(x_2 - x_1), \end{aligned} \tag{4.2}$$

for $z_2 - r_1 \le x_1 < x_2 \le z_2$. We can assume that the functions f_k^n defining the domains D_n satisfy (4.1) for $z_1^n \le x_1 < x_2 \le z_1^n + r_1/2$ and (4.2) for $z_2^n - r_1/2 \le x_1 < x_2 \le z_2^n$.

Let $R_+ = \{(s,t) : s \ge 0, |t| \le c_0\, s\}$ and $R_- = \{(s,t) : s \le 0, |t| \le c_0\, |s|\}$.

Proof of Theorem 1.1 (ii). *Step 1.* Let $\mu_2 < 0$ be the second eigenvalue of the Laplacian in D with the Neumann boundary conditions. In this step, we will show that there exists an eigenfunction $\varphi_2(x)$ corresponding to μ_2, such that for every x we have $\nabla \varphi_2(x) \in R_+$.

Let $u_n(t,x)$ be the solution of the heat equation in D_n with the Neumann boundary conditions and the initial condition $u_n(0,x) = x_1$ for $x = (x_1, x_2) \in D_n$. Then $\nabla_x u_n(t,x) \in R_+$ for every $t \ge 0$. This can be proved in the same way as Theorem 3.1 of Bañuelos and Burdzy (1999) (see also their Example 3.3).

We define $u(t,x)$ for $x \in D$ in a way completely analogous to that for $u_n(t,x)$, namely, we let $u(t,x)$ be the solution of the heat equation in D with the Neumann boundary conditions and $u(0,x) = x_1$ for $x = (x_1, x_2) \in D$. By Theorem 2.1 and Lemma 2.2, functions $u_n(t,x)$ converge pointwise to $u(t,x)$ on compact subsets of $(0,\infty) \times D$ as $n \to \infty$. It follows that $\nabla_x u(t,x) \in R_+$ for every $t > 0$, just like in the case of $\nabla u_n(t,x)$. This combined with the methods from the proof of Theorem 3.3 of Bañuelos and Burdzy (1999) shows that there exists an eigenfunction $\varphi_2(x)$ such that $\nabla\varphi_2(x) \in R_+$ for every x. Let us fix an eigenfunction φ_2 with this property and let $v(t,x) = \varphi_2(x)e^{-\mu_2 t}$.

Step 2. Define $V_+ = \{(s,t) : s \geq 0, |t| \leq s/c_0\}$, and $V_- = \{(s,t) : s \leq 0, |t| \leq |s|/c_0\}$. Since $0 < c_0 < 1$, clearly $R_+ \subset V_+$ and $R_- \subset V_-$. We will show that for any fixed $a_2 > 0$ there exists $a_3 > 0$ such that if $x, y \in D$, $|x - y| \geq a_2$, $\text{dist}\,(x, \partial D) \geq a_2$, $\text{dist}\,(y, \partial D) \geq a_2$, and $y - x \in R_+ \cup R_-$, then $|\varphi_2(x) - \varphi_2(y)| \geq a_3$. If this is not true then there exist sequences of points x_k and y_k satisfying the above conditions but such that $|\varphi_2(x_k) - \varphi_2(y_k)| \leq 1/k$. By compactness, we can extract convergent subsequences with limits x_∞ and y_∞. We have $|x_\infty - y_\infty| \geq a_2$. We will assume without loss of generality that $y_\infty - x_\infty \in R_+$. By the continuity of φ_2, we have $\varphi_2(x_\infty) = \varphi_2(y_\infty)$. The eigenfunction φ_2 satisfies $\nabla\varphi_2 \in R_+$ so if $z \in x_\infty + V_+$ then $\varphi_2(z) \geq \varphi_2(x_\infty)$. Similarly, $\varphi_2(z) \leq \varphi_2(y_\infty)$ for $z \in y_\infty + V_-$. It follows that $\varphi_2(z) = \varphi_2(x_\infty)$ for all z in the intersection of $x_\infty + V_+$ and $y_\infty + V_-$. The intersection of these sets is nonempty because $y_\infty - x_\infty \in R_+$ and $c_0 < 1$. Hence, the function φ_2 is constant on a nonempty open set. Since it is real analytic in D, it must be constant on the whole set D. This is impossible, so our claim must be true.

