AN ASYMPTOTICALLY 4-STABLE PROCESS

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ABSTRACT

An asymptotically 4-stable process is constructed. The model identifies the 4-stable process with a sequence of processes converging in a very weak sense. It is proved that the 4-th variation of the process is a linear function of time and its quadratic variation may be identified with a Brownian motion.

1. Introduction. The purpose of this article is to develop rigorous mathematical foundations for a model which has been applied in mathematical physics ([MR1], [MR2]).

We will define and examine a process with values in $\mathbf{C}^{\mathbf{N}}$ which is, in an appropriate sense, 4-stable. Suppose that X_t is a stable symmetric process with values in $\mathbf{R}$ and index α, i.e., $\{X(t), t \geq 0\}$ has homogeneous independent increments and $a^{-1/\alpha}X(at)$ has the same distribution as $X(t)$ for each $a > 0$. Then the characteristic function $\psi(u)$ of X_1 satisfies

$$\psi(u) = E\exp(iuX_1) = \exp(c|u|^\alpha).$$

It is known that only $\alpha \in (0, 2]$ correspond to stable processes. The function $\exp(c|u|^\alpha)$ is not a characteristic function for any $\alpha > 2$. However, we will construct a process Z_t with values in $\mathbf{C}^{\mathbf{N}}$ whose increments have, in some sense, 4-stable distributions, i.e., their characteristic functions have the form $\exp(c|u|^4)$. This stability property is achieved in the limit; the n-th component $Z_t(n)$ of Z_t is not 4-stable but when $n \to \infty$, $Z_t(n)$ become 4-stable in an appropriate sense. Our model was inspired by the Mikusiński-Sikorski approach to distributions in which, for example, the delta function is identified

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with any sequence of functions such that their integrals converge to what is considered as the integral of the delta function itself (see, e.g., [A]).

Our process is an attempt to provide a probabilistic model corresponding to

$$\partial_t u = -\partial_x^4 u. \tag{1.1}$$

There is no Markov process which corresponds to "squared Laplacian" in the same sense as Brownian motion corresponds to Laplacian. However, there are several models which attack (1.1) from the probabilistic point of view. Krylov [Kr] and later Hochberg [Ho] considered a signed, finitely additive measure with infinite variation on $C[0,1]$ which may be viewed as the distribution of a process corresponding to (1.1). The most significant disadvantage of the Krylov-Hochberg model is that it uses a finitely additive measure. This leads to paradoxes which are typical in such context, e.g., Hochberg [Ho] shows that his process is, in a sense, Hölder continuous and at the same time it is discontinuous (the contradiction holds only at the intuitive level as the continuity of the trajectory is not a measurable event). More importantly, many standard tools of the probability theory cannot be applied in situations when finitely additive measures are used.

Funaki [Fu] showed that some solutions of (1.1) may be probabilistically represented using the composition of two independent Brownian motions. Recently Burdzy [B1, B2] studied a very similar process under the name of "iterated Brownian motion" or IBM. The main drawback of the IBM model is that the increments of IBM are not independent.

Our model may be presented as a compromise between the Krylov-Hochberg model and IBM. We deal with a genuine probability space and the increments of our process are independent in an asymptotic sense. The lack of an interpretation for our process as the trajectory of a moving particle is the price we have to pay for having the other two desirable features.

It should be mentioned that there exists a completely different probabilistic approach to the "squared Laplacian" problem, see, e.g., Helms [He].

The referee has pointed out to us an article of Sainty [S] which is very close in spirit to ours. Sainty's approach seems to lack mathematical rigour as his Definition 3.1 can only be interpreted as that of an n-stable process with homogeneous, independent increments which does not exist for $n > 2$. It is possible that his analysis can be made rigorous using the same approach as ours. Our model is similar to his in several respects: the complex analytic setting, the use of multiple roots of unity (see our Example 3.4 below) and the use of a weak mode of convergence.

This paper is devoted to the construction and elementary properties of a 4-stable process Z. One of our main tools is a weak version of the characteristic function. Among other things, we will prove that the fourth variation of Z is a linear function of time very much the same as the quadratic variation is a linear function of time for the standard Brownian motion. The quadratic variation of Z may be identified with a Brownian motion. We prove these properties with view towards developing stochastic integrals and an Itô-type formula for Z. A similar program was started in [B2] for the iterated Brownian motion. The mode of convergence in our theorems is rather weak. This is to be expected as one cannot develop a theory of 4-stable processes as strong as that for α-stable processes with $\alpha \leq 2$. Comparable results in [Ho] and [B2] also hold in a very weak sense. Berger and Sloan [BS] showed that one may even go as far as studying differential equations using probabilistic methods without using probability.

Papers of Mądrecki and Rybaczuk [MR1] and [MR2] contain, among other results, an Itô type formula for Z, a representation for the wave function of a Schrödinger equation in terms of Z and the solution of a fourth-order evolution equation using Z.

The paper is organized as follows. Section 3 contains definitions of $\mathbf{C}^{\mathbf{N}}$-valued random elements, their (asymptotic) moments and characteristic functions, and p-stable distributions. A 4-stable motion is constructed and analyzed in Section 4. Section 5 is devoted to higher order variation of this process.

The present article is based to a large extent on an unpublished manuscript of Mądrecki [M]. We are grateful to the refree for pointing out to us the article of Sainty [S] and for many helpful comments.

2. Preliminaries. The sets of all natural, real and complex numbers will be denoted $\mathbf{N}, \mathbf{R}$ and $\mathbf{C}$, resp. It will be convenient to identify the complex plane $\mathbf{C}$ with $\mathbf{R}^2$ and occasionally switch from complex notation to vector notation.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. In this paper we will concentrate on $\mathbf{C}^{\mathbf{N}}$-valued random variables $Z : (\Omega, \mathcal{F}, P) \to (\mathbf{C}^{\mathbf{N}}, \mathcal{B}^{\mathbf{N}})$, where $\mathcal{B}^{\mathbf{N}}$ is the Borel σ-algebra of $\mathbf{C}^{\mathbf{N}}$. Clearly, Z is a $\mathbf{C}^{\mathbf{N}}$-valued random variable iff Z is a sequence $\{Z_n\}$ of complex random variables.

If $Z = \{Z_n\}$ is a $\mathbf{C}^{\mathbf{N}}$-valued random variable and f is a function defined on $\mathbf{C}$ then

$f(Z)$ will stand for the sequence $\{f(Z_n)\}$.

3. Asymptotic moments and distributions.

Definition 3.1. *Let $cL^1(\Omega \times \mathbf{N})$ be the $\mathbf{C}$-linear space of all $\mathbf{C}^{\mathbf{N}}$-valued random variables $Z = \{Z_n\}$ on $(\Omega, \mathcal{F}, P)$ with the following properties:*

(i) for each $n \in \mathbf{N}$, the random variable Z_n belongs to $L^1(\Omega)$, and

(ii) the sequence $\{E(Z_n)\}$ is convergent.

Definition 3.2. *Suppose that $Z = \{Z_n\} \subset cL^1(\Omega \times \mathbf{N})$, i.e., $\lim_{n\to\infty} EZ_n$ exists. The limit will be denoted $\mathcal{E}Z$ and called the first (asymptotic) moment of Z or (asymptotic) expected value of Z. For example, if $Z^k = \{Z_n^k\} \subset cL^1(\Omega \times \mathbf{N})$ for some $k \in \mathbf{N}$, then the k-th (asymptotic) moment of Z is given by $\mathcal{E}Z^k = \lim_{n\to\infty} EZ_n^k$.*

It is easy to check that the map $\mathcal{E} : cL^1(\Omega \times \mathbf{N}) \to \mathbf{C}$ is a $\mathbf{C}$-linear functional on $cL^1(\Omega \times \mathbf{N})$.

