The Radiative Transfer Equation in Photoacoustic Imaging

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Abstract

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Photoacoustic tomography is a rapidly developing medical imaging technique that combines optical and ultrasound imaging to exploit the high contrast and high resolution of the respective individual modalities. Mathematically, photoacoustic tomography is divided into two steps. In the first step, one solves an inverse problem for the wave equation to determine how tissue absorbs light as a result of a boundary illumination. The second step is generally modeled by either diffusion or transport equations, and involves recovering the optical properties of the region being imaged.

In this paper we, we provide an overview of mathematical progress in photoacoustic and thermoacoustic tomography, as well as inverse problems for the radiative transfer equation. We then address quantitative photoacoustic tomography modeled by the radiative transfer equation, and in particular, we show that the absorption coefficient in the stationary transport equation can be recovered given certain internal information about the solution. Our new result will consider the variable index of refraction case, which will correspond to an inverse transport problem on a Riemannian manifold with internal data and a known metric. We will prove a stability estimate for a functional of the absorption coefficient of the medium by finding a singular decomposition for the distribution kernel of the measurement operator. Finally, we will use this estimate to recover the desired absorption properties.

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Chapter 1

INTRODUCTION

Photoacoustic and thermoacoustic tomography (PAT and TAT respectively) are rapidly developing medical imaging modalities that combine electromagnetic imaging with acoustic or ultrasound imaging to achieve both high resolution and high contrast. Both imaging methods are based on the *thermoacoustic effect*. The thermoacoustic effect is the property that thermal expansion can result in generation of an acoustic pressure wave, which was first observed by Alexander Graham Bell in 1880.

In photoacoustics, the tissue to be imaged is illuminated with visible or near-infra-red light causing a small amount of thermal expansion (in thermoacoustics, radio frequency electromagnetic waves are used). Different tissues in the region expand at different rates, and this difference in expansion rates generates an acoustic wave. One can then use ultrasound transducers to measure the generated acoustic wave.

The ultimate goal is then to obtain useful information about the body being imaged from knowledge of the illumination and measurements of the acoustics waves. There are two inverse problems currently being studied to that end. First, using the measured acoustic data, can we recover the deposited thermal energy map? That is, can we determine how thermal expansion occurred as a result of a given illumination? It is known, for example, that cancerous tissue tends to be hypervascular, which causes the tissue to preferentially absorb electromagnetic energy. Thus it would be expected that tumors would be visible as “hot spots” on an image constructed from the thermal deposition map. The second inverse problem is to recover the optical properties of the medium once we have the knowledge of the thermal deposition map in order to obtain more accurate and detailed images. The recovery of the thermal deposition map has been studied in numerous publications including, [13], [17], [18], [20], and [28]. The second problem, called quantitative photo- or thermo-acoustics, has not yet been studied as thoroughly, but results can be found in [2], [4], and

[5].

Based on early results allowing for reconstruction of the thermal deposition map via filtered backprojection algorithms, many bioengineers and scientists have already begun to experiment with PAT and TAT. In [19] the author notes that thermoacoustic computed tomography has produced images with 1 to 2 mm of spatial resolution through as much as 10 cm of soft tissue in vivo in the breast. He also notes that while mammogram images tend to degrade as breast density increases, initial findings seem to indicate that TAT breast images improve as density increases. This might indicate that TAT imaging could be useful for early detection of breast cancer in women who are not able to get clear mammography results.

In addition, in [11], it is observed that PAT can allow doctors to "examine blood vessels at a depth depth of a few millimeters with a spatial resolution of tens of micrometers." This is particularly useful for diagnosis of cancers near the surface of the skin, such as melanoma. For some examples of the outstanding images that can be obtained, we encourage readers to look at the work of Lihong Wang at the Optical Imaging Laboratory at Washington University. He has produced a number of images demonstrating both the clarity of microvascular structures and the detection of melanomas. We also refer readers to the collection of work done by the Photoacoustic Imaging group at University College London for further information about photo- and thermo- acoustic imaging from the biomedical engineering perspective (see [6, 9, 10, 27], for example).

From a mathematical point of view, TAT and PAT provide several interesting inverse problems. Various models involve inverse problems for Maxwell's equations, the scalar Helmholtz equation, the diffusion equation, the wave equation, and the radiative transfer equation. The inverse problems related to quantitative thermo- and photo- acoustic tomography are unique in that they involve internal data. In a typical inverse problem, data is available only on the boundary as obtaining internal data would involve invasive procedures, but in PAT and TAT, the first step (recovery of the thermal deposition map) is an inverse problem with boundary data resulting in the thermal deposition map, which is used as internal data for QTAT and QPAT.

In chapter 2, we will discuss results related to recovery of the thermal deposition map as well as the diffusion model of QPAT and results in QTAT. In chapter 3, we will present previous work in inverse problems for the radiative transfer equation, and in chapter 4, we will bring these ideas together to discuss the radiative transfer model of quantitative photoacoustic tomography. We prove a new result showing that the absorption coefficient may be recovered in the transport model of QTAT under the assumption that the index of refraction varies in the medium. In particular, this type of situation arises when imaging regions with combinations of bone, soft tissue, and air cavities, although we should note that further work will still be required to extend the results to the case in which the index of refraction is discontinuous.

Chapter 2

AN OVERVIEW OF PHOTOACOUSTIC AND THERMOACOUSTIC TOMOGRAPHY

As discussed in the introduction, photoacoustic tomography consists of illuminating tissue with near-infra-red light and making measurements of acoustic waves generated by the resulting thermal expansion. The analysis is then roughly divided into two steps. First, we solve an inverse wave problem to recover the thermal deposition map from knowledge of the boundary illuminations and the measured acoustic signals. This gives us information about how heat was absorbed as a result of the illumination. The second step, quantitative photoacoustics, then involves determining the optical properties of the tissue based on the knowledge of the thermal deposition map. In this chapter, we will begin by briefly discussing work that has been done regarding recovery of the thermal deposition map. We will then discuss one model for quantitative photoacoustics and results related to thermoacoustics.

2.1 Recovering the Thermal Deposition Map

Ultimately, our goal in chapter 4 will be to address quantitative photoacoustic tomography. Both quantitative photo- and thermo- acoustic tomography require the assumption that the thermal deposition map has already been recovered. Therefore it is appropriate for us to include a brief discussion of what progress has been made in the recovery of the thermal deposition map. We will discuss several different methods that have been used to address this first step. The pressure wave generated by the thermal expansion in both thermoacoustic and photoacoustic tomography is modeled by the following boundary value problem for the wave equation:

$$\begin{aligned} p_{tt} - c^2(x)\Delta_x p =& 0 \quad\quad t \geq 0, \quad x \in \mathbb{R}^n \\ p(x,0) =& f(x) \\ p_t(x,0) =& 0 \\ p(y,t) =& g(y,t) \quad\quad y \in S, \quad t \geq 0. \end{aligned}$$

Here S is the observation surface and $c(x)$ is the sound speed in the medium. The objective is to recover the initial acoustic pressure, which corresponds to the initial condition $f(x)$.

To begin, we remark that there have been two main categories of techniques to approach this problem. The first is to rephrase the problem in terms a generalized Radon transform and then apply Radon transform type methods. The second category takes advantage of the properties of the wave equation and uses time reversal to recover the initial condition. [18] provides a good comparison of some of the methods in use. We note, however, that [18] does not discuss the most recent developments in the time reversal techniques, which we will include later in this section and can also be found in [33].

First, we consider the Radon transform methods. Informally, the spherical Radon transform arises from the following idea: if an ultrasound signal is emitted at time 0, and if the speed of ultrasound propagation is constant and equal to c, then a signal that reaches a detector at time T was generated by a source on the sphere of radius cT centered on the detector. Thus any given detector on the observation surface is measuring the integral of the source function, f, over a sphere centered on S. More rigorously, the spherical Radon transform arises because there is an explicit solution for the wave equation in terms of spherical means in the case of constant sound speed. We have $p(x,t) = \frac{\partial}{\partial t}[tM(f)(x,t)]$ where $(Mf)(x,t) := \frac{1}{4\pi t^2}\int_{\partial B_t(x)} f(z)dS(z)$. Thus the measured data $g(y,t)$ corresponds to knowledge of $(Mf)(y,t)$ for $y \in S$ and $t > 0$, and the inverse problem becomes reconstruction of f from knowledge of its mean values over spheres centered on the observation surface, S. There have been several papers devoted to generalized Radon transforms, and we recommend [13] as a good starting point for the spherical Radon transform in photo- and thermo- acoustic tomography.

In [13], the authors establish the following results for uniqueness and reconstruction respectively.

Theorem 1. *Suppose that D is a bounded open subset of $\mathbb{R}^n$ with $n \geq 2$. Further suppose that D has smooth boundary, S, and that $\overline{D}$ is strictly convex. If Γ is any relatively open subset of S, and if f is a smooth function on $\mathbb{R}^n$ such that $supp(f) \subset \overline{D}$ and $(Mf)(p,r) = 0$ for all $p \in \Gamma$ and all r, then $f = 0$.*

We note that this is actually a partial data result that requires measurements only on an open subset, Γ, of the boundary. On the other hand, if we have measurements over the entire observation surface, but for a finite length of time, we have the following result.

Theorem 2. *Suppose that D is a bounded, open, connected subset of $\mathbb{R}^n$, n odd and $n \geq 3$, with a smooth boundary S. If f is a smooth function on $\mathbb{R}^n$ supported in $\overline{D}$, and $(Mf)(p,r)$ is known for all p in S and for all $r \in [0, diam(D)/2]$, then one may stably recover f.*

Additionally, the authors produce closed form filtered backprojection type formulas for the odd dimensional case similar to the type encountered when working with the standard Radon transform. In [12], those formulas are extended to the even dimensional case. Despite the existence of closed form inversion formulas, there are significant limitations to the filtered backprojection methods. Specifically, the sound speed must be constant, which may not be a reasonable assumption for certain types of imaging, the observation surface, S, must be a sphere, and there cannot be any ultrasound sources outside of $\overline{D}$ (i.e. $supp(f) \subset \overline{D}$), see [18].

The second method used to reconstruct the thermal deposition map is time reversal in the wave equation. The time reversal method is based on Huygens' principle. When the sound speed is constant and the dimension of the space is odd, Huygens' principle tells us that if the initial source has bounded support (which is assumed to be the case), then the pressure wave will exit any bounded domain in finite time. The consequence of this is that there is some time T such that $p(x,T) = 0 = p_t(x,T)$. Then one can find $f(x)$ by solving

$$\begin{aligned} u_{tt} - c^2\Delta_x u =& 0 \qquad D \times [0,T] \\ u(x,T) =& 0 \\ u_t(x,T) =& 0 \\ u(x,t) =& g(x,t) \qquad S \times [0,T] \end{aligned}$$

and setting $f(x) = u(x,0)$.

When the sound speed is variable and/or the dimension is even, Huygens' principle no longer holds, however, one can show that the solution, $p(x,t)$ decays in time, with the type of decay depending on the properties of $c(x)$ (in particular, if the associated metric is non-trapping, the solution will decay faster and uniformly). As a consequence, we can choose a time T at which we estimate $p(x,t)$ and $p_t(x,t)$ as zero, and then proceed with time reversal as in the constant speed odd dimension case. One difficulty that arises is that the measured data g may not decay to zero at T. As a result, it is conventional to include a smooth cut-off function. The model is then

$$\begin{aligned} u_{tt} - c^2\Delta_x u =& 0 \qquad B \times [0,T] \\ u(x,T) =& 0 \\ u_t(x,T) =& 0 \\ u(x,t) =& g(x,t)\phi_\epsilon(t) \qquad S \times [0,T] \end{aligned}$$

where $\phi_\epsilon(t)$ is a smooth cut-off function that is equal to 0 for $t \geq T$, and is equal to 1 for $t \leq T - \epsilon$. [17] provides error estimates on this approximation in the case of non-trapping sound speed.

We end this section with a discussion of a variation of the time reversal method that actually provides an exact solution rather than the approximate solution obtained from traditional time reversal. This is based on [33].

Let g be a Riemannian metric on $\mathbb{R}^n$ such that g is Euclidean outside of a large compact set, and let $c > 0$ be a smooth real-valued function with $c = 1$ outside of a large compact set. Let P be the differential operator

$$P = c^2 \frac{1}{\sqrt{\det g}} \left(\frac{1}{i}\frac{\partial}{\partial x^i}\right) g^{ij} \sqrt{\det g} \left(\frac{1}{i}\frac{\partial}{\partial x^j}\right).$$

Finally, let Ω be a smooth bounded domain and $T>0$ fixed. If u solves the problem

$$\begin{aligned} (\partial_t^2+P)u&=0 \qquad (0,T)\times\mathbb{R}^n\\ u|_{t=0}&=f \\ \partial_t u|_{t=0}&=0 \end{aligned} \tag{2.1}$$

then the measured data for thermoacoustic tomography has the form $\Lambda f := u|_{[0,T]\times\partial\Omega}$, and the goal is to recover f. The result presented in [33] shows that if the metric $c^{-2}g$ is non-trapping in Ω, then one can recover f via a Neumann series provided that T is selected to be larger than the length of the longest geodesic of $c^{-2}g$ in Ω. If we let $T(\Omega)$ be the supremum of the lengths of geodesics in Ω, the non-trapping condition simply means that $T(\Omega)<\infty$.

