On the integral Chow rings of various moduli stacks of curves

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Abstract

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The contents of this thesis are focused on the intersection theory of the stack of marked (stable or smooth) elliptic curves. We first recount some results on equivariant intersection theory, then give an exposition of some known results for the $n = 1, 2$ cases. Along the way, we provide some original proofs of these results as well as compute the higher Chow groups with ℓ-adic coefficients for many of these stacks. The primary results, the integral Chow rings of $\mathcal{M}_{1,n}$ for $n = 3, \ldots, 10$, are contained in the last chapter.

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Contents

1 Introduction **9**

2 Equivariant intersection theory **13**
2.1 Equivariant intersection theory 13
2.2 Some Chow rings of quotient stacks 17
2.3 Higher Chow groups 19
2.3.1 Introduction: the patching problem 19
2.3.2 Solving the patching problem: higher Chow groups . . 20
2.3.3 Some first higher Chow groups 22
2.3.4 Patching techniques 22

3 The $\mathcal{M}_{1,1}$ and $\mathcal{M}_{1,2}$ cases **25**
3.1 The Weierstrass form 25
3.2 The Chow rings of $\widetilde{\mathcal{M}}_{1,n}$, $\overline{\mathcal{M}}_{1,n}$, and $\mathcal{M}_{1,n}$ for $n = 1, 2$ 27
3.2.1 The $n = 1$ case 27
3.2.2 The $n = 2$ case 28

4 The $\mathcal{M}_{1,n}$ case for $n = 3, \ldots, 10$ **35**
4.1 The integral Chow ring of $\mathcal{M}_{1,3}$ 35
4.2 The case $4 \leq n \leq 10$ 40

A Structure of $\mathcal{M}_{1,1}$ **45**
A.1 Defining the morphism 45
A.2 Verifying it is an isomorphism 46

B Automorphisms of elliptic curves **51**

Chapter 1

Introduction

Intersection theory is one of the oldest areas of study in algebraic geometry. The goal of intersection theory, loosely stated, is to understand the geometry of a space by studying how closed subspaces intersect with each other. The first great success in this field is the classically known *Bezout's Theorem*, which states that if C_1 and C_2 are general curves in $\mathbb{P}^2$ of degrees d_1 and d_2, then

$$|C_1 \cap C_2| = d_1 d_2.$$

This formula is very suggestive. On the left hand side lies something geometric, the intersection of two closed subspaces. On the right hand side lies something algebraic, multiplication. This gives the following idea: Put an algebraic structure on the set of closed subspaces so that intersection corresponds to multiplication.

In the early 20^{th} century, this structure was discovered for smooth varieties and is called the *integral Chow ring* of X, denoted $\mathrm{CH}(X)$. Having a presentation of the Chow ring of a space in terms of geometrically significant generators gives one a great understanding of the geometry of the space. For example, the Chow ring of $\mathbb{P}^2$ is $\mathrm{CH}(\mathbb{P}^2) = \mathbb{Z}[x]/(x^3)$, where $x = [L]$ is the class of a line. The statement of Bezout's Theorem follows trivially from this, as we have $[C_1 \cap C_2] = [C_1][C_2] = d_1[L]d_2[L] = d_1 d_2 [L]^2 = d_1 d_2 [\mathrm{pt}]$.

However, many important spaces are not varieties, but rather schemes, or even worse (or better, depending on your perspective), stacks. One of the most famous stacks, and the central object in this thesis, is the moduli stack of stable pointed curves. The moduli stack of stable pointed curves parametrizes families of at worst nodal curves with marked points, where the curves have only finitely many automorphisms and the marked points are required to be in the smooth locus.

Intuitively speaking, the difference between a stack and a scheme is analogous to the difference between a scheme and a variety. One of the great strengths of scheme theory is its ability to track nilpotent elements. Take, for example, the line $y = a$ and the parabola $y = x^2$. In the real plane $\mathbb{R}^2$, these intersect in exactly two points for $a > 0$. At $a = 0$, there is one point of intersection. However, we intuitively understand this to be a double point (in Precalculus or Algebra 2 one might say that the equation $x^2 = 0$ has a double root at $x = 0$). Scheme theory allows us to formalize this by putting the scheme structure of a double point on this intersection: the intersection is the scheme $\operatorname{Spec} \mathbb{R}[x]/(x^2)$, since we are solving the equation $x^2 = 0$. The spectrum of this ring is one single point, but it carries the additional information of the nilpotent element x, information which is not picked up by varieties.

In a similar fashion, stacks are the result of attempting to package additional information into schemes, information which arises naturally in the context of moduli problems. This information is the data of *automorphisms*. As an example, let's consider $\overline{\mathcal{M}}_{1,1}$, the stack of one-pointed stable elliptic curves.

Example 1.0.1. Any one-pointed, possibly nodal elliptic curve (E, p) comes with a double cover of $\mathbb{P}^1$, from the linear system $|2p|$. Riemann-Hurwitz shows that this branches at exactly four points, which via an automorphism of $\mathbb{P}^1$ we may assume to be $0, 1, \infty, \lambda$. We then define the *j-invariant* of E to be

$$j(E) = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}.$$

This defines a map $j : \overline{\mathcal{M}}_{1,1} \to \mathbb{P}^1$ by $E \mapsto j(E)$ which expresses $\mathbb{P}^1$ as the *coarse moduli space* of $\overline{\mathcal{M}}_{1,1}$ – essentially its best approximation as a scheme. But what information is $\mathbb{P}^1$ missing? As hinted above, it's missing the information of the automorphisms of elliptic curves.

Every point in $\mathbb{P}^1$ "looks the same". More precisely, there is an automorphism of $\mathbb{P}^1$ interchanging any two points. However, not all elliptic curves are interchangeable. For instance, the elliptic curves $C_{(1,0)}$ and $C_{(0,1)}$, given by the equations $y^2 = x^3 + x$ and $y^2 = x^3 + 1$ have automorphism groups μ_4 and μ_6, so there is no automorphism of $\overline{\mathcal{M}}_{1,1}$ interchanging the points corresponding to these curves.

Moreover, every elliptic curve comes equipped with a hyperelliptic involution: when written in Weierstrass form, it is the map $y \mapsto -y$ – more abstractly it comes from exchanging the points in the fibers of the double cover of $\mathbb{P}^1$. This is to say that the automorphism group of every elliptic curve contains at least a μ_2. But when we look at $\mathbb{P}^1$, there is no such information to be found. If we knew nothing about elliptic curves and were handed the scheme $\mathbb{P}^1$, we would have no way of knowing what the automorphism groups are. Therefore we should seek to create a new object which bundles the information of the automorphism groups.

A stack is such an object. So in the same way that schemes enrich varieties by adding the information of nilpotents, stacks enrich schemes by adding the information of automorphisms.

Frequently, when there is a stack parametrizing some moduli problem, important classes of objects are parametrized by closed or open substacks, and so understanding the geometry of the stack sheds light on the corresponding moduli problem. Therefore having an intersection theory for stacks is a desirable goal. Thankfully such a theory exists for what are known as quotient stacks. We will give an exposition of this theory, along with computations of the integral Chow rings and higher Chow groups of various moduli stacks of one and two-pointed elliptic curves, before describing in the complete detail the intersection theory of $\mathcal{M}_{1,n}$ for $3 \leq n \leq 10$, as follows:

Theorem 1.0.2. *Let λ_1 be the first Chern class of the Hodge bundle. Then over a field of characteristic not equal to 2 or 3:*

(a) $\mathrm{CH}(\mathcal{M}_{1,3}) = \mathbb{Z}[\lambda_1]/(12\lambda_1, 6\lambda_1^2)$

(b) $\mathrm{CH}(\mathcal{M}_{1,4}) = \mathbb{Z}[\lambda_1]/(12\lambda_1, 2\lambda_1^2)$

(c) $\mathrm{CH}(\mathcal{M}_{1,n}) = \mathbb{Z}[\lambda_1]/(12\lambda_1, \lambda_1^2)$*, for $n = 5, \ldots, 10$.*

Chapter 2

Equivariant intersection theory

In this chapter we will develop the basics of equivariant intersection theory and close with a discussion of higher Chow groups. Along the way, we compute many examples which will be helpful in our later computations.

2.1 Equivariant intersection theory

In [Mum83] Mumford investigated the Chow ring of $\overline{M}_g$, the coarse moduli space of $\overline{\mathcal{M}}_g$. This space is singular, but is locally the quotient of a smooth variety by a finite group, and globally the quotient of a Cohen-Macauley variety by a finite group. Using this, he defined in characteristic 0 an intersection product with rational coefficients on $\mathrm{CH}(\overline{M}_g)$ and explicitly computed it in the case of $\overline{M}_2$.

Gillet [Gil84] and Vistoli [Vis89] extended this notion to algebraic stacks of finite type over a field of any characteristic in an analogous way to classical Chow groups, using integral substacks. However, there are not enough such substacks to define an intersection product with integer coefficients [EG98, Section 3.5]. Using rational coefficients, most classical intersection theory transfers directly over. Moreover, Vistoli showed [Vis89, Proposition 6.1] that the rational Chow ring of an algebraic stack agrees with the rational Chow ring of its coarse moduli space, when one exists.

Therefore research on the Chow ring of $\overline{\mathcal{M}}_{g,n}$ was limited to finding the rational Chow ring of the coarse moduli space $\overline{M}_{g,n}$. Even now, most Chow rings of $\overline{\mathcal{M}}_{g,n}$ are still only known rationally. The issue with this is that having $\mathbb{Q}$ coefficients eliminates all torsion, and so ignores a rich part of the structure of the Chow ring. For instance, it is shown by Belorousski in his thesis [Bel98] that $\mathrm{CH}(M_{1,n})_{\mathbb{Q}} = \mathbb{Q}$ for $n = 1, \ldots, 10$, whereas Mumford showed in [Mum65] that the Picard group of $\mathcal{M}_{1,1}$ is isomorphic to $\mathbb{Z}/12$. The following lemma, due to Edidin and Graham [EG98] and Totaro [Tot99], allows us to extend the definition of *integral* Chow rings to include quotient stacks, a class of spaces which includes $\overline{\mathcal{M}}_{g,n}$.

Lemma 2.1.1. *Let X be an n-dimensional, quasi-separated algebraic space of finite type over a field $\Bbbk$. Let G be a g-dimensional linear algebraic group over $\Bbbk$ acting on X. Let V be a representation of G and $S \subseteq V$ a codimension s G-invariant closed subset such that G acts freely on the complement $V - S$ and the quotient $[(V - S)/G]$ is a scheme (such an S exists by [EG98, Lemma 8]). Then the Chow groups*

$$\mathrm{CH}^k\left(\left[\frac{X \times (V - S)}{G}\right]\right)$$

are indpendent of V and S for $k < s$.

Proof. We first show invariance of S for a fixed V. So let S' be another such set of codimension greater than or equal to s. Without loss of generality we may assume that $S' \supseteq S$ (else replace S' with $S \cup S'$). Indeed, if the Chow groups upon excising S and $S \cup S'$ are the same, then by symmetry so are the Chow groups upon excising S' and $S \cup S'$. This gives the isomorphism between the excisions of S and S', and so we have reduced to the case $S' \supseteq S$. So suppose $S' \supseteq S$.

We use the excision exact sequence

$$\mathrm{CH}(C) \to \mathrm{CH}(B) \to \mathrm{CH}(B - C) \to 0$$

for $C = (S' - S)/G$ and $B = (V - S)/G$. Since the Chow groups of $(S' - S)/G$ vanish in codimension less than s, we see that we have an isomorphism of $\mathrm{CH}((V - S)/G)$ and $\mathrm{CH}(((V - S) - (S' - S))/G) = \mathrm{CH}((V - S')/G)$ in codimension less than s, as required.

We now show invariance of V. So let (V, S_V) and (W, S_W) be two such representations and closed subsets. Consider

$$\frac{(V - S_V) \times W}{G}$$

which is a vector bundle over $(V - S_V)/G$ (and hence they have isomorphic Chow groups) and

$$\frac{V \times (W - S_W)}{G},$$

which is a vector bundle over $(W - S_W)/G$ (and hence they have isomorphic Chow groups).

Note that $(V - S_V) \times W = V \times W - (S_V \times W)$ and $V \times (W - S_W) = V \times W - (V \times S_W)$. Invariance of S implies that these have the same Chow groups in codimension less than s, which completes the proof. □

We refer to a pair (V, U) with $U = V - S$ as a *good representation in codimension less than s*. This leads us to the notion of *G-equivariant Chow group*.

Definition 2.1.2. Let X be an algebraic space of finite type over a field $\Bbbk$, and G a linear algebraic group acting on X. We define the *G-equivariant Chow group of X*, written $\mathrm{CH}_G(X)$, to be the graded abelian group which in degree i is $\mathrm{CH}^i((X \times U)/G)$ for a good representation U.

Remark 2.1.3. Note that if X is smooth, $\mathrm{CH}_G(X)$ inherits a ring structure. To see this, let $\alpha \in \mathrm{CH}^a_G(X)$ and $\beta \in \mathrm{CH}^b_G(X)$. Pick a (V, U) that is a good representation in codimension less than $a + b + 1$. Then α and β correspond to cycles in $\mathrm{CH}^a((X \times U)/G)$ and $\mathrm{CH}^b((X \times U)/G)$, and hence

$$\alpha \cdot \beta \in \mathrm{CH}^{a+b}((X \times U)/G) = \mathrm{CH}^{a+b}_G(X).$$

Theorem 2.1.4. *Suppose G and H are linear algebraic groups acting on algebraic spaces X and Y of finite type over a field $\Bbbk$ with $[X/G] \cong [Y/H]$. Then $\mathrm{CH}_G(X) \cong \mathrm{CH}_H(Y)$.*

Proof. Let U_1 and U_2 be good representations of G and H, respectively. Write $X_G = (X \times U_1)/G$ and $Y_G = (Y \times U_2)/H$, and consider $Z = X_G \times_{[X/G]} Y_H$. Then $Z \to [X/G]$ is a U_2-bundle and $Z \to [Y/H]$ is a U_1-bundle, and so $\mathrm{CH}^i_G(X) = \mathrm{CH}^i(X_G) \cong \mathrm{CH}^i(Z) \cong \mathrm{CH}^i(Y_H) = \mathrm{CH}^i_H(Y)$ for appropriate i. Hence $\mathrm{CH}_G(X) \cong \mathrm{CH}_H(Y)$. □

This theorem tells us, in particular, that we may make a well-defined notion of the integral Chow ring of a quotient stack: if $\mathcal{F}$ is a quotient stack and

$[X/G]$ and $[Y/H]$ are two different presentations of $\mathcal{F}$, then we must have $\mathrm{CH}_G(X) \cong \mathrm{CH}_H(Y)$.