Definition 3.3. *We will say that a $\mathbf{C}^{\mathbf{N}}$-valued random variable $Z = \{Z_n\}$ is admissible if the following two conditions are satisfied.*

(i) Suppose that $Z_n = U_n + iV_n = (U_n, V_n)$ and let μ_n be the distribution of V_n. Then μ_n has the two-sided Laplace transform, i.e.,

$$\int_{\mathbf{R}} e^{-tv} \mu_n(dv) < \infty,$$

for every $t \in \mathbf{R}$ and $n \in \mathbf{N}$, and,

(ii) the complex sequence $\{E\exp(itZ_n)\}$ is convergent.

The set of all admissible random variables will be denoted $\mathcal{R}(\Omega, \mathbf{C}^{\mathbf{N}})$.

Suppose that X and X_m are random elements with values in $\mathbf{C}^{\mathbf{N}}$. We will say that X_m converge to X in $L^p_{\mathcal{E}}$ if $\lim_{m\to\infty} \mathcal{E}(X_m - X)^p = 0$.

For each $Z = \{Z_n\} \in \mathcal{R}(\Omega, \mathbf{C}^{\mathbf{N}})$ we define the (asymptotic) characteristic function $\psi_Z(t) = \psi(Z, t)$ of Z by the formula

$$\psi_Z(t) = \mathcal{E}e^{itZ} = \lim_{n\to\infty} E\exp(itZ_n), \quad t \in \mathbf{R}.$$

The distribution of a $\mathbf{C}^{\mathbf{N}}$-valued random variable Z will be denoted $\mathcal{L}_Z$, i.e., $\mathcal{L}_Z(B) = P\{Z^{-1}(B)\}$ for all Borel sets $B \in \mathbf{C}^{\mathbf{N}}$.

Proposition 3.1. *(i) If* $\alpha_1, \alpha_2 \in \mathbf{R}$ *and* Z_1, Z_2 *are two independent random elements from* $\mathcal{R}(\Omega, \mathbf{C}^{\mathbf{N}})$ *then* $\alpha_1 Z_1 + \alpha_2 Z_2$ *also belongs to* $\mathcal{R}(\Omega, \mathbf{C}^{\mathbf{N}})$ *and*

$$\psi(\alpha_1 Z_1 + \alpha_2 Z_2, t) = \psi(Z_1, \alpha_1 t)\psi(Z_2, \alpha_2 t)$$

for $t \in \mathbf{R}$. *Moreover,* $\psi_Z(0) = 1$ *for any* $Z \in \mathcal{R}(\Omega, \mathbf{C}^{\mathbf{N}})$. *If* $Z \in \mathcal{R}(\Omega, \mathbf{C}^{\mathbf{N}})$ *is such that* $Z : \Omega \to \mathbf{R}^{\mathbf{N}}$ *then* $\sup_{t \in \mathbf{R}} |X(t)| = 1$.

(ii) Let $X = \{X_n\}$ *be a sequence of real random variables. If the sequence* $\{X_n\}$ *tends to* Y *in the sense of distribution then* $\psi_X(t) = Ee^{itY}$, *i.e.,* $\psi_X(t)$ *is the characteristic function of* Y *in the classical sense.*

(iii) If $Z \in \mathcal{R}(\Omega, \mathbf{C}^{\mathbf{N}})$ *and* Z *takes values in the space of convergent real sequences, then*

$$\psi_Z(t) = \int \exp(it \lim_{n\to\infty} x_n) d\mathcal{L}_Z(\{x_n\})$$

where $\mathcal{L}_Z$ *is the distribution of* Z.

Proof: The proof is easy and so it is omitted. ■

Proposition 3.1 shows that the (asymptotic) characteristic functions ψ_Z have similar properties to the classical characteristic functions of real-valued random variables. However, the following two examples will demonstrate that ψ_Z does not determine $\mathcal{L}_Z$.

Example 3.1. For each $X \in \mathcal{R}(\Omega, \mathbf{C}^{\mathbf{N}})$ which takes values in the space of all sequences which tend to zero we have $\psi_X(t) = 1$ for all $t \in \mathbf{R}$ (see Proposition 3.1 (iii)).

Example 3.2. Suppose that $X = \{X_n\}$ is a sequence of real random variables with $EX_n = 0$ and $EX_n^2 = 1$ for each n. Let $S(X) = \{S_n(X)\}$ where $S_n(X) = (X_1 + \ldots + X_n)/n^{1/2}$, $n \geq 1$. Then clearly $S(X)$ belongs to $\mathcal{R}(\Omega, \mathbf{C}^{\mathbf{N}})$ and it follows from the Central Limit Theorem and Proposition 3.1 (ii) that

$$\psi_{S(X)}(t) = e^{-t^2/2}, \quad t \in \mathbf{R}.$$

We will write $Z \sim W$ if $\psi_Z(t) = \psi_W(t)$ for all t. The last two examples illustrate the fact that we may have $Z \sim W$ for some $Z \neq W$. Let $\mathcal{L}_Z^a = \mathcal{L}^a(Z)$ denote the set $\{\mathcal{L}_W : W \sim Z\}$. We will call $\mathcal{L}_Z^a$ an "asymptotic distribution" of Z.

Example 3.3. Fix some $z \in \mathbf{C}$ and suppose that for each n, $Z_n = z$ with probability 1. Let δ_z denote the asymptotic distribution $\mathcal{L}_Z^a$ of $Z = \{Z_n\}$.

Suppose that for every n, U_n and V_n are independent standard normal distributions and let $W = \{U_n + iV_n\}$. Then $\mathcal{L}_W^a = \delta_0$. In order to see this, note that $\psi_Z(t) = 1$ if $\mathcal{L}_Z^a = \delta_0$. On the other hand,

$$\psi_W(t) = E\exp(it(U_n + iV_n)) = E\exp(itU_n)E\exp(-tV_n)) = e^{-t^2/2}e^{t^2/2} = 1.$$

Lemma 3.1. *Suppose that $Z = \{Z_n\} \in \mathcal{R}(\Omega, \mathbf{C}^{\mathbf{N}})$ and let $\psi = \psi_Z$ be its characteristic function. Let $\psi_{Z_n}^{(m)}(t)$ denote the m-th derivative of the characteristic function $E\exp(itZ_n)$. Assume that*

(i) $\mathcal{E}|Z|^m < \infty$,

(ii) the sequence $\{\psi_{Z_n}^{(m)}\}$ is uniformly convergent on some open neighborhood of 0, *and*

(iii) the limit $\lim_{n\to\infty} \psi_{Z_n}^{(m-1)}(0)$ exists.