In order to state the result rigorously, we need to define an appropriate space in which to work. Let U be a domain in $\mathbb{R}^n$. Then we define the space $H_D(U)$ as the completion of $C_0^\infty(U)$ under the norm

$$||f||^2_{H_D}=\int_U |Df|^2 dVol$$

where $dVol=(\det g)^{1/2}dx$. We will also need to work with the energy, given by

$$E_U(t,u)=\int_U (|Du|^2+c^{-2}|u_t|^2)dVol$$

Given some function, h, solve

$$\begin{aligned} (\partial_t^2+P)v&=0 \qquad \text{in } (0,T)\times\Omega\\ v|_{[0,T]\times\partial\Omega}&=h \\ v|_{t=T}&=\phi \\ \partial_t v|_{t=T}&=0 \end{aligned} \tag{2.2}$$

where ϕ satisfies $P\phi=0$, $\phi|_{\partial\Omega}=h(T,\cdot)$. Then define $Ah:=v(0,\cdot)$ in $\overline{\Omega}$. Then the result from [33] is as follows:

Theorem 3. *Let $(\Omega,c^{-2}g)$ be non-trapping, and let $T>T(\Omega)$. Then $A\Lambda=Id-K$ where K is compact in $H_D(\Omega)$ and $||K||_{H_D(\Omega)}<1$. In particular, $Id-K$ is invertible on $H_D(\Omega)$, and we have*

$$f=\sum_{m=0}^{\infty}K^m A(\Lambda f).$$

Proof. Suppose that u solves (2.1), and v solves (2.2) with $h = \Lambda f$. Further consider the solution, w, to

$$\begin{aligned}(\partial_t^2 + P)w &= 0 \quad (0,T)\times\Omega \\ w|_{[0,T]\times\partial\Omega} &= 0 \\ w|_{t=T} &= u|_{t=T} - \phi \\ w_t|_{t=T} &= u_t|_{t=T}.\end{aligned} \tag{2.3}$$

Then observe that $v + w$ also solves (2.1) so that by uniqueness, $u = v + w$. Since $f = u(0,\cdot)$, we have $f = A\Lambda f + w(0,\cdot)$, and therefore $Kf = w(0,\cdot)$. Since $P\phi = 0$ on Ω and $u_T := u(T,\cdot) = \phi$ on $\partial\Omega$, we have

$$(u_T - \phi, \phi)_{H_D(\Omega)} = 0.$$

Then

$$\begin{aligned}(u_T, u_T)_{H_D(\Omega)} =& (u_T - \phi + \phi, u_T - \phi + \phi)_{H_D(\Omega)} \\ =& (u_T - \phi, u_T - \phi)_{H_D(\Omega)} + (u_T - \phi, \phi)_{H_D(\Omega)} + (\phi, u_T - \phi)_{H_D(\Omega)} + (\phi,\phi)_{H_D(\Omega)} \\ =& (u_T - \phi, u_T - \phi)_{H_D(\Omega)} + (\phi,\phi)_{H_D(\Omega)},\end{aligned}$$

and it follows that

$$\begin{aligned}||u_T - \phi||^2_{H_D(\Omega)} =& ||u_T||^2_{H_D(\Omega)} - ||\phi||^2_{H_D(\Omega)} \\ \le& ||u_T||^2_{H_D(\Omega)}.\end{aligned}$$

Now we consider the energy. From the definitions, we have

$$E_\Omega(T,w) = ||u_T - \phi||^2_{H_D(\Omega)} + ||\partial_t u_T||^2_{L^2(\Omega)},$$

and this together with the previous inequality gives us

$$E_\Omega(T,w) \le E_\Omega(T,u). \tag{2.4}$$

Conservation of energy is a well known property of the wave equation, and this tells us that $E_\Omega(0,w) = E_\Omega(T,w)$, and therefore $E_\Omega(0,w) \le E_\Omega(T,u)$ by (2.4). Furthermore,

$$E_\Omega(T,u) \le E_{\mathbb{R}^n}(T,u) = E_\Omega(0,u) = ||f||^2_{H_D(\Omega)}.$$

Then

$$||Kf||^2_{H_D(\Omega)} = ||w(0,\cdot)||^2_{H_D(\Omega)} \le E_\Omega(0,w) \le ||f||^2_{H_D(\Omega)}.$$

This final inequality establishes that the operator norm of K is less than or equal to 1. In order to arrive at a Neumann series expression for f, we will need the operator norm of K to be strictly less than 1, and thus it remains to show that there is no nonzero f such that $||Kf||_{H_D(\Omega)} = ||f||_{H_D(\Omega)}$. Suppose that there is some $f \neq 0$ such that $||Kf||_{H_D(\Omega)} = ||f||_{H_D(\Omega)}$. Then tracing back through the argument above, it must be the case that $E_\Omega(T, u) = E_{\mathbb{R}^n}(T, u)$, and therefore $u(T, x) = 0$ for $x \in \Omega^c$.

The idea is to apply Tataru's unique continuation theorem to see that $f = 0$, resulting in a contradiction. Indeed, the finite domain of dependence property for the wave equation implies that $u(t, x) = 0$ for $\text{dist}(x, \Omega) > |T - t|$ and for $\text{dist}(x, \Omega) > |t|$. Then $u(x, t) = 0$ for x such that $\text{dist}(x, \Omega) > T/2$ and t in the interval $t \in [-T/2, 3T/2]$. Since u is a solution of the wave equation, it can be extended as an even function of time to obtain a solution with $u(t, x) = 0$ for x such that $\text{dist}(x, \Omega) > T/2$ and $|t| < 3T/2$. Now Tataru's theorem does allow us to conclude that $f = 0$.

Finally, we must show that K is compact. The key point here is that T was selected to be large enough that all geodesics exit $\overline{\Omega}$ by time T, and hence all singularities have left $\overline{\Omega}$. In particular, $u(T, \cdot)$ and $u_t(T, \cdot)$ are smooth on $\overline{\Omega}$. Then elliptic regularity implies that ϕ is also smooth on $\overline{\Omega}$. From this information, one concludes that the maps that take $f \mapsto u(T, \cdot) - \phi$ and $f \mapsto u_t(T, \cdot)$ have smooth Schwartz kernels from $H_D(\Omega)$ to $H_D(\Omega)$, and hence are compact. Then it follows that the operator $f \mapsto w(0, \cdot) = Kf$ mapping $H_D(\Omega)$ to $H_D(\Omega)$ is compact as the composition of a compact map and the bounded solution operator of (2.3). □

We note that the authors of [33] have also considered a model with discontinuous sound speed in [34].This situation arises in brain imaging and other types of imaging that involve a combination of bone and soft tissue, and is therefore important in medical applications.

For the purposes of our discussion of quantitative photo- and thermo- acoustic tomography, the key information is that the amount of energy deposited is proportional to attenuation. In our discussion of quantitative photoacoustics, we will use that the thermal deposition map has the form $H(t, x) = \int_{\mathbb{S}^{n-1}} \sigma_a(x, v')u(t, x, v')dv'$ (see [2]).

2.2 Modeling Quantitative Photoacoustic Tomography via the Diffusion Equation

In order to model both the absorption and scattering properties of photons in photoacoustic tomography, it is natural to use the radiative transfer equation. The diffusion approximation to radiative transfer can be used in highly scattering media, however, and is often valid in situations where radiation will be traveling more than a couple of centimeters into the tissue. The discussion in this section is based on [5]. In this setting, one hopes to recover the diffusion and absorption coefficients, $(D(x), \sigma_a(x))$ in the equation

$$-\nabla \cdot D(x)\nabla u + \sigma_a(x)u = 0.$$

The setup is as follows: we take X to be an open bounded domain with C^2 boundary. We assume that the thermal deposition map has been recovered. In this setting, we take the our data to have the form $H_j(x) = \sigma_a(x)u_j(x)$ (Note that some simplification is required to reduce the data to this form. See [2] and [5] for a discussion of how this data is related to the thermal deposition map). Here u_j is the solution to

$$\begin{aligned} -\nabla \cdot D(x)\nabla u_j + \sigma_a(x)u_j &= 0 \quad \textit{in} \;\; X \\ u_j &= g_j \quad \textit{on} \;\; \partial X. \end{aligned}$$

The main result is that $(D(x), \sigma_a(x))$ can be recovered if we know $H_j(x)$ for two well chosen boundary conditions, g_1 and g_2 (with, of course, some a priori assumptions on $D(x)$ and $\sigma_a(x)$). To state the result rigorously, let us set

$$Y := H^{n/2+k+2+\epsilon}(X)$$

and

$$\mathcal{M} := \{(D, \sigma_a) : (\sqrt{D}, \sigma_a) \in Y \times C^{k+1}(\bar{X}), ||\sqrt{D}||_Y + ||\sigma_a||_{C^{k+1}(\bar{X})} \leq M\}$$

with $\epsilon > 0$ and $k \geq 1$.

Theorem 4. *Let X be an open bounded domain with C^2 boundary. Assume that (D, σ_a) and $(\tilde{D}, \tilde{\sigma}_a)$ belong to $\mathcal{M}$ and that $D|_{\partial X} = \tilde{D}|_{\partial X}$. There is an open set of illuminations $\Omega \subset C^{1,\alpha}(\partial X)$ for some $\alpha > 1/2$ such that if $g_1, g_2 \in \Omega$, and $H_i(x) = \tilde{H}_i(x)$, $i = 1, 2$, then $(D, \sigma_a) = (\tilde{D}, \tilde{\sigma}_a)$.*

One appealing aspect of this result is that it requires knowledge of the thermal deposition map for only two illuminations. The problems we will consider for the transport equation, on the other hand, will assume knowledge of the thermal deposition map for all possible illuminations, g. One weakness of this result, however, is that it does not imply stability. In fact, one must either use more illuminations or impose geometric constraints on the domain to insure stability of the reconstruction of the coefficients. The heart of the proof technique is the construction of a vector field whose integral curves connect every point in the interior of the domain to a point in the boundary of the domain, which will then allow us to use the method of characteristics to find a unique solution to a related Schrödinger equation. The stability issue comes about at points where the integral curves are "nearly tangent" to the boundary. In this situation, small perturbations in the constructed vector field can result in large variations in the solution. There are actually two ways to handle this problem in order to obtain stable reconstruction. First, one can make convexity assumptions on the domain X to eliminate the tangency issue. Using this method, one can obtain a stable reconstruction while still requiring only two boundary illuminations. The second option is to use $2n$ well chosen boundary illuminations, where n is the spatial dimension. This allows one to construct n vector fields, that are ultimately used to construct an invertible matrix that depends stably on the data. We will only give an overview of the proof methods here, as our focus is on the transport equation. For completeness, we will state the two stability results.

Theorem 5. *Let X be a bounded domain with C^{k+1} boundary, $k \geq 2$. Assume that (D, σ_a) and $(\tilde{D}, \tilde{\sigma}_a)$ belong to $\mathcal{M}$ and that $D|_{\partial X} = \tilde{D}|_{\partial X}$. Let $d = (H_1, H_2, ..., H_{2n})$ and $\tilde{d} = (\tilde{H}_1, ..., \tilde{H}_{2n})$ be the internal data for the coefficients (D, σ_a) and $(\tilde{D}, \tilde{\sigma}_a)$ respectively with the boundary conditions $g = (g_j)_{1 \leq j \leq 2n}$. Then there is an open set of illuminations $g \in (C^{k,\alpha}(\partial X))^{2n}$ and a constant C such that*

$$||D - \tilde{D}||_{C^k(X)} + ||\sigma_a - \tilde{\sigma}_a||_{C^k(X)} \leq C||d - \tilde{d}||_{(C^{k+1}(X))^{2n}}.$$

To state the final result, we must establish the appropriate notion of convexity on X.

Hypothesis 6. *There exists $R < \infty$ such that for each $x_0 \in \partial X$, we have $X \subset B_{x_0}(R)$, where $B_{x_0}(R)$ is a ball of radius R that is tangent to ∂X at x_0.*

Theorem 7. *Let X satisfy hypothesis 6 with C^{k+1} boundary, ∂X, $k \geq 3$. Assume that (D, σ_a) and $(\tilde{D}, \tilde{\sigma}_a)$ belong to $\mathcal{M}$ and that $D|_{\partial X} = \tilde{D}|_{\partial X}$. There is an open set of illuminations $\Omega \subset C^{1,\alpha}(\partial X)$ and a constant, C, such that if $g_1, g_2 \in \Omega$, then*

$$||D - \tilde{D}||_{C^{k-1}(X)} + ||\sigma_a - \tilde{\sigma}_a||_{C^{k-1}(X)} \leq C||d - \tilde{d}||_{(C^k(X))^2}.$$

The proof technique for theorem 5 with the diffusion approximation is very different from what we will discuss in the case of the radiative equation. The proof relies on methods somewhat similar to those seen in [35] for approaching Calderón's problem: one reduces the problem to an inverse Schrödinger equation, and then applies the theory of complex geometrical optics solutions.

To make the reduction to the Scrödinger equation, one begins with the equation $-\nabla \cdot D\nabla u + \sigma_a u = 0$ and makes the change of variables $v = \sqrt{D}u$. Then v satisfies the Schrödinger equation $\Delta v + qv = 0$ with potential $q = -\frac{\Delta\sqrt{D}}{\sqrt{D}} - \frac{\sigma_a}{D}$. The internal data is given by $d = \sigma_a u = \mu v$ where $\mu := \frac{\sigma_a}{\sqrt{D}}$. The goal is then to recover the unknown potential q and the function μ from the internal measurements, d.

Now, let us suppose that u_1 and u_2 are solutions of

$$\begin{aligned} \Delta u_j + qu_j &= 0 \\ u_j|_{\partial X} &= g_j \end{aligned}$$

where $g_j \in C^{k,\alpha}(\partial X; \mathbb{C})$. Note that we are working with complex valued boundary values and solutions at the moment, but we will restrict to real and imaginary parts later to return to real valued functions. As before, let $d_j = \mu u_j$. By replacing u_j by d_j/μ in the Schrödinger equation, one can verify that μ satisfies

$$\beta_d \cdot \nabla\mu + \gamma_d \mu = 0 \tag{2.5}$$

with

$$\begin{aligned} \beta_d &:= \chi(x)(d_1\nabla d_2 - d_2\nabla d_1) \\ \gamma_d &:= \frac{-\beta_d \cdot \nabla\mu}{\mu}. \end{aligned}$$

Here χ is a known function, the form of which will depend on the solution, u. We will provide an explicit formula for χ in the case where u is taken to be a CGO solution in a moment.

The thrust of the argument is then as follows: we would like to solve (2.5) for μ. Succeeding at that, we could then solve for u_j using the internal data, $d_j = \mu u_j$. Finally, if u_j is known, then q can be recovered from the Schrödinger equation. Thus our primary concern is to ensure that the equation for μ has a unique solution (note that the boundary values of μ are known since d_j is known and $u_j|_{\partial X} = g_j$ is known). If we knew that the integral curves of β_d connect any point in the interior of X to a point on the boundary, then the method of characteristics would provide us with an explicit formula for the (unique) solution, μ. This is where CGO solutions come into play. In the case where u_1 and u_2 are CGO solutions with $u_2 = \bar{u}_1$, then we can normalize the vector field β_d to have the desired property. We now come to the characterization of the "open set of illuminations" on which uniqueness is guaranteed: we simply require that the illumination, g, be sufficiently close to the boundary values of a CGO solution.

