A boundary Harnack principle in twisted Hölder domains[1]

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Abstract. The boundary Harnack principle for the ratio of positive harmonic functions is shown to hold in twisted Hölder domains of order α for $\alpha \in (1/2, 1]$. For each $\alpha \in (0, 1/2)$, there exists a twisted Hölder domain of order α for which the boundary Harnack principle fails. Extensions are given to L-harmonic functions for uniformly elliptic operators L in divergence form.

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1. Introduction

Let D be a Hölder domain of order α. One of the main purposes of this paper is to prove

Theorem 1.1. *Suppose $\alpha \in (1/2, 1]$. Let V be an open set, K a compact set, $K \subseteq V$. There exists a constant c_1 such that if u and v are positive harmonic functions in D that vanish continuously at the regular points of $(\partial D) \cap V$ and are bounded in a neighborhood of $(\partial D) \cap V$, then*

$$u(x)/v(x) \le c_1 u(y)/v(y) \qquad \text{for all } x, y \in K \cap D,$$

together with an extension to a class of domains called twisted Hölder domains.

In the case when D is Lipschitz, i.e., $\alpha = 1$, this theorem was first proved by Ancona [A] and Dahlberg [Da]. Alternate proofs have been given by Wu [Wu], Jerison and Kenig [JK1], and Bass and Burdzy [BB1]. Our Theorem 1.1 is the exact analog of the corresponding theorem for the Lipschitz case, except that account must be taken of the fact that for Hölder domains not every point of the boundary need be regular (in the sense of the Dirichlet problem). The result in Lipschitz domains has been extended to L-harmonic functions for elliptic operators in divergence form [CFMS] and in nondivergence form [FGMS], and to parabolic functions [FGS]. Extensions to nontangentially accessible domains have been given by [JK2] and [Wi].

We assume throughout that $D \subseteq \mathbb{R}^d, d \ge 3$. For two dimensions complex analytic methods can be used to show easily that the boundary Harnack principle holds under much weaker assumptions on the domain.

In Section 2 we begin by estimating some hitting probabilities for d-dimensional Brownian motion killed on exiting D. In Section 3 we obtain some estimates for conditioned Brownian motion and then use these to obtain the boundary Harnack principle for Hölder domains. The proofs in Sections 2 and 3 are highly probabilistic. The ones in Section 2 could be readily translated into purely analytic ones without much trouble. The proofs in Section 3 could also be translated, but less easily, since they use notions of Brownian path decomposition, time reversal, and the like.

In Section 4 we obtain two extensions of our results. In [JK2] Jerison and Kenig introduced a class of domains called nontangentially accessible (NTA) domains. These are domains that share many of the potential-theoretic properties of Lipschitz domains, but they can have quite wild boundaries. Here and in [BB2] we discuss a class of domains called twisted Hölder domains; these are extensions of Hölder domains in an analogous way. In

[BB2] we showed that the class of twisted Hölder domains of order 1, i.e., twisted Lipschitz domains, is equivalent to the class of John domains; hence in particular, every NTA domain and every uniform domain is a twisted Lipschitz domain. Our first extension is to twisted Hölder domains of order α, $\alpha \in (1/2, 1]$. Our second extension is to L-harmonic functions where L is a uniformly elliptic operator in divergence form.

A natural question is, how generally does the boundary Harnack principle hold? In Section 5 we give two examples. The first is of a domain lying above the graph of a continuous function (not a Hölder continuous one, however) for which the boundary Harnack principle fails. The second shows that, given $\alpha < 1/2$, there exists a twisted Hölder domain of order α for which the boundary Harnack principle does not hold.

This, of course, leads to another question: setting aside twisted Hölder domains, does the boundary Harnack principle hold for Hölder domains when $\alpha \in (0, 1/2]$? Recent results ([BBB]) show that the answer is yes.

Besides the boundary Harnack principle discussed in this paper, one can also ask about the parabolic boundary Harnack principle for parabolic functions. One motivation for studying the parabolic version is that there is a close relationship between it and the question of when the expected lifetimes of conditioned Brownian motion are finite. In [BB2] we prove that the parabolic boundary Harnack principle holds (and also the finiteness of the expected lifetimes) in twisted Hölder domains if $\alpha > 1/3$, and that $1/3$ is the critical value. We also prove there that the parabolic boundary Harnack principle holds in L^p domains, $p > d - 1$; these are domains whose boundary may be represented locally as the graph of an L^p function.

There is another related problem. In Lipschitz domains the Martin boundary may be identified with the Euclidean one. This was first proved by Hunt and Wheeden [HW]. Alternate proofs [JK1, JK2, BB1] have shown that this property is intimately related to the boundary Harnack principle. One might ask whether the Martin boundary may be identified with the Euclidean one in domains whose boundary is locally given by the graph of functions with modulus of continuity $\omega(\delta)$ weaker than Lipschitz, Hölder say. In [BB3] we show that (a) yes, the identification holds for $\omega(\delta)$ weaker than Lipschitz, and (b) no, Hölder is too weak, as can be shown by counterexamples. The critical function is close to $\omega(\delta) = \delta \ln \ln(1/\delta)$.

This paper uses extensively the relationship between Brownian motion and harmonic functions. See [Do] or [PS] for further information.

2. Hitting probabilities

In this section we obtain an estimate on hitting probabilities for standard d-dimensional Brownian motion in the region above the graph of a Hölder function. We write $x = (x^1, \ldots, x^d) = (\widetilde{x}, \widehat{x})$, where $\widetilde{x} = (x^1, \ldots, x^{d-1})$ and $\widehat{x} = x^d$ for points in $\mathbb{R}^d$. Suppose $\alpha \in (0,1]$, $M > 1$, and $\Gamma : \mathbb{R}^{d-1} \to [-M, M]$ is a function satisfying

$$|\Gamma(\widetilde{x}) - \Gamma(\widetilde{y})| \le M(|\widetilde{x} - \widetilde{y}|^\alpha \wedge 1), \qquad \widetilde{x}, \widetilde{y} \in \mathbb{R}^{d-1}. \tag{2.1}$$

Let $D = \{x \in \mathbb{R}^d : \widehat{x} > \Gamma(\widetilde{x})\}$. Write

$$d(x) = \operatorname{dist}(x, \partial D), \qquad \delta(x) = \widehat{x} - \Gamma(\widetilde{x}),$$

$$F_a = \{x \in D : \delta(x) < a\}.$$

Let $B(x,r) = \{y \in \mathbb{R}^d : |y - x| < r\}$, and let $|A|$ denote the Lebesgue measure of the Borel set A.

We also introduce

$$\begin{aligned} \Delta(x,a,r) &= \{y \in D : \Gamma(\widetilde{y}) < \widehat{y} < \Gamma(\widetilde{y}) + a, |\widetilde{y} - \widetilde{x}| < r\}, \\ \partial^u \Delta(x,a,r) &= \{y \in \partial\Delta(x,a,r) : \widehat{y} = \Gamma(\widetilde{y}) + a\}, \qquad (\text{“u” for upper}) \end{aligned} \tag{2.2}$$

and

$$\partial^s \Delta(x,a,r) = \{y \in \partial\Delta(x,a,r) : |\widetilde{y} - \widetilde{x}| = r\} \qquad (\text{“s” for side}).$$

Let (X_t, P^x) be standard Brownian motion in $\mathbb{R}^d$. For any Borel set A, let

$$T(A) = \inf\{t > 0 : X_{t-} \in A\}.$$

Remark. Since Brownian motion has continuous paths, $T(A)$ is just the hitting time of A. We write the definition in this form so that it will also apply in the next section where we consider killed Brownian motion.

