Boundary Harnack Principle for Stable-Like Processes

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Abstract

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We establish the boundary Harnack principle for certain classes of symmetric stable-like processes in $\mathbf{R}^d$ on arbitrary open sets as well as censored stable-like processes on $\mathcal{C}^{1,1}$-domains. Using those results, we derive Dirichlet heat kernel estimates for killed stable-like processes and killed censored stable-like processes in κ-fat domains in terms of the surviving probabilities and the global transition density of the processes. For $\mathcal{C}^{1,1}$-domains, we derive explicit estimates of the Dirichlet heat kernel.

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Chapter 1

INTRODUCTION

The first chapter provides the necessary background for this dissertation. We will introduce the Harnack principle as well as the boundary Harnack princple, symmetric stable-like processes, and so on.

1.1 Harmonic Functions and Harnack Inequalities

Let $D \subseteq \mathbf{R}^d$ be a connected open set. A locally bounded function u is called harmonic in D if the value of the function at a point x equals the mean of the function over concentric balls in D. That is, for any $x \in \mathbf{R}^d$ and any $r > 0$ such that $B(x,r) \subseteq D$,

$$u(x) = \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y) \mathrm{d}y.$$

Here, $|A|$ denotes the Lebesgue measure of a measurable set $A \subseteq \mathbf{R}^d$. There is an equivalent probabilistic characterization of harmonic functions: Let B_t be Brownian motion. Then u is harmonic in D if for every $x \in D$ and $r > 0$ such that $B(x,r) \subseteq D$,

$$u(x) = \mathbf{E}_x \left[u\left(B_{\tau_{B(x,r)}} \right) \right]. \tag{1.1}$$

We denote by $\tau_A = \inf \{t > 0 : X_t \notin A\}$ the first exit time of B_t from the set A. It can be shown that this implies that equation (1.1) holds for any open, relatively compact set $V \subseteq D$ with $\overline{V} \subseteq D$, for every $x \in V$,

An important property of harmonic functions is the Harnack Inequality which states that in a compact set $K \subseteq D$, the values of harmonic functions are comparable:

Theorem 1.1 (Harnack inequality)**.** *Let* $D \subset \mathbf{R}^d$ *be open and connected, and* $K \subset D$ *be compact. Then there exist positive constants* $C_1 = C_1(D,K)$ *and* $C_2 = C_2(D,K)$ *such that* *for all nonnegative harmonic functions* u *in* D*,*

$$C_1 u(y) \leq u(x) \leq C_2 u(y)$$

for all $x, y \in K$*.*

Note that the constants C_1 and C_2 are universal for all functions which are harmonic in D. The proof of the Harnack inequality is elementary and can be found, for example, in [24]. If one restricts attention to open balls rather than general open sets, one can quantify the constants C_1 and C_2.

Theorem 1.2 (Harnack inequality for balls)**.** *Let* $x_0 \in \mathbf{R}^d$*,* $0 < r < r_0 < \infty$*. Then for all nonnegative harmonic functions* u *in* $B(x_0, r_0)$*,*

$$\frac{1 - \frac{r}{r_0}}{\left(1 - \frac{r}{r_0}\right)^{d-1}} u(y) \leq u(x) \leq \frac{1 + \frac{r}{r_0}}{\left(1 - \frac{r}{r_0}\right)^{d-1}} u(y)$$

for all $x, y \in \overline{B(x_0, r)}$*.*

Here, $D = B(x_0, r_0)$ and $K = \overline{B(x_0, r)}$. Regarding the constants, we see that $C_1 = C_1(d, r, r_0) = (1 - r/t_0) \,/\, (1 - r/r_0)^{d-1}$ and similarly for $C_2 = C_2(d, r, r_0)$. Note that in this case, the constants C_1 and C_2 both depend on r and r_0 only through the ratio r/r_0. This type of Harnack inequality is referred to as scale-invariant.

Since constant functions are harmonic, an equivalent formulation of the Harnack inequality reads as

$$u(x)u(y) \leq C u(y) v(x)$$

for all non-negative harmonic functions u and v on D, and all x, y in K.

Note that the condition that K is a compact subset of D implies that there is a positive distance between K and ∂D. Naturally, the question arises whether the result can be extended to subsets which may contain parts of the boundary.

Theorem 1.3 (Boundary Harnack principle)**.** *Let* $D \subset \mathbf{R}^d$ *be a Lipschitz open set. If* $Q \in \partial D$ *and* $r \in (0, r_0)$*, then for any nonnegative functions* u, v *which are not identically*

0, harmonic in $D \cap B(Q,r)$*, and vanish continuously on* $\partial D \cap B(Q,r)$*, there exists* $C = C(d, r, r_0) > 0$ *such that*

$$\frac{u(x)}{v(x)} \le C \frac{u(y)}{v(y)},$$

for all $x, y \in D \cap B(Q,r)$*.*

As with the Harnack inequality, if the constant C depends on r and r_0 only through their ratio, so $C = C(d, r/r_0)$, we call it the scale-invariant boundary Harnack principle.

The boundary Harnack principle is a much deeper result and was first proven independently by Ancona [1], Dahlberg [22], and Wu [32] for Lipschitz domains in the late 70s. Bass and Burdzy [2] proved the same result using probabilistic techniques in the early 90s. The technique they had used is now referred to as the *box method.* Their result was extended to Hölder domains of order $r > \frac{1}{2}$ [3] and it was shown that there are Hölder domains of order $r < \frac{1}{2}$ for which the boundary Harnack principle fails [4].

1.2 Probabilistic Notion of Harmonicity

The equivalent probabilistic definition of harmonicity, equation (1.1), allows the definition of harmonic functions with respect to general strong Markov processes. Let X_t be a Hunt process. We say a function u is harmonic with respect to the process X_t if for every $V \subseteq D$ such that $\overline{V} \subseteq D$, and every $x \in V$,

$$\mathbf{E}_x \left| u(X_{\tau_V}) \right| < \infty \quad \text{and} \quad u(x) = \mathbf{E}_x \left[u(X_{\tau_V}) \right]$$

See [11] for an analytic characterization of harmonicity for general symmetric Markov processes. As before, one can ask whether the harmonic functions corresponding to a given Hunt process satisfy the Harnack and boundary Harnack principle. The first family of discontinous processes studied were rotationally symmetric stable processes, a class of Lévy processes which have characteristic function

$$\mathbf{E}_0 \left[e^{i\xi X_t} \right] = e^{-t|\xi|^\alpha}, \quad \xi \in \mathbf{R}^d, \quad t \ge 0,$$

for $0 < \alpha \leq 2$, called the index of the stable process. Note that for $\alpha = 2$, we obtain Brownian motion and the usual notion of harmonicity. When referring to stable processes, we will always exclude the Brownian motion case since is significant different. For example, whereas Brownian motion has continuous sample paths, stable processes are pure jump processes.

The Harnack inequality can easily be established for rotationally symmetric stable processes due to the explicit formula for the Poisson kernel on ball due to Riesz, see for example [30]. The boundary Harnack principle has been established for rotationally symmetric stable processes by various different techniques. Bogdan proved the boundary Harnack principle for Lipschitz domains using the analytic techniques in [6], and later using probabilistic techniques in joint work with Byczkowski [8]. Song and Wu [31] extended the results in [6] to κ-fat open sets. Finally, Bogdan, Kulczycki and Kwaśnicki [9] established the scale-invariant boundary Harnack principle for arbitrary open sets. The approaches in [6,9,31] were mainly analytic and the key step involves the regularization of the Poisson kernel for the ball, an analogue of volume averaging in classical potential theory.

Recently, Bogdan, Kumagai and Kwaśnicki [10] extended this appraoch to cover a large class of Hunt processes with an associated dual process on metric measure spaces. We will state their result only for symmetric processes for which one can always associate a dual process with the Lebesgue measure as their symmetrizing measure defined on open sets $D \subseteq \mathbf{R}^d$. So let X_t be a symmetric Hunt process on D. Bogdan, Kumagai, and Kwaśnicki [10] use the more restrictive notion of regular harmonic functions. A function u is said to be regular harmonic in an open set $V \subseteq \mathbf{R}^d$ if

$$\mathbf{E}_x \left|u(X_{\tau_V})\right| < \infty \quad \text{and} \quad u(x) = \mathbf{E}_x \left[u\left(X_{\tau_V}\right)\right]$$

for all $x \in D$. They provide certain conditions under which any positive regular harmonic function satisfies the boundary Harnack principle.

We will denote by $r_0 \in (0, \infty]$ the localization radius of the process which is defined as any number such that $D \setminus B(x, 2r) \neq \emptyset$ for all $x \in D$ and $r < r_0$. It is easy to see that

$r_0 = \operatorname{diam}(D)/2$, where diam denoted the diameter of the set.

Assumption A. The transition semigroup of X_t is both Feller and strong Feller. Morevoer, Hunt's hypothesis is satisfied: Every semi-polar set is polar. Recall that we say that a strong Markov process is Feller if its transition semigroup p_t maps $\mathcal{C}_c(\mathbf{R}^d)$, the space of continuous functions with compact support, into itself for any $t > 0$. A strong Markov process is strong Feller if $p_t f$ is continuous for every bounded function f.

Assumption B. There is a linear subspace $\mathcal{D}$ of the domain of the Feller generator of X_t that satisfies the following condition: If K is compact, and V is open with $K \subseteq V \subseteq D$, then there is $f \in \mathcal{D}$ such that $f(x) = 1$ for $x \in K$, $f(x) = 0$ for $x \in D \backslash V$, $0 \le f(x) \le 1$ for $x \in D$, and the boundary of the set $\{x : f(x) > 0\}$ has Lebesgue measure zero. This can be summarized as to say that there is a Urysohn bump function in the domain which separates the sets K and $D \setminus V$. For later reference, denote

$$\delta(K, V) = \inf_{f} \sup_{x \in \mathbf{R}^d} \mathcal{A} f(x)$$

where the infimum is taken over all functions f described above.

Assumption C. The Lévy kernel of the process X_t has the form $\nu(x, y) \mathrm{d}y$, where $\nu(x, y) = \nu(y, x) > 0$ for all $x, y \in D$, $x \neq y$. For every $x_0 \in D$, $0 < r < R < R_0$, $x \in B(x_0, r)$ and $y \in D \setminus B(x_0, R)$,

$$\frac{1}{c} \nu(x_0, y) \le \nu(x, y) \le c \nu(x_0, y),$$

where $c = c(x_0, r, R)$.

Assumption D. If $x_0 \in D$, $0 < r < p < R < R_0$, then the Green's function for the ball satisfies

$$\sup_{x \in B(x_0, r)} \sup_{y \in D \backslash B(x_0, p)} G_{B(x_0, R)}(x, y) < \infty.$$

The theorem is now as follows:

Theorem 1.4 (Bogdan, Kumagai, Kwaśnicki, 2013)**.** *Suppose a symmetric Hunt process* X_t *satisfies Assumptions A to D. Assume D is any open set and let* $Q \in \partial D$ *and* $r \in (0, r_0)$*. Then for any nonnegative functions* u, v *which are not identically 0, harmonic in* $D \cap B(Q, r)$*, and vanish continuously on* $D^c \cap B(Q, r_0)$*, there exists* $C = C(r, r_0) > 0$ *such that*

$$\frac{u(x)}{v(x)} \leq C \frac{u(y)}{v(y)}$$

for all $x, y \in D \cap B(Q, r)$*.*

Under additional conditions, the authors establish the scale-invariant version of the boundary Harnack principle (see below). The proof is based on an approximate factorization of $f(x) = \mathbf{P}_x \left(X_{\tau_V} \in E \right)$ where E is a Borel set in $\mathbf{R}^d$ as

$$f(x) \approx \mathbf{E}_x \left[\tau_{V \cap B(x_0, \rho)} \right] \int_{B(x_0, q)^c} f(y) \nu(x_0, y) m(\mathrm{d}y), \qquad x \in B(x_0, r) \cap D.$$

The notation $f \approx g$ means that there exists $C \geq 1$ such that $C^{-1} f \leq g \leq C f$.

There are two main steps in proving this approximate factorization. Deducing the approximate factorization from a local supremum estimate, and then establishing said local supremum estimate. The first step can be proved using an estimate of the harmonic measure. If $x_0 \in D$ and $0 < r < R < \tilde{R} < R_0$, then for all $V \subseteq B(x_0, R)$ we have

$$\mathbf{P}_x \left(X_{\tau_V} \in \bar{A}(x_0, R, \tilde{R}) \right) \leq c \mathbf{E}_x [\tau_V], \qquad x \in B(x_0, r) \cap V, \tag{1.2}$$

where $c = c(x_0, r, R, \tilde{R})$ and $A(x_0, r, R)$ denotes the annulus centered at x_0 with inner radius r and outer radius R. The proof of this fact uses the Urysohn bump functions from assumption B, and does not rely on any of the other assumptions.

The second step is the proof of a local supremum estimate (see Theorem 3.4): Given the same conditions as in the approximate factorization, for any nonnegative function f which is regular harmonic function in $B(x_0, R)$.

$$f(x) \leq \int_{B(x_0, q)^c} f(y) \pi_{x_0, r, q, R}(y) \mathrm{d}y, \qquad x \in B(x_0, r) \cap D,$$

where

$$\pi_{x_0,r,q,R}(y) = \begin{cases} C_1 & y \in B(x_0,R) \setminus B(x_0,q) \\ C_2 \cdot (1 \wedge \nu\,(y, B(x_0,R))) & y \in B(x_0,R)^c \end{cases} .$$

To prove the local supremum estimate, one cannot generally proceed by volume averaging as in the case of stable processes: To obtain the estimates, one needs to bound the harmonic measure outside a ball. For regions far from the ball, one can obtain sufficiently good estimates using the Ikeda-Watanabe formula, but near the boundary of the ball one would need the exact decay rate of the Green's function to obtain a sufficiently good estimate.

As a remedy, the authors mollify the harmonic measure. Recall that the harmonic measure of a set U is the distribution of X_{τ_U},

$$\omega_{x,U}(A) = \mathbf{P}_x\left(X_{\tau_U} \in A\right).$$

Suppose we attach a certain mass to a particle traveling along a path of the process. The harmonic measure corresponds to the case that its mass remains constant until it exits V and then loses all of its mass. The mollification of the harmonic measure will result in a gradual decrease of the mass as the particle moves towards the boundary.

1.3 Symmetric Stable Processes and Generalizations

Even though the boundary Harnack principle has been established for several Markov processes including stable processes, it still remains an open problem for a large class of symmetric stable-like processes. Let us consider a different way of defining stable processes by the means of (regular) Dirichlet forms. A Dirichlet form is a symmetric form $\mathcal{E}(\cdot,\cdot)$ defined on a dense subset $\mathcal{F}$ of $L^2(E, \mathrm{d}m)$, called the domain of $\mathcal{E}$, where E is a locally compact separable metric space which satisfies the following conditions: The symmetric form is *closed* in the sense that the set $\mathcal{F}$ is a Hilbert space with respect to the metric $\mathcal{E}_1(\cdot,\cdot) = \mathcal{E}(\cdot,\cdot) + (\cdot,\cdot)_{L^2(E;m)}$. The symmeric form is *Markovian*, if for each $\varepsilon > 0$, there exist a function $\varphi_\varepsilon : \mathbf{R} \to \mathbf{R}$ which satisfies

$$\varphi_\varepsilon(t) = t, \qquad\qquad t \in [0,1]$$

$$\begin{aligned} -\varepsilon \leq \varphi_\varepsilon(t) &\leq 1+\varepsilon, && t \in \mathbf{R} \\ 0 \leq \varphi_\varepsilon(t') - \varphi_\varepsilon(t) &\leq t'-t, && t < t'. \end{aligned}$$

such that

$$f \in \mathcal{F} \quad \Rightarrow \quad \varphi_\varepsilon \circ f \in \mathcal{F}$$

with $\mathcal{E}(\phi_\varepsilon \circ f, \phi_\varepsilon \circ f) \leq \mathcal{E}(f,f)$.

A Dirichlet form is *regular* if there is a subset $\mathcal{C}$ of $\mathcal{F} \cap \mathcal{C}_0(E)$ such that $\mathcal{C}$ is dense in $\mathcal{F}$ in the $\mathcal{E}_1$-norm and dense in $\mathcal{C}_0(E)$ in the uniform norm. Here, $\mathcal{C}_0(E)$ denotes the set of continuous functions vanishing at infinity. Given a regular Dirichlet space, Fukushima's theorem states that there exists a symmetric Hunt process starting from any $x \in E$ except for a set of zero capacity whose Dirichlet space is the given one [12, 25].

Consider the symmetric form defined by

$$\mathcal{E}(u,v) = \frac{1}{2}\mathcal{A}(d,-\alpha) \iint_{\mathbf{R}^d \times \mathbf{R}^d} (u(x)-u(y))\,(v(x)-v(y))\,\frac{1}{|x-y|^{d+\alpha}}\,\mathrm{d}x\mathrm{d}y$$

where

$$\mathcal{A}(d,-\alpha) = \frac{|\alpha| 2^{\alpha-1}\Gamma\left(\frac{d+\alpha}{2}\right)}{\pi^{\frac{d}{2}}\Gamma\left(1-\frac{\alpha}{2}\right)}$$

and

$$\mathcal{F} = \left\{ u \in L^2(\mathbf{R}^d, \mathrm{d}x) : \iint\limits_{\mathbf{R}^d \times \mathbf{R}^d} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\,\mathrm{d}x\mathrm{d}y < \infty \right\} = W^{\alpha/2,2}(\mathbf{R}^d).$$

It can be easily checked that this defines a regular Dirichlet form, and the process corresponding to it is the *rotationally symmetric stable process* of index α.

We can now define more general stable-like processes: First, we can define the *symmetric stable-like process* by perturbing the Dirichlet form of a symmetric stable process. Let $\kappa(x,y)$ be a symmetric function which satisfies

$$\kappa_1 \leq \kappa(x,y) \leq \kappa_2, \qquad x,y \in \mathbf{R}^d$$

for some positive constants $0 < \kappa_1 \leq \kappa_2 < \infty$. The Dirichlet form of symmetric stable-like processes is given by

$$\mathcal{E}(u,v) = \frac{1}{2}\mathcal{A}(d,-\alpha) \iint\limits_{\mathbf{R}^d\times\mathbf{R}^d} \left(u(x)-u(y)\right)\left(v(x)-v(y)\right)\frac{\kappa(x,y)}{|x-y|^{d+\alpha}}\,\mathrm{d}x\mathrm{d}y,$$

and the domain is identical to that of the symmetric stable process with the same index. We denote the symmetric stable-like process by X_t. Unlike the rotationally symmetric stable processes, symmetric stable-like processes are state-dependent, they do not need to be Lévy processes, and they do not satisfy a simple scaling condition which makes their analysis more subtle.

Second, we can define the rotationally symmetric stable process, or more genrerally, symmetric stable-like processes, on subsets $D \subseteq \mathbf{R}^d$. There are generally three ways of constructing a subprocess which takes values in D. The first is the *killed stable-like process.* We can kill the process upon leaving D by sending it to a cemetery state ∂. Thekilled stable-like process can therefore defined by

$$X_t^D = \begin{cases} X_t & t < \tau_D \\ \partial & t \geq \tau_D \end{cases}.$$

Alternatively, the process can defined through Dirichlet forms. The is the same as above, but its domain is given by

$$\mathcal{F}^D = \left\{u \in \mathcal{F} : u = 0 \text{ q.e. on } D^c\right\}.$$

where the notation q. e. means that the statement has to hold everywhere on a set except on a set of zero capacity. Note that for $u, v \in \mathcal{F}^D$, the Dirichlet form can be rewritten as

$$\begin{aligned}\mathcal{E}(u,v) &= \frac{1}{2}\mathcal{A}(d,-\alpha)\iint\limits_{D\times D}\left(u(x)-u(y)\right)\left(v(x)-v(y)\right)\frac{\kappa(x,y)}{|x-y|^{d+\alpha}}\,\mathrm{d}x\mathrm{d}y \\ &\quad + \int_D u(x)v(x)\kappa_D(x)\mathrm{d}y\mathrm{d}x,\end{aligned}$$

where

$$\kappa_D(x) = \mathcal{A}(d,-\alpha)\int_{D^c}\frac{\kappa(x,y)}{|x-y|^{d+\alpha}}\mathrm{d}y.$$

This quantity is called the killing density of the killed process X_t^D.

The second process we will consider is the *censored stable-like process* on D. It is obtained by suppressing jumps of the process out of D. Effectively, this is done by restricting the Lévy kernel of the symmetric stable-like process, $\nu(x,y) = \kappa(x,y)|x-y|^{-d-\alpha}$, to $D \times D$. Therefore, the Lévy kernel of the censored stable-like process is given by

$$\nu^c(x,y) = \mathbf{1}_{D\times D}(x,y)\frac{\kappa(x,y)}{|x-y|^{d+\alpha}}.$$

The associated Dirichlet form is

$$\mathcal{E}^D(u,v) = \frac{1}{2}\mathcal{A}(d,-\alpha) \iint\limits_{D\times D} (u(x)-u(y))\,(v(x)-v(y))\,\frac{\kappa(x,y)}{|x-y|^{d+\alpha}}\,\mathrm{d}x\mathrm{d}y$$

for $u, v \in \mathcal{C}_c^\infty(D)$, the set of smooth functions of compact support on D. The domain is given by the closure of $\mathcal{C}_c^\infty(D)$ in the $\mathcal{E}^1$-norm, denoted by $\mathcal{F}^c = \overline{\mathcal{C}_c^\infty(D)}^{\mathcal{E}_1}$. We will denote censored stable-like processes by Y_t.

The last process is the *reflected stable-like process* on $\overline{D}$. Its Dirichlet form is

$$\mathcal{E}^{\overline{D}}(u,v) = \frac{1}{2}\mathcal{A}(d,-\alpha) \iint\limits_{\overline{D}\times\overline{D}} \frac{\kappa(x,y)\,(u(x)-u(y))\,(v(x)-v(y))}{|x-y|^{d+\alpha}}\,\mathrm{d}x\mathrm{d}y,$$

and its domain is

$$\mathcal{F}^{\overline{D}} = \left\{ u \in L^2(\overline{D},\mathrm{d}x) : \iint\limits_{\overline{D}\times\overline{D}} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\,\mathrm{d}x\mathrm{d}y < \infty \right\}.$$

The three processes that take valued in D are related as follows: The killed stable-like process can be obtained from the censored stable-like process by killing inside D through a potential. The censored stable-like process is obtained from the reflected stable-like process by killing it upon leaving D. The censored stable-like process is a proper subprocess of the reflected stable-like process in the case of $1 < \alpha < 2$, whereas both processes are the same if $0 < \alpha \leq 1$ and D is, for example, a bounded Lipschitz domain [7,27].

Symmetric stable-like processes have been investigated before by several authors. Chen and Kumagai [19,20] have derived several key properties of reflected stable-like processes on a large class of closed d-sets. A set $F \subset \mathbf{R}^n$, $n \geq 2$, is a *d-set* if there is d satisfying $0 < d \leq n$ for which there exists a positive Borel measure μ on F such that there exist $0 < b_1 \leq b_2 < \infty$ so that

$$b_1 r^d \leq \mu(B(x,r) \cap F) \leq b_2 r^d, \qquad x \in F, 0 < r \leq 1. \tag{1.3}$$

Important examples of closed d sets are $\mathbf{R}^d$ and the closure of any κ-fat open set.

Theorem 1.5 (Chen & Kumagai, 2003)**.** *Let X_t be a reflected stable-like process of index $\alpha \in (0,2)$ on a closed d-set F such that for all $r > 0$,*

$$\mu(B(x,r)) \leq b_2 r^d.$$

Then the process X_t is a Feller process, and given $T > 0$, there exists a constant $C_1 = C_1(d, \alpha, \kappa_1, \kappa_2, b_1, b_2, T) \geq 1$ such that its transition density function $p(t,x,y)$ is continuous and satisfies that for all $(t,x,y) \in [0,T] \times F \times F$, we have

$$C_1^{-1}\left(t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x-y|^{d+\alpha}}\right) \leq p(t,x,y) \leq C_1\left(t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x-y|^{d+\alpha}}\right).$$

If equation (1.3) holds for all $r > 0$, then T can taken to be infinite.

Besides the main result, Chen and Kumagai [19] obtain other important results in their paper. They include an important estimate on the exit times from balls, the parabolic Harnack inequality, the Lévy systems formula, and Dynkin's formula. The exit time estimates are as follows.

Proposition 1.6 (Chen & Kumagai, 2003)**.** *Let X_t be a reflected stable-like process of index $\alpha \in (0,2)$ on a closed d-set F such that for all $r > 0$,*

$$\mu(B(x,r)) \leq b_2 r^d.$$

Then, for each $r_0 > 0$, $A > 0$, and $0 < B < 1$, there exists $\gamma = \gamma(r_0, A, B, d, \alpha, \kappa_1, \kappa_2, b_1, b_2) \in (0,1)$ such that for every $r \in (0, r_0]$,

$$\mathbf{P}_x\left(\tau_{B(x,Ar)} < \gamma r^\alpha\right) \leq B. \tag{1.4}$$

Before we state the parabolic Harnack inequality, we need to introduce some terminology: Let $Z_t = (V_0 + t, X_t)$ be the space-time process associated to X_t. We denote the filtration generated by Z_t by $\{\tilde{\mathcal{F}}_s, s \geq 0\}$, and the law of the space time process starting from (t,x) by $\mathbf{P}_{(t,x)}$. A non-negative Borel function $q(t,y)$ on $[0,\infty) \times D$ is *parabolic* in a relatively open set $V \subseteq [0,\infty) \times D$ if for every relatively compact open subset $V_1 \subseteq V$,

$$q(t,x) = \mathbf{E}_{(t,x)}[q(Z_{\tau_{V_1}})]$$

for every $(t,x) \in V_1$. For each $R_0 > 0$, denote by γ_{R_0} the constant in the previous proposition corresponding to $r_0 = R_0$, and $A = B = \frac{1}{2}$. For $t \leq 1$ and $r \leq R_0$ define

$$Q_{R_0}(t,x,r) := [t, t + \gamma_{R_0} r^\alpha] \times B(x,r)$$

Proposition 1.7 (Parabolic Harnack Inequality, Chen & Kumagai, 2003)**.** *Let X_t be a reflected stable-like process of index $\alpha \in (0,2)$ on a closed d-set F such that for all $r > 0$,*

$$\mu(B(x,r)) \leq b_2 r^d.$$

For every $\tilde{R} > 0$, $0 < \delta \leq \gamma_{\tilde{R}}$, there exists $c = c(\tilde{R}, \delta, d, \alpha, \kappa_1, \kappa_2) > 0$ such that for every $z \in D$, $0 < R \leq \tilde{R}$ and every non-negative function q on $[0,\infty) \times D$ that is parabolic and bounded on $[0, 3\gamma_{\tilde{R}} R^\alpha] \times B(z,R)$,

$$\sup_{(t,y) \in Q_{R_0}(\delta,z,R/3)} q(t,y) \leq c \inf_{y \in B(z,R/3)} q(0,y). \tag{1.5}$$

Next, the Lévy systems formula is valid for these processes.

Proposition 1.8 (L evy System Formula, Chen & Kumagai, 2003)**.** *Let X_t be a reflected stable-like process of index $\alpha \in (0,2)$ on a closed d-set F such that for all $r > 0$,*

$$\mu(B(x,r)) \leq b_2 r^d.$$

Suppose f be a non-negative measurable function on $[0,\infty) \times D \times D$ vanishing on the diagonal and T be a predictable stopping time of $\{\mathcal{F}_t\}_{t \geq 0}$. Then for every $t \geq 0$ and $x \in D$,

$$\mathbf{E}_x\left[\sum_{s \leq T} f(s, Y_{s-}, Y_s)\right] = \mathbf{E}_x\left[\int_0^T \int_D \frac{\kappa(Y_s,y) f(s,Y_s,y)}{|Y_s - y|^{d+\alpha}} \mathrm{d}y \mathrm{d}s\right]. \tag{1.6}$$

Besides these statements which were derived specifically for stable-like processes, the following theorems is known to be valid for more general processes. Dynkin's formula is a fundamental theorem of calculus-type theorem which holds for any Feller process [23, Formula 5.8]. Denote the Feller generator by $\mathcal{A}$.

Theorem 1.9 (Dynkin's formula)**.** *Let X_t be a Feller process. For any function f in the domain of the Feller generator, $\mathcal{D}(\mathcal{A})$, and any stopping time T with $\mathbf{E}_x[T] < \infty$,*

$$\mathbf{E}_x\left[f\left(X_T\right)\right] - f(x) = \mathbf{E}_x\left[\int_0^T \mathcal{A}f(X_t)\,\mathrm{d}t\right], \qquad x \in D.$$

1.4 Domains of interest

In what follows, we will encounter processes which are defined on sets with varying degree of regularity. Three important types of domains that we will encounter are κ-fat domains, Lipschitz domains, and $\mathcal{C}^{1,\beta}$-domains.

1.4.1 $\mathcal{C}^{1,\beta}$-domains

We start with the definition of a global version of a $\mathcal{C}^{1,\beta}$-domain.

Let $\beta \in (0,1]$. An open set $D \subset \mathbf{R}^d$ is called a *special $\mathcal{C}^{1,\beta}$-domain* if there exists a $\mathcal{C}^{1,\beta}$-function $\Gamma : \mathbf{R}^{d-1} \to \mathbf{R}$ satisfying

$$|\nabla\Gamma(\tilde{x}) - \nabla\Gamma(\tilde{y})| \leq \Lambda|\tilde{x} - \tilde{y}|^\beta, \qquad \tilde{x}, \tilde{y} \in \mathbf{R}^{d-1}$$

for some $\Lambda > 0$ and such that

$$D = D_\Gamma := \left\{x = (\tilde{x}, x_d) \in \mathbf{R}^d : x_d > \Gamma(\tilde{x})\right\}.$$

In other words, a special $\mathcal{C}^{1,\beta}$-domain can represented as the points lying above a given $\mathcal{C}^{1,\beta}$-function.

A $\mathcal{C}^{1,\beta}$-domain is a local version of the a special $\mathcal{C}^{1,\beta}$-domain. Let $\beta \in (0,1]$. An open set $D \subset \mathbf{R}^d$ is called a *$\mathcal{C}^{1,\beta}$-domain* if there exists $\Lambda > 0$ and $r_0 > 0$ such that for every boundary point $Q \in \partial D$, there exists a $C^{1,\beta}$-function $\Gamma = \Gamma_Q : \mathbf{R}^{d-1} \to \mathbf{R}$ which satisfies $\Gamma(\tilde{0}) = 0$, has an essentially bounded gradient, $\|\nabla\phi\|_\infty \leq \Lambda$, and

$$|\nabla\Gamma(\tilde{x}) - \nabla\Gamma(\tilde{y})| \leq \Lambda|\tilde{x} - \tilde{y}|^\beta, \qquad \tilde{x}, \tilde{y} \in \mathbf{R}^{d-1}$$

as well as an orthogonal coordinate system such that

$$B(Q, r_0) \cap D = B(Q, r_0) \cap \left\{x = (\tilde{x}, y_d) \in \mathbf{R}^d : x_d > \Gamma(\tilde{x})\right\}.$$

The pair (Λ, r_0) is called the *characteristic* of the set.

For our purposes, the main property of $\mathcal{C}^{1,\beta}$ domains is that given a point on the boundary, say $Q \in \partial D$, we can find a parabola-like sets which are rooted at Q, tangent to the boundary, and lie within the domain D (we prove this below). Next, we define these parabola-like sets. For $C \geq 1$, set

$$\mathcal{P} := \left\{x = (\tilde{x}, x_d) \in \mathbf{R}^d : C|\tilde{x}|^{\beta+1} < x_d < C^{-1}\right\}.$$

Some algebra shows that this regions can be equivalently expressed as

$$\mathcal{P} = \left\{x = (\tilde{x}, x_d) \in \mathbf{R}^d : C\left(|x|^2 - (x \cdot e_d)^2\right)^{(\beta+1)/2} < x \cdot e_d < C^{-1}\right\}.$$

where $e_d = (0, \ldots, 0, 1)$ is the d-th unit vector in $\mathbf{R}^d$. This formulation allows us to define rotated versions of these regions. For any unit vector b we can define

$$\mathcal{P}_b := \left\{x = (\tilde{x}, x_d) \in \mathbf{R}^d : C\left(|x|^2 - (x \cdot b)^2\right)^{(\beta+1)/2} < x \cdot b < C^{-1}\right\}.$$

Every shift of such a set, $\mathcal{P}_b + x$ for some $x \in \mathbf{R}^d$, will be called a region of $(1+\beta)$-tangential approach of size C^{-1}.

Lemma 1.10. *Let* $\beta \in (0, 1]$. *If* $0 < v \leq u$, $x \geq 1$, *and*

$$c(u^2 - v^2)^{(\beta+1)/2} < v < c^{-1}, \tag{1.7}$$

then

$$v > \frac{c^\beta}{2} u^{\beta+1}. \tag{1.8}$$

Proof. From the assumption, equation (1.7), we see that

$$u^2 - v^2 < c^{-4/(\beta+1)}$$

and therefore

$$u^2 < c^{-4/(\beta+1)} + c^{-2} \leq 2c^{-2}.$$

Now suppose that the conclusion of the lemma, equation (1.8), does not hold. Then

$$\begin{aligned} u^2 - v^2 &\geq u^2 - \left(\frac{c^\beta}{2}\right)^2 u^{2(\beta+1)} \\ &= u^2 - u^2 \frac{c^{2\beta}}{4} u^{2\beta} \\ &> u^2 = u^2 \frac{c^{2\beta}}{4} (2c^{-1})^{2\beta} \\ &= u^2(1 - 2^{\beta-2}) \\ &> \frac{u^2}{2}. \end{aligned}$$

Plugging this into equation (1.7) yields $c\left(\frac{u^2}{2}\right)^{(\beta+1)/2} < v$ and thus

$$v > cu^{\beta+1}/2 \geq c^\beta u^{\beta+1}/2.$$

Therefore equation (1.8) holds. □

With this lemma, we can now show the following:

Lemma 1.11. *Let the deimension be $d \geq 2$ and $\beta \in (0, 1]$. Moreover, let $\Gamma : \mathbf{R}^{d-1} \to \mathbf{R}$ be a $\mathcal{C}^{1,\beta}$ function uch that $D = D_\Gamma$. Then there exist $C = C(\beta, \|\Gamma\|_{1,\beta}) \geq 1$ such that for every $Q \in \partial D$, the region of β-tangential approach $\mathcal{P}_n + Q$ satisfies $\mathcal{P}_n + Q \subset D$, where n is the unit inward normal. Furthermore, we also have that $\mathcal{P}_{-n} + Q \subset D^c$.*

Proof. We only prove the first statement since the second one can be proved ina similar fashion. Without loss of generality, we willassume that $Q = 0 \in \partial D$. First, note that by the mean value theorem applied to the function Γ,

$$\left|\Gamma(\tilde{x}) - \left(\Gamma(\tilde{0}) + \nabla\Gamma(\tilde{0}) \cdot \tilde{x}\right)\right| \leq \|\Gamma\|_{1,\beta} |\tilde{x}|^{\beta+1}.$$

Therefore, it suffices to prove that for any $x \in D$, that is, any $x = (\tilde{x}, x_d)$ satisfying

$$C\left(|x|^2 - (x \cdot b)^2\right)^{(\beta+1)/2} < x \cdot b < C^{-1}, \tag{1.9}$$

it follows that

$$x_d > \Gamma(\tilde{x}) > \nabla\Gamma(\tilde{0}) \cdot \tilde{x} + \|\Gamma\|_{1,\beta} |\tilde{x}|^{\beta+1}.$$

If x satisfies equation (1.9), then

$$x_d - \nabla\Gamma(\tilde{0}) \cdot \tilde{x} = x \cdot \left(-\nabla\Gamma(\tilde{0}), 1\right) = \sqrt{|\nabla\Gamma(\tilde{Q})|^2 + 1}\,(x \cdot b) \geq x \cdot b.$$

Now Lemma 1.10, applied with $u = |x|$, $v = x \cdot b$ and a yet to be specified c yields

$$y_d - \nabla\Gamma(\tilde{0}) \cdot \tilde{y} > \frac{c^\beta}{2}|y|^{\beta+1}.$$

We therefore see that if we choose $C = \left(2\|\Gamma\|_{1,\beta}\right)^\beta \vee 1$, then we obtain

$$\begin{aligned} y_d - \nabla\Gamma(\tilde{0}) \cdot \tilde{y} &> \left(2\|\Gamma\|_{1,\beta} \vee 1\right) |y|^{\beta+1}/2 \\ &\geq \|\Gamma\|_{1,\beta} |\tilde{y}|^{\beta+1}. \end{aligned}$$

□

Moreover, the two following the two geometric results hold on $\mathcal{C}^{1,1}$-domains, see [7, Lemma 6.1, Lemma 6.2].