Then ψ_Z has a continuous m-th derivative and

$$\psi^{(m)}(0) = i^m \mathcal{E}(Z^m).$$

Proof: First let us consider the case $m = 1$. By assumptions (ii) and (iii),

$$\psi'(u) = (\lim_{n\to\infty} \psi_{Z_n}(u))' = \lim_{n\to\infty} \psi'_{Z_n}(u).$$

According to (i), $E|Z_n|$ exists and is finite for every n. Therefore, for each n there exists a continuous derivative ψ'_{Z_n} and

$$\psi'(u) = \lim_{n\to\infty} \psi'_{Z_n}(u) = i \lim_{n\to\infty} E(Z_n \exp(iuZ_n)) = i\mathcal{E}(Z\exp(iuZ)).$$

Finally, putting $u = 0$ we obtain Lemma 3.1. The argument for general $m \geq 1$ uses induction. ■

Example 3.4. Suppose that $Z_0 = Z_1 = Z_2 = \ldots$ and Z_0 takes values $\exp(k\pi i/4)$, $k = 0, 1, \ldots, 7$ with equal probabilities. Then

$$\psi_Z(t) = (1/8)\sum_{k=0}^{7} \exp(it\exp(k\pi i/4)).$$

It is elementary to check that

$$\psi_Z^{(m)}(0) = i^m \mathcal{E}(Z^m) = 0$$

for $m = 1, 2, 3$ and

$$\psi_Z^{(4)}(0) = \mathcal{E}(Z^4) = 1.$$

Definition 3.4. *We will say that a* $\mathbf{C}^{\mathbf{N}}$*-valued random variable* Z *has a* p*-stable (asymptotic) distribution with* $0 < p \leq 4$ *if there exist two complex numbers* m *and* σ *and a real number* $\tau > 0$ *such that*

$$\psi_Z(t) = \exp(imt + \sigma|t|^{p/2} + \tau|t|^p), \quad t \in \mathbf{R}. \tag{3.1}$$

The asymptotic distribution $\mathcal{L}_Z^a$ *of such* Z *will be denoted* $S_p(m, \sigma, \tau)$*. We will also write* $Z \sim S_p(m, \sigma, \tau)$*.*

The set $S_p(m, \sigma, \tau)$ is non-empty for each p with $0 < p \leq 4$ and each triplet $(m, \sigma, \tau) \in \mathbf{C}^2 \times \mathbf{R}_+$ (see Remark 4.1 below).

Proposition 3.2. *Suppose that* $Z \sim S_p(m_1, \sigma_1, \tau_1)$*,* $Y \sim S_p(m_2, \sigma_2, \tau_2)$ *and* $\alpha_1, \alpha_2 \in \mathbf{R}$*. If* Z *and* Y *are independent then*

$$(\alpha_1 Z + \alpha_2 Y) \sim S_p(\alpha_1 m_1 + \alpha_2 m_2, \alpha_1^{p/2}\sigma_1 + \alpha_2^{p/2}\sigma_2, \alpha_1^p \tau_1 + \alpha_2^p \tau_2).$$

Proof: The result follows immediately from (3.1) and Proposition 3.1 (i). ■

Proposition 3.2 and (3.1) justify the name "stable distribution" for $S_p(m, \sigma, \tau)$ as they show that these distributions have similar properties to the classical stable distributions. We consider the 4-stable asymptotic distributions especially interesting. Before we review their properties in Theorem 3.1 below, we renormalize their parameters m, σ and τ. Namely, we will write $N_4(m, \sigma, \tau)$ instead of $S_4(m, \sqrt{3}\sigma/2, \tau/8)$, i.e., $Z \sim N_4(m, \sigma, \tau)$ iff

$$\psi_Z(t) = \exp(imt + \sqrt{3}\sigma t^2/2 + \tau t^4/8), \quad t \in \mathbf{R}.$$

We will call $N_4(m, \sigma, \tau)$ a "normal" (asymptotic) distribution. The distribution $N_4(0, 0, 1)$ will be called the standard 4-stable distribution or standard "normal" (asymptotic) distribution.

Theorem 3.1. *(see [MR1, Theorem 1]) (i) Assume that Z and Y are independent, $Z \sim N_4(m_1, \sigma_1, \tau_1)$, $Y \sim N_4(m_2, \sigma_2, \tau_2)$ and $\alpha_1, \alpha_2, \in \mathbf{R}$. Then*

$$(\alpha_1 Z + \alpha_2 Y) \sim N_4(\alpha_1 m_1 + \alpha_2 m_2, \alpha_1^2 \sigma_1 + \alpha_2^2 \sigma_2, \alpha_1^4 \tau_1 + \alpha_2^4 \tau_2).$$

(ii) Suppose that $Z \sim N_4(m, \sigma, \tau)$. Then for each $j \in \mathbf{N}$ there exists j-asymptotic moment $\mathcal{E}(Z^j)$ and $\mathcal{E}Z = m$, $\mathcal{E}(Z - m)^2 = \sigma$ and $\mathcal{E}((Z - m)^2 - \sigma)^2 = 3\tau$. Moreover, if $m = \sigma = 0$ then the j-th derivative of $\psi_Z(t)$ at zero exists and

$$\psi_Z^{(j)}(0) = i^j \mathcal{E}(Z^j).$$

(iii) (Central Limit Theorem) Let $\{Z_k^n : k, n \in \mathbf{N}\}$ be a family of complex random variables and let $Z^n = (Z_1^n, Z_2^n, ..., Z_k^n, ...)$. Assume that for all $m, n \in \mathbf{N}$

(1) Z^m and Z^n have the same asymptotic distribution, i.e., $\mathcal{L}^a_{Z^m} = \mathcal{L}^a_{Z^n}$,

(2) for a fixed n, the random variables $\{Z_n^k\}_{k \geq 1}$ are jointly independent,

(3) Z^n has the first four asymptotic moments,

$$\mathcal{E}(Z^n) = \mathcal{E}(Z^n)^2 = \mathcal{E}(Z^n)^3 = 0$$

and for some $\tau > 0$ independent of n

$$\mathcal{E}(Z^n)^4 = \tau > 0,$$

(4) Z^n satisfies condition (ii) of Lemma 3.1 with $m = 4$.

Then

$$\lim_{n \to \infty} \psi([Z^1 + Z^2 + ... + Z^n]/n^{1/4}, u) = \exp(\tau u^4/8), \quad u \in \mathbf{R},$$

i.e., the distributions of the normalized sums $S_n = (Z^1 + ... + Z^n)/n^{1/4}$ tend in a very weak sense to the "normal" distribution $N_4(0, 0, \tau)$.

Proof: Part (i) is a special case of Proposition 3.2. Part (ii) follows easily from Lemma 3.1.

It remains to prove (iii). Let ϕ be the asymptotic characteristic function of Z^n (it does not depend on n by (1)). Then by the definition of the asymptotic characteristic function and by the assumption (2),

$$\begin{aligned}\psi([Z^1 + ... + Z^n]/n^{1/4}, u) &= \mathcal{E}\exp(iuS_n) = \lim_{k \to \infty} E\exp(iu[(Z_k^1 + ... + Z_k^n)/n^{1/4}]) \\ &= \lim_{k \to \infty} (E\exp(iuZ_k^n/n^{1/4}))^n = [\phi(u/n^{1/4})]^n.\end{aligned}$$

By (3), (4) and Lemma 3.1 we have the following Taylor expansion:

$$\phi(u) = 1 + (\tau u^4/8) + o(u^4).$$

Hence

$$\psi_{S_n}(u) = \lim_{n\to\infty} \phi(u/n^{1/4})^n = \lim_{n\to\infty}(1 + \tau u^4/8n + o(1/n))^n = \exp(\tau u^4/8).$$

This proves the theorem. ■

Condition (3) of Theorem 3.1 (iii) is satisfied by random elements discussed in Example 3.4.

A Central Limit Theorem for sums of i.i.d. random variables with signed distributions normalized by $n^{-1/4}$ was proved by Hochberg [Ho].

4. A 4-stable process. This section is devoted to the definition, construction and the simplest properties of a 4-stable process.

It will be convenient to work with a product of two probability spaces $(\Omega, \mathcal{F}, P) = (\Omega_1, \mathcal{F}_1, P_1) \times (\Omega_2, \mathcal{F}_2, P_2)$. First we will show that we can define on $(\Omega_2, \mathcal{F}_2, P_2)$ a standard Brownian motion $\{b_t : t \geq 0\}$ and two families $\{b_t^+(n) : t \geq 0\}_{n\geq 1}$ and $\{b_t^-(n) : t \geq 0\}_{n\geq 1}$ of processes which satisfy the following conditions (the space $(\Omega_2, \mathcal{F}_2, P_2)$ has to be sufficiently rich).

