Delta Matroids and the Geometry of Polynomials

Tracy Chin

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Reading Committee:

Cynthia Vinzant, Chair

Gaku Liu

Rekha Thomas

Program Authorized to Offer Degree:

Department of Mathematics

University of Washington

Abstract

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Chair of the Supervisory Committee:
Cynthia Vinzant
Department of Mathematics

Dating back to Newton, mathematicians have long studied the interplay between the properties of polynomials as functions and their combinatorial properties. Recently, this has taken the form of stable and Lorentzian polynomials. In this thesis, we explore various facets of the connection between continuous and discrete properties of polynomials, with a particular focus on Δ-matroids.

This thesis is comprised of four projects, all related to Δ-matroids or the geometry of polynomials. In the first project, we show that it is coNP-hard to decide whether a polynomial of fixed degree is real stable or log concave, while we can decide Lorentzianity in polynomial time.

Next, we turn our attention to the intersection of Lorentzianity and symmetric functions. Our main result is a reduction scheme that significantly reduces the complexity of testing for Lorentzianity. Using this method, we provide explicit semialgebraic descriptions of the spaces of Lorentzian symmetric polynomials and functions for degrees up to six. These techniques can also be applied to simplify the proofs to known cases of Lorentzian symmetric functions. We conclude by showing that some natural symmetric operators fail to preserve Lorentzianity which in turn highlights an inherent tension between symmetry in variables and the Lorentzian property.

In the third chapter, we introduce valuated Δ-matroids, a natural generalization of two objects of study in matroid theory: valuated matroids and Δ-matroids. We show that these objects exhibit nice properties analogous to ordinary valuated matroids. We also show that these objects arise as the valuations of principal minors of a Hermitian matrix over a valued field, generalizing other forms of Δ-matroid representability.

Finally, we conclude with ongoing work trying to establish a theory analogous to Lorentzian polynomials for Δ-matroids. We describe the motivations and applications that would come from such a result, and we discuss some partial progress towards it. This chapter aims to be a guide for any future research in the area, with emphasis placed on likely avenues for attacking the problem as well as warnings for potential pitfalls.

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Finally, I would like to thank my family, for all of the love and support they have given me over the years, even (and especially) when they didn't really understand the math I was trying to tell them about.

DEDICATION

To my parents, Bryan and Amanda,

and especially to their cats, Turbo and Sammy

Chapter 1

INTRODUCTION

Dating back to Newton's ultra log concavity inequalities for real-rooted polynomials, mathematicians have long studied the interplay between geometric properties of a polynomial and combinatorial properties of its coefficients.

Theorem (Newton's Inequalities)**.** *If $f(x) = \sum_{k=0}^{d} c_k x^k$ is real-rooted, then*

$$\left(\frac{c_k}{\binom{d}{k}}\right)^2 \geq \frac{c_{k-1}}{\binom{d}{k-1}} \frac{c_{k+1}}{\binom{d}{k+1}}$$

for all $1 \leq k \leq d-1$.

More recently, these questions have seen a resurgence of activity in the form of stable and Lorentzian polynomials. Stable polynomials are a natural multivariate generalization of real-rootedness, while Lorentzian polynomials are a further generalization of stable polynomials that maintain many of the same combinatorial properties. For background on stable and Lorentzian polynomials, refer to Section 2.4 and Section 2.5, respectively. While these classes are defined purely in algebraic and geometric terms, they have nontrivial combinatorial consequences.

For example, stability is defined purely in terms of the polynomial being nonvanishing in a certain region, but implies a fairly strong combinatorial condition on what monomials can appear [Brä07].

Definition 1.0.1. Let $f = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} c_\alpha \left(\prod_{i=1}^n x_i^{\alpha_i}\right) \in R[x_1, \ldots, x_n]$ be a polynomial in n variables. Then the *support* of f is the subset of $\mathbb{Z}_{\geq 0}^n$ corresponding to monomials appearing with nonzero coefficient in f:

$$\operatorname{supp}(f) = \{\alpha : c_\alpha \neq 0\}.$$

Namely, the support of a stable polynomial is always a jump system, which is a combinatorial object generalizing matroids and Δ-matroids. (Jump systems are discussed in more detail in Section 2.3.2.)

Real stability also implies discrete inequalities, similar to how real-rootedness implies Newton's inequalities. For example, if $f = \sum_{S \subseteq [n]} c_S \left(\prod_{i \in S} x_i\right)$ has nonnegative coefficients and degree one in each variable, one can view it as a generating function for a probability distribution on its support, where $\mathbb{P}[S] \propto c_S$. If f is real stable, then this probability distribution exhibits negative dependence: for all $i, j \in [n]$, $\mathbb{P}[i, j \in S] \leq \mathbb{P}[i \in S]\mathbb{P}[j \in S]$ [BBL09].

However, while the support of any homogeneous multiaffine stable polynomial is a matroid [COSW04, Theorem 7.1], the converse does not hold; there are matroids that are not the support of any stable polynomial [Brä07, Section 6].

Lorentzianity, also known as complete log-concavity, is a generalization of real stability. First defined simultaneously in [BH20] and [AOV21], it preserves many of the combinatorial properties of stable polynomials. For example, the support of a Lorentzian polynomial is always M-convex, which is a combinatorial property generalizing the basis exchange axiom for matroids. (See Section 2.2.2 for more about M-convexity).

Moreover, as with Newton's inequalities and as with real stable polynomials, the coefficients of a Lorentzian polynomial also exhibit discrete log concavity [BH20, Proposition 4.4]. This fact was instrumental in one of the first applications of Lorentzianity, proving the strongest form Mason's conjecture for independent sets of matroids [ALOV24b, Theorem 1.2] [BH20, Theorem 4.14]. More recently, this property was used to resolve a special case of Fox's conjecture [HMV25], and to prove ultra log concavity of the W polynomial for certain classes of posets [AJ25].

Lorentzian polynomials also have algorithmic applications. In particular, if the generating polynomial of a probability distribution of Lorentzian, then there is a polynomial-time algorithm to sample from the distribution using a down-up random walk [AOV21]. In particular, since the basis generating polynomial of any matroid is Lorentzian, one can approximately uniformly sample from the bases of a matroid in polynomial time. By the equivalence of (approximate) sampling and counting [JVV86], this also gives an algorithm to approximately count the number of bases of any matroid.

On the other hand, Lorentzian polynomials no longer exhibit the negative dependence properties we see in real stability. However, Lorentzianity still implies bounded positive correlations [BH20, Proposition 2.17].

One advantage of Lorentzianity over stability is that we have a full characterization of all possible supports: a nonempty set $J \subseteq \mathbb{N}^n$ is the support of a Lorentzian polynomial if and only if it is M-convex [BH20, Theorem 3.10]. This connection extends further to valuated matroids and M-convex functions: tropicalizations of Lorentzian polynomials are M-convex functions, and all M-convex functions are limits of tropicalizations of Lorentzian polynomials [BH20, Corollary 3.28].

Lorentzian polynomials also have the advantage that we can check whether a polynomial is Lorentzian in polynomial time, while there is no such algorithm for checking real stability unless $P = NP$. The details of both coNP-hardness of testing real stability and the polynomial-time algorithm for Lorentzianity can be found in [Chi24] and Chapter 3.

The contents of this thesis are arranged as follows. In Chapter 2, we give an overview of some of the objects used throughout this work, including matroids, Δ-matroids, stable polynomials, and Lorentzian polynomials. We cover background and results about each of these objects, generalizations thereof, and relationships between them.

In Chapter 3, we cover results from [Chi24] about the computational complexity of testing stability and log concavity. This section also further explores the relationship between real stability, Lorentzianity, and log concavity.

In Chapter 4, we study Lorentzian symmetric polynomials and Lorentzian symmetric functions, covering results from [CQ25]. In particular, the symmetric structure allows us to simplify the conditions for checking Lorentzianity considerably.

In Chapter 5, we cover results from [CCLV25] defining and studying valuated Δ-matroids. Of particular interest is the connection between valuated Δ-matroids and principal minors and isotropic linear spaces.

Finally, in Chapter 6, we discuss the state-of-the-art in our ongoing work to generalize Lorentzian polynomials to the Δ-matroid setting. In particular, we seek to prove that (homogenized) generating polynomials of even Δ-matroids are fractionally log concave. We outline the motivation for trying to prove this result arising from mixing times and give some partial progress towards proving it. Specifically, we begin by demonstrating some general results on fractional log concavity, including various necessary and sufficient conditions. We then describe a few attempts at generalizing the story of Lorentzianity from matroids to even Δ-matroids and propose a general plan of attack towards resolving the conjecture. We conclude by enumerating a list of some interesting examples and counterexamples and discuss how they should guide any future attempts at proving this conjecture.

Chapter 2

PRELIMINARIES

2.1 Notation and Basic Definitions

We begin by fixing some notation and definitions which will be used throughout this thesis.

First, we establish some shorthand.

- We write $[n]$ to denote the set of numbers $\{1, \dots, n\}$.

- For $S \subseteq [n]$, we write $\mathbb{1}_S$ or e_S for the indicator vector of S. That is $\mathbb{1}_S(i) = 1$ if $i \in S$ and zero otherwise.

- We write $2^{[n]}$ to denote the power set of $[n]$, i.e. the set of all subsets of $[n]$.

- We use the notation $\binom{[n]}{k}$ to denote all size-k subsets of $[n]$.

- Unless otherwise specified, we use Δ to denote the symmetric difference $S \Delta T = (S \setminus T) \cup (T \setminus S) = (S \cup T) \setminus (S \cap T)$.

- For a vector $\alpha \in \mathbb{R}^n$, unless otherwise specified, we use $|\alpha|$ to denote the 1-norm: $|\alpha| \coloneqq \sum_{i=1}^n |\alpha_i|$.

Next, some notational conventions for polynomials.

- We write $\mathbb{R}[x_1, \dots, x_n]$ and $\mathbb{C}[x_1, \dots, x_n]$ for polynomials with real and complex coefficients respectively. Moreover, we write $\mathbb{R}_{\geq 0}[x_1, \dots, x_n]$ to denote the set of real polynomials with nonnegative coefficients.

- Similarly, we write $\mathbb{R}\{\{t\}\}[x_1, \dots, x_n]$ and $\mathbb{C}\{\{t\}\}[x_1, \dots, x_n]$ for polynomials over real (resp. complex) Puiseux series. We refer the reader to [MS15] for relevant background on Puiseux series.

- We write $R[x_1, \dots, x_n]_d$ for the set of homogeneous polynomials of degree d over R.

- For $\kappa \in \mathbb{Z}^n_{\geq 0}$, we write $R[x_1, \ldots, x_n]_{\leq \kappa}$ to denote the set of polynomials over R such that $\deg_{x_i}(f) \leq \kappa_i$ for all $i = 1, \ldots, n$.

- For $\alpha \in \mathbb{Z}^n_{\geq 0}$, we use x^α to denote the monomial $x^\alpha := \prod_{i=1}^n x_i^{\alpha_i}$. Similarly, for $S \subseteq [n]$, we use the shorthand $x^S := \prod_{i \in S} x_i$.

- For derivatives, we use the shorthand $\partial_i := \frac{\partial}{\partial x_i}$ and similarly $\partial^2_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$. For $\alpha \in \mathbb{Z}^n_{\geq 0}$, we use the shorthand $\partial^\alpha := (\prod_{i=1}^n \partial_i^{\alpha_i})$.

- For $v \in \mathbb{R}^n$, we denote the derivative in direction v by $D_v := \sum_{i=1}^n v_i \partial_i$.

- We denote the gradient of a function by $\nabla f := (\partial_1 f, \ldots, \partial_n f)$. We also recall the Hessian, denoted $\nabla^2 f$, which is the $n \times n$ symmetric matrix with $(\nabla^2 f)_{ij} := \frac{\partial^2}{\partial x_i \partial x_j} f$.

2.2 Matroids

Matroids are a well-studied combinatorial object that generalizes the notion of linear independence. In this section, we briefly recall some foundational results about matroids and introduce some important generalizations. For further information on matroids, we refer the reader to [Oxl11].

2.2.1 Introduction to Matroids

There are several equivalent definitions that characterize matroids. For the purposes of this thesis, we mostly use the definition in terms of bases.

Definition 2.2.1. A *matroid* is a pair $(E, \mathcal{B})$ of a (finite) ground set E and collection $\mathcal{B} \subseteq 2^E$ such that

(B1) $\mathcal{B} \neq \varnothing$, and

(B2) for all $A, B \in \mathcal{B}$ and all $i \in A \setminus B$, there exists $j \in B \setminus A$ such that $A \setminus \{i\} \cup \{j\} \in \mathcal{B}$.

One can check that it follows from the definition that all elements of $\mathcal{B}$ have the same cardinality, which we call the *rank* of M.

The latter condition is known as the *basis exchange axiom*. A foundational fact in matroid theory is that (B2) is equivalent to the seemingly stronger *strong basis exchange axiom*:

(B2') for all $A, B \in \mathcal{B}$ and all $i \in A \setminus B$, there exists $j \in B \setminus A$ such that $A \setminus \{i\} \cup \{j\} \in \mathcal{B}$ and $B \setminus \{j\} \cup \{i\} \in \mathcal{B}$.

One important class of matroids is those arising from linear independence.

Example 2.2.2. Let $v_1, \ldots, v_n \in K^d$ be vectors such that $\mathrm{span}\{v_1, \ldots, v_n\} = K^d$, where K is any field. Let $\mathcal{B} = \{T \subseteq [n] : \{v_i : i \in T\}$ forms a basis for $K^d\}$. Then $M = ([n], \mathcal{B})$ is a matroid.

Put another way, if

$$A = \begin{pmatrix} | & & | \\ v_1 & \ldots & v_n \\ | & & | \end{pmatrix}$$

is the $d \times n$ matrix with columns $v_1, \ldots, v_n$, then the bases of M are exactly the sets $S \in \binom{[n]}{d}$ such that $\det((v_i : i \in S)) \neq 0$.

Matroids arising from this construction are called *linear* or *representable matroids*.

For a concrete example, consider

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

This gives a matroid with bases $\mathcal{B} = \{12, 13, 14, 23, 24\}$.

Checking the basis exchange axiom for a matroid can also be characterized in terms of polytopes.

Proposition 2.2.3 ([GGMS87]). *Suppose $\mathcal{B} \subseteq 2^{[n]}$ is nonempty, and let $P_{\mathcal{B}} = \mathrm{conv}\{\mathbb{1}_B : B \in \mathcal{B}\}$ be the convex hull of the indicator vectors of the members of $\mathcal{B}$. Then $([n], \mathcal{B})$ is a matroid if and only if all edges of $\mathcal{B}$ are in directions $e_i - e_j$.*

Example 2.2.4. Consider the matroid on ground set $[4]$ with bases $\mathcal{B} = \{12, 13, 14, 23, 24\}$, as described in Example 2.2.2. The convex hull of these bases is the square pyramid shown in Figure 2.1.

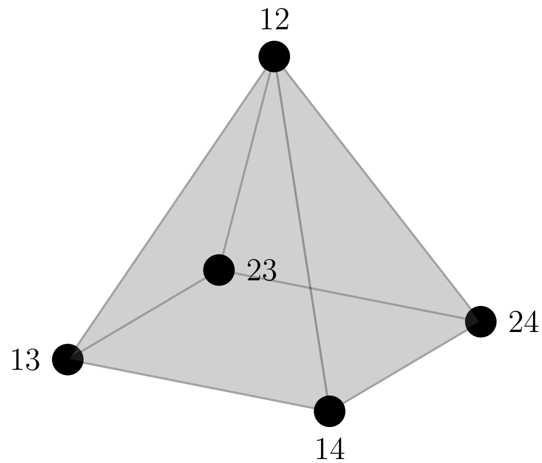

Figure 2.1: Basis polytope for the matroid in Example 2.2.2.

One can check that all of the edges of this polytope are in directions $e_i - e_j$.

2.2.2 M-convex Sets

Matroids are fundamentally based on subsets of the ground set, which we can identify with vectors in $\{0, 1\}^n \subset \mathbb{Z}^n$. M-convex sets are a generalization of matroids that allow us to extend matroid-like properties beyond just 0/1-vectors.

Definition 2.2.5 ([Mur03, Chapter 4])**.** We say a set $J \subseteq \mathbb{Z}^n$ is *M-convex* if

(M1) $J \neq \varnothing$, and

(M2) for all $\alpha, \beta \in J$ and all $i \in [n]$ such that $\alpha_i > \beta_i$, there exists j such that $\alpha_j < \beta_j$ and $\alpha - e_i + e_j, \beta + e_i - e_j \in J$.

As with matroids, one can check that it follows from this definition that $\sum \alpha_i = \sum \beta_i$ for all $\alpha, \beta \in J$.

M-convex sets generalize matroids in the sense that the indicator vectors of the bases of any matroid form an M-convex set.

Example 2.2.6. Let $M = ([n], \mathcal{B})$ be a matroid, and let $J = \{\mathbb{1}_B : B \in \mathcal{B}\}$. Then, by the strong basis exchange axiom, J is M-convex.

We can also characterize M-convex sets in terms of polytopes.

Theorem 2.2.7 ([Mur03, Chapter 4])**.** *A set $J \subseteq \mathbb{Z}^n$ if and only if $J = \operatorname{conv}(J) \cap \mathbb{Z}^n$ and all edges of $\operatorname{conv}(J)$ are in directions $e_i - e_j$.*

Example 2.2.8. Consider

$$J = \{(2,1,0), (2,0,1), (1,2,0), (1,1,1), (1,0,2), (0,2,1), (0,1,2), (0,0,3)\} \subseteq \mathbb{Z}^3.$$

One can check from the definition that J is an M-convex set.

Moreover, we note that the convex hull of J, as depicted in Figure 2.2, has all edges in directions $e_i - e_j$, and that J is exactly the integer points of its convex hull.

2.2.3 Valuated Matroids

In a different direction, one can generalize matroids to valuated matroids. Note that our definition differs from that in [DW90] by a sign.

Definition 2.2.9 ([DW90])**.** A *valuated matroid* of rank r on ground set $[n]$ is a function $\rho : \binom{[n]}{r} \to \mathbb{R} \cup \{\infty\}$ such that for all $A, B \in \binom{[n]}{r}$ and all $i \in A \setminus B$, there exists $j \in B \setminus A$ such that

$$\rho(A) + \rho(B) \geq \rho(A \setminus \{i\} \cup \{j\}) + \rho(B \setminus \{j\} \cup \{i\}).$$

The *support* of ρ is $\operatorname{supp}(\rho) = \{B \in \binom{[n]}{r} : \rho(B) < \infty\}$. It follows from the definition that the support of any valuated matroid must be a matroid.

The converse also holds in the sense that every matroid can be made into a valuated matroid.

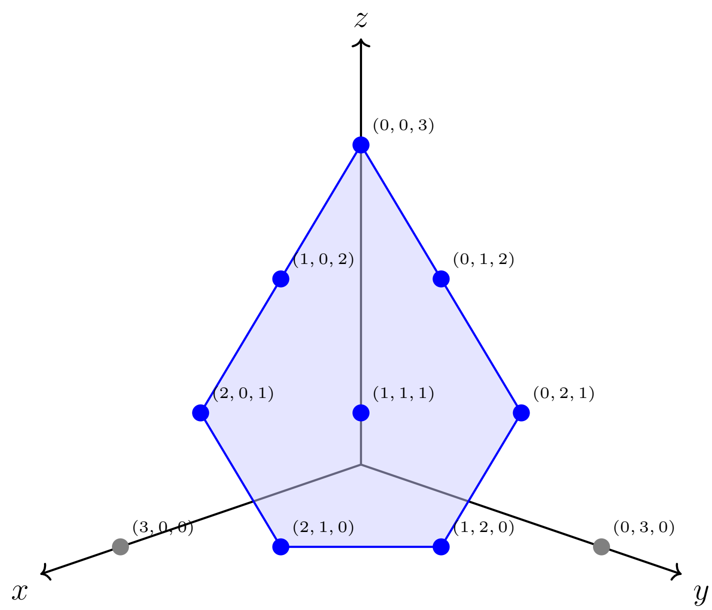

Figure 2.2: Convex hull of the M-convex set from Example 2.2.8.

Example 2.2.10. Let $M = ([n], \mathcal{B})$ be a matroid of rank r. Define $\rho_M : \binom{[n]}{r} \to \mathbb{R} \cup \{\infty\}$ by

$$\rho_M(B) = \begin{cases} 0, & \text{if } B \in \mathcal{B} \\ \infty, & \text{otherwise} \end{cases}.$$

Then ρ_M is a valuated matroid.

There is also a less-trivial construction involving rank functions.

Example 2.2.11 ([Shi12, Theorem 3])**.** Let $M = ([n], \mathcal{B})$ be a matroid of rank r, and define $\mathrm{rk}_M : \binom{[n]}{r} \to \mathbb{Z}_{\geq 0}$ by

$$\mathrm{rk}_M(S) = \max_{B \in \mathcal{B}} |S \cap B|.$$

Then $-\mathrm{rk}_M$ is a valuated matroid.

We can also construct valuated matroids by looking at representable matroids over valued fields.

Example 2.2.12. Let K be a field with non-Archimedean valuation $\nu : K \to \mathbb{R} \cup \{\infty\}$, and let $v_1, \ldots, v_n \in K^d$ such that $\operatorname{span}\{v_1, \ldots, v_n\} = K^d$. Define $\rho : \binom{[n]}{d} \to \mathbb{R} \cup \{\infty\}$ by

$$\rho(S) = \nu(\det(v_i : i \in S)).$$

Then ρ is a valuated matroid.

For a concrete example, consider

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1+t \end{pmatrix} \in \mathbb{R}\{\{t\}\}^{2\times 4}.$$

Then

$$\rho(\{i,j\}) = \begin{cases} 1, & \text{if } \{i,j\} = \{3,4\} \\ 0, & \text{otherwise} \end{cases}.$$

One can check that this function satisfies the valuated exchange axiom.

Finally, we can again characterize valuated matroids in terms of polytopes.

We recall that for a finite set $\{p_1, \ldots, p_m\} \subseteq \mathbb{R}^n$, a height function w induces a *regular subdivision* of $\operatorname{conv}(p_1, \ldots, p_m)$. Specifically, the cells of the subdivision are given by the lower faces of the lifted polytope $\operatorname{conv}((p_1, w(p_1)), \ldots, (p_n, w(p_n))) \subseteq \mathbb{R}^{n+1}$. (Recall that a lower face is a face that can be defined by minimizing a linear functional whose last coordinate is positive.) For further background on regular subdivisions, we refer the reader to [DRS10].

Proposition 2.2.13 (See, e.g., [MS15, Lemma 4.4.6])**.** *Let $\rho : \binom{[n]}{r} \to \mathbb{R} \cup \{\infty\}$ be a function. Then ρ is a valuated matroid if and only if all edges in the regular subdivision of* $\operatorname{conv}(\operatorname{supp}(\rho))$ *induced by ρ are in directions $e_i - e_j$.*

Example 2.2.14. Consider the valuated matroid from Example 2.2.12. The subdivision induced by this height function is shown in Figure 2.3.

One can check that all edges in this subdivision are in directions $e_i - e_j$, as desired.

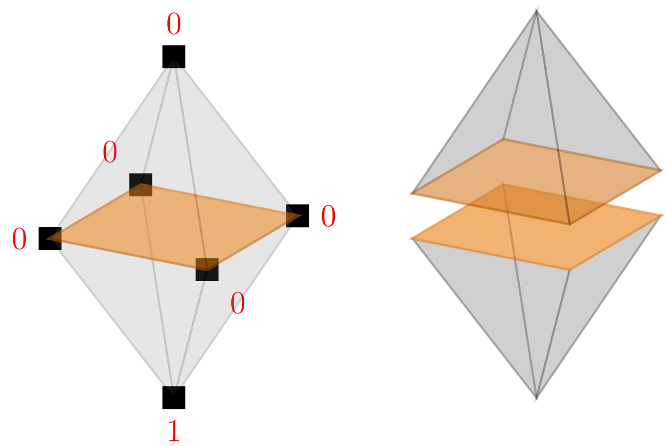

Figure 2.3: The regular subdivision induced by the height function in Example 2.2.12.

2.2.4 M-convex Functions

Finally, we consider M-convex functions, which generalize both valuated matroids and M-convex sets.

Definition 2.2.15. A function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{\infty\}$ is *M-convex* if $\operatorname{supp}(f) \neq \varnothing$ and for all $\alpha, \beta \in \mathbb{Z}^n$ and all i such that $\alpha_i > \beta_i$, there exists j such that $\alpha_j < \beta_j$ and

$$f(\alpha) + f(\beta) \geq f(\alpha - e_i + e_j) + f(\beta + e_i - e_j).$$

M-convex functions are an extension of both M-convex sets and valuated matroids. In particular, a valuated matroid is exactly an M-convex function supported on a subset of $\{0,1\}^n$. On the other hand, much as every matroid can naturally become a valuated matroid (Example 2.2.10), a similar construction allows us to turn any M-convex set into an M-convex function.

The relation between these four objects can be summarized in the following diagram.

$$\begin{tikzcd}
\text{matroids} \arrow[r, hook] \arrow[d, hook] & M\text{-convex sets} \arrow[d, hook] \\
\text{valuated matroids} \arrow[r, hook] & M\text{-convex functions}
\end{tikzcd}$$

2.3 Delta Matroids

Delta matroids (also written Δ-matroids) generalize matroids in a different direction, where we allow our bases to not all be the same size. For an introductory survey on Δ-matroids, we refer the reader to [Mof19].

Definition 2.3.1. A Δ-*matroid* is a pair $(E, \mathcal{B})$ of a (finite) ground set E and a collection $\mathcal{B} \subseteq 2^E$ such that

(B1) $\mathcal{B} \neq \varnothing$, and

(B2) for all $S, T \in \mathcal{B}$ and $i \in S\Delta T$, there exists $j \in S\Delta T$ such that $S\Delta\{i, j\} \in \mathcal{B}$.

Note we allow for the case $i = j$ in (B2). In particular, the size of S after doing the exchange may decrease by one or two, stay the same, or increase by one or two. The cases of the exchange axiom are illustrated in Figure 2.4.

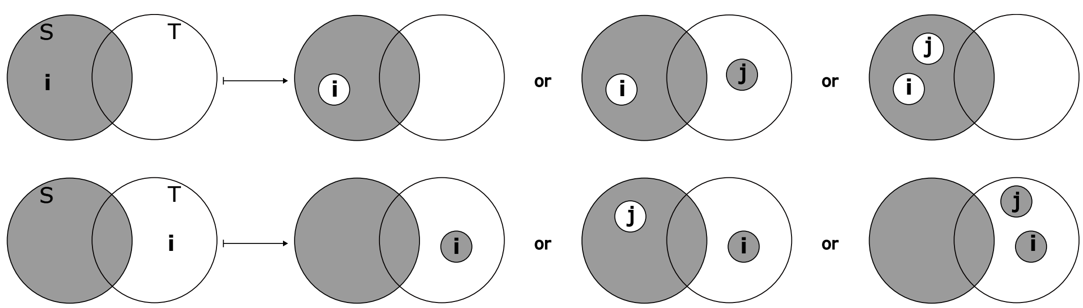

Figure 2.4: The different cases that can occur in the Δ-matroid exchange axiom. Adapted from [Mof19, Figure 1].

Example 2.3.2. If $([n], \mathcal{B})$ is a matroid, then it is also a Δ-matroid, since the basis exchange condition for matroids is strictly stronger than the Δ-matroid condition.

Example 2.3.3 ([Mof19, Section 2.4.6])**.** Let $G = (V, E)$ be a simple graph. For a matching $M \subseteq E$, let $V(M) \subseteq V$ denote the set of vertices that are incident to an edge in M. Then $(V, \mathcal{B})$ is a Δ-matroid, where $\mathcal{B} = \{V(M) : M \text{ matching on } G\}$. This Δ-matroid is known as the *matching Δ-matroid* of G.

Example 2.3.4 ([Bou88])**.** Let $M \in K^{n\times n}$ be a symmetric or skew-symmetric matrix over a field K. Let $\mathcal{B} = \{S \subseteq [n] : \det(M[S,S]) \neq 0\}$ denote the set of nonvanishing principal minors of M. Then $([n], \mathcal{B})$ is a Δ-matroid. A Δ-matroid arising in this way is called *representable.*

This coincides with the notion of matroid representability.

Proposition 2.3.5 ([Bou88, Theorem 4.4])**.** *A matroid $([n], \mathcal{B})$ is representable over a field K in the sense of matroid representability (Example 2.2.2) if and only if it is representable as a Δ-matroid by a skew-symmetric matrix over K in the sense of Example 2.3.4.*

Example 2.3.6. Consider the matroid with bases $\mathcal{B} = \{12, 13, 14, 23, 24\}$ over ground set $[4]$. As we saw in Example 2.2.2, this matroid is representable over any field of characteristic $\neq 2$.

It also a representable Δ-matroid, since $\mathcal{B}$ is exactly the nonvanishing principal minors of the skew-symmetric matrix

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix}.$$

Similarly to matroids, we can characterize Δ-matroids in terms of polytopes.

Proposition 2.3.7. *Suppose $\mathcal{B} \subseteq 2^{[n]}$ is nonempty and let $P_{\mathcal{B}} = \operatorname{conv}\{\mathbb{1}_B : B \in \mathcal{B}\}$. Then $([n], \mathcal{B})$ is a Δ-matroid if and only if all edges of $P_{\mathcal{B}}$ are in directions $\pm e_i, e_i \pm e_j$.*

Also like matroids, there is a natural strong exchange version of (B2).

(B2') for all $S, T \in \mathcal{B}$ and $i \in S\Delta T$, there exists $j \in S\Delta T$ such that $S\Delta\{i,j\}, T\Delta\{i,j\} \in \mathcal{B}$.

However, unlike matroids, (B2') is strictly stronger than (B2).

Example 2.3.8. Consider the Δ-matroid $\mathcal{B} = \{\varnothing, 12, 13, 23, 123\}$ on ground set $[3]$. One can verify that this is a Δ-matroid either by directly checking the axioms or by observing that the basis polytope edges are all in directions $\pm e_i$ or $e_i \pm e_j$ (Figure 2.5).

However, this Δ-matroid does not satisfy (B2'). Indeed, if we take $S = \varnothing, T = 123, i = 1$, then the only way we have $T\Delta\{1,j\} \in \mathcal{B}$ is if $j = 1$. On the other hand, if $j = 1$, then $S\Delta\{i,j\} = \{1\} \notin \mathcal{B}$. Hence, there does not exist a $j \in S\Delta T$ such that $S\Delta\{1,j\}, T\Delta\{1,j\}$ are both in $\mathcal{B}$.

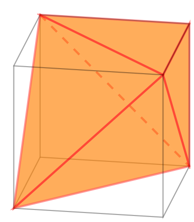

Figure 2.5: Basis polytope for the Δ-matroid in Example 2.3.8.

A natural operation on Δ-matroids is twisting.

Definition 2.3.9. Let $D = (E, \mathcal{B})$ be a Δ-matroid and fix $X \subseteq E$. The *twist* of D by X is the Δ-matroid on ground set E with bases

$$\mathcal{B}\Delta X \coloneqq \{B\Delta X : B \in \mathcal{B}\}.$$

It is easily verified from the definition that the twist of a Δ-matroid by any X is again a Δ-matroid. One can also see this geometrically by noting that twisting by X corresponds to applying some reflective symmetries of the $[0,1]^E$ hypercube to $P_{\mathcal{B}}$, which preserves the fact that all edges are in directions $\pm e_i, e_i \pm e_j$.

It is easy to check that the maximum cardinality bases of a Δ-matroid form the bases of a matroid. Indeed, if $S, T \in \mathcal{B}$ have maximum cardinality, let $i \in T \setminus S \subseteq S \Delta T$. Since S has maximum cardinality, we know that if $S\Delta\{i,j\} \in \mathcal{B}$, then $j \in S \setminus T$. Hence, for all $i \in T \setminus S$, there exists $j \in S \setminus T$ such that $(S \cup \{i\}) \setminus \{j\}$ is a maximum-cardinality basis in $\mathcal{B}$, so we satisfy the matroid basis exchange axiom. This fact, together with twisting, is sufficient to fully characterize Δ-matroids.

Theorem 2.3.10 ([BGW03, Theorem 4.1.1])**.** *A pair $([n], \mathcal{B})$ is a Δ-matroid if and only if for all $X \subseteq [n]$, the maximum cardinality sets in $\mathcal{B}\Delta X$ form the bases of matroid.*

2.3.1 Even Δ-matroids

One special class of Δ-matroids is *even Δ-matroids*.

Definition 2.3.11. A Δ-matroid $([n], \mathcal{B})$ is *even* if all elements of $\mathcal{B}$ have the same parity, either all even size or all odd size.

Example 2.3.12. Since all bases of a matroid are the same size and hence have the same parity, all matroids are even Δ-matroids.

Example 2.3.13. Since any matching saturates an even number of vertices, matching Δ-matroids (Example 2.3.3) are always even Δ-matroids.

Example 2.3.14. Representable Δ-matroids may or may not be even. For example, if M is skew-symmetric, we know that all odd-size minors of a skew-symmetric matrix are singular, so we get an even Δ-matroid with all even-size bases.

On the other hand, the Δ-matroid from Example 2.3.8 is representable and is not even. To see that it is representable, consider the matrix

$$M = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

over a field K with $\mathrm{char}(K) \neq 2$. We note that all principal minors are nonsingular except for the 1×1 principal minors, which gives us exactly the Δ-matroid from Example 2.3.8.

This can also be characterized in terms of exchange axioms, which many sources use as the definition.

Proposition 2.3.15 ([Wen93, Theorem 1])**.** *Let $\mathcal{B} \subseteq 2^{[n]}$ be nonempty. Then $([n], \mathcal{B})$ is an even Δ-matroid if and only if*

(E2) for all $S, T \in \mathcal{B}$ and $i \in S\Delta T$, there exists $j \in (S\Delta T) \setminus \{i\}$ such that $S\Delta\{i, j\} \in \mathcal{B}$.

Proof. ($\Rightarrow$) First, suppose that $\mathcal{B}$ is an even Δ-matroid, and let $S, T \in \mathcal{B}, i \in S\Delta T$. By the Δ-matroid exchange axiom, we know there exists $j \in S\Delta T$ such that $S\Delta\{i, j\} \in \mathcal{B}$. Moreover, since $\mathcal{B}$ is even, we know that $S\Delta\{i\} \notin \mathcal{B}$, since S and $S\Delta\{i\}$ have opposite parities. Hence, since we are guaranteed the existence of a j, and since $j = i$ does not satisfy the exchange axiom, we conclude there exists $j \in (S\Delta T) \setminus \{i\}$ such that $S\Delta\{i, j\} \in \mathcal{B}$, as desired.

($\Leftarrow$) On the other hand, suppose (E2) holds. We note that (E2) implies (B2), so $([n], \mathcal{B})$ is a Δ-matroid. Hence it just remains to show that all elements of $\mathcal{B}$ have the same parity.

Indeed, suppose for the sake of contradiction that $\mathcal{B}$ is not constant parity. Let $S, T \in \mathcal{B}$ such that S and T have opposite parities, and choose S, T such that $|S\Delta T|$ is minimal. Since S and T have opposite parities, we know that $S\Delta T \neq \varnothing$, so we may choose an arbitrary $i \in S\Delta T$. Then, by assumption, there exists $j \in (S\Delta T) \setminus \{i\}$ such that $S' \coloneqq S\Delta\{i, j\} \in \mathcal{B}$. We note that since $i \neq j$, the parity of S' is the same as the parity of S, so S' and T have opposite parities. Moreover, we note $S'\Delta T = S\Delta\{i, j\}\Delta T = (S\Delta T)\Delta\{i, j\} = (S\Delta T)\setminus\{i, j\}$. Hence, we have $S', T \in \mathcal{B}$ of opposite parities such that $|S'\Delta T| < |S\Delta T|$, which contradicts the minimality of $|S\Delta T|$.

Hence, if (E2) holds, then $([n], \mathcal{B})$ is a Δ-matroid and all elements of $\mathcal{B}$ have the same parity, so in particular it is an even Δ-matroid, as desired. $\square$

Unlike the general Δ-matroid case, even Δ-matroids admit a strong exchange axiom.

Theorem 2.3.16 ([Wen93, Theorem 1])**.** *A pair* $([n], \mathcal{B})$ *is an even* Δ*-matroid if and only if*

(E1) $\mathcal{B} \neq \varnothing$*, and*

(E2') for all $S, T \in \mathcal{B}$ *and all* $i \in S\Delta T$*, there exists* $j \in (S\Delta T) \setminus \{i\}$ *such that* $S\Delta\{i,j\}$ *and* $T\Delta\{i,j\}$ *are both in* $\mathcal{B}$*.*

2.3.2 Jump Systems

Like matroids, Δ-matroids are confined to subsets of $[n]$, or equivalently to the vertices of the n-hypercube, $\{0,1\}^n$. Also like matroids, we want to generalize this story to $\mathbb{Z}^n$. In the Δ-matroid case, this takes the form of *jump systems*.

For integer vectors $x, y \in \mathbb{Z}^n$, an (x, y)-increment is a unit vector $s = \pm e_i$ such that $\min\{x_i, y_i\} \leq (x+s)_i \leq \max\{x_i, y_i\}$. Intuitively, it is a unit step from x toward y.

Definition 2.3.17. A nonempty set $J \subseteq \mathbb{Z}^n$ is a *jump system* if it satisfies the two-step axiom:

> (two-step axiom) if $x, y \in J$ and s is an (x, y)-increment, then either $x + s \in J$ or there exists an $(x + s, y)$-increment t such that $x + s + t \in J$.

In the case where $J \subseteq \{0,1\}^n$, this is essentially a direct translation of the basis exchange axiom for Δ-matroids.

Example 2.3.18. Let $([n], \mathcal{B})$ be a Δ-matroid, and let $J = \{\mathbb{1}_B : B \in \mathcal{B}\}$. Then J is a jump system.

The two-step axiom also implies the (M2) axiom for M-convex sets, similar to how Δ-matroids generalize matroids.

Example 2.3.19. If $J \subseteq \mathbb{Z}^n$ is an M-convex set, then J is also a jump system.

Analogous to M-convex sets, convex hulls of jump systems give rise to a certain kind of polytope, called *bisubmodular polyhedra*. We refer the interested reader to [BC95] for details.

Note that again, there is a natural strengthening of the exchange assumption, analogous to the strong exchange axiom for matroids.

($J^\natural$) if $x, y \in J$ and s is an (x, y)-increment, we have either (i) $x + s \in J$ and $y - s \in J$, or (ii) there exists an $(x + s, y)$-increment t such that $x + s + t \in J$ and $y - s - t \in J$.

Definition 2.3.20 ([Mur21])**.** If $J \subseteq \mathbb{Z}^n$ is nonempty and satisfies ($J^\natural$), then we call J a *simultaneous exchange jump system*.

We note that Example 2.3.8 again shows that ($J^\natural$) is strictly stronger than the two-step axiom. However, like the strong exchange property for even Δ-matroids, we get nicer results if J has constant parity.

Proposition 2.3.21 ([Mur06, Lemma 2.1])**.** *A nonempty subset $J \subseteq \mathbb{Z}^n$ is a constant parity jump system if and only if for all $x, y \in J$ and any (x, y)-increment s, there exists an $(x + s, y)$-increment t such that $x + s + t \in J$ and $y - s - t \in J$.*

In particular, Proposition 2.3.21 implies that all constant parity jump systems are simultaneous exchange jump systems.

2.3.3 Jump M-convex Functions and Valuated Δ-matroids

As with matroids, we can again generalize Δ-matroids to a valuated version. In the prior literature, this had only been done for even Δ-matroids (or, slightly more generally, Δ-matroids satisfying the strong exchange axiom (B2')). The general Δ-matroid case is novel; our definition and results for non-even valuated Δ-matroids can be found in Chapter 5 and [CCLV25].

Once we assume the strong exchange property, it is simple to formulate valuated versions of the exchange axiom. Note that the definition below is equivalent to [Rin12, Definition 4.2].

Definition 2.3.22. A vector $p = (p_S) \in (\mathbb{R} \cup \{\infty\})^{2^{[n]}}$ is a *valuated even Δ-matroid* if for all $S, T \subseteq [n]$ and all $i \in S\Delta T$, there exists $j \in (S\Delta T)\setminus\{i\}$ such that $p_{S\Delta\{i,j\}} + p_{T\Delta\{i,j\}} \leq p_S + p_T$.

As with valuated matroids, this is also equivalent to a condition on regular subdivisions.

Theorem 2.3.23 ([Rin12, Theorem 5.4])**.** *The vector $p \in (\mathbb{R} \cup \{\infty\})^{2^{[n]}}$ is a valuated even Δ-matroid if and only if the regular subdivision induced by p is a subdivision of an even Δ-matroid polytope into even Δ-matroid polytopes.*

Analogous to M-convex functions, we can generalize this definition to functions on constant parity jump systems.

Definition 2.3.24 ([Mur21])**.** We say $f : \mathbb{Z}^n \to \mathbb{R} \cup \{\infty\}$ is *jump M-convex* if for all $x, y \in \mathbb{Z}^n$ and any (x, y)-increment s, there exists an $(x + s, y)$-increment t such that

$$f(x + s + t) + f(y - s - t) \leq f(x) + f(y).$$

In fact, we can generalize it to all simultaneous exchange jump systems.

Definition 2.3.25 ([Mur21])**.** We say $f : \mathbb{Z}^n \to \mathbb{R} \cup \{\infty\}$ is *jump $M^\natural$-convex* if for all $x, y \in \mathbb{Z}^n$ and any (x, y)-increment s, at least one of the following holds:

(i) $f(x + s) + f(y - s) \leq f(x) + f(y)$, or

(ii) there exists an $(x + s, y)$-increment t such that $f(x + s + t) + f(y - s - t) \leq f(x) + f(y)$.

It turns out that simultaneous exchange jump systems are closely related to constant parity jump systems, and similarly jump $M^\natural$-convex functions are closely related to jump M-convex functions.

Following [Mur21, Section 3], for any $x \in \mathbb{Z}^n$, we define $\pi(x) = 0$ if $\sum x_i$ is even, and $\pi(x) = 1$ otherwise, so $\pi(x) + \sum x_i$ is always even. For any $J \subseteq \mathbb{Z}^n$, we define

$$\widetilde{J} = \{(\pi(x), x) : x \in J\} \subseteq \mathbb{Z}^{n+1}.$$

Theorem 2.3.26 ([Mur21, Theorem 3.1])**.** *A subset $J \subseteq \mathbb{Z}^n$ is a simultaneous exchange jump system if and only if $\widetilde{J}$ is a constant parity jump system.*

Again following [Mur21, Section 3], for any $f : \mathbb{Z}^n \to \mathbb{R} \cup \{\infty\}$, we similarly define $\widetilde{f} : \mathbb{Z}^{n+1} \to \mathbb{R} \cup \{\infty\}$ such that

$$\widetilde{f}(x_0, x) = \begin{cases} f(x), & \text{if } x_0 = \pi(x) \\ \infty, & \text{otherwise} \end{cases}.$$

Theorem 2.3.27 ([Mur21, Theorem 3.2])**.** *A function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{\infty\}$ is jump $M^\natural$-convex if and only if $\widetilde{f}$ is jump M-convex.*

These operations of transforming a jump system to be constant parity will arise again in the form of parity homogenization in Chapter 6.

2.4 Stable Polynomials

Stable polynomials are a natural multivariate analogue to real rooted univariate polynomials. For a survey of stable polynomials, as well as the closely related class of hyperbolic polynomials (Section 2.4.3), we refer the reader to [Pem12].

Definition 2.4.1. We say a polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ is *stable* if $f(z_1, \dots, z_n) \neq 0$ whenever $\mathrm{Im}(z_1), \dots, \mathrm{Im}(z_n) > 0$. If additionally f has real coefficients, we say that f is *real stable.*

Remark 2.4.2. *In Chapter 6, we will deal with another form of stability known as* Hurwitz stability*. A polynomial is Hurwitz stable if $f(z) \neq 0$ whenever $\mathrm{Re}(z_1), \dots, \mathrm{Re}(z_n) > 0$. While this definition is equivalent up to a change of variables, as we will see in that chapter, some of the remarkable properties of real stability do not translate nicely to the Hurwitz stable case.*

Remark 2.4.3. *In general, for any region $\Omega \subseteq \mathbb{C}^n$, we say a polynomial is Ω-stable if it is nonvanishing on Ω. Another common form of stability is* Schur stability*, where $\Omega = \{z \in \mathbb{C} : |z| < 1\}^n$. In Chapter 6, we also touch on* sector stability*, as defined in [AASV21].*

Real stability is closely related to real-rootedness.

Proposition 2.4.4. *Let $f \in \mathbb{R}[x]$ be a univariate polynomial. Then f is real-rooted if and only if it is real stable.*

Proof. Since f has real coefficients, its roots come in conjugate pairs. Hence, f has a root with positive imaginary part if and only if it has a non-real root. □

The following equivalent definition is well known in the literature.

Theorem 2.4.5 ([BB10, Lemma 2.1])**.** *Let $f \in \mathbb{R}[x_1, \ldots, x_n]$. Then f is real stable if and only if for all $v \in \mathbb{R}^n_{>0}$ and all $w \in \mathbb{R}^n$, the univariate polynomial $f(tv + w) \in \mathbb{R}[t]$ is real rooted.*

While Theorem 2.4.5 only applies to real stable polynomials, it turns out that we can essentially reduce general stability to the real stable case. To do this, we require the theory of interlacing polynomials.

Definition 2.4.6 ([Brä07, Section 5])**.** Let $g, h \in \mathbb{R}[x]$ be polynomials with $\deg(g) = d$ and $\deg(h) = d - 1$ or d. We say h and g are in proper position (written $h \ll g$) if both h and g are real-rooted, their Wronskian $W[h, g] := h'g - hg'$ is nonpositive on $\mathbb{R}$, and their zeros interlace, meaning that the roots of g and h alternate as we go along $\mathbb{R}$.

Concretely, suppose that $\alpha_1 \leq \cdots \leq \alpha_d$ are the roots of g. If $\deg(h) = d - 1$, let $\beta_1 \leq \cdots \leq \beta_{d-1}$ be the roots of h. The interlacing condition is equivalent to

$$\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \beta_2 \leq \cdots \leq \beta_{d-1} \leq \alpha_d.$$

If $\deg(h) = d$, let $\beta_1 \leq \cdots \leq \beta_d$ be the roots of h. Then the interlacing condition says that either $\alpha_1 \leq \beta_1 \leq \cdots \leq \alpha_d \leq \beta_d$ or $\beta_1 \leq \alpha_1 \leq \cdots \leq \beta_d \leq \alpha_d$.

We can then extend this to real stable polynomials by considering the univariate restrictions from Theorem 2.4.5.

Definition 2.4.7 ([Brä07, Section 5])**.** Let $g, h \in \mathbb{R}[x_1, \ldots, x_n]$ be real stable polynomials with $\deg(g) = d$ and $\deg(h) = d - 1$ or d. We say h interlaces g (again written $h \ll g$) if $h(tv + w) \ll g(tv + w)$ for all $v \in \mathbb{R}^n_{>0}, w \in \mathbb{R}^n$.

Geometrically, this means that the zero sets of g and h alternate along any line in a positive direction. See Figure 2.6 for an example.

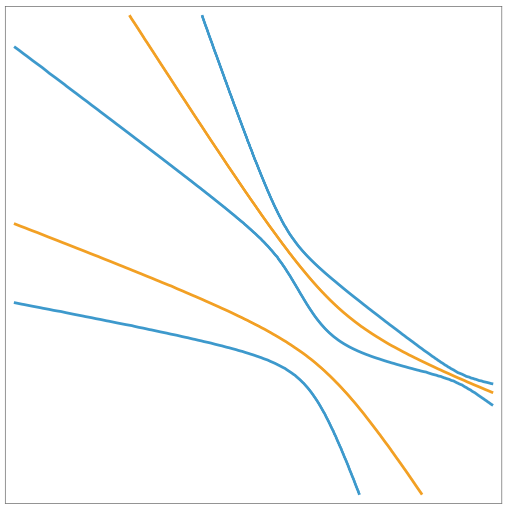

Figure 2.6: Zero sets of a cubic real stable polynomial f (blue) and quadratic real stable polynomial g (orange) such that $g \ll f$.

This then lets us convert general stability into real stability.

Theorem 2.4.8 ([BB09, Lemma 1.8])**.** *Let* $f = g + ih \in \mathbb{C}[x_1, \ldots, x_n]$ *where* $g, h \in \mathbb{R}[x_1, \ldots, x_n]$*. Then the following are equivalent:*

(i) f *is stable.*

(ii) $g + x_0 h \in \mathbb{R}[x_0, x_1, \ldots, x_n]$ *is real stable*

(iii) $h \ll g$*.*

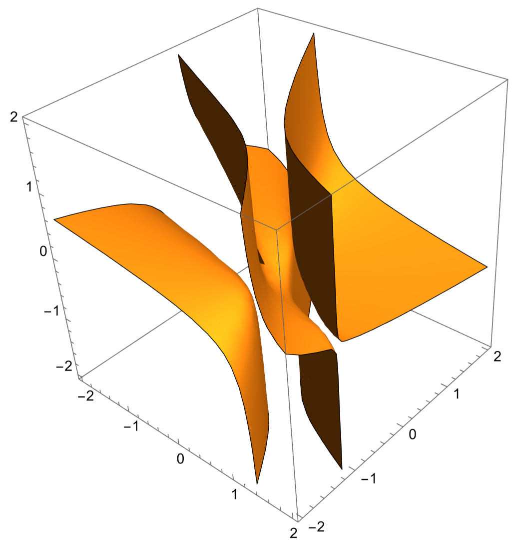

Figure 2.7: Zero set of $f + zg$ for the polynomials f and g from Figure 2.6. Observe that along any positive ray, we intersect the zero set three times, so all univariate restrictions in positive directions are real rooted and hence this polynomial is real stable.

We remark that stable polynomials are closed under several natural operations. Each of these is easily verified from the definition.

Proposition 2.4.9. *The following operations preserve stability:*

1. *Permutation:* $f \mapsto f(x_{\pi(1)}, \ldots, x_{\pi(n)})$ *for any permutation* $\pi : [n] \to [n]$.

2. *Specialization:* $f \mapsto f(a, x_2, \ldots, x_n)$ *where* $\mathrm{Im}(a) \geq 0$.

3. *Scaling:* $f \mapsto c \cdot f(\lambda_1 x_1, \ldots, \lambda_n x_n)$ *where* $c \in \mathbb{C}$ *and* $\lambda \in \mathbb{R}^n_{\geq 0}$.

4. *Multiplication:* $f, g \mapsto f \cdot g$ *where* f, g *are both stable.*

5. *Projection:* $f \mapsto f(x_1, x_2, \ldots, x_{n-1}, x_{n-1})$

One class of polynomials of particular interest is *multiaffine* polynomials, which are polynomials that are degree at most one in each variable. The *polarization* operator transforms non-multiaffine polynomials into multiaffine ones, and preserves stability.

Definition 2.4.10. For $\kappa \in \mathbb{N}^n$, the *polarization map* is the ring homomorphism $\Pi_\kappa^\uparrow : \mathbb{C}[x_1, \dots, x_n]_{\leq \kappa} \to \mathbb{C}[x_{ij} : 1 \leq i \leq n, 1 \leq j \leq \kappa_i]$ such that

$$\Pi_\kappa^\uparrow(x_i^k) = \frac{1}{\binom{\kappa_i}{k}} e_k(x_{i,1}, \dots, x_{i,\kappa_i}),$$

where e_k denotes the elementary symmetric polynomial of degree k.

In particular, $\Pi_\kappa^\uparrow$ is the unique linear map $\mathbb{C}[x_1, \dots, x_n]_{\leq \kappa} \to \mathbb{C}[x_{ij} : 1 \leq i \leq n, 1 \leq j \leq \kappa_i]$ such that $\Pi_\kappa^\uparrow(f)$ is multiaffine, symmetric in $x_{i,1}, \dots, x_{i,\kappa_i}$ for all i, and satisfies that we recover f if we set $x_{i,1} = \cdots = x_{i,\kappa_i} = x_i$ for all i. Since specializing variables to be equal to one another, this latter property immediately implies that if $\Pi_\kappa^\uparrow(f)$ is stable, then so is f. More surprisingly, the converse also holds.

Theorem 2.4.11 ([BB09, Proposition 2.4])**.** *Let $f \in \mathbb{C}[x_1, \dots, x_n]_{\leq \kappa}$. Then f is stable if and only if $\Pi_\kappa^\uparrow(f)$ is stable.*

We also note that the set of stable polynomials is closed in the Euclidean topology, so the limit of a sequence of stable polynomials is also stable.

One important construction of (real) stable polynomials is determinantal polynomials.

Example 2.4.12 (Determinantal Polynomials, [BBL09, Proposition 3.2])**.** Let

$$f(x_1, \dots, x_n) = \det\left(x_1 A_1 + \cdots + x_n A_n + B\right),$$

where $A_i \succeq 0$ are positive semidefinite Hermitian matrices and $B \in \mathbb{C}^{m \times m}$ is Hermitian. Then f is real stable.

For a concrete example, let

$$M = x_1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + x_2 \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 - 1 & x_1 + x_2 + 2 \\ x_1 + x_2 + 2 & x_1 + x_2 \end{pmatrix}.$$

Then $\det(M) = x_2^2 + x_1x_2 - 5x_2 - 5x_1 - 4$. We plot the zero set of this polynomial in Figure 2.8, and observe that every line with nonnegative slope intersects both components of the zero locus and hence the polynomial is real stable.

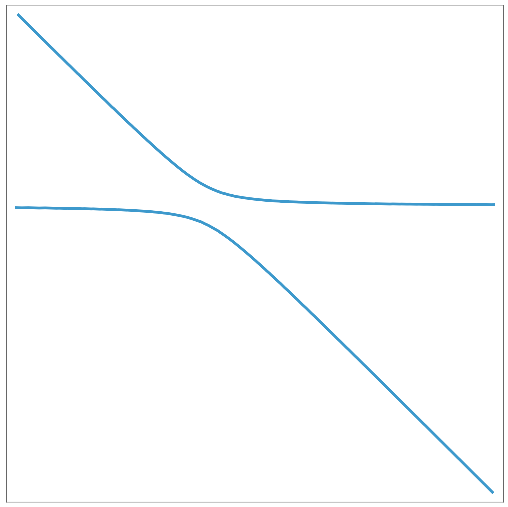

Figure 2.8: Zero set of the determinantal polynomial computed in Example 2.4.12

Stable polynomials are also closely connected to matroids and Δ-matroids. A result in [Brä07] shows that the support of any stable polynomial is a jump system. In particular, the support of any multiaffine stable polynomial is a Δ-matroid, and the support of any homogeneous multiaffine stable polynomial is a matroid.

Homogeneous stable polynomials have an additional nice property, which is that all of their coefficients have the same complex phase.

Theorem 2.4.13 ([COSW04, Theorem 6.1]). *Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a homogeneous stable polynomial. Then the coefficients of f have constant complex phase. That is, there exists $\theta \in [0, 2\pi)$ such that $e^{i\theta} f \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$.*

Moreover, real stable polynomials also give rise to negative dependence. In fact, multiaffine real stable polynomials have an even stronger property, namely globally nonnegative Rayleigh differences.

Definition 2.4.14 (Rayleigh difference, [Brä07, Section 5])**.** For $f \in \mathbb{C}[x_1, \ldots, x_n]$ and $1 \le i, j \le n$, we define

$$\Delta_{ij} f = (\partial_i f)(\partial_j f) - (f)(\partial_{ij}^2 f).$$

We note that if f is multiaffine, then we may write

$$f = a + bx_i + cx_j + dx_i x_j,$$

where

$$a = f|_{x_i = x_j = 0} \qquad b = \partial_i f|_{x_j=0}$$
$$c = \partial_j f|_{x_i=0} \qquad d = \partial_{ij}^2 f$$

are independent of x_i, x_j, and we have

$$\Delta_{ij} f = bc - ad = \left(\partial_i f|_{x_j=0}\right)\left(\partial_j f|_{x_i=0}\right) - \left(f|_{x_i=x_j=0}\right)\left(\partial_{ij}^2 f\right).$$

Proposition 2.4.15 ([Brä07, Theorem 5.6])**.** *Let $f \in \mathbb{R}[x_1, \ldots, x_n]$ be multiaffine. The following are equivalent:*

(1) f is real stable

(2) for all $x \in \mathbb{R}^n$ and $1 \le i, j \le n$,

$$\Delta_{ij} f(x) \ge 0.$$

Evaluating Proposition 2.4.15 at $x = \mathbb{1}$ shows that real stability implies negative dependence.

Proposition 2.4.16 ([BBL09])**.** *Let $f = \sum_{S \subseteq [n]} c_S x^S \in \mathbb{R}_{\ge 0}[x_1, \ldots, x_n]$ be a multiaffine real stable polynomial. Define a probability distribution μ on $2^{[n]}$ by $\mu(S) \propto c_S$. Then μ is globally negatively correlated. That is, for all $i, j \in [n]$, $(\mathbb{P}_{S\sim\mu}[i \in S])(\mathbb{P}_{S\sim\mu}[j \in S]) \ge \mathbb{P}_{S\sim\mu}[i, j \in S]$.*

2.4.1 Real Stability and Directional Derivatives

In the case of univariate polynomials, it follows easily from Rolle's theorem that the derivative of a real-rooted polynomial is also real rooted. This closure under taking derivatives carries over into the real stability setting.

Theorem 2.4.17 ([Pem12]). *Let $f \in \mathbb{R}[x_1, \ldots, x_n]$ be a stable polynomial and let $w \in \mathbb{R}^n_{\geq 0}$ be a nonnegative vector. Then $D_w f$ is stable.*

As an easy corollary, it follows that $D_v f$ interlaces f.

Corollary 2.4.18. *If $f \in \mathbb{R}[x_1, \ldots, x_n]$ is real stable, then $D_v f \ll f$ for any $v \in \mathbb{R}^n_{\geq 0}$.*

Proof. By Theorem 2.4.8, it suffices to show that $f + x_0 D_v f$ is real stable for all $v \in \mathbb{R}^n_{\geq 0}$. Since products of real stable polynomials are real stable, $h(x_0, x) \coloneqq x_0 \cdot f(x)$ is real stable. Moreover, $D_{(1,v)} h = f(x) + x_0 D_v f(x)$. Since nonnegative directional derivatives preserve real stability, this implies that $f(x) + x_0 D_v f(x)$ is real stable, as desired. □

2.4.2 Symbols and Stability-Preserving Operators

In this subsection, we discuss a remarkable property of stable polynomials, which is that we have a complete characterization of linear operators that preserve stability. This characterization comes in the form of a *symbol theory*.

Definition 2.4.19 ([BB09]). Let $T : \mathbb{C}[x_1, \ldots, x_n]_{\leq \kappa} \to \mathbb{C}[x_1, \ldots, x_m]$ be a linear operator on the space of polynomials of multidegree at most κ. The *symbol* of T is defined as

$$\mathrm{symb}_T(x, y) = T[(x + y)^\kappa] = \sum_{\alpha \leq \kappa} \binom{\kappa}{\alpha} T(x^\alpha) y^{\kappa - \alpha},$$

where $\binom{\kappa}{\alpha} \coloneqq \prod_{i=1}^n \binom{\kappa_i}{\alpha_i}$.

The symbol is the key object in a theorem of Borcea and Brändén.

Theorem 2.4.20 ([BB09, Theorems 1.1 and 1.2]). *Let $T : \mathbb{C}[x_1, \ldots, x_n]_{\leq \kappa} \to \mathbb{C}[x_1, \ldots, x_m]$ be a linear operator on the space of polynomials of multidegree $\leq \kappa$. Then T preserves stability if and only if either*

1. T has range of dimension at most one and is of the form $T(f) = \alpha P$, where α is a linear functional on $\mathbb{C}[x_1, \ldots, x_n]_{\leq\kappa}$ and P is a stable polynomial, or

2. $\operatorname{symb}_T(x, y)$ *is stable.*

Moreover, T preserves real stability if and only if T preserves real coefficents, and

1. T has range of dimension at most two and is of the form $T(f) = \alpha(f)P + \beta(f)Q$ where α, β are linear functionals and P, Q are real stable polynomials such that $P \ll Q$, or

2. $\operatorname{symb}_T(z, w)$ *is real stable, or*

3. $\operatorname{symb}_T(z, -w)$ *is real stable.*

The necessity of the symbol condition is somewhat technical to prove. However, the sufficiency portion is quite elegant, and we reproduce the argument for sufficiency for the general (non-real) stability case from [BB09] here.