Lemma 1.12. *Let $b \in \mathbf{R}^d$ be such that $|b| = 1$. Then the region of 2-tangential approach $\mathcal{P}_b$ defined above satisfies*

$$B(Rb, R) \subset \mathcal{P}_b$$

where $R = (4\|\Gamma\|_{1,1} \vee 2)^{-1}$.

Lemma 1.13. *Assume that $\Gamma : \mathbf{R}^{d-1} \to \mathbf{R}$ is a $\mathcal{C}^{1,1}$ function and let $D = D_\Gamma$. Then for every $Q \in \partial D$,*

$$B(Q + Rn, R) \subset D, \qquad B(Q - Rn, R) \subset D^c,$$

where b is the unit inward normal at Q and $R = (4\|\Gamma\|_{1,1} \vee 2)^{-1}$.

1.4.2 κ-fat domains

Let $\kappa \in (0,1]$. An open set $D \subset \mathbf{R}^d$ is *κ-fat* if there exists $r_0 > 0$ such that for all $x \in \overline{D}$ and all $r \in (0, r_0]$, there exist $A_r(x) \in D$ such that

$$B\left(A_r(x), \kappa r\right) \subset D \cap B(x,r).$$

We say that pair (κ, r_0) is the *characteristic* of the κ-fat open set. Note that $\mathcal{C}^{1,\beta}$-domains are examples of κ-fat domains.

1.5 Notation

In what follows, we will use some notation which might be unknown to the reader and is therefore explained in this section. We use $f \approx Cg$ if there exists $C \geq 1$ such that

$$C^{-1}g(x) \leq f(x) \leq Cg(x).$$

Chapter 2

BOUNDARY HARNACK PRINCIPLE FOR SYMMETRIC STABLE-LIKE PROCESSES

This section serves three objectives. First, we will derive boundary Harnack principle for symmetric stable-like processes satisfying additional assumptions for the perturbation $\kappa(x,y)$. Second, we will derive the scale-invariant boundary Harnack principle under the same assumptions on $\kappa(x,y)$. Finally, we will use the boundary Harnack principle as a main tool to derive Dirichlet heat kernel estimates for the killed symmetric stable-like process.

2.1 Boundary Harnack Principle

In this section we will prove the following theorem:

Theorem 2.1. *Let X_t be a symmetric stable-like process of index α in $\mathbf{R}^d$ and assume $\kappa(x,y)$ is a measurable symmetric function with $0 < \kappa_1 \le \kappa(x,y) \le \kappa_2 < \infty$. Suppose that additionally, one of the following conditions holds:*

(1) $0 < \alpha < 1$, and $\kappa(x,y)$ is continuous.

(2) $1 \le \alpha < 2$, and $\kappa(x,y)$ is continuous and for a. e. $x \in \mathbf{R}^d$,

$$|\kappa(x,x) - \kappa(x,y)|\, \mathbf{1}_{\{|x-y|<1\}} \le M_\kappa(x)\, |x-y|^\gamma .$$

where $\gamma > \alpha - 1$ and $M_\kappa \in L^1_{\text{loc}}(\mathbf{R}^d)$.

Then X_t satisfies the boundary Harnack principle: If $Q \in \partial D$ and $r \in (0, r_0)$, then for any nonnegative functions u, v which are not identically 0, regular harmonic in $D \cap B(Q,r)$, and vanish continuously on $\partial D \cap B(Q,r)$, there exists $C_1 = C_1(d, \alpha, \kappa_1, \kappa_2, Q, r, r_0) > 0$ such that

$$\frac{u(x)}{v(x)} \le C_1 \frac{u(y)}{v(y)}, \qquad x, y \in B(Q,r) \cap D.$$

We will establish the boundary Harnack principle for stable-like processes by showing that assumptions A-D in Theorem 1.4 are satisfied. Indeed, assumptions A, C, and D can be verified whenever the state space F is sufficiently nice without additional assumptions on $\kappa(x,y)$.

Regarding assumption A, Chen and Kumagai [19] establish that the stable-like process on d-sets (including $\mathbb{R}^d$) as well as the closure $\overline{G}$ of a Lipschitz open G set can be refined to a Feller process. From here on we use F to denote $F = \mathbf{R}^d$ in the case of the symmetric stable-like process on $\mathbf{R}^d$ or $F = \overline{G}$ in the case of a reflected process on $\overline{G}$.

Proposition 2.2. *Let X_t be the symmetric stable-like process or the reflected stable-like process on the closure of a Lipschitz open set of index $\alpha \in (0,2)$. Then the transition semigroup of X_t is both Feller and strong Feller, and the process satisfies Hunt's hypothesis.*

Proof. This is essentially established in [19]. The strong Feller property can be obtained from the dominated convergence theorem. □

Regarding Assumption C, it is straightforward to check the asserted relative constancy of the Lévy kernel.

Proposition 2.3. *Let X_t be the symmetric stable-like process or the reflected stable-like process on the closure of a Lipschitz open set of index $\alpha \in (0,2)$. Then its Lévy kernel has a symmetric density,*

$$\nu(x,y) = \frac{\kappa(x,y)}{|x-y|^{d+\alpha}}.$$

Moreover, for any $x_0 \in F$, $0 < r < R < R_0$, $x \in B(x_0,r)$ and $y \in F \setminus B(x_0,R)$,

$$\frac{\kappa_1}{\kappa_2}\left(1-\frac{r}{R}\right)^{d+\alpha}\nu(x_0,y) \le \nu(x,y) \le \frac{\kappa_2}{\kappa_1}\left(\frac{1}{1-\frac{r}{R}}\right)^{d+\alpha}\nu(x_0,y).$$

Proof. The structure of the Lévy kernel follows from Dirichlet form theory, see for example [12]. Its symmetry is inherited from the symmetry of $\kappa(x,y)$. To prove the remaining assertion, note that for $x_0 \in \mathbf{R}^d$, $x \in B(x_0,r)$, and $y \in B(x_0,r)^c$,

$$\frac{1}{|x_0-y|^{d+\alpha}} \quad \ge \quad \left(\frac{|y-x_0|-|x_0-x|}{|x_0-y|}\right)^{d+\alpha}\frac{1}{|x-y|^{d+\alpha}}$$

$$\geq \left(1-\frac{r}{R}\right)^{d+\alpha}\frac{1}{|x-y|^{d+\alpha}}.$$

Therefore, we obtain

$$\begin{aligned}\nu(x,y) &= \mathcal{A}(n,-\alpha)\frac{\kappa(x,y)}{|x-y|^{d+\alpha}}\\ &\leq \frac{\kappa_2}{\kappa_1}\mathcal{A}(n,-\alpha)\left(1-\frac{r}{R}\right)^{-(d+\alpha)}\frac{\kappa(x_0,y)}{|x_0-y|^{d+\alpha}}\\ &= \frac{\kappa_2}{\kappa_1}\left(\frac{1}{1-\frac{r}{R}}\right)^{d+\alpha}\nu(x_0,y).\end{aligned}$$

Similarly,

$$\frac{1}{|x_0-y|^{d+\alpha}} \leq \left(1+\frac{r}{R}\right)^{d+\alpha}\frac{1}{|x-y|^{d+\alpha}}$$

implies that

$$\nu(x,y) \geq \left(\frac{\kappa_2}{\kappa_1}\left(1+\frac{r}{R}\right)^{d+\alpha}\right)^{-1}\nu(x_0,y) \geq \left(\frac{\kappa_2}{\kappa_1}\left(\frac{1}{1-\frac{r}{R}}\right)^{d+\alpha}\right)^{-1}\nu(x_0,y).$$

□

Proposition 2.4. *Let X_t be the symmetric stable-like process or the reflected stable-like process on the closure of a Lipschitz open set of index $\alpha \in (0,2)$. Then If $x_0 \in \mathbf{R}^d$, $0 < r < p < R < R_0$, then the Green's function for the ball $B(x_0,r)$ satisfies*

$$\sup_{x\in B(x_0,r)}\ \sup_{y\in\backslash B(x_0,p)^c} G_{B(x_0,R)}(x,y) < \infty.$$

Proof. Bogdan, Kumagai, and Kwaśnicki [10, Proposition 5.3] show that assumption D can be verified whenever the following heat kernel estimate holds: For some $\alpha > 0$ and $r_0 > 0$, there exists c such that

$$c^{-1}\left(t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x-y|^{d+\alpha}}\right) \leq p(t,x,y) \leq c\left(t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x-y|^{d+\alpha}}\right)$$

for $x,y \in F$ with $|x-y| < r_0$, and any $t \in (0,r_0^{\alpha})$. As mentioned before, this has been established by Chen and Kumagai [19], see Theorem 1.5. □

It remains to verify assumption B. We will start by investigating the L^2-generator of the process X_t which is associated with its Dirichlet form (see equation (1.3)) and given by

$$\Delta_F^{\frac{\alpha}{2},\kappa} u(x) = \lim_{\varepsilon \to 0^+} \mathcal{A}(d,-\alpha) \int_{\{y \in F : |x-y| > \varepsilon\}} (u(x) - u(y)) \frac{\kappa(x,y)}{|x-y|^{d+\alpha}} \, \mathrm{d}y.$$

If $F = \mathbf{R}^d$, we denote this operator by

$$\Delta^{\frac{\alpha}{2},\kappa} := \Delta_{\mathbf{R}^d}^{\frac{\alpha}{2},\kappa}.$$

We proceed to show that for certain nice classes of functions, $\Delta_F^{\frac{\alpha}{2},\kappa} u$ exists and is in the domain of both the L^2 and Feller generators. We will denote the domain of the Feller generator of the symmetric stable-like process by $\mathcal{D}(\mathcal{A})$, and the domain of the L^2-generator by $\mathcal{D}(\Delta_F^{\frac{\alpha}{2},\kappa})$. As mentioned above, the conditions that we need to impose on $\kappa(x,y)$ depend on whether $0 < \alpha < 1$ or $1 \le \alpha < 2$, that is why we treat both cases separately.

A function u is uniformly η-Hölder continuous on F if

$$\|u\|_{\mathcal{C}^\eta} := \sup_{(x,y) \in F \times F, |x-y| < 1} \frac{|u(x) - u(y)|}{|x-y|^\eta} < \infty.$$

Lemma 2.5. *Let X_t be the symmetric stable-like process or the reflected stable-like process on the closure of a Lipschitz open set of index $\alpha \in (0,1)$. For $\eta > \alpha$ and $u \in \mathcal{C}^\eta(F)$,*

1. $$\lim_{\varepsilon \to 0^+} \Delta_{F,\varepsilon}^{\frac{\alpha}{2},\kappa} u = \Delta_F^{\frac{\alpha}{2},\kappa} u$$

   locally uniformly in F. In particular, $\Delta_F^{\frac{\alpha}{2},\kappa} u(x)$ exists for every $x \in F$.

2. *If, furthermore, $u \in L^\infty(F)$, then $\Delta_F^{\frac{\alpha}{2},\kappa} u$ is uniformly bounded on F,*

   $$\left| \Delta_F^{\frac{\alpha}{2},\kappa} u(x) \right| \le \kappa_2 \mathcal{A}(d,-\alpha) d\omega_d \left(\frac{\|u\|_{\mathcal{C}^\eta}}{\eta - \alpha} + 2 \frac{\|u\|_{L^\infty}}{\alpha} \right).$$

Proof. 1. Assume that $u \in \mathcal{C}^\eta(F)$. Let $x \in F$ and $0 < \delta < \varepsilon < 1$. Then,

$$\left| \Delta_{F,\delta}^{\frac{\alpha}{2},\kappa} u(x) - \Delta_{F,\varepsilon}^{\frac{\alpha}{2},\kappa} u(x) \right|$$

$$\begin{aligned}
&= \mathcal{A}(d,-\alpha)\left|\int_{\{y\in F:\varepsilon\geq|x-y|>\delta\}} (u(x)-u(y))\,\frac{\kappa(x,y)}{|x-y|^{d+\alpha}}\,\mathrm{d}y\right| \\
&\leq \kappa_2\mathcal{A}(d,-\alpha)\|u\|_{\mathcal{C}^\eta}\int_{\{y\in F:\varepsilon\geq|x-y|>\delta\}} |x-y|^\eta\,\frac{1}{|x-y|^{d+\alpha}}\,\mathrm{d}y \\
&\leq \kappa_2\mathcal{A}(d,-\alpha)\|u\|_{\mathcal{C}^\eta}\int_{\{y\in\mathbf{R}^d:\varepsilon\geq|y|>\delta\}} \frac{1}{|y|^{d+\alpha-\eta}}\,\mathrm{d}y.
\end{aligned}$$

The last integral can be computed using polar coordinates which yields

$$\sup_{x\in F}\left|\Delta_{F,\delta}^{\frac{\alpha}{2},\kappa}u(x)-\Delta_{F,\varepsilon}^{\frac{\alpha}{2},\kappa}u(x)\right|\leq\frac{\kappa_2 d\omega_d\mathcal{A}(d,-\alpha)\|u\|_{\mathcal{C}^\eta}}{\eta-\alpha}\left(\varepsilon^{\eta-\alpha}-\delta^{\beta-\alpha}\right),$$

where ω_d is the volume of the unit ball in $\mathbf{R}^d$. Since $\eta>\alpha$, the right hand side converges to zero as $\delta,\varepsilon\to 0^+$. We conclude that $\Delta_{F,\varepsilon}^{\frac{\alpha}{2},\kappa}u(x)$ is uniformly Cauchy and therefore converges uniformly to $\Delta_F^{\frac{\alpha}{2},\kappa}u(x)$.

2. Now assume additionally that $u\in L^\infty(F)$. Then

$$\begin{aligned}
\left|\Delta_F^{\frac{\alpha}{2},\kappa}u(x)\right| &\leq \mathcal{A}(d,-\alpha)\left(\left|\int_{F\cap B(x,1)}(u(x)-u(y))\,\frac{\kappa(x,y)}{|x-y|^{d+\alpha}}\,\mathrm{d}y\right|\right. \\
&\quad\left.+\left|\int_{F\cap B(x,1)^c}(u(x)-u(y))\,\frac{\kappa(x,y)}{|x-y|^{d+\alpha}}\,\mathrm{d}y\right|\right) \\
&\leq \mathcal{A}(d,-\alpha)\left(\kappa_2\|u\|_{\mathcal{C}^\eta}\int_{B(x,1)}|x-y|^\eta\frac{1}{|x-y|^{d+\alpha}}\mathrm{d}y+\right. \\
&\quad\left.2\kappa_2\|u\|_{L^\infty}\int_{B(x,1)^c}\frac{1}{|x-y|^{d+\alpha}}\mathrm{d}y\right) \\
&= \kappa_2\mathcal{A}(d,-\alpha)d\omega_d\left(\frac{\|u\|_{\mathcal{C}^\eta}}{\eta-\alpha}+2\frac{\|u\|_{L^\infty}}{\alpha}\right).
\end{aligned}$$

□

To show that a suitable subclass of these functions is in the domain of the L^2 generator is relatively straightforward in the case $D=\mathbf{R}^d$.

Lemma 2.6. *Let X_t be the symmetric stable-like process of index $\alpha\in(0,1)$. If $\eta>\alpha$, then the set of square-integrable, η-Hölder continuous functions is a subset of the L^2-generator of the Dirichlet form,*

$$\mathcal{C}^\eta(\mathbf{R}^d)\cap L^2(\mathbf{R}^d)\subseteq\mathcal{D}\left(\Delta^{\frac{\alpha}{2},c}\right).$$

Proof. Let $u \in \mathcal{C}^\beta(\mathbf{R}^d)$. Since $\mathcal{C}_c^\infty(\mathbf{R}^d)$ is dense in $\mathcal{F}$ in the $\mathcal{E}_1$-norm, it suffices to show that for all $v \in \mathcal{C}_c^\infty(\mathbf{R}^d)$,

$$\mathcal{E}(u,v) = -(\Delta^{\frac{\alpha}{2},\kappa} u, v)$$

to conclude that u is in the domain of the L^2-generator. A straightforward computation shows that

$$\mathcal{E}(u,v) = \lim_{\varepsilon \to 0^+} \int_{\mathbf{R}^d} \int_{\{y:|y|>\varepsilon\}} \frac{u(x)-u(x+y)}{|y|^{d+\alpha}} \kappa(x,x+y)\,\mathrm{d}y v(x)\,\mathrm{d}x.$$

Since v is compactly supported, there exists a ball B such that v vanishes outside of B, and it suffices to take the outer integral over B. Since v is bounded, we only need to show that

$$\int_{\{y:|y|>\varepsilon\}} \frac{u(x)-u(x-y)}{|y|^{d+\alpha}} \kappa(x,x-y)\,\mathrm{d}y$$

is bounded by a constant independent of ε and apply the bounded convergence theorem. Now,

$$\begin{aligned} & \left| \int_{\{y:|y|>\varepsilon\}} \frac{u(x)-u(x-y)}{|y|^{d+\alpha}} \kappa(x,x-y)\,\mathrm{d}y \right| \qquad (2.1) \\ \leq & \left| \int_{B(0,1)} \frac{u(x)-u(x-y)}{|y|^{d+\alpha}} \kappa(x,x-y)\,\mathrm{d}y \right| \\ & + \left| \int_{B(0,1)^c} \frac{u(x)-u(x-y)}{|y|^{d+\alpha}} \kappa(x,x-y)\,\mathrm{d}y \right| \\ \leq & \kappa_2 \|u\|_{\mathcal{C}^\eta} \int_{B(0,1)} \frac{1}{|y|^{d+\alpha-\eta}}\,\mathrm{d}y \\ & + 2\kappa_2 \|u\|_{L^\infty} \int_{B(0,1)^c} \frac{1}{|y|^{d+\alpha}}\,\mathrm{d}y. \qquad (2.2) \end{aligned}$$

Both integrals can be computed using polar coordinates, whence

$$\left| \int \frac{u(x)-u(x-y)}{|y|^{d+\alpha}} \kappa(x,x-y)\,\mathrm{d}y \right| \leq \frac{d\omega_d \kappa_2 \|u\|_{\mathcal{C}^\eta}}{\eta - \alpha} + \frac{2d\omega_d \kappa_2 \|u\|_{L^\infty}}{\alpha}.$$

Recall that any square-integrable, uniformly continuous function is vanishing at infinity, and therefore bounded. □

Proposition 2.7. *Let X_t be a Feller process which acts on L^2 as a strongly continuous contraction semigroup. Let $\mathcal{L}$ denote the L^2 generator and $\mathcal{A}$ the Feller generator. If f is in* *the domain of the L^2-generator and $\mathcal{L}f \in \mathcal{C}_0(\mathbf{R}^d)$, then f is also in the domain of the Feller generator and $\mathcal{A}f = \mathcal{L}f$.*

Proof. Let $f \in \mathcal{D}(\mathcal{L})$. Then by Dynkin's formula for functions in the L^2-generator,

$$\frac{\mathbf{E}_x[f(X_t)] - f(x)}{t} = \frac{1}{t}\int_0^t \mathbf{E}_x\left[\mathcal{L}f(X_s)\right] \mathrm{d}s = \frac{1}{t}\int_0^t p_s\mathcal{L}f(x)\,\mathrm{d}s.$$

Since X_t is a Feller process, p_t is a strongly continuous semigroup acting on $\mathcal{C}_0(\mathbf{R}^d)$. In particular, for any $f \in \mathcal{C}_0(\mathbf{R}^d)$, $p_t f \to f$ uniformly as $t \to 0^+$. By assumption, $\mathcal{L}f \in \mathcal{C}_0(\mathbf{R}^d)$, so $p_t\mathcal{L}f(x) \to \mathcal{L}f(x)$ uniformly. This clearly implies that the time average converges uniformly to the same limit, so for the Feller generator $\mathcal{A}$ of X_t we obtain that

$$\mathcal{A}f(x) = \lim_{t\to 0^+} \frac{\mathbf{E}_x[f(X_t)] - f(x)}{t} = \lim_{t\to 0^+} \frac{1}{t}\int_0^t p_s\mathcal{L}f(x)\,\mathrm{d}s = \mathcal{L}f(x)$$

converges uniformly, and hence $f \in \mathcal{D}(\mathcal{A})$ and $\mathcal{A}f = \mathcal{L}f$. □

Lemma 2.8. *Let X_t be the symmetric stable-like process of index $\alpha \in (0,1)$. Furthermore, assume that $\kappa(x,y)$ is continuous. If u is bounded and Hölder continuous with Hölder exponent $\eta > \alpha$, then $\Delta^{\frac{\alpha}{2},\kappa}u$ is continuous.*

Proof. Let $x_n \to x$. Then

$$\Delta_{\varepsilon}^{\frac{\alpha}{2},\kappa}u(x_n) = \mathcal{A}(d,-\alpha)\int_{|y|>\varepsilon} (u(x_n) - u(y+x_n))\,\frac{\kappa(x_n, y+x_n)}{|y|^{d+\alpha}}\,\mathrm{d}y.$$

The integrand converges pointwise to $(u(x) - u(y+x))\,\frac{\kappa(x,y+x)}{|y|^{d+\alpha}}$ as $n \to \infty$ since both κ and u are continuous. Now,

$$\begin{aligned} & \mathbf{1}_{\{|y|>\varepsilon\}}\,(u(x_n) - u(y+x_n))\,\frac{\kappa(x_n, y+x_n)}{|x_n - y|^{d+\alpha}} \\ \leq\ & 2\kappa_2\|u\|_{L^\infty}\mathbf{1}_{\{|y|>\varepsilon\}}\frac{1}{|y|^{d+\alpha}} \end{aligned}$$

which is integrable. Therefore, the dominated convergence theorem implies that

$$\lim_{n\to\infty} \Delta_{\varepsilon}^{\frac{\alpha}{2},\kappa}u(x_n) = \Delta_{\varepsilon}^{\frac{\alpha}{2},\kappa}u(x),$$

that is, continuity of $\Delta_{\varepsilon}^{\frac{\alpha}{2},\kappa}u$. Since by Lemma 2.5, this quantity converges uniformly to $\Delta^{\frac{\alpha}{2},\kappa}u$ as $\varepsilon \to 0^+$, we can conclude that $\Delta^{\frac{\alpha}{2},\kappa}u$ is continuous, being the uniform limit of continuous functions. □

Lemma 2.9. *Let X_t be the symmetric stable-like process of index $\alpha \in (0,1)$. If $u \in \mathcal{D}(\Delta^{\frac{\alpha}{2},\kappa})$ is compactly supported, then $\Delta^{\frac{\alpha}{2},\kappa}u$ vanishes at infinity. Indeed, there exists $M = M(u) > 0$ and $C = C(d,\alpha,\kappa_2)$ such that*

$$|\Delta^{\frac{\alpha}{2},\kappa}u(x)| \leq C\frac{\|u\|_{L^1}}{|x|^{d+\alpha}}, \qquad |x| \geq M$$

Proof. Let $u \in \mathcal{D}(\Delta^{\frac{\alpha}{2},\kappa})$ be compactly supported. Its support is contained in $B(0,R/2)$ for some $R > 0$. Then for $|x| > R$,

$$\left|\Delta^{\frac{\alpha}{2},\kappa}u(x)\right| \leq \kappa_2\mathcal{A}(d,-\alpha)\int_{B(0,R/2)} |u(y)|\,\frac{1}{|x-y|^{d+\alpha}}\,\mathrm{d}y$$

Since $|y| \leq R/2$ and $|x| > R$

$$|x-y| \geq |x| - \frac{R}{2} = \frac{|x|}{2} + \frac{R}{2} - \frac{R}{2} = \frac{|x|}{2},$$

and therefore

$$\left|\Delta^{\frac{\alpha}{2},\kappa}u(x)\right| \leq \frac{2^{d+\alpha}\kappa_2\mathcal{A}(d,-\alpha)\,\|u\|_{L^1}}{|x|^{d+\alpha}}.$$

□

Corollary 2.10. *Let X_t be a stable-like process with index $0 < \alpha < 1$ and assume $\kappa(x,y)$ is continuous. Then for any $\eta > \alpha$, the η-Hölder continuous, square-integrable functions are in the Feller generator of the process.*

Proof. This follows from Lemmata 2.6, 2.8, and 2.9, as well as Proposition 2.7. □

We may therefore prove the theorem in case (1).

Proof of Theorem 2.1, case (1). We have shown previously that assumptions A, C, and D are satisfied. By the preceding corollary, the class of smooth functions of compact support is a subset of the Feller generator, a class of functions which satisfies assumption B. Therefore, we can apply Bogdan, Kumagai, and Kwaśnickis' result [10]. □

Next, we treat the case where $1 \leq \alpha < 2$. If we strengthen the requirements on the regularity of $\kappa(x,y)$, we can still establish the boundary Harnack principle in the same way.

Lemma 2.11. *Let X_t be the symmetric stable-like process or the reflected stable-like process on the closure of a Lipschitz open set of index $\alpha \in [1,2)$. Furthermore, assume that for some $\gamma > \alpha - 1$, the function $\kappa(x,y)$ satisfies*

$$|\kappa(x,x) - \kappa(x,y)| \, \mathbf{1}_{\{|x-y|<1\}} \leq M_\kappa(x) \, |x-y|^\gamma , \qquad x,y \in \mathbf{R}^d.$$

Let $\eta > \alpha - 1$. If u is differentiable with η-Hölder continuous and bounded gradient, then $\Delta^{\frac{\alpha}{2},\kappa} u(x)$ exists for every x for which $M_\kappa(x) < \infty$.

Proof. Let $1 > \varepsilon > \delta > 0$. Then

$$\begin{aligned}
& \left| \Delta_\delta^{\frac{\alpha}{2},\kappa} u(x) - \Delta_\varepsilon^{\frac{\alpha}{2},\kappa} u(x) \right| \\
= \; & \mathcal{A}(d,-\alpha) \left| \int_{\varepsilon \geq |x-y| > \delta} (u(x) - u(y)) \frac{\kappa(x,y)}{|x-y|^{d+\alpha}} \, \mathrm{d}y \right| \\
\leq \; & \mathcal{A}(d,-\alpha) \left(\left| \int_{\varepsilon \geq |x-y| > \delta} (u(x) - u(y)) \frac{\kappa(x,y) - \kappa(x,x)}{|x-y|^{d+\alpha}} \, \mathrm{d}y \right| \right. \\
& \left. + \kappa(x,x) \left| \int_{\varepsilon \geq |x-y| > \delta} (u(x) - u(y)) \frac{1}{|x-y|^{d+\alpha}} \, \mathrm{d}y \right| \right) \\
=: \; & I + II
\end{aligned} \tag{2.3}$$

For the left integral, we use the Hölder type condition for $\kappa(x,y)$, and the fact that we assumed u to be η-Hölder continuous,

$$\begin{aligned}
I & \leq M_\kappa(x) \left| \int_{\varepsilon \geq |x-y| > \delta} \frac{u(x) - u(y)}{|x-y|^{d+\alpha-\gamma}} \, \mathrm{d}y \right| \\
& \leq M_\kappa(x) \|\nabla u\|_{L^\infty} \int_{\varepsilon \geq |x-y| > \delta} \frac{1}{|x-y|^{d+\alpha-1-\gamma}} \, \mathrm{d}y \\
& = M_\kappa(x) \|\nabla u\|_{L^\infty} d\omega_d \frac{\varepsilon^{d+\alpha-\gamma} - \delta^{d+\alpha-\gamma}}{d+\alpha-\gamma}.
\end{aligned}$$

For the right integral in equation (2.3), we use the mean value theorem which asserts that there exists $\xi \in (x,y)$ such that $u(x) - u(y) = \nabla u(\xi) \cdot (x-y)$ and the Hölder continuity of the gradient,

$$\begin{aligned} II &= \left|\int_{\varepsilon\geq|x-y|>\delta} \frac{(u(x)-u(y))-\nabla u(x)\cdot(x-y)}{|x-y|^{d+\alpha}}\,\mathrm{d}y\right| \\ &= \left|\int_{\varepsilon\geq|x-y|>\delta} \frac{(\nabla u(\xi)-\nabla u(x))\cdot(x-y)}{|x-y|^{d+\alpha}}\,\mathrm{d}y\right| \\ &\leq \|\nabla u\|_{\mathcal{C}^\eta} \int_{\varepsilon\geq|x-y|>\delta} \frac{1}{|x-y|^{d+\alpha-1-\eta}}\,\mathrm{d}y \\ &= \|\nabla u\|_{\mathcal{C}^\eta}\, d\omega_d \frac{\varepsilon^{d+\alpha-\eta}-\delta^{d+\alpha-\eta}}{d+\alpha-\eta}. \end{aligned}$$

We conclude that

$$\begin{aligned} &\left|\Delta_\delta^{\frac{\alpha}{2},\kappa}u(x)-\Delta_\varepsilon^{\frac{\alpha}{2},\kappa}u(x)\right| \\ \leq\ & d\omega_d\left(M_\kappa(x)\|\nabla u\|_{L^\infty}\frac{\varepsilon^{d+\alpha-\gamma}-\delta^{d+\alpha-\gamma}}{d+\alpha-\gamma}+\|\nabla u\|_{\mathcal{C}^\eta}\,\kappa_2\frac{\varepsilon^{d+\alpha-\eta}-\delta^{d+\alpha-\eta}}{d+\alpha-\eta}\right), \end{aligned}$$

which converges to zero as $\delta,\varepsilon\to 0^+$ since we have assumed that $M_\kappa(x)<\infty$. □

Lemma 2.12. *Let X_t be the symmetric stable-like process or the reflected stable-like process on the closure of a Lipschitz open set of index $\alpha\in[1,2)$. Furthermore, assume that for some $\gamma>\alpha-1$, and some $M_\kappa\in L^1_{\mathrm{loc}}(\mathbf{R}^d)$, we have*

$$|\kappa(x,x)-\kappa(x,y)|\,\mathbf{1}_{\{|x-y|<1\}}\leq M_\kappa(x)\,|x-y|^\gamma, \qquad x,y\in\mathbf{R}^d.$$

Let $\eta>\alpha-1$. Then the set of differentiable, square-integrable functions whose gradient is η-Hölder continuous and bounded is a subset of the generator of the Dirichlet form,

$$\mathcal{C}_b^{1,\eta}(\mathbf{R}^d)\cap L^2(\mathbf{R}^d)\subseteq\mathcal{D}\left(\Delta^{\frac{\alpha}{2},\kappa}\right).$$

Proof. Let u be a differentiable function whose gradient is β-Hölder continuous and bounded. Similarly to Proposition 2.6, it suffices to show that for all $v\in\mathcal{C}_c^\infty(\mathbf{R}^d)$,

$$\left|\int_{|x-y|>\varepsilon}\frac{u(x)-u(y)}{|x-y|^{d+\alpha}}\kappa(x,y)\,\mathrm{d}y v(x)\right| \tag{2.4}$$

is dominated by a function which is integrable on the support of v, denoted by D, and apply the dominated convergence theorem. Now,

$$\left|\int_{|x-y|>\varepsilon}\frac{u(x)-u(y)}{|x-y|^{d+\alpha}}\kappa(x,y)\,\mathrm{d}y v(x)\right|$$

$$\leq \left|\int_{1\geq|x-y|>\varepsilon} \frac{u(x)-u(y)}{|y|^{d+\alpha}}\kappa(x,y)\,\mathrm{d}y v(x)\right|$$
$$+\left|\int_{|x-y|>1} \frac{u(x)-u(x)}{|x-y|^{d+\alpha}}\kappa(x,y)\,\mathrm{d}y v(x)\right|$$

The second integral can be handled exactly as in Proposition 2.6,

$$\left|\int_{|x-y|>1} \frac{u(x)-u(y)}{|x-y|^{d+\alpha}}\kappa(x,y)\,\mathrm{d}y\right| \leq \frac{2d\omega_d\kappa_2\|u\|_{L^\infty}}{\alpha}.$$

For the first integral, we repeat the computations from Proposition 2.11 which yield

$$\begin{aligned}\left|\int_{1\geq|x-y|>\varepsilon} \frac{u(x)-u(y)}{|x-y|^{d+\alpha}}\kappa(x,y)\,\mathrm{d}y\right| &\leq \pi^{d-1}\left(\frac{2^\gamma\kappa_2\|\nabla u\|_{L^\infty}}{\gamma-\alpha+1}\right.\\ &\qquad\left.+\frac{2^\eta M_\kappa(x)\|u\|_{\mathcal{C}^\eta}}{\eta-\alpha+1}\right).\end{aligned}$$

Since $M_\kappa(x)$ locally is integrable, this shows that the function in equation (2.4) is dominated by a function which is integrable on D. □

The respective versions of Lemma 2.8 and Theorem 2.1 can now be proven as before.

Lemma 2.13. *Let X_t be the symmetric stable-like process or the reflected stable-like process on the closure of a Lipschitz open set of index $\alpha \in [1,2)$. Furthermore, assume that for some $\gamma > \alpha - 1$, and some $M_\kappa \in L^1(\mathbf{R}^d)$, we have*

$$|\kappa(x,x)-\kappa(x,y)|\,\mathbf{1}_{\{|x-y|<1\}} \leq M_\kappa(x)\,|x-y|^\gamma, \qquad x,y\in\mathbf{R}^d.$$

If u is bounded, square-integrable, and has a Hölder continuous gradient with Hölder exponent $\eta > \alpha - 1$, then $\Delta^{\frac{\alpha}{2},\kappa}u$ is continuous.

Proof. This follows from Lemmata 2.9 to 2.12 and Proposition 2.7. □

This, finally, yields the boundary Harnack principle just as before.

Proof of Theorem 2.1, case (2). We have shown previously that assumptions A, C, and D are satisfied. By the preceding corollary, the class of smooth functions of compact support is a subset of the Feller generator, a class of functions which satisfies assumption B. Therefore, we can apply Bogdan, Kumagai, and Kwaśnickis' result [10]. □

2.2 Scale-invariant Boundary Harnack Principle

Aside from their main result, Bogdan, Kumagai, and Kwaśnicki [10] also provide sufficient conditions for a process to satisfy the scale-invariant version of the boundary Harnack inequality.

