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# Results on singularities of pairs

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**Abstract**

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Singularities of algebraic varieties have been studied extensively, and recently also the properties of singularities of pairs have been investigated. This thesis presents several results on singularities of different kinds of pairs.

The simplest pairs are of the form  $(\mathbb{P}^2, C)$  where  $C$  is a plane curve: the first chapter presents an approach to classifying isomorphism classes for ordinary multiple points of plane curves.

The next chapter is an introduction to singularities in algebraic geometry, including the fundamental definitions and properties of rational pairs, introduced by Kollàr and Kovács in 2013.

The final three chapters contain novel results on log canonical centers, rationalizing divisors and images of rational pairs under certain morphisms.

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## Chapter 1

### INTRODUCTION

Among all the algebraic varieties, particular attention has been given to smooth varieties, as they have undoubtedly important properties: for example, they are local complete intersections at every point and when  $k = \mathbb{C}$  they have the structure of complex holomorphic manifolds. Nevertheless, singular varieties arise naturally in many important areas of geometry: moduli problems and deformation theory are two fundamental examples. It appears reasonable to try to classify the singularities of algebraic varieties and establish what “nice” singularities should be. For example, if  $X$  be a smooth variety of general type (i.e.  $K_X$  is big), then the image  $X^{can}$  of the pluricanonical maps need not be smooth, but one can control the type of singularities it has.

In the framework of the Minimal Model Program it became more and more clear that special attention should be given to pairs of the form  $(X, D)$ , where  $X$  is a normal variety and  $D$  is a divisor on  $X$ . Inductive arguments are a prototypical example of usage of pairs, for example. Therefore singularities can be defined and studied for pairs.

In this thesis we will present several results about the properties of different types of singularities of pairs.

The first chapter is an introduction to classifying singularities: we consider multiple ordinary points on plane curve and describe the corresponding isomorphisms classes. The chapter requires very little background and culminates with a description of the isomorphisms classes as a quotient of the action of an algebraic group.

The second chapter contains an introduction to the basic definitions and concepts of singularities in algebraic geometry. The first part presents the fundamental singularities of pairs of the Minimal Model Program (terminal, canonical, log terminal, log canonical) and

the second part describes rational pairs, a recent addition to the literature by Kollár and Kovács.

Log canonical centers are the subject of the third chapter, the two main results being the following.

**Theorem A** (Theorem 4.2.1). *Let  $X$  a  $\mathbb{Q}$ -Gorenstein normal variety and let  $Y$  be any irreducible subvariety not entirely contained in  $\text{Sing } X$ . Then there is a boundary  $\Delta$  such that  $Y$  is a log canonical center for the pair  $(X, \Delta)$ .*

**Theorem B** (Theorem 4.4.1). *Let  $X$  be a Gorenstein variety with lc singularities, and let  $W$  be a reduced irreducible special complete intersection in  $X$  (i.e. cut out by homogeneous forms of the same degree) with log canonical singularities, not contained in  $\text{Sing } X$ . Then there exist a boundary  $\Delta$  such that the pair  $(X, \Delta)$  has log canonical singularities and  $W$  is a log canonical center for  $(X, \Delta)$ .*

The fourth chapter deals with the problem of determining whether a variety  $X$  can have a reduced divisor  $D$  such that the pair  $(X, D)$  is rational. After focusing on the case of cones, the principal result is this:

**Theorem C** (Theorem 5.3.2). *Let  $(X, D)$  be a rational pair. Then the non rational locus of  $X$  has codimension at least 3.*

In the fifth and final chapter we examine quotients and images of rational pairs, and introduce the concept of lift via thrifty resolutions to give a sufficient condition for the quotient of a rational pair by the action of a finite group to be rational. The main theorem is a rational pairs analogue of a theorem of Fujino:

**Theorem D** (Theorem 6.2.4). *Let  $f : (Y, B) \rightarrow (X, D)$  be a surjective, proper morphism of pairs with  $(Y, B)$  rational and  $D$  Cartier. Assume  $K_Y + B$  is Cartier and  $-K_Y - 2B$  is  $f$ -ample. If  $f$  lifts via thrifty resolutions then  $(X, D)$  is a rational pair.*

## Chapter 2

## ANALYTIC ISOMORPHISM CLASSES OF PLANE CURVES

## 2.1 Motivation and basic definitions

**Definition 2.1.1.** Let  $k$  be an algebraically closed field. A plane affine curve  $\mathcal{C}$  over  $k$  is the zero set of an irreducible polynomial  $f(x, y) \in k[x, y]$ .

A point  $P \in \mathcal{C}$  is *singular* if  $f_x(P) = f_y(P) = 0$  and *smooth* otherwise.

Note that the origin  $O$  is singular iff  $f = f_r + f_{r+1} + \dots$ , with  $f_j$  homogeneous of degree  $j$ ,  $n \geq 2$ . The initial form  $f_r$  splits into linear factors, called the *principal tangents* at  $O$ .

In other words, singular curves have more than one tangent at a point.

**Example 2.1.2.** Let  $f(x, y) = x^2 - y^2 + x^3$ . The curve  $\mathcal{C} : f = 0$  has principal tangents  $x + y = 0, x - y = 0$  at the origin. Intuitively, it should look like two lines near  $O$ , but  $f$  does not factor in  $k[x, y]$ .

Nevertheless, one can factor  $f$  in the power series ring  $k[[x, y]]$ : write

$$x^2 - y^2 + x^3 = (x + y + a_2 + a_3 + \dots)(x - y + b_2 + b_3 + \dots)$$

where  $a_i, b_i$  are homogeneous polynomials of degree  $i$ . Then it's easy to see that the degree  $n + 1$  piece of the product on the right hand side is given by

$$(x + y)b_n + (x - y)a_n + \text{terms involving } a_2, b_3, \dots, a_{n-1}, b_{n-1}$$

It's easy to see that for each polynomial  $G$  of degree  $n + 1$ , we can solve for  $a_n, b_n$  in the equation  $(x + y)b_n + (x - y)a_n = G$ . Therefore the following infinite system has a solution

$$\begin{aligned} (x + y)b_2 + (x - y)a_2 &= x^3 \\ (x + y)b_3 + (x - y)a_3 &= -a_2b_2 \\ &\dots \end{aligned}$$

thus showing that we can factor  $f$  in  $k[[x, y]]$ .

It seems like power series will be important to analyze singularities of plane curves. We will now introduce notation and state definitions that will allow us to use them properly in our context.

Let  $P$  be a point on a plane curve  $\mathcal{C}$ . The completion of the local ring  $\mathcal{O}_{P,\mathcal{C}}$  w.r.t its maximal ideal will be denoted by  $\hat{\mathcal{O}}_{P,\mathcal{C}}$ .

**Definition 2.1.3** ([Har77]). Let  $P, Q$  be points on plane curves  $\mathcal{C}, \mathcal{D}$ . We say that  $P$  and  $Q$  are *analytically isomorphic* if there is a  $k$ -algebra isomorphism  $\hat{\mathcal{O}}_{P,\mathcal{C}} \simeq \hat{\mathcal{O}}_{Q,\mathcal{D}}$

One sees easily that if  $P = O$  and  $\mathcal{C}$  is defined by  $f = 0$  then  $\hat{\mathcal{O}}_{P,\mathcal{C}} = k[[x, y]]/(f)$ .

In this chapter we will focus on the following

**Question.** *Fixed  $r$  distinct lines  $L_1, \dots, L_r$  through the origin in  $\mathbb{A}^2$ , classify isomorphism classes of curves having  $O$  as an  $r$ -fold ordinary point, and  $L_1, \dots, L_r$  as principal tangents.*

It turns out that we can exploit the ideas in Example 2.1.2 to answer the question.

More precisely, in this chapter we will discuss:

- A concrete way to test whether two curves have the same isomorphism class at the origin
- The explicit description of the isomorphism classes for  $r \leq 5$
- A characterization of the isomorphism classes (for arbitrary  $r$ ) as quotient of the action of an algebraic group on an affine space

Throughout the rest of this chapter we will assume  $\text{char } k$  does not divide  $r$ .

## 2.2 The mechanics of isomorphisms of power series rings

Here and throughout the rest of the chapter we will use this

**Setup 2.2.1.** Let  $f, \tilde{f}$  be irreducible polynomials in  $k[x, y]$ , singular at the origin. In particular  $f, \tilde{f} \in (x, y)^2 \subseteq k[x, y]$ . We will write  $f \sim \tilde{f}$  iff they have analytically isomorphic singularities at the origin.

We want to investigate the properties of isomorphisms of power series rings. First of all, note that as a direct consequence of [Eis95, Ex. 7.25] we have:

**Proposition 2.2.2.** *A  $k$ -linear algebra morphism  $\psi : k[[x, y]] \rightarrow k[[x, y]]$  such that*

$$x \mapsto L_1 + \dots, y \mapsto L_2 + \dots$$

*(with  $L_i$  linear forms) is an isomorphism iff the map  $x \mapsto L_1, y \mapsto L_2$  induces an isomorphism of the degree 1 part of  $k[x, y]$ .*

We can now relate isomorphisms between of power series rings to isomorphisms of their quotients.

**Proposition 2.2.3.** *Any isomorphism  $\psi : k[[x, y]]/(f) \rightarrow k[[x, y]]/(\tilde{f})$  between the completions of the local rings lifts to an isomorphism  $\Psi : k[[x, y]] \rightarrow k[[x, y]]$ .*

*Proof.* Let  $R = k[[x, y]]/(f)$  (resp.  $\tilde{R} = k[[x, y]]/(\tilde{f})$ ): it is a local ring with maximal ideal  $\mathfrak{m}$  (resp.  $\tilde{\mathfrak{m}}$ ), generated by the images of  $x, y$ . Since  $\psi$  is an isomorphism of local  $k$ -algebras, it induces an isomorphism

$$\mathfrak{m}/\mathfrak{m}^2 \rightarrow \tilde{\mathfrak{m}}/\tilde{\mathfrak{m}}^2$$

Now, it's easy to see that the images of  $x$  and  $y$  actually form a basis for both vector spaces: this implies that  $\psi$  maps  $x, y$  to independent linear combinations of  $x, y$  (modulo  $\tilde{f}$ ). By Proposition 2.2.2,  $\psi$  lifts to an isomorphism  $\Psi : k[[x, y]] \rightarrow k[[x, y]]$ .  $\square$

**Corollary 2.2.4.** *Let  $Q$  (resp.  $\tilde{Q}$ ) be the initial form of  $f$  (resp.  $\tilde{f}$ ). If  $f \sim \tilde{f}$  then  $Q = \tilde{Q}$ , possibly after a linear change of coordinates in  $\mathbb{A}^2$ .*

*Proof.* Indeed, by Proposition 2.2.3 an isomorphism  $R \rightarrow \tilde{R}$  has the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto A \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \text{higher order terms}$$

where  $A \in GL_2(k)$ . By composing with the linear morphism given by  $A^{-1}$  we can assume  $\psi$  has the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix} + \text{higher order terms}$$

The element  $\eta = \psi(f)$  clearly has the same initial form as  $f$ , and it generates the ideal  $(\tilde{f})$ . Upon multiplying by a unit scalar, then we can assume that  $\eta$  has the same initial form as  $\tilde{f}$ , as units in  $k[[x, y]]$  are series with non-zero constant term.  $\square$

**Remark 2.2.5.** From a geometric perspective, this implies that if two points are analytically isomorphic, then the respective curves have the same principal tangents, up to a linear automorphism of  $\mathbb{A}^2$ . This seems a reasonable condition, as the curve should “look like” its principal tangents near any of its points. Nevertheless, as we will see later, this generally does not suffice to characterize the analytic isomorphism classes of multiple points.

### 2.3 A criterion for comparing isomorphism classes

In this section we will prove a criterion to establish whether two ordinary multiple points have the same analytic isomorphism class.

First of all, we show that the image of a polynomial under a morphism has a particular form.

**Proposition 2.3.1.** *Assume  $f = Q + f_{r+1} + \dots$  is a polynomial whose initial form  $Q$  factors as a product of  $r$  linear forms. Let  $\Psi : k[[x, y]] \rightarrow k[[x, y]]$  be a  $k$ -algebra morphism of the form*

$$\begin{aligned} x &\mapsto x + g_2 + g_3 + \dots \\ y &\mapsto y + h_2 + h_3 + \dots \end{aligned}$$

where  $g_i, h_i$  are homogeneous polynomials in  $x, y$  of degree  $i$ .

Then there are polynomials  $T_{r+n}$  in the coefficients of  $f_{r+1}, \dots, f_{r+n-1}$  and  $g_2, h_2, \dots, g_n, h_n$  such that the  $(r+n)$ th homogeneous part of  $\Psi(f)$  is given by

$$f_{r+n} + T_{r+n} + g_{n+1}Q_x + h_{n+1}Q_y$$

*Proof.* Write the initial form as

$$Q = L_1 \cdot \dots \cdot L_r \text{ with } L_i = a_i x + b_i y$$

Then the power series  $\Psi(f)$  is

$$\prod_i (a_i(x + g_2 + \dots) + b_i(y + h_2 + \dots)) + \dots$$

Clearly, the degree  $r$  part of  $\Psi(f)$  is exactly  $Q$ . As for the degree  $r + 1$ , note that the terms  $f_{r+i}(\Psi(x), \Psi(y))$  all start in degree  $> r + 1$  for  $i > 1$ , and the degree  $r + 1$  of  $f_{r+1}(\Psi(x), \Psi(y))$  is exactly  $r + 1$ . As for the  $r + 1$ -degree terms that come from  $Q(\Psi(x), \Psi(y))$ , they are exactly the expansion of the products of the degree 1 part of  $r - 1$  factors of  $Q(\Psi(x), \Psi(y))$  and the degree 2 part of the remaining factor, that is

$$\sum_j \left( (a_j g_2 + b_j h_2) \prod_{i \neq j} (a_i x + b_i y) \right) = Q_x g_2 + Q_y h_2$$

So the  $r + 1$ st degree part of  $\Psi(f)$  is

$$f_{r+1}(x, y) + Q_x g_2 + Q_y h_2$$

In the same fashion, the only terms of degree  $r + n$  in  $\Psi(f)$  that involve  $g_{n+1}, h_{n+1}$  are those coming from the expansion of  $Q(\Psi(x), \Psi(y))$ , and they are precisely the product of  $r - 1$  linear forms with the degree  $n + 1$  part of the remaining factor, i.e.

$$\sum_j \left( (a_j g_{n+1} + b_j h_{n+1}) \prod_{i \neq j} (a_i x + b_i y) \right) = Q_x g_{n+1} + Q_y h_{n+1}$$

The other degree  $r + n$  terms consist of  $f_{r+n}(x, y)$  and other terms depending on  $f_{r+1}, \dots, f_{r+n-1}$  and  $g_2, h_2, \dots, g_n, h_n$ .  $\square$

Summing up,  $f \sim \tilde{f}$  iff we can find a change of variables in  $k[[x, y]]$  that maps  $f$  to  $\tilde{f}$ . By combining Proposition 2.3.1 and Proposition 2.2.3 we can get the following criterion:

**Corollary 2.3.2.** *Assume:*

$$\begin{aligned} f &= Q + f_{r+1} + \dots \\ \tilde{f} &= Q + \tilde{f}_{r+1} + \dots \end{aligned}$$

where  $Q$  is the product of linear forms corresponding to the principal tangents. Then  $f \sim \tilde{f}$  iff for some  $g_i, h_i$  we have

$$\begin{aligned} f_{r+1} - \tilde{f}_{r+1} &= g_2 Q_x + h_2 Q_y \\ f_{r+2} - \tilde{f}_{r+2} + T_{r+2} &= g_3 Q_x + h_3 Q_y \\ &\dots \\ f_{r+n} - \tilde{f}_{r+n} + T_{r+n} &= g_{n+1} Q_x + h_{n+1} Q_y \\ &\dots \end{aligned}$$

where  $T_{r+n}$  is the same as in Proposition 2.3.1.

As in the Example 2.1.2, the obstruction for  $f, \tilde{f}$  to be in the same analytic isomorphism class lies in whether we can express certain polynomials in the form  $g_n Q_x + h_n Q_y$ . To make this more precise, we introduce the following

**Definition 2.3.3.** Let  $f = Q + f_{r+1} + \dots$  as above and let  $S = k[x, y]$ ; denote by  $S_d$  the degree  $d$  piece of  $S$ . The  $n$ th multiplication map is the  $k$ -linear map

$$\begin{aligned} \mu_n : S_{n+1} \oplus S_{n+1} &\rightarrow S_{r+n} \\ (A, B) &\mapsto A Q_x + B Q_y \end{aligned}$$

Note that the dimension of the domain of  $\mu_n$  is  $2(n+2)$  and the dimension of its codomain is  $r + n + 1$ .