(i) For each $n \in \mathbf{N}$, $b_0^+(n) = b_0^-(n) = 0$ and the processes $b^+(n)$ and $-b^-(n)$ have non-decreasing paths, i.e., $0 \leq b_s^+(n) \leq b_t^+(n)$ and $b_t^-(n) \leq b_s^-(n) \leq 0$ for all $0 < s < t < \infty$.

(ii) For all $t \geq 0$,

$$\lim_{n\to\infty} (b_t^+(n) + b_t^-(n)) = b_t \quad P_2\text{−a.s.}$$

(iii) Let $b_t(n) = b_t^+(n) + b_t^-(n)$. For every pair $(n, t) \in \mathbf{N} \times \mathbf{R}_+$, the random variable $b_t(n)$ has a Gaussian density on $\mathbf{R}$.

Here is one way to construct such processes. We will limit ourselves to the interval $[0, 1]$. Recall that a Haar function is given by

$$H(\tau) = \begin{cases} 2^{(m-1)/2} & \text{if } (k-1)/2^m \leq \tau < k/2^m, \\ -2^{(m-1)/2} & \text{if } k/2^m \leq \tau < (k+1)/2^m, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\{H_n\}$ be the sequence of all Haar functions for $m \geq 0$ and $k = 1, 3, 5, ..., 2^m - 1$. Let X_n be i.i.d. real standard normal random variables. Let

$$b_t(\omega) = \sum_{n=0}^{\infty} X_n(\omega) \int_0^t H_n(\tau) d\tau, \quad 0 \leq t \leq 1.$$

It is well known that thus defined b_t is a Brownian motion (see [KS]). Let $\mathrm{sgn}(x)$ denote the sign of x with the convention $\mathrm{sgn}0 = 0$ and let $\chi_1(.)$, $\chi_{-1}(.)$ be the indicator functions of the sets $\{0,1\}$ and $\{-1\}$, resp. Then let

$$b_t^+(n) = \sum_{k=0}^{2^n} X_k \left[\chi_1(\mathrm{sgn}X_k) \int_0^t \max(H_k(\tau),0)d\tau + \chi_{-1}(\mathrm{sgn}X_k) \int_0^t \min(H_k(\tau),0)d\tau\right]$$

and

$$b_t^-(n) = \sum_{k=0}^{2^n} X_k \left[\chi_1(\mathrm{sgn}X_k) \int_0^t \min(H_k(\tau),0)d\tau + \chi_{-1}(\mathrm{sgn}X_k) \int_0^t \max(H_k(\tau),0)d\tau\right].$$

It is easy to check that these processes satisfy conditions (i)-(iii).

We will also need 2 independent Brownian motions $w_x^+ = (w_x^+(t) : t \geq 0)$ and $w_x^- = (w_x^-(t) : t \geq 0)$ starting from $x \in \mathbf{R}$ and defined on $(\Omega_1, \mathcal{F}_1, P_1)$. The processes w^+, w^- and b may be looked upon as three independent processes defined on the product space $(\Omega, \mathcal{F}, P)$. A generic element of Ω will be denoted $\omega = (\omega_1, \omega_2)$.

Definition 4.1. (cf. [MR2, Definition 2.1]) *Suppose that $a = a_1 + ia_2 \in \mathbf{C}$. A $\mathbf{C}^{\mathbf{N}}$-valued process $\{Z_t^a : t \geq 0\}$ defined by the formula*

$$Z_t^a(\omega, n) = Z_t^a(\omega_1, \omega_2, n) = w_{a_1}^+(\omega_1, b_t^+(\omega_2, n)) + iw_{a_2}^-(\omega_1, -b_t^-(\omega_2, n))$$

will be called 4-stable motion starting from a.

In the sequel we will write Z instead of Z^0 and we will call Z the standard 4-stable motion.

Since our probability space is a product space, the expected value functional $\mathcal{E}$ for $\mathbf{C}^{\mathbf{N}}$-valued random variables may be written as

$$\mathcal{E}Z = \lim_{n\to\infty} (E_2(E_1 Z(n)))$$

where E_j is the expected value on the probability space Ω_j.

Theorem 4.1. *The 4-stable motion $Z^a = \{Z_t^a : t \geq 0\}$ has the following properties.*

(i) For each $(t, n) \in \mathbf{R}_+ \times \mathbf{N}$ the function $\omega \to Z_t^a(\omega, n)$ is $\mathcal{F}$-measurable, i.e., it is a random variable.

(ii) $Z_0^a(n) = a$ *a.s.*

(iii) *For any* $t > s \geq 0$ *the* $\mathbf{C}^{\mathbf{N}}$*-valued random variable* $(Z_t^a - Z_s^a)$ *has the 4-stable asymptotic distribution* $N_4(0, 0, t-s)$.

(iv) *For each natural number* $l \in \mathbf{N}$ *and every* $0 \leq s < t$*, the expectation* $\mathcal{E}(Z_t^a - Z_s^a)^l$ *exists. If* l *is divisible by* 4 *and* $l = 4p$ *then*

$$\mathcal{E}(Z_t^a - Z_s^a)^l = (4p-1)!!(2p-1)!!(t-s)^p,$$

where $k!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot k$*. If* l *is not divisible by* 4 *then* $\mathcal{E}(Z_t^a - Z_s^a)^l = 0$*. In particular* $\mathcal{E}(Z_t^a - Z_s^a)^j = 0$ *if* $j = 1, 2, 3$ *and* $\mathcal{E}(Z_t^a - Z_s^a)^4 = 3(t-s)$.

(v) *For each* $p \in \mathbf{N}$*, every sequence* $t_0 < t_1 < \ldots < t_p$*, for arbitrary multi-indices* $\alpha = (\alpha_1, \ldots, \alpha_p) \in \mathbf{N}^p$ *and for arbitrary complex numbers* c_α*, the asymptotic expectation*

$$\mathcal{E}\Bigg(\sum_{\alpha=(\alpha_1,\ldots,\alpha_p)} c_\alpha \prod_{i=1}^{p} (Z_{t_{i+1}}^a - Z_{t_i}^a)^{\alpha_i} \Bigg)$$

exists, provided the sum extends over a finite set of multi-indices α.

(vi) *The increments of* Z^a *are asymptotically uncorrelated, i.e., for every sequence* $t_0 < t_1 < \ldots < t_p$ *and each multi-index* $\alpha = (\alpha_1, \ldots, \alpha_p)$,

$$\mathcal{E}\Bigg(\prod_{i=1}^{p} (Z_{t_{i+1}}^a - Z_{t_i}^a)^{\alpha_i} \Bigg) = \prod_{i=1}^{p} \mathcal{E}(Z_{t_{i+1}}^p - Z_{t_i}^a)^{\alpha_i}.$$

(vii) *For every* $\beta_1, \beta_2 \in \mathbf{N}$ *and all disjoint intervals* (s,t) *and* (u,v) *we have*

$$\mathcal{E}[(Z_t^a - Z_s^a)^{\beta_1}(b_v - b_u)^{\beta_2}] = \mathcal{E}(Z_t^a - Z_s^a)^{\beta_1}\mathcal{E}(b_v - b_u)^{\beta_2}.$$

(viii) *The paths* $t \longrightarrow Z_t^a(\omega_1, \omega_2) \in \mathbf{C}^{\mathbf{N}}$ *are continuous assuming that* $\mathbf{C}^{\mathbf{N}}$ *is endowed with the product topology.*

(ix) *For each fixed* $(\omega_2, n) \in \Omega_0 \times \mathbf{N}$*, the stochastic process* $\{Z_t^a(\,\cdot\,, \omega_2, n) : t \geq 0\}$ *has independent increments.*

Proof: (i) We may assume that $(\omega_1, t) \to w_{a_1}^+(\omega_1, t)$, $(\omega_1, t) \to w_{a_2}^-(\omega_1, t)$, $(\omega_2, t) \to b_t^+(\omega_2, n)$ and $(\omega_2, t) \to b_t^-(\omega_2, n)$ are measurable functions. The result follows since the composition of measurable functions is measurable.