Theorem 2.14. *Let X_t be a symmetric stable-like process of index α in $\mathbf{R}^d$ and assume $\kappa(x, y)$ is a measurable function with $0 < \kappa_1 \leq \kappa(x, y) \leq \kappa_2 < \infty$. Suppose that additionally, one of the following conditions holds:*

(1) The index satisfies $0 < \alpha < 1$, and $\kappa(x, y)$ is a continuous.

(2) The index satisfies $1 \leq \alpha < 2$, and $\kappa(x, y)$ is such that

$$|\kappa(x,x) - \kappa(x, x+y)| \, \mathbf{1}_{\{|y|<1\}} \leq M_\kappa(x) \, |y|^\gamma , \qquad x, y \in \mathbf{R}^d.$$

where $\gamma > \alpha - 1$ and $M_\kappa \in L^1_{\text{loc}}(\mathbf{R}^d)$.

Then X_t satisfies the scale-invariant boundary Harnack principle: Let $x_0 \in \mathbf{R}^d$ and $0 < r < R < \infty$. Let $D \subset B(x_0, R)$ be any open set. Suppose f and g are functions which are nonnegative, regular harmonic with respect to X_t, and vanish in $B(x_0, R) \setminus D$. Then there exist $C_1 = C_1(d, \alpha, \kappa_1, \kappa_2, x_0, r/R) > 0$ such that

$$f(x)g(y) \leq C_1 f(y)g(x), \qquad x, y \in B(x_0, r).$$

Not that the only difference to the ordinary boundary Harnack principle is that the constant C_1 may no longer depend on r and R independently, but only through their ratio r/R. Bogdan, Kumagai, and Kwaśnicki [10] show that there are several additional assumptions needed which they refer to as *stable-like scaling.*

Definition 2.1. A symmetric Hunt process X_t on $\mathbf{R}^d$ is said to have stable-like scaling if the following conditions are satisfied:

(a) The Lévy measure of the process $\nu(x,y)$ satisfies that for every $x_0 \in \mathbf{R}^d$, $0 < r < R < \infty$, $x \in B(x_0, r)$ and $y \in \mathbf{R}^d \setminus B(x_0, R)$.

$$c^{-1}\nu(x_0, y) \le \nu(x, y) \le c\nu(x_0, y)$$

with

$$c = c(x_0, r, R) \le C\left(\frac{r}{R}, x_0\right).$$

(b) For $x_0 \in \mathbf{R}^d$ and $0 < r < R < \infty$,

$$\inf_{y \in \overline{A}(x_0, r, R)} \nu(x_0, y) \ge C\left(\frac{r}{R}, x_0\right) R^{-d-\alpha}.$$

(c) For $0 < r < \infty$ and $x_0 \in \mathbf{R}^d$,

$$\sup_{x \in B(x_0, r)} \mathbf{E}_x\left[\tau_{B(x_0, r)}\right] \le Cr^{\alpha}.$$

(d) For $x_0 \in \mathbf{R}^d$, $0 < r < p < R < \infty$,

$$\sup_{x \in B(x_0, r)} \sup_{y \in \mathbf{R}^d \setminus B(x_0, p)} G_{B(x_0, r)}(x, y) \le C\left(\frac{r}{R}, \frac{p}{R}\right) R^{\alpha-d}.$$

(e) If $0 < r < R < \infty$ and $x_0 \in \mathbf{R}^d$, then

$$\delta\left(\overline{B(x_0, r)}, B(x_0, R)\right) \le C\left(\frac{r}{R}\right) R^{-\alpha}.$$

when $0 < r < R < \infty$ and $x_0 \in \mathbf{R}^d$; if $0 < r < p < R < \tilde{r}$ and $x_0 \in \mathbf{R}^d$, then

$$\delta\left(\overline{A(x_0, p, R)}, A(x_0, r, \tilde{r})\right) \le C\left(\frac{r}{R}, \frac{p}{R}, \frac{R}{\tilde{r}}\right) R^{-\alpha}.$$

Naturally, one it is interested in a more stringent criterion than stable-like scaling. The authors provide several criteria that imply some of the conditions. For our purpose, the following version of their results is most useful:

Proposition 2.15. *Suppose that X_t is a stochastic process satisfying assumptions A to D. If its Lévy kernel satisfies*

$$c^{-1}|x-y|^{-(d+\alpha)}e^{-q|x-y|} \le \nu(x,y) \le c|x-y|^{-(d+\alpha)}e^{-q|x-y|}. \tag{2.5}$$

for some $q \ge 0$ and all $x, y \in \mathbf{R}^d$, and condition (e) of stable-like scaling is satisfied, then the process X_t satisfies the scale-invariant version of BHP.

Proof. In [10, Proposition 5.2], they show that the condition on the Lévy kernel implies properties (a) to (c) directly. Moreover, [20, Theorem 1.2] show that it implies the following heat kernel estimate: There exists $c_1 = c_1(d, \alpha, c)$, such that for all $(t, x, y) \in (0, \infty) \times \mathbf{R}^d \times \mathbf{R}^d$,

$$c_1^{-1}\left(t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x-y|^{d+\alpha}}\right) \le p_t(x,y) \le c_1\left(t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x-y|^{d+\alpha}}\right).$$

This, however, implies condition (d) as is shown in [10, Proposition 5.3]. □

It is now easy to check that scale-invariant BHP holds for stable-like processes satisfying the conditions detailed in the previous section.

Proof of Theorem 2.14. Since we assume that $\kappa_1 \le \kappa(x,y) \le \kappa_2$, the Lévy kernel of a stable-like process,

$$\nu(x,y) = \mathcal{A}(d,-\alpha)\frac{\kappa(x,y)}{|x-y|^{d+\alpha}},$$

clearly satisfies equation (2.5). To use the previous proposition, it remains to show the validity of condition (e). A refinement of the estimate given in equation (2.2) where the region is split up according to the $B(0,R)$ and its complement yields

$$\begin{aligned}\left|\int_{\{y:|x-y|>\varepsilon\}} \frac{u(x)-u(y)}{|x-y|^{d+\alpha}}\kappa(x,y)\,\mathrm{d}y\right| \quad &\le \quad \kappa_2\|\nabla u\|_{L^\infty}\frac{d\omega_d}{1-\alpha}R^{1-\alpha} \\ &+ \quad 2\kappa_2\|u\|_{L^\infty}\frac{d\omega_d}{\alpha}R^{-\alpha}.\end{aligned}$$

On the one hand, we can choose v to be radial with

$$v(|x|) = \begin{cases} 1 & |x| \le r \\ \frac{R-|x|}{R-r} & r < |x| \le R \\ 0 & |x| > R \end{cases}.$$

Then $u(x) = v(x - x_0)$ is a Urysohn bump function for $\overline{B(x_0, r)}$ and $B(x_0, R)$ with

$$\|u\|_{L^\infty} = 1, \qquad \|\nabla u\|_\infty = \frac{1}{R - r}.$$

We conclude that

$$\varrho\left(B(x_0, r), B(x_0, R)\right) \le c_2 d\omega_d \left(\frac{1}{1-\alpha}\frac{R}{R-r} + \frac{2}{\alpha}\right) R^{-\alpha}.$$

Note that $\frac{R}{R-r} = \frac{1}{1-r/R}$ which is a function of $\frac{r}{R}$.

On the other hand, we can choose v to be radial with

$$v(|x|) = \begin{cases} 0 & |x| \le r \\ \frac{|x|-r}{p-r} & r < |x| \le p \\ 1 & p < |x| \le R \\ \frac{\tilde{r}-|x|}{\tilde{r}-R} & R < |x| \le \tilde{r} \\ 0 & |x| > \tilde{r} \end{cases}.$$

Then $u(x) = v(x - x_0)$ is a Urysohn bump function for $\overline{A(x_0, p, R)}$ and $A(x_0, r, \tilde{r})$ with

$$\|u\|_{L^\infty} = 1, \qquad \|\nabla u\|_\infty = \frac{1}{\tilde{r} - R} \vee \frac{1}{p - r}.$$

We conclude that

$$\varrho\left(\overline{A(x_0, p, R)}, A(x_0, r, \tilde{r})\right) \le \kappa_2 d\omega_d \left(\frac{1}{1-\alpha}\left(\frac{R}{\tilde{r}-R} \vee \frac{R}{p-r}\right) + \frac{2}{\alpha}\right) R^{-\alpha}.$$

Note that the functions u and v are Lipschitz-functions with bounded gradient and therefore in the generator of stable-like processes with index $0 < \alpha < 1$, so this establishes condition (e) in that case. If $1 \le \alpha < 2$, we can mollify the functions u and v to obtain smooth functions of compact support which satisfy similar estimates and are in the generator of the stable-like process which establishes condition (e) in the remaining case. $\square$

2.3 Dirichlet Heat Kernel Estimates on κ-fat Domains

Given a symmetric stable-like process X_t and an open set D, we have defined the killed process upon leaving D, denoted by X_t^D in section 1.3. Recall that we adjoin a cemetery state ∂ and set

$$X_t^D = \begin{cases} X_t & t < \tau_D \\ \partial & t \geq \tau_D \end{cases} .$$

Denote by $p_D(t,x,y)$ the transition density function of the killed symmetric stable process. In this section, we will derive heat kernel estimates for this density, so called *Dirichlet heat kernel estimates*, in terms of the survival probability $\mathbf{P}_x(\tau_D > t)$. Our main tool in this endeavor will be the boundary Harnack principle derived in the previous section, Theorem 2.14.

Theorem 2.16. *Let X_t^D be a killed stable-like process of index $\alpha \in (0,2)$ on a κ-fat domain D and assume $\kappa(x,y)$ is a measurable function with $0 < \kappa_1 \leq \kappa(x,y) \leq \kappa_2 < \infty$. Suppose that additionally, one of the following conditions holds:*

(1) The index satisfies $0 < \alpha < 1$, and $\kappa(x,y)$ is a continuous.

(2) The index satisfies $1 \leq \alpha < 2$, and $\kappa(x,y)$ is such that

$$|\kappa(x,x) - \kappa(x,y)| \, \mathbf{1}_{\{|x-y|<1\}} \leq M_\kappa(x) \, |x-y|^\gamma , \qquad x,y \in \mathbf{R}^d.$$

where $\gamma > \alpha - 1$ and $M_\kappa \in L^1_{\text{loc}}(\mathbf{R}^d)$.

Fix $T > 0$. Then there is $C_2 = C_2(d, \alpha, \kappa_1, \kappa_2, C_\kappa, \kappa, R, T) > 0$ such that for all $t \in (0,T] \times D \times D$,

$$\begin{gathered} C_2^{-1} \mathbf{P}_x(\tau_D > t) \, \mathbf{P}_y(\tau_D > t) \left(t^{-\frac{d}{\alpha}} \wedge \tfrac{t}{|x-y|^{d+\alpha}} \right) \\ \leq p_D(t,x,y) \leq C_2 \mathbf{P}_x(\tau_D > t) \, \mathbf{P}_y(\tau_D > t) \left(t^{-\frac{d}{\alpha}} \wedge \tfrac{t}{|x-y|^{d+\alpha}} \right), \end{gathered}$$

where $p_D(t,x,y)$ is the transition density function of the killed stable-like process X_t^D.

For the proof of this theorem, we follow the approach of [16, Theorem 1.3].

2.3.1 Preliminary results

Before we start with the core estimates of the transition density function, we will obtain some auxilliary results that we will need throughout the proof of the main theorem.

Lemma 2.17. *Fix $r_0 < \infty$. Then there exists $C_1 = C_1(d, \alpha, \kappa_1, \kappa_2, r_0) \geq 1$ such that for all $r \in (0, r_0]$ and $x \in \mathbf{R}^d$,*

$$C_1^{-1} r^\alpha \leq \mathbf{E}_x \left[\tau_{B(x,r)}\right] \leq C_1 r^\alpha.$$

Proof. By Proposition 1.6, there is $\varepsilon = \varepsilon(d, \alpha, \kappa_1, \kappa_2, r_0) < 1$ such that for all $r \in (0, r_0]$ and $x \in \mathbf{R}^d$,

$$\mathbf{P}_x \left(\tau_{B(x,r)} > \varepsilon r^\alpha\right) \geq \frac{1}{2}.$$

Then Markov's inequality impies that

$$\mathbf{E}_x \left[\tau_{B(x,t)}\right] \geq c r^\alpha \mathbf{P}_x \left(\tau_{B(x,t)} > \varepsilon r^\alpha\right) \geq \frac{\varepsilon}{2} r^\alpha.$$

For the reverse inequality, the Lévy system formula, Proposition 1.8, and the doubling property imply that

$$\begin{aligned} 1 &\geq \mathbf{P}_x \left(X_{\tau_{B(x,r)}} \in B(x, 2r)^c\right) \\ &= \mathbf{E}_x \left[\int_0^{\tau_{B(x,r)}} \int_{B(x,2r)^c} \frac{\kappa(X_s, y)}{|X_s - y|^{d+\alpha}} \,\mathrm{d}y\mathrm{d}s\right] \\ &\geq c_1 \mathbf{E}_x \left[\int_0^{\tau_{B(x,r)}} \int_{B(x,3r)\setminus B(x,2r)} \frac{1}{|X_s - y|^{d+\alpha}} \,\mathrm{d}y\mathrm{d}s\right] \\ &\geq c_2 \frac{m\left(B(x,3r) \setminus B(x,2r)\right)}{r^{d+\alpha}} \mathbf{E}_x \left[\tau_{B(x,r)}\right] \\ &= c_3 r^{-\alpha} \mathbf{E}_x \left[\tau_{B(x,r)}\right]. \end{aligned}$$

where c_1, c_2, and c_3 depend only on d, α, and κ_1. Thus the claim follows with $C_1 = c_3^{-1} \vee (\varepsilon/2)$. □

Lemma 2.18. *Let a and r_0 be positive constants. There exists $C_2 = C_2(d, \alpha, \kappa_1, \kappa_2, a, r_0) > 0$ such that for all $x \in \mathbf{R}^d$ and $r \in (0, r_0]$,*

$$\inf_{y \in B(x, r/2)} \mathbf{P}_y \left(\tau_{B(x,r)} > a r^{1/\alpha}\right) \geq C_2.$$

Proof. By Proposition 1.6, there exists $\varepsilon = \varepsilon(d, \alpha, \kappa_1, \kappa_2, r_0) < 1$ such that for all $r \in (0, r_0]$, we have that $\mathbf{P}_x\left(\tau_{B(x,r)} > \varepsilon r^\alpha\right) \geq 1/2$. Hence, the doubling property implies

$$\inf_{y \in \mathbf{R}^d} \mathbf{P}_y\left(\tau_{B(y,r/2)} > \varepsilon r^\alpha\right) \geq \frac{1}{2}.$$

This proves the Lemma for $a \leq \varepsilon$. For $a > \varepsilon$, apply the parabolic Harnack inequality, Proposition 1.7. After at most $2 + \lceil a/\varepsilon \rceil$ applications, we see that there exists $c_1 = c_1(d, \alpha, \kappa_1, \kappa_2, a, r_0) > 0$ such that for all $w, y \in B(x, r/2)$,

$$c_1 p_{B(x,r/2)}(\varepsilon r^\alpha, x, w) \leq p_{B(x,r)}(a r^\alpha, y, w).$$

We conclude that for every $y \in B(x, r/2)$,

$$\begin{aligned}
\mathbf{P}_x\left(\tau_{B(x,r)} > a r^\alpha\right) &= \int_{B(x,r)} p_{B(x,r)}(a r^\alpha, y, w)\,\mathrm{d}w \\
&\geq \int_{B(x,r/2)} p_{B(x,r)}(a r^\alpha, y, w)\,\mathrm{d}w \\
&\geq c_1 \int_{B(x,r/2)} p_{B(x,r)}(\varepsilon r^\alpha, y, w)\,\mathrm{d}w \\
&= c_1 \mathbf{P}_x\left(\tau_{B(z,r/2)} > \varepsilon r^\alpha\right) \\
&\geq \frac{c_1}{2}.
\end{aligned}$$

This proves the lemma with $C_2 = c_1/2$. □

2.3.2 Upper bound estimate

In what follows, we will assume that D is a fixed κ-fat open set with characteristics (R_1, κ). Recall that given $x \in \bar{D}$ and $r \in (0, R_1]$, we can find $A_r(x) \in D$ such that $A_r(x), \kappa r) \subset D \cap B(x, r)$. We can then define

$$U(x,t) := D \cap B(x, |x - A_r(x)| + \kappa r/3), \qquad V(x,t) := D \cap B(x, |x - A_r(x)| + \kappa r).$$

Finally we can pick $A_r'(x) \in D$ which satisfies

$$B(A_r(x), \kappa r/3) \subset B(A_r(x), \kappa r) \setminus U(x,t).$$

Finally, note that $B(A_r(x), \kappa r/3) \subset U(x,t)$ and $B(A'_r(x), \kappa r/3) \subset V(x,t) \setminus U(x,t)$. For the following section, given $(t,x) \in (0,T] \times \bar{D}$, we fix $r = r(t) = R_1(t/T)^{1/\alpha} \leq R_1$.

Our base tool to derive the upper bound is the following lemma which holds for general symmetric Hunt processes.

Lemma 2.19. *Let Z_t be a symmetric Hunt process, and let $U_1, U_3 \subset E \subset \mathbf{R}^d$ all be open with* $\operatorname{dist}(U_1, U_3) > 0$. *Set $U_2 = E \setminus (U_1 \cup U_3)$. If $x \in U_1$ and $y \in U_3$, then for all $t > 0$,*

$$p_E(t,x,y) \quad \leq \quad \mathbf{P}_x\left(Z_{\tau_{U_1}} \in U_2\right) \sup_{s<t, z \in U_2} p_E(s,z,y) + (t \wedge \mathbf{E}_x\left[\tau_{U_1}\right]) \sup_{u \in U_1, z \in U_3} J(u,z).$$

Proof. See [17, Lemma 3.1]. □

The next lemma establishes the relationship between certain quantities involving the exit times from the sets D, $U(x,t)$, and $V(x,t)$.

Lemma 2.20. *Fix $T > 0$ and $M \geq 1$. Then for any $(t,x) \in (0,T] \times D$,*

$$\begin{aligned} \mathbf{P}_x\left(\tau_D > t/M\right) &\asymp \mathbf{P}_x\left(\tau_{V(x,t)} > Mt\right) \asymp \mathbf{P}_x\left(\tau_{V(x,t)} > t/M\right) \asymp \mathbf{P}_x\left(\tau_D > Mt\right) \\ &\asymp \mathbf{P}_x\left(X_{\tau_{U(x,t)}} \in D\right) \asymp t^{-1}\mathbf{E}_x\left[\tau_{U(x,t)}\right]. \end{aligned}$$

Each two of those quantities can be bounded it terms of each other by a constant $C_3 = C_3(d, \alpha, \kappa_1, \kappa_2, M, T, R_1, \kappa) \geq 1$.

Proof. We follow the proof in [16, Lemma 4.1]. We may assume without loss of generality that $r_0 \leq 1$. Fix $(t,x) \in (0,T] \times D$ and recall the definitions of r, $U(x,t)$, and $V(X,t)$ at the beginning of this section.

First, note that

$$\mathbf{P}_x\left(\tau_{V(x,t)} > Mt\right) \leq \mathbf{P}_x\left(\tau_{V(x,t)} > t/M\right) \wedge \mathbf{P}_x\left(\tau_D > Mt\right) \leq \mathbf{P}_x\left(\tau_D > t/M\right). \tag{2.6}$$

Next, the Lévy system formula, Proposition 1.8, implies

$$\mathbf{P}_x\left(X_{\tau_{U(x,t)}} \in B(A'(r)(x), \kappa r/6)\right) \quad = \quad \mathbf{E}_x\left[\int_0^{\tau_{U(x,t)}} \int_{B(A'_r(x), \kappa r/6)} \frac{\kappa(X_s, y)}{|X_s - y|^{d+\alpha}} \, \mathrm{d}y \mathrm{d}s\right]$$

$$\begin{aligned} &\asymp r^d r^{-d-\alpha}\mathbf{E}_x\left[\tau_{U(x,t)}\right] \\ &\asymp t^{-1}\mathbf{E}_x\left[\tau_{U(x,t)}\right]. \end{aligned} \tag{2.7}$$

The last inequality follows since $r = R_1(t/T)^{1/\alpha}$. The remainder we split up into two cases.

Case 1. Assume that $|x - A_r(x)| < \kappa r/2$. Then $B(x, \kappa r/3) \subset U(x,t) \subset V(x,t)$. Along with Lemma 2.18, this implies that there exists $c_1 = c_1(d, \alpha, \kappa_1, \kappa_2, M, T, \kappa) > 0$ such that

$$1 \geq \mathbf{P}_x\left(\tau_D > t/M\right) \geq \mathbf{P}_x\left(\tau_D > Mt\right) \geq \mathbf{P}_x\left(\tau_D > Mt\right) \geq c_1.$$

Since $\mathbf{P}_x\left(\tau_{V(x,t)} > t/M\right) \geq \mathbf{P}_x\left(\tau_{V(x,t)} > Mt\right)$, running the same argument establishes that and establishes that

$$\mathbf{P}_x\left(\tau_D > t/M\right) \asymp \mathbf{P}_x\left(\tau_{V(x,t)} > Mt\right) \asymp \mathbf{P}_x\left(\tau_{V(x,t)} > t/M\right) \asymp \mathbf{P}_x\left(\tau_D > Mt\right) \asymp 1.$$

The assumption $|x - A_r(x)| < \kappa r/2$ also implies that $B(x, \kappa r/3) \subset U(x,t) \subset B(x,r)$, hence Lemma 2.17 and the fact that $r = R_1(t/T)^{1/\alpha}$ implies that

$$c_2 t = C_1^{-1}\left(\frac{\kappa r}{3}\right)^\alpha \leq \mathbf{E}_x\left[\tau_{B(x,\kappa r/3)}\right] \leq \mathbf{E}_x\left[\tau_{U(x,t)}\right] \leq \mathbf{E}_x\left[\tau_{B(x,r)}\right] \leq C_1 r^\alpha = c_3 t$$

where c_2 and c_3 depend only on d, α, κ_1, κ_2, R_1, T, and κ. Using the previous equations as well as (2.7), we can deduce that

$$1 \geq \mathbf{P}_x\left(X_{\tau_{U(x,t)}} \in D\right) \geq \mathbf{P}_x\left(X_{\tau_{U(x,t)}} \in B(A_r'(x), \kappa r/6)\ \right) \geq c_4 t^{-1}\mathbf{E}_x\left[\tau_{U(x,t)}\right] \geq c_2 c_4$$

for some $c_4 = c_4(d, \alpha, \kappa_1, \kappa_2, M, T, \kappa) > 0$. which establishes the remainder of the lemma,

$$\left(X_{\tau_{U(x,t)}} \in D\right) \asymp \mathbf{E}_x\left[\tau_{B(x,r)}\right] \asymp 1,$$

in the present case.

Case 2. Now we will assume that $|x - A_r(x)| \geq \kappa r/2$. Note that

$$\mathbf{P}_x\left(\tau_D > t/M\right) \leq \mathbf{P}_x\left(\tau_{U(x.t)} > t/M\right) + \mathbf{P}_x\left(X_{\tau_{U(x,t)}} \in D\right). \tag{2.8}$$

By the boundary Harnack inequality, Theorem 2.14, there exists $c_5 = c_5(d, \alpha, \kappa_1, \kappa_2, \kappa, R_1) \geq 1$ such that

$$\begin{aligned}
\mathbf{P}_x\left(X_{\tau_{U(x,t)}} \in D\right) &\leq c_5 \mathbf{P}_{A_r(x)}\left(X_{\tau_{U(x,t)}} \in D\right) \frac{\mathbf{P}_x\left(X_{\tau_{U(x,t)}} \in B(A'_r(x), \kappa r/6)\right)}{\mathbf{P}_{A_r(x)}\left(X_{\tau_{U(x,t)}} \in B(A'_r(x), \kappa r/6)\right)} \\
&\leq c_5 \frac{\mathbf{P}_x\left(X_{\tau_{U(x,t)}} \in B(A'_r(x), \kappa r/6)\right)}{\mathbf{P}_{A_r(x)}\left(X_{\tau_{B(A_r(x), \kappa r/3)}} \in B(A'_r(x), \kappa r/6)\right)}. \qquad (2.9)
\end{aligned}$$

If $(w, y) \in B(A_r(x), \kappa r/3) \times B(A'_r(x), \kappa r/6)$, then $|y - w| \leq 2\kappa r$, and thus the Lévy systems formula, Proposition 1.8, and Lemma 2.17 yield

$$\begin{aligned}
&\mathbf{P}_{A_r(x)}\left(X_{\tau_{B(A_r(x),\kappa r/3)}} \in B(A'_r(x), \kappa r/6)\right) \\
&= \mathbf{E}_{A_r(x)}\left[\int_0^{\tau_{B(A_r(x),\kappa r/3)}} \int_{B(A'_r(x),\kappa r/6)} \frac{\kappa(X_s, y)}{|X_s - y|^{d+\alpha}} \,\mathrm{d}y\mathrm{d}s\right] \\
&\geq c_6 r^d r^{-d-\alpha} \mathbf{E}_{A_r(x)}\left[\tau_{B(A_r(x),\kappa r/3)}\right] \\
&= c_6 r^{-\alpha} \left(\frac{\kappa r}{3}\right)^{\alpha} = c_7 \qquad (2.10)
\end{aligned}$$

where c_6 and c_7 depend only on d, α, κ_1, κ_2, κ, and R_1. Combining equations (2.9) and (2.10) show that

$$\mathbf{P}_x\left(X_{\tau_{U(x,t)}} \in D\right) \leq c_5 c_7 \mathbf{P}_x\left(X_{\tau_{U(x,t)}} \in B(A'_r(x), \kappa r/6)\right). \qquad (2.11)$$

By Markov's inequality, $\mathbf{P}_x(\tau_{U(x,t)} > t/M) \geq t^{-1}\mathbf{E}_x[\tau_{U(x,t)}]$, and therefore by equations (2.8), (2.7), and (2.9)

$$\begin{aligned}
\mathbf{P}_x\left(\tau_D > t/M\right) &\leq \mathbf{P}_x\left(\tau_{U(x.t)} > t/M\right) + \mathbf{P}_x\left(X_{\tau_{U(x,t)}} \in D\right) \\
&\leq t^{-1}\mathbf{E}_x\left[\tau_{U(x,t)}\right] + c_5 c_7 \mathbf{P}_x\left(X_{\tau_{U(x,t)}} \in B(A'_r(x), \kappa r/6)\right) \\
&\leq c_8 t^{-1} \mathbf{E}_x\left[\tau_{U(x,t)}\right].
\end{aligned}$$

for some $c_8 = c_8(d, \alpha, \kappa_1, \kappa_2, R_1, T, \kappa, M) \geq 1$.

Next, the strong Markov property implies that

$$\mathbf{P}_x\left(X_{\tau_{U(x,t)}} \in B(A'_r(x), \kappa r/6)\right)$$

$$\begin{aligned}
&\leq \mathbf{E}_x\left[\mathbf{P}_{X_{\tau_{U(x,t)}}}\left(\tau_{B(X_{\tau_{U(x,t)}},\kappa r/6)} > Mt\right); X_{\tau_{U(x,t)}} \in B(A_r'(x),\kappa r/6)\right] \\
&\quad +\mathbf{E}_x\left[\mathbf{P}_{X_{\tau_{U(x,t)}}}\left(\tau_{B(X_{\tau_{U(x,t)}},\kappa r/6)} \leq Mt\right); X_{\tau_{U(x,t)}} \in B(A_r'(x),\kappa r/6)\right] \\
&\leq \mathbf{E}_x\left[\mathbf{P}_{X_{\tau_{U(x,t)}}}\left(\tau_{V(x,t)} > Mt\right); X_{\tau_{U(x,t)}} \in B(A_r'(x),\kappa r/6)\right] \\
&\quad +\mathbf{P}_0\left(\tau_{B(0,\kappa r/6)} \leq Mt\right)\mathbf{P}_x\left(X_{\tau_{U(x,t)}} \in B(A_r'(x),\kappa r/6)\right) \\
&= \mathbf{P}_x\left(\tau_{V(x,t)} > Mt, X_{\tau_{U(x,t)}} \in B(A_r'(x),\kappa r/6)\right) \\
&\quad +\left(1-\mathbf{P}_0\left(\tau_{B(0,\kappa r/6)} > Mt\right)\right)\mathbf{P}_x\left(X_{\tau_{U(x,t)}} \in B(A_r'(x),\kappa r/6)\right) \\
&\leq \mathbf{P}_x\left(\tau_{V(x,t)} > Mt\right) \\
&\quad +\left(1-\mathbf{P}_0\left(\tau_{B(0,\kappa r/6)} > Mt\right)\right)\mathbf{P}_x\left(X_{\tau_{U(x,t)}} \in B(A_r'(x),\kappa r/6)\right).
\end{aligned}$$

Rearranging this inequality yields

$$\mathbf{P}_0\left(\tau_{B(0,\kappa r/6)} > Mt\right)\mathbf{P}_x\left(X_{\tau_{U(x,t)}} \in B(A_r'(x),\kappa r/6)\right) \leq \mathbf{P}_x\left(\tau_{V(x,t)} > Mt\right). \tag{2.12}$$

Note that $\mathbf{P}_0\left(\tau_{B(0,\kappa r/6)} > Mt\right) \geq c_9$ for some $c_9 = c_9(d,\alpha,\kappa_1,\kappa_2,T,R_1,M) \geq 1$ by Lemma 2.18 and the doubling property. Using this along with equations (2.8) and (2.12), we get

$$t^{-1}\mathbf{E}_x\left[\tau_{U(x,t)}\right] \leq c_{10}\mathbf{P}_x\left(X_{\tau_{U(x,t)}} \in B(A_r'(x),\kappa r/6)\right) \leq c_{11}\mathbf{P}_x\left(\tau_{V(x,t)} > Mt\right),$$

where the constants c_{10} and c_{11} may depend on d, α, κ_1, κ_2, T, R_1, κ, and M. In combination with the previous equation and (2.6) this completes the proof in case 2. □

We are finally set to give the proof of the upper bound for the transition density function of the killed process.

Proof of upper bound in Theorem 2.16. By Lemma 2.20 and the the semigroup property, we only need to establish the claim for $T \leq 1$. Fix $t \in (0,T]$. We will prove the result in two cases. Recall tha we defined $r = R_1(t/T)^{1/\alpha}$.

Case 1. Assume that $|x-y| \leq 8r$. The heat kernel estimate for the symmetric stable process, Theorem 1.5, implies that

$$c_1^{-1}t^{-d/\alpha} \leq p(t/2,x,y) \leq c_1 t^{-d/\alpha} \tag{2.13}$$

where $c_1 = c_1(d, \alpha, \kappa_1, \kappa_2, T) \geq 1$. Therefore, the semigroup property, the doubling property, Lemma 2.20, and equation (2.13) imply that

$$\begin{aligned} p_D(t/2, x, y) &= \int_D p_D(t/4, x, z) p_D(t/4, z, y)\, \mathrm{d}z \\ &\leq \sup_{z \in \mathbf{R}^d} p(t/2, z, y) \mathbf{P}_x \left(\tau_D > t/4 \right) \\ &\leq c_2 t^{-d/\alpha} \mathbf{P}_x \left(\tau_D > t \right) \\ &\leq c_2 c_1 p(t/2, x, y) \mathbf{P}_x \left(\tau_D > t \right). \end{aligned}$$

Case 2. Now we will assume $|x - y| > 8r$. With reference to the setup for thie section, we set $D_1 := U(x, t)$, $D_3 := \{z \in D : |z - x| > |x - y|/2\}$, and

$$D_2 := D \setminus (D_1 \cup D_3) = \{z \in D \setminus U(x, t) : |z - x| > |x - y|/2\}.$$

Next, using the heat kernel estimates for the uncensored process, Theorem 1.5,

$$\begin{aligned} \sup_{s < t/2, z \in D_2} p(s, z, y) &\leq c_1 \sup_{s < t/2, |z-y| \geq |x-y|/2} \left(s^{-d/\alpha} \wedge \frac{s}{|z - y|^{d+\alpha}} \right) \\ &\leq c_1 \sup_{s < t/2, |z-y| \geq |x-y|/2} \left(s^{-d/\alpha} \wedge \frac{s}{(|x - y|/2)^{d+\alpha}} \right) \\ &\leq c_1^2 \sup_{s < t/2} p(s, z/2, y/2). \end{aligned} \tag{2.14}$$

We now extend the definition of $p(t, x, y)$ be setting $p(t, x, y) = 0$ for ant $t < 0$. Then the function $(s, w) \mapsto p(s, w/2, y/2)$ is parabolic in $(-\infty, T] \times B(x, 2r)$. The parabolic Harnack inequality, Proposition 1.7, then implies there exists a constant $c_3 = c_3(d, \alpha, \kappa_1, \kappa_2) \geq 1$ such that for every $t \in (0, T]$,

$$\sup_{s < t/2} p(s, z/2, y/2) \leq c_3 p(t/2, z/2, y/2).$$

This equation and a similar computation as in equation (2.14) yields

$$\sup_{s < t/2, z \in D_2} p(s, z, y) \leq c_1^2 \sup_{s < t/2} p(s, z/2, y/2) \leq c_1^2 c_3 p(t/2, z/2, y/2) \leq c_3 c_1^4 p(t/2, z/4, y/4). \tag{2.15}$$

Next, if $u \in D_1$, then $|x-u| \leq |x - A_r(x)| + \kappa r/3$. If, additionally, $z \in D_3$, then

$$\begin{aligned} |u-z| &\geq |z-x| - |x-u| \geq |z-x| - |x-A_r(x)| - \kappa r/3 \\ &\geq |z-x| - r \geq \frac{1}{2}|x-z| \geq \frac{1}{4}|x-y|. \end{aligned}$$

The heat kernel estimates for the uncensored process, Theorem 1.5, then implies that

$$\begin{aligned} t \sup_{u\in D_1, z\in D_3} J(u,z) &= t \sup_{u\in D_1, z\in D_3} \frac{\kappa(u,z)}{|u-z|^{d+\alpha}} \leq \kappa_2 t \sup_{|u-z|\geq|x-y|/4} \frac{1}{|u-z|^{d+\alpha}} \\ &\leq \kappa_2 \frac{t}{(|x-z|/4)^{d+\alpha}} \leq \kappa_2 c_1 p(t/2, x/4, z/4). \end{aligned}$$

Now, equation (2.15) as well as Lemma 2.20, Lemma 2.19, and Markov's inequality show that

$$\begin{aligned} p_D(t/2,x,y) &\leq \mathbf{P}_x\left(Y_{\tau_{D_1}} \in D_2\right) \sup_{s<t/2, z\in D_2} p_D(s,z,y) \\ &\quad + (t/2 \wedge \mathbf{E}_x\left[\tau_{D_1}\right]) \sup_{u\in D_1, z\in D_3} J(u,z) \\ &\leq \left(\mathbf{P}_x\left(X_{\tau_{U(x,t)}} \in D\right) + t^{-1}\mathbf{E}_x\left[\tau_{U(x,t)}\right]\right) c_3 c_1^4 p(t/2, z/4, y/4) \\ &\quad + \mathbf{P}_x\left(\tau_D > t\right) \kappa_2 c_1 p(t/2, x/4, z/4) \\ &\leq c_6 \mathbf{P}_x\left(\tau_D > t\right) p(t/2, x/4, z/4), \end{aligned}$$

where $c_6 = c_6(d, \alpha, \kappa_1, \kappa_2, T, R_1, \kappa) > 0$. Using the heat kernel estimates for the uncensored process, Theorem 1.5, we can show that $p(t/2, x/4, z/4) \leq c_1^2 p(t/2, x, z)$, and therefore

$$p_D(t/2,x,y) \leq c_6 p(t/2,x,y) \mathbf{P}_x\left(\tau_D > t\right). \tag{2.16}$$

for some $c_6 = c_6(d, \alpha, \kappa_1, \kappa_2, R_1, T) \geq q$ holds, in both cases.