Definition 2.1.5. Let $\mathcal{F}$ be a global quotient stack. We define the *integral Chow ring* of $\mathcal{F}$ to be $\mathrm{CH}(\mathcal{F}) = \mathrm{CH}_G(X)$ for any X and G with $\mathcal{F} \cong [X/G]$.

Proposition 2.1.6 ([EG98, Proposition 15])**.** *If X is smooth and $[X/G]$ is a locally separated Deligne-Mumford stack, then the intersection products on $\mathrm{CH}([X/G])_{\mathbb{Q}}$ defined by Vistoli and Gillet are the same as the equivariant product on $\mathrm{CH}_G(X)_{\mathbb{Q}}$.*

Observe that our definition of integral Chow rings of quotient stacks allows for elements of arbitrarily high degree (see Section 2.2). That is, we may have elements in codimension greater than the dimension of the stack. But the integral substack definition of Chow ring allows only for elements in degree up to the dimension of the stack, as in classical intersection theory. Therefore the proposition yields the following corollary.

Corollary 2.1.7. *If X is smooth and $[X/G]$ is a locally separated Deligne-Mumford stack of dimension n, then the integral Chow ring of $[X/G]$ is torsion in degree greater than n.*

The next two lemmas tell us about the kernel of the excision sequence, and apply in the non-equivariant case as well.

Lemma 2.1.8. *Let $p : Z \hookrightarrow X$ be a closed immersion. If $p^* : \mathrm{CH}(X) \to \mathrm{CH}(Z)$ is surjective, then the kernel of $\mathrm{CH}(X) \to \mathrm{CH}(X \setminus Z)$ is the ideal generated by $[Z]$.*

Proof. Let $j : U \to X$ be the inclusion. Clearly we must have $([Z]) \subseteq \ker j^*$. Let $\beta \in \ker j^* = \operatorname{im} p_*$. Then there exists $\alpha \in \mathrm{CH}(Z)$ such that $\beta = p_*(\alpha)$. Since p^* is surjective, there exists $\gamma \in \mathrm{CH}(X)$ with $p^*(\gamma) = \alpha$. Then $\beta = p_*(\alpha) = p_*p^*(\gamma) = \gamma \cdot p_*(1) = \gamma[Z]$, by the push-pull formula. □

Corollary 2.1.9 ([MRV06, Lemma 2.2])**.** *Let G be an affine linear group acting on a smooth scheme X, $\pi : E \to X$ an equivariant vector bundle of rank r. Call $E_0 \subseteq E$ the complement of the zero section of E. Then the pullback homomorphism $\mathrm{CH}^*_G(X) \to \mathrm{CH}^*_G(E_0)$ is surjective, and its kernel is generated by the top Chern class $c_r(E) \in \mathrm{CH}^r_G(X)$.*

Proof. Let $j : E_0 \to E$ be the inclusion and $s : X \to E$ the zero section. Since $\pi^* : \mathrm{CH}(X) \to \mathrm{CH}(E)$ is an isomorphism with inverse s^*, excision gives that $\mathrm{CH}(E_0)$ is a quotient of $\mathrm{CH}(X)$. As s^* is an isomorphism, and in particular surjective, by Lemma 2.1.8 we have that the kernel of $\mathrm{CH}(E) \to \mathrm{CH}(E_0)$ is generated by $[X]$. Pulling this back to $\mathrm{CH}(X)$ via s^* gives that $\mathrm{CH}(E_0) = \mathrm{CH}(X)/(s^*s_*(1)) = \mathrm{CH}(X)/(c_r(E))$, by the self-intersection formula [LMS75, Theorem 1] and the fact that the normal bundle to the zero section of a vector bundle is the vector bundle itself. □

2.2 Some Chow rings of quotient stacks

The following four Chow rings are excellent first examples of equivariant Chow rings, and will also be very useful to us later.

Proposition 2.2.1. *The Chow ring of* $B\mathbb{G}_m$ *is* $\mathrm{CH}(B\mathbb{G}_m) = \mathbb{Z}[x]$.

Proof. Note that $\mathbb{A}^n$ is a representation of $\mathbb{G}_m$, free outside of $\{0\}$. Then $\mathrm{CH}^k(B\mathbb{G}_m) = \mathrm{CH}^k_{\mathbb{G}_m}(\operatorname{Spec} \Bbbk) = \mathrm{CH}^k((\operatorname{Spec} \Bbbk \times (\mathbb{A}^n \setminus 0))/\mathbb{G}_m) = \mathrm{CH}^k(\mathbb{P}^{n-1})$ for $k < n$.

So define a homomorphism

$$\begin{aligned} \mathbb{Z}[x] &\to \mathrm{CH}(B\mathbb{G}_m) \\ x &\mapsto [H] \end{aligned}$$

where $[H]$ is the class of a hyperplane in any projective space. This is surjective, as any $r \in \mathrm{CH}^k(B\mathbb{G}_m)$ is equal to $d[H]^k \in \mathrm{CH}^k(\mathbb{P}^k)$, and injective, for if $f(x) \mapsto 0$ and $\deg f = e$, then we must have $f(x) \mapsto 0$ in the map $\mathbb{Z}[x] \to \mathrm{CH}(\mathbb{P}^e) = \mathbb{Z}[x]/(x^{e+1})$, implying $f(x) = 0$. □

Note that this also proves that $\mathrm{CH}([\mathbb{A}^n/\mathbb{G}_m]) \cong \mathbb{Z}[x]$.

Definition 2.2.2. Let $\mathbb{G}_m$ act on $\mathbb{A}^{n+1}$ with weights $a_0, \ldots, a_n$. We denote this by $\mathbb{A}^{n+1}_{a_0,\ldots,a_n}$. We call the stack $[(\mathbb{A}^{n+1}_{a_0,\ldots,a_n} \setminus 0)/\mathbb{G}_m]$ *weighted projective space* and denote it $\mathcal{P}(a_0, \ldots, a_n)$.

Proposition 2.2.3. *The Chow ring of weighted projective space is*

$$\mathrm{CH}(\mathcal{P}(a_0, \ldots, a_n)) \cong \mathbb{Z}[x]/(a_0 \cdots a_n x^{n+1}).$$

Proof. Note that $\pi : [\mathbb{A}^{n+1}/\mathbb{G}_m] \to B\mathbb{G}_m$ is a vector bundle with zero section $s : B\mathbb{G}_m \to [\mathbb{A}^{n+1}/\mathbb{G}_m]$, and so by Corollary 2.1.9 we have that

$$\mathrm{CH}([(\mathbb{A}^{n+1} \setminus 0)/\mathbb{G}_m]) \cong \mathrm{CH}(B\mathbb{G}_m)/c_n([\mathbb{A}^{n+1}/\mathbb{G}_m]).$$

Since $\mathbb{A}^{n+1}$ splits as $\mathbb{A}^1 \oplus \cdots \oplus \mathbb{A}^1$, we have

$$c_n\left([\mathbb{A}^{n+1}/\mathbb{G}_m]\right) = c_n^{\mathbb{G}_m}(\mathbb{A}^{n+1}) = c_1^{\mathbb{G}_m}(\mathbb{A}^1)\cdots c_1^{\mathbb{G}_m}(\mathbb{A}^1),$$

so it suffices to compute $c_1^{\mathbb{G}_m}(\mathbb{A}^1)$ where $\mathbb{G}_m$ acts with weight a. We claim that $c_1^{\mathbb{G}_m}(\mathbb{A}^1) = ax \in \mathrm{CH}(B\mathbb{G}_m) \cong \mathbb{Z}[x]$, where x is the class of a hyperplane in $\mathbb{P}^k$ for any $k \geq 1$.

Following [EG98], consider

$$\left[\frac{\mathbb{A}^1_a \times (\mathbb{A}^2 \setminus 0)}{\mathbb{G}_m}\right]$$

where the subscript a indicates that this copy of $\mathbb{A}^1$ is being acted on with weight a. This is a line bundle over

$$\left[\frac{\mathrm{Spec}\,\Bbbk \times (\mathbb{A}^2 \setminus 0)}{\mathbb{G}_m}\right] \cong \mathbb{P}^1,$$

and so corresponds to $\mathcal{O}(d)$ for some d. We claim that $d = a$. To see this, note that for every k, $\mathbb{G}_m$ acts on $\mathcal{O}(k)$ with weight k. Thus d is determined by the weight of the $\mathbb{G}_m$ action, and so $d = a$.

Therefore $c_1^{\mathbb{G}_m}(\mathbb{A}^1) := c_1\mathcal{O}(a) = ax$. This completes the computation of the top Chern class and of the proposition. □

Note that setting $a_0 = \cdots = a_n = 1$ recovers the Chow ring of projective space,

$$\mathrm{CH}(\mathbb{P}^n) = \mathbb{Z}[x]/(x^{n+1}).$$

Proposition 2.2.4. *The Chow ring of $B\mu_n$ is $\mathrm{CH}(B\mu_n) = \mathbb{Z}[x]/(nx)$.*

Proof. This follows from the previous proposition, as $B\mu_n \cong \mathcal{P}(n)$. □

Proposition 2.2.5 ([EG98, Section 3.3])**.** *Let $\mathbb{P}^n_{a_0,\ldots,a_n}$ denote $\mathbb{P}^n$ with a weighted*

$\mathbb{G}_m$ action, i.e. $\mathbb{P}(\mathbb{A}^{n+1}_{a_0,\dots,a_n})$. Let $p(x,y) = \sum_{i=0}^{n+1} e_i(a_0 x, \dots, a_n x) y^{n+1-i}$ Then

$$\mathrm{CH}([\mathbb{P}^n_{a_0,\dots,a_n}/\mathbb{G}_m]) \cong \frac{\mathbb{Z}[x,y]}{p(x,y)}.$$

Proof. This follows from the projective bundle formula, but we may also see it equivariantly. Let $U = \mathbb{A}^l \setminus 0$ be a good representation for $\mathbb{G}_m$, and set $X = \mathbb{P}(\mathbb{A}^{n+1}_{a_0,\dots,a_n})$. Then we can compute the Chow ring in degree less than l as the Chow ring of $[(X \times U)/\mathbb{G}_m]$. This is a projective bundle over $[U/\mathbb{G}_m] = \mathbb{P}^{l-1}$. In particular, since $\mathbb{G}_m$ acts with weights $a_0, \dots, a_n$, we know that it is $\mathbb{P}(\mathcal{O}(a_0), \dots, \mathcal{O}(a_n))$. The projective bundle formula for schemes gives us

$$\mathrm{CH}([(X \times U)/\mathbb{G}_m]) \cong \mathbb{Z}[x,y]/(p(x,y), x^{n+1}).$$

Allowing the dimension of U to go to infinity gives us our result. □

Note 2.2.6. Consider the special case of $n = 1$. Then

$$\mathrm{CH}([\mathbb{P}^1_{a,b}/\mathbb{G}_m]) \cong \frac{\mathbb{Z}[x,y]}{(y^2 + (a+b)xy + abx^2)} = \frac{\mathbb{Z}[x,y]}{((y+ax)(y+bx))} \cong \frac{\mathbb{Z}[x,y]}{y(y+(b-a)x)}.$$

Observe that the inclusion $i_\infty : B\mathbb{G}_m \to [\mathbb{P}^1_{a,b}/\mathbb{G}_m]$ of the point $[1:0]$ has pushforward $i_{\infty,*}(f(x)) = f(x)(ax+y)$ and i_0, the inclusion of the point $[0:1]$, has pushforward $i_{0,*}(f(x)) = f(x)(bx+y)$.

2.3 Higher Chow groups

2.3.1 Introduction: the patching problem

As just shown above, the excision exact sequence

$$\mathrm{CH}(Z) \to \mathrm{CH}(X) \to \mathrm{CH}(X \setminus Z) \to 0,$$

is an extremely useful tool, allowing us to compute the Chow ring of $X \setminus Z$ given the Chow rings of X and Z. However, we frequently find ourselves in the opposite scenario. When dealing with complicated objects which decompose into simpler ones, we may be able to compute the Chow rings of a closed locus Z and its complement $X \setminus Z$, and find ourselves needing to patch these Chow rings together to get the Chow ring of X.

The issue is that the excision exact sequence says nothing of the kernel of the map $\mathrm{CH}(Z) \to \mathrm{CH}(X)$. Without some knowledge of this, there's no clear way to go about deducing the ring structure of $\mathrm{CH}(X)$. As an example, here are two misguided attempts at patching.

Example 2.3.1. Suppose we only know the Chow rings of $\operatorname{Spec} \Bbbk$ and $\mathbb{A}^1 \setminus 0$ and we wish to find the Chow ring of $\mathbb{A}^1$. We may try to patch these together using the excision exact sequence

$$\begin{array}{ccccccc} \mathrm{CH}(\operatorname{Spec} \Bbbk) & \longrightarrow & \mathrm{CH}(\mathbb{A}^1) & \longrightarrow & \mathrm{CH}(\mathbb{A}^1 \setminus 0) & \longrightarrow & 0 \\ & & & & & & \\ \mathbb{Z} & \longrightarrow & \mathrm{CH}(\mathbb{A}^1) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

but all we can conclude from this is that $\mathrm{CH}(\mathbb{A}^1) \cong \mathbb{Z} \oplus \mathbb{Z}/n$ for some n (possibly 0). The knowledge that the left map is trivial would allow us to correctly conclude that $\mathrm{CH}(\mathbb{A}^1) = \mathbb{Z}$.

Example 2.3.2. We may also try to find the Chow ring of $\mathbb{P}^1$ from the Chow rings of $\mathbb{A}^1$ and $\operatorname{Spec} \Bbbk$. Here excision gives

$$\begin{array}{ccccccc} \mathrm{CH}(\operatorname{Spec} \Bbbk) & \longrightarrow & \mathrm{CH}(\mathbb{P}^1) & \longrightarrow & \mathrm{CH}(\mathbb{A}^1) & \longrightarrow & 0 \\ & & & & & & \\ \mathbb{Z} & \longrightarrow & \mathrm{CH}(\mathbb{P}^1) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

which alone is not enough for us to find the Chow ring of $\mathbb{P}^1$. Knowing that the left map is injective, however, would allow us to correctly find that $\mathrm{CH}(\mathbb{P}^1) \cong \mathbb{Z}[x]/(x^2)$.