Let $\rho = p + ip^{\perp}$ with $|p| = |p^{\perp}|$, $p \cdot p^{\perp} = 0$, and $|p|$ sufficiently large. Define the CGO solution $u_\rho := e^{\rho\cdot x}(1+\psi_\rho)$ where $\psi_\rho \in H_\delta^{n/2+k+\epsilon}$ is a weak solution of $\Delta\psi_\rho + 2\rho\cdot\nabla\psi_\rho = -q(1+\psi_\rho)$. That such a ψ_ρ exists is discussed in [35], and that it has the stated regularity is discussed in [5]. Then we define the set of allowable boundary conditions in $C^{k,\alpha}(\partial X;\mathbb{C})$ by

$$||g - u_\rho|_{\partial X}||_{C^{k,\alpha}(\partial X;\mathbb{C})} \leq \epsilon$$

for some sufficiently small, positive ϵ.

Now, for such an illumination, g, we let u be the corresponding solution to the Schrödinger equation, and d the corresponding internal data. As before, we obtain β_d and γ_d with $\chi(x) = e^{-2p\cdot x}$. Further, we construct normalized, real-valued counter parts:

$$\begin{aligned} \beta &= \frac{1}{2|p|}\Im\beta_d \\ \gamma &= \frac{1}{2|p|}\Im\gamma_d. \end{aligned}$$

Then it is easy to see that μ satisfies the equation $\beta\cdot\nabla\mu + \gamma\mu = 0$. Our reconstruction will be complete provided that the vector field β has the correct behavior. Thanks to our requirement that g be close to the CGO solution, one can show that

$$\left\|\beta - \mu^2 \frac{p^\perp}{|p^\perp|}\right\|_{C^k(\bar{X})} \leq C\frac{1+\epsilon}{|p^\perp|}.$$

In particular, this inequality implies that all integral curves of β exit the domain in finite time, which therefore implies that μ and q can be reconstructed as discussed above.

The final question then, is how one use knowledge of μ and q to recover the desired functions, D and σ_a. We must assume that $\sqrt{D}$ is known on the boundary, and then the result follows quickly. One simply observes that

$$-\Delta\sqrt{D} - q\sqrt{D} = \mu.$$

Since μ and q are known, and because of our assumptions on D, this equation can be solved for $\sqrt{D}$. Finally, $\sigma_a = \mu\sqrt{D}$.

We will end our discussion of the diffusion approximation here, however, readers are encouraged to refer to [5] for more details and further analysis.

2.3 Quantitative Thermoacoustic Tomography

Although recovery of the thermal deposition map is identical for thermoacoustic and photoacoustic tomography, the second step, in which we attempt to recover the optical parameters of the medium, differs. We take a moment here to describe reconstruction and uniqueness in quantitative thermoacoustic tomography. The appropriate model for thermoacoustic tomography involves time harmonic Maxwell's equations, with the inverse problem being to recover the coefficient $\sigma(x)$ from knowledge of $H(x) := \sigma(x)|E(x)|^2$ where $E(x)$ is the solution to Maxwell's equations, and the quantity $H(x)$ is the thermal deposition map recovered in the first step of thermoacoustic tomography. Following [4], one can make a scalar approximation of Maxwell's equations which ultimately leads to a model for quantitative thermoacoustic tomography via the scalar Helmholtz equation. Then for a boundary illumination $g \in H^{1/2}(\partial\Omega)$ the model becomes

$$\begin{aligned} &\Delta u + k^2 u + ik\sigma(x)u = 0 \qquad && X \\ &u = g && \partial X \end{aligned}$$

where g and k are controlled experimentally. The available data corresponds to the reconstructed thermal deposition map, given in the scalar approximation by $H(x) = \sigma(x)|u|^2(x)$ in Ω, and again, one wishes to recover $\sigma(x)$.

As in the diffusion model for quantitative photoacoustic tomography, the solution to the inverse problem is based on complex geometric optics (CGO) solutions. In this setting, we define a potential $q = k^2 + ik\sigma(x)$ so that the model becomes

$$\begin{aligned} \Delta u + q(x)u &= 0 \qquad && X \\ u &= g && \partial X, \end{aligned}$$

and one aims to recover $q(x)$. The idea is fairly straightforward. If σ (and hence q) is assumed to be sufficiently smooth, and if the illumination g is selected correctly, then a specific functional of σ admits a unique fixed point, and one can obtain an explicit reconstruction algorithm using a standard fixed point iteration scheme. We will provide an overview here, and detailed discussion can be found in [4].

To begin, recall that a CGO solution, u_ρ, for the Schrödinger equation is a function of the form $u_\rho = e^{\rho\cdot x}(1+\psi_\rho)$ such that u_ρ is a solution to $\Delta u_\rho + q(x)u_\rho = 0$ on $\mathbb{R}^n$ and $\psi_\rho \in L^2_\delta$ satisfies $\Delta\psi_\rho + 2\rho\cdot\nabla\psi_\rho = -q(x)(1+\psi_\rho)$ on $\mathbb{R}^n$. We note that this requires q to be extended to all of $\mathbb{R}^n$, and that standard extension theorems allow us to extend q from X to $\mathbb{R}^n$ in such a way that

$$||q||_{H^p(\mathbb{R}^n)} \leq C||q|_X||_{H^p(X)}$$

with C independent of q.

A simple computation shows that when u is replaced by u_ρ in the data $H(x) = \sigma(x)|u|^2(x)$, we have

$$e^{-(\rho+\overline{\rho})\cdot x}H(x) = \sigma(x) + \sigma(x)(\psi_\rho + \overline{\psi}_\rho + \psi_\rho\overline{\psi}_\rho).$$

Then we set $\mathcal{H}[\sigma](x) := \sigma(x)(\psi_\rho + \overline{\psi}_\rho + \psi_\rho\overline{\psi}_\rho)$, and show that $\mathcal{H}$ is a contraction as a function of σ. In the proof method presented in [4], one of the most important details is the choice of the function space on which to show that $\mathcal{H}$ is a contraction. In particular, the proof relies on the property that the selected function space is an algebra. For that reason, we choose $\sigma \in H^{n/2+\epsilon}(X)$ and show that $\mathcal{H}$ is a contraction on this space. We will also assume that σ satisfies $0 < \sigma \leq \sigma_M$ for some positive constant, σ_M. For a fixed $M > 0$, set $\mathcal{M} := \{f \in Y = H^{n/2+\epsilon}(X) : ||f||_Y \leq M\}$. Then a reconstruction algorithm for σ will follow easily from the following lemma:

Lemma 8. *Suppose that q and $\tilde{q}$ are two potentials corresponding to σ and $\tilde{\sigma}$ respectively. Let ψ and $\tilde{\psi}$ be solutions of*

$$\Delta\psi + 2\rho \cdot \nabla\psi = -q(1+\psi)$$

and

$$\Delta\tilde{\psi} + 2\rho \cdot \nabla\tilde{\psi} = -\tilde{q}(1+\tilde{\psi})$$

respectively. If q and $\tilde{q}$ are in $\mathcal{M}$, then there is a constant C such that for all ρ with $|\rho| \geq |\rho_0|$, we have

$$||\psi - \tilde{\psi}||_Y \leq \frac{C}{|\rho|}||\sigma - \tilde{\sigma}||_Y.$$

Now we observe that

$$\begin{aligned}\mathcal{H}[\sigma] - \mathcal{H}[\tilde{\sigma}] =& \sigma(x)(\psi + \overline{\psi} + \psi\overline{\psi}) - \tilde{\sigma}(x)(\tilde{\psi} + \overline{\tilde{\psi}} + \tilde{\psi}\overline{\tilde{\psi}}) \\ =& (\sigma - \tilde{\sigma})\psi + (\psi - \tilde{\psi})\tilde{\sigma} + (\sigma - \tilde{\sigma})\overline{\psi} + (\overline{\psi} - \overline{\tilde{\psi}})\tilde{\sigma} + \sigma\psi\overline{\psi} - \tilde{\sigma}\tilde{\psi}\overline{\tilde{\psi}}.\end{aligned}$$

Using the lemma together with the fact that Y is an algebra and that σ is bounded in Y, we are able to deduce that

$$||(\psi - \tilde{\psi})\tilde{\sigma}||_Y \leq ||(\psi - \tilde{\psi})||_Y||\tilde{\sigma}||_Y \leq \frac{C\sigma_M}{|\rho|}||\sigma - \tilde{\sigma}||_Y,$$

and similarly

$$||(\overline{\psi} - \overline{\tilde{\psi}})\tilde{\sigma}||_Y \leq \frac{C\sigma_M}{|\rho|}||\sigma - \tilde{\sigma}||_Y.$$

In addition, it is known that the remainder term of a CGO solution satisfies $||\psi||_Y \leq \frac{C}{|\rho|}$, and thus

$$\begin{aligned}||(\sigma - \tilde{\sigma})\psi||_Y \leq& \frac{C}{|\rho|}||\sigma - \tilde{\sigma}||_Y \\ ||(\sigma - \tilde{\sigma})\overline{\psi}||_Y \leq& \frac{C}{|\rho|}||\sigma - \tilde{\sigma}||_Y.\end{aligned}$$

Finally, we consider the terms $\sigma\psi\overline{\psi} - \tilde{\sigma}\tilde{\psi}\overline{\tilde{\sigma}}$. We have

$$\begin{aligned}
||\sigma\psi\overline{\psi} - \tilde{\sigma}\tilde{\psi}\overline{\tilde{\sigma}}||_Y =&||(\sigma\psi - \tilde{\sigma}\tilde{\psi})\overline{\psi} + (\overline{\psi} - \overline{\tilde{\psi}})(\tilde{\sigma}\tilde{\psi})||_Y \\
\leq&||(\sigma\psi - \tilde{\sigma}\tilde{\psi})||_Y \frac{C}{|\rho|} + ||(\overline{\psi} - \overline{\tilde{\psi}})||_Y \frac{C\sigma_M}{|\rho|} \\
\leq&\frac{C}{|\rho|}||(\sigma - \tilde{\sigma})\psi||_Y + \frac{C}{|\rho|}||(\psi - \tilde{\psi})\tilde{\sigma}||_Y + \frac{C\sigma_M}{|\rho|}||(\overline{\psi} - \overline{\tilde{\psi}})||_Y \\
\leq&\frac{C}{|\rho|}||\sigma - \tilde{\sigma}||_Y
\end{aligned}$$

for $|\rho|$ sufficiently large.

Putting all of this together, we deduce that

$$||\mathcal{H}[\sigma] - \mathcal{H}[\tilde{\sigma}]||_Y \leq \frac{C}{|\rho|}||\sigma - \tilde{\sigma}||_Y$$

so that $\mathcal{H}$ is a contraction in σ for ρ sufficiently large. In conclusion, if the coefficient σ satisfies $\sigma \in H^{n/2+\epsilon}(X)$ and $0 < \sigma(x) \leq \sigma_M$, and if the illumination g is selected to be the trace on ∂X of a CGO solution, then one can reconstruct σ via the algorithm

$$\begin{aligned}
\sigma_0 =&0 \\
\sigma_j =&e^{-(\rho-\overline{\rho})\cdot x}H(x) - \mathcal{H}[\sigma_{j-1}](x) \\
\sigma =& \lim_{j\to\infty} \sigma_j.
\end{aligned}$$

In [4], the authors extend this result to show that if g is sufficiently 'close' to the trace of a CGO solution, then the same conclusion and reconstruction algorithm hold. In particular, there is an open subset of illuminations in $H^{n/2-1/2+\epsilon}(\partial X)$ for which reconstruction is possible.

We'll now address the proof of the lemma.

Proof. First, we note that in [35] (in which CGO solutions were used to address Calderón's problem), it was shown that ψ can be constructed recursively by setting $\psi_{-1} = 1$, $(\Delta + 2\rho\cdot\nabla)\psi_j = -q\psi_{j-1}$ for $j \geq 0$, and $\psi = \sum_{j\geq 0}\psi_j$, where the series will be summable for ρ sufficiently large.

It was shown in [5] that

$$||\psi_j||_{H^s_\delta} \leq \frac{C}{\rho}||q||_{H^s_1}||\psi_{j-1}||_{H^s_\delta}$$

where C does not depend on ρ, and H^s_δ is the completion of $C_0^\infty(\mathbb{R}^n)$ under the norm

$$||u||^2_{H^s_\delta} = \int_{\mathbb{R}^n} (1+|x|^2)^\delta |(I-\Delta)^{s/2} u|^2 dx.$$

Here $||\psi_{-1}||_{H^s_\delta}$ is taken to be 1.

Since the estimate we ultimately want involves the norm on $Y = H^{n/2+\epsilon}(X)$, we would now like to consider the restriction of ψ_j to X. Recall that the extension of q to $\mathbb{R}^n$ was selected so that

$$||q||_{H^p(\mathbb{R}^n)} \leq C||q|_X||_{H^p(X)}.$$

Then we have

$$||\psi_j||_Y \leq C|\rho|^{-1}||q||_Y||\psi_{j-1}||_Y.$$

From the definition of the functions ψ_j, it is easy to see that

$$(\Delta + 2\rho\cdot\nabla)(\psi_j - \tilde{\psi}_j) = -((q-\tilde{q})\psi_{j-1} + \tilde{q}(\psi_{j-1} - \tilde{\psi}_{j-1})),$$

from which we conclude that

$$|\rho|\,||\psi_j - \tilde{\psi}_j||_Y \leq C||\psi_{j-1}||_Y||q-\tilde{q}||_Y + CM||\psi_{j-1} - \tilde{\psi}_{j-1}||_Y.$$

The key is then to sum over j so that for ρ sufficiently large,

$$\sum_{j\geq 0} ||\psi_j - \tilde{\psi}_j||_Y \leq \frac{C||q-\tilde{q}||_Y}{(1-CM|\rho|^{-1})|\rho|}\sum_{j\geq 0}||\psi_{j-1}||_Y \leq \frac{C'}{(1-CM|\rho|^{-1})|\rho|}(1+||q||_Y)||q-\tilde{q}||_Y.$$

Then we have

$$||\psi - \tilde{\psi}||_Y \leq \frac{C}{|\rho|}||q-\tilde{q}||_Y \leq \frac{Ck}{|\rho|}||\sigma - \tilde{\sigma}||_Y.$$

□

One interesting open problem in QTAT is to attempt to reduce the regularity requirement on σ. If σ is in $H^{n/2+\epsilon}$, then by Sobolev embedding, it must be at least continuous. This level of regularity can be an acceptable assumption in some imaging applications; however, for imaging in regions of the body that involve significantly different tissue types (such as soft tissue and bone), continuity is often an unreasonable assumption. The difficulty in reducing the regularity is primarily caused by the requirement that we have "well-behaved" multiplication in the space. In our discussion, we used that $H^{n/2+\epsilon}$ is an algebra. This property is ultimately more than is required, but one must still be able to obtain an estimate along the lines of $||\sigma\psi|| \leq ||\sigma|| \cdot ||\psi||$. One idea is to attempt to use the new Bourgain style spaces adapted to the Schrödinger equation introduced in [15], but this work is still in progress.