With the notation of the previous Propositions, determining whether  $f \sim \tilde{f}$  boils down to checking when the differences  $f_{r+n} - \tilde{f}_{r+n} + T_{r+n}$  lie in the image of the  $\mu_n$  for all  $n$ , and this is a matter of linear algebra.

We still apparently have to deal with the fact that there are infinitely many degrees to consider. This actually not the case, as the next proposition shows.

**Proposition 2.3.4.** *If  $Q$  factors as the product of  $r$  distinct linear forms, the multiplication maps  $\mu_n$  are surjective for  $n \geq r - 2$ .*

Before proceeding, we make a useful observation.

**Lemma 2.3.5.** *If  $Q = L_1 \dots L_r$  where the  $L_i = a_i x + b_i y$ 's are distinct, then  $Q_y, Q_x$  are coprime.*

*Proof of the lemma.* Indeed, by Euler's formula  $rQ = xQ_x + yQ_y$ . Since  $\text{char } k \nmid r$ , if  $Q_x, Q_y$  shared a factor, it would have to be a linear form dividing  $Q$ , say  $L_1$ . But

$$Q_x = \sum_i a_i \prod_{j \neq i} L_j = a_1 \prod_{j \neq 1} L_j + L_1 \left( \sum_{i=2}^r a_i \prod_{j \neq i, j \neq 1} L_j \right)$$

$$Q_y = \sum_i b_i \prod_{j \neq i} L_j = b_1 \prod_{j \neq 1} L_j + L_1 \left( \sum_{i=2}^r b_i \prod_{j \neq i, j \neq 1} L_j \right)$$

Since  $L_1 \neq 0$ , at least one of  $Q_x, Q_y$  is not divisible by  $L_1$ , contradiction.  $\square$

*Proof.* We know the dimensions of the domain and codomain of  $\mu_n$ , so the strategy is to compute the dimension of the kernel  $K_n$ . Let  $(A, B) \in K_n$ . Then  $AQ_x = -BQ_y$  and since  $Q_x, Q_y$  are coprime then necessarily  $Q_x | B$ .

If we write  $B = \beta Q_x$ , where  $\beta$  is a polynomial of degree  $n - r + 2$ , then necessarily

$$(A, B) = (\beta(-Q_y), \beta Q_x) = \beta(-Q_y, Q_x)$$

If the degree of  $Q_x$  is strictly greater than the degree of  $B$ , then  $\beta = 0$ , and in this case  $K_n = 0$ . Otherwise, that is if  $n \geq r - 2$ , the kernel has dimension  $\deg \beta + 1 = n - r + 3$ . It follows that in this latter case the image of  $\mu_n$  has dimension  $2(n+2) - (n - r + 3) = n + r + 1$ , so the map  $\mu_n$  is surjective.  $\square$

Note that this is the only result for which we need an ordinary multiple point (i.e. the principal tangents are all distinct): if the linear forms in  $Q$  are not all distinct, then Proposition 2.3.4 is clearly false, as  $Q_x$  and  $Q_y$  would share a factor  $L$  in this case, hence the image of  $\mu_n$  would consist of multiples of  $L$  for all  $n$ .

The proof also shows that  $\mu_n$  is injective for  $n < r - 2$ .

## 2.4 Isomorphism classes for low numbers of tangents

It is a well-known fact [Har77, Ex I.5.14] that all nodes (ordinary double points) are analytically isomorphic to each other: indeed, if  $f = L_1L_2 + \dots$  with  $L_i$  distinct linear forms, then one can write down an explicit isomorphism  $k[[x, y]]/(f) \xrightarrow{\sim} k[[x, y]]/(xy)$ .

We can also use prove it by using the machinery developed in the previous sections. One can observe that any pair of distinct lines in  $\mathbb{A}^2$  can be mapped to the coordinate axis via a linear morphism, and for any pair of polynomials  $f, \tilde{f}$  all the maps  $\mu_n$  are surjective. The same argument shows that there is only one isomorphism class of triple ordinary points.

As for quadruple ordinary points, first of all observe that it is not true that every quadruple of line can be mapped to any other quadruple by a linear transformation. Therefore, fix 4 distinct lines through the origin. The maps  $\mu_n$  are surjective for  $n \geq 2$  and one can check that  $\mu_1$  is an isomorphism: this implies that once the principal tangents are fixed, there is only one isomorphism class for quadruple points.

The case of 5 tangents is a little more complicated. If one fixes different 5 lines through the origin, the maps  $\mu_n$  are surjective for  $n \geq 3$ , and one can check that  $\mu_2$  is an isomorphism. This means that two polynomials  $f, \tilde{f}$  have the same analytic isomorphism class iff there are quadratic homogeneous polynomials  $g_2(x, y), h_2(x, y)$  such that

$$\tilde{f}_6 = f_6 + \mu_1(g_2, h_2)$$

The image of the map  $\mu_1$  is a 6-dimensional subspace of the 7-dimensional, so its cokernel is one dimensional, and it parametrizes the isomorphism classes in this case.

The case  $r \geq 6$  is more complicated to describe, as one can no longer check if  $f \sim \tilde{f}$  by considering vector spaces and their quotients, and it inspired the ideas in the next section.

## 2.5 An approach using quotients

In this section we will describe a group structure on the affine space and an action of this group on an affine space whose orbits are exactly the isomorphism classes of certain singularities.

First of all we state a simple lemma on the composition of power series, whose proof is left to the interested reader.

**Lemma 2.5.1.** *Let  $\alpha, \beta$  be power series in  $\mathfrak{m} \subseteq k[[x, y]]$ . The degree  $n$  part of  $\alpha \circ \beta$  only depends on the degree  $< n$  terms of  $\alpha, \beta$ .*

*If  $\alpha$  has a compositional inverse  $\gamma$ , then the degree  $n$  part of  $\gamma$  only depends on the degree  $< n$  terms of  $\alpha$ .*

The basic idea is to give the affine space a group structure by composing power series and then truncating them: the lemma guarantees that we can keep the group structure. We will now describe the process in detail.

Let  $S = k[[x, y]]$  and let  $S = \bigoplus_i S_i$  be its decomposition into homogeneous pieces. Define

$$G_n := S_2 \oplus S_2 \oplus \dots \oplus S_n \oplus S_n$$

Clearly  $G_n$  is an affine space, and we can endow it with a group structure as follows. Interpret two elements

$$a = (g_2, h_2, \dots, g_n, h_n), b = (G_2, H_2, \dots, G_n, H_n)$$

as the two power series automorphisms

$$\begin{aligned} \alpha : x &\mapsto x + g_2 + \dots + g_n & \beta : x &\mapsto x + G_2 + \dots + G_n \\ y &\mapsto y + h_2 + \dots + h_n & y &\mapsto y + H_2 + \dots + H_n \end{aligned}$$

The composition  $\alpha \circ \beta$  has the form

$$\begin{aligned} x &\mapsto x + A_2 + \dots + A_n + \dots \\ y &\mapsto y + B_2 + \dots + B_n + \dots \end{aligned}$$

Then define

$$a \cdot b = (A_2, B_2, \dots, A_n, B_n)$$

The inverse of an element in  $G_n$  is computed in a similar way.

Let

$$A_m = S_{r+1} \oplus \dots \oplus S_m$$

There is a natural right action of  $G_n$  on  $S_m$ : indeed, interpret as before an element  $a \in G_n$  as a power series automorphism

$$\alpha : x \mapsto x + g_2 + \dots, \quad y \mapsto y + h_2 + \dots$$

and think of an element  $b \in A_n m$  as the polynomial  $B(x, y) = Q + b_{r+1} + \dots + b_m$ . Then define  $b \cdot a$  to be the degree  $r + 1, \dots, m$  pieces of  $B(\alpha(x, y))$ .

Recall that the maps  $\mu_n$  are surjective if  $n \geq r - 2$ . This means that the action of  $G_{r-2}$  on  $A_{r+n}$  has as orbits the isomorphism classes of multiple points whose principal tangents are the factors of  $Q$ .

This gives a geometric description of the isomorphism classes.

## 2.6 Some further questions

It is not known whether the quotient  $A_m/G_n$  is a scheme. A related question is: are the groups  $G_n$  reductive? Even if these groups are not reductive, is the coordinate ring  $k[A_m]$  finitely generated as an algebra over the invariant subring  $k[A_m]^{G_n}$ ?

## Chapter 3

### SINGULARITIES IN ALGEBRAIC GEOMETRY

This chapter is dedicated to presenting some important classes of singularities, and the definition stated here will be used throughout the rest of this dissertation.

#### 3.1 Normality

Given any variety  $X$  over a field of characteristic 0 there is a resolution of singularities, i.e. a birational morphism  $X' \rightarrow X$  with  $X'$  smooth, by a celebrated theorem by Hironaka: unfortunately, there cannot be a resolution functor, as the example in [Kol05, Example 3.4] shows. Nevertheless, the normalization is a characteristic free, functorial procedure.

**Definition 3.1.1.** An irreducible variety  $X$  is *normal* if for each point  $x \in X$  the local ring  $\mathcal{O}_{X,x}$  is integrally closed in the fraction field  $K(X)$ .

By the general properties of integral closure of finitely generated algebras over fields, we have that for any variety  $X$  there is a normal variety  $X^n$  and a morphism  $N : X^n \rightarrow X$  such that every morphism from a normal variety  $Y$  to  $X$  factors through  $N$ .

The definition does not really cast much light on the meaning of normality, apart from the fact that normalization exists: the following theorem, due to Serre, gives a geometric interpretation.

**Theorem 3.1.2** (Serre). *A variety  $X$  is normal if and only if  $\text{codim Sing}(X) \geq 2$  and for every  $x \in X$  with  $\text{codim } x \geq 2$  we have  $\text{depth } \mathcal{O}_{X,x} \geq 2$ .*

The first condition says that the singular locus is smaller than a hypersurface, whereas the second one, also referred to as Serre's  $S_2$  condition, implies that regular functions defined

away from a codimension  $\geq 2$  set are everywhere defined (see for example [Har77, Ex. III.3.5]).

It follows that normal curves are actually smooth, and normal surfaces have at worst isolated singularities.

### 3.2 The canonical sheaf

If  $X$  is a smooth variety, then the top dimensional exterior power of the sheaf of differentials  $\Omega_{X,k}$  is an invertible sheaf called *the canonical sheaf* and it's usually denoted by  $\omega_X$ . Any representative  $K_X$  in the corresponding Weil divisor class is called a *canonical divisor*.

It turns out that one can define a canonical divisor on normal (but not necessarily smooth) varieties. Indeed, let  $X$  be normal and let  $W$  be its singular locus. Then  $W$  has codimension at least 2, and we can consider the sheaf  $\omega_{X \setminus W}$ . This is an invertible sheaf on the open set  $X \setminus W$ , hence it corresponds to a Weil divisor  $K_{X \setminus W}$ . By taking its closure we get a Weil divisor  $K_X$  on  $X$  that we call the canonical divisor of  $X$ . Note that since the class groups of  $X$  and  $X \setminus W$  are isomorphic, the linear equivalence class of  $K_X$  is well defined. One can also see that the corresponding divisorial sheaf  $\mathcal{O}_X(K_X)$  is the pushforward  $j_*\omega_{X \setminus W}$ , where  $j : X \setminus W \hookrightarrow X$  is the canonical inclusion. In general  $\mathcal{O}_X(K_X)$  has rank one, but need not be invertible, therefore  $K_X$  is a Weil divisor but not necessarily Cartier.

**Example 3.2.1.** Let  $X$  be the cone over the twisted cubic  $C$  in  $\mathbb{P}^3$ , with vertex  $P$ . Then its coordinate ring is  $S := k[x_0, \dots, x_4]/(Q_1, Q_2, Q_3)$ , where the  $Q_i$ s are the 2 by 2 minors of the matrix

$$\begin{bmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{bmatrix}$$

It is easy to see that if  $L_1, L_2$  are the two lines joining the vertex  $P$  with the two points  $[1 : 0 : 0 : 0 : 0]$  and  $[0 : 0 : 0 : 1 : 0]$ , then a canonical divisor for  $X$  is  $-L_1 - L_2$  (see for example [Har77, Ex. II.8.4]). We will prove that  $D := L_1 + L_2$  is not Cartier.

Let  $I = (x_0x_3, x_1, x_2)$  be the ideal of  $D$  in  $S$ . If  $D$  were Cartier, then in a neighborhood of  $P$  it would be defined by one equation, hence the image of  $I$  in  $\mathfrak{m}_P/\mathfrak{m}_P^2$  would be at most one

dimensional. But  $\mathfrak{m}_P/\mathfrak{m}_P^2$  has a basis consisting of  $x_0, \dots, x_4$ , and the image of  $I$  contains  $x_1$  and  $x_3$ , so it can't be one dimensional. Therefore  $K_X = -D$  is not Cartier.

On the other hand, a generic hyperplane section of  $C$  in  $\mathbb{P}^3$  consists of 3 points, therefore if  $H$  is a hyperplane in  $\mathbb{P}^4$  passing through  $P$ , then  $3K_X = -2H|_X$ , so a multiple of  $K_X$  is actually Cartier.

◇

### 3.3 (Log) canonical and (log) terminal singularities

Let  $X$  be a normal variety, and let  $f : Y \rightarrow X$  be a proper birational morphism from a smooth variety  $Y$ . We would like to compare the canonical divisors of  $X$  and  $Y$ , and to do the most intuitive way is to pull back  $K_X$  along  $f$ . In order for this to be possible,  $K_X$  (or at least one of its multiples) has to be Cartier. This leads us to the following

**Definition 3.3.1.** A normal variety  $X$  is  $\mathbb{Q}$ -Gorenstein if  $K_X$  is  $\mathbb{Q}$ -Cartier, i.e.  $mK_X$  is Cartier for some integer  $m > 0$ .

**Remark 3.3.2.** Although the previous definition reflects the current usage, when first introduced in the literature the expression “ $\mathbb{Q}$ -Gorenstein” indicated a Cohen-Macaulay variety such that  $K_X$  is  $\mathbb{Q}$ -Cartier: see for example [KSB88, Def. 3.1].

Suppose then that  $mK_X$  is Cartier. Let  $E$  be a codimension one component of  $\text{Ex}(f)$ , and let  $e$  be its generic point. Let  $\gamma$  be a local generator for  $\mathcal{O}_X(mK_X)$ . If  $\{y_i\}$  is a local coordinate system near  $e$  then we have that

$$f^*\gamma = h(y)(dy_1 \wedge \dots \wedge dy_n)^{\otimes m}$$

for some rational function  $h$ . If  $E$  has the local equation  $y_1 = 0$  then  $h$  can have only poles or zeros at  $E$ , since the map  $f$  is an isomorphism away from  $E$ . Therefore we have

$$f^*\gamma = y_1^r u(y)(dy_1 \wedge \dots \wedge dy_n)^{\otimes m} \tag{*}$$

for some unit  $u$  and some integer  $r$ . It's relatively easy to check that the rational number  $a(E, X) := r/m$  does not depend on  $m$ , and it's called the *discrepancy* of  $E$  with respect to  $X$ .

Although the above definition is quite concrete, as it relates the discrepancies with zeroes and poles of differential forms, there is an equivalent definition. Let  $C$  be the exceptional locus of the proper birational morphism  $f : Y \rightarrow X$ : if  $c \in C$  then  $f^{-1}(f(c))$  is positive dimensional. Hence the fibers of  $C \rightarrow f(C)$  are positive dimensional, therefore  $\text{codim}_X f(C) \geq 2$ . Let  $K_X$  be a canonical (Weil) divisor: then the set  $\text{supp} K_X \cap (X \setminus f(C))$  is not empty for dimension reasons and its closure  $K_Y$  is a canonical divisor for  $Y$ . It follows that the (Cartier) divisor  $mK_Y - f^*(mK_X)$  is trivial outside  $C$ . If  $E_i$  are the codimension one components of  $C$  then we have

$$mK_Y \sim f^*(mK_X) + \sum m \cdot a(E_i, X)E_i$$

and we can formally divide by  $m$  to get the cleaner formula

$$K_Y \equiv f^*K_X + \sum a(E_i, X)E_i$$

This is essentially the global version of (\*), and it's immediate to see that the numbers  $a(E_i, X)$  are the same.