2.3.2 Solving the patching problem: higher Chow groups

Thankfully, there is an effective strategy for solving this problem. In [Blo86], Bloch introduced *higher Chow groups*, extending the excision exact sequence into a long exact sequence. In [EG98, Section 2.7], Edidin and Graham extended this definition to quotient stacks using the same equivariant methods as usual Chow groups.

Proposition 2.3.3 ([EG98, Proposition 5])**.** *Let X be equidimensional and quasiprojective with a linearized G-action, and let $Y \subset X$ be an invariant equidimensional subscheme. There is a long exact sequence of higher Chow groups*

$$\begin{aligned} \cdots \to \mathrm{CH}_G(Y,k) \to \quad & \mathrm{CH}_G(X,k) \to \mathrm{CH}_G(X \setminus Y, k) \to \\ \cdots \to \mathrm{CH}_G(Y) \to \quad & \mathrm{CH}_G(X) \to \mathrm{CH}_G(X \setminus Y) \to 0. \end{aligned}$$

Higher Chow groups enjoy similar properties to Chow groups, such as functoriality for proper morphisms and flat morphisms, invariance under taking vector bundles, and the projective bundle formula [Blo86]. But we still have another issue: higher Chow groups are quite difficult to compute. In [Lar21, Section 9], Larson worked around this issue by computing *higher Chow groups with ℓ-adic coefficients.*

Definition 2.3.4. Let X be a scheme over $\Bbbk$. Then the *higher Chow groups of X with ℓ-adic coefficients* are

$$\mathrm{CH}(X,n;\mathbb{Z}_\ell) := H_n\left(\lim z^*(X,\cdot) \otimes^L \mathbb{Z}/l^m\mathbb{Z}\right),$$

where $z^*(X,\cdot)$ is as defined in [Blo86].

One key feature of this definition is that, when ℓ is coprime to the characteristic of $\Bbbk$, [MVW06, Theorems 10.2 and 19.1] we have $\mathrm{CH}(\operatorname{Spec}\Bbbk, 1; \mathbb{Z}_\ell) = 0$, and hence $\mathrm{CH}(X, 1; \mathbb{Z}_\ell) = 0$ for any combination of vector and projective bundles over $\operatorname{Spec}\Bbbk$. In particular, we have that $\mathrm{CH}(\mathbb{P}^n, 1; \mathbb{Z}_\ell) = 0$. The other key feature is the following proposition.

Proposition 2.3.5. *If $Z \to X$ is a closed immersion with complement U and:*

- $\mathrm{CH}(Z)$ *and* $\mathrm{CH}(U)$ *are finitely generated,*
- $\mathrm{CH}(Z) \to \mathrm{CH}(Z_{\bar{\Bbbk}})$ *is injective,*
- *there exists at least one ℓ for which* $\mathrm{CH}(U, 1; \mathbb{Z}_\ell) = 0$,
- *and* $\mathrm{CH}(U, 1; \mathbb{Z}_\ell) = 0$ *whenever* $\mathrm{CH}(Z)$ *has ℓ-torsion,*

then $\mathrm{CH}(Z) \to \mathrm{CH}(X)$ *is injective.*

Proof. Notice that, at first glance, ℓ-adic higher Chow groups tell us about the injectivity of the excision sequence with all spaces base-changed to $\bar{\Bbbk}$. However, we can infer the injectivity of $\mathrm{CH}(Z) \to \mathrm{CH}(X)$ via the following diagram

$$\begin{array}{ccc} \mathrm{CH}(Z) & \longrightarrow & \mathrm{CH}(X) \\ \downarrow & & \downarrow \\ \mathrm{CH}(Z_{\bar{\Bbbk}}) & \longrightarrow & \mathrm{CH}(X_{\bar{\Bbbk}}) \end{array}$$

Let $\alpha \in \mathrm{CH}(Z)$, and, abusing notation, refer to its image in $\mathrm{CH}(Z_{\bar{\Bbbk}})$ as α as well. Pick an ℓ such that α is ℓ-torsion (if α is not torsion, then pick any ℓ where U's first ℓ-adic higher Chow group vanishes). This gives

$$\mathrm{CH}(Z) \otimes \mathbb{Z}_\ell \hookrightarrow \mathrm{CH}(Z_{\bar{\Bbbk}}) \otimes \mathbb{Z}_\ell \hookrightarrow \mathrm{CH}(X_{\bar{\Bbbk}}) \otimes \mathbb{Z}_\ell,$$

and so the image of α under $\mathrm{CH}(Z) \to \mathrm{CH}(X)$ cannot vanish. □

Note that the first higher Chow group being trivial ℓ-adically does not tell us that the first higher Chow group is trivial, just that the left map is injective (and so the image of the connecting homomorphism is trivial). In fact, by [Blo86, (iii)], $\mathrm{CH}^1(\operatorname{Spec} \Bbbk, 1) = \Bbbk^*$, while, as above, $\mathrm{CH}^1(\operatorname{Spec} \Bbbk, 1; \mathbb{Z}_\ell) = 0$ for ℓ coprime to $\operatorname{char} \Bbbk$.

2.3.3 Some first higher Chow groups

Proposition 2.3.6. *All of $B\mathbb{G}_m$, $B\mu_n$, $\mathcal{P}(a_0, \ldots, a_n)$, and $[\mathbb{P}^n/\mathbb{G}_m]$ have trivial first higher Chow groups with ℓ-adic coefficients.*

Proof. We proceed equivariantly, as in the computation of the Chow ring of $B\mathbb{G}_m$. For $j < k$, we have

$$\mathrm{CH}^j\left(\left[\frac{\operatorname{Spec} \Bbbk \times \mathbb{A}^k \setminus 0}{\mathbb{G}_m}\right], 1; \mathbb{Z}_\ell\right) = \mathrm{CH}^j(\mathbb{P}^{k-1}, 1; \mathbb{Z}_\ell) = 0$$

which gives $\mathrm{CH}(B\mathbb{G}_m, 1; \mathbb{Z}_\ell) = 0$. This also gives $\mathrm{CH}([\mathbb{P}^n/\mathbb{G}_m], 1; \mathbb{Z}_\ell) = 0$ by the projective bundle formula.

Now let $\mathbb{G}_m$ act on $\mathbb{A}^{n+1}$ with weights $a_0, \ldots, a_n$. Then as above the long excision sequence gives

$$\begin{array}{cccc} \mathrm{CH}([\mathbb{A}^{n+1}/\mathbb{G}_m],1;\mathbb{Z}_\ell) & \to \mathrm{CH}(\mathcal{P}(a_0,\ldots,a_n),1;\mathbb{Z}_\ell) & \to \mathrm{CH}(B\mathbb{G}_m,\mathbb{Z}_\ell) & \to \mathrm{CH}([\mathbb{A}^{n+1}/\mathbb{G}_m],\mathbb{Z}_\ell) \\ 0 & \to \mathrm{CH}(\mathcal{P}(a_0,\ldots,a_n),1;\mathbb{Z}_\ell) & \to \mathbb{Z}[x]\otimes\mathbb{Z}_\ell & \to \mathbb{Z}[x]\otimes\mathbb{Z}_\ell \end{array}$$

But the right map, as shown above, is injective, and we conclude $\mathrm{CH}(\mathcal{P}(a_0,\ldots,a_n),1;\mathbb{Z}_\ell) = 0$. Since $B\mu_n \cong \mathcal{P}(n)$, we are done. □

2.3.4 Patching techniques

There are select situations where we may patch Chow rings together with no a priori knowledge of the first higher Chow groups.

Lemma 2.3.7 (The patching lemma, [DV21]). *Let X be a smooth variety endowed with the action of a group G, and let $i : Y \hookrightarrow X$ be a smooth, closed, and G-invariant subvariety, with normal bundle $\mathscr{N}$. Suppose that $c^G_{top}(\mathscr{N})$ is not a zero-divisor in $\mathrm{CH}_G(Y)$. Then the following diagram of rings is cartesian:*

$$\begin{array}{ccc} \mathrm{CH}_G(X) & \xrightarrow{i^*} & \mathrm{CH}_G(Y) \\ \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle q} \\ \mathrm{CH}_G(X\setminus Y) & \xrightarrow{p} & \mathrm{CH}_G(Y)/(c^G_{top}(\mathscr{N})) \end{array}$$

where the bottom horizontal arrow p sends the class of a variety V to the equivalence class $i^*\eta$ where η is any element in the set $(j^*)^{-1}([V])$.

This technique is even more powerful than patching with higher Chow groups, as it gives us the ring structure. However, it comes with the caveat that the top Chern class of the normal bundle cannot be a zero-divisor. This condition is actually never met for the case of schemes or even DM-stacks: the Chow ring of a scheme is zero in degree higher than the dimension of the scheme, so every element is nilpotent, and the Chow ring of a DM-stack is torsion in degree higher than the dimension of the stack (Corollary 2.1.7), so every element raised to a high enough power becomes torsion and hence a zero-divisor. Therefore the utility of this lemma isn't seen until one works with non-DM stacks, such as $B\mathbb{G}_m$, whose Chow ring we showed to have no zero-divisors. This idea informed the approach of [DPV21].

To see how we may patch in a more general setting, we return to Example 2.3.2.

Example 2.3.8. Since $\mathrm{CH}(\mathbb{A}^1, 1; \mathbb{Z}_\ell) = 0$, we have the short exact sequence

$$0 \to \mathbb{Z} = \mathrm{CH}(\operatorname{Spec} \Bbbk) \to \mathrm{CH}(\mathbb{P}^1) \to \mathbb{Z} = \mathrm{CH}(\mathbb{A}^1) \to 0$$

from which we can conclude that $\mathrm{CH}(\mathbb{P}^1) = \mathbb{Z} \oplus \mathbb{Z}$ as abelian groups. In fact, we can deduce its ring structure as well. Letting x be a degree one generator, we see that we must have $x^2 = 0$, as the Chow ring of $\mathbb{P}^1$ has no degree two component. There cannot be any other relations, as they would have to be in degree one and we know that x is not torsion. Therefore $\mathrm{CH}(\mathbb{P}^1) = \mathbb{Z}[x]/(x^2)$.

Of course, this is not how one usually computes the Chow ring of $\mathbb{P}^1$, but it demonstrates our technique. If we can stratify a stack into stacks whose first higher Chow groups are ℓ-adically trivial, then we may patch the components together, each step of the way using geometric and algebraic arguments to find relations for the new elements. That each strata has first higher Chow group ℓ-adically 0 allows us to patch together any amount of strata and be assured of the injectivity of the left map.

Lemma 2.3.9. *Assume that the conditions of Proposition 2.3.5 are met. Additionally assume that* $\mathrm{CH}(Z, 1; \mathbb{Z}_\ell) = 0$ *whenever* $\mathrm{CH}(U, 1; \mathbb{Z}_\ell) = 0$. *Then* $\mathrm{CH}(X, 1; \mathbb{Z}_\ell) = 0$.

Proof. This follows from the long excision sequence:

$$\begin{array}{ccccccccc} \dots & \longrightarrow & \mathrm{CH}(Z,1;\mathbb{Z}_\ell) & \longrightarrow & \mathrm{CH}(X,1;\mathbb{Z}_\ell) & \longrightarrow & \mathrm{CH}(X \setminus Z,1;\mathbb{Z}_\ell) & \longrightarrow & \dots \\ & & \| & & \|\therefore & & \| & & \\ \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots \end{array}$$

□

This discussion and the previous lemma proves the following.

Theorem 2.3.10. *Let X be a stack stratified into stacks X_i each of whose first higher Chow group with ℓ-adic coefficients is trivial. Then we may compute the Chow ring of X by patching together the Chow rings of the X_i, and the first higher Chow group of X is ℓ-adically trivial itself.*

Chapter 3

The $\mathcal{M}_{1,1}$ and $\mathcal{M}_{1,2}$ cases

3.1 The Weierstrass form

We open with the classically known Weierstrass form for elliptic curves.

Theorem 3.1.1 (Weierstrass)**.** *Any one-pointed smooth elliptic curve over a field $\Bbbk$ of characteristic not equal to 2 or 3 can be written in the form $y^2z = x^3 + axz^2 + bz^3$, where the marked point is the point at infinity $[0 : 1 : 0]$. Moreover, if we denote such a curve by $C_{(a,b)}$, then*

$$C_{(a,b)} \cong C_{(a',b')} \quad \textit{if and only if} \quad (a',b') = (t^{-4}a, t^{-6}b).$$

The isomorphism between these curves is given by

$$[x : y : z] \mapsto [t^{-2}x : t^{-3}y : z].$$

An elliptic curve is smooth if and only if $D = 4a^3 + 27b^2 = 0$, nodal if and only if $D = 0$ and $(a,b) \neq (0,0)$, and cuspidal if and only if $(a,b) = (0,0)$. Lastly, we have

$$H^0(\omega_C) = \left\langle \frac{dx}{y} \right\rangle.$$

Rephrasing this gives the following corollaries:

Corollary 3.1.2. *The Weierstrass form gives isomorphisms*

$$\widetilde{\mathcal{M}}_{1,1} \cong \left[\frac{\mathbb{A}^2}{\mathbb{G}_m}\right],$$

$$\overline{\mathcal{M}}_{1,1} \cong \left[\frac{\mathbb{A}^2 \setminus 0}{\mathbb{G}_m}\right],$$

and

$$\mathcal{M}_{1,1} \cong \left[\frac{\mathbb{A}^2 \setminus V(D)}{\mathbb{G}_m}\right],$$

where the $\mathbb{G}_m$ *action has weight* $(-4, -6)$.

Proof. See Appendix A. □

Corollary 3.1.3. *We have that* $\mathcal{M}_{1,2}$ *is isomorphic to an open substack of a vector bundle over* $B\mathbb{G}_m$.

Proof. From the Weierstrass form, a two-pointed smooth elliptic curve is determined, up to scaling, by a choice of (a, b) and (x, y) such that

$$y^2 = x^3 + ax + b \quad \text{and} \quad D \neq 0.$$

We can solve for b to see that a, x, y vary freely, provided that $D \neq 0$, where

$$D = 4a^3 + 27b^2 = 4a^3 + 27(y^2 - (x^3 + ax))^2.$$

Since $\mathbb{G}_m$ acts with weights $-4, -2, -3$ on a, x, y, we conclude that $\mathcal{M}_{1,2}$ is open in $\left[\frac{\mathbb{A}^3_{a,x,y}}{\mathbb{G}_m}\right]$, where $\mathbb{G}_m$ acts with the above weights. □

Corollary 3.1.4. *The rings* $\mathrm{CH}(\mathcal{M}_{1,1})$ *and* $\mathrm{CH}(\mathcal{M}_{1,2})$ *are both quotients of* $\mathbb{Z}[x]/(12x)$.