(ii) $Z_0^a = w_{a_1}^+(0) + iw_{a_2}^-(0) = a_1 + ia_2 = a$.

(iii) Since

$$Z_t^a(n) - Z_s^a(n) = \big(w_{a_1}^+(b_t^+(n)) - w_{a_1}^+(b_s^+(n))\big) + i\big(w_{a_2}^-(-b_t^-(n)) - w_{a_2}^-(-b_s^-(n))\big)$$

then

$$\begin{aligned} &E_1 \exp(iu(Z_t^a(n) - Z_s^a(n))) \\ &\quad = E_1 \exp\Big(iu\big(w_{a_1}^+(b_t^+(n)) - w_{a_1}^+(b_s^+(n))\big)\Big)\times \\ &\qquad \times E_1 \exp\Big(-u\big(w_{a_2}^-(-b_t^-(n)) - w_{a_2}^-(-b_s^-(n))\big)\Big) \\ &\quad = \exp(-(u^2/2)(b_t^+(n) - b_s^+(n)) \exp((u^2/2)(-b_t^-(n) + b_s^-(n)) \\ &\quad = \exp(-(u^2/2)(b_t(n) - b_s(n)). \end{aligned}$$

Hence

$$\begin{aligned} \mathcal{E} \exp(iu(Z_t^a(n) - Z_s^a(n))) &= \lim_{n\to\infty} E_2(E_1 \exp(iu(Z_t^a(n) - Z_s^a(n)))) \\ &= \lim_{n\to\infty} E_2 \exp(-(u^2/2)(b_t(n) - b_s(n)) \\ &= E_2 \exp(-(u^2/2)(b_t(n) - b_s(n)) \\ &= \exp((u^4/8)(t-s)) \end{aligned}$$

for all $t > s$. Thus $Z_t^a - Z_s^a \sim N_4(0,0,t-s)$.

(iv) We will consider the case when $a = 0$ and $s = 0$. The general case can be treated in a similar way.

Let $l = 4p$ where $p \geq 1$ is an integer and let $k \in \mathbf{N}$. We have

$$E_1\big(w_0^+(b_t^+(k)) + iw_0^-(-b_t^-(k))\big)^{4p} = \sum_{j=0}^{4p} \binom{4p}{j} E_1(w_0^+(b_t^+(k)))^j E_1(w_0^-(-b_t^-(k)))^{(4p-j)} i^{(4p-j)}$$

$$= \sum_{m=0}^{2p} \binom{4p}{2m} E_1(w_0^+(b_t^+(k)))^{2m} E_1(w_0^-(-b_t^-(k)))^{2(2p-m)} (-1)^{(2p-m)}. \tag{4.1}$$

But

$$E_1(w_0^+(b_t^+(k)))^{2m} = (2m-1)!!(b_t^+(k))^m \tag{4.2}$$

and

$$E_1(w_0^-(-b_t^-(k)))^{2(2p-m)} = (4p-2m-1)!!(-b_t^-(k))^{(2p-m)}. \tag{4.3}$$

Moreover, since $(2\nu)! = (2\nu-1)!!2^\nu \nu!$,

$$\binom{4p}{2m}(2m-1)!!(4p-2m-1)!! = (4p-1)!!\binom{2p}{m}. \tag{4.4}$$

Now, combining (4.1)-(4.4) we obtain

$$\begin{aligned} E_1(Z_t^a(k))^{4p} &= (4p-1)!! \sum_{m=0}^{2p} \binom{2p}{m} b_t^+(k)^m b_t^-(k)^{(2p-m)} = \\ &= (4p-1)!!(b_t^+(k) \ + \ b_t^-(k))^{2p} \\ &= (4p-1)!!(b_t(k))^{2p}. \end{aligned}$$

It is routine to check that for a fixed t, the expectations $E_2(b_t(k))^{4p}$ are uniformly bounded in k and so the random variables $(b_t(k))^{2p}$ are uniformly integrable. It follows that

$$\begin{aligned} \mathcal{E}(Z_t^0)^{4p} &= \lim_{k\to\infty} E_2(E_1(Z_t^{4p}(k))) \\ &= (4p-1)!! \lim_{k\to\infty} E_2(b_t(k))^{2p} \\ &= (4p-1)!! E_2 b_t^{2p} \\ &= (4p-1)!!(2p-1)!! t^p \end{aligned}$$

which gives (iv) in the case when $a = 0$, $s = 0$ and l is divisible by 4.

If l is odd then $\mathcal{E}(Z_t^0)^l = 0$ since either $E_1(w_0^+(u))^j = 0$ or $E_1(w_0^-(v))^{(l-j)} = 0$ for each $j = 0, 1, 2, ..., l$ and for arbitrary $u, v > 0$.

If $l = 2h$ and 2 does not divide h then similar calculations yield

$$E_2 E_1(Z_t^0(k))^{2h} = (2h-1)!! E_2(b_t(k))^h = 0.$$

Hence, in this case $\mathcal{E}(Z_t^0)^l = 0$ which completes the proof of (iv).

(v) We will write $E_{2,1}Z$ instead of $E_2(E_1(Z))$. It is clear that for each k the expectations

$$E_{2,1}\bigg(\sum_\alpha c_\alpha \prod_{i=1}^p (Z_{t_{i+1}}^a(k) - Z_{t_i}^a(k))^{\alpha_i}\bigg) = \sum_\alpha c_\alpha E_{2,1}\bigg(\prod_{i=1}^p (Z_{t_{i+1}}^a(k) - Z_{t_i}^a(k))^{\alpha_i}\bigg).$$

exist and are finite. Thus, it suffices to show that the limit

$$\lim_{k\to\infty} E_{2,1}\bigg(\prod_{i=1}^p (Z_{t_{i+1}}^a(k) \ - \ Z_{t_i}^a(k))^{\alpha_i}\bigg) \tag{4.5}$$

exists for each $\alpha = (\alpha_1, ..., \alpha_p)$ and every sequence $(t_0, ..., t_p)$ with $t_0 < t_1 < ... < t_p$. Now observe that if at least one α_i is odd then the expectation in (4.5) exists and is equal to zero. Thus without loss of generality we can assume that $\alpha_i = 2\beta_i$, $\beta_i \in \mathbf{N}$, $\beta_i \neq 0$. Then

$$\lim_{k\to\infty} E_{2,1}\bigg(\prod_{i=1}^p (Z_{t_{i+1}}^a(k) - Z_{t_i}^a(k))^{\alpha_i}\bigg) = \lim_{k\to\infty} E_2\bigg(\prod_{i=1}^p E_1(Z_{t_{i+1}}^a(k) - Z_{t_i}^a(k))^{2\beta_i}\bigg).$$

Using the same identities as in the proof of (iv) we see that

$$
\begin{aligned}
&E_1(Z^a_{t_{i+1}}(k)-Z^a_{t_i}(k))^{2\beta_i} = \\
&\qquad = \sum_{j=0}^{\beta_i}\binom{2\beta_i}{2j} E_1(w^+_{a-1}(b^+_{t_{i+1}}(k)) - w^+_{a_2}(b^+_{t_i}(k)))^{2j}\times \\
&\qquad\qquad \times E_1(w^-_{a_2}(-b^-_{t_{i+1}}(k)) - w^-_{a_2}(-b^-_{t_i}(k)))^{(2\beta_i-2j)}(-1)^{(\beta_i-j)} \\
&\qquad = (2\beta_i-1)!!\sum_{j=0}^{\beta_i}\binom{\beta_i}{j}(b^+_{t_{i+1}}(k)-b^+_{t_i}(k))^j(b^-_{t_{i+1}}(k)-b^-_{t_i}(k))^{(\beta_i-j)} \\
&\qquad = (2\beta_i-1)!!(b_{t_{i+1}}(k)-b_{t_i}(k))^{\beta_i}. \qquad (4.6)
\end{aligned}
$$