Since the development of the minimal model program showed the importance of working with *pairs*  $(X, \Delta)$ , where  $\Delta$  is a divisor on  $X$ , we will see now that the definition of discrepancy readily generalizes to pairs.

**Definition 3.3.3.** Let  $(X, \Delta)$  be a pair where  $X$  is normal and  $\Delta = \sum a_i D_i$  is a sum of distinct prime divisors with rational coefficients. Assume that  $m(K_X + \Delta)$  is Cartier for some  $m > 0$ . Let  $f : Y \rightarrow X$  be a birational morphism with  $Y$  smooth and let  $E_i$  be the codimension one components of the exceptional locus  $E$ . Then the birational transform of  $D_i$  is the closure of  $f_{|Y \setminus E}^{-1}(D_i)$ , and it's denoted by  $f_*^{-1}D_i$ . The birational transform of  $\Delta$  is then

$$f_*^{-1}\Delta := \sum a_i f_*^{-1}D_i$$

As before, one can choose the Weil divisors  $K_X, K_Y$  such that the Cartier divisors  $mK_Y + mf_*^{-1}\Delta$  and  $f^*(m(K_X + \Delta))$  are trivial outside the exceptional locus, hence with a slight abuse of notation we can write

$$K_Y + f_*^{-1}\Delta \sim f^*(K_X + \Delta) + \sum a(E_i, X, \Delta)E_i$$

where the rational numbers  $a(E_i, X, \Delta)$  are independent of  $m$  and are called the *discrepancies* of  $E_i$  w.r.t  $(X, \Delta)$ .

If  $D$  is a divisor on  $X$ , then we define  $a(D, X, \Delta) = -\text{coeff}_D\Delta$ .

Now we are ready to classify the singularities of  $(X, \Delta)$  by looking at the discrepancies.

**Definition 3.3.4.** Let  $\Delta$  be a subboundary, i.e. a formal sum  $\sum a_i D_i$  with  $a_i \leq 1$ , such that  $K_X + \Delta$  is  $\mathbb{Q}$ -Cartier. We say that the singularities of the pair  $(X, \Delta)$  are

$$\left. \begin{array}{l} \text{terminal} \\ \text{canonical} \\ \text{Kawamata log terminal} \\ \text{purely log terminal} \\ \text{log canonical} \end{array} \right\} \text{if } a(E, X, \Delta) \left\{ \begin{array}{l} > 0 \text{ for all exceptional } E \\ \geq 0 \text{ for all exceptional } E \\ > -1 \text{ for all } E \\ > -1 \text{ for all exceptional } E \\ \geq -1 \text{ for all } E \end{array} \right.$$

The threshold  $-1$  is justified by the fact that if for some  $E$  one has  $a(E, X, \Delta) < -1$  then there are exceptional divisors with arbitrarily negative discrepancy [KM08, Cor. 2.31]. Therefore the class of log canonical singularities is the largest class in which we have some control over the discrepancies. Also, it is not necessary to check every resolution to establish the singularities of a pair  $(X, \Delta)$ : indeed, it is sufficient to look at a single log resolution (see for example [Kol13, Cor. 2.12, Cor. 2.13]).

There is a notion analogous to smoothness for pairs:

**Definition 3.3.5.** Let  $(X, \Delta)$  be a pair, with  $\Delta = \sum a_i D_i$ . We say that  $(X, \Delta)$  is *simple normal crossings* (snc) at  $P$  if  $X$  is smooth at  $P$  and there are local coordinates  $x_1, \dots, x_n$  such that in a neighborhood of  $P$  the set  $\text{Supp } \Delta$  is given by  $x_1 \cdots x_r = 0$ . A pair is snc if it's snc at every point.

One can prove that a pair  $(X, D)$  is snc iff  $X$  is smooth, every component  $D_i$  of  $D$  is smooth and the  $D_i$ 's intersect transversely.

Although it's not trivial, one can prove that smooth varieties have terminal singularities and snc pairs have klt singularities.

**Example 3.3.6.** Let  $C \subseteq \mathbb{P}^n$  be a complete intersection smooth curve of degree  $d > 1$  and genus  $g$ , and let  $X \subseteq \mathbb{P}^{n+1}$  be the projective cone over  $X$ . We want to classify the singularities of  $X$ . It will be sufficient to look at one resolution of  $X$ .

We know that  $X$  is normal by Serre's criterion and has an isolated singularity at the vertex  $P$ . Moreover  $X$  is  $\mathbb{Q}$ -Gorenstein as  $K_C$  is a multiple of the ample sheaf  $\mathcal{O}_C(1)$  that defines the cone: an easy computation shows that actually  $K_X$  is Cartier.

Let  $f : Y \rightarrow X$  be the blow up of  $X$  at  $P$ . Clearly, this is a resolution of singularities and by [Har77, Example V.2.11.4]  $Y$  is a (smooth) ruled surface over  $C$ ; moreover, the exceptional divisor  $E$  is a section with self intersection  $E^2 = -d$ . We have

$$K_Y \sim f^*K_X + rE$$

for some rational number  $r$ . By taking the intersection of both sides with  $E$  we get

$$K_Y.E = r(-d)$$

On the other hand, by the adjunction formula

$$2g - 2 = E(K_Y + E) = E.K_Y - d$$

This implies that

$$r + 1 = \frac{2}{d} - 2\frac{g}{d}$$

So the singularities of  $X$  are:

$$\begin{aligned} \text{canonical} & \quad \text{if} \quad d = 2 - 2g \\ \text{purely log terminal} & \quad \text{if} \quad g = 0 \\ \text{log canonical} & \quad \text{if} \quad g \leq 1 \end{aligned}$$



It's important to point out that although the definition looks global, having (log) canonical, (log) terminal singularities is a local property.

Since a divisor  $E$  over  $X$  that has discrepancy  $-1$  seems like a limit case for the singularities we want to handle, then it makes sense to study the properties of their images in  $X$ : they are called log canonical centers and they will be the main topic of one of the next chapters.

### 3.4 Rational pairs

Since the definition of discrepancies so easily generalizes to pairs, one can ask whether other types of singularities, originally defined for algebraic varieties, can be defined for pairs. This section describes the case of rational singularities, for which a notion of rational pair has been introduced by Kollár and Kovács: the main references are [Kol13], [Eri14b].

#### 3.4.1 Rational singularities of algebraic varieties

**Definition 3.4.1.** A normal variety  $X$  has rational singularities if for some resolution of singularities  $f : Y \rightarrow X$  we have

1. the inclusion  $\mathcal{O}_X \hookrightarrow f_*\mathcal{O}_Y$  is an isomorphism
2.  $R^i f_*\mathcal{O}_Y = 0$  for all  $i > 0$
3. the natural map  $\omega_X \rightarrow f_*\omega_Y$  is an isomorphism

Note that conditions 1 and 2 in the definition above are equivalent to the natural map of complexes  $\mathcal{O}_X \rightarrow Rf_*\mathcal{O}_Y$  being a quasi isomorphism, and condition 1 is equivalent to  $X$  being normal.

In the case where  $X$  is a normal surface, then isolated rational singularities have a resolution whose exceptional divisor is a tree of curves isomorphic to  $\mathbb{P}^1$ , hence the name.

In general, we can say that varieties with rational singularities are cohomologically smooth: more generally one can easily check the following (see for example [Har77, Ex. III.8.1]).

**Proposition 3.4.2.** *Let  $X$  be a variety with rational singularities and let  $f : Y \rightarrow X$  be a resolution of singularities. Then for every line bundle  $\mathcal{L}$  on  $X$  there is a natural isomorphism  $H^i(Y, f^*\mathcal{L}) \xrightarrow{\sim} H^i(X, \mathcal{L})$*

Rational singularities have many interesting properties. For example, flat families of varieties with rational singularities deform: more precisely, Renée Elkik showed in [Elk78] that if  $X \rightarrow S$  is a flat morphism and  $S$  is smooth, if the fiber  $X_s$  over  $s \in S$  has rational singularities then  $X$  has rational singularities in a neighbourhood of  $X_s$ .

Furthermore, the quotient of an algebraic variety with rational singularities by the action of a finite group still has rational singularities, as one can see in [KM08].

In the next sections, we will describe a suitable generalization of rational singularities to pairs and see which of the above properties generalize and how.

### 3.4.2 Rational resolutions and thrifty resolutions

In this section we present the basic definitions and results on rational pairs. We will work on an algebraically closed field  $k$  of characteristic zero.

**Definition 3.4.3.** A *reduced pair*  $(X, D)$  consists of the datum of a normal variety  $X$  and a reduced Weil divisor  $D$  on  $X$ .

As the reader might expect, we require maps between varieties to respect the pair structure. More precisely, we will use the following terminology:

**Definition 3.4.4.** Let  $(X, D), (Y, B)$  be reduced pairs. A *birational morphism of pairs*  $f$  is a birational morphism  $f : Y \rightarrow X$  such that  $B = f_*^{-1}D$ , where  $f_*^{-1}D$  denotes the strict transform of  $D$ .

**Definition 3.4.5.** A *resolution* of a pair  $(X, D)$  is a birational morphism of pairs  $(Y, B) \rightarrow (X, D)$ , with  $(Y, B)$  snc.

We can now define a class of resolutions whose properties closely resemble the behavior of the resolution of a variety with rational singularities.

**Definition 3.4.6.** A resolution of pairs  $f : (Y, B) \rightarrow (X, D)$  is called *rational* if

1. The natural injection  $\mathcal{O}_X(-D) \hookrightarrow f_*\mathcal{O}_Y(-B)$  is an isomorphism
2.  $R^i f_*\mathcal{O}_Y(-B) = 0$  for  $i > 0$
3.  $R^i f_*\omega_Y(B) = 0$  for  $i > 0$

It is worth pointing out that over a field of characteristic zero condition (3) is automatically satisfied by an analogue of the Grauert-Riemenschneider vanishing theorem [Eri14b, Prop. 3.6]. Moreover, we will show in one of the next chapters that condition (1) always holds if  $X$  is normal. This implies that most of the times, in order to prove rationality of a resolution, we will concentrate our efforts in showing that (2) holds.

One could try to define a rational pair as a reduced pair that has a rational resolution. However, this would lead to some ambiguities, as the following example will show.

**Example 3.4.7.** Let  $X = \mathbb{P}^2$  and let  $D = L_1 + L_2$  be a reduced divisor on  $X$ , where  $L_1, L_2$  are two different lines meeting at a point  $x$ . Clearly  $(X, D)$  is snc and the identity  $(X, D) \rightarrow (X, D)$  is a rational resolution.

Let  $f : Y \rightarrow X$  be the blow up of  $X$  at  $x$  and let  $B$  be the birational transform of  $D$ . It is easy to see that  $f$  is a thrifty resolution: we will show that  $f$  is *not* a rational resolution.

Consider the exact sequence

$$0 \rightarrow \mathcal{O}_Y(-B) \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_B \rightarrow 0$$

By taking the pushforward by  $f$  we get the exact sequence

$$0 \rightarrow f_*\mathcal{O}_Y(-B) \rightarrow f_*\mathcal{O}_Y \rightarrow f_*\mathcal{O}_B \rightarrow R^1 f_*\mathcal{O}_Y(-B) \rightarrow R^1 f_*\mathcal{O}_Y$$

Since  $X$  is smooth then it has rational singularities, hence  $R^1 f_* \mathcal{O}_Y = 0$ . If  $f$  were a rational resolution then the map on stalks  $(f_* \mathcal{O}_Y)_x \rightarrow (f_* \mathcal{O}_B)_x$  would be surjective, but this is impossible since  $(f_* \mathcal{O}_B)_x$  requires at least two generators over  $\mathcal{O}_{X,x}$  as  $x$  has two distinct closed points in its preimage. Therefore  $f$  is a thrifty non rational resolution of a snc pair.

◇

So far the generalization of smoothness and resolutions was relatively straightforward. In order to define singularities of pairs that behave similarly to rational singularities, we have to restrict to a particular class of resolutions.

**Definition 3.4.8.** If  $(X, D)$  is snc, any intersection of the components  $D_i$  of  $D$  is called a *stratum* of  $(X, D)$ .

**Definition 3.4.9.** A resolution  $f : (Y, B) \rightarrow (X, D)$  is called *thrifty* if it satisfies the following conditions:

1.  $f$  is an isomorphism over the generic point of any stratum of  $snc(X, D)$
2.  $f$  is an isomorphism over the generic point of any stratum of  $(Y, B)$

The conditions in the above definition are equivalent, respectively, to these two:

1.  $f(\text{Ex}f)$  does not contain any stratum of  $snc(X, D)$ .
2.  $\text{Ex}f$  does not contain any stratum of  $(Y, B)$

Fortunately, thrifty resolutions always exist for any pair in characteristic zero [Kol13, Thm 10.45].

We can now state our main definition:

**Definition 3.4.10.** A reduced pair is a *rational pair* if it has a rational thrifty resolution.

One can prove that if a reduced pair has a rational thrifty resolution, then every thrifty resolution is rational. It is not known whether rational resolutions are necessarily thrifty; it is true for log resolutions, as shown in [Eri14b, Prop. 3.4].

### 3.4.3 *Properties of rational pairs*

Since rational pairs were introduced in the literature relatively recently, it is not completely known which properties of rational singularities generalize to rational pairs in a straightforward way.

It is known that rational pairs deform: Lindsay Erickson proved in [Eri14b] that given a flat morphism  $X \rightarrow S$  with  $(X, D)$  a rational pair,  $S$  smooth and  $D$  Cartier then if a fiber  $(X_s, D_s)$  is rational then  $(X, D)$  is rational in a neighborhood of the fiber. An analogous result holds for rational singularities (see for example [Elk78, Théorème 2]).

We will see in the last chapter that quotient of rational pairs need not be rational pairs: we will give a sufficient condition for this to happen.

## Chapter 4

## POTENTIAL LOG CANONICAL CENTERS

Log canonical centers are special subvarieties of a pair  $(X, \Delta)$ , and their behavior is related to the properties of log canonical singularities; they also provide a framework for inductive arguments in higher dimensional algebraic geometry.

Most of the results in the literature consider a pair  $(X, \Delta)$  and draw conclusions on its lc centers. In this chapter, we fix an ambient variety  $X$  and a subvariety  $Z$  and we try to answer the following:

**Question.** *Is there a divisor  $\Delta$  such that  $Z$  is a log canonical center for  $(X, \Delta)$ ? What can be said about the pair  $(X, \Delta)$ ?*

If such a divisor exists, we call  $Z$  a *potential* lc center for  $X$ .

As it turns out, quite a large class of subvarieties  $Z$  are potential lc centers, at least with reasonable hypotheses on  $X$ . The first result is the following:

**Theorem A** (Theorem 4.2.1). *Let  $X$  a  $\mathbb{Q}$ -Gorenstein normal variety and let  $Y$  be any irreducible subvariety not entirely contained in  $\text{Sing } X$ . Then there is a boundary  $\Delta$  such that  $Y$  is a log canonical center for the pair  $(X, \Delta)$ .*

If we make stronger assumptions on the singularities of both the subvariety and the ambient variety, we can even classify the singularities of  $(X, \Delta)$ .

**Theorem B** (Theorem 4.4.1). *Let  $X$  be a Gorenstein variety with lc singularities, and let  $W$  be a reduced irreducible special complete intersection in  $X$  (i.e. cut out by homogeneous forms of the same degree) with log canonical singularities, not contained in  $\text{Sing } X$ . Then there exist a boundary  $\Delta$  such that the pair  $(X, \Delta)$  has log canonical singularities and  $W$  is a log canonical center for  $(X, \Delta)$ .*

This chapter is structured as follows: the first section is devoted to the basic properties of lc centers. The second states and proves the first theorem. The third section is a brief introduction to the theory of Du Bois singularities, as they are used extensively in the fourth section to prove the second theorem.

#### 4.1 Background and notations

**Definition 4.1.1.** Let  $X$  be a normal variety over an algebraically closed field  $k$  of characteristic zero, and let  $\Delta$  be an effective Weil  $\mathbb{R}$ -divisor such that  $K_X + \Delta$  is  $\mathbb{R}$ -Cartier. Assume furthermore that  $\Delta$  is a subboundary, i. e. all of its coefficients are at most 1. A subvariety  $Z \subseteq X$  is called a *log canonical center* for the pair  $(X, \Delta)$  if:

1. The pair  $(X, \Delta)$  has log canonical singularities at the generic point of every component of  $Z$
2. There is a birational morphism  $f : Y \rightarrow X$  and a prime divisor  $E$  in  $Y$  such the discrepancy of  $E$  with respect to  $\Delta$  is  $-1$  and  $f(E) = Z$

If the pair  $(X, \Delta)$  has log canonical singularities, more of the geometry of its lc centers is known.