Proof. This follows from Corollaries 3.1.2 and 3.1.3, along with the fact that D has weight 12 under the $\mathbb{G}_m$ action. □

Corollary 3.1.5. *The generator of the Chow ring of* $\mathcal{M}_{1,1}$ *and* $\mathcal{M}_{1,2}$ *is* λ_1, *the first Chern class of the Hodge bundle.*

Proof. Since $\mathbb{G}_m$ acts with weights -2 and -3 on x and y, respectively, we see that $\frac{dx}{y}$ has weight 1 under the $\mathbb{G}_m$ action. Hence under the pullback map $\mathrm{CH}(B\mathbb{G}_m) \to \mathrm{CH}(\widetilde{\mathcal{M}}_{1,1})$ we have $x \mapsto \lambda_1$. Since by the previous Corollary $\mathrm{CH}(\mathcal{M}_{1,1})$ and $\mathrm{CH}(\mathcal{M}_{1,2})$ are generated by the pullback of x, we see that they are generated by λ_1. □

Corollary 3.1.6. *The pullback of $x \in \mathrm{CH}(B\mathbb{G}_m)$ to any moduli stack of pointed elliptic curves is λ_1.*

Proof. This follows since x pulls-back to λ_1 in $\mathrm{CH}(B\mathbb{G}_m) \to \mathrm{CH}(\widetilde{\mathcal{M}}_{1,1})$ and the Hodge bundle pulls-back to the Hodge bundle. □

3.2 The Chow rings of $\widetilde{\mathcal{M}}_{1,n}$, $\overline{\mathcal{M}}_{1,n}$, and $\mathcal{M}_{1,n}$ for $n = 1,2$

3.2.1 The $n = 1$ case

Theorem 3.2.1. *Let λ_1 be the first Chern class of the Hodge bundle. Over a field of characteristic not equal to 2 or 3*

(a) $\mathrm{CH}(\widetilde{\mathcal{M}}_{1,1}) \cong \mathbb{Z}[\lambda_1]$ *and* $\mathrm{CH}(\widetilde{\mathcal{M}}_{1,1}, 1; \mathbb{Z}_\ell) = 0$

(b) $\mathrm{CH}(\overline{\mathcal{M}}_{1,1}) \cong \mathbb{Z}[\lambda_1]/(24\lambda_1^2)$ *and* $\mathrm{CH}(\overline{\mathcal{M}}_{1,1}, 1; \mathbb{Z}_\ell) = 0$

(c) $\mathrm{CH}(\mathcal{M}_{1,1}) \cong \mathbb{Z}[\lambda_1]/(12\lambda_1)$ *and* $\mathrm{CH}(\mathcal{M}_{1,1}, 1; \mathbb{Z}_\ell) = 0$.

Proof. Statements (a) and (b) follow from vector bundle invariance, the computations of the relevant Chow rings and higher Chow groups in 2.2.1, 2.2.3, and 2.3.6, as well as Corollary 3.1.6, which tells us that the generator is λ_1.

To show (c), let $\nu : \mathbb{A}^1 \to V(4a^3 + 27b^2)$ be the normalization of the affine cuspidal cubic. Then ν induces an equivariant map $\nu : [\mathbb{A}^1/\mathbb{G}_m] \to [V(4a^3 + 27b^2)/\mathbb{G}_m]$. This map is a *Chow envelope*, as defined below.

Definition 3.2.2 ([Ful98, Definition 18.3], [EG98, Lemma 3])**.** We call a morphism $\pi : \tilde{X} \to X$ of schemes a *Chow envelope* if it is proper, $\tilde{X}$ is quasiprojective, and for all subvarieties W of X there exists a subvariety $\tilde{W}$ of $\tilde{X}$ mapping birationally to W. We call a morphism $\pi : [\tilde{X}/G] \to [X/G]$ over BG a Chow envelope if $\tilde{X} \to X$ is one. In this case, $[(\tilde{X} \times U)/G] \to [(X \times U)/G]$ is a Chow envelope for any good representation U. Notice that π induces a surjection on Chow rings via the pushforward.

Returning to the proof, we see that f then induces a surjection on Chow rings. Let $p : V(4a^3 + 27b^2) \to \mathbb{A}^2$ be the inclusion. Then the composition $(\nu \circ p)_* : \mathrm{CH}([\mathbb{A}^1/\mathbb{G}_m]) \to \mathrm{CH}([\mathbb{A}^2/\mathbb{G}_m])$ has image $(12x)$, and in particular is injective. Since $(\nu \circ p)^*(x) = x$, we see that Lemma 2.1.8 applies, and we conclude that p_* has image $(12x)$. This shows the first part of (c). Note further that since f_* is surjective and $(\nu \circ p)_*$ is injective, we must have that p_* is injective as well.

Then the long excision sequence gives

$$\mathrm{CH}(\widetilde{\mathcal{M}}_{1,1}, 1; \mathbb{Z}_\ell) \longrightarrow \mathrm{CH}(\mathcal{M}_{1,1}, 1; \mathbb{Z}_\ell) \longrightarrow \mathrm{CH}(V(4a^3 + 27b^2))_{\mathbb{Z}_\ell} \longrightarrow \mathrm{CH}(\widetilde{\mathcal{M}}_{1,1})_{\mathbb{Z}_\ell}$$

Since $\mathrm{CH}(\widetilde{\mathcal{M}}_{1,1}, 1; \mathbb{Z}_\ell) = 0$ and the right map is injective, we conclude that $\mathrm{CH}(\mathcal{M}_{1,1}, 1; \mathbb{Z}_\ell) = 0$. □

3.2.2 The $n = 2$ case

We seek to prove the following:

Theorem 3.2.3. *Let λ_1 be the first Chern class of the Hodge bundle. For part (a) and (b), let μ_1 be the fundamental class of the universal section $p : \widetilde{\mathcal{M}}_{1,1} \to \widetilde{\mathcal{M}}_{1,2}$, respectively $p : \overline{\mathcal{M}}_{1,1} \to \overline{\mathcal{M}}_{1,2}$. Over a field of characteristic not equal to 2 or 3:*

(a)

$$\mathrm{CH}(\widetilde{\mathcal{M}}_{1,2}) = \frac{\mathbb{Z}[\lambda_1, \mu_1]}{(\mu_1(\lambda_1 + \mu_1), 24\lambda_1^2(\lambda_1 + \mu_1))} \quad \textit{and} \quad \mathrm{CH}(\widetilde{\mathcal{M}}_{1,2}, 1; \mathbb{Z}_\ell) = 0.$$

(b) [DPV21, Theorem 2.6]

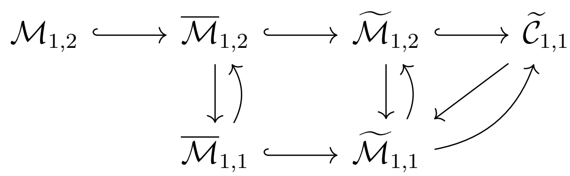
$$\mathrm{CH}(\overline{\mathcal{M}}_{1,2}) = \frac{\mathbb{Z}[\lambda_1, \mu_1]}{(\mu_1(\lambda_1 + \mu_1), 24\lambda_1^2)} \quad \textit{and} \quad \mathrm{CH}(\overline{\mathcal{M}}_{1,2}, 1; \mathbb{Z}_\ell) = 0.$$

(c)

$$\mathrm{CH}(\mathcal{M}_{1,2}) = \frac{\mathbb{Z}[\lambda_1]}{(12\lambda_1)}.$$

Our method will be as follows. First, we will consider $\widetilde{\mathcal{C}}_{1,1}$, the universal curve over $\widetilde{\mathcal{M}}_{1,1}$. We will compute its integral Chow ring by stratifying it into $\widetilde{\mathcal{C}}_{1,1} \setminus \widetilde{\mathcal{M}}_{1,1}$ and $\widetilde{\mathcal{M}}_{1,1}$, and solving the patching problem in two different ways. We will then identify and excise the locus of cuspidal curves in $\widetilde{\mathcal{C}}_{1,1}$ to obtain the integral Chow ring of $\overline{\mathcal{M}}_{1,2}$. In addition, we will compute the integral Chow ring of $\widetilde{\mathcal{M}}_{1,2}$, we will show that the first higher Chow group with ℓ-adic coefficients of all of these previous stacks is trivial, and we will compute the integral Chow ring of $\mathcal{M}_{1,2}$. The Chow rings of $\overline{\mathcal{M}}_{1,2}$ and $\mathcal{M}_{1,2}$ were also computed by Inchiostro in [Inc22] using different methods.

Here is a diagram containing the relevant stacks and maps between them:



Each horizontal arrow is the natural open inclusion, each downwards arrow is forgetting the last marked point, and each upwards arrow is the universal section. Notice that, unlike in the case of stable curves, $\widetilde{\mathcal{M}}_{1,2} \ncong \widetilde{\mathcal{C}}_{1,1}$ since the stack $\widetilde{\mathcal{C}}_{1,1}$ contains the cuspidal cubic with the cusp marked, which is not A_2-stable.

Proposition 3.2.4. *We have*

$$\operatorname{CH}(\widetilde{\mathcal{C}}_{1,1} \setminus \widetilde{\mathcal{M}}_{1,1}) = \mathbb{Z}[\lambda_1] \quad \textit{and} \quad \operatorname{CH}(\widetilde{\mathcal{C}}_{1,1} \setminus \widetilde{\mathcal{M}}_{1,1}, 1; \mathbb{Z}_\ell) = 0.$$

Proof. Earlier, we used the Weierstrass form to see that $\widetilde{\mathcal{M}}_{1,1} \cong [\mathbb{A}^2_{4,6}/\mathbb{G}_m]$, and so $\operatorname{CH}(\widetilde{\mathcal{M}}_{1,1}) = \mathbb{Z}[\lambda_1]$ and $\operatorname{CH}(\widetilde{\mathcal{M}}_{1,1}, 1; \mathbb{Z}_\ell) = 0$. The universal curve $\widetilde{\mathcal{C}}_{1,1}$ is the stack whose fiber over a scheme S is the groupoid of families of one-marked elliptic curves over S with an additional section (with no stability requirements on this section). Therefore the Weierstrass form again tells us that

$$\widetilde{\mathcal{C}}_{1,1} \cong \left[\frac{W}{\mathbb{G}_m}\right]$$

where $W = V(y^2 z - (x^3 + ax^2 z + bz^3)) \subseteq \mathbb{A}^2 \times \mathbb{P}^2$, and the action has weight $(-4, -6, -2, -3, 0)$.

With this explicit presentation, the maps

$$\begin{array}{c} \widetilde{\mathcal{C}}_{1,1} \\ {\scriptstyle p}\uparrow\downarrow{\scriptstyle \pi} \\ \widetilde{\mathcal{M}}_{1,1} \end{array}$$

are the stackifications of the prestack maps $\pi : (a, b, x, y, z) \mapsto (a, b)$ and $p : (a, b) \mapsto (a, b, 0, 1, 0)$. Since removing the marked point from a marked projective elliptic curve yields an affine curve with equation $y^2 = x^3 + ax^2 + b$, excising the section p yields $\widetilde{\mathcal{C}}_{1,1} \setminus \widetilde{\mathcal{M}}_{1,1} \cong [W'/\mathbb{G}_m]$, where $W' = V(y^2 - (x^3 + ax^2 + b)) \subseteq \mathbb{A}^2 \times \mathbb{A}^2$ and the $\mathbb{G}_m$ action has weight $(-4, -6, -2, -3)$.

We may solve for b in the above expression, $b = y^2 - x^3 - ax^2$. Then a, x, y may vary freely, and so $W' \cong \mathbb{A}^1 \times \mathbb{A}^2$. Then $[W'/\mathbb{G}_m]$ is a vector bundle over $B\mathbb{G}_m$, and so the proposition follows by vector bundle invariance. □

Proposition 3.2.5. *We have*

$$\mathrm{CH}(\widetilde{\mathcal{C}}_{1,1}) = \frac{\mathbb{Z}[\lambda_1, \mu_1]}{(\mu_1(\lambda_1 + \mu_1))} \quad \textit{and} \quad \mathrm{CH}(\widetilde{\mathcal{C}}_{1,1}, 1; \mathbb{Z}_\ell) = 0.$$

Note 3.2.6. This Chow ring is due to [DPV21]. We first present an alternate proof and then, for the sake of completeness, present the original proof, which utilizes Lemma 2.3.7.

Proof. Note that the second assertion follows from Lemma 2.3.9. Let $p : \widetilde{\mathcal{M}}_{1,1} \to \widetilde{\mathcal{C}}_{1,1}$ be the universal section. Then $p_*(\lambda_1^k) = p_*p^*(\lambda_1^k) = \lambda_1^k p_*(1) = \lambda_1^k \mu_1$, and $\mu_1^2 = p_*(1)p_*(1) = p_*p^*(p_*(1)) = p_*(c_1\mathscr{N}) = p_*(-\lambda_1) = -\lambda_1\mu_1$, and so we have the relation $\mu_1(\mu_1 + \lambda_1)$. Since the excision sequence is injective with $\mathrm{im}(p_*) = (\mu_1)$, we see that $\mathrm{CH}(\widetilde{\mathcal{C}}_{1,1})$ is generated by λ_1, μ_1 with the only relation given by $\mu_1(\lambda_1 + \mu_1)$.

□

Proof due to [DPV21]. Note that the second assertion follows from Lemma 2.3.9. Let ψ_1 denote the first Chern class of the conormal bundle of the universal section. By definition, $c_1(\mathscr{N}) = -\psi_1$. However, we also have $\psi_1 = \lambda_1$ (see [Koc], for instance), and so $c_1(\mathscr{N}) = -\lambda_1$. Then the patching lemma says we have a cartesian diagram of rings

$$\begin{array}{ccc} \mathrm{CH}(\widetilde{\mathcal{C}}_{1,1}) & \xrightarrow{p^*} & \mathrm{CH}(\widetilde{\mathcal{M}}_{1,1}) \cong \mathbb{Z}[\lambda_1] \\ \downarrow{\scriptstyle j^*} & & \downarrow \\ \mathrm{CH}(\widetilde{\mathcal{C}}_{1,1} \setminus p) \cong \mathbb{Z}[\lambda_1] & \longrightarrow & \mathrm{CH}(\widetilde{\mathcal{M}}_{1,1})/(c_1(\mathscr{N})) \cong \mathbb{Z}. \end{array}$$

Since λ_1 pulls back to λ_1 and the kernel of j^* is the ideal generated by μ_1, we see that $\widetilde{\mathcal{C}}_{1,1}$ is generated by μ_1 and λ_1. So we can write $\widetilde{\mathcal{C}}_{1,1} \cong \mathbb{Z}[\lambda_1, \mu_1]/I$ for some ideal I.