Finally, let $(t,x,y) \in (0,T] \times D \times D$. Then equation (2.16), the semigroup property, and symmetry of $p_D(t,x,y)$ show that

$$\begin{aligned} p_D(t,x,y) &= \int_D p_D(t/2,x,z) p_D(t/2,z,y)\, \mathrm{d}z \\ &\leq c_6^2 \mathbf{P}_x\left(\tau_D > t\right) \mathbf{P}_y\left(\tau_D > t\right) \int_{\mathbf{R}^d} p(t/2,x,z) p(t/2,z,y)\, \mathrm{d}z \\ &= c_6^2 \mathbf{P}_x\left(\tau_D > t\right) \mathbf{P}_y\left(\tau_D > t\right) p(t,x,y). \end{aligned}$$

□

2.3.3 Lower bound estimate

Next, we will derive the lower bound for the density. We will start by deriving a preliminary lower bound which actually holds on an arbitrary open set D.

Proposition 2.21. *Let $T > 0$ and $a > 0$ be constants. There is $C = C(d, \alpha, \kappa_1, \kappa_2, T, a) > 0$ such that for all $(t, x, y) \in (0, T] \times D \times D$ with $\delta_D(x) \wedge \delta_D(y) \geq at^{1/\alpha}$,*

$$p_D(t, x, y) \geq C\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right).$$

Note that the right hand side is comparable to $p(t, x, y)$.

To prove this proposition, we need several premiminary steps.

Proposition 2.22. *Let $T > 0$ and $a > 0$ be constants. There is $C_3 = C_3(d, \alpha, \kappa_1, \kappa_2, a) > 0$ such that for all $(t, x, y) \in (0, T] \times D \times D$ with $\delta_D(x) \wedge \delta_D(y) \geq at^{1/\alpha} \geq 4|x - y|$,*

$$p_D(t, x, y) \geq C_3 t^{-d/\alpha}.$$

Proof. Fix $(t, x, y) \in (0, T] \times D \times D$ such that $\delta_D(x) \wedge \delta_D(y) \geq at^{1/\alpha} \geq 4|x - y|$. The second condition implies $|x - y| \leq at^{1/\alpha}/4 \leq aT^{1/\alpha}/4$ and

$$B(x, at^{1/\alpha}/4) \subset B(y, at^{1/\alpha}/2) \subset B(y, 2at^{1/\alpha}/3) \subset D.$$

Therefore, the parabolic Harnack inequality, Proposition 1.7, implies that there exists $c_1 = c_1(d, \alpha, \kappa_1, \kappa_2, T) > 0$ such that for all $w \in B(x, at^{1/\alpha}/4)$,

$$c_1 p_D(t/2, x, w) \leq p_D(t, x, y).$$

Along with the doubling property and Lemma 2.18 this implies

$$\begin{aligned}
p_D(t, x, y) &\geq \frac{c_1}{m(B(x, at^{1/\alpha})/4)} \int_{B(x, at^{1/\alpha}/4)} p_D(t/2, x, w)\,\mathrm{d}w \\
&\geq c_2 t^{1/\alpha} \int_{B(x, at^{1/\alpha}/4)} p_{B(x, at^{1/\alpha}/4)}(t/2, x, w)\,\mathrm{d}w \\
&= c_2 t^{1/\alpha} \mathbf{P}_x\left(\tau_{B(x, at^{1/\alpha}/4)} > t/2\right) \geq c_3 t^{1/\alpha}.
\end{aligned}$$

where $c_i = c_i(d, \alpha, \kappa_1, \kappa_2, T, a) > 0$ for $i = 2, 3$. □

Lemma 2.23. *Let $T > 0$ and $a > 0$ be constants. Then there exists $C_4 = C_4(d, \alpha, \kappa_1, \kappa_2) > 0$ such that for all $(t, x, y) \in (0, T] \times D \times D$ satisfying $\delta_D(x) \wedge \delta_D(y) \geq at^{1/\alpha}$ and $at^{1/\alpha} \leq 4|x-y|$.*

$$\mathbf{P}_x \left(X_t^D \in B(y, at^{1/\alpha}/2) \right) \geq C_4 t^{-d/\alpha} \frac{t}{|x-y|^{d+\alpha}}.$$

Proof. Lemma 2.18 implies that if we start the process at $z \in B(y, at^{1/\alpha}/4)$, then with probability at least $c_1 = c_1(d, \alpha, \kappa_1, \kappa_2, a, T) > 0$, the process X_t does not move more than $at^{1/\alpha}/6$ by time t. Consequently, it suffices to show that there exists $c_2 = c_2(d, \alpha, \kappa_1, \kappa_2, a, T) > 0$ such that for every $(t, x, y) \in (0, T] \times D \times D$ with $\delta_D(x) \wedge \delta_D(y) \geq at^{1/\alpha}$ and $at^{1/\alpha} \leq 4|x-y|$, it holds that

$$\mathbf{P}_x \left(X^D_{\sigma_{B(y, at^{1/\alpha}/4)}} < t \right) \geq c_2 t^{-d/\alpha} \frac{t}{|x-y|^{d+\alpha}}.$$

Let $B_z := B(z, at^{1/\alpha}/9)$ for $z \in D$, and denote $\tau_z =: \tau_{B_z}$. Markov's inequality, the doubling property and Lemma 2.18 show that there exists $c_3 = c_3(d, \alpha, \kappa_1, \kappa_2, a, T) > 0$ such that for all $t \in (0, T]$,

$$\mathbf{E}_x \left[t \wedge \tau_x \right] \geq t \mathbf{P}_x \left(\tau_x \geq t \right) \geq x_3 t. \tag{2.17}$$

Since $\delta_D(x) \wedge \delta_D(y) \geq at^{1/\alpha}$, $B_x \cap B_y = \emptyset$, and therefore the Lévy systems formula of X, Proposition 1.8, shows that

$$\begin{aligned} \mathbf{P}_x \left(X^D_{\sigma_{B(y, at^{1/\alpha}/4)}} < t \right) &\geq \mathbf{P}_x \left(X_{t \wedge \tau_x} \in B(y, at^{1/\alpha}/4) \right) \\ &\geq c_4 \mathbf{E}_x \left[\int_{t \wedge \tau_x} \int_{B_y} \frac{\kappa(u, X_s)}{|u - X_s|^{d+\alpha}} \, \mathrm{d}u \mathrm{d}s \right] \end{aligned} \tag{2.18}$$

where $c_4 = c_4(d, \alpha, \kappa_1, \kappa_2, T, a) > 0$.

Next, we consider two cases. *Case 1.* Suppose that $|x - y| \leq aT^{1/\alpha}$. Then, since $|x - y| \geq at^{1/\alpha}$, for any $s < \tau_x$ and $u \in B_y$,

$$|X_s - u| \leq |X_s - x| + |x - y| + |y - u| \leq 2|x - y|.$$

Thus, equations (2.17) and (2.18) imply that

$$\mathbf{P}_x \left(X^D_{\sigma_{B(y, at^{1/\alpha}/4)}} < t \right) \geq c_4 \mathbf{E}_x \left[t \wedge \tau_x \right] m(B_y) \frac{1}{|2(x-y)|^{d+\alpha}} \geq c_5 t^{-d/\alpha} \frac{t}{|x-y|^{d+\alpha}}.$$

for some $c_5 = c_5(d, \alpha, \kappa_1, \kappa_2, a, T) > 0$.

Case 2. Now suppose that $|x-y| > aT^{1/\alpha}$. Then, for $s < \tau_x$ and $u \in B_y$,

$$|X_s - u| \leq |X_s - x| + |x-y| + |y-u| \leq |x-y| + at^{1/\alpha}/4 \leq |x-y| + aT^{1/\alpha}/4.$$

Thus, equations (2.17) and (2.18) imply that

$$\begin{aligned}
\mathbf{P}_x\left(X^D_{\sigma_{B(y,at^{1/\alpha}/4)}} < t\right) &\geq c_6 \mathbf{E}_x\left[t \wedge \tau_x\right] \int_{B_y} \frac{1}{\left(|x-y| + aT^{1/\alpha}/4\right)^{d+\alpha}} \,\mathrm{d}u \\
&\leq c_7 t m(B_y) \frac{1}{\left(|x-y| + aT^{1/\alpha}/4\right)^{d+\alpha}} \\
&\leq c_8 t^{-d/\alpha} \frac{t}{\left(|x-y| + aT^{1/\alpha}/4\right)^{d+\alpha}} \\
&\leq c_9 t^{-d/\alpha} \frac{t}{|x-y|^{d+\alpha}}
\end{aligned}$$

for some constants $c_i(d, \alpha, \kappa_1, \kappa_2, T, a) > 0$ for $i = 6, 7, 8, 9$. This compltes the proof. □

Proposition 2.24. *Let $T > 0$ and $a > 0$ be constants. Then there exists a constant $C_5 = C_5(d, \alpha, \kappa_1, \kappa_2, T, a) > 0$ such that for all $(t, x, y) \in (0, T] \times D \times D$ satisfying $\delta_D(x) \wedge \delta_D(y) \geq at^{1/\alpha}$ as well as $at^{1/\alpha} \leq 4|x-y|$,*

$$p_d(t, x, y) \geq C_5 \frac{t}{|x-y|^{d+\alpha}}.$$

Proof. The semigroup property, Proposition 2.22, and Lemma 2.23 imply

$$\begin{aligned}
p_d(t, x, y) &= \int_D p(t/2, x, z) p_D(t/2, z, y) \,\mathrm{d}z \\
&\geq \int_{B(y, a(t/2)^{1/\alpha}/2)} p(t/2, x, z) p_D(t/2, z, y) \,\mathrm{d}z \\
&\geq c_1 \left(\frac{t}{2}\right)^{-d/\alpha} \mathbf{P}_x\left(X^D_{t/2} \in B(y, a(t/2)^{1/\alpha}/2)\right) \\
&\geq c_2 \frac{t}{|x-y|^{d+\alpha}}.
\end{aligned}$$

for some constants $c_i = c_i(d, \alpha, \kappa_1, \kappa_2, a, T) > 0$. □

Proof of Proposition 2.21. This follows immediately from Propositions 2.22 and 2.24. $\square$

Finally, we prove the lower bound. As for the upper bound, there is a lemma which provides an abstract lower bound in a general setting.

Lemma 2.25. *Let* $E_1, E_2 \subset E \subset \mathbf{R}^d$ *be such that* $\operatorname{dist}(E_1, E_2) > 0$. *If* $x \in E_1$ *and* $y \in E_2$, *then for all* $t \geq 0$,

$$p_E(t,x,y) \geq t\mathbf{P}_x\left(\tau_{E_1} > t\right)\mathbf{P}_x\left(\tau_{E_2} > t\right) \inf_{(u,w)\in E_1\times E_2} J(w,z)$$

where $J(w,z)$ *is the jumping kernel of the process.*

Proof. See [15, Lemma 3.3]. $\square$

Proof of lower bound in Theorem 2.16. Fix $(t,x,y) \in (0,T]\times D\times D$ and recall the definitions at the beginning of the section. Then the semigroup property implies that

$$\begin{aligned}
p_D(t,x,y) &= \int_{D\times D} p_D(t/3,x,u)p_D(t/3,u,v)p_D(t/3,v,y)\,\mathrm{d}u\mathrm{d}v \\
&\geq \int_{B(A'_r(x),\kappa r/6)\times B(A'_r(y),\kappa r/6)} p_D(t/3,x,u)p_D(t/3,u,v)p_D(t/3,v,y)\,\mathrm{d}u\mathrm{d}v \\
&\geq \inf_{(u,v)\in B(A'_r(x),\kappa r/6)\times B(A'_r(y),\kappa r/6)} p_D(t/3,u,v) \\
&\qquad \cdot \int_{B(A'_r(x))} p_D(t/3,x,u)\,\mathrm{d}u \int_{B(A'_r(x))} p_D(t/3,v,y)\,\mathrm{d}v \qquad (2.19)
\end{aligned}$$

For $(u,v) \in B(A'_r(x),\kappa r/6) \times B(A'_r(y),\kappa r/6)$, Proposition 2.21 shows that

$$\begin{aligned}
&\inf_{(u,v)\in B(A'_r(x),\kappa r/6)\times B(A'_r(y),\kappa r/6)} p_D(t/3,u,v) \qquad (2.20) \\
\geq\ & c_1 \inf_{(u,v)\in B(A'_r(x),\kappa r/6)\times B(A'_r(y),\kappa r/6)} (t/3)^{-d/\alpha} \wedge \frac{t}{3|u-v|^{d+\alpha}} \\
\geq\ & c_2\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right), \qquad (2.21)
\end{aligned}$$

for some $c_i = c_i(d,\alpha,\kappa_1,\kappa_1,T,a) > 0$, $i = 1,2$, where we estimated the last inequality separately for $|x-y| \geq \kappa r$ and $|x-y| < \kappa r$.

Next, for $u \in B(A'_r(x), \kappa r/6)$, Lemma 2.25 with $E_1 := U(x,t)$ and $E_3 = B(A'_r(x), \kappa r/6)$ yields

$$p_D(t/2, x, u) \geq t\mathbf{P}_x\left(\tau_{U(x,t)} > t/3\right)\mathbf{P}_x\left(\tau_{(A'_r(x), \kappa r/6)} > t/3\right) \inf_{w \in U(x,t), z \in U_3} \frac{\kappa(w,z)}{|w-z|^{d+\alpha}}.$$

Since for $(w,z) \in U(x,t) \times \in U_3$, we have that $|w-z| \geq r = R_1(t/T)^{1/\alpha}$, and by Lemma 2.18

$$p_D(t/2, x, u) \geq c_3 t^{-d/\alpha}\mathbf{P}_x\left(\tau_{U(x,t)} > t/3\right).$$

for some $c_3 = c_3(d, \alpha, \kappa_1, \kappa_2, T, a) > 0$.

Now we make a choice for a. Set $a := (\kappa/3)^{1/\alpha}$. Then $(at)^{1/\alpha} = (\kappa/3)\, t^{1/\alpha}$ for all $t \in (0, T]$. Hence $V(x, at) \subset U(x,t)$, and Lemma 2.20 implies that

$$\mathbf{P}_x\left(\tau_{U(x,t)} > t/3\right) \geq \mathbf{P}_x\left(\tau_{V(x,t)} > t/3\right) \geq c_5\mathbf{P}_x\left(\tau_D > t\right)$$

for some $c_6 = c_6(d, \alpha, \kappa_1, \kappa_2, \kappa, T, a) > 0$. Therefore,

$$\int_{B(A'_r(x), \kappa r/6)} p_D(t/3, x, u)\, \mathrm{d}u \geq c_6 t^{-d/\alpha}\mathbf{P}_x\left(\tau_D > t\right) m\left(B(A'_r(x), \kappa r/6)\right) \geq c_7\mathbf{P}_x\left(\tau_D > t\right)$$

for some $c_i = c_i(d, \alpha, \kappa_1, \kappa_2, \kappa, T) > 0$. By symmetry, we also obtain

$$\int_{B(A'_r(y), \kappa r/6)} p_D(t/3, y, v)\, \mathrm{d}v \geq c_8 t^{-d/\alpha}\mathbf{P}_v\left(\tau_D > t\right) m\left(B(A'_r(x), \kappa r/6)\right) \geq c_9\mathbf{P}_v\left(\tau_D > t\right)$$

for some $c_i = c_i(d, \alpha, \kappa_1, \kappa_2, \kappa, T) > 0$. Finally, equations (2.19) and (2.21) show that

$$p_D(t, x, y) \geq c_9\mathbf{P}_x\left(\tau_D > t\right)\mathbf{P}_y\left(\tau_D > t\right)\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right)$$

for some $c_9 = c_9(d, \alpha, \kappa_1, \kappa_2, \kappa, T) > 0$ □

2.4 Dirichlet Heat Kernel Estimates on $\mathcal{C}^{1,1}$ Domains

In this chapter we refined the Dirichlet heat kernel estimates obtained in the previous chapter if D is s $\mathcal{C}^{1,1}$ domain.

Theorem 2.26. *Let X_t^D be a killed stable-like process of index $\alpha \in (0,2)$ on a $\mathcal{C}^{1,1}$-domain D and assume $\kappa(x,y)$ is a measurable function with $0 < \kappa_1 \leq \kappa(x,y) \leq \kappa_2 < \infty$. Suppose that additionally, one of the following conditions holds:*

(1) The index satisfies $0 < \alpha < 1$, and $\kappa(x,y)$ is a continuous.

(2) The index satisfies $1 \leq \alpha < 2$, and $\kappa(x,y)$ is such that

$$|\kappa(x,x) - \kappa(x,x+y)| \, \mathbf{1}_{\{|y|<1\}} \leq C_\kappa(x) \, |y|^\gamma \,, \qquad x,y \in \mathbf{R}^d.$$

where $\gamma > \alpha - 1$ and $C_\kappa \in L^1_{\text{loc}}(\mathbf{R}^d)$.

Fix $T > 0$. Then there is $C_3 = C_3(d, \alpha, \kappa_1, \kappa_2, C_\kappa, \kappa, R, T) > 0$ such that for all $t \in (0,T] \times D \times D$,

$$\begin{aligned} &C_3^{-1} \tfrac{\delta_D^{\alpha/2}(x)}{\sqrt{t}} \tfrac{\delta_D^{\alpha/2}(y)}{\sqrt{t}} \left(t^{-\frac{d}{\alpha}} \wedge \tfrac{t}{|x-y|^{d+\alpha}} \right) \\ &\leq p_D(t,x,y) \leq C_3 \tfrac{\delta_D^{\alpha/2}(x)}{\sqrt{t}} \tfrac{\delta_D^{\alpha/2}(y)}{\sqrt{t}} \left(t^{-\frac{d}{\alpha}} \wedge \tfrac{t}{|x-y|^{d+\alpha}} \right), \end{aligned}$$

where $p_D(t,x,y)$ is the transition density function of the killed stable-like process X_t^D.

Remark 2.1. Since $\alpha - 1 < \alpha/2$ for $\alpha < 2$, this result not only recovers, but improves the main result of Kim & Kim [29].

In view of Theorem 2.16, it suffices to show the following proposition.

Proposition 2.27. *Let X_t^D be the killed stable-like process satifying the conditions from Theorem 2.26. Fix $t > 0$. Then there exist $C_4 = C_4(d, \alpha, \kappa_1, \kappa_2, C_\kappa, \kappa, T) > 0$ such that for all $(t,x) \in (0,T] \times D$,*

$$C_4^{-1} \frac{\delta_D^{\alpha/2}(x)}{\sqrt{t}} \leq \mathbf{P}_x(\tau_D > t) \leq C_4 \frac{\delta_D^{\alpha/2}(x)}{\sqrt{t}}.$$

Proof of upper bound. Assume without loss of generality that the localization radius of D is $R_1 \leq 1$. Fix $(t,x) \in (0,T] \times D$. As in the previous section, we take $r = r(t) = R_1 (t/T)^\alpha \leq 1$.

Case 1. Assume $\delta_D(x) \geq r/16$. This inequality is equivalent to $1 \leq (R_1 T^{1/2}/16) \cdot (\delta_D^{\alpha/2}(x)/t^{1/2})$, and therefore

$$\mathbf{P}_x\left(\tau_D > t\right) \leq 1 \leq \frac{R_1 T^{1/2}}{16}\frac{\delta_D^{\alpha/2}(x)}{\sqrt{t}} \leq \frac{T^{1/2}}{16}\frac{\delta_D^{\alpha/2}(x)}{\sqrt{t}}.$$

Case 2. Now assume that $\delta_D(x) < r/16$. Let $Q \in \partial D$ be such that $\delta_D(Q) = |x - Q|$. Next, we define $U_1 = B(Q, r/8) \cap D$ and let n be the inward unit normal of ∂D at Q. Let $x_0 = x + (r/16)n$. Then $\delta_D(x_0) = r/16$. The boundary Harnack principle, Theorem 2.14, implies that

$$\begin{aligned} \mathbf{P}_x\left(X_{\tau_{U_1}} \in D \setminus U_1\right) &\leq c_1 \mathbf{P}_{x_0}\left(X_{\tau_{U_1}} \in D \setminus U_1\right)\frac{\delta_D^{\alpha/2}(x)}{\delta_D^{\alpha/2}(x_0)} \\ &= c_2 \mathbf{P}_{x_0}\left(X_{\tau_{U_1}} \in D \setminus U_1\right)\frac{\delta_D^{\alpha/2}(x)}{\sqrt{t}} \leq c_2 \frac{\delta_D^{\alpha/2}(x)}{\sqrt{t}}. \end{aligned} \tag{2.22}$$

where $c_i = c_i(d, \alpha, \kappa_1, \kappa_2, C_\kappa, \gamma, T) > 0$ for $i = 1, 2$. Next, pick $x_1 \in \mathbf{R}^d$ such that $B(x_1, r) \subset B(Q, 4r) \setminus B(Q, r)$. Then the Lévy system formula implies that

$$\begin{aligned} \mathbf{P}_x\left(X_{\tau_{U_1}} \in B(x_1, r)\right) &= \mathbf{E}_x\left[\int_0^{\tau_{U_1}} \int_{B(x_1,r)} \frac{\kappa(X_s, y)}{|X_s - y|^{d+\alpha}}\,\mathrm{d}y\mathrm{d}s\right] \\ &\geq c_3 m\left(B(x_1, r)\right)\frac{1}{(5r)^{d+\alpha}}\mathbf{E}_x[\tau_{U_1}] = c_4 t^{-1}\mathbf{E}_x[\tau_{U_1}], \end{aligned}$$

where $c_i = c_i(d, \alpha, \kappa_1, T) > 0$ for $i = 3, 4$. Now, as in equation (2.22), we obtain that

$$t^{-1}\mathbf{E}_x[\tau_{U_1}] \leq c_4^{-1}\mathbf{P}_x\left(X_{\tau_{U_1}} \in B(x_1, r)\right) \leq c_5\sqrt{t}\delta_D^{\alpha/2}(x). \tag{2.23}$$

for some $c_5 = c_5(d, \alpha, \kappa_1, \kappa_2, C_\kappa, \gamma, T) > 0$. Finally, Markov's inequality and equations (2.22) and (2.23) yield

$$\begin{aligned} \mathbf{P}_x\left(\tau_D > t\right) &\leq \mathbf{P}_x\left(\tau_{U_1} > t\right) + \mathbf{P}_x\left(X_{\tau_{U_1}} \in D \setminus U_1\right) \\ &\leq t^{-1}\mathbf{E}_x\left[\tau_{U_1}\right] + \mathbf{P}_x\left(X_{\tau_{U_1}} \in D \setminus U_1\right) \leq (c_2 + c_5)\frac{\delta_D^{\alpha/2}(x)}{\sqrt{t}}. \end{aligned}$$

The claim follows since $\mathbf{P}_x\left(\tau_D > t\right) \leq 1$. □

To prove the lower bound of Theorem 2.26, we need to recall some basic geometric properties of $\mathcal{C}^{1,1}$-sets. Let the characteristic of the set be (R_2, Λ). Then it satisfies the *uniform interior ball condition*: There exists $r_0 = r_0(R_2, \Lambda) \leq R_2$ such that for all $x \in D$ with $\delta_D(x) < r_0$, there exist $Q_x \in \partial D$ such that

$$|x - Q_x| = \delta_D(x), \qquad B(x_0, r_0) \subset D$$

where $x_0 = Q_x + r_0(x - Q_x)/|x - Q_x|$. It also satisfies the similar *uniform exterior ball condition*: There exists $r_1 = r_1(R_2, \Lambda) \leq R_2$ such that for all $y \in \overline{D}^c$ with $\delta_{\overline{D}^c}(y) < r_2$, there exist $Q_y \in \partial D$ such that

$$|y - Q_y| = \delta_{\overline{D}^c}(y), \qquad B(y_0, r_0) \subset \overline{D}^c$$

where $y_0 = Q_y + r_1(y - Q_y)/|y - Q_y|$.

For the proof of the lower bound of Theorem 2.26, we will fix r_2 such that both the uniform interior and exterior ball conditions are satisfied. Moreover, we set $T_2 := (r_0/16)^{1/\alpha}$. Given $x \in D$ with $\delta_D(x) < r_2$, let Q_x be a point on ∂D such that $|Q_z - x| = \delta_D(x)$ and denote $n(Q_x) = (x - Q_x)/|Q_x - x|$. We need the following lemma to prove the lower bound.

Lemma 2.28. *Let $\kappa_0 \in (0,1)$ and $a > 0$. Then there exists $C_3 = C_3(d, \alpha, \kappa_0, r_2, a) > 0$ such that for every $(t, x) \in (0, T_0] \times D$ with $\delta_D(x) \leq 3t^{1/\alpha} < r_2/4$ and $\kappa_0 \in (0,1)$,*

$$\mathbf{P}_x\left(X^D_{at} \in B(x_0, \kappa_0 t^{1/\alpha})\right) \geq C_3 \frac{\delta_D^{\alpha/2}(x)}{\sqrt{t}}. \tag{2.24}$$

where $x_0 = Q_x + 9n(Q_x)t^{1/\alpha}/2$.

Proof. Let $k \in (0, \kappa_0)$. We will prove the claim in two separate cases.

Case 1. Assume that $kt^{1/\alpha}/16 < \delta_D(x)$. As in the proof of Lemma 2.23, we can deduce from $3\kappa_0 t^{1/\alpha}/2 \leq |x - x_0| \leq 6t^{1/\alpha}$ that for all $t \leq T_2$,

$$\mathbf{P}_x\left(X^D_{at} \in B(x_0, \kappa_0 t^{1/\alpha})\right) \geq c_1 \left(t^{1/\alpha}\right)^d \frac{t}{|x - x_0|^{d+\alpha}} \geq c_2 > 0 \tag{2.25}$$

for some $c_i = c_i(d, \alpha, \kappa_1, \kappa_2, k, r_0, a) > 0$, $i = 1, 2$. By taking $k = \kappa_0$, we see that equation (2.24) holds in this case for all $a > 0$.

Case 2. Now assume that $\delta_D(x) \le \kappa_0 t^{1/\alpha}/16$. First, we show that there exists $a_0 > 1$ such that equation (2.24) holds for every $a \ge a_0$. Without loss of generality, we may assume that $x_0 = 0$. Set $B := B(0, \kappa_0 t^{1/\alpha})$, and $U := B(Q_x, \kappa_0 t^{1/\alpha}) \cap D$. Using Lemma 2.18 and the strong Markov property of X_t^D, there exists a constant $c_3 = c_3(d, \alpha, \kappa_1, \kappa_2, a) > 0$ such that

$$\begin{aligned}
\mathbf{P}_x\left(X_{at}^D \in B\right) &\le \mathbf{P}_x\big(\tau_U < at, X_{\tau_U} \in B(0, \kappa_0 t^{1/\alpha}/2), \\
&\qquad |X_s^D - X_{\tau_U}| < \kappa_0 t^{1/\alpha}/2 \text{ for } \tau_U \le s \le \tau_U + at\big) \\
&\le c_3 \mathbf{P}_x\left(\tau_U < at, X_{\tau_U} \in B(0, \kappa_0 t^{1/\alpha}/2)\right). \qquad (2.26)
\end{aligned}$$

Next, set $x_1 := Q_x + \kappa_0 n(Q_x) t^{1/\alpha}/4$ and $B_1 := B_1(\kappa_0 t^{1/\alpha}/4)$. The boundary Harnack inequality, Theorem 2.14, implies that there exist $c_i = c_i(d, \alpha, \kappa_1, \kappa_1, R_2, \Lambda, \gamma) > 0$, $i = 4, 5$ such that for all $t \in (0, T_0]$,

$$\begin{aligned}
\mathbf{P}_x\left(X_{\tau_U} \in B(0, \kappa_0 t^{1/\alpha}/2)\right) &\ge \mathbf{P}_{x_1}\left(X_{\tau_U} \in B(0, \kappa_0 t^{1/\alpha}/2)\right) \frac{\delta_D^{\alpha/2}(x)}{\delta_D^{\alpha/2}(x_1)} \\
&\ge c_5 \mathbf{P}_{x_1}\left(X_{\tau_{B_1}} \in B(0, \kappa_0 t^{1/\alpha}/2)\right) \frac{\delta_D^{\alpha/2}(x)}{\sqrt{t}}. \qquad (2.27)
\end{aligned}$$

The Lévy systems formula, Proposition 1.8, and Lemma 2.17 now impliy that

$$\begin{aligned}
\mathbf{P}_x\left(X_{\tau_{B_1}} \in B(0, \kappa_0 t^{1/\alpha}/2)\right) &= \mathbf{E}_x\left[\int_0^{\tau_{B_1}} \int_{B(0, \kappa_0 t^{1/\alpha}/2)} \frac{\kappa(y, X_s)}{|y - X_s|^{d+\alpha}} \, \mathrm{d}y \mathrm{d}s\right] \\
&\ge c_6 \left(\kappa_0 t^{1/\alpha}\right)^{-d-\alpha} m\left(B(0, \kappa_0 t^{1/\alpha}/2)\right) \mathbf{E}_x\left[\tau_{B_1}\right] \\
&\ge c_7 \left(\kappa_0 t^{1/\alpha}\right)^{-d-\alpha} m\left(B(0, \kappa_0 t^{1/\alpha}/2)\right) \left(\kappa_0 t^{1/\alpha}/4\right)^{\alpha} \\
&= c_8
\end{aligned}$$

for some positive constants $c_i = c_i(d, \alpha, \kappa_1, \kappa_1, R_2, \Lambda, \gamma, a) > 0$, $i = 6, 7, 8$. Combined with equation (2.27), we obtain

$$\mathbf{P}_x\left(X_{\tau_U} \in B(0, \kappa_0 t^{1/\alpha}/2)\right) \ge c_5 c_8 \frac{\delta_D^{\alpha/2}(x)}{\sqrt{t}}. \qquad (2.28)$$

Now, Markov's inequality and equation (2.23) imply that there exists some positive constant $c_9 = c_9(d, \alpha, \kappa_1, \kappa_1, R_2, \Lambda, \gamma, a) > 0$, we have that

$$\mathbf{P}_x\left(\tau_U \ge at\right) \le \frac{\mathbf{E}_x\left[\tau_U\right]}{at} \le a^{-1} \frac{\delta_D^{\alpha/2}(x)}{\sqrt{t}}.$$

Set $a_0 = 2c_9/(c_5c_8)$. Then the previous equation along with equations (2.26) and (2.28) show that for $a \geq a_0$,

$$\begin{aligned}\mathbf{P}_x\left(X_{at}^D \in B\right) &\geq c_3\Big(\mathbf{P}_x\left(X_{\tau_U} \in B(0, \kappa_0 t^{1/\alpha}/2)\right) - \mathbf{P}_x\left(\tau_U \geq at\right)\Big) \\ &\geq c_3(c_9/2)\frac{\delta_D^{\alpha/2}(x)}{\sqrt{t}}.\end{aligned}$$

This shows along with equation (2.25) that equation (2.24) holds for all $a \geq a_0$.

It therefore remains to show that in this case, that is, when $\delta_D(x) \leq \kappa_0 t^{1/\alpha}/16$, the claim holds for $a < a_0$. If $\delta_D(x) \leq 3(at/a_0)^{1/\alpha}$, then we can use the fact that we have proven equation (2.24) in the case $a = a_0$ to obtain a $c_i = c_i(d, \alpha, \kappa_1, \kappa_1, R_2, \Lambda, \gamma, a) > 0$, $i = 10, 11$ such that

$$\begin{aligned}\mathbf{P}_x\left(X_{at}^D \in B(x_0, \kappa_0 t^{1/\alpha})\right) &\geq \mathbf{P}_x\left(X_{a_0(at/a_0)}^D \in B\left(x_0, (at/a_0)^{1/\alpha}\right)\right) \\ &\geq c_{10}\frac{\delta_D^{\alpha/2}(x)}{\sqrt{at/a_0}} = c_{11}\frac{\delta_D^{\alpha/2}(x)}{\sqrt{t}}.\end{aligned}$$

If $3(at/a_0)^{1/\alpha} < \delta_D(x) \leq \kappa_0 t^{1/\alpha}/16$, then we have that $1 > \kappa_0 > 48(a/a_0)^{1/\alpha}$, then we get equation (2.24) from equation (2.25) by setting $\kappa_1 := (a/a_0)^{1/\alpha}$. This proves the lemma. $\square$

We can now proceed to prove the lower bound of Theorem 2.26.

Proof of lower bound. Let $(t, x) \in (0, T] \times D$. Recall that D satisfies the uniform interior ball condition with radius r_0 and $0 < (T_0/T)t \leq T_0$.

Case 1. Assume that $\delta_D(x) \leq 3((T_0/T)t)^{1/\alpha}$. Set $z_x^t := Q_x + 9((T_0/T)t)^{a/\alpha}n(Q_x)/2$. This implies that

$$B\left(z_x^t, \frac{3}{2}\left(\frac{T_0}{T}t\right)^{1/\alpha}\right) \subset B\left(Q_x + 3\left(\frac{T_0}{T}t\right)^{1/\alpha} n(Q_x), 3\left(\frac{T_0}{T}t\right)^{1/\alpha}\right) \setminus \{x\}$$

and $\delta_D(z) \geq 3((T_0/T)t)^{1/\alpha}$ for all $z \in B(z_x^t, 3((T_0/T)t)^{1/\alpha}/2)$. Analogously, define z_y^t.

Lemma 2.20 with $M = T/T_0$ if $T \geq T_0$ and the previous lemma, Lemma 2.28, with $a = 1$ and $\kappa = 1/2$ imply that

$$\mathbf{P}_x\left(\tau_D > t\right) \geq c_1\mathbf{P}_x\left(\tau_D > (T_0/T)t\right)$$

$$\geq \quad c_1\mathbf{P}_x\left(X^D_{(T_0/T)t} \in B(z^t_x, ((T_0/T)t)^{1/\alpha}/2)\right) \geq c_2\frac{\delta_D^{\alpha/2}(x)}{\sqrt{t}}.$$

for some constants $c_i = c_i(d, \alpha, \kappa_1, \kappa_2, C_\kappa, \gamma, r_0, \Lambda, T) > 0$, $i = 1, 2$.

Case 2. Now, assume that $\delta_D(x) > 3((T_0/T)t)^{1/\alpha}/2$. Since D satisfies the uniform interior ball condition, we can pick $z^t_x \in B(x, \delta_D(x))$ such that $|x - z^t_x| = 3((T_0/T)t)^{1/\alpha}/2$. Then it follows that

$$B\left(z^t_x, \frac{3}{2}\left(\frac{T_0}{T}t\right)^{1/\alpha}\right) \subset B(x, \delta_D(x)) \setminus \{x\}$$

and $\delta_D(z) \geq ((T_0/T)t)^{1/\alpha}$ for all $z \in B(z^t_x, ((T_0/T)t)^{1/\alpha})$. Analogously, define z^t_y.