**Theorem 4.1.2** (Theorem 5.14, [Kol13]). *If  $(X, \Delta)$  is a lc pair, then*

1. *Any intersection of lc centers is a union of lc centers*
2. *Every union of lc centers is seminormal*
3. *Every lc center that is minimal w. r. t. inclusion is normal*

Based on the definition, it seems like the nature of the lc centers of a variety is hard to grasp. Let's start by characterizing them when  $(X, \Delta)$  is as "nice" as possible.

**Lemma 4.1.3.** *Let  $(X, \Delta)$  be a snc pair with  $\Delta = \sum a_i D_i$  a subboundary. Then the log canonical centers of  $(X, \Delta)$  are the components of the intersections of the divisors  $D_i$  such that  $a_i = 1$ .*

*Proof.* Let  $W$  any reduced subscheme of  $X$ , and let  $f : Y \rightarrow X$  be a proper birational morphism such that a prime divisor  $E \subset Y$  dominates  $W$ . By replacing  $X$  with  $X \setminus \text{Sing } W$  we may assume that  $W$  is smooth. It follows from [Koll13, Lemma 2.22] that  $f$  factors as a tower of blow ups along smooth centers

$$Y = X_n \rightarrow X_{n-1} \rightarrow \dots \rightarrow X_0 = X$$

Since in the blow up of a smooth variety along a smooth center the exceptional divisor only has one component, there is  $X_N$  such that  $X_{X+1} \rightarrow X_N$  is the blow up along a subscheme  $\overline{W}_N$  birational to  $W$ . The morphism  $X_N \rightarrow X$  is then birational on  $\overline{W}_N$ , hence if  $E_n$  denotes the irreducible exceptional divisor in  $X_n \rightarrow X_{n-1}$  we have

$$a(E_n, X, \Delta) = a(E, X, \Delta)$$

This implies that in order to compute  $a(E, X, \Delta)$  we can blow  $W$  up and compute the discrepancy of the exceptional divisor thusly obtained. If  $\pi : Z := \text{Bl}_W X \rightarrow X$  is the blow up map with exceptional divisor  $C$ , then as a consequence of [Har77] we have

$$\pi^*(K_X + \Delta) \sim K_Z + \pi_*^{-1} \Delta + (1 - \text{codim}_X W + \text{mult}_W \Delta) C$$

Since  $\Delta$  is snc, then

$$\begin{aligned} \text{mult}_W \Delta &= \sum_i \text{coeff}_{D_i} \Delta \\ 1 + a(C, X, \Delta) &= \text{codim}_X W - \sum_i \text{coeff}_{D_i} \Delta \end{aligned}$$

Now,  $W$  is a lc center iff  $1 + a(C, X, \Delta) = 0$ , which implies

$$\text{codim}_X W = \sum_{D_i \supseteq W} a_i$$

If  $k$  is the number of  $D_i$ 's containing  $W$  then since  $\Delta$  is snc then  $k = \text{codim}_X W$ . On the other hand, since  $\Delta$  is a subboundary, then  $(\sum_{D_i \supseteq W} a_i) < k$  unless all the  $a_i$  in the sum are equal to 1.

This concludes the proof. □

As a corollary, we get that

**Lemma 4.1.4.** *A pair  $(X, \Delta)$ , where  $\Delta$  is a subboundary, has finitely many log canonical centers.*

*Proof.* By definition,  $(X, \Delta)$  has to be lc at the generic point of each of its lc centers. Therefore we can replace  $X$  with its lc locus and assume that  $(X, \Delta)$  is lc. Let  $f : Y \rightarrow X$  be a log resolution and define

$$\Delta_Y := f^*(K_X + \Delta) - K_Y$$

In other words,  $\Delta_Y$  is the birational transform of  $\Delta$  together with a divisor supported on  $\text{Ex}(f)$ . Since  $f$  is a log resolution,  $(Y, \Delta_Y)$  is snc and since  $(X, \Delta)$  is lc then  $\Delta_Y$  is a subboundary. Moreover for any divisor over  $X$ , we have  $a(E, X, \Delta) = a(E, Y, \Delta_Y)$ . This implies that the lc centers of  $(X, \Delta)$  are the images of the lc centers of  $(Y, \Delta_Y)$ , and we know there are finitely many of these by Lemma 4.1.3. □

In this chapter, we will make use of the following

**Definition 4.1.5.** Let  $X$  be a variety in  $\mathbb{P}^n$ , and let  $Y$  be a subvariety of  $X$  whose codimension in  $X$  is  $r$ . Then we say that  $Y$  is a *complete intersection* in  $X$  if there are  $r$  homogeneous forms  $F_1, \dots, F_r$  such that  $Y$  is the subscheme theoretic of  $X$  defined by the ideal  $(F_1, \dots, F_r)$ . If  $d_i = \deg F_i$ , then we say that  $Y$  is of type  $(d_1, \dots, d_r)$  in  $X$ .

If  $d_1 = \dots = d_r$  we say that  $Y$  is a *special complete intersection* in  $X$ .

We will also need some well known results from commutative algebra.

**Definition 4.1.6.** Let  $A$  be a Noetherian ring. We say that  $A$  is  $R_n$  if for every prime  $\mathfrak{p} \subset A$  of height  $\leq n$  the local ring  $A_{\mathfrak{p}}$  is regular. We say that  $A$  is  $S_n$  if for every prime  $\mathfrak{p} \subset A$  of height  $\leq n$  we have  $\text{depth } A_{\mathfrak{p}} \geq n$ .

**Proposition 4.1.7** (Serre's criterion, [Eis95]). *A ring  $A$  is normal iff it's  $R_1$  and  $S_2$ . A ring  $A$  is reduced iff  $R_0$  and  $S_1$ .*

Finally, we will use the following notation: if  $A$  is a graded ring and  $I$  a homogeneous ideal,  $V(I)$  will denote the closed subscheme of  $\text{Proj} A$  cut out by the ideal  $I$ .

## 4.2 Log canonical centers of $\mathbb{Q}$ -Gorenstein varieties

The main result of this section is:

**Theorem 4.2.1.** *Let  $X$  a  $\mathbb{Q}$ -Gorenstein normal variety and let  $Z \subseteq X$  be any irreducible subvariety not entirely contained in  $\text{Sing } X$ . Then there is a boundary  $\Delta$  such that  $Z$  is a log canonical center for the pair  $(X, \Delta)$ .*

Note that Theorem 4.2.1 implies that any subvariety of a smooth variety is a log canonical center.

The structure of the proof can be summarized as follows: first, we will embed  $Z$  in a complete intersection  $W$  and consider the birational morphism  $f : Y \rightarrow X$ , where  $Y$  is the normalization of the blow up of  $X$  at  $W$ . Then we will produce a divisor  $\Delta$  such that the unique component of the exceptional divisor of  $f$  that dominates  $Z$  has discrepancy  $-1$  for the pair  $(X, \Delta)$ .

Let's focus on the Cohen-Macaulay case first.

**Lemma 4.2.2.** *Every subvariety  $Z$  of a Cohen-Macaulay variety  $X$  is a component of a reduced complete intersection in  $X$ .*

*Proof.* Let  $S$  be the homogeneous coordinate ring of  $X$  in  $\mathbb{P}^n$ . The ideal  $I_Z \subseteq S$  is prime, hence we can apply Lemma 4.2.3 to get a (homogeneous) non zerodivisor  $g_1 \in I_Z$  such that

$S/g_1$  is reduced and CM. Continuing in this fashion one gets a regular sequence  $g_1, \dots, g_r$  contained in  $I_Z$ , where  $r = \text{codim } Z$ . Since the ideal is radical and equidimensional, then it's the intersection of the minimal prime ideals containing it. By construction, the height of  $I_Z$  is the same as the height of  $(g_1, \dots, g_r)$ , so  $I_Z$  has to be one of its minimal primes, hence  $Z$  is a component of the complete intersection cut out in  $X$  by the  $g_i$ 's.  $\square$

**Lemma 4.2.3.** *Let  $R$  be a finite type reduced graded Cohen-Macaulay  $k$ -algebra, with  $\text{char } k = 0$ , and let  $\mathfrak{p}$  be a prime in  $R$  of height  $\geq 1$ . Then there is a homogenous element  $g \in \mathfrak{p}$  such that  $g$  is a non zerodivisor and  $R/g$  is reduced and CM.*

*Proof.* Write  $R = S/I$  where  $S$  is a graded polynomial ring over  $k$ , and let  $f_1, \dots, f_N$  be homogeneous polynomials of the same degree  $d$  such that  $\mathfrak{p}_{\geq d}$  is generated by the images of  $f_i$ 's in  $S/I$ . The idea is to look for an element of the form

$$g = \sum \lambda_i f_i, \quad \lambda_i \in k$$

Let's first establish that at least a non zerodivisor of that form exists: if all linear combinations were zero divisors, then they would lie in the union of the minimal primes  $\mathfrak{p}_i$  of  $R$ , thus implying that  $\mathfrak{p}_{\geq d} \subseteq \bigcup \mathfrak{p}_i$ . If  $a \in \mathfrak{p}$  then for some  $q$  we would have  $a^q \in \mathfrak{p}_{\geq d}$ , thus  $a^q \in \bigcup \mathfrak{p}_i$ , which in turn forces  $a \in \bigcup \mathfrak{p}_i$ . This would show that  $\mathfrak{p} \subseteq \bigcup \mathfrak{p}_i$  and by the prime avoidance lemma then  $\mathfrak{p} \subseteq \mathfrak{p}_j$  for some  $j$ , which is impossible by the assumption on the height of  $\mathfrak{p}$ : therefore at least one of the  $f_i$ 's is not a zero divisor.

Now we shall show that there is a non empty open set  $U \subseteq k^N$  such that  $g$  is a non zerodivisor in  $R$  for all  $(\lambda_i) \in U$ . Indeed,  $g$  is a zerodivisor iff it is contained in one of the minimal primes  $\mathfrak{p}_i$ 's, and this happens precisely when  $(\lambda_i)$  is in the kernel of the  $k$ -linear map

$$\psi_i : k^N \rightarrow R_{\mathfrak{p}_i}/\mathfrak{p}_i R_{\mathfrak{p}_i}, (\lambda_i) \mapsto g$$

By the reasoning in the previous paragraph, the kernel of  $\psi_i$  is not the whole of  $k^N$ , and setting  $U = k^N \setminus \bigcup \ker \psi_i$  yields the desired open set.

Since  $\text{char } k = 0$ , then by Bertini's theorem the generic element  $g$  of the linear system generated by the  $f_i$ 's is smooth on the nonsingular locus of  $Z(I)$  away from  $Z(\mathfrak{p})$ , and from

the discussion above  $g$  is also a non zerodivisor. In particular,  $R/g$  is reduced at the generic point of each component, hence it's  $R_0$ . Since  $R$  is CM and  $g$  a non zerodivisor, we have that  $R/g$  is CM too: in particular it's  $S_1$ . It follows that  $R/g$  is reduced. Finally, note that since  $R$  is CM and  $g$  is a non zerodivisor, then  $R/g$  is CM as well.  $\square$

Note that this implies that every subvariety of  $\mathbb{P}^n$  is a component of a reduced complete intersection, a fact that is quite interesting on its own.

For the general case, let  $W$  the complete intersection cut out in  $X$  by  $\text{codim}_X Z$  forms in  $I_X$  of sufficiently high degree. We know that  $\dim Z = \dim W$  and if  $x \in Z\text{Sing}X$  then by the same argument as in Lemma 4.2.2 we have that  $W$  is reduced at  $x$ . This implies that  $W$  is reduced near  $x$ , and by equidimensionality of complete intersections we have that  $Z$  is a component of  $W$ .

We can now lay out the notation and the setup used for the rest of the section.

**Setup 4.2.4.** Let  $X$  be  $\mathbb{Q}$ -Gorenstein and let  $W$  be a complete intersection in  $X$  cut out by the polynomials  $F_1, \dots, F_r$ . Let  $f : Y \rightarrow X$  be the normalization of the blow up of  $X$  at  $W$ .

Now we will proceed to compute the multiplicities of the exceptional divisors of the blow up of  $X$  at a complete intersection.

**Lemma 4.2.5.** *Write  $W = W_1 \cup \dots \cup W_q$  as the union of its irreducible components. Then the exceptional divisor  $E$  has exactly  $r$  components  $E_i$  and moreover we have*

$$K_Y \sim f^*K_X + (r-1)E_1 + \dots + (r-1)E_q$$

*Proof.* Let  $Y'$  be the blow up of  $X$  at  $W$ , with exceptional locus  $E$ . Since  $W$  is a complete intersection, then the ideal of  $W$  in  $X$  is locally generated by a regular sequence, as one can easily check. Therefore, the very same proof of [Har77, Theorem II.8.24(b)] gives that the exceptional divisor is a projective bundle over  $W$ , hence  $E$  has exactly as many components as  $W$ . It follows that the preimage  $V$  of the smooth locus of  $W$  is smooth in  $Y'$ : in particular  $Y'$  is regular in codimension one.

Let  $n : Y \rightarrow Y'$  be the normalization map and let  $f : Y \rightarrow X$  be the composition of  $Y$  with the blow up map. The R1 condition on  $Y'$  implies that  $n$  is an isomorphism in codimension one, therefore the exceptional locus has codimension at least 2 and  $n : n^{-1}E \rightarrow E$  induces an isomorphism over the smooth locus of each component  $E_i$  of  $E$ . Therefore, in order to compute the discrepancies of the  $E_i$ 's we can restrict to looking at the smooth locus of  $W_i$ . Let  $U_i$  be a neighborhood of the smooth locus of  $W_i$  in  $X$  on which  $X$  is smooth. By [Har77, Ex. II.8.5] we have that

$$K_Y|_{f^{-1}U_i} \sim f^*K_X|_{f^{-1}U_i} + (r-1)E_i|_{f^{-1}U_i}$$

Since  $U_i$  cannot contain any other components of  $W$  other than  $W_i$ , this concludes the proof.  $\square$

Now we need to produce a boundary  $\Delta$ : if  $D_i$  denotes the divisor determined by  $F_i$ , then a natural choice is to set  $\Delta = D_1 + \dots + D_r$ . It turns out that this actually works. First of all we check that  $(X, \Delta)$  satisfies the condition (1) of the definition of lc center.

**Lemma 4.2.6.** *The pair  $(X, \Delta)$  is snc away from the union of the singular loci of  $W$  and  $X$ .*

*Proof.* Let  $U = X \setminus (\text{Sing } X \cup \text{Sing } W)$ , and let  $p \in U$ . We will first prove that every component  $D_i$  of  $\Delta$  is smooth at  $p$ .

Let  $(G_1, \dots, G_s)$  be the ideal of  $X$  in  $\mathbb{P}^n$ , and assume that  $W$  is cut out by the regular sequence  $F_1, \dots, F_r$ . Then the Jacobian of  $W$  as a subvariety of  $\mathbb{P}^n$  is given by

$$J = \begin{bmatrix} \nabla F_1 \\ \dots \\ \nabla F_r \\ \nabla G_1 \\ \dots \\ \nabla G_s \end{bmatrix}$$

where the zero locus of  $F_i$  is the component  $D_i$  of  $\Delta$ .

Let  $r' = n - \dim X$ : since  $X$  is smooth, then the rank of the submatrix of  $J(p)$  obtained by considering the last  $s$  rows is  $r'$ . If any of the first  $r$  rows were zero, then the rank of  $J(p)$  would be less than  $r + r'$ , and this would contradict the smoothness of  $p$ . Therefore  $\nabla G_i \neq 0$  for all  $i$ 's: this shows that the divisors  $D_i$  are smooth at  $p$ .