In fact, the above diagram implies that I is precisely the polynomials $p(\lambda_1, \mu_1)$ that are in the kernel of both j^* and p^*. So then assume that $j^*(p(\lambda_1, \mu_1)) = 0$. Since $j^*\lambda_1 = \lambda_1$ and $j^*\mu_1 = 0$, we see that we must have $p(\lambda_1, \mu_1) = \mu_1 q(\lambda_1, \mu_1)$. Applying p^* to this we have

$$p^*(\mu_1 q(\lambda_1, \mu_1)) = -\lambda_1(q(\lambda_1, -\lambda_1)),$$

since $p^*\mu_1 = p^*p_*1 = c_1(\mathscr{N}) = -\lambda_1$. But if $-\lambda_1 q(\lambda_1, -\lambda_1) = 0$ in $\mathbb{Z}[\lambda_1]$, then $q(\lambda_1, -\lambda_1) = 0$, that is, $\mu_1 = -\lambda_1$ is a solution to q. Therefore $\lambda_1 + \mu_1$ divides q. Thus we see that I is generated by $\mu_1(\lambda_1 + \mu_1)$. □

We now prove Theorem 3.2.3. We begin with (b), which we use to prove (a) and (c).

Proof of 3.2.3(b). Now that $\mathrm{CH}(\widetilde{\mathcal{C}}_{1,1})$ is known, to compute $\mathrm{CH}(\overline{\mathcal{M}}_{1,2})$ we must identify and excise the locus of cuspidal curves using the excision sequence corresponding to $\overline{\mathcal{M}}_{1,2} \subseteq \widetilde{\mathcal{C}}_{1,1}$.

Let C be the projective cuspidal cubic $y^2 z = x^3$, and let $\nu : \mathbb{P}^1 \to C$ be its normalization. Notice that the locus of cuspidal curves in $\widetilde{\mathcal{C}}_{1,1}$ is isomorphic to $[C/\mathbb{G}_m]$, under the morphism $[x : y : z] \mapsto ((0, 0), [x : y : z])$. Consider the quotient stack

$$[\mathbb{P}^1_{-2,-3}/\mathbb{G}_m] = [\mathbb{P}(\mathbb{A}^2_{-2,-3})/\mathbb{G}_m].$$

By the projective bundle formula,

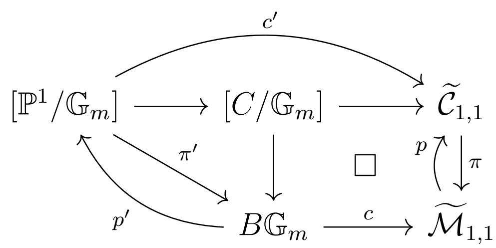
$$\begin{aligned} \mathrm{CH}([\mathbb{P}^1_{-2,-3}/\mathbb{G}_m]) &\cong \mathbb{Z}[x, y]/((y - 2x)(y - 3x)) \\ &\cong \mathbb{Z}[x, y]/((y - x)y). \end{aligned}$$

Since the normalization $\nu : \mathbb{P}^1 \to C$ is a $\mathbb{G}_m$-equivariant Chow envelope, $[\mathbb{P}^1/\mathbb{G}_m] \to [C/\mathbb{G}_m]$ is a Chow envelope as well.

Let c' be the composition $[\mathbb{P}^1/\mathbb{G}_m] \to [C/\mathbb{G}_m] \to \widetilde{\mathcal{C}}_{1,1}$. Since $[\mathbb{P}^1/\mathbb{G}_m] \to [C/\mathbb{G}_m]$ is a Chow envelope, to excise the image of $[C/\mathbb{G}_m]$ from $\widetilde{\mathcal{C}}_{1,1}$ it suffices to excise the image of c'. That is, we have the following diagram:

$$\mathrm{CH}([\mathbb{P}^1/\mathbb{G}_m]) \twoheadrightarrow \mathrm{CH}([C/\mathbb{G}_m]) \to \mathrm{CH}(\widetilde{\mathcal{C}}_{1,1}) \to \mathrm{CH}(\overline{\mathcal{M}}_{1,2}) \to 0.$$

We get a commutative diagram



where the sections are given by:

$$p' : \cdot \mapsto [1:0] = \infty$$

and

$$p : (a,b) \mapsto ((a,b), [0:1:0]).$$

Recall that $c_* : \mathbb{Z}[x] \to \mathbb{Z}[\lambda_1]$ is given by $f(x) \mapsto 24\lambda_1^2 f(\lambda_1)$, that $\pi^*(f(\lambda_1)) = f(\lambda_1)$, and that $p^*(f(\lambda_1)) = f(\lambda_1)$. Notice that since the square is cartesian and c' is a Chow envelope for the image of $[C/\mathbb{G}_m] \to \widetilde{\mathscr{C}}_{1,1}$, we get that $c'_* \circ \pi'^* = \pi^* \circ c_*$. Then

$$c'_*(x^k) = c'_* \circ \pi'^*(x^k) = \pi^* \circ c_*(x^k) = \pi^*(\lambda_1^k \cdot 24\lambda_1^2) = \lambda_1^k \cdot 24\lambda_1^2.$$

We seek to show that the image of c'_* is the free abelian group generated by $24\lambda_1^k$ and $24\lambda_1^k\mu_1$ for $k \geq 2$. Since we've just shown $c'_*(x^k) = 24\lambda_1^{k+2}$ for all $k \geq 0$, it now suffices to show that $c'_*(x^k y) = 24\lambda_1^{k+2}\mu_1$ for all $k \geq 0$.

Now consider $p'_* : \mathrm{CH}(B\mathbb{G}_m) \to \mathrm{CH}([\mathbb{P}^1/\mathbb{G}_m])$. We have $p'_*(1) = y$, and

$p'^*(x^k) = x^k$, therefore $p'_*(x^k) = x^k y$ for all k. Then

$$\begin{aligned} c'_*(x^k y) &= c'_* \circ p'_*(x^k) \\ &= p_* \circ c_*(x^k) \\ &= p_*(24\lambda_1^{k+2}) \\ &= p_*(p^*(24\lambda_1^{k+2})) \\ &= 24\lambda_1^{k+2} \cdot p_*(1) \\ &= 24\lambda_1^{k+2}\mu_1. \end{aligned}$$

Therefore the image of c'_* is generated by $24\lambda_1^k$ and $24\lambda_1^k\mu_1$ for $k \geq 2$, and the first assertion is proven.

To see that $\mathrm{CH}(\overline{\mathcal{M}}_{1,2}, 1; \mathbb{Z}_\ell) = 0$, observe that since c'_* is injective and the pushforward from $[\mathbb{P}^1/\mathbb{G}_m] \to [C/\mathbb{G}_m]$ is surjective, we must have that the pushforward from $[C/\mathbb{G}_m] \to \widetilde{\mathcal{C}}_{1,1}$ is injective, and so our excision sequence becomes

$$0 \longrightarrow \mathrm{CH}(\overline{\mathcal{M}}_{1,2}, 1; \mathbb{Z}_\ell) \longrightarrow \mathrm{CH}([C/\mathbb{G}_m]) \otimes \mathbb{Z}_\ell \hookrightarrow \mathrm{CH}(\widetilde{\mathcal{C}}_{1,1}) \otimes \mathbb{Z}_\ell$$

Thus $\mathrm{CH}(\overline{\mathcal{M}}_{1,2}, 1; \mathbb{Z}_\ell) = 0$. □

Proof of 3.2.3(a). Notice that $\widetilde{\mathcal{C}}_{1,1}$ and $\widetilde{\mathcal{M}}_{1,2}$ differ only in that $\widetilde{\mathcal{C}}_{1,1}$ allows the cusp to be marked. Therefore to obtain $\widetilde{\mathcal{M}}_{1,2}$ from $\widetilde{\mathcal{C}}_{1,1}$ we need only excise this curve, which is the image of the point $[0 : 1]$ in $[\mathbb{P}^1/\mathbb{G}_m]$ under the map $c' : [\mathbb{P}^1/\mathbb{G}_m] \to \widetilde{\mathcal{C}}_{1,1}$.

Recall from Note 2.2.6 the map $i_0 : B\mathbb{G}_m \to [\mathbb{P}^1/\mathbb{G}_m]$ corresponding to including the point $[0 : 1]$. Then the inclusion of the marked cusp in $\widetilde{\mathcal{C}}_{1,1}$ is the composition of i_0 with c'. Therefore $(c' \circ i_0)_*(x^k) = c'_*(x^k(x + y)) = 24\lambda_1^{k+2}(\lambda_1 + \mu_1)$, and so the image of the pushforward is the ideal generated by $24\lambda_1^2(\lambda_1 + \mu_1)$, which proves the first claim.

For the higher Chow group, observe that since the pushforward map is injective we can conclude exactly the same as in the previous proposition that $\mathrm{CH}(\widetilde{\mathcal{M}}_{1,2}, 1; \mathbb{Z}_\ell) = 0$. □

Proof of 3.2.3(c). From Corollary 3.1.4 we know that $\mathrm{CH}(\mathcal{M}_{1,2})$ is abstractly a quotient of $\mathbb{Z}[x]/(12x)$ and that the Picard group is generated by λ_1. Now consider any two-pointed elliptic curve with μ_3 automorphisms, such as $(C_{(0,1)}, \infty, [0:1:1])$ and with μ_4 automorphisms, such as $(C_{(1,0)}, \infty, [0:0:1])$. These induce residual gerbes

$$\begin{array}{ccc} B\mu_n & \longrightarrow & \mathcal{M}_{1,2} \\ & \searrow \quad \swarrow & \\ & B\mathbb{G}_m & \end{array}$$

for $n = 3, 4$. Since $\mathrm{CH}(B\mathbb{G}_m) \to \mathrm{CH}(B\mu_n)$ is surjective, we see that $\mathrm{CH}(\mathcal{M}_{1,2})$ surjects onto $\mathbb{Z}[x]/(nx)$ for $n = 3, 4$. Therefore $\mathrm{CH}(\mathcal{M}_{1,2}) = \mathbb{Z}[\lambda_1]/(12\lambda_1)$. $\square$

Chapter 4

The $\mathcal{M}_{1,n}$ case for $n = 3, \ldots, 10$

We will now compute the integral Chow rings of $\mathcal{M}_{1,n}$ for $n = 3, \ldots, 10$. The overall structure of the computation is to stratify $\mathcal{M}_{1,n}$ into an open whose complement stratifies into closed substacks which are isomorphic to opens inside of $\mathcal{M}_{1,n-1}$.

4.1 The integral Chow ring of $\mathcal{M}_{1,3}$

We first stratify $\mathcal{M}_{1,n}$ into the open locus where $p_2 \neq \iota(p_3)$ and the divisor where $p_2 = \iota(p_3)$.

Definition 4.1.1. For $n \geq 2$:

(a) Let $U_n \subseteq \mathcal{M}_{1,n}$ be the locus where $p_2 \neq \iota(p_3)$.

(b) Let $U'_n \subseteq \mathcal{M}_{1,n}$ be the locus where $p_2 \neq \iota(p_i)$ for any i.

Note 4.1.2. Note that the condition for U_2 is ill-defined. We use the convention that this is an empty condition, so that $U_2 = \mathcal{M}_{1,2}$.

Observation 4.1.3. Where π is, as usual, the map $\pi : \mathcal{M}_{1,n} \to \mathcal{M}_{1,n-1}$ forgetting the last marked point, we always have $\pi(U_{n+1}) \subseteq U_n$ and $\pi(U'_{n+1}) \subseteq U'_n$, and for $n \geq 4$ we have $\pi^{-1}(U_{n-1}) = U_n$.

Therefore we have induced pullback maps on Chow rings given by π^*. Since $\pi^*(\lambda_1) = \lambda_1$, we see that π pulls relations back to relations: if $a\lambda_1 = 0$ in $\mathrm{CH}(U_m)$ or $\mathrm{CH}(U'_m)$ for some m, then $a\lambda_1 = 0$ on that same locus for all $n \geq m$.

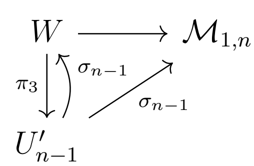
Definition 4.1.4. For $n \geq 3$, define the morphism of stacks $\sigma_{n-1} : U'_{n-1} \to \mathcal{M}_{1,n}$ by

$$(C, p_i) \mapsto (C, p_1, p_2, \iota(p_2), p_3, \ldots, p_{n-1}).$$

This map sheds light on why the loci in Definition 4.1.1 were defined: the defining conditions for U'_n are precisely the conditions needed to insure that this map exists.