Proof of sufficiency of Theorem 2.4.20. Suppose $T : \mathbb{C}[x_1, \ldots, x_n]_{\leq\kappa} \to \mathbb{C}[x_1, \ldots, x_m]_{\leq\gamma}$ is a linear operator. Define the polarization of T as the linear operator $\Pi(T) := \Pi^{\uparrow}_{\gamma} \circ T \circ \Pi^{\downarrow}_{\kappa}$. Then $\Pi(T)(\Pi^{\uparrow}_{\kappa} f) = \Pi^{\uparrow}_{\gamma}(T(f))$, so $T(f)$ is stable if and only if $\Pi(T)(\Pi^{\uparrow}_{\kappa} f)$ is stable. Moreover, by [BB09, Lemma 2.5], $\operatorname{symb}_{\Pi(T)} = \Pi^{\uparrow}_{\gamma\oplus\kappa} \operatorname{symb}_T(z, w)$, so it suffices to prove the claim for the case where $\kappa = (1, \ldots, 1)$.

Thus, we may suppose that $T : \mathbb{C}[x_1, \ldots, x_n]_{\leq(1,\ldots,1)} \to \mathbb{C}[x_1, \ldots, x_n]$ is a linear operator such that $\operatorname{symb}_T \in \mathbb{C}[z_1, \ldots, z_m, w_1, \ldots, w_n]$ is stable, and suppose that $f = \sum_{S\subseteq[n]} c_S x^S \in \mathbb{C}[x_1, \ldots, x_n]_{\leq(1,\ldots,1)}$ is stable. Since stability is preserved under products, then

$$\operatorname{symb}_T(z, w) \cdot f(x) = \sum_{\substack{S\subseteq[n] \\ S'\subseteq[n]}} c_S T(z^{S'}) w^{[n]\setminus S'} x^S$$

is stable.

Following [BB09], let $D_{ij} : \mathbb{C}[x_1, \ldots, x_n]_{\leq(1,\ldots,1)} \to \mathbb{C}[x_1, \ldots, x_n]$ be the linear operator defined by

$$D_{ij} : x^S \mapsto \begin{cases} x^{S \setminus i}, & \text{if } i \in S, j \notin S \\ x^{S \setminus j}, & \text{if } j \in S, i \notin S \\ 0, & \text{otherwise} \end{cases}.$$

We note that $D_{ij}(f) = ((\partial_i + \partial_j)f)|_{x_i = x_j = 0}$. Since nonnegative directional derivatives and specializing variables to zero both preserve stability (Theorem 2.4.17 and Proposition 2.4.9, respectively), D_{ij} preserves stability.

Moreover, we note that

$$\left(\prod_{i=1}^n D_{w_i,x_i}\right) w^{S_1} x^{S_2} = \begin{cases} 1, & \text{if } S_2 = [n] \setminus S_1 \\ 0, & \text{otherwise} \end{cases}.$$

Hence, applying $\prod_{i=1}^n D_{w_i,x_i}$ annihilates all terms where $S \neq S'$, leaving us with

$$\left(\prod_{i=1}^n D_{w_i,x_i}\right) (\mathrm{symb}_T(z,w) f(x)) = \sum_{S \subseteq [n]} c_S T(z^S) = T(f),$$

and since D_{w_i,x_i} preserves stability, we conclude that $T(f)$ is stable, as desired. $\square$

Remark 2.4.21. *It follows from the proof above that if $\mathcal{P}$ is a class of polynomials that is closed under polarization and specialization, products on disjoint sets of variables, and the D_{ij} operator, and if T is a linear operator such that $\mathrm{symb}_T \in \mathcal{P}$, then T preserves $\mathcal{P}$. This fact will be key to our discussions in Section 6.3.3.*

We also note that the conditions for real stability preservers and general stability preservers are very closely related. In particular, $\mathrm{symb}_T(z, -w) = \mathrm{symb}_{T'}(z, w)$, where we define $T'(f(z)) = (-1)^\kappa T(f(-z))$ [BB09]. Further, we note that T preserves real stability if and only if T' does. Hence, up to this minor transformation (and outside of the low-dimensional case), real stability preservers are essentially the same as stability preservers that also preserve real coefficients. However, this does not translate to other forms of half-plane stability, such as Hurwitz stability.

In particular, if T preserves real coefficients and real stability, then either T preserves stability, or $T'(f(z)) \coloneqq T(f(-z))$ does. By tracing through the definitions, T preserves Hurwitz stability if and only if $f(x) \mapsto [T(f(ix))]_{x=-iz}$ preserves stability, so the analogous statement for Hurwitz stability would be that if T preserves real Hurwitz stability, then either T or $\widetilde{T}(f(x)) \coloneqq [T(f(ix))]_{x=iz}$ preserves Hurwitz stability. However, this is not the case.

Proposition 2.4.22. *Let T be the parity homogenization linear operator, sending*

$$T : x^\alpha \mapsto \begin{cases} x_0 x^\alpha, & \text{if } |\alpha| \text{ odd} \\ x^\alpha, & \text{if } |\alpha| \text{ even} \end{cases}.$$

Then T preserves real Hurwitz stability, but neither T nor $\widetilde{T}$ preserves Hurwitz stability in general.

Proof. First, we claim that T preserves real Hurwitz stability. That is, if $f \in \mathbb{R}[x_1, \ldots, x_n]$ is Hurwitz stable, then $T(f) \in \mathbb{R}[x_1, \ldots, x_n]$ is also Hurwitz stable. Indeed, we note that by definition, f is Hurwitz stable if and only if $g(x) \coloneqq f(-ix)$ is stable.

Write $f = f_e + f_o$, where f_e consists of the terms of f with even degree and f_o is the terms of f with odd degree. Then by definition, $T(f) = f_e + x_0 f_0$. Furthermore, we note that $g = \mathrm{Re}(g) + i\,\mathrm{Im}(g) \in \mathbb{C}[x_1, \ldots, x_n]$, with $\mathrm{Re}(g) = f_e(-ix)$ and $\mathrm{Im}(g) = -if_o(-ix)$. By Theorem 2.4.8, we have that $\widetilde{g}(x_0, x) = \mathrm{Re}(g)(x) + x_0\,\mathrm{Im}(g)(x) = f_e(-ix) - ix_0 f_o(-ix)$ is real stable. Tranforming back, this implies that $\widetilde{g}(ix_0, ix)$ is Hurwitz stable. Finally, we note that $\widetilde{g}(ix_0, ix) = f_e(x) + x_0 f_o(x) = T(f)$. Hence T preserves real Hurwitz stable polynomials, as desired.

On the other hand, consider $f(x) = -x^2 + 2ix + 1$. We note that $f(-ix) = x^2 + 2x + 1$, which is real stable, so f is Hurwitz stable. On the other hand, $T(f)(x_0, x) = -x^2 + 2ix_0 x + 1$, so $T(f)(z, z) = (-1 + 2i)z^2 + 1$. We note that $az^2 + 1$ is Hurwitz stable if and only if $a \in \mathbb{R}_{\geq 0}$, so $T(f)$ is not Hurwitz stable, and hence T does not preserve general Hurwitz stability.

Moreover, we compute

$$\widetilde{T} : f \mapsto f_e(z) - iz_0 f_o(z),$$

so taking $f(x) = -x^2 + 2ix + 1$ as above, we have $\widetilde{T}(f) = -x^2 + 2x_0x + 1$, which has a zero at $x = 1 + \sqrt{2}, x_0 = 1$. Thus, $\widetilde{T}(f)$ is also not Hurwitz stable, as desired. $\square$

There is one additional approach to addressing parity homogenization with complex coefficients. By mimicking the proof with real coefficients, we see that if $f = g + ih$ is Hurwitz stable, then $T_e(g) + iT_o(h)$ is Hurwitz stable, where T_e is the (even) parity homogenization operator described above, and

$$T_o(x^\alpha) = \begin{cases} x^\alpha, & \text{if } |\alpha| \text{ odd} \\ x_0 x^\alpha, & \text{if } |\alpha| \text{ even} \end{cases}$$

is the "odd" parity homogenization operator. However, this operation is not linear over $\mathbb{C}$, so none of the theorems described above apply.

2.4.3 Hyperbolic Polynomials

As seen in Theorem 2.4.5, one can view real stability as saying that for any $v \in \mathbb{R}^n_{>0}$, any univariate restriction of f to a line in direction v is real-rooted. By allowing ourselves to pick other directions, we get *hyperbolic polynomials*.

Definition 2.4.23. Let $f \in \mathbb{R}[x_1, \ldots, x_n]$ be a homogeneous polynomial. For $e \in \mathbb{R}^n$, we say that f is *hyperbolic with respect to* e if $f(e) > 0$ and for all $a \in \mathbb{R}^n$, the univariate polynomial $f(te - a)$ is real-rooted.

If f is hyperbolic with respect to e, this defines the (open) *hyperbolicity cone*

$$\Lambda_+(f, e) = \{a \in \mathbb{R}^n : \text{all roots of } f(te - a) \text{ are positive}\}.$$

In fact, if f is hyperbolic with respect to e, then it is also hyperbolic with respect to any point in its hyperbolicity cone. Moreover, its hyperbolicity cone is convex and connected.

Proposition 2.4.24 ([Går59])**.** *Suppose f is hyperbolic with respect to e. Then*

(i) $\Lambda_+(f, e)$ *is a convex cone.*

(ii) If $a \in \Lambda_+(f,e)$, then f is hyperbolic with respect to a, and $\Lambda_+(f,a) = \Lambda_+(f,e)$.

(iii) The closure of the hyperbolicity cone $\overline{\Lambda_+(f,e)}$ is exactly the closure of the connected component of $\mathbb{R}^n \setminus V_{\mathbb{R}}(f)$ containing e.

If K is a convex cone and f is hyperbolic with respect to every $a \in K$, then as shorthand we say that f is hyperbolic with respect to K.

As one might guess from the similarity of the definitions, hyperbolicity is closely related to real stability. For a (non-homogeneous) degree d polynomial f, we define the homogenization $f^{\mathrm{hom}}(x_0, x) \coloneqq x_0^d f(x_1/x_0, \ldots, x_n/x_0)$.

Proposition 2.4.25 ([BB10, Proposition 1.1])**.** *Let $f \in \mathbb{R}[x_1, \ldots, x_n]$ such that $f(t, \ldots, t) \to \infty$ as $t \to \infty$. Then f is real stable if and only if f^{hom} is hyperbolic with respect to $\{0\} \times \mathbb{R}^n_{>0}$.*

One consequence of Proposition 2.4.25 is that if f has nonnegative coefficients, then f is real stable if and only if f^{hom} is real stable.

Proposition 2.4.26 ([Gur09, Proposition 5.3])**.** *Let $f \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$. Then f is real stable if and only if f^{hom} is real stable.*

Proof. The reverse direction is trivial: if f^{hom} stable, then so is f, since $f = f^{\mathrm{hom}}\big|_{x_0=1}$, and specializing variables preserves stability.

On the other hand, suppose that f is real stable. By Proposition 2.4.25, f^{hom} is hyperbolic with respect to $\{0\} \times \mathbb{R}^n_{>0}$. Since f has nonnegative coefficients, it is also positive (and hence nonvanishing) on $\mathbb{R}^{n+1}_{>0}$, so by Proposition 2.4.24, f^{hom} is hyperbolic with respect to $\mathbb{R}^{n+1}_{>0}$. Thus, f^{hom} is stable, as desired. $\square$

2.4.4 Real Stable Polynomials and Log Concavity

We say a polynomial $f \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ is *log concave* if $\log(f)$ is concave as a function on $\mathbb{R}^n_{>0}$. It is a well-known fact in the literature that if f is real stable with nonnegative coefficients, it is log concave. For example, the version of this statement for hyperbolic polynomials can be found in [Gül97].

Importantly, the proof of this fact uses only real algebraic techniques. This is in contrast to an analogous fact about Hurwitz stable polynomials (Theorem 6.1.14), for which the only known proof requires complex analysis.

Theorem 2.4.27 (See, e.g., [Gül97])**.** *Suppose f is a real stable polynomial with nonnegative coefficients. Then f is log concave.*

Proof. By Proposition 2.4.26, f^{hom} is also real stable. Moreover, since $f(x) = f^{\text{hom}}(1, x)$, it suffices to show that f^{hom} is log concave. Fix $v \in \mathbb{R}^n$ and $a \in \mathbb{R}^n_{>0}$. We wish to show that $D_v^2 \log(f^{\text{hom}})(a) \leq 0$.

Indeed, by real stability of f^{hom}, we know that $f^{\text{hom}}(v + ta)$ is real-rooted. Since f^{hom} is homogeneous, we have $f^{\text{hom}}(v + ta) = t^d f^{\text{hom}}((1/t)v + a)$, so $f^{\text{hom}}(tv + a)$ is also real-rooted. Thus,

$$f^{\text{hom}}(tv + a) = f^{\text{hom}}(v) \prod_{i=1}^{d} (t - \alpha_i)$$

for some $\alpha_i \in \mathbb{R}$. Furthermore, we note that $f^{\text{hom}}(a) > 0$, so the α_i are all nonzero. We then compute

$$\log f^{\text{hom}}(tv + a) = \log(f^{\text{hom}}(v)) + \sum_{i=1}^{d} \log(t - \alpha_i),$$

so

$$D_v^2 \log f^{\text{hom}}(a) = \frac{d^2}{dt^2}\bigg|_{t=0} \log(f^{\text{hom}}(tv + a)) = \left[\sum_{i=1}^{d} \frac{-1}{(t - \alpha_i)^2}\right]_{t=0} = \sum_{i=1}^{d} \frac{-1}{\alpha_i^2} \leq 0,$$

as desired. ∎

2.5 *Lorentzian Polynomials*

Lorentzian polynomials are a generalization of real stable polynomials retaining many of the same combinatorial properties.

Definition 2.5.1 ([BH20, Definition 2.6])**.** Let $f \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree d. We say f is *Lorentzian* if

(i) $\mathrm{supp}(f)$ is M-convex, and

(ii) for all $\alpha \in \mathbb{N}^n$, $|\alpha| = d - 2$, $\nabla^2 \partial^\alpha f$ has at most one positive eigenvalue.

Example 2.5.2 (Matroid Basis Generating Polynomials; [AOV21, Theorem 4.2],[BH20, Theorem 3.10])**.** If $M = ([n], \mathcal{B})$ is a matroid, then the basis generating polynomial

$$f = \sum_{B \in \mathcal{B}} x^B$$

is Lorentzian.

Remark 2.5.3. *In fact, one can show that* $\nu : \binom{[n]}{d} \to \mathbb{R} \cup \{\infty\}$ *is a valuated matroid if and only if*

$$f_q^\nu = \sum_{S \in \mathrm{supp}(\nu)} q^{\nu(S)} x^S$$

is Lorentzian for all $0 < q \leq 1$ *[BH20, Theorem 3.14].*

We can take this connection further by working over the field of real Puiseux series, $\mathbb{R}\{\{t\}\}$*. A function* $\nu : \binom{[n]}{d} \to \mathbb{Q} \cup \{\infty\}$ *is a valuated matroid if and only if there exists a Lorentzian polynomial*

$$f = \sum_{S \in \binom{[n]}{d}} c_S(t) x^S \in \mathbb{R}\{\{t\}\}_{\geq 0}[x_1, \ldots, x_n]$$

such that $\mathrm{val}(c_S(t)) = \nu(S)$ *for all* S *[BH20, Theorem 3.20].*

Example 2.5.4 (Matroid Independence Polynomials [ALOV24b, Theorem 4.1])**.** If M is a matroid on ground set $[n]$ with independent sets $\mathcal{I}$, then

$$f = \sum_{I \in \mathcal{I}} x^I y^{n-|I|}$$

is Lorentzian.

Remark 2.5.5. *Note that the generating polynomial of the independent sets of a rank r matroid has degree r, so it would seem more natural to homogenize it to degree r. However, this polynomial is not, in general, Lorentzian. For example, consider the uniform matroid of rank three on four elements, so*

$$f^{\text{hom}} = e_3(x_1, x_2, x_3, x_4) + ye_2(x_1, x_2, x_3, x_4) + y^2(x_1 + x_2 + x_3 + x_4) + y^3.$$

Then

$$\nabla^2 \partial_y f^{\text{hom}} = \begin{pmatrix} 0 & 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 & 2 \\ 2 & 2 & 2 & 2 & 6 \end{pmatrix},$$

which has two positive eigenvalues, so the degree three homogenization f^{hom} is not Lorentzian.

Lorentzian polynomials are closely related to two other classes, namely *strongly log-concave* and *completely log-concave* polynomials.

Definition 2.5.6 ([Gur09])**.** A polynomial $f \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ is *strongly log-concave* if for all $\alpha \in \mathbb{N}^n$, $\partial^\alpha f$ is either identically zero or log-concave on $\mathbb{R}^n_{>0}$.

Definition 2.5.7 ([AOV21])**.** A polynomial $f \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ is *completely log-concave* if for all $v_1, \ldots, v_k \in \mathbb{R}^n_{\geq 0}$, $D_{v_1} \ldots D_{v_k} f$ is either identically zero or log-concave on $\mathbb{R}^n_{>0}$.

By taking $\alpha = 0$ or $k = 0$, respectively, it is clear that strongly log-concave and completely log-concave polynomials are both in particular log concave. In fact, these classes are actually equivalent to Lorentzianity, so Lorentzian polynomials are also log concave.

Theorem 2.5.8 ([BH20, Theorem 2.30])**.** *The following are equivalent for any homogeneous polynomial f.*

1. *f is completely log-concave.*

2. f is strongly log-concave.

3. f is Lorentzian.

The main non-trivial implication is that Lorentzianity implies complete log concavity. A very similar statement appears as [ALOV24b, Theorem 3.2].

Theorem 2.5.9 ([ALOV24b, Theorem 3.2]). *Let $f \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ be homogeneous of degree $d \geq 2$. Then f is completely log concave if and only if*

(i) for all $\alpha \in \mathbb{Z}^n_{\geq 0}$ with $|\alpha| \leq d-2$, the polynomial $\partial^\alpha f$ cannot be written as the sum of two nonzero polynomials on disjoint variables, and

(ii) for all $\alpha \in \mathbb{Z}^n_{\geq 0}$ with $|\alpha| = d-2$, either $\partial^\alpha f$ is identically zero or $\nabla^2 \partial^\alpha f$ has exactly one positive eigenvalue.

The equivalence with complete log concavity also makes it clear that homogeneous real stable polynomials are Lorentzian.

Proposition 2.5.10 ([BH20, Proposition 2.2]). *Let $f \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ be homogeneous of degree $d \geq 2$. If f is real stable, then f is Lorentzian.*

Proof. It suffices to show that $D_{v_1} \ldots D_{v_k} f$ is log concave for all $v_1, \ldots, v_k \in \mathbb{R}^n_{>0}$. Indeed, by Theorem 2.4.17, we know that $D_{v_1} \ldots D_{v_k} f$ is real stable for all such v_i, so it follows from Theorem 2.4.27 that $D_{v_1} \ldots D_{v_k} f$ is log concave, as desired. □

Remark 2.5.11. *The above proposition implies that if f is real stable with non-negative coefficients, then $f^{\text{hom}} := y^d f(x_1/y, \ldots, x_n/y)$ is Lorentzian. This follows from the fact that if f is real stable with nonnegative coefficients, then f^{hom} is also real stable by Proposition 2.4.26.*

It is worth noting that the second condition in the definition of Lorentzianity is equivalent to both real stability and log-concavity. Depending on the reference one uses, it may vary which of the below equivalences may be definitional versus proven as a theorem. In particular, for homogeneous quadratics, real stability, Lorentzianity, and log concavity are all equivalent. In degree three, Lorentzianity and log concavity are equivalent, but real stability is strictly stronger. In degree four, all the containments are strict. This dependence on degree for the heirarchy underlies many of our results in Chapter 3.

Proposition 2.5.12. *Let $f \in \mathbb{R}_{\geq 0}[x_1, \dots, x_n]$ be homogeneous of degree two. Then the following are equivalent:*

(i) f is real stable.

(ii) f is Lorentzian.

(iii) f is log concave.

(iv) $\nabla^2 f$ has exactly one positive eigenvalue.

2.5.1 Application: Discrete Log Concavity

One application of Lorentzian polynomials of interest to combinatorialists is that the coefficients of a Lorentzian polynomial are log concave.

Proposition 2.5.13 ([BH20, Proposition 4.4])**.** *If $f = \sum_{\alpha \in \Delta_n^d} \frac{c_\alpha}{\alpha!} x^\alpha$ is Lorentzian, then*

$$c_\alpha^2 \geq c_{\alpha - e_i + e_j} c_{\alpha + e_i - e_j}$$

for all $\alpha \in \Delta_n^d$ and $i, j \in [n]$

Proof. If $\alpha_i = 0$ or $\alpha_j = 0$, then the right hand side is zero, and we are done. Otherwise, let $\gamma \coloneqq \alpha - e_i - e_j$. Since f is Lorentzian, $\partial^\gamma f$ is Lorentzian and homogeneous of degree two, so $\nabla^2 \partial^{\alpha - e_i - e_j} f$ is either identically zero or has one positive eigenvalue. If it is identically zero, then $c_\alpha = c_{\alpha - e_i + e_j} = c_{\alpha + e_i - e_j} = 0$ and we are done.

Otherwise, by Cauchy interlacing, the 2×2 submatrix indexed by the i, jth rows and columns also has at most one positive eigenvalue. Since it has nonnegative entries, it cannot be negative semidefinite, so

$$\det \begin{pmatrix} \partial_{ii}^2 \partial^\gamma f & \partial_{ij}^2 \partial^\gamma f \\ \partial_{ij}^2 \partial^\gamma f & \partial_{jj}^2 \partial^\gamma f \end{pmatrix} \le 0.$$

Moreover, we note that $\partial_{ii}^2 \partial^\gamma f = c_{\alpha+e_i-e_j}, \partial_{jj}^2 \partial^\gamma f = c_{\alpha-e_i+e_j}, \partial_{ij}^2 \partial^\gamma f = c_\alpha$. Hence

$$c_{\alpha-e_i+e_j} c_{\alpha+e_i-e_j} - c_\alpha^2 = \det \begin{pmatrix} \partial_{ii}^2 \partial^\gamma f & \partial_{ij}^2 \partial^\gamma f \\ \partial_{ij}^2 \partial^\gamma f & \partial_{jj}^2 \partial^\gamma f \end{pmatrix} \le 0,$$

as desired. □

This fact about Lorentzian polynomials was key to proving Mason's log concavity conjecture on the numbers of independent sets of a given rank in a matroid.

Theorem 2.5.14 (Mason's conjecture; [ALOV24b, Theorem 1.2] [BH20, Theorem 4.14])**.** *Let M be a matroid on n elements and let $\mathcal{I}_k$ denote the number of independent sets of M of size k. For any $1 < k < n$,*

$$\left(\frac{\mathcal{I}_k}{\binom{n}{k}} \right)^2 \ge \frac{\mathcal{I}_{k-1}}{\binom{n}{k-1}} \frac{\mathcal{I}_{k+1}}{\binom{n}{k+1}}.$$

2.5.2 Application: Bounded Correlations

As we saw in Proposition 2.4.16, real stable polynomials define probability distributions with global negative correlation. While this is no longer true for Lorentzian polynomials, we do get bounded positive correlations.

Proposition 2.5.15 ([BH20, Proposition 2.17])**.** *If f is homogeneous of degree d and log concave, then for all $w \in \mathbb{R}^n_{\ge 0}$ and all $i, j \in [n]$,*

$$f(w) \partial_{ij}^2 f(w) \le 2 \left(1 - \frac{1}{d}\right) \partial_i f(w) \partial_j f(w).$$

By evaluating Proposition 2.5.15 at $w = \mathbb{1}$, we obtain the following.

Theorem 2.5.16 (cf. [BH20, Proposition 2.19])**.** *Let $\mu : 2^{[n]} \to \mathbb{R}_{\ge 0}$ be a probability distribution such that $\mu(S) = 0$ whenever $|S| > d$. If*

$$f = \sum_{S \subseteq [n]} c_S x_0^{d-|S|} x^S$$

is Lorentzian, then

$$\mathbb{P}[i,j \in S] \leq 2\left(1-\frac{1}{d}\right)\mathbb{P}[i \in S]\mathbb{P}[j \in S]$$

for all $i, j \in [n]$.

2.5.3 Application: Trickle-Down and Mixing Time

In the theoretical computer science community, Lorentzian polynomials are of interest due to their connections with high dimensional expanders and mixing times.

Suppose we wish to sample from a distribution $\mu : \binom{[n]}{k} \to \mathbb{R}_{\geq 0}$. One way to do this is with a *down-up random walk*. The algorithm for the $k \leftrightarrow k-1$ dowwn-up random walk is as follows. Starting from a set $S_t \in \binom{[n]}{k}$, we pick $i \in S_t$ uniformly at random and set $T_t \coloneqq S_t \setminus \{i\}$. Then we pick $S_{t+1} \supset T_t$ with probability $\frac{\mu(S_{t+1})}{\sum_{k=1}^{n} \mu(T_t \cup \{k\})}$.

Proposition 2.5.17 ([ALOV24a, Section 3])**.** *The down-up random walk described above has stationary distribution μ.*

Remark 2.5.18. *In general, we can define the $k \leftrightarrow k-\ell$ down-up random walk for any $1 \leq \ell \leq k$, where we now remove ℓ elements of S_t uniformly at random to get T_t, and add ℓ elements with probability proportional to μ for S_{t+1}. An identical argument again shows that $\mu(S)\mathbb{P}[S \to T] = \mu(T)\mathbb{P}[T \to S]$, and hence that μ is the stationary distribution for this random walk.*

In [ALOV24a], Anari et al. use results from [Opp18] and [KO20] to show that if $\mu : \binom{[n]}{k} \to \mathbb{R}_{\geq 0}$ is a probability distribution such that $f_\mu = \sum_{S \in \binom{[n]}{k}} \mu(S)x^S$ is Lorentzian, then the $k \leftrightarrow k-1$ down-up random walk described above has spectral gap at least $1/d$. In particular, by taking μ to be the uniform distribution on the bases of some matroid of rank r, then the down-up random walk on the bases of a matroid mixes to within ε total variation distance of the uniform distribution in time $O(r^2 \log(1/\varepsilon))$. The mixing time bound for the $k \leftrightarrow k-1$ down-up walk for log-concave distributions was later improved to $O(k \log(k/\varepsilon))$ in [AVLOV21, Theorem 1].

Chapter 3

COMPUTATIONAL COMPLEXITY OF REAL STABILITY AND LOG CONCAVITY

In this chapter, we cover results from [Chi24] showing that it is coNP-hard to test real stability and log concavity, while by contrast, we can test Lorentzianity in polynomial time.

3.1 Introduction

The theory of stable polynomials has been a key ingredient in seemingly unrelated mathematical advances. In the past few years, they were used to construct infinite families of Ramanujan graphs [MSS15a; MSS18], and to resolve the Kadison-Singer problem [MSS15b].

Hyperbolic polynomials are a closely related class. They first arose in PDEs [Går51], but have also gained traction in the optimization community, as they provide a class of optimization problems well-suited to interior point methods [Gül97].

Beyond hyperbolic polynomials are the classes of Lorentzian and log-concave polynomials. Lorentzian polynomials in particular have close ties to combinatorial Hodge theory [Huh23] and matroids [ALOV24b; BH20]. They are also called completely log-concave polynomials, and they provide a bridge between continuous log concavity of a polynomial and discrete log concavity of its coefficients [BH20]. Because of this connection to discrete log concavity, they have been used to prove log concavity of several famous combinatorial sequences, including Kostka numbers [HMMS22] and the coefficients of the Alexander polynomial for certain types of links [HMV24].

While stable, completely log-concave, and log-concave polynomials are powerful tools, less is known about how to test if a given polynomial has these properties. In [RRS17], Raghavendra, Ryder, and Srivastava give a polynomial time algorithm for checking real stability of bivariate polynomials, but their techniques do not generalize easily to more variables. Instead, the results using stable polynomials in [MSS15b; MSS15a; MSS18] often start with a known construction for stable polynomials, then use a series of stability-preserving operations to reach the desired polynomial, rather than checking stability directly. It is relatively easy to generate particular stable polynomials, and Borcea and Brändén give a full classification of stability-preserving operations [BB09], enabling this style of argument for proving particular polynomials are real stable. Similarly, to prove Lorentzianity in [HMV24], Hafner, Mészáros, and Vidinas construct a multivariate generalization of the Alexander polynomial in order to show that the single variable specialization has log-concave coefficients.

In this paper, we study the hierarchy of stable, completely log-concave, and log-concave polynomials through the lens of computational complexity. Building on works of Saunderson [Sau19] and Ahmadi et al. [AOPT11], we obtain computational complexity results on testing real stability and log concavity.

In Section 3.2, we show that it is coNP-hard to determine if a homogeneous polynomial is real stable. The main result is Theorem 3.2.9, which states that it is coNP-hard to decide if a homogeneous cubic with rational coefficients is real stable. As an easy consequence, we obtain Corollary 3.2.11, which states that deciding if a homogeneous polynomial is real stable is coNP-hard in all degrees $d \geq 3$.

By contrast, deciding whether a polynomial is completely log concave can be done in polynomial time (Section 3.3). Specifically, in Proposition 3.3.2 we give an explicit algorithm to decide if a homogeneous polynomial of degree d is completely log concave that runs in time $O(n^{d+1})$.

On the other hand, deciding whether an arbitrary polynomial is log concave is again coNP-hard (Section 3.4). In Theorem 3.4.1, we show that it is coNP-hard to decide if a homogeneous polynomial of degree four is log concave on $\mathbb{R}^n_{\geq 0}$, and we generalize this result to all polynomials of degree ≥ 4 in Corollary 3.4.7.

Finally, in Section 3.5, we study log concavity properties of cubic polynomials. Specifically, we show that we can decide if a homogeneous cubic polynomial is log concave in polynomial time, but that the weaker property of directional log concavity is coNP-hard.

3.2 Computational Complexity of Stability

In [RRS17], Raghavendra, Ryder, and Srivastava gave a deterministic algorithm that runs in time $O(d^5)$ to determine if a bivariate polynomial $p \in \mathbb{R}[x, y]$ of degree d is real stable. They also ask whether their algorithm can be generalized to three or more variables.

In this section, we show that it is coNP-hard to test whether a homogeneous cubic with rational coefficients is real stable. To do this, we will leverage the following result from [Sau19]:

Theorem 3.2.1 ([Sau19], Proposition 5.1)**.** *If* $p(x_0, x) = x_0^3 - 3x_0\|x\|^2 + 2q(x)$, *where* $q \in \mathbb{R}[x_1, \ldots, x_n]_3$, *then* p *is hyperbolic with respect to* $e_0 = (1, 0, \ldots, 0)$ *if and only if* $\max_{\|x\|^2=1} |q(x)| \leq 1$.

In [Sau19], Saunderson proves this result and uses it to show that it coNP-hard to decide if a homogeneous cubic polynomial is hyperbolic in a given direction. Specifically, by a clever choice of q, one can compute the clique number of a graph by optimizing a cubic polynomial. Computing the clique number of a graph is one of Karp's original NP-complete problems, so this implies that it is hard to test if a cubic polynomial is hyperbolic in a given direction. In this section, we use similar techniques to show that stability is also coNP-hard.

To apply this result, we will construct a polyhedral cone K such that p is hyperbolic with respect to e_0 if and only if it is hyperbolic with respect to K. After an appropriate change of variables, this is equivalent to being stable. Note that throughout this section, we use $e_0, e_1, \ldots, e_n \in \mathbb{R}^{n+1}$ to denote the standard coordinate vectors.

Remark 3.2.2. *Given the close relationship between stability and hyperbolicity, it is tempting to prove the hardness of testing stability directly from the coNP-hardness of testing hyperbolicity. However, this approach can run into difficulties showing that all the numbers in our reduction are of polynomial size. By starting our reduction one step further back, we allow ourselves to restrict our attention to a particular form of polynomial, making the analysis much cleaner.*

For $\varepsilon > 0$, let $L_\varepsilon \subseteq \mathbb{R}^{n+1}$ denote the Lorentz cone

$$L_\varepsilon = \{(x_0, x) \in \mathbb{R}^{n+1} : x_0 \geq \frac{1}{\varepsilon}\|x\|_2\},$$

where we let $x = (x_1, \ldots, x_n)$.

Lemma 3.2.3. *Let $p(x_0, x) = x_0^3 - 3x_0\|x\|^2 + 2q(x)$, where $q(x) \in \mathbb{R}[x_1, \dots, x_n]_3$ is a homogeneous cubic. Let N be the largest absolute value of a coefficient of q, and let $0 < \varepsilon < \min(\frac{1}{2Nn^3}, \frac{1}{2})$. Then $p(\|x\|, \varepsilon x) > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$.*

Proof. By homogeneity,

$$\begin{aligned} p(\|x\|, \varepsilon x) &= \|x\|^3 - 3\|x\|(\varepsilon^2\|x\|^2) + 2q(\varepsilon x) \\ &= \|x\|^3 \left(1 - 3\varepsilon^2 + 2\varepsilon^3 q\left(x/\|x\|\right)\right). \end{aligned}$$

Therefore, $p(\|x\|, \varepsilon x) > 0$ if and only if $1 - 3\varepsilon^2 + 2\varepsilon^3 q(\frac{x}{\|x\|}) > 0$, and so $p(\|x\|, \varepsilon x) > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$ if and only if

$$1 - 3\varepsilon^2 + 2\varepsilon^3 \left(\min_{\|x\|=1} q(x)\right) > 0.$$

Since q is homogeneous of degree three, we have $\min_{\|x\|=1} q(x) = -\max_{\|x\|=1} q(x)$, so $p(\|x\|, \varepsilon x) > 0$ for all $x \neq 0$ if and only if

$$1 - 3\varepsilon^2 - 2\varepsilon^3 \left(\max_{\|x\|=1} q(x)\right) > 0.$$

Since q is homogeneous of degree three, it has at most $\binom{n+2}{3} \le n^3$ terms. Moreover, if $\|x\| = 1$, each monomial has absolute value at most one, so each term is $\le N$. Thus, $\max_{\|x\|=1} q(x) \le Nn^3$.

Therefore,

$$1 - 3\varepsilon^2 - 2\varepsilon^3 \left(\max_{\|x\|=1} q(x)\right) \ge 1 - 3\varepsilon^2 - 2\varepsilon^3 Nn^3.$$

Since $\varepsilon < \frac{1}{2Nn^3}$ and $\varepsilon < \frac{1}{2}$,

$$1 - 3\varepsilon^2 - 2\varepsilon^3 Nn^3 > 1 - 3\varepsilon^2 - 2\varepsilon^2 \frac{1}{2Nn^3}(Nn^3) = 1 - 4\varepsilon^2 > 0.$$

Therefore, for any $0 < \varepsilon < \min(\frac{1}{2Nn^3}, \frac{1}{2})$, $p(\|x\|, \varepsilon x) > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$. $\square$

Proposition 3.2.4. *Let $p(x_0, x) = x_0^3 - 3x_0\|x\|^2 + 2q(x)$, where $q(x)$ is a homogeneous polynomial of degree three. Let N be the largest absolute value of a coefficient of q, and let $0 < \varepsilon < \min(\frac{1}{2Nn^3}, \frac{1}{2})$. Then p is hyperbolic with respect to e_0 if and only if it is hyperbolic with respect to L_ε.*

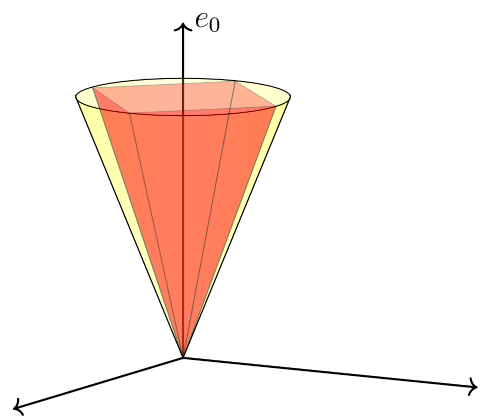

Figure 3.1: Cones $L_\varepsilon \supseteq K$, both containing e_0. These cones are constructed so that $p > 0$ on L_ε and K, so p is hyperbolic with respect to e_0 if and only if it is hyperbolic with respect to K.

Proof. For one direction, we note that $e_0 \in L_\varepsilon$. Thus, if p is hyperbolic with respect to L_ε, then in particular it is hyperbolic with respect to e_0.

Now, suppose that p is hyperbolic with respect to e_0. Since the hyperbolicity cone $\Lambda_+(p, e_0)$ is exactly the connected component of $\{(x_0, x) \in \mathbb{R}^{n+1} : p(x_0, x) \neq 0\}$ containing e_0, it suffices to show that $p(x_0, x) > 0$ on L_ε.

Let $(x_0, x) \in L_\varepsilon$, so that $x_0 = \frac{1}{\delta}\|x\| \geq \frac{1}{\varepsilon}\|x\|$ for some $0 < \delta \leq \varepsilon$. Then in particular, $0 < \delta < \min(\frac{1}{2Nn^3}, \frac{1}{2})$, so by Lemma 3.2.3,

$$p(x_0, x) = p(\tfrac{1}{\delta}\|x\|, x) = p\left(\left\|\frac{1}{\delta}x\right\|, \delta\left(\frac{1}{\delta}x\right)\right) > 0.$$

Therefore, $p(x_0, x) > 0$ for all $(x_0, x) \in L_\varepsilon$, so L_ε is contained in the connected component of $\mathbb{R}^{n+1} \setminus \{x \in \mathbb{R}^{n+1} : p(x) = 0\}$ containing e_0. Thus, if p is hyperbolic with respect to e_0, then it is also hyperbolic with respect to L_ε. $\square$

Proposition 3.2.5. *Let p and ε be as above, and let $K = \operatorname{cone}\{e_0 \pm \varepsilon e_i : i = 1, \ldots, n\}$. Then p is hyperbolic with respect to e_0 if and only if p is hyperbolic with respect to K.*

Proof. Since $e_0 \in K$, the if direction is immediate.

Conversely, by Proposition 3.2.4, if p is hyperbolic with respect to e_0, then it is hyperbolic with respect to L_ε. Since $K \subseteq L_\varepsilon$, it follows that p is hyperbolic with respect to K. $\square$

Theorem 3.2.6. *Let $p(x_0, x) = x_0^3 - 3x_0\|x\|^2 + 2q(x)$, where $q \in \mathbb{R}[x_1, \ldots, x_n]_3$ is homogeneous of degree three. Let N be the largest absolute value of a coefficient of q, let $0 < \varepsilon < \min(\frac{1}{2Nn^3}, \frac{1}{2})$, and let M be the $(n+1) \times 2n$ matrix whose columns are $e_0 \pm \varepsilon e_i$ for $i = 1, \ldots, n$. Define*

$$\widetilde{p}(x_1, \ldots, x_{2n}) = p(Mx) \in \mathbb{R}[x_1, \ldots, x_{2n}]_3.$$

Then p is hyperbolic with respect to e_0 if and only if $\widetilde{p}$ is real stable.

Proof. First, suppose that $\widetilde{p}$ is real stable. We wish to show that for every $z \in \mathbb{R}^{n+1}$, $p(te_0+z)$ is real rooted. We note that M is rank $n+1$, so there exists some $x \in \mathbb{R}^{2n}$ such that $z = Mx$. Moreover, $M\mathbb{1} = 2ne_0$, where $\mathbb{1}$ denotes the vector of all ones. Then,

$$p(te_0 + z) = p\left(\frac{t}{2n}M\mathbb{1} + Mx\right) = \widetilde{p}\left(\frac{t}{2n}\mathbb{1} + x\right).$$

Since $\widetilde{p}$ is real stable and $\frac{1}{2n}\mathbb{1} \in \mathbb{R}^{2n}_{>0}$, then this polynomial is real rooted, so p is hyperbolic with respect to e_0, as desired.

Now suppose that p is hyperbolic with respect to e_0. By Proposition 3.2.5, p is hyperbolic with respect to $K = \operatorname{cone}\{e_0 \pm \varepsilon e_i\}$. To show that $\widetilde{p}$ is stable, we wish to show that for any $v \in \mathbb{R}^{2n}_{>0}$ and $x \in \mathbb{R}^{2n}$, $\widetilde{p}(tv + x)$ is real rooted. Indeed, by definition,

$$\widetilde{p}(tv + x) = p(t(Mv) + Mx).$$

Since $v \in \mathbb{R}^{2n}_{>0}$, then $Mv \in K$, and so $p(t(Mv) + Mx)$ is real rooted. Therefore, $\widetilde{p}$ is stable, as desired. $\square$

We are now ready to show that testing stability of cubics is coNP-hard. We require the following result of Nesterov:

Theorem 3.2.7 ([Nes03, Theorem 4], [Sau19, Theorem 5.2])**.** *Let $G = (V, E)$ be a simple graph and let $\omega(G)$ be the size of a largest clique in G. Define*

$$q_G(x, y) = \sum_{(i,j) \in E} x_i x_j y_{ij}.$$

Then

$$\max_{\|x\|^2 + \|y\|^2 = 1} q_G(x, y) = \sqrt{\frac{2}{27}} \sqrt{1 - \frac{1}{\omega(G)}}.$$

Since we want the cubic we are testing to have rational coefficients, we first need a quick technical lemma.

Lemma 3.2.8. *Let G be a graph on n vertices, and let $\omega(G)$ denote its clique number. Let*

$$q_G(x, y) = \sum_{ij \in E} x_i x_j y_{ij}.$$

For any integer $k \le n$, there exists a rational number $\ell(k)$, computable in polynomial time in n, such that $\omega(G) \le k$ if and only if

$$\max_{\|x\|^2 + \|y\|^2 = 1} q_G(x, y) \le \ell(k).$$

Proof. For ease of notation, let $a(k) = \sqrt{\frac{2}{27}} \sqrt{1 - \frac{1}{k}}$. By Theorem 3.2.7, we know that $\max_{\|(x,y)\|^2 = 1} q_G(x, y) = a(\omega(G))$. Moreover, $a(k)$ is an increasing function, so we have $\max_{\|(x,y)\|^2 = 1} q_G(x, y) \le a(k)$ if and only if $\omega(G) \le k$.

Define

$$\ell(k) = \frac{\lceil 8n^2 a(k) \rceil}{8n^2}.$$

We note that $\ell(k)$ is rational. Moreover, since $a(k) < \sqrt{\frac{2}{27}}$ for all $k \ge 1$, its numerator and denominator are both polynomial in n, so $\ell(k)$ can be computed in polynomial time.

Since $\ell(k) \ge a(k)$, one direction is immediate. Namely, if $\omega(G) \le k$, then

$$\max_{\|x\|^2 + \|y\|^2 = 1} q_G(x, y) \le a(k) \le \ell(k).$$

For the converse, suppose that $\omega(G) > k$. Since $\omega(G)$ is always an integer, this implies that $\omega(G) \ge k + 1$, and so $\max_{\|(x,y)\|^2 = 1} q_G(x, y) \ge a(k + 1)$.

In order to complete the proof, it remains to show that $\ell(k) < a(k+1)$. Indeed, for any $1 < x \leq n$,

$$a'(x) = \sqrt{\frac{2}{27}}\frac{1}{2\sqrt{x^3(x-1)}} > \frac{1}{\sqrt{54}}\frac{1}{x^2} \geq \frac{1}{\sqrt{54}}\frac{1}{n^2} > \frac{1}{8n^2},$$

so

$$a(k+1) = a(k) + \int_k^{k+1} a'(x)dx \geq a(k) + \frac{1}{8n^2}.$$

Since $\lceil 8n^2 a(k) \rceil < 8n^2 a(k) + 1$, it follows that $\ell(k) < a(k) + \frac{1}{8n^2} \leq a(k+1)$.

Thus, if $\omega(G) > k$, then $\max_{\|(x,y)\|^2=1} q_G(x,y) > \ell(k)$, and so $\omega(G) \leq k$ if and only if $\max_{\|(x,y)\|^2=1} q_G(x,y) \leq \ell(k)$, as desired. $\square$

Theorem 3.2.9. *Given a homogeneous cubic polynomial p with rational coefficients, it is coNP-hard to decide if p is stable.*

Proof. Given a graph G and integer k, it is coNP-hard to decide if $\omega(G) \leq k$ [Kar72]. Given G and k, define $\ell = \ell(k)$ as in Lemma 3.2.8, and consider the cubic polynomial

$$p(x_0, x, y) = x_0^3 - 3x_0\left(\|x\|^2 + \|y\|^2\right) + 2\frac{q_G(x,y)}{\ell}.$$

By Theorem 3.2.1, this polynomial is hyperbolic with respect to e_0 if and only if

$$\begin{aligned} \max_{\|(x,y)\|^2=1} \left|\frac{q_G(x,y)}{\ell}\right| \leq 1 &\Leftrightarrow \max_{\|(x,y)\|^2=1} q_G(x,y) \leq \ell \\ &\Leftrightarrow \omega(G) \leq k, \end{aligned}$$

so it is coNP-hard to decide if this polynomial (with rational coefficients) is hyperbolic with respect to e_0. By Theorem 3.2.6, deciding whether p is hyperbolic with respect to e_0 is equivalent to checking stability of $\widetilde{p} = p(Mx)$.

By choosing ε to be rational, then $\widetilde{p}$ will have rational coefficients, so if we could check stability of homogeneous cubics with rational coefficients, we could also check hyperbolicity of p with respect to e_0.

Since N is the largest norm of a coefficient of $q(x)$, it is already encoded in the input, so computing a positive rational number $\varepsilon < \min(\frac{1}{2Nn^3}, \frac{1}{2})$ takes polynomial time, as does performing the necessary change of variables by M.

Therefore, we have a polynomial time reduction, and so deciding stability for a homogeneous cubic with rational coefficients is coNP-hard. $\square$

Example 3.2.10. Let G be the path graph on three vertices, and let $k = 2$. We construct $q_G(x_1, x_2, x_3, y_{12}, y_{23}) = x_1x_2y_{12} + x_2x_3y_{13}$. Then $a(k) = \sqrt{1/27}$, and $\ell(k) = \frac{\lceil 8(25)/\sqrt{27}\rceil}{8(25)} = \frac{39}{200}$. Following the construction in Theorem 3.2.9 gives

$$p(x_0, x_1, x_2, x_3, y_{12}, y_{23}) = x_0^3 - 3x_0(x_1^2 + x_2^2 + x_3^2 + y_{12}^2 + y_{23}^2) + \frac{400}{39}(x_1x_2y_{12} + x_2x_3y_{23}).$$

Thus, $N = \frac{400}{39}$, so we may take $\varepsilon = 10^{-4} < \frac{1}{2(\frac{400}{39})5^3} = \frac{39}{50000}$. Finally, we construct $\widetilde{p} \in \mathbb{R}[z_1, z_2, z_3, z_{12}, z_{23}, w_1, w_2, w_3, w_{12}, w_{13}]$, where $z_i \mapsto x_0 + 10^{-4}x_i$, $z_{ij} \mapsto x_0 + 10^{-4}y_{ij}$, $w_i \mapsto x_0 - 10^{-4}x_i$, and $w_{ij} \mapsto x_0 - 10^{-4}y_{ij}$. Then,

$$\begin{aligned}\widetilde{p}(z, w) = {} & (z_1 + w_1 + z_2 + w_2 + z_3 + w_3 + z_{12} + w_{12} + z_{23} + w_{23})^3 \\ & - 3 \cdot 10^{-8} (z_1 + w_1 + z_2 + w_2 + z_3 + w_3 + z_{12} + w_{12} + z_{23} + w_{23}) \|z - w\|^2 \\ & + \frac{400}{39} \cdot 10^{-12} \left((z_1 - w_1)(z_2 - w_2)(z_{12} - w_{12}) + (z_2 - w_2)(z_3 - w_3)(z_{23} - w_{23})\right).\end{aligned}$$

Since $\omega(G) \leq 2$, $\widetilde{p}$ is real stable.

As an easy corollary to Theorem 3.2.9, it is also coNP-hard to test for stability in higher degrees.

Corollary 3.2.11. *Fix a degree $d \geq 3$. Given a homogeneous polynomial p with rational coefficients of degree d, it is coNP-hard to decide if p is stable.*

Proof. Given a homogeneous cubic $p \in \mathbb{R}[z_1, \ldots, z_n]$, we introduce a new variable y and consider the polynomial $q = y^{d-3}p \in \mathbb{R}[y, z_1, \ldots, z_n]$. Then q is a homogeneous polynomial of degree d, and q is stable if and only if p is stable.

Thus, if we can test stability in degree $d \geq 3$, we can also test stability of homogeneous cubics, so stability is coNP-hard for all $d \geq 3$. $\square$

3.3 Lorentzianity

While all homogeneous stable polynomials are Lorentzian, the converse is not true. In fact, while deciding if a polynomial is real stable is coNP-hard, deciding complete log concavity for polynomials of fixed degree d can be done in polynomial time in the number of variables n. Note that by fixing the degree, we ensure that our input size is polynomial in n, which is important for complexity results. Moreover, our hardness results are also for fixed degrees, so this result showing polynomial time is still in contrast to those results.

The key is the following result from [ALOV24b]. We say a polynomial $f \in \mathbb{R}[z_1, \dots, z_n]$ is *indecomposable* if it cannot be written as $f = f_1 + f_2$, where f_1, f_2 are nonzero polynomials in disjoint sets of variables.

Theorem 3.3.1 ([ALOV24b, Theorem 3.2])**.** *Let $f \in \mathbb{R}[z_1, \dots, z_n]$ be a homogeneous polynomial of degree $d \geq 2$ with nonnegative coefficients. Then f is completely log concave if and only if the following two conditions hold:*

(i) For all $\alpha \in \mathbb{Z}^n_{\geq 0}$ with $|\alpha| \leq d-2$, the polynomial $\partial^\alpha f$ is indecomposable.

(ii) For all $\alpha \in \mathbb{Z}^n_{\geq 0}$ with $|\alpha| = d-2$, the quadratic polynomial $\partial^\alpha f$ is log concave over $\mathbb{R}^n_{\geq 0}$.

The following proposition uses Theorem 3.3.1 to give a polynomial time algorithm to check complete log concavity.

Proposition 3.3.2. *For fixed degree d, there is an algorithm, which runs in time $O(n^{d+1})$, that decides if a homogeneous polynomial $f \in \mathbb{R}_{\geq 0}[z_1, \dots, z_n]_d$ is completely log concave on $\mathbb{R}^n_{\geq 0}$.*

Proof. Since d is fixed, there are $O(n^{d-2})$ partial derivatives to check. In order to check if a polynomial is indecomposable, we form a graph with vertices $\{i \in [n] : \partial_i f \neq 0\}$ and add edge (i, j) if and only if there is a monomial of f that contains both x_i and x_j. Then f is indecomposable if and only if this graph is connected, and we can construct this graph and check if it is connected in time $O(n^2)$. Since there are $O(n^{d-2})$ derivatives to check, each of which takes $O(n^2)$ time, we can check the first condition of Theorem 3.3.1 in time $O(n^d)$.

Moreover, if $|\alpha| = d-2$, then $\partial^\alpha f$ is quadratic, so the entries of $\nabla^2 \partial^\alpha f(z)$ do not depend on z, and $\partial^\alpha f$ is log concave if and only if $\nabla^2 \partial^\alpha f$ has exactly one positive eigenvalue. We can compute the signs of the eigenvalues of an $n \times n$ matrix in $O(n^3)$ time using Gaussian elimination [BW66], so it follows that checking whether $\partial^\alpha f$ is log concave can be accomplished in time at most $O(n^3)$. There are $O(n^{d-2})$ derivatives to check, each of which takes time $O(n^3)$, so we can check the second condition in time $O(n^{d+1})$.

Therefore, we can check both conditions of Theorem 3.3.1 in time at most $O(n^{d+1})$, as desired. □

3.4 Computational Complexity of Log Concavity

In Section 3.3, we showed that we can decide if a homogeneous polynomial is completely log concave in polynomial time in the number of variables. However, unlike complete log concavity, we cannot reduce checking if a polynomial is log concave to checking its quadratic derivatives, and in fact deciding if a polynomial is log concave is coNP-hard in degrees $d \geq 4$.

Recall that we say a polynomial $f \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ is log-concave if $\log(f)$ is concave as a function on $\mathbb{R}^n_{>0}$.

Theorem 3.4.1. *It is coNP-hard to decide if a homogeneous polynomial of degree four with nonnegative coefficients is log concave on* $\mathbb{R}^n_{\geq 0}$.

To prove this, we first give a reduction from checking if a homogeneous quartic is convex on $\mathbb{R}^n_{\geq 0}$. Then, we refine the main result from [AOPT11] to show that it is coNP-hard to decide if a polynomial is convex on the nonnegative orthant.

There is some subtlety in that log concavity is defined over the positive orthant $\mathbb{R}^n_{>0}$, while in our construction, we will use points in the nonnegative orthant $\mathbb{R}^n_{\geq 0}$. For the sake of completeness, we prove the equivalence here.

Lemma 3.4.2. *Let $g \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree $d \geq 2$. Then g is log concave if and only if $\nabla^2 g(x)$ has at most one positive eigenvalue for all $x \in \mathbb{R}^n_{\geq 0}$.*

Proof. First, suppose that $\nabla^2 g(x)$ has at most one positive eigenvalue for all $x \in \mathbb{R}^n_{\geq 0}$. Then for any $x \in \mathbb{R}^n_{>0}$, we have $g(x) > 0$ and $\lambda_2(\nabla^2 g(x)) \leq 0$. Hence, by [BH20, Proposition 2.33], g is log concave on $\mathbb{R}^n_{>0}$.

On the other hand, suppose that $\lambda_2(\nabla^2 g(x)) > 0$ for some $x \in \mathbb{R}^n_{\geq 0}$. Since x is in the nonnegative orthant, we know that $x + \varepsilon \mathbb{1} \in \mathbb{R}^n_{>0}$ for all $\varepsilon > 0$. Since eigenvalues are continuous in the entries of a matrix, for sufficiently small $\varepsilon > 0$, $\lambda_2(\nabla^2 g(x + \varepsilon \mathbb{1})) > 0$, so there exists a point in $\mathbb{R}^n_{>0}$ where g is not log concave, as desired. □

Next, we use this result to relate log concavity to convexity on $\mathbb{R}^n_{\geq 0}$.

Lemma 3.4.3. *Let $f \in \mathbb{R}[x_1, \ldots, x_n]_4$ be a homogeneous quartic polynomial, and let $N > 0$ be at least as large as the largest coefficient of f. Let*

$$g(x_1, \ldots, x_n, z) = N \left(z + \sum_{i=1}^{n} x_i \right)^4 - f(x_1, \ldots x_n) \in \mathbb{R}[x_1, \ldots, x_n, z].$$

Then g is a homogeneous quartic with nonnegative coefficients, and f is convex on $\mathbb{R}^n_{\geq 0}$ if and only if g is log concave.

Proof. First, we claim that g has nonnegative coefficients. Indeed, every degree four monomial in $x_1, \ldots, x_n$ appears in $N(z + x_1 + \cdots + x_n)^4$ with coefficient at least N. Since we chose N to be at least as large as the largest coefficient of f, then the coefficient of every monomial of g is at least zero, so g has nonnegative coefficients.

Next, we claim that if f is convex on $\mathbb{R}^n_{\geq 0}$, then g is log concave. We note that by construction,

$$\nabla^2 g(x, z) = 12N(z + x_1 + \cdots + x_n)^2 \mathbb{1}\mathbb{1}^T + \left(\begin{array}{c|c} & 0 \\ -\nabla^2 f(x) & \vdots \\ & 0 \\ \hline 0 \quad \ldots \quad 0 & 0 \end{array} \right), \tag{3.1}$$

where $\mathbb{1} \in \mathbb{R}^{n+1}$ denotes the vector of all ones. By Lemma 3.4.2, g is log concave if and only if $\nabla^2 g(x,z)$ has at most one positive eigenvalue for all $(x,z) \in \mathbb{R}^{n+1}_{\geq 0}$. If f is convex on $\mathbb{R}^n_{\geq 0}$, then $-\nabla^2 f(x) \preceq 0$, and since the first term is rank one, this implies that $\nabla^2 g(x,z)$ has at most one positive eigenvalue, so g is log concave, as desired.

Now, suppose that f is not convex on $\mathbb{R}^n_{\geq 0}$. Then we claim that g is not log concave. Indeed, suppose that f is not convex at $x \in \mathbb{R}^n_{\geq 0}$, and let $Q = \nabla^2 g(x,1)$. By the Courant-Fisher theorem, in order to show that Q has at least two positive eigenvalues, it suffices to show that $Q \succ 0$ on a two-dimensional subspace of $\mathbb{R}^{n+1}$.

Since f is not convex at $x \in \mathbb{R}^n_{\geq 0}$, then $\nabla^2 f(x) \not\succeq 0$, so there exists some $v \in \mathbb{R}^n \setminus \{0\}$ and $\lambda > 0$ such that $-\nabla^2 f(x)v = \lambda v$. We claim that $Q \succ 0$ on $L = \operatorname{span}\left\{\begin{pmatrix}\vec{v}\\0\end{pmatrix}, \begin{pmatrix}\vec{0}\\1\end{pmatrix}\right\} \subseteq \mathbb{R}^{n+1}$.

Indeed, let $w = \begin{pmatrix}\alpha v\\ \beta\end{pmatrix} \in L \subset \mathbb{R}^{n+1}$. Then

$$\begin{aligned} w^T Q w &= 12N(1 + x_1 + \cdots + x_n)^2(\mathbb{1}^T w)^2 + \alpha^2 v^T(-\nabla^2 f(x))v \\ &= 12N(1 + x_1 + \cdots + x_n)^2(\mathbb{1}^T w)^2 + \alpha^2 \lambda \|v\|^2. \end{aligned}$$

Since both terms are nonnegative, $w^T Q w \geq 0$, with equality if and only if both terms are zero. It thus just remains to show that $w^T Q w = 0$ implies that $\alpha = \beta = 0$.

Since $\lambda > 0$ and $v \neq 0$, $\alpha^2 \lambda \|v\|^2 = 0$ if and only if $\alpha = 0$. Thus, if $w^T Q w = 0$, then $\alpha = 0$, so $w = \begin{pmatrix}0\\ \beta\end{pmatrix}$ and

$$0 = \begin{pmatrix}0\\ \beta\end{pmatrix}^T Q \begin{pmatrix}0\\ \beta\end{pmatrix} = 12N(1 + x_1 + \cdots + x_n)^2 \beta^2.$$

Since $x \in \mathbb{R}^n_{\geq 0}$, we have $x_1 + \cdots + x_n + 1 > 0$. Additionally, we chose $N > 0$, so this implies that $\beta = 0$. Thus, if $w \in L \subset \mathbb{R}^{n+1}$, then $w^T Q w \geq 0$ with equality if and only if $w = 0$. Hence, $Q \succ 0$ on L, which implies that g is not log-concave at $(x, 1)$.

Hence, g is log concave if and only if f is convex on $\mathbb{R}^n_{\geq 0}$, as desired. $\square$

Next, we show that it is coNP-hard to decide if a quartic polynomial is convex on $\mathbb{R}^n_{\geq 0}$. We use the construction from [AOPT11] and refine their argument to show that for a graph G and integer k, we can construct a quartic polynomial f so that f is convex on the nonnegative orthant if and only if G has no clique of size larger than k.

We say a biquadratic form $b(x; y)$ is positive semidefinite (PSD) if $b(x; y) \geq 0$ for all $x, y \in \mathbb{R}^n$.

Theorem 3.4.4 ([AOPT11], Theorem 2.3)**.** *Given a biquadratic form*

$$b(x; y) = \sum_{i,j=1}^{n} \alpha_{ij} x_i x_j y_i y_j,$$

define the $n \times n$ polynomial matrix $C(x, y)$ by setting $[C(x, y)]_{ij} = \frac{\partial^2 b(x;y)}{\partial x_i \partial y_j}$ and let γ be the largest coefficient, in absolute value, of any monomial present in some entry of the matrix $C(x, y)$. Let f be the form given by

$$f(x, y) = b(x; y) + \frac{n^2\gamma}{2} \left(\sum_{i=1}^{n} x_i^4 + \sum_{i=1}^{n} y_i^4 + \sum_{\substack{i,j=1,\ldots,n \\ i<j}} x_i^2 x_j^2 + \sum_{\substack{i,j=1,\ldots,n \\ i<j}} y_i^2 y_j^2 \right). \tag{3.2}$$

Then $b(x; y)$ is PSD if and only if $f(x, y)$ is convex.

In [AOPT11], given a graph G and integer k, they use a result from [LNQY10] to construct a biquadratic form that is PSD if and only if $\omega(G) \leq k$, which implies that deciding convexity of quartics is coNP-hard.