We now claim that the  $D_i$ 's intersect transversely at  $p$ : this amounts to showing that

$$\text{codim}_{T_p X}(T_p D_1 \cap \dots \cap T_p D_r \cap T_p X) = \text{codim}_{X,p}(D_1) + \dots + \text{codim}_{X,p}(D_r)$$

Observe that each term in the right hand side of the above equality is 1, hence the r.h.s. is equal to  $r$ . On the other hand, since  $T_p D_1 \cap \dots \cap T_p D_r \cap T_p X$  is the tangent space to  $W$  at  $p$ , counting (co)dimensions yields that the l.h.s. is equal to  $r$ , too.  $\square$

Now we compute the pullbacks of the components of  $\Delta$ . The idea is that the fact that they meet transversely forces each  $D_i$  to contain each component of  $W$  with multiplicity one. More precisely, we have the following

**Lemma 4.2.7.** *Let  $\tilde{D}_i$  the proper transform of  $D_i$ . Then*

$$f^*(D_i) \sim \tilde{D}_i + E_1 + \dots + E_q$$

*Proof.* The proof of Lemma 4.2.5 implies that the  $E_i$ 's are the only exceptional divisors of  $f$ . Therefore

$$f^*(D_i) \sim \tilde{D}_i + \mu_1 E_1 + \dots + \mu_r E_q$$

for some  $\mu_i \in \mathbb{Z}$ . The idea is to compute the  $\mu_j$ 's locally on the the smooth locus of the  $W_j$  and to show that they are all equal to 1.

We also know from Lemma 4.2.6 that every  $D_i$  is smooth on an open subset  $U$  of  $W_j$ . Let's then work locally and assume that  $W \subset X = \text{Spec}R$  is a smooth integral complete intersection given by the ideal  $I = (f_1, \dots, f_r)$  of the ring  $R$  and  $D_i = D$  is the zero set of an element  $g \in I$ , with  $D$  smooth along  $W$ .

With this setup, the blow up  $Y = Bl_W X \subset X \times \mathbb{P}^{r-1}$  is given by taking Proj of the graded ring

$$S = R[t_1, \dots, t_r]/(t_i f_j - t_j f_i | i, j = 1, \dots, r)$$

and the ideal  $J$  of the exceptional divisor  $E$  is the ideal generated by  $I$  in  $S$ . We want to show that  $g \in J \setminus J^2$ , where  $g$  is the image of  $g$  under the natural injection  $R \hookrightarrow S$ , as this will ensure that  $f^*(D) \sim \tilde{D} + E$ .

Assume not, i.e. suppose we can write

$$g = \sum \alpha_{ij}(x, t) f_i f_j, \quad \alpha_{ij}(x, t) \in S$$

Then this would imply

$$g = \sum \alpha_{ij}(x, 0) f_i f_j, \quad \alpha_{ij}(x, 0) \in R$$

thus  $g \in I^2$ , which contradicts the assumption that  $D$  is smooth along  $W$ .  $\square$

Putting everything together, we can prove the main result of this section.

*Proof of Theorem 4.2.1.* By using Lemma 4.2.2 we can assume that  $Z$  is a component of a complete intersection  $W$  in  $X$ , so  $Z = W_i$  for some  $i$ . Using the same notation as before, with  $\Delta = D_1 + \dots + D_r$  we get

$$\begin{aligned} K_Y - f^*(K_X + \Delta) &\sim (r-1) \sum E_i - f^* \Delta \\ &\sim (r-1) \sum E_i - \sum \tilde{D}_i - r \sum E_i \\ &\sim - \sum \tilde{D}_i - \sum E_i \end{aligned}$$

from which it follows that the discrepancy of  $E_i$  with respect to  $(X, \Delta)$  is  $-1$  for all  $i$ 's. This implies that every component  $W_i$  of  $W$  satisfies the condition (2) of Definition 4.1.1 for the pair  $(X, \Delta)$ . Moreover,  $(X, \Delta)$  satisfies (1) by Lemma 4.2.7.  $\square$

It is worth to point out that with these hypotheses on  $X$  and  $Y$  the pair  $(X, \Delta)$  might not be lc at every point of  $X$ , since if the pair  $(X, \Delta)$  is lc then every log canonical center has to be seminormal and Du Bois (see [Kol13, Section 4.3 and Theorem 5.14]).

### 4.3 Du Bois singularities

If  $X$  is a smooth variety, then one can construct a special resolution of the constant sheaf, namely the DeRham complex. It turns out that an analogue construction can be carried out for an arbitrary normal variety  $X$  over  $\mathbb{C}$ , yielding the *Deligne-Du Bois* complex: it is an object  $\underline{\Omega}_X^\bullet$  in  $D_{coh}^b(X)$  that has a natural map  $\mathcal{O}_X \rightarrow \underline{\Omega}_X^\bullet$ . If this last morphism is a quasi-isomorphism, then we say that  $X$  has *Du Bois singularities*.

The technical details of the definition can be found in [Kol13, Chapter 6]: here we state Schwede's equivalent definition ([Sch07]).

**Definition 4.3.1.** Let  $X$  be normal and let  $X \subset Y$  be an embedding in a smooth scheme. Let  $f : Z \rightarrow Y$  be a log resolution that is an isomorphism outside of  $X$ , and let  $E$  be the pre image with the reduced induced subscheme structure. Then  $X$  has *Du Bois* singularities if the natural map  $\mathcal{O}_X \rightarrow Rf_*\mathcal{O}_E$  is a quasi-isomorphism.

We are interested in Du Bois singularities since in many cases of interest they are closely related to log canonical singularities.

In one direction we have:

**Theorem 4.3.2** ([Kov99]). *Let  $X$  be a normal variety with DB singularities such that  $K_X$  is Cartier. Then  $X$  is log canonical.*

Conversely,

**Theorem 4.3.3** ([KK10], Theorem 1.4). *Let  $(X, \Delta)$  be a log canonical pair. Then  $X$  is Du Bois.*

We now state a few results that will be used in the next section. The first one is a deformation result:

**Theorem 4.3.4** ([KS11], Theorem 4.1). *Let  $X$  be a scheme of finite type over  $\mathbb{C}$  and  $H$  a reduced Cartier divisor on  $X$ . If  $H$  has Du Bois singularities, then  $X$  has Du Bois singularities near  $H$ .*

A substantial advantage of Du Bois singularities over lc singularities is the notion of a Du Bois pair. Given any reduced subscheme  $Z \subseteq X$ , there's a natural map  $\underline{\Omega}_X^\bullet \rightarrow \underline{\Omega}_Z^\bullet$ , that can be completed to a distinguished triangle

$$\underline{\Omega}_{Z,X}^\bullet \rightarrow \underline{\Omega}_X^\bullet \rightarrow \underline{\Omega}_Z^\bullet \xrightarrow{+}$$

and one can prove that there is a natural map  $\mathcal{I}_Z \rightarrow \underline{\Omega}_{Z,X}^\bullet$ .

**Definition 4.3.5.** A pair  $(X, Z)$  is a *Du Bois pair* if  $\mathcal{I}_Z \rightarrow \underline{\Omega}_{Z,X}^\bullet$  is a quasi-isomorphism.

One would expect that if both  $X$  and  $Z$  are Du Bois then the pair  $(X, Z)$  is, too. This is true, by the following very useful

**Lemma 4.3.6** ([Kol13], Prop. 6.15). *Let  $(X, Z)$  be a reduced pair. If two of  $\{X, Z, (X, Z)\}$  are Du Bois, so is the third.*

Another effective way to decide whether a pair is Du Bois or not, is the next

**Lemma 4.3.7** ([Kol13], Excision lemma, Proposition 6.17). *Let  $X = (Y \cup Z)_{red}$  be the union of closed reduced subschemes with  $W = (Y \cap Z)_{red}$ . Then  $(X, Y)$  is a Du Bois pair iff  $(Z, W)$  is a Du Bois pair.*

One might wonder about the relationship between being a Du Bois pair and being an lc pair. The following theorem provides an answer.

**Theorem 4.3.8** ([GK14], Theorem 1.4). *Let  $X$  be a normal complex variety. Let  $(X, \Sigma)$  be a Du Bois pair and  $\Delta$  a reduced effective divisor on  $X$  such that  $\text{supp} \Delta \subseteq \Sigma$  and  $K_X + \Delta$  is Cartier. Then  $(X, \Delta)$  is a log canonical pair.*

The theorems above will be used extensively in the next section. Meanwhile, as an example of an immediate application, one can see that a normal lc center  $Z$  of  $X$  such that  $K_Z$  is Cartier has lc singularities, as any union of lc centers is Du Bois ([Kol13, Thm. 5.14]).

#### 4.4 A global lc condition for special lc complete intersections

The aim of this section is to show that under certain hypotheses one can prove the existence of a *log canonical* pair  $(X, \Delta)$  such that  $Z$  is a log canonical center for  $(X, \Delta)$ .

The precise statement is the following:

**Theorem 4.4.1.** *Let  $X$  be a normal variety with lc singularities such that  $K_X$  is Cartier, and let  $W$  be a reduced irreducible special complete intersection in  $X$  such that  $W$  has lc singularities and it is not contained in  $\text{Sing } X$ . Then there exist a boundary  $\Delta$  such that the pair  $(X, \Delta)$  has log canonical singularities and  $W$  is a lc center for  $(X, \Delta)$ .*

Inspired by the proof of Proposition 4.2.1, the idea is to produce new generators for the ideal of  $I_W$ , then use Bertini's theorem to control the singularities of their zero sets and finally to rely on results from the theory of Du Bois pairs to draw conclusions on the pair  $(X, \Delta)$ .

We will break down the proof in a few technical lemmas. For the rest of the section, we will employ the following

**Setup 4.4.2.** The variety  $W$  will be a reduced complete intersection in  $X$  given by the ideal  $I_W = (F_1, \dots, F_r) \subseteq S(X)$ , where  $S(X)$  is the coordinate ring of  $X$  in  $\mathbb{P}^n$ .

Since we'll be using partial complete intersections in inductive arguments, we start by stating a lemma about reducedness:

**Lemma 4.4.3.** *Using the notation of Setup 4.4.2, for any  $s \leq r$  the partial complete intersection  $(F_1, \dots, F_s)$  is reduced.*

*Proof.* It's enough to show that  $J = (F_1, \dots, F_{r-1})$  is reduced. Assume towards contradiction that there is a homogeneous element  $G \in S(X)$  such that  $G \notin J$  but  $G^q \in J$  for some  $q > 0$ . Among all such  $G$ 's, pick one with the minimal degree. By hypothesis,  $G^q \in J \subset I$  so  $G \in I$ . We can write

$$G = H + MF_r$$

where  $H$  and  $M$  are homogeneous elements with  $H \in J$  and  $\deg M < \deg G$ . Then

$$0 \equiv G^q \equiv M^q F_r^q \pmod{J}$$

Since the  $F_i$ 's form a regular sequence,  $F_r^r$  is not a zerodivisor modulo  $J$ , hence  $M^q \in J$ . By minimality of the degree of  $G$ ,  $M$  has to be an element of  $J$ , and this forces  $G \in J$ , contradiction.  $\square$

Many proofs rely on the fact that  $I_W$  is generated by a regular sequence. Since we want to consider a (possibly) different set of generators, we have to make sure they still form a regular sequence: this turns out to be the case.

**Lemma 4.4.4.** *Using the notation of Setup 4.4.2, every generating set  $G_1, \dots, G_r$  for  $I_W$  consisting of  $r$  elements is a regular sequence.*

*Proof.* Let  $J = (G_1, \dots, G_{s-1})$  and assume towards contradiction that  $G_s$  is a zerodivisor modulo  $J$ . Then  $G_s$  is contained in a minimal prime of  $J$ , which are the same as the minimal primes of  $\sqrt{J}$ . This implies that  $V(G_s)$  contains an irreducible component of  $V(J)$ , so that  $\dim V(J + (G_s)) = \dim V(J)$ . Since intersecting with a hypersurface makes the dimension go down by at most one, it would follow that  $\dim V(I_W) > n - r$ , contradiction.  $\square$

Before proving the main lemma, we need a way to control the singularities of a complete intersection.

**Lemma 4.4.5.** *Let  $(R, \mathfrak{m})$  be a local Noetherian ring,  $f \in \mathfrak{m}$  a non zero-divisor and assume that  $(R/f, \mathfrak{m}/f)$  is regular. Then  $(R, \mathfrak{m})$  is a regular ring.*

*Proof.* Let  $n = \dim R$ . Then since  $f$  is a regular element  $\dim R/f = n - 1$ , hence there are  $n - 1$  elements  $a_i \in \mathfrak{m}$  such that

$$(a_1, \dots, a_{n-1}) = \mathfrak{m}/f \text{ in } R/f$$

hence

$$(a_1, \dots, a_{n-1}, f) = \mathfrak{m}$$

This proves that  $\mathfrak{m}$  can be generated by  $n$  elements, hence  $R$  is regular.  $\square$

Now we are ready to state and prove the main technical tool of this section: basically, it says that a complete intersection can be reached by a sequence of complete intersections with mild singularities.

**Lemma 4.4.6.** *Using the notation of Setup 4.4.2, and assume furthermore that all the degrees of the  $F_i$ 's are the same. Then there is a set of generators  $G_1, \dots, G_r$  for  $I_W$  such that for every subset  $S \subseteq \{1, \dots, r\}$  the complete intersection*

$$D_S = \bigcap_{i \in S} V(G_i)$$

*is reduced, normal and smooth away from the singular locus of  $W$ .*

*Proof.* Let  $G_i = \sum \mu_{ij} F_j$  where  $\mu_{ij} \in k$ . Let  $U \subset \mathbb{A}^{r^2}$  be the open subset where the matrix  $\Xi = (\mu_{ij})$  is invertible. It's easy to check that for any choice of  $(\mu_{ij}) \in U$  the  $G_i$ 's generate the ideal  $I_W$ , so they form a regular sequence by Lemma 4.4.4.

It follows that the  $D_S$ 's are complete intersections, and the fact that they are reduced follows directly from Lemma 4.4.3.

Let  $D_i = V(G_i)$ . We claim that there is an open set  $V \subset \mathbb{A}^{r^2}$  such that all the intersections

$$D_1, D_1 \cap D_2, \dots, D_1 \cap \dots \cap D_{r-1}$$

are smooth away from  $\text{Sing } W$ . Note that by applying the same reasoning to any permutation of the set  $\{1, \dots, r\}$  and intersecting all the  $V$ 's thusly obtained we can prove the claim for an arbitrary  $D_S$ .

Consider the linear system  $\mathfrak{d}$  on  $X$  generated by the  $F_i$ 's: by Bertini's theorem (cfr. [Har77, Rem. III.10.9.2]), for a generic choice of the first row of  $\Xi$  the scheme  $V(G_1)$  is smooth away from  $W$  and  $G_1, F_2, \dots$  generate  $I_W$ . Let  $w \in W$  be a smooth point: then by Lemma 4.4.5, it's also a smooth point of  $V(G_1, \dots, F_{r-1})$ . By applying the same lemma repeatedly, we get that  $D_1$  is smooth at  $w$ .

Having chosen  $G_1$ , by restricting  $\mathfrak{d}$  to  $D_1$  we can apply Bertini's theorem again to get that for a generic choice of the second row of  $\Xi$  the zero set of  $G_2$  is smooth away from  $\text{Sing } D_1$ . Continuing in this fashion we get the desired open set  $V$ .

Moreover each  $D_S$  is a complete intersection that is smooth away from  $\text{Sing } W$ , whose codimension in  $D_S$  is at least 1 since  $W$  is normal. Therefore every  $D_S$  is normal.  $\square$

Note that the fact that every  $D_S$  is normal implies that it is actually irreducible, as complete intersections are connected. We can even say more about the singularities of  $D_S$ .

**Lemma 4.4.7.** *Let  $W$  be an irreducible reduced complete intersection in  $X$  with lc singularities and let  $G_1, \dots, G_r$  be generators for  $I_W$  such that each partial complete intersection  $D_S$  is smooth away from  $W$ . Then the pair*

$$(D_S, D_{S'}), \quad \text{where } S' = S \cup \{j\} \text{ for some } j \notin S$$

*is a Du Bois pair.*

Note that the  $H_s$ 's are reduced by Lemma 4.4.3, so the pair  $(H_s, H_{s+1})$  is actually a reduced pair.

*Proof.* The idea is to use descending induction on  $|S|$ . Assume that  $D_{S'} = D_S \cap D_j$  has Du Bois singularities: then  $D_S$  has Du Bois singularities near  $W$  by [KS11, Theorem 4.1]. Moreover,  $H_s$  is smooth away from  $W$ , therefore actually  $D_S$  has Du Bois singularities. By [Kol13, Prop. 6.15] then the pair  $(D_S, D_{S'})$  is Du Bois: in particular  $D_S$  has Du Bois singularities.