Proposition 4.1.5. *For $n \geq 3$, the map $\sigma_{n-1} : U'_{n-1} \to \mathcal{M}_{1,n}$ is a closed immersion.*

Proof. Let W be the locus inside of $\mathcal{M}_{1,n}$ where $p_2 \neq \iota(p_2)$. Observe that σ_{n-1} factors through this locus: the image of σ_{n-1} is the locus where $p_3 = \iota(p_2)$, hence in particular $p_2 \neq \iota(p_2)$. Let $\pi_3 : \mathcal{M}_{1,n} \to \mathcal{M}_{1,n-1}$ be the map which forgets the third marked point. Then we have the following diagram:



where $\pi_3 \circ \sigma_{n-1} = \text{id}$. Therefore $\sigma_{n-1} : U'_{n-1} \to \mathcal{M}_{1,n}$ factors as a closed immersion into W followed by an open immersion into $\mathcal{M}_{1,n}$. Since its image in $\mathcal{M}_{1,n}$ is the closed locus of curves with $p_3 = \iota(p_2)$, we see that $\sigma_{n-1} : U'_{n-1} \to \mathcal{M}_{1,n}$ is a closed immersion. □

Corollary 4.1.6. *For $n \geq 3$, the stack $\mathcal{M}_{1,n}$ stratifies into the disjoint union*

$$\mathcal{M}_{1,n} = U_n \sqcup \operatorname{im} \sigma_{n-1} \cong U_n \sqcup U'_{n-1}.$$

Lemma 4.1.7. *The stack U_3 is isomorphic to an open substack of a vector bundle $\mathcal{U}_3$ over $B\mu_2$.*

Proof. A smooth three-pointed elliptic curve is determined, up to scaling, by a choice of (a, b), $p_2 = (x_2, y_2)$, and $p_3 = (x_3, y_3)$ such that

$$y_i^2 = x_i^3 + a x_i + b \quad \text{and} \quad D \neq 0.$$

Solving for b and then a gives

$$a = \frac{(y_3^2 - x_3^3) - (y_2^2 - x_2^3)}{x_3 - x_2}.$$

Therefore we see that x_2, x_3, y_2, y_3 may vary freely, provided $x_2 \neq x_3$ and $D \neq 0$. But the condition that $x_2 \neq x_3$ is precisely the condition that p_2 and p_3 do not overlap and are not involutions of each other involutions of each other (the defining condition for U_3), and so U_3 is open inside of

$$\mathcal{U}_3 := \left[\frac{\mathbb{A}^2_{y_i} \times (\mathbb{A}^2_{x_i} \setminus \Delta)}{\mathbb{G}_m}\right],$$

where Δ is the diagonal and $\mathbb{G}_m$ acts with weight -2 on x_i and -3 on y_i. This is a vector bundle over

$$\left[\frac{\mathbb{A}^2_{x_i} \setminus \Delta}{\mathbb{G}_m}\right]$$

which is a vector bundle over

$$\left[\frac{\mathbb{A}^1 \setminus 0}{\mathbb{G}_m}\right] \cong B\mu_2$$

since $\mathbb{G}_m$ acts with weight -2 on x_i. □

Lemma 4.1.8. *The stack* $\operatorname{im}\sigma_2$ *is isomorphic to an open substack of a vector bundle* $\mathcal{U}'_2$ *over* $B\mu_3$.

Proof. Since $\operatorname{im}\sigma_2$ is isomorphic to the locus U'_2 in $\mathcal{M}_{1,2}$, we just need to analyze two-pointed elliptic curves where $\iota(p_2) \neq p_2$. Recall from Corollary 3.1.3 that $\mathcal{M}_{1,2}$ is open inside of a vector bundle over $B\mathbb{G}_m$. More specifically, $\mathcal{M}_{1,2}$ is open inside of $[\mathbb{A}^3/\mathbb{G}_m]$ with coordinates a, x, y. The condition that $\iota(p_2) \neq p_2$ is equivalent to the condition $y \neq 0$, since $\iota(p_2) = \iota([x : y : z]) = [x : -y : z]$. Therefore U'_2 is open inside of

$$\mathcal{U}'_2 := \left[\frac{\mathbb{A}^3 \setminus \{y = 0\}}{\mathbb{G}_m}\right] \cong \left[\frac{\mathbb{A}^2_{a,x} \times (\mathbb{A}^1_y \setminus 0)}{\mathbb{G}_m}\right]$$

which is a vector bundle over

$$\left[\frac{\mathbb{A}^1_y \setminus 0}{\mathbb{G}_m}\right] \cong B\mu_3,$$

since $\mathbb{G}_m$ acts with weight -3 on y. □

Now we compute the integral Chow ring of $\mathcal{M}_{1,3}$ by first observing that the vector bundles $\mathcal{U}_3$ and $\mathcal{U}_2'$ of the previous section naturally live inside of an enlargement of the stack $\widetilde{\mathcal{M}}_{1,3}$, where the second and third marked points are not required to be in the nodal locus. We patch those vector bundles together inside of this stack using higher Chow groups with ℓ-adic coefficients, and from there deduce $\mathrm{CH}(\mathcal{M}_{1,3})$.

Lemma 4.1.9. *Let $\mathcal{X} = \mathcal{U}_3 \sqcup \mathcal{U}_2'$. Then*

$$\mathrm{CH}(\mathcal{X}) = \frac{\mathbb{Z}[x]}{(6x^2)} \quad \textit{and} \quad \mathrm{CH}(\mathcal{X}, 1; \mathbb{Z}_\ell) = 0$$

for ℓ coprime to $\operatorname{char} \Bbbk$.

Proof. Recall that because $\mathcal{U}_3$ and $\mathcal{U}_2'$ are both quotients by $\mathbb{G}_m$ and vector bundles over $B\mu_2$ and $B\mu_3$, respectively, that their first higher Chow groups with ℓ-adic coefficients vanish for ℓ co-prime to $\operatorname{char} \Bbbk$ and that their Chow rings are

$$\mathrm{CH}(\mathcal{U}_3) = \frac{\mathbb{Z}[x]}{(2x)} \quad \text{and} \quad \mathrm{CH}(\mathcal{U}_2') = \frac{\mathbb{Z}[x]}{(3x)}$$

where in both rings x denotes the pullback of the generator $x \in \mathrm{CH}(B\mathbb{G}_m) = \mathbb{Z}[x]$.

Consider the following diagram

$$\begin{array}{ccccc} \mathcal{U}_2' & \xrightarrow{\sigma_2} & \mathcal{X} & \xleftarrow{j} & \mathcal{U}_3 \\ & \searrow^{\pi_2} & \downarrow^{\pi} & \swarrow^{\pi_3} & \\ & & B\mathbb{G}_m & & \end{array}$$

Denote the pullback of $x \in \mathrm{CH}(B\mathbb{G}_m)$ to $\mathrm{CH}(\mathcal{X})$ by x as well, so that the pullback of x along any map is again x.

Since $\mathrm{CH}(\mathcal{U}_3, 1; \mathbb{Z}_\ell)$ vanishes for ℓ co-prime to char $\Bbbk$, the excision sequence for $\mathcal{U}_3$ and $\mathcal{U}_2'$ gives

$$0 \to \mathrm{CH}(\mathcal{U}_2') \xrightarrow{\sigma_{2*}} \mathrm{CH}(\mathcal{X}) \to \mathrm{CH}(\mathcal{U}_3) \to 0$$

$$0 \to \mathbb{Z}[x]/(3x) \xrightarrow{\sigma_{2*}} \mathrm{CH}(\mathcal{X}) \to \mathbb{Z}[x]/(2x) \to 0.$$

Moreover, since $\mathrm{CH}(\mathcal{U}_2', 1; \mathbb{Z}_\ell) = 0$ for ℓ coprime to char $\Bbbk$, we see that $\mathrm{CH}(\mathcal{X}, 1; \mathbb{Z}_\ell) = 0$.

In all degrees $k \geq 2$, the above sequence looks like

$$0 \to \mathbb{Z}/3 \to \mathrm{CH}^k(\mathcal{X}) \to \mathbb{Z}/2 \to 0,$$

and so $\mathrm{CH}^k(\mathcal{X}) \cong \mathbb{Z}/6 \cong \mathrm{CH}^k(\mathbb{Z}[x]/(6x^2))$ for $k \geq 2$. In degree one the sequence looks like

$$0 \to \mathbb{Z} \xrightarrow{\sigma_{2*}} \mathrm{CH}^1(\mathcal{X}) \xrightarrow{j^*} \mathbb{Z}/2 \to 0.$$

We now have either $\mathrm{CH}^1(\mathcal{X}) \cong \mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}/2$, and we seek to show that $\mathrm{CH}^1(\mathcal{X}) \cong \mathbb{Z}$.

Since $j^*(x) = x$ in any case, if the sequence splits then x has order 2 in $\mathrm{CH}^1(\mathcal{X})$. However we see that in $\mathrm{CH}^1(\mathcal{U}_2')$

$$0 \neq 2x = \pi_2^*(2x) = (\pi \circ \sigma_2)^*(2x) = \sigma_2^* \circ \pi^*(2x) = \sigma_2^*(2x),$$

and so $2x \neq 0 \in \mathrm{CH}^1(\mathcal{X})$. Therefore the sequence does not split and $\mathrm{CH}^1(\mathcal{X}) \cong \mathbb{Z}$.

Since $2x \in \ker j^* = \operatorname{im} \sigma_{2*}$, we see that $\sigma_{2*}(a) = 2x$ for some a, necessarily $a = \pm 1$. Therefore x is the generator of $\mathrm{CH}(\mathcal{X})$ as desired. □

Corollary 4.1.10. *The Chow ring of $\mathcal{M}_{1,3}$ is a quotient of $\mathbb{Z}[\lambda_1]/(6\lambda_1^2)$.*

Proof. This follows from the excision sequence, since $\mathcal{M}_{1,3}$ is open in $\mathcal{X}$, namely it is the complement of the loci of cuspidal curves, nodal curves, and curves where the marked points coincide. The fact that it is generated by λ_1 is a consequence of Corollary 3.1.6. □

To conclude precisely which quotient, we use the following Theorem.

Theorem 4.1.11 ([FV20]). *The Picard group of $\mathcal{M}_{1,n}$ is isomorphic to $\mathbb{Z}/12$ for all n, generated by the Hodge bundle.*

Theorem 4.1.12. *The integral Chow ring of $\mathcal{M}_{1,3}$ is*

$$\mathrm{CH}(\mathcal{M}_{1,3}) = \frac{\mathbb{Z}[\lambda_1]}{(12\lambda_1, 6\lambda_1^2)}.$$

Proof. The inclusion of any three-pointed curve with μ_2 or μ_3 automorphisms (which exists by Appendix B) shows that $\mathrm{CH}(\mathcal{M}_{1,3})$ surjects onto $\mathbb{Z}[x]/(2x)$ and $\mathbb{Z}[x]/(3x)$, respectively. Since $\mathrm{Pic}(\mathcal{M}_{1,3}) = \mathbb{Z}/12$, generated by λ_1, the theorem is proven. □

Note 4.1.13. Our argument can in fact be modified to not use higher Chow groups, in a similar fashion as the argument for $\mathcal{M}_{1,2}$. However, this version of the argument allows us to conclude that the first higher Chow group of the stack $\mathcal{X}$ with ℓ-adic coefficients vanishes, which may prove useful in future computations.

4.2 The case $4 \leq n \leq 10$

We first make an analogous definition of the *tautological ring* in the integral case.

Definition 4.2.1. The *integral tautological ring* of $\mathcal{M}_{1,n}$, written $\mathcal{R}(\mathcal{M}_{1,n})$, is the subring of the Chow ring generated by λ_1.

The remainder of this section has the following structure: first, we compute the integral tautological ring of $\mathcal{M}_{1,n}$ for $n \geq 4$, then we show that the full Chow ring is indeed generated by λ_1 for $4 \leq n \leq 10$.

Corollary 4.2.2. *For $n \geq 3$:*

(a) the integral tautological ring $\mathcal{R}(U_n)$ is a quotient of $\mathbb{Z}[x]/(2x)$.

(b) the integral tautological ring $\mathcal{R}(U_n')$ is equal to $\mathbb{Z}$.

Proof. We showed in Lemma 4.1.7 that $2\lambda_1 = 0$ on U_3, and so this relation holds on U_n for all $n \geq 3$. We also showed in Lemma 4.1.8 that $3\lambda_1 = 0$ on U_2', and so this relation holds on U_3' and hence on U_n' for all $n \geq 3$.

Since $U'_n \subseteq U_n$ for all n, we see that for all $n \geq 3$ both relations $2\lambda_1 = 0$ and $3\lambda_1 = 0$ hold on U'_n. Therefore $\lambda_1 = 0$ on U'_n for $n \geq 3$. □

Lemma 4.2.3. *For all $n \geq 4$, the integral tautological ring of $\mathcal{M}_{1,n}$ is a quotient of $\mathbb{Z}[\lambda_1]/(12\lambda_1, 2\lambda_1^2)$.*

Proof. Since by Corollary 4.2.2 the tautological ring of U_n is a quotient of $\mathbb{Z}[x]/(2x)$, we can write

$$\mathcal{R}(U_n) = \frac{\mathbb{Z}[x]/(2x)}{I}$$

for some ideal I. The excision sequence for $\operatorname{im}\sigma_{n-1} \cong U'_{n-1}$ gives

$$\mathcal{R}(U'_{n-1}) \to \mathcal{R}(\mathcal{M}_{1,n}) \to \mathcal{R}(U_n) \to 0$$

$$\mathbb{Z} \to \mathcal{R}(\mathcal{M}_{1,n}) \to \frac{\mathbb{Z}[x]/(2x)}{I} \to 0.$$

Since the image of the morphism lands in degree one and the Picard group of $\mathcal{M}_{1,n}$ is known to be $\mathbb{Z}/12$, the lemma follows. □

Proposition 4.2.4. *The integral tautological ring of $\mathcal{M}_{1,4}$ is*

$$\mathcal{R}(\mathcal{M}_{1,4}) = \frac{\mathbb{Z}[\lambda_1]}{(12\lambda_1, 2\lambda_1^2)}.$$

Proof. Observe that by Appendix B there still exists four-pointed smooth elliptic curves with automorphisms: $n = 4$ is the largest n for which such a curve exists, and all such curves have μ_2 automorphisms, generated by the involution. Moreover, such a curve is necessarily contained inside of U_4, the locus where the second and third points are not involutions of each other, since each marked point is fixed by the involution. Therefore we get a surjection

$$\mathrm{CH}(U_4) \twoheadrightarrow \mathbb{Z}[x]/(2x).$$

However, since the degree one generator of $\mathrm{CH}(U_4)$ is λ_1, this morphism in fact factors as

$$\mathcal{R}(U_4) \hookrightarrow \mathrm{CH}(U_4) \twoheadrightarrow \mathbb{Z}[x]/(2x)$$

Therefore $\mathcal{R}(U_4) = \mathbb{Z}[\lambda_1]/(2\lambda_1)$, and so $\mathcal{R}(\mathcal{M}_{1,4}) = \mathbb{Z}[\lambda_1]/(12\lambda_1, 2\lambda_1^2)$. □

Before we can compute the integral tautological ring for $n \geq 5$, we must analyze $\mathcal{M}_{1,4}$ more thoroughly.

Definition 4.2.5. Let $Z_n \subseteq \mathcal{M}_{1,n}$ be the locus of curves with non-trivial automorphisms.