Then Lemma 2.20 and Proposition 2.21 show that

$$\begin{aligned}
\mathbf{P}_x(\tau_D > t) \quad &\geq \quad c_1\mathbf{P}_x\left(X^D_{(T_0/T)t} \in B(z^t_x, ((T_0/T)t)^{1/\alpha}/2)\right) \\
&= \quad c_1\int_{B(z^t_x, ((T_0/T)t^{1/\alpha})/2} p_D((T_0/T)t, x, u)\,du \geq c_3 \\
&\geq \quad c_4\left(1 \wedge \frac{\delta_D^{\alpha/2}(x)}{\sqrt{t}}\right).
\end{aligned}$$

This completes the proof.

□

Chapter 3

BOUNDARY HARNACK PRINCIPLE FOR CENSORED STABLE-LIKE PROCESSES

3.1 Boundary Harnack Principle for $\mathcal{C}^{1,1}$ Domains

This section will be devoted to establishing the boundary Harnack principle for censored stable-like processes on $\mathcal{C}^{1,1}$-domains. For $0 < \alpha \leq 1$, [7] have shown that if D is a bounded Lipschitz domain, then the censored stable-like process does not approach the boundary, see [7, Remark 2.4]. Therefore, we wil focus on the case where $1 < \alpha < 2$. Our main results will be valid if D is a $\mathcal{C}^{1,\beta}$-domain, see subsection 1.4. Some progress towards the boundary Harnack inequality for censored stable-like processes has been made in [26].

To state the theorem, we define

$$D_r := \{x \in D : \delta_D(x) \geq r\}$$

to be the set of points in D which have at least distance r from the boundary. We wil prove the following boundary Harnack principle:

Theorem 3.1. *Let $D \subset \mathbf{R}^d$, $d \geq 2$, be an open set with $\mathcal{C}^{1,1}$ boundary with characteristics $R \geq 1$ and Λ. Let Y be the censored stable-like process in D with index $\alpha \in (1,2)$, $Q \in \partial D$, and $r \in (0,R)$. Furthermore, assume that $\kappa(x,y)$ satisfies the following conditions: There is a function $\phi : \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}_+$ and constants $\delta > 0$ and $0 < \eta < \gamma \leq 1$ such that*

$$0 < \kappa_1 \leq \kappa(x,y) \leq \kappa_2 < \infty, \qquad x, y \in D$$

$$|\kappa(x,y) - \kappa(x,x)| \leq C_\kappa |x-y|^\gamma, \quad x \in D \setminus \overline{D}_\delta, y \in \overline{D}_\delta$$

$$\left|\kappa(x,y) - \kappa(x,x) - \phi(x,x)\tfrac{|y-x|^{d+\alpha}}{|y-\bar{x}|^{d+\alpha}}\right| \leq C_{\kappa,\phi}\delta_D^{-\eta}(x)|x-y|^\gamma, \quad x, y \in D \setminus \overline{D}_\delta$$

$$|\phi(x,x) - \phi(x,y)| \leq C_\phi |x-y|^\gamma, \quad x, y \in D \setminus \overline{D}_\delta$$

$$|\phi(x,x)| \le \kappa(x,x), \quad x \in D \setminus \overline{D}_\delta.$$

Then, if u is a function on D which is not identically equal to 0, harmonic on $D \cap B(Q,r)$ for the censored process Y_t, and vanishes continuously on $\partial D \cap B(Q,r)$, there exists a constant $C_1 = C_1(d,\alpha,\gamma,\Lambda,\kappa_1,\kappa_2,C_\kappa,C_\phi) > 1$ such that

$$\frac{u(x)}{u(y)} \le C_1 \frac{\delta_D^{\alpha-1}(x)}{\delta_D^{\alpha-1}(y)}, \quad x,y \in D \cap B(Q,r/2).$$

The main theme of the proof is to construct explicit functions which are sub- and superharmonic near the boundary of D and follows [7] and [18]. The appropriate functions to consider are

$$h_p(x) = \mathbf{1}_{\{0 \le \delta_D(x) \le \delta\}} \delta_D^p(x),$$

for $p \in (0,\alpha-1]$. We start by considering the case where $D = \mathbf{R}_+^d$ is the upper half space. In that particular case, $\delta_D(x) = x_d$, so that for $x = (\tilde{x}, x_d)$ with $x_d > 0$,

$$h_p(x) = \mathbf{1}_{\{0 \le x_d \le \delta\}} x_d^p.$$

We first look at the generator of the symmetric stable process, and then use the those results to derive the result for general symmetric stable-like processes. We can write

$$\begin{aligned}
A_D^{\alpha,\kappa} u(x) &= \mathcal{A}(d,-\alpha) \lim_{\varepsilon \to 0^+} \int_{D \setminus B(x,\varepsilon)} \frac{u(y)-u(x)}{|y-x|^{d+\alpha}} \kappa(x,y)\,\mathrm{d}y \\
&= \kappa(x,x)\mathcal{A}(d,-\alpha) \lim_{\varepsilon \to 0^+} \int_{D \setminus B(x,\varepsilon)} \frac{u(y)-u(x)}{|y-x|^{d+\alpha}}\,\mathrm{d}y \\
&\quad + \mathcal{A}(d,-\alpha) \lim_{\varepsilon \to 0^+} \int_{D \setminus B(x,\varepsilon)} \frac{u(y)-u(x)}{|y-x|^{d+\alpha}} \left(\kappa(x,y)-\kappa(x,x)\right)\mathrm{d}y \\
&=: \kappa(x,x) A_D^{\alpha} u(x) + \qquad (3.1) \\
&\quad \mathcal{A}(d,-\alpha) \lim_{\varepsilon \to 0^+} \int_{D \setminus B(x,\varepsilon)} \frac{u(y)-u(x)}{|y-x|^{d+\alpha}} \left(\kappa(x,y)-\kappa(x,x)\right)\mathrm{d}y \qquad (3.2)
\end{aligned}$$

Similarly, we define

$$\bar{A}_D^{\alpha,\kappa} u(x) = \lim_{\varepsilon \to 0^+} \int_{D \setminus B(x,\varepsilon)} \frac{u(y)-u(x)}{|y-\bar{x}|^{d+\alpha}} \kappa(x,y)\,\mathrm{d}y$$

where $\bar{x}$ is the reflection point of x across ∂D. We will start with an analysis of $A_{\mathbf{R}_+^d}^{\alpha} h_p(x)$, tha is we take $D = \mathbf{R}_+^d$ to be the upper half space.

Lemma 3.2. *Let* $p \in (0, \alpha - 1)$*. Then there exists* $C_2 = C_2(d, \alpha, p, \delta) > 0$ *such that for all* $x = (\tilde{x}, x_d) \in \mathbf{R}^d_+$ *with* $0 < x_d < \delta/2$*,*

$$\left| A^{\alpha}_{\mathbf{R}^d_+} h_p(x) - \mathcal{A}(d, -\alpha) \frac{\omega_{d-1}}{2} \mathcal{B}\left(\frac{\alpha+1}{2}, \frac{d-1}{2}\right) \gamma(p, \alpha) x^{p-\alpha} \right| \leq C_2$$

where

$$\gamma(p, \alpha) = \int_0^1 \frac{(t^p - 1)(1 - t^{\alpha - p - 1})}{(1-t)^{\alpha+1}} \, \mathrm{d}t.$$

Proof. Let $x = (\tilde{x}, x_d) \in \mathbf{R}^d_+$ with $0 < x_d < \delta/2$. We compute

$$\begin{aligned} A^{\alpha}_{\mathbf{R}^d_+} h_p(x) &= \mathcal{A}(d, -\alpha) \int_{\mathbf{R}^d_+} \frac{y_d^p \mathbf{1}_{\{y_d \leq \delta\}} - x_d^p}{|y - x|^{d+\alpha}} \, \mathrm{d}y \\ &= \mathcal{A}(d, -\alpha) \left(\int_{\mathbf{R}^d_+ \cap \{y_d \leq \delta\}} \frac{y_d^p - x_d^p}{|y - x|^{d+\alpha}} \, \mathrm{d}y + \int_{\mathbf{R}^d_+ \cap \{y_d > \delta\}} \frac{-x_d^p}{|y - x|^{d+\alpha}} \, \mathrm{d}y \right) \\ &=: \mathcal{A}(d, -\alpha) \left(I + II \right). \end{aligned} \tag{3.3}$$

Now we can estimate the integrals as follows:

$$\begin{aligned} I &= \int_{\mathbf{R}^d_+ \cap \{y_d \leq \delta\}} \frac{y_d^p - x_d^p}{|y - x|^{d+\alpha}} \, \mathrm{d}y \\ &= \int_0^{\delta} (y_d^p - x_d^p) \int_{\mathbf{R}^{d-1}} \frac{1}{|y - x|^{d+\alpha}} \, \mathrm{d}\tilde{y} \, \mathrm{d}y_d \\ &= \int_0^{\delta} \frac{y_d^p - x_d^p}{|y_d - x_d|^{d+\alpha}} \int_{\mathbf{R}^{d-1}} \left(\left| \frac{\tilde{y} - \tilde{x}}{y_d - x_d} \right|^2 + 1 \right)^{-\frac{d+\alpha}{2}} \, \mathrm{d}\tilde{y} \, \mathrm{d}y_d \\ &\overset{\tilde{u} = \frac{\tilde{y} - \tilde{x}}{|y_d - x_d|}}{=} \int_0^{\delta} \frac{y_d^p - x_d^p}{|y_d - x_d|^{\alpha+1}} \int_{\mathbf{R}^{d-1}} \left(|\tilde{u}|^2 + 1 \right)^{-\frac{d+\alpha}{2}} \, \mathrm{d}\tilde{u} \, \mathrm{d}y_d \\ &= \frac{\omega_{d-1}}{2} \mathcal{B}\left(\frac{\alpha+1}{2}, \frac{d-1}{2}\right) x_d^{p-\alpha+1} \cdot \int_0^{\delta} \frac{\left(\frac{y_d}{x_d}\right)^p - 1}{|\frac{y_d}{x_d} - 1|^{\alpha+1}} \, \mathrm{d}y_d \\ &\overset{t = \frac{y_d}{x_d}}{=} \frac{\omega_{d-1}}{2} \mathcal{B}\left(\frac{\alpha+1}{2}, \frac{d-1}{2}\right) x_d^{p-\alpha} \cdot \int_0^{\delta/x_d} \frac{t^p - 1}{|t - 1|^{\alpha+1}} \, \mathrm{d}t \\ &= \frac{\omega_{d-1}}{2} \mathcal{B}\left(\frac{\alpha+1}{2}, \frac{d-1}{2}\right) x_d^{p-\alpha} \cdot \left(\int_0^{\infty} \frac{t^p - 1}{|t - 1|^{\alpha+1}} \, \mathrm{d}t - \int_{\delta/x_d}^{\infty} \frac{t^p - 1}{|t - 1|^{\alpha+1}} \, \mathrm{d}t \right). \end{aligned}$$

Recall that ω_{d-1} denotes the volume of the unit sphere in $\mathbf{R}^{d-1}$. We can further compute

$$\int_0^{\infty} \frac{t^p - 1}{|t - 1|^{\alpha+1}} \, \mathrm{d}t = \lim_{\varepsilon \to 0} \int_0^{1-\varepsilon} \frac{t^p - 1}{(1-t)^{\alpha+1}} \, \mathrm{d}t + \lim_{\varepsilon \to 0} \int_{1+\varepsilon}^{\infty} \frac{t^p - 1}{(t-1)^{\alpha+1}} \, \mathrm{d}t$$

$$\begin{aligned}
&\overset{t\mapsto\frac{1}{t}}{=} \lim_{\varepsilon\to 0}\int_0^{1-\varepsilon} \frac{t^p-1}{(1-t)^{\alpha+1}}\,\mathrm{d}t + \lim_{\varepsilon\to 0}\int_0^{\frac{1}{1+\varepsilon}} \frac{t^{-p}-1}{(t^{-1}-1)^{\alpha+1}}t^{-2}\,\mathrm{d}t \\
&= \lim_{\varepsilon\to 0}\int_0^{1-\varepsilon} \frac{t^p-1}{(1-t)^{\alpha+1}}\,\mathrm{d}t + \lim_{\varepsilon\to 0}\int_0^{1-\varepsilon} \frac{1-t^p}{(1-t)^{\alpha+1}}t^{\alpha-p-1}\,\mathrm{d}t \\
&\quad + \lim_{\varepsilon\to 0}\int_{1-\varepsilon}^{\frac{1}{1+\varepsilon}} \frac{1-t^p}{(1-t)^{\alpha+1}}t^{\alpha-p-1}\,\mathrm{d}t \\
&= \int_0^1 \frac{(t^p-1)(1-t^{\alpha-p-1})}{(1-t)^{\alpha+1}}\,\mathrm{d}t \\
&\quad + \lim_{\varepsilon\to 0}\int_{1-\varepsilon}^{\frac{1}{1+\varepsilon}} \frac{1-t^p}{(1-t)^{\alpha+1}}t^{\alpha-p-1}\,\mathrm{d}t
\end{aligned}$$

The remainder tends to zero as $\varepsilon \to 0$, and hence

$$I = \frac{\omega_{d-1}}{2}\mathcal{B}\left(\frac{\alpha+1}{2}, \frac{d-1}{2}\right)\left(\gamma(p,\alpha) - \int_{\delta/x_d}^{\infty} \frac{t^p-1}{|t-1|^{\alpha+1}}\,\mathrm{d}t\right)x_d^{p-\alpha}. \tag{3.4}$$

For the second integral, a similar computation shows that

$$\begin{aligned}
II &= -\int_{\mathbf{R}^d_+\cap\{y_d\leq\delta\}} \frac{x_d^p}{|y-x|^{d+\alpha}}\,\mathrm{d}y \\
&= -x_d^p\int_\delta^\infty\int_{\mathbf{R}^{d-1}} \frac{1}{|y-x|^{d+\alpha}}\,\mathrm{d}\tilde{y}\,\mathrm{d}y_d \\
&= -\frac{\omega_{d-1}}{2}\mathcal{B}\left(\frac{\alpha+1}{2}, \frac{d-1}{2}\right)x_d^{p-\alpha}\cdot\int_{\delta/x_d}^{\infty} \frac{1}{|t-1|^{\alpha+1}}\,\mathrm{d}t. \qquad (3.5)
\end{aligned}$$

Therefore, using equations (3.3) to (3.5) shows that

$$\begin{aligned}
&\left|A^\alpha_{\mathbf{R}^d_+}h_p(x) - \mathcal{A}(d,-\alpha)\frac{\omega_{d-1}}{2}\mathcal{B}\left(\frac{\alpha+1}{2}, \frac{d-1}{2}\right)\gamma(p,\alpha)x_d^{p-\alpha}\right| \\
&= \mathcal{A}(d,-\alpha)\frac{\omega_{d-1}}{2}\mathcal{B}\left(\frac{\alpha+1}{2}, \frac{d-1}{2}\right)\int_{\delta/x_d}^{\infty} \frac{t^p}{(t-1)^{\alpha+1}}\,\mathrm{d}t\cdot x_d^{p-\alpha} \\
&\leq \mathcal{A}(d,-\alpha)\frac{\omega_{d-1}}{2}\mathcal{B}\left(\frac{\alpha+1}{2}, \frac{d-1}{2}\right)2^{\alpha+1}\int_{\delta/x_d}^{\infty} t^{p-\alpha-1}\,\mathrm{d}t\cdot x_d^{p-\alpha} \\
&= \mathcal{A}(d,-\alpha)\frac{\omega_{d-1}}{2}\mathcal{B}\left(\frac{\alpha+1}{2}, \frac{d-1}{2}\right)\frac{2^{\alpha+1}}{(\alpha-p)\delta^{\alpha-p}}.
\end{aligned}$$

We have used that that $x_d < \delta/2$ and $t > \delta/x_d$ imply $(t-1)/t = 1-(1/t) \geq 1-(x_d/\delta) \geq 1/2$, and therefore $1/(t-1) \leq 2/t$. This completes the proof. □

After having obtained an estimate of the first term in equation (3.2), we start estimating the second term.

Lemma 3.3. *Let* $p \in (0, \alpha - 1)$, $\gamma \in (0, 1]$, *and* $\eta \geq 0$ *such that* $\gamma > \alpha - 1 \wedge \eta$. *Assume that additionally, there exists a function* $\phi : \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}_+$ *such that the following properties are satisfied:*

$$0 < \kappa_1 \leq \kappa(x, y) \leq \kappa_2 < \infty, \qquad x, y \in \mathbf{R}^d_+$$

$$|\kappa(x, y) - \kappa(x, x)| \leq C_\kappa |x - y|^\gamma, \quad x, y \in \mathbf{R}^d_+, x_d < \delta/2, y_d > \delta \tag{3.6}$$

$$\left|\kappa(x, y) - \kappa(x, x) - \phi(\bar{x}, y)\tfrac{|x-y|^{d+\alpha}}{|\bar{x}-y|^{d+\alpha}}\right| \leq C_{\kappa,\phi} x_d^{-\eta} |x - y|^\gamma, \quad x, y \in \mathbf{R}^d_+, x_d < \delta/2, y_d \leq \delta \tag{3.7}$$

$$|\phi(\bar{x}, y) - \phi(\bar{x}, \bar{x})| \leq C_\phi x_d^{-\eta} |\bar{x} - y|^\gamma, \qquad x, y \in \mathbf{R}^d_+, x_d < \delta/2, y_d > \delta \tag{3.8}$$

Then there exists $C_3 = C_3(d, \alpha, \kappa_1, \kappa_2, \gamma, C_\kappa, C_{\kappa,\phi}, C_\phi, \delta) > 0$ *such that for all* $x = (\tilde{x}, x_d)$ *with* $0 < x_d < \delta/2$,

$$\begin{aligned} & \left| A^{\alpha,\kappa}_{\mathbf{R}^d_+} h_p(x) - \mathcal{A}(d, -\alpha) \frac{\omega_{d-1}}{2} \mathcal{B}\left(\frac{\alpha+1}{2}, \frac{d-1}{2}\right) (\kappa(x, x)\gamma(p, \alpha) + \phi(\bar{x}, \bar{x})\bar{\gamma}(p, \alpha)) x_d^{p-\alpha} \right| \\ \leq \ & C_3 x_d^{p-\alpha+\gamma-\eta}, \end{aligned}$$

where

$$\bar{\gamma}(p, \alpha) = \int_0^1 \frac{(t^p - 1)(1 - t^{\alpha-p-1})}{(1 + t)^{\alpha+1}} \, \mathrm{d}t.$$

Proof. Let $x = (\tilde{x}, x_d) \in \mathbf{R}^d_+$ with $0 < x_d < \frac{\delta}{2}$. To apply the result from Lemma 3.2, we use equation (3.2) and write

$$\begin{aligned} & \left| A^{\alpha,\kappa}_{\mathbf{R}^d_+} h_p(x) - \mathcal{A}(d, -\alpha) \frac{\omega_{d-1}}{2} \mathcal{B}\left(\frac{\alpha+1}{2}, \frac{d-1}{2}\right) (\kappa(x, x)\gamma(p, \alpha) + \phi(\bar{x}, \bar{x})\bar{\gamma}(p, \alpha)) x_d^{p-\alpha} \right| \\ \leq \ & \left| \kappa(x, x) A^{\alpha}_{\mathbf{R}^d_+} h_p(x) - \mathcal{A}(d, -\alpha) \frac{\omega_{d-1}}{2} \mathcal{B}\left(\frac{\alpha+1}{2}, \frac{d-1}{2}\right) \kappa(x, x)\gamma(p, \alpha) x_d^{p-\alpha} \right| \\ & + \mathcal{A}(d, -\alpha) \left| \int_{\mathbf{R}^d_+} \frac{h_p(y) - h_p(x)}{|y - x|^{d+\alpha}} (\kappa(x, y) - \kappa(x, x)) \, \mathrm{d}y \right. \\ & \left. - \frac{\omega_{d-1}}{2} \mathcal{B}\left(\frac{\alpha+1}{2}, \frac{d-1}{2}\right) \phi(\bar{x}, \bar{x})\bar{\gamma}(p, \alpha) x_d^{p-\alpha} \right| \\ \leq \ & \kappa_2 C_2 + \mathcal{A}(d, -\alpha) \left| \int_{\mathbf{R}^d_+} \frac{y_d^p \mathbf{1}_{\{y_d \leq \delta\}} - x_d^p}{|y - x|^{d+\alpha}} (\kappa(x, y) - \kappa(x, x)) \, \mathrm{d}y \right. \\ & \left. - \frac{\omega_{d-1}}{2} \mathcal{B}\left(\frac{\alpha+1}{2}, \frac{d-1}{2}\right) \phi(\bar{x}, \bar{x})\bar{\gamma}(p, \alpha) x_d^{p-\alpha} \right| \end{aligned} \tag{3.9}$$

We now focus on the the integral in the last line.

$$\begin{aligned}\int_{\mathbf{R}^d_+} \frac{y_d^p \mathbf{1}_{\{y_d\le\delta\}} - x_d^p}{|y-x|^{d+\alpha}} \left(\kappa(x,y)-\kappa(x,x)\right) \mathrm{d}y &= \int_{\mathbf{R}^d_+\cap\{y_d\ge\delta\}} \frac{-x_d^p}{|y-x|^{d+\alpha}} \left(\kappa(x,y)-\kappa(x,x)\right) \mathrm{d}y \\ &\quad + \int_{\mathbf{R}^d_+\cap\{y_d<\delta\}} \frac{y_d^p-x_d^p}{|y-x|^{d+\alpha}} \left(\kappa(x,y)-\kappa(x,x)\right) \mathrm{d}y \\ &= I + II.\end{aligned}$$

Using the assumption given in equation (3.6), we obtain

$$\begin{aligned}|I| &\le C_\kappa x_d^{p-\eta} \int_\delta^\infty \int_{\mathbf{R}^{d-1}} \frac{1}{|y-x|^{d+\alpha-\gamma}} \,\mathrm{d}\tilde{y}\,\mathrm{d}y_d \\ &= C_\kappa x_d^p \int_\delta^\infty \frac{1}{|x_d-y_d|^{\alpha+1-\gamma}} \int_{\mathbf{R}^{d-1}} \frac{1}{(|\tilde{u}|^2+1)^{\frac{d+\alpha-\gamma}{2}}} \,\mathrm{d}\tilde{u}\,\mathrm{d}y_d \\ &= C_\kappa x_d^{p-\alpha+\gamma} \frac{\omega_{d-1}}{2} \mathcal{B}\left(\frac{d-1}{2}, \frac{\alpha-\gamma+1}{2}\right) \int_{\delta/x_d}^\infty \frac{1}{|t-1|^{\alpha+1-\gamma}} \,\mathrm{d}t \\ &\le C_\kappa \frac{\omega_{d-1}}{2} \mathcal{B}\left(\frac{d-1}{2}, \frac{\alpha-\gamma+1}{2}\right) \int_2^\infty \frac{1}{(t-1)^{\alpha+1-\gamma}} \,\mathrm{d}t \cdot x_d^{p-\alpha+\gamma} \\ &= \frac{C_\kappa}{\alpha-\gamma} \frac{\omega_{d-1}}{2} \mathcal{B}\left(\frac{d-1}{2}, \frac{\alpha-\gamma+1}{2}\right) x_d^{p-\alpha+\gamma}.\end{aligned}$$

The second last inequality follows since $x_d \le \delta/2$, so $\delta/x_d \ge 2$.

The second term, we split further up:

$$\begin{aligned}II &= \int_{\mathbf{R}^d_+\cap\{y_d<\delta\}} \frac{y_d^p-x_d^p}{|y-x|^{d+\alpha}} \left(\kappa(x,y)-\kappa(x,x)\right) \mathrm{d}y \\ &= \int_{\mathbf{R}^d_+\cap\{y_d<\delta\}} \frac{y_d^p-x_d^p}{|y-x|^{d+\alpha}} \left(\kappa(x,y)-\kappa(x,x)-\phi(\bar{x},y)\frac{|y-x|^{d+\alpha}}{|y-\bar{x}|^{d+\alpha}}\right) \mathrm{d}y \\ &\quad + \int_{\mathbf{R}^d_+\cap\{y_d<\delta\}} \frac{y_d^p-x_d^p}{|y-\bar{x}|^{d+\alpha}} \phi(\bar{x},y)\,\mathrm{d}y \\ &= III + IV.\end{aligned}$$

For the third term, we compute using the assumption given in equation (3.7)

$$\begin{aligned}|III| &= \left| \int_{\mathbf{R}^d_+\cap\{y_d<\delta\}} \frac{y_d^p-x_d^p}{|y-x|^{d+\alpha}} \left(\kappa(x,y)-\kappa(x,x)-\phi(\bar{x},y)\frac{|y-x|^{d+\alpha}}{|y-\bar{x}|^{d+\alpha}}\right) \mathrm{d}y \right| \\ &\le C_{\kappa,\phi} x_d^{-\eta} \int_{\mathbf{R}^d_+\cap\{y_d<\delta\}} \frac{|y_d^p-x_d^p|}{|y-x|^{d+\alpha-\gamma}} \,\mathrm{d}y\end{aligned}$$

$$= \quad C_{\kappa,\phi} x_d^{-\eta} \left(\int_{\mathbf{R}_+^d \cap \{y_d < \delta\} \cap B(x, x_d/2)^c} \frac{|y_d^p - x_d^p|}{|y-x|^{d+\alpha-\gamma}} \, \mathrm{d}y + \int_{B(x,x_d/2)} \frac{|y_d^p - x_d^p|}{|y-x|^{d+\alpha-\gamma}} \, \mathrm{d}y \right)$$

For the first summand,

$$\begin{aligned} \int_{\mathbf{R}_+^d \cap \{y_d < \delta\} \cap B(x, x_d/2)^c} \frac{|y_d^p - x_d^p|}{|y-x|^{d+\alpha-\gamma}} \, \mathrm{d}y &\leq \int_{B(x,x_d/2)^c} \frac{1}{|y-x|^{d+\alpha-\gamma-p}} \, \mathrm{d}y \\ &= \frac{\omega_{d-1} 2^{\alpha-p-\gamma}}{(\alpha-p-\gamma)} x_d^{p-\alpha+\gamma}, \end{aligned}$$

and for the second summand, since

$$|y_d^p - x_d^p| \leq \sup_{x_d/2 \leq z \leq 3x_d/2} p z^{p-1} \, |x_d - y_d| \leq \frac{p}{2^{p-1}} x_d^{p-1} |x-y|$$

we obtain the estimate

$$\begin{aligned} \int_{B(x,x_d/2)} \frac{|y_d^p - x_d^p|}{|y-x|^{d+\alpha-\gamma}} \, \mathrm{d}y &\leq \frac{p}{2^{p-1}} x_d^{p-1} \int_{B(x,x_d/2)} \frac{1}{|y-x|^{d+\alpha-\gamma-1}} \, \mathrm{d}y \\ &= \frac{p\omega_{d-1}}{2^{p-1}} x_d^{p-1} \frac{x_d^{\gamma+1-\alpha}}{(\gamma+1-\alpha) 2^{\gamma+1-\alpha}} \\ &= \frac{p\omega_{d-1} 2^{\alpha-p-\gamma}}{(\gamma+1-\alpha)} x_d^{p-\alpha+\gamma}. \end{aligned}$$

Thus,

$$|III| \leq 2^{\alpha-p-\gamma} \omega_{d-1} \left(\frac{1}{\alpha-p-\gamma} + \frac{p}{\gamma+1-\alpha} \right) C_{\kappa,\phi} x_d^{p-\alpha+\gamma-\eta}.$$

The fourth term, we split up once again:

$$\begin{aligned} IV &= \int_{\mathbf{R}_+^d \cap \{y_d < \delta\}} \frac{y_d^p - x_d^p}{|y-\bar{x}|^{d+\alpha}} \phi(\bar{x}, y) \, \mathrm{d}y \\ &= \phi(\bar{x}, \bar{x}) \int_{\mathbf{R}_+^d \cap \{y_d < \delta\}} \frac{y_d^p - x_d^p}{|y-\bar{x}|^{d+\alpha}} \, \mathrm{d}y \\ &\quad + \int_{\mathbf{R}_+^d \cap \{y_d < \delta\}} \frac{y_d^p - x_d^p}{|y-\bar{x}|^{d+\alpha}} \left(\phi(\bar{x}, y) - \phi(\bar{x}, \bar{x}) \right) \, \mathrm{d}y \\ &=: V + VI. \end{aligned}$$

The second term can be estimated similarly to the first term,

$$|VI| \quad = \quad \left| \int_{\mathbf{R}_+^d \cap \{y_d < \delta\}} \frac{y_d^p - x_d^p}{|y-\bar{x}|^{d+\alpha}} \left(\phi(\bar{x}, y) - \phi(\bar{x}, \bar{x}) \right) \, \mathrm{d}y \right|$$

$$\begin{aligned}
&\leq C_\phi \mathcal{B}\left(\frac{d-1}{2}, \frac{\alpha-\gamma+1}{2}\right) \frac{\omega_{d-1}}{2} x_d^{p-\alpha+\gamma-\eta} \int_0^{\delta/x_d} \frac{|t^p-1|}{(t+1)^{\alpha-\gamma+1}} \,\mathrm{d}t \\
&\leq C_\phi \mathcal{B}\left(\frac{d-1}{2}, \frac{\alpha-\gamma+1}{2}\right) \frac{\omega_{d-1}}{2} x_d^{p-\alpha+\gamma-\eta} \int_0^{\infty} \frac{|t^p-1|}{(t+1)^{\alpha-\gamma+1}} \,\mathrm{d}t \\
&= c_1 x_d^{p-\alpha+\gamma-\eta}
\end{aligned}$$

for some $c_1 = c_1(d, \alpha, \gamma, C_\phi, p) > 0$. Term V on the other hand, can be computed directly using the same techniques that we have employed in the proof of Lemma 3.2. Indeed,

$$\int_{\mathbf{R}^d_+ \cap \{y_d < \delta\}} \frac{y_d^p - x_d^p}{|y-\bar{x}|^{d+\alpha}} \,\mathrm{d}y = \frac{\omega_{d-1}}{2} \mathcal{B}\left(\frac{\alpha+1}{2}, \frac{d-1}{2}\right) \left(\bar{\gamma}(p,\alpha) x_d^{p-\alpha} - \int_{\delta/x_d}^{\infty} \frac{1}{(t+1)^{\alpha+1}} \,\mathrm{d}t\right).$$

Finally, we can combine all these estimates to obtain

$$\begin{aligned}
&\left| \int_{\mathbf{R}^d_+} \frac{h_p(y)-h_p(x)}{|y-x|^{d+\alpha}} \left(\kappa(x,y)-\kappa(x,x)\right) \mathrm{d}y - \frac{\omega_{d-1}}{2} \mathcal{B}\left(\frac{\alpha+1}{2}, \frac{d-1}{2}\right) \phi(\bar{x},\bar{x}) \bar{\gamma}(p,\alpha) x_d^{p-\alpha} \right| \\
= &\left| I + III + V + VI - \frac{\omega_{d-1}}{2} \mathcal{B}\left(\frac{\alpha+1}{2}, \frac{d-1}{2}\right) \phi(\bar{x},\bar{x}) \bar{\gamma}(p,\alpha) x_d^{p-\alpha} \right| \\
\leq &\ |I| + |III| + |VI| + \left| \frac{\omega_{d-1}}{2} \mathcal{B}\left(\frac{\alpha+1}{2}, \frac{d-1}{2}\right) \phi(\bar{x},\bar{x}) \bar{\gamma}(p,\alpha) x_d^{p-\alpha} \int_{\delta/x_d}^{\infty} \frac{t^p}{(t+1)^{\alpha+1}} \,\mathrm{d}t \right| \\
\leq &\ \frac{C_\kappa}{\alpha-\gamma} \frac{\omega_{d-1}}{2} \mathcal{B}\left(\frac{d-1}{2}, \frac{\alpha-\gamma+1}{2}\right) x_d^{p-\alpha+\gamma} \\
&+ 2^{\alpha-p-\gamma} \omega_{d-1} C_{\kappa,\phi} \left(\frac{1}{\alpha-p-\gamma} + \frac{p}{\gamma+1-\alpha}\right) x_d^{p-\alpha+\gamma-\eta} \\
&+ c_1 x_d^{p-\alpha+\gamma-\eta} + \frac{\omega_{d-1}}{2} \mathcal{B}\left(\frac{\alpha+1}{2}, \frac{d-1}{2}\right) \phi(\bar{x},\bar{x}) \bar{\gamma}(p,\alpha) x_d^{p-\alpha} \int_{\delta/x_d}^{\infty} \frac{t^p}{t^{\alpha+1}} \,\mathrm{d}t \\
\leq &\ c_2 x_d^{p-\alpha+\gamma-\eta}
\end{aligned}$$

where $c_2 = c_2(d, \alpha, \gamma, C_\kappa, C_{\kappa,\phi}, C_\phi, p, \delta) > 0$. We used that $p - \alpha < 0$ since $p \in (0, \alpha - 1)$ and that $\eta > 0$ to estimate $1 \vee x_d^{p-\alpha+\gamma} \leq x_d^{p-\alpha+\gamma-\eta}$. This completes the proof in view of equation (3.9). □

The previous proposition can be used to produce sub- and superharmonic function near the boundary of the upper half space. Next, we will use this result to produce sub- and superharmonic functions near the boundary of a sppecial $\mathcal{C}^{1,\beta}$-domain. Recall that a special $\mathcal{C}^{1,\beta}$ domain can be represented as the set lying above the graph of a $\mathcal{C}^{1,\beta}$ function Γ,

$$D = D_\Gamma := \left\{x = (\tilde{x}, x_d) \in \mathbf{R}^d : x_d > \Gamma(\tilde{x})\right\},$$

see section 1.4.

The main idea is to note that, locally, the boundary of such a domain can be approximated by that of a half-space as follows. Let $x \in D$ be sufficiently close to the boundary of D, say $\delta_D(x) < \delta$, and pick $Q \in \partial D$ such that $|x - Q| = \delta_D(x)$. Denote by n the inward normal vector at the boundary point Q. Finally, consider the region $\Pi_+ = \big\{x \in \mathbf{R}^d : (x - Q) \cdot n > 0\big\}$. Set

$$h^p_{\Pi_+}(y) = \mathbf{1}_{\left\{\delta_{\pi_+}(y) \le \delta\right\}} \operatorname{dist}\left(y, (\Pi_+)^c\right)^p = \begin{cases} ((y - Q) \cdot n)^p & y \in \Pi_+, \delta_{\Pi_+}(y) \le \delta, \\ 0 & \text{otherwise} \end{cases} .$$

Note that for the $x \in D$ which we fixed above,

$$h^p_{\Pi_+}(x) = |x - Q|^p = h^p_D(x).$$

We can now decompose

$$A^{\alpha,\kappa}_D h^p_D(x) = A^{\alpha,\kappa}_D \left(h^p_D - h^p_{\Pi_+}\right)(x) + \left(A^{\alpha,\kappa}_D - A^{\alpha,\kappa}_{\Pi_+}\right) h^p_{\Pi_+}(x) + A^{\alpha,\kappa}_{\Pi_+} h^p_{\Pi_+}(x) \tag{3.10}$$

and treat each of the summands separately. Note that we can use Lemma 3.3 for the last term. For the middle term, note that we can express $A^{\alpha,\kappa}_D$ as the sum of the fractional Laplacian $\Delta^{\alpha/2}$ and the killing κ_D. Indeed, for $u \in \mathcal{F}^D$ which in particular satisfy $u = 0$ a. e. on D^c,

$$\begin{aligned} A^{\alpha,\kappa}_D u(x) &= \lim_{\varepsilon \to 0^+} \int_{D \setminus B(x,\varepsilon)} \frac{u(y) - u(x)}{|y - x|^{d+\alpha}} \kappa(x, y)\, \mathrm{d}y \\ &= \lim_{\varepsilon \to 0^+} \int_{\mathbf{R}^d \setminus B(x,\varepsilon)} \frac{u(y) - u(x)}{|y - x|^{d+\alpha}} \kappa(x, y)\, \mathrm{d}y - \lim_{\varepsilon \to 0^+} \int_{D^c \setminus B(x,\varepsilon)} \frac{-u(x)}{|y - x|^{d+\alpha}} \kappa(x, y)\, \mathrm{d}y \\ &= \Delta^{\alpha/2,\kappa} u(x) + \kappa_D(x) u(x) \end{aligned}$$

As an immediate consequence,

$$\left(A^{\alpha,\kappa}_D - A^{\alpha,\kappa}_{\Pi_+}\right) u(x) = \left(\kappa_D(x) - \kappa_{\Pi_+}(x)\right) u(x).$$

The following lemma gives an estimate on $\kappa_D(x) - \kappa_{\Pi_+}(x)$ that we will need.