Lemma 3.4.5 ([AOPT11; LNQY10])**.** *Let $G = ([n], E)$ be a graph and let $1 \leq k \leq n$. Define the biquadratic form*

$$b_G(x; y) = -2k \sum_{ij \in E} x_i x_j y_i y_j - (1 - k) \left(\sum_{i=1}^{n} x_i^2 \right) \left(\sum_{i=1}^{n} y_i^2 \right). \tag{3.3}$$

Then $b_G(x; y)$ is PSD if and only if $\omega(G) \leq k$, where $\omega(G)$ denotes the size of the largest clique in G.

We use the same biquadratic form and construction, and refine the result to show that it is coNP-hard to test if a quartic is convex on the nonnegative orthant, rather than on all of $\mathbb{R}^n$.

Lemma 3.4.6. *Let $G = ([n], E)$ be a graph, and let $1 \leq k \leq n$. Define the biquadratic form $b_G(x; y)$ as in Equation (3.3), and construct f as in Theorem 3.4.4. Then f is convex on $\mathbb{R}^{2n}_{\geq 0}$ if and only if $\omega(G) \leq k$.*

Proof. One direction is immediate from [AOPT11]. Namely, if $\omega(G) \leq k$, then Lemma 3.4.5 implies that $b_G(x; y)$ is PSD. Theorem 3.4.4 then says that f is convex, so in particular, it is convex on $\mathbb{R}^{2n}_{\geq 0}$.

Now, suppose that $\omega(G) > k$. Following the constructions in [LNQY10, Theorem 2.2] and [MS65, Theorem 1], let $\mathcal{C} \subseteq [n]$ be a clique of size $\omega(G)$ in G, and define $x^*, y^* \in \mathbb{R}^n_{\geq 0}$ by

$$x_i^* = y_i^* = \begin{cases} \frac{1}{\sqrt{\omega(G)}}, & i \in \mathcal{C} \\ 0, & \text{otherwise} \end{cases}.$$

We claim that f is not convex at $(x^*, 0) \in \mathbb{R}^{2n}_{\geq 0}$.

We note that

$$b_G(x^*; y^*) = -2k\binom{\omega(G)}{2}\frac{1}{\omega(G)^2} - (1-k) = \frac{k}{\omega(G)} - 1 < 0.$$

Moreover, we note that for any $x \in \mathbb{R}^n$, $\frac{\partial^2}{\partial y_i \partial y_j} f(x, 0) = \frac{\partial^2}{\partial y_i \partial y_j} b_G(x, 0)$. Furthermore, $\frac{\partial^2}{\partial y_i \partial y_j} b_G(x, y)$ does not depend on y, so $\frac{\partial^2}{\partial y_i \partial y_j} b_G(x, 0) = \frac{\partial^2}{\partial y_i \partial y_j} b_G(x, y^*)$. Thus, since b_G is homogeneous of degree 2 in the y variables,

$$\begin{pmatrix} 0 \\ y^* \end{pmatrix}^T \nabla^2 f(x^*, 0) \begin{pmatrix} 0 \\ y^* \end{pmatrix} = \sum_{i,j=1}^{n} y_i^* y_j^* \frac{\partial^2}{\partial y_i \partial y_j} b_G(x^*, y^*) = 2b_G(x^*, y^*) < 0.$$

Therefore, $\nabla^2 f(x^*, 0) \not\succeq 0$, and so f is not convex on $\mathbb{R}^{2n}_{\geq 0}$, as desired. $\square$

Finally, we combine Lemma 3.4.3 and Lemma 3.4.6 to show that deciding log concavity is coNP-hard.

Proof of Theorem 3.4.1. Let $G = ([n], E)$ be a graph on n vertices, and let $1 \leq k \leq n$. Construct $b_G(x; y)$ as in Equation (3.3) and f as in Equation (3.2). Then we have a quartic polynomial in $2n$ variables, and by Lemma 3.4.6, f is convex on $\mathbb{R}^{2n}_{\geq 0}$ if and only if $\omega(G) \leq k$. Moreover, constructing f takes polynomial time in n [AOPT11, Theorem 2.1].

Then, using Lemma 3.4.3, we get a homogeneous quartic polynomial g in $2n+1$ variables with nonnegative coefficients so that g is log concave if and only if f convex on $\mathbb{R}^{2n}_{\geq 0}$. Thus, g is log concave if and only if $\omega(G) \leq k$, so it just remains to show that constructing g from f takes polynomial time.

Indeed, to get from f to g, we add a polynomial with $\binom{n+4}{4}$ monomials, each with coefficient at most $24N$. The size of N is polynomial in our input, since we may simply choose it to be the largest coefficient of f.

Thus, our construction runs in polynomial time, so it is coNP-hard to decide if a homogeneous quartic polynomial is log concave. $\square$

Corollary 3.4.7. *For fixed degree $d \geq 4$, it is coNP-hard to decide if a homogeneous polynomial of degree d with nonnegative coefficients is log concave.*

Proof. We give a reduction from the problem of deciding log concavity of homogeneous quartic polynomials. Let $f \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ be a homogeneous quartic, and define $g = z^{d-4} f(x) \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n, z]$. We note that g is a homogeneous polynomial of degree d with nonnegative coefficients, and we claim that g is log concave if and only if f is log concave.

Indeed, we note that $\log g = (d-4)\log(z) + \log f$, so

$$\nabla^2 \log g(x,z) = (d-4)\left(\begin{array}{ccc|c} 0 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & 0 \\ \hline 0 & \ldots & 0 & \frac{-1}{z^2} \end{array}\right) + \left(\begin{array}{ccc|c} & & & 0 \\ & \nabla^2 \log f(x) & & \vdots \\ & & & 0 \\ \hline 0 & \ldots & 0 & 0 \end{array}\right).$$

Then $\nabla^2 \log g(x,z) \preceq 0$ if and only if $\nabla^2 \log f(x) \preceq 0$, so g is log concave if and only if f is log concave.

This construction takes polynomial time and only increases the size of the input by a polynomial amount. Therefore, since it is coNP-hard to decide if a homogeneous quartic is log concave, it is also coNP-hard to decide if a homogeneous polynomial of degree $d \geq 4$ is log concave, as desired. □

3.5 Log Concavity For Cubics

For a polynomial f of degree $d \geq 3$, we know that it is coNP-hard to decide if f is real stable (Section 3.2) or hyperbolic with respect to a given direction [Sau19, Theorem 5.4]. However, our results in Section 3.4 only apply to polynomials of degree four or more.

In this section, we examine the reason for this gap by studying log concavity of cubic polynomials from two different perspectives. First, we examine the weaker property of directional log concavity, where we restrict ourselves to testing whether a polynomial is log concave in a given direction, and show that testing this property is coNP-hard for homogeneous cubics. On the other hand, in Section 3.5.2, we show that we can decide if a homogeneous cubic is log concave in the usual sense in polynomial time.

3.5.1 Directional Log Concavity

First we give a formal definition of directional log concavity.

Definition 3.5.1. We say that a polynomial $f \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ is *log concave in direction* $v \in \mathbb{R}^n$ if for all $x \in \mathbb{R}^n_{\geq 0}$, $D_v^2 \log f(x) \leq 0$.

To show that testing directional log concavity is coNP-hard, we first relate testing directional log concavity to maximizing a homogeneous cubic.

Theorem 3.5.2. *Let* $q(x) \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]_3$. *Then*

$$f(x, z) = z^3 + 3\|x\|^2 z + 2q(x)$$

is log concave in direction $e_z = (0, \ldots, 0, 1)$ *at all* $(x, z) \in \mathbb{R}^{n+1}_{\geq 0}$ *if and only if* $\max_{\|x\|=1} q(x) \leq 1$.

To prove this, we first characterize log concavity for depressed cubics (i.e. univariate cubics where the coefficient of z^2 is zero).

Lemma 3.5.3. *Let $f(z) = z^3 + bz + c \in \mathbb{R}_{\geq 0}[z]$. Then f is log concave on $\mathbb{R}_{\geq 0}$ if and only if $4b^3 \geq 27c^2$.*

Proof. Define $g(z) = f^2 \frac{d^2}{dz^2} \log f = f \cdot f'' - (f')^2$. Then f is log concave at z if and only if $g(z) \leq 0$.

By direct computation,

$$g(z) = (z^3 + bz + c)(6z) - (3z^2 + b)^2 = -3z^4 + 6cz - b^2.$$

Then

$$g'(z) = -12z^3 + 6c,$$

so g is minimized at $z = \left(\frac{c}{2}\right)^{1/3} \geq 0$, and

$$g\left(\left(\frac{c}{2}\right)^{1/3}\right) = 9\left(\frac{c}{2}\right)^{4/3} - b^2.$$

Thus, f is log concave if and only if $b^2 \geq 9\left(\frac{c}{2}\right)^{4/3}$, which happens if and only if $4b^3 \geq 27c^2$, as desired. □

Proof of Theorem 3.5.2. Fix $x \in \mathbb{R}^n_{\geq 0}$. By Lemma 3.5.3, f is log concave (as a function of z) at x if and only if $\|x\|^6 \geq q(x)^2$. Since q is homogeneous of degree three, this happens if and only if $q(\frac{x}{\|x\|})^2 \leq 1$.

Since $\{x \in \mathbb{R}^n : \|x\| = 1\}$ is compact, we know that q achieves its maximum on this set at some x^*. Moreover, since q has nonnegative coefficients, we may assume without loss of generality that $x^* \in \mathbb{R}^n_{\geq 0}$. If $q(x^*) > 1$, then $q(x^*)^2 > \|x^*\|^6$, and so f is not log concave in the z direction at (x^*, z^*) for some $z^* \geq 0$.

Conversely, if $q(x^*) \leq 1$, then for any $x \in \mathbb{R}^n_{\geq 0}$, $q(x) \leq \|x\|^3$, and so f, as a function of z, satisfies the conditions of Lemma 3.5.3 and is therefore log concave in the z direction at all $(x, z) \in \mathbb{R}^{n+1}_{\geq 0}$. □

By the same argument as we used in Theorem 3.2.9, it is coNP-hard to decide if $q(x) \leq 1$ on the unit sphere, so we get the following hardness result for directional log concavity.

Theorem 3.5.4. *It is coNP-hard to determine if a homogeneous cubic is log concave on the positive orthant in a given direction.*

3.5.2 Log Concavity for Homogeneous Cubics

Although it is coNP-hard to test whether a homogeneous cubic is log concave in a particular direction, we can test log concavity on $\mathbb{R}^n_{>0}$ for homogeneous cubics in polynomial time.

Since we can test complete log concavity in polynomial time (Proposition 3.3.2), it suffices to show that a homogeneous cubic is log concave if and only if it is completely log concave.

Theorem 3.5.5. *Let $f \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]_3$ be a homogeneous cubic. Then f is log concave on $\mathbb{R}^n_{>0}$ if and only if it is completely log concave with respect to $\mathbb{R}^n_{>0}$.*

Proof. One direction is immediate: if f is completely log concave then it is by definition also log concave.

It remains to show that if f is log concave, then it is also completely log concave. First, since f is log concave, it is indecomposable [ALOV24b, Theorem 3.2]. Therefore, in order to show it is completely log concave, it suffices to show that $\partial_1 f, \ldots, \partial_n f$ are all log concave [ALOV24b, Lemma 3.4].

Since f is homogeneous of degree three, by Euler's identity,

$$3f(x) = \sum_{i=1}^{n} x_i \partial_i f(x).$$

Let E_{ij} denote the $n \times n$ matrix with a one in the (i, j)th entry and zeros elsewhere.

Using the product rule for the Hessian,

$$\begin{aligned}3\nabla^2 f(x) &= \sum_{i=1}^n x_i \nabla^2 \partial_i f(x) + \sum_{i=1}^n (e_i(\nabla \partial_i f(x))^T + (\nabla \partial_i f(x)) e_i^T) \\ &= \sum_{i=1}^n x_i \nabla^2 \partial_i f(x) + \sum_{i=1}^n \sum_{j=1}^n \partial_{ij}^2 f(x) E_{ij} + \partial_{ij}^2 f(x) E_{ji} \\ &= \sum_{i=1}^n x_i \nabla^2 \partial_i f(x) + 2\nabla^2 f(x),\end{aligned}$$

so

$$\nabla^2 f(x) = \sum_{i=1}^n x_i \nabla^2 \partial_i f(x).$$

Since $\partial_i f$ is a homogeneous polynomial of degree two, $\nabla^2 \partial_i f$ does not depend on x. Let $M_i = \nabla^2 \partial_i f$, and let $Q(x) = \sum_{i=1}^n x_i M_i$, so $\nabla^2 f(x) = Q(x)$. It remains to show that $\lambda_2(M_i) \le 0$ for all $i = 1, \ldots, n$.

Since f is log concave by assumption, then $\lambda_2(Q(x)) \le 0$ for all $x \in \mathbb{R}^n_{>0}$. Since eigenvalues are continuous functions in the entries of a matrix, then $\lambda_2(Q(x)) \le 0$ for all $x \in \mathbb{R}^n_{\ge 0}$. Since $M_i = Q(e_i)$, this implies that $\lambda_2(M_i) \le 0$, so $\partial_i f$ is log concave, as desired. □

Since we can decide if a polynomial is completely log concave in polynomial time, this has immediate implications for testing log concavity of homogeneous cubics.

Corollary 3.5.6. *There is a polynomial time algorithm to decide if a homogeneous cubic polynomial is log concave on* $\mathbb{R}^n_{\ge 0}$.

3.6 Conclusion and Future Work

This paper discusses stable, Lorentzian, and log-concave polynomials. In particular, we show that it is coNP-hard to test if polynomials are stable or log concave. While stable and log-concave polynomials are useful tools, this implies that (assuming P $\neq$ NP), it is intractible to test if a given polynomial is stable or log concave unless additional structure is present.

Unfortunately, while all stable polynomials are Lorentzian and all Lorentzian polynomials are log concave, testing Lorentzianity doesn't give any meaningful approximation for the max clique problem. Indeed, if we want to check if $\omega(G) \leq k$ for any $k \geq 2$, then the polynomial from Section 3.2 is always Lorentzian. On the other hand, the polynomial from Section 3.4 is never Lorentzian. Thus, the most that testing Lorentzianity can tell us is that the clique number of any graph is at least one.

Throughout this paper, we used reductions from the max clique problem to show that testing our desired properties is coNP-hard. However, our reductions do not use the full strength of max clique: the max clique problem is not only NP-hard, but also NP-hard to approximate. This hardness of approximation should allow us to strengthen our results, but the exact form of this conjectured stronger result remains open.

Moreover, while we have given lower bounds on the complexity of deciding stability and log concavity, we do not have upper bounds on their complexity. While there are obvious certificates that a given polynomial is not real stable or not log concave, we cannot guarantee that these certificates are polynomial size, so we cannot conclude that these problems are in coNP. Instead, we would need to broaden our scope to other complexity classes, such as the universal theory of the reals.

Chapter 4

LORENTZIAN SYMMETRIC FUNCTIONS

In this chapter, we cover results from [CQ25] discussing the interplay between symmetric polynomials and Lorentzianity. The results presented in this chapter are based on joint work with Daniel Qin. The only new material not from [CQ25] is the brief discussion of the topology of the space of Lorentzian symmetric functions, including the fact that this space has nonempty interior (Proposition 4.3.12).

4.1 Introduction

Lorentzian polynomials have been an extremely active area of research over the past few years. They have been a particularly useful tool in proving conjectures about (ultra) log-concavity of sequences, as they provide a link between the continuous log-concavity of a polynomial as a function on $\mathbb{R}^n_+$ and the discrete log-concavity of its coefficients.

The emergence of Lorentzian polynomials traces back to the independent works of Anari, Liu, Oveis Gharan, and Vinzant [ALOV24b] and Brändén and Huh [BH19], who simultaneously used the theory to prove Mason's ultra log-concavity conjecture on the number of independent sets of a matroid. Since then, Lorentzian polynomials have been used in obtaining results towards long-standing log-concavity conjectures. For example, in [HMV25], Hafner, Mészáros, and Vidinas used Lorentzian polynomials to prove a special case of Fox's conjecture, and in [AJ25], Alexandersson and Jal used them to prove the log-concavity consequence of the Neggers-Stanley conjecture for naturally labeled width two posets.

Additionally, the support of a Lorentzian polynomial is always an M-convex set, meaning that the monomials appearing in a Lorentzian polynomial are always the integer points of some generalized permutahedron. This fact was used, for example, to show that the mixed volume polynomial of any collection of convex bodies has M-convex support [BH20, Corollary 4.2].

In this paper, we explore the intersection of Lorentzian and symmetric polynomials, with an eye towards establishing useful techniques and general theory. One major paper in this intersection is [HMMS22], in which Huh, Matherne, Mészáros, and St. Dizier show that Schur polynomials are denormalized Lorentzian, and, as a corollary, show that Kostka numbers are log concave in root directions. Their proof relies heavily on tools from algebraic geometry rather than combinatorics. A second noteworthy result in the same paper is that weight multiciplicities of shifted Verma modules over $\mathfrak{sl}_{n+1}(\mathbb{C})$ are log-concave. More recently Khare, Matherne, and St. Dizier extended the latter result to the class of parabolic Verma modules [KMS25].

A large part of this paper is devoted to showing that symmetric polynomials are particularly amenable to using combinatorial techniques to test for Lorentzianity. While testing Lorentzianity for a fixed degree is always polynomial in the number of variables [Chi24], for symmetric polynomials, the number of arithmetic operations to check the Lorentzian property is independent of the number of variables (see Theorem 4.3.20). In Section 4.3, we give general results on the space of Lorentzian symmetric polynomials. In Section 4.3.2, we study the topology of the space of Lorentzian symmetric polynomials, and specifically show that it is homeomorphic to a closed Euclidean ball (Theorem 4.3.11). Then, in Section 4.3.3, we give structural results on Lorentzian symmetric polynomials, including describing the structures that enable constant time algorithms for checking Lorentzianity. In Section 4.4, we then use these structures to give explicit semialgebraic descriptions of the space of Lorentzian symmetric polynomials for small degrees.

In Section 4.5, we apply our techniques to simplify the proofs of two known results on Lorentzian symmetric polynomials. First, we tackle the result of Matherne, Morales, and Selover showing that chromatic symmetric polynomials of certain graphs are Lorentzian; in Section 4.5.1, we use our techniques to simplify the computations in their proof considerably. We also improve on the work of Qin in [Qin23], in which he uses combinatorial techniques to show that Schur polynomials of two-column tableaux are denormalized Lorentzian. In Section 4.5.2, we use our techniques to give a simpler proof, again combinatorial in nature, of this special case.

Finally, in Section 4.6, we explore the ways in which symmetric polynomials and the Lorentzian property may be incompatible. Specifically, we show that two natural operations on symmetric polynomials, the ω-involution and Hall inner product, do not preserve

Lorentzianity.

4.2 Background and Notation

4.2.1 Lorentzian Polynomials

In addition to the background established in Section 2.5, in this section, we work with denormalized Lorentzian polynomials. The normalization operator takes generating functions to exponential generating functions. The interactions between this operation and the Lorentzian property was first explored in [BH20], and further expanded in [HMMS22; BLP23].

Definition 4.2.1. Given a polynomial $f = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} c_\alpha x^\alpha$, we define its *normalization* as

$$N(f) := \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} c_\alpha \frac{x^\alpha}{\alpha!},$$

where $\alpha! := \prod_{i=1}^n \alpha_i!$. We say f is *denormalized Lorentzian* if $N(f)$ is Lorentzian.

We will also find the following result useful for analyzing the signatures of matrices, though it is purely linear algebra and already known as folklore. For example, a similar statement for strictly positive matrices appears in [BR97, Theorem 4.4.6]. However, for the sake of completeness, we reproduce it here.

Proposition 4.2.2. *Let $A \in \mathbb{R}_{\geq 0}^{n \times n}$ be a symmetric matrix with nonnegative entries. Then A has at most one positive eigenvalue if and only if for all nonempty subsets $S \subseteq [n]$, $(-1)^{|S|-1} A_S \geq 0$, where $A_S = \det(A[S])$ denotes the principal minor with rows and columns indexed by S.*

Proof. **(⇒)** Let $A \in \mathbb{R}_{\geq 0}^{n \times n}$ be a symmetric matrix and assume that A has at most one positive eigenvalue. By Cauchy interlacing, for any $S \subseteq [n]$, $A[S]$ also has at most one positive eigenvalue. If $A[S]$ is not invertible, then $(-1)^{|S|-1} A_S = 0$, and the desired inequality holds. Otherwise, we have $A_S \neq 0$, and in particular $A[S]$ is not the zero matrix. Since A

has nonnegative entries, this implies that $\mathbb{1}^T A[S]\mathbb{1} > 0$, so $A[S]$ has at least one positive eigenvalue. Therefore, $A[S]$ has exactly one positive eigenvalue, and since we assumed it was invertible, it must have exactly $|S|-1$ negative eigenvalues. Thus, $\text{sign}(A_S) = (-1)^{|S|-1}$, so $(-1)^{|S|-1}A_S \geq 0$, as desired.

($\Leftarrow$) We proceed by induction. As a base case, let $n = 2$. By assumption, we have $\det(A) \leq 0$. If $\det(A) < 0$, then A must have exactly one positive and one negative eigenvalue, as desired. If $\det(A) = 0$, then $\lambda = 0$ must be an eigenvalue of A. Since A only has two eigenvalues, then it can have at most one positive eigenvalue, as desired.

Now, suppose the result holds for $k \times k$ matrices for all $2 \leq k < n$, and let A be an $n \times n$ symmetric matrix satisfying the assumptions. By induction, we know that all $(n-1)\times(n-1)$ principal submatrices of A have at most one positive eigenvalue, so by Cauchy interlacing, A has at most two positive eigenvalues. Suppose for the sake of contradiction that A has two positive eigenvalues. This implies that $(-1)^{n-2}\det(A) \geq 0$, but by assumption we have $(-1)^{n-1}\det(A) \geq 0$, so we must have $\det(A) = 0$. Therefore $\text{rank}(A) = r < n$, and, writing $\lambda_1 \geq \cdots \geq \lambda_n$ for the eigenvalues of A, we have

$$\lambda_3(A) = \cdots = \lambda_{n-r+2}(A) = 0.$$

Since A is symmetric, $\text{rank}(A)$ is equal to the maximum size of an invertible principal submatrix of A. Let $B = A[S]$ be an invertible $r \times r$ principal submatrix of A. By the inductive hypothesis, we know that $\lambda_2(B) \leq 0$. On the other hand, by Cauchy interlacing, we have

$$\lambda_2(B) \geq \lambda_{n-r+2}(A) = 0.$$

However, this implies that $\lambda_2(B) = 0$, which contradicts the invertibility of B. Hence, A has at most one positive eigenvalue, as desired. $\square$

4.2.2 Symmetric Polynomials

A symmetric function over a ring R is an element $f \in R[[x]]$ such that for any permutation $\sigma : \mathbb{N} \to \mathbb{N}$, $f(x_1, x_2, \ldots) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots)$. We write $\Lambda_{\mathbb{R}}$ (or simply Λ) for the ring of symmetric functions over $\mathbb{R}$. The ring of symmetric functions naturally admits a grading by degree; we write Λ^d for the degree d graded piece.

Similarly, a symmetric polynomial is an element $f \in R[x_1, \dots, x_n]$ such that for any $\sigma \in S_n$, $f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$. We write Λ_n to denote the ring of symmetric polynomials over $\mathbb{R}$ in n variables. We note that Λ_n is again graded by degree, and we write Λ_n^d for the degree d graded piece.

A simple basis for Λ as an $\mathbb{R}$-vector space is the set of monomial symmetric functions

$$m_\lambda = \sum_\alpha x_1^{\alpha_1} x_2^{\alpha_2} \dots,$$

where α ranges over all permutations of the vector $\lambda = (\lambda_1, \lambda_2, \dots)$. In this paper, we primarily work with the normalized monomial basis

$$\widetilde{m}_\lambda := N(m_\lambda) = \frac{m_\lambda}{\lambda!}$$

because the normalization makes studying derivatives much cleaner.

We will also work with another basis for symmetric functions, namely Schur symmetric functions. These are again indexed by partitions and defined by

$$s_\lambda = \sum_\mu K_{\lambda,\mu} m_\mu,$$

where the Kostka number $K_{\lambda,\mu}$ is the number of semistandard Young tableaux (SSYT) of shape λ and content μ.

4.2.3 Partitions

We will also require some background on integer partitions. Recall that a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ of a nonnegative integer n is a non-increasing sequence of nonnegative numbers $\lambda_1 \geq \lambda_2 \geq \dots$ such that $\sum \lambda_i = n$. The length of λ, denoted $\ell(\lambda)$, is the number of nonzero entries in λ.

We recall the dominance order on partitions, which puts a lattice structure on the set of partitions of a given integer n.

Definition 4.2.3. *Dominance order* is the partial order on partitions of the same size defined as follows. If $\lambda, \mu \vdash n$, we say $\lambda \preceq \mu$ if $\sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{k} \mu_i$ for all k.

We also work with certain structural properties of partitions. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ be a partition of length $\ell(\mu) = k$. Suppose μ has the structure

$$\mu = (\mu_1 = \ldots = \mu_{m_2-1} > \mu_{m_2} = \ldots = \mu_{m_3-1} > \ldots > \mu_{m_\ell} = \ldots = \mu_{m_{\ell+1}-1}, 0, \ldots, 0).$$

This gives us a set partition $(\beta_1, \ldots, \beta_\ell)$ of $[k]$ with $\beta_t = [m_t, m_{t+1})$, where $m_1 = 1$ and $m_{\ell+1} = k+1$. We define $n_t := m_{t+1} - m_t$ to be the size of block β_t.

By construction, $\mu_i = \mu_j$ if and only if i and j are in the same part of this set partition. Hence, n_t is equivalently the number of times the tth largest distinct entry appears in μ. We note that μ also induces a set partition of $[n]$ by adding one more block $\beta_{\ell+1} = [k+1, n]$ to this partition. This new partition again has the property that $\mu_i = \mu_j$ if and only if i and j are in the same block.

Every partition λ is also associated to a natural polytope called a permutohedron.

Definition 4.2.4. Let $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n)$ be a partition of d. The *permutohedron* $P(\lambda) \subseteq \{x_1 + \cdots + x_n = d\} \subseteq \mathbb{R}^n$ is given by $\mathrm{conv}\{\lambda^\sigma : \sigma \in S_n\}$ where λ^σ denotes the composition $\lambda^\sigma = (\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n)})$ for all $\sigma \in S_n$.

A classical result of Rado [Rad52] characterizes integer points of permutohedra.

Proposition 4.2.5. *Let $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n)$ be a partition of d. A point $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$ belongs to $P(\lambda_1, \ldots, \lambda_n)$ if and only if t is a weak composition of d and for any nonempty subset $\{i_1, \ldots, i_k\} \subseteq [n]$ we have*

$$t_{i_1} + \ldots + t_{i_k} \leq \lambda_1 + \ldots + \lambda_k.$$

In particular, the second condition simplifies to dominance when t is a partition. A short proof of the simplification can be found in [Qin23]. Later on, this will give us an immediate alternative interpretation of supports of symmetric M-convex sets.

Proposition 4.2.6. *A partition $\mu \vdash d$ belongs to $P(\lambda)$ if and only if $\mu \preceq \lambda$.*

4.3 General Theory on Lorentzian Symmetric Polynomials

In this section, we establish some of the general properties of symmetric functions that make them particularly amenable to testing the Lorentzian property. First, however, we need to address the subtlety that a symmetric function is not a polynomial, since they are traditionally defined on an infinite set of variables. However, by restricting to finite numbers of variables, we can think of a symmetric function as defining a family of symmetric polynomials, and we define a symmetric function to be Lorentzian if all polynomials in this family are Lorentzian.

Definition 4.3.1. We say a symmetric function $f \in \Lambda^d$ is Lorentzian if for all n, the symmetric polynomial $f_n(x_1, \dots, x_n) := f(x_1, \dots, x_n, 0, \dots) \in \Lambda_n^d$ is Lorentzian.

For example, normalized Schur polynomials are one family of Lorentzian symmetric functions [HMMS22], as are chromatic symmetric functions of certain graphs arising from abelian Dyck paths [MMS24].

The finite polynomial specializations in this definition are also naturally related to one another. In particular, if f_n is Lorentzian, then so is f_k for all $k \leq n$.

Proposition 4.3.2. *Let $\{m_\alpha(n)\}_{\alpha \vdash d} \subseteq \mathbb{R}[x_1, \dots, x_n]$ denote the monomial basis for Λ_n^d, and let*

$$A_n = \left\{ (c_\alpha)_{\alpha \vdash d} \mid f_n(x_1, \dots, x_n) := \sum c_\alpha m_\alpha(n) \text{ is Lorentzian} \right\}.$$

Then these sets form a descending chain $A_d \supseteq A_{d+1} \supseteq \dots$, and the intersection $\bigcap_{n=d}^{\infty} A_n$ is non-empty.

Proof. First, we note that $m_\alpha(n-1) = m_\alpha(n)|_{x_n=0}$. Thus, $f_{n-1} = f_n|_{x_n=0}$. Since specialization preserves Lorentzianity, this implies that $A_n \subseteq A_{n-1}$, as desired.

Moreover, the elementary symmetric polynomial $e_d(x_1, \dots, x_n)$ is Lorentzian for all n [BH20, Example 2.27], so the intersection $\bigcap_{n=d}^{\infty} A_n$ is nonempty. □

Lorentzian symmetric functions inherit many useful properties from Lorentzian polynomials. For example, products of Lorentzian symmetric functions are again Lorentzian.

Proposition 4.3.3. *Suppose $f \in \Lambda^d$, $g \in \Lambda^e$ are Lorentzian symmetric functions. Then $fg \in \Lambda^{d+e}$ is again a Lorentzian symmetric function.*

Proof. We note that $(fg)_n = f_n g_n$, so it suffices to show that $f_n g_n$ is Lorentzian for all n. Indeed, f_n and g_n are Lorentzian by definition, and the product of Lorentzian polynomials is Lorentzian [BH20, Corollary 2.32], which completes the proof. $\square$

We can also study the class of linear operators that preserve Lorentzianity of symmetric functions. For example, we consider the symmetric functions analog of taking partial derivatives.

Example 4.3.4. Let $\frac{\partial}{\partial p_1} : \Lambda^d \to \Lambda^{d-1}$ be the derivative with respect to p_1, acting on symmetric functions expressed as polynomials in the power sum basis. We claim that this is a Lorentzianity-preserving linear operator on symmetric functions.

Indeed, we claim that $\left(\frac{\partial}{\partial p_1} f\right)_n = \frac{\partial}{\partial x_{n+1}}\Big|_{x_{n+1}=0} f_{n+1}$. The right-hand side of this equation is Lorentzian whenever f_{n+1} is Lorentzian, so proving this equality is sufficient to show that $\frac{\partial}{\partial p_1}$ is Lorentzianity-proving.

By linearity, it suffices to show that $\left(\frac{\partial}{\partial p_1} p_\lambda\right)_n = \frac{\partial}{\partial x_{n+1}}\Big|_{x_{n+1}=0} p_\lambda(x_1, \ldots, x_{n+1})$. Let $\lambda = (1^{m_1} 2^{m_2} \ldots)$ be the partition with m_1 ones, m_2 twos, and so on. Then we have

$$\left(\frac{\partial}{\partial p_1} p_\lambda\right)_n = m_1 \cdot p_{(1^{m_1 - 1} 2^{m_2} \ldots)}(x_1, \ldots, x_n).$$

On the other hand, we note that

$$\frac{\partial}{\partial x_{n+1}}\Big|_{x_{n+1}=0} p_r = \begin{cases} 1, & \text{if } r = 1 \\ 0, & \text{otherwise} \end{cases}.$$

Therefore,

$$\frac{\partial}{\partial x_{n+1}}\Big|_{x_{n+1}=0} p_\lambda(x_1, \ldots, x_{n+1}) = m_1 p_{(1^{m_1 - 1} 2^{m_2} \ldots)},$$

which completes the proof.

We also get some version of a symbol theory for linear operators preserving Lorentzian symmetric functions, though the statement is not nearly as neat as it is for the polynomial case.

Proposition 4.3.5. *Let $T : \Lambda^d \to \Lambda^\ell$ be a linear map on symmetric functions. For $n \geq \ell$, define $T_n : \Lambda_n^d \to \Lambda_n^\ell$ by $T_n(m_\lambda(x_1, \ldots, x_n)) = T(m_\lambda)|_{x_{n+1}=x_{n+2}=\cdots=0}$. Suppose that for all $n \geq \ell$,*

$$\mathrm{Symb}_{T,n}(w, z) := \sum_{\substack{\lambda \text{ partition} \\ \ell(\lambda) \leq n, \lambda_1 \leq d}} \binom{\mathbf{d}}{\lambda} \frac{T_n(m_\lambda(w_1, \ldots, w_n))}{m_\lambda(1^n)} m_{\mathbf{d}-\lambda}(z_1, \ldots, z_n)$$

is Lorentzian, where we use $\mathbf{d}$ as shorthand for $(d, \ldots, d) \in \mathbb{Z}_{\geq 0}^n$. Then T is a Lorentzianty-preserving operator on symmetric functions.

Proof. By construction, $(T(f))_n = T_n(f_n)$, so it suffices to show that $T_n : \Lambda_n^d \to \Lambda_n^\ell$ is Lorentzianity-preserving for all sufficiently large n. We note that we can extend T_n to a linear map $\widetilde{T}_n : \mathbb{R}[x_1, \ldots, x_n] \to \mathbb{R}[x_1, \ldots, x_n]$ by defining $\widetilde{T}_n(x^\alpha) := \frac{1}{m_{[\alpha]}(1^n)} T_n(m_{[\alpha]}(x_1, \ldots, x_n))$, and that it suffices to show that $\widetilde{T}_n$ preserves Lorentzianity.

By [BH20, Theorem 3.2], it suffices to show that

$$\mathrm{Symb}_{\widetilde{T}_n} = \sum_{0 \leq \alpha \leq \mathbf{d}} \binom{\mathbf{d}}{\alpha} \widetilde{T}_n(w^\alpha) z^{\mathbf{d}-\alpha}$$

is Lorentzian.

By construction, $\widetilde{T}_n(w^\alpha)$ depends only on the partition $[\alpha]$ obtained by reordering the parts of α, so

$$\begin{aligned}
\mathrm{Symb}_{\widetilde{T}_n} &= \sum_{\substack{\lambda \text{ partition:}\\ \ell(\lambda)\leq n, \lambda_1 \leq d}} \binom{\mathbf{d}}{\lambda} \sum_{\alpha \widetilde{\lambda}} \widetilde{T}_n(w^\alpha) z^{\mathbf{d}-\alpha} \\
&= \sum_{\substack{\lambda \text{ partition:}\\ \ell(\lambda)\leq n, \lambda_1 \leq d}} \binom{\mathbf{d}}{\lambda} \sum_{\alpha \sim \lambda} \widetilde{T}_n(w^\alpha) z^{\mathbf{d}-\alpha} \\
&= \sum_{\substack{\lambda \text{ partition:}\\ \ell(\lambda)\leq n, \lambda_1 \leq d}} \binom{\mathbf{d}}{\lambda} \frac{T_n(m_\lambda(w_1, \ldots, w_n))}{m_\lambda(1^n)} \sum_{\alpha \sim \lambda} z^{\mathbf{d}-\alpha} \\
&= \sum_{\substack{\lambda \text{ partition:}\\ \ell(\lambda)\leq n, \lambda_1 \leq d}} \binom{\mathbf{d}}{\lambda} \frac{T_n(m_\lambda(w_1, \ldots, w_n))}{m_\lambda(1^n)} m_{\mathbf{d}-\lambda}(z_1, \ldots, z_n),
\end{aligned}$$

which is exactly $\mathrm{Symb}_{T,n}(w, z)$. Hence, if $\mathrm{Symb}_{T,n}$ is Lorentzian for all n, then T_n is Lorentzianity-preserving for all n, and so T preserves Lorentzianity of symmetric functions, as desired. □

While we can naturally extend any symmetric polynomial to a symmetric function by simply taking the same coefficients in the monomial basis, this extension need not preserve the Lorentzian property. The semialgebraic descriptions we find in Section 4.4 include the number of variables as an input; by taking $n \to \infty$ in these inequalities, one recovers a semialgebraic description for Lorentzian symmetric functions.

4.3.1 Symmetric M-Convex Sets

One necessary condition for a polynomial to be Lorentzian is that its support must be M-convex, in the sense of [Mur03]. In fact, any M-convex set is the support of some Lorentzian polynomial [BH20, Theorem 3.10]. Recall that we say a set $J \subseteq \mathbb{Z}^n$ is *M-convex* if for all $x, y \in J$ and all $i \in [n]$ such that $x_i > y_i$, there exists an index j satisfying $x_j < y_j$ such that $x - e_i + e_j$ and $y + e_i - e_j$ are both in J.

If f is also symmetric, this puts additional constraints on its support. In this section, we fully characterize the possible supports of Lorentzian symmetric functions and polynomials.

Definition 4.3.6. Let $f = \sum_{k=0}^{d} \sum_{\lambda \vdash k} c_\lambda m_\lambda$. We define the *$m$-support* to be $\mathrm{supp}_m(f) = \{\lambda : c_\lambda \neq 0\}$.

Specifically, we show that if f is a Lorentzian symmetric function or polynomial, then the m-support of f is an order ideal of the dominance lattice with a unique maximal element.

Theorem 4.3.7. *If f is a Lorentzian symmetric polynomial of degree d on $n \geq d$ variables, then there is some $\lambda \vdash d$ such that $\mathrm{supp}_m(f) = \{\mu : \mu \preceq \lambda\}$. That is, the m-support of f is an interval $[1^d, \lambda]$ in dominance order.*

We prove this theorem in two steps. First we show that $\mathrm{supp}_m(f)$ has a unique maximal element. Then, we show that M-convexity implies that all partitions beneath this maximal element in dominance order must also be in the m-support of f.

Lemma 4.3.8. *Suppose $f = \sum_{\lambda \vdash d} c_\lambda m_\lambda$ is a Lorentzian symmetric polynomial. Then $\mathrm{supp}_m(f)$ has a unique maximal element with respect to dominance order.*

Proof. Suppose for the sake of contradiction that there are two distinct maximal elements λ and μ. Since $\mu \not\succeq \lambda$, we have $\lambda_1 + \cdots + \lambda_k > \mu_1 + \cdots + \mu_k$ for some k. By picking the smallest such k, we have $\lambda_k > \mu_k$.

Since f is Lorentzian, $\mathrm{supp}(f)$ is M-convex, so there exists some j such that $\mu_j > \lambda_j$ and $\lambda - e_k + e_j$ and $\mu + e_k - e_j$ are in $\mathrm{supp}(f)$. However, we claim that the maximality of λ and μ means that no such j can exist.

First, if $j < k$, then we claim that $\lambda - e_k + e_j \succ \lambda$, so λ is not maximal. Indeed, for any r such that $1 \leq r < j$ or $k \leq r \leq \ell(\lambda)$, $\sum_{i=1}^{r}(\lambda - e_k + e_j)_i = \sum_{i=1}^{r} \lambda_i$. Moreover, for any $j \leq r < k$, $\sum_{i=1}^{r}(\lambda - e_k + e_j)_i = \sum_{i=1}^{r} \lambda_i + 1 > \sum_{i=1}^{r} \lambda_i$. Thus, λ is not maximal. On the other hand, if $j > k$, then by the same logic, $\mu + e_k - e_j \succ \mu$, so μ is not maximal.

Thus, no such j can exist, so f does not have M-convex support, which contradicts our assumption that f is Lorentzian. $\square$

To prove Theorem 4.3.7, it just remains to show that all partitions below the unique maximal element are also in the m-support of f.

Proof of Theorem 4.3.7. By Lemma 4.3.8, we know that if f is Lorentzian, then $\operatorname{supp}_m(f)$ has a unique maximal element λ, so $\operatorname{supp}_m(f) \subseteq [1^d, \lambda]$. It remains to show that if $\mu \preceq \lambda$, then $\mu \in \operatorname{supp}_m(f)$.

Suppose f is a symmetric function of degree d on n variables, and let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n) \vdash d$. As in Definition 4.2.4, for any $\sigma \in S_n$, let λ^σ denote the composition $\lambda^\sigma = (\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n)})$. Since $c_\lambda \neq 0$, $\lambda^\sigma \in \operatorname{supp}(f)$ for all $\sigma \in S_n$. Since $\operatorname{supp}(f)$ is M-convex, this implies that $\operatorname{supp}(f) \supseteq P(\lambda) \cap \mathbb{Z}^n$.

Moreover, by Proposition 4.2.6, if we have a partition $\mu \vdash d$ such that $\mu \preceq \lambda$ in dominance order, then $\mu \in P(\lambda) \cap \mathbb{Z}^n$. Hence $\mu \in \operatorname{supp}(f)$, and so $\mu \in \operatorname{supp}_m(f)$ whenever $\mu \preceq \lambda$, as desired. $\square$

Following [BH20, Theorem 3.10], this gives us exactly the set of all m-supports of Lorentzian symmetric functions and polynomials. This result can also be found scattered throughout [Qin23]. We consolidate it below.

Theorem 4.3.9. *Let $J \subseteq \{\mu : \mu \vdash d\}$ be a set of partitions of d. The following are equivalent.*

(i) There is a Lorentzian symmetric function whose m-support is J.

(ii) There is a Lorentzian symmetric polynomial on $n \geq d$ variables whose m-support is J.

(iii) J is an interval $[1^d, \lambda]$ in dominance order.

(iv) J is the m-support of some Schur polynomial s_λ.

Furthermore, if we define $S_n \cdot J := \bigcup_{\sigma \in S_n} \{\mu^\sigma : \sigma \in S_n\}$, then these are also equivalent to

(v) $S_n \cdot J$ is the set of integer points of a permutohedron $P(\lambda)$.

Proof. We note that (i) $\Rightarrow$ (ii) follows immediately from the definition of Lorentzian symmetric functions, while (ii) $\Rightarrow$ (iii) follows from Theorem 4.3.7. We also note that $\operatorname{supp}_m(s_\lambda) = [1^d, \lambda]$, so (iii) and (iv) are equivalent. Thus, to prove the equivalence of (i) - (iv), it just remains to show that if $J = [1^d, \lambda]$, then there exists a Lorentzian symmetric function $f \in \Lambda^d$ with $\mathrm{supp}_m(f) = J$.

To show this, we claim that for any $\lambda \vdash d$, $f = \sum_{\mu \preceq \lambda} \widetilde{m}_\mu$ is a Lorentzian symmetric function. Indeed, by the argument in the proof of Theorem 4.3.7, $\mathrm{supp}(f_n) = \mathrm{conv}\{\lambda^\sigma : \sigma \in S_n\} \cap \mathbb{Z}^n$, which is exactly the integer points of a (generalized) permutahedron. Moreover, the coefficient of x^α in f_n is exactly $\frac{1}{\alpha!}$, so f_n is the generating function of an M-convex set, and hence Lorentzian by [BH20, Theorem 3.10].

Finally, note that (iii) $\Leftrightarrow$ (v) follows easily from Proposition 4.2.6. $\square$

We note that for any $\lambda \vdash d$, $\mathrm{supp}_m(s_\lambda) = \mathrm{supp}_m(N(s_\lambda)) = [1^d, \lambda]$, so if f is (normalized) Schur positive with a unique maximal element in its (normalized) Schur expansion, then it has M-convex support.

In Section 4.4.1, we show that normalized Schur positivity is equivalent to Lorentzianity in degree two. However, for higher degrees, (normalized) Schur positivity is in general neither a necessary nor sufficient condition for Lorentzianity. In Section 4.4.2, we show that starting in degree three, normalized Schur positivity is neither necessary nor sufficient.

We also know that Schur positivity is not a sufficient condition because, in general, s_λ is not Lorentzian, even in degree two [HMMS22, Example 8]. On the other hand, the normalized Schur polynomial Ns_{33} is Lorentzian but not Schur positive, so Schur positivity is also not a necessary condition.

4.3.2 Topology of the Space of Lorentzian Symmetric Polynomials

In [Brä21], Brändén shows that the projective space of Lorentzian polynomials is homeomorphic to a closed Euclidean ball. In this section, we make a small adaptation to Brändén's proof in order to show that the space of Lorentzian symmetric polynomials is also homeomorphic to a closed Euclidean ball.

Theorem 4.3.10 (cf. [Brä21, Theorem 3.1]). *Let $S \subseteq S_n$ be any subgroup of the symmetric group generated by elements of order 2. Let $\underline{L}_S^d$ denote the set of multiaffine Lorentzian degree*

d polynomials that are invariant under S. Identify $\mathbb{P}\underline{L}_S^d$ with $\{f \in \underline{L}_S^d : f(1,\ldots,1) = 1\}$. Then $\mathbb{P}\underline{L}_S^d$ is homeomorphic to a closed Euclidean ball.

Proof. Borrowing the notation from [Brä21], let $\underline{L}_n^d$ denote the set of multiaffine Lorentzian polynomials of degree d on n variables, let $\mathcal{L}$ be the linear operator defined by $\mathcal{L} = \frac{1}{\binom{n}{2}} \sum_\tau \tau$, where the sum runs over all transpositions τ, and define $T_s = e^s e^{-s\mathcal{L}}$ for $s \geq 0$. By [Brä21, Lemma 2.4], we know that $T_s(\mathbb{P}\underline{L}_n^d) \subseteq \mathrm{Int}(\mathbb{P}\underline{L}_n^d)$ for all $s > 0$.

We claim that for all $s > 0$ and $f \in \mathbb{P}\underline{L}_S^d$, we have $T_s(f) \in \mathbb{P}\underline{L}_S^d$. By [Brä21, Proposition 2.3], we know that $T_s(f) \in \mathbb{P}\underline{L}_n^d$, so it just remains to show that $T_s(f)$ is invariant under S. Indeed, let σ be a generator of S. By assumption $\sigma^2 = \mathrm{id}$, so $\sigma = \tau_1 \cdot \cdots \cdot \tau_\ell$ for some disjoint transpositions $\tau_1, \ldots, \tau_\ell$. We note that for any transposition τ, we have $\tau\mathcal{L} = \mathcal{L}\tau$, so

$$\sigma T_s(f) = T_s(\tau_\ell \cdot \cdots \cdot \tau_1 f) = T_s(\sigma f) = T_s(f),$$

so $T_s(f)$ is invariant under S, as desired.

In particular, for any $s > 0$, we have $T_s(\mathbb{P}\underline{L}_S^d) \subseteq \mathrm{Int}(\mathbb{P}\underline{L}_n^d) \cap \mathbb{P}\underline{L}_S^d$. Since $\mathbb{P}\underline{L}_S^d = \mathbb{P}\underline{L}_n^d \cap H$ for some hyperplane H, we have $\mathrm{Int}(\mathbb{P}\underline{L}_S^d) \supseteq \mathrm{Int}(\mathbb{P}\underline{L}_n^d) \cap H = \mathrm{Int}(\mathbb{P}\underline{L}_n^d) \cap \mathbb{P}\underline{L}_S^d$. Therefore, $T_s(\mathbb{P}\underline{L}_S^d) \subseteq \mathrm{Int}(\mathbb{P}\underline{L}_S^d)$ for all $s > 0$, which implies that $\mathbb{P}\underline{L}_S^d$ is homeomorphic to a closed Euclidean ball by the contractive flow argument in [Brä21, Theorem 3.1] and [GKL18, Lemma 2.3]. □

Theorem 4.3.11. *For all n, d, the projective space of Lorentzian symmetric polynomials of degree d on n variables is homeomorphic to a closed Euclidean ball.*

Proof. Consider the polynomial ring $\mathbb{R}[x_{ij} : 1 \leq i \leq n, 1 \leq j \leq d]$, and let S be the subgroup of the symmetric group generated by the transpositions $(x_{i,j}, x_{i,j'})$ for all $i \in [n], j, j' \in [d]$, as well as the order two elements of the form $\prod_{j=1}^d (x_{i,j}, x_{i',j})$ for all $1 \leq i, i' \leq n$.

We note that $\underline{L}_S^d$ is exactly the polarizations of Lorentzian symmetric polynomials of degree d in n variables. Since polarization is a linear map, the set of Lorentzian symmetric polynomials is homeomorphic to $\underline{L}_S^d$, so the projective space of Lorentzian symmetric polynomials is homeomorphic to $\mathbb{P}\underline{L}_S^d$ and therefore homeomorphic to a closed Euclidean ball by Theorem 4.3.10. □

Similar to the projective space of Lorentzian polynomials, this ball is also naturally stratified by support. Specifically, we get one stratum for each $\lambda \vdash d$, consisting of Lorentzian symmetric polynomials whose m-support is $[1^d, \lambda]$.

Theorem 4.3.11 shows that for any n, d, the projective space of Lorentzian symmetric polynomials of degree d on n variables is homeomorphic to a closed Euclidean ball. It is natural to ask whether we can say anything about the topology of the space of Lorentzian symmetric functions of degree d.

By definition, this is the intersection over all $n \geq d$ of the spaces of Lorentzian symmetric polynomials on n variables, so in particular by Theorem 4.3.11 and Proposition 4.3.2, it is the intersection of a descending chain of sets homeomorphic to closed Euclidean balls. This immediately gives some topological consequences, such as the fact that it is closed and connected. However, the proof technique we used above runs into issues with convergence and does not immediately translate to the setting of symmetric functions.

On the other hand, we can show that the space of Lorentzian symmetric functions has nonempty interior.

Proposition 4.3.12. *For any $d \geq 2$, let*

$$f = p_1^d - \varepsilon \sum_{k=1}^{d} \binom{d}{k} p_k p_1^{d-k}.$$

For sufficiently small $\varepsilon > 0$, f is a strictly Lorentzian symmetric function.

Proof. Let $\varepsilon > 0$ be small enough such that f has full m-support and strictly positive coefficients. This is possible since p_1^d has full m-support and positive coefficients.

We claim that for all n and all $\alpha \vdash (d-2)$, $\lambda_2(\nabla^2 \partial^\alpha f_n) \leq -d!\varepsilon$. In particular, this means that $\lim_{n\to\infty} \lambda_2(\nabla^2 \partial^\alpha f_n) \leq -d!\varepsilon < 0$, so f is a strictly Lorentzian symmetric function.

Thus, it just remains to show that for all n and all $\alpha \vdash (d-2)$, $\lambda_2(\nabla^2 \partial^\alpha f_n) \leq -d!\varepsilon$.

Indeed, we note that

$$\begin{aligned}
f_n &= (x_1 + \cdots + x_n)^d - \varepsilon \sum_{k=1}^{d} \binom{d}{k} (x_1^k + \cdots + x_n^k)(x_1 + \cdots + x_n)^{d-k} \\
&= (x_1 + \cdots + x_n)^d - \varepsilon \sum_{j=1}^{n} \left(\sum_{k=1}^{d} \binom{d}{k} x_j^k (x_1 + \cdots + x_n)^{d-k} \right) \\
&= (x_1 + \cdots + x_n)^d - \varepsilon \left(\sum_{j=1}^{n} \left((x_j + (x_1 + \cdots + x_n))^d - (x_1 + \cdots + x_n)^d \right) \right).
\end{aligned}$$

Therefore, for any $\alpha \vdash (d-2)$,

$$\partial^\alpha f_n = \frac{d!}{2}(x_1 + \cdots + x_n)^2 - \frac{d!}{2}\varepsilon \left(\sum_{j=1}^{n} \left(2^{\alpha_j}(x_j + (x_1 + \cdots + x_n))^2 - (x_1 + \cdots + x_n)^2 \right) \right),$$

so

$$\begin{aligned}
\frac{1}{d!}\nabla^2 \partial^\alpha f_n &= \mathbb{1}\mathbb{1}^T - \varepsilon \left(\sum_{j=1}^{n} \left(2^{\alpha_j}(\mathbb{1} + e_j)(\mathbb{1} + e_j)^T - \mathbb{1}\mathbb{1}^T \right) \right) \\
&\preceq \mathbb{1}\mathbb{1}^T - \varepsilon \left(\sum_{j=1}^{n} \left((\mathbb{1} + e_j)(\mathbb{1} + e_j)^T - \mathbb{1}\mathbb{1}^T \right) \right) \\
&= \mathbb{1}\mathbb{1}^T - \varepsilon \left(2\mathbb{1}\mathbb{1}^T + I \right) = (1 - 2\varepsilon)\mathbb{1}\mathbb{1}^T - \varepsilon I.
\end{aligned}$$

We note that $\lambda_2((1-2\varepsilon)\mathbb{1}\mathbb{1}^T - \varepsilon I) = -\varepsilon$, so it follows that $\lambda_2(\nabla^2 \partial^\alpha f_n) \leq -d!\varepsilon$, as desired. □

4.3.3 Hessians of Symmetric Polynomials

In this section, we examine the structures appearing in the Hessians of quadratic derivatives of symmetric polynomials. Specifically, these Hessians will have a very particular block matrix structure that makes it easy to explicitly compute some of their eigenvalues and greatly simplifies computing their signatures. The results in this section underpin our characterizations of low-degree Lorentzian symmetric polynomials in Section 4.4 as well as our simplifications in Section 4.5.

Let f be a homogeneous symmetric polynomial of degree d with nonnegative coefficients. We study the Lorentzian property of such polynomials f by their $\widetilde{m}_\alpha$-expansions:

$$f = \sum_{\alpha \vdash d} c_\alpha \widetilde{m}_\alpha.$$

Since f is symmetric, we claim it suffices to analyze quadratic forms associated to all *partitions* of $d-2$, rather than needing to study all compositions as we would for general Lorentzian polynomials. Indeed, we know that reordering the variables preserves Lorentzianity, and we note that for any composition $\gamma \in \mathbb{Z}^n_{\geq 0}$ and permutation $\sigma \in S_n$, we have

$$\partial^{\sigma\cdot\gamma} f = \partial^{\sigma\cdot\gamma}(\sigma^{-1}\cdot f) = \sigma^{-1}\cdot \partial^{\gamma} f,$$

where $\sigma \cdot \gamma = (\gamma_{\sigma(1)}, \ldots, \gamma_{\sigma(n)})$ and $\sigma^{-1} \cdot f = f(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})$. Since permuting the variables does not affect the signature of the Hessian, this means that $\nabla^2\partial^{\sigma\cdot\gamma} f$ has the same signature as $\nabla^2\partial^{\gamma} f$. In particular, any composition of $d-2$ is a permutation of some partition $\mu \vdash d-2$, so it suffices to check that $\nabla^2\partial^{\mu} f$ has Lorentzian signature for all $\mu \vdash d-2$.

We will find that for such quadratics, each Hessian is a block matrix where each off diagonal block is a scaled all 1's matrix and each diagonal block will be of the form

$$\begin{pmatrix} p & q & \cdots & q \\ q & p & \ddots & \vdots \\ \vdots & \ddots & \ddots & q \\ q & \cdots & q & p \end{pmatrix}.$$

We call symmetric matrices with such a block structure *Haynsworth matrices*. In this section, we show that Hessians of quadratic derivatives of symmetric polynomials are Haynsworth matrices, and that Haynsworth matrices are particularly amenable to analyzing the eigenvalue signature.

Definition 4.3.13. Let $\Gamma = (P_1, \ldots, P_k)$ be a collection of intervals that partitions $[n]$. A symmetric matrix $M \in \mathbb{R}^{n\times n}$ is *Haynsworth of type* Γ if $M \in \operatorname{span}(\{\mathbf{1}_{P_i}\mathbf{1}_{P_j}^T + \mathbf{1}_{P_j}\mathbf{1}_{P_i}^T : i, j = 1, \ldots, k\} \cup \{\operatorname{diag}(\mathbf{1}_{P_j}) : j = 1, \ldots, k\})$.

We claim that if $f \in \Lambda_n^d$, then for any $\mu \vdash d-2$, $\nabla^2\partial^{\mu} f$ is such a Haynsworth matrix. At the heart of this fact is the following lemma.

Lemma 4.3.14. *Let $\mu, \nu \in \mathbb{Z}^n_{\geq 0}$ be compositions, and let $[\mu], [\nu]$ be the partitions obtained by ordering the parts of μ and ν, respectively. If $[\mu] = [\nu]$ and $\mu_i = \nu_j$, then $[\mu + e_i] = [\nu + e_j]$.*

Proof. We note that $[\mu] = [\nu]$ if and only if their associated multisets $\{\{\mu\}\}$ and $\{\{\nu\}\}$ are equal. Further, we note that the only difference between $\{\{\mu\}\}$ and $\{\{\mu + e_i\}\}$ is that μ_i appears one fewer time while $\mu_i + 1$ appears one more time. Hence, we have

$$\{\{\mu + e_i\}\} = \{\{\mu\}\} \setminus \{\mu_i\} \cup \{\mu_i + 1\} = \{\{\nu\}\} \setminus \{\nu_j\} \cup \{\nu_j + 1\} = \{\{\nu + e_j\}\}$$

and so $[\mu + e_i] = [\nu + e_j]$, as desired. □

Lemma 4.3.15. *Given $f \in \Lambda_n^d$ with coefficients c_α in the $\widetilde{m}_\alpha$-basis and a partition $\mu \vdash d-2$, the Hessian associated to the $\partial^\mu f$ is a Haynsworth matrix H, where the blocks of the Haynsworth matrix correspond to the blocks in the set partition of $[n]$ defined by μ.*

Proof. We partition $\nabla^2 \partial^\mu f$ into a block matrix

$$M := \nabla^2 \partial^\mu f = \left(\begin{array}{c|c|c|c} A_1 & B_{1,2} & \cdots & B_{1,\ell} \\ \hline B_{1,2}^T & A_2 & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & B_{\ell-1,\ell} \\ \hline B_{1,\ell}^T & \cdots & B_{\ell-1,\ell}^T & A_\ell \end{array}\right)$$

where $i, j \in [n]$ are in the same block of this matrix if and only if $\mu_i = \mu_j$. We wish to show that all $B_{s,t}$ blocks are constant multiples of the all ones matrix, and that each $A_t = p_t I + q_t(\mathbb{1} - I)$ for some p_t, q_t.

Indeed, first suppose that $(i, j), (i', j') \in B_{s,t}$ for some $s \neq t$. Then $\mu_i = \mu_{i'}$, so by Lemma 4.3.14, we have $[\mu + e_i] = [\mu + e_{i'}]$. Since $j, j' \in \beta_t$, we have $\mu_j = \mu_{j'}$, and since $s \neq t$, we know $j \neq i$ and $j' \neq i'$. Hence $(\mu + e_i)_j = \mu_j = \mu_{j'} = (\mu + e_{i'})_{j'}$, and therefore $[\mu + e_i + e_j] = [\mu + e_{i'} + e_{j'}]$ by Lemma 4.3.14. Thus,

$$M_{ij} = \partial^{\mu + e_i + e_j} f = c_{[\mu + e_i + e_j]} = c_{[\mu + e_{i'} + e_{j'}]} = M_{i'j'},$$

as desired.

Next, suppose that $(i, j), (i', j') \in A_t$. To prove that all off-diagonal entries of A_t are equal, we need to show that if $i \neq j$ and $i' \neq j'$, then $M_{ij} = M_{i'j'}$. Indeed, we again have

$\mu_i = \mu_{i'}$ and $(\mu + e_i)_j = (\mu + e_{i'})_{j'}$, so again by Lemma 4.3.14, $[\mu + e_i + e_j] = [\mu + e_{i'} + e_{j'}]$, and hence

$$M_{ij} = c_{[\mu+e_i+e_j]} = c_{[\mu+e_{i'}+e_{j'}]} = M_{i'j'}.$$

Finally, if $i = j$ and $i' = j'$, then $[\mu+e_i] = [\mu+e_{i'}]$ and $(\mu+e_i)_i = \mu_i+1 = \mu_{i'}+1 = (\mu+e_{i'})_{i'}$ so Lemma 4.3.14 again applies, and we have

$$M_{ii} = c_{[\mu+2e_i]} = c_{[\mu+2e_{i'}]} = M_{i'i'},$$

as desired. $\square$

Notably, when Haynsworth matrices carry nontrivial blocks of size greater than one, we can immediately compute a special class of eigenvalues which we call *linear eigenvalues.*

Lemma 4.3.16. *Let M be a Haynsworth matrix with block sizes $n_1, n_2, \ldots, n_\ell$ of the form:*

$$M = \left(\begin{array}{c|c|c|c} A_1 & B_{1,2} & \cdots & B_{1,\ell} \\ \hline B_{1,2}^T & A_2 & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & B_{\ell-1,\ell} \\ \hline B_{1,\ell}^T & \cdots & B_{\ell-1,\ell}^T & A_\ell \end{array}\right)$$

where $A_t = p_t I + q_t(\mathbb{1} - I)$ and $B_{s,t} = b_{s,t}\mathbb{1}$. Whenever $n_t \geq 2$, then $p_t - q_t$ is an eigenvalue of M with multiplicity $n_t - 1$, giving us a total of $n - \ell$ linear eigenvalues. Moreover, the signature of the remaining eigenvalues is determined by the signature of the $\ell \times \ell$ matrix $\widetilde{M}$ with entries

$$\widetilde{M}_{st} = \begin{cases} n_s^2 q_s + n_s(p_s - q_s), & s = t \\ n_s n_t b_{s,t}, & s \neq t \end{cases}.$$

Proof. Index the entries of the t-th block by $\{m_t, \ldots, m_{t+1}-1\}$ as in the previous subsection, i.e., $m_{t+1}-1-m_t = n_t$. Observe that for each block A_t with $n_t \geq 2$, we can immediately find n_t-1 eigenvectors $e_j - e_{m_t-1}$ with eigenvalue $p_t - q_t$ for $m_t < j < m_{t+1}$. Thus, for $t = 1, \ldots, \ell$, the tth block contributes $n_t - 1$ linear eigenvalues, giving us a total of $\sum_{t=1}^{\ell}(n_t - 1) = n - \ell$ linear eigenvalues.