Now observe that since  $D_S = W$  has lc singularities by hypothesis when  $|S| = r$ , then by [KK10, Theorem 1.4]  $D_S$  is Du Bois when  $|S| = r$ .  $\square$

Note that this proof implies that all the  $D_S$ 's are Du Bois, hence if  $S'$  is any superset of  $S$ , every pair  $(D_S, D_{S'})$  is Du Bois.

An interesting corollary of the proof is that the pairs  $(X, D_i)$  are lc by [GK14, Thm. 1.4], as both  $K_X$  and  $D_i$  are Cartier. It turns out that we can exploit this fact to get a stronger result.

**Lemma 4.4.8.** *Assume  $Y$  is reduced, Du Bois and  $B_1, \dots, B_r$  are reduced, Du Bois Cartier divisors such that for all  $q \leq r$  every possible intersection*

$$B_{s_1} \cap \dots \cap B_{s_q}$$

*is reduced and Du Bois. Then the pair  $(Y, B_1 \cup \dots \cup B_r)$  is Du Bois.*

*Proof.* The idea is to use induction and the excision lemma for Du Bois singularities (see [Kol13, Prop. 6.17]). If  $r = 1$ , the statement is clearly true, since  $Y$  Du Bois and  $B_1$  Du Bois implies that the pair  $(Y, B_1)$  is Du Bois as well. Assume the claim is true for  $r - 1$ . It's enough to show that  $B_1 \cup \dots \cup B_r$  is Du Bois, since we already know that  $Y$  is Du Bois. By the induction hypothesis we know that  $B_1 \cup \dots \cup B_{r-1}$  is Du Bois. Then the claim will follow if we can prove that

$$(B_1 \cup \dots \cup B_r, B_1 \cup \dots \cup B_{r-1})$$

is a Du Bois pair. By the excision lemma, this is equivalent to showing that

$$(B_r, (B_1 \cap B_r) \cup \dots \cup (B_{r-1} \cap B_r))$$

is Du Bois, and this last claim follow from the inductive hypothesis, as  $B_r$  is reduced and Du Bois and  $B_j \cap B_r$  is a reduced Du Bois Cartier divisor on  $B_r$  for each  $j$ .  $\square$

*Proof of Theorem 4.4.1.* By Lemmas 4.4.6 and 4.4.7, there is a sequence of generators  $G_i$  for  $I_W$  such that all the possible partial intersections have Du Bois singularities. Then Lemma 4.4.8 yields that the pair

$$(X, \bigcup D_i)$$

is Du Bois. If  $\Delta = D_1 + \dots + D_r$  then, since  $K_X$  is Cartier by [GK14, Thm. 1.4] we have that  $(X, \Delta)$  is a log canonical pair. The other part of the statement follows from Proposition 4.2.1.  $\square$

**Remark 4.4.9.** Actually the proof of this last theorem shows that if a subvariety  $Z$  of  $X$  is a component of a special Du Bois complete intersection, then  $Z$  is a lc center for a *log canonical* pair  $(X, \Delta)$ .

## Chapter 5

**RATIONALIZING DIVISORS****Introduction**

In this chapter, we focus on the following:

**Question.** *Given a normal variety  $X$ , when is there a reduced divisor  $D$  such that  $(X, D)$  is rational?*

We will call such a divisor a *rationalizing divisor*.

The first result is a condition for cones on rational pairs:

**Theorem A** (Theorem 5.2.2). *Let  $(X, D)$  be a rational pair, and let  $L$  be an ample line bundle. Then the cone  $(C(X, L), C(D, L|_D))$  is a rational pair if and only if*

$$H^i(X, L^m(-D)) = 0 \text{ for } i > 0, m \geq 0$$

As a consequence of the above theorem, we get the following

**Corollary B** (Corollary 5.2.5). *Let  $X$  be a normal variety of dimension  $n$  and let  $L$  be an ample line bundle on  $X$ . If the cone  $C(X, L)$  has a rationalizing divisor, then*

$$H^n(X, L^m) = 0 \text{ for } m \geq 0$$

Then we give a necessary condition for the existence of a rationalizing divisor:

**Theorem C** (5.3.2). *Let  $(X, D)$  be a rational pair. Then the non rational locus of  $X$  has codimension at least 3.*

We conclude the chapter with an example that shows that the above theorems provide necessary but not sufficient conditions to guarantee the existence of a rationalizing divisor.

### 5.1 An observation on the normality condition

In the usual definition of rational singularities, one requires that for a resolution of singularities  $f : Y \rightarrow X$  the natural map  $\mathcal{O}_X \rightarrow f_*\mathcal{O}_Y$  is an isomorphism: this is equivalent to  $X$  being normal.

The first condition for  $(X, D)$  to be a rational pair is that  $\mathcal{O}_X(-D) \rightarrow f_*\mathcal{O}_Y(-B)$  is an isomorphism for a thrifty resolution of pairs  $f : (Y, B) \rightarrow (X, D)$ . We'll show that it's always true if  $X$  is normal. More precisely:

**Lemma 5.1.1.** *Let  $X$  be normal and let  $D$  be a reduced divisor on  $X$ . Let  $f : (Y, B) \rightarrow (X, D)$  be a binational morphism of pairs, with  $Y$  normal. Then the natural map  $\mathcal{O}_X(-D) \rightarrow f_*\mathcal{O}_Y(-B)$  is an isomorphism.*

*Proof.* The natural injection  $a : \mathcal{O}_X(-D) \rightarrow f_*\mathcal{O}_Y(-B)$ , induced by the inclusion  $\mathcal{O}_X \hookrightarrow f_*\mathcal{O}_Y$ , fits into the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_X(-D) & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_D & \longrightarrow & 0 \\ & & \downarrow a & & \simeq \downarrow & & & & \\ 0 & \longrightarrow & f_*\mathcal{O}_Y(-B) & \longrightarrow & f_*\mathcal{O}_Y & \longrightarrow & f_*\mathcal{O}_B & & \end{array}$$

and a bit of diagram chasing shows that there's a morphism  $b : \mathcal{O}_D \rightarrow f_*\mathcal{O}_B$  that fits into the above commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_X(-D) & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_D & \longrightarrow & 0 \\ & & \downarrow a & & \simeq \downarrow & & \downarrow b & & \\ 0 & \longrightarrow & f_*\mathcal{O}_Y(-B) & \longrightarrow & f_*\mathcal{O}_Y & \longrightarrow & f_*\mathcal{O}_B & & \end{array}$$

We know that the middle arrow is an isomorphism by normality of  $X$ . We want to show that  $a$  is surjective.

First of all, we claim that the map  $b : \mathcal{O}_D \rightarrow f_*\mathcal{O}_B$  is injective. Suppose it had a non trivial kernel  $K$ : this is a sheaf of ideals on  $D$ . Since  $D$  is reduced, then this sheaf of ideals defines a proper closed subscheme  $W$  through which  $f$  factors: this is impossible since  $f$  is dominant.

Now we can take the stalk at a point  $x \in X$  and prove surjectivity of  $a$  via a diagram chase. Let  $\alpha \in f_*\mathcal{O}_Y(-B)_x$ , and let  $\beta$  its image in  $f_*\mathcal{O}_{Y,x}$ . Then this last element corresponds to  $\gamma$  via the isomorphism  $\mathcal{O}_X \simeq f_*\mathcal{O}_Y$ : let  $\delta$  be its image in  $\mathcal{O}_D$ . Then  $b(\delta) = 0$  in  $f_*\mathcal{O}_B$ : since  $b$  is injective,  $\delta = 0$ . This implies that  $\gamma$  is the image of a (unique) element  $\eta \in \mathcal{O}_X(-D)$ , that maps to  $\alpha$ . This concludes the proof.  $\square$

## 5.2 A rationality criterion for cones over pairs

**Definition 5.2.1** ([Kol13]). Let  $X$  a projective variety and let  $L$  be an ample line bundle on  $X$ . The *affine cone* over  $X$  with conormal bundle  $L$  is

$$C_a(X, L) = \text{Spec} \bigoplus_{m \geq 0} H^0(X, L^m)$$

Note that this is the normalization of the projective cone over  $X$  relative to the embedding defined by some power of  $L$ .

Let  $(X, D)$  be a rational pair, and let  $L$  be an ample line bundle on  $X$ . Let  $C = C_a(X, L)$  be the affine cone over  $X$ , and let  $\Delta = C_a(D, L|_D)$ . In this section, we will find a criterion to find out whether  $(C, \Delta)$  is a rational pair, thus generalizing [Kol13, Prop. 3.13].

A suggestion for a necessary condition comes from the following observation: consider the exact sequence

$$0 \rightarrow \mathcal{O}_X(-D) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_D \rightarrow 0$$

After twisting by  $L^m$  and taking cohomology we have the long exact sequence

$$\begin{array}{ccccccc} & & & \dots & \longrightarrow & H^{i-1}(D, L^m) & \longrightarrow \\ \rightarrow & H^i(X, L^m(-D)) & \longrightarrow & H^i(X, L^m) & \longrightarrow & H^i(D, L^m) & \longrightarrow \dots \end{array}$$

Assume now  $D$  is Cartier and reduced. Then both  $X$  and  $D$  are Cohen-Macaulay, as  $\mathcal{O}_X(-D)$  is Cohen-Macaulay (see [Kol13, Prop. 2.82]).

For the same reason, if  $(C, \Delta)$  is a rational pair then  $C$  and  $\Delta$  are CM, hence the intermediate dimensional cohomology of  $L^m$  on  $X$  and  $D$  vanishes (see [Kol13, Prop. 3.11]).

The above long exact sequence implies that

$$H^i(X, L^m(-D)) = 0$$

at least for  $m \geq 0$ ,  $1 \leq i \leq \dim X - 1$ .

As it turns out, this is a necessary and sufficient condition for the cone pair  $(C, \Delta)$  to be rational.

**Theorem 5.2.2.** *Let  $(X, D)$  be a rational pair, and let  $L$  be an ample line bundle. Then the cone  $(C(X, L), C(D, L|_D))$  is a rational pair if and only if*

$$H^i(X, L^m(-D)) = 0 \text{ for } i > 0, m \geq 0$$

Before the proof, we will need a couple of lemmas. The first one is about rational pairs and bundles with a smooth fiber, even though we just state it for projective and affine line bundles. The second one is a simple observation on the composition of derived functors.

**Lemma 5.2.3.** *Let  $\pi : Y \rightarrow X$  be an  $F$ -bundle, where  $F$  is either  $\mathbb{A}^1$  or  $\mathbb{P}^1$  and  $X$  is normal. Let  $D$  be a reduced divisor on  $X$  and let  $B = \pi^{-1}D$  be its scheme-theoretic preimage on  $Y$ . Then  $(X, D)$  is a rational pair if and only if  $(Y, B)$  is.*

*Proof.* Since the question is local on  $X$ , we can assume  $Y = F \times X$  is the trivial bundle. We will start the proof by relating the properties of a resolution of  $X$  to the property of a resolution of  $Y$

Let  $\varepsilon : \tilde{X} \rightarrow X$  be a thrifty resolution of  $X$ : let  $\tilde{Y} = F \times \tilde{X}$ , let  $\eta = id \times \varepsilon$  and consider the commutative diagram

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{\pi}} & \tilde{X} \\ \eta \downarrow & & \varepsilon \downarrow \\ Y & \xrightarrow{\pi} & X \end{array}$$

Note that the thriftiness of  $\varepsilon$  clearly implies thriftiness of  $\eta$ . Moreover, both  $\pi$  and  $\tilde{\pi}$  are faithfully flat maps, as they are obtained by base change of the faithfully flat maps  $Y \rightarrow \text{Spec } k, \tilde{Y} \rightarrow \text{Spec } k, .$

By [Har77, Prop. III.9.3], for any coherent sheaf  $\mathcal{F}$  on  $\tilde{X}$  and for any  $i \geq 0$  we have a natural isomorphism

$$\pi^* R^i \varepsilon_* \mathcal{F} \xrightarrow{\sim} R^i \eta_* (\tilde{\pi}^* \mathcal{F})$$

In particular, if  $\mathcal{F} = \mathcal{O}_{\tilde{X}}(-\tilde{D})$  then one can check that  $\tilde{\pi}^* \mathcal{F} \simeq \mathcal{O}_{\tilde{Y}}(-\tilde{B})$ , so

$$\pi^* R^i \varepsilon_* \mathcal{O}_{\tilde{X}}(-\tilde{D}) = 0 \Leftrightarrow R^i \eta_* \mathcal{O}_{\tilde{Y}}(-\tilde{B}) = 0$$

Moreover, since  $\pi$  is faithfully flat,

$$\pi^* R^i \varepsilon_* \mathcal{O}_{\tilde{X}}(-\tilde{D}) = 0 \Leftrightarrow R^i \varepsilon_* \mathcal{O}_{\tilde{X}}(-\tilde{D}) = 0$$

Hence  $\varepsilon$  is a rational (thrifty) resolution iff  $\eta$  is.

Now assume  $(X, D)$  is a rational pair: then  $\varepsilon$  is rational, so  $\eta$  is rational and thrifty hence  $(Y, B)$  is a rational pair. Conversely, if  $(Y, B)$  is a rational pair, then the thrifty resolution  $\eta$  has to be rational, so  $\varepsilon$  is a rational thrifty resolution of  $(X, D)$ .

□

**Lemma 5.2.4.** *Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be morphisms of varieties, and let  $\mathcal{F}$  be a coherent sheaf on  $X$ , and let  $\mathcal{G} = f_* \mathcal{F}$ . If  $R^i f_* \mathcal{F} = 0$  for  $i > 0$ , then*

$$R^i (g \circ f)_* \mathcal{F} = R^i g_* \mathcal{G}$$

*Proof.* The Grothendieck-Serre spectral sequence gives the following identities in the derived category of sheaves over  $Z$ :

$$R(g \circ f)_* \mathcal{F} = Rg_* \circ Rf_* \mathcal{F} = Rg_* (f_* \mathcal{F}) = Rg_* \mathcal{G}$$

and taking homology in the  $i$ th degree yields the desired result. □

*Proof of Theorem 5.2.2.* The basic idea is the following: blow up the vertex of the normalized cone to get a map

$$p : (Y, B) \rightarrow (C, \Delta)$$

where  $B$  is the birational transform of  $\Delta$ . Since locally the blow up is a  $\mathbb{P}^1$ -bundle over  $X$ , then  $(Y, B)$  is still a rational pair (cfr. Lemma 5.2.3).

Let  $g : (\tilde{Y}, \tilde{B}) \rightarrow (Y, B)$  be a rational thrifty resolution of  $(Y, B)$ , and let  $h = f \circ g$ .

The strategy for the proof is the following:

1. Show the map  $h$  is a thrifty resolution.
2. Prove that the pair  $(C, \Delta)$  is rational if and only if  $R^j p_* \mathcal{O}_Y(-B) = 0$  for  $j > 0$ .
3. Show that  $R^j p_* \mathcal{O}_Y(-B) = \bigoplus_{m \geq 0} H^j(X, L^m(-D))$  for  $j > 0$

1. (THRIFTINESS OF  $h$ ) By construction,  $h(\text{Ex}(h))$  contains the vertex  $P$  of  $C$ , that is not in the snc locus of  $(C, \Delta)$ . This means that condition 1 is satisfied. As for condition 2, let  $\tilde{E}$  be the strict transform of  $E$  in  $\tilde{Y}$ . If the intersection of some components  $\tilde{B}_j$  of  $\tilde{B}$  is contained in  $\text{Ex}(h)$ , then it has to be contained in  $\tilde{E}$  by the thriftiness of  $g$ . In other words,

$$\tilde{E} \supseteq \bigcap \tilde{B}_i$$

and  $\bigcap \tilde{B}_i$  is disjoint from  $\text{Ex}(g)$ . This implies that

$$E \supseteq \bigcap B_i$$

Nevertheless, if two (or more) of the  $B_i$ 's meet in  $E$ , since  $Y$  is locally a product, then the corresponding components of  $\Delta$  meet in  $C$ . This implies that the intersection of the  $B_i$ 's cannot be mapped to a point by  $p$  (indeed, its image is the cone over the corresponding components of  $\Delta$ ). Therefore  $h$  is thrifty.