Lemma 2.1. *There exist $c_1, c_2 > 0$ such that if $a < c_1$ and $x \in F_a$, then*

$$|B(x,2a) \cap F_a^c| / |B(x,2a)| \ge c_2.$$

Proof. Fix x and let $R = \{y : |\widetilde{y} - \widetilde{x}| \le 5a/4, |\widehat{y} - \widehat{x}| \le 5a/4\}$. Note $R \subseteq B(x,2a)$. If m denotes one-dimensional Lebesgue measure, then for each y,

$$m\big(\{\widetilde{y}\} \times (\widehat{x} - 5a/4, \widehat{x} + 5a/4)\big) = 5a/2,$$

while

$$m\big((\{\widetilde{y}\}\times\mathbb{R})\cap F_a\big)=a.$$

Hence

$$m\bigg(\big(\{\widetilde{y}\}\times(\widehat{x}-5a/4,\widehat{x}+5a/4)\big)\cap F_a^c\bigg)\ge 3a/2.$$

Integrating,

$$\begin{aligned}|B(x,2a)\cap F_a^c|\ge|R\cap F_a^c|&=\int_{|\widetilde{y}-\widetilde{x}|\le 5a/4} m\bigg(\big(\{\widetilde{y}\}\times(\widehat{x}-5a/4,\widehat{x}+5a/4)\big)\cap F_a^c\bigg)d\widetilde{y}\\&\ge c_3a^d,\end{aligned}$$

which proves the lemma. □

We use Lemma 2.1 to get

Lemma 2.2. *There exists $c_4\in(0,1)$ such that if k is an integer >0, $b>0, r\ge bk$, and $y\in\Delta(x,b,r)$ with $\widetilde{y}=\widetilde{x}$, then*

$$P^y\big(X_{T(\partial\Delta(x,b,r))}\in\partial^s\Delta(x,b,r)\big)\le c_4^k. \tag{2.3}$$

Proof. We first get a lower bound on the probability of hitting sets of positive measure. Suppose $A\subseteq B(0,1)$ with $|A|>\delta$. Take ϵ small enough so that $|B(0,1)-B(0,1-\epsilon)|\le\delta/2$. Writing $A'=A\cap B(0,1-\epsilon)$, noting that $|A'|\ge\delta/2$, and recalling that the transition densities $p_t^0(x,y)$ for Brownian motion killed on exiting $B(0,1)$ are bounded below on $B(0,1-\epsilon)\times B(0,1-\epsilon)$ for each ϵ and t, we have

$$\begin{aligned}P^0\big(T(A)<T(\partial B(0,1))\big)&\ge\int_{A'}p_1^0(0,y)dy\\&\ge c_5|A'|\ge c_5\delta/2.\end{aligned}\tag{2.4}$$

Hence the probability on the left hand side of (2.4) is bounded below by a constant depending only on δ.

Now let $U_1=\inf\{t:|X_t-X_0|=2b\}, U_{i+1}=U_i+U_1\circ\theta_{U_i}, i=1,2,\ldots$, where θ is the usual shift operator. Using the strong Markov property,

$$\begin{aligned}P^y\big(T(F_b^c)\ge U_j\big)&=P^y\big(T(F_b^c)\ge U_{j-1},T(F_b^c)\ge U_1\circ\theta_{U_{j-1}}\big)\\&=E^y\big(P^{X_{U_{j-1}}}(T(F_b^c)\ge U_1);T(F_b^c)\ge U_{j-1}\big)\\&\le\sup_{w\in F_b}P^w\big(T(F_b^c)\ge U_1\big)P^y\big(T(F_b^c)\ge U_{j-1}\big).\end{aligned}\tag{2.5}$$

Using Lemma 2.1 and then using scaling with (2.4), the right hand side of (2.5) is less than or equal to $c_6 P^y\big(T(F_b^c) \ge U_{j-1}\big)$, with $c_6 \in (0,1)$ depending only on c_5 and c_2. So by induction,

$$P^y\big(T(F_b^c) \ge U_{[k/2]}\big) \le c_6^{[k/2]},$$

which implies (2.3), since $T(\partial\Delta(x,b,r)) \ge U_{[k/2]}$. □

Lemma 2.2 gives an upper bound on escaping $\Delta(x,b,r)$ through the sides. We also need a lower bound on escaping upwards.

Lemma 2.3. *Suppose $x \in D$ and $y \in \Delta(x,2M,a)$. Then there exist $c_7, c_8 > 0$ depending only on a and M such that*

$$P^y\big(X_{T(\partial\Delta(x,4M,2a))} \in \partial^u\Delta(x,4M,2a)\big) \ge c_7 \exp(-c_8 \delta(y)^{1-1/\alpha}). \tag{2.6}$$

Proof. Let $y_0 = y$. We construct a sequence of points $y_i \in D, 1, \dots, n$, by setting $\widetilde{y}_i = \widetilde{y}$, $\widehat{y}_{i+1} = \widehat{y}_i + r_i$, where $r_i = \frac{1}{2}\min\big(a, d(y_i)\big)$. Since $\widehat{y}_i$ is increasing, so is $d(y_i)$. Hence $|y_{i+1} - y_i| \le \frac{1}{2}\min\big(a, d(y_i), d(y_{i+1})\big)$.

Let n be the first integer for which $\delta(y_n) \ge 4M + a$. We claim that

$$n \le c_9 \delta(y)^{1-1/\alpha} \tag{2.7}$$

for a constant c_9 depending on a and M. Note that $d(y_0) \ge c_{10}\delta(y_0)^{1/\alpha}$ by (2.1). If $d(y_0) \ge a$, we can just take $r_i = a/2$ for each i, and then $n \le 2(4M+a)/a + 1$. If $d(y_0) < a$, let $m_1 = [2c_{10}^{-1}\delta(y_0)^{1-1/\alpha}] + 1$, and then

$$\delta(y_{m_1}) \ge m_1 r_0 + \delta(y_0) \ge 2\delta(y_0).$$

We then argue by induction. Assume $\delta(y_{m_j}) \ge 2^j\delta(y_0)$. If $d(y_{m_j}) \ge a$, we can take $n \le m_j + 2(4M+a)/a + 1$. If $d(y_{m_j}) < a$, then let $m_{j+1} = m_j + [2c_{10}^{-1}\big(2^j\delta(y_0)\big)^{1-1/\alpha}] + 1$. Observe that $d(y_{m_j}) \ge c_{10}\delta(y_{m_j})^{1/\alpha}$ and so

$$\begin{aligned}\delta(y_{m_{j+1}}) &\ge \delta(y_{m_j}) + (m_{j+1} - m_j) r_{m_j} \ge \delta(y_{m_j}) + 2^j\delta(y_0)\\ &\ge 2^{j+1}\delta(y_0).\end{aligned}$$

Pick j_0 to be the first j such that $2^j\delta(y_0) \ge 4M + a$. Then we see that

$$\begin{aligned} n &\le 2(4M+a)/a + 1 + j_0 + \sum_{j=1}^{\infty}[2c_{10}^{-1}2^j\delta(y_0)^{1-1/\alpha}] \\ &\le c_{11} + |\ln\ \delta(y_0)| + c_{12}\delta(y_0)^{1-1/\alpha}\end{aligned} \tag{2.8}$$

since $1-\alpha^{-1}<0$. This proves the claim (2.7).

Now let $B_i = B(y_i, 2r_i), B = \bigcup_{i=1}^n B_i$. Let h be the harmonic function that has boundary value 1 on $\partial B \cap \partial B_n$ and value 0 on the remainder of ∂B. Clearly there exists a constant c_{13} depending only on a and M such that $h(y_n) \geq c_{13}$. By the usual Harnack inequality in the ball $B(y_i, 2r_i)$, we get

$$h(y_i)/h(y_{i+1}) \geq c_{14}.$$

So we have $h(y_0) \geq c_{13}c_{14}^n$. But

$$P^y(X_{T(\partial\Delta(x,4M,2a))} \in \partial^u\Delta(x,4M,2a)) \geq P^y(X_{T(\partial B)} \in \partial B_n) = h(y_0),$$

which, combined with (2.8), gives our result. □

We now prove the main estimate of this section. Let $A>0$, $R>2A$. Let

$$H_1 = \{X_{T(\partial\Delta(0,2A,2R))} \in \partial^s\Delta(0,A,2R)\},$$

$$\Delta^g(0,2A,2R) = \partial\Delta(0,2A,2R) - \big(\partial D \cup \partial^s\Delta(0,A,2R)\big) \qquad (\text{“g” for good}),$$

and

$$H_2 = \{X_{T(\partial\Delta(0,2A,2R))} \in \partial^g\Delta(0,2A,2R)\}.$$

Theorem 2.4. *Suppose $\alpha \in (\frac{1}{2}, 1]$. Then there exists $c_{15}>0$ depending only on α, A, M and R such that for all $x \in \Delta(0,A,R)$,*

$$P^x(H_1) \leq c_{15}P^x(H_2).$$

Proof. Write $b_k = 2^{-k}$, $r_k = 2R - (R/8)\sum_{i=0}^{k}(1+i)^{-2}$. Note $r_0 = \frac{15}{8}R$ and $\inf_k r_k > R$.