Moreover, since our eigenvectors form an orthonormal basis, the remaining signature is determined by the $\ell \times \ell$ matrix $\widetilde{M}$ which is obtained by a congruence transform that collapses each of the blocks uniformly:

$$\widetilde{M} := \left(\begin{array}{c|c|c|c} 1 \ \cdots \ 1 & \mathbf{0} & \cdots & \mathbf{0} \\ \hline \mathbf{0} & 1 \ \cdots \ 1 & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & \mathbf{0} \\ \hline \mathbf{0} & \cdots & \mathbf{0} & 1 \ \cdots \ 1 \end{array} \right) M \left(\begin{array}{c|c|c|c} \begin{matrix} 1 \\ \vdots \\ 1 \end{matrix} & \mathbf{0} & \cdots & \mathbf{0} \\ \hline \mathbf{0} & \begin{matrix} 1 \\ \vdots \\ 1 \end{matrix} & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & \mathbf{0} \\ \hline \mathbf{0} & \cdots & \mathbf{0} & \begin{matrix} 1 \\ \vdots \\ 1 \end{matrix} \end{array} \right).$$

In particular, $\widetilde{M}$ has entries

$$\widetilde{M}_{s,t} = \begin{cases} n_s^2 q_s - n_s(q_s - p_s) & \text{if } s = t, \\ n_s n_t b_{st} & \text{if else.} \end{cases}$$

□

Note that if we wish to have a nonnegative Haynsworth matrix with Lorentzian signature, then we must require $p_t \leq q_t$ whenever $n_t \geq 2$, since $\widetilde{M}$ will have nonnegative entries and hence at least one positive eigenvalue. In the case of a Lorentzian symmetric polynomial f, these linear inequalities on the quadratic Hessians are between coefficients in the $\widetilde{m}$-basis. Furthermore, by studying the 2×2-minors, one finds that the coefficients respect the dominance order on partitons.

Theorem 4.3.17. *Let $f = \sum_{\alpha \vdash d} c_\alpha \widetilde{m}_\alpha$ be a Lorentzian symmetric polynomial of degree d, with c_α its $\widetilde{m}$-coefficients in the normalized monomial basis. If $\lambda, \mu \vdash d$ such that $\mu \preceq \lambda$ in dominance order, then $c_\mu \geq c_\lambda$.*

Proof. By Theorem 4.3.7, if $\mu \preceq \lambda$ and $c_\lambda > 0$, then $c_\mu > 0$, so we may assume that $c_\lambda, c_\mu > 0$. Moreover, it suffices to prove the result for covering relations $\mu \lessdot \lambda$.

We proceed by induction on the dominance order. For the base case, the unique minimal element in dominance order is $\mu = 1^d$, and the unique covering relation over μ is $\lambda = (2, 1^{d-2})$. Then we compute

$$\partial^{(1^{d-2})} f = c_\mu \widetilde{m}_{11} + c_\lambda(\widetilde{m}_2 + (x_1 + \cdots + x_{d-2})\widetilde{m}_1),$$

where the $\widetilde{m}$ are now symmetric functions in $x_{d-1}, \ldots, x_n$.

The 2×2 submatrix of $\nabla^2 \partial^{(1^{d-2})} f$ indexed by x_d, x_{d+1} is

$$\begin{pmatrix} c_\lambda & c_\mu \\ c_\mu & c_\lambda \end{pmatrix},$$

and by Lorentzianity it must have at most one positive eigenvalue, which implies that $c_\mu \geq c_\lambda$, as desired.

For the inductive step, suppose we have $\mu \vdash d$ such that $c_\nu \geq c_\mu$ for all $\nu \preceq \mu$, and suppose that $\lambda \gtrdot \mu$. Recall that $\lambda \gtrdot \mu$ if and only if there exists $i > k$ such that $\lambda_i = \mu_i + 1$, $\lambda_k = \mu_k - 1$, and $\lambda_j = \mu_j$ for all $j \neq i, k$; and either

(1) $\mu_i = \mu_k$, or

(2) $\mu_i > \mu_k$ and $k = i + 1$.

We obtain desired inequalities by considering $\partial^\gamma f$ where $\gamma = \mu - e_i - e_k$. We note that by construction, $\gamma + e_i + e_k = \mu$ and $\gamma + 2e_i = \mu + e_i - e_k = \lambda$. Now, consider the 2×2 minor of $\nabla^2 \partial^\gamma f$ indexed by x_i, x_k:

$$Q := \begin{pmatrix} \partial_i^2 \partial^\gamma f & \partial_i \partial_k \partial^\gamma f \\ \partial_i \partial_k \partial^\gamma f & \partial_k^2 \partial^\gamma f \end{pmatrix} = \begin{pmatrix} c_\lambda & c_\mu \\ c_\mu & c_{[\gamma+2e_k]} \end{pmatrix},$$

where we use $[\cdot]$ to denote the partition associated to a composition. By Lorentzianity, this submatrix must have at most one positive eigenvalue, which happens if and only if $c_\mu^2 \geq c_\lambda c_{[\gamma+2e_k]}$. We consider two cases: either $\mu_i = \mu_k$, or $\mu_i > \mu_k$ and $k = i + 1$.

In the former case, $\mu_i = \mu_k$, so $[\gamma + 2e_k] = [\mu - e_i + e_k] = [\mu + e_i - e_k] = \lambda$. Therefore, since Q has Lorentzian signature, we have $c_\mu^2 \geq c_\lambda^2$, and since $c_\mu, c_\lambda > 0$, we have $c_\mu \geq c_\lambda$, as desired.

If $k = i + 1$ and $\mu_i > \mu_{i+1}$, then we have $c_\mu^2 \geq c_\lambda c_\nu$, where $\nu = [\mu - e_i + e_{i+1}]$. We note that $\nu \preceq \mu$, so by induction we have $c_\nu \geq c_\mu$. Therefore, $c_\mu^2 \geq c_\lambda c_\nu \geq c_\lambda c_\mu$, so $c_\mu \geq c_\lambda$, as desired. □

Remark 4.3.18. *If we apply the linear inequalities from Theorem 4.3.17 to normalized Schur polynomials, we recover the monotonicity of Kostka coefficients proved in [Whi80, Theorem 2].*

From Lemmas 4.3.15 and 4.3.16, we immediately have the following characterization of Lorentzian symmetric functions, which is the main theorem of this section.

Theorem 4.3.19. *Let $f = \sum_{\alpha \vdash d} c_\alpha \widetilde{m}_\alpha \in \Lambda^d$. Then f is a Lorentzian symmetric function if and only if*

(M) $\operatorname{supp}_m(f)$ *has a unique maximal element in dominance order,*

(D) $c_\lambda \leq c_\mu$ *whenever* $\mu \preceq \lambda$ *in dominance order, and*

(H) for all partitions $\mu \vdash d-2$ *where*

- $\ell(\mu) = k$,
- μ *induces a set partition* $(\beta_1, \ldots, \beta_\ell)$ *of* $[k]$

the $(\ell+1) \times (\ell+1)$ *symmetric matrix* $M(\mu)$ *has at most one positive eigenvalue:*

$$M(\mu) = \begin{pmatrix} a_1 & b_{1,2} & \cdots & b_{1,\ell} & r_1 \\ b_{1,2} & a_2 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & b_{\ell-1,\ell} & r_{\ell-1} \\ b_{1,\ell} & \cdots & b_{\ell-1,\ell} & a_\ell & r_\ell \\ r_1 & \cdots & r_{\ell-1} & r_\ell & q \end{pmatrix},$$

where

$$\begin{aligned}
a_t &= n_t \left((c_{\mu+2e_{m_t}} + (n_t - 1)c_{\mu+e_{m_t}+e_{m_t+1}} \right) \\
b_{s,t} &= n_s n_t c_{\mu+e_{m_s}+e_{m_t}} \\
r_s &= n_s c_{\mu+e_{m_s}+e_{k+1}} \\
q &= c_{\mu+e_{k+1}+e_{k+2}}.
\end{aligned}$$

Moreover, for any $f \in \Lambda_n^d$ with $n \geq d$, f is a Lorentzian symmetric polynomial if and only if (M), (D), and (H') hold, where (H') is identical to (H) except q is replaced by $q' = c_{\mu+e_{k+1}+e_{k+2}} - \frac{1}{n-k}(c_{\mu+e_{k+1}+e_{k+1}} - c_{\mu+2e_{k+1}})$.

Proof. We treat only the symmetric function case of the above theorem. The proof for symmetric polynomials follows nearly identically.

First, if f is Lorentzian, then (M) holds by Lemma 4.3.8, and (D) holds by Theorem 4.3.17. To see that (H) holds, we first apply the Haynsworth reduction technique from Lemma 4.3.16, then apply a congruence transform by $\mathrm{diag}(1, \ldots, 1, \frac{1}{n-k})$ and take the limit as $n \to \infty$, which gives the desired $M(\mu)$.

Now, suppose that (M), (D), and (H) hold. We note that (D) implies that $\mathrm{supp}_m(f)$ is downward closed, so (M) and (D) together imply that f has M-convex support by Theorem 4.3.7.

Thus, it just remains to show that $\nabla^2 \partial^\mu f$ has Lorentzian signature for all $\mu \vdash d - 2$. Indeed, we note that for any diagonal block in the Haynsworth structure described in Lemma 4.3.15, if $A_{t,t} = c_\nu I + c_\lambda(\mathbb{1} - I)$, then $\lambda \preceq \nu$. In particular, (D) implies that all of the linear eigenvalues described in Lemma 4.3.16 are nonpositive, and so $\nabla^2 \partial^\mu f_n$ has Lorentzian signature if and only if

$$Q(\mu) = \begin{pmatrix} a_1 & b_{1,2} & \cdots & b_{1,\ell} & (n-k)r_1 \\ b_{1,2} & a_2 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & b_{\ell-1,\ell} & (n-k)r_{\ell-1} \\ b_{1,\ell} & \cdots & b_{\ell-1,\ell} & a_\ell & (n-k)r_\ell \\ (n-k)r_1 & \cdots & (n-k)r_{\ell-1} & (n-k)r_\ell & (n-k)\left((n-k-1)c_{\mu+e_{k+1}+e_{k+2}} + c_{\mu+2e_{k+1}}\right) \end{pmatrix}$$

has at most one positive eigenvalue. Since $c_{\mu+e_{k+1}+e_{k+2}} \geq c_{\mu+2e_{k+1}}$ by (D), we note that

$$\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \frac{1}{n-k} \end{pmatrix} Q(\mu) \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \frac{1}{n-k} \end{pmatrix} = M(\mu) - \frac{c_{\mu+e_{k+1}+e_{k+2}} - c_{\mu+2e_{k+1}}}{n-k} e_{\ell+1}e_{\ell+1}^T \preceq M(\mu).$$

Since $M(\mu)$ has Lorentzian signature by (H), $Q(\mu)$ has Lorentzian signature as well, and hence $\nabla^2\partial^\mu f_n$ has at most one positive eigenvalue for all n. Hence, if (M), (D), and (H) hold, then f is a Lorentzian symmetric function, as desired. □

As seen in [Chi24] and Chapter 3, for fixed degree d, one can test if a homogeneous polynomial of degree d is Lorentzian using a polynomial number of operations in the number of variables. For symmetric polynomials, we can do even better: the number of arithmetic operations is now independent of the number of variables.

Theorem 4.3.20. *For fixed degree d, we can test Lorentzianity of $f \in \Lambda^d$ or $f \in \Lambda_n^d$ in a constant number of arithmetic operations. Specifically, the number of operations is independent of the number of variables.*

Proof. Since d is fixed, we can enumerate partitions of d and $d-2$ in constant time. Checking dominance order to verify conditions (M) and (D) from Theorem 4.3.19 therefore takes constant time, as does computing the reduced Hessian $M(\mu)$ for each $\mu \vdash d-2$. (Note that for the symmetric polynomials case, the parameter n may appear in some entries of $M(\mu)$.)

If $M(\mu)$ is size $(\ell+1)\times(\ell+1)$, then $1+\cdots+\ell \le d-2$, so $\ell \le O(\sqrt{d})$, which we regard as a constant. Then, by Proposition 4.2.2, we may test the signature of each $M(\mu)$ in a constant number of operations, as desired. □

4.4 Applying Reductions for Small Degrees

In this section, we demonstrate the techniques outlined in Section 4.3.3 and give explicit complete characterizations of Lorentzian symmetric polynomials of degree up to four and Lorentzian symmetric functions for degrees five and six.

4.4.1 Symmetric Quadratics

For quadratic polynomials, we only have a single Haynsworth block, so quadratic Lorentzian symmetric polynomials of degree two are characterized by a single linear inequality. In fact, quadratic symmetric functions are Lorentzian if and only if their normalized Schur expansion is nonnegative.

Theorem 4.4.1. *Let $f = a\widetilde{m}_2 + b\widetilde{m}_{11} \in \Lambda^2$. The following are equivalent:*

(i) The symmetric function f is Lorentzian.

(ii) The symmetric polynomial $f_n \in \Lambda_n^2$ is Lorentzian for some $n \ge 2$.

(iii) The coefficients of f satisfy $0 \le a \le b$.

Proof. The implication (i) ⇒ (ii) holds by the definition of Lorentzian symmetric functions.

To prove (ii) ⇒ (iii), we note that by direct computation,

$$\nabla^2 f_n = \begin{pmatrix} a & b & \dots & b \\ b & a & \ddots & \vdots \\ \vdots & \ddots & \ddots & b \\ b & \dots & b & a \end{pmatrix} = b\mathbb{1}\mathbb{1}^T + (a-b)I,$$

which has eigenvalues $\lambda_1 = a+(n-1)b$ with multiplicity one and $\lambda_2 = a-b$ with multiplicity $n-1$. If f_n is Lorentzian, then $a, b \geq 0$ and at least one of a, b is strictly positive, so $\lambda_1 > 0$. Since $n \geq 2$, λ_2 has multiplicity at least one, so we must have $\lambda_2 = a - b \leq 0$, and hence $b \geq a$, as desired.

On the other hand, if $0 \leq a \leq b$, then f has nonnegative coefficients and by the same computation as above, $\nabla^2 f_n$ has at most one positive eigenvalue for all n. Moreover, since a and b are not both zero, this implies that $b > 0$, so f_n is indecomposable for all n. Hence, f is Lorentzian, so (iii) $\Rightarrow$ (i), as desired. □

In fact, this condition coincides exactly with f having a nonnegative expansion in the normalized Schur basis. This was first observed in [Qin23].

Proposition 4.4.2. *Let f be a symmetric polynomial of degree two in $n \geq 2$ variables. Then f is Lorentzian if and only if it is normalized Schur positive. That is, $f \in L_n^2 \cap \Lambda_n^2$ if and only if $f = aNs_2 + bNs_{11}$ with $a, b \geq 0$.*

Proof. Let $f = aNs_2 + bNs_{11}$. We wish to show that f is Lorentzian if and only if $a, b \geq 0$. We note that $Ns_{11} = \widetilde{m}_{11}$ and $Ns_2 = \widetilde{m}_2 + \widetilde{m}_{11}$, so $f = a\widetilde{m}_2 + (a+b)\widetilde{m}_{11}$. Thus, by Theorem 4.4.1, f is Lorentzian if and only if $0 \leq a \leq a + b$, which happens if and only if $a, b \geq 0$, as desired. □

4.4.2 Symmetric Cubics

Next, we analyze the semialgebraic set of Lorentzian symmetric polynomials of degree three. We show that Lorentzian cubics are defined by a chain of linear inequalities, corresponding to the inequalities from Theorem 4.3.17, as well as a single quadratic inequality.

The computations in Theorem 4.4.3 are essentially those described in the structural results in Lemmas 4.3.15 and 4.3.16. However, we will go through the steps in detail in the proof, so as to have an explicit illustrative example of how our techniques work.

Theorem 4.4.3. *Let $f = a\widetilde{m}_3 + b\widetilde{m}_{21} + c\widetilde{m}_{111} \in \mathbb{R}_{\geq 0}[x_0, \ldots, x_n]$ be a symmetric homogeneous cubic polynomial on $n+1 \geq 3$ variables. Then f is Lorentzian if and only if $0 \leq a \leq b \leq c$ and $ab + (n-1)ac \leq nb^2$.*

Proof. First, we consider the case where $b = 0$. In this case, the claim reduces to showing that f is Lorentzian if and only if $a = 0$. By Theorem 4.3.7, if f is Lorentzian, then $\mathrm{supp}_m(f)$ is an interval $[111, \mu]$ in dominance order, and if $b = 0$, then $(2,1) \notin \mathrm{supp}_m(f)$, which implies that $\mathrm{supp}_m(f) = \{111\}$ and so $a = 0$. On the other hand, if $a = 0$, then $f = c\widetilde{m}_{111} = ce_3$, which is Lorentzian [BH20, Example 2.27], as desired.

It now remains to consider the case where $b > 0$. We note that the fact that $(2,1) \in \mathrm{supp}_m(f)$ implies that f is indecomposable. As noted in Section 4.3.3, all derivatives of the same type are equivalent up to permuting the variables. The only partition of $d - 2 = 1$ is (1), so it suffices to check that $M := \nabla^2 \partial_0 f$ has Lorentzian signature.

By the definition of the symmetric normalized monomial basis,

$$\partial_0 f = a\frac{x_0^2}{2} + b\sum_{j=1}^{n} x_0 x_j + b\sum_{j=1}^{n} \frac{x_j^2}{2} + c\sum_{1 \leq j < k \leq n} x_j x_k,$$

so

$$M = \left(\begin{array}{c|ccccc} a & b & b & \ldots & b \\ \hline b & b & c & \ldots & c \\ b & c & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & c \\ b & c & \ldots & c & b \end{array}\right) \in \mathbb{R}^{(n+1)\times(n+1)}.$$

It remains to show that M has at most one positive eigenvalue if and only if $a \leq b \leq c$ and $ab + (n-1)ac \leq nb^2$.

The bottom right block of M gives us a linear eigenvalue $\lambda_1 = b - c$ with multiplicity $n - 1$ and associated eigenspace $V_1 = \mathrm{span}\{e_1 - e_k : k = 2, \ldots, n\}$.

Since $\nabla^2(\partial_0 f)$ is symmetric, then the remaining eigenvectors may be chosen to be in $V_1^\perp = \mathrm{span}\{e_0, e_1 + \cdots + e_n\}$. Let U be the $(n+1)\times 2$ matrix with columns $\begin{bmatrix} e_0 & e_1 + \cdots + e_n \end{bmatrix}$.

Then the remaining eigenvectors lie in the column span of U, so the signs of the remaining eigenvalues are the same as the signs of the eigenvalues of U^TMU.

We compute

$$U^TMU = \begin{pmatrix} a & nb \\ nb & n(b+(n-1)c) \end{pmatrix}$$

Since this is a 2×2 matrix with nonnegative entries, and since it is not the zero matrix, it has at least one positive eigenvalue. Moreover, it has exactly one positive eigenvalue if and only if its determinant is nonpositive, which happens if and only if

$$ab + (n-1)ac \leq nb^2.$$

Therefore, M has Lorentzian signature if and only if $b \leq c$ and $ab + (n-1)ac \leq nb^2$. To complete the proof, we note that if M is Lorentzian, then $a \leq b$ by Theorem 4.3.17, since $(3) \succ (2,1)$ in dominance order. ☐

We can then use the conditions derived in Theorem 4.4.3 and take $n \to \infty$ to obtain the following characterization of cubic Lorentzian symmetric functions. Interestingly, note that it suffices to check the dominance linear inequalities from Theorem 4.3.17 and log-concavity of coefficients with respect to dominance:

Corollary 4.4.4. *The symmetric function* $f = a\widetilde{m}_3 + b\widetilde{m}_{21} + c\widetilde{m}_{111} \in \Lambda^3$ *is Lorentzian if and only if* $0 \leq a \leq b \leq c$ *and* $ac \leq b^2$.

We also note that cubic Lorentzian symmetric polynomials are Schur-positive.

Corollary 4.4.5. *If* $f \in L_n^3 \cap \Lambda_n^3$, *then* f *is* s*-positive.*

Proof. Suppose $f = a\widetilde{m}_3 + b\widetilde{m}_{21} + c\widetilde{m}_{111}$ is Lorentzian. By direct computation, we note that the expansion of f in the Schur basis is

$$f = \left(\frac{a}{6} - b + c\right) s_{1,1,1} + \frac{1}{2}\left(b - \frac{a}{3}\right) s_{2,1} + \frac{a}{6} s_3.$$

This immediately implies s-positivity since $b \leq c$ and $b \geq a \geq \frac{a}{3}$. ☐

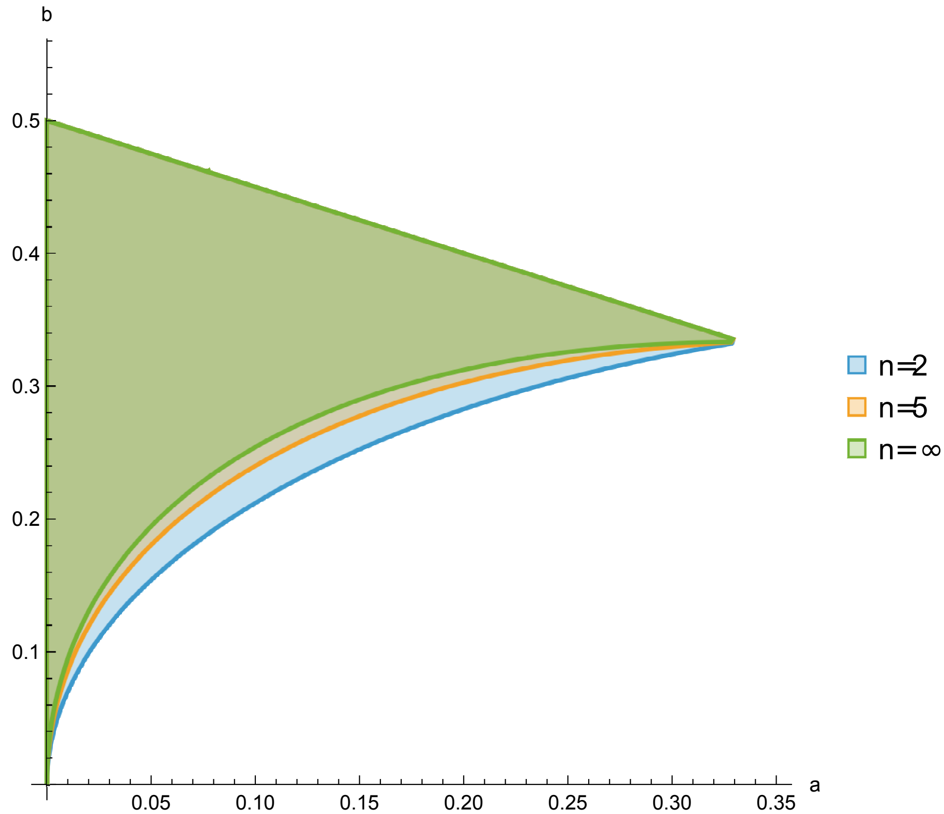

Figure 4.1: Regions defined by the conditions of Theorem 4.4.3 for $n = 2, 5$ and as $n \to \infty$, intersected with the hyperplane $a + b + c = 1$. Note that each region is homeomorphic to a closed Euclidean ball, as shown in Theorem 4.3.11.

However, unlike the quadratic case, Lorentzianity is no longer equivalent to normalized Schur positivity. For example, $4Ns_{111} + Ns_{21} + Ns_3 = \widetilde{m}_3 + 2\widetilde{m}_{21} + 7\widetilde{m}_{111}$ is normalized Schur positive but not Lorentzian, even when restricted to three variables. Normalized Schur positivity is also no longer a necessary condition, as one can see from the corollary below.

Corollary 4.4.6. *Let* $f = aNs_3 + bNs_{2,1} + cNs_{1,1,1}$*. Then,* f *is Lorentzian if and only if* $a, b, b + c \geq 0$ *and* $ac - \frac{1}{n}a(b + c) \leq b^2$.

Proof. Changing to the normalized monomial basis, we have

$$\begin{aligned} f &= a(\widetilde{m}_{111} + \widetilde{m}_{21} + \widetilde{m}_3) + b(2\widetilde{m}_{111} + \widetilde{m}_{21}) + c(\widetilde{m}_{111} + \widetilde{m}_{21} + \widetilde{m}_3) \\ &= a\widetilde{m}_3 + (a+b)\widetilde{m}_{21} + (a+2b+c)\widetilde{m}_{111}. \end{aligned}$$

By Theorem 4.4.3, f is Lorentzian if and only if $0 \leq a \leq a+b \leq a+2b+c$ and $a(a+b)+(n-1)a(a+2b+c) \leq n(a+b)^2$. The first chain of inequalities gives us exactly $a, b, b+c \geq 0$. The latter inequality can be rearranged to $ac - \frac{1}{n}a(b+c) \leq b^2$, as desired. ☐

This implies that Lorentzian cubics need not be normalized Schur positive. For example, $f = Ns_3 + 2Ns_{21} - Ns_{111} = \widetilde{m}_3 + 3\widetilde{m}_{21} + 4\widetilde{m}_{111}$ is not normalized Schur positive, but it satisfies the conditions of Corollary 4.4.6 and is therefore Lorentzian.

4.4.3 Symmetric Quartics

We continue our study of low-degree polynomials by fully classifying Lorentzian symmetric polynomials of degree four.

Theorem 4.4.7. *Let $f = a\widetilde{m}_4 + b\widetilde{m}_{31} + c\widetilde{m}_{22} + d\widetilde{m}_{211} + e\widetilde{m}_{1111} \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ be a homogeneous symmetric quartic polynomial in $n \geq 4$ variables. Then f is Lorentzian if and only if the following coefficient inequalities hold:*

(1) $0 \leq a \leq b \leq c \leq d \leq e$

(2) $a(c + d(n-2)) \leq (n-1)b^2$

(3) $(b+c)(d + e(n-3)) \leq 2(n-2)d^2$

Proof. By Theorem 4.3.7, f satisfies the support condition if and only if its m-support is downward closed, since dominance order on partitions of $d = 4$ is totally ordered. We note that the set of linear inequalities (1) implies this condition is satisfied. Therefore, it suffices to show that f satisfies (H') if and only if inequalities (2) and (3) are satisfied.

By definition, f satisfies (H') if and only if

$$M(2) = \begin{pmatrix} a & (n-1)b \\ (n-1)b & (n-1)(c+(n-2)d) \end{pmatrix},$$
$$M(11) = \begin{pmatrix} \frac{1}{2}(b+c) & (n-2)d \\ (n-2)d & (n-2)(d+(n-3)e) \end{pmatrix}$$

both have Lorentzian signature. This happens if and only if

$$a(c+(n-2)d) \leq (n-1)b^2 \quad \text{and} \quad b(d+(n-3)e) \leq 2(n-2)d^2.,$$

as desired. □

By taking $n \to \infty$, we also get a semialgebraic description for the set of Lorentzian symmetric functions of degree four.

Corollary 4.4.8. *The symmetric function $f = a\widetilde{m}_4 + b\widetilde{m}_{31} + c\widetilde{m}_{22} + d\widetilde{m}_{211} + e\widetilde{m}_{1111} \in \Lambda^4$ is Lorentzian if and only if all of the following hold:*

(1) $0 \leq a \leq b \leq c \leq d \leq e$

(2) $ad \leq b^2$

(3) $(b+c)e \leq 2d^2$.

Example 4.4.9. Observe that there are quartic Lorentzian symmetric polynomials that do not extend to Lorentzian symmetric functions. For example, consider

$$f = \widetilde{m}_4 + 2\widetilde{m}_{31} + 2\widetilde{m}_{22} + 5\widetilde{m}_{211} + 5\widetilde{m}_{1111}.$$

Then f_n is Lorentzian if and only if $n \leq 4$, so some restrictions of f are Lorentzian polynomials, but f itself is not a Lorentzian symmetric function.

We now briefly examine the relationship between Lorentzian symmetric polynomials and M-concave functions. Suppose $f = \sum \frac{c_\alpha}{\alpha!} x^\alpha$. By [BH20, Corollary 3.16], we know that if $\nu_f : \alpha \mapsto \log(c_\alpha)$ is an M-concave function, then f is Lorentzian. In general, the converse fails even for quadratics. Starting in degree four, the converse does not hold for symmetric polynomials.

Example 4.4.10. Let

$$f = \frac{1}{256}\widetilde{m}_4 + \frac{1}{16}\widetilde{m}_{31} + \frac{3}{8}\widetilde{m}_{22} + \frac{1}{2}\widetilde{m}_{211} + \widetilde{m}_{1111}.$$

The coefficients satisfy the conditions in Corollary 4.4.8, so f is Lorentzian.

However, we claim that ν_f is not M-concave. Indeed, consider $\alpha = 2e_1 + 2e_2, \beta = e_1 + e_2 + e_3 + e_4$, so $\alpha_1 > \beta_1$. Then in order for ν_f to be M-concave, we must have

$$\nu_f(\alpha) + \nu_f(\beta) \leq \max\{\nu_f(\alpha - e_1 + e_3) + \nu_f(\beta + e_1 - e_3), \nu_f(\alpha - e_1 + e_4) + \nu_f(\beta + e_1 - e_4)\}.$$

We note that $\alpha - e_1 + e_3 = (1,2,1,0)$ and $\alpha - e_1 + e_4 = (1,2,0,1)$. Since f is symmetric, this implies that $x^{\alpha - e_1 + e_3}$ and $x^{\alpha - e_1 + e_4}$ have the same coefficient, and hence $\nu_f(\alpha - e_1 + e_3) = \nu_f(\alpha - e_1 + e_4) = \log(c_{211})$. A similar computation shows that $\nu_f(\beta + e_1 - e_3) = \nu_f(\beta + e_1 - e_4) = \log(c_{211})$.

Hence, in order for ν_f to be M-concave, we must have $c_{22}c_{1111} \leq c_{211}^2$. However, $c_{22}c_{1111} = \frac{3}{8}$ while $c_{211}^2 = \frac{1}{4}$, so ν_f is not M-concave.

Remark 4.4.11. *This only starts to happen in degree four. In degree three, one can check that a symmetric cubic is Lorentzian if and only if ν_f is M-concave. This differs from the non-symmetric case, where M-concavity may fail even for quadratics, as shown in [BH20].*

4.4.4 Symmetric Quintics

Next, we classify Lorentzian symmetric functions of degree five. Given the complexity of the inequalities, we do not treat the polynomial case separately here, though the computations are essentially identical.

Theorem 4.4.12. *Let* $p = a\widetilde{m}_5 + b\widetilde{m}_{41} + c\widetilde{m}_{32} + d\widetilde{m}_{311} + e\widetilde{m}_{221} + f\widetilde{m}_{2111} + g\widetilde{m}_{11111}$ *be a symmetric function of degree five with nonnegative coefficients. Then* f *is Lorentzian if and only if the following coefficient inequalities hold:*

(1) $0 \leq a \leq b \leq c \leq d \leq e \leq f \leq g$

(2) $ad \leq b^2$

(3) $(d + 2e)g \leq 3f^2$

(4) $bf \leq d^2$

(5) $cf \leq e^2$

(6) $\det \begin{pmatrix} b & c & d \\ c & c & e \\ d & e & f \end{pmatrix} \geq 0$

Proof. We again apply Theorem 4.3.19. Since the dominance order on partitions of $d = 5$ is a total order, $\mathrm{supp}_m(f)$ vacuously must have a unique maximal element. Moreover, inequality (1) is exactly condition (D). Thus, it just remains to check the Hessian condition on $\partial^3 f$, $\partial^{21} f$, and $\partial^{111} f$. Condition (H) says that f is Lorentzian if $M(3), M(21), M(111)$ all have Lorentzian signature. Using the definition of $M(\mu)$, we have

$$M(3) = \begin{pmatrix} a & b \\ b & d \end{pmatrix} \qquad M(111) = \begin{pmatrix} 3(d+2e) & 3f \\ 3f & g \end{pmatrix} \qquad M(21) = \begin{pmatrix} b & c & d \\ c & c & e \\ d & e & f \end{pmatrix}$$

Taking the determinants of $M(3)$ and $M(111)$ give inequalities (2) and (3), so it just remains to study the signature of $M(21)$.

By Proposition 4.2.2, $M(21)$ has Lorentzian signature if and only if the following four inequalities all hold:

(i) $b^2 - bc \leq 0$

(ii) $bf \leq d^2$

(iii) $cf \leq e^2$

(iv) $\det(M(21)) \geq 0$

Already (i) fits into the linear chain given in (1), and the remaining inequalities are exactly conditions (4)-(6), which completes the proof. □

4.4.5 Symmetric Sextics

As our final low-degree case, we fully characterize Lorentzian symmetric functions of degree six.

Theorem 4.4.13. *Let $f = \sum_{\alpha \vdash 6} c_\alpha \widetilde{m}_\alpha$ be a symmetric function of degree six with nonnegative coefficients. Then f is Lorentzian if and only if all of the following hold:*

- *Whenever $\lambda \preceq \mu$ in dominance order, we have $c_\mu \geq c_\lambda$.*
- $c_6 c_{411} \leq c_{51}^2$
- $c_{2211}(c_{42} + c_{33}) \leq 2c_{321}^2$
- $c_{1^6}(c_{3111} + 3c_{2211}) \leq 4c_{21111}^2$
- *Each of the following matrices has exactly one positive eigenvalue*[1]

$$Q_{31} = \begin{pmatrix} c_{51} & c_{42} & c_{411} \\ c_{42} & c_{33} & c_{321} \\ c_{411} & c_{321} & c_{311} \end{pmatrix} \qquad Q_{211} = \begin{pmatrix} c_{411} & c_{321} & c_{3111} \\ c_{321} & \frac{1}{2}(c_{321} + c_{222}) & c_{2211} \\ c_{3111} & c_{2211} & c_{21111} \end{pmatrix}.$$

[1]By Proposition 4.2.2, testing the signature of either of these matrices reduces to checking three quadratic inequalities and one cubic inequality, but for the sake of compactness and clarity, we leave it in matrix form.

Proof. By Theorem 4.3.19, we know that f is Lorentzian if and only if it satisfies conditions (M), (D), and (H). We note that (D) is also a condition above, so it remains to consider (M) and (H). After applying our reduction, we are left with

$$M(4) = \begin{pmatrix} c_6 & c_{51} \\ c_{51} & c_{411} \end{pmatrix} \qquad M(31) = \begin{pmatrix} c_{51} & c_{42} & c_{411} \\ c_{42} & c_{33} & c_{321} \\ c_{411} & c_{321} & c_{3111} \end{pmatrix}$$

$$M(22) = \begin{pmatrix} c_{32} + c_{42} & 2c_{321} \\ 2c_{321} & c_{2211} \end{pmatrix} \qquad M(211) = \begin{pmatrix} c_{411} & 2c_{321} & c_{3111} \\ 2c_{321} & c_{321} + c_{222} & 2c_{2211} \\ c_{3111} & 2c_{2211} & c_{21111} \end{pmatrix}$$

$$M(1111) = \begin{pmatrix} 4(3c_{2211} + c_{3111}) & 4c_{21111} \\ 4c_{21111} & c_{1^6} \end{pmatrix}$$

First, suppose f is Lorentzian. The determinants of $M(4), M(22)$, and $M(1111)$ give the desired quadratic inequalities. Moreover, $Q_{31} = M(31)$, and Q_{211} is obtained from $M(211)$ by scaling the second row and column by $1/2$. Hence Q_{31} and Q_{211} both have Lorentzian signature, as desired.

On the other hand, suppose that the conditions above hold. We note that (D) holds by assumption. Moreover, as noted above, the quadratic and matrix inequalities exactly give us (H). Thus, it just remains to show that $\mathrm{supp}_m(f)$ has a unique maximal element in dominance order. Indeed, the only incomparable partitions of six are $\{(4,1,1),(3,3)\}$ and $\{(3,1,1,1),(2,2,2)\}$. If the former pair are both maximal in $\mathrm{supp}_m(f)$, then

$$\det(Q_{31}) = \det \begin{pmatrix} 0 & 0 & c_{411} \\ 0 & c_{33} & c_{321} \\ c_{411} & c_{321} & c_{311} \end{pmatrix} = -c_{411}^2 c_{33} < 0,$$

so Q_{31} does not have Lorentzian signature. Similarly, if the latter pair are both maximal, then

$$\det(Q_{211}) = \det \begin{pmatrix} 0 & 0 & c_{3111} \\ 0 & \frac{1}{2}c_{222} & c_{2211} \\ c_{3111} & c_{2211} & c_{21111} \end{pmatrix} = -\frac{1}{2}c_{3111}^2 c_{222} < 0,$$

so Q_{211} does not have Lorentzian signature. Hence, if $\mathrm{supp}_m(f)$ does not have a unique maximal element, then either Q_{31} or Q_{211} does not have Lorentzian signature, as desired. □

One surprising observation is that the condition in Theorem 4.3.19 that $\mathrm{supp}_m(f)$ has a unique maximal element is redundant in the degree six case, which is the smallest degree in which it is not vacuously satisfied. It is not known if this condition continues to be redundant in higher degrees.

4.5 Applications to Known Results

In addition to fully characterizing Lorentzian symmetric polynomials in low degrees, our techniques also allow us to significantly simplify some known proofs that certain symmetric functions are Lorentzian. In Section 4.5.1, we simplify the linear algebra in the proof for chromatic symmetric functions, originally proved in [MMS24]. We also give a simplified combinatorial proof for two-column Schur functions. The full Schur function case was proved in [HMMS22] using volume polynomials of varieties. The two-column case was proved combinatorially in [Qin23], and we give a simplified version of this proof in Section 4.5.2.

4.5.1 Lorentzianity of Chromatic Symmetric Functions

In [MMS24, Theorem 6.8], Matherne, Morales, and Selover prove that the chromatic symmetric functions of indifference graphs of abelian Dyck paths are Lorentzian. In this section, we use the computational reductions found in Section 4.3.3 to give a shorter proof of this fact. Note that we still use the same underlying combinatorics and inequalities, but we are now able to forgo Schur complements entirely and to simplify the linear algebra considerably.

We begin by recalling some definitions.

Definition 4.5.1. A *Dyck path d* of length n is a lattice path from $(0,0)$ to (n,n) that does not go below the diagonal $y=x$. For a Dyck path d, we define the indifference graph $G(d)$ as the graph with vertex set $[n]$ where there is an edge between i and j, with $i<j$ if the square in column i and row $n+1-j$ is between the path and the diagonal. We say that d is abelian if $G(d)$ is cobipartite.

Definition 4.5.2. For a graph G on vertex set $[n]$, the *chromatic symmetric polynomial* of G is

$$X_G := \sum_{\substack{\kappa: V(G)\to\mathbb{Z}_+ \\ \text{proper coloring}}} x_{\kappa(1)}x_{\kappa(2)}\dots x_{\kappa(n)}.$$

In [MMS24], Matherne, Morales, and Selover prove that chromatic symmetric polynomials of the indifference graphs of abelian Dyck paths are Lorentzian.

Theorem 4.5.3 ([MMS24, Section 6.2])**.** *Let d be an abelian Dyck path. Then $X_{G(d)}$ is a Lorentzian symmetric function.*

Proof. Following [MMS24, Theorem 6.8], we know that

$$X_{G(d)} = \sum_i i!(n-2i)!r_i m_{2^i 1^{n-2i}},$$

where r_i is the number of non-attacking rook placements on the Ferrers board B_μ associated to $G(d)$.

Since $X_{G(d)}$ has degree at most two in each variable, it suffices to consider $\partial^\alpha X_{G(d)}$ for all partitions α of the form $\alpha=(2^{k-1},1^{n-2k})$.

Partition the variables into sets Y_1,Y_2,Y_3, where

$$Y_1=\{x_1,\dots,x_{n-2k}\}, Y_2=\{x_{n-2k+1},\dots,x_{n-k+1}\}, Y_3=\{x_{n-k+2},\dots,x_n\},$$

and consider $\partial_{Y_1}\partial^2_{Y_3}X_{G(d)}$. By direct computation,

$$\begin{aligned}\partial_{Y_1}\partial^2_{Y_3}X_{G(d)} = 2^{k+1}&(k+1)!(n-2k-2)!r_{k+1}m_{11}(Y_1)\\ &+2^{k-1}k!(n-2k)!r_k(m_2(Y_2)+2m_1(Y_1)m_1(Y_2))\\ &\qquad+2^{k-1}(k-1)!(n-2k+2)!r_{k-1}m_{11}(Y_2),\end{aligned}$$

so

$$\nabla^2\partial^\alpha X_{G(d)} = \left(\begin{array}{ccc|ccc} 0 & & a & & & \\ & \ddots & & & b & \\ a & & 0 & & & \\ \hline & & & b & & c \\ & b & & & \ddots & \\ & & & c & & b \end{array}\right),$$

where

$$\begin{aligned} a &= 2^{k+1}(k+1)!(n-2k-2)!r_{k+1} \\ b &= 2^k k!(n-2k)!r_k \\ c &= 2^{k-1}(k-1)!(n-2k+2)!r_{k-1}. \end{aligned}$$

Using our block matrix techniques, if $X_{G(d)}$ has r variables, this matrix has Lorentzian signature if and only if $a \geq 0$, $b \leq c$, and

$$\det \begin{pmatrix} (n-2k)(n-2k-1)a & (n-2k)(r-n+k+1)b \\ (n-2k)(r-n+k+1)b & (r-n+k)(r-n+k+1)c+(r-n+k+1)b \end{pmatrix} \leq 0,$$

which happens if and only if

$$(n-2k-1)a(b+(r-n+k)c)-(n-2k)(r-n+k+1)b^2 \leq 0.$$

These inequalities are verified in [MMS24, Propositions 6.10, 6.12] using properties of rook numbers on Ferrers boards. □

4.5.2 Normalized Two-Column Schur Functions

In [Qin23], Qin gives a combinatorial proof that two-column Schur functions are denormalized Lorentzian. In this section we give a simplified version of this proof, again eliminating the need for Schur complements while proving the same underlying inequalities.

First, we recall the *ballot numbers* $C_{k,\ell}$ where

$$C_{k,\ell} = \frac{(k+\ell)!}{\frac{(k+1)!}{k-\ell+1}\cdot \ell!} = \frac{k-\ell+1}{k+1}\binom{k+\ell}{\ell} = \binom{k+\ell}{\ell} - \binom{k+\ell}{\ell-1}.$$

By the hook length formula, $C_{k,\ell}$ is the number of standard Young tableaux on shape (k,ℓ) and also the number of standard Young tableaux on $(k,\ell)'$.

Theorem 4.5.4. *Let $s \geq t$ such that $s+t=d$, and let $\gamma = (s,t)'$ be the two columned tableau with columns of length s and t. Then Ns_γ is Lorentzian.*

Proof. Since Schur polynomials of two-column tableaux have degree at most two in each variable, it is sufficient to consider $\mu = (2^p, 1^q, 0^m) \in \mathbb{N}^n$ where $\mu \vdash d-2$. For such a partition μ, define $\ell = t-p$ and $k = s-p$.

Choose variables $z_1 < \ldots < z_p < x_1 < \ldots < x_q < y_1 < \ldots < y_m$. For ease of notation, we will denote $X = \{x_i\}_{i=1}^q$, $Y = \{y_i\}_{i=1}^m$, and $Z = \{z_i\}_{i=1}^p$. By direct computation,

$$\partial^\mu Ns_{(s,t)'} = a \sum_{x_i<x_j\in X} x_ix_j + b \sum_{x_i\in X y_j\in Y} x_iy_j + c_1 \sum_{y_i<y_j\in Y} y_iy_j + c_2 \sum_{y_i\in Y}\frac{y_i^2}{2}$$

for some a, b, c_1, c_2.

First, we find the inequalities necessary to show that $\nabla^2\partial^\mu Ns_\gamma$ has Lorentzian signature. By direct computation, the Hessian is of the form

$$H = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & H' \end{array}\right) \quad \text{with} \quad H' = \left(\begin{array}{c|c} A & B \\ \hline B^T & D \end{array}\right)$$

where

$$A = a(\mathbf{1}_{q\times q} - I)$$

$$B = b\cdot \mathbf{1}_{q\times m}$$

$$D = c_1\cdot \mathbf{1}_{m\times m} + (c_2-c_1)I$$

Using the techniques from Section 4.3.3, H' has Lorentzian signature for all n if and only if the linear eigenvalues $\lambda = -a$ and $\lambda = c_2 - c_1$ are nonpositive and the remaining matrix

$$\widetilde{H} = \begin{pmatrix} q(q-1)a & qb \\ qb & c_1 \end{pmatrix}$$

has Lorentzian signature. Therefore, in order to show that $\nabla^2\partial^\mu N s_\lambda$ has Lorentzian signature, it just remains to show

$$c_2 \leq c_1 \tag{4.1}$$

$$(q-1)ac_1 \leq qb^2. \tag{4.2}$$

We note that if $\ell = 0, 1$, then $q+2 > t$ and hence $[\mu+e_{x_i}+e_{x_j}] = (2^{q+2}, 1^{p-2}) \succ \gamma = (2^t, 1^s)$, so $a = 0$. Thus, if $\ell = 0, 1$, then (4.2) holds automatically. If $\ell = 0$, then additionally $[\mu + e_{x_i} + e_{y_j}] = (2^{q+1}, 1^p) \succ \gamma$, so $b = 0$, in which case the former inequality holds as well. Thus, it just remains to verify that $c_1 \geq c_2$ whenever $\ell \geq 1$, and that $(q-1)ac_1 - qb^2 \leq 0$ whenever $\ell \geq 2$.

To show this, we enumerate the coefficients. Note that a is the number of semistandard fillings of $(s,t)'$ with content $\mu + e_{x_1} + e_{x_2}$. Given any such filling f, it must satisfy the following:

- The first $2 \times p$ boxes contain exactly the $z_i's$,
- There are two x_1 boxes and two x_2 boxes, and these boxes must be next to each other and immediately below the Z boxes, so the next two rows of f must be a row of two x_1s and a row of two x_2s.

Therefore, the first $p+2$ rows of such a semistandard tableau are forced. Moreover, each of $x_3, \ldots, x_q$ must appear at least once in the tableau. These $q-2$ boxes plus the $2(p+2)$ boxes from the first rows total to $2(p+2)+(q-2) = 2p+q+2 = d$ boxes, so the remaining boxes of the tableau after the first $p+2$ rows are exactly a standard Young tableau of shape $(s-(p+2), t-(p+2))' = (k-2, \ell-2)'$. Therefore, by the hook length formula,

$$a = C_{k-2,\ell-2}.$$

To compute b, we compute the number of semistandard Young tableaux of type $\mu + e_{x_1} + e_{y_1}$. By the same logic as above, the first $p+1$ rows are forced to be filled with

$(z_1, z_1), \ldots, (z_p, z_p), (x_1, x_1)$, and each of $x_2, \ldots, x_d$ must appear at least once, and y_1 must appear exactly once. This sums to $2(p+1) + (q-1) + 1 = d$, so the remaining part of the tableau is a standard Young tableau of shape $(s-(p+1), t-(p+1))' = (k-1, \ell-1)'$. Thus,

$$b = C_{k-1,\ell-1}.$$

By similar logic, c_1 is the number of semistandard Young tableaux of type $\mu + e_{y_1} + e_{y_2}$. Such a tableau must have the first p rows filled with the z_is, and the remaining boxes form a standard Young tableau of shape $(2^{s-p}, 1^{t-p})$, so

$$c_1 = C_{k,\ell}.$$

Finally, c_2 is the number of semistandard Young tableaux of shape $(s,t)'$ with content $\mu + 2e_{y_2} = (2^p, 1^q, 2)$, which, by symmetry of Schur polynomials, must be the same as the number of tableaux of type $\mu + e_{x_1} + e_{y_1} = (2^p, 2, 1^{q-1}, 1)$. Therefore, $c_2 = b$, so

$$c_2 = C_{k-1,\ell-1}.$$

By direct computation, for $\ell \geq 1$,

$$C_{k,\ell} = \frac{(k+\ell)(k+\ell-1)}{(k+1)\ell} C_{k-1,\ell-1}.$$

Since $k \geq \ell \geq 1$, we have

$$\frac{k+\ell-1}{\ell} \geq 1 \qquad \text{and} \qquad \frac{k+\ell}{k+1} \geq 1,$$

so $C_{k,\ell} \geq C_{k-1,\ell-1}$. Hence, $c_1 \geq c_2$ whenever $\ell \geq 1$, as desired.

It remains to verify that $(q-1)ac_1 - qb^2 \leq 0$ whenever $\ell \geq 2$. By definition, $k = s-p$, $\ell = t-p$, $s+t = d$, and $2p+q = d-2$, so $q = k+\ell-2$. Thus, (4.2) can be equivalently written as

$$(k+\ell-3)C_{k,\ell}C_{k-2,\ell-2} - (k+\ell-2)C_{k-1,\ell-1}^2 \leq 0.$$

Since $k \geq \ell \geq 2$, then $C_{k-2,\ell-2}, C_{k-1,\ell-1}, C_{k,\ell} > 0$, so it suffices to show that

$$\frac{k+\ell-2}{k+\ell-3} \frac{C_{k-1,\ell-1}^2}{C_{k,\ell}C_{k-2,\ell-2}} \geq 1.$$

By the same computation as above, we have

$$\frac{C_{k-1,\ell-1}}{C_{k,\ell}} = \frac{(k+1)\ell}{(k+\ell)(k+\ell-1)} \qquad \frac{C_{k-1,\ell-1}}{C_{k-2,\ell-2}} = \frac{(k+\ell-2)(k+\ell-3)}{k(\ell-1)},$$

so this reduces to showing that

$$(k+1)\ell(k+\ell-2)^2 \geq k(\ell-1)(k+\ell)(k+\ell-1). \tag{4.3}$$

Indeed, we note that

$$(k+1)(k+\ell-2) = k(k+\ell-1) - k + (k+\ell-2) = k(k+\ell-2) + \ell - 2 \geq k(k+\ell-2), \tag{4.4}$$

where the final inequality comes from the assumption that $\ell \geq 2$.

Similarly,

$$\ell(k+\ell-2) = (\ell-1)(k+\ell) + (k+\ell) - 2\ell = (\ell-1)(k+\ell) + k - \ell \geq (\ell-1)(k+\ell). \tag{4.5}$$

Multiplying the inequalities from (4.4) and (4.5) gives the desired inequality in (4.3), which completes the proof. □

4.6 Nonresults: Symmetric Function Operations and Lorentzianity

One of the main breakthroughs in the study of Lorentzian symmetric polynomials is that normalized Schur polynomials are Lorentzian [HMMS22]. In the study of symmetric functions, Schur polynomials behave particularly nicely under operations such as the ω-involution and Hall inner product. Inspired by this, it is natural to ask whether this nice behavior extends to all symmetric denormalized Lorentzian polynomials, but unfortunately, the answer is no.

4.6.1 ω-involution is not Positivity Preserving

The ω-involution is an algebra endomorphism $\Lambda \to \Lambda$ defined by $\omega(e_\lambda) = h_\lambda$, where e_λ and h_λ are the elementary symmetric function and complete homogeneous symmetric function indexed by a partition λ, respectively. This map is an involution, and sends Schur polynomials to Schur polynomials.

Theorem 4.6.1 ([Sta99, Theorem 7.14.5])**.** *For every partition λ, we have $\omega(s_\lambda) = s_{\lambda'}$.*

Since $N(s_\mu)$ is Lorentzian for all partitions μ, and since $N \circ \omega \circ N^{-1} : N(s_\lambda) \mapsto N(s_{\lambda'})$, it is reasonable to conjecture that $N \circ \omega \circ N^{-1}$ might preserve Lorentzianity (or, equivalently, that ω might preserve denormalized Lorentzianity), but this is not the case.

Proposition 4.6.2. *The operator $N \circ \omega \circ N^{-1}$ is not Lorentzian-preserving. In fact, it does not preserve nonnegativity of coefficients when applied to Lorentzian polynomials.*

Proof. Consider the symmetric function $f = \widetilde{m}_{211} + \widetilde{m}_{1111}$. We have that f is Lorentzian by the characterization of Lorentzian symmetric quartics in Section 4.4.3.

However,

$$N(\omega(N^{-1}(f))) = \widetilde{m}_{1111} - \widetilde{m}_{22} - \widetilde{m}_{31} - 2\widetilde{m}_4,$$

so $N(\omega(N^{-1}(f)))$ does not have nonnegative coefficients and therefore cannot be Lorentzian. □

4.6.2 $\langle -, Ns_\lambda \rangle$ is not Denormalized Lorentzian Preserving

One tempting way to find a combinatorial proof that normalized Schur polynomials are Lorentzian is via an alternative form of the dual Cauchy identity. The following lemma is widely known in the symmetric functions community; for example, it is used in the original proof of the Lorentzianity of normalized Schur polynomials in [HMMS22], cited as [FH91, Exercise 15.50]. For the sake of completeness, we produce the proof below.

Lemma 4.6.3. *For all $m, n \in \mathbb{Z}_+$, we have*

$$\prod_{i=1}^{n} \prod_{j=1}^{m} (x_i + y_j) = \sum_{\substack{\lambda : \ell(\lambda) \le n \\ \lambda_1 \le m}} s_\lambda(x) s_{\widetilde{\lambda}'}(y),$$

where $\widetilde{\lambda}'$ denotes the complementary partition to λ' in the $n \times m$ rectangle, i.e. $\widetilde{\lambda}'_k = n - \lambda'_{m+1-k}$.

Proof. We recall the dual Cauchy identity [Sta99, Theorem 7.14.3]:

$$\prod_{i,j}(1+x_iy_j) = \sum_{\lambda} s_\lambda(x)s_{\lambda'}(y).$$

By specializing $x_i = y_j = 0$ for all $i > n, j > m$, we have

$$\begin{aligned}\prod_{i=1}^{n}\prod_{j=1}^{m}(1+x_iy_j) &= \sum_{\lambda} s_\lambda(x_1,\dots,x_n,0,\dots)s_{\lambda'}(y_1,\dots,y_m,0,\dots)\\ &= \sum_{\substack{\lambda:\ell(\lambda)\le n\\ \lambda_1\le m}} s_\lambda(x_1,\dots,x_n)s_{\lambda'}(y_1,\dots,y_m).\end{aligned}$$

For compactness, we will omit $\ell(\lambda)\le n$ and $\lambda_1 \le m$ from the sum going forward.

Using the above identity, we find

$$\begin{aligned}\prod_{i=1}^{n}\prod_{j=1}^{m}(x_i+y_j) &= \prod_{i=1}^{n}\prod_{j=1}^{m} y_j(1+x_iy_j^{-1})\\ &= \left(\prod_{j=1}^{m} y_j^n\right)\prod_{i=1}^{n}\prod_{j=1}^{m}(1+x_iy_j^{-1})\\ &= \left(\prod_{j=1}^{m} y_j^n\right)\sum_{\lambda} s_\lambda(x_1,\dots,x_n)s_{\lambda'}(y_1^{-1},\dots,y_m^{-1})\\ &= \sum_{\lambda} s_\lambda(x_1,\dots,x_n)\left(y_1^n\dots y_m^n s_{\lambda'}(y_1^{-1},\dots,y_m^{-1})\right).\end{aligned}$$

Thus, in order to prove the desired identity, it just remains to show that for any λ with $\ell(\lambda)\le m$ and $\lambda_1 \le n$, we have

$$s_{\widetilde{\lambda}}(y_1,\dots,y_m) = y_1^n\dots y_m^n s_\lambda(y_1^{-1},\dots,y_m^{-1}),$$

where $\widetilde{\lambda}$ is the partition defined by $\widetilde{\lambda}_k = n - \lambda_{m+1-k}$.

Indeed, we recall the bialternant formula [Sta99, Theorem 7.15.1]:

$$s_\lambda(x_1,\dots,x_n) = \frac{\det\left(x_i^{\lambda_j+n-j}\right)_{1\le i,j\le n}}{\det\left(x_i^{n-j}\right)_{1\le i,j\le n}}.$$

Using this formula, we have

$$y_1^n \dots y_m^n s_\lambda(y_1^{-1}, \dots, y_m^{-1}) = \frac{y_1^{n+m-1} \dots y_m^{n+m-1} \det\left(y_i^{-\lambda_j - m + j}\right)_{1 \le i,j \le m}}{y_1^{m-1} \dots y_m^{m-1} \det\left(y_i^{-n+j}\right)_{1 \le i,j \le m}}$$
$$= \frac{\det\left(\left(y_i^{-\lambda_j - m + j}\right)_{1 \le i,j \le m} \operatorname{diag}\left(y_i^{n+m-1}\right)\right)}{\det\left(\left(y_i^{-m+j}\right)_{1 \le i,j \le m} \operatorname{diag}\left(y_i^{m-1}\right)\right)}$$
$$= \frac{\det\left(y_i^{n-\lambda_j + j - 1}\right)}{\det\left(y_i^{j-1}\right)} = \frac{\det\left(y_i^{\widetilde{\lambda}_{m+1-j} + m - (m+1-j)}\right)}{\det\left(y_i^{m-(m+1-j)}\right)}.$$

Let σ be the permutation on $[m]$ that transposes j with $m+1-j$ for all j, and let P_σ be its permutation matrix. Then, by construction, $\left(y_i^{\widetilde{\lambda}_{m+1-j} + m - (m+1-j)}\right) = \left(y_i^{\widetilde{\lambda}_j + m - j}\right) \cdot P_\sigma$ and similarly $\left(y_i^{m-(m+1-j)}\right)_{1 \le i,j \le m} = \left(y_i^{m-j}\right) \cdot P_\sigma$. Plugging this into the equation above, we get

$$y_1^n \dots y_m^n s_\lambda(y_1^{-1}, \dots, y_m^{-1}) = \frac{\det\left(y_i^{n-\lambda_j + j - 1}\right)}{\det\left(y_i^{j-1}\right)} = \frac{\det\left(y_i^{\widetilde{\lambda}_{m+1-j} + m - (m+1-j)}\right)}{\det\left(y_i^{m-(m+1-j)}\right)}$$
$$= \frac{\det\left(\left(y_i^{\widetilde{\lambda}_j + m - j}\right) \cdot P_\sigma\right)}{\det\left(\left(y_i^{m-j}\right) \cdot P_\sigma\right)}$$
$$= \frac{\det\left(y_i^{\widetilde{\lambda}_j + m - j}\right)_{1 \le i,j \le m}}{\det\left(y_i^{m-j}\right)_{1 \le i,j \le m}}$$
$$= s_{\widetilde{\lambda}}(y_1, \dots, y_m),$$

which completes the proof. $\square$

Using the fact that Schur polynomials are an orthonormal basis with respect to the Hall inner product, this gives us a tantalizing approach towards proving that Schur polynomials are denormalized Lorentzian. We note that $\prod_{i,j}(x_i + y_j)$ is a product of linear forms, so it is real stable and therefore Lorentzian [BH20, Proposition 2.2]. Moreover, by the orthonormality of Schur polynomials, we have

$$s_\lambda(x) = \left\langle \prod_{i,j}(x_i + y_j), s_{\widetilde{\lambda}'}(y) \right\rangle,$$

so if $N \circ \langle -, s_\lambda(y)\rangle : \Lambda(x,y) \to \Lambda(x)$ were to preserve Lorentzianity, this would be sufficient to prove that Schur polynomials are denormalized Lorentzian.

Unfortunately, $\langle -, s_\lambda(x)\rangle$ does not, in general, send Lorentzian polynomials to denormalized Lorentzian polynomials.

Example 4.6.4. Consider

$$\begin{aligned} f = \frac{1}{2}m_{211}(\vec{x})m_1(\vec{y}) + m_{1111}(\vec{x})m_1(\vec{y}) \\ + \frac{1}{2}m_{21}(\vec{x})m_{11}(\vec{y}) + \frac{1}{2}m_{111}(\vec{x})m_2(\vec{y}) + m_{111}(\vec{x})m_{11}(\vec{y}) \\ + \frac{1}{2}m_{11}(\vec{x})m_{21}(\vec{y}) + m_{11}(\vec{x})m_{111}(\vec{y}). \end{aligned}$$

Since $\langle m_{21}, s_{111}\rangle = -2$ and $\langle m_{111}, s_{111}\rangle = 1$,

$$\langle f, s_{111}(\vec{x})\rangle = -2\left(\frac{1}{2}m_{11}(\vec{y})\right) + \frac{1}{2}m_2(\vec{y}) + m_{11}(\vec{y}) = \frac{1}{2}m_2(\vec{y}),$$

so $\langle f, s_{111}(\vec{x})\rangle$ is not denormalized Lorentzian.