2. CLAIM:  $(C, \Delta)$  is rational if and only if  $R^j p_* \mathcal{O}_Y(-B) = 0$  for  $j \geq 0$

Assume first  $(C, \Delta)$  is a rational pair: then every thrifty resolution is rational [Kol13, Cor. 2.86], so  $Rh_* \mathcal{O}_{\tilde{Y}}(-\tilde{B}) \sim \mathcal{O}_C(-\Delta)$ . This implies, by Lemma 5.2.4, that

$$Rp_* \mathcal{O}_Y(-B) \sim \mathcal{O}_C(-\Delta)$$

Conversely, assume the above condition holds. Then since  $h$  is thrifty and  $\text{char } k = 0$ , then  $R^i h_* \omega_{\tilde{Y}}(\tilde{B}) = 0$  for  $i > 0$ , by rational pair analogue of Grauert-Riemenschneider vanishing theorem (see [Eri14b, Prop. 3.6]). This proves that  $h$  is a rational thrifty resolution, hence  $(C, \Delta)$  is a rational pair.

3. CLAIM:  $R^j p_* \mathcal{O}_Y(-B) = \bigoplus_{m \geq 0} H^j(X, L^m(-D))$

*Proof.* First of all, note that  $R^i p_* \mathcal{O}_Y(-B)$  is a skyscraper sheaf supported on the cone point  $P \in C(X)$ , and its stalk is  $H^i(Y, \mathcal{O}_Y(-B))$ .

Let  $\pi : Y \rightarrow X$  the projection: note that it's an affine morphism, therefore

$$H^i(Y, \mathcal{O}_Y(-B)) \simeq H^i(X, \pi_* \mathcal{O}_Y(-B))$$

We will now express  $\pi_* \mathcal{O}_Y(-B)$  in terms of  $L$ .

Consider the exact sequence

$$0 \rightarrow \pi_* \mathcal{O}_Y(-B) \rightarrow \pi_* \mathcal{O}_Y \rightarrow \pi_* \mathcal{O}_B \rightarrow 0$$

(observe that exactness on the right follows from the fact that  $\pi$  is an affine morphism)

By construction,  $\pi_* \mathcal{O}_Y \simeq \bigoplus_{m \geq 0} L^m$ , and since  $B$  is an affine bundle over  $D$ , it's easy to check that  $\pi_* \mathcal{O}_B \simeq \bigoplus_{m \geq 0} L^m|_D$ . This implies that  $\pi_* \mathcal{O}_Y(-B) \simeq \bigoplus_{m \geq 0} L^m(-D)$ : taking cohomology of both sides yields  $H^i(X, \pi_* \mathcal{O}_Y(-B)) \simeq \bigoplus_{m \geq 0} H^i(X, L^m(-D))$ . □

Combining everything together, the pair  $(C, \Delta)$  is rational if and only if  $H^i(X, L^m(-D))$  vanishes for every  $i > 0, m \geq 0$ . □

Theorem 5.2.2 has an interesting consequence for the existence of rationalizing divisors on cones.

**Corollary 5.2.5.** *Let  $X$  be a normal variety of dimension  $n$  and let  $L$  be an ample line bundle on  $X$ . If the cone  $C(X, L)$  has a rationalizing divisor, then*

$$H^n(X, L^m) = 0 \text{ for } m \geq 0$$

*Proof.* First of all, observe that there's a natural surjection  $\text{Cl } X \rightarrow \text{Cl } C(X, L)$ , taking a divisor to the cone over it (see [Har77, Ex. II.6.3]). This means that every divisor on  $C(X)$  is the cone over a divisor on  $X$ , up to linear equivalence.

Now suppose  $(C(X), C(D))$  is a rational pair, and let  $U$  be  $C(X)$  minus the cone point. Since  $(U, C(D)|_U)$  is a rational pair and  $U \rightarrow X$  is a  $\mathbb{A}^1$ -bundle over  $X$ , it follows from Lemma 5.2.3 that  $(X, D)$  is a rational pair, too.

Consider now the exact sequence

$$0 \rightarrow \mathcal{O}_X(L^m(-D)) \rightarrow \mathcal{O}_X(L^m) \rightarrow \mathcal{O}_D(L^m) \rightarrow 0$$

Then taking cohomology in degree  $n$  yields

$$H^n(X, L^m(-D)) \rightarrow H^n(X, L^m) \rightarrow H^n(D, \mathcal{O}_D(L^m)) \rightarrow 0$$

The first term vanishes by Proposition 5.2.2, and the rightmost group is zero for dimension reasons. This gives  $H^n(X, L^m) = 0$  for all  $m \geq 0$ .  $\square$

**Remark 5.2.6.** This necessary condition for cones to have a rationalizing divisor implies Erickson's condition on the vanishing of the top-dimensional cohomology [Eri14a, Prop. 3.5.2].

### 5.3 Rationalizing divisors and non-rational locus

In this section we will give a necessary condition for a variety to have a rationalizing divisor. Many of the statements hold in the context of Du Bois pairs (see [GK14, Theorem 1.4.1]).

First of all, we will investigate the case of surfaces. The simplest case is the cone over a curve: let  $C$  be a smooth projective curve with an effective divisor  $F$  and let  $L$  be an ample

line bundle on it. If  $X$  is the cone over  $C$  and  $D$  the cone over  $F$ , then we know by Theorem 5.2.2 that if the pair  $(X, D)$  is rational then  $H^1(C, L^m(-F)) = 0$  for all  $m \geq 0$ . It's easy to see that  $H^1(C, L^m(-F))$  surjects onto  $H^1(C, L^m)$ : using again Theorem 5.2.2, it follows that  $X$  has rational singularities.

This holds in general. Indeed, we have the following

**Lemma 5.3.1.** *If  $(X, D)$  is a rational pair and  $X$  is a normal surface, then  $X$  has rational singularities.*

*Proof.* Since singularities are isolated on normal surfaces, then we can work locally and assume that  $X$  has one isolated singularity  $x \in X$ . Moreover, if  $x \notin \text{Supp}D$  then  $X$  is rational at  $D$  and smooth away from it, hence the claim is trivially true. Therefore we will assume  $x \in \text{Supp}D$ .

First of all we claim that if  $\pi : (Y, B) \rightarrow (X, D)$  is a thrifty rational resolution of  $X$  with exceptional locus  $E$ , then  $X$  has rational singularities iff

$$R^1\pi_*\mathcal{O}_Y(-E) = 0$$

We will prove this first claim by establishing some vanishing results. First of all let's consider the long exact sequence

$$\begin{aligned} 0 \longrightarrow \pi_*\mathcal{O}_Y(-E - B) &\longrightarrow \pi_*\mathcal{O}_Y(-B) \longrightarrow \pi_*\mathcal{O}_E(-B|_E) \longrightarrow \\ &\longrightarrow R^1\pi_*\mathcal{O}_Y(-E - B) \longrightarrow R^1\pi_*\mathcal{O}_Y(-B) \longrightarrow R^1\pi_*\mathcal{O}_E(-B|_E) \longrightarrow 0 \end{aligned}$$

By rationality of the pair  $(X, D)$  we have that  $R^1\pi_*\mathcal{O}_Y(-B) = 0$ , which implies

$$R^1\pi_*\mathcal{O}_E(-B|_E) = 0$$

Consider now the exact sequence

$$\begin{aligned} 0 \longrightarrow \pi_*\mathcal{O}_E(-B|_E) &\longrightarrow \pi_*\mathcal{O}_E \longrightarrow \pi_*\mathcal{O}_{B|_E} \longrightarrow \\ &\longrightarrow R^1\pi_*\mathcal{O}_E(-B|_E) \longrightarrow R^1\pi_*\mathcal{O}_E \longrightarrow R^1\pi_*\mathcal{O}_{B|_E} \longrightarrow 0 \end{aligned}$$

we already know that  $R^1\pi_*\mathcal{O}_E(-B|_E) = 0$ . Moreover, note that since  $E$  is exceptional,  $\pi : B|_E \rightarrow x$  has a zero-dimensional image, hence  $R^1\pi_*\mathcal{O}_{B|_E} = 0$ . This implies that  $R^1\pi_*\mathcal{O}_E = 0$ .

Finally, consider the exact sequence

$$0 \rightarrow \mathcal{O}_Y(-E) \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_E \rightarrow 0 \quad (5.1)$$

and apply the functor  $\pi_*$  to get

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_*\mathcal{O}_Y(-E) & \longrightarrow & \mathcal{O}_X & \longrightarrow & \pi_*\mathcal{O}_E \longrightarrow \\ & & \longrightarrow & & R^1\pi_*\mathcal{O}_Y & \longrightarrow & R^1\pi_*\mathcal{O}_E \longrightarrow 0 \end{array}$$

The exceptional locus  $E$  is connected by Zariski's main theorem, hence the sheaf  $\pi_*\mathcal{O}_E$  is a skyscraper sheaf supported on  $x \in X$  and therefore the map  $\mathcal{O}_X \rightarrow \pi_*\mathcal{O}_E$  is a surjection.

This yields the exact sequence

$$0 \longrightarrow R^1\pi_*\mathcal{O}_Y(-E) \longrightarrow R^1\pi_*\mathcal{O}_Y \longrightarrow R^1\pi_*\mathcal{O}_E \longrightarrow 0$$

By the vanishing results above, the last sheaf is zero, hence  $R^1\pi_*\mathcal{O}_Y(-E) \simeq R^1\pi_*\mathcal{O}_Y$ . It follows that  $X$  has rational singularities iff  $R^1\pi_*\mathcal{O}_Y = 0$  iff  $R^1\pi_*\mathcal{O}_Y(-E) = 0$ .

We now claim that

$$R^1\pi_*\mathcal{O}_Y(-E - B) = 0$$

Indeed, twisting the sequence (5.1) by  $-B$  and taking  $\pi_*$  we get an exact sequence

$$\begin{array}{ccccccc} & & \pi_*\mathcal{O}_Y(-B) & \xrightarrow{\alpha} & \pi_*\mathcal{O}_E(-B|_E) & \longrightarrow & \\ \longrightarrow & R^1\pi_*\mathcal{O}_Y(-E - B) & \longrightarrow & & 0 & & \end{array}$$

It suffices to show that the map  $\alpha$  is surjective, and we can do it by checking surjectivity at the stalk at  $x$ . Clearly  $\mathcal{O}_E(-B) \subseteq \mathcal{O}_E$ , thus  $(\pi_*\mathcal{O}_E(-B))_x \subseteq (\pi_*\mathcal{O}_E)_x$ . We already observed that  $\pi_*\mathcal{O}_E \simeq k_x$ , so  $(\pi_*\mathcal{O}_E(-B))_x$  is either 0 or  $k$ . Since we have

$$(\pi_*\mathcal{O}_E(-B))_x = \Gamma(E, \mathcal{O}_E(-B|_E))$$

and given that  $x \in D$  ensures that  $\mathcal{O}_E(-B|_E)$  is a proper ideal of  $\mathcal{O}_E$ , we can conclude that  $\mathcal{O}_E(-B|_E) = 0$ .

Finally, we will be done if we can show that  $R^1\pi_*\mathcal{O}_Y(-E - B) = 0$  implies  $R^1\pi_*\mathcal{O}_Y(-E) = 0$ . This follows easily by considering

$$0 \rightarrow \mathcal{O}_Y(-B - E) \rightarrow \mathcal{O}_Y(-E) \rightarrow \mathcal{O}_B(-E) \rightarrow 0$$

which yields an exact sequence

$$R^1\pi_*\mathcal{O}_Y(-B-E) \rightarrow R^1\pi_*\mathcal{O}_Y(-E) \rightarrow R^1\pi_*\mathcal{O}_B(-E)$$

Since  $\pi|_B$  is finite, then  $R^1\pi_*\mathcal{O}_B(-E) = 0$ , and this concludes the proof. □

Now we are ready to state the main result of the section:

**Theorem 5.3.2.** *Let  $(X, D)$  be a rational pair. Then the non-rational locus of  $X$  has codimension at least 3.*

Before proving the proposition, we have to establish a deformation result for rational singularities.

**Lemma 5.3.3.** *Let  $D$  be a Cartier divisor of  $X$  that has rational singularities. Then on an open neighborhood of  $D$  the ambient space  $X$  has rational singularities.*

*Proof.* Since the question is local, after possibly shrinking we can think of  $D$  as being the zero set of a regular function  $f$ . Consider the map

$$\varphi : X \rightarrow \mathbb{A}^1, x \mapsto f(x)$$

We have that  $\varphi$  is flat, as it's dominant onto a one dimensional integral regular scheme and  $X$  is reduced [Har77, Prop. III.9.7]. The special fiber  $X_0 \simeq D$  has rational singularities, hence by Elkik's theorem [Elk78, Théorème 2]  $X$  has rational singularities in a neighborhood of  $D$ . □

Now we can proceed with the proof of the proposition.

*Proof.* We will prove the claim by showing that the non rational locus does not intersect a general complete intersection surface in  $X$ .

Let  $S$  be the intersection of  $(\dim X - 2)$  general hyperplanes in  $X$ . Then  $X$  is normal (see for example [Sei50]), and by using the Bertini property for rational pairs [Eri14b, Theorem 4.4], the pair  $(S, D|_S)$  is rational. It follows from Lemma 5.3.3 that  $S$  has rational

singularities, so by repeatedly applying the deformation lemma 5.3.3 we conclude that  $X$  has rational singularities near  $S$ .  $\square$

The proposition implies, for example, that a cone over a normal surface with a non rational isolated singularity cannot be a part of a rational pair, as the non rational locus is a line, which has codimension 2.

#### 5.4 An interesting example

In this section we will show that the necessary conditions in the previous sections (Propositions 5.2.5, 5.3.2) are necessary for the existence of a rationalizing divisor, but not sufficient. We will construct a threefold with an isolated non-rational singularity at a point, that does not have a rationalizing divisor.

We will achieve this by considering a cone  $Y$  over a polarized surface  $(X, L)$  with zero top-dimensional cohomology of  $L^m$  for  $m \geq 0$ : to be consistent with the previous sections,  $L$  will denote an ample line bundle.

Let  $C$  be a smooth curve of genus at least 2 and let  $X$  be the ruled surface  $C \times \mathbb{P}^1$ . Let  $\pi : X \rightarrow C, \xi : X \rightarrow \mathbb{P}^1$  be the projections. Then  $\pi_*\mathcal{O}_X = \mathcal{O}_C \oplus \mathcal{O}_C$  (see for example [Har77, Example V.2.11]). Let  $P \in \mathbb{P}^1$  and  $Q \in C$  be arbitrary points, and let  $C_0 = \xi^*P$  be the canonical section,  $f = \pi^*Q$  the fiber.

The effective divisor  $L := \mathcal{O}_X(C_0 + f)$  is ample by [Har77, Prop. V.2.20]. Note also that by [Har77, Prop. V.2.8] we have

$$L = \mathcal{O}_X(C_0) \otimes \mathcal{O}_X(f) = \mathcal{O}_X(1) \otimes \pi^*\mathcal{O}_C(Q)$$

It follows that

$$\pi_*L = \pi_*(\mathcal{O}_X(1) \otimes \pi^*\mathcal{O}_C(Q))$$

By [Har77, Lemma V.2.4] , since  $D.f = 1 \geq 0$  then

$$H^1(X, L) = H^1(C, \pi_*L) = H^1(C, \mathcal{O}_X(Q)) \oplus H^1(C, \mathcal{O}_X(Q)) \neq 0$$

where the last claim follows from Riemann-Roch and the assumption on the genus of  $C$ .

Since  $H^1(X, L) \neq 0$ , then we can deduce that the cone  $Y = C(X, L)$  has an isolated, non-rational singularity by applying Proposition 5.2.2 with  $D = 0$ . Note that by construction  $Y$  is normal.

As for the cohomology of  $X$ , notice that by Lemma 2.4

$$H^2(X, \mathcal{O}_X) \simeq H^2(C, \pi_* \mathcal{O}_X) = 0$$

where the last equality holds for dimension reasons.

We will now prove that  $Y$  does not have any rationalizing divisors. We have to prove that there is no effective (Weil) divisor  $D = aC_0 + bf$  on  $X$  such that  $(Y, C(D, L|_D))$  is a rational pair, and by the criterion in section one this amounts to showing that there are no integers  $a, b$  such that

$$H^1(X, L^m(-D)) = 0 \text{ for all } m \geq 0$$

First note that since  $D$  is effective, by taking the intersection with  $C_0$  and  $f$  necessarily we have  $a \geq 0, b \geq 0$ . Moreover, by Künneth formula we have

$$\begin{aligned} H^1(X, L^m(-D)) &= H^1(X, \mathcal{O}_X((m-a)C_0 + (m-b)f)) & (5.2) \\ &= H^1(X, \xi^* \mathcal{O}_{\mathbb{P}^1}((m-a)P) \otimes \pi^* \mathcal{O}_X((m-b)Q)) \\ &= H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}((m-a)P)) \otimes H^1(C, \mathcal{O}_C((m-b)Q)) \oplus \\ &\quad H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}((m-a)P)) \otimes H^0(C, \mathcal{O}_C((m-b)Q)) \end{aligned}$$

We claim that for any choice of  $a, b$  there is an  $m$  such that  $H^1(X, L^m(-D)) \neq 0$ : this will prove that there is no rationalizing divisor for  $Y$ . The basic idea is to leverage the fact that  $H^1(C, \mathcal{O}_C(Q)) \neq 0$ , which follows from Riemann-Roch.