Observation 4.2.6. Since $\mathcal{M}_{1,4} \setminus Z_4$ is a four-dimensional variety, we must have $\lambda_1^5 = 0$ on this locus, and hence on any locus inside of it. Moreover, observe that every curve in Z_4 must have p_2, p_3, and p_4 collinear: the only four-pointed smooth elliptic curves with automorphisms are the ones where $y_i = 0$ for $i = 2, 3, 4$, and hence p_2, p_3, and p_4 lie on the line $y = 0$ (see Appendix B).

We now give a second stratification of $\mathcal{M}_{1,n}$ for $n \geq 3$ as follows:

- the open locus where p_2, p_3, and p_4 are not colinear under the Weierstrass embedding.
- and the divisor where p_2, p_3, and p_4 *are* colinear.

Definition 4.2.7. For $n \geq 3$ define the loci:

(a) Let $V_n \subseteq \mathcal{M}_{1,n}$ be the locus where $p_2 + p_3 \neq \iota(p_4)$.

(b) Let $V_n' \subseteq \mathcal{M}_{1,n}$ be the locus where $p_2 + p_3 \neq \iota(p_i)$ for any $i = 1, \ldots, n$.

Observation 4.2.8. As before, π pulls relations back to relations.

Definition 4.2.9. Define the morphism of stacks $\tau_{n-1} : V_{n-1}' \to \mathcal{M}_{1,n}$ by

$$(C, p_i) \mapsto (C, p_1, p_2, p_3, \iota(p_2 + p_3), \ldots, p_{n-1}).$$

Proposition 4.2.10. *The map $\tau_{n-1} : V_{n-1}' \to \mathcal{M}_{1,n}$ is a closed immersion.*

Proof. Similar to Proposition 4.1.5. □

Corollary 4.2.11. *For $n \geq 3$, the stack $\mathcal{M}_{1,n}$ stratifies into the disjoint union*

$$\mathcal{M}_{1,n} = V_n \sqcup \operatorname{im} \tau_{n-1} \cong V_n \sqcup V_{n-1}'.$$

Proposition 4.2.12. *For $n = 2, \dots, 10$ the stacks $\mathcal{M}_{1,n}$ are rational. Moreover, for $n = 4, \dots, 10$ the open in $\mathcal{M}_{1,n}$ which exhibits this rationality is $U_n \cap V_n$.*

Proof. This was proven by Belorousski in the case where $\Bbbk$ is algebraically closed and characteristic zero in [Bel98] by constructing a bijective morphism between $U_n \cap V_n$ and an open subset of $\mathbb{P}^n$. He concludes that it is an isomorphism since $\mathbb{P}^n$ is normal. This proof does not work in arbitrary characteristic (for example the Frobenius morphism). However, Belorousski's argument showing that the morphism is bijective is in fact functorial and works in families, therefore directly establishing that the moduli stacks are isomorphic. □

Proposition 4.2.13. *The Chow ring of $\mathcal{M}_{1,n}$ is tautological for $n = 1, \dots, 10$.*

Proof. We have already shown this for $n = 1, 2, 3$. For $n \geq 4$, stratify $\mathcal{M}_{1,n}$ into $\mathcal{M}_{1,n} = (U_n \cap V_n) \sqcup \operatorname{im} \sigma_{n-1} \sqcup \operatorname{im} \tau_{n-1}$. That is, $\mathcal{M}_{1,n}$ is the union of the open locus where p_2 and p_3 are not involutions and p_2, p_3, and p_4 are not colinear along with the divisors where these conditions do hold. But $U_4 \cap V_4$ is rational by the above Proposition and hence generated in degree one, hence generated by λ_1, hence tautological. Since $\operatorname{im} \sigma_{n-1}$ and $\operatorname{im} \tau_{n-1}$ are isomorphic to opens in $\mathcal{M}_{1,n-1}$, $\mathcal{M}_{1,n}$ is inductively built out of tautological pieces, hence itself tautological. This breaks at $n = 11$ since $U_{11} \cap V_{11}$ is not birational to an open in $\mathbb{P}^{11}$ by [Bel98]. □

Theorem 4.2.14. *The integral Chow ring of $\mathcal{M}_{1,4}$ is given by*

$$\mathrm{CH}(\mathcal{M}_{1,4}) = \frac{\mathbb{Z}[\lambda_1]}{(12\lambda_1, 2\lambda_1^2)}.$$

Proof. Since $\mathcal{M}_{1,4}$ is tautological by the above Proposition, we have

$$\mathrm{CH}(\mathcal{M}_{1,4}) = \mathcal{R}(\mathcal{M}_{1,4}) = \frac{\mathbb{Z}[\lambda_1]}{(12\lambda_1, 2\lambda_1^2)}$$

by Proposition 4.2.4. □

Proposition 4.2.15. *The Chow ring of V_4 and V_4' is $\mathbb{Z}$.*

Proof. The excision sequence gives

$$\mathrm{CH}(V_3') \xrightarrow{\tau_{3*}} \mathrm{CH}(\mathcal{M}_{1,4}) \to \mathrm{CH}(V_4) \to 0$$

$$\mathbb{Z}[\lambda_1]/(2\lambda_1) \to \frac{\mathbb{Z}[\lambda_1]}{(12\lambda_1, 2\lambda_1^2)} \to \mathrm{CH}(V_4) \to 0.$$

Since by Observation 4.2.6 $\lambda_1^5 = 0$ on V_4 and $\lambda_1^5 \neq 0$ on $\mathcal{M}_{1,4}$, we see that λ_1^5 must be in the image of τ_{3*}. Hence $\tau_{3*}(\lambda_1^4) = \lambda_1^5$ in $\mathrm{CH}(\mathcal{M}_{1,4})$. But we also have $\tau_{3*}(\lambda_1^4) = \tau_{3*}(\tau_3^*(\lambda_1^4)) = \lambda_1^4\tau_{3*}(1)$. Therefore we must have $\tau_{3*}(1) = \lambda_1$, and so $\mathrm{CH}(V_4) = \mathbb{Z}$. Since $V_4' \subseteq V_4$, we then also have $\mathrm{CH}(V_4') = \mathbb{Z}$. □

Corollary 4.2.16. *The integral tautological ring of V_n and V_n' is $\mathbb{Z}$ for all $n \geq 4$.*

Proof. Since $\mathrm{CH}(V_4) = \mathrm{CH}(V_4') = \mathbb{Z}$, we in particular have $\lambda_1 = 0$ on V_4 and V_4', hence on V_n and V_n' for all $n \geq 4$. Therefore $\mathcal{R}(V_n) = \mathcal{R}(V_n') = \mathbb{Z}$. □

Proposition 4.2.17. *For $n \geq 5$, the integral tautological ring of $\mathcal{M}_{1,n}$ is*

$$\mathcal{R}(\mathcal{M}_{1,n}) = \frac{\mathbb{Z}[\lambda_1]}{(12\lambda_1, \lambda_1^2)}.$$

Proof. The excision sequence gives

$$\mathcal{R}(V_{n-1}') \to \mathcal{R}(\mathcal{M}_{1,n}) \to \mathcal{R}(V_n) \to 0$$

$$\mathbb{Z} \to \mathcal{R}(\mathcal{M}_{1,n}) \to \mathbb{Z} \to 0.$$

Since $\mathcal{R}(V_n)$ is $\mathbb{Z}$ and we are patching in the divisor $\operatorname{im} \tau_{n-1} \cong V_{n-1}'$, whose integral tautological ring is $\mathbb{Z}$, we see that the integral tautological ring of $\mathcal{M}_{1,n}$ is concentrated in degrees 0 and 1. Therefore $\mathcal{R}(\mathcal{M}_{1,n}) = \mathbb{Z}[\lambda_1]/(12\lambda_1, \lambda_1^2)$. □

Theorem 4.2.18. *For $5 \leq n \leq 10$, the integral Chow ring of $\mathcal{M}_{1,n}$ is*

$$\mathrm{CH}(\mathcal{M}_{1,n}) = \frac{\mathbb{Z}[\lambda_1]}{(12\lambda_1, \lambda_1^2)}.$$

Proof. The Chow ring of $\mathcal{M}_{1,n}$ is tautological for $5 \leq n \leq 10$ by Proposition 4.2.13 and the tautological ring was computed in the above Proposition. □

Appendix A

Structure of $\mathcal{M}_{1,1}$

We claim that there are isomorphisms

- $[\mathbb{A}_{-4,-6}/\mathbb{G}_m] \to \widetilde{\mathcal{M}}_{1,1}$,
- $[(\mathbb{A}_{-4,-6} \setminus 0)/\mathbb{G}_m] \to \overline{\mathcal{M}}_{1,1}$,
- $[(\mathbb{A}_{-4,-6} \setminus V(D))/\mathbb{G}_m] \to \mathcal{M}_{1,1}$,

where the subscript $-4, -6$ indicates that $\mathbb{G}_m$ acts with weights -4 and -6, i.e. $t \cdot (a, b) = (t^{-4}a, t^{-6}b)$, and D is the form $4a^3 + 27b^2$. We start with $\widetilde{\mathcal{M}}_{1,1}$.

A.1 Defining the morphism

By the universal property of stackification, it suffices to define a morphism

$$[\mathbb{A}_{-4,-6}/\mathbb{G}_m]^{\mathrm{pre}} \to \widetilde{\mathcal{M}}_{1,1}.$$

The objects of the prestack $[\mathbb{A}_{-4,-6}/\mathbb{G}_m]^{\mathrm{pre}}$ over a scheme S are pairs of global sections (a, b) of $\mathcal{O}_S$. There is a morphism

$$\begin{array}{ccc} (a', b') & \longrightarrow & (a, b) \\ \Big| & & \Big| \\ S' & \xrightarrow{f} & S \end{array}$$

if and only if $(f^\# a, f^\# b) = (t^{-4}a', t^{-6}b')$ for some $t \in \mathbb{G}_m(S')$. Define a morphism
$\varphi : [\mathbb{A}_{-4,-6}/\mathbb{G}_m]^{\mathrm{pre}} \to \widetilde{\mathcal{M}}_{1,1}$ by:

Objects over S: $(a,b) \mapsto C_{(a,b)} \subseteq \mathbb{P}^2_S$, where $C_{(a,b)}$ is the curve given by $y^2z = x^3 + axz^2 + bz^3$.

Morphisms over $S' \xrightarrow{f} S$: $[(a',b') \to (a,b)] \mapsto \left[C_{(a',b')} \to C_{(a,b)}\right]$. We must justify why such a morphism exists. We have such a morphism in the stack $\widetilde{\mathcal{M}}_{1,1}$ if and only if the following diagram is cartesian

$$\begin{array}{ccc} C_{(a',b')} & \longrightarrow & C_{(a,b)} \\ \downarrow & & \downarrow \\ S' & \xrightarrow{\quad f \quad} & S \end{array}$$

But the pullback of $C_{(a,b)}$ along f is the curve $C_{(f^\# a, f^\# b)}$ over S', which is isomorphic to $C_{(a',b')}$ because $(t^{-4}a', t^{-6}b') = (f^\# a, f^\# b)$.

A.2 Verifying it is an isomorphism

Now to show that φ induces an isomorphism, it suffices to show that it is fully faithful and essentially surjective. But the full faithfulness follows from what was said above: morphisms $C_{(a',b')} \to C_{(a,b)}$ in $\widetilde{\mathcal{M}}_{1,1}$ correspond precisely to $t \in \mathbb{G}_m(S')$ such that $(t^{-4}a', t^{-6}b') = (f^\# a, f^\# b)$, which correspond precisely to morphisms in $[\mathbb{A}_{-4,-6}/\mathbb{G}_m]^{\text{pre}}$.

All that remains is verifying essential surjectivity, which requires substantially more work. Here we follow [Ols16, 13.1.6.] and [Har77, Proposition IV.4.6.].

First note that it suffices to check essential surjectivity locally: if φ is essentially surjective on an open cover $\{S_i\}$ of a scheme S, then the stack map it induces is essentially surjective on S because objects of stacks glue. So let $\pi : E \to S$ be an elliptic curve with a section $e : S \to E$. Note that e is a closed immersion, as it is a section of a proper (in particular separated) map, so we may consider S as a closed subscheme of E with a sheaf of ideals $\mathscr{I}$. We will show that $\mathscr{I}$ is a Cartier divisor.

To do so, we may proceed locally and set $S = \operatorname{Spec} A$ for A a local ring with maximal ideal $\mathfrak{m}$. Letting $x \in E$ be the image of the closed point of S, we get an exact sequence

$$0 \to \mathscr{I}_x \to \mathcal{O}_{E,x} \to e_*(\mathcal{O}_S)_x \cong A \to 0$$

of stalks. Since E is flat over A and A is flat over itself, $\mathscr{I}_x$ is flat over A.

Thus tensoring with $A/\mathfrak{m}$ preserves exactness and gives

$$0 \to \mathscr{I}_x/\mathfrak{m}\mathscr{I}_x \to \mathcal{O}_{E,x}/\mathfrak{m}\mathcal{O}_{E,x} \to A/\mathfrak{m} \to 0.$$

This tells us that $\mathscr{I}_x/\mathfrak{m}\mathscr{I}_x$ is the ideal sheaf of the image of $\operatorname{Spec} A/\mathfrak{m}$ in the elliptic curve E_x. Hence it is a free $\mathcal{O}_{E,x}/\mathfrak{m}\mathcal{O}_{E,x}$ module of rank 1, as it is the maximal ideal of a DVR (the local ring is a DVR because the image of the section e lands in the smooth locus of E, DVRs are PIDs hence the ideal is principally generated, DVRs are torsion free hence the ideal is free).

Then by Nakayama's Lemma, $\mathscr{I}_x$ is generated by one element as an $\mathcal{O}_{E,x}$-module, hence isomorphic to $\mathcal{O}_{E,x}$. Letting $f \in \mathcal{O}_{E,x}$ be a generator of $\mathscr{I}_x$, in order for $\mathscr{I}_x$ to be a Cartier divisor we need f to not be a zero-divisor. In other words, we need

$$\ker\left(\cdot f : \mathcal{O}_{E,x} \to \mathcal{O}_{E,x}\right) = 0.$$

We will show a stronger statement, that the inclusion map

$$\ker\left(\cdot f : \mathscr{I}_x^i \to \mathscr{I}_x^i\right) \to \ker\left(\cdot f : \mathcal{O}_{E,x} \to \mathcal{O}_{E,x}\right)$$

is an isomorphism for each i, so that $\ker\left(\cdot f : \mathcal{O}_{E,x} \to \mathcal{O}_{E,x}\right)$ is isomorphic to a subset of the intersection of the $\mathscr{I}_x^i$'s and hence is 0.