Lemma 3.4. *Given $x \in D$, let $Q \in \partial D$ be the point on the boundary closest to x,so that $|x - Q| = \delta_D(x)$. Let n be the unit inward normal for D at Q, then set*

$$\Pi_+ := \left\{x \in \mathbf{R}^d : (x - Q) \cdot n > 0\right\}.$$

There exists $C_4 = C_4(d, \alpha, \beta, \|\Gamma\|_{1,\beta}, \kappa_2)$ such that for all $x \in D$,

$$\left|\kappa_D(x) - \kappa_{\Pi_+}(x)\right| \leq C_4 \delta_D^{-\alpha}(x) \left(1 \wedge \delta_D^{\beta}(x)\right).$$

Proof. Let $c_1 = c_1(\beta, \|\Gamma\|_{1,\beta}) > 0$ be the constant from Lemma 1.11, and pick $x \in D$.

Under the assumption $\delta_D(x) \geq (2c_1)^{-1}$ the lemma is not hard to prove. In this case,

$$\kappa_D(x) \leq \kappa_2 \mathcal{A}(d, -\alpha) \int_{B(x,\delta_D(x))^c} \frac{\mathrm{d}y}{|y - x|^{d+\alpha}} = \kappa_2 \mathcal{A}(d, -\alpha) \frac{\omega_d}{\alpha} \delta_D^{-\alpha}(x)$$

and similarly

$$\kappa_{\Pi^+}(x) \leq \kappa_2 \mathcal{A}(d, -\alpha) \frac{\omega_d}{\alpha} \delta_D^{-\alpha}(x),$$

so

$$|\kappa_D(x) - \kappa_{\Pi^+}(x)| \leq \kappa_2 \mathcal{A}(d, -\alpha) \frac{\omega_d}{\alpha} \delta_D^{-\alpha}(x).$$

Next, we assume that $\delta_D(x) < (2c_1)^{-1}$ and denote the regions of $(1 + \beta)$-tangential approach by

$$\mathcal{P}_+ = \mathcal{P}_n + Q, \qquad \mathcal{P}_- = \mathcal{P}_{-n} + Q.$$

Without loss of generality, assume that $Q = 0$. Since the regions of $(1 + \beta)$-tangential approach satisfy $\mathcal{P}_n \subset D$ and $\mathcal{P}_{-n} \subset D^c$ and $\kappa(x, y) > 0$, we see that

$$\begin{aligned} \kappa_{(\mathcal{P}_{-b})^c}(x) \leq &\quad \kappa_D(x) &\leq \kappa_{\mathcal{P}_b}(x) \\ \kappa_{(\mathcal{P}_{-b})^c}(x) \leq &\quad \kappa_{\Pi^+}(x) &\leq \kappa_{\mathcal{P}_b}(x). \end{aligned}$$

Consequently,

$$\left|\kappa_D(x) - \kappa_{\Pi_+}(x)\right| \leq \kappa_{\mathcal{P}_b}(x) - \kappa_{(\mathcal{P}_{-b})^c}(x) =: R.$$

Without loss of generality assume that $n = e_d$ is the d-th unit vector. With the notation above, we now have $\mathcal{P} = \mathcal{P}_n$ and $\mathcal{P}_{-n} = -\mathcal{P}$. Furthermore, $x = (\tilde{0}, x_d)$ and $\delta_D(x) = x_d$. Therefore,

$$R = \mathcal{A}(d, -\alpha) \int_{\mathcal{P}^c \cap (-\mathcal{P})^c} \frac{\kappa(x,y)}{|y-x|^{d+\alpha}} \, \mathrm{d}y.$$

To estimate this integral, we define the set

$$T = \left\{ y \in \mathbf{R}^d : |\tilde{y}| < C^{-2/(\beta+1)}, |y_d| < C^{-1} \right\} \supset \mathcal{P} \cup (-\mathcal{P})$$

and split up the intgral accordingly,

$$\begin{aligned} R &= \mathcal{A}(d, -\alpha) \left(\int_{T^c} \frac{\kappa(x,y)}{|y-x|^{d+\alpha}} \, \mathrm{d}y + \int_{T \setminus (\mathcal{P} \cup (-\mathcal{P}))} \frac{\kappa(x,y)}{|y-x|^{d+\alpha}} \, \mathrm{d}y \right) \\ &=: \mathcal{A}(d, -\alpha) \left(I + II \right). \end{aligned}$$

The first integral can be easily estimated and computed using polar coordinates,

$$I \leq \int_{B(x, C^{-2/(\beta+1)}/2)^c} \frac{\kappa(x,y)}{|y-x|^{d+\alpha}} \leq \kappa_2 \frac{\omega_d}{\alpha} 2^\alpha C^{2\alpha/(\beta+1)} < \infty.$$

To deal with the second integral II, we further split the region of integratio up into two different sets. Let $T_1 = \left\{ y \in T \setminus (\mathcal{P} \cup -\mathcal{P}) : |\tilde{y}| \leq C^{-2/(\beta+1)} x_d/2 \right\}$ and $T_2 = T \setminus (\mathcal{P} \cup (-\mathcal{P}) \cup T_1)$. Since by assumption $\delta_D(x) \leq (2c_1)^{-1}$, we have that $x_d < 1$. Moreover, if $y \in T_1$, then

$$x_d - C \left| \tilde{y} \right|^{\beta+1} \geq x_d - x_d^{\beta+1} / \left(C 2^{\beta+1} \right) \geq x_d - x_d / \left(C 2^{\beta+1} \right) > x_d/2.$$

Using the co-area formula,, we see that

$$\begin{aligned} &\int_{T_1} \frac{\kappa(x,y)}{\left(|\tilde{y}| + (x_d - y_d)^2 \right)^{(d+\alpha)/2}} \, \mathrm{d}y \\ \leq \; & \kappa_2 \int_{\left\{ \tilde{u} \in \mathbf{R}^{d-1} : |\tilde{u}| < C^{-2/(\beta+1)} x_d/2 \right\}} \int_{-C|\tilde{u}|^{\beta+1}}^{C|\tilde{u}|^{\beta+1}} 2^{d+\alpha} x_d^{-d-\alpha} \, \mathrm{d}t \mathrm{d}\tilde{u} \\ \leq \; & \frac{4C\omega_{d-1}\kappa_2}{d+\beta} x_d^{-\alpha+\beta}. \end{aligned} \tag{3.11}$$

and

$$\int_{T_2} \frac{\kappa(x,y)}{\left(|\tilde{y}| + (x_d - y_d)^2 \right)^{(d+\alpha)/2}} \, \mathrm{d}y$$

$$
\begin{aligned}
&\le \kappa_2 \int_{\left\{\tilde{u}\in\mathbf{R}^{d-1}:C^{-2/(\beta+1)}x_d/2<|\tilde{u}|<C^{-2/(\beta+1)}\right\}} \int_{-C|\tilde{u}|^{\beta+1}}^{C|\tilde{u}|^{\beta+1}} |\tilde{u}|^{-d-\alpha}\,\mathrm{d}t\mathrm{d}\tilde{u} \\
&\le 2\kappa_2 C\omega_{d-1}\int_{C^{-2/(\beta+1)}x_d/2}^{C^{-2/(\beta+1)}} r^{\beta+1-\alpha-2}\,\mathrm{d}r \\
&\le 2^{1+\alpha-\beta}\frac{\omega_{d-1}}{\alpha-\beta}C^{(2\alpha+1-\beta)/\beta}x_d^{-\alpha+\beta}.
\end{aligned}
$$

Therefore, the conclusion holds in the case $\delta_D(x) \le (2c_1)^{-1}$, too. $\square$

In view of the decomposition of $A_D^{\alpha,\kappa}h_p(x)$ given in equation (3.10) and the results in Lemmata 3.3 and 3.4, we only need to derive an estimate for one term to show that $A_D^{\alpha,\kappa}h_p(x)$ is subharmonic near the ∂D.

Theorem 3.5. *Let* $p \in (0, \alpha-1)$ *and* $0 < \eta < \gamma \le 1$. *Then there exists* $C_5 = C_5(d, \alpha, \beta, \gamma, \|\Gamma\|_{1,\beta}, p) \le 1$ *such that for any* $x \in D$ *with* $0 < \delta_D(x) < C_5$,

$$A_D^{\alpha,\kappa}h_D(x) < 0.$$

In other words, $h_D(x)$ *is subharmonic with respect to* Y_t *near the boundary of* D.

Proof. Let $x \in D$ be such that $\delta_D(x) \le (2c_1)^{-1}$ where c_1 is the constant from Lemma 1.11. As before, pick $Q \in \partial D$ such that $|x-Q| = \delta_D(x)$, denote by n the inward normal vector at the boundary point Q, and the regions of β-tangential approach

$$\mathcal{P}_+ = \mathcal{P}_n + Q, \qquad \mathcal{P}_- = \mathcal{P}_{-n} + Q.$$

Recall the decomposition from equation (3.10),

$$
\begin{aligned}
A_D^{\alpha,\kappa}h_D^p(x) &= A_D^{\alpha,\kappa}\left(h_D^p - h_{\Pi_+}^p\right)(x) + \left(A_D^{\alpha,\kappa} - A_{\Pi_+}^{\alpha,\kappa}\right)h_{\Pi_+}^p(x) + A_{\Pi_+}^{\alpha,\kappa}h_{\Pi_+}^p(x) \\
&=: J_1 + J_2 + J_3. \qquad (3.12)
\end{aligned}
$$

By Lemma 3.3 we obtain

$$
\begin{aligned}
J_3 \le\ & \mathcal{A}(d,-\alpha)\frac{\omega_{d-1}}{2}\mathcal{B}\left(\frac{\alpha+1}{2}, \frac{d-1}{2}\right)\left(\kappa(x,x)\gamma(p,\alpha) + \phi(\bar{x},\bar{x})\bar{\gamma}(p,\alpha)\right)\delta_D^{p-\alpha}(x) \\
&+C_3\delta_D^{p-\alpha+\gamma-\eta}(x),
\end{aligned}
$$

and Lemma 3.4 shows that

$$|J_2| = \left|\left(A_D^{\alpha,\kappa} - A_{\Pi_+}^{\alpha,\kappa}\right) h_{\Pi_+}^p(x)\right| \le \left|\kappa_D(x) - \kappa_{\Pi_+}(x)\right| h_{\Pi^+}^p(x) \le C_4 \delta_D^{p-\alpha+\beta}(x).$$

Next, we will derive an estimate for J_1 by further subdividing the integral. We compute

$$\begin{aligned}
J_1 &= \mathcal{A}(d,\alpha)\, P.V.\int_{\mathbf{R}^d} \frac{\left(h_D^p(y) - h_{\Pi_+}^p(x)(y) - \left(h_D^p(x) - h_{\Pi_+}^p(x)(x)\right)\right)}{|y-x|^{d+\alpha}} \kappa(x,y)\,\mathrm{d}y \\
&= \mathcal{A}(d,\alpha)\, P.V.\int_{\mathbf{R}^d} \frac{h_D^p(y) - h_{\Pi_+}^p(x)(y)}{|y-x|^{d+\alpha}} \kappa(x,y)\,\mathrm{d}y \\
&= \mathcal{A}(d,\alpha) \left(P.V.\int_{\mathcal{P}_+} \frac{h_D^p(y) - h_{\Pi_+}^p(x)(y)}{|y-x|^{d+\alpha}} \kappa(x,y)\,\mathrm{d}y \right. \\
&\quad + \int_{\mathcal{P}_-} \frac{h_D^p(y) - h_{\Pi_+}^p(x)(y)}{|y-x|^{d+\alpha}} \kappa(x,y)\,\mathrm{d}y \\
&\quad \left. + \int_{\mathbf{R}^d \setminus (\mathcal{P}_+ \cup \mathcal{P}_-)} \frac{h_D^p(y) - h_{\Pi_+}^p(x)(y)}{|y-x|^{d+\alpha}} \kappa(x,y)\,\mathrm{d}y \right) \\
&=: \mathcal{A}(d,-\alpha)\left(K_1 + K_2 + K_3\right).
\end{aligned}$$

To compute K_2, it suffices to note that by definition $h_D^p(y) = h_{\Pi_+}^p(y) = 0$ for $y \in \mathcal{P}_- \subset D^c \cap (\Pi_+)^c$, and therefore

$$K_2 = \int_{\mathcal{P}_-} \frac{h_D^p(y) - h_{\Pi_+}^p(x)(y)}{|y-x|^{d+\alpha}} \kappa(x,y)\,\mathrm{d}y = 0.$$

To estimate the remaining integrals we will assume that, without loss of generlity, $Q = 0$ and $n = e_d$ which implies that $x = \left(\tilde{0}, \delta_D(x)\right)$ and $\Pi^+ = \mathbf{R}_d^+$ We denote $\mathcal{P}_+ =: \mathcal{P}$ and therefore $\mathcal{P}_- = -\mathcal{P}$. We will split the integral up in the same way as in Lemma 3.4. Therefore,

$$\begin{aligned}
K_3 &= \int_{\mathbf{R}^d \setminus (\mathcal{P} \cup -\mathcal{P})} \frac{h_D^p(y) - h_{\Pi_+}^p(y)}{|y-x|^{d+\alpha}} \kappa(x,y)\,\mathrm{d}y \\
&= \int_{T^c} \frac{h_D^p(y) - h_{\Pi_+}^p(y)}{|y-x|^{d+\alpha}} \kappa(x,y)\,\mathrm{d}y + \int_{T_1} \frac{h_D^p(y) - h_{\Pi_+}^p(y)}{|y-x|^{d+\alpha}} \kappa(x,y)\,\mathrm{d}y \\
&\quad + \int_{T_2} \frac{h_D^p(y) - h_{\Pi_+}^p(y)}{|y-x|^{d+\alpha}} \kappa(x,y)\,\mathrm{d}y.
\end{aligned}$$

To estimate the first integral, recall that

$$T = \left\{ y \in \mathbf{R}^d : |\tilde{y}| < C^{-2/(\beta+1)}, |y_d| < C^{-1} \right\} \supset \mathcal{P} \cup (-\mathcal{P}).$$

If $y \in T^c$, then

$$|y - x| \geq C^{-2/(\beta+1)}/2 \geq C^{1-2/(\beta+1)}|x| \geq C^{-2/(\beta+1)}|x|/2, \tag{3.13}$$

and therefore

$$h_D^p(x) \leq |y|^p \leq (|y-x| + |x|)^p \leq \left(1 + 2C^{2/(\beta+1)}\right)^p |y-x|^p \tag{3.14}$$

and

$$h_{\Pi_+}^p(y) \leq |y|^p \leq \left(1 + 2C^{2/(\beta+1)}\right)^p |y-x|^p.$$

Thus

$$\begin{aligned} \int_{T^c} \frac{h^p(y) - h_{\Pi_+}^p(y)}{|y-x|^{d+\alpha}} \kappa(x,y)\,\mathrm{d}y &\leq \kappa_2 \int_{T^c} \frac{|h_D^p(y)| + |h_{\Pi_+}^p(y)|}{|y-x|^{d+\alpha}}\,\mathrm{d}y \\ &\leq 2\left(1 + 2C^{2/(\beta+1)}\right)^p \int_{T^c} |y-x|^{-d-\alpha+p}\,\mathrm{d}y \\ &\leq c_2 \\ &\leq c_2 x_d^{p-\alpha+\beta}. \end{aligned}$$

since $p - \alpha + \beta < 0$. Here, $c_2 = c_2(d, \alpha, \beta, \|\Gamma\|_{1,\beta}, p)$.

For the second integral, recall that

$$T_1 = \left\{ y \in T \setminus (\mathcal{P} \cup (-\mathcal{P})) : |\tilde{y}| \leq C^{-2/(\beta+1)} x_d/2 \right\}.$$

Therefore, it $y \in T_1$, then $|h_D^p(y) - h_{\Pi_+}^p(y)| \leq 2|y|^p \leq 2x_d^p$, and the computation we performed earlier to obtain equation (3.11) now yields

$$\begin{aligned} \int_{T_1} \frac{h_D^p(y) - h_{\Pi_+}^p(y)}{|y-x|^{d+\alpha}} \kappa(x,y)\,\mathrm{d}y &\leq \kappa_2 \int_{T^1} \frac{|h_D^p(y) - h_{\Pi_+}^p(y)|}{|y-x|^{d+\alpha}} \kappa(x,y)\,\mathrm{d}y \\ &\leq 2\kappa_2 x_d^p \int_{T_1} \frac{\mathrm{d}y}{|y-x|^{d+\alpha}} \end{aligned}$$

$$\begin{aligned} &\leq \frac{8C\kappa_2}{d+\beta}\omega_{d-1}x_d^{p-\alpha+\beta} \\ &\leq c_3 x_d^{p-\alpha+\beta}. \end{aligned}$$

where $c_3 = c_3(d,\alpha,\beta,\kappa_2)$.

For the third integral, recall that $T_2 = T \setminus (\mathcal{P} \cup (-\mathcal{P}) \cup T_1)$ and therefore if $y \in T_2$, then $|y-x| > C^{-2/(\beta+1)}|x|/2$, then equations (3.13) and (3.14) imply that

$$\begin{aligned} \int_{T^2} \frac{h_D^p(y) - h_{\Pi_+}^p(y)}{|y-x|^{d+\alpha}}\kappa(x,y)\,\mathrm{d}y &\leq 2\left(1+2C^{2/(\beta+1)}\right)^p \kappa_2 \int_{T^1} |y-x|^{-d-\alpha+p}\,\mathrm{d}y \\ &\leq c_4 x_d^{p-\alpha+\beta}, \end{aligned}$$

where $c_4 = c_4(d,\alpha,\beta,\|\Gamma\|_{1,\beta},p)$.

Finally, to estimate K_1, we define

$$K_1^\varepsilon = \int_{\mathcal{P}\setminus B(x,\varepsilon)} \frac{h_D^p(y) - h_{\Pi_+}^p(y)}{|y-x|^{d+\alpha}}\kappa(x,y)\,\mathrm{d}y.$$

for $\varepsilon < \operatorname{dist}(x,\mathcal{P}^c)$. We can estimate

$$\begin{aligned} K_1^\varepsilon &= \int_{\mathcal{P}\setminus B(x,\varepsilon)} \frac{h_D^p(y) - h_{\Pi_+}^p(x)(y)}{|y-x|^{d+\alpha}}\kappa(x,y)\,\mathrm{d}y \\ &\leq \int_{\mathcal{P}} \frac{h_{-\mathcal{P}}^p(y) - h_{\Pi_+}^p(y)}{|y-x|^{d+\alpha}}\kappa(x,y)\,\mathrm{d}y \\ &\leq \kappa_2 \int_{\mathcal{P}} \frac{h_{-\mathcal{P}}^p(y) - h_{\Pi_+}^p(y)}{|y-x|^{d+\alpha}}\,\mathrm{d}y. \end{aligned}$$

where the last line follows since $h_{-\mathcal{P}}(y) \geq h_{\Pi_+}(y)$ for $y \in \mathcal{P}$ and $0 < \kappa(x,y) \leq \kappa_2$. Recalling that

$$\mathcal{P} = \left\{x = (\tilde{x}, x_d) \in \mathbf{R}^d : C|\tilde{x}|^{\beta+1} < x_d < C^{-1}\right\},$$

we see that

$$K_1^\varepsilon \leq \kappa_2 \int_{\mathcal{P}\setminus B(x,\varepsilon)} \frac{\left(y_d + C|\tilde{y}|^{\beta+1}\right)^p - y_d^p}{|y-x|^{d+\alpha}}\,\mathrm{d}y$$

The mean value theorem now implies that

$$K_1^\varepsilon \leq \kappa_2 \int_{\mathcal{P}\setminus B(x,\varepsilon)} \frac{2pC|\tilde{y}|^{\beta+1}y_d^{p-1}}{|y-x|^{d+\alpha}}\,\mathrm{d}y$$

$$\leq \ 2\kappa_2 pC\int_{B(0,2)\cap\mathbf{R}^d_+}\frac{|\tilde{y}|^{\beta+1}y_d^{p-1}}{|y-x|^{d+\alpha}}\,\mathrm{d}y$$

Note that

$$\frac{|\tilde{y}|^{\beta+1}y_d^{p-1}}{|y-x|^{d+\alpha}}\leq\frac{y_d^{p-1}}{|y-x|^{d+\alpha-\beta-1}}$$

and since $\alpha<\beta$, if the expression is viewed as a function of y is both integrable in a neighborhood of x and in a neighborhood of $\partial\mathbf{R}^d_+$. We continue to estimate K_1^ε,

$$\begin{aligned}
K_1^\varepsilon \ &\leq \ 2\kappa_2 pCx_d^{p-\alpha+\beta}\int_{B(0,2/x_d)\cap\mathbf{R}^d_+}\frac{|\tilde{z}|^{\beta+1}z_d^{p-1}}{|z-e_d|^{d+\alpha}}\,\mathrm{d}z\\
&\leq \ 2\kappa_2 pCx_d^{p-\alpha+\beta}\left(\int_{B(0,2)\cap\mathbf{R}^d_+}\frac{|\tilde{z}|^{\beta+1}z_d^{p-1}}{|z-e_d|^{d+\alpha}}\,\mathrm{d}z\right.\\
&\quad\left.+\,2^{d+\alpha}\kappa_2\int_{(B(0,2/x_d)\setminus B(0,2))\cap\mathbf{R}^d_+}\frac{|\tilde{z}|^{\beta+1}z_d^{p-1}}{|z|^{d+\alpha}}\,\mathrm{d}z\right)\\
&\leq \ c_5x_d^{p-\alpha+\beta}.
\end{aligned}$$

where $c_5=c_5(d,\alpha,\beta,p,\kappa_2)$. We may conclude that the integral K_1 exists, and moreover can obtain the estimate

$$K_1=\lim_{\varepsilon\to0^+}K_1^\varepsilon\leq c_5x_d^{p-\alpha+\beta}.$$

Combining all these estimates shows that if $\delta_D(x)\leq(2c_1)^{-1}$, then

$$\begin{aligned}
A_D^{\alpha,\kappa}h_D^p(x) \ &= \ \mathcal{A}(d,-\alpha)\left(K_1+K_2+K_3\right)+J_2+J_3\\
&\leq \ \mathcal{A}(d,-\alpha)\left(c_5+c_2+c_3+c_4\right)\delta_D^{p-\alpha+\beta}(x)+C_4\delta_D^{p-\alpha+\beta}(x)\\
&\quad+\mathcal{A}(d,-\alpha)\frac{\omega_{d-1}}{2}\mathcal{B}\left(\frac{\alpha+1}{2},\frac{d-1}{2}\right)\left(\kappa(x,x)\gamma(p,\alpha)+\phi(\bar{x},\bar{x})\bar{\gamma}(p,\alpha)\right)\delta_D^{p-\alpha}(x)\\
&\quad+C_3\delta_D^{p-\alpha+\gamma-\eta}(x).
\end{aligned}$$

Finally, note that since $p<\alpha-1$, $\gamma(p,\alpha)<0$ and $\bar{\gamma}(p,\alpha)<0$. Since $|\phi(\bar{x},\bar{x})|\leq\kappa(x,x)$, we see that coefficient of $\delta_D^{p-\alpha}(x)$ is negative. Since $\gamma-\eta>0$ and $\beta>0$ by assumption, this term decays faster than all other terms, so there is a constant $C_5=C_5(d,\alpha,\beta,\gamma,\|\Gamma\|_{1,\beta},p)$ such that $A_D^{\alpha,\kappa}h_D^p(x)<0$ for $x\in D$ with $\delta_D(x)<C_5$. □

Proposition 3.6. *Let* $0 < p < \alpha - 1 < \beta < 1$. *Then there exists a constant* $C_6 = C_6(d, \alpha, \beta, \gamma, \eta, p, \|\Gamma\|_{1,\beta}$ *such that for any special* $\mathcal{C}^{1,\beta}$ *domain* $D = D_\Gamma$, *for all* $x \in D$

$$P_x\left(Y_{\tau_{D\setminus\overline{D}_1}} \in \overline{D}_1\right) \le C_6 \delta_D^p(x).$$

Proof. By Theorem 3.5, the function $\delta_D^p(x)$ is subharmonic in $D \setminus \overline{D}_{C_5}$ where C_5 is a constant defined in the said theorem. Let E be an open precompact subset of $D \setminus \overline{D}_{C_5}$. Then by the equivalent probabilistic notion of subharmonicity, see [11, Theorem 2.11],

$$\delta_D^p(x) \ge \mathbf{E}_x\left[Y_{\tau_E}^p\right].$$

Now define $E_n = \left\{x \in D : \frac{1}{n} < x_d < C_5 + \frac{1}{n}\right\} \cap B(0, n)$. Note that these sets are open and precompact subsets of $D \setminus \overline{D}_{C_5}$ and that since the process Y_t is quasi-left continuous, $Y_{\tau_{E_n}} \to Y_{\tau_{D\setminus\overline{D}_{C_5}}}$. Then Fatou's lemma and Markov's inequality imply that

$$\begin{aligned}
\delta_D^p(x) &\ge \lim_{n\to\infty} \mathbf{E}_x\left[Y_{\tau_{E_n}}^p\right] \ge \mathbf{E}_x\left[\lim_{n\to\infty} Y_{\tau_{E_n}}^p\right] = \mathbf{E}_x\left[Y_{\tau_{D\setminus\overline{D}_{C_5}}}^p\right] \\
&\ge C_5^p \mathbf{P}_x\left(Y_{\tau_{D\setminus\overline{D}_{C_5}}}^p \in \overline{D}_{C_5}\right) \ge C_5^p \mathbf{P}_x\left(Y_{\tau_{D\setminus\overline{D}_1}}^p \in \overline{D}_1\right).
\end{aligned}$$

This finishes the proof with $C_6 = C_5^{-p}$. □

From now on, we restrict our attention to special $\mathcal{C}^{1,1}$-domains.

Proposition 3.7 (case $p = \alpha - 1$)**.** *Suppose that* $\beta = 1$. *Then there exists* $C_6 = C_6\left(d, \alpha, , \gamma, \eta, \|\Gamma\|_{1,1}, \kappa_2\right) > 1$ *such that for every* $x \in D$ *satisfying* $\delta_D(x) \le C_6^{-1}$, *we have*

$$A_D^{\alpha,\kappa} h_D^{\alpha-1}(x) \le C_7 \log \frac{1}{\delta_D(x)}$$

Proof. The proof is a refinement of the estimates we performed in Theorem 3.5 and we will therefore adopt the notation from the said theorem. Following equation (3.12), we only need to estimate J_1, J_2, and J_3. Recall that we showed $J_2 = 0$. Next, the same computation as before shows that for $x \in D$ with $\delta_D(x)$ small,

$$|J_3| \le C_2 \delta_D^{\beta-2}(x).$$

where $C_2 = C_2(d, \alpha, \beta, \|\Gamma\|_{1,\beta})$. Finally, we split J_1 up in the same fashion as before,

$$J_1 = \mathcal{A}(d, -\alpha)\left(K_1 + K_2 + K_3\right).$$

The same argument as before shows that $K_2 = 0$. As in the proof of Theorem 3.5, we compute

$$\begin{aligned} K_1^\varepsilon &= \int_{\mathcal{P}\setminus B(x,\varepsilon)} \frac{h_D^p(y) - h_{\Pi_+}^p(y)}{|y-x|^{d+\alpha}}\,\mathrm{d}y \\ &\leq 2(\alpha-1)Cx_d^{\beta-1}\left(\int_{B(0,2)\cap\mathbf{R}_+^d} \frac{|\tilde{z}|^{\beta+1} z_d^{\alpha-2}}{|z-e_d|^{d+\alpha}}\,\mathrm{d}z \right. \\ &\quad \left. + 2^{d+\alpha}\int_{(B(0,2/x_d)\setminus B(0,2))\cap\mathbf{R}_+^d} \frac{|\tilde{z}|^{\beta+1} z_d^{\alpha-2}}{|z|^{d+\alpha}}\,\mathrm{d}z\right) \end{aligned}$$

Using the co-area formula, we obtain

$$K_1 = \lim_{\varepsilon\to 0^+} K_1^\varepsilon \leq \begin{cases} c_1 x_d^{\beta-1} & \beta < 1 \\ c_1 \log\frac{1}{x_d} & \beta = 1 \end{cases}$$

where $c_1(d, \alpha, \beta)$. Finally, we estimate K_3 as previously, except that for the third term, in the case $\beta = 1$,

$$\begin{aligned} \int_{T^2} \frac{h_D^p(y) - h_{\Pi_+}^p(y)}{|y-x|^{d+\alpha}}\kappa(x,y)\,\mathrm{d}y &\leq 2\left(1 + 2C^{2/(\beta+1)}\right)^p \kappa_2 \int_{T^1} |y-x|^{-d-\alpha+p}\,\mathrm{d}y \\ &\leq c_4 \log\frac{1}{x_d} \end{aligned}$$

where $c_4 = c_4(d, \alpha, \|\Gamma\|_{1,1})$. □

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Lemma 3.8.

$$\begin{aligned} &\kappa_1 \frac{\mathcal{A}(d,-\alpha)}{\alpha}\frac{\omega_{d-1}}{2}\mathcal{B}\left(\frac{\alpha+1}{2}, \frac{d-1}{2}\right)\delta_D^{-\alpha}(x) \\ \leq\ & \kappa_{\Pi_+}(x) \leq \kappa_2 \frac{\mathcal{A}(d,-\alpha)}{\alpha}\frac{\omega_{d-1}}{2}\mathcal{B}\left(\frac{\alpha+1}{2}, \frac{d-1}{2}\right)\delta_D^{-\alpha}(x). \end{aligned}$$

Proof. Without loss of generality assume that $Q = 0$ and $n = e_d$. Then note that

$$\kappa_1 \int_{\left(\mathbf{R}^d_+\right)^c} \frac{1}{|y-x|^{d+\alpha}} \,\mathrm{d}y \leq \kappa_{\Pi_+}(x) \leq \kappa_2 \int_{\left(\mathbf{R}^d_+\right)^c} \frac{1}{|y-x|^{d+\alpha}} \,\mathrm{d}y.$$

The claim now follows since

$$\begin{aligned} &\int_{\left(\mathbf{R}^d_+\right)^c} \frac{1}{|y-x|^{d+\alpha}} \,\mathrm{d}y \\ = &\int_{-\infty}^{0} \int_{\mathbf{R}^{d-1}} \left(|\tilde{x}-\tilde{y}|^2 + |x_d - y_d|^2\right)^{-\frac{d+\alpha}{2}} \,\mathrm{d}\tilde{y}\,\mathrm{d}y_d \\ = &\int_{-\infty}^{0} \frac{1}{|y_d - x_d|^{\alpha+1}} \mathrm{d}y_d \cdot \int_{\mathbf{R}^{d-1}} \frac{1}{\left(|\tilde{u}|^2+1\right)^{d+\alpha}} \,\mathrm{d}\tilde{u} \\ = &\frac{\mathcal{A}(d,-\alpha)}{\alpha} \frac{\omega_{d-1}}{2} \mathcal{B}\left(\frac{\alpha+1}{2}, \frac{d-1}{2}\right) x_d^{-\alpha}. \end{aligned}$$

□

Recall that

$$\overline{D}_r := \{x \in D : \delta_D(x) \geq r\}$$

is the set of points which have at least distance r from the boundary.

Proposition 3.9. *Let $1 < \alpha < 2$ and $0 < \beta < \alpha - 1$. For any special $\mathcal{C}^{1,1}$ domain $D = D_\Gamma$ there is a constant $C_6 = C_6(d, \alpha, \|\Gamma\|_{1,1})$ such that for all $x \in D$,*

$$\mathbf{P}_x\left(Y_{\tau_{D\setminus\overline{D}_1}} \in \overline{D}_1\right) \leq C\delta_D^{\alpha-1}(x).$$

Remark 3.1. This is the equivalent of Proposition 3.6 for the case $p = \alpha - 1$.

Proof. Denote the uncensored α-stable-like process by X_t and note that by the boundary Harnack principle for the uncensored process, Theorem 2.1, there exists $c = c(d, \alpha, \gamma, \kappa_1, \kappa_2, C_\kappa) > 0$ such that

$$c^{-1}\delta_D^{\alpha/2}(x) \leq \mathbf{P}_x\left(X_{\tau_{D\setminus\overline{D}_1}} \in \overline{D}_1\right) \leq c\delta_D^{\alpha/2}(x).$$

Now define

$$u(x) = \delta_D^{\alpha-1}(x) - \frac{1}{2c}\mathbf{P}_x\left(X_{\tau_{D\setminus\overline{D}_1}} \in \overline{D}_1\right).$$

and note that for $x \in D^c$, $u(x) = 0$. Lemmata 3.4 and 3.8 imply that

$$\kappa_D(x) = \big(\kappa_D(x) - \kappa_{\Pi_+}(x)\big) + \kappa_{\Pi_+}(x) \le c_1 \delta_D^{-\alpha}(x).$$

Therefore, by Proposition 3.7,

$$\begin{aligned} A_D^{\alpha,\kappa} u(x) &= A_D^{\alpha,\kappa} \delta_D^{\alpha-1}(x) - \frac{1}{2c} \kappa_D(x) \mathbf{P}_x \left(X_{\tau_{D \setminus \overline{D}_1}} \in \overline{D}_1 \right) \\ &\le A_D^{\alpha,\kappa} \delta_D^{\alpha-1}(x) - \frac{1}{2c} c_1 \delta_D^{-\alpha}(x) c^{-1} \delta_D^{\alpha/2}(x) \\ &\le \begin{cases} C_5 \delta_D^{\beta-1}(x) - \delta_D^{-\alpha/2}(x)/(2c^2 c_1) & \beta < 1 \\ C_5 \log \frac{1}{\delta_D(x)} - \delta_D^{-\alpha/2}(x)/(2c^2 c_1) & \beta = 1 \end{cases} \end{aligned}$$

where $c_1 = c_1(d, \alpha, \beta, \|\Gamma\|_{1,\beta})$. Thus there exists $C_6 = C_6(d, \alpha, \beta, \|\Gamma\|_{1,\beta}, \kappa_2) \ge 1$ such that for all $\delta_D(x) < C_6$,

$$A_D^{\alpha,\kappa} u(x) < 0,$$

i. e. $u(x)$ is subharmonic near the boundary of D. We can now obtain the conclusion as in the Proposition 3.6. □

Next, we prove a reverse inequality of Proposition 3.7.