Chapter 3
THE RADIATIVE TRANSFER EQUATION

3.1 Introduction: the Forward Problem

In addition to modeling near-infra-red photons in photoacoustic tomography, the radiative transfer equation has a number of other applications including modeling neutron densities and energy density waves with applications in nuclear and atmospheric science as well as other disciplines. As a result, inverse problems for the radiative transfer equation have been studied in a number of settings. In the next two sections, we will address the results from [8] and [23], as the ideas and proof method will guide our discussion in chapter 4. We note that we use the terms radiative transfer equation and transport equation interchangeably, and in searching through the mathematical references, the reader will frequently see the model referred to as the transport equation.

For a brief discussion of how the transport equation arises in physics, we refer the reader to section XI.12 of [29]. The authors address the use of the time dependent linear transport equation as a model for the scattering of a low-density beam of neutrons off of an obstacle in free space (such as uranium in a nuclear reactor). For mathematical inverse problems related to the transport equation, see [1, 2, 3, 8, 7, 21, 22, 23, 24, 25, 31, 32, 37]. The equation of study in each of these references is the stationary linear transport equation given by:

$$-v \cdot \nabla_x f(x,v) - \sigma_a(x,v) f(x,v) + \int_V k(x,v',v) f(x,v')dv' = 0,$$

and what distinguishes the various problems considered is generally the type of data available and the presence or absence of a source term. The problems we will consider in this chapter may be considered as the "simplest" in some sense as they involve knowledge of the solution, f, on the full boundary of the domain and do not involve a source term. Problems related to medical imaging often involve more complicated averaged data, as we will see in the next chapter.

Before addressing the inverse problems, we begin with a brief discussion of the forward problem for the transport equation. The resulting solution operator for the forward problem turns out to be a valuable tool for addressing several inverse problems in transport theory. The approach we present here for the forward problem is based on [8] and is also presented more generally in [23]. The forward boundary value problem is

$$\begin{aligned} -v\cdot\nabla_x f(x,v) - \sigma_a(x,v)f(x,v) + \int_V k(x,v',v)f(x,v')dv' &= 0 \quad \text{ in } \ X\times V \\ f|_{\Gamma_-} &= f_- \end{aligned} \tag{3.1}$$

Here we assume that the differential equation holds in $X \times V$, where $X \subset \mathbb{R}^n$ is a bounded open set with C^1 boundary ($n \geq 2$), and V is a subset of $\mathbb{R}^n$, called the velocity space. We will assume that $V = \mathbb{S}^{n-1}$, however, other types of velocity spaces can also be used. Γ_- is defined to be the set of all points (x, v) such that x is in the boundary and v "points into the domain." Similarly, we can define a set Γ_+ as the set of points (x, v) such that $x \in \partial X$ and v "points out of the domain." More rigorously, $\Gamma_\pm = \{(x,v) \in \partial X \times V : \pm n(x)\cdot v > 0\}$, where $n(x)$ is the unit outer normal to ∂X at x. We will want to work with function spaces on Γ_-, and thus we define the measure $d\xi = |n(x)\cdot v|d\mu(x)dv$, where $d\mu(x)$ and dv are the standard Lebesgue measures on ∂X and V respectively.

In order to address the forward problem, we will assume that (σ_a, k) is an *admissible pair*, meaning that $0 \leq \sigma_a \in L^\infty(X \times V)$, $0 \leq k(x, v', \cdot) \in L^1(V)$ for a.e. $(x, v') \in X \times V$, and $\sigma_p(x, v') \in L^\infty(X \times V)$ where $\sigma_p := \int_V k(x, v', v)dv$.

We also introduce several operators (following the notation of [8]) that will arise throughout our analysis. For $f \in L^1(X \times V)$, set

$$\begin{aligned} T_0 f &= -v\cdot\nabla_x f \\ A_1 f &= -\sigma_a f \\ A_2 f &= \int_V k(x,v',v)f(x,v')dv' \\ T_1 &= T_0 + A_1 \\ T &= T_0 + A_1 + A_2. \end{aligned}$$

We will also make use of two operators that arise as part of the solution operator:

$$Jf_- = e^{\int_0^{\tau_-(x,v)} \sigma_a(x-sv,v)ds} f_-(x - \tau_-(x,v)v, v)$$
$$Kf = -\int_0^{\tau_-(x,v)} e^{-\int_0^t \sigma_a(x-sv,v)ds}(A_2 f)(x - tv, d)dt.$$

Here we again have $f \in L^1(X \times V)$, and $f_- \in L^1(\Gamma_-, d\xi)$.

It is straightforward to show that the boundary value problem (3.1) is equivalent to the integral equation $(I+K)f = Jf_-$. Indeed, if $Tf = 0$, then $\int_0^{\tau_-(x,v)} e^{-\int_0^t \sigma_a(x-sv,v)ds}(Tf)(x-tv,v)dt = 0$. Recall that$Tf = (T_0 + A_1 + A_2)f$. Inserting the definitions of T_0, A_1, and A_2 and performing an integration by parts on the A_1 term yields $(I + K)f = Jf_-$, and therefore the forward problem is solved if $(I + K)$ is invertible. It is indeed the case that (I+K) is invertible, and a unique solution exists in the space

$$\mathcal{W} := \{f : T_0 f \in L^1(X \times V), \tau^{-1} f \in L^1(X \times V)\}$$

with $||f||_{\mathcal{W}} := ||T_0 f|| + ||\tau^{-1} f||$.

Theorem 9. *Assume that (σ_a, k) is an admissible pair and that $||\tau\sigma_a||_{L^\infty} < \infty$ and $||\tau\sigma_p||_{L^\infty} < 1$. Then K is a bounded operator in $L^1(X\times V; \tau^{-1}dxdv)$ with $||K|| \le ||\tau\sigma_p||_{L^\infty} < 1$ so that $(I+K)^{-1}$ exists in $L^1(X \times V; \tau^{-1}dxdv)$. The integral equation $(I+K)f = Jf_-$ is uniquely solvable for any $f \in L^1(\Gamma_-, d\xi)$, and for the resulting solution, f, we have $f \in \mathcal{W}$.*

The idea of the proof is to recognize that $K = T_1^{-1}A_2$ and that $\tau^{-1}T_1^{-1}$ and $A_2\tau$ are bounded operators on $L^1(X \times V)$. These observations allow us to conclude that

$$||\tau^{-1}Kf|| = ||\tau^{-1}T_1^{-1}A_2 f|| \le ||A_2 f|| \le ||\tau\sigma_p||_{L^\infty}||\tau^{-1}f||.$$

Since we have assumed that $||\tau\sigma_p||_{L^\infty} < 1$, $(I + K)$ is invertible, and the forward problem is uniquely solvable with $f \in \mathcal{W}$.

3.2 The full Albedo Operator in Euclidean Space

Some of the most widely used methods for approaching inverse problems in transport theory involve analyzing the singular decomposition of the solution operator for the forward transport problem. In particular, this type of method was used by Choulli and Stefanov in [8]. These authors had also employed such methods for the time dependent transport equation in [7]. In [8], the authors consider the stationary (i.e. time independent) linear transport equation introduced above:

$$-v \cdot \nabla_x f(x,v) - \sigma_a(x,v) f(x,v) + \int_V k(x,v',v) f(x,v') dv' = 0 \quad \text{in} \quad X \times V$$
$$f|_{\Gamma_-} = f_-$$

The coefficient σ_a is called the absorption (or attenuation) coefficient, and k is called the scattering coefficient. In the inverse problems we consider, we hope to recover σ_a and k from information about the solution, f.

In the case of [8], the measured data is encoded in the *albedo operator*:

$$\mathcal{A} : f_- \mapsto f|_{\Gamma_+}.$$

Essentially, this map takes the prescribed condition on the incoming set, Γ_-, and maps it to the outgoing flux on the boundary. The goal is to recover the absorption and scattering coefficients from knowledge of the albedo operator, $\mathcal{A}$.

Before we state the main result of [8], we must discuss necessary *apriori* assumptions on σ_a and k. First, it is conventional to assume that (σ_a, k) is an admissible pair as defined in the previous section (see [29]). We recall that this means that σ_a is a nonnegative function in $L^\infty(X \times V)$, $0 \leq k(x,v',\cdot) \in L^1(v)$ a.e. in $X \times V$, and that the *production rate*, defined as $\sigma_p(x,v') = \int_V k(x,v',v)dv$, is in $L^\infty(X \times V)$. In order to insure well-posedness of the direct problem, it is necessary to make at least one more assumption on the absorption and production rates. There are several common (physically reasonable assumptions), one of which is simply that the absorption rate is strictly larger than the production rate ($\sigma_a - \sigma_p \geq c > 0$ a.e. in $X \times V$). For our purposes, we will instead assume that

$$||\tau\sigma_a||_{L^\infty} < \infty \quad \text{and} \quad ||\tau\sigma_p||_{L^\infty} < 1. \tag{3.2}$$

Physically, the purpose of these conditions is to insure a “subcritical” dynamic, which is required for the energy of the system to remain finite [29]. The main result obtained in [8] is then as follows:

Theorem 10. *Let* (σ_a, k) *and* $(\hat{\sigma}_a, \hat{k})$ *be two admissible pairs with* $\sigma_a = \sigma_a(x, |v|)$ *and* $\hat{\sigma}_a = \hat{\sigma}_a(x, |v|)$*. Further assume* (3.2)*. If* $\mathcal{A} = \hat{\mathcal{A}}$*, then* $\sigma_a = \hat{\sigma}_a$*, and when* $n \geq 3$*, we also have* $k = \hat{k}$*.*

We will primarily be interested in the method used to prove this result, but let us first briefly remark on the requirement that the absorption coefficient depend only on the magnitude of the velocity. In [8], the authors provide a counterexample to uniqueness when σ_a is allowed to depend arbitrarily on v. Specifically, if $p(x, v)$ is a continuous function such that $(x + p(x, v)v, v)$ is in $X \times V$ whenever (x, v) is, then the distinct absorption coefficients $\sigma_a(x, v)$ and $\sigma_a(x + p(x, v)v, v)$ result in the same albedo operator (with the scattering coefficient, k, taken to be 0). Readers may refer to [25] for additional results concerning the dependence of σ_a on v.

We will state here a brief overview of the proof method used in [8], which we will return to when we discuss the specific application of the transport equation in photoacoustic tomography. The heart of the argument presented by Choulli and Stefanov is to obtain a singular decomposition of the albedo operator, $\mathcal{A}$.

In order to accomplish this, we begin by solving the following boundary value problem:

$$-v \cdot \nabla_x f(x, v) - \sigma_a(x, v) f(x, v) + \int_V k(x, v', v) f(x, v') dv' = 0 \quad \text{in} \quad X \times V$$

$$f|_{\Gamma_-} = \delta(x - x')\delta(v - v').$$

Here the delta distributions are defined as expected:

$$\int_{\Gamma_-} f(x', v')\delta(y' - x')\delta(w', v') d\mu(x') dv' = f(y', w').$$

Since f is to be a weak (distribution) solution, we find an expression for f by choosing $\phi_- \in C_0^1(\Gamma_-)$ and considering the problem

$$-v \cdot \nabla_x \phi(x, v) - \sigma_a(x, v)\phi(x, v) + \int_V k(x, v', v)\phi(x, v') dv' = 0 \quad \text{in} \quad X \times V$$

$$\phi|_{\Gamma_-} = \phi_-.$$

Here we use test functions in $C_0^1(\Gamma_-)$ rather than $C_0^\infty(\Gamma_-)$ because the boundary is only assumed to be C^1. Since f is simply the kernel of the solution operator, finding an expression for ϕ will allow us to determine f. As seen in the previous section, the solution, ϕ, can be written as a sum of three terms:

$$\phi = J\phi_- - KJ\phi_- + T^{-1}A_2KJ\phi_-.$$

The analysis proceeds by considering the distribution kernel of each term individually. Thus we will decompose f as $f = f_1 + f_2 + f_3$ where

$$\begin{aligned} J\phi_- &= \int_{\Gamma_-} f_1(x,v,x',v')\phi_-(x',v')d\mu(x')dv', \\ -KJ\phi_- &= \int_{\Gamma_-} f_2(x,v,x',v')\phi_-(x',v')d\mu(x')dv', \qquad \text{and} \\ T^{-1}A_2KJ\phi_- &= \int_{\Gamma_-} f_2(x,v,x',v')\phi_-(x',v')d\mu(x')dv'. \end{aligned}$$

The results are summarized in the following theorem.

Theorem 11. $f = f_1 + f_2 + f_3$, *where*

$$\begin{aligned} f_1 &= |n(x')\cdot v'| \int_0^{\tau_+(x',v')} e^{-\int_0^{\tau_-(x,v)} \sigma_a(x-pv,v)dp}\delta(x-x'-tv)\delta(v-v')dt \\ f_2 &= |n(x')\cdot v'| \int_0^{\tau_-(x,v)} \int_0^{\tau_+(x',v')} e^{-\int_0^s \sigma_a(x-pv,v)dp} \\ &\times\ e^{-\int_0^{\tau_-(x-sv,v')} \sigma_a(x-sv-pv',v')dp} k(x-sv,v',v)\delta(x-x'-sv-tv')dtds \\ & (\min\{\tau,\lambda\})^{-1}|n(x')\cdot v'|^{-1} f_3 \in L^\infty(\Gamma_-;\mathcal{W}) \end{aligned}$$

We will include the proof of the formula for f_2 here.