However, f is Lorentzian. The quadratic derivatives of f that are not identically zero are as follows. Note that we abuse notation slightly by writing m_λ for the symmetric polynomial in all the non-differentiated variables.

$$\begin{aligned} \partial_{x_1}^2\partial_{x_2} f &= m_1(\vec{x})m_1(\vec{y}) + m_{11}(\vec{y}) \\ \partial_{x_1}\partial_{x_2}\partial_{x_3} f &= (x_1+x_2+x_3)m_1(\vec{y}) + m_1(\vec{x})m_1(\vec{y}) + \frac{1}{2}m_2(\vec{y}) + m_{11}(\vec{y}) \\ \partial_{x_1}^2\partial_{y_1} f &= m_{11}(\vec{x}) + m_1(\vec{x})m_1(\vec{y}) \\ \partial_{x_1}\partial_{x_2}\partial_{y_1} &= \frac{1}{2}m_2(\vec{x}) + (x_1+x_2)m_1(\vec{x}) + m_{11}(\vec{x}) \\ &\quad + (x_1+x_2)m_1(\vec{y}) + m_1(\vec{x})y_1 + m_1(\vec{x})m_1(\vec{y}) \\ &\quad + y_1 m_1(\vec{y}) + \frac{1}{2}m_2(\vec{y}) + m_{11}(\vec{y}) \\ \partial_{x_1}\partial_{y_1}^2 f &= m_{11}(\vec{x}) + m_1(\vec{x})m_1(\vec{y}) \\ \partial_{x_1}\partial_{y_1}\partial_{y_2} f &= x_1 m_1(\vec{x}) + \frac{1}{2}m_2(\vec{x}) + m_{11}(\vec{x}) + m_1(\vec{x})(y_1+y_2) + m_1(\vec{x})m_1(\vec{y}). \end{aligned}$$

One can check that all of these derivatives have Lorentzian signature.

The fact that $\langle \prod(x_i + y_j), s_\lambda(y)\rangle$ is denormalized Lorentzian seems to be more a coincidence than a mathematically meaningful fact. A fundamental limitation of this inner product as an operator is that the output only depends on the part of f that is homogeneous of degree $|\lambda|$ in the y variables and degree $m+n-|\lambda|$ in the x variables. We know that initial forms of Lorentzian polynomials are Lorentzian, so if we take only the terms of f with maximal (resp. minimal) degree in y, then the result is Lorentzian, but in general there are no guarantees on how the intermediate degree parts of f must behave. As such, any operator that only depends on one graded piece of f should not be Lorentzianity-preserving, and as we showed above, restricting ourselves to symmetric polynomials does not fix this problem. Restricting to stable polynomials as our input is also likely insufficient to guarantee Lorentzianity of the output, as stability again only gives guarantees on the top- and bottom- degree parts.

4.7 Conclusion and Future Directions

This paper explores the intersection of Lorentzian polynomials and symmetric functions. We give some general theorems on their topology and structure, and we exploit this structure to give complete characterizations in small degree and to simplify some known proofs of Lorentzianity. While we show that many natural operations on symmetric functions do not interact nicely with the Lorentzian property, there are still several interesting directions to explore.

One future direction might be to explore the Lorentzianity of chromatic symmetric polynomials, following on the work in [MMS24], and in particular to try to characterize which graphs have Lorentzian chromatic symmetric functions. By analyzing the support conditions, we know that the graph must be claw-free. From [MMS24], we know that abelian Dyck paths give Lorentzian chromatic symmetric functions, but that not all non-abelian Dyck paths have this property.

From a slightly different direction, we know that the chromatic polynomial of any graph has log-concave coefficients [AHK18, Theorem 9.9 (4)]. Historically, the chromatic symmetric function has been regarded as the natural multivariate lift of the chromatic polynomial. Since not all chromatic symmetric functions are Lorentzian, it is natural to ask if there is another multivariate lift that does preserve the Lorentzian property.

It is also natural to study certain subtypes of Lorentzian symmetric polynomials, such as M-convex functions. As shown in [BH20], M-convex functions give rise to a special, restrictive class of Lorentzian polynomials. In future work, we are looking to study the intersection of this class with symmetric polynomials. M-convexity imposes further restrictions on the coefficients of the polynomial, which should have interesting consequences when combined with the symmetry restrictions.

Last but not least, a sensible direction is to continue studying the obvious stratification of the projective space of Lorentzian symmetric polynomials in the same spirit as recent work of Baker, Huh, Kummer, and Lorscheid [BHKL25]. Answering general questions like computing the dimension and other topological invariants of each piece would be a natural next step.

Chapter 5

VALUATED DELTA MATROIDS

In this chapter, we cover results from [CCLV25] defining and studying valuated Δ-matroids. This chapter is based on joint work with Nathan Cheung, Gaku Liu, and Cynthia Vinzant

We call a function $p : \{0,1\}^n \to \mathbb{R} \cup \{\infty\}$ a *valuated Δ-matroid* if every polytope in the induced subdivision is a Δ-matroid polytope (not necessarily even). In other words, every edge of the induced regular subdivision has direction $\pm e_i$ or $e_i \pm e_j$ for some $i, j \in [n]$. We note that this conflicts with other recent terminology as mentioned above. However, we believe our definition is the natural definition of a valuated Δ-matroid which is consistent with the history and other terminology of Δ-matroids. As we demonstrate in this chapter, these objects are interesting in their own right and generalize previous concepts in the study of Δ-matroids, including non-even Δ-matroids.

A summary of our results is as follows:

- As mentioned in Section 2.2, valuated matroids can be characterized as functions $\binom{[n]}{r} \to \mathbb{R} \cup \{\infty\}$ satisfying a set of local relations called the tropical Plücker relations. Geometrically, this is equivalent to saying a function is a valuated matroid if and only if its induced subdivision has no edges of length 3, where the *length* of a vector $(x_1, \ldots, x_n)$ is defined to be its L_1-length $\sum_{i=1}^n |x_i|$.

  In analogy to this, we prove that a function $\{0,1\}^n \to \mathbb{R} \cup \{\infty\}$ is a valuated Δ-matroid if and only if its induced subdivision has no edges of length 3 or 4. This means that determining whether a function is a valuated Δ-matroid can be done by checking a list of local conditions which is bounded up to symmetry, as is the case with ordinary valuated matroids. The exact conditions are given in Lemma 5.2.1 and Theorem 5.2.3.

- We discuss the rank function, and use it to prove that any Δ-matroid can appear as the cell of a Δ-matroid subdivision.

- The set of all functions $\binom{[n]}{r} \to \mathbb{R} \cup \{\infty\}$ that are valuated matroids form a polyhedral fan called the *Dressian*. Analogously, we call the set of all functions $\{0,1\}^n \to \mathbb{R} \cup \{\infty\}$

that are valuated Δ-matroids the Δ-*Dressian*, $\Delta\text{-Dr}(n)$. (This conflicts slightly with the usage in [Rin12].) In Theorem 5.5.2, we establish bounds on the dimension of this object, showing that is exponential in n.

- Given a symmetric or skew-symmetric matrix over any field [Bou88] or a Hermitian matrix in $\mathbb{C}^{n\times n}$ [Brä07], the nonzero principal minors of the matrix form a Δ-matroid. We generalize this result to show that given any valued field equipped with an automorphic involution preserving the valuation, and any matrix over this field which is Hermitian or skew-Hermitian with respect to the involution, the valuations of the principal minors of the matrix form a valuated Δ-matroid.

  We show that in dimension at most three, all valuated Δ-matroids can be realized in this way up to symmetry. We show this is not true in dimension four. Finally, we conjecture that if a matrix is the sum of a skew-Hermitian matrix and a rank one Hermitian matrix, then the valuations of its principal minors form a Δ-matroid. We prove this is several cases, including the case of a skew-symmetric matrix plus a rank one symmetric matrix.

- We show how our results on Hermitian matrices can be rewritten in terms of sesquilinear forms and isotropic subspaces, and that this generalizes previous ways of representing Δ-matroids through these subspaces.

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5.1 Δ-matroid polytopes and regular subdivisions

Given a line segment with endpoints in $\{0,1\}^n$, we define its *length* to be the Hamming distance between its endpoints, i.e. the number of coordinates at which the endpoints differ. Hence, a 0-1 polytope is a Δ-matroid if and only if all its edges have length at most 2.

Let $\mathbb{R}\cup\{\infty\}$ be the real number system extended by positive ∞. The only operations we will perform involving ∞ will be $+$, max, min, and multiplication by a positive scalar, in which case ∞ behaves as one would expect (e.g., $a + \infty = \infty$ for all $a \in \mathbb{R} \cup \{\infty\}$).

A *polytopal complex* is a collection $\mathcal{S}$ of polytopes in $\mathbb{R}^n$ satisfying the following properties.

1. If $P \in \mathcal{S}$ and F is a face of P, then $F \in \mathcal{S}$.

2. If P, $Q \in \mathcal{S}$ and $P \neq Q$, then the relative interiors of P and Q are disjoint.

The elements of $\mathcal{S}$ are called *cells*. The *support* of $\mathcal{S}$ is $\bigcup_{P \in \mathcal{S}} P$. A polytopal complex whose support is a polytope Q is called a *polytopal subdivision*, or *subdivision*, of Q.

Let $X \subset \mathbb{R}^n$ be a finite set of points and $p : X \to \mathbb{R} \cup \{\infty\}$ any function. The *effective domain* of p is $\operatorname{dom} p := \{x \in X : p(x) \neq \infty\}$. Recall that a function p induces a regular subdivision on $\operatorname{conv}(\operatorname{dom}(p))$, which we will denote $\mathcal{S}_p$.

We next give a characterization of when a simplex appears in $\mathcal{S}_p$. Recall that if I is an affinely independent set of points and x is in the relative interior of $\operatorname{conv} I$, then x can be written uniquely as $x = \sum_{i \in I} \lambda_i i$ where $0 < \lambda_i < 1$ and $\sum_{i \in I} \lambda_i = 1$. We call this the *convex circuit representation* of x with support I. If the elements of I have weights given by a function p, we define

$$p(x, I) := \sum_{i \in I} \lambda_i p(i).$$

Proposition 5.1.1. *Let $X \subset \mathbb{R}^n$ be a finite set of points in convex position, and let $p : X \to \mathbb{R} \cup \{\infty\}$ be any function. Let I be an affinely independent subset of X. The following are equivalent.*

1. *I is the set of vertices of a simplex in $\mathcal{S}_p$.*

2. *For any x in the relative interior of* $\operatorname{conv} I$, *and for any convex circuit representation of x with different support $J \subseteq X$, we have $p(x, I) < p(x, J)$.*

3. *There exists x in the relative interior of* $\operatorname{conv} I$, *such that for any convex circuit representation of x with different support $J \subseteq X$, we have $p(x, I) < p(x, J)$.*

Proof. (1) $\implies$ (2). Suppose I is the set of vertices of a simplex in $\mathcal{S}_p$. Then $I_p := \{(x, p(x)) : x \in I\}$ is the set of vertices of a lower face of P_p. Hence, there is a linear functional $(\varphi, 1) \in (\mathbb{R}^n)^* \times \mathbb{R}$ and real number b such that $(\varphi, 1)(x, y) = b$ for all $(x, y) \in \operatorname{conv}(I_p) \subset \mathbb{R}^n \times \mathbb{R}$, and $(\varphi, 1)(x, y) > b$ for all $(x, y) \in P_p \setminus \operatorname{conv}(I_p)$.

Let x be in the relative interior of $\operatorname{conv} I$, and let $\sum_{i\in I} \lambda_i i$ be its convex circuit representation with support I. We have

$$\sum_{i\in I} \lambda_i(\varphi, 1)(i, p(i)) = b \implies \varphi(x) + p(x, I) = b \implies p(x, I) = -\varphi(x) + b.$$

Suppose that x is also in the relative interior of $\operatorname{conv} J$, where $J \subseteq X$ and $J \neq I$. Since X is in convex position, at least one element of J is not in $\operatorname{conv} I$. Let $\sum_{j\in J} \lambda'_j j$ be the convex support representation of x with support J. Then

$$\sum_{j\in J} \lambda'_j(\varphi, 1)(j, p(j)) > b \implies \varphi(x) + p(x, J) > b \implies p(x, J) > -\varphi(x) + b.$$

It follows that $p(x, I) < p(x, J)$, as desired. (If any element of J is not in $\operatorname{dom} p$, then this inequality is obvious as the right side is finite and the left side is ∞.)

(2) $\implies$ (3). Immediate.

(3) $\implies$ (1). We prove the contrapositive. Suppose I is not the set of vertices of a simplex in $\mathcal{S}_p$. Choose any x in the relative interior of $\operatorname{conv} I$. Let $F \subset X$ be the set of vertices of the cell of $\mathcal{S}_p$ whose relative interior contains x. Since x is in the relative interior of $\operatorname{conv} I$, F is not a proper subset of I. In addition, $F \neq I$ by assumption. Hence F contains a point not in I. It follows that there exists $J \subset F$, $J \neq I$, such that J is affinely independent and x is in the relative interior of $\operatorname{conv} J$. (For example, one can consider a pulling triangulation of $\operatorname{conv} F$ starting with a vertex not in I.) Using the same argument as above, we have $p(x, J) \leq p(x, I)$, and thus (3) is not true. □

We call a function $p : \{0, 1\}^n \to \mathbb{R} \cup \{\infty\}$ a *valuated* Δ-*matroid* if every cell in the induced subdivision $\mathcal{S}_p$ is a Δ-matroid. Equivalently, a function $p : \{0, 1\}^n \to \mathbb{R} \cup \{\infty\}$ is a valuated Δ-matroid if and only if all the edges (1-dimensional cells) of its induced subdivision have length at most 2. Slightly more generally, given a subset M of $\{0,1\}^n$ such that M is a Δ-matroid, a function $p : M \to \mathbb{R} \cup \{\infty\}$ is a *valuated Δ-matroid* if every cell in the induced subdivision $\mathcal{S}_p$ is a Δ-matroid. Crucially, if $p : \{0,1\}^n \to \mathbb{R} \cup \{\infty\}$ is a valuated Δ-matroid, then the restriction of p to any face of $[0,1]^n$ is also a valuated Δ-matroid.

We set some final notation. Given a function $p : \{0,1\}^n \to \mathbb{R} \cup \{\infty\}$, we will often abbreviate the value $p(e_S)$ as p_S. In addition, we will often abbreviate a set $\{a_1, \ldots, a_k\}$ as $a_1 \ldots a_k$; for example $p(e_{\{1,2,3\}})$ would be written p_{123}. For a set $S \subseteq [n]$, we define $S^C := [n] \setminus S$ to be its complement in $[n]$.

5.2 The cases $n = 3$ and $n = 4$

Before we describe a characterization for valuated Δ-matroids on $\{0,1\}^n$ for all n, we first work with $n = 3$ and 4. In Section 5.3, we will show that our constraints for $n = 3$ and 4 are enough to characterize valuated Δ-matroids for all n.

Lemma 5.2.1. *A function $p : \{0,1\}^3 \to \mathbb{R} \cup \{\infty\}$ is a valuated Δ-matroid on $[3]$ if and only if the minimum*

$$\min(2p_\emptyset + 2p_{123}, 2p_1 + 2p_{23}, 2p_2 + 2p_{13}, 2p_3 + 2p_{12},$$
$$p_\emptyset + p_{12} + p_{13} + p_{23}, p_1 + p_2 + p_3 + p_{123})$$

is achieved on at least two of its arguments, or on one of the last two arguments.

Proof. We will show the contrapositive in both directions.

First, suppose that $p : \{0,1\}^3 \to \mathbb{R} \cup \{\infty\}$ is not a valuated Δ-matroid. Then the induced subdivision $\mathcal{S}_p$ has an edge of length 3, so $[e_S, e_{S^C}]$ is an edge for some $S \subseteq [3]$. Without loss of generality, assume $[0, e_{123}]$ is an edge. Let $x = (1/2, 1/2, 1/2)$ be the midpoint of this edge. We list all of the convex circuit representations of x using subsets of $\{0,1\}^3$:

$$\frac{1}{2}(e_\emptyset + e_{123}), \frac{1}{2}(e_1 + e_{23}), \frac{1}{2}(e_2 + e_{13}), \frac{1}{2}(e_3 + e_{12}),$$
$$\frac{1}{4}(e_\emptyset + e_{12} + e_{13} + e_{23}), \frac{1}{4}(e_1 + e_2 + e_3 + e_{123})$$

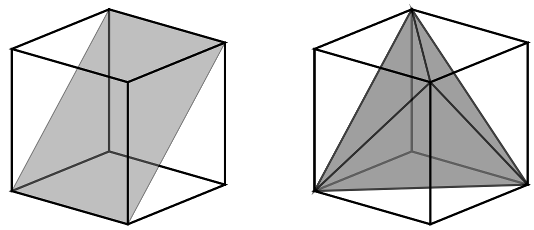

Figure 5.1: Illustration of two cases in Lemma 5.2.1. The left cube is the case where the minimum is achieved on two of the first four arguments, and the right cube is the case where the minimum is achieved uniquely on one of the last two arguments.

By Proposition 5.1.1, $[0, e_{123}]$ being an edge means that in the list of numbers

$$\frac{1}{2}(p_\emptyset + p_{123}), \frac{1}{2}(p_1 + p_{23}), \frac{1}{2}(p_2 + p_{13}), \frac{1}{2}(p_3 + p_{12}), \\ \frac{1}{4}(p_\emptyset + p_{12} + p_{13} + p_{23}), \frac{1}{4}(p_1 + p_2 + p_3 + p_{123})$$

the minimum is achieved uniquely at $\frac{1}{2}(p_\emptyset + p_{123})$. In particular, the minimum is not achieved twice nor at one of the last two arguments, as desired.

Conversely, suppose the above minimum is not achieved on at least two arguments nor on one of the last two arguments. Then it must be achieved uniquely on one of the first four arguments, that is, $\frac{1}{2}(p_S + p_{S^C})$ for some $S \subseteq [3]$. By Proposition 5.1.1(3) applied to the point $x = (1/2, 1/2, 1/2)$, it follows that $[e_S, e_{S^C}]$ is an edge in $\mathcal{S}_p$. Thus $\mathcal{S}_p$ is not a Δ-matroid. $\square$

Before stating the characterization for $n = 4$, we first need to describe the convex circuit representations of $(1/2, 1/2, 1/2, 1/2)$ with support in $\{0, 1\}^4$. Let B_n denote the symmetry group of the n-cube. It is generated by permutations of the coordinates and flipping the bits of coordinates. The following can be verified computationally:

Proposition 5.2.2. *The convex circuit representations of* $(1/2, 1/2, 1/2, 1/2)$ *are exactly the images of the following formal linear combinations under* B_4*:*

$$\frac{1}{2}(e_\emptyset + e_{1234}),\ \frac{1}{4}(e_\emptyset + e_{14} + e_{123} + e_{234}),\ \frac{1}{3}e_\emptyset + \frac{1}{6}(e_{123} + e_{124} + e_{134} + e_{234}).$$

Let $\mathcal{J}$ be the set of all supports of convex circuit representations from Proposition 5.2.2. We now state the result for $n = 4$.

Theorem 5.2.3. *A function $p : \{0,1\}^4 \to \mathbb{R} \cup \{\infty\}$ is a valuated Δ-matroid on $[4]$ if and only if both of the following hold:*

1. *p restricts to a valuated Δ-matroid on each facet of $[0,1]^4$.*

2. *For $x = (1/2, 1/2, 1/2, 1/2)$, the minimum*

$$\min_{J \in \mathcal{J}} p(x, J)$$

is achieved on at least two of its arguments.

Proof. Let $\mathcal{J} = \mathcal{J}_1 \sqcup \mathcal{J}_2 \sqcup \mathcal{J}_2$, where $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}_3$ are the orbits of $\{\emptyset, 1234\}$, $\{\emptyset, 14, 123, 234\}$, and $\{\emptyset, 123, 124, 134, 234\}$, respectively, under the action of B_4. We again prove the contrapositive in both directions.

Suppose $p : \{0,1\}^4 \to \mathbb{R} \cup \{\infty\}$ is not a valuated Δ-matroid. Then its induced subdivision $\mathcal{S}_p$ either it has an edge of length 3 or an edge of length 4. If it has an edge of length 3, then this edge is contained in some facet of $[0,1]^4$. Therefore (1) is not true, as desired. Suppose $\mathcal{S}_p$ has an edge of length 4. The set of endpoints of this edge is an element $J \in \mathcal{J}_1$. Therefore, by Proposition 5.1.1, the minimum $\min_{J \in \mathcal{J}} p(x, J)$ is achieved uniquely on J. Hence (2) is not true, proving one direction of Theorem 5.2.3.

For the other direction, suppose that one of (1) or (2) is not true. If (1) is not true, then some facet of $[0,1]^4$ contains an edge of length 3, so p is not a valuated Δ-matroid. Suppose (2) is not true. Let $J \in \mathcal{J}$ be the argument at which the minimum is achieved. If $J \in \mathcal{J}_1$, then by Proposition 5.1.1 $\mathcal{S}_p$ contains an edge of length 4, so p is not a valuated Δ-matroid. If $J \in \mathcal{J}_2$ or $\mathcal{J}_3$, then $\mathcal{S}_p$ contains an image of $[e_\emptyset, e_{123}]$ under B_n, and hence an edge of length 3. So p is not a valuated Δ-matroid, completing the proof. $\square$

5.3 *A finite characterization for all* n

Here we show that in order to check if $(p_S)_{S\subseteq[n]}$ is a valuated Δ-matroid, it is sufficient to only consider the 4-dimensional faces of $[0,1]^n$. This is analogous to ordinary valuated matroids, where only the 3-dimensional faces of the hypersimplex need to be considered.

Proposition 5.3.1. *Let $\mathcal{S}$ be a subdivision of $[0,1]^n$ with vertices in $\{0,1\}^n$. Let k be a positive integer. If $\mathcal{S}$ has no edges of length ℓ for all $k < \ell \leq 2k$, then it has no edges of length $> 2k$.*

Proof. Suppose for the sake of contradiction that $\mathcal{S}$ has no edges of length $k < \ell \leq 2k$ and an edge of length $m > 2k$. Choose the smallest such edge e. By applying symmetry, we may assume e has endpoints $e_\emptyset$ and $e_{[m]}$. Now let $\mathcal{S}'$ be the restriction of $\mathcal{S}$ to the face $[0,1]^m$ of $[0,1]^n$. Consider a 2-dimensional cell F of $\mathcal{S}'$ which contains e. The polygon F contains an edge between $e_\emptyset$ and some point of $\{0,1\}^m$ other than $e_{[m]}$, say e_S. By assumption e is the shortest edge of length $> 2k$ and $\mathcal{S}$ has no edges of length $k < \ell \leq 2k$, so $|S| \leq k$. Since $\dim F = 2$, we have that F lies in the linear span of e_S and $e_{[m]}$. Hence, the only other possible vertex of F is $e_{[m]\setminus S}$. Since $[0, e_{[m]}]$ is an edge, F cannot contain $e_{[m]\setminus S}$, since $[e_\emptyset, e_{[m]}]$ and $[e_S, e_{[m]\setminus S}]$ have the same midpoint. Thus F only has three vertices, so it contains the edge $[e_S, e_{[m]}]$. This edge has length $m - |S| > 2k - |S| \geq k$, and also length strictly less than m, a contradiction. □

Applying Proposition 5.3.1 with $k = 2$ gives the following immediate corollary.

Corollary 5.3.2. *If a subdivision of $[0,1]^n$ with vertices in $\{0,1\}^n$ has no edges of length 3 or 4, then it has no edges of length > 4.*

This gives us a characterization of valuated Δ-matroids whose effective domain is the full cube.

Corollary 5.3.3. *A function $p : \{0,1\}^n \to \mathbb{R}$ is a valuated Δ-matroid if and only if its restriction to every 4-dimensional face of $[0,1]^n$ is a valuated Δ-matroid.*

For general valuated Δ-matroids, we have the following characterization.

Theorem 5.3.4. *A function $p : \{0,1\}^n \to \mathbb{R} \cup \{\infty\}$ is a valuated Δ-matroid if and only if $\operatorname{dom} p$ is a Δ-matroid and the restriction of p to every 4-dimensional face of $[0,1]^n$ is a valuated Δ-matroid.*

Proof. One direction is immediate: if p is a valuated Δ-matroid, then $\operatorname{conv} \operatorname{dom} p$ has no edges of length > 2 and hence is a Δ-matroid polytope. Moreover, the restriction of p to any 4-face of the cube is also a valuated Δ-matroid.

To show the reverse direction, let $p : \{0,1\}^n \to \mathbb{R} \cup \{\infty\}$ be a function such that $\operatorname{dom} p$ is a Δ-matroid and the restriction of p to every 4-dimensional face of $[0,1]^n$ is a valuated Δ-matroid. Suppose $\mathcal{S}_p$ has an edge of length greater than or equal to 5, and let e be such an edge of minimum length m. We may assume e has endpoints $e_\emptyset$ and $e_{[m]}$. Let $\mathcal{S}'$ be the restriction of $\mathcal{S}$ to the face $[0,1]^m$ of $[0,1]^n$. Since $\operatorname{dom} p$ is a Δ-matroid and $\varnothing$, $[m] \in \operatorname{dom} p$, by the symmetric exchange property there exists some $j \in [m]$ such that $\{1, j\} \in \operatorname{dom} p$. In particular, $\mathcal{S}'$ is not the single edge e, and therefore e is contained in a two-dimensional cell of $\mathcal{S}'$. The same argument as in Proposition 5.3.1 leads to a contradiction. $\square$

5.4 The rank function

One way to characterize matroids is via rank functions. In fact, the rank function of any matroid gives a valuated matroid [Spe08, Proposition 4.4], so in particular, any matroid polytope appears as a cell in some matroidal subdivision.

Let $M = ([n], \mathcal{F})$ be a Δ-matroid, where $\mathcal{F}$ is the set of bases. Following [Lar25, Section 2.2], we define the rank function for a M as $r_M : 2^{[n]} \to \mathbb{Z}_{\geq 0}$ where

$$r_M(S) = \max_{B \in \mathcal{F}} (|B \cap S| + |B^C \cap S^C|).$$

By construction, $r_M(S) \leq n$ for all $S \subseteq [n]$, with equality if and only if $S \in \mathcal{F}$.

In this section, we show that if M is a Δ-matroid, then $-r_M$ is a valuated Δ-matroid. Thus, just as in the matroid case, any Δ-matroid polytope appears as a cell in some Δ-matroidal subdivision.

First, we give a geometric interpretation of r_M.

Proposition 5.4.1. *The rank function $r_M(S)$ is equal to*

$$r_M(S) = n - \min_{B\in\mathcal{F}} |e_B - e_S|_1.$$

That is, $r_M(S)$ also has a geometric interpretation in terms of how far e_S is from the closest vertex of P_M.

Proof. We note that $|e_B - e_S|_1 = |B\setminus S| + |S\setminus B|$. Since $|B\cap S| + |B^C\cap S^C| + |B\setminus S| + |S\setminus B| = n$, then

$$\begin{aligned} r_M(S) &= \max_{B\in\mathcal{F}} (|B\cap S| + |B^C \cap S^C|) \\ &= \max_{B\in\mathcal{F}} (n - (|B\setminus S| + |S\setminus B|)) \\ &= n - \min_{B\in\mathcal{F}} (|B\setminus S| + |S\setminus B|) \\ &= n - \min_{B\in\mathcal{F}} |e_B - e_S|_1. \end{aligned}$$

□

In particular, this means that $|r_M(S) - r_M(S\Delta i)| \leq 1$ by the triangle inequality, so adding or removing a single element from S changes the rank by at most one.

Lemma 5.4.2. *Suppose $n \geq 3$, and let $M = ([n], \mathcal{F})$ be a Δ-matroid. If $\max_{S\subseteq[n]}(r_M(S) + r_M(S^C))$ is attained uniquely by some $S \subseteq [n]$, then $S, S^C \in \mathcal{F}$.*

Proof. Assume $r_M(S) + r_M(S^C)$ attains its maximum at S and suppose $S \notin \mathcal{F}$. We wish to show that $\max_{S\subseteq[n]}(r_M(S) + r_M(S^C))$ is not attained uniquely.

Let $B \in \mathcal{F}$ be a set in $\mathcal{F}$ closest to S (i.e. B attains $\min_{B\in\mathcal{F}} |e_B - e_S|_1$), and let $i \in S\Delta B$. Then $|e_B - e_{S\Delta i}| = |e_B - e_S| - 1$, so $r(S\Delta i) = r(S) + 1$. Moreover, adding or removing an element can drop the rank by at most one, so $r(S^C\Delta i) \geq r(S^C) - 1$. Since $S^C\Delta i = (S\Delta i)^C$,

$$r(S\Delta i) + r((S\Delta i)^C) \geq r(S) + r(S^C).$$

We know that $S\Delta i \neq S$, and since $n \geq 3$, we have $S\Delta i \neq S^C$, so $r_M(S) + r_M(S^C)$ cannot attain $\max_{S\subseteq[n]}(r_M(S) + r_M(S^C))$ uniquely. □

Now, we are ready to show that $-r_M$ is a valuated Δ-matroid.

Theorem 5.4.3. *Let $M = ([n], \mathcal{F})$ be a Δ-matroid and let $r_M : 2^{[n]} \to \mathbb{Z}_{\geq 0}$ be its rank function. Then $-r_M$ is a valuated Δ-matroid.*

Proof. If $n = 2$, all functions on $2^{[2]}$ are valuated Δ-matroids, so it suffices to consider the case where $n \geq 3$.

Let $M = ([n], \mathcal{F})$ be a Δ-matroid and suppose that the subdivision induced by $-r_M$ has an edge of length greater than 2. By replacing M with its restriction to the face of $[0, 1]^n$ containing this edge, we may assume that this subdivision contains an edge of length n. It follows that $\max_{S\subseteq[n]}(r_M(S) + r_M(S^C))$ is attained uniquely. By Lemma 5.4.2, this implies that it is attained uniquely by some $T, T^C \in \mathcal{F}$.

Since $r_M(S) \leq n$ for all S with equality if and only if $S \in \mathcal{F}$, the Δ-matroid polytope $P_M = \mathrm{conv}\{e_S : S \in \mathcal{F}\}$ must appear as a cell in the induced subdivision, and this cell contains both e_T and e_{T^C}. Since M is a Δ-matroid, all edges of P_M have length at most two, so $[e_T, e_{T^C}]$ cannot be an edge of the subdivision, a contradiction. Thus, $-r_M$ cannot induce an edge of length n, completing the proof. □

5.5 *Dimension*

We define Δ-Dr(n) to be the set of all functions $p : \{0, 1\}^n \to \mathbb{R}$ which are valuated Δ-matroids. We call this the Δ-*Dressian* on $[n]$. Δ-Dr(n) is a polyhedral conical complex in the vector space of all functions $p : \{0, 1\}^n \to \mathbb{R}$.

In this section we consider the dimension of Δ-Dr(n); that is, the largest d such that Δ-Dr(n) contains a subset affinely equivalent to a d-dimensional ball. We note that for $n \geq 3$, Δ-Dr(n) is not a pure polyhedral complex; that is, the maximal cones are not all the same dimension. For example, consider the two subdivisions in Figure 5.2. The functions which induce these subdivisions form (the relative interiors of) two maximal cones in Δ-Dr(3). However, the cone for the subdivision on the left has dimension seven, while the subdivision on the right is a triangulation, so its cone is full- (eight-) dimensional.

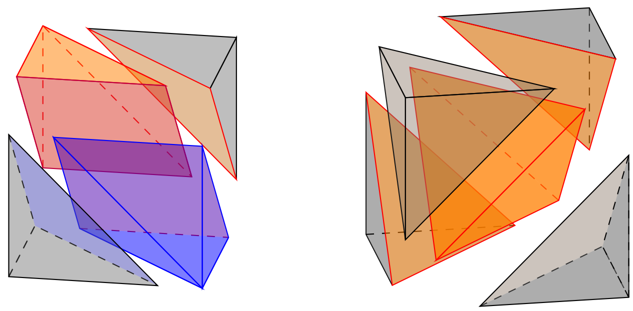

Figure 5.2: Two Δ-matroidal subdivisions of the cube, each corresponding to a maximal cone in Δ-Dr(3).

By the above discussion, the dimension of Δ-Dr(3) is eight, i.e. Δ-Dr(3) is full-dimensional. Now, we show that the codimension of Δ-Dr(4) is 1, and that in general, the dimension of Δ-Dr(n) is exponential in n.

Proposition 5.5.1. Δ-Dr(4) *has codimension* 1 *in* $\mathbb{R}^{2^4} \cong \mathbb{R}^{16}$.

Proof. Consider the subdivision induced by $p_\emptyset = p_{12} = p_{34} = p_{1234} = 0$, $p_i = p_{ijk} = 100$ for all distinct $i, j, k \in [4]$, and $p_{ij} = 1$ for all $ij \neq 12, 34$. We claim that this induces a Δ-matroid subdivision, and that the associated cone in the Δ-Dressian has codimension 1.

First, we claim that this induces a Δ-matroid subdivision. Indeed, since $\min_{S \subseteq [4]} p_S + p_{S^C}$ is obtained by $[e_\emptyset, e_{1234}]$ and $[e_{12}, e_{34}]$, there are no edges of length 4. Moreover, on any 3-dimensional face of the form $x_l = 0$, setting $\{i, j, k\} = [4] \setminus l$, we have

$$2(p_\emptyset + p_{ijk}) = 200$$

$$2(p_m + p_{\{i,j,k\} \setminus m}) \geq 200 \text{ for all } m \in \{i, j, k\}$$

$$p_\emptyset + p_{ij} + p_{ik} + p_{jk} \leq 3$$

$$p_i + p_j + p_k + p_{ijk} = 400,$$

so the minimum is attained uniquely by $p_\emptyset + p_{ij} + p_{ik} + p_{jk}$. Similarly, on the 3-dimensional face $x_l = 1$, setting $\{i, j, k\} = [4] \setminus l$, we have

$$\begin{aligned} 2(p_l + p_{1234}) &= 200 \\ 2(p_{lm} + p_{[4]\setminus m}) &\geq 200 \text{ for all } m \in \{i, j, k\} \\ p_l + p_{ijl} + p_{jkl} + p_{ikl} &= 400 \\ p_{il} + p_{jl} + p_{kl} + p_{1234} &\leq 3, \end{aligned}$$

so the minimum is attained uniquely by $p_{il} + p_{jl} + p_{kl} + p_{1234}$. Thus, by Lemma 5.2.1, there are no edges of length 3, so p is a valuated Δ-matroid.

Moreover, we claim that the functions which induce $\mathcal{S}_p$ form a cone of codimension 1 in Δ-Dr(4). Let $x = (1/2, 1/2, 1/2, 1/2)$. First, we claim that $\min_{J\in\mathcal{J}} p(x, J)$ is attained exactly twice. Indeed, since $p_S > 0$ for all $S \neq \varnothing, 12, 34, 1234$, then $p(x, J) > 0$ for all $J \in \mathcal{J}$ except for $[e_\emptyset, e_{1234}]$ and $[e_{12}, e_{34}]$, and so the minimum is attained exactly twice, as desired.

Moreover, we showed above that the minimum from Lemma 5.2.1 on each three dimensional face is attained uniquely. Hence, the only linear relation satisfied by functions which induce $\mathcal{S}_p$ is $p_\emptyset + p_{1234} = p_{12} + p_{34}$. Thus, the cone of such functions has codimension 1.

Finally, Δ-Dr(4) is not full-dimensional, because any Δ-matroid must satisfy (2) from Theorem 5.2.3. $\square$

In fact, the dimension of the Δ-Dressian is exponential in n.

Theorem 5.5.2. *The dimension of the Δ-Dressian on $[n]$ is between $\frac{1}{2}2^n$ and $\frac{15}{16}2^n$.*

Proof. First, we claim that the Δ-Dressian has codimension at least 2^{n-4}, which gives the upper bound of $\frac{15}{16}2^n$ on the dimension. Indeed, we note that we can find 2^{n-4} disjoint 4-faces of $[0, 1]^n$. For each $b_5, \ldots, b_n \in \{0, 1\}$, we consider the 4-dimensional face $\{x \in [0, 1]^n : x_5 = b_5, \ldots, x_n = b_n\}$. Each of these faces contributes a codimension-1 constraint on the Δ-Dressian, and these constraints are independent since the faces are pairwise disjoint. Thus, the dimension of the Δ-Dressian is at most $2^n - 2^{n-4} = \frac{15}{16}2^n$.

Next, we claim the dimension is at least $\frac{1}{2}2^n$. Consider the subdivision induced by taking $p_S = 0$ whenever $|S|$ is even, and $p_S > 0$ otherwise. We claim that this is a Δ-matroid subdivision. Indeed, on any 3-face or 4-face of the cube, the minimum of p is attained uniquely on the convex hull of all vertices with an even sum of coordinates, which is a polytope of dimension 3 or 4 respectively. Hence the induced subdivision has no edges of length 3 or 4, so it Δ-matroid subdivision as claimed. Since we allow any positive value on the odd vertices, the cone corresponding to this subdivision has dimension at least 2^{n-1}, as desired. $\square$

5.6 Representability via Hermitian matrices

Let K be a field with a nonarchimedean valuation $\nu : K \to \mathbb{R} \cup \{\infty\}$. The image Γ of K^* under valuation is an additive subgroup of $\mathbb{R}$ called the *value group*. We use $\Bbbk$ to denote the *residue field* of K, which is the quotient of the valuation ring $\mathcal{O} = \{a \in K : \nu(a) \geq 0\}$ by its unique maximal ideal $\mathfrak{m} = \{a \in K : \nu(a) > 0\}$. Given an element $a \in \mathcal{O}$, we use $[a]_\mathfrak{m}$ to denote its image in $\Bbbk = \mathcal{O}/\mathfrak{m}$.

Throughout this section, we will assume that the field K has an automorphic involution $a \mapsto \overline{a}$ satisfying $\nu(a) = \nu(\overline{a})$ with fixed field F. We include the possibility of this involution being the identity map and $F = K$. Since the valuation is invariant under the involution $a \mapsto \overline{a}$, the valuation ring $\mathcal{O}$ and maximal ideal $\mathfrak{m}$ are invariant under this involution. The involution on K therefore induces an automorphic involution of the residue field $\Bbbk$, which we also denote by $a \mapsto \overline{a}$.

We furthermore assume that the value group Γ is dense in $\mathbb{R}$. Since $\nu(a\overline{a}) = 2\nu(a)$, this implies that the image of F^* under the valuation is also dense in $\mathbb{R}$. Without loss of generality we can do this as follows. If the original valuation ν on K is trivial, then we can replace K with the field $K\{\{t\}\}$ of Puiseux series over K, with the automorphic involution $\overline{\sum_q a_q t^q} = \sum_q \overline{a_q} t^q$ and the valuation $\nu(\sum_q a_q t^q) = \min\{q : a_q \neq 0\}$. If Γ is non-zero but discrete then $\Gamma = \alpha\mathbb{Z}$ for some $\alpha \in \mathbb{R}_{>0}$. Choose $\beta \in \mathbb{R}\backslash(\alpha\mathbb{Q})$. By [EP05, Theorem 2.2.1.] we can extend the valuation on K to a transcendental extension $K(t)$ by taking $\nu(\sum_i a_i t^i) = \min_i\{\nu(a_i) + i\beta\}$ and $\nu(f/g) = \nu(f) - \nu(g)$. The automorphic involution also extends to $K(t)$ acting on coefficients and fixing t. The image of the valuation is then

$\alpha\mathbb{Z} + \beta\mathbb{Z}$, which is dense in $\mathbb{R}$.

The main result of the section is that, with the expected definition of Hermitian, the valuations of principal minors of Hermitian and skew-Hermitian matrices give a valuated Δ-matroid. Specifically, we call a matrix $A \in K^{n\times n}$ *Hermitian* if $\overline{A} = A^T$. Similarly, A is *skew-Hermitian* if $\overline{A} = -A^T$. For an $n \times n$ matrix A and subset $S \subseteq [n]$, we use A_S to denote the determinant of the $|S| \times |S|$ principal submatrix of A whose rows and columns are indexed by S.

Theorem 5.6.1. *If A is an $n \times n$ Hermitian or skew-Hermitian matrix over K, then the function $p : \{0,1\}^n \to \mathbb{R} \cup \{\infty\}$ given by $p_S = \nu(A_S)$ is a valuated Δ-matroid.*

We need to build up to the proof of this theorem. The main idea resembles the proof of [Brä07, Corollary 4.3] that for any Hermitian or skew-Hermitian matrix over $\mathbb{C}$, the subsets corresponding to non-zero principal minors form a Δ-matroid.

We work with a generating polynomial for the principal minors,

$$f_A = \det(\mathrm{diag}(x_1, \dots, x_n) + A) = \sum_{S\subseteq[n]} A_S x^{[n]\setminus S} \in K[x_1, \dots, x_n]. \tag{5.1}$$

In [AAV24a; AAV24b], the principal minor map was studied via *Rayleigh differences*. Formally, for $i, j \in [n]$ and $f \in K[x_1, \dots, x_n]$ define

$$\Delta_{ij}(f) = \frac{\partial f}{\partial x_i} \cdot \frac{\partial f}{\partial x_j} - f \cdot \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

If we write $f = x_i x_j f_{ij} + x_i f_i^j + x_j f_j^i + f^{ij}$ for some polynomials f_{ij}, f_i^j, f_j^i, f^{ij} that do not involve x_i and x_j, this expression simplifies to $f_i^j f_j^i - f_{ij} f^{ij}$. We say a polynomial is *multiaffine* if it has degree ≤ 1 in each variable.

In fact, we prove the following slightly stronger version of Theorem 5.6.1:

Theorem 5.6.2. *Let A be an $r \times r$ Hermitian or skew-Hermitian matrix over K and let $v_1, \dots, v_n$ be vectors K^r. Define $f \in F[x_1, \dots, x_n]$ to be the determinantal polynomial*

$$f = \sum_{S\subseteq[n]} c_S x^{[n]\setminus S} = \det\left(\sum_{i=1}^{n} x_i v_i \overline{v_i}^T + A\right).$$

Then the function $p : \{0,1\}^n \to \mathbb{R} \cup \{\infty\}$ given by $p_S = \nu(c_S)$ is a valuated Δ-matroid.

Note that Theorem 5.6.1 follows by taking $r = n$ and $v_i = e_i$ for $i = 1, \dots, n$.

Lemma 5.6.3. *Let $A \in K^{r\times r}$ be with $\overline{A} = \sigma A^T$ with $\sigma \in \{\pm 1\}$ and $v_1, \dots, v_n \in K^r$. Define*

$$f = \sum_{S\subseteq[n]} c_S x^{[n]\setminus S} = \det\left(\sum_{i=1}^n x_i v_i \overline{v_i}^T + A\right).$$

For any $i, j \in [n]$, $\Delta_{ij}(f)$ factors as $\sigma^{n-1} g_{ij}(x)\overline{g_{ij}}(\sigma x)$ for some multiaffine polynomial $g_{ij} \in K[x_1, \dots, x_n]$.

Proof. Without loss of generality, we take $i = 1$ and $j = 2$. First suppose that v_1 and v_2 are linearly independent in K^r. Then they can be extended to a basis of K^r that we collect into the columns of a matrix U. The determinant of linear matrices M and $U^{-1}M(\overline{U}^{-1})^T$ differ by only a constant scalar, so we may assume without loss of generality that $v_1 = e_1$ and $v_2 = e_2$. Dodgson condensation states that for any $r \times r$ matrix M,

$$\det(M(1,1))\det(M(2,2)) - \det(M)\det(M(12,12)) = \det(M(1,2))\det(M(2,1))$$

where $M(S,T)$ is obtained from M by *dropping* rows S and columns T. Consider the linear matrix $M = \sum_{k=1}^n x_k v_k \overline{v_k}^T + A$ and polynomial $f = \det(M)$. For $S \subseteq \{1,2\}$, $\det(M(S,S))$ is precisely $\left(\prod_{k\in S} \frac{\partial}{\partial x_k}\right) f$. Therefore $\Delta_{12}(f)$ factors as $g_{12} \cdot g_{21}$ where $g_{12} = \det(M(1,2))$ and $g_{21} = \det(M(2,1))$.

If $\overline{A} = \sigma A^T$, then

$$M^T = \sum_{k=1}^n x_k \overline{v_k} v_k^T + \sigma\overline{A} = \sigma\left(\sum_{k=1}^n \sigma x_k \overline{v_k} v_k^T + \overline{A}\right) = \sigma\overline{M}(\sigma x).$$

It follows that

$$g_{21} = \det(M(2,1)) = \det(M^T(1,2)^T) = \sigma^{n-1}\overline{g_{12}}(\sigma x).$$

Finally, if v_1 and v_2 are linearly dependent, then, up to relabeling, $v_1 = \lambda v_2$ for some $\lambda \in K$. Following the proof of [AAV24a, Theorem 6.1], $v_2\overline{v_2}^T = \lambda\overline{\lambda} v_1\overline{v_1}^T$, showing that $f(x_1, \dots, x_n) = f(0, \lambda\overline{\lambda}x_1 + x_2, x_3, \dots, x_n)$. Taking partial derivatives shows that $\frac{\partial f}{\partial x_1} = \lambda\overline{\lambda}\frac{\partial f}{\partial x_2}$ and that $\frac{\partial^2 f}{\partial x_1 \partial x_2} = 0$. Then $\Delta_{12}(f) = \lambda\overline{\lambda}(\frac{\partial f}{\partial x_2})^2$. If $\overline{A} = \sigma A^T$, we note that since

$\frac{\partial f}{\partial x_2}$ has a determinantal representation of the same form as f but of size $n-1$, we have $\frac{\partial f}{\partial x_2} = \sigma^{n-1}\overline{\frac{\partial f}{\partial x_2}}(\sigma x)$ by the computation in (5.6). Thus $\Delta_{12}(f) = \sigma^{n-1}g_{ij}(x)\overline{g_{ij}}(\sigma x)$ with $g_{12} = \lambda\frac{\partial f}{\partial x_2}$. $\square$

We will show that such factorizations pass to factorizations in $\Bbbk[x_1,\dots,x_n]$. When K has a splitting this can be interpreted as factorizations passing to initial forms, but we avoid the assumption that K has a splitting.

For $\lambda = (\lambda_1,\dots,\lambda_n) \in (K^*)^n$ and $g = \sum c_\alpha x^\alpha \in K[x_1,\dots,x_n]$, define

$$\lambda \star g = g(\lambda_1 x_1,\dots,\lambda_n x_n) = \sum c_\alpha \lambda^\alpha x^\alpha.$$

Lemma 5.6.4. *Let $f \in K[x_1,\dots,x_n]$, $\lambda \in (K^*)^n$ and $i,j \in [n]$. Then*

$$\Delta_{ij}(\lambda \star f) = \lambda_i\lambda_j(\lambda \star \Delta_{ij}(f)).$$

Proof. This follows from the chain rule, since $\frac{\partial}{\partial x_i}(\lambda \star f) = \lambda_i(\lambda \star \frac{\partial}{\partial x_i} f)$. $\square$

Lemma 5.6.5. *Let $f \in \mathcal{O}[x_1,\dots,x_n]$ be multiaffine and $i,j \in [n]$. Then*

$$[\Delta_{ij}(f)]_{\mathfrak{m}} = \Delta_{ij}([f]_{\mathfrak{m}}).$$

Suppose further that $[\Delta_{ij}(f)]_{\mathfrak{m}}$ is nonzero and that $\Delta_{ij}(f)$ factors as $g \cdot h$ in $K[x_1,\dots,x_n]$ where $g = \sum_S a_S x^S$, $h = \sum_T b_T x^T$ are multiaffine with $\operatorname{argmin}_S \nu(a_S) = \operatorname{argmin}_T \nu(b_T)$. Then $\Delta_{ij}([f]_{\mathfrak{m}})$ has factorization $\tilde{g} \cdot \tilde{h}$ in $\Bbbk[x_1,\dots,x_n]$ where $\tilde{g}$ and $\tilde{h}$ are multiaffine polynomials with $\operatorname{supp}(\tilde{g}) = \operatorname{supp}(\tilde{h})$.

Proof. As above, we write $f = x_i x_j f_{ij} + x_i f_i^j + x_j f_j^i + f^{ij}$ for some polynomials f_{ij}, f_i^j, f_j^i, f^{ij} in $K[x_1,\dots,x_n]$ that do not involve x_i and x_j and $\Delta_{ij}(f) = f_i^j f_j^i - f_{ij} f^{ij}$.

Since all of the coefficients of f belong to $\mathcal{O}$, the polynomials f_{ij}, f_i^j, f_j^i, f^{ij}, and $\Delta_{ij}(f)$ all belong to $\mathcal{O}[x_1,\dots,x_n]$. Moreover, $[f]_{\mathfrak{m}} = x_i x_j [f_{ij}]_{\mathfrak{m}} + x_i [f_i^j]_{\mathfrak{m}} + x_j [f_j^i]_{\mathfrak{m}} + [f^{ij}]_{\mathfrak{m}}$. Since the map taking $h \in \mathcal{O}[x_1,\dots,x_n]$ to $[h]_{\mathfrak{m}} \in \Bbbk[x_1,\dots,x_n]$ is a ring homomorphism, we see that

$$\Delta_{ij}([f]_{\mathfrak{m}}) = \left[f_i^j\right]_{\mathfrak{m}} \left[f_j^i\right]_{\mathfrak{m}} - [f_{ij}]_{\mathfrak{m}} \left[f^{ij}\right]_{\mathfrak{m}} = \left[f_i^j f_j^i - f_{ij} f^{ij}\right]_{\mathfrak{m}} = [\Delta_{ij}(f)]_{\mathfrak{m}}.$$

For the second claim suppose $[\Delta_{ij}(f)]_{\mathfrak{m}}$ is nonzero and that $\Delta_{ij}(f)$ factors as $g \cdot h$ where $g = \sum_S a_S x^S$, $h = \sum_T b_T x^T$ are multiaffine with $\mathrm{argmin}_S \nu(a_S) = \mathrm{argmin}_T \nu(b_T)$ and assume that $[\Delta_{ij}(f)]_{\mathfrak{m}}$ is nonzero. First, we claim that, after rescaling, we can take both g and h in $\mathcal{O}[x_1, \dots, x_n]$.

Indeed, suppose that $U \in \mathrm{argmin}_S \nu(a_S) = \mathrm{argmin}_T \nu(b_T)$. Then both $a_U^{-1} g(x)$ and $b_U^{-1} h(x)$ belong to $\mathcal{O}[x_1, \dots, x_n]$ and have nonzero image mod $\mathfrak{m}$. Therefore so does their product $a_U^{-1} b_U^{-1} g(x) h(x) = a_U^{-1} b_U^{-1} \Delta_{ij}(f)$.

Since $[\Delta_{ij}(f)]_{\mathfrak{m}}$ is nonzero, the minimum valuation of the coefficients of $\Delta_{ij}(f)$ is zero. The above argument shows that the minimum valuation of the coefficients of $a_U^{-1} b_U^{-1} \Delta_{ij}(f)$ is also zero, implying that $\nu(a_U^{-1} b_U^{-1}) = 0$. Therefore $\nu(a_U) = \nu(b_U^{-1})$.

It follows that both $a_U^{-1} g$ and $a_U h$ belong to $\mathcal{O}[x_1, \dots, x_n]$ and have nonzero image mod $\mathfrak{m}$. We can then take $\tilde{g} = [a_U^{-1} g]_{\mathfrak{m}}$ and $\tilde{h} = [a_U h]_{\mathfrak{m}}$. The support of $\tilde{g}$ and $\tilde{h}$ will be $\mathrm{argmin}_S \nu(a_S)$ and $\mathrm{argmin}_T \nu(b_T)$, respectively, which are assumed to be equal. $\square$

Proof of Theorem 5.6.2. Let $f = \sum_{S \subseteq [n]} c_S x^{[n] \setminus S} \in K[x_1, \dots, x_n]$ and $p : \{0,1\}^n \to \mathbb{R} \cup \{\infty\}$ be as given in the statement of theorem. Let $[e_T, e_{T'}]$ be an edge of the regular subdivision of $[0,1]^n$ induced by the function p, and suppose that $(w, 1) \in \mathbb{R}^{n+1}$ is the inner normal vector of a hyperplane supporting the corresponding lower face of P_p. That is,

$$\langle (w,1), (e_T, p_T) \rangle = \langle (w,1), (e_{T'}, p_{T'}) \rangle < \langle (w,1), (e_S, p_S) \rangle \tag{5.2}$$

for all $S \neq T, T'$.

We claim that we can choose $w \in \nu(F^*)^n$. Without loss of generality, we can assume that $1 \in T \setminus T'$. Note that $w = (p_{T'} - p_T) e_1$ satisfies the equation $\langle (w,1), (e_T, p_T) \rangle = \langle (w,1), (e_{T'}, p_{T'}) \rangle$ and that any other solution differs by an vector u with $\langle u, e_T - e_{T'} \rangle = 0$. Since $e_T - e_{T'} \in \{0, \pm 1\}^n$, the solutions $u = (u_1, \dots, u_n)$ of $\langle u, e_T - e_{T'} \rangle = 0$ over $\nu(F^*)^n$ are dense in those over $\mathbb{R}^n$. Indeed, we can explicitly parametrize solutions by choosing arbitrary values for $u_2, \dots, u_n$ and writing u_1 as the appropriate integer combination of these values. Since $(p_{T'} - p_T) e_1 = \nu(c_{T'}/c_T, 1, \dots, 1)$ also belongs to $\nu(F^*)^n$, we see that the solutions to

$\langle (w,1),(e_T,p_T)\rangle = \langle (w,1),(e_{T'},p_{T'})\rangle$ over $\nu(F^*)$ are dense in those over $\mathbb{R}$. In particular, we can find a solution $w \in \nu(F^*)^n$ to (5.2).

It follows that there exists $\lambda \in (F^*)^n$ with $\nu(\lambda) = w$. Suppose, for the sake of contradiction, that $|T\Delta T'| \geq 3$. Without loss of generality, we can take distinct elements $i,j \in T' \setminus T$. Note that $\tilde{f} = (\lambda^{-T}c_T^{-1})(\lambda \star f)$ belongs to $\mathcal{O}[x_1,\ldots,x_n]$. Its image under the ring homomorphism $\mathcal{O}[x_1,\ldots,x_n] \to \Bbbk[x_1,\ldots,x_n]$ induced by $\mathcal{O} \to \Bbbk = \mathcal{O}/\mathfrak{m}$ has the form $[\tilde{f}]_{\mathfrak{m}} = a_T x^T + a_{T'}x^{T'}$ where $a_T, a_{T'} \in \Bbbk$ are nonzero.

By Lemma 5.6.3, the Rayleigh difference of f factors as $\Delta_{ij}(f) = \sigma^{n-1}g_{ij}(x)\overline{g_{ij}}(\sigma x)$ for some multiaffine polynomial g_{ij} and $\sigma \in \{\pm 1\}$. Then

$$\begin{aligned}\Delta_{ij}(\tilde{f}) &= \mu^2 \Delta_{ij}(\lambda \star f)\\ &= \mu^2\lambda_i\lambda_j(\lambda \star \Delta_{ij}(f))\\ &= \sigma^{n-1}\mu^2\lambda_i\lambda_j(\lambda \star g_{ij}(x))(\lambda \star \overline{g_{ij}}(\sigma x))\end{aligned}$$

where $\mu = \lambda^{-T}c_T^{-1}$. Note that because ν is invariant under the involution $a \mapsto \overline{a}$, the set of terms in $\sigma^{n-1}\mu^2\lambda_i\lambda_j(\lambda \star g_{ij})$ and $(\lambda \star \overline{g_{ij}}(\sigma x))$ achieving the minimum valuation will be the same.

By assumption $[\tilde{f}]_{\mathfrak{m}}$ has the form $a_T x^T + a_{T'}x^{T'}$ for some $a_T, a_{T'} \in \Bbbk^*$. Then

$$\begin{aligned}[\Delta_{ij}(\tilde{f})]_{\mathfrak{m}} &= \Delta_{ij}([\tilde{f}]_{\mathfrak{m}})\\ &= \Delta_{ij}(a_T x^T + a_{T'}x^{T'})\\ &= (a_T x^T)\cdot(a_{T'}x^{T'\setminus\{i,j\}}) = a_T a_{T'} x^{2(T\cap T')} x^{T\Delta T' \setminus\{i,j\}}\end{aligned}$$

Since $T\Delta T' \setminus \{i,j\}$ has at least one element, we see that this polynomial cannot factor as $\tilde{g}\tilde{h}$ for multiaffine polynomials $\tilde{g}$, $\tilde{h}$ with the same support, contradicting Lemma 5.6.5. $\square$

5.6.1 Examples

Example 5.6.6. One motivating example is the field $K = \mathbb{C}\{\{t\}\}$ of complex Puiseux series with complex conjugation, where $\overline{t} = t$. Here the fixed field is $F = \mathbb{R}\{\{t\}\}$. The value group is $\Gamma = \mathbb{Q}$. Consider the 3×3 Hermitian matrix

$$A = \begin{pmatrix} t & i & i \\ -i & t & i \\ -i & -i & t \end{pmatrix}$$

with entries in $K = \mathbb{C}\{\{t\}\}$. The principal minors of A are encoded by the coefficients of

$$f = \det(\mathrm{diag}(x_1, x_2, x_3)+A) = x_1x_2x_3+t(x_1x_2+x_1x_3+x_2x_3)+(-1+t^2)(x_1+x_2+x_3)+(-3t+t^3).$$

In particular, $p_S = \nu(A_S)$ equals 0 for $|S| \in \{0, 2\}$, 1 for $|S| \in \{1, 3\}$. In this case $\Delta_{12}(f)$ factors as

$$\Delta_{12}(f) = x_3^2 + 2tx_3 + 1 + t^2 = (-1 + it + ix_3)(-1 - it - ix_3)$$

Its image modulo $\langle t \rangle$ in $\mathbb{C}[x_1, x_2, x_3]$ factors as $(-1 + ix_3)(-1 - ix_3)$.

Example 5.6.7. Consider $K = \mathbb{Q}[\alpha]/\langle \alpha^2 + \alpha + 1 \rangle$ with the 2-adic valuation on $\mathbb{Q}$ extended by $\nu(\alpha) = 0$. That is, $\nu(a + b\alpha) = \min\{\nu(a), \nu(b)\}$ where $a, b \in \mathbb{Q}$. This has an involution given by $\overline{\alpha} = -1 - \alpha$ with fixed field $F = \mathbb{Q}$. The value group is $\Gamma = \mathbb{Z}$. As discussed in the introduction to this section, we could take a transcendental extension of K if needed to make the value group dense in $\mathbb{R}$. Note that $\alpha \mapsto -1-\alpha$ gives a ring automorphism of the valuation ring $\mathcal{O}$ under which $\mathfrak{m} = \langle 2 \rangle$ is invariant. The residue field $\Bbbk$ is then $\mathbb{F}_2[\alpha]/\langle \alpha^2 + \alpha + 1 \rangle$ with induced involution $\alpha \mapsto 1 + \alpha$. Consider the 3×3 Hermitian matrix

$$A = \begin{pmatrix} 1 & 1 + 2\alpha & 1 + 2\alpha \\ -1 - 2\alpha & 1 & 1 + 2\alpha \\ -1 - 2\alpha & -1 - 2\alpha & 1 \end{pmatrix}$$

with entries in $K = \mathbb{Q}[\alpha]/\langle \alpha^2 + \alpha + 1 \rangle$. The principal minors of A are encoded by the coefficients of

$$f = \det(\mathrm{diag}(x_1, x_2, x_3) + A) = x_1x_2x_3 + x_1x_2 + x_1x_3 + x_2x_3 - 2(x_1 + x_2 + x_3) - 8.$$

In particular, $p_S = \nu(A_S)$ equals 0 for $|S| \in \{0, 1\}$, 1 for $|S| = 2$ and 3 for $|S| = 3$.

In this case $\Delta_{12}(f)$ factors as

$$\Delta_{12}(f) = 3(x_3^2 + 2x_3^2 + 4) = ((1+2\alpha)x_3 - 2 + 2\alpha)((-1-2\alpha)x_3 - 4 - 2\alpha)$$

Its image modulo 2 in $\Bbbk[x_1, x_2, x_3]$ factors as $(x_3)^2$.