Let

$$J_k = \{y \in D : \widehat{y} \in [\Gamma(\widetilde{y}) + Ab_{k+1}, \Gamma(\widetilde{y}) + Ab_k], |\widetilde{y}| \leq r_k\}, \quad k=0,1,\ldots$$

By Lemma 2.3, $P^z(H_2) \geq c_{16}$ for $z \in J_0$. So for $z \in J_0$,

$$P^z(H_1) \leq 1 \leq c_{16}^{-1}P^z(H_2).$$

Let $d_m = \sup_{z\in J_m} P^z(H_1)/P^z(H_2)$. From what we just observed, $d_0 \leq c_{16}^{-1}$. Since $\Delta(0,A,R) \subseteq \bigcup_{k=0}^\infty J_k$, to prove the theorem it suffices to prove $\sup_m d_m < \infty$.

Fix m and suppose $z \in J_{m+1}$. For the remainder of the proof, write

$$\Delta_m = \Delta(z, Ab_{m+1}, r_m - r_{m+1}), \quad U_m = T(\partial \Delta_m).$$

By the strong Markov property,

$$P^z(H_1) \le E^z\big(P^{X_{U_m}}(H_1); X_{U_m} \in \partial^u \Delta_m\big) + P^z(X_{U_m} \in \partial^s \Delta_m) \tag{2.9}$$

and

$$P^z(H_2) \ge E^z\big(P^{X_{U_m}}(H_2); X_{U_m} \in \partial^u \Delta_m\big). \tag{2.10}$$

Since $\partial^u \Delta_m \subseteq J_m$ when $z \in J_{m+1}$, the first term on the right hand side of (2.9) is bounded by

$$d_m E^z\big(P^{X_{U_m}}(H_2); X_{U_m} \in \partial^u \Delta_m\big) \le d_m P^z(H_2).$$

By Lemma 2.2 the second term on the right of (2.9) is bounded above by $c_4^{[(r_m - r_{m+1})/Ab_{m+1}]}$. On the other hand, by Lemma 2.3 with $a = R/16$, the right hand side of (2.10) is bounded below by $c_7 \exp\big(-c_8 (Ab_{m+1})^{1-1/\alpha}\big)$. So, since $\alpha^{-1} < 2$, our choice of r_m leads to

$$\begin{aligned} P^z(H_1) &\le d_m P^z(H_2) + c_4^{[(r_m - r_{m+1})/Ab_{m+1}]} \\ &\le d_m P^z(H_2) + \exp(-c_{17}/Am^2 b_{m+1}) \\ &\le d_m P^z(H_2) + c_{18} m^{-2}\big(c_7 \exp(-c_8 (Ab_{m+1})^{1-1/\alpha}\big) \\ &\le d_m P^z(H_2) + c_{18} m^{-2} P^z(H_2). \end{aligned}$$

Hence

$$d_{m+1} \le d_m + c_{18} m^{-2},$$

or $\sup_m d_m \le d_0 + c_{18} \sum_{m=1}^{\infty} m^{-2} < \infty$, as required. □

3. Conditioned Brownian motion

In this section we obtain the analog of Theorem 2.4 for conditioned Brownian motion. If h is a positive harmonic function in a domain D, let (X_t, P_h^x) be the h–path transform of Brownian motion in D or in other words, Brownian motion conditioned by h. See [Do] for details.

Let $G_v(\cdot) = G(v,\cdot)$ be the usual Green function in $\mathbb{R}^d$, i.e., $G(v,y) = c_1|v-y|^{2-d}$, where $c_1 = \Gamma(n/2-1)/(2\pi)^{n/2}$. Let $Q^x = P^x_{G(v,\cdot)}$ be the probabilities for Brownian motion in $\mathbb{R}^d - \{v\}$ that are conditioned by the harmonic function $G(v,\cdot)$. Let D be as in Section 2 and $v \in D$. Write $G_D(v,\cdot)$ for the Green function for the domain D and let g be the Q^x–harmonic function in $D - \{v\}$ that has boundary values 1 on $\{v\}$ and 0 on ∂D. So $g(x) = Q^x(X_{T(\partial D\cup\{v\})} = v)$.

Lemma 3.1. $Q^x_g = P^x_{G_D(v,\cdot)}$.

Proof. Fix $v \in D$ and write $G_D(\cdot)$ for $G_D(v,\cdot), G(\cdot)$ for $G(v,\cdot)$. Since g is harmonic on $D-\{v\}$ for Q^x, $\frac{1}{2}\Delta g + \frac{\nabla G}{G}\nabla g = 0$ in $D-\{v\}$. Then gG is harmonic on $D-\{v\}$ for P^x, since

$$\frac{1}{2}\Delta(gG) = \frac{1}{2}(\Delta g)G + \frac{1}{2}(\Delta G)g + \nabla g\nabla G = 0$$

in $D-\{v\}$ by the above and the fact that $\Delta G = 0$ on $\mathbb{R}^d - \{v\}$. Since g is bounded, gG has a pole at v. Since $g = 0$ on the regular points (for the Dirichlet problem) of ∂D, then gG vanishes at the regular points of ∂D. Therefore gG is a constant multiple of G_D.

If $A \subseteq D$, then

$$\begin{aligned} Q^x_g(X_{t\wedge T(\partial D)} \in A) &= E^x_G(g1_A(X_{t\wedge T(\partial D)}))/g(x) \\ &= E^x(gG1_A(X_{t\wedge T(\partial D)}))/G(x)g(x) = P^x_{G_D}(X_{t\wedge T(\partial D)} \in A). \end{aligned}$$

Hence, using the Markov property and the continuity of the paths of X_t, $Q^x_g = P^x_{gG} = P^x_{G_D}$. □

Theorem 3.2. *Suppose A, R, H_1, H_2 are defined as in Theorem 2.4 and $\alpha \in (1/2, 1]$. Fix $v \in D - \Delta(0, 5A, 5R)$ and let $h(x) = G_D(v,x)$. Then there exists a constant c_2 depending only on α, M, A and R such that for all $x \in \Delta(0, A, R)$,*

$$P^x_h(H_2) \geq c_2.$$

Proof. Let us denote

$$\begin{aligned} &\Delta_1 = \Delta(0, 2A, 2R), \quad \partial^b\Delta_1 = \{y \in \partial\Delta(0,2A,2R) : 0 < \delta(y) \leq A\}, \\ &\partial^g\Delta_1 = \partial\Delta_1 - (\partial D \cup \partial^b\Delta_1); \end{aligned}$$

and

$$\begin{aligned} \Delta_2 = \Delta\big(v, \delta(v) + A, 2R\big) - \Delta\big(v, \delta(v) - A, 2R\big), \\ \partial^b\Delta_2 = \partial\Delta_2 \cap F_A, \qquad \partial^g\Delta_2 = \partial\Delta_2 - (\partial D \cup \partial^b\Delta_2). \end{aligned}$$

Write D^0 for $D - \{v\}$. Let

$$L(A) = \sup\{t : X_t \in A\}$$

denote the last exit of A for Borel sets A.

We begin by getting an upper bound on $Q^x(X_{T(\partial D^0)} = v)$. Since Q^x-a.s. all paths eventually hit v, we have for $x \in \Delta(0, A, R)$,

$$Q^x(X_{T(\partial D^0)} = v) \le \int_{\partial^b \Delta_1 \cup \partial^g \Delta_1} Q^x(X_{T(\partial\Delta_1)} \in dz) Q^z(X_{L(\partial\Delta_2)} \in \partial^b \Delta_2 \cup \partial^g \Delta_2). \tag{3.1}$$

Since $G(z, w)/G(z, v)$ is bounded above by a constant $c_3 = c_3(A, R)$ for $z \in \partial\Delta_1$, $w \in \partial\Delta_2$, we have, using the symmetry of G and time reversal,

$$\begin{aligned} Q^z(X_{L(\partial\Delta_2)} \in \partial^b \Delta_2 \cup \partial^g \Delta_2) &= P^z_{G_v}(X_{L(\partial\Delta_2)} \in \partial^b \Delta_2 \cup \partial^g \Delta_2) \\ &= P^v_{G_z}(X_{T(\partial\Delta_2)} \in \partial^b \Delta_2 \cup \partial^g \Delta_2) \\ &= E^v(G_z(X_{T(\partial\Delta_2)}); X_{T(\partial\Delta_2)} \in \partial^b \Delta_2 \cup \partial^g \Delta_2)/G_z(v) \\ &\le c_3 P^v(X_{T(\partial\Delta_2)} \in \partial^b \Delta_2 \cup \partial^g \Delta_2). \end{aligned} \tag{3.2}$$

By Theorem 2.4, the last line of (3.2) is less than or equal to

$$c_3 c_4 P^v(X_{T(\partial\Delta_2)} \in \partial^g \Delta_2),$$

where c_4 is one plus the constant c_{15} of Section 2.