One can check that

$$E_2\bigg(\prod_{i=1}^p (b_{t_{i+1}}(k)-b_{t_i}(k))^{2\beta_i}\bigg)$$

are uniformly bounded in k and so $\prod_{i=1}^p (b_{t_{i+1}}(k)-b_{t_i}(k))^{\beta_i}$ are uniformly integrable. Hence we can pass to the limit under the expectation sign to obtain

$$
\begin{aligned}
\lim_{k\to\infty} E_{2,1}\bigg(\prod_{i=1}^p (Z^a_{t_{i+1}}(k)-Z^a_{t_i}(k))^{\alpha_i}\bigg) &= \lim_{k\to\infty} E_2\bigg(\prod_{i=1}^p (2\beta_i-1)!!(b_{t_{i+1}}(k)-b_{t_i}(k))^{\beta_i}\bigg) \\
&= E_2\bigg(\prod_{i=1}^p (2\beta_i-1)!!(b_{t_{i+1}}-b_{t_i})^{\beta_i}\bigg).
\end{aligned}
$$

The last expectation is finite and this completes the proof of (v).

(vi) It follows from the proof of (v) that

$$\mathcal{E}\bigg(\prod_{i=1}^p (Z^a_{t_{i+1}}-Z^a_{t_i})^{\alpha_i}\bigg) = 0$$

if at least one α_i is odd. Then (vi) follows in view of (iv).

Suppose that $\alpha_i = 2\beta_i$, $\beta_i \in \mathbf{N}$. Then we obtain from the proof of (v)

$$
\begin{aligned}
\mathcal{E}\bigg(\prod_{i=1}^p (Z^a_{t_{i+1}}-Z^a_{t_i})^{\alpha_i}\bigg) &= \lim_{k\to\infty} E_2\bigg(\prod_{i=1}^p (2\beta_i-1)!!(b_{t_{i+1}}(k)-b_{t_i}(k))^{\beta_i}\bigg) \\
&= E_2\bigg(\prod_{i=1}^p (2\beta_i-1)!!(b_{t_{i+1}}-b_{t_i})^{\beta_i}\bigg) \\
&= \prod_{i=1}^p (2\beta_i-1)!!(2\gamma_i-1)!!(t_{i+1}-t_i)^{\gamma_i}
\end{aligned}
$$

if $\gamma_i = \beta_i/2$ is an integer; otherwise the expression is equal to 0. The result follows now easily from (iv).

(vii) First suppose that $\beta_1 = 2\beta$ for some $\beta \in \mathbf{N}$. Since b_t depends only on ω_2 we can write as in (4.6)

$$E_1[(Z_t^a(k) - Z_s^a(k))^{\beta_1}(b_v - b_u)^{\beta_2}] = (2\beta - 1)!!(b_t(k) - b_s(k))^{\beta}(b_v - b_u)^{\beta_2}.$$

As in the proof of part (v) we use uniform convergence to see that

$$\begin{aligned}\lim_{k\to\infty} E_{2,1}[(Z_t^a(k) - Z_s^a(k))^{\beta_1}(b_v - b_u)^{\beta_2}] &= \lim_{k\to\infty} E_2[(2\beta - 1)!!(b_t(k) - b_s(k))^{\beta}(b_v - b_u)^{\beta_2}] \\ &= E_2[(2\beta - 1)!!(b_t - b_s)^{\beta}(b_v - b_u)^{\beta_2}] \\ &= \mathcal{E}(Z_t^a - Z_s^a)^{\beta_1}\mathcal{E}(b_v - b_u)^{\beta_2}.\end{aligned}$$

If β_1 is an odd number then one can argue just like in the proof of (iv) that both sides of the formula in (vii) are equal to 0.

(viii) It follows immediately from the definition of $(Z_t^a : t \geq 0)$ and the continuity of the paths of $w^+, w^-, b^+(n)$ and $b^-(n)$ for $n \geq 1$, that Z is continuous.

(ix) The proof is easy and so it is omitted. ■

Remark 4.1. If in Definition 4.1 we replace w^+ and w^- with two independent copies s^+ and s^- of the standard q-stable Levy motion for some $0 < q < 2$ then we obtain a family $(\tilde{Z}^a : a \in \mathbf{C})$ of (asymptotic) p-stable motions with $0 < p = 2q < 4$. The family has the property that

$$\mathcal{E}\exp(iu(\tilde{Z}_t^a - \tilde{Z}_s^a)) = \exp((t - s)u^p/2)$$

for $u \in \mathbf{R}$.

Remark 4.2. Observe that $(Z_t^a - Z_s^a)$ does not have absolute asymptotic moments, i.e., for every $p = 1, 2, ...$ and every $t > s \geq 0$, $\mathcal{E}|Z_t^a - Z_s^a|^p = \infty$. For simplicity, we will show that $\mathcal{E}|Z_t^a - Z_s^a|^2 = \infty$. We have

$$\begin{aligned}&\mathcal{E}|Z_t^a - Z_s^a|^2 \\ &\quad= \lim_{n\to\infty} E_2(E_1[(w^+(b_t^+(n)) - w^+(b_s^+(n)))^2 + (w^-(-b_t^-(n)) - w^-(-b_s^-(n)))^2]) \\ &\quad= \lim_{n\to\infty} E_2\{(b_t^+(n) - b_s^+(n)) - (b_t^-(n) - b_s^-(n))\} \\ &\quad= \lim_{n\to\infty} E_2\Big\{\sum_{k=0}^{2^n} X_k[\chi(\mathrm{sgn}X_k)\int_s^t H_k^+(\tau)d\tau + \chi_{-1}(\mathrm{sgn}X_k)\int_s^t H_k^-(\tau)d\tau] \\ &\qquad\qquad - \sum_{k=0}^{2^n} X_k[\chi_1(\mathrm{sgn}X_k)\int_s^t H_k^-(\tau)d\tau + \chi_{-1}(\mathrm{sgn}X_k)\int_s^t H_k^+(\tau)d\tau]\Big\},\end{aligned}$$

where H_k^+ and H_k^- denote the positive and negative parts of H_k. Note that

$$E_2(X_k\chi_1(\text{sgn}X_k)) = E_2(X_1\chi_1(\text{sgn}X_1)) = c_1 > 0$$

and

$$E_2(X_k\chi_{-1}(\text{sgn}X_k)) = E_2(X_1\chi_{-1}(\text{sgn}X_1)) = -c_1 < 0.$$

Therefore we have

$$\mathcal{E}|Z_t^a - Z_s^a|^2 = 2c_1 \lim_{n\to\infty}\left(\sum_{k=0}^{2^n}\int_s^t |H_k|(\tau)d\tau\right) = 2c_1 \lim_{n\to\infty}\int_s^t\left(\sum_{k=0}^{2^n}|H_k(\tau)|d\tau\right).$$

Since

$$\sum_{k=0}^{2^n}|H_k(\tau)| = \sum_{k=1}^{n-1}\sum_{i=2^k}^{2^{k+1}}|H_i(\tau)|$$

and

$$\sum_{i=2^k}^{2^{k+1}}|H_i(\tau)| = 2^{k/2}, \quad \tau\in[0,1],$$

it follows that

$$\sum_{k=0}^{2^n}|H_k(\tau)| = 1+1+\sum_{k=1}^{n-1}2^{k/2} = 2+\sqrt{2}^{n+1}/(\sqrt{2}-1).$$

We conclude that

$$\lim_{n\to\infty}\int_s^t\left(\sum_{k=0}^{2^n}|H_k(\tau)|d\tau\right) = \lim_{n\to\infty}\int_s^t(2+\sqrt{2}^{n+1}/(\sqrt{2}-1))d\tau = \infty$$

and, therefore, $\mathcal{E}|Z_t^a - Z_t^a|^2 = \infty$. As for higher moments, one can show that

$$\mathcal{E}(|Z_t^a - Z_s^a|^n) \geq (\mathcal{E}|Z_t^a - Z_s^a|^2)^{n/2} = \infty$$

if $n \geq 2$.