Proof. To establish the formula for f_2, we analyze the distribution kernel of $-KJ\phi_-$. Based on the definitions of K and J, we have

$$\begin{aligned} -KJ\phi_- = \int_0^{\tau_-(x,v)} \int_V e^{-\int_0^s \sigma_a(x-pv,v)dp} k(x-sv,v',v)E(x-sv,v') \\ \times\ \phi_-(x-sv-\tau_-(x-sv,v')v',v')dv'ds. \end{aligned}$$

We introduce the distribution δ on X characterized by

$$\int_X \delta(y-x)f(y)dy = f(x)$$

to obtain

$$-KJ\phi_- = \int_X \int_0^{\tau_-(x,v)} \int_V e^{-\int_0^s \sigma_a(x-pv,v)dp} k(x-sv,v',v)E(x-sv,v') \times \phi_-(y-\tau_-(y,v')v',v')\delta(y-x+sv)dv'dsdy.$$

Next we make the observation that if $f \in L^1(X \times V)$, then

$$\int_{X\times V} f(x,v)dxdv = \int_{\Gamma_-} \int_0^{\tau_+(x',v)} f(x'+tv,v)dtd\xi(x',v).$$

This can be seen by a straightforward change of variables, and is a special case of the more general formula known as Santaló's formula. Applying this formula in the variables (y,v'), we have

$$-KJ\phi_- = \int_{\Gamma_-} \int_0^{\tau_+(x',v')} \int_0^{\tau_-(x,v)} e^{-\int_0^s \sigma_a(x-pv,v)dp} k(x-sv,v',v)E(x-sv,v') \times \phi_-(x',v')\delta(x'+tv'-x+sv)dsdt|n(x')\cdot v'|d\mu(x')dv'.$$

From this, we conclude that

$$-KJ\phi_- = \int_{\Gamma_-} f_2(x,v,x',v')\phi_-(x',v')d\mu(x')dv'$$

with f_2 as claimed. □

This provides useful information about f, however, recall that we are ultimately hoping to obtain information about the distribution kernel, α, of the albedo operator, $\mathcal{A}$. Since the albedo operator simply maps the values of the solution on Γ_- to the values of the solution on Γ_+, we formally have

$$\alpha(x,v,x',v') = f(x,v,x',v')|_{(x,v)\in\Gamma_+,(x',v')\in\Gamma_-}.$$

Consequently, we get the following expression for the kernel of the albedo operator.

Theorem 12. $\alpha = \alpha_1 + \alpha_2 + \alpha_3$, *where*

$$\begin{aligned}
\alpha_1 &= e^{-\int_0^{\tau_-(x,v)} \sigma_a(x-pv,v)dp}\delta(x-x'-\tau_-(x,v)v)\delta(v-v') \\
\alpha_2 &= \int_0^{\tau_-(x,v)} e^{-\int_0^s \sigma_a(x-pv,v)dp} e^{-\int_0^{\tau_-(x-sv,v')} \sigma_a(x-sv-pv',v')dp} \\
&\times k(x-sv,v',v)\delta(x-x'-sv-\tau_-(x-sv,v')v')ds \\
&\quad (\min\{\tau(x',v'),\lambda\})^{-1}|n(x')\cdot v'|^{-1} \\
\alpha_3 &\in L^\infty(\Gamma_-;L^1(\Gamma_+,d\tilde{\xi}))
\end{aligned}$$

This result indicates that α_1 and α_2 are distributions with different degrees of singularity, while α_3 is an L^∞ function. This means that it may be possible to distinguish α_1, α_2, and α_3 from one another, which is vital for the recovery of σ_a and k. Now it only remains to show that the coefficients σ_a and k can be recovered from knowledge of α. The results obtained [8] show that in any dimension, $n \geq 2$, α_1 can be distinguished from α_2 and α_3. Furthermore, α_1 determines the x-ray transform of σ_a, which in turn uniquely determines σ_a (provided that $\sigma_a = \sigma_a(x, |v|)$, see [16]). Set

$$\phi_\epsilon(x, v, x', v') = \phi\left(\frac{x - x' - \tau_-(x, v)}{\epsilon}\right)\phi\left(\frac{v - v'}{\epsilon}\right)$$

where ϕ is a smooth, compactly supported function on $\mathbb{R}^n$, taking values between 0 and 1, and satisfying $\phi(0) = 1$ and $\int \phi(x)dx = 1$.

Theorem 13.

$$\lim_{\epsilon \to 0} \int_{\Gamma_-} \alpha(x, v, x', v')\phi_\epsilon(x, v, x', v')d\mu(x')dv' = e^{-\int_0^{\tau_-(x,v)} \sigma_a(x - pv, v)dp}$$

where the integral is to be considered in distribution sense and the limit holds in $L^1_{loc}(\Gamma_+, d\xi)$.

For recovery of k, the situation is not quite as good. In three dimensions or higher, α_2 can be distinguished from α_3, however, in two dimensions, this is not the case. In fact, when $n = 2$, α_2 is a function rather than a singular distribution, and hence cannot be distinguished from α_3. The consequence is that this method can only be used to recover k for $n \geq 3$. The following theorem shows that once σ_a is known, one can use α_2 to recover k. Take ϕ as above and choose ϕ_1 a smooth compactly supported function on $\mathbb{R}^n$ taking values between 0 and 1 and with total integral equal to 1. Then set

$$\phi_{\epsilon_1\epsilon_2}(x', v, v') = \frac{1}{\epsilon_1}\phi_1\left(\frac{x' \cdot m(v, v')}{\epsilon_1 v \cdot m(v, v')}\right)\phi\left(\frac{x' - \pi_{v,v'}(x')}{\epsilon_2}\right)$$

where $\pi_{v,v'}(x)$ is the projection of x onto the plane spanned by v and v' and $m(v, v') = (v \cdot v'/|v'|^2)v' - v$.

Theorem 14. *Assume that* $n \geq 3$. *Then for* $x \in X$, *we have*

$$\lim_{\epsilon_1 \to 0} \lim_{\epsilon_2 \to 0} \int_{\partial X} \alpha(x + \tau_+(x, v)v, v, x', v')$$
$$\times \phi_{\epsilon_1\epsilon_2}(x' - x + \tau_-(x, v')v', v, v')d\mu(x')$$
$$= e^{-\int_0^{\tau_-(x,v')} \sigma_a(x - pv, v)dp} e^{-\int_0^{\tau_-(x - sv, v')} \sigma_a(x - sv - pv', v')dp} k(x, v', v)$$

where the limit holds in $L^1_{loc}(X \times (V^2 \backslash D))$.

Since $e^{-\int_0^{\tau_-(x,v')} \sigma_a(x-pv,v)dp}$ and $e^{-\int_0^{\tau_-(x-sv,v')} \sigma_a(x-sv-pv',v')dp}$ are determined once σ_a is known, this results in the recovery of k.

3.3 The Full Albedo Operator on a Riemannian Manifold

In some applications modeled by the radiative transfer equation, there may be an ambient field or inhomogeneous media that influences the behavior of the photons or neutrons being studied. In this situation, we must switch from Euclidean space to a Riemannian manifold. When we discuss photoacoustic tomography, the Riemannian manifold case will actually correspond to a situation in which the index of refraction varies in the body to be imaged.

We will begin here with a brief discussion of *An Inverse Problem for the Transport Equation in the Presence of a Riemannian Metric*, by Stephen McDowall. This paper translates the work of Choulli and Stefanov discussed above into the Riemannian manifold setting in which the particles travel along the geodesics of the metric (in the Euclidean case, the geodesics are straight lines, and the analysis assumed that the ballistic particles traveled along these lines).

The underlying method used by McDowall in [23] is the same as that used in [8], however, one must use some care in properly defining objects in the manifold setting. To begin, we let $M \subset \mathbb{R}^n$ be a smooth bounded domain ($n \geq 2$) and g a Riemannian metric on M. We will assume that the velocity space, V, is equal to the tangent space, TM, however, the results are equally valid when one considers the unit sphere bundle in the tangent space. In this setting, the transport equation becomes:

$$-\mathcal{D}f(x,v) - \sigma_a(x,v)f(x,v) + \int_{T_xM} k(x,v',v)f(x,v')dv'_x = 0.$$

Here $\mathcal{D}$ is the derivative along the geodesic flow of the metric, and is defined by

$$\mathcal{D}f(x,v) = \frac{\partial}{\partial t}|_{t=0} f(\gamma_{(x,v)}(t), \dot{\gamma}_{(x,v)}(t)).$$

We use $\gamma_{(x,v)}(t)$ to denote the geodesic with initial point x and initial velocity v. Following [23], we will also use the notation $\vec{\gamma}_{(x,v)}(t) = (\gamma_{(x,v)}(t), \dot{\gamma}_{(x,v)}(t))$. The sets $\Gamma_\pm$ are defined as before with the standard dot product replaced by the inner product with respect to the metric g. Then provided that the forward boundary value problem has unique solution, we can define the albedo operator as $\mathcal{A} : f_- = f|_{\Gamma_-} \mapsto f|_{\Gamma_+}$. The inverse problem, as before, is the recovery of σ_a and k from full knowledge of the albedo operator.

To begin, we must start by making some assumptions on the metric, g. One advantage of working with the Euclidean metric is that any geodesic that begins in the interior of a bounded domain will reach the boundary in finite time. For general Riemannian metrics, this may not be the case. In addition, we recall that in [8], the authors showed that one could recover the x-ray transform of σ_a from knowledge of the albedo operator. In this setting, one recovers the geodesic ray transform. In order to insure invertibility of this transform, assumptions on the metric are necessary. The primary assumption that we will need to make here is that g be *simple*. This means that M is strictly convex with respect to g and that the exponential map $\mathrm{Exp} : \mathrm{Exp}^{-1}(\bar{M}) \to \bar{M}$ is a diffeomorphism at each point $x \in \bar{M}$. Additionally, one must assume that the full set of geodesics join boundary points.

The approach is similar to that for the Euclidean case. One begins by considering the boundary value problem

$$-\mathcal{D}f(x,v) - \sigma_a(x,v)f(x,v) + \int_{T_xM} k(x,v'',v)f(x,v'')dv''_x = 0 \qquad M \times TM$$

$$f|_{\Gamma_-} = \delta_{\{\hat{x},\hat{v}\}}(x',v').$$

In this case, the boundary is assumed to be smooth, and thus one searches for the distribution solution by solving

$$-\mathcal{D}\phi(x,v) - \sigma_a(x,v)\phi(x,v) + \int_{T_xM} k(x,v'',v)\phi(x,v'')dv''_x = 0$$

$$\phi|_{\Gamma_-} = \phi_-$$

for $\phi_- \in C_0^\infty(\Gamma_-)$. Just as in the Euclidean case, the analysis of the forward problem shows that

$$\phi = J\phi_- - KJ\phi_- + \mathrm{T}^{-1}T_1KJ\phi_-$$

where

$$\begin{aligned} Jf_-(x,v) =& E(x,v,0,-\tau_-(x,v))f_-(\vec{\gamma}_{(x,v)}(-\tau_-(x,v))) \\ E(x,v,s,t) =& \exp\left(\int_s^t \sigma_a(\vec{\gamma}_{(x,v)}(p))dp\right) \\ Kf(x,v) =& -\int_0^{\tau_-(x,v)} E(x,v,0,t-\tau_-(x,v))(T_1 f)(\vec{\gamma}_{(x,v)}(t-\tau_-(x,v)))dt \\ T_1 f(x,v) =& \int_{T_xM} k(x,v',v)f(x,v')dv' \quad \text{ and} \\ Tf =& -\mathcal{D}f - \sigma_a f + T_1 f. \end{aligned}$$

The analysis presented in [23] of the terms in the expression for ϕ results in the following formulas for the distribution kernel, f, of the solution operator, and consequently α, the distribution kernel of the albedo operator.

Theorem 15. *The distribution kernel, f, of the solution operator is given by $f = f_0+f_1+f_2$, where*

$$f_0(x,v,x',v') = \int_0^{\tau_+(x',v')} E(x,v,0,-\tau_-(x,v))\delta_{(x,v)}(\vec{\gamma}_{(x',v')}(t))dt$$

$$\begin{aligned} f_1(x,v,x',v') \quad = \quad & \int_0^{\tau_+(x',v')}\int_0^{\tau_-(x,v)} E(x,v,0,s-\tau_-(x,v))E(x',v',0,r) \\ & \times k(\vec{\gamma}_{(x',v')}(r), \mathcal{P}(\dot{\gamma}_{(x,v)}(s-\tau_-(x,v)); \gamma_{(x,v)}(s-\tau_-(x,v)), \gamma_{(x',v')}(r))) \\ & \times \delta_{\{\gamma_{(x,v)}(s-\tau_-(x,v))\}}(\gamma_{(x',v')}(r))dsdr \end{aligned}$$

and $f_2 \in L^\infty(\Gamma_-;\mathcal{W})$.

We will include the proof of the formula for f_0 here to demonstrate the ideas that will appear in chapter 4. The results for f_1 and f_2 use similar techniques. We omit that portion of the proof here and refer the reader to [23] for details.

Proof. To find a formula for f_0, we need to find the distribution kernel of the operator J. To accomplish this, we will consider the behavior of $J\phi_-$ when paired with an element of $C_0^\infty(TM)$. If $\psi \in C_0^\infty(TM)$, we have

$$\begin{aligned} \langle J\phi_-, \psi\rangle =& \int_M \int_{T_xM} J\phi_-(x,v)\psi(x,v)dv_x d\omega \\ =& \int_{\Gamma_-}\int_0^{\tau_+(x',v')} J\phi_-(\vec{\gamma}_{(x',v')}(t))\psi(\vec{\gamma}_{(x',v')}(t))dtd\mu(x',v'). \end{aligned}$$

The second line here comes from the observation that if $f \in L^1(TM)$, then

$$\int_M \int_{T_xM} f(x,v)dv_x dx = \int_{\Gamma_-} \int_0^{\tau_+(x',v')} f(\vec{\gamma}_{(x',v')}(t))dtd\mu(x',v').$$

Inserting the definition of J, we have

$$\begin{aligned}\langle J\phi_-, \psi\rangle = &\int_{\Gamma_-} \int_0^{\tau_+(x',v')} E(\vec{\gamma}_{(x',v')}(t), 0, -\tau_-(\vec{\gamma}_{(x',v')}(t))) \\ &\times \phi_-(\vec{\gamma}_{(\vec{\gamma}_{(x',v')}(t))}(-\tau_-(\vec{\gamma}_{(x',v')}(t))))\psi(\vec{\gamma}_{(x',v')}(t))dtd\mu(x',v').\end{aligned}$$

We can make two straightforward simplifications here. First, since $(x',v') \in \Gamma_-$, we note that $-\tau_-(\vec{\gamma}_{(x',v')}(t)) = -t$. Second, $\vec{\gamma}_{(\vec{\gamma}_{(x',v')}(t))}(-t) = \vec{\gamma}_{(x',v')}(t+(-t)) = (x',v')$. Thus we have

$$\langle J\phi_-, \psi\rangle = \int_{\Gamma_-} \phi_-(x',v') \int_0^{\tau_+(x',v')} E(\vec{\gamma}_{(x',v')}(t), 0, -t)\psi(\vec{\gamma}_{(x',v')}(t))dtd\mu(x',v').$$

Now let $\delta_{(y,w)}$ be the distribution on TM characterized by

$$\left\langle \delta_{(y,w)}, f \right\rangle = \int_M \int_{T_xM} \delta_{(y,w)}(x,v) f(x,v) dv_x d\omega = f(y,w).$$

Then we have

$$\begin{aligned}\langle J\phi_-, \psi\rangle &= \int_M \int_{T_xM} \int_{\Gamma_-} \phi_-(x',v') \int_0^{\tau_+(x',v')} E(x,v,0,-\tau_-(x,v))\delta_{\vec{\gamma}_{(x',v')}(t)}(x,v) \\ &\quad \times dtd\mu(x',v')\psi(x,v)dv_x d\omega \\ &= \left\langle \int_{\Gamma_-} f_0(x,v,x',v')\phi(x',v')d\mu(x',v'), \psi \right\rangle\end{aligned}$$

with f_0 as in the statement of the theorem. Thus we have

$$J\phi_-(x,v) = \int_{\Gamma_-} f_0(x,v,x',v')\phi_-(x',v')d\mu(x',v')$$

with f_0 as claimed. □

As before, we obtain information about α by looking at $f(x,v,x',v')$ for $(x,v) \in \Gamma_+$ and $(x',v') \in \Gamma_-$.