Substituting in (3.1),

$$Q^x(X_{T(\partial D^0)} = v) \le c_3 c_4 Q^x(X_{T(\partial\Delta_1)} \in \partial^b \Delta_1 \cup \partial^g \Delta_1) P^v(X_{T(\partial\Delta_2)} \in \partial^g \Delta_2). \tag{3.3}$$

We also have

$$\begin{aligned} &Q^x(X_{T(\partial\Delta_1)} \in \partial^g \Delta_1, X_{T(\partial D^0)} = v) \\ &\quad \ge \int_{\partial^g \Delta_1} Q^x(X_{T(\partial\Delta_1)} \in dz) Q^z(T(\partial D) = \infty). \end{aligned} \tag{3.4}$$

For all $z \in \partial^g \Delta_1$ and all $w \in \partial^g \Delta_2$, we have

$$P^w_{G_z}\big(T(\partial D) = \infty\big) \ge c_5, \tag{3.5),}$$

since Brownian motion has positive probability of going from w to a small neighborhood of z without first hitting ∂D. Moreover, c_5 depends only on α, M, A and R. So, using time reversal and symmetry again and (3.5), for $z \in \partial^g \Delta_1$,

$$
\begin{aligned}
Q^z\big(T(\partial D)=\infty\big) &= P^v_{G_z}\big(T(\partial D)=\infty\big) \\
&\geq \int\limits_{\partial^g \Delta_2} P^v_{G_z}(X_{T(\partial\Delta_2)} \in dw) P^w_{G_z}\big(T(\partial D)=\infty\big) \\
&\geq c_5 P^v_{G_z}(X_{T(\partial\Delta_2)} \in \partial^g \Delta_2).
\end{aligned}
\tag{3.6}
$$

Since $G(z,w)/G(z,v) \geq c_6 = c_6(A,R)$ for $z \in \partial^g \Delta_1$, $w \in \partial^g \Delta_2$, then (3.6) leads to

$$
\begin{aligned}
Q^z\big(T(\partial D)=\infty\big) &\geq c_5 E^v(G(z, X_{T(\partial\Delta_2)}); X_{T(\partial\Delta_2)} \in \partial^g \Delta_2)/G(z,v) \\
&\geq c_5 c_6 P^v(X_{T(\partial\Delta_2)} \in \partial^g \Delta_2).
\end{aligned}
\tag{3.7}
$$

Substituting in (3.4),

$$
Q^x(X_{T(\partial\Delta_1)} \in \partial^g \Delta_1, X_{T(\partial D^0)} = v) \geq c_7 P^v(X_{T(\partial\Delta_2)} \in \partial^g \Delta_2) Q^x(X_{T(\partial\Delta_1)} \in \partial^g \Delta_1). \tag{3.8}
$$

Now by Lemma 3.1, the strong Markov property, and the definition of h–path transform,

$$
P^x_h(H_2) = \frac{Q^x(X_{T(\partial\Delta_1) \in \partial^g \Delta_1}, X_{T(\partial D^0)} = v)}{Q^x(X_{T(\partial D^0)} = v)} \quad . \tag{3.9}
$$

Substituting (3.3) and (3.8) in (3.9),

$$
P^x_h(H_2) \geq c_8 \frac{Q^x(X_{T(\partial\Delta_1)} \in \partial^g \Delta_1)}{Q^x(X_{T(\partial\Delta_1)} \in \partial^g \Delta_1 \cup \partial^b \Delta_1)} \quad . \tag{3.10}
$$

For $x \in \Delta(0,A,R)$, $z \in \partial\Delta_1$, we have

$$
c_9^{-1} \leq G(v,z)/G(v,x) \leq c_9, \qquad c_9 \in (0,\infty).
$$

So, similarly to (3.2) and (3.7), (3.10) leads to

$$
P^x_h(H_2) \geq c_8 c_9^{-2} \frac{P^x(X_{T(\partial\Delta_1)} \in \partial^g \Delta_1)}{P^x(X_{T(\partial\Delta_1)} \in \partial^g \Delta_1 \cup \partial^b \Delta_1)} \quad .
$$

Using Theorem 2.4 again,

$$
P^x_h(H_2) \geq c_8 c_9^{-2} c_4^{-1}.
$$

□

Theorem 3.3. *Let $\alpha \in (1/2, 1]$. Suppose u and v are positive and harmonic on D, u and v both vanish continuously on the regular points of $\partial D \cap \overline{\Delta(0, 6A, 6R)}$, and u and v both are bounded in a neighborhood of $\partial D \cap \overline{\Delta(0, 6A, 6R)}$. Then there exists $c_{10} = c_{10}(\alpha, M, A, R) < \infty$ such that*

$$u(x)/v(x) \le c_{10} u(y)/v(y) \quad \textit{whenever} \quad x, y \in \Delta(0, A, R).$$

Proof. Suppose first that D is smooth. Pick $x_0 \in \partial^g \Delta(0, 2A, 2R)$. Suppose $z \in D - \Delta(0, 5A, 5R)$ and $v(x) = G_D(z, x)$. By the usual Harnack inequality, there exists c_{11} such that $u(y) \ge c_{11} u(x_0)$, $v(y) \le c_{11}^{-1} v(x_0)$ if $y \in \partial^g \Delta(0, 2A, 2R)$, c_{11} depending only on α, M, A and R.

Since u is harmonic then u/v is harmonic with respect to P_v^x. So if $x \in \Delta(0, A, R)$,

$$\begin{aligned}(u/v)(x) &\ge E_v^x\big((u/v)\big(X_{T(\partial\Delta(0,2A,2R))}\big); X_{T(\partial\Delta(0,2A,2R))} \in \partial^g \Delta(0, 2A, 2R)\big) \\ &\ge \inf_{y \in \partial^g \Delta(0,2A,2R)} (u/v)(y) P_v^x(H_2) \\ &\ge c_{11}^2 c_2 (u/v)(x_0),\end{aligned}$$

or

$$u(x)v(x_0) \ge c_{12} u(x_0) v(x). \tag{3.11}$$

Since D is smooth, the Martin boundary for D may be identified with the Euclidean one (see [JK1] or [BB1]). Fix $x_1 \in D$. If $K_D(x, z_0)$ is the Martin kernel for D with pole at $z_0 \in \partial\big(D - \overline{\Delta(0, 5A, 5R)}\big)$, then $K_D(x, z_0) = \lim_{z \to z_0} G_D(x, z)/G_D(x_1, z)$, up to a constant multiple. So multiplying (3.11) by $G_D(x_1, z)^{-1}$ and letting $z \to z_0$, we obtain (3.11) for $v = K_D(x, z_0)$.

If v vanishes continuously on $\partial D \cap \overline{\Delta(0, 5A, 5R)}$, then, since D is smooth, $v(x) = \int K_D(x, z_0) \mu(dz_0)$ for some measure μ supported on $\partial(D - \overline{\Delta(0, 5A, 5R)})$. So integrating (3.11) with respect to $\mu(dz)$ again gives (3.11) for v vanishing continuously on $\partial D \cap \overline{\Delta(0, 5A, 5R)}$.

Now suppose D is given as the region above the graph of a Hölder function Γ. Let $\Gamma_n \downarrow \Gamma$, where each Γ_n is smooth, $\Gamma_n \to \Gamma$ uniformly, and every Γ_n satisfies the same modulus of continuity as Γ. Let D_n be the region above Γ_n. Define $\Delta_n(0, 5A, 5R)$, etc., as in (2.2). Let $\varphi : \mathbb{R}^{d-1} \to [0, 1]$ be smooth such that $\varphi = 0$ on $\{x : |\widetilde{x}| \le 5R\}$, $\varphi = 1$ on

$\{x : |\widetilde{x}| \geq 6R\}$. Let $u_n(x) = E^x\big(u\varphi(X_{T(D_n)})\big)$, and similarly for v_n. Take n large enough so that $x_0, x \in D_n$. Since u_n vanishes continuously on $\partial D_n \cap \overline{\Delta_n(0, 5A, 5R)}$, by the above

$$u_n(x)v_n(x_0) \geq c_{12} u_n(x_0)v_n(x). \tag{3.12}$$

If we knew $u_n(x) \to u(x), v_n(x) \to v(x)$, and similarly with x replaced by x_0, we could take the limit in (3.12) to get

$$u(x)/v(x) \geq c_{12} u(x_0)/v(x_0). \tag{3.13}$$

Then reversing the roles of u and v would give the converse inequality. And combining the converse inequality with (3.13) with x replaced by y would prove the theorem.