Remark 4.3. For a fixed $t \geq 0$, the sequence $Z_t(n)$ of complex random variables does not converge in the sense of distribution. Calculations similar to those in the proof of Theorem 4.1 (iii) show that the (2-dimensional) characteristic function of the vector $Z_t(n) = (Z_t^1(n), Z_t^2(n))$ is given by

$$\psi_{(Z_t^1(n),Z_t^2(n))}(u,v) = E_{2,1}\exp(iuZ_t^1(n)+ivZ_t^2(n)) = E_2\exp(-u^2b_t^+(n)/2 - v^2b_t^-(n)/2).$$

As $n \to \infty$, this expression converges to 0 for all $(u,v) \neq (0,0)$. The limit is not a characteristic function.

5. Variation of Z. In this section we will suppress the starting point of Z in the notation. The notation for Brownian motion b_t will be changed to B_t (see the definition in Section 4). The symbol i will stand for the imaginary unit in formula (5.3) and its proof.

Suppose that X and X_m are random elements with values in $\mathbf{C}^{\mathbf{N}}$. Recall that X_m converge to X in $L^p_{\mathcal{E}}$ if $\lim_{m\to\infty} \mathcal{E}(X_m - X)^p = 0$.

Theorem 5.1. *Let $a = t_0(m) < t_1(m) < \ldots < t_{N(m)}(m) = b$ be a sequence of partitions of $[a,b]$ and let*

$$\Delta_m = \max_{0\le j\le N(m)-1} (t_{j+1}(m) - t_j(m)).$$

Suppose that $\lim_{m\to\infty} \Delta_m = 0$. Then the following limits exist in the sense of $L^p_{\mathcal{E}}$ for every integer $p \geq 1$.

$$\lim_{m\to\infty} \sum_{j=0}^{N(m)-1} (Z_{t_{j+1}(m)} - Z_{t_j(m)})^4 = 3(b-a), \tag{5.1}$$

$$\lim_{m\to\infty} \sum_{j=0}^{N(m)-1} (Z_{t_{j+1}(m)} - Z_{t_j(m)})^3 = 0, \tag{5.2}$$

$$\lim_{m\to\infty} \sum_{j=0}^{N(m)-1} (Z_{t_{j+1}(m)} - Z_{t_j(m)})^2 = (1+i\sqrt{2})(B_b - B_a). \tag{5.3}$$

First we prove a lemma.

Lemma 5.1. *Suppose that $\beta_1, \beta_2 \in \mathbf{N}$ and $\beta_1+\beta_2$ is even. Then for $s<t$,*

$$\mathcal{E}[(Z_t - Z_s)^{2\beta_1}(B_t - B_s)^{\beta_2}] = (\beta_1+\beta_2-1)!!(t-s)^{(\beta_1+\beta_2)/2}.$$

Proof: Recall that b_t depends only on ω_2. A calculation analogous to (4.6) gives

$$\begin{aligned} E_1[(Z_t(n) - Z_s(n))^{2\beta_1}(B_t - B_s)^{\beta_2}] &= E_1[(Z_t(n) - Z_s(n))^{2\beta_1}(b_t - b_s)^{\beta_2}] \\ &= (b_t(n) - b_s(n))^{\beta_1}(b_t - b_s)^{\beta_2}. \end{aligned}$$

As in the proof of Theorem 4.1 (v) we use uniform convergence to conclude that

$$\begin{aligned}\lim_{n\to\infty} E_{2,1}[(Z_t(n)-Z_s(n))^{2\beta_1}(B_t-B_s)^{\beta_2}] &= \lim_{n\to\infty} E_2[(b_t(n)-b_s(n))^{\beta_1}(b_t-b_s)^{\beta_2}] \\ &= E_2(b_t-b_s)^{\beta_1+\beta_2} \\ &= (\beta_1+\beta_2-1)!!(t-s)^{(\beta_1+\beta_2)/2}.\end{aligned}$$

■

Proof of Theorem 5.1: We will prove only convergence in the sense of $L^4_{\mathcal{E}}$. The other cases may be treated in a similar way.

Let $U_j = (Z_{t_{j+1}} - Z_{t_j})^4 - 3(t_{j+1}-t_j)$. In view of Theorem 4.1 (vi) we have

$$\begin{aligned}\mathcal{E}\Bigg(\Bigg(\sum_{j=0}^{N(m)-1}(Z_{t_{j+1}}-Z_{t_j})^4\Bigg)-3(b-a)\Bigg)^4 &= \mathcal{E}\Bigg(\sum_{j=0}^{N(m)-1}[(Z_{t_{j+1}}-Z_{t_j})^4-3(t_{j+1}-t_j)]\Bigg)^4 \\ &= \mathcal{E}\Bigg(\sum_{j=0}^{N(m)-1}U_j\Bigg)^4 \\ &= \sum_{k_0+\ldots+k_{N(m)-1}=4} P_4(k_1,\ldots,k_{N(m)})\mathcal{E}(U_0^{k_0}\ldots U_{N(m)-1}^{k_{N(m)-1}}) \\ &= \sum_{k_0+\ldots+k_{N(m)-1}=4} P_4(k_1,\ldots,k_{N(m)})\mathcal{E}(U_0^{k_0})\ldots\mathcal{E}(U_{N(m)-1}^{k_{N(m)-1}}), \qquad (5.4)\end{aligned}$$

where $P_4(k_1,\ldots,k_N(m)) = 4!/k_1!\ldots k_{N(m)}!$. We obtain from Theorem 4.1 (iv) that for some constants A_2 and A_4,

$$\mathcal{E}(U_j) = \mathcal{E}[(Z_{t_{j+1}(m)}-Z_{t_j(m)})^4 - 3(t_{j+1}(m)-t_j(m))] = 0,$$

$$\begin{aligned}\mathcal{E}(U_j^2) &= \mathcal{E}(Z_{t_{j+1}(m)}-Z_{t_j(m)})^8 - 6(t_{j+1}(m)-t_j(m))\mathcal{E}(Z_{t_{j+1}(m)}-Z_{t_j(m)})^4 \\ &\quad + 9(t_{j+1}(m)-t_j(m))^2 \\ &= A_2(t_{j+1}(m)-t_j(m))^2,\end{aligned}$$

$$\begin{aligned}\mathcal{E}(U_j^4) &= \mathcal{E}(Z_{t_{j+1}(m)}-Z_{t_j(m)})^{16} - 12\mathcal{E}(Z_{t_{j+1}(m)}-Z_{t_j(m)})^{12}(t_{j+1}(m)-t_j(m)) \\ &\quad + 54\mathcal{E}(Z_{tj+1(m)}-Z_{t_j(m)})^8(t_{j+1}(m)-t_j(m))^2 \\ &\quad - 98\mathcal{E}(Z_{t_{j+1}(m)}-Z_{t_j(m)})^4(t_{j+1}(m)-t_j(m))^3 + 3^4(t_{j+1}(m)-t_j(m))^4 \\ &= A_4(t_{j+1}(m)-t_j(m))^4.\end{aligned}$$