Lemma 3.10. *Suppose that $\beta = 1$. Then there is $C_7 = C_7(d, \alpha, \gamma, \kappa_1, \kappa_2) < 1$ such that for every $x \in D$ with $\delta_D(x) \le C_7$,*

$$A_D^{\alpha,\kappa} \delta_D^{\alpha-1}(x) \ge \log C_7 \delta_D(x).$$

Proof. The proof uses the same technique as the proof of Theorem 3.5. Following the notation in that proof, it suffices to derive a lower bound for K_1, or equivalently, K_1^ε. As usual, we may assume without loss of generality that $Q = 0$ and $n = e_d$. Let $B = B(Re_d, R) \subseteq D)$ be the inner tangent ball where $R = 1/(4\|\Gamma\|_{1,1} \vee 2)$. Then, since $\kappa(x, y) > 0$,

$$\begin{aligned} K_1^\varepsilon &= \int_{\mathcal{P} \setminus B(x,\varepsilon)} \frac{\delta_D^{\alpha-1}(y) - \delta_{\Pi_+}^{\alpha-1}(y)}{|y - x|^{d+\alpha}} \kappa(x, y) \, \mathrm{d}y \\ &\ge \int_{\mathcal{P} \setminus B(x,\varepsilon)} \frac{\delta_B^{\alpha-1}(y) - y_d^{\alpha-1}(y)}{|y - x|^{d+\alpha}} \kappa(x, y) \, \mathrm{d}y \end{aligned}$$

Note that we can easily compute the distance to the tangent ball provided that $y \in \mathcal{P} \subset B$. This is the case when $C \geq 1/R$. Since we can change the value of C in the definition of $\mathcal{P}$, we redefine it to be $C = 2/R = (8\|\Gamma\|_{1,1}) \vee 4$. Then

$$\delta_B(y) = R - \sqrt{(R-y_d)^2 + |\tilde{y}|^2}.$$

Moreover, by the definition of $\mathcal{P}$, any $y \in \mathcal{P}$ satisfies $y_d < C^{-1} = R/2$. With those assumptions, $\delta_B(y) \leq y_d$, and therefore

$$K_1^{\varepsilon} \geq \kappa_1 \int_{\mathcal{P}\setminus B(x,\varepsilon)} \frac{\delta_B^{\alpha-1}(y) - y_d^{\alpha-1}(y)}{|y-x|^{d+\alpha}} \, \mathrm{d}y.$$

To finish the proof, we consider the function $s \mapsto (R - \sqrt{(R-y_d)^2 - |\tilde{y}|^2})^{\alpha-1} - y_d^{\alpha-1}$. Applying the mean value theorem yields the existence of some $\theta \in (0,1)$ such that

$$\begin{aligned}
\left(R - \sqrt{(R-y_d)^2 + |\tilde{y}|^2}\right)^{\alpha-1} - y_d^{\alpha-1} \quad &= \quad -|\tilde{y}|^2 \frac{\alpha-1}{2} \frac{\left(R - \sqrt{(R-y_d)^2 + \theta|\tilde{y}|^2}\right)^{\alpha-1}}{R - \sqrt{(R-y_d)^2 + \theta|\tilde{y}|^2}} \\
&\geq \quad \frac{\alpha-1}{R} |\tilde{y}|^2 \left(R - \sqrt{(R-y_d)^2 + \theta|\tilde{y}|^2}\right)^{\alpha-2} \\
&\geq \quad -\frac{2^{\alpha}(\alpha-1)}{4R} |\tilde{y}|^2 y_d^{\alpha-2},
\end{aligned}$$

and

$$K_1^{\varepsilon} \geq -\kappa_1 \frac{2^{\alpha}(\alpha-1)}{4R} \int_{\mathcal{P}\setminus B(x,\varepsilon)} \frac{|\tilde{y}|^2 y_d^{\alpha-2}}{|y-x|^{d+\alpha}} \, \mathrm{d}y = c_1 \log\left(1/x_d\right),$$

where $c_1 = c_1(d, \alpha, \kappa_1, \|\Gamma\|_{1,1}) > 0$. The estimate now follows as in Theorem 3.5.

□

This enables us to prove the reverse inequality of equation (3.9).

Theorem 3.11. *There exists* $C_8 = C_8(d, \alpha, \gamma, \kappa_1, \kappa_2) > 0$ *such that*

$$\mathbf{P}_x\left(Y_{\tau_{D\setminus\overline{D}_1}} \in \overline{D}_1\right) \geq C_8 \delta_D^{\alpha-1}(x).$$

Proof. As before, we consider the uncensored process X_t and define

$$u(x) = \delta_D^{\alpha-1}(x) + \mathbf{P}_x\left(X_{\tau_{D\setminus\overline{D}_1}} \in \overline{D}_1\right).$$

By the previous lemma as well as the exit time estimate for the uncensored process,

$$\begin{aligned} A_D^{\alpha,\kappa} u(x) &= \left(\Delta^{\alpha/2} + \kappa_D\right) \delta_D^{\alpha-1}(x) + \kappa_D \mathbf{P}_x \left(X_{\tau_{D\setminus\overline{D}_1}} \in \overline{D}_1\right) \\ &\geq \log C_7 \delta_D(x) + C\delta_D^{-\alpha}(x)\delta_D^{\alpha/2}(x) \\ &= \log C_7 \delta_D(x) + C\delta_D^{-\alpha/2}(x) \end{aligned}$$

and since the second term decays faster than the first term as $\delta_D(x) \to 0$, we can find C_9 such that for $\delta_D(x) < C_9$, $A_D^{\alpha,\kappa} u(x) > 0$. In other words, $u(x)$ is subharmonic near the boundary of D.

Next, note that $u(x) = 0$ in D^c, and that $u(x) \leq C\delta_D^{\alpha/2}(x)$ by the exit time estimate for the uncensored process. By reducing C_9 is necessary, we may assume that $u(x) \leq 1$ for $\delta_D(x) < C_9$. Continuing the computation from before, we obtain

$$\delta_D^{\alpha-1}(x) \leq u(x) \leq \mathbf{E}_x \left[u\left(Y_{\tau_{D\setminus\overline{D}_c}}\right)\right] \leq \mathbf{P}_x \left(Y_{\tau_{D\setminus\overline{D}_c}} \in \overline{D}_c\right).$$

Finally, obtain a similar estimate for the $\overline{D}_1$.

$$\begin{aligned} \mathbf{P}_{Y_{\tau_{D\setminus\overline{D}_c}}} \left(Y_{\tau_{D\setminus\overline{D}_c}} \in \overline{D}_c\right) &\geq \mathbf{P}_{Y_{\tau_{D\setminus\overline{D}_c}}} \left(Y_{B(Y_{\tau_{D\setminus\overline{D}_c}}, C_9/2)} \in \overline{D}_c\right) \\ &\geq \mathbf{P}_{X_{\tau_{D\setminus\overline{D}_c}}} \left(Y_{B(Y_{\tau_{D\setminus\overline{D}_c}}, C_9/2)} \in \overline{D}_c\right) \geq c(d,\alpha) > 0 \end{aligned}$$

and by the strong Markov property

$$\begin{aligned} \mathbf{P}_x \left(Y_{D\setminus\overline{D}_1} \in \overline{D}_1\right) &\geq \mathbf{E}_x \left[Y_{\tau_{D\setminus\overline{D}_c}} \in \overline{D}; \mathbf{P}_{Y_{\tau_{D\setminus\overline{D}_c}}} \left(Y_{\tau_{D\setminus\overline{D}_1}} \in \overline{D}_1\right)\right] \\ &\geq c\mathbf{P}_x \left(Y_{\tau_{D\setminus\overline{D}_c}} \in \overline{D}_c\right). \end{aligned}$$

$\square$

Since we are working on a $\mathcal{C}^{1,1}$ domain, we can write $D = D_\Gamma$ where Γ is a $\mathcal{C}^{1,1}$-function satiafying

$$|\Gamma(\tilde{x}) - \Gamma(\tilde{y})| \leq \lambda|\tilde{x} - \tilde{y}|, \qquad \tilde{x}, \tilde{y} \in \mathbf{R}^{d-1}.$$

For $x = (\tilde{x}, x_d) \in \mathbf{R}^d$, we set $\eta(x) = (x_d - \Gamma(\tilde{x})) \vee 0$ which represents the vertical distance of x to the complement of D. The pythagoren theorem shows that this quantity is comparable to the Euclidean distance of x to the complement of D,

$$\delta_D(x) \leq \eta(x) \leq \sqrt{\lambda^2+1}\delta_D(x), \qquad x \in \mathbf{R}^d$$

For the next steps, we need to define a box with bottom on ∂D,

$$\Delta(x,a,r) := \{y \in D : 0 < \eta(y) < a, |\tilde{x}-\tilde{y}| < r\}.$$

where $x \in D$, and $a, r > 0$.

Before we prove the next theorem, we need the following lemma:

Next, we will derive a preliminary estimate on $A_D^{\alpha,\kappa}$ for $\phi \in C^{1,1}$.

Proposition 3.12. *Let $\gamma > \alpha - 1$ and $\phi \in \mathcal{C}^{1,1}(\mathbf{R}^d)$. Then there exists a constant $C = C(d,\alpha,\gamma,\kappa_2,C_\kappa) > 0$ such that*

$$|A_D^{\alpha,\kappa}\phi(x)| \leq C \|\phi\|_{1,1}\left(1+\delta_D^{1-\alpha}(x)\right), \qquad x \in \mathbf{R}^d$$

Proof.

$$\begin{aligned}
\Delta^{\alpha/2,\kappa}\phi(x) &= P.V.\int_{\mathbf{R}^d} \frac{\phi(x)-\phi(y)}{|x-y|^{d+\alpha}}\kappa(x,y)\,\mathrm{d}y \\
&= P.V.\int_{B(x,1)^c} \frac{\phi(x)-\phi(y)}{|x-y|^{d+\alpha}}\kappa(x,y)\,\mathrm{d}y \\
&\quad +\kappa(x,x)P.V.\int_{B(x,1)} \frac{\phi(x)-\phi(y)}{|x-y|^{d+\alpha}}\,\mathrm{d}y \\
&\quad +P.V.\int_{B(x,1)} \frac{\phi(x)-\phi(y)}{|x-y|^{d+\alpha}}\left(\kappa(x,x)-\kappa(x,y)\right)\mathrm{d}y \\
&=: I + II + III.
\end{aligned}$$

We estimate each term separatetly: First,

$$\begin{aligned}
I &\leq \int_{B(x,1)^c} \frac{|\phi(x)-\phi(y)|}{|x-y|^{d+\alpha}}|\kappa(x,y)|\,\mathrm{d}y \\
&\leq 2\|\phi\|_\infty c_2 \int_{B(0,1)^c} \frac{1}{|y|^{d+\alpha}}\,\mathrm{d}y \\
&= \frac{2d\omega_d c_2}{\alpha}\|\phi\|_\infty.
\end{aligned}$$

Next,

$$\begin{aligned}
II &= \kappa(x,x)P.V.\int_{B(x,1)} \frac{\phi(x)-\phi(y)}{|x-y|^{d+\alpha}}\,\mathrm{d}y \\
&= \kappa(x,x)P.V.\int_{B(x,1)} \frac{\phi(x)-\phi(y)-\nabla\phi(x)(x-y)}{|x-y|^{d+\alpha}}\,\mathrm{d}y \\
&\leq \kappa_2\left\|\nabla\phi\right\|_\infty \int_{B(x,1)} \frac{|x-y|^2}{|x-y|^{d+\alpha}}\,\mathrm{d}y \\
&= \kappa_2\left\|\nabla\phi\right\|_\infty \int_{B(0,1)} \frac{1}{|y|^{d+\alpha-2}}\,\mathrm{d}y \\
&= \frac{\kappa_2 d\omega_d}{2-\alpha}\left\|\nabla\phi\right\|_\infty
\end{aligned}$$

Finally,

$$\begin{aligned}
III &\leq \left\|\phi\right\|_\infty C_\kappa \int_{B(x,1)} \frac{|x-y|}{|x-y|^{d+\alpha}}|x-y|^\gamma\,\mathrm{d}y \\
&= \left\|\phi\right\|_\infty C_\kappa \int_{B(0,1)} \frac{1}{|y|^{d+\alpha-1-\gamma}}\mathrm{d}y \\
&= \frac{d\omega_d C_\kappa}{1+\gamma-\alpha}\left\|\phi\right\|_\infty.
\end{aligned}$$

The killing term can be estimated by

$$\begin{aligned}
\kappa_D\phi(x) &= \int_{D^c} \frac{\phi(x)-\phi(y)}{|x-y|^{d+\alpha}}\kappa(x,y)\,\mathrm{d}y \\
&\leq 2\kappa_2\left\|\nabla\phi\right\|_\infty \int_{D^c} \frac{|x-y|}{|x-y|^{d+\alpha}}\,\mathrm{d}y \\
&\leq 2\kappa_2\left\|\nabla\phi\right\|_\infty \int_{\{|x-y|>\delta_D(x)\}} \frac{1}{|x-y|^{d+\alpha-1}}\,\mathrm{d}y \\
&= \frac{2\kappa_2 d\omega_d}{1-\alpha}\left\|\nabla\phi\right\|_\infty \delta_D^{1-\alpha}(x),
\end{aligned}$$

so that, overall, we obtain Thus

$$\begin{aligned}
|A_D^{\alpha,\kappa}\phi(x)| &\leq \left|\Delta^{\alpha/2}\phi(x)\right| + |\kappa_D\phi(x)| \\
&\leq \frac{2d\omega_d\kappa_2}{\alpha}\left\|\phi\right\|_\infty + \frac{d\omega_d}{2-\alpha} + \frac{d\omega_d C_\kappa}{2-\alpha}\left\|\phi\right\|_\infty \\
&\quad + \frac{2\kappa_2 d\omega_d}{1-\alpha}\left\|\nabla\phi\right\|_\infty \delta_D^{1-\alpha}(x)
\end{aligned}$$

$$\leq \; C(d,\alpha,\kappa_2,C_\kappa)\, \|\phi\|_{1,1} \left(1+\delta_D^{1-\alpha}(x)\right),$$

□

Theorem 3.13. *There exists* $C_{10} = C_{10}(d,\alpha,\gamma) > 0$ *such that for* $x \in \Delta(0,1,1)$ with $\tilde{x} = 0$,

$$\mathbf{P}_x \left(Y_{\tau_{\Delta(0,1,1)}} \in D\right) \leq C_{10}\mathbf{P}_x \left(Y_{\tau_{\Delta(0,1,1)}} \in \Delta(0,2,1)\right).$$

Proof. Let

$$\begin{aligned} u^Y(x) &= \mathbf{P}_x \left(Y_{\tau_{\Delta(0,1,\infty)}} \in D\right), \qquad x \in D \\ u^X(x) &= \mathbf{P}_x \left(X_{\tau_{\Delta(0,1,\infty)}} \in D\right), \qquad x \in \mathbf{R}^d \end{aligned}$$

and note that by Proposition 3.9 and the results for the uncensored process, there are positive constants c_1 and c_2 which depend only on d, α, and γ such that

$$c_1^{-1}\left(\delta_D^{\alpha-1}(x) \wedge 1\right) \leq \; u^Y(x) \; \leq c_1 \left(\delta_D^{\alpha-1}(x) \wedge 1\right), \qquad x \in D \tag{3.15}$$

$$c_2^{-1}\left(\delta_D^{\alpha/2}(x) \wedge 1\right) \leq \; u^X(x) \; \leq c_2 \left(\delta_D^{\alpha/2}(x) \wedge 1\right), \qquad x \in \mathbf{R}^d \tag{3.16}$$

Let ϕ be a $\mathcal{C}^2$ function which satisfies the followiing:

$$\begin{aligned} \|\phi\|_{\mathcal{C}^2} &< \infty \\ \phi(x) &= |\tilde{x}| = x_1^2 + x_2^2 + \ldots + x_{d-1}^2, \qquad |\tilde{x}| < 1 \\ \phi(x) &\geq 1, \qquad |\tilde{x}| \geq 1 \end{aligned}$$

Since $\alpha - 1 < \alpha/2$,

$$\frac{u^Y(x) - u^X(x)}{2c_1c_2} \geq \frac{\delta_D^{\alpha-1}(x) \wedge 1}{2c_2}, \qquad x \in \mathbf{R}^d_+$$

and we therefore define

$$v(x) = \frac{u^Y(x) - u^X(x)}{2c_1c_2} + 8c_1^2\phi(x), \qquad x \in \mathbf{R}^d_+.$$

Then, for x near ∂D,

$$A_D^{\alpha,\kappa} v(x) \;\; = \;\; -\frac{\kappa_D(x)u^X(x)}{1c_1c_2} + 8c_1^2 A_D^{\alpha,\kappa}\phi(x)$$

$$\approx \quad -C_1\delta_D^{-\alpha/2}(x) + C_2\delta_D^{1-\alpha}(x).$$

For sufficiently small $\delta_D(x)$ the quantity on the right hand side is negative, and we can therefore conclude that for small m, v is superharmonic in $\Delta(0,m,\infty)$. By the mean-value characterization of superharmonic functions, for every $x = (\tilde{0}, x_d) \in D$,

$$c_1\left(\delta_D^{\alpha-1}(x) \wedge 1\right) \geq v(x) \geq 2c_1^2\mathbf{P}_x\left(Y_{\tau_{\Delta(0,m,1/2)}} \in D \setminus \Delta(0,\infty,1/2)\right).$$

By equation (3.16),

$$\mathbf{P}_x\left(Y_{\tau_{\Delta(0,m,1/2)}} \in D \setminus \Delta(0,\infty,1/2)\right) \geq \frac{\delta_D^{\alpha-1}(x) \wedge 1}{2c_1} \geq \frac{1}{2}u^Y(x).$$

Therefore,

$$\begin{aligned}
\mathbf{P}_x\left(Y_{\tau_{\Delta(0,m,1/2)}} \in D\right) &= \mathbf{P}_x\left(Y_{\tau_{\Delta(0,m,1/2)}} \in D \setminus \Delta(0,\infty,1/2)\right) \\
&\quad +\mathbf{P}_x\left(Y_{\tau_{\Delta(0,m,1/2)}} \in \Delta(0,\infty,1/2)\right) \\
&\leq \frac{1}{2}v(x) + \mathbf{P}_x\left(Y_{\tau_{\Delta(0,m,1/2)}} \in \Delta(0,\infty,1/2)\right) \\
&= \frac{1}{2}\mathbf{P}_x\left(Y_{\tau_{\Delta(0,m,1/2)}} \in D\right) \\
&\quad +\mathbf{P}_x\left(Y_{\tau_{\Delta(0,m,1/2)}} \in \Delta(0,\infty,1/2)\right) \\
&\leq \frac{1}{2}\mathbf{P}_x\left(Y_{\tau_{\Delta(0,1,\infty)}} \in D\right) \\
&\quad +\mathbf{P}_x\left(Y_{\tau_{\Delta(0,m,1/2)}} \in \Delta(0,\infty,1/2)\right),
\end{aligned}$$

and subtracting the left summand from both sides and rearranging terms yields

$$\mathbf{P}_x\left(Y_{\tau_{\Delta(0,m,1/2)}} \in D\right) \leq 2\mathbf{P}_x\left(Y_{\tau_{\Delta(0,m,1/2)}} \in \Delta(0,\infty,1/2)\right). \tag{3.17}$$

Also, we have that for some constant C which depends only d, α, γ such that

$$\begin{aligned}
&\mathcal{A}(d,-\alpha)\int_{\Delta(0,\infty,1/2)\setminus\Delta(0,m,1/2)} \frac{\kappa(y,z)}{|z-y|^{d+\alpha}}\,\mathrm{d}y \\
\leq\ & C\frac{c_2}{c_1}\mathcal{A}(d,-\alpha)\int_{\Delta(0,3m/2,1/2)\setminus\Delta(0,m,1/2)} \frac{\kappa(y,z)}{|z-y|^{d+\alpha}}\,\mathrm{d}y,
\end{aligned}$$

so that by equation (3.17), for x with $\tilde{x}=0$,

$$\mathbf{P}_x\left(Y_{\tau_{\Delta(0,m,1/2)}}\in D\right)\le C\mathbf{P}_x\left(Y_{\tau_{\Delta(0,m,1/2)}}\in\Delta(0,3m/2,1/2)\right).$$

The reverse inequality with D replaced by $\Delta(0,2,1)$ holds if we replace C by a new constant C_1 since this inequality is true for the uncensored process X^D, so that for x with $\tilde{x}=0$,

$$\begin{aligned}\mathbf{P}_x\left(Y_{\tau_{\Delta(0,1,1)}}\in\Delta(0,2,1)\right) &\ge C_1\mathbf{P}_x\left(Y_{\tau_{\Delta(0,m,1/2)}}\in\Delta(0,3m/2,1/2)\right)\\ &\ge \frac{C_1}{C}\mathbf{P}_x\left(Y_{\tau_{\Delta(0,m,1/2)}}\in D\right)\\ &\ge \frac{C_1}{C}\mathbf{P}_x\left(Y_{\tau_{\Delta(0,1,1)}}\in D\right),\end{aligned}$$

proving our claim. □

Now we are ready to prove the Carleson estimate.

Theorem 3.14 (Carleson estimate)**.** *Assume* $F=\left\{x\in\mathbf{R}^d:|\tilde{x}|<2,x_d=0\right\}\subset\partial D$ *and* $A=\left(\tilde{0},1/2\right)\in\Delta(0,1,1)$. *Let u be nonnegative function on* $\mathbf{R}^d$ *which vanishes cotinuously at every point of F and is harmonic on* $\Delta(0,2,2)$ *or the censored process Y_t on D. Then there is a constant C_{11} which depends only on d, α, and γ such that*

$$u(x)\le C_{11}u(A),\qquad x\in\Delta(0,1,1).\tag{3.18}$$

Proof. By the Harnack inequality, there exists $M=M(d,\alpha,c_1,c_2)$ such that

$$u(x)\le 2^{kM}u(A),\qquad x\in\Delta(0,1,1),\quad x_d\ge 2^{-k}.$$

for some $k\in\mathbf{N}$. Assume that at some $x_0\in\Delta(0,1,1)$ we have that $u(x_0)>2^{k_0M}u(A)$; we will show that if this integer k_0, depending on d, α, γ is large enough then the continuous decay of u on F will be contradicted.

By equation (3.18), $x_{0d}\le 2^{-k_0}$. Let $\varepsilon=\frac{\alpha-1}{2\alpha}\in(0,1)$ and define $\Delta_0=\Delta\left(x_0,2^{-\varepsilon k_0},2^{-\varepsilon k_0}\right)$. The harmonicity of u implies that $u(x_0)=\mathbf{E}_{x_0}\left[u(Y_{\tau_{B_\ell}})\right]$ where $B_\ell=\Delta_0\cap\{x_d>\ell^{-1}\}$ for $\ell\in\mathbf{N}$. Take limits and use the continuity of u on F and the quasi-continuity of Y to obtain

$$u(x_0)=\mathbf{E}_{x_0}\left[u\left(Y_{\tau_{\Delta_0}}\right)\right].$$

Now write

$$\begin{aligned} u(x_0) &= \mathbf{E}_{x_0}\left[Y_{\tau_{\Delta'}} \in 2\Delta_0; u\left(Y_{\tau_{\Delta_0}}\right)\right] + \mathbf{E}_{x_0}\left[Y_{\tau_{\Delta_0}} \in D \setminus 2\Delta_0; u\left(Y_{\tau_{\Delta_0}}\right)\right] \\ &=: E_1 + E_2. \end{aligned}$$

We will produce a sequence of points x_k which are significantly increasing by considering two cases. First, assume $E_2 \geq E_1$, so that $E_2 \geq u(x_0)/2$. Denote by $\mathbf{E}_x^v e_K(\tau_B)$ the conditional gaude function and by $G_D^X(x,y)$ the Green's function of D for the uncensored process. Finally, write $G^X(x,y) = G_{\mathbf{R}^d}(x,y)$, see [7]. Then

$$\begin{aligned} \frac{1}{2}u(x_0) &\leq \mathcal{A}(d,-\alpha)\int_{\mathbf{R}_+^d \setminus 2\Delta_0}\int_{\Delta_0} \frac{G_{\Delta_0}^X(x_0,z)\mathbf{E}_{x_0}^z\left[e_\kappa\left(\tau_{\Delta_0}\right)\right]\kappa(y,z)}{|y-z|^{d+\alpha}}\,\mathrm{d}v\, u(y)\mathrm{d}y \\ &\leq c_2\mathcal{A}(d,-\alpha)\int_{\mathbf{R}_+^d \setminus 2\Delta_0}\int_{\Delta_0} \frac{G_{\Delta_0}^X(x_0,z)\mathbf{E}_{x_0}^z\left[e_\kappa\left(\tau_{\Delta_0}\right)\right]}{\left|\left(y-(\tilde{x}_0,0)\right)/4\right|^{d+\alpha}}\,\mathrm{d}v\, u(y)\mathrm{d}y \\ &= c_2\mathcal{A}(d,-\alpha)4^{d+\alpha}\int_{\Delta_0} G_{\Delta_0}^X(x_0,z)\mathbf{E}_{x_0}^z\tau_{\Delta_0}^Y\int_{\mathbf{R}_+^d \setminus 2\Delta_0}\frac{u(y)}{|y-(\tilde{x}_0,0)|^{d+\alpha}}\,\mathrm{d}y \\ &= c_2\mathcal{A}(d,-\alpha)4^{d+\alpha}\mathbf{E}_{x_0}\tau_{\Delta_0}^Y\int_{\mathbf{R}_+^d \setminus 2\Delta_0}\frac{u(y)}{|y-(\tilde{x}_0,0)|^{d+\alpha}}\,\mathrm{d}y \end{aligned}$$

Now define $x_1 = \left(\tilde{x}_0, 2^{1-\varepsilon k_0}\right)$. Then

$$\begin{aligned} u(x_1) &\geq \mathcal{A}(d,-\alpha)\int_{\mathbf{R}_+^d \setminus 2\Delta_0}\int_{\Delta_0} \frac{G_{\Delta_0}^X(x_1,z)\mathbf{E}_{x_1}^z\left[e_\kappa\left(\tau_{\Delta_0}\right)\right]\kappa(y,z)}{|y-z|^{d+\alpha}}\,\mathrm{d}v\, u(y)\mathrm{d}y \\ &\geq c_1\mathcal{A}(d,-\alpha)\int_{\mathbf{R}_+^d \setminus 2\Delta_0}\int_{\Delta_0} \frac{G_{\Delta_0}^X(x_1,z)\mathbf{E}_{x_1}^z\left[e_\kappa\left(\tau_{\Delta_0}\right)\right]\kappa(y,z)}{2|y-(\tilde{x}_0,0)|^{d+\alpha}}\,\mathrm{d}v\, u(y)\mathrm{d}y \\ &= c_1\mathcal{A}(d,-\alpha)2^{-d-\alpha}\mathbf{E}_{x_0}\tau_{\Delta_0}^Y\int_{\mathbf{R}_+^d \setminus 2\Delta_0}\frac{u(y)}{|y-(\tilde{x}_0,0)|^{d+\alpha}}\,\mathrm{d}y \end{aligned}$$

Combining both equalities yields

$$\frac{u(x_1)}{u(x_0)} \geq 2^{-3(d+\alpha)}\frac{c_1}{c_2}\frac{\mathbf{E}_{x_1}\tau_{\Delta_0}^Y}{\mathbf{E}_{x_2}\tau_{\Delta_0}^Y}$$

By Theorem 3.11,

$$C_8\delta_D^{\alpha-1}(x) \geq \mathbf{P}_{x_0}\left(Y_{\tau_{D\setminus\overline{D}_1}} \in \overline{D}_1\right)$$

$$= \mathcal{A}(d,-\alpha)\int_{\mathbf{R}_+^d\setminus S}\int_S \frac{G_S^X(x_0,z)\mathbf{E}_{x_0}^z e_k(\tau_S)\,\kappa(y,z)}{|y-z|^{d+\alpha}}\,\mathrm{d}z\mathrm{d}y,$$

and therefore, since there is S_1 which depends only on d such that for every $z \in S$, $|B(z,S_1)\cap D\setminus \overline{D}_1| > 1$,

$$c_8\delta_D^{\alpha-1}(x) \geq S_1^{-d-\alpha}\mathcal{A}(d,-\alpha)\mathbf{E}_{x_0}\tau_S^Y \geq S_1^{-d-\alpha}\mathcal{A}(d,-\alpha)\mathbf{E}_{x_0}\tau_{\Delta_0}^Y$$

However, since $B_1 \subset \Delta_0$, we obtain

$$\mathbf{E}_{x_1}\tau_{B_1}^Y \geq \mathbf{E}_{x_1}\tau_{B_1}^X \geq C(d,\alpha)2^{-\varepsilon k_0\alpha}.$$

Combining the By the previous equations, we see that there is a number $C = C(d,\alpha,\gamma,c_1,c_2)$ such that

$$u(x_1) \geq C2^{-k_0(\alpha\varepsilon+1-\alpha)}u(x_0) = Ce^{k_0(1-\alpha)/2}u(x_0).$$

because of the choice of ε that has been made previously which implies that $\alpha\varepsilon+1-\alpha = (1-\alpha)/2$.

Next, let's cover the case where $E_2 \leq E_1$. Then

$$u(x_0) \leq 2\mathbf{E}_{x_0}\left[Y_{\tau_{\Delta_0}} \in 2\Delta_0; u\left(Y_{\tau_{\Delta_0}}\right)\right] \leq 2\sup_{y\in\Delta_0} u(y)\mathbf{P}_{x_0}\left[y_{\tau_{\Delta_0}} \in 2\Delta_0\right]$$

An inspection of the proof of the previous theorem shows that for $x \in D$,

$$\mathbf{P}_x\left(Y_{\tau_{\Delta(0,1,1)}} \in \Delta(0,2,1)\right) \leq \mathbf{P}_x\left(Y_{\tau_{\Delta(0,1,\infty)}} \in D\right) \leq C_{10}\left(x_d^{\alpha-1}\wedge 1\right),$$

so that the previous two theorems,

$$\begin{aligned}\mathbf{P}_{x_0}\left(Y_{\tau_{\Delta_0}} \in 2\Delta_0\right) &\leq \mathbf{P}_{x_0}\left(Y_{\tau_{\Delta_0}} \in D\right) \leq C_8C_{10}\left(x_{0d}/2^{-\varepsilon k_0}\right)^{\alpha-1}\\ &\leq C_8C_{10}2^{-k_0(\alpha^2-1)/(2\alpha)}.\end{aligned}$$

Thus $u(x_0) \leq 2C_8C_{10}2^{-k_0(\alpha^2-1)/(2\alpha)}\sup_{y\in 2\Delta_0}u(y)$, and we may conclude that there exists $x_1 \in 2\Delta_0$ such that

$$u(x_1) \geq \frac{2^{k_0(\alpha^2-1)/(2\alpha)}}{4C_8C_10}u(x_0).$$

Note that

$$\operatorname{dist}(x_0, 2\Delta_0) \le 2^{-\varepsilon k_0} 2\sqrt{2},$$

so that in both cases we have treated, any x_1 satisfies

$$|x_1 - x_0| \le 2^{-\varepsilon k_0} 2\sqrt{2}.$$

Moreover, for large k_0, the estimates that we have derived in both cases can be combined to a single one,

$$u(x_1) \ge 2^{k_0(\alpha-1)/4} u(x_0) > 2^{M(k_0 + k_0(\alpha-1)/(4M))} u(A),$$

that is,

$$u(x_1) \ge 2^{k_1 M} u(A),$$

where k_1 is the smallest integer exceeding $k_0 + k_0(\alpha-1)/(4M) - 1$. At the same time, we may choose to enlarge k_0 until $k_1 \ge k_0 + 1$.

The result now follows from induction. Given a point $x_n \in \Delta(0,2,2)$ and an integer $k_n > k_{n-1}$ by repeating the steps above with x_n and k_n replaced by x_1 and k_1, we obtain that

$$u(x_{n+1}) \ge 2^{Mk_{n+1}} u(A),$$

where $k_{n+1} \ge k_n \ge k_n + n$. Moreover,

$$|x_{n+1} - x_n| \le 2^{-k_n(\alpha-1)/2} 2\sqrt{2} \le 2^{-k_0(\alpha-1)/2} 2^{-(n+1)(\alpha-1)/2} 2\sqrt{2}.$$

Summing both sides of this inequality yields

$$|x_n - x_0| \le \sum_{j=1}^{\infty} |x_j - x_{j-1}|,$$

and if k_0 is large enough, then the sum of the series is smaller than $2\sqrt{2}$ so that $x_n \in \Delta(0, 3/2, 3/2)$ for all n. This, in combination with the result we had derived at the beginning of the proof,

$$u(x) \ge 2^{kM} u(A)$$

for $x \in \Delta(0,1,1)$ with $x_d \ge 2^{-k}$, shows that $x_{nd} \le 2^{-k_n}$ for all n which contradicts the continuous decay of u at F. We conclude that $u(x) < 2^{Mk_0}$ for $x \in \Delta(0,1,1)$. $\square$

We can now, finally, prove the boundary Harnack inquality for censored stable-like processes in the upper half space:

Theorem 3.15. *Assume that $u \geq 0$ on $\mathbf{R}^d$ and $u = 0$ on D^c. Furthermore, assume that u is regular harmonic on $\Delta(0,4,4)$ for the censored process Y on D, that is,*

$$u(x) = \mathbf{E}_x \left[u \left(Y_{\tau_{\Delta(0,4,4)}} \right); Y_{\tau_{\Delta(0,4,4)}} \in D \right]$$

Let $A = (\tilde{0}, 1/2) \in \Delta(0,1,1)$. Then there is C_{12} which depends only on d, α, and γ such that

$$C_{12}^{-1} u(A) \delta_D^{\alpha-1}(x) \leq u(x) \leq C_{12} u(A) \delta_D^{\alpha-1}(x), \qquad x \in \Delta(0,1,1).$$

Proof. We will first assume that $u(x)$ takes on the specific form

$$u(x) = \begin{cases} \mathbf{P}_x \left(Y_{\tau_{\Delta(0,3,3)}} \in T \right), & x \in D \\ 0, & x \in D^c \end{cases},$$

where $T \subset \Delta(0,3,3)$. For $x \in \Delta(0,2,2)$, by Theorem 3.13 and the fact that for $x \in D$,

$$\mathbf{P}_x \left(Y_{\tau_{\Delta(0,1,1)}} \in \Delta(0,2,1) \right) \leq \mathbf{P}_x \left(Y_{\tau_{\Delta(0,1,\infty)}} \in D \right) \leq C \left(\delta_D(x) \wedge 1 \right),$$

it follows that

$$\begin{aligned} u(x) &= \mathbf{P}_x \left(Y_{\tau_{\Delta(0,3,3)}} \in T \right) \leq \mathbf{P}_x \left(Y_{\tau_{\Delta(x,1,1)}} \in D \right) \\ &\leq C_8 \mathbf{P}_x \left(Y_{\tau_{\Delta(x,1,1)}} \in \Delta(x,2,1) \right) \leq C_8 \mathbf{P}_x \left(Y_{\tau_{\Delta(x,1,\infty)}} \in D \right) \\ &\leq C_8 C_{10} \left(\delta_D^{\alpha-1}(x) \wedge 1 \right) \end{aligned}$$

Therefore, $u(x)$ decays continuously at the bottom part of $\partial\Delta(0,2,2)$. Using the string Markov property of Y, we obtain

$$\begin{aligned} u(x) &= \mathbf{E}_x \left[u \left(Y_{\tau_{\Delta(0,1,1)}} \right); Y_{\tau_{\Delta(0,1,1)}} \in \Delta(0,2,1) \right] \\ &\quad + \mathbf{E}_x \left[u \left(Y_{\tau_{\Delta(0,1,1)}} \right); Y_{\tau_{\Delta(0,1,1)}} \in \Delta(0,3,2) \setminus \Delta(0,2,1) \right] \\ &\quad \mathbf{E}_x \left[u \left(Y_{\tau_{\Delta(0,1,1)}} \right); Y_{\tau_{\Delta(0,1,1)}} \in D \setminus \Delta(0,3,2) \right] \end{aligned}$$

$$=: \quad E_1 + E_2 + E_3.$$

For the moment, let us assume that $x = (\tilde{0}, x_d)$ with $0 < x_d < 1$. By the Harnack principle, there exists C which depends only on d and α such that

$$C^{-1}u(A) \le u(y) \le Cu(A), \qquad y \in \Delta(0,2,1) \setminus \Delta(0,1,1).$$

Therefore, by Theorem 3.13 and the estimate

$$c_1^{-1}\left(x_d^{\alpha-1} \wedge 1\right) \le u^Y(x) \le c_1\left(x_d^{\alpha-1} \wedge 1\right), \qquad x \in D \tag{3.19}$$

we deduce that

$$\begin{aligned} u(x) \;&\ge\; E_1 \ge C^{-1}\mathbf{P}_x\left(Y_{\tau_{\Delta(0,1,1)}} \in \Delta(0,2,1)\right)u(A) \\ &\ge\; C^{-1}C_8^{-1}\mathbf{P}_x\left(Y_{\tau_{\Delta(0,1,1)}} \in D\right)u(A) \\ &\ge\; C^{-1}C_8^{-1}\mathbf{P}_x\left(Y_{\tau_{\Delta(0,1,\infty)}} \in D\right)u(A) \\ &\ge\; C_1'\left(\delta_D^{\alpha-1}(x) \wedge 1\right)u(A) \end{aligned}$$

where the constant C_1' depends on d, α, and γ.