On the other hand, consider the 3×3 skew-Hermitian matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1+2\alpha \\ -1 & 1+2\alpha & 0 \end{pmatrix}$$

with entries in $K = \mathbb{Q}[\alpha]/\langle \alpha^2 + \alpha + 1 \rangle$. Note that $\overline{1+2\alpha} = 1 + 2(-1-\alpha) = -(1+2\alpha)$. The principal minors of A are encoded by the coefficients of

$$f = \det(\operatorname{diag}(x_1, x_2, x_3) + A) = x_1x_2x_3 + 3x_1 + x_2 + x_3 + 2 - 4\alpha.$$

In particular, $p_S = \nu(A_S)$ equals 0 for $|S| \in \{0, 2\}$, ∞ for $|S| = 1$ and 1 for $|S| = 3$.

Example 5.6.8. To see that the assumption $\nu(a) = \nu(\overline{a})$ is necessary for Theorem 5.6.1, consider the field $K = \mathbb{Q}[\alpha]/\langle \alpha^2 + \alpha + 2 \rangle$ with the 2-adic valuation. For any extension of the 2-adic valuation on $\mathbb{Q}$ to K, $x^2 + x + 2$ has one root of valuation 1, which we can take to be α, and one of valuation 0, which is then $-1-\alpha$. Then $\overline{\alpha} = -1-\alpha$ extends to an automorphic involution on K. We see that $\nu(\alpha) \neq \nu(\overline{\alpha})$. Consider the matrix

$$A = \begin{pmatrix} 4 & 4+\alpha & 3-\alpha \\ 3-\alpha & 4 & 4+\alpha \\ 4+\alpha & 3-\alpha & 4 \end{pmatrix}.$$

Note that $\overline{A} = A^T$. The principal minors of A are encoded by the coefficients of

$$f = \det(\operatorname{diag}(x_1, x_2, x_3) + A) = x_1x_2x_3 + 4(x_1x_2 + x_1x_3 + x_2x_3) + 2(x_1 + x_2 + x_3) - 55.$$

Here $\nu(A_S)$ equals 0 for $|S| \in \{0, 3\}$, 2 for $|S| = 1$ and 1 for $|S| = 2$. The regular subdivision induced by the function $S \mapsto \nu(A_S)$ has an edge $\{(0,0,0), (1,1,1)\}$, showing that $p_S = \nu(A_S)$ is not a valuated Δ-matroid.

5.6.2 Representability in low dimensions

We next discuss representability of valuated Δ-matroids in low dimensions.

Proposition 5.6.9. *Every element $p \in \Delta\text{-Dr}(3)$ with $p_\emptyset = 0$ is realizable as the valuation of the principal minors of a Hermitian matrix.*

Proof. Suppose $p \in \Delta\text{-Dr}(3)$ and $p_\emptyset = 0$. By Lemma 5.2.1, the minimum of $2p_\emptyset + 2p_{123}$, $2p_1 + 2p_{23}$, $2p_2 + 2p_{13}$, $2p_3 + 2p_{12}$, $p_\emptyset + p_{12} + p_{13} + p_{23}$, and $p_1 + p_2 + p_3 + p_{123}$ is achieved on at least two of its arguments, or on one of the last two arguments.

To obtain the result, we can work over the field of real Hahn series $F = \mathbb{R}[\![t^{\mathbb{R}}]\!]$, whose nonzero elements consist of formal power series $\gamma = \sum_{r\in\mathbb{R}} c_r t^r$ for which $c_r \in \mathbb{R}$ and $\operatorname{supp}(\gamma) = \{r : c_r \neq 0\}$ is well ordered. This is a real closed field whose value group is all of $\mathbb{R}$. See, e.g., [Rib92, (6.10)]. The valuation of an element $\sum_{r\in\mathbb{R}} c_r t^r$ is $\min\{r : c_r \neq 0\}$. Such an element is positive (or negative) if the coefficient of its term with minimum valuation is positive (or negative, respectively). This is the fixed field of $K = \mathbb{C}[\![t^{\mathbb{R}}]\!]$ under the automorphic involution induced by usual complex conjugation on $\mathbb{C}$.

By [AAV24a], the image of 3×3 Hermitian matrices (over $\mathbb{C}[\![t^{\mathbb{R}}]\!]$) under the principal minor map is defined by the equation $a_\emptyset = 1$ and the inequalities $a_i a_j - a_{ij} \geq 0$, $a_{ik}a_{jk} - a_k a_{ijk} \geq 0$ for all distinct i, j, k, and $\operatorname{HypDet}(\mathbf{a}) \leq 0$ where

$$\begin{aligned}\operatorname{HypDet}(\mathbf{a}) = {} & a_\emptyset^2 a_{123}^2 + a_1^2 a_{23}^2 + a_2^2 a_{13}^2 + a_3^2 a_{12}^2 - 2a_\emptyset a_1 a_{23} a_{123} - 2a_\emptyset a_2 a_{13} a_{123} - 2a_\emptyset a_3 a_{12} a_{123} \\ & - 2a_1 a_2 a_{13} a_{23} - 2a_1 a_3 a_{12} a_{23} - 2a_2 a_3 a_{12} a_{13} + 4a_\emptyset a_{12} a_{13} a_{23} + 4a_1 a_2 a_3 a_{123}.\end{aligned}$$

Thus it suffices to show there exists $(a_S)_{S\subseteq 3}$ with $a_S \in \mathbb{R}[\![t^{\mathbb{R}}]\!]$ such that $\nu(a_S) = p_S$ for all S and such that $a_\emptyset = 1$, $a_i a_j - a_{ij} \geq 0$, and $a_{ik}a_{jk} - a_k a_{ijk} \geq 0$ for all i, j, k, and $\operatorname{HypDet}(\mathbf{a}) \leq 0$. We construct this explicitly as follows. Define $a_i = -t^{p_i}$ and $a_{ij} = -t^{p_{ij}}$ for all i, j, and define $a_\emptyset = 1$ and $a_{123} = t^{p_{123}}$. Note that $\nu(a_S) = p_S$ for all S. By our choice of signs,

$$a_i a_j - a_{ij} = t^{p_i + p_j} + t^{p_{ij}} \geq 0 \quad \text{and} \quad a_{ik}a_{jk} - a_k a_{ijk} = t^{p_{ik}+p_{jk}} + t^{p_k + p_{123}} \geq 0.$$

Finally, we claim that $\mathrm{HypDet}(\mathbf{a}) \leq 0$. Note that for our choice of $\mathbf{a}$, $\mathrm{HypDet}(\mathbf{a}) =$

$$t^{2p_{123}} + t^{2p_1+2p_{23}} + t^{2p_2+2p_{13}} + t^{2p_3+2p_{23}} - 2t^{p_1+p_{23}+p_{123}} - 2t^{p_2+p_{13}+p_{123}} - 2t^{p_3+p_{12}+p_{123}}$$
$$- 2t^{p_1+p_2+p_{13}+p_{23}} - 2t^{p_1+p_3+p_{12}+p_{23}} - 2t^{p_2+p_3+p_{12}+p_{13}} - 4t^{p_{12}+p_{13}+p_{23}} - 4t^{p_1+p_2+p_3+p_{123}}.$$

Let m be the minimum from Lemma 5.2.1. Note that m is also the minimum valuation of the terms appearing in the above expression for $\mathrm{HypDet}(\mathbf{a})$. Consider the set $\mathcal{S} = \{S \subset [3] : |S| \leq 1, p_S + p_{S^C} = m\}$. If one of the last two terms of the minimum appearing in Lemma 5.2.1 has valuation m, then the coefficient t^m in the expression for $\mathrm{HypDet}(\mathbf{a})$ above is at most $-4 + |\mathcal{S}| - 2\binom{|\mathcal{S}|}{2} < 0$, showing that $\mathrm{HypDet}(\mathbf{a}) < 0$.

If neither of the last two terms of the minimum appearing in Lemma 5.2.1 has valuation m, then, since $p \in \Delta\text{-Dr}(3)$, $|\mathcal{S}| \geq 2$ and the coefficient of t^m in $\mathrm{HypDet}(\mathbf{a})$ is $|\mathcal{S}| - 2\binom{|\mathcal{S}|}{2}$. This is strictly negative when $|\mathcal{S}| \geq 3$, implying $\mathrm{HypDet}(\mathbf{a}) < 0$. If $|\mathcal{S}| = 2$, then the coefficient of t^m in $\mathrm{HypDet}(\mathbf{a})$ is zero. The only remaining terms with positive coefficients are $t^{2(p_S+p_{S^C})}$ for $S \subseteq [3]$ with $S \notin \mathcal{S}$. Let $m' = \min\{2(p_S + p_{S^C}) : S \subseteq [3], |S| \leq 1, S \notin \mathcal{S}\}$. Note that $-2t^{(m+m')/2}$ appears in the above expression for $\mathrm{HypDet}(\mathbf{a})$ and has valuation strictly less than that of $t^{m'}$ and therefore less that of any term with positive coefficient. It follows that the leading term of $\mathrm{HypDet}(\mathbf{a})$ has negative coefficient and thus $\mathrm{HypDet}(\mathbf{a}) < 0$. □

We note that for $n \geq 4$, there are valuated Δ-matroids that are not representable as the valuations of principal minors of any matrix. This follows from the dimension counts in Section 5.5.

Corollary 5.6.10. *There exists $p \in \Delta\text{-Dr}(4)$ with $p_\emptyset = 1$ that is not achieved as the valuations of principal minors of a 4×4 matrix over K.*

Proof. By [Sto24], over any field K, the image of the principal minors of an $n \times n$ matrix are contained in a variety of dimension $n^2 - n + 1$. By the structure theorem in tropical geometry [MS15, Thm 3.3.5], the image of this set under coordinate-wise valuation has dimension at most $n^2 - n + 1$. In particular, for $n = 4$, this has dimension 13.

On the other hand, by Proposition 5.5.1, Δ-Dr(4) has codimension one in $\mathbb{R}^{16}$. It is not difficult to check that the set of $p \in \Delta$-Dr(4) with $p_\emptyset = 0$ therefore has dimension 14. $\square$

5.6.3 Extensions of Theorem 5.6.1 and conjectures

By taking any valued field with the trivial involution $\overline{a} = a$, we obtain the following from Theorem 5.6.1.

Corollary 5.6.11. *The valuations of the principal minors of symmetric and skew-symmetric matrices over K are valuated Δ-matroids.*

In fact, the above result can be extended to matrices which are skew-symmetric plus a rank one symmetric matrix.

Corollary 5.6.12. *Let $A = B + C$ where $B \in K^{n\times n}$ is skew-symmetric, and $C \in K^{n\times n}$ is symmetric of rank one. Then the valuations of the principal minors of A is a valuated Δ-matroid.*

Proof. Let $K(t)$ be the transcendental field extension of K by an element t. Invoking [EP05, Theorem 2.2.1], we can extend ν to a valuation on $K(t)$ by defining $\nu(\sum_i a_i t^i) = \min_i\{\nu(a_i)\}$ and $\nu(f/g) = \nu(f) - \nu(g)$. The field $K(t)$ has an automorphic involution $t \mapsto -t$ that preserves the valuation.

Define $A' = tB + C$. This is a matrix over $K(t)$ which is Hermitian with respect to the above involution. We claim that $\nu(A_S) = \nu(A'_S)$ for all $S \subset [n]$. Since the valuations of principal minors of A' is a valuated Δ-matroid by Theorem 5.6.1, this will prove the desired result.

We can write $C = \alpha v v^T$ where $\alpha \in K$ and $v \in K^n$. Let $B[S]$ denote the principal submatrix of B with rows and columns indexed by S, so that $B_S = \det(B[S])$. Similarly, for the vector $v \in K^n$, let $v[S]$ denote the subvector consisting of with entries of v indexed by S. Then $A[S] = B[S] + \alpha v[S]v[S]^T$ and $A'[S] = tB[S] + \alpha v[S]v[S]^T$. Using the matrix determinant lemma, we have

$$A_S = B_S + \alpha v[S]^T \mathrm{adj}(B[S])v[S] \quad \text{and} \quad A'_S = t^{|S|}B_S + \alpha t^{|S|-1}v[S]^T \mathrm{adj}(B[S])v[S].$$

where $\operatorname{adj}(M)$ is the adjugate matrix of M. If $|S|$ is odd, then $B_S = 0$ since $B[S]$ is skew-symmetric. In this case $A_S = \alpha v[S]^T \operatorname{adj}(B[S]) v[S]$ and $A'_S = t^{|S|-1} A_S$, showing that $\nu(A_S) = \nu(A'_S)$, as desired. If $|S|$ is even, then $\operatorname{adj}(B[S])$ is skew-symmetric, so $v[S]^T \operatorname{adj}(B[S]) v[S] = 0$. In this case $A_S = B_S$ and $A'_S = t^{|S|} B_S$, again showing that $\nu(A_S) = \nu(A'_S)$. □

We conjecture that we can extend Corollary 5.6.12 as follows.

Conjecture 5.6.13. *Let K be a valued field with an automorphic involution preserving the valuation. Let $A = B + C$ where $B \in K^{n\times n}$ is skew-Hermitian, and $C \in K^{n\times n}$ is Hermitian of rank one. Then the valuations of the principal minors of A form a valuated Δ-matroid.*

As discussed in the next section, this conjecture has implications for the representability of Δ-matroids by isotropic subspaces of Hermitian spaces. We will prove the conjecture in the case where the residue field of K does not have characteristic 2.

Proposition 5.6.14. *Conjecture 5.6.13 is true if the residue field of K does not have characteristic 2.*

We begin by proving the following.

Lemma 5.6.15. *Let K be a field with nonarchimedean valuation ν and automorphic involution $a \mapsto \overline{a}$ preserving the valuation. Assume the residue field of K does not have characteristic 2. Let $a, b \in K$ such that $\overline{a} = a$ and $\overline{b} = -b$. Then $\nu(a+b) = \min(\nu(a), \nu(b))$.*

Proof. We only need to prove this when $\nu(a) = \nu(b)$. Since the residue field of K does not have characteristic 2, we have $\nu(2) = 0$. Therefore,

$$\nu(a) = \nu(2a) = \nu((a+b) + (a-b)) \geq \min(\nu(a+b), \nu(a-b)) = \nu(a+b)$$

where the last equality holds since $\overline{a+b} = a - b$. Thus $\nu(a+b) \leq \min(\nu(a), \nu(b))$ since $\nu(a) = \nu(b)$. Hence $\nu(a+b) = \min(\nu(a), \nu(b))$. □

Proof of Proposition 5.6.14. Let $K(t)$ be the transcendental field extension of K by an element t, and extend the valuation on K to $K(t)$ as in the proof of Corollary 5.6.12. Equip

$K(t)$ with the automorphic involution sending t to $-t$ and a to $\overline{a}$ for all $a \in K$. This involution preserves the valuation on $K(t)$.

Define $A' = tB + C$. This is a matrix over $K(t)$ which is Hermitian with respect to the above involution. We claim that $\nu(A_S) = \nu(A'_S)$ for all $S \subset [n]$, which would give the desired result by Theorem 5.6.1.

Let F be the fixed field of the involution on K. We can write $C = \alpha v \overline{v}^T$ where $\alpha \in F$ and $v \in K^n$. As in the proof of Corollary 5.6.12, we have

$$A_S = B_S + \alpha\overline{v}[S]^T \mathrm{adj}(B[S])v[S] \quad \text{and} \quad A'_S = t^{|S|}B_S + \alpha t^{|S|-1}\overline{v}[S]^T \mathrm{adj}(B[S])v[S].$$

Let G denote the set of all $a \in K$ such that $\overline{a} = -a$. If $|S|$ is odd, then since B is skew-Hermitian, we have $B_S \in G$ and $\alpha\overline{v}[S]^T \mathrm{adj}(B[S])v[S] \in F$. If $|S|$ is even, then we have $B_S \in F$ and $\alpha\overline{v}[S]^T \mathrm{adj}(B[S])v[S] \in G$. Either way, by Lemma 5.6.15, we have

$$\nu(A_S) = \min(\nu(B_S), \nu(\alpha\overline{v}[S]^T \mathrm{adj}(B[S])v[S])).$$

By definition, this also the valuation of A'_S, completing the proof. □

When $\operatorname{char} K = 2$, Conjecture 5.6.13 is true because then A is Hermitian. In this case we make another conjecture. Assume $\operatorname{char} K = 2$. Let F be the fixed field of the involution on K, and assume $F \neq K$. By standard Galois theory, we have $K = F[\omega]$ where $\omega^2 + \omega + \beta = 0$ for some $\beta \in F$, and $\overline{\omega} = \omega + 1$.

Conjecture 5.6.16. *Let $K = F[\omega]$ be as above. Let $A = B + \omega C$ where B, $C \in K^{n\times n}$ are Hermitian and C has rank one. Then the valuations of the principal minors of A form a valuated Δ-matroid.*

Proposition 5.6.17. *Conjecture 5.6.16 is true if the valuation is trivial.*

Proof. Assume the valuation ν is trivial. Then for all a, $b \in F$ we have $\nu(a + b\omega) = \min(\nu(a), \nu(b\omega))$. Indeed, $a + b\omega = 0$ if and only if a and $b\omega$ are both 0. The result then follows from an argument similar to the proof of Proposition 5.6.14. □

5.7 Representability via sesquilinear forms

We now restate Theorem 5.6.1 in the language of sesquilinear forms and isotropic Grassmannians. This work was initiated by Gelfand and Serganova, who proved the below results for symplectic and quadratic forms on trivially-valued complex vector spaces. Their results were extended to arbitrary fields in [BGW03] and [EFLS24]. The purpose of this section is to show that these results extend to non-trivial valuations and Hermitian forms.

We begin by recalling some polar geometry. We refer to [Cam00] for further information. Let K be a field equipped with an automorphic involution $\iota : K \to K$, possibly the identity. We write $\overline{a}$ for $\iota(a)$. Let V be a finite-dimensional vector space over K. A *sesquilinear form* on V is a biadditive function $b : V \times V \to K$ satisfying $b(\alpha x, \beta y) = \alpha\overline{\beta}b(x, y)$ for all α, $\beta \in K$ and x, $y \in V$. A sesquilinear form is *nondegenerate* if for all $x \in V \setminus \{0\}$ neither of the functions $b(x, \cdot)$ and $b(\cdot, x)$ are identically 0. A sesquilinear form is *reflexive* if $b(x, y) = 0$ implies $b(y, x) = 0$. A sesquilinear form is *Hermitian* if $b(x, y) = \overline{b(y, x)}$.

We have the following characterization of nondegenerate reflexive sesquilinear forms.

Theorem 5.7.1. *[Cam00, Thm 3.3] Let b be a nondegenerate reflexive sesquilinear form over a field with automorphic involution ι. Then one of the following is true.*

1. *ι is the identity, and b is alternating ($b(x, x) = 0$).*

2. *ι is the identity, and b is symmetric ($b(x, y) = b(y, x)$).*

3. *ι is not the identity, and b is a scalar multiple of a Hermitian sesquilinear form.*

In Case 1 we call b a *symplectic form*. Case 2 is usually not dealt with in full generality due to issues when $\operatorname{char} K = 2$. Instead, we introduce quadratic forms. A *quadratic form* on a vector space V over K is a function $q : V \to K$ for which there exists a bilinear form b on V satisfying

$$q(\alpha x + \beta y) = \alpha^2 q(x) + \alpha\beta b(x, y) + \beta^2 q(y)$$

for all α, $\beta \in K$ and x, $y \in V$. We necessarily have

$$b(x, y) = q(x + y) - q(x) - q(y)$$

so b is a symmetric bilinear form determined by q. We call b the bilinear form associated to q. If $\operatorname{char} K \neq 2$, then q is also determined by b as $q(x) = \frac{1}{2}b(x, x)$. However, if $\operatorname{char} K = 2$ there can be multiple quadratic forms associated to a bilinear form. A quadratic form q with associated bilinear form b is *nondegenerate* if there is no $x \in V \setminus \{0\}$ such that $b(x, \cdot) = 0$ is identically 0 and $q(x) = 0$. (If $\operatorname{char} K = 2$, a nondegenerate quadratic form can have a degenerate associated bilinear form.)

From now on, we will use the term "Hermitian" to mean we are in Case 3, that is the involution ι is not the identity.

Let V be a finite-dimensional vector space equipped with a (nondegenerate) symplectic form b, a nondegenerate quadratic form q with associated bilinear form b, or a nondegenerate Hermitian form b. In what follows, we will make statements for the symplectic and Hermitian cases, followed in parentheses by the corresponding statement for the quadratic case if different. An *isotropic subspace* of V is a linear subspace $W \leq V$ such that b (respectively, q) vanishes on W. Witt's theorem implies that all maximal isotropic subspaces of V have the same dimension, called the *Witt index* of V. If the Witt index is n, then $n \leq \dim V/2$, and we can write

$$V = H_1 \oplus \cdots \oplus H_n \oplus W$$

where the summands are pairwise orthogonal with respect to b, W is anisotropic (that is, $b(x, x) \neq 0$ (respectively, $q(x) \neq 0$) for all $x \in W \setminus \{0\}$), and each H_i has a basis $\{e_i, f_i\}$ with $b(e_i, e_i) = b(f_i, f_i) = 0$ ($q(e_i) = q(f_i) = 0$) and $b(e_i, f_i) = 1$ for all i. The space of all maximal isotropic subspaces of V is called the *maximal isotropic Grassmannian.*

From now on assume V has Witt index $n = \lfloor \dim V/2 \rfloor$. Let $\{e_1, \ldots, e_n, f_1, \ldots, f_n\} \cup G$ be a basis of V where e_i and f_i are as described above and G is a basis of W. If $\dim V$ is odd, then we let $G = \{g\}$ where $b(g, g) = \alpha \neq 0$ ($q(g) = \alpha \neq 0$). Any maximal isotropic subspace L can be represented by a $n \times (\dim V)$ full-rank matrix where the rows of the matrix form a basis for L in the coordinates $(e_1, \dots, e_n, f_1, \dots, f_n) \cup G$. Let $\nu : K \to \mathbb{R} \cup \{\infty\}$ be a nonarchimedian valuation with $\nu(a) = \nu(\overline{a})$. For each $S \subseteq [n]$, let

$$p_S(L) := \nu\left(\det M[\{e_i : i \in S\} \cup \{f_i : i \in [n] \setminus S\}]\right),$$

that is, the valuation of the minor of M using the columns corresponding to e_i with $i \in S$ and f_i with $i \in [n] \setminus S$. Our result is as follows.

Theorem 5.7.2. *Let V be as above. If ι is not the identity, assume V has even dimension. Then for any maximal isotropic subspace L, the function $S \mapsto p_S(L)$ is a valuated Δ-matroid.*

For example, if ν is trivial, then $p_S(L)$ defines an ordinary Δ-matroid. These Δ-matroids in the cases where V is quadratic of odd dimension, symplectic, or quadratic of even dimension are known in the literature as representable of type B, C, and D respectively. For V Hermitian of even dimension, the Δ-matroids we obtain are, up to a symmetry of the hypercube, those obtained from Hermitian matrices as described in the previous section.

We conjecture that Theorem 5.7.2 can be extended to odd-dimensional Hermitian V.

Conjecture 5.7.3. *The conclusion of Theorem 5.7.2 holds when ι is not the identity and V has odd dimension.*

We do not have a full proof of this. See Proposition 5.7.4 for partial progress.

We split the proof of Theorem 5.7.2 into the symplectic, quadratic, and Hermitian cases. In all cases, we write $V = H_1 \oplus \cdots \oplus H_n \oplus W$ as above.

Symplectic case

In this case, we necessarily have $W = \{0\}$ and $\dim V = 2n$. Let M be a matrix representing L. Note that the projection of L onto any H_i is nontrivial, because otherwise L is contained in a space of smaller Witt index, contradicting the fact that L is a maximal isotropic subspace. Hence, by swapping e_i and f_i for some set of i, the last n columns of M can be made linearly independent. Without loss of generality we may do this, because this corresponds to precomposing the function $p_S(L)$ with $S \mapsto S\Delta i$, and the new function is a Δ-matroid if and only if the old one is. Performing row operations, we may assume M is of the form $\begin{pmatrix} A & I \end{pmatrix}$, where I is the $n \times n$ identity matrix. Then b vanishes on L if and only if A is symmetric. Moreover, $p_S(L)$ is exactly $\nu(A_S)$. Hence, $p_S(L)$ is the valuation of the principal minors of a symmetric matrix A, which is a valuated Δ-matroid by Theorem 5.6.11.

Quadratic case

Let M be a matrix representing L. Assume first that $\dim V$ is even. Then in the coordinates of the chosen basis, we have

$$q(x_1, \ldots, x_n, y_1, \ldots, y_n) = \sum_{i=1}^{n} x_i y_i.$$

As above we may assume $M = \begin{pmatrix} A & I \end{pmatrix}$. Let A_i denote the i-th column of A. Then q vanishes on L if and only if for all $u \in K^n$,

$$q(u^T M) = 0 \iff \sum_{i=1}^{n} (u^T A_i) u_i = 0 \iff u^T A u = 0.$$

The last equation is true for all u if and only if A is skew-symmetric. Hence $p_S(L)$ is the valuation of the principal minors of a skew-symmetric matrix A, which is a valuated Δ-matroid by Theorem 5.6.11.

Now suppose $\dim V$ is odd. In the coordinates of the chosen basis, we have

$$q(x_1, \ldots, x_n, y_1, \ldots, y_n, z) = \sum_{i=1}^{n} x_i y_i + \alpha z^2.$$

where $\alpha \neq 0$. We may assume $M = \begin{pmatrix} A & I & v \end{pmatrix}$ where v is a column vector. Then q vanishes on L if and only if for all $u \in K^n$,

$$q(u^T M) = 0 \iff \sum_{i=1}^{n} (u^T A_i) u_i + \alpha (u^T v)^2 = 0 \iff u^T (A + \alpha v v^T) u = 0$$

The last equation is true for all u if and only if $A + \alpha v v^T$ is skew-symmetric. Hence we can write $A = B - \alpha v v^T$ where B is a skew-symmetric matrix. The result then follows from Corollary 5.6.12.

Hermitian case

First assume $\dim V$ is even. Let M be the matrix representing L. We may assume $M = \begin{pmatrix} A & I \end{pmatrix}$. Then b vanishes on L if and only if A is skew-Hermitian. The result then follows from Theorem 5.6.11. This concludes the proof of Theorem 5.7.2.

5.7.1 Partial results for odd-dimensional Hermitian

We make the following partial progress toward proving Conjecture 5.7.3.

Proposition 5.7.4. *Conjecture 5.7.3 is true in the following cases.*

1. *The residue field of K does not have characteristic 2.*

2. *The valuation is trivial.*

Proof. Recall that V is an odd-dimensional vector space over K with non-degenerate Hermitian form b. In the coordinates of the chosen basis, we have

$$b((x_1,\dots,x_n,y_1,\dots,y_n,z),(x_1',\dots,x_n',y_1',\dots,y_n',z')) = \sum_{i=1}^n x_i\overline{y_i'} + \sum_{i=1}^n y_i\overline{x_i'} + \alpha z\overline{z'}$$

for some α in the fixed field F. Let M be the matrix representing L. We may assume $M = \begin{pmatrix} A & I & v \end{pmatrix}$ where v is a column vector. Then b vanishes on L if and only if $A + \overline{A^T} + \alpha v\overline{v^T} = 0$.

First assume $\operatorname{char} \Bbbk \neq 2$. It follows that $\operatorname{char} K \neq 2$. Thus b vanishes on L if and only if $A + \frac{1}{2}\alpha v\overline{v^T}$ is skew-Hermitian. Hence $A = B - \frac{1}{2}\alpha v\overline{v^T}$ for some skew-Hermitian B, and the result follows by Proposition 5.6.14.

Now assume the valuation ν is trivial. By the previous case, we only need to consider the case where $\operatorname{char} K = 2$. As discussed before Conjecture 5.6.16, since ι is not the identity we have $K = F[\omega]$ where $\overline{\omega} = \omega + 1$. Then b vanishes on L if and only if $A + \omega\alpha v\overline{v^T}$ is Hermitian. Thus $A = B + \omega\alpha v\overline{v^T}$ for some Hermitian matrix B. The result then follows from Proposition 5.6.17. □

Remark 5.7.5. *As seen from the above proof, Conjecture 5.7.3 is equivalent to Conjecture 5.6.13 when* $\operatorname{char} K \neq 2$*, and is equivalent to Conjecture 5.6.16 when* $\operatorname{char} K = 2$*.*

Chapter 6

(MIS)ADVENTURES IN FRACTIONAL LOG CONCAVITY

In this chapter, we study even Δ-matroids, Hurwitz stability, and fractional log-concavity. Fractionally log-concave polynomials (and their connection to sector-stable polynomials) were first introduced in [AASV21]. Similar to the results in [AVLOV21], fractional log-concavity gives us bounds on the mixing time of a down-up random walk. Also, similar to discrete ultra log-concavity of the coefficients of Lorentzian polynomials (see, e.g. [Gur09, Proposition 2.7] [ALOV24b, Proposition 5.1]), a version of strong 1/2-log concavity implies a version discrete log-concavity on the coefficients.

The ultimate motivation is to prove Conjecture 6.1.4 stating that homogenized generating polynomials of even Δ-matroids are 1/2-log concave, which remains open. We begin by defining the objects under study as well as some motivation for these definitions, including applications to mixing times of Markov chains (Section 6.1.1) and discrete log-concavity (Section 6.1.3). In Section 6.2, we then discuss some positive results that may serve as stepping stones towards a proof of Conjecture 6.1.4. Finally, in Section 6.3, we discuss some of the difficulties that arise in trying to prove the full conjecture, and in particular, we highlight a few of the ways in which the story of Lorentzian polynomials fails to generalize to fractional log-concavity.

6.1 Preliminaries

For a Δ-matroid with bases $\mathcal{B}$, we consider its generating polynomial

$$f_{\mathcal{B}} = \sum_{S \in \mathcal{B}} x^S.$$

Definition 6.1.1. If $f \in \mathbb{R}[x_1, \ldots, x_n]$ is a multiaffine polynomial, we define the *multiaffine homogenization* of f,

$$f^{\text{hom}} = y_1 \ldots y_n f(x_1/y_1, \ldots, x_n/y_n).$$

Note that for Δ-matroid generating polynomials, we have

$$f_{\mathcal{B}}^{\text{hom}} = \sum_{S \in \mathcal{B}} x^S y^{[n] \setminus S}.$$

In particular, twisting $\mathcal{B}$ by i simply interchanges the roles of x_i and y_i in $f_{\mathcal{B}}^{\text{hom}}$.

Remark 6.1.2. *As a notational convention, in this section, we will use $\partial_{\bar{i}}$ to denote both $\partial_{\bar{i}} f := f|_{x_i=0}$ as an operator on $\mathbb{R}[x_1, \ldots, x_n]$, as well as $\partial_{\bar{i}} := \frac{\partial}{\partial y_i}$ as an operator on $\mathbb{R}[x_1, \ldots, x_n, y_1, \ldots, y_n]$.*

The aim of this project concerns fractional log-concavity of the multiaffine homogenizations of generating polynomials of even Δ-matroids. (For a discussion on why we limit ourselves to only even Δ-matroids, see Section 6.3.1.)

Definition 6.1.3. For $0 < \alpha \leq 1$, we say $f \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ is *α-log concave* if

$$f_\alpha(x_1, \ldots, x_n) := f(x_1^\alpha, \ldots, x_n^\alpha)$$

is log concave as a function on $\mathbb{R}^n_{>0}$.

Note that this is a generalization of log-concavity, which corresponds to the case $\alpha = 1$.

It is conjectured that generating polynomials of even Δ-matroids give rise to 1/2-log concave polynomials. A weaker form of this conjecture with simply $\alpha = O(1)$ independent of the size of the ground set n first appeared in [AASV21].

Conjecture 6.1.4. *If $([n], \mathcal{B})$ is an even Δ-matroid, then*

$$f_{\mathcal{B}}^{\text{hom}} = \sum_{B \in \mathcal{B}} x^B y^{[n] \setminus B}$$

is 1/2-log concave.

As of now, the main tool we have for understanding fractional log-concavity is sector stability, which gives a sufficient condition for fractional log-concavity analogous to the connection between real stable and Lorentzian polynomials.

Definition 6.1.5. We say $f \in \mathbb{R}[x_1, \ldots, x_n]$ is α-sector stable if $f(x_1, \ldots, x_n) \neq 0$ whenever $x_1, \ldots, x_n \in \Gamma_\alpha := \{re^{i\theta} : r > 0, -\alpha\pi/2 < \theta < \alpha\pi/2\}$.

If $\alpha = 1$, this means that f is nonvanishing whenever $\text{Re}(x_1), \ldots, \text{Re}(x_n) > 0$, which is also known as *Hurwitz stability*.

For the case of multiaffine polynomials with even support on four variables, we already have a full characterization for Hurwitz stability in terms of four-term triangle inequalities. In Theorem 6.1.17, we show that in this setting, Hurwitz stability is actually equivalent to 1/2-log concavity.

Proposition 6.1.6 ([Vuo23])**.** *Let $f = \sum_{S\subseteq[4],|S|\ even} c_S x^S \in \mathbb{R}_{\geq 0}[x_1, x_2, x_3, x_4]$. Then f is Hurwitz stable if and only if*

$$\begin{aligned}
\sqrt{c_0 c_{1234}} &\leq \sqrt{c_{12}c_{34}} + \sqrt{c_{13}c_{24}} + \sqrt{c_{14}c_{23}} \\
\sqrt{c_{12}c_{34}} &\leq \sqrt{c_0 c_{1234}} + \sqrt{c_{13}c_{24}} + \sqrt{c_{14}c_{23}} \\
\sqrt{c_{13}c_{24}} &\leq \sqrt{c_0 c_{1234}} + \sqrt{c_{12}c_{34}} + \sqrt{c_{14}c_{23}} \\
\sqrt{c_{14}c_{23}} &\leq \sqrt{c_0 c_{1234}} + \sqrt{c_{12}c_{34}} + \sqrt{c_{13}c_{24}}.
\end{aligned}$$

6.1.1 Applications to Mixing Times

Lorentzian polynomials have found applications in the theoretical computer science community due to their connections to high-dimensional expanders. In particular, [ALOV24a, Proposition 4.1] shows that the conditions required for a (multiaffine) polynomial to be Lorentzian, when translated into the language of simplicial complexes, are exactly the conditions of a 0-local spectral expander, in the sense of [Opp18]. In particular, since basis generating polynomials of matroids are Lorentzian, then the down-up random walk on the bases of a rank-d matroid has spectral gap at least $1/d$, which implies that it has mixing time at most $t_{\text{mix}}(\varepsilon) \leq d\log(\frac{|\mathcal{B}|}{\varepsilon})$ [ALOV24a, Theorem 1.1]. In particular, because of the equivalence of approximate counting and sampling [JVV86], this gives us efficient, polynomial-time algorithms for both sampling and counting the bases of a matroid.

One primary motivation for studying fractional log-concavity is that it, too, allows us to bound the mixing time for down-up random walks. Given a probability distribution

$\mu : 2^{[n]} \to \mathbb{R}_{\geq 0}$, we get an equivalent probability distribution $\overline{\mu}$ on $\binom{[n]\sqcup[n^*]}{n}$ given by

$$\overline{\mu}(S \sqcup T^*) = \begin{cases} \mu(S), & \text{if } T = [n] \setminus S \\ 0, & \text{otherwise} \end{cases}.$$

Since $\overline{\mu}$ is now a homogeneous distribution, we can then define down-up walks as before.

We note that a version of the following theorem first appeared in [AASV21, Theorem 24], proved in Section 3 of that paper. However, the argument we present below is from the later paper [AJKPV21], as it gives more explicit constants.

Theorem 6.1.7. *Suppose $\mu : 2^{[n]} \to \mathbb{R}_{\geq 0}$ is a probability distribution such that the generating polynomial*

$$g_\mu(x_1, \ldots, x_n) := \sum_{S \subseteq [n]} \mu(S) x^S$$

is α-log concave, where $1/\alpha \in \mathbb{Z}$. Then the $n \leftrightarrow (n - 1/\alpha)$ down-up walk defined by $\overline{\mu}$ starting from τ has mixing time at most

$$t_{\text{mix}}(\tau, \varepsilon) \leq O\left(n^{1/\alpha} \log(1/\varepsilon) \log\log(1/\mu(\tau))\right).$$

In particular, if μ is uniform over its support, then

$$t_{\text{mix}}(\varepsilon) \leq O\left(n^{1/\alpha} \log(1/\varepsilon) \log(n)\right).$$

Before we prove Theorem 6.1.7, we first recall some classical definitions and results on Markov chains, which we learned from [Ove24].

Recall that the *total variation distance* between two distributions on a finite state space $\mathfrak{X}$ is given by

$$\|\nu - \mu\|_{\text{TV}} = \frac{1}{2} \sum_{x \in \mathfrak{X}} |\nu(x) - \mu(x)|.$$

Definition 6.1.8 (Mixing Time)**.** For $\varepsilon > 0$, the *mixing time* of a Markov chain P with stationary distribution μ, with respect to starting distribution ν, is

$$t_{\text{mix}}(\nu, \varepsilon) = \min\{t : \|\nu P^t - \mu\|_{\text{TV}} \leq \varepsilon\}.$$

Definition 6.1.9 (KL-divergence)**.** For two discrete probability distributions ν, μ on $\mathfrak{X}$, the *KL-divergence* or *relative entropy* of ν with respect to μ is

$$\mathcal{D}_{\mathrm{KL}}(\nu\|\mu) \coloneqq \mathbb{E}_{x\sim\mu}\left[\frac{\nu(x)}{\mu(x)}\log\frac{\nu(x)}{\mu(x)}\right] = \sum_{x\in\mathfrak{X}}\nu(x)\log\frac{\nu(x)}{\mu(x)}.$$

Proposition 6.1.10 (Data Processing Inequality)**.** *For all row-stochastic matrices $P \in \mathbb{R}^{\mathfrak{X}\times\mathfrak{X}'}$ and all distributions $\nu, \mu : \mathfrak{X} \to \mathbb{R}_{\geq 0}$, we have*

$$\mathcal{D}_{\mathrm{KL}}(\nu P\|\mu P) \leq \mathcal{D}_{\mathrm{KL}}(\nu\|\mu).$$

Proposition 6.1.11 (Pinsker's Inequality)**.** *For any distributions $\nu, \mu : \mathfrak{X} \to \mathbb{R}_{\geq 0}$, the total variation distance between ν and μ is bounded by*

$$\|\nu - \mu\|_{\mathrm{TV}} \leq \sqrt{\frac{1}{2}\mathcal{D}_{\mathrm{KL}}(\nu\|\mu)}.$$

Definition 6.1.12. We say that a row-stochastic matrix $P \in \mathbb{R}_{\geq 0}^{\mathfrak{X}\times\mathfrak{X}'}$ contracts KL-divergence by a factor of α with respect to a distribution μ if for all distributions $\nu : \mathfrak{X} \to \mathbb{R}_{\geq 0}$,

$$\mathcal{D}_{\mathrm{KL}}(\nu P\|\mu P) \leq \alpha \cdot \mathcal{D}_{\mathrm{KL}}(\nu\|\mu).$$

The following lemma is well-known to experts. We first learned of it and its proof from [Ove24].

Lemma 6.1.13. *If a Markov kernel P contracts KL-divergence by a factor of $1 - \kappa$ with respect to its stationary distribution μ, then the Markov chain has mixing time*

$$t_{\mathrm{mix}}(\nu, 1/4) \leq O\left(\frac{\log(\mathcal{D}_{\mathrm{KL}}(\nu\|\mu))}{\kappa}\right).$$

Proof. We have

$$\mathcal{D}_{\mathrm{KL}}(\nu P^t\|\mu) = \mathcal{D}_{\mathrm{KL}}((\nu P^{t-1})P\|\mu P) \leq (1-\kappa)\mathcal{D}_{\mathrm{KL}}(\nu P^{t-1}\|\mu),$$

whence it follows by induction that

$$\mathcal{D}_{\mathrm{KL}}(\nu P^t\|\mu) \leq (1-\kappa)^t\mathcal{D}_{\mathrm{KL}}(\nu\|\mu).$$

By Pinsker's inequality, it follows that

$$\|\nu P^t - \mu\|_{\mathrm{TV}} \leq \sqrt{\frac{1}{2}\mathcal{D}_{\mathrm{KL}}(\nu P^t \| \mu)} \leq \sqrt{\frac{1}{2}(1-\kappa)^t \mathcal{D}_{\mathrm{KL}}(\nu\|\mu)} = \frac{1}{\sqrt{2}}(1-\kappa)^{t/2}\sqrt{\mathcal{D}_{\mathrm{KL}}(\nu\|\mu)}.$$

If $\mathcal{D}_{\mathrm{KL}}(\nu\|\mu) \leq 1/8$, then $\|\nu - \mu\|_{\mathrm{TV}} \leq 1/4$ and we are done. Otherwise, define $t^* = \frac{2}{\kappa}\ln(\sqrt{8\mathcal{D}_{\mathrm{KL}}(\nu\|\mu)}) \leq O(\frac{\log(\mathcal{D}_{\mathrm{KL}}(\nu\|\mu))}{\kappa})$. Then

$$(1-\kappa)^{t^*/2} = ((1-\kappa)^{1/\kappa})^{\ln(\sqrt{8\mathcal{D}_{\mathrm{KL}}(\nu\|\mu)})} \leq \frac{1}{2\sqrt{2\mathcal{D}_{\mathrm{KL}}(\nu\|\mu)}},$$

where the inequality follows from the fact that $\log(\sqrt{8\mathcal{D}_{\mathrm{KL}}(\nu\|\mu)}) > 0$ and $(1-\kappa)^{1/\kappa} \leq 1/e$ for all $0 < \kappa < 1$.

Therefore,

$$\|\nu P^{t^*} - \mu\|_{\mathrm{TV}} \leq 1/4$$

and hence $t_{\mathrm{mix}}(\nu, 1/4) \leq t^* \leq O(\frac{\log(\sqrt{\mathcal{D}_{\mathrm{KL}}(\nu\|\mu)})}{\kappa})$, as desired. □

We are now ready to prove our bound on mixing times of fractionally log concave distributions.

Proof of Theorem 6.1.7, following [Ove24]. It is a classical result in Markov chains (see, e.g., [LPW09, Section 4.5]) that $t_{\mathrm{mix}}(\varepsilon) \leq \lceil \log_2(1/\varepsilon)\rceil t_{\mathrm{mix}}(1/4)$, so it suffices to show that

$$t_{\mathrm{mix}}(\tau, 1/4) \leq O\left(n^{1/\alpha}\log(1/\mu(\tau))\right).$$

We note that starting our walk from state τ is the same as setting $\nu = 1_\tau$. Moreover, we note that $\mathcal{D}_{\mathrm{KL}}(1_\tau\|\mu) = \log(1/\mu(\tau))$.

Since the $k \leftrightarrow k-\frac{1}{\alpha}$ down-up random walk transition matrix is given by $P = D_{k\to k-\frac{1}{\alpha}}U_{k-\frac{1}{\alpha}\to k}$, by the data-processing inequality, we have

$$\begin{aligned}\mathcal{D}_{\mathrm{KL}}(\nu P\|\mu) &= \mathcal{D}_{\mathrm{KL}}(\nu P\|\mu P)\\ &= \mathcal{D}_{\mathrm{KL}}(\nu D_{k\to k-1/\alpha}U_{k-1/\alpha\to k}\|\mu D_{k\to k-1/\alpha}U_{k-1/\alpha\to k})\\ &\leq \mathcal{D}_{\mathrm{KL}}(\nu D_{k\to k-1/\alpha}\|\mu D_{k\to k-1/\alpha}).\end{aligned}$$

By [AJKPV22, Theorem 5] (proved in [AJKPV21]), we know that if μ is α-log concave, then for all distributions ν, we have

$$\mathcal{D}_{\mathrm{KL}}(\nu D_{k\to k-1/\alpha} \| \mu D_{k\to k-1/\alpha}) \leq \left(1 - \frac{1}{\binom{n}{1/\alpha}}\right) \mathcal{D}_{\mathrm{KL}}(\nu\|\mu) \leq \left(1 - \frac{1}{n^{1/\alpha}}\right) \mathcal{D}_{\mathrm{KL}}(\nu\|\mu).$$

Combining this with the inequality above, we get

$$\mathcal{D}_{\mathrm{KL}}(\nu P\|\mu) \leq \left(1 - \frac{1}{n^{1/\alpha}}\right) \mathcal{D}_{\mathrm{KL}}(\nu\|\mu),$$

or, in other words, P contracts KL-divergence by a factor of $1 - 1/n^{1/\alpha}$ with respect to its stationary distribution μ. Hence, by Lemma 6.1.13, we have

$$t_{\mathrm{mix}}(1_\tau, 1/4) \leq O\left(n^{1/\alpha} \log\log(1/\mu(\tau))\right),$$

as desired. □

6.1.2 Connections with Hurwitz Stability

In [AASV21], they show that sector-stability implies fractional log-concavity. In particular, a special case of their results implies that if $f \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ is Hurwitz stable, then f and its multiaffine homogenization are both 1/2-log concave. The proof in [AASV21] uses heavily probabilistic language. We reproduce their argument for the Hurwitz stable case below, rephrased into the language of polynomials and derivatives.

Theorem 6.1.14 ([AASV21, Lemma 71])**.** *If $f \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ is Hurwitz stable, then f^{hom} is 1/2-log concave.*

We first recall the following version of the Schwarz-Pick Theorem.

Theorem 6.1.15 (Schwarz-Pick Theorem)**.** *Let $f : \mathcal{H} \to \mathcal{H}$ be a holomorphic function from the upper half plane to itself. Then, for all $z \in \mathcal{H}$,*

$$\frac{|f'(z)|}{\mathrm{Im}(f(z))} \leq \frac{1}{\mathrm{Im}(z)}.$$

The bulk of the proof of Theorem 6.1.14 is in the following lemma.

Lemma 6.1.16. *Suppose $f \in \mathbb{R}[x_1, \ldots, x_n]$ is Hurwitz stable and multiaffine with even support, and let $v \in \mathbb{R}^n_+$. Suppose further that neither $\partial_j f$ nor $\partial_{\bar{j}} f$ is identically zero. Then $D_v \left(\frac{\partial_{\bar{j}} f}{\partial_j f} \right)(v) \leq \frac{\partial_{\bar{j}} f}{\partial_j f}(v)$.*

Proof. Let $g = f(-ix)$, so g is real stable. Define $h(t) = \frac{\partial_{\bar{j}} g}{\partial_j g}(tv)$. For ease of notation, let $\mathcal{H}_+$ denote the complex upper half plane. Since $\partial_j g$ is also real stable, $\partial_j g(tv) \neq 0$ whenever $t \in \mathcal{H}_+$, so h is holomorphic on the upper half plane.

Moreover, since g is real stable and multiaffine, we know that $\mathrm{Im}\left(\frac{\partial_{\bar{j}} g}{\partial_j g}(z) \right) > 0$ for all $z \in \mathcal{H}^n_+$, so h is a holomorphic function from $\mathcal{H}_+ \to \mathcal{H}_+$. In particular, by the Schwarz-Pick theorem, for any $t \in \mathcal{H}_+$, we have $\frac{|h'(t)|}{\mathrm{Im}(h(t))} \leq \frac{1}{\mathrm{Im}(t)}$.

Note that

$$\frac{\partial_{\bar{j}} g(x)}{\partial_j g(x)} = \frac{(\partial_{\bar{j}} f)(-ix)}{-i(\partial_j f)(-ix)} = i \left(\frac{\partial_{\bar{j}} f}{\partial_j f}(-ix) \right),$$

so $\partial_k \left(\left(\frac{\partial_{\bar{j}} g}{\partial_j g} \right)(x) \right) = \left(\partial_k \left(\frac{\partial_{\bar{j}} f}{\partial_j f} \right) \right)(-ix)$. Hence,

$$h'(t) = \frac{d}{dt}\left[\frac{\partial_{\bar{j}} g}{\partial_j g}(tv) \right] = \left(D_v \left(\frac{\partial_{\bar{j}} g}{\partial_j g} \right) \right)(tv) = \left(D_v \left(\frac{\partial_{\bar{j}} f}{\partial_j f} \right) \right)(-itv).$$

Now take $t = i \in \mathcal{H}_+$. We compute

$$h'(i) = D_v \left(\frac{\partial_{\bar{j}} f}{\partial_j f} \right)(v).$$

By similar logic,

$$h(i) = \frac{\partial_{\bar{j}} g(iv)}{\partial_j g(iv)} = \frac{\partial_{\bar{j}} f(v)}{-i \partial_j f(v)} = i \left(\frac{\partial_{\bar{j}} f}{\partial_j f}(v) \right).$$

Since f has real coefficients, $\frac{\partial_{\bar{j}} f}{\partial_j f}(v)$ is real, so so $\mathrm{Im}(h(i)) = \frac{\partial_{\bar{j}} f}{\partial_j f}(v)$.

Moreover, we note that $D_v \left(\frac{\partial_{\bar{j}} f}{\partial_j f} \right)(v)$ is also real. Applying the Schwarz-Pick theorem, we conclude

$$D_v \left(\frac{\partial_{\bar{j}} f}{\partial_j f} \right)(v) \leq \frac{\partial_{\bar{j}} f}{\partial_j f}(v),$$

as desired. ∎

We now use this lemma to show that Hurwitz stability implies 1/2-log concavity.

Proof of Theorem 6.1.14. Since positive rescaling of variables preserves Hurwitz stability, it suffices to check that f^{hom} is 1/2-log concave at $x = \mathbb{1}$. (For brevity, we may sometimes omit evaluation at $x = \mathbb{1}$.) Moreover, since polarizations and parity homogenezations of real Hurwitz stable polynomials are again Hurwitz stable, it suffices to consider the case where f is multiaffine with even support.

From Theorem 6.2.7, we know that f^{hom} is α-log concave at $x = \mathbb{1}$ if and only if

$$Q \coloneqq \left[f\nabla^2 f - (\nabla f)(\nabla f)^T + \operatorname{diag}\left((\partial_i f)^2\right)\right]_{x=\mathbb{1}} \preceq (1/\alpha - 1)\operatorname{diag}\left(\partial_i f \partial_{\bar{i}} f\right)\big|_{x=\mathbb{1}}.$$

Therefore, to prove that f is 1/2-log concave, it suffices to show that

$$\sum_{j \neq i} \left| f\partial_{ij}^2 f - \partial_i f \partial_j f \right| \leq (\partial_i f)(\partial_{\bar{i}} f)$$

for all $i = 1, \dots, n$.

Consider the ith row of this matrix. We note that inverting x_j preserves Hurwitz stability and negates the jth column of Q. Moreover, this simply interchanges the roles of x_j and y_j in f^{hom}, so it does not affect whether f^{hom} is 1/2-log concave at $\mathbb{1}$. Thus, we may assume without loss of generality that for all $j \neq i$, $|Q_{ij}| = \partial_i f \partial_j f - f\partial_{ij}^2 f$. Further, we note that by direct computation,

$$\partial_j \left(\frac{\partial_{\bar{i}} f}{\partial_i f}\right) = \frac{\partial_{\bar{i}} f}{\partial_i f}\frac{1}{\partial_i f \partial_{\bar{i}} f}\left(\partial_i f \partial_j f - f\partial_{ij}^2 f\right).$$

Taking $v = \mathbb{1}$ in Lemma 6.1.16 and noting that $\partial_i(\partial_{\bar{i}} f/\partial_i f) \equiv 0$, we have

$$D_{\mathbb{1}}\left(\frac{\partial_{\bar{i}} f}{\partial_i f}\right)(\mathbb{1}) = \frac{\partial_{\bar{i}} f}{\partial_i f}(\mathbb{1})\left(\frac{1}{\partial_i f \partial_{\bar{i}} f}\sum_{j \neq i}\left(\partial_i f \partial_j f - f\partial_{ij}^2 f\right)\right) \leq \frac{\partial_{\bar{i}} f}{\partial_i f}(\mathbb{1}).$$

Hence,

$$\sum_{j \neq i}(\partial_i f \partial_j f - f\partial_{ij}^2 f) \leq (\partial_i f)(\partial_{\bar{i}} f),$$

so f^{hom} is 1/2-log concave at $\mathbb{1}$, as desired.

$\square$

Moreover, for multiaffine polynomials with even support on four variables, the converse also holds.

Theorem 6.1.17. *Let $f(x_1, \dots, x_4) = \sum c_S x^S$ be a polynomial supported on $\{S \subseteq [4] : |S| \text{ even}\}$. Then f is Hurwitz stable if and only if f^{hom} is $1/2$-log concave.*

By Theorem 6.1.14, we know that if f is Hurwitz stable, then f^{hom} is 1/2-log concave, so we just need to show that if f is not Hurwitz stable, then f^{hom} is not 1/2-log concave.

We begin by proving a special case where f has a particular form, then show that the special case is enough to prove the general case.

Lemma 6.1.18. *Let*

$$f = \sum_{\substack{S \subseteq [4] \\ |S| even}} c_S x^S$$

be a polynomial such that $\sqrt{c_0 c_{1234}} > \sqrt{c_{12}c_{34}} + \sqrt{c_{13}c_{24}} + \sqrt{c_{14}c_{23}}$ and $c_{12} = c_{34}, c_{13} = c_{24}$, and $c_{14} = c_{23}$. Then f^{hom} is not $1/2$-log concave at $\mathbb{1}$.

Proof. Let $Q = \nabla^2 f^{\text{hom}}_{1/2}(\mathbb{1})$. Let $\mathbb{1}_x \in \mathbb{R}^8$ denote the vector with ones in the indices corresponding to x variables and zeros in the y variables, and let $\mathbb{1}_y$ denote the vector with ones in the y indices and zeros in the x indices.

We note that

$$Q_{ij} = \begin{cases} -\frac{1}{4}\sum_{S \ni i} c_S, & i = j \\ \frac{1}{4}\sum_{S \ni i,j} c_S, & i \neq j \end{cases}$$

$$Q_{i\bar{j}} = \begin{cases} 0, & i = j \\ \frac{1}{4}\sum_{S \ni i, S \not\ni j} c_S, & i \neq j \end{cases}$$

$$Q_{\bar{i}\bar{j}} = \begin{cases} -\frac{1}{4}\sum_{S \not\ni i} c_S, & i = j \\ \frac{1}{4}\sum_{S \not\ni i,j} c_S, & i \neq j \end{cases}.$$

Hence,

$$
\begin{aligned}
[Q\mathbb{1}_x]_i &= \frac{1}{4}\sum_{j\neq i}\sum_{S\ni i,j} c_S - \frac{1}{4}\sum_{S\ni i} c_S \\
&= \frac{1}{4}\sum_{S\ni i}\sum_{j\in S\setminus i} c_S - \frac{1}{4}\sum_{S\ni i} c_S \\
&= \frac{1}{4}\sum_{S\ni i}(|S|-2)c_S \\
[Q\mathbb{1}_x]_{\bar{i}} &= \frac{1}{4}\sum_{j\neq i}\sum_{\substack{S\subseteq[4] \\ i\notin S, j\in S}} c_S \\
&= \frac{1}{4}\sum_{S\not\ni i}\sum_{j\in S} c_S \\
&= \frac{1}{4}\sum_{S\not\ni i}|S|c_S.
\end{aligned}
$$

An analogous calculation shows that

$$
\begin{aligned}
[Q\mathbb{1}_y]_i &= \frac{1}{4}\sum_{S\ni i}(4-|S|)c_S \\
[Q\mathbb{1}_y]_{\bar{i}} &= \frac{1}{4}\sum_{S\not\ni i}((4-|S|)-2)c_S.
\end{aligned}
$$

Hence,

$$\begin{aligned}
\mathbb{1}_x^T Q \mathbb{1}_x &= \frac{1}{4}\sum_{i=1}^{4}\sum_{S\ni i}(|S|-2)c_S \\
&= \frac{1}{4}\sum_{S\subseteq[4]}\sum_{i\in S}(|S|-2)c_S = \frac{1}{4}\sum_{S\subseteq[4]}|S|(|S|-2)c_S \\
&= 2c_{1234} \\
\mathbb{1}_y^T Q \mathbb{1}_y &= \frac{1}{4}\sum_{i=1}^{4}\sum_{S\not\ni i}((4-|S|)-2)c_S \\
&= \frac{1}{4}\sum_{S\subseteq[4]}\sum_{i\notin S}((4-|S|)-2)c_S = \frac{1}{4}\sum_{S\subseteq[4]}(4-|S|)((4-|S|)-2)c_S \\
&= 2c_0 \\
\mathbb{1}_x^T Q \mathbb{1}_y &= \frac{1}{4}\sum_{i=1}^{4}\sum_{S\ni i}(4-|S|)c_S \\
&= \frac{1}{4}\sum_{S\subseteq[4]}\sum_{i\in S}(4-|S|)c_S = \frac{1}{4}\sum_{S\subseteq[4]}|S|(4-|S|)c_S \\
&= c_{12}+c_{13}+c_{14}+c_{23}+c_{24}+c_{34}.
\end{aligned}$$

Then

$$(\alpha\mathbb{1}_x+\beta\mathbb{1}_y)^T Q(\alpha\mathbb{1}_x+\beta\mathbb{1}_y) = 2(\alpha^2 c_{1234}+\alpha\beta(c_{12}+c_{13}+c_{14}+c_{23}+c_{24}+c_{34})+\beta^2 c_0).$$

Since $(\mathbb{1}_x+\mathbb{1}_y)^T Q(\mathbb{1}_x+\mathbb{1}_y) > 0$, if $(c_{12}+c_{13}+c_{14}+c_{23}+c_{24}+c_{34})^2 < 4c_0c_{1234}$, then $Q \succ 0$ on $\mathrm{span}\{\mathbb{1}_x, \mathbb{1}_y\}$.

By assumption,

$$\sqrt{c_0c_{1234}} > \sqrt{c_{12}c_{34}}+\sqrt{c_{13}c_{24}}+\sqrt{c_{14}c_{23}} = \frac{1}{2}(c_{12}+c_{13}+c_{14}+c_{23}+c_{24}+c_{34}),$$

so $(c_{12}+c_{13}+c_{14}+c_{23}+c_{24}+c_{34})^2 < 4c_0c_{1234}$. Hence, Q has at least two positive eigenvalues, so f^{hom} is not 1/2-log concave at $\mathbb{1}$, as desired. □

This now allows us to prove the general case.

Proposition 6.1.19. *Let $f(x_1, \dots, x_4) = \sum c_S x^S$ be a polynomial such that* $\operatorname{supp}(f) = \{S \subseteq [4] : |S| \text{ even}\}$. *If f^{hom} is 1/2-log concave, then f is Hurwitz stable.*

Proof. Suppose f is not Hurwitz stable. Then it violates one of the four term triangle inequalities in Proposition 6.1.6. Since inverting variables preserves Hurwitz stability, we may assume without loss of generality that $\sqrt{c_0 c_{1234}} > \sqrt{c_{12}c_{34}} + \sqrt{c_{13}c_{24}} + \sqrt{c_{14}c_{23}}$.

Define $a_i = \frac{1}{\sqrt{\prod_{j \neq i} c_{ij}}}$, and let

$$\widetilde{f}(x_1, x_2, x_3, x_4) = f(a_1 x_1, a_2 x_2, a_3 x_3, a_4 x_4) = \sum_{S \subseteq [4]} \widetilde{c}_S x^S.$$

By construction, for distinct i, j, k, l,

$$\widetilde{c}_{ij} = a_i a_j c_{ij} = \frac{1}{\sqrt{c_{ik} c_{il} c_{jk} c_{jl}}} = a_k a_l c_{kl} = \widetilde{c}_{kl}.$$

Moreover,

$$\begin{aligned}
\sqrt{\widetilde{c}_0 \widetilde{c}_{1234}} &= \sqrt{a_1 a_2 a_3 a_4} \sqrt{c_0 c_{1234}} \\
&> \sqrt{a_1 a_2 a_3 a_4} (\sqrt{c_{12}c_{34}} + \sqrt{c_{13}c_{24}} + \sqrt{c_{14}c_{23}}) \\
&= \sqrt{\widetilde{c}_{12}\widetilde{c}_{34}} + \sqrt{\widetilde{c}_{13}\widetilde{c}_{24}} + \sqrt{\widetilde{c}_{14}\widetilde{c}_{23}},
\end{aligned}$$

so $\widetilde{f}$ satisfies the assumptions of Lemma 6.1.18 and so $\widetilde{f}^{\text{hom}}$ is not 1/2-log concave at $\mathbb{1}$.