These formulae and (5.4) imply that

$$\begin{aligned}
&\mathcal{E}\Bigg(\Bigg(\sum_{j=0}^{N(m)-1}(Z_{t_{j+1}}-Z_{t_j})^4\Bigg)-3(b-a)\Bigg)^4\\
&=\sum_{k_0+\ldots+k_{N(m)-1}=4,k_j\neq 1} P_4(k_0,\ldots,k_{N(m)-1})A_{k_0}\ldots A_{k_{N(m)-1}}\prod_{j=0}^{N(m)-1}(t_{j+1}(m)-t_j(m))^{k_j}\\
&=A_4\sum_{j=0}^{N(m)-1}(t_{j+1}(m)-t_j(m))^4+6A_2^2\sum_{j\neq\ell}(t_{j+1}(m)-t_j(m))^2(t_{\ell+1}(m)-t_\ell(m))^2\\
&\leq \max\{A_4,3A_2^2\}\Bigg(\sum_{j=0}^{N(m)-1}(t_{j+1}(m)-t_j(m))^2\Bigg)^2\leq \max\{A_4,3A_2^2\}(b-a)^2\cdot\Delta_m^2\to 0,
\end{aligned}$$

which completes the proof of (5.1).

The proof of (5.2) proceeds along the same lines. Let $V_j=(Z_{t_{j+1}(m)}-Z_{t_j(m)})^3$, $j=0,1,\ldots,N(m)-1$. Then

$$\begin{aligned}
\mathcal{E}\Bigg(\sum_{j=0}^{N(m)-1}(Z_{t_{j+1}(m)}-Z_{t_j(m)})^3\Bigg)^4&=\mathcal{E}\Bigg(\sum_{j=0}^{N(m)-1}V_j\Bigg)^4\\
&=\sum_{k_0+\ldots+k_{N(m)-1}=4}P_4(k_1,\ldots,k_{N(m)})\mathcal{E}(V_0^{k_0})\ldots\mathcal{E}(V_{N(m)-1}^{k_{N(m)-1}})
\end{aligned}$$

Since

$$\begin{aligned}
\mathcal{E}(V_j)&=\mathcal{E}(Z_{t_{j+1}(m)}-Z_{t_j(m)})^3=0,\\
\mathcal{E}(V_j^2)&=\mathcal{E}(Z_{t_{j+1}(m)}-Z_{t_j(m)})^6=0,\\
\mathcal{E}(V_j^3)&=\mathcal{E}(Z_{t_{j+1}(m)}-Z_{t_j(m)})^9=0,
\end{aligned}$$

and

$$\mathcal{E}(V_j^4)=\mathcal{E}(Z_{t_{j+1}(m)}-Z_{t_j(m)})^{12}=C_4(t_{j+1}(m)-t_j(m))^3,$$

it follows that

$$\begin{aligned}
\mathcal{E}\Bigg(\sum_{j=0}^{N(m)-1}(Z_{t_{j+1}(m)}-Z_{t_j(m)})^3\Bigg)^4&=\sum_{j=0}^{N(m)-1}C_4(t_{j+1}(m)-t_j(m))^3\\
&\leq C_4(b-a)\Delta_m^2\longrightarrow_m 0.
\end{aligned}$$

The proof of (5.2) is complete.

Let $Y_j = [(Z_{t_{j+1}(m)} - Z_{t_j(m)})^2 - (1+i\sqrt{2})(B_{t_{j+1}(m)} - B_{t_j(m)})]$. Then parts (vi) and (vii) of Theorem 4.1 imply that

$$\begin{aligned}\mathcal{E}\Big(\sum_{j=0}^{N(m)-1} (Z_{t_{j+1}(m)} - Z_{t_j(m)})^2 - (1+i\sqrt{2})(B_b - B_a)\Big)^4 &= \mathcal{E}\Big(\sum_{j=0}^{N(m)-1} Y_j\Big)^4 \\ &= \sum_{k_0+\ldots+k_{N(m)-1}=4} P_4(k_1,\ldots,k_{N(m)})\mathcal{E}(Y_0^{k_0}\ldots Y_{N(m)-1}^{k_{N(m)-1}}) \\ &= \sum_{k_0+\ldots+k_{N(m)-1}=4} P_4(k_1,\ldots,k_{N(m)})\mathcal{E}(Y_0^{k_0})\ldots\mathcal{E}(Y_{N(m)-1}^{k_{N(m)-1}}).\end{aligned}$$

But

$$\mathcal{E}(Y_j) = \mathcal{E}[(Z_{t_{j+1}(m)} - Z_{t_j(m)})^2 - (1+i\sqrt{2})(B_{t_{j+1}(m)} - B_{t_j(m)})] = 0.$$

Lemma 5.1 yields

$$\begin{aligned}\mathcal{E}(Y_j^2) &= \mathcal{E}(Z_{t_{j+1}(m)} - Z_{t_j(m)})^4 - 2(1+i\sqrt{2})\mathcal{E}[(Z_{t_{j+1}(m)} - Z_{t_j(m)})^2(B_{t_{j+1}(m)} - B_{t_j(m)})] \\ &\quad + (1+i\sqrt{2})^2\mathcal{E}(B_{t_{j+1}(m)} - B_{t_j(m)})^2 \\ &= 3(t_{j+1}(m) - t_j(m)) - 2(1+i\sqrt{2})(t_{j+1}(m) - t_j(m)) \\ &\quad + (1+i\sqrt{2})^2(t_{j+1}(m) - t_j(m)) = 0.\end{aligned}$$

We obtain from Theorem 4.1 (iv) and Lemma 5.1 for some constants c_k

$$\begin{aligned}\mathcal{E}(Y_j^4) &= \mathcal{E}(Z_{t_{j+1}(m)} - Z_{t_j(m)})^8 + c_1\mathcal{E}[(Z_{t_{j+1}(m)} - Z_{t_j(m)})^6(B_{t_{j+1}(m)} - B_{t_j(m)})] \\ &\quad + c_2\mathcal{E}[(Z_{t_{j+1}(m)} - Z_{t_j(m)})^4(B_{t_{j+1}(m)} - B_{t_j(m)})^2] \\ &\quad + c_3\mathcal{E}[(Z_{t_{j+1}(m)} - Z_{t_j(m)})^4(B_{t_{j+1}(m)} - B_{t_j(m)})^2] + 9E(B_{t_{j+1}(m)} - B_{t_j(m)})^4 \\ &= c_4(t_{j+1}(m) - t_j(m))^2.\end{aligned}$$

It follows that

$$\begin{aligned}\lim_{m\to\infty}\mathcal{E}\Big(\sum_{j=0}^{N(m)-1} (Z_{t_{j+1}(m)} - Z_{t_j(m)})^2 - (1+i\sqrt{2})(B_b - B_a)\Big)^2 &= \sum_{j=0}^{N(m)-1}\mathcal{E}(Y_j^4) \\ &= c_4\sum_{j=0}^{N(m)-1}(t_{j+1}(m) - t_j(m))^2 \le c_4(b-a)\Delta_m \longrightarrow 0.\end{aligned}$$

■

Remark 5.1. We would like to point out a paradoxical fact. The same proof which shows (5.3) also gives

$$\lim_{m\to\infty} \sum_{j=0}^{N(m)-1} (Z_{t_{j+1}(m)} - Z_{t_j(m)})^2 = (1 - i\sqrt{2})(B_b - B_a).$$

Remark 5.2. Theorem 5.1 is very close in spirit to results of [B2]. Formula (3.16) in [Ho] indicates that the quadratic variation of Hochberg's process may be also interpreted as a Brownian motion.

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