Theorem 16. *The distribution kernel, $\alpha(x,v,x',v')$ of the albedo operator, A, is given by $\alpha = \alpha_0 + \alpha_1 + \alpha_2$ where*

$$\begin{aligned}
\alpha_0 &= E(x,v,-\tau_-(x,v),0)\delta_{\vec{\gamma}_{(x,v)}(-\tau_-(x,v))}(x',v') \\
\alpha_1 &= \int_0^{\tau_+(x',v')}\int_0^{\tau_-(x,v)} E(x,v,0,s-\tau_-(x,v))E(x',v',0,r) \\
&\quad \times k(\vec{\gamma}_{(x',v')}(r), \mathcal{P}(\dot{\gamma}_{(x,v)}(s-\tau_-(x,v)); \gamma_{(x,v)}(s-\tau_-(x,v)), \gamma_{(x',v')}(r))) \\
&\quad \times \delta_{\{\gamma_{(x,v)}(s-\tau_-(x,v))\}}(\gamma_{(x',v')}(r))dsdr \\
\alpha_2 &\in L^\infty(\Gamma_-; L^1(\Gamma_+, d\mu)).
\end{aligned}$$

Now that we have an expression for the distribution kernel, we may use an approximate identity to recover σ_a. We start with the following function: let $\psi \in C_0^\infty(\mathbb{R})$ satisfy $0 \leq \psi \leq 1$, $\psi(0) = 1$, and $\int \psi(x)dx = 1$, and set $\psi_\epsilon(x) = \psi(x/\epsilon)$. Further, let $\phi(x,v,x',v')$ be a defining function for the support of the distribution $\delta_{\{\vec{\gamma}_{(x,v)}(-\tau_-(x,v))\}}(x',v')$ (that is, a function ϕ such that $supp(\delta_{\{\vec{\gamma}_{(x,v)}(-\tau_-(x,v))\}}(x',v')) = \{\phi(x,v,x',v') = 0\}$). For the sake of brevity, we will not discuss how such a ϕ is found, however, one may refer to [23] for the explicit formula. Then we have the following:

Theorem 17.

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_{\Gamma_-} \alpha(x,v,x',v')(\psi_\epsilon\circ\phi)(x,v,x',v')d\mu(x',v') \\
&\qquad = \exp\left(\int_{-\tau_-(x,v)}^0 \sigma_a(\vec{\gamma}_{(x,v)}(r))dr\right)
\end{aligned}$$

where the limit holds in $L^1_{loc}(\Gamma_+, d\mu(x,v))$ and the first integral is to be interpreted in the sense of distributions.

This theorem tells us that one can recover the geodesic ray transform of σ_a from knowledge of A. Under our assumptions, the geodesic ray transform is invertible (see [30]), and thus we have recovered σ_a.

Similarly, McDowall shows that for $n \geq 3$, one can recover

$$E(\vec{\gamma}_{(y,w)}(\tau_+(y,w)),0,-\tau_+(y,w))E(\vec{\gamma}_{(y,w')}(\tau_-(y,w')),0,\tau_-(y,w'))k(y,w,w').$$

Since the two exponential terms are known once σ_a is known, this gives the recovery of k.

At the beginning of this chapter, we provided a list of references for inverse transport problems. Many of these problems involve similar techniques but differ based on the form of the albedo operator to be analyzed. Interested readers may refer specifically to [21], [22], and [24] for similar types of problems with slight variations on the albedo operator.

Chapter 4

THE RADIATIVE TRANSFER MODEL OF QUANTITATIVE PHOTOACOUSTIC TOMOGRAPHY

4.1 Transport Equation in Photoacoustics

Finally, we will bring together our discussions of the radiative transfer equation and quantitative photoacoustics. We will begin by discussing the model presented in [2]. This is an alternative to the diffusion approximation that is useful when scattering is low enough that ballistic and single scattering photons play a significant role. From a purely theoretical point of view, the transport approach is accurate in any setting, however, from a physical perspective, if scattering is very great, then the number of ballistic and single scattering photons will be too small relative to the number of multiply scattered photons to be reasonably recovered, and in such a setting, the diffusion approximation may be appropriate.

In [2], the authors again consider the stationary transport equation:

$$v \cdot \nabla_x u + \sigma(x,v)u - \int_{\mathbb{S}^{n-1}} k(x,v',v)u(x,v')dv' = 0 \qquad X \times \mathbb{S}^{n-1}$$

$$u(x,v)|_{\Gamma_-} = \phi(x,v)$$

The primary deviation from the result discussed in [8] is the definition of the albedo operator. As in [5], one assumes that the data for the inverse problem is the thermal deposition map. Thus the albedo operator is here defined as

$$A : \phi(x,v) \mapsto \int_{\mathbb{S}^{n-1}} \sigma_a(x,v)u(x,v)dv.$$

Here we assume that σ is the total attenuation coefficient, and $\sigma_a = \sigma - \sigma_s$ where $\sigma_s := \int_{\mathbb{S}^{n-1}} k(x,v,v')dv'$. Ultimately, the authors use the same type of method used in [8], however, the data we have has been angularly averaged, with the consequence that the results obtained are a bit weaker and the analysis a bit more challenging.

A first step is to show that the albedo operator is well-defined. The appropriate spaces to work in are

$$A : L^1(\Gamma_-, d\xi) \to L^1(X)$$

where $d\xi = |v \cdot \nu(x)| d\mu(x) dv$ with $d\mu(x)$ being the surface measure on ∂X. To insure that A is well-defined and simplify analysis, several assumptions are made. First, the domain, X is assumed to be an open, bounded, convex domain in $\mathbb{R}^n$ with C^1 boundary. We assume that $k(x, v', v) = 0$ outside X and that $\sigma_a(x, v)$ is known outside of X. Both σ and k are assumed to be nonnegative and bounded by a constant M. We also assume that σ_a is bounded above and below by positive constants.

The authors begin by decomposing the albedo operator as $A = A_0 + A_1 + G_2$. Informally, A_0 is the contribution of the ballistic particles to A, A_1 is the contribution of the single scattered particles, and G_2 accounts for the multiply scattered particles. Formally, these terms are defined by

$$A_0\phi(x) = \int_{\mathbb{S}^{n-1}} \sigma(x, v) u_0(x, v) dv$$

where u_0 is the solution of

$$v \cdot \nabla_x u_0 + \sigma(x, v) u_0 = 0$$

$$u_0 = \phi.$$

A_1 is defined similarly with u_0 replaced by u_1, the solution to

$$v \cdot \nabla_x u_1 + \sigma(x, v) u_1 = \int k(x, v', v) u_0(x, v') dv'$$

$$u_1 = 0.$$

Then we simply have $G_2 = A - A_0 - A_1$. The authors proceed by seeking information about the kernels α_0, α_1, and Γ_2 of the operators A_0, A_1, and G_2 respectively.

Along the lines of the work done in [8], the authors obtain the following information about the kernels.

Theorem 18.

$$\alpha_0(x, x', v') = \sigma_a(x, v') \exp\left(- \int_0^{\tau_-(x,v')} \sigma(x - sv', v') ds \right) \delta_{\{x - \tau_-(x,v')v'\}}(x')$$

$$\begin{aligned}\alpha_1(x,x',v') &= |\nu(x')\cdot v'| \\ &\times \int_0^{\tau_+(x',v')} \sigma_a(x,v)\frac{E(x,x'+t'v',x')}{|x-x'-t'v'|^{n-1}}k(x'+t'v',v',v)|_{v=\frac{x-x'-t'v'}{|x-x'-t'v'|}}dt'\end{aligned}$$

$$\begin{aligned}\frac{\Gamma_2(x,x',v')}{|\nu(x')\cdot v'|} &\in L^\infty(X\times\Gamma_-) \quad \textit{when} \quad n=2 \\ \frac{\Gamma_2(x,x',v')}{|\nu(x')\cdot v'|\ln(|x-x'-((x-x')\cdot v')v'|)} &\in L^\infty(X\times\Gamma_-) \quad \textit{when} \quad n=3 \\ \frac{|x-x'-((x-x')\cdot v')v'|^{n-3}\Gamma_2(x,x',v')}{|\nu(x')\cdot v'|} &\in L^\infty(X\times\Gamma_-) \quad \textit{when} \quad n\geq 4.\end{aligned}$$

We can see from this result that α_0 is more singular than α_1 and Γ_2, however, unlike the results obtained by Choulli and Stefanov, it is not clear that α_1 can be distinguished from Γ_2. Furthermore, in [8], the most singular term of the decomposition of the distribution kernel determined the x-ray transform of the absorption coefficient. Here the behavior is more complicated as α_0 includes the product of the unknown function σ_a and the x-ray transform of the (unknown) total attenuation coefficient. Determining σ from this term is not nearly as clear cut as it was in [8]. In fact, Bal, Jollivet, and Jugnon do not actually prove that σ can be recovered in general. The attenuation can be recovered in some specific instances, and in more general cases, we only recover limited information about the coefficient. In order to see what information can be obtained, the authors prove the following stability estimate.

Theorem 19. *Let A and $\tilde{A}$ be two albedo operators and $(x',v')\in\Gamma_-$. Then*

$$\int_0^{\tau_+(x',v')} |\sigma_a(x'+tv',v')e^{-\int_0^t \sigma(x'+sv',v')ds} - \tilde{\sigma}_a(x'+tv',v')e^{-\int_0^t \tilde{\sigma}(x'+sv',v')ds}|dt$$

$$\leq ||A-\tilde{A}||_{\mathcal{L}(L^1(\Gamma_-,d\xi);L^1(X))}.$$

This theorem tells us that we can uniquely recover $\sigma_a(x'+tv',v')e^{-\int_0^t \sigma(x'+sv',v')ds}$, but not necessarily σ or σ_a. The authors are able to show, however, that there are at least two situations in which this information is sufficient: when $k=0$ or when the absorption and attenuation satisfy $\sigma_a(x,v)=\sigma_a(x,-v)$ and $\sigma(x,v)=\sigma(x,-v)$ (note that this requirement on σ is implied by the assumption of [8] that the absorption coefficient depend only on speed, and not direction).

In the no-scattering case ($k = 0$), the authors simply observe that $\sigma = \sigma_a$, and thus

$$\sigma_a(x' + tv', v')e^{-\int_0^t \sigma(x'+sv',v')ds} = -\frac{d}{dt}\left(e^{-\int_0^t \sigma(x'+sv',v')ds}\right)$$

The stability estimate therefore implies that $-\frac{d}{dt}\left(e^{-\int_0^t \sigma(x'+sv',v')ds}\right)$ is known. Integrating this expression in t using the fact that $e^{-\int_0^t \sigma(x'+sv',v')ds} = 1$ when $t = 0$, we see that $e^{-\int_0^t \sigma(x'+sv',v')ds}$ is known. Then we have

$$\sigma(x' + tv', v') = -\frac{d}{dt}\left(\ln(e^{-\int_0^t \sigma(x'+sv',v')ds})\right).$$

This expression holds for any $(x', v') \in \Gamma_-$ and $t > 0$, and thus we have determined σ on $X \times \mathbb{S}^{n-1}$.

In the case where $k \neq 0$ and we assume that $\sigma_a(x, v) = \sigma_a(x, -v)$ and $\sigma(x, v) = \sigma(x, -v)$, we again start by observing that $\sigma_a(x' + tv', v')e^{-\int_0^t \sigma(x'+sv',v')ds}$ is known. Then we have determined

$$\frac{\sigma_a(x' + tv', v')e^{-\int_0^t \sigma(x'+sv',v')ds}}{\sigma_a(x' + tv', v')e^{-\int_0^{\tau_+(x',v')-t} \sigma(x'+\tau_+(x',v')v'-sv',-v')ds}}$$

since the denominator is equal to

$$\sigma_a(x' + tv', v')e^{-\int_0^t \sigma(x'+sv',v')}|_{(x',v',t)=(x'+\tau_+(x',v')v',-v',\tau_+(x',v')-t)}.$$

A little simplification then shows that we have determined

$$\exp\left(-\int_0^t \sigma(x' + sv', v')ds + \int_t^{\tau_+(x',v')} \sigma(x' + sv', v')ds\right).$$

As in the non-scattering case, we simply take the derivative of the natural logarithm to determine σ. Once σ is known, σ_a can be recovered from the original expression, $\sigma_a(x' + tv', v')e^{-\int_0^t \sigma(x'+sv',v')}$. Recall that $\sigma - \sigma_a = \int_{\mathbb{S}^{n-1}} k(x, v, v')dv'$ so that knowledge of the absorption and attenuation gives us some information about scattering, k.

Of course, the next question is what other information can be gathered about k. The current state of this problem is not very satisfying. We will recall that it was not clear from our expressions and estimates for α_1 and Γ_2 that the two kernels could be distinguished from one another. To that end, the authors prove an asymptotic expansion for α_1 which shows that α_1 is indeed more singular than Γ_2 (at least in a neighborhood of the support of the ballistic term, which is where the analysis is to be done). We will omit the asymptotic expansion here. Let $C_b(Y)$ denote the set of bounded continuous functions from Y to $\mathbb{R}$ for a topological space Y, and let $\Gamma_1 = \alpha - \alpha_0$. The relevant stability estimate for k is then as follows.