Assume first  $a \neq b + 2$ . Then we set  $m - b = 1$ , that is  $m = b + 1$ . We get that

$$\begin{aligned} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}((b-a+1)P)) \otimes H^1(C, \mathcal{O}_C(Q)) &\neq 0 \quad \text{if } a \leq b + 1 \\ H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}((b+1-a)P)) \otimes H^0(C, \mathcal{O}_C(Q)) &\simeq \\ H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}((a-b-3)P)) \otimes H^0(C, \mathcal{O}_C(Q)) &\neq 0 \quad \text{if } a \geq b + 3 \end{aligned}$$

Since  $a \neq b + 2$ , then at least one of the two tensor products is non zero.

Assume now  $a = b + 2$ . Then the second term in the direct sum of (5.2) is then

$$\begin{aligned} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}((m - b - 2)P)) \otimes H^0(C, \mathcal{O}_C((m - b)Q)) &\simeq \\ H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}((b - m)P)) \otimes H^0(C, \mathcal{O}_C((m - b)Q)) \end{aligned}$$

Setting  $m = b$  yields  $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \otimes H^0(C, \mathcal{O}_C) \neq 0$ .

This shows that the threefold  $Y$  has a non rational singularity at the cone point, yet does not have a rationalizing divisor.

## Chapter 6

## MORE RESULTS ON RATIONAL PAIRS

## 6.1 Quotients of rational pairs

It is well known that if  $G$  is a finite group acting on a variety with rational singularities then the quotient still has rational singularities. Given that rational pairs generalize the definition of rational singularities, a natural question is whether this still holds for the action of a finite group  $G$  on a rational pair. More precisely, this section will use the following

**Setup 6.1.1.** Given a rational pair  $(X, D)$  and an action of a finite group  $G$  on  $X$  such that  $D$  is  $G$ -invariant, let  $Y$  be the quotient  $X/G$ , let  $f : X \rightarrow Y$  be the quotient map and let  $B = f_*(D)_{red}$ . We want to investigate under which conditions the pair  $(Y, B)$  is rational.

## 6.1.1 An example: what can go wrong and how

Unlike the case of rational singularities, not all the quotients of rational pairs are rational: indeed, let  $X = \mathbb{A}^2 = \text{Spec}k[x, y]$  and let  $G = \{Id, \sigma\}$  be the cyclic group of order 2 acting on  $X$  as:

$$\sigma \cdot x = -x, \quad \sigma \cdot y = -y$$

The quotient  $Y = X/G$  is a quadric cone: indeed, the invariant subring  $R$  of  $k[x, y]$  is

$$R = k[x^2, xy, y^2] \simeq k[\alpha, \beta, \gamma]/(\beta^2 - \alpha\gamma) \hookrightarrow k[x, y]$$

where the last inclusion is given by

$$\alpha \mapsto x^2, \quad \beta \mapsto xy, \quad \gamma \mapsto y^2$$

Let  $D$  be the union of the coordinate axes of  $X$ , and define  $B$  on  $Y$  as in the introduction. Then  $B$  consists of two lines through the cone point.

It's clear that  $(X, D)$  is snc, hence rational, but  $(Y, B)$  is not a rational pair. Indeed, let  $\tilde{Y}$  be the blow up of  $Y$  at the cone point: the blow-up map  $\varepsilon : \tilde{Y} \rightarrow Y$  is a thrifty resolution. There is an exact sequence of sheaves on  $Y$

$$\begin{array}{ccccccc} 0 & \longrightarrow & \varepsilon_* \mathcal{O}_{\tilde{Y}}(-\tilde{B}) & \longrightarrow & \varepsilon_* \mathcal{O}_{\tilde{Y}} & \longrightarrow & \varepsilon_* \mathcal{O}_{\tilde{B}} \longrightarrow \\ & & \longrightarrow & & R^1 \varepsilon_* \mathcal{O}_{\tilde{Y}}(-\tilde{B}) & \longrightarrow & R^1 \varepsilon_* \mathcal{O}_{\tilde{Y}} \end{array}$$

and note that  $Y$  is the quotient of a smooth variety by the action of a finite group, hence it has rational singularities, so  $\varepsilon_* \mathcal{O}_{\tilde{Y}} = 0$ . Let  $V$  be the cone point in  $Y$ : if the resolution  $\varepsilon$  is rational, then the map of stalks

$$(\varepsilon_* \mathcal{O}_{\tilde{Y}})_V \longrightarrow (\varepsilon_* \mathcal{O}_{\tilde{B}})_V$$

is surjective. This is impossible since  $(\varepsilon_* \mathcal{O}_{\tilde{Y}})_V \simeq \mathcal{O}_{Y,V}$  and  $(\varepsilon_* \mathcal{O}_{\tilde{B}})_V$  requires at least two generators over  $\mathcal{O}_{Y,V}$ , since there are two points in the fiber over  $V$  of the map  $\tilde{B} \rightarrow B$ : the same phenomenon appeared in Example 3.4.7.

Let's try to gain some insight on what just happened. Let  $\tilde{X} = X \times_Y \tilde{Y}$  and consider the following cartesian square

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\eta} & X \\ \downarrow & & \downarrow f \\ \tilde{Y} & \xrightarrow{\varepsilon} & Y \end{array}$$

The pullback map  $\eta : \tilde{X} \rightarrow X$  is the blow up of the plane at the origin and note that the birational transform  $\tilde{D}$  of  $D$  is a pair of two non-intersecting lines in  $\tilde{X}$ . Let  $E$  be the exceptional divisor of  $\eta$ : then its image  $\eta(E)$  is the origin in  $X$ , which is a stratum of  $D$ . Therefore  $\eta$  is *not* a thrifty map.

We will see in the rest of the section that thriftiness is crucial in the rationality of the quotient pair.

### 6.1.2 Some preliminary remarks

When  $Y$  is the quotient of  $X$  by a finite group, one can average over the elements of the group to get a splitting or the natural injection  $\mathcal{O}_Y \hookrightarrow f_* \mathcal{O}_X$ : this is the case even when working with pairs.

**Lemma 6.1.2.** *With the notation introduced in Setup 6.1.1, the injection  $\iota : \mathcal{O}_Y \hookrightarrow f_*\mathcal{O}_X$  induces a split injection  $\mathcal{O}_Y(-B) \hookrightarrow f_*\mathcal{O}_X(-D)$ .*

*Proof.* Since we know that  $\mathcal{O}_Y(-B) \subseteq \mathcal{O}_Y$  and  $f_*\mathcal{O}_X(-D) \subseteq f_*\mathcal{O}_X$ , then all we have to show is that  $\iota(\mathcal{O}_Y(-B)) \subseteq f_*\mathcal{O}_X(-D)$ .

Let  $U = \text{Spec } A$  be a  $G$ -invariant affine subset of  $X$ : then locally  $f$  corresponds to the inclusion of rings  $A^G \subseteq A$ , whereas the map  $\iota$  is the same inclusion, but seen as an  $A^G$ -linear map. Let  $I_D$  be the ideal of  $D$  in  $A$ , which is  $G$ -invariant since  $D$  is, and let  $I_B$  be the ideal of  $B$  in  $A^G$ . Then we have that  $I_B = \sqrt{I_D^G}$ .

Now let  $\xi \in I_B$ . Then  $\xi^n \in I_D$  for some  $n$ , and since  $I_D$  is radical,  $\xi \in I_D$ , which proves the first part of the statement.

We are now going to define a splitting  $f_*\mathcal{O}_X(-D) \rightarrow \mathcal{O}_Y(-B)$ . We will work locally, and it will be clear that the local splittings glue together.

More specifically, define

$$\Xi : I_D \rightarrow A^G, \quad \xi \mapsto \frac{1}{|G|} \sum_{g \in G} g \cdot \xi$$

The map  $\Xi$  is clearly  $A^G$ -invariant, and clearly fixes  $I_B$  pointwise. We just have to show that the image of  $I_D$  lies in  $I_B$ , but this is immediate from the fact that  $\Xi(I_D) \subseteq I_D^G \subseteq I_B$ .  $\square$

### 6.1.3 Quotients of rational pairs

Since the behavior of thrifty resolutions with pull backs determines whether a quotient is a rational pair, it is important to state the following

**Definition 6.1.3.** A map  $f : X \rightarrow Y$  *lifts via thrifty resolutions* if there are thrifty resolutions  $Y' \rightarrow Y, X' \rightarrow X$  such that the composition  $X' \rightarrow X \rightarrow Y$  factors through  $Y' \rightarrow Y$ .

We can now state the main result of this section

**Theorem 6.1.4.** *Let  $G$  be a finite group acting on a rational pair and let  $f : (X, D) \rightarrow (Y, B)$  be the quotient map. If  $f$  lifts via thrifty resolutions, then the pair  $(Y, B)$  is rational.*

*Proof.* Assume we have the following thrifty lift of  $f$

$$\begin{array}{ccc} (\tilde{X}, \tilde{D}) & \xrightarrow{\Sigma} & (X, D) \\ f' \downarrow & & \downarrow f \\ (Y', B') & \xrightarrow{\sigma} & (Y, B) \end{array}$$

where  $\sigma$  and  $\Sigma$  are thrifty resolutions. We know that the inclusion  $\mathcal{O}_Y(-B) \hookrightarrow f_*\mathcal{O}_X(-D)$  splits (see above). If  $\pi$  is the splitting  $f_*\mathcal{O}_X(-D) \rightarrow \mathcal{O}_Y(-B)$ , then the following composition is an isomorphism

$$\mathcal{O}_Y(-B) \longrightarrow Rf_*\mathcal{O}_X(-D) \simeq f_*\mathcal{O}_X(-D) \xrightarrow{\pi} \mathcal{O}_Y(-B)$$

Also note that we can complete the above to a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_Y(-B) & \longrightarrow & Rf_*\mathcal{O}_X(-D) \simeq f_*\mathcal{O}_X(-D) \xrightarrow{\pi} \mathcal{O}_Y(-B) \\ \downarrow & & \simeq \downarrow \\ R\sigma_*\mathcal{O}_{Y'}(-B') & \longrightarrow & Rf_*R\Sigma_*\mathcal{O}_{\tilde{X}}(-\tilde{D}) \end{array}$$

where the second vertical arrow is an isomorphism because of Grothendieck spectral sequence.

Then we have a splitting of the natural map  $\mathcal{O}_Y(-B) \rightarrow R\sigma_*\mathcal{O}_{Y'}(-B')$ : by Corollary 2.75 of [Kol13] then  $R^i\sigma_*\mathcal{O}_{Y'}(-B') = 0$  for  $i > 0$ , hence the pair  $(Y, B)$  is rational.  $\square$

## 6.2 A theorem about rationality and morphisms of pairs

The following recent theorem of Fujino relates the singularities of a variety and its image via a projective morphism under some assumptions:

**Theorem 6.2.1** ([Fuj16]). *Let  $(Y, \Delta)$  be a log canonical pair where  $Y$  has rational singularities. Let  $f : Y \rightarrow X$  be a projective surjective morphism such that  $f_*\mathcal{O}_Y \simeq \mathcal{O}_X$  and  $-(K_Y + \Delta)$  is  $f$ -ample. Then  $X$  has rational singularities.*

In this we will generalize Theorem 6.2.1 to rational pairs.

We will begin by

**Lemma 6.2.2.** *Let  $X$  be normal and let  $\phi : (Y, B) \rightarrow (X, D)$  be a thrifty resolution, with  $D$  Cartier, and assume  $\omega_X(D) \xrightarrow{\sim} \phi_*\omega_Y(B)$ . Then  $(X, D)$  is a rational pair.*

*Proof.* We have to show that  $\phi$  is a rational resolution. By [Eri14b, Prop. 3.6] (Erickson's version of Grauert-Riemenschneider vanishing for pairs), our hypothesis implies  $\omega_X(D)^\bullet \simeq R\phi_*\omega_Y(B)^\bullet$ . Moreover

$$\begin{aligned} \mathcal{O}_X(-D) &\simeq R\mathcal{H}om_X(\omega_X, \omega_X(-D)) \\ &\simeq R\mathcal{H}om_X(\omega_X(D), \omega_X) \\ &\simeq R\mathcal{H}om_X(R\phi_*\omega_Y(B)^\bullet, \omega_X) \\ &\simeq R\phi_*R\mathcal{H}om_Y(\omega_Y(B), \omega_Y) \\ &\simeq R\phi_*\mathcal{O}_Y(-B) \end{aligned}$$

which is what we wanted to show. □

Now we state a proposition similar to the characterization of rational singularities given in [Kov00].

**Proposition 6.2.3.** *Let  $\phi : (Y, B) \rightarrow (X, D)$  be a surjective, proper morphism of pairs, with  $(Y, B)$  rational pair and  $D$  Cartier. Assume the map  $\mathcal{O}_X(-D) \rightarrow R\phi_*\mathcal{O}_Y(-B)$  splits, and assume there is a thrifty resolution  $\pi : \tilde{X} \rightarrow X$  such that the composition  $\tilde{X} \rightarrow Y$  factors through a thrifty resolution  $\sigma : \tilde{Y} \rightarrow Y$ . Then  $(X, D)$  is a rational pair.*

*Proof.* We have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_X(-D) & \xrightarrow{\rho} & R\phi_*\mathcal{O}_Y(-B) \\ \alpha \downarrow & & \beta \downarrow \\ R\mathcal{O}_{\tilde{X}}(-\tilde{X}) & \xrightarrow{\gamma} & R\phi_*R\sigma_*\mathcal{O}_{\tilde{Y}}(-\tilde{B}) \end{array}$$

and observe that  $\beta$  is a quasi-isomorphism since  $(Y, B)$  is a rational pair. This implies that  $\alpha$  splits, too, therefore we can reduce to the case where  $\phi : Y \rightarrow X$  is a thrifty resolution.

Now consider the sequence

$$\mathcal{O}_X(-D) \rightarrow R\phi_*\mathcal{O}_Y(-B) \rightarrow \mathcal{O}_X(-D)$$

By hypothesis the composition is an isomorphism. By applying the functor  $R\mathcal{H}om_X(-, \omega_X^\bullet)$  we get the sequence

$$\omega_X(D)^\bullet \rightarrow \omega_Y(B)^\bullet \rightarrow \omega_X(D)^\bullet$$

Taking cohomology in degree  $-\dim X$  yields that

$$\omega_X(D) \rightarrow \phi_*\omega_Y(B) \rightarrow \omega_X(D)$$

is an isomorphism, therefore  $\phi_*\omega_Y(B) \simeq \omega_X(D) \oplus \mathcal{F}$  for some coherent sheaf  $\mathcal{F}$  on  $X$ . Since  $\phi$  is birational then  $\mathcal{F}$  is a torsion sheaf, and since  $\phi_*\omega_Y(B)$  is a torsion-free sheaf necessarily  $\phi_*\omega_Y(B) \simeq \omega_X(D)$ . By Lemma 6.2.2,  $(X, D)$  is a rational pair.  $\square$

Finally we can state and prove the main theorem of the section.

**Theorem 6.2.4.** *Let  $f : (Y, B) \rightarrow (X, D)$  be a surjective, proper morphism of pairs with  $(Y, B)$  rational and  $D$  Cartier. Assume  $K_Y + B$  is Cartier and  $-K_Y - 2B$  is  $f$ -ample. If  $f$  lifts via thrifty resolutions then  $(X, D)$  is a rational pair.*

*Proof.* Since  $K_Y + B$  is Cartier, then  $(Y, B)$  is along canonical pair by [GK14]. Moreover, since  $-(K_Y + 2B)$  is  $f$ -ample then by [Fuj09, Thm 8.1] we can observe that  $R^i f_* \mathcal{O}_Y(-B) = 0$  for  $i > 0$ . It follows that the map  $\mathcal{O}_X(-D) \rightarrow Rf_* \mathcal{O}_Y(-B)$  splits trivially, hence by Proposition 6.2.3  $(X, D)$  is a rational pair.  $\square$

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