But by the continuity of Brownian motion, $X_{T(D_n)} \to X_{T(D)}$, a.s. Since the set of irregular points of ∂D is polar, $u(1-\varphi)(X_{T(D_n)}) \to 0$, a.s., by the hypotheses on u. And since u is bounded in a neighborhood of $\partial D \cap \overline{\Delta(0, 6A, 6R)}$, we get $u(x) - u_n(x) = E^x\big(u(1-\varphi)(X_{T(D_n)})\big) \to 0$ by dominated convergence. □

Remark. The assumption that u and v be bounded in a neighborhood cannot be dispensed with. Otherwise, if we had a domain with an irregular point $z \in \partial D$, we could let $u(x) = K_D(x, z)$, and then $u(x)/v(x)$ could blow up as $x \to z$. Nevertheless, u vanishes continuously on the regular points of D.

¿From Theorem 3.3, it is easy to get one of our main theorems.

Definition 3.4. *D is a Hölder domain of order α, $\alpha \in (0, 1]$, if for all $x \in D$ there exists $r_x > 0$, a coordinate system CS_x, and a bounded function Γ_x that is Hölder continuous of order α such that*

$$D \cap B(x, r_x) = \{y : y = (y^1, \dots, y^{d-1}, y^d) \text{ in } CS_x, y^d > \Gamma_x(y^1, \dots, y^{d-1})\} \cap B(x, r_x).$$

Remark. The term "Hölder domain" has also been used to describe other types of domains; see [SS].

Theorem 3.5. *Suppose D is a Hölder domain of order α, $\alpha \in (\frac{1}{2}, 1]$. Suppose V is open and K compact is contained in V. Then there exists a constant c_{13} such that whenever u and v are positive and harmonic in D, u and v vanish continuously on the regular points of $\partial D \cap V$, and u and v are bounded in a neighborhood of $\partial D \cap V$, then*

$$u(x)/v(x) \leq c_{13} u(y)/v(y), \qquad x, y \in K \cap D.$$

Proof. This follows by repeated applications of Theorem 3.3 and the usual Harnack inequality. $\square$

Corollary 3.6. *Let D, K, V be as above. Suppose f and g are two positive functions on D, vanishing on $\partial D \cap V$. Let $u(x) = E^x f(X_{T(D)})$, $v(x) = E^x g(X_{T(D)})$. Then*

$$u(x)/v(x) \leq c_{13} u(y)/v(y), \qquad x, y \in K \cap D.$$

Proof. We need to show $u(x)v(y) \leq c_{13} v(x)u(y)$. By monotone convergence, we may truncate f and g, and so it suffices to consider f, g bounded. If f and g are bounded, so are u and v. Moreover it is standard that $u(x), v(x) \to 0$ as $x \to z \in \partial D \cap V$ if z is a regular point (see [PS]). Now apply Theorem 3.5. $\square$

4. Extensions

In this section we consider two extensions to Theorem 3.5. The first is to domains more general than Hölder domains, the second to operators more general than the Laplacian.

We introduce a class of domains called twisted Hölder domains (see also [BB2]). These are extensions of Hölder domains similar to the way nontangentially accessible domains and uniform domains are extensions of Lipschitz domains.

Definition 4.1. *A bounded domain $D \subseteq \mathbb{R}^d$ is a twisted Hölder domain of order α, $\alpha \in (0,1]$, if there exist constants $c_1, c_2, c_3, c_4 \in (0,\infty)$, a point $x_0 \in D$, and a bounded continuous function $\delta : D \to (0,\infty)$, such that*

a) $\delta(x) \leq c_1 \big(d(x)\big)^\alpha$ for all $x \in D$, where $d(x) = dist(x, \partial D)$;

b) for all $x \in D$, there exists a rectifiable curve γ lying in D joining x and x_0 such that $\delta(x_1) \geq c_2 \Big(\ell\big(\gamma(x_1,x)\big) + \delta(x)\Big)$ for all x_1 lying on γ, where $\ell\big(\gamma(x_1,x)\big)$ is the length of the piece of γ connecting x_1 and x; and

c) $\mathrm{Cap}\big(B(x, c_3 a) \cap F_a^c\big)/\mathrm{Cap}\big(B(x, c_3 a)\big) \geq c_4$ for all $x \in F_a$, where $F_a = \{y : \delta(y) \leq a\}$ and Cap is Newtonian capacity.

We have the following properties:

(4.2) if D is a Hölder domain of order α, then D is a twisted Hölder domain of order α.

(4.3) if D is a John domain (in particular, if D is a nontangentially accessible domain or a uniform domain), then D is a twisted Hölder domain of order 1.

(4.4) given $a>0$, there exists $c_5=c_5(a,D)$ such that if $y\in D$, we can find a sequence of points $y_0=y,\ y_1,y_2,\dots,y_n=x_0$ with $n\le c_5\delta(y)^{1-1/\alpha}$, such that $|y_{i+1}-y_i|\le \frac12\min\big(a,d(y_i),d(y_{i+1})\big), i=0,1,\cdots,n-1$.

See [BB2] for proofs of these and other properties of twisted Hölder domains.

We have

Lemma 4.2. *Suppose* $\operatorname{Cap}\big(A\cap B(x,c_3a)\big)\ge c_4\ \operatorname{Cap}\big(B(x,c_3a)\big)$. *Then* $P^x\big(T(A)<T(\partial B(x,2c_3a))\big)\ge c_6$, *where* $c_6>0$ *depends only on* c_4.

Remark. Recall that $\operatorname{Cap}(A)=\sup\{\mu(A):\mu$ is a measure supported on A with $\int G(x,y)\mu(dy)\le 1$ for all $x\in\mathbb{R}^d\}$.

Note also that if $|A|>0$, and we take $\mu(dx)=c_7 1_A(x)dx$, we have by Hölder's inequality that $c_7\int G(x,y)1_A(y)dy\le c_7\|G(x,\cdot)\|_p|A|^{1/q}\le c_7c_8|A|^{1/q}\le 1$ if $p<d/(d-2)$, $p^{-1}+q^{-1}=1$ and $c_7=c_8^{-1}|A|^{-1/q}$. Hence we have the crude estimate $\operatorname{Cap}(A)\ge c_8^{-1}|A|^{1-1/q}$.

Proof. By scaling and translation invariance, we may assume $x=0$, $A\subseteq B(0,1)$, and $\operatorname{Cap}(A)\ge c_4\ \operatorname{Cap}(B(0,1))$ and we must show $P^0\big(T(A)<T(\partial B(0,2)\big)\ge c_6$.

If μ is the capacitary measure for A, we have $G_{B(0,2)}\mu(0)=\int_A G_{B(0,2)}(0,y)\mu(dy)\ge \mu(A)\inf_{y\in B(0,1)} G_{B(0,2)}(0,y)\ge c_9\ \operatorname{Cap}(A)\ge c_4c_9\ \operatorname{Cap}(B(0,1))$, while by the strong Markov property,

$$\begin{aligned}G_{B(0,2)}\mu(0)&=E^0\big(G_{B(0,2)}\mu(X_{T(A)});T(A)<T(\partial B(0,2))\big)\\&\le \sup_{y\in\mathbb{R}^d} G\mu(y)P^0\big(T(A)<T(\partial B(0,2))\big)\\&\le P^0\big(T(A)<T(\partial B(0,2))\big).\end{aligned}$$

Hence $P^0\big(T(A)<T(\partial B(0,2))\big)\ge c_4c_9\ \operatorname{Cap}(B(0,1))$. □

Let $\Delta(x,b,r)=\{y\in D:0<\delta(y)<b,|y-x|<r\}, \partial^u\Delta(x,b,r)=\{y\in D:\delta(y)=b,|y-x|<r\}$, and $\partial^s\Delta(x,b,r)=\partial\Delta(x,b,r)-\big(\partial^u\Delta(x,b,r)\cup\partial D\big)$.