To prove the reverse inequality, we need to estimate each of the three terms E_1, E_2, and E_3. First, we can use again equation (3.19) to deduce

$$E_1 \ge C\mathbf{P}_x\left(Y_{\tau_{\Delta(0,1,1)}} \in \Delta(0,2,1)\right)u(A) \le C_2'\left(\delta_D^{\alpha-1}(x) \wedge 1\right)u(A)$$

where the constant C_2' depends on d, α, and γ. Second, By the Carleson estimate er have derived previously, scaling, and Theorem 3.13, we get

$$E_2 \le C_3'\mathbf{P}_x\left(Y_{\tau_{\Delta(0,1,1)}} \in D\right)u(A) \le C_3'C_8C_{10}\left(x_d^{\alpha-1} \wedge 1\right)u(A).$$

Finally, since $S \subset \Delta(0,3,3)$, we set

$$I(z) = \mathcal{A}(d,-\alpha)\int_{\mathbf{R}_+^d \setminus \Delta(0,3,2)} \frac{u(y)\kappa(z,y)}{|y-z|^{d+\alpha}}\,\mathrm{d}y, \qquad z \in \Delta(0,3,2).$$

One can easily check that there exists C_4' which depends only on d and α such that

$$C_4'^{-1} I(z) \le I(A) \le C_4' I(z), \qquad z \in \Delta(0, 5/2, 3/2).$$

Therefore,

$$E_3 \le C_4' \mathbf{E}_x \left[\tau^Y_{\Delta(0,1,1)}\right] I(A).$$

For every $w \in \Delta(0,2,1) \setminus \Delta(0,1,1)$, define $B = B(w, 1/4) \subset \Delta(0, 5/2, 3/2)$. Therefore,

$$u(z) \ge C_4'^{-1} I(A) \mathbf{E}_z \left[\tau^Y_B\right] \ge C_4'^{-1} I(A) \mathbf{E}_z \tau^X_B = C_6 I(A)$$

for some constant $C_6 = C_6(d, \alpha)$. Next, there is C_5' which depends only on d, α, and γ such that

$$J(z) = \mathcal{A}(d, -\alpha) \int_{\Delta(0,2,1)\setminus\Delta(0,1,1)} \frac{\kappa(y,z)}{|y-z|^{d+\alpha}} \, \mathrm{d}y \ge C_5', \qquad z \in \Delta(0,1,1),$$

so the previous equations imply that

$$E_1 \ge \mathbf{E}_x \tau^Y_{\tau_{\Delta(0,1,1)}} C_5' C_4'^{-1} I(A) \ge C_7 E_3.$$

Combining this with the estimates on E_1 and E_2, we see that

$$u(x) = E_1 + E_2 + E_3 \ge C \left(\delta_D^{\alpha-1}(x) \wedge 1\right) u(A).$$

This yields that there exists some $C_{12} = C_{12}(d, \alpha, \gamma) > 0$ such that for all $x = (\tilde{0}, x_d)$ with $0 < x_d < 1$,

$$C_{12}^{-1} u(A) \left(\delta_D^{\alpha-1} \wedge 1\right) \le u(x) \le C_{12} u(A) \left(\delta_D^{\alpha-1} \wedge 1\right)$$

Finally, we can use functions of this type to approximate any harmonic function since it is immediate that the result holds for positive linear combinations of functions

$$u_i(x) = \mathbf{P}_x \left(Y_{\tau_{\Delta(0,3,3)}} \in T_i\right).$$

where $T_i \subset D \setminus \Delta(0,3,3)$ is measurable. Using the convential Harnack inequality we can replace the interior point A by any point y which satisfies $|\tilde{y}| \ge 1$ and $y_d = 1/2$, possibly by adjusting the constant C_{12}. We can now obtain the final result by applying the equation above in $\Delta(0,3,3)$ with $x \in \Delta(0,1,1)$.

$\square$

Finally, we prove the main theorem.

Proof of Theorem 3.1. We will first assume that $d \geq 2$. Since u decays continuously at $\partial D \cap B(Q, r)$, it follows that u is bounded and regular harmonic for Y_t in the region $D \cap B(0, 3r/4)$. As before, we will assume without loss of generality that $Q = 0$ and $r = 1$. Finally, by an isometric mapping of D, we may assume that

$$D \cap B(0,1) = D_0 \cap B(0,1)$$

where D_0 now is a *special* $C^{1,1}$-domain with defining functions Γ which satisfies

$$\|\nabla\Gamma\|_\infty \leq \Lambda, \qquad \|\Gamma\|_{1,1} \leq \Lambda.$$

By the Harnack inequality it sufficed to verify that if $u \geq 0$ in $\mathbf{R}^d$, $u = 0$ in D^c, and there are constants $a = a(d, \alpha, \Lambda, \kappa_1, \kappa_2)$ and $c = c(d, \alpha, \Lambda, \kappa_1, \kappa_2)$ such that

$$u(x) = \mathbf{E}_x\left[u\left(Y_{\tau_{\Delta(0,17a,17a)}}\right)\right], \qquad x \in D.$$

implies

$$c^{-1}u(A)\delta_D^{\alpha-1} \leq u(x) \leq cu(A)\delta_D^{\alpha-1}(x), \qquad x \in \Delta(0, a, a), \tag{3.20}$$

where, $A = (\tilde{0}, a/2)$.

To prove equation (3.20), we introduce the sets D_- and D_+ which satisfy

$$D_+ \subset D \subset D_-,$$

which can be expressed as $D_- = D_{\Gamma_-}$ and $D_+ = D_{\Gamma_+}$ where Γ_- and Γ_+ are functions from $\mathbf{R}^{d-1}$ to $\mathbf{R}$, and satisfy

$$\kappa_D(x) - c_1 \leq \kappa_{D_-}(x) \leq \kappa_D(x) \leq \kappa_{D_+}(x) \leq \kappa_D(x) + c_1, \qquad x \in \Delta(0, b, b) \tag{3.21}$$

where $c_1 = c_1(d, \alpha, \Lambda) > 0$ and $b = b(d, \alpha, \Lambda) > 0$ are constants. Next, fix a function $\phi \in \mathcal{C}^\infty\left(\mathbf{R}^{d-1}\right)$ which is nonnegative, supported in $\left\{\tilde{x} \in \mathbf{R}^{d-1} : 3/4 < |\tilde{x}| < 1\right\}$, and satisfies

$$\int_{\mathbf{R}^{d-1}} \phi(\tilde{x})\,\mathrm{d}\tilde{x} = 1.$$

Moreover, define for $j \in \mathbf{N}$

$$\phi_j(\tilde{x}) = 2^{-2j}\phi(2^j\tilde{x}), \qquad \tilde{x} \in \mathbf{R}^{d-1}.$$

It is immediate that $\|\phi_j\|_{1,1} = \|\phi\|_{1,1}$ and

$$\int_{\mathbf{R}^{d-1}} \phi_j(\tilde{x})\,\mathrm{d}\tilde{x} = 2^{-j(d+1)}. \tag{3.22}$$

Set $\Gamma_j = \Gamma - \phi_j$ and $D = D_{\Gamma_j}$ and note that $D_0 \subset D_{\Gamma_j}$. Let $x \in D \cap B(0, 2^{-j-1}$. We write

$$\kappa_D(x) - \kappa_{D_j}(x) = (\kappa_D(x) - \kappa_{D_0}(x)) + \left(\kappa_{D_0}(x) - \kappa_{D_j}(x)\right).$$

to get an estimate for the left hand side. For the first term, note that since $D \cap B(0,1) = D_0 \cap B(0,1)$, we see that there exists $c_2 = c_2(d, \alpha, \kappa_1, \kappa_2)$ such that

$$-c_2 \le \kappa_D(x) - \kappa_{D_0}(x) \le c_2. \tag{3.23}$$

For the second term, (3.22) implies that

$$\begin{aligned} \kappa_{D_0}(x) - \kappa_{D_j}(x) &= \int_{D_0^c \setminus D_j^c} \frac{\kappa(x,y)}{|y-x|^{d+\alpha}}\,\mathrm{d}y \\ &\ge c_3^{-1} 2^{-j(d+1)} 2^{-j(-d-\alpha)} = c_3^{-1} 2^{j(\alpha-1)} \end{aligned}$$

and

$$\kappa_{D_0}(x) - \kappa_{D_j}(x) \le c_3 2^{j(\alpha-1)}$$

for some $c_3 = c_3(d, \alpha, \Lambda, \kappa_1, \kappa_2)$. By equation (3.23), we may choose $j = j(d, \alpha, \Lambda, \kappa_1, \kappa_2)$ such that the first two inequalities in equation (3.21) are still valid in the larger set $D \cap B(0, 2^{-j-1})$ for $\Gamma_j =: \Gamma_-$.

Similarly, we can choose j such that if $\Gamma_+ := \Gamma + \phi_j$, then the last two inequalities in equation (3.21) are still valid in the larger set $D \cap B(0, 2^{-j-1})$. Note that with these choices,

$$\|\Gamma_-\|_{1,1} \le \|\Gamma\|_{1,1} + \|\phi\|_{1,1}, \qquad \|\nabla\Gamma_-\|_\infty \le \|\nabla\Gamma\|_\infty + \|\nabla\phi\|_\infty$$

which provides us with control of the characteristics of D_- in terms of the characteristics of D. A similar statement holds for D_+ and D. Note that $\Gamma_- \le \Gamma \le \Gamma_+$ and that

$$\Gamma_-(\tilde{x}) = \Gamma(\tilde{x}) = \Gamma_+(\tilde{x}), \qquad |\tilde{x}| \le 2^{-j-1},$$

and hence the domains D_-, D, and D_+ coincide near 0; more precisely, the boxes $\Delta(0,s,s)$ defined by the graphs Γ_-, Γ, and Γ_+ are identical if $s \leq 2^{-j-1}$. Finally, we choose $b = 2^{-j-1}/\left(2\sqrt{\Lambda^2+1}\right)$ which guarantees $\Delta(0,b,b) \subset D \cap B(0,2^{-j-1})$.

Let $Y-_t$ and Y_t^+ denote the censored processes on D^- and D^+, respectively. We will consider two cases:

Case 1. Let $a := b/19 \wedge r/26$. Note that u is bounded in $\Delta(0,19a,19a)$. We will furthermore assume that $u = 0$ on $D \setminus \Delta(0,19a,19a)$. Let

$$U(x) := \begin{cases} \mathbf{E}_x\left[u\left(Y^+_{\tau_{\Delta(0,17a,17a)}}\right)\right] & x \in D_+ \\ 0 & x \in D_+^c \end{cases}.$$

By equation (3.21) and the fact that the censored process Y_t on a set E can be expressed in terms of the killed process X_t^E and the Faynman-Kac transform by (see Theorem 2.1 and Remark 2.4 of [7])

$$Y_t = e^{\int_0^t \kappa_E(X_s^E)\,\mathrm{d}s},$$

we see that

$$u(x) \leq U(x), \qquad x \in \mathbf{R}^d.$$

By the previous theorem, Theorem 3.15, and scaling applied to U we see that U, and hence u, decay continuously at the boundary ∂D in a neighborhood of the closure of $\Delta(0,4a,4a)$. Consequently, u is continuous in a neighborhood of $\Delta(0,4a,4a)$.

Next, define u_- and u_+ by

$$u_\pm(x) = \begin{cases} \mathbf{E}_x\left[u\left(Y^\pm_{\tau_{\Delta(0,4a,4a)}}\right)\right] & x \in D_\pm \\ 0 & x \in D_\pm^c \end{cases}.$$

The continuity of u in a neighborhood of $\Delta(0,4a,4a)$ implies that u_- and u_+ are continuous in the closure of $\Delta(0,4a,4a)$. The regularity of points in the bottom part of $\partial\Delta(0,4a,4a)$ follow from Theorem 3.15.

As before, the representation of the censored process with the Feynman-Kac transform as well as equation (3.21) imply

$$u_-(x) \leq u(x) \leq u_+(x), \qquad x \in \mathbf{R}^d, \tag{3.24}$$

whence Theorem 3.15 and scaling let us deduce that there is $c_4 = c_4(d, \alpha, \Lambda, \kappa_1, \kappa_2)$ such that

$$c_4^{-1} u_-(A) \delta_D^{\alpha-1}(x) \leq u(x) \leq c_4 u_+(A) \delta_D^{\alpha-1}(x), \qquad x \in \Delta(0, a, a),$$

where $A = (\tilde{0}, 1/2)$.

We therefore see that to prove equation (3.20), it sufficed to verify the existance of some $c_5 = c_5(d, \alpha, \Lambda, \kappa_1, \kappa_2)$ such that

$$u_+(A) \leq c_5 u_-(A). \tag{3.25}$$

Set $\bar{u} = u_+ - u_-$. Note that $\bar{u} \in \mathcal{C}_0(\Delta(0, 4a, 4a))$ and $\bar{u} \geq 0$. Let

$$\bar{u}(x_0) = \max_{x \in \Delta(0,4a,4a)} \bar{u}(x).$$

Then,

$$\begin{aligned} A_D^{\alpha,\kappa} u(x_0) &= \mathcal{A}(d, -\alpha) P.V. \int_D \frac{\bar{u}(y) - \bar{u}(x_0)}{|y-x|^{d+\alpha}} \kappa(x_0, y) \, \mathrm{d}y \\ &\leq -\bar{u}(x_0) \mathcal{A}(d, -\alpha) \kappa_1 \int_D \frac{1}{|y-x|^{d+\alpha}} \, \mathrm{d}y \\ &\leq -u(x_0) c_6 a^{-\alpha} \end{aligned} \tag{3.26}$$

where $c_6 = c_6(d, \alpha, \Lambda, \kappa_1)$. Conversely, we can write

$$A_D^{\alpha,\kappa} u(x_0) = \big(\kappa_D(x_0) - \kappa_{D_+}(x)\big) u_+(x_0) - \big(\kappa_D(x_0) - \kappa_{D_-}(x_0)\big) u_-(x_0).$$

Using equations (3.21), (3.24), and the Carelon estimate, Proposition 3.14, we see that there is $c_7 = c_7(d, \alpha, \Lambda) > 0$ such that

$$A_D^{\alpha,\kappa} u(x_0) \geq -2c_1 u_+(x_0) \geq -c_7 u_+(A).$$

Therefore, equation (3.26) implies

$$\bar{u}(x_0) \geq c_7 c_6^{-1} u_+(A) a^{\alpha}.$$

By choosing $a = a(d, \alpha, \Lambda)$ sufficiently small, we have that

$$\bar{u}(A) \leq \bar{u}(x_0) \leq u_+(A)/2,$$

and

$$u_-(A) = u_+(A) - \bar{u}(A) \geq u_+(A)/2.$$

This is the estimate we required in (3.25) and therefore completes the proof in case 1.

Case 2. Now, assume that $u = 0$ on $\Delta(0, 18a, 18a) \setminus \Delta(0, 17a, 17a)$. As before, we may assume that $u \geq 0$ on $\mathbf{R}^d$ and

$$u(x) = \begin{cases} \mathbf{E}_x \left[u\left(Y^+_{\tau_{\Delta(0,17a,17a)}} \right) \right] & u \in D \\ 0 & x \in D^c \end{cases}.$$

The constant $a = a(d, \alpha, \Lambda)$ is as determined in the end of case 1. As in the proof of the Carleson estimate, Theorem 3.14, we see that there exists $c_8 = c_8(d, \alpha, \Lambda, \kappa_1, \kappa_2) > 1$ such that

$$c_8^{-1} u(x) \leq \mathbf{E}_x \left[\tau^Y_{\Delta(0,17a,17a)} \right] \int_{\Delta(0,17a,17a)^c} \frac{u(y)}{(1 + |y|)^{d+\alpha}} \,\mathrm{d}y \leq c_8 u(x). \tag{3.27}$$

Therefore, there exists $c_9 = c_9(d, \alpha, \Lambda, \kappa_1, \kappa_2) \geq 1$ such that

$$\begin{aligned} c_9^{-1} \mathbf{E}_x \left[\tau^Y_{\Delta(0,17a,17a)} \right] &\leq \mathbf{P}_x \left(Y_{\tau_{\Delta(0,17a,17a)}} \in \Delta(0, 19a, 19a) \setminus \Delta(0, 18a, 18a) \right) \\ &\leq c_9 \mathbf{E}_x \left[\tau^Y_{\Delta(0,17a,17a)} \right]. \end{aligned} \tag{3.28}$$

Now, the result from case 1 shows that (3.20) can be applied to the function

$$x \mapsto \mathbf{P}_x \left(Y_{\tau_{\Delta(0,17a,17a)}} \in \Delta(0, 19a, 19a) \setminus \Delta(0, 18a, 18a) \right).$$

Along with equations (3.27) and (3.28), this implies that there is a $c_{10} = c_{10}(d, \alpha, \Lambda, \kappa_1, \kappa_2) \geq 1$ such that

$$c_{10}^{-1} \delta_D^{\alpha-1}(x) \int_{\Delta(0,17a,17a)^c} \frac{u(y)}{(1 + |y|)^{d+\alpha}} \,\mathrm{d}y \leq u(x) \leq c_{10} \delta_D^{\alpha-1}(x) \int_{\Delta(0,17a,17a)^c} \frac{u(y)}{(1 + |y|)^{d+\alpha}} \,\mathrm{d}y.$$

In particular, this holds for $x = A$, that is, by including c_{10} and $\delta_D^{\alpha-1}(A)$ into a new constant c_{11},

$$c_{11}^{-1} \int_{\Delta(0,17a,17a)^c} \frac{u(y)}{(1 + |y|)^{d+\alpha}} \,\mathrm{d}y \leq u(x) \leq c_{11} \int_{\Delta(0,17a,17a)^c} \frac{u(y)}{(1 + |y|)^{d+\alpha}} \,\mathrm{d}y.$$

These last two equtions imply equation (3.20). This equation will hold for general u be combining both cases. $\square$

3.2 Examples (in construction)

In this section, we will show that the $\alpha/2$-subordination of reflected Brownian motion is a reflected stable-like process. The function $\kappa(x,y)$ associated to this process satisfies the conditions of the boundary Harnack principle derived in the previous section. Let B_t be reflected Brownian motion on a $\mathcal{C}^{1,1}$ domain D, and S_t be a $\alpha/2$-subordinator. Recall that this process is defined by

$$Y_t = B_{S_t}$$

see for example [21]. One can show that the jumping kernel of Y_t is given by

$$J(\mathrm{d}x, \mathrm{d}y) = \frac{1}{2}\mathrm{d}x \int_0^\infty p(s,x,dy)\nu(\mathrm{d}s),$$

if the subordinator S_t is an increasing Lévy process taking values in $[0,\infty)$ with $S_0 = 0$ and the distribution of the process is characterized by

$$\mathbf{E}\left[e^{-\lambda S_t}\right] = e^{-t\phi(\lambda)},$$

where ϕ satisfies

$$\phi(\lambda) = b\lambda + \int_0^\infty \left(1 - e^{\lambda t}\right)\nu(\mathrm{d}t).$$

Here $p(t,x,y)$ is the transition density function of reflected Brownian motion on D.

For an $\alpha/2$-subordinator, $b = 0$ and

$$\nu(\mathrm{d}t) = \frac{\mathrm{d}t}{t^{1+\alpha/2}},$$

so the jumping measure has density

$$J(x,y) = \frac{1}{2}\int_0^\infty p(s,x,y)t^{-1-\alpha/2}\,\mathrm{d}t.$$

Finally, recall that for stable-like process,

$$J(x,y) = \frac{\kappa(x,y)}{|x-y|^{d+\alpha}},$$

so we get the following formula for $\kappa(x,y)$:

$$\kappa(x,y) = \frac{|x-y|^{d+\alpha}}{2} \int_0^\infty p(s,x,y) t^{-1-\alpha/2}\,\mathrm{d}t.$$

To show that $\kappa(x,y)$ satisfies the required properties, we will use the following heat kernel estimates for $p(t,x,y)$ which are given in [28, Theorem 3.10] in a more general setting.

Theorem 3.16. *Neumann Heat Kernel Estimates for Reflected Brownian Motion Let D be a Lipschitz domain and $\overline{B}_t$ be reflected Brownian motion on D. Let $p(t,x,y)$ be its transition density function. Then there exists $c_i > 0$, $i = 1,2,3,4$ such that for all $(t,x,y) \in (0,\infty) \times D \times D$,*

$$\frac{c_1}{m(B(x,\sqrt{t}) \cap D)} e^{c_2|x-y|^2/t} \le p(t,x,y) \le \frac{c_3}{m(B(x,\sqrt{t}) \cap D)} e^{c_4|x-y|^2/t}.$$

The result was proved for special Lipschitz domains in [5].

Therefore,

$$\begin{aligned}
\kappa(x,y) \quad &= \quad |x-y|^{d+\alpha} \int_0^\infty p(t,x,y) t^{-1-\alpha/2}\,\mathrm{d}t \\
&\ge \quad c_1 |x-y|^{d+\alpha} \int_0^\infty \frac{c_1}{m(B(x,\sqrt{t}) \cap D)} e^{c_2|x-y|^2/t} t^{-1-\alpha/2}\,\mathrm{d}t \\
&\ge \quad c_1 |x-y|^{d+\alpha} \int_0^\infty \frac{c_1}{m(B(x,\sqrt{t}))} e^{c_2|x-y|^2/t} t^{-1-\alpha/2}\,\mathrm{d}t \\
&\overset{s=c_2|x-y|^2/t}{=} \quad \frac{\omega_{d-1} c_1}{d} |x-y|^{d+\alpha} \left(\int_0^\infty \left(\frac{|x-y|^2}{c_2^{-1} s} \right)^{-d/2} e^{-s} \left(\frac{|x-y|^2}{c_2^{-1} s} \right)^{-1-\alpha/2} \frac{|x-y|^2}{c_2^{-1} s^2}\,\mathrm{d}s \right) \\
&= \quad \frac{\omega_{d-1} c_1}{d} c_2^{-(d+\alpha)/2} \int_0^\infty s^{(d+\alpha)/2-1} e^{-s}\,\mathrm{d}s \\
&= \quad \frac{\omega_{d-1} c_1}{d} c_2^{-(d+\alpha)/2} \Gamma\left(\frac{d+\alpha}{2} \right) > 0.
\end{aligned}$$

3.3 Dirichlet Heat Kernel Estimates for Censored Stable-like Processes

In the previous chapter, we established the boundary Harnack principle for symmetric stable-like processes on $\mathbf{R}^d$ and used it to derive Dirchlet heat kernel estimates for the associated killed process. Analogously, we will use the boundary Harnack principle for censored stable-like processes for the previous section to prove Dirichlet heat kernel estimates for killed censored stable-like processes on a domain D.

Theorem 3.17. *Let $E \subset \mathbf{R}^d$ be a $\mathcal{C}^{1,1}$-domain and fix a subset κ-fat subset $E \subset D$. Let Y_t^D be a censored stable-like process of index $1 < \alpha < 2$ which is being killed upon leaving E. Suppose that for some $\gamma > \alpha - 1$ and some function ϕ we have*

$$\begin{gathered} 0 < \kappa_1 \leq \kappa(x,y) \leq \kappa_2 < \infty \\ |\kappa(x,y) - \kappa(x,x)| \leq C_\kappa |x-y|^\gamma \\ |\kappa(x,y) - \kappa(x,x) - \phi(x,y)\tfrac{|x-y|^{d+\alpha}}{|\bar{x}-y|^{d+\alpha}}| \leq C_{\kappa,\phi}|x-y|^\gamma \\ |\phi(x,x) - \phi(x,y)| \leq C_\phi |x-y|^\gamma \\ |\phi(x,x)| \leq \kappa_2. \end{gathered}$$

Fix $T > 0$. Then there exists $C_1 = C_1(d, \alpha, \gamma, \kappa_1, \kappa_2, C_\kappa, C_\phi, C_\kappa, \Lambda, r_0, T) > 0$ such that the transition density function $p_D(t,x,y)$ of the killed censored stable-like process satisfies that for all $(t,x,y) \in (0,T] \times D \times D$,

$$\begin{aligned} & C_1^{-1} \mathbf{P}_x(\tau_D > t)\, \mathbf{P}_y(\tau_D > t) \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right) \\ \leq \;\; & p_D(t,x,y) \leq C_1 \mathbf{P}_x(\tau_D > t)\, \mathbf{P}_y(\tau_D > t) \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right). \end{aligned}$$

For the proof, we need the analogue of several lemmata that were available for symmetric stable-like processes. First, we obtain a variant of the exit time estimate from Proposition 1.6.

Proposition 3.18. *Let Y_t be a censored stable-like process of index $\alpha \in (0,2)$ on a subset $D \subset \mathbf{R}^d$. Given $a > 0$ and $r_0 > 0$, there exists $C_2 = C_2(d, \alpha, \kappa_1, \kappa_2, a, r_0) > 0$ such that for every $x \in D$ and $r \in (0, r_0]$ with $B(x,r) \subset D$,*

$$\inf_{y \in D \cap B(x,r/2)} \mathbf{P}_y\left(\tau_{B(x,r)} > a r^{1/\alpha}\right) \geq C_2.$$

Proof. We provide the proof given in [14, Lemma 4.1] for the convenience of the reader. Denote by X_t the symmetric α-stable like process on $\mathbf{R}^d$, and for any set U the killed process by X_t^U. Denote the transition density function of the killed process by $p_U(t,x,y)$.

Theorem 2.1 in [7] shows that

$$\begin{aligned}
\inf_{y\in B(z,r/2)} \mathbf{P}_y\left(\tau_{B(x,r)} > ar^{1/\alpha}\right) &\geq \inf_{y\in B(z,r/2)} \mathbf{P}_y\left(\tau^X_{B(x,r)} > ar^{1/\alpha}\right) \\
&\geq \inf_{y\in \mathbf{R}^d} \mathbf{P}_y\left(\tau^X_{B(x,r)} > ar^{1/\alpha}\right).
\end{aligned}$$

The exit time estimate for the symmetric stable-like process, Proposition 1.6, implies that there exists $\varepsilon > 0$ such that

$$\inf_{y\in \mathbf{R}^d} \mathbf{P}_y\left(\tau^X_{B(x,r)} > \varepsilon r^{1/\alpha}\right) \geq \frac{1}{2}.$$

Let $a > \varepsilon$. By the parabolic Harnack inequality for the symmetric stable-like process, Proposition 1.7, there is $c_1 = c_1(d,\alpha,\kappa_1,\kappa_2,a,r_0) > 0$ such that

$$c_1 p^X_{B(x,r)}(\varepsilon r^{1/\alpha}, y, w) \leq p^X_{B(x,r)}(\varepsilon r^{1/\alpha}, y, w), \qquad w \in B(y, r/2).$$

Therefore,

$$\begin{aligned}
\mathbf{P}_y\left(\tau^X_{B(x,r)} > ar^{1/\alpha}\right) &= \int_{B(x,r)} p^X_{B(x,r)}(ar^{1/\alpha}, y, w)\,\mathrm{d}w \\
&\geq \int_{B(x,r/2)} p^X_{B(x,r)}(ar^{1/\alpha}, y, w)\,\mathrm{d}w \\
&\geq c_1 \int_{B(x,r/2)} p^X_{B(x,r)}(\varepsilon r^{1/\alpha}, y, w)\,\mathrm{d}w \\
&\geq \frac{c_1}{2}.
\end{aligned}$$

This proves the lemma with $C_2 = c_1/2$. □

We can now proceed to prove the theorem

Proof. The proof is virtually identical to the proof of the Dirichlet heat kernel estimates for the killed symmetric stable-like process. We established the boundary Harnack principle for censored stable-like processes in the previous section, see Theorem 3.1 which replaces the boundary Harnack principle for the symmetric stable-like process (Theorem 1.3). The parabolic Harnack inequality follows from the heat kernel estimates, see [13, Theorem 4.1]. The Lévy system still holds for killed censored stable-like processes, see for example [20, Appendix A]. Finally, the exit time estimate in Proposition 3.18 replaces Proposition 1.6. □

BIBLIOGRAPHY

[1] A. Ancona, *Principe der Harnack à la frontière et théoremè de Fatou pour un opérateur elliptique dans un domaine lipschitzien*, Annales de l'institut Fourier **28** (1978), no. 4, 169–213.

[2] R. Bass and K. Burdzy, *A Probabilistic Proof of the Boundary Harnack Inequality*, Seminar on Stochastic Processes, 1990, pp. 1–16.

[3] ———, *A boundary Harnack principle in twisted Hölder domains*, Annals of Mathematics **134** (1991), no. 2, 253–276.

[4] ———, *The boundary Harnack principle for nondeivergence form elliptic operators*, Journal of the London Mathematical Society (2) **150** (1994), no. 1, 157–169.

[5] R. Bass and P. Hsu, *Some potential theory for reflecting Browninan motion in Hölder and Lipschitz domains*, The Annals of Probability **19** (1991), no. 2, 486–508.

[6] K. Bogdan, *The boundary Harnack principle for the fractional Laplacian*, Studia Mathematica **123** (1997), no. 1, 43–80.

[7] K. Bogdan, K. Burdzy, and Z.-Q. Chen, *Censored stable processes*, Probability Theory and Related Fields **127** (2003), no. 1, 89–152.

[8] K. Bogdan and T. Byczkowski, *Probabilistic proof of boundary Harnack principle for α-harmonic functions*, Potential Analysis **11** (1999), no. 2, 135–156.

[9] K. Bogdan, T. Kulczycki, and M. Kwaśnicki, *Estimates and structure of α-harmonic functions*, Probability Theory and Related Fields **140** (2008), no. 3-4, 345–380.

[10] K. Bogdan, T. Kumagai, and M. Kwaśnicki, *Boundary Harnack inequality for Markov processes with jumps*, Transactions of the American Mathematical Society **367** (2015), no. 1, 477–517.

[11] Z.-Q. Chen, *On notions of harmonicity*, Proceedings of the American Mathematical Society **137** (2009), no. 10, 3497–3510.

[12] Z.-Q. Chen and M. Fukushima, *Symmetric Markov Processes, Time Change, and Boundary Theory*, Princeton University Press, Princeton, NJ, 2011.

[13] Z.-Q. Chen, P. Kim, and T. Kumagai, *On heat kernel estimates and parabolic harnack inequality for jump processes on metric measure spaces*, Acta Mathematica Sinica, English Series **25** (2009), 1067–1086.

[14] Z.-Q. Chen, P. Kim, and R. Song, *Two-sided heat kernel estimates for censored stable-like processes*, Probability Theory and Related Fields **146** (2010), no. 3/4, 361–399.

[15] ______, *Sharp heat kernel estimates for relativistic stable processes in open sets*, Annals of Probability **40** (2012), no. 1, 213–244.

[16] ______, *Dirichlet heat kernel estimates for rotationally symmetric Lévy processes*, Proceedings of the London Mathematical Society **109** (2014), no. 1, 90–120.

[17] ______, *Dirichlet heat kernel estimates for subordinate Brownian motions with Gaussian components*, Journal für die Angewandte und Reine Mathematik **711** (2016), 111–138.

[18] Z.-Q. Chen, P. Kim, R. Song, and Z. Vondraček, *Boundary Harnack Principle for* $\Delta+\Delta^{\alpha/2}$, Transactions of the American Mathematical Society **364** (2012), no. 8, 4169–4205.

[19] Z.-Q. Chen and T. Kumagai, *Heat kernel estimates for stable-like processes on d-sets*, Stochastic Processes and their Applications **108** (2003), no. 1, 27–62.

[20] ______, *Heat kernel estimates for jump processes of mixed types on metric measure spaces*, Probability Theory and Related Fields **140** (2008), no. 1–2, 277–317.

[21] Z.-Q. Chen and R. Song, *Two-sided eigenvalue estimates for subordinate processes in domains*, Journal of Functional Analysis **226** (2005), 90–113.

[22] B. E. J. Dahlberg, *Estimates of harmonic measuer*, Archive for Rational Mechanics and Analysis **28** (1977), no. 4, 169–213.

[23] E. B. Dynkin, *Markov Processes*, Academic Press, New York, 1965.

[24] L. Evans, *Partial differential equations*, AMS, Providence, RI, 1998.

[25] M. Fukushima, Y. Oshima, and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, De Gruyter, Berlin, 2011.

[26] Q.-Y. Guan, *Boundary Harnack inequalities for regional fractional Laplacian*, 2009. arXiv:0705.1614.

[27] Q.-Y. Guan and Z.-M. Ma, *Reflected symmetric α-stable processes and regional fractional Laplacian*, Probability Theory and Related Fields **134** (2006), no. 4, 649–694.

[28] P. Gyrya and L. Saloff-Coste, *Neumann and Dirichlet Heat Kernel Estimates in Inner Uniform Domains*, Société Mathématique de France, Paris, France, 2011.

[29] K.-Y. Kim and P. Kim, *Two-sided estimates for the transition densities of symmetric Markov processes dominated by stable-like processes in $C^{1,\eta}$ open sets*, Stochastic processes and their applications **124** (2014), no. 9, 3055–3083.

[30] N S Landkof, *Foundations of Modern Potential Theory*, Springer, New York, 1972.

[31] R. Song and J.-M. G. Wu, *Boundary Harnack principle for symmetric stable processes*, Journal of Functional Analysis **168** (1999), no. 2, 403–427.

[32] J.-M. G. Wu, *Comparisons of kernel functions boundary Harnack principle and relative Fatou theorem on Lipschitz domains*, Annales de l'institut Fourier **28** (1978), no. 4, 147–167.