However,

$$\nabla^2 \widetilde{f}^{\text{hom}}_{1/2}(\mathbb{1}) = D \nabla^2 f^{\text{hom}}_{1/2}(a, \mathbb{1}) D,$$

where $D = \operatorname{diag}(\sqrt{a}, \mathbb{1})$. Hence $\nabla^2 f^{\text{hom}}_{1/2}(a, \mathbb{1})$ also has at least two positive eigenvalues, so f^{hom} is not 1/2-log concave, as desired. $\square$

In the case of real polynomials, one can prove directly that if $f \in \mathbb{R}_{\geq 0}[x_1, \dots, x_n]$ is real stable, then $\log(f)$ is concave on $\mathbb{R}^n_{>0}$ using only techniques from the geometry of polynomials, rather than complex analysis. For Hurwitz stable polynomials, no such proof is known. However, in the univariate case, we can prove directly that Hurwitz stable polynomials are 1/2-log concave.

Proposition 6.1.20. *If $f \in \mathbb{R}_{\geq 0}[t]$ is Hurwitz stable, then $f(\sqrt{t})$ is log concave on $\mathbb{R}_{>0}$.*

Proof. By direct computation,

$$\begin{aligned}\frac{d^2}{dt^2}\log(f(\sqrt{t})) &= \frac{1}{4}t^{-1}\left(\frac{f(\sqrt{t})f''(\sqrt{t}) - (f'(\sqrt{t}))^2}{f(\sqrt{t})^2} - \frac{1}{\sqrt{t}}\frac{f'(\sqrt{t})}{f(\sqrt{t})}\right) \\ &= \frac{1}{4t}\left(\frac{d^2}{ds^2}\log(f(s)) - \frac{1}{s}\frac{d}{ds}\log(f(s))\right)\Bigg|_{s=\sqrt{t}}.\end{aligned}$$

Hence f is 1/2-log concave if and only if

$$g(t) := \frac{d^2}{dt^2}\log(f) - \frac{1}{t}\frac{d}{dt}\log(f) \leq 0$$

for all $t > 0$.

Without loss of generality, we may rescale so that f is monic. Since f has real coefficients, its complex roots must come in conjugate pairs, and since f is Hurwitz stable, the real parts of its roots must all be nonpositive. Hence we can write

$$f(t) = \prod(t + r_j)\prod(t^2 + 2a_jt + b_j)$$

where $r_j, a_j, b_j \geq 0$ and $a_j^2 < b_j$. Thus, $\log(f) = \sum\log(t + r_j) + \sum\log(t^2 + 2a_jt + b_j)$, and so

$$\begin{aligned}g(t) &= \sum\left(\left(\frac{d^2}{dt^2} - \frac{1}{t}\frac{d}{dt}\right)\log(t + r_j)\right) + \sum\left(\left(\frac{d^2}{dt^2} - \frac{1}{t}\frac{d}{dt}\right)\log(t^2 + 2a_jt + b_j)\right) \\ &= \sum\left(\frac{-1}{(t + r_j)^2} - \frac{1}{t(t + r_j)}\right) + \sum\left(\frac{2(t^2 + 2a_jt + b) - (2(t + a_j))^2}{(t^2 + 2a_jt + b)^2} - \frac{2(t + a_j)}{t(t^2 + 2a_jt + b)}\right).\end{aligned}$$

We note that $\frac{-1}{(t+r_j)^2} - \frac{1}{t(t+r_j)} \leq 0$ for all $t > 0$, so it suffices to show that each term of the second sum is nonpositive whenever $t > 0$. To see this, we compute

$$\frac{2(t^2 + 2a_jt + b) - (2(t + a_j))^2}{(t^2 + 2a_jt + b)^2} - \frac{2(t + a_j)}{t(t^2 + 2a_jt + b)} = \frac{-2t^3 - 6a_jt^2 - 6a_j^2t - 2a_jb_j}{t(t^2 + 2a_jt + b_j)^2}.$$

Since $a_j, b_j \geq 0$, this expression is nonpositive whenever $t > 0$, as desired. $\square$

6.1.3 Connections with Discrete Log Concavity

One important application of Lorentzian polynomials is that Lorentzianity of a polynomial implies that its coefficients form a log-concave sequence. This fact was at the heart of much of the development of the theory of Lorentzian polynomials, including their key role in the proof of Mason's conjecture [ALOV24b; BH19]. Namely, by showing that independence polynomials of matroids are Lorentzian, we can conclude that $I_k := \#\{I \in \mathcal{I}(M) : |I| = k\}$ forms an ultra-log concave sequence. More recently, this property of Lorentzian polynomials has been used to investigate Fox's conjecture on the coefficients of the Alexander polynomial [HMV25].

It remains an open question whether we can find similar log-concavity constraints on the number of bases of a Δ-matroid of a given size. However, if we can prove (complete) 1/2-log concavity of the homogenized generating polynomials of even Δ-matroids, then we get log concavity of the coefficients, up to a factor of 1/3.

Lemma 6.1.21. *If $f = \sum_{k=0}^{d} a_k x^k \in \mathbb{R}_{\geq 0}[x]$ is log concave, then $a_1^2 \geq 2a_0a_2$.*

Proof. Note that if $a_0 = 0$, then the inequality holds, so it suffices to consider the case where $f(0) > 0$. In this case, f is log-concave at $x = 0$ if and only if $[f \cdot f'' - (f')^2]_{x=0} \leq 0$. Thus, we have

$$0 \geq f(0)f''(0) - (f'(0))^2 = 2a_0a_2 - a_1^2,$$

as desired. □

We now consider a class of "completely 1/2-log concave" polynomials (cf. Section 6.3.3), and use the above lemma to derive bounds analogous to the discrete log concavity property for Lorentzian (aka completely log concave) polynomials.

Assumption 6.1.22. Let $\mathcal{P} \subseteq \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ be a family of multiaffine polynomials such that

- (1/2-log concavity) for all $f \in \mathcal{P}$, $\mathrm{supp}(f)$ has constant parity and f is 1/2-log concave
- (directional derivatives) for all $f \in \mathcal{P}$ and $v \in \mathbb{R}^n_{>0}$, $D_v f \in \mathcal{P}$.

For example, the set of constant-parity, multiaffine, Hurwitz stable polynomials with nonnegative coefficients is such a family.

Theorem 6.1.23. *Suppose $\mathcal{P}$ has the $1/2$-log concavity and directional derivative properties from above, and suppose $f \in \mathcal{P}$ has even parity support. Suppose $f(x,\ldots,x) = \sum_{k=0}^{\lfloor n/2\rfloor} c_k x^{2k}$. Then $c_k^2 \geq \frac{1}{3}c_{k-1}c_{k+1}$ for all $1 \leq k \leq \lfloor n/2\rfloor - 1$.*

Proof. Fix some $1 \leq \ell \leq \lfloor n/2\rfloor - 1$. We wish to show that $c_\ell^2 \geq \frac{1}{3}c_{\ell-1}c_{\ell+1}$. Indeed, we know that $D_{\mathbb{1}}^{2\ell-2} f \in \mathcal{P}$, so in particular $(D_{\mathbb{1}}^{2\ell-2} f)(x,\ldots,x)$ is $1/2$-log concave. By the multivariate chain rule, we have

$$(D_{\mathbb{1}}^{2\ell-2} f)(x,\ldots,x) = \frac{d^{2\ell-2}}{dx^{2\ell-2}} f(x,\ldots,x) = \frac{d^{2\ell-2}}{dx^{2\ell-2}}\left(\sum_{k=0}^{\lfloor n/2\rfloor} c_k x^{2k}\right),$$

and by direct computation, we have

$$\begin{aligned}\frac{d^{2\ell-2}}{dx^{2\ell-2}}\left(\sum_{k=0}^{\lfloor n/2\rfloor} c_k x^{2k}\right) &= \sum_{k=\ell-1}^{\lfloor n/2\rfloor} c_k \frac{(2k)!}{(2(k-\ell+1))!} x^{2(k-\ell+1)} \\ &= \sum_{k=0}^{\lfloor n/2\rfloor-(\ell-1)} c_{k+\ell-1}\frac{(2k+2\ell-2)!}{(2k)!} x^{2k}.\end{aligned}$$

Since $\mathcal{P}$ is closed under directional derivatives, and since all elements of $\mathcal{P}$ are $1/2$-log concave, this implies that

$$g(x) = \sum_{k=0}^{\lfloor n/2\rfloor-(\ell-1)} c_{k+\ell-1}\frac{(2k+2\ell-2)!}{(2k)!} x^{k}$$

is log concave.

We now apply Lemma 6.1.21, which gives

$$\left(c_\ell \frac{(2\ell)!}{2!}\right)^2 \geq 2\left(c_{\ell-1}\frac{(2\ell-2)!}{0!}\right)\left(c_{\ell+1}\frac{(2\ell+2)!}{4!}\right).$$

After rearranging, we have

$$c_\ell^2 \geq \frac{1}{3}c_{\ell-1}c_{\ell+1}\frac{(2\ell+2)(2\ell+1)}{(2\ell)(2\ell-1)} \geq \frac{1}{3}c_{\ell-1}c_{\ell+1},$$

as desired. $\square$

6.2 Partial Progress

In this section, we discuss the state-of-the-art of known results on fractionally log-concave polynomials. We begin with a section of basic, easily-verified properties. We then discuss how the multihomogeneous structure of f^{hom} interacts with fractional log-concavity. Finally, we explore results on fractional log concavity of derivates, as well as the limitations of such inductive techniques.

6.2.1 Basic Facts about Fractional Log Concavity

We begin by establishing some basic facts about fractionally log concave polynomials, as well as some simple operations that preserve fractional log-concavity.

Proposition 6.2.1. *The space of α-log concave polynomials is closed in the Euclidean topology.*

Proof. For each $a \in \mathbb{R}^n_{>0}$, define $V_a = \{f \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n] : f \text{ is } \alpha\text{-log concave at } a\}$. We note that the eigenvalues of $\nabla^2 f_\alpha(a)$ are continuous in the coefficients of f and that $\nabla^2 f_\alpha(a) \preceq 0$ is a closed condition. Hence each V_a is closed.

Moreover, the set of α-log concave polynomials is exactly $\bigcap_{a \in \mathbb{R}^n_{>0}} V_a$, so it is an intersection of closed sets and hence closed, as desired. $\square$

The following properties of α-log concavity are easily verified.

Proposition 6.2.2. *The following operations preserve α-log concavity:*

1. *Permutation:* $f \mapsto f(x_{\pi(1)}, \ldots, x_{\pi(n)})$ *for any permutation* $\pi : [n] \to [n]$
2. *Specialization:* $f \mapsto f(a, x_2, \ldots, x_n)$, *where* $a \in \mathbb{R}_{\geq 0}$
3. *Scaling:* $f \mapsto c \cdot f(\lambda_1 x_1, \ldots, \lambda_n x_n)$, *where* $c \geq 0, \lambda \in \mathbb{R}^n_{\geq 0}$
4. *Multiplication:* $f, g \mapsto f \cdot g$ *where* f, g *are* α*-log concave.*

Together with Proposition 6.2.1, this gives us the following useful fact.

Proposition 6.2.3. *The space of α-log concave polynomials is closed under taking initial forms.*

Proof. Suppose $f \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ is α-log concave, and let $w \in \mathbb{R}^n$. Let $m = \max\{w^T\alpha : \alpha \in \operatorname{supp}(f)\}$ and define $f_t := t^{-m} f(t^{w_1}x_1, \ldots, t^{w_n}x_n)$. By Proposition 6.2.2, f_t is α-log concave for all $t > 0$. Moreover, $\operatorname{in}_w(f) = \lim_{t\to\infty} f_t$. Thus, $\operatorname{in}_w(f)$ can be written as the limit of α-log concave polynomials. Since the space of α-log concave polynomials is closed, this implies that $\operatorname{in}_w(f)$ is α-log concave, as desired. $\square$

We now turn towards studying how we can check if a given polynomial is α-log concave. One key insight of Lorentzian polynomials is that for homogeneous polynomials, one can study log concavity of f through the signature of $\nabla^2 f$ rather than $\nabla^2 \log f$. This observation actually carries through to all homogeneous functions, not just polynomials, allowing us to apply it in the fractional log concavity setting as well.

Proposition 6.2.4 (cf. [BH20, Proposition 2.33], [ALOV24b, Lemma 2.2]). *Let $f \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ be homogeneous of degree $d > 1$, and suppose that $1/d \leq \alpha \leq 1$. Then f is α-log concave at $w \in \mathbb{R}^n_{>0}$ if and only if $\nabla^2 f_\alpha(w)$ has at most one positive eigenvalue.*

Proof. First, we note that

$$\partial^2_{ij} \log f_\alpha = \frac{f_\alpha \partial^2_{ij} f_\alpha - (\partial_i f_\alpha)(\partial_j f_\alpha)}{f_\alpha^2}$$

so

$$\nabla^2 \log f_\alpha(w) = \frac{1}{f_\alpha(w)^2}\left(f_\alpha(w)\nabla^2 f_\alpha(w) - (\nabla f_\alpha(w))(\nabla f_\alpha(w))^T\right).$$

Hence, if f is α-log concave at w, then $f_\alpha(w)\nabla^2 f_\alpha(w) - (\nabla f_\alpha(w))(\nabla f_\alpha(w))^T \preceq 0$. Since $(\nabla f_\alpha(w))(\nabla f_\alpha(w))^T$ is rank one and since $f_\alpha(w) > 0$, this implies that $\nabla^2 f_\alpha(w)$ has at most one positive eigenvalue.

Conversely, suppose that $\nabla^2 f_\alpha(w)$ has at most one positive eigenvalue. For ease of notation, let $Q = \nabla^2 f_\alpha(w)$. Then by homogeneity, $w^T Q w = (\alpha d)(\alpha d - 1) f_\alpha(w)$, and $Qw = (\alpha d - 1)\nabla f_\alpha(w)$. Thus, to prove that f is α-log concave at w, it suffices to show that

$$(\alpha d - 1)(w^T Q w)Q - (\alpha d)(Qw)(Qw)^T \preceq 0.$$

Since the space of negative semidefinite matrices is a convex cone, and since $(\alpha d - 1) \geq 0$ by assumption, it suffices to show that

$$(w^T Q w)Q - (Qw)(Qw)^T \preceq 0.$$

Let $v \in \mathbb{R}^n$, and let P be the $2 \times n$ matrix with rows v^T and w^T. Then

$$PQP^T = \begin{bmatrix} v^T Q v & v^T Q w \\ w^T Q v & w^T Q w \end{bmatrix}.$$

By Cauchy interlacing, if Q has at most one positive eigenvalue, then so does PQP^T. Moreover, since $w^T Q w = (\alpha d)(\alpha d - 1) f_\alpha(w) \geq 0$, we know PQP^T has at least one non-negative eigenvalue, so it must either must be singular or have exactly one positive and one negative eigenvalue. Hence $\det(PQP^T) \leq 0$. Thus,

$$\det(PQP^T) = (w^T Q w)(v^T Q v) - (v^T Q w)(w^T Q v) = v^T \left((w^T Q w)Q - (Qw)(Qw)^T\right) v \leq 0$$

for all $v \in \mathbb{R}^n$, so $(w^T Q w)Q - (Qw)(Qw) \preceq 0$, as desired. □

Using these propositions, the converse to Conjecture 6.1.4 is fairly straightforward, and can even be generalized to the valuated setting. Note that a non-valuated version of this statement appears for multiaffine polynomials appears in [AASV21, Lemma 68].

Theorem 6.2.5. *Let* $f = \sum c_\gamma x^\gamma \in \mathbb{R}\{\{t\}\}_{\geq 0}[x_1, \ldots, x_n]$ *be a multiaffine polynomial and suppose that* f^{hom} *is* $1/2$*-log concave. Then* $\mathrm{val}(f) : \gamma \mapsto \mathrm{val}(c_\gamma)$ *is a valuated* Δ*-matroid.*

Proof. By degenerating to initial forms (Proposition 6.2.3), it suffices to consider the case of binomials. In particular, we wish to show that if $b_S, b_T > 0$ and $g = b_S x^S y^{[n]\setminus S} + b_T x^T y^{[n]\setminus T}$ is $1/2$-log concave, then $|S \Delta T| \leq 2$.

By specializing $x_i = y_i = 1$ for all $i \notin S\Delta T$, and by interchanging x_i with y_i for all $i \in T \setminus S$, we may assume that our binomial is of the form $g = b_S x^{S\Delta T} + b_T y^{S\Delta T}$. Finally, by specializing $x_i = x$ and $y_i = y$ for all $i \in S\Delta T$, we are left with $g = b_S x^m + b_T y^m$, where $m = |S\Delta T|$. We note that

$$\nabla^2 g_{1/2}(1,1) = \begin{pmatrix} b_S(m/2)(m/2-1) & 0 \\ 0 & b_T(m/2)(m/2-1) \end{pmatrix}.$$

Since $b_S, b_T > 0$, this matrix has at most one positive eigenvalue if and only if $m \leq 2$. Hence, if f is $1/2$-log concave, then $\mathrm{val}(f)$ is a valuated Δ-matroid, as desired. $\square$

We now continue our discussion of conditions for testing α-log concavity of a polynomial.

Proposition 6.2.6. *Let $f \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ be a non-zero d-homogeneous polynomial and let $1/d \leq \alpha \leq 1$. Then f is α-log concave at $x = a$ if and only if*

$$\left[\nabla^2 f - \frac{1-\alpha}{\alpha} \operatorname{diag}\left(\frac{\partial_i f}{x_i}\right)\right]_{x=a^\alpha}$$

has at most one positive eigenvalue.

Proof. By Proposition 6.2.4, it suffices to show that $\nabla^2 f_\alpha(a)$ has the same signature as the matrix above.

By direct computation,

$$\partial_{ij}^2 = \alpha^2 \frac{1}{x_i^{1-\alpha}} \frac{1}{x_j^{1-\alpha}} \partial_{ij}^2 f(x^\alpha)$$

whenever $i \neq j$, and

$$\partial_{ii}^2 = \alpha^2 \left(\frac{1}{x_i^{1-\alpha}}\right)^2 \partial_{ii}^2 f(x^\alpha) - \alpha(1-\alpha)\frac{1}{x_i^{2-\alpha}} \partial_i f(x^\alpha).$$

Thus,

$$\nabla^2 f_\alpha = \alpha^2 \operatorname{diag}\left(\frac{1}{x_i^{1-\alpha}}\right) \left[\nabla^2 f(x^\alpha) - \frac{1-\alpha}{\alpha} \operatorname{diag}\left(\frac{\partial_i f(x^\alpha)}{x_i^\alpha}\right)\right] \operatorname{diag}\left(\frac{1}{x_i^{1-\alpha}}\right).$$

By Sylvester's law of inertia, this implies that $\nabla^2 f_\alpha(a)$ has the same signature as $\nabla^2 f(a^\alpha) - \frac{1-\alpha}{\alpha} \operatorname{diag}\left(\frac{\partial_i f(a^\alpha)}{a_i^\alpha}\right)$, as desired. $\square$

If f is a multiaffine polynomial on n variables, then in order to test α-log concavity of f^{hom}, a priori we would need to compute the signature of the $2n \times 2n$ matrix $\nabla^2 f_\alpha^{\text{hom}}$. However, using the multihomogeneous structure of f^{hom}, we are able to construct a $n \times n$ matrix such that f^{hom} is α-log concave if and only if this $n \times n$ matrix is negative semidefinite. This result is similar to the technique seen in the proof of [AASV21, Lemma 71].

Theorem 6.2.7. *Let $f \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]_{MA}$ be a multiaffine polynomial, and let f^{hom} be its multiaffine homogenization. If $\alpha > \frac{1}{n}$, then f^{hom} is α-log concave at $(1, \ldots, 1)$ if and only if*

$$\left(f \left(\nabla^2 f + \operatorname{diag}(\partial_i(f)) \right) - (\nabla f)(\nabla f)^T - \frac{1}{\alpha} \operatorname{diag}((\partial_i f)(\partial_{\bar{i}} f)) \right)\Big|_{x=\mathbb{1}} \preceq 0,$$

where $\partial_{\bar{i}} f = f|_{x_i=0}$.

Proof. We know that f^{hom} is α-log concave at $\mathbb{1}$ if and only if $\nabla^2 f_\alpha^{\text{hom}}(\mathbb{1})$ has at most one positive eigenvalue. By direct computation,

$$\nabla^2 f_\alpha^{\text{hom}}(\mathbb{1}) = \alpha^2 \left(\nabla^2 f^{\text{hom}}(\mathbb{1}) - \frac{1-\alpha}{\alpha} \operatorname{diag}(\nabla f^{\text{hom}}(\mathbb{1})) \right).$$

Let $D = \operatorname{diag}(\nabla f^{\text{hom}}(\mathbb{1}))$. Then, we know that $\nabla^2 f_\alpha^{\text{hom}}(\mathbb{1})$ has the same signature as

$$D^{-1/2} \nabla^2 f^{\text{hom}}(\mathbb{1}) D^{-1/2} - \frac{1-\alpha}{\alpha} I.$$

(For ease of notation, from this point forward, everything will be implicitly evaluated at $\mathbb{1}$.)

We claim that $\sqrt{\nabla f^{\text{hom}}} := (\sqrt{\partial_1 f^{\text{hom}}}, \ldots, \sqrt{\partial_n f^{\text{hom}}}, \sqrt{\partial_{\bar{1}} f^{\text{hom}}}, \ldots, \sqrt{\partial_{\bar{n}} f^{\text{hom}}})^T$ is an eigenvector of $D^{-1/2} \nabla^2 f^{\text{hom}} D^{-1/2}$.

Indeed, we compute

$$D^{-1/2} \nabla^2 f^{\text{hom}} D^{-1/2} \sqrt{\nabla f^{\text{hom}}} = D^{-1/2} (\nabla^2 f^{\text{hom}}) \mathbb{1} = D^{-1/2} (n-1) \nabla f^{\text{hom}} = (n-1) \sqrt{\nabla f^{\text{hom}}},$$

so $\sqrt{\nabla f^{\text{hom}}}$ is an eigenvector of $D^{-1/2} \nabla^2 f^{\text{hom}} D^{-1/2}$ with eigenvalue $n-1$. Thus, it is also an eigenvector of $D^{-1/2} \nabla^2 f^{\text{hom}} D^{-1/2} - \frac{1-\alpha}{\alpha} I$ with eigenvalue $\lambda_1 = n-1-\frac{1-\alpha}{\alpha} = n - \frac{1}{\alpha}$. Since $\alpha > 1/n$, then $\lambda_1 > 0$, so f^{hom} is α-log concave at $\mathbb{1}$ if and only if $D^{-1/2} \nabla^2 f^{\text{hom}} D^{-1/2} - \frac{1-\alpha}{\alpha} I \preceq 0$ on $\left(\sqrt{\nabla f^{\text{hom}}} \right)^{\perp}$.

Next, we claim that for any $i \neq j$, $D^{1/2}\left((e_i + e_{\bar{\imath}}) - (e_j + e_{\bar{\jmath}})\right)$ is also an eigenvector of $D^{-1/2}\nabla^2 f^{\text{hom}} D^{-1/2}$. Indeed, we note that $(\nabla^2 f^{\text{hom}} + D)(e_i + e_{\bar{\imath}}) = \nabla f^{\text{hom}}$ for all i, so $\nabla^2 f^{\text{hom}}(e_i + e_{\bar{\imath}}) = \nabla f^{\text{hom}} - D(e_i + e_{\bar{\imath}})$. Thus,

$$\begin{aligned} D^{-1/2}\nabla^2 f^{\text{hom}} D^{-1/2}\left(D^{1/2}\left((e_i + e_{\bar{\imath}}) - (e_j + e_{\bar{\jmath}})\right)\right) &= D^{-1/2}\nabla^2 f^{\text{hom}}\left((e_i + e_{\bar{\imath}}) - (e_j + e_{\bar{\jmath}})\right) \\ &= D^{-1/2}\left(-D(e_i + e_{\bar{\imath}}) + D(e_j + e_{\bar{\jmath}})\right) \\ &= -D^{1/2}\left((e_i + e_{\bar{\imath}}) - (e_j + e_{\bar{\jmath}})\right). \end{aligned}$$

Thus, $D^{1/2}\left((e_i + e_{\bar{\imath}}) - (e_j + e_{\bar{\jmath}})\right)$ is an eigenvector of $D^{-1/2}\nabla^2 f^{\text{hom}} D^{-1/2}$ with eigenvalue -1, so it is also an eigenvector of $D^{-1/2}\nabla^2 f^{\text{hom}} D^{-1/2} - \frac{1-\alpha}{\alpha} I$ with eigenvalue $-1 - \frac{1-\alpha}{\alpha} < 0$.

Therefore, $D^{-1/2}\nabla^2 f^{\text{hom}} D^{-1/2} - \frac{1-\alpha}{\alpha} I$ has at most one positive eigenvalue if and only if it is negative semidefinite on $\operatorname{span}\left\{\sqrt{\nabla f^{\text{hom}}}, D^{1/2}\left((e_i + e_{\bar{\imath}}) - (e_j + e_{\bar{\jmath}})\right) : i \neq j\right\}^{\perp}$.

We note that $D^{-1/2}(e_i - e_{\bar{\imath}}) \in \operatorname{span}\left\{\sqrt{\nabla f^{\text{hom}}}, D^{1/2}\left((e_i + e_{\bar{\imath}}) - (e_j + e_{\bar{\jmath}})\right) : i \neq j\right\}^{\perp}$ for all $i = 1, \ldots, n$. Moreover, these vectors are linearly independent, and

$$\dim\left(\operatorname{span}\left\{\sqrt{\nabla f^{\text{hom}}}, D^{1/2}\left((e_i + e_{\bar{\imath}}) - (e_j + e_{\bar{\jmath}})\right) : i \neq j\right\}^{\perp}\right) = 2n - n = n,$$

so they form a basis for this subspace.

Therefore, if B is the $2n \times n$ matrix with columns $e_i - e_{\bar{\imath}}$, then f^{hom} is α-log concave at $\mathbb{1}$ if and only if

$$(D^{-1/2}B)^T\left(D^{-1/2}\nabla^2 f^{\text{hom}} D^{-1/2} - \frac{1-\alpha}{\alpha} I\right)(D^{-1/2}B) \preceq 0.$$

First, we compute $Q_1 = (D^{-1/2}B)^T\left(D^{-1/2}\nabla^2 f^{\text{hom}} D^{-1/2}\right)(D^{-1/2}B) = B^T D^{-1}\nabla^2 f^{\text{hom}} D^{-1} B$. Since f is multiaffine, $[\nabla^2 f^{\text{hom}}]_{ii} = [\nabla^2 f^{\text{hom}}]_{i\bar{\imath}} = 0$ for all i, so $[B^T D^{-1}\nabla^2 f^{\text{hom}} D^{-1} B]_{ii} = 0$

for all $i = 1, \dots, n$. On the other hand, for any $i \neq j$,

$$\begin{aligned}
[Q_1]_{ij} &= (e_i - e_{\bar{\imath}})^T D^{-1} \nabla^2 f^{\mathrm{hom}} D^{-1} (e_j - e_{\bar{\jmath}}) \\
&= \frac{1}{\partial_i f} \left(\frac{\partial^2_{ij} f}{\partial_j f} - \frac{\partial^2_{i\bar{\jmath}} f}{\partial_{\bar{\jmath}} f} \right) - \frac{1}{\partial_{\bar{\imath}} f} \left(\frac{\partial^2_{\bar{\imath}j} f}{\partial_j f} - \frac{\partial^2_{\bar{\imath}\bar{\jmath}} f}{\partial_{\bar{\jmath}} f} \right) \\
&= \frac{1}{(\partial_j f)(\partial_{\bar{\jmath}} f)} \left[\frac{1}{\partial_i f} \left(\partial_{\bar{\jmath}} f \partial^2_{ij} f - \partial_j f \partial^2_{i\bar{\jmath}} f \right) - \frac{1}{\partial_{\bar{\imath}} f} \left(\partial_{\bar{\jmath}} f \partial^2_{\bar{\imath}j} f - \partial_j f \partial^2_{\bar{\imath}\bar{\jmath}} f \right) \right] \\
&= \frac{1}{(\partial_j f)(\partial_{\bar{\jmath}} f)} \left[\frac{1}{\partial_i f} \left((f - \partial_j f) \partial^2_{ij} f - \partial_j f (\partial_i f - \partial^2_{ij}) f \right) - \frac{1}{\partial_{\bar{\imath}} f} \left((f - \partial_j f) \partial^2_{\bar{\imath}j} f - \partial_j f (\partial_{\bar{\imath}} f - \partial^2_{\bar{\imath}j} f) \right) \right] \\
&= \frac{1}{(\partial_j f)(\partial_{\bar{\jmath}} f)} \left[\frac{1}{\partial_i f} (f \partial^2_{ij} f - \partial_i f \partial_j f) - \frac{1}{\partial_{\bar{\imath}} f} (f \partial^2_{\bar{\imath}j} f - \partial_{\bar{\imath}} f \partial_j f) \right] \\
&= \frac{1}{(\partial_i f)(\partial_{\bar{\imath}} f)(\partial_j f)(\partial_{\bar{\jmath}} f)} \left[(f - \partial_i f)(f \partial^2_{ij} f - \partial_i f \partial_j f) - \partial_i f (f(\partial_j f - \partial^2_{ij} f) - (f - \partial_i f)(\partial_j f)) \right] \\
&= \frac{f}{(\partial_i f)(\partial_{\bar{\imath}} f)(\partial_j f)(\partial_{\bar{\jmath}} f)} (f \partial^2_{ij} f - (\partial_i f)(\partial_j f)).
\end{aligned}$$

Thus,

$$Q_1 = f \operatorname{diag}\left(\frac{1}{(\partial_i f)(\partial_{\bar{\imath}} f)} \right) \left[f \nabla^2 f - (\nabla f)(\nabla f)^T + \operatorname{diag}((\partial_i f)^2) \right] \operatorname{diag}\left(\frac{1}{(\partial_i f)(\partial_{\bar{\imath}} f)} \right)$$

Next, we compute $Q_2 = (D^{-1/2} B)^T (D^{-1/2} B)$. We note that $[Q_2]_{ij} = 0$ whenever $i \neq j$, and

$$\begin{aligned}
[Q_2]_{ii} &= \frac{1}{\partial_i f} + \frac{1}{\partial_{\bar{\imath}} f} \\
&= \frac{1}{(\partial_i f)^2 (\partial_{\bar{\imath}} f)^2} \left[(\partial_i f)(\partial_{\bar{\imath}} f)(\partial_i f + \partial_{\bar{\imath}} f) \right] \\
&= \frac{f}{(\partial_i f)^2 (\partial_{\bar{\imath}} f)^2} (\partial_i f)(\partial_{\bar{\imath}} f),
\end{aligned}$$

so

$$Q_2 = f \operatorname{diag}\left(\frac{1}{(\partial_i f)(\partial_{\bar{\imath}} f)} \right) (\operatorname{diag}((\partial_i f)(\partial_{\bar{\imath}} f)) \operatorname{diag}\left(\frac{1}{(\partial_i f)(\partial_{\bar{\imath}} f)} \right).$$

After conjugating by $\operatorname{diag}((\partial_i f)(\partial_{\bar{\imath}} f))$ and dividing by f, we see that f^{hom} is α-log concave at $\mathbb{1}$ if and only if

$$f \nabla^2 f - (\nabla f)(\nabla f)^T + \operatorname{diag}((\partial_i f)^2) - \frac{1 - \alpha}{\alpha} \operatorname{diag}((\partial_i f)(\partial_{\bar{\imath}} f)) \preceq 0.$$

To get the expression in the statement of the theorem, we note that

$$(\partial_i f)^2 - \frac{1-\alpha}{\alpha}(\partial_i f)(\partial_{\bar{i}} f) = f\partial_i f - \frac{1}{\alpha}(\partial_i f)(\partial_{\bar{i}} f).$$

Thus, f^{hom} is α-log concave at $(x, y) = \mathbb{1}$ if and only if

$$f\left(\nabla^2 f + \operatorname{diag}(\partial_i(f))\right) - (\nabla f)(\nabla f)^T - \frac{1}{\alpha}\operatorname{diag}((\partial_i f)(\partial_{\bar{i}} f)) \preceq 0,$$

as desired. □

This gives a necessary condition for multihomogeneous fractional log concavity. For ease of notation, if g is a multihomogeneous multiaffine polynomial, for a subset $T \subseteq [n]$, let g_T denote g after specializing $x_i = 1$ for all $i \notin T$ and $y_i = 1$ for all $y \in T$, so $g_T \in \mathbb{R}_{\geq 0}[x_i : i \in T, y_i : i \notin T] \cong \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ is multiaffine and $g_T^{\text{hom}} = g$.

Corollary 6.2.8. *Let $g \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be multiaffine and multihomogeneous and suppose that g is $1/2$-log concave at $\mathbb{1}$. Then there exists $T \subseteq [n]$ such that g_T is log concave at $\mathbb{1}$.*

Proof. Rearranging the result of Theorem 6.2.7, we see that f^{hom} is $1/2$-log concave at $\mathbb{1}$ if and only if

$$M_f := \left[f\nabla^2 f - (\nabla f)(\nabla f)^T + \operatorname{diag}\left((\partial_i f)(\partial_i f - \partial_{\bar{i}} f)\right)\right]_{x=\mathbb{1}} \preceq 0.$$

Let $T = \{i \in [n] : \partial_i g(\mathbb{1}) \geq \partial_{\bar{i}} g(\mathbb{1})\}$. Then by construction, $\partial_i g_T(\mathbb{1}) \geq \partial_{\bar{i}} g_T(\mathbb{1})$ for all i, and so $\operatorname{diag}\left((\partial_i g_T)(\partial_i g_T - \partial_{\bar{i}} g_T)\right)\big|_{x=\mathbb{1}} \succeq 0$.

Since g_T^{hom} is $1/2$-log concave at $\mathbb{1}$ by assumption, it follows that

$$\left[g_T \nabla^2 g_T - (\nabla g_T)(\nabla g_T)^T\right]_{x=\mathbb{1}} \preceq M_{g_T^{\text{hom}}} = M_g \preceq 0.$$

Since $\nabla^2 \log g_T(\mathbb{1}) = \frac{1}{g_T(\mathbb{1})^2}\left[g_T \nabla^2 g_T - (\nabla g_T)(\nabla g_T)^T\right]_{x=\mathbb{1}}$, we conclude that g_T is log concave at $\mathbb{1}$, as desired.

□

In particular, if Conjecture 6.1.4, then every even Δ-matroid must have a twist whose generating polynomial is log concave at $\mathbb{1}$.

6.2.2 Connection to Transversals

We know that f^{hom} is not 1/2-log concave at $a \in \mathbb{R}^{2n}_{>0}$ if and only if there exists a two-dimensional subspace $L \subset \mathbb{R}^{2n}$ such that $\nabla^2 f^{\mathrm{hom}}_{1/2}(a) \succ 0$ on L. By exploiting the structure of $\nabla^2 f^{\mathrm{hom}}_{1/2}$, we can further show that if f^{hom} is not 1/2-log concave, then we can find such a subspace L that also respects the transversal structure of the variables.

Consider the symmetric set $[n] \sqcup [n^*]$. A set $S \subseteq [n] \sqcup [n^*]$ is *admissible* if $\{i, i^*\} \not\subseteq S$ for all $i = 1, \dots, n$. A *transversal* is an admissible set of size n. In terms of our polynomials, a transversal corresponds to picking either x_i or y_i for each $i = 1, \dots, n$.

Since testing fractional log-concavity at $a \in \mathbb{R}^{2n}_{>0}$ is equivalent, up to rescaling variables, to testing fractional log-concavity at $(1, \dots, 1)$ of the rescaled polynomial, it suffices to prove our claim only for $a = (1, \dots, 1)$.

Theorem 6.2.9. *Let $f \in \mathbb{R}_{\geq 0}[x_1, \dots, x_n]$ be a multiaffine polynomial. The following are equivalent:*

(a) The multiaffine homogenization f^{hom} is 1/2-log concave at $(1, \dots, 1)$.

(b) For all $v, w \in \mathbb{R}^{2n}$ such that $v_i w_i = 0$ for all $i = 1, \dots, 2n$, $\nabla^2 f^{\mathrm{hom}}_{1/2}(\mathbb{1}) \not\succ 0$ on $\mathrm{span}\{v, w\}$.

(c) For all transversals T and all $w \in \mathbb{R}^n_{\geq 0}$, $\nabla^2 f^{\mathrm{hom}}_{1/2}(\mathbb{1}) \not\succ 0$ on $\mathrm{span}\{D_{T^C} w_T, D_T w_{T^C}\}$, where

$$(D_{T^C} w_T)(i) = \begin{cases} \partial_{\bar{i}} f^{\mathrm{hom}}(\mathbb{1}) \cdot w_i, & \text{if } i \in T \\ 0, & \text{otherwise} \end{cases},$$

$$(D_T w_{T^C})(i) = \begin{cases} \partial_i f^{\mathrm{hom}}(\mathbb{1}) \cdot w_i, & \text{if } i \notin T \\ 0, & \text{otherwise} \end{cases}.$$

Proof. **(a) ⇒ (b)** If (b) does not hold, then in particular there is a two-dimensional subspace on which $\nabla^2 f^{\mathrm{hom}}_{1/2}(\mathbb{1}) \succ 0$. In particular, this implies that $\nabla^2 f^{\mathrm{hom}}_{1/2}(\mathbb{1})$ has at least two positive eigenvalues, and hence f^{hom} is not 1/2-log concave at $\mathbb{1}$.

(b) $\Rightarrow$ (c) We note that $(D_{T^C}w_T)(i) = 0$ whenever $i \notin T$ and $(D_T w_{T^C})(i) = 0$ whenever $i \in T$, so these vectors satisfy the condition in (b).

(c) $\Rightarrow$ (a) It just remains to show that (c) $\Rightarrow$ (a).

Suppose f is not 1/2-log concave at $(1, \ldots, 1)$. By Theorem 6.2.7,

$$M_f := f\nabla^2 f - ((\nabla f)(\nabla f)^T - \mathrm{diag}((\partial_i f)^2)) - \mathrm{diag}((\partial_i f)(\partial_{\bar{i}} f)) \not\preceq 0,$$

so there exist $\lambda > 0$ and $w \in \mathbb{R}^n$ such that $M_f w = \lambda w$. One can check that twisting f by x_i conjugates M_f by the diagonal matrix that negates the ith row/column. Then $(w_1, \ldots, -w_i, \ldots, w_n)^T$ is an eigenvector of $M_{f\Delta i}$ with the same eigenvalue $\lambda > 0$. Twisting f corresponds to taking different transversals in f^{hom}, so up to our choice of T, we may assume without loss of generality that M_f has a positive eigenvalue $\lambda > 0$ with a corresponding eigenvector $w \in \mathbb{R}^n_{\geq 0}$, and we consider the transversal $T = [n]$ corresponding to taking all of the x variables in f^{hom}.

Write $\nabla^2 f^{\mathrm{hom}}_{1/2} = \frac{1}{4}\left(\begin{array}{c|c} Q_1 & Q_2 \\ \hline Q_2^T & Q_3 \end{array}\right)$. In order to show $\nabla^2 f^{\mathrm{hom}}_{1/2} \succ 0$ on $\mathrm{span}\{D_{T^C}w_T, D_T w_{T^C}\}$, we show that

$$\left(\begin{array}{c|c} w^T D_{T^C} Q_1 D_{T^C} w & w^T D_{T^C} Q_2 D_T w \\ \hline w^T D_T Q_2^T D_{T^C} w & w^T D_T Q_3 D_T w \end{array}\right) \succ 0.$$

Since $w \in \mathbb{R}^n_{\geq 0}$ and $D_{T^C}, Q_2, D_T \in \mathbb{R}^{n\times n}_{\geq 0}$, we know that $w^T D_{T^C} Q_2 D_T w, w^T D_T Q_2^T D_{T^C} w \geq 0$, so it suffices to show that $w^T D_{T^C} Q_1 D_{T^C} w > w^T D_{T^C} Q_2 D_T w$ and $w^T D_T Q_3 D_T w^T > w^T D_T Q_2^T D_{T^C} w$.

Similar to the proof of Theorem 6.2.7, we compute

$$\begin{aligned}
[Q_1 D_{T^C} - Q_2 D_T]_{ii} &= (-\partial_i f)(\partial_{\bar{i}} f) - (0)(\partial_i f) \\
&= -(\partial_i f)(\partial_{\bar{i}} f) \\
[Q_1 D_{T^C} - Q_2 D_T]_{ij} &= (\partial^2_{ij} f)(\partial_{\bar{j}} f) - (\partial^2_{i\bar{j}} f)(\partial_j f) \\
&= f\partial^2_{ij} f - (\partial_i f)(\partial_j f).
\end{aligned}$$

Hence,

$$Q_1 D_{T^C} - Q_2 D_T = f\nabla^2 f - \left((\nabla f)(\nabla f)^T - \mathrm{diag}\left((\partial_i f)^2\right)\right) - \mathrm{diag}\left((\partial_i f)(\partial_{\bar{i}} f)\right) = M_f,$$

and so

$$
\begin{aligned}
w^T D_{T^C} Q_1 D_{T^C} w - w^T D_{T^C} Q_2 D_T w &= w^T D_{T^C} M_f w \\
&= \lambda w^T D_{T^C} w > 0.
\end{aligned}
$$

A similar computation shows that

$$
Q_3 D_T - Q_2^T D_{T^C} = M_f
$$

and therefore

$$
w^T D_T Q_3 D w - w^T D_T Q_2^T D_{T^C} w = \lambda w^T D_T w > 0.
$$

Thus,

$$
\left(\begin{array}{c|c}
w^T D_{T^C} Q_1 D_{T^C} w & w^T D_{T^C} Q_2 D w \\
\hline
w^T D Q_2^T D_{T^C} w & w^T D Q_3 D w
\end{array}\right) \succ 0,
$$

as desired. □

6.2.3 Interlacing for Fractionally Log-Concave Polynomials

Interlacing polynomials are closely connected to half-plane stability, including real stability and Hurwitz stability; see, e.g. [Brä07]. More recently, the Lorentzian analogue of interlacing has received some attention, including connections to the incidence geometry of tropical linear spaces [Wan26].

In this section, we find the analogue of interlacing for fractionally log-concave polynomials and demonstrate a connection with interlacing of Hurwitz stable polynomials.

Theorem 6.2.10. *Let $f, g \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ and consider $yf + zg \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n, y, z]$. Let $0 < \alpha < 1$. Then $\nabla^2(yf+zg)_\alpha$ has one positive eigenvalue for all $(x_1, \ldots, x_n, y, z) \in \mathbb{R}_{>0}^{n+2}$ if and only if $a\nabla^2 f_\alpha^{\frac{1}{1-\alpha}} + b\nabla^2 g_\alpha^{\frac{1}{1-\alpha}}$ has one positive eigenvalue for all $a, b > 0, x \in \mathbb{R}^n_{>0}$.*

Proof. We note that

$$\nabla^2(y^\alpha f_\alpha + z^\alpha g_\alpha) = \left(\begin{array}{c|cc} y^\alpha \nabla^2 f_\alpha + z^\alpha \nabla^2 g_\alpha & \alpha y^{\alpha-1}(\nabla f_\alpha) & \alpha z^{\alpha-1}(\nabla g_\alpha) \\ \hline \alpha y^{\alpha-1}(\nabla f_\alpha)^T & \alpha(\alpha-1)y^{\alpha-2}f_\alpha & 0 \\ \alpha z^{\alpha-1}(\nabla g_\alpha)^T & 0 & \alpha(\alpha-1)z^{\alpha-2}g_\alpha \end{array}\right)$$

Since $0 < \alpha < 1$, the bottom right corner is negative definite, so by the Haynsworth inertia addivity formula, $\nabla^2(yf+zg)_\alpha$ has the same number of positive eigenvalues as M', where

$$\begin{aligned} M' &= y^\alpha \nabla^2 f_\alpha + z^\alpha \nabla^2 g_\alpha - \frac{\alpha}{\alpha-1}\begin{pmatrix} y^{\alpha-1}\nabla f_\alpha & z^{\alpha-1}\nabla g_\alpha \end{pmatrix}\begin{pmatrix} \frac{1}{y^{\alpha-2}f_\alpha} & \\ & \frac{1}{z^{\alpha-2}g_\alpha} \end{pmatrix}\begin{pmatrix} y^{\alpha-1}\nabla f_\alpha^T \\ z^{\alpha-1}\nabla g_\alpha^T \end{pmatrix} \\ &= y^\alpha\left(\nabla^2 f_\alpha + \frac{1}{f_\alpha}\left(\frac{1}{1-\alpha}-1\right)(\nabla f_\alpha)(\nabla f_\alpha)^T\right) + z^\alpha\left(\nabla^2 g_\alpha + \frac{1}{g_\alpha}\left(\frac{1}{1-\alpha}-1\right)(\nabla g_\alpha)(\nabla g_\alpha)^T\right). \end{aligned}$$

Since we may choose y, z to be any positive real numbers, this means that $aQ_1 + bQ_2$ has exactly one positive eigenvalue for all $a, b > 0$, where $Q_1 := f_\alpha \nabla^2 f_\alpha + \left(\frac{1}{1-\alpha} - 1\right)(\nabla f_\alpha)(\nabla f_\alpha)^T$ and $Q_2 := g_\alpha \nabla^2 g_\alpha + \left(\frac{1}{1-\alpha} - 1\right)(\nabla g_\alpha)(\nabla g_\alpha)^T$.

For any $p : \mathbb{R}^n \to \mathbb{R}$ and $\beta > 0$, we note that

$$\nabla^2(p^\beta) = \beta p^{\beta-2}(p\nabla^2 p + (\beta-1)(\nabla p)(\nabla p)^T),$$

so $Q_1 = (1-\alpha)f_\alpha^{\frac{1}{1-\alpha}-2}\nabla^2 f_\alpha^{\frac{1}{1-\alpha}}$ and $Q_2 = (1-\alpha)g_\alpha^{\frac{1}{1-\alpha}-2}\nabla^2 g_\alpha^{\frac{1}{1-\alpha}}$. Hence, by rescaling, $aQ_1 + bQ_2$ has one positive eigenvalue for all $a, b > 0$ if and only if $a\nabla^2(f_\alpha^{\frac{1}{1-\alpha}}) + b\nabla^2(g_\alpha^{\frac{1}{1-\alpha}})$ has one positive eigenvalue for all $a, b > 0$, as desired. □

Applying the above theorem to f^{hom} and g^{hom} with $\alpha = 1/2$ gives the following corollary.

Corollary 6.2.11. *Let $f, g \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$, and define $h(x_0, x_1, \ldots, x_n) := x_0 f + g$. Then $h^{\text{hom}} = x_0 f^{\text{hom}} + y_0 g^{\text{hom}} \in \mathbb{R}_{\geq 0}[x_0, \ldots, x_n, y_0, \ldots, y_n]$ is 1/2-log concave if and only if $a(f^{\text{hom}})^2 + b(g^{\text{hom}})^2$ is 1/2-log concave for all $a, b > 0$.*

Proof. We know that h^{hom} is 1/2-log concave if and only if $\nabla^2(x_0 f^{\text{hom}} + y_0 g^{\text{hom}})_{1/2}$ has one positive eigenvalue for all $(x_0, \vec{x}, y_0, \vec{y}) \in \mathbb{R}^{2n}_{>0}$. By Theorem 6.2.10, this happens if and only if $a\nabla^2((f^{\text{hom}})^2_{1/2}) + b\nabla^2((g^{\text{hom}})^2_{1/2}) = \nabla^2(a(f^{\text{hom}})^2 + b(g^{\text{hom}})^2)_{1/2}$ has one positive eigenvalue for all $a, b > 0$, i.e. if and only if $a(f^{\text{hom}})^2 + b(g^{\text{hom}})^2$ is 1/2-log concave for all $a, b > 0$. $\square$

This condition also aligns with interlacing for Hurwitz stable polynomials.

Theorem 6.2.12. *Let $f, g \in \mathbb{R}[x_1, \ldots, x_n]$ such that $f(t\mathbb{1}), g(t\mathbb{1}) \to \infty$ as $t \to \infty$. (This assumption can always be achieved by replacing f with $-f$ and/or g with $-g$ as necessary.) Then $f + zg$ is Hurwitz stable if and only if $af^2 + bg^2$ is Hurwitz stable for all $a, b > 0$.*

Proof. First, suppose that $f + zg$ is Hurwitz stable and consider $af^2 + bg^2$. By rescaling, we may assume without loss of generality that $a = 1$, and since $b > 0$, we may replace b by b^2. Thus, it suffices to show that $f^2 + b^2g^2$ is Hurwitz stable for all $b \in \mathbb{R}$. Indeed, we note that $f^2 + b^2g^2 = (f + ibg)(f - ibg)$. Since $f + zg$ is Hurwitz stable, so is $f + z_0 g$ for all z_0 such that $\operatorname{Re}(z_0) \geq 0$. Hence, $f + ibg$ and $f - ibg$ are both Hurwitz stable, so $f^2 + b^2g^2$ is as well.

Now, suppose that $f + zg$ is not Hurwitz stable. We note that if either f or g is not Hurwitz stable, then we are done, as then either $f^2 + \varepsilon g^2$ or $\varepsilon f^2 + g^2$ is not Hurwitz stable for sufficiently small $\varepsilon > 0$.

Furthermore, we note that for sufficiently large $N \gg 0$, we have $\frac{f}{g}(N\mathbb{1}) > 0$.

Since $f + zg$ is not Hurwitz stable, there exists some $x \in \mathcal{H}^n$ such that $\operatorname{Re}\left(\frac{f}{g}(x)\right) < 0$, where $\mathcal{H}$ denotes the right half plane. Define $x(t) := (1 - t)x + t(N\mathbb{1})$. We note that $x(t) \in \mathcal{H}^n$ for all $0 \leq t \leq 1$, so $\frac{f}{g}(x(t))$ is defined and nonzero for all $0 \leq t \leq 1$. Moreover, we know that $\operatorname{Re}\left(\frac{f}{g}(x(0))\right) < 0$ and $\operatorname{Re}\left(\frac{f}{g}(x(1))\right) > 0$, so there exists some $0 < t^* < 1$ and $x^* := x(t^*)$ such that $\operatorname{Re}\left(\frac{f(x^*)}{g(x^*)}\right) = 0$. Since $f(x^*) \neq 0$, this implies that $\frac{f(x^*)}{g(x^*)} = ib$ for some $b \in \mathbb{R} \setminus \{0\}$, and hence $\frac{f^2(x^*)}{g^2(x^*)} = -b^2$. Thus, $f^2(x^*) + b^2g^2(x^*) = 0$. Since $x^* \in \mathcal{H}^n$, this implies that $f^2 + b^2g^2$ is not Hurwitz stable, as desired. $\square$

The condition in Theorem 6.2.10 seems to be a strictly stronger condition than requiring that $af^{\text{hom}} + bg^{\text{hom}}$ is 1/2-log concave for all $a, b > 0$, at least on the level of matrices. Note that there is nothing inherent to the structure of $\nabla^2 f_{1/2}^{\mathrm{hom}}$ that should prevent finding similar counterexamples with explicit f, g.

Example 6.2.13. Let $Q_1 = \nabla^2 f_{1/2}^{\mathrm{hom}}$, $Q_2 = \nabla^2 g_{1/2}^{\mathrm{hom}}$, $\widetilde{Q}_1 = \frac{1}{2}\nabla^2 (f_{1/2}^{\mathrm{hom}})^2$, $\frac{1}{2}\widetilde{Q}_2 = \nabla^2 (g_{1/2}^{\mathrm{hom}})^2$, all evaluated at $a \in \mathbb{R}_+^{2n}$. By direct computation and homogeneity,

$$\begin{aligned}\widetilde{Q}_1 &= f_{1/2}^{\mathrm{hom}}(a)\nabla^2 f_{1/2}^{\mathrm{hom}}(a) + (\nabla f_{1/2}^{\mathrm{hom}}(a))(\nabla f_{1/2}^{\mathrm{hom}}(a))^T \\ &= \frac{1}{(n/2)(n/2-1)}(a^T Q_1 a)Q_1 + \left(\frac{1}{n/2-1}\right)^2 (Q_1 a)(Q_1 a)^T,\end{aligned}$$

and similarly for Q_2. Thus, on the level of matrices, the statement that $\lambda f^{\mathrm{hom}} + \mu g^{\mathrm{hom}}$ is 1/2-log concave for all $\lambda, \mu > 0$ says that $\lambda Q_1 + \mu Q_2$ has one positive eigenvalue for all $\lambda, \mu > 0$, while the condition from Theorem 6.2.10 says that

$$\lambda((a^T Q_1 a)Q_1 + \left(1 + \frac{2}{n-2}\right)(Q_1 a)(Q_1 a)^T) + \mu((a^T Q_2 a)Q_2 + \left(1 + \frac{2}{n-2}\right)(Q_2 a)(Q_2 a)^T)$$

has one positive eigenvalue for all $a, b > 0$, where we know that $a^T Q_1 a, a^T Q_2 a > 0$.

Consider

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad Q_2 = \begin{pmatrix} \frac{21}{100} & -\frac{4}{5} & -\frac{3}{5} \\ -\frac{4}{5} & -1 & 0 \\ -\frac{3}{5} & 0 & -1 \end{pmatrix}$$

with $a = e_1$. We note that $Q_1 \begin{pmatrix} 21/100 \\ 4/5 \\ 3/5 \end{pmatrix} = Q_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, and that $\begin{pmatrix} 21/100 \\ 4/5 \\ 3/5 \end{pmatrix}^T Q_1 \begin{pmatrix} 21/100 \\ 4/5 \\ 3/5 \end{pmatrix}, e_1^T Q_2 e_1 > 0$, and Q_1 and Q_2 both have strictly Lorentzian signature, so $\lambda Q_1 + \mu Q_2$ has one positive eigenvalue for all $\lambda, \mu > 0$.

On the other hand, we compute

$$\widetilde{Q}_1 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \widetilde{Q}_2 = \begin{pmatrix} \frac{441}{2500} & -\frac{84}{125} & -\frac{63}{125} \\ -\frac{84}{125} & \frac{171}{100} & \frac{36}{25} \\ -\frac{63}{125} & \frac{36}{25} & \frac{87}{100} \end{pmatrix}$$

We note that

$$\frac{1}{2}\widetilde{Q}_1 + \frac{1}{2}\widetilde{Q}_2 = \begin{pmatrix} \frac{10441}{5000} & -\frac{42}{125} & -\frac{63}{250} \\ -\frac{42}{125} & \frac{71}{200} & \frac{18}{25} \\ -\frac{63}{250} & \frac{18}{25} & -\frac{13}{200} \end{pmatrix},$$

which has two positive eigenvalues. Hence, there exist $\lambda, \mu > 0$ such that $\lambda\widetilde{Q}_1 + \mu\widetilde{Q}_2$ has more than one positive eigenvalue, so it is not sufficient to check the condition on only Q_1, Q_2.

6.2.4 Fractional Log Concavity and Derivatives

In this section, we explore the connection between fractional log-concavity of a polynomial f and fractional log-concavity if its derivatives. Here, we present some positive results bounding fractional log-concavity of f in terms of its derivatives and vice versa. These results originally came about while attempting to generalize the story of completely log-concave polynomials to the fractional log concavity setting. For more details on the limitations of such a generalization, see Section 6.3.3.

One elegant fact about (non-fractional) log-concavity is that for f homogenenous of degree $d \geq 3$ and any $a \in \mathbb{R}^n_{>0}$, f is log concave at a if and only if $D_a f$ is log concave at a. Unfortunately, the analogous statement for fractional log-concavity requires changing the parameter as we go from f to its derivative.

Proposition 6.2.14. *Let f be a homogeneous polynomial of degree $d > 2$ with positive coefficients and $a \in \mathbb{R}^n_+$ with $f(a) > 0$. Then the following are equivalent*

- *f is α-log-concave at $x = a^{1/\alpha}$*
- *$D_a f$ is β-log-concave at $x = a^{1/\beta}$*

where $\alpha, \beta \in (0, 1]$ satisfy

$$\left(\frac{1-\beta}{\beta}\right) \cdot \left(\frac{d-1}{d-2}\right) = \frac{1-\alpha}{\alpha}.$$

Proof. Recall from Proposition 6.2.6 that f is α-log concave at $a^{1/\alpha}$ if and only if $Q := \nabla^2 f(a) - \frac{1-\alpha}{\alpha} \operatorname{diag}\left(\frac{\partial_i f(a)}{a_i}\right)$ has at most one positive eigenvalue. Similarly, $D_a f$ is β-log concave at $a^{1/\beta}$ if and only if $Q' := \nabla^2 D_a f(a) - \frac{1-\beta}{\beta}\operatorname{diag}\left(\frac{\partial_i D_a f(a)}{a_i}\right)$ has at most one positive eigenvalue. Thus, it just remains to show that Q and Q' have the same signature.

Indeed, since f is homogeneous of degree d, we know by Euler's identity that $\nabla^2 D_a f(a) = (d-2)\nabla^2 f(a)$ and $\partial_i(D_a f)(a) = (d-1)\partial_i f(a)$. Hence,

$$Q' = (d-2)\nabla^2 f(a) - \frac{1-\beta}{\beta}(d-1)\operatorname{diag}\left(\frac{\partial_i f(a)}{a_i}\right).$$

Since we chose β such that $\left(\frac{1-\beta}{\beta}\right)\left(\frac{d-1}{d-2}\right) = \frac{1-\alpha}{\alpha}$, $Q' = (d-2)Q$. Since $d > 2$, it follows that Q and Q' have the same signature, as desired. $\square$

We make a few brief remarks on this result.

Remark 6.2.15. *We remark that the identity* $\left(\frac{1-\beta}{\beta}\right) \cdot \left(\frac{d-1}{d-2}\right) = \frac{1-\alpha}{\alpha}$ *resembles the relationship in the spectral decay found in the trickle-down theorem [Opp18]. This is likely not a coincidence, although it also does not allow us to prove fast mixing for* Δ*-matroids, as we are not able to prove any spectral gap on the two-dimensional links.*

Remark 6.2.16. *When applying Proposition 6.2.14, one must be careful about taking limits where* $a_i \to 0$ *for some* i*. For example, for* $f = y^3 + x_1x_2y$, $\alpha = 1/2$, $\beta = 2/3$, $a = (s, s, 1)$, $D_a f$ *is* β*-log concave at* $(t^{1/\beta}, t^{1/\beta}, 1)$ *whenever* $s = t > 0$*. But, for* $s = 0$, $D_a f = 3y^2 + x_1x_2$ *fails to be* $2/3$*-log concave at* $(t^{1/\beta}, t^{1/\beta}, 1)$ *for all* $t > 0$*. In particular, we must be careful when trying to apply this result to points in* $\mathbb{R}^n_{\geq 0}$ *instead of* $\mathbb{R}^n_{>0}$*.*

As the above remark shows, there are subtleties to address for checking fractional log concavity at points where some coordinates are zero. However, in the case of multiaffine polynomials, we can sidestep some of these difficulties. In particular, if f is multiaffine and α-log concave, then so is $\partial_i f$ for all i. This is because for multiaffine polynomials, one can interpret partial derivatives as taking initial forms, which preserves fractional log-concavity (Proposition 6.2.3).

Conversely, in the case of non-fractional log-concavity, we can use log concavity of partial derivatives to prove log concavity of the original polynomial. In particular, if $f \in$

$\mathbb{R}_{\geq 0}[x_1, \dots, x_n]_d$ is indecomposable and homogeneous of degree $d \geq 3$, and if $\partial_1 f, \dots, \partial_n f$ are all log concave, then $D_a f$ is log concave for all $a \in \mathbb{R}^n_{\geq 0}$[ALOV24b, Lemma 3.4], and hence f is also log concave.

In the case of fractional log-concavity of multihomogeneous polynomials, we get an analogue of [ALOV24b, Lemma 3.4] for directional derivatives of f^{hom}, but we lose the multiplicative factor from Proposition 6.2.14 for fractional log-concavity of f^{hom} itself.

Theorem 6.2.17. *Suppose f is multiaffine and multihomogeneous on $2n$ variables and $\alpha > \frac{1}{n-1}$, and let $a \in \mathbb{R}^{2n}_{>0}$. Suppose that $a_i \partial_i f + a_{\bar{i}} \partial_{\bar{i}} f$ is α-log concave at $a^{1/\alpha}$ for all $i = 1, \dots, n$. Then $D_a f$ is α-log concave at $a^{1/\alpha}$.*

Before the proof, we first note that Theorem 6.2.17 together with Proposition 6.2.14 gives the following immediate corollary.

Corollary 6.2.18. *Suppose f is multiaffine on $n \geq 4$ variables and $\sqrt{a_i} \partial_i f^{\text{hom}} + \sqrt{a_{\bar{i}}} \partial_{\bar{i}} f^{\text{hom}}$ is $1/2$-log concave at a for all $a \in \mathbb{R}^{2n}_{>0}$. Then f is $\frac{n-2}{2n-3}$-log concave.*

To prove Theorem 6.2.17, we claim it suffices to show the following proposition.