Lemma 4.3. *Let R be fixed. There exist constants $c_{10},c_{11},c_{12}>0$ such that if $A\le c_{10}$, then*

$$P^y\big(X_{T(\partial\Delta(x,A,R))}\in\partial^u\Delta(x,A,R)\big)\ge c_{11}\exp\big(-c_{12}\delta(y)^{1-1/\alpha}\big),\qquad y\in\Delta(x,A,R/4).$$

Proof. Let $y_0,y_1,\dots,y_n$ be the sequence of points given by (4.4) with $a\in(0,R/8)$ sufficiently small so that $|\delta(z_1)-\delta(z_2)|<c_2R/8$ whenever $|z_1-z_2|<a$. Let $j_0=\inf\{j:$

$|y_j - y| \geq R/4\}$. If γ is the curve connecting y and x_0, then $\ell\big(\gamma(y_{j_0}, y)\big) \geq R/4$, hence by Definition 4.1(b), $\delta(y_{j_0}) \geq c_2 R/4$. So by the choice of a and taking $c_{10} = c_2/16$, we have $\delta(z) \geq c_2 R/8 \geq 2A$ if $|z - y_{j_0}| < a$.

We now follow the proof of Lemma 2.3 by constructing the sequence of balls B_i and letting $B = \bigcup_{i=0}^{j_0 \wedge n} B_i$. We note that $B \subseteq \Delta(x, 2R, R/2)$ while $B_{j_0 \wedge n} \cap \Delta(x, A, R) = \emptyset$, hence

$$P^y\big(X_{T(\partial\Delta(x,A,R))} \in \partial^u \Delta(x, A, R)\big) \geq P^{y_0}\big(T(\partial B) > T(\partial \Delta(x, A, R))\big).$$

Then as in the proof of Lemma 2.3, we get our desired estimate. □

Theorem 4.4. *Theorem 3.5 still holds if "Hölder domain of order α, $\alpha \in (1/2, 1]$" is replaced by "twisted Hölder domain of order α, $\alpha \in (1/2, 1]$".*

Proof. We let $J_m = \{y \in D : y \in F_{2^{-m}} - F_{2^{-(m+1)}}, |y - x| \leq r_m\}$ where r_m is as in Theorem 2.4. We follow the proof of Theorem 2.4 closely, except that (i) we get the analog of Lemma 2.2 by using balls of radius $2c_3 a$ and Lemma 4.2 in place of (2.4), and (ii) we use Lemma 4.3 in place of Lemma 2.3. We then can follow Section 3 closely to complete the proof. □

The other extension we consider is to L-harmonic functions, where L is a uniformly elliptic operator in divergence form. We let L be the operator defined by

$$Lf(x) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x^i}\big(a_{ij}(x) \frac{\partial f}{\partial x^j}\big)(x),$$

where we assume the a_{ij} are bounded and uniformly elliptic:

(4.5) there exists c_{13} such that

$$c_{13}^{-1} \sum_{i=1}^{d} (y^i)^2 \leq \sum_{i,j=1}^{d} a_{ij}(x) y^i y^j \leq c_{13} \sum_{i=1}^{d} (y^i)^2, \qquad (y^1, \dots, y^d) \in \mathbb{R}^d.$$

We also assume the a_{ij} are smooth (so that h path transforms, etc. make sense), but the estimates we obtain will depend on the a_{ij}'s only through the bound c_{13} and not on the smoothness of the a_{ij}'s.

Recall that

(4.6) the Markov process X corresponding to L is time reversible, has a symmetric Green function (see [Fu]), and by the estimates of [LSW], there exists c_{14} such that

$$c_{14}|x-y|^{2-d} \le G(x,y) \le c_{14}^{-1}|x-y|^{2-d},$$

where here G is the Green function for $\mathbb{R}^d$ corresponding to L;

(4.7) by [LSW] for each $\epsilon > 0$ there exists $c_{15} = c_{15}(\epsilon)$ such that

$$c_{15} G^{\Delta}_{B(0,2)} \le G^{L}_{B(0,2)} \le c_{15}^{-1} G^{\Delta}_{B(0,2)}, \qquad x, y \in B(0, 2-\epsilon),$$

where $G^{\Delta}_{B(0,2)}$ is the Green function for $B(0,2)$ for the Laplacian, and similarly for $G^{L}_{B(0,2)}$; and

(4.8) Moser's Harnack inequality takes the place of the usual Harnack inequality for L-harmonic functions ([M] or [FS]).

Capacity can be defined analogously to the definition in the Remark following Lemma 4.2, and by [LSW] or (4.6), $\mathrm{Cap}_L A/\mathrm{Cap}_\Delta A$ is bounded above and below. By (4.6), sets of positive Lebesgue measure have positive capacity, just as in the Remark following Lemma 4.1.

Theorem 4.5. *Theorems 3.5 and 4.4 hold if "harmonic" is replaced by "L-harmonic."*

Proof. We first observe that the analog of Lemma 4.2 holds. In fact, by scaling and translation we obtain a process corresponding to $\overline{L}$, an operator of the same type as L with the same ellipticity bounds c_{13}. So we can follow the proof of Lemma 4.2 exactly, using (4.7) at the appropriate place. Using (4.6) and (4.8), we can then follow the proofs of Theorems 3.5 and 4.4. □

5. Examples

One might wonder whether the boundary Harnack principle always holds in domains that lie above the graph of a continuous function. This is not the case. We give an example of a domain lying above the graph of a continuous function (but not a Hölder function) for which the boundary Harnack principle fails.

Let (X_t, Y_t, Z_t) be three dimensional Brownian motion and

$$S_a = S(a) = \inf\{t : Y_t = a\}.$$

Lemma 5.1. *Suppose $a < y < b$. There exist $c_1, c_2 > 0$ such that*

$$\begin{aligned} c_1(b-y)t^{-3/2}\exp\bigl(-(b-y)^2/2t\bigr)dt &\le P^y(S_b < S_a, S_b \in dt) \\ &\le c_2(b-y)t^{-3/2}\exp\bigl(-(b-y)^2/2t\bigr)dt, \end{aligned}$$

provided $t/(y-a)^2$ and $t/(b-y)^2$ are sufficiently small.

Proof. By [K, p. 279] we have

$$P^y(S_b \in dt)/dt = (2\pi)^{-1/2}(b-y)t^{-3/2}\exp(-(b-y)^2/2t). \tag{5.1}$$

Since $P^y(S_b < S_a, S_b \in dt) \le P^y(S_b \in dt)$, the right hand estimate is immediate. Writing

$$\begin{aligned} P^y(S_b < S_a, S_b \in dt) &= P^y(S_b \in dt) - P^y(S_a < S_b, S_b \in dt) \\ &\ge P^y(S_b \in dt) - \int_0^t P^y(S_a \in ds)P^a(S_b \in dt - s) \end{aligned}$$

and applying (5.1) gives the left hand inequality. □

We construct our region in a series of steps. Let $\epsilon_1 = \epsilon_2 < \epsilon_3 < \epsilon_4 < \cdots < \epsilon_n < 1/4$ be an increasing sequence to be chosen later, L a real > 2 to be chosen later. Let

$$A = A_n = \cup_{i=1}^n [-\epsilon_i, \epsilon_i] \times [i-1, i) \times [-L, 0]. \tag{5.2}$$

Let $p = p_n = (0, 1, -L/2)$. We let $a_i = 9^{-1}(n-i+1)^{-2}, i = 1, \ldots, n$, so that $\sum_{i=1}^n a_i \le 1$. And we let $b_i = n\epsilon_i$. Finally, let $\tau(A) = \inf\{t > 0 : (X_t, Y_t, Z_t) \notin A\}$.

We show that we can choose L (independent of n) and a sequence ϵ_i so that the probability of exiting A through the plane $z = 0$ is much less than the probability of exiting through the plane $y = n$.