Theorem 20. *Assume that* $(\sigma, \tilde{\sigma}) \in C_b(X \times \mathbb{S}^{n-1})^2$ *and* $(k, \tilde{k}) \in C_b(X \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1})^2$*. Let* $(x, x') \in X \times \partial X$ *and set* $v' = \frac{x-x'}{|x-x'|}$*. When* $n = 2$*, we have*

$$|E(x,x')(\chi(x,v',v') + \chi(x,v',-v')) - \tilde{E}(x,x')(\tilde{\chi}(x,v',v') + \tilde{\chi}(x,v',-v'))|$$
$$\leq \left|\left| \frac{(\Gamma_1 - \tilde{\Gamma_1})(x,x',v')}{|\nu(x') \cdot v'|w_2(x,x',v')} \right|\right|_{L^\infty(X \times \Gamma_-)}$$

where $\chi(x,v',v) = \sigma_a(x,v)k(x,v',v)$ *and*

$$w_2(x,x',v') = 1 + \ln\left(\frac{|x - x' - \tau_+(x',v')v'| - (x - x' - \tau_+(x',v')v') \cdot v'}{|x-x'| - (x-x') \cdot v'}\right).$$

When $n \geq 3$*, we have*

$$\left|\int_0^\pi \sin^{n-3}(\theta)(E(x,x')\chi(x,v',v(\theta)) - \tilde{E}(x,x')\tilde{\chi}(x,v',v(\theta))d\theta\right|$$
$$\leq \left|\left| \frac{(\Gamma_1 - \tilde{\Gamma}_1)(x,x',v')}{|\nu(x') \cdot v'|w_n(x,x',v')} \right|\right|_{L^\infty(X \times \Gamma_-)}$$

where χ *is as above,* $v(\theta) = \cos(\theta)v' + \sin(\theta)v'^{\perp}$*, and* $w_n(x,x',v') = |x - x' - ((x-x') \cdot v')v'|^{2-n}$*.*

The consequence of this theorem is that when $n = 2$, we can recover $k(x,v',v') - k(x,v',-v')$, and when $n \geq 3$ we can recover information about $\int_0^\pi \chi(x,v',\cos(\theta)v' + \sin(\theta)v'^{\perp})d\theta$. The authors go on to prove that for certain types of scattering kernels (called Henyey-Greenstein kernels), some additional information can be recovered (see [2]).

4.2 Quantitaive Photoacoustics on a Riemannian Manifold

The results discussed in the previous section assume that the index of refraction is constant in the medium. Our aim is to extend the results pertaining to recovery of the absorption coefficient to the case of variable smooth index of refraction. The problem of QPAT with variable refractive index is of particular interest because the speed at which waves propagate changes depending upon the medium. In medical imaging for example, it is known that acoustic waves travel faster through bone than through soft tissue by a factor of approximately two. Thus most medical imaging will require a that such transitions between different tissues be taken into account. We must also observe, however, that transitions between these types of regions in the body are 'sharp' in the sense that they will introduce jump discontinuities in the index of refraction. As such, smooth index of refraction does not provide a truly realistic model, however, this can be seen as a first step toward considering more general refractive indices. We note that in [34], the authors address the first step of PAT under the assumption of variable sound speed with discontinuities as would arise in brain imaging applications.

We will begin in the next section by stating our assumptions and defining the operators necessary for our analysis. We will then work to find information about the singular components of the distribution kernel of the measurement operator. Finally, we will show that the most singular term of the decomposition can be isolated and used to extract information about the absorption properties of the medium.

We remark that we will not discuss recovery of the scattering coefficient, k, or the index of refraction of the medium here. We will, however, assume that the scattering information is unknown. Typically, information about k is recovered from the second most singular term in the decomposition of the distribution kernel once the absorption coefficient has been recovered. There are only limited results up to this point concerning the recovery of k in the QPAT setting (see [2]), and in general, recovery of k is more difficult than recovery of σ. On the other hand, we must assume that the the index of refraction is known *a priori* for our analysis. It should be noted that the issue of recovering the index of refraction is an interesting problem in itself.

4.2.1 Assumptions and Definitions

As discussed previously, the model for QPAT can be reduced to solving an inverse problem for the recovery of σ and k in the equation

$$v \cdot \nabla_x u + \sigma(x, v)u - \int_{\mathbb{S}^{n-1}} k(x, v', v)u(x, v')dv' = 0,$$

given some internal information about u. In this model, information travels along straight lines through the domain. Allowing the index of refraction to vary corresponds to studying the QPAT problem on a Riemannian manifold. In this case the metric is determined by the index of refraction, and information will travel along the geodesics of that metric.

To define our problem, let $M \subset \mathbb{R}^n$ $(n \geq 2)$ be a smooth bounded domain and g a given Riemannian metric on M. We must, of course, place restrictions on the metric for the problem to be well-defined. To that end, we will assume that g is simple, meaning that $\bar{M}$ is strictly convex with respect to g and that the exponential map, $\mathrm{Exp}_x : \mathrm{Exp}^{-1}(\bar{M}) \to \bar{M}$, is a diffeomorphism at each point. In particular, this insures that for any points $x, y \in \bar{M}$, there is a unique geodesic in $\bar{M}$ connecting x and y, and that the maximum length of all geodesics in M is finite. We will denote the unit sphere bundle on (M, g) by ΩM, and $\Omega_x M$ will refer to the intersection of ΩM with the tangent space to M at x . Following the notation of [23], set

$$\Gamma_- = \{(x, v) \in \Omega M | x \in \partial M, -v \cdot n(x) > 0\}$$

where the inner product is understood to be with respect to g, and $n(x)$ is the unit outer normal to $\partial\Omega$ at x. We will also write $\gamma_{(x,v)}(t)$ to denote the geodesic with initial point x and initial velocity v, and set $\vec{\gamma}_{(x,v)}(t) := (\gamma_{(x,v)}(t), \dot{\gamma}_{(x,v)}(t))$.

On M, integration will be with respect to the standard Riemannian volume form, while integration on $\Omega_x M$ will be with respect to the Euclidean volume form determined by g. For simplicity, these forms will be denoted by dx and dv_x. The volume form on Γ_- will be denoted $d\xi(x', v')$, and will be defined in terms of the pullback of the product measure $dv_x dx$ by the map $F(x', v', t) : \Gamma_- \times \mathbb{R} \to \Omega M$; $F(x', v', t) = \vec{\gamma}_{(x',v')}(t)$. Thus $d\xi(x', v')dt = F^*(dv_x dx)$ (see [23]).

Then our goal is to recover the absorption coefficient, σ, in

$$-\mathcal{D}\phi(x, v) - \sigma(x, v)\phi(x, v) + \int_{\Omega_x M} k(x, v', v)\phi(x, v')dv'_x = 0,$$

where the leading term, $v \cdot \nabla_x \phi$, of the Euclidean transport equation has been replaced by the derivative along the geodesic flow, $\mathcal{D}$. This is defined as $\mathcal{D}\phi(x, v) = \frac{\partial}{\partial t}|_{t=0}\phi(\vec{\gamma}_{(x,v)}(t))$.

In order to define our data, we must also work with the attenuation coefficient $\sigma_a(x, v) := \sigma(x, v) - \int_{\Omega_x M} k(x, v, v')dv'_x$, which is unknown. The data for QPAT is then encoded in the *albedo operator*:

$$A : L^1(\Gamma_-, d\xi) \to L^1(M)$$
$$\phi_- \mapsto \int_{\Omega_x M} \sigma_a(x,v)\phi(x,v)dv_x$$

where ϕ is the solution to

$$\begin{aligned} &\mathcal{D}\phi(x,v) + \sigma(x,v)\phi(x,v) - \int_{\Omega_x M} k(x,v',v)\phi(x,v')dv'_x = 0 \\ &\phi|_{\Gamma_-} = \phi_- \in L^1(\Gamma_-, d\xi). \end{aligned} \tag{4.1}$$

This is the data obtained by solving the first step of PAT: recovery of the thermal deposition map. We assume that this step has already been completed exactly, so that we have full knowledge of A.

We recall that the forward problem for the transport equation on a Riemannian manifold was addressed in [23], in which the author showed that (4.1) is uniquely solvable for $\phi_- \in L^1(\Gamma_-, d\xi)$. Our assumptions on the metric (together with *a priori* assumptions on σ and k that we will state below) imply that the solution is in $L^1(\Omega M)$ so that A has the stated mapping properties.

Before moving to the analysis, we must define several operators and quantities, and discuss our assumptions on σ and k. In addressing the forward problem, it is shown that the solution can be written in terms of the boundary values as $\phi = J\phi_- - KJ\phi_- + T^{-1}T_1KJ\phi_-$. The operators appearing in this decomposition are what we will define first. For this, we will need the 'time to boundary functions' given by

$$\tau_\pm(x,v) = \inf\{t \geq 0 : \gamma_{(x,v)}(\pm t) \notin \bar{M}\}, \qquad \tau = \tau_- + \tau_+.$$

Thanks to our assumptions on the metric and the domain, τ is bounded by the maximum length of all geodesics in M, which is finite.

We set:

$$J\phi_-(x,v) = e^{-\int_0^{\tau_-(x,v)} \sigma(\vec{\gamma}_{(x,v)}(-p))dp}\phi_-(\vec{\gamma}_{(x,v)}(-\tau_-(x,v)))$$
$$K\phi(x,v) = -\int_0^{\tau_-(x,v)} e^{-\int_0^t \sigma(\vec{\gamma}_{(x,v)}(-p)dp)}(T_1\phi)(\vec{\gamma}_{(x,v)}(-t))dt$$
$$T_1\phi(x,v) = \int_{\Omega_x M} k(x,v',v)\phi(x,v')dv'_x$$

and $$T\phi = -\mathcal{D}\phi - \sigma\phi + T_1\phi$$

on the domain

$$D(T) = \{\phi \in L^1(\Omega M) | T\phi \in L^1(\Omega M), \phi|_{\Gamma_-} = 0\}.$$

Finally, we state our assumptions on σ and k. We will assume that $\sigma(x, v)$ and $k(x, v', v)$ are both nonnegative and bounded by a constant $C < \infty$. Additionally, we define $\sigma_a(x, v) = \sigma - \int_{\Omega_x M} k(x, v, v') dv'_x$, and for all $(x, v) \in \Omega M$, we assume that $0 < \sigma_0 \leq \sigma_a(x, v) \leq \sigma_1$.

The next section will be devoted to analysis of the albedo operator, which will lead to our discussion of recovery of the absorption coefficients in the final section.

4.2.2 Decomposition of the Albedo Operator

First, we recall our definition of the albedo operator:

$$\begin{aligned} A : L^1(\Gamma_-, d\xi) &\to L^1(M) \\ \phi_- &\mapsto \int_{\Omega_x M} \sigma_a(x, v)\phi(x, v) dv_x. \end{aligned}$$

Our task will be to find a distribution kernel for this operator. That is, we seek to find a distribution α such that

$$A\phi_-(x) = \int_{\Gamma_-} \alpha(x, x', v')\phi_-(x', v') d\xi(x', v').$$

To accomplish this, we will use the expression for the solution ϕ given in the previous section to rewrite the albedo operator as follows:

$$\begin{aligned} A\phi_-(x) =& \int_{\Omega_x M} \sigma_a(x, v)[(J - KJ + T^{-1}T_1 KJ)\phi_-](x, v) dv_x \\ =& \int_{\Omega_x M} \sigma_a(x, v)(J\phi_-)(x, v) dv + \int_{\Omega_x M} \sigma_a(x, v)(-KJ\phi_-)(x, v) dv_x \\ &+ \int_{\Omega_x M} \sigma_a(x, v)(T^{-1}T_1 KJ\phi_-)(x, v) dv_x \\ =& E_1\phi_-(x) + E_2\phi_-(x) + E_3\phi_-(x). \end{aligned}$$

The task of finding the kernel α is thus reduced to finding three kernels, α_1, α_2, and α_3 for the operators E_1, E_2, and E_3. That will be the subject of the next three lemmas.

Lemma 21. *We have*

$$\int_{\Omega_x M} \sigma_a(x, v)(J\phi_-)(x, v) dv_x = \int_{\Gamma_-} \alpha_1(x, x', v')\phi_-(x', v') d\xi(x', v')$$

where

$$\alpha_1(x,x',v') = \int_{\Omega_x M} \int_0^{\tau_+(x',v')} \sigma_a(x,v) e^{-\int_0^{\tau_-(x,v)} \sigma(\vec{\gamma}_{(x,v)}(-s))ds} \times \delta_{(\vec{\gamma}_{(x',v')}(t))}(x,v)dtdv_x.$$

Proof. Let $\psi \in C_0^\infty(M)$. For $E_1\phi_-(x) = \int_{\Omega_x M} \sigma_a(x,v) J\phi_-(x,v)dv_x$, we have:

$$\begin{aligned}\langle E_1\phi_-(x), \psi(x)\rangle &= \int_M \int_{\Omega_x M} \sigma_a(x,v)J\phi_-(x,v)\psi(x)dv_x dx \\ &= \int_M \int_{\Omega_x M} \sigma_a(x,v) e^{-\int_0^{\tau_-(x,v)} \sigma(\vec{\gamma}_{(x,v)}(-s))ds} \\ &\quad \times \phi_-(\vec{\gamma}_{(x,v)}(-\tau_-(x,v)))\psi(x)dv_x dx.\end{aligned}$$

Now we perform the change of variables $x' = \gamma_{(x,v)}(-\tau_-(x,v))$, $v' = \dot{\gamma}_{(x,v)}(-\tau_-(x,v))$, $t = \tau_-(x,v)$, $dv_x dx = d\xi(x',v')dt$. This yields:

$$\int_{\Gamma_-} \int_0^{\tau_+(x',v')} \sigma_a(\vec{\gamma}_{(x',v')}(t)) e^{-\int_0^{\tau_-(\vec{\gamma}_{(x',v')}(t))} \sigma(\vec{\gamma}_{(x',v')}(t-s))ds} \times \phi_-(x',v')\psi(\gamma_{(x',v')}(t))dtd\xi(x',v').$$

Now we introduce the distribution $\delta_{(y,w)}(x,v)$ characterized by

$$\int_M \int_{\Omega_x M} f(x,v)\delta_{(y,w)}(x,v)dv_x dx = f(y,w).$$

The expression then becomes

$$\langle E_1\phi_-(x), \psi(x)\rangle = \int_M \int_{\Omega_x M} \int_{\Gamma_-} \int_0^{\tau_+(x',v')} \sigma_a(x,v) e^{-\int_0^{\tau_-(x,v)} \sigma(\vec{\gamma}_{(x,v)}(-s))ds} \times \phi_-(x',v')\psi(x)\delta_{\vec{\gamma}_{(x',v')}(t)}(x,v)dtd\xi(x',v')dv_x dx.$$

Changing the order of integration, we see that

$$\alpha_1(x,x',v') = \int_{\Omega_x M} \int_0^{\tau_+(x',v')} \sigma_a(x,v) e^{-\int_0^{\tau_-(x,v)} \sigma(\vec{\gamma}_{(x,v)}(-s))ds} \times \delta_{(\vec{\gamma}_{(x',v')}(t))}(x,v)dtdv_x.$$

□

Since our information about the absorption kernel will come from α_1, that is the only piece of the kernel for which we need an explicit formula. For α_2 and α_3, we will simply establish regularity results.