Proposition 6.2.19. *Suppose f is multiaffine and multihomogeneous on $2n$ variables and $\alpha > \frac{1}{n-1}$. If $\partial_i f + \partial_{\bar{i}} f$ is α-log concave at $\mathbb{1}$ for all $i \in [n]$, then $D_{\mathbb{1}} f$ is α-log concave at $\mathbb{1}$.*

Proof of Theorem 6.2.17 from Proposition 6.2.19. For $a \in \mathbb{R}^{2n}_{>0}$, define

$$(x_1, \dots, x_n, y_1, \dots, y_n) = f(a_1 x_1, \dots, a_n x_n, a_{\bar{1}} y_1, \dots, a_{\bar{n}} y_n).$$

We note that

$$(\partial_i g + \partial_{\bar{i}} g)(x_1, \dots, y_n) = \left.(a_i \partial_i f + a_{\bar{i}} \partial_{\bar{i}} f)\right|_{(a_1 x_1, \dots, a_{\bar{n}} y_n)}.$$

Further, we note that for any polynomial h,

$$h(a_1 x_1^\alpha, \dots, a_n x_n^\alpha) = h((a_1^{1/\alpha} x_1)^\alpha, \dots, (a_n^{1/\alpha} x_n)^\alpha) = h_\alpha(a_1^{1/\alpha} x_1, \dots, a_n^{1/\alpha} x_n).$$

Hence $h(a_1 x_1, \dots, a_n x_n)$ is α-log concave at $\mathbb{1}$ if and only if h is α-log concave at $a^{1/\alpha}$.

Applying this to our expression above, we see that $a_i\partial_i f + a_{\bar{\imath}} f$ is α-log concave at $a^{1/\alpha}$ if and only if $\partial_i g + \partial_{\bar{\imath}} g$ is α-log concave at $\mathbb{1}$.

We can then apply Proposition 6.2.19, so $D_{\mathbb{1}} g$ is α-log concave at $\mathbb{1}$. We again note that

$$D_{\mathbb{1}} g = (D_a f)|_{(a_1 x_1, \dots, a_{\bar{n}} y_n)},$$

so this is equivalent to α-log concavity of $D_a f$ at $a^{1/\alpha}$, as desired. □

Thus, it just remains to prove Proposition 6.2.19. We prepare the proof with a few computations.

Lemma 6.2.20. *Let $M_i = \nabla^2(\partial_i f + \partial_{\bar{\imath}} f)_\alpha|_{\mathbb{1}}$. Then for any $j \neq i$,*

$$[M_i(e_j + e_{\bar{\jmath}})]_\ell = \begin{cases} -\alpha(1-\alpha)\partial_\ell f(\mathbb{1}), & \ell = j, \bar{\jmath} \\ 0, & \ell = i, \bar{\imath} \\ \alpha^2 dl_\ell f(\mathbb{1}), & \textit{otherwise} \end{cases}$$

Proof. First, we note that $\partial_i f + \partial_{\bar{\imath}} f$ does not depend on x_i, y_i, so $[M_i v]_i = [M_i v]_{\bar{\imath}} = 0$ for all $v \in \mathbb{R}^{2n}$.

Next, we consider $\ell = j$. Since no terms of $(\partial_i f + \partial_{\bar{\imath}} f)_\alpha$ depend on both x_j and y_j, then $\partial^2_{j\bar{\jmath}}(\partial_i f + \partial_{\bar{\imath}} f)_\alpha = 0$. Moreover, since f is multiaffine, then $\partial^2_{jj} f = 0$. Thus, we compute

$$\begin{aligned} [M_i(e_j + e_{\bar{\jmath}})]_j &= \partial^2_{jj}(\partial_i f + \partial_{\bar{\imath}} f)_\alpha|_{\mathbb{1}} + \partial^2_{j\bar{\jmath}}(\partial_i f + \partial_{\bar{\imath}} f)_\alpha|_{\mathbb{1}} \\ &= \partial^2_{jj}(\partial_i f + \partial_{\bar{\imath}} f)_\alpha|_{\mathbb{1}} \\ &= \partial^2_{jj}(x_i\partial_i f + y_i\partial_{\bar{\imath}} f)_\alpha|_{\mathbb{1}} \\ &= \partial^2_{jj}(f_\alpha)|_{\mathbb{1}} \\ &= -\alpha(1-\alpha)\partial_j f(\mathbb{1}). \end{aligned}$$

An analogous computation shows the result for $\ell = \bar{\jmath}$.

Finally, we consider $\ell \neq i, \bar{\imath}, j, \bar{\jmath}$. By multihomogeneity,

$$\begin{aligned}
[M_i(e_j + e_{\bar{\jmath}})]_\ell &= \partial^2_{\ell j}(\partial_i f + \partial_{\bar{\imath}} f)_\alpha|_{\mathbb{1}} + \partial^2_{\ell \bar{\jmath}}(\partial_i f + \partial_{\bar{\imath}} f)_\alpha|_{\mathbb{1}} \\
&= ((\partial_j + \partial_{\bar{\jmath}})\partial_\ell((x_i\partial_i f + y_i\partial_{\bar{\imath}} f)_\alpha))|_{\mathbb{1}} \\
&= ((x_j\partial_j + y_j\partial_{\bar{\jmath}})\partial_\ell(f_\alpha))|_{\mathbb{1}} \\
&= (\alpha\partial_\ell(f_\alpha))|_{\mathbb{1}} \\
&= \alpha^2\partial_\ell f(\mathbb{1})
\end{aligned}$$

□

Lemma 6.2.21. *Suppose that $f \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ is homogenous of degree 1 in (x_i, y_i) for all $i \in [n]$. Let $M_i = \nabla^2(\partial_i f + \partial_{\bar{\imath}} f)|_{\mathbb{1}}$, and let $a_k = (n-2)\frac{\alpha}{1-\alpha}(e_k + e_{\overline{k}}) + \sum_{j \neq k} e_j + e_{\bar{\jmath}}$. Then, for any $k \neq i$,*

$$[M_i a_k]_\ell = \begin{cases} 0, & \text{if } \ell = i, \bar{\imath}, k, \overline{k} \\ \alpha\left((n-2)\frac{\alpha}{1-\alpha} - 1\right)\partial_\ell f(\mathbb{1}), & \text{otherwise} \end{cases}.$$

In particular, $a_k^T M_i a_k > 0$.

Proof. By Lemma 6.2.20, we see that $[M_i a_k]_i = [M_i a_k]_{\bar{\imath}} = 0$. The remaining cases also follow from linearity and Lemma 6.2.20.

Inded, next consider the case where $\ell = k, \overline{k}$. We compute

$$\begin{aligned}
[M_i a_k]_k &= (n-2)\frac{\alpha}{1-\alpha}[M_i(e_k + e_{\overline{k}})]_k + \sum_{j \neq k}[M_i(e_j + e_{\bar{\jmath}})]_k \\
&= (n-2)\frac{\alpha}{1-\alpha}\left((-\alpha)(1-\alpha)\partial_k f(\mathbb{1})\right) + \sum_{j \neq k, i} \alpha^2\partial_k f(\mathbb{1}) = 0,
\end{aligned}$$

and an analogous computation holds for $[M_i a_k]_{\overline{k}}$.

Now consider the case where $\ell \neq i, \bar{\imath}, k, \overline{k}$. Applying Lemma 6.2.20 once more,

$$\begin{aligned}
[M_i a_k]_\ell &= (n-2)\frac{\alpha}{1-\alpha}[M_i(e_k + e_{\overline{k}})]_\ell + \sum_{j \neq i, \ell, k}[M_i(e_j + e_{\bar{\jmath}})]_\ell + [M_i(e_\ell + e_{\bar{\ell}})]_\ell \\
&= (n-2)\frac{\alpha}{1-\alpha}\left(\alpha^2\partial_\ell f(\mathbb{1})\right) + (n-3)\left(\alpha^2\partial_\ell f(\mathbb{1})\right) - \alpha(1-\alpha)\partial_\ell f(\mathbb{1}) \\
&= \alpha\left((n-2)\frac{\alpha}{1-\alpha} - 1\right)\partial_\ell f(\mathbb{1}),
\end{aligned}$$

as desired.

Finally, we note that

$$a_k^T M_i a_k = \alpha \left((n-2)\frac{\alpha}{1-\alpha} - 2 \right) (n-2) f(\mathbb{1}).$$

Since $\alpha > \frac{1}{n-1}$ by assumption, then $\frac{1-\alpha}{\alpha} > n-2$, so $(n-2)\frac{\alpha}{1-\alpha} > 1$. Hence $a_k^T M_i a_k > 0$, which completes the proof. $\square$

With these lemmata, we are now ready to prove Proposition 6.2.19.

Proof of Proposition 6.2.19. Let $M_i = \nabla^2(\partial_i f + \partial_{\bar{i}} f)_\alpha|_{\mathbb{1}}$, and define $Q_k = \sum_{i=1}^k M_i$. Then $Q_n = \sum_{i=1}^n \nabla^2(\partial_i f + \partial_{\bar{i}} f)_\alpha|_{\mathbb{1}} = \nabla^2(D_{\mathbb{1}} f)_\alpha|_{\mathbb{1}}$, so it suffices to show that Q_n has at most one positive eigenvalue. We show by induction on k that Q_k has at most one positive eigenvalue for all k.

As a base case, $Q_1 = M_1$ has at most one positive eigenvalue since we assumed that $\partial_1 f + \partial_{\bar{1}} f$ is α-log concave at $\mathbb{1}$.

Now suppose that Q_{k-1} has at most one positive eigenvalue, where $k \geq 2$. Defining $a_k = (n-2)\frac{\alpha}{1-\alpha}(e_k + e_{\bar{k}}) + \sum_{j \neq k} e_j + e_{\bar{j}}$ as above, we note that

$$[Q_{k-1}a_k]_\ell = \begin{cases} (k-2)\alpha\left((n-2)\frac{\alpha}{1-\alpha} - 1\right)\partial_\ell f(\mathbb{1}), & \ell \leq k-1 \text{ or } \bar{\ell} \leq k-1 \\ 0, & \ell = k, \bar{k} \\ (k-1)\alpha\left((n-2)\frac{\alpha}{1-\alpha} - 1\right)\partial_\ell f(\mathbb{1}), & \ell \geq k+1 \text{ or } \bar{\ell} \geq k+1 \end{cases}.$$

Thus,

$$\begin{aligned} a_k^T Q_{k-1} a_k &= ((k-1)(k-2) + (n-k)(k-1))\,\alpha\left((n-2)\frac{\alpha}{1-\alpha} - 1\right) f(\mathbb{1}) \\ &= (n-2)(k-1)\alpha\left((n-2)\frac{\alpha}{1-\alpha} - 1\right) f(\mathbb{1}) > 0. \end{aligned}$$

Since Q_{k-1} has at most one positive eigenvalue, this implies that $Q_{k-1} \preceq 0$ on $(Q_{k-1}a_k)^\perp$.

Now consider $b = ((n-2)\frac{\alpha}{1-\alpha} + k - 2)\sum_{i=1}^{k-1}(e_i + e_{\bar{i}}) + (k-1)\sum_{i=k+1}^n (e_i + e_{\bar{i}})$. By construction, we chose our coefficients so that $M_k b = Q_{k-1}a_k$. Moreover, since $k \geq 2$, b is nonnegative, and in fact b_ℓ is positive for all $\ell \neq k, \overline{k}$. Since $[M_k b]_\ell > 0$ for all $\ell \neq k, \overline{k}$, we have $b^T M_k b > 0$. Since M_k has at most one positive eigenvalue, it follows that $M_k \preceq 0$ on $(M_k b)^\perp = (Q_{k-1} a_k)^\perp$.

Hence, Q_{k-1} and M_k are both negative semidefinite on $L = (Q_{k-1} a_k)^\perp$, so $Q_k = Q_{k-1} + M_k$ is also negative semidefinite on L. Therefore, Q_k has at most one positive eigenvalue, as desired.

Then, by induction, Q_n has at most one positive eigenvalue, so $D_{\mathbb{1}} f$ is α-log concave at $\mathbb{1}$, as desired. $\square$

6.2.5 Fractional Log-Concavity of Homogenized Quadratics

In this section, we study conditions for f^{hom} to be 1/2-log concave when $\deg(f) = 2$. Besides polynomials on $n = 4$ variables, this is the other natural base case for any inductive argument for proving 1/2-log concavity.

The main result is the following theorem, which reduces the question to homogeneous quadratics, and gives an explicit hyperplane to test.

Theorem 6.2.22. *Let $f = f_0 + c \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$, where f_0 is homogeneous of degree two and $c \geq 0$. Then the following are equivalent.*

(i) f^{hom} *is 1/2-log concave.*

(ii) f_0^{hom} *is 1/2-log concave.*

(iii) For all $(a, \overline{a}) \in \mathbb{R}^{2n}_{>0}$, $\nabla^2 (f_0^{\text{hom}})_{1/2}(a, \overline{a}) \preceq 0$ on $(0, w)^\perp$, where $w_i = \partial_{\overline{i}}((f_0^{\text{hom}})_{1/2})\big|_{(a,\overline{a})} = \frac{1}{2\sqrt{a_{\overline{i}}}}(\partial_{\overline{i}} f_0^{\text{hom}}(\sqrt{a}, \sqrt{\overline{a}}))$.

(iv) For all $(a, \overline{a}) \in \mathbb{R}^{2n}_{>0}$ and all $c' \geq 0$, $\nabla^2 (f_0 + c')^{\text{hom}}_{1/2}(a, \overline{a}) \preceq 0$ on $(0, w)^\perp$, where w is the same as above.

(v) For all $c' \geq 0$, $(f_0 + c')^{\text{hom}}$ is 1/2-log concave.

Proof. (i) $\Rightarrow$ (ii) follows by taking initial forms. (v) $\Rightarrow$ (i) is clear by taking $c' = c$. To show (iv) $\Rightarrow$ (v), we note that (iv) implies that $\nabla^2(f_0 + c')^{\mathrm{hom}}_{1/2}$ has at most one positive eigenvalue, which implies that $(f_0 + c')^{\mathrm{hom}}$ is 1/2-log concave (Proposition 6.2.4). Thus, it just remains to show (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).

(ii) $\Rightarrow$ (iii): Fix $(a, \overline{a}) \in \mathbb{R}^{2n}_{>0}$, and for ease of notation, define $Q := \nabla^2(f_0^{\mathrm{hom}})_{1/2}(a, \overline{a})$. Let $v = (a, 0)$ and consider Qv. Using the fact that f_0 is homogeneous of degree two, and hence $(f_0^{\mathrm{hom}})_{1/2}$ is homogeneous of degree one in the x variables,

$$
\begin{aligned}
[Qv]_i &= \sum_{j=1}^{n} a_j \partial^2_{ij}((f_0^{\mathrm{hom}})_{1/2})(a, \overline{a}) \\
&= \sum_{j=1}^{n} a_j \partial_j \left(\partial_i((f_0^{\mathrm{hom}})_{1/2})\right)(a, \overline{a}) = 0 \\
[Qv]_{\overline{\imath}} &= \sum_{j=1}^{n} a_j \partial^2 \overline{\imath} j((f_0^{\mathrm{hom}})_{1/2})(a, \overline{a}) \\
&= \sum_{j=1}^{n} a_j \partial_j (\partial_{\overline{\imath}}((f_0^{\mathrm{hom}})_{1/2}))(a, \overline{a}) \\
&= \partial_{\overline{\imath}}((f_0^{\mathrm{hom}})_{1/2})(a, \overline{a}).
\end{aligned}
$$

Hence, $Qv = (0, w)$, $v^T Qv = 0$, and it just remains to show that $Q \preceq 0$ on $(0, w)^\perp$.

Now, define $v_\varepsilon := v + \varepsilon(a, \overline{a})$. We note that

$$
v_\varepsilon^T Q v_\varepsilon = 2\varepsilon \overline{a}^T(\nabla_y((f_0^{\mathrm{hom}})_{1/2})(a, \overline{a})) + \varepsilon^2 \begin{pmatrix} a \\ \overline{a} \end{pmatrix}^T Q \begin{pmatrix} a \\ \overline{a} \end{pmatrix} > 0
$$

for all $\varepsilon > 0$, so by 1/2-log concavity of f_0, $Q \preceq 0$ on $(Qv_\varepsilon)^\perp$ for all $\varepsilon > 0$. Taking the limit $\varepsilon \to 0$, we conclude that $Q \preceq 0$ on $(Qv)^\perp = (0, w)^\perp$, as desired.

(iii) $\Rightarrow$ (iv): We note that $\nabla^2(f_0 + c')^{\mathrm{hom}}_{1/2} = \nabla^2(f_0^{\mathrm{hom}})_{1/2} + c'\nabla^2(\sqrt{y_1, \ldots, y_n})$, so it suffices to show that $\nabla^2(\sqrt{y_1 \ldots y_n})(a, \overline{a}) \preceq 0$ on $w^\perp$.

Indeed, we note that

$$
\nabla^2 \log(\sqrt{y_1, \ldots, y_n}) = \frac{1}{2} \begin{pmatrix} -1/y_1^2 & & \\ & \ddots & \\ & & -1/y_n^2 \end{pmatrix} \preceq 0,
$$

so $\sqrt{y_1 \ldots y_n}$ is log concave, and hence $\nabla^2 \sqrt{y_1 \ldots y_n}$ has at most one positive eigenvalue.

By direct computation,

$$\nabla^2(\sqrt{y_1 \ldots y_n})(a, \overline{a}) = \frac{1}{4}\sqrt{a_{\overline{1}} \ldots a_{\overline{n}}}\, \mathrm{diag}(1/a_{\overline{1}}, \ldots, 1/a_{\overline{n}})(\mathbb{1}\mathbb{1}^T - 2I)\, \mathrm{diag}(1/a_{\overline{1}}, \ldots, 1/a_{\overline{n}}).$$

Hence, we wish to show that $M \coloneqq \mathrm{diag}(1/a_{\overline{\imath}} : i \in [n])(\mathbb{1}\mathbb{1}^T - 2I)\, \mathrm{diag}(1/a_{\overline{\imath}} : i \in [n]) \preceq 0$ on $w^{\perp}$. Since we know that M has at most one positive eigenvalue, it suffices to show that $w^T M^{-1} w > 0$.

Indeed, we claim that $M^{-1}w \in \mathbb{R}^n_{>0}$. Since $w \in \mathbb{R}^n_{\geq 0} \setminus \{0\}$, this proves the claim. We note that

$$M^{-1} = \frac{1}{2n-4}\, \mathrm{diag}(a_{\overline{1}}, \ldots, a_{\overline{n}})(\mathbb{1}\mathbb{1}^T - (n-2)I)\, \mathrm{diag}(a_{\overline{1}}, \ldots, a_{\overline{n}}).$$

Hence,

$$[M^{-1}w]_i = a_{\overline{\imath}}\left(\left(\sum_{j=1}^n a_{\overline{\jmath}}(f^{\mathrm{hom}}_{1/2})(a, \overline{a})\right) - (n-2)a_{\overline{\imath}}\partial_{\overline{\imath}}(f^{\mathrm{hom}}_{1/2})(a, \overline{a})\right).$$

We note that

$$a_{\overline{\imath}}\partial_{\overline{\imath}}(f^{\mathrm{hom}}_{1/2})(a, \overline{a}) = \frac{1}{2}\sqrt{a_{\overline{\imath}}}\partial_{\overline{\imath}}(f^{\mathrm{hom}})(\sqrt{a}, \sqrt{\overline{a}}).$$

Since f has nonnegative coefficients, we know that $y_i \partial_{\overline{\imath}} f^{\mathrm{hom}}(x, y) < f^{\mathrm{hom}}(x, y)$ for all $(x, y) \in \mathbb{R}^{2n}_{>0}$, so it follows that $a_{\overline{\imath}}\partial_{\overline{\imath}}(f^{\mathrm{hom}}_{1/2})(a, \overline{a}) < \frac{1}{2} f^{\mathrm{hom}}(\sqrt{a}, \sqrt{\overline{a}})$.

On the other hand, $f^{\mathrm{hom}}_{1/2}$ is homogeneous of degree $\frac{n-2}{2}$ in the y variables, so

$$\sum_{j=1}^n a_{\overline{\jmath}}\partial_{\overline{\jmath}}(f^{\mathrm{hom}}_{1/2})(a, \overline{a}) = \frac{n-2}{2} f^{\mathrm{hom}}_{1/2}(a, \overline{a}) = \frac{n-2}{2} f^{\mathrm{hom}}(\sqrt{a}, \sqrt{\overline{a}}).$$

Combining these, we find

$$(n-2)a_{\overline{\imath}}\partial_{\overline{\imath}}(f^{\mathrm{hom}}_{1/2})(a, \overline{a}) < \sum_{j=1}^n a_{\overline{\jmath}}\partial_{\overline{\jmath}}(f^{\mathrm{hom}}_{1/2})(a, \overline{a}),$$

so $[M^{-1}w]_i > 0$, as desired. $\square$

6.3 A Gallery of Ideas and Counterexamples

6.3.1 Fractional Log-Concavity and the Strong Exchange Axiom

The evenness of the Δ-matroid in Conjecture 6.1.4 is important. There exist non-even Δ-matroids on as few as three elements that are not 1/2-log concave.

Proposition 6.3.1. *Let $f = a + x_1 + x_2 + x_3 + x_1x_2x_3$ where $a > 0$. Then f^{hom} is not $1/2$-log concave.*

Proof. Let $v > 0$, and define $f_v(x) = f(vx_1, vx_2, vx_3)$. We note that f^{hom} is 1/2-log concave at $(v^2, v^2, v^2, 1, 1, 1)$ if and only if f_v^{hom} is 1/2-log concave at $(1, 1, 1, 1, 1, 1)$. By Theorem 6.2.7, this happens if and only if

$$M(v) = \begin{pmatrix} -v\left(v^2+1\right)(a+2v) & v^2\left(av+v^2-1\right) & v^2\left(av+v^2-1\right) \\ v^2\left(av+v^2-1\right) & -v\left(v^2+1\right)(a+2v) & v^2\left(av+v^2-1\right) \\ v^2\left(av+v^2-1\right) & v^2\left(av+v^2-1\right) & -v\left(v^2+1\right)(a+2v) \end{pmatrix} \preceq 0.$$

We compute

$$\det(M(v)) = v^3(-a - 4v + av^2)(a + v + 2av^2 + 3v^3)^2.$$

If $a > 0$, then $\det(M(v)) > 0$ for sufficiently large v, which implies that $M(v) \not\preceq 0$, and hence that f^{hom} is not 1/2-log concave. $\square$

Example 6.3.2. Consider the (non-even) Δ-matroid with bases $\mathcal{B} = \{\varnothing, 1, 2, 3, 123\}$. Then $f_{\mathcal{B}}$ is exactly of the form in Proposition 6.3.1 with $a = 1$. In particular, $\mathcal{B}$ are the bases of a Δ-matroid, but $f_{\mathcal{B}}^{\mathrm{hom}}$ is not 1/2-log concave.

In fact, Proposition 6.3.1 tells us that if $\mathcal{B}$ does not have strong exchange, then $f_{\mathcal{B}}^{\mathrm{hom}}$ is not 1/2-log concave. This is because we have a forbidden minor characterization for strong exchange.

Theorem 6.3.3. *Let $([n], \mathcal{B})$ be a Δ-matroid that does not have the strong exchange property. Then there exists $S \in \mathcal{B}$ and distinct elements $\{i, j, k\} \in [n]$ such that $S\Delta\{i, j, k\} \in \mathcal{B}$ but*

$S\Delta i, S\Delta j, S\Delta k \notin \mathcal{B}$. In particular, by taking the three-dimensional restriction to the face spanned by $[S, S\Delta\{i,j,k\}]$, $\mathcal{B}$ has a minor isomorphic to $\{\varnothing, 12, 13, 23, 123\}$.

Proof. Suppose $([n], \mathcal{B})$ is a Δ-matroid that does not satisfy the strong exchange property. Then, by [Mur21, Theorem 3.1], the parity homogenization $([n] \cup \{*\}, \widetilde{\mathcal{B}})$ is not an even Δ-matroid.

Suppose $\widetilde{S}, \widetilde{T} \in \widetilde{\mathcal{B}}$ fail the even Δ-matroid exchange axiom, chosen so that $|S\Delta T|$ is minimal among all such pairs of bases. Then we claim that $|S\Delta T|$ is odd, and that $\widetilde{S}\Delta\{i,*\} \notin \widetilde{\mathcal{B}}$ for all $i \in (\widetilde{S}\Delta\widetilde{T}) \setminus \{*\}$.

Indeed, first suppose that $|S\Delta T|$ is even, so $\widetilde{S}\Delta\widetilde{T} = S\Delta T$. Suppose that $\widetilde{S}, \widetilde{T}$ fail the exchange axiom with respect to i. Since $\mathcal{B}$ is a Δ-matroid, then either $S\Delta\{i\} \in \mathcal{B}$ or there exists $j \in (S\Delta T) \setminus \{i\}$ usch that $S\Delta\{i,j\} \in \mathcal{B}$. In the latter case, we would have $\widetilde{S}\Delta\{i,j\} \in \widetilde{\mathcal{B}}$, so we must have $S\Delta\{i\} \in \mathcal{B}$. By our minimality assumption, there exists $j \in (\widetilde{S\Delta\{i\}}\Delta\widetilde{T}) \setminus \{*\} = (\widetilde{S}\Delta\widetilde{T}) \setminus \{i\}$ such that $\widetilde{S\Delta\{i\}}\Delta\{j,*\} \in \widetilde{\mathcal{B}}$. But we note that $\widetilde{S\Delta\{i\}}\Delta\{j,*\} = \widetilde{S}\Delta\{i,j\}$, which cannot be in $\widetilde{\mathcal{B}}$. Hence, we have reached a contradiction, so we must have that $|S\Delta T|$ is odd.

Next, we claim that for all $i \in \widetilde{S}\Delta\widetilde{T}$ such that $i \neq *$, there exists $j \in (\widetilde{S}\Delta\widetilde{T}) \setminus \{i\}$ such that $\widetilde{S}\Delta\{i,j\} \in \widetilde{\mathcal{B}}$. Indeed, we know that either $S\Delta\{i\} \in \mathcal{B}$ or $S\Delta\{i,j\} \in \mathcal{B}$ for some $j \in (S\Delta T) \setminus \{i\}$. In the former case, we have $\widetilde{S}\Delta\{i,*\} \in \widetilde{\mathcal{B}}$, and in the latter, $\widetilde{S}\Delta\{i,j\} \in \widetilde{\mathcal{B}}$. Therefore, the only way $\widetilde{S}$ and $\widetilde{T}$ can fail the Δ-matroid exchange axiom is if $\widetilde{S}\Delta\{i,*\} \notin \widetilde{\mathcal{B}}$ for all $i \in (\widetilde{S}\Delta\widetilde{T}) \setminus \{*\}$. Thus, $S\Delta\{i\} \notin \mathcal{B}$ for all $i \in S\Delta T$. Conversely, we note that if $|S\Delta T|$ is odd and $S\Delta\{i\} \notin \mathcal{B}$ for all $i \in S\Delta T$, then $\widetilde{S}, \widetilde{T}$ will fail the Δ-matroid exchange axiom with respect to $*$.

Next, we claim that if we have $X \subsetneq S\Delta T$ with $|X|$ odd, then $S\Delta X \notin \mathcal{B}$. This follows directly from our observation above: $|S\Delta(S\Delta X)| = |X|$ is odd, and we have $S\Delta\{i\} \notin \mathcal{B}$ for all $i \in X \subset (S\Delta T)$, so if $S\Delta X$ were in $\mathcal{B}$, then $\widetilde{S}, \widetilde{S\Delta X}$ would fail the Δ-matroid exchange axiom with respect to $*$, which would contradict the minimality of $|S\Delta T|$.

On the other hand, we know that for all $i \in S\Delta T$, there exists $j \in S\Delta T$ such that

$S\Delta\{i,j\} \in \mathcal{B}$, and since $S\Delta\{i\} \notin \mathcal{B}$, we must have $j \neq i$. Since $|(S\Delta\{i,j\})\Delta T|$ is odd and strictly less than $|S\Delta T|$, then by our minimality assumption, there must exist $k \in (S\Delta\{i,j\})\Delta T = (S\Delta T) \setminus \{i,j\}$ such that $(S\Delta\{i,j\})\Delta\{k\} = S\Delta\{i,j,k\} \in \mathcal{B}$. Since i,j,k are all distinct and $\{i,j,k\} \subseteq S\Delta T$, this then implies that $\{i,j,k\} = S\Delta T$, so $|S\Delta T| = 3$. Moreover, we also have that $S\Delta\{i\}, S\Delta\{j\}, S\Delta\{k\} \notin \mathcal{B}$, which completes the proof. □

Corollary 6.3.4. *If $\mathcal{B}$ does not have the strong exchange property, then $f_{\mathcal{B}}^{\text{hom}}$ is not $1/2$-log concave.*

Proof. Let S, i, j, k be as in Theorem 6.3.3, and let $g^{\text{hom}} = \text{in}_w(f^{\text{hom}})$ be the initial form of f^{hom} found by degenerating to the face spanned by $[S, S\Delta\{i,j,k\}]$. Then, g^{hom} is exactly the polynomial from Example 6.3.2, and hence g^{hom} is not $1/2$-log concave. Since $1/2$-log concavity is closed under initial forms, we conclude that $f_{\mathcal{B}}^{\text{hom}}$ is not $1/2$-log concave, as desired. □

By restricting ourselves to even Δ-matroids, we ensure that we always have the strong exchange property [Mur06], so we avoid this issue. It remains open if there exists some constant α such that for every Δ-matroid $\mathcal{B}$, $f_{\mathcal{B}}^{\text{hom}}$ is α-log concave. However, we do know that if such an α exists, we can bound it strictly away from $1/2$.

Proposition 6.3.5. *For all $\alpha > \frac{1}{2.160755}$, there exists a Δ-matroid $\mathcal{B}$ such that $f_{\mathcal{B}}^{\text{hom}}$ is not α-log concave.*

Proof. Consider a family of Δ-matroids $(\mathcal{B}_n)$, where $\mathcal{B}_n = 2^{[n]} \setminus \binom{[n]}{1}$ consist of all subsets of $[n]$ besides the singletons.

Since f is symmetric, then the eigenvalues of the matrix from Theorem 6.2.7 are exactly $A - B$ and $A + (n-1)B$, where $A = f^{(t)}(\partial_1 f^{(t)}) - (\partial_1 f^{(t)})^2 - a(\partial_1 f^{(t)})(\partial_{\bar{1}} f^{(t)})$ and $B = f^{(t)}(\partial_{12}^2 f^{(t)}) - (\partial_1 f^{(t)})^2$. Thus, f^{hom} is α-log concave at $(t\mathbb{1}, \mathbb{1})$ for all $t > 0$ if and only if $\lambda(a,n,t) = A + (n-1)B \leq 0$ for all $t > 0$.

We compute

$$\begin{aligned} f^{(t)} &= (1+t)^n - nt \\ \partial_1 f^{(t)} &= t((1+t)^{n-1} - 1) \\ \partial_{\bar{1}} f^{(t)} &= (1+t)^{n-1} - (n-1)t \\ \partial_{12}^2 f^{(t)} &= t^2(1+t)^{n-2}. \end{aligned}$$

After simplifying,

$$\begin{aligned} \lambda(a,n,t) = t(1+t)^{n-2}\left[(1+t)^n - (n-1)^2t^2 + (n-2)t - 1\right] \\ - a\,t\left((1+t)^{n-1} - 1\right)\left((1+t)^{n-1} - (n-1)t\right). \end{aligned}$$

Let a_n be the root of $\mathrm{Disc}_t(\lambda(a,n,t))$ so that $\lambda(a_n,n,t)$ has a positive root of multiplicity two. We claim that as $n \to \infty$, this positive root goes to zero. For ease of notation, write $\lambda(a,n,t) = t(\lambda_0(n,t) - a\lambda_1(n,t))$.

Indeed, $\lambda(a,n,t)$ has a positive root with multiplicity ≥ 2 if and only if there exists some $t^* > 0$ such that $h_n(t^*) := \lambda_0(n,t^*)\lambda_1'(n,t^*) - \lambda_1(n,t^*)\lambda_0'(n,t^*) = 0$, and that in this case, t_0 is a double root of $\lambda(a_n',n,t)$, where a_n' is chosen such that $\lambda_0(n,t_0) + a_n\lambda_1(t_0) = \lambda_0'(n,t_0) + a_n\lambda_1'(n,t_0) = 0$. Thus, it suffices to that there exists a sequence (t_n) such that $t_n \to 0$ and $h_n(t_n) = 0$ for all n.

By direct computation, we can show that $h_n(0) = h_n'(0) = 0$ for all n, and $h_n''(0) = -(n+2)(n-1)^2 < 0$. Hence, for all n, we have $h_n(t) < 0$ for sufficiently small $t > 0$. Moreover, also by direct computation,

$$h_n\left(\frac{1}{n-1}\right) = \frac{\left(1+\frac{1}{n-1}\right)^n (n-2)(n-1)^2\left(n\left(1+\frac{1}{n-1}\right)^n - n - \left(1+\frac{1}{n-1}\right)^n\right)^2}{n^4} > 0.$$

Hence, each h_n has a root in the open interval $0 < t_n < \frac{1}{n-1}$, so we have our desired sequence.

Again writing $\lambda(a,n,t) = t(\lambda_0(n,t) - a\lambda_1(n,t))$, we have

$$
\begin{aligned}
\lambda_0 &= (1+t)^{n-2}\left((1+t)^n - (n-1)^2t^2 + (n-2)t - 1\right) \\
&= \left(1 + (n-2)t + \binom{n-2}{2}t^2 + o(t^3)\right)\left((2n-2)t + \left(\binom{n}{2} - (n-1)^2\right)t^2 + \binom{n}{3}t^3 + o(t^4)\right) \\
&= 2(n-1)t + 3\binom{n-1}{2}t^2 + \left(\binom{n}{3} + 2(n-1)\binom{n-2}{2}\right)t^3 + o(t^4) \\
\lambda_1 &= \left((n-1)t + \binom{n-1}{2}t^2 + \binom{n-1}{3}t^3 + o(t^4)\right)\left(1 + \binom{n-1}{2}t^2 + o(t^3)\right) \\
&= (n-1)t + \binom{n-1}{2}t^2 + \left(\binom{n-1}{3} + (n-1)\binom{n-1}{2}\right)t^3 + o(t^4).
\end{aligned}
$$

Thus, for $0 < t \ll 1$,

$$\frac{\lambda(a,n,t)}{t} = (n-1)t\left((2-a) + \frac{1}{2}(3-a)(n-2)t + \frac{1}{6}\left((4n-9)(2-a) - n\right)t^2\right) + o(t^4),$$

and since our $t_n \to 0$, this is a good approximation near t_n for each $\lambda(a,n,t)$.

Letting $a = 2+\varepsilon$, then

$$\frac{\lambda(2+\varepsilon,n,t)}{t} = (n-1)t\left(-\varepsilon + \frac{1}{2}(1-\varepsilon)(n-2)t + \frac{1}{6}\left(-(4n-9)\varepsilon - n\right)t^2\right) + o(t^4).$$

Define $g(\varepsilon,n,t)$ to be the Taylor polynomial above. Then g has a positive real root if and only if $\mathrm{Disc}_t(g) \geq 0$, and we compute

$$\mathrm{Disc}_t(g) = \frac{1}{12}(n-2)\left((66-29n)\varepsilon^2 + (12-14n)\varepsilon + (3n-6)\right).$$

Hence, as $n \to \infty$, $\mathrm{Disc}_t(g) \to -\frac{1}{12}n^2\left(29\varepsilon^2 + 14\varepsilon - 3\right)$, which has a positive root at $\varepsilon \approx 0.160755$. We also note that the constant term of g does not depend on n, but the linear and quadratic terms grow with n, so as $n \to \infty$, the roots (if any) of g go to zero, so g remains a good approximation of λ/t near its roots. Thus, if g has a positive real root, then f is not $1/a$-log concave.

Thus, for any $\alpha > \frac{1}{2.160755}$, there exists a Δ-matroid in this family whose homogenized generating polynomial is not α-log concave, as desired. $\square$

6.3.2 Traversal Lorentzianity

There is an alternative definition for Δ-matroids involving transversals and independent sets. Consider the symmetric set $[n] \sqcup [n^*]$. A set $S \subseteq [n] \sqcup [n^*]$ is *admissible* if $\{i, i^*\} \not\subseteq S$ for all $i = 1, \ldots, n$. A *transversal* is an admissible set of size n.

Definition 6.3.6 ([BGW03, Theorem 4.1.1])**.** A set of transversals $\mathcal{B}$ form the bases of a Δ-matroid if and only if for all transversals T, the set

$$\mathcal{I}_T := \{I \subseteq T : \exists B \in \mathcal{B} \text{ such that } I \subseteq B \cap T\}$$

forms the independent sets of a matroid on ground set T.

The bases in this new (equivalent) definition are a homogenized version of the bases in our original definition. Moreover, this new definition aligns more closely with our generating polynomial construction: if we associate i to x_i and i^* to y_i, then the multiaffine homogenized polynomial we have been studying is exactly the generating polynomial of $\mathcal{B}$ in the transversal definition.

We know that if $\mathcal{I}$ form the independent sets of a matroid on ground set $[n]$, then

$$\sum_{I \in \mathcal{I}} z^{n-|I|} \prod_{i \in I} x_i$$

is Lorentzian [ALOV24b, Theorem 4.1].

By Definition 6.3.6, $\mathcal{B}$ form the bases of a Δ-matroid if and only if for all transversals T, if we replace

$$x_i = \begin{cases} x_i + z, & i \in T \\ z, & i \notin T \end{cases} \qquad\qquad y_i = \begin{cases} y_i + z, & i^* \in T \\ z, & i^* \notin T \end{cases}$$

in the generating polynomial for $\mathcal{B}$, we get a polynomial whose support is exactly the support of a (homogenized) generating polynomial for the independent sets of a matroid (i.e the support of a Lorentzian polynomial).

Given this, it might be reasonable to hope that applying this transversal operation to 1/2-log concave polynomials gives us Lorentzian polynomials, but this is unfortunately not the case.

Example 6.3.7. Let $f = x_1x_2 + y_1y_2$. (Note that f is the multiaffine homogenization of the Hurwitz stable polynomial $x_1x_2 + 1$.) Then f is 1/2-log concave. However, consider the transversal $T = \{1, 2\}$. Then $f_T = (x_1 + z)(x_2 + z) + z^2 = x_1x_2 + (x_1 + x_2)z + 2z^2$, so

$$\nabla^2 f_T = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 4 \end{pmatrix},$$

which has two positive eigenvalues.

In fact, $f^{(c)} = x_1x_2 + cy_1y_2$ is 1/2-log concave for all $c \geq 0$, but $\nabla^2 f_T^{(c)}$ does not have Lorentzian signature for any $c \geq 0$. Note that this freedom to choose c also means that amending our definition to instead require denormalized Lorentzianity would not fix the underlying problem.

Furthermore, there is no linear operation we can use to transform our homogenized generating polynomials into the corresponding independence generating polynomials, as different Δ-matroids can give rise to the same collection of independent sets. For example, $\mathcal{B}_1 = \{12, 13, 24, 34\}$ and $\mathcal{B}_2 = \{\varnothing, 12, 13, 24, 34\}$ both give $\mathcal{I} = \{\varnothing, 1, 2, 3, 4, 12, 13, 24, 34\}$ with respect to the transversal $T = [4]$, so such an operation would have to send both $f_1 = x_1x_2y_3y_4 + x_1x_3y_2y_4 + x_2x_4y_1y_3 + x_3x_4y_1y_2$ and $f_2 = f_1 + y_1y_2y_3y_4$ to the same independence polynomial $g_{\mathcal{I}} = (x_1x_2 + x_1x_3 + x_2x_4 + x_3x_4)y^2 + (x_1 + x_2 + x_3 + x_4)y^3 + y^4$.

6.3.3 Complete Fractional Log Concavity

Lorentzian polynomials were key in proving log-concavity results on generating polynomials of matroids, so it is natural that we might try to attack Conjecture 6.1.4 by defining a generalization of Lorentzianity that is appropriate for generating polynomials of even Δ-

matroids. Much of the work in this subsection came out of discussions at the AIM workshop "The geometry of polynomials in combinatorics and sampling."

Among many nice properties, some of the key facts about Lorentzian polynomials that made them particularly amenable to studying log concavity in the matroid setting include:

- Lorentzian polynomials are always log concave.
- All (valuated) matroid basis generating polynomials are Lorentzian.
- Lorentzian polynomials admit a symbol theory [BH20, Theorem 3.1]. That is, if T is a linear operator whose symbol is Lorentzian, then T preserves Lorentzianity.
- Lorentzianity can be checked inductively on quadratic derivatives.

In this section, we survey various attempts to generalize the class of Lorentzian polynomials to the setting of even Δ-matroids and 1/2-log concavity. We begin by enumerating some desired properties of our class $\mathcal{P}$. We then discuss various attempts to define such classes $\mathcal{P}$, mostly inspired by Lorentzianity and complete log concavity. Finally, we survey some notable counterexamples that arose in studying these attempted definitions and consider their implications on how a class satisfying our desired properties must behave.

For comparison, we briefly recall the definitions of Lorentzian and completely log concave polynomials.

Definition 6.3.8 ([BH20, Definition 2.6])**.** We say a homogeneous polynomial $f \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]_d$ is *Lorentzian* if $\operatorname{supp}(f)$ is M-convex and for all $\alpha \in \Delta_n^{d-2}$, $\partial^\alpha f$ is real stable.

Definition 6.3.9 ([AOV21])**.** We say a polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ is *completely log-concave* if for all $v_1, \ldots, v_k \in \mathbb{R}^n_{\geq 0}$, $D_{v_1} \ldots D_{v_k} f$ is nonnegative and log concave as a function on $\mathbb{R}^n_{>0}$.

For homogeneous polynomials, these two definitions are equivalent [BH20, Theorem 2.30].

We seek to define a class of polynomials $\mathcal{P}$, analogous to Lorentzian and completely log concave polynomials, but for even Δ-matroids and 1/2-log concavity. In order to obtain results similar to the Lorentzian case, we would like this class $\mathcal{P}$ to satisfy the following assumptions.

Assumption 6.3.10.
1. Implies 1/2-log concavity: if $f \in \mathcal{P}$, then f^{hom} is 1/2-log concave
2. Includes Hurwitz stable polynomials: if $f \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ is a multiaffine, Hurwitz stable polynomial, then $f \in \mathcal{P}$.
3. Includes even Δ-matroids: if $([n], \mathcal{B})$ is an even Δ-matroid, then $f_{\mathcal{B}} \in \mathcal{P}$.
4. Symbol theory: if T is a linear operator and $\operatorname{symb}(T) \in \mathcal{P}$, then for any $f \in \mathcal{P}$, $T(f) \in \mathcal{P}$.

Before we discuss potential candidates for $\mathcal{P}$, we make a few remarks about the symbol condition. First, we recall from Remark 2.4.21 sufficient conditions for $\mathcal{P}$ to satisfy the symbol condition.

Proposition 6.3.11. *Suppose $\mathcal{P}$ is a class of polynomials such that $x_i, x_i + x_j \in \mathcal{P}$ for all i, j. Further suppose that $\mathcal{P}$ is closed under the following operations:*

(i) If $f, g \in \mathcal{P}$ are polynomials on disjoint variables, then their product fg is in $\mathcal{P}$.

(ii) If $a + bx_i + cx_j + dx_ix_j \in \mathcal{P}$, where a, b, c, d are independent of x_i, x_j, then $b + c \in \mathcal{P}$.

(iii) Polarization and projection $\Pi^{\uparrow}$ and $\Pi^{\downarrow}$.

Then a linear operator T preserves $\mathcal{P}$ whenever $\operatorname{symb}(T) \in \mathcal{P}$.

We also make some brief remarks on why the symbol theory condition is valuable. Since we assume that Hurwitz stable polynomials are in our class, and since T preserves Hurwitz stability if and only if $\operatorname{symb}(T)$ is Hurwitz stable [BB09], this means that any Hurwitz stability-preserving operator that also preserves multiaffine polynomials and nonnegative coefficients will preserve $\mathcal{P}$. This is analogous to the result for Lorentzian polynomials [BH20, Theorem 3.4], and it unlocks a variety of natural operations that preserve $\mathcal{P}$, including nonnegative directional derivatives and twisting.

Note that throughout this section, we will use f^{ph} to denote the parity homogenization of f.

Attempt 1: Fractional Lorentzianity

Analogous to the definition of Lorentzian polynomials in [BH20, Definition 2.6], we define $\mathcal{P}_1$ to be the space of multiaffine polynomials $f \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_{n-1}]$ such that $f^{\mathrm{ph}} \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ satisfies the following:

- $\mathrm{supp}(f^{\mathrm{ph}})$ is an even Δ-matroid, and
- for all $\alpha \in \binom{[n] \sqcup [\overline{n}]}{(n-4)}$, $\partial^\alpha f^{\mathrm{ph}}$ is Hurwitz stable, where $\partial_{\bar{i}} f \coloneqq f|_{x_i=0}$.

This is the most obvious analogous definition to Lorentzian polynomials, building on the fact that a multiaffine polynomial with even support on four variables is Hurwitz stable if and only if its multiaffine homogenization is 1/2-log concave (Theorem 6.1.17). Unfortunately, there exist $f \in \mathcal{P}$ such that f^{hom} is not 1/2-log concave. The following counterexample was first found by June Vuong.

Example 6.3.12 ([Vuo23]). Consider the polynomial

$$\begin{aligned} f = 0.64 &+ 0.69x_1x_2 + 0.56x_1x_3 + 0.91x_2x_3 + 0.73x_1x_4 + 1.18x_2x_4 + 1.1x_3x_4 \\ &+ 0.35x_1x_2x_3x_4 + 1.58x_1x_5 + 0.94x_2x_5 + 4.62x_3x_5 + 0.13x_1x_2x_3x_5 + 0.18x_4x_5 \\ &+ 1.35x_1x_2x_4x_5 + 0.72x_1x_3x_4x_5 + 1.91x_2x_3x_4x_5 + 0.78x_1x_6 + 0.74x_2x_6 \\ &+ 0.22x_3x_6 + 0.22x_1x_2x_3x_6 + 0.1x_4x_6 + 3.64x_1x_2x_4x_6 + 0.62x_1x_3x_4x_6 \\ &+ 1.26x_2x_3x_4x_6 + 0.6x_5x_6 + 1.06x_1x_2x_5x_6 + 0.88x_1x_3x_5x_6 + 0.7x_2x_3x_5x_6 \\ &+ 0.7x_1x_4x_5x_6 + 1.13x_2x_4x_5x_6 + x_3x_4x_5x_6 + 0.49x_1x_2x_3x_4x_5x_6. \end{aligned}$$

One can check that this polynomial's support is all even-parity subsets of [6], which is an even Δ-matroid, and, using Proposition 6.1.6, one sees that all degree four derivatives outlined above are Hurwitz stable. However, by direct computation, the multiaffine homogenization of this polynomial is not 1/2-log concave at $\mathbb{1}$.

Moreover, one can also check that the mixed partial derivatives as outlined in Proposition 6.2.19 are all 1/2-log concave at $\mathbb{1}$, so this example also shows that we cannot strengthen the conclusion of Corollary 6.2.18 to imply that f^{hom} is 1/2-log concave.

Attempt 2: Homogenized Complete Fractional Log Concavity

The next most obvious thing to try, since the obvious generalization of Lorentzianity fails, is to try the obvious generalization of complete log concavity. Let $\mathcal{P}_2$ be the class of multiaffine polynomials such that $D_{v_1} \dots D_{v_k}(f^{\mathrm{hom}})$ is 1/2-log concave for all $v_1, \dots, v_k \in \mathbb{R}^{2n}_{>0}$.

By taking $k = 0$, this vacuously implies 1/2-log concavity of f^{hom}. However, this class does not include all Hurwitz stable polynomials nor generating polynomials of even Δ-matroids. The following construction was first brought to our attention by Nima Anari.

Example 6.3.13. Let

$$f = \prod_{i=1}^{n} (x_i y_{i+1} + 1) \in \mathbb{R}_{\geq 0}[x_1, \dots, x_n, y_1, \dots, y_n],$$

where we interpret the indices modulo n, i.e. $y_{n+1} \coloneqq y_1$. We note that this is a product of Hurwitz stable polynomials, and hence Hurwitz stable. Moreover, we note that every variable appears in exactly one factor of the product, so this polynomial is multiaffine and has 0/1 coefficients. Hence, it is the generating polynomial of an even Δ-matroid.

We claim that $f \notin \mathcal{P}_2$. Indeed, we compute

$$f^{\mathrm{hom}} = \prod_{i=1}^{n} (x_i y_{i+1} + z_i w_{i+1}).$$

Note that by taking limits, it suffices to provide $v_1, \dots, v_k \in \mathbb{R}^{2n}_{\geq 0}$ such that $D_{v_1} \dots D_{v_k} f^{\mathrm{hom}}$

is not 1/2-log concave. Indeed,

$$\left(\prod_{i=1}^{n}(\partial_{x_i}+\partial_{w_i})\right) f^{\mathrm{hom}} = y^{[n]} + z^{[n]},$$

which is α-log concave if and only if $\alpha \le 1/n$. Hence, for $n \ge 3$, $f \notin \mathcal{P}_2$, as desired.

Attempt 3: Non-Homogenized Fractional Log Concavity

One issue with our previous attempt is that directional derivatives and homogenization do not commute. However, since positive directional derivatives preserve Hurwitz stability, perhaps if we take derivatives first and then homogenize, we get a better class of polynomials.

To this end, we define $\mathcal{P}_3$ to be the class of multiaffine polynomials such that for all $v_1, \ldots, v_k \in \mathbb{R}^n_{>0}$, $(D_{v_1} \ldots D_{v_k} f)^{\mathrm{hom}}$ is 1/2-log concave. Again by taking $k = 0$, this class vacuously implies that f^{hom} is 1/2-log concave. Moreover, since positive directional derivatives preserve Hurwitz stability, this class also includes our desired Hurwitz stable polynomials.

However, this class is not closed under parity homogenization, though we might expect our desired class to be closed under this operation, in keeping with the connection with strong exchange and $M^\natural$-convexity.

Example 6.3.14. Consider

$$f = x_1x_2x_3 + \frac{1}{4}e_2 + \frac{1}{4}e_1 + 1,$$

where e_i denotes the ith elementary symmetric polynomial. For $a \in \mathbb{R}^3_{>0}$, let $f^{(a)} = f(a_1x_1, a_2x_2, a_3x_3)$, so f^{hom} is 1/2-log concave if and only if $(f^{(a)})^{\mathrm{hom}}$ is 1/2-log concave at $\mathbb{1}$ for all $a \in \mathbb{R}^3_{>0}$. One shows computationally that $\det(\nabla^2 M_{f^{(a)}}(\mathbb{1}))$ is nonvanishing on the positive orthant as a function of a, where M_f is the matrix from Theorem 6.2.7. One can also check directly that f is 1/2-log concave at $\mathbb{1}$, so this implies that $\nabla^2 M_{f^{(a)}} \prec 0$ for all $a \in \mathbb{R}^3_{>0}$. Hence f^{hom} is 1/2-log concave.

Moreover, we note that $f - f(0)$ is Hurwitz stable. Hence, since $D_v f = D_v(f - f(0))$, we conclude that all positive directional derivatives of f are Hurwitz stable, so $f \in \mathcal{P}$.

However,

$$f^{\mathrm{ph}} = x_0x_1x_2x_3 + \frac{1}{4}e_2(x_0, \dots, x_3) + 1.$$

By Proposition 6.1.6, this is not Hurwitz stable, and hence it is not 1/2-log concave by Theorem 6.1.17. Thus, $\mathcal{P}_3$ is not closed under parity homogenization.

While it is clear that this class satisfies condition (ii) of Proposition 6.3.11, it is not clear that it is closed under products on disjoint sets of variables.

Additionally, in order to make this condition useful for proving 1/2-log concavity of even Δ-matroids, we would like to have an inductive argument. Currently, our only possible base case involves polynomials on four variables. In order to make this class work, we would need a base case based on degree, since we are not reducing the number of variables at any step. To this end, we make the following conjecture.

Conjecture 6.3.15. *Let $f \in \mathbb{R}_{\geq 0}[x_1, \dots, x_n]$ be degree two with even parity support (i.e. $f = f_0 + c$, where f_0 is homogeneous of degree two and $c \geq 0$). Then f^{hom} is 1/2-log concave if and only if f_0 is Hurwitz stable.*

We remark that one direction of this conjecture is clear: if f_0 is Hurwitz stable, then so is $f_0 + c$ for any $c \geq 0$, and hence f^{hom} is 1/2-log concave by Theorem 6.1.14.

We also know that the reverse direction holds combinatorially. If $\mathrm{val}(f_0)$ is not a valuated matroid, then $\mathrm{val}(f)$ is not a valuated Δ-matroid, and hence f^{hom} is not 1/2-log concave.

Attempts 4 & 5: Mixed Derivatives

Given that fractional log-concavity of mixed derivatives gives us a bound on fractional log-concavity of the polynomial itself, another natural option is to try to impose a slightly stronger condition on mixed derivatives and try to derive a 1/2-log concavity of the original polynomial.

First, we consider $\mathcal{P}_4$ to be the class of polynomials such that for all $i \in [n]$ and all $a_{i_j}, b_{i_j} > 0$, $\left(\prod_{j=1}^{k}(a_{i_j}y_{i_j}\partial_{i_j} + b_{i_j}x_{i_j}\partial_{\overline{i_j}})\right) f^{\mathrm{hom}}$ is 1/2-log concave. One can check that $(a_iy_i\partial_i+$

$b_i x_i \partial_{\bar{i}}) f^{\text{hom}} = (a_i \partial_i f + b_i \partial_{\bar{i}} f)^{\text{hom}}$, so this includes all Hurwitz stable polynomials. It also implies 1/2-log concavity. The problem is that this class does not simplify our problem at all. One can also interpret $a_i y_i \partial_i f^{\text{hom}} + b_i x_i \partial_{\bar{i}} f^{\text{hom}}$ as scaling x_i and y_i by a_i and b_i, respectively, and then swapping x_i with y_i. Hence, this new polynomial is 1/2-log concave if and only if the original was, so this class is simply the class of multiaffine polynomials whose multihomogenizations are 1/2-log concave.

We now consider a slight strengthening of the assumptions of Corollary 6.2.18. Let $\mathcal{P}_5$ be the class of polynomials such that for all i and all $a_{i_j}, b_{i_j} > 0$, $\left(\prod_{j=1}^{k} (a_{i_j} \partial_{i_j} + b_{i_j} \partial_{\overline{i_j}})\right) f^{\text{hom}}$ is 1/2-log concave. Unfortunately, $\mathcal{P}_5$ is again simply the class of multiaffine polynomials whose multihomogenizations are 1/2-log concave. Indeed, by taking $k = 0$, if $f \in \mathcal{P}$, then f^{hom} is 1/2-log concave. On the other hand, $a_i \partial_i f^{\text{hom}} + b_i \partial_{\bar{i}} f^{\text{hom}} = f^{\text{hom}}\big|_{x_i = a_i, y_i = b_i}$, so if f^{hom} is 1/2-log concave, then $f \in \mathcal{P}_5$.

6.3.4 Interesting Counterexamples

Here we list some of the interesting counterexamples we found in our exploration of fractional log-concavity, as well as some of their consequences for future attempts at attacking this problem.

Example (Example 6.3.12)**.** Consider the polynomial

$$\begin{aligned} f = 0.64 &+ 0.69x_1x_2 + 0.56x_1x_3 + 0.91x_2x_3 + 0.73x_1x_4 + 1.18x_2x_4 + 1.1x_3x_4 \\ &+ 0.35x_1x_2x_3x_4 + 1.58x_1x_5 + 0.94x_2x_5 + 4.62x_3x_5 + 0.13x_1x_2x_3x_5 + 0.18x_4x_5 \\ &+ 1.35x_1x_2x_4x_5 + 0.72x_1x_3x_4x_5 + 1.91x_2x_3x_4x_5 + 0.78x_1x_6 + 0.74x_2x_6 \\ &+ 0.22x_3x_6 + 0.22x_1x_2x_3x_6 + 0.1x_4x_6 + 3.64x_1x_2x_4x_6 + 0.62x_1x_3x_4x_6 \\ &+ 1.26x_2x_3x_4x_6 + 0.6x_5x_6 + 1.06x_1x_2x_5x_6 + 0.88x_1x_3x_5x_6 + 0.7x_2x_3x_5x_6 \\ &+ 0.7x_1x_4x_5x_6 + 1.13x_2x_4x_5x_6 + x_3x_4x_5x_6 + 0.49x_1x_2x_3x_4x_5x_6, \end{aligned}$$

which is multiaffine on $n = 6$ variables, with support $\text{supp}(f) = \{S \subseteq [6] : |S| \text{ even}\}$, which is an even Δ-matroid.

This counterexample demonstrates the difficulties in applying inductive techniques to proving fractional log-concavity. In particular, as described in Section 6.3.3, all degree four partial derivatives are Hurwitz stable, but f^{hom} is not 1/2-log concave. Additionally, $(\partial_i + \partial_{\bar{i}})f^{\mathrm{hom}}$ is 1/2-log concave at $\mathbb{1}$ for all $i = 1, \ldots, 6$, but f^{hom} is not 1/2-log concave at $\mathbb{1}$, so we cannot strengthen the conclusion of Corollary 6.2.18.

Example (Example 6.3.13)**.** Consider

$$f = \prod_{i=1}^{n}(x_i y_{i+1} + 1),$$

where we consider the indices modulo n.

This example is Hurwitz stable, so it must be included in any class $\mathcal{P}$ satisfying Assumption 6.3.10. However, by taking appropriate directional derivatives $v_1, \ldots, v_n \in \mathbb{R}^{2n}_{\geq 0}$, we get $D_{v_1} \ldots D_{v_n} f^{\mathrm{hom}} = x^{[n]} + z^{[n]}$, which is α-log concave if and only if $\alpha \leq 1/n$. Hence, directional derivatives of homogenizations of Hurwitz stable polynomials can be arbitrarily far from 1/2-log concavity. In particular, it is vital to bear in mind the distinction between $(D_a f)^{\mathrm{hom}}$, which should be 1/2-log concave whenever $f \in \mathcal{P}$, and $D_{(a,\bar{a})}(f^{\mathrm{hom}})$, which may not be 1/2-log concave.

Example (Example 6.3.2)**.** Consider $f = 1 + x_1 + x_2 + x_3 + x_1x_2x_3$.

This example is the generating polynomial of the Δ-matroid with bases $\mathcal{B} = \{\varnothing, 1, 2, 3, 123\}$, but f^{hom} is not 1/2-log concave. As discussed in Section 6.3.1, if a Δ-matroid does not satisfy the strong exchange axiom, then it has a minor isomorphic to this Δ-matroid. On the other hand, [Mur21, Theorem 3.1] says that a (valuated) Δ-matroid satisfies the strong exchange property if and only if its parity homogenization is an even (valuated) Δ-matroid. Combining these two facts may suggest that our desired class of polynomials should be closed under parity homogenization, though we did not include this in our requirements.

Example (Example 6.3.14)**.** Let

$$f = x_1x_2x_3 + \frac{1}{4}(x_1x_2 + x_1x_3 + x_2x_3) + \frac{1}{4}(x_1 + x_2 + x_3) + 1.$$

As discussed in Section 6.3.3, $(D_{v_1} \dots D_{v_k} f)^{\mathrm{hom}}$ is 1/2-log concave for all $v_1, \dots, v_k \in \mathbb{R}^3_{\geq 0}$. However, the parity homogenization of f is not Hurwitz stable, and hence $(f^{\mathrm{ph}})^{\mathrm{hom}}$ is not 1/2-log concave. This means that any class that is closed under parity homogenization cannot include this polynomial. As discussed above, parity homogenization is a natural operation on Δ-matroids. Moreover, requiring parity homogenization is desirable as it excludes generating polynomials of any Δ-matroids that do not have the strong exchange property, which we showed above do not have 1/2-log concave homogenized generating polynomials.

6.4 Conclusion and Future Directions

As seen throughout this chapter, the question of fractional log concavity and Δ-matroids is subtle, and many of the tools that were effective for matroids and Lorentzianity fail in this setting.

For now, the most promising avenue seems to be $\mathcal{P}_3$ from Section 6.3.3. To prove a symbol theory for this class, it suffices to show that it is closed under products of polynomials on disjoint variables, as the other conditions of Proposition 6.3.11 are easily verified. However, even in the case of multiplying by a single new variable, it is not obvious that $(D_{(1,w)}(x_0 f))^{\mathrm{hom}} = y_0 f^{\mathrm{hom}} + x_0 (D_w f)^{\mathrm{hom}}$ should be 1/2-log concave whenever $f \in \mathcal{P}_3$.

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