Proceeding by induction, note that the base case $i = 0$ is a tautology. There is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{I}_x^{i+1} & \longrightarrow & \mathscr{I}_x^{i} & \longrightarrow & \mathcal{O}_S \cdot f^i & \longrightarrow & 0 \\
 & & \downarrow{\cdot f} & & \downarrow{\cdot f} & & \downarrow{\cdot 0} & & \\
0 & \longrightarrow & \mathscr{I}_x^{i+1} & \longrightarrow & \mathscr{I}_x^{i} & \longrightarrow & \mathcal{O}_S \cdot f^i & \longrightarrow & 0
\end{array}$$

A diagram chase shows that the connecting homomorphism $\ker(\cdot 0) \to \operatorname{coker}(\cdot f)$ is injective, and hence $\ker(\cdot f : \mathscr{I}_x^{i+1} \to \mathscr{I}_x^{i+1}) \to \ker(\cdot f : \mathscr{I}_x^{i} \to \mathscr{I}_x^{i})$ is an isomorphism, as required.

Therefore f is not a zero-divisor, so $\mathscr{I}_x$ is a Cartier divisor, so $\mathscr{I}$ is a Cartier divisor. Define $\mathcal{O}_S(ne) := \mathscr{I}^{-n}$.

We return to the setting where S is an arbitrary scheme. We claim that for all positive n, $\pi_*\mathcal{O}_E(ne)$ is locally free of rank n on S and that its formation commutes with base change. We will use cohomology and base change, as in [Har77, IV.12.11.]. We refer to Theorem IV.12.11(a) and (b) as CBC1 and CBC2, respectively. See also [Vak17, 28.1.6.] for a statement without the projectivity hypothesis. We reduce to the case S Noetherian using Noetherian Approximation (see [Alp]).

Notice that on the fiber of each point $s \to S$, we have

$$H^1(E_s, \mathcal{O}(ne)_s) \cong H^0(E_s, \omega \otimes \mathcal{O}(-ne)_s) = 0$$

where the isomorphism is due to Serre duality and the equality to the fact that $\omega \otimes \mathcal{O}(-ne)_s$ is of negative degree on a curve. Therefore the natural map

$$\varphi^1(s) : R^1\pi_*\mathcal{O}(ne) \otimes k(s) \to H^1(E_s, \mathcal{O}(ne)_s)$$

is trivially surjective at each point, and hence by CBC1 $R^1\pi_*\mathcal{O}(ne) \otimes k(s) = 0$ for each $s \in S$. Then by Nakayama's Lemma $R^1\pi_*\mathcal{O}(ne)$ is the zero sheaf, and in particular is locally free of rank 0. Therefore by CBC2, $\varphi^0(s)$ is also surjective for each s. But trivially $\varphi^{-1}(s)$ is surjective for each s, and so by CBC2 $R^0\pi_*\mathcal{O}(ne) = \pi_*\mathcal{O}(ne)$ is locally free. By CBC1, on each fiber $\pi_*\mathcal{O}(ne)_s \cong H^0(E_s, \mathcal{O}(ne)_s)$, so we have reduced to showing that the dimension of this is n on each fiber.

Because $e_s : \operatorname{Spec} k(s) \to E_s$ is a section of π_s on each fiber, we see that $\deg \mathcal{O}(ne)_s = n$ (a priori the degree could have been higher than n as $k(s)$ is not necessarily algebraically closed, but the presence of the section assures that the image of $\operatorname{Spec} k(s)$ is a $k(s)$-rational point). Therefore by Riemann-Roch,

$$H^0(E_s, \mathcal{O}(ne)_s) = H^0(E_s, \mathcal{O}(ne)_s) - \underbrace{H^1(E_s, \mathcal{O}(ne)_s)}_{0} = \underbrace{\deg \mathcal{O}(ne)_s}_{n} - 1 + \underbrace{g}_{1} = n,$$

which shows that $\pi_*\mathcal{O}(ne)$ is locally free of rank n.

We claim that the adjunction map $\pi^*\pi_*\mathcal{O}(3e) \to \mathcal{O}(3e)$ is surjective. Once this is established, it follows by [Har77, II.7.12.] that there is a morphism $E \to \mathbb{P}(\pi_*\mathcal{O}(3e))$ over S. Given the existence of this morphism, we know that its restriction to each fiber is a closed immersion because $\deg \pi_*\mathcal{O}(3e)_s = 3$ [Har77, IV.3.2.]. Then by [Sta18, 04XV] and [Gro67, 17.2.6] it is a closed immersion.

To establish surjectivity of $\pi^*\pi_*\mathcal{O}(3e) \to \mathcal{O}(3e)$, note that since on each fiber $\mathcal{O}(3e)_s$ is generated by global sections, the map $\pi^*\pi_*\mathcal{O}(3e)_s \to \mathcal{O}(3e)_s$ is surjective (if f is a global section of $\mathcal{O}(3e)_s$, then $f \otimes 1$ is a global section of $\pi^*\pi_*\mathcal{O}(3e)_s = \pi^*(H^0(E_s, \mathcal{O}(3e)_s)$ which maps to f). Therefore by Nakayama's Lemma the map is surjective.

This gives an embedding into a relative projective space. By shrinking S to a sufficiently small affine, we may assume that it is an embedding into $\mathbb{P}^2_A$.

Next, we find an equation for this embedding, and, using classical methods, show that $E \to S$ may be locally written in Weierstrass form.

Cohomology and base change also gives that each inclusion quotient

$$\pi_*\mathcal{O}(e) \hookrightarrow \pi_*\mathcal{O}(2e) \hookrightarrow \pi_*\mathcal{O}(3e)$$

is locally free of rank 1. Replacing S with an appropriate free neighborhood, pick a suggestively named basis

$$1 \in \Gamma(S, \pi_*\mathcal{O}(e)) \quad 1, x \in \Gamma(\pi_*\mathcal{O}(2e), S) \quad 1, x, y \in \Gamma(\pi_*\mathcal{O}(3e), S)$$

(where the inclusion maps are given by $f \mapsto f \otimes 1$). Because $\pi_*\mathcal{O}(6e)$ is locally free of rank 6 the following seven sections are linearly dependent after shrinking to a small enough neighborhood

$$1, x, y, x^2, xy, y^2, x^3 \in \Gamma(S, \pi_*\mathcal{O}(6e))$$

(where, for instance, by xy we mean $x \otimes y \otimes 1$ and by y^2 we mean $y \otimes y$). Let $\alpha_i \in \Gamma(S, \mathcal{O}_S)$ be coefficients such that

$$\alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 + \alpha_5 y + \alpha_6 xy + \alpha_6 y^2 = 0.$$

By [Har77, IV.4.6], $\alpha_4, \alpha_7 \neq 0$ on each fiber, implying that they are units in $\Gamma(S, \mathcal{O}_S)$. So without loss of generality assume that $\alpha_4 = 1$.

Let

$$G = \alpha_1 Z^3 + \alpha_2 XZ^2 + \alpha_3 X^2 Z + X^3 + \alpha_5 YZ^2 + \alpha_6 XYZ + \alpha_7 Y^2 Z.$$

By construction, $\iota : E \to \mathbb{P}^2$ factors through $V(G) \subseteq \mathbb{P}^2$, as $1, x, y$ are the coordinates of the embedding, so when they vanish so must E. By the proof of [Har77, IV.4.6] this is an isomorphism on each fiber. Let $\mathscr{J}$ be the ideal sheaf of E in $V(G)$. Since $\mathcal{O}_E$ is flat over $\mathcal{O}_S$, the sequence

$$0 \to \mathscr{J} \to \mathcal{O}_{V(G)} \to \mathcal{O}_E \to 0$$

is exact after base changing to any fiber. Since it is an isomorphism on each fiber we have $\mathscr{J} = 0$ by Nakayama's Lemma. Therefore $E \cong V(G)$.

We may now perform a change of variables so that E is in the form $V(X^3 + aXZ^2 + bZ^3)$. Then E is in the image of φ.

To summarize, we constructed a map $\varphi : [\mathbb{A}_{4,6}/\mathbb{G}^m]^{\text{pre}} \to \widetilde{\mathcal{M}}_{1,1}$. We observed that this map is fully faithful, so that all that remains is showing that it is locally essentially surjective. To show this, for any elliptic curve $E \xrightarrow{\pi} S$ we constructed Cartier divisors $\mathcal{O}_E(ne)$ which we pushed forward to S. We then embedded E into the projectivization of $\pi_*\mathcal{O}_E(3e)$ and showed that we may write E as the vanishing of the polynomial $X^3 + aX^2Z + bZ^3$, which is precisely the image of φ. The universal property of stackification and the gluing properties of stacks show that the stack map induced by φ is essentially surjective, and hence an isomorphism. This establishes

$$\widetilde{\mathcal{M}}_{1,1} \cong [\mathbb{A}_{-4,-6}/\mathbb{G}_m].$$

To see the other isomorphisms, note that an elliptic curve is cuspidal precisely when $(a, b) = (0, 0)$ and singular precisely when $D = 0$. Therefore $\overline{\mathcal{M}}_{1,1} \cong [(\mathbb{A}_{-4,-6} \setminus 0]/\mathbb{G}_m]$ and $\mathcal{M}_{1,1} \cong [(\mathbb{A}_{-4,-6} \setminus V(D))/\mathbb{G}_m]$.

Appendix B

Automorphisms of elliptic curves

In this Appendix we note the following fact about automorphisms of marked elliptic curves.

Proposition B.0.1. *Over a field $\Bbbk$ of characteristic not equal to 2 or 3, there exists: one-pointed elliptic curves with automorphism groups μ_2, μ_4, and μ_6; two-pointed elliptic curves with automorphism groups μ_2, μ_3, and μ_4; three-pointed elliptic curves with automorphism groups μ_2 and μ_3; and four-pointed elliptic curves with automorphism group μ_2. Every four-pointed elliptic curve with μ_2 automorphisms has p_2, p_3, p_4 colinear, and every n-pointed elliptic curve with $n \geq 5$ has no (non-trivial) automorphisms.*

Proof. Recall the Weierstrass form for elliptic curves:

Theorem B.0.2 (Weierstrass)**.** *Any one-pointed smooth elliptic curve over a field $\Bbbk$ of characteristic not equal to 2 or 3 can be written in the form $y^2z = x^3 + axz^2 + bz^3$, where the marked point is the point at infinity $[0 : 1 : 0]$. Moreover, if we denote such a curve by $C_{(a,b)}$, then $C_{(a,b)} \cong C_{(a',b')}$ if and only if $(a', b') = (t^4a, t^6b)$. The isomorphism between these curves is given by $[x : y : z] \mapsto [t^2x : t^3y : z]$. Lastly, an elliptic curve is smooth if and only if $D = 4a^3 + 27b^2 = 0$, nodal if and only if $D = 0$ and $(a, b) \neq (0, 0)$, and cuspidal if and only if $(a, b) = (0, 0)$.*

From this we see that an elliptic curve with n marked points over $\Bbbk$ is determined by a choice of (a, b) and $p_2, \ldots, p_n$, $p_i = (x_i, y_i)$, and that the automorphisms of this curve are given by the $t \in \mathbb{G}_m$ such that $t \cdot (a, b) = (t^4a, t^6b) = (a, b)$ and $t \cdot p_i = (t^2x_i, t^3y_i) = (x_i, y_i)$.

Now for each $m > 1$ let ζ_m denote a primitive m^{th} root of unity. From $(t^4a, t^6b) = (a, b)$ we see that the automorphism group of every one-pointed elliptic curve contains a copy of μ_2 corresponding to $t = \zeta_2 = -1$, the involution. Additionally, the curves $C_{(1,0)}$ and $C_{(0,1)}$ are fixed by $\mu_4 = \langle\zeta_4\rangle$ and $\mu_6 = \langle\zeta_6\rangle$. Since any automorphism of an n-pointed elliptic elliptic curve $(C, p_1, \ldots, p_n)$ is in particular an automorphism of (C, p_1), they must all correspond to elements of μ_2, μ_4, or μ_6.

The element ζ_2 is an automorphism of every elliptic curve and induces the map $\zeta_2 : [x : y : z] \mapsto [x : -y : z]$, and so for a point $p_i \neq \infty$ to be fixed by this we must have $p_i = [x : 0 : 1]$. Then we have

$$y^2 = x^3 + ax + b$$

$$0 = x^3 + ax + b,$$

which has at most three solutions. Therefore the involution $\iota = \zeta_2$ fixes at most four points in total. An example of a four-pointed elliptic curve with automorphism group μ_2 is $(C_{(-1,0)}, \infty, [1 : 0 : 1], [0 : 0 : 1], [-1 : 0 : 1])$. Notice that any four-pointed elliptic curve fixed by the involution must have p_2, p_3, p_4 colinear, as each point lies on the line $y = 0$.

The element ζ_4 is an automorphism of the curve corresponding to $(1, 0)$ and induces the map $\zeta_4 : [x : y : z] \mapsto [-x : \zeta_4^3 y : z]$, and so for a point $p_i \neq \infty$ to be fixed by this we must have $p_i = [0 : 0 : 1]$, which is indeed a point on the curve $C_{(1,0)}$. Therefore there is exactly one two-pointed elliptic curve with automorphism group μ_4, the curve $(C_{(1,0)}, \infty, [0 : 0 : 1])$.

The element ζ_6 is an automorphism of the curve corresponding to $(0, 1)$ and induces the map $\zeta_6 : [x : y : z] \mapsto [\zeta_3 x : -y : z]$, and so for a point $p_i \neq \infty$ to be fixed by this we must have $p_i = [0 : 0 : 1]$, which is *not* a point on the curve $C_{(0,1)}$. Therefore there is no n-pointed elliptic curve with automorphism group μ_6 for $n \geq 2$.

Lastly, the element $\zeta_6^2 = \zeta_3$ is an automorphism of the curve corresponding to $(0, 1)$ and induces the map $\zeta_3 : [x : y : z] \mapsto [\zeta_3^2 x : y : z]$, and so for a point $p_i \neq \infty$ to be fixed by this we must have $p_i = [0 : y : z]$. Then we have

$$y^2 = x^3 + ax + b$$

$$y^2 = 1.$$

Therefore an example of a three pointed elliptic curve with automorphism group μ_3 is $(C_{(0,1)}, \infty, [0 : 1 : 1], [0 : -1 : 1])$.

This exhausts all possible automorphisms, and so there are no n-pointed elliptic curves with non-trivial automorphisms for $n \geq 5$. $\square$

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