Proposition 5.2. *There exists L (independent of n) and a sequence $\epsilon_1, \ldots, \epsilon_n$ such that*

$$P^p(Z_{\tau(A)} = 0) \le n^{-1} P^p(Y_{\tau(A)} = n).$$

Proof. Let $c_3 = \pi^2/8$ and $c_4 = \sqrt{2c_3}$. Recall that by [Fe, pp. 341-343], if $t/\epsilon^2 \ge 1$, then there exists c_5 such that

$$c_5 \exp(-c_3 t/\epsilon^2) \le P^0(\sup_{s \le t} |X_s| < \epsilon) \le c_5^{-1} \exp(-c_3 t/\epsilon^2). \tag{5.3}$$

Also, if $a^2/t \ge 4$, then $P^0(\sup_{s \le t} |Z_s| \le a) \ge 1/2$. So provided D/ϵ, a/ϵ, and b/ϵ are all larger than 1 and $D \le 2$, then

$$P^{(0,0,0)}\big(Y_s > -b, |X_s| < \epsilon, |Z_s| \le a \quad \text{for all } s < S(D)\big) \tag{5.4}$$

$$
\begin{aligned}
&\geq \int_0^\infty P^0\big(S(D)\in dt, S(D)<S(-b)\big)P^0(\sup_{s\leq t}|X_s|<\epsilon)P^0(\sup_{s\leq t}|Z_s|\leq a)\\
&\geq \frac{c_6}{2}(\epsilon D)^{-3/2}D\int_{\epsilon D/c_4}^{2\epsilon D/c_4}\exp(-D^2/2t)\exp(-c_3t/\epsilon^2)dt\\
&\geq \frac{c_6}{2}(\epsilon D)^{-3/2}D\exp(-c_4D/\epsilon)\\
&\qquad\times\int_{\epsilon D/c_4}^{2\epsilon D/c_4}\exp\big(-(D^2/2)(t^{-1}-c_4/\epsilon D)\big)\exp\big(-(c_3/\epsilon^2)(t-\epsilon D/c_4)\big)dt\\
&\geq \frac{c_6}{2}(\epsilon D)^{-3/2}D\exp(-c_4D/\epsilon)\int_{\epsilon D/c_4}^{2\epsilon D/c_4}\exp\big(-(c_3/\epsilon^2)(t-\epsilon D/c_4)\big)dt\\
&\geq c_7\epsilon^2\exp(-c_4D/\epsilon).
\end{aligned}
$$

Now, P^p–a.s.,

$$
\begin{aligned}
\{Y_{\tau(A)}=n\}\supseteq \big\{&|X_s|<\epsilon_1, |Z_s|\leq a_1, Y_s>0\\
&\text{for all } s\leq S(2+b_2)\big\}\cap\\
\bigcap_{i=3}^n\big\{&|X_s|<\epsilon_i-\epsilon_{i-1}, |Z_s|\leq a_i, Y_s>i-1\\
&\text{for all } s\in[S(i-1+b_{i-1}), S(i+b_i)]\big\}.
\end{aligned}
$$

So using the strong Markov property repetitively at times $S(i+b_i)$ and using (5.4),

$$
\begin{aligned}
P^p(Y_{\tau(A)}=n)\geq (c_7)^n\epsilon_2^2\prod_{i=3}^n(\epsilon_i-\epsilon_{i-1})^2\exp\big(-c_4(1+b_2)/\epsilon_1\big)\times\\
\exp\big(-c_4\sum_{i=3}^n(1+b_i-b_{i-1})/(\epsilon_i-\epsilon_{i-1})\big).
\end{aligned}
\tag{5.5}
$$

Let $\rho_i=\exp(-c_4/\epsilon_i)$. Suppose that

$$
\epsilon_{i-1}\leq\epsilon_i^2, \qquad i=3,\ldots,n;
\tag{5.6}
$$

this implies that $2\epsilon_{i-1}\leq\epsilon_i$, $i=3,\ldots,n$. Suppose also that $\epsilon_n\leq n^{-12}/4$ so that

$$
\prod_{j=i}^n\rho_j\geq\exp(-c_4/n^5\epsilon_{i-1}).
\tag{5.7}
$$

Recalling the definition of b_i, we have

$$
\exp(-c_4(b_i-b_{i-1})/(\epsilon_i-\epsilon_{i-1}))\geq\exp(-2c_4n), \qquad i=3,\ldots,n.
\tag{5.8}
$$

Since

$$(\epsilon_i - \epsilon_{i-1})^{-1} = \epsilon_i^{-1}(1 - \epsilon_{i-1}/\epsilon_i)^{-1} \le \epsilon_i^{-1}(1 + 2\epsilon_{i-1}/\epsilon_i),$$

then recalling that $2\epsilon_{i-1} \le \epsilon_i$,

$$\exp(-c_4/(\epsilon_i - \epsilon_{i-1})) \ge \rho_i \exp(-4c_4). \tag{5.9}$$

Combining (5.5), (5.8) and (5.9) we get the lower bound

$$P^p(Y_{\tau(A)} = n) \ge c_7^n \epsilon_1^{2n} \exp(-c_8 n^2) \prod_{i=2}^{n} \rho_i. \tag{5.10}$$

We have the upper bound

$$\begin{aligned} P^{(0,0,0)}(\sup_{s \le S(1)} |X_s| < \epsilon) &= \int_0^\infty P^0\big(S(1) \in dt\big) P^0(\sup_{s \le t} |X_s| < \epsilon) \\ &\le c_9 \epsilon^{-4} \exp(-c_4/\epsilon) \end{aligned} \tag{5.11}$$

if ϵ is sufficiently small, using (5.3). We also have

$$\begin{aligned} &P^{(0,0,0)}(\sup_{s \le S(1)} |X_s| < \epsilon, \sup_{s \le S(1)} |Z_s| \ge a) \\ &\qquad = \int_0^\infty P^0\big(S(1) \in dt\big) P^0(\sup_{s \le t} |X_s| < \epsilon) P^0(\sup_{s \le t} |Z_s| \ge a) \\ &\qquad \le c_9 \epsilon^{-4} \exp(-c_4/\epsilon) \exp(-a^2/c_{10}\epsilon), \end{aligned} \tag{5.12}$$

and if $U = \inf\{t : |Z_t| > L/4\}$, then similarly,

$$P^{(0,0,0)}(\sup_{s \le U} |X_s| < \epsilon) \le c_9 \epsilon^{-4} \exp(-c_4 L/4\epsilon). \tag{5.13}$$

To bound the probability of exiting A from the top, note that if (X_t, Y_t, Z_t) exits A from the top, either it does so before time $S_0 \wedge S_2$, or it does so at some time between S_i and $S_{i+1}, i = 2, \dots, n-1$. If it does so between S_i and S_{i+1}, then either

(a) $\sup_{S(j) \le s < S(j+1)} |Z_s - Z_{S(j)}| \le a_j, j = 1, \dots, i-1$ and $|Z_{\tau(A)} - Z_{S(i)}| > L/4$, or else

(b) for some $j = 1, \dots, i-1$ we have $\sup_{S(j) \le s < S(j+1)} |Z_s - Z_{S(j)}| > a_j$.

So P^p–a.s.,

$$\{Z_{\tau(A)} = 0\} \subseteq C_1 \cup \bigcup_{i=2}^{n-1} (C_i \cup D_i), \tag{5.14}$$

where $C_1 = \{\sup_{s\le S(2)} |X_s - X_0| < \epsilon_2,\ \sup_{s\le S(2)} |Z_s - Z_0| \ge L/4\}$,

$$C_i = \{\sup_{S(j)\le s\le S(j+1)} |X_s - X_{S(j)}| \le \epsilon_{j+1}, j = 1, 2, \ldots, i-1, \sup_{S(i)\le s<\tau(A)} |X_s| < \epsilon_{i+1},$$
$$\sup_{S(i)\le s<\tau(A)} |Z_s - Z_{S(i)}| > L/4\}$$

and

$$D_i = \bigcup_{j=1}^{i-1} \{\sup_{S(k)\le s<S(k+1)} |X_s| < \epsilon_{k+1}, k = 1, \ldots, j, \text{ and } \sup_{S(j)\le s<S(j+1)} |Z_s - Z_{S(j)}| \ge a_j\}.$$

Using (5.11) we get

$$P^p(C_1) \le c_9 \epsilon_2^{-4} \exp(-c_4 L^2/16\epsilon_2), \qquad c_9 > 1. \tag{5.15}$$

Using the strong Markov property repetitively at times S_2, S_3, ..., and (5.13) we get

$$\begin{aligned} P^p(C_i) &\le (c_9)^i \Big(\prod_{j=1}^{i-1} \epsilon_{j+1}^{-4}\Big)\Big(\prod_{j=1}^{i-1} \rho_{j+1}\Big) \exp(-c_4 L/16\epsilon_i + 1) \\ &\le c_9^n \epsilon_1^{-4n} \exp(-c_4 L/16\epsilon_{i+1}) \prod_{j=2}^{i} \rho_j. \end{aligned} \tag{5.16}$$