Step 3. Recall the definitions of the wedges R_+ and R_-. First we will show that we can assume that $y_0 - x_0 \in R_+ \cup R_-$. Let (X_t^n, Y_t^n) be a synchronous coupling of reflecting Brownian motions in D_n. In each piecewise smooth domain D_n, the reflecting Brownian motions X_t^n and Y_t^n are strong solutions to the stochastic differential equation (2.1), so if $X_{t_0}^n = Y_{t_0}^n$ for some $t_0 \geq 0$ then $X_t^n = Y_t^n$ for all $t \geq t_0$. If (X_t^n, Y_t^n) starts from (x_n, y_n) such that $y_n - x_n \in R_+$ (respectively, $x_n - y_n \in R_+$) then $Y_t^n - X_t^n \in R_+$ (respectively, $X_t^n - Y_t^n \in R_+$) for all $t \geq 0$, by the argument used in Theorem 3.1 of Bañuelos and Burdzy (1999). If $x_n \to x_\infty$ and $y_n \to y_\infty$ then (X_t^n, Y_t^n) converge weakly (along a subsequence) to (X_t, Y_t), a synchronous coupling in D starting from (x_∞, y_∞). The above mentioned monotonicity property carries over to the limit and we conclude that for any synchronous coupling (X_t, Y_t) starting from any (x_∞, y_∞) with $y_\infty - x_\infty \in R_+$ (respectively, $x_\infty - y_\infty \in R_+$), we have $Y_t - X_t \in \mathbf{R}_+$ (respectively, $X_t - Y_t \in R_+$) for all $t \geq 0$.

Let T_1^n be the first hitting time of the lower part of the boundary of D_n by the process X_t^n, and let T_2^n be the first time after T_1^n when X_t^n hits the upper part of ∂D_n. The vertical component of the normal vector points down at every point of the upper part of ∂D_n. Hence, the vertical component of X_t^n is a Brownian motion plus a nonincreasing process, until time T_1^n. This easily implies that the distributions of T_1^n's are stochastically majorized by a single distribution independent of n. The same applies to T_2^n. Easy geometry shows that for some random time $T_0^n < T_2^n$, we must have either $Y^n_{T_0^n} - X^n_{T_0^n} \in R_+$ or $X^n_{T_0^n} - Y^n_{T_0^n} \in R_+$ This also holds for all $t \geq T_0^n$, by the argument mentioned in the last paragraph. It is now easy to deduce from the weak convergence that there is an almost surely finite random time T_0 such that $Y_t - X_t \in R_+$ for all $t \geq T_0$, or $Y_t - X_t \in R_-$ for all $t \geq T_0$. Thus the probability $P(Y_t - X_t \in R_+ \cup R_-)$ increases to 1 as $t \to \infty$. This and a "weak" Markov property presented in Remark 2.5 (ii) show that it is enough to prove that $|X_t - Y_t| \to 0$ as $t \to \infty$, assuming that (X_t, Y_t) starts from (x_∞, y_∞) with $y_\infty - x_\infty \in R_+ \cup R_-$.

Assume without loss of generality that (X_t, Y_t) starts from (x_0, y_0) with $y_0 - x_0 \in R_+$. Recall the function $v(t, x) = \varphi_2(x)e^{-\mu_2 t}$ from Step 1. The function $|\varphi_2(x)|$ is bounded on $\overline{D}$ and so, for a fixed $s > 0$, $|v(t, x)|$ is bounded on $[0, s] \times \overline{D}$. It follows that the processes $t \to v(t, X_t)$ and $t \to v(t, Y_t)$ are martingales. As we observed in the last paragraph, $Y_t - X_t \in R_+$ for all $t \geq 0$. This, together with the fact that for every x, $\nabla \phi_2(x) \in R_+$, implies that $\varphi_2(Y_t) - \varphi_2(X_t) \geq 0$ for all $t > 0$. Hence, the process

$$v(t, Y_t) - v(t, X_t) = (\varphi_2(Y_t) - \varphi_2(X_t))e^{-\mu_2 t}$$

is a non-negative martingale and, therefore, it converges a.s. as $t \to \infty$. Since $\mu_2 < 0$, this implies that, with probability one, $\varphi_2(Y_t) - \varphi_2(X_t) \to 0$, as $t \to \infty$. In view of Step 2 and the fact that both X and Y spend zero Lebesgue amount of time on the boundary of D, this shows that $|Y_t - X_t| \to 0$, a.s. □

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Department of Mathematics
University of Washington
Box 354350
Seattle, WA 98195-4350, USA

Email: burdzy@math.washington.edu
http://www.math.washington.edu/~burdzy/

Email: zchen@math.washington.edu
http://www.math.washington.edu/~zchen/