THREE–DIMENSIONAL BROWNIAN PATH REFLECTED ON BROWNIAN PATH IS A FREE BROWNIAN PATH

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Abstract. Three-dimensional Brownian path reflected on Brownian path is a free Brownian path.

Research partially supported by NSF grant DMS 91-00244 .

The purpose of this short note is to present a simple but significant — in the author's opinion — corollary of some very recent results on reflected Brownian motion. The result announced in the title is related to the following model introduced in physics literature. Amit, Parisi and Peliti defined in [1] a "True Self-Avoiding Random Walk" (TSARW) on the d-dimensional lattice $\mathbb{Z}^d$ parameterized by a certain coefficient g and discussed the asymptotics when $g \to \infty$. For $g = \infty$, one may describe TSARW as a process which goes from a site to one of its nearest neighbors but cannot go to a site that has been previously visited. At each stage the next site is chosen uniformly from the sites which are not "forbidden." (The process cannot make a step in such a way that it would have to self-intersect at a future time, i.e., the process is prohibited from "trapping" itself. This additional restriction does not arise in [1] as only the case when $g < \infty$ is considered.) In view of our description of TSARW it seems that the name "Self-Reflecting Random Walk" would be more accurate. It is argued in [1] that for $d > 2$, with suitable space and time rescaling, TSARW converges to a free Brownian motion in the same manner as the standard symmetric nearest neighbor random walk converges to free Brownian motion. The non-rigorous argument is supported by computer simulations. At the intuitive level, the result suggests that the "self-reflecting Brownian motion" and free Brownian motion have the same distributions. Theorem 1 below is a rigorous statement which strongly supports this intuitive idea.

Let $D \subset \mathbb{R}^3$ be the complement of the trace $X([0,\infty))$ of 3-dimensional Brownian motion X. The set D is a.s. open and connected. For a domain A, let σ_A be its surface area measure (provided it is well defined). Let $B(x,r) = \{y \in \mathbb{R}^3 : |x-y| < r\}$.

Lemma 1. *With probability 1 there exists an increasing sequence of domains D_n with smooth boundaries whose union is equal to D and such that for each $r > 0$,*

$$\lim_{n\to\infty} \sigma_{D_n}(D_n \cap B(0,r)) = 0.$$

For each fixed point $x_0 \in D$ there is m_0 such that $x_0 \in D_n$ for all $n > m_0$. For each $x_0 \in D$ and $n > m_0$ there exists a reflected Brownian motion Y_n in D_n with the normal reflection on the boundary ∂D_n and starting from x_0 (the reflected Brownian motion in smooth domains is well-understood; see, e.g., [2]) In view of our Lemma 1, Theorem 4.2 of [2] shows that the processes Y_n converge weakly to a process Y as $n \to \infty$. We may call Y a reflected Brownian motion in D. Even more is true—formulae (4.2) and (4.3) of [2] combined with Lemma 1 prove that Y is a standard (free) Brownian motion. Hence we have the following result.

Theorem 1. *With probability 1 three-dimensional Brownian motion Y reflected on Brownian motion X is a free Brownian motion.*

It should be pointed out that the result is not completely obvious. Three-dimensional Brownian paths X have positive capacity and are hit by Brownian motion Y a.s. (Another manifestation of this phenomenon is the existence of double points on three-dimensional Brownian paths.) It is somewhat surprising that the hitting of the path of X by the path of Y does not introduce any drift in Y.

The proof of Lemma 1 below uses only the fact that the 2-dimensional measure of the path of X is equal to 0. Hence, Theorem 1 may generalized as follows: 3-dimensional Brownian motion reflected on a set whose 2-dimensional measure is 0 is a free Brownian motion.

Proof of Lemma 1. The 2-dimensional measure of the path of three-dimensional Brownian motion X is equal to 0 (see [4]). Hence, for each n, the set $X([0,\infty))$ may be covered by open balls $B(z_k, r_k)$, $k \geq 1$, such that

$$\sum_{k\geq 1} r_k^2 < 1/2n. \tag{1}$$

For each m, the set $X([0,\infty)) \cap \overline{B(0,2^m) \setminus B(0,2^{m-1})}$ is compact so it has a finite subcover B_j^m, $1 \leq j \leq s(m)$. Moreover, we can assume that B_j^m are disjoint from $B^c(0,2^{m+1}) \cup B(0,2^{m-2})$. Let D_n^* be the unbounded component of the complement of $\overline{\bigcup_{m,j} B_j^m}$. Note that the surface area of D_n^* is bounded by the sum of surface areas of the balls $B(z_k, r_k)$ and, therefore, it is less than $c/2n$ in view of (1). Since each ball $B(0,r)$ intersects only a finite number of balls B_j^m, it is easy to modify the domains D_n^* so that the resulting domains D_n have smooth boundaries, have surface area bounded by c/n, and increase to D as $n \to \infty$. $\square$

The proof of Theorem 1 requires a localization argument which we have omitted. It is discussed in [3, Section 4.4]. See also [3, Section 4.5] for a rigorous treatment of Brownian motion reflected on the sets of measure zero.

This article is to large extent a result of discussions with Z.-Q. Chen, G. Lawler, S.J. Taylor and R.J. Williams. I am grateful for their most useful advice.

References

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