Similarly, using (5.12) we get

$$\begin{aligned} P^p(D_i) &\le \sum_{j=1}^{i-1} \Big\{ c_9^k \Big(\prod_{k=1}^{j-1} \epsilon_{k+1}^{-4}\Big)\Big(\prod_{k=1}^{j-1} \rho_{k+1}\Big) \\ &\qquad \times \big(c_9 \epsilon_{j+1}^{-4} \rho_{j+1} \exp(-a_j^2/c_{10}\epsilon_{j+1})\big)\Big\} \\ &\le \sum_{j=1}^{i-1} c_9^n \epsilon_1^{-4} \exp(-a_1^2/c_{10}\epsilon_{j+1}) \prod_{k=2}^{j+1} \rho_k. \end{aligned} \tag{5.17}$$

We now take $\epsilon_i = \epsilon_n^{(2^{n-i})}$ so that (5.6) is satisfied. If $\epsilon_n \in (0, n^{-12}/4)$ is taken sufficiently small, then $\exp(-a_1^2/4c_{10}\epsilon_{j+1})$ will be smaller than $n^{-4}(c_7/c_9)^n \exp(-c_8 n^2)$, smaller than ϵ_1^{6n}, and by (5.7), smaller than $\prod_{k=j+2}^{n} \rho_k$. Using this in (5.17) and comparing to (5.10),

$$P^p(D_i) \le n^{-3} P^p(Y_{\tau(A)} = n).$$

We can get a similar bound for $P^p(C_1)$ and $P^p(C_i)$ if L is sufficiently large. This with (5.14) gives our result. □

We now use the A_n's to construct our domain. Fix n for the moment and let $A(1) = A$,

$$A(\beta) = \bigcup_{i=1}^{n} [-(1-\beta)\epsilon_i, (1-\beta)\epsilon_i] \times [i+\beta-1, (i+\beta)\wedge n) \times [-L, 0].$$

Then $A(\beta) \uparrow \mathrm{int}(A)$ as $\beta \downarrow 0$. It is easy to see that

$$P^p(Y_{\tau(A(\beta))} = n) \to P^p(Y_{\tau(A)} = n) \text{ as } \beta \to 0.$$

Choose $\beta = \beta(n)$ small enough so $P^p(Y_{\tau(A(\beta))} = n) \geq \frac{1}{2}P^p(Y_{\tau(A)} = n)$. Then we choose $f : [-1, 1] \times [0, n) \to [-L, 0]$ to be a smooth function such that $f(x, y) = -L$ if $(x, y, 0) \in A(\beta)$, $f(x, y) = 0$ if $(x, y, 0) \notin A$. Finally let $B = B_n = \{(x, y, z) : 0 \geq z > f(x, y)\}$. Note B is the region above the graph of a smooth function and that

$$\text{(5.18)} \quad P^p(Y_{\tau(B)} = n) \geq P^p(Y_{\tau(A(\beta))} = n) \geq \frac{1}{2}P^p(Y_{\tau(A)} = n) \geq \frac{n}{2}P^p(Z_{\tau(A)} = 0) \\ \geq \frac{n}{2}P^p(Z_{\tau(B)} = 0).$$

We now let D be the interior of $[0, 2] \times [0, 2] \times [0, 2] \cup \bigcup_{m=1}^{\infty} \big[(1+2^{-m}, 1, 0) + 2^{-m}B_{2^m}\big]$, where, of course, $q + rA = \{q + (rx, ry, rz) : (x, y, z) \in A\}$. Let $q_m = (1 + 2^{-m}, 1, 0) + 2^{-m}p_{2^m}, m = 1, 2, \ldots$. Let $q_0 = (1, 1, 1)$.

Clearly D is the region above the graph of a function intersected with a rectangular solid, and since the trough $2^{-m}B_{2^m}$ has depth $2^{-m}L \to 0$, the function is continuous. It is not hard to see that every point of the boundary of D is regular for D in the sense of the Dirichlet problem (see [Do]). Let $V = (-2, 7/4) \times (-2, 7/4) \times (-2, 7/4)$ and $K = [-1, 3/2] \times [-1, 3/2] \times [-1, 3/2]$.

Proposition 5.3. *The boundary Harnack principle does not hold for D,V, K.*

Proof. Let v be the harmonic function that has boundary values 1 if $(x, y, z) \in \partial D$ and $z \geq 0$ and boundary values 0 elsewhere, and let u be the harmonic function that has boundary values 1 if $(x, y, z) \in \partial D$ and either $z \geq 0$ or $y = 2$ and boundary values 0 elsewhere. Clearly $u(q_0)$ and $v(q_0)$ are both nonzero and finite. But

$$v(q_m)/(u - v)(q_m) \leq P^{p_{2^m}}(Z_{T(A_2 m)} = 0)/P^{p_{2^m}}(Y_{T(A_2 m)} = 2^m) \to 0,$$

hence $(u/v)(q_m) = \big((u-v)/v\big)(q_m) + 1 \to +\infty$. □

Our second example is an example to show that if $\alpha \in (0, \frac{1}{2})$, then there exists a twisted Hölder domain of order α for which the boundary Harnack principle does not hold.

Fix $\alpha \in (0, \frac{1}{2})$. Let $A_\epsilon = \{(x,z) : |x| < (4\epsilon)^{1/\alpha}, |x|^\alpha - 4\epsilon < z < 0\} \cup B\big((0,-3\epsilon), \epsilon\big)$. Then A_ϵ is a set lying in $\mathbb{R}^2$. Let $B_\epsilon = \{(x,y,z) : (x,z) \in A_\epsilon, 0 \le y < 2\}$. Let $p_\epsilon = (0, 1, -3\epsilon)$.

Lemma 5.4. *$P^{p_\epsilon}(Z_{\tau(B_\epsilon)} = 0)/P^{p_\epsilon}(Y_{\tau(B_\epsilon)} = 2) \to 0$ as $\epsilon \to 0$.*

Proof. We have

$$
\begin{aligned}
(5.19)\; P^{p_\epsilon}(Y_{\tau(B_\epsilon)} = 2) &\ge \int_0^\infty P^1(S_2 \in dt) P^0(\sup_{s \le t} |X_s| \le \epsilon/\sqrt{2}) P^0(\sup_{s \le t} |Z_s - Z_0| \le \epsilon/\sqrt{2}) \\
&\ge \int_0^\infty P^0(S_1 \in dt) \exp\big(-2c_3 t/(\epsilon^2/2)\big) \\
&\ge 4c_5 \epsilon^2 \exp(-2c_4/\epsilon)
\end{aligned}
$$

as in the proof of Proposition 5.2.

Let $D_\epsilon = [-2(4\epsilon^{1/\alpha}), 2(4\epsilon)^{1/\alpha}] \times (-\infty, \infty) \times [-3\epsilon/2, 0]$. Then

$$
\begin{aligned}
(5.20) \qquad P^{p_\epsilon}(Z_{\tau(B_\epsilon)} = 0) &\le \sup_{z = -\epsilon, (x,y,z) \in B_\epsilon} P^{(x,1,z)}(Z_{\tau(B_\epsilon)} = 0) \\
&\le P^{(0,0,-\epsilon)}(Z_{\tau(D_\epsilon)} = -3\epsilon/2 \text{ or } 0).
\end{aligned}
$$

But by Lemma 2.2 and proof, the right side of (5.20) is less than or equal to

$$\exp(-c_{11}\epsilon^{1-1/\alpha})$$

if ϵ is small enough. Since $\alpha < \frac{1}{2}$, this is much smaller than the right hand side of (5.19) if ϵ is small. □

Now let D be the interior of

$$[0,2] \times [0,2] \times [0,2] \cup \bigcup_{m=1}^\infty \big[(1 + 2^{-m}, 0, 0) + B_{9^{-m}}\big],$$

let $q_m = (1 + 2^{-m}, 0, 0) + p_{9^{-m}}$. Let $q_0 = (1,1,1)$ Let V and K be as in Proposition 5.3.

Proposition 5.5. *D is a twisted Hölder domain of order α and the boundary Harnack principle does not hold for D, V, and K.*

Proof. That the boundary Harnack principle does not hold for D, V, K follows by the same argument as in the proof of Proposition 5.3, using Lemma 5.4 in place of Lemma 5.2.

So it remains to show D is a twisted Hölder domain of order α. We leave the proof of this to the reader. See also [BB2, Section 4] for a related construction. □

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