Lemma 22. *We have*

$$\int_{\Omega_x M} \sigma_a(x,v)(-KJ\phi_-)(x,v)dv_x = \int_{\Gamma_-} \alpha_2(x,x',v')\phi_-(x',v')d\xi(x',v')$$

where $\alpha_2 \in L^\infty(\Gamma_-(x',v'); L^1(M))$.

Proof. Let V denote the operator $Vf(x) = \int_{\Omega_x M} \sigma_a(x,v)f(x,v)dv_x$ so that $E_2\phi_-(x) = -VKJ\phi_-(x,v)$. We observe that

$$||VKJ\phi_-(x)||_{L^1(M)} = ||(V\tau)(\tau^{-1}K\tau)(\tau^{-1}J)\phi_-||_{L^1(M)}.$$

If $\phi_- \in L^1(\Gamma_-, d\xi)$, then $\tau^{-1}J\phi_- \in L^1(\Omega M)$ since

$$\begin{aligned}
&||\tau^{-1}J\phi_-||_{L^1(\Omega M)} \\
&= \int_M \int_{\Omega_x M} |\tau^{-1}(x,v)e^{-\int_0^{\tau_-(x,v)} \sigma(\vec{\gamma}_{(x,v)}(-p))dp}\phi_-(\vec{\gamma}_{(x,v)}(-\tau_-(x,v)))|dv_x dx \\
&\leq \int_M \int_{\Omega_x M} \tau^{-1}(x,v)|\phi_-(\vec{\gamma}_{(x,v)}(-\tau_-(x,v)))|dv_x dx.
\end{aligned}$$

Making the change of variables $x' = \gamma_{(x,v)}(-\tau_-(x,v))$, $v' = \dot{\gamma}_{(x,v)}(-\tau_-(x,v))$, $t = \tau_-(x,v)$, $dv_x dx = d\xi(x',v')dt$, we have:

$$\begin{aligned}
||\tau^{-1}J\phi_-||_{L^1(\Omega M)} &\leq \int_{\Gamma_-} \int_0^{\tau_+(x',v')} \tau_+^{-1}(x',v')|\phi_-(x',v')|dtd\xi(x',v') \\
&= ||\phi_-||_{L^1(\Gamma_-,d\xi)}.
\end{aligned}$$

Furthermore, $\tau^{-1}K\tau$ is a bounded operator from $L^1(\Omega M)$ to itself by [23] proposition 2.6 (ii). Last, $V\tau$ is a bounded operator from $L^1(\Omega M)$ to $L^1(M)$ since

$$\begin{aligned}
||V\tau f||_{L^1(M)} &= \int_M \left| \int_{\Omega_x M} \tau(x,v)\sigma_a(x,v)f(x,v)dv \right| dx \\
&\leq C||f||_{L^1(\Omega M)}
\end{aligned}$$

by our assumptions on the metric and σ_a. Thus we have

$$||VKJ\phi_-(x)||_{L^1(M)} = ||(V\tau)(\tau^{-1}K\tau)(\tau^{-1}J)\phi_-||_{L^1(M)} \leq C||\phi_-||_{L^1(\Gamma_-,d\xi)}.$$

We have shown that E_2 is a bounded operator from $L^1(\Gamma_-, d\xi)$ to $L^1(M)$. Now we will show that $E_2\phi_-$ can be written as integration of ϕ_- against a measurable function on Γ_-, from which the regularity of the kernel will follow.

From the definitions, we have

$$E_2\phi_-(x) = \int_{\Omega_x M}\int_0^{\tau_-(x,v)}\int_{\Omega_{\hat{x}} M} \sigma_a(x,v)e^{-\int_0^t \sigma(\vec{\gamma}_{(x,v)}(-p))dp}k(\hat{x},v',\hat{v}) \\ \times e^{-\int_0^{\tau_-(\hat{x},v')}\sigma(\vec{\gamma}_{(\hat{x},v')}(-p))dp}\phi_-(\vec{\gamma}_{(\hat{x},v')}(-\tau_-(\hat{x},v')))dv'_{\hat{x}}dtdv_x$$

where $(\hat{x},\hat{v}) = \vec{\gamma}_{(x,v)}(-t)$.

We formally compute the distribution kernel of E_2 by considering $E_2\delta_{(y',w')}$ where $\delta_{(y',w')}$ is the distribution characterized by $\int_{\Gamma_-} f(x',v')\delta_{(y',w')}(x',v')d\xi(x',v') = f(y',w')$. As above, we begin with

$$E_2\delta_{(y',w')}(x) = \int_{\Omega_x M}\int_0^{\tau_-(x,v)}\int_{\Omega_{\hat{x}} M} \sigma_a(x,v)e^{-\int_0^t \sigma(\vec{\gamma}_{(x,v)}(-p))dp}k(\hat{x},v',\hat{v}) \\ \times e^{-\int_0^{\tau_-(\hat{x},v')}\sigma(\vec{\gamma}_{(\hat{x},v')}(-p))dp}\delta_{(y',w')}(\vec{\gamma}_{(\hat{x},v')}(-\tau_-(\hat{x},v')))dv'_{\hat{x}}dtdv_x.$$

We would like to make the change of variables $y = \gamma_{(x,v)}(-t)$ with Jacobian J so that $dtdv_x = |J|dy$. To do this, we must rewrite v, $\hat{v}$, and t in terms of y. Let $\gamma_{\{y\to x\}}$ be the unique unit speed geodesic from y to x. Then under the change of variables, we have

$$E_2\delta_{(y',w')}(x) = \int_M\int_{\Omega_y M} \sigma_a(x,\dot{\gamma}_{\{y\to x\}}(t(y)))e^{-\int_0^{t(y)}\sigma(\vec{\gamma}_{(x,\dot{\gamma}_{\{y\to x\}}(t(y)))}(-p))dp} \\ \times k(y,v',\dot{\gamma}_{\{y\to x\}}(0))e^{-\int_0^{\tau_-(y,v')}\sigma(\vec{\gamma}_{(y,v')}(-p))dp} \\ \times \delta_{(y',w')}(\vec{\gamma}_{(y,v')}(-\tau_-(y,v')))|J|dv'_y dy$$

where $t(y)$ is the distance from x to y along $\gamma_{\{y\to x\}}$. We then make another change of variables $(y,v') = \vec{\gamma}_{(a',b')}(s)$ to obtain

$$E_2\delta_{(y',w')}(x) = \int_{\Gamma_-}\int_0^{\tau_+(a',b')} \sigma_a(x,\dot{\gamma}_{\{\gamma_{(a',b')}(s)\to x\}}(t)) \\ \times e^{-\int_0^t \sigma(\vec{\gamma}_{(x,\dot{\gamma}_{\{\gamma_{(a',b')}(s)\to x\}}(t))}(-p))dp}k(\vec{\gamma}_{(a',b')}(s),\dot{\gamma}_{\{\gamma_{(a',b')}(s)\to x\}}(0)) \\ \times e^{-\int_0^s \sigma(\vec{\gamma}_{(a',b')}(-p))dp}\delta_{(y',w')}(a',b')|J|dsd\xi(a',b')$$

where we have compressed the notation $t = t(\gamma_{(a',b')}(s))$.

Finally, this yields

$$E_2\delta_{(y',w')}(x) = \int_0^{\tau_+(y',w')} \sigma_a(x, \dot{\gamma}_{\{\gamma_{(y',w')}(s)\to x\}}(t))$$
$$\times e^{-\int_0^t \sigma(\vec{\gamma}_{(x,\dot{\gamma}_{\{\gamma_{(y',w')}(s)\to x\}}(t))}(-p))dp} k(\vec{\gamma}_{(y',w')}(s), \dot{\gamma}_{\{\gamma_{(y',w')}(s)\to x\}}(0))$$
$$\times e^{-\int_0^s \sigma(\vec{\gamma}_{(y',w')}(-p))dp}|J|ds.$$

The final step of the proof is then to show that we indeed have

$$E_2\phi_-(x) = \int_{\Gamma_-} E_2\delta_{(y',w')}(x)\phi_-(y',w')d\xi(y',w')$$

for all $\phi_- \in L^1(\Gamma_-, d\xi)$. It is easy to see that this is the case by writing out the definition of $E_2\phi_-(x)$ and manipulating the expression in precisely the same way that we approached $E_2\delta_{(y',w')}(x)$.

As claimed, E_2 is a bounded operator from $L^1(\Gamma_-)$ to $L^1(M)$ that can be expressed as integration against a measurable function, and therefore it follows that the kernel, α_2, of the operator $-VKJ$ is in $L^\infty(\Gamma_-(x',v'); L^1(M))$. □

For our regularity result on the final kernel, we will need the following space:

$$\mathcal{W} = \{f : \mathcal{D}f \in L^1(\Omega M), \tau^{-1}f \in L^1(\Omega M)\}$$

with $||f||_{\mathcal{W}} = ||\mathcal{D}f||_{L^1(\Omega M)} + ||\tau^{-1}f||_{L^1(\Omega M)}$. This is in fact the space containing the solution to (4.1) with $\phi_- \in L^1(\Gamma_-, d\xi)$. We remark that because of our assumptions on the domain and the metric, τ is bounded, and therefore τ^{-1} is bounded below by a positive constant. It follows then that $f \in \mathcal{W}$ implies $f \in L^1(\Omega M)$.

Lemma 23. *We have*

$$\int_{\Omega_x M} \sigma_a(x,v)(T^{-1}T_1KJ\phi_-)(x,v)dv_x = \int_{\Gamma_-} \alpha_3(x,x',v')\phi_-(x',v')d\xi(x',v')$$

where α_3 satisfies

$$\alpha_3(x,x',v') = \int_{\Omega_x M} \sigma_a(x,v)f_3(x,v,x',v')dv_x,$$

with $f_3 \in L^\infty(\Gamma_-; \mathcal{W})$.

Proof. For this result, we refer to [23], Proposition 3.3. The statement of that proposition is that

$$T^{-1}T_1KJ\phi_- = \int_{\Gamma_-} f_3(x,v,x',v')\phi_-(x',v')d\xi(x',v')$$

where $f_3 \in L^\infty(\Gamma_-;\mathcal{W})$.

For our albedo operator, we must consider

$$\begin{aligned} E_3\phi_-(x) &= \int_{\Omega_x M} \sigma_a(x,v)T^{-1}T_1KJ\phi_(x,v)dv_x \\ &= \int_{\Omega_x M} \sigma_a(x,v)\int_{\Gamma_-} f_3(x,v,x',v')\phi_-(x',v')d\xi(x',v')dv_x. \end{aligned}$$

We may change the order of integration since σ_a is bounded. This yields

$$E_3\phi_-(x) = \int_{\Gamma_-}\int_{\Omega_x M} \sigma_a(x,v)f_3(x,v,x',v')dv_x\phi_-(x',v')d\xi(x',v'),$$

from which the result follows. □

4.2.3 Recovery of the Absorption Coefficient

The key step in the recovery of the absorption coefficient is to prove a stability estimate for a functional of σ and σ_a. We will begin with this theorem, after which we will demonstrate how σ and σ_a can be recovered under certain conditions.

Theorem 24. *Let A and $\tilde{A}$ be two albedo operators corresponding to (σ_a, k) and $(\tilde{\sigma}_a, \tilde{k})$. Then we have*

$$\begin{aligned} \int_0^{\tau_+(x',v')} |\sigma_a(\vec{\gamma}_{(x',v')}(t))e^{-\int_0^t \sigma(\vec{\gamma}_{(x',v')}(s))ds} - \tilde{\sigma}_a(\vec{\gamma}_{(x',v')}(t))e^{-\int_0^t \tilde{\sigma}(\vec{\gamma}_{(x',v')}(s))ds}|dt \\ \leq ||A-\tilde{A}||_{\mathcal{L}(L^1(\Gamma_-,d\xi);L^1(M))}. \end{aligned}$$

for a.e. $(x',v') \in \Gamma_-$.

Proof. We will proceed as in [2] in which the authors prove a similar estimate for the Euclidean case. Let $\phi \in L^\infty(M)$ with $||\phi||_{L^\infty(M)} \leq 1$ and $\psi \in L^1(\Gamma_-,d\xi)$ with $||\psi||_{L^1(\Gamma_-,d\xi)} \leq 1$. Since ϕ and ψ each have corresponding norm less than one, we have

$$\left|\int_M \phi(x)[(A-\tilde{A})\psi](x)dx\right| \leq ||A-\tilde{A}||_{\mathcal{L}(L^1(\Gamma_-,d\xi),L^1(M))}.$$

Following the notation of [2], set

$$\Delta_0(\phi,\psi) = \int_M \phi(x) \int_{\Gamma_-} (\alpha_1 - \tilde{\alpha}_1)(x,x',v')\psi(x',v')d\xi(x',v')dx$$

and

$$\Delta_1(\phi,\psi) = \int_M \phi(x) \int_{\Gamma_-} ((\alpha_2+\alpha_3) - (\tilde{\alpha}_2+\tilde{\alpha}_3))(x,x',v')\psi(x',v')d\xi(x',v')dx.$$

Then we have $|\Delta_0(\phi,\psi)| \le ||A - \tilde{A}||_{\mathcal{L}(L^1(\Gamma_-,d\xi),L^1(M))} + |\Delta_1(\phi,\psi)|$.

To begin, we examine Δ_0. Recall that

$$\begin{aligned}\alpha_1(x,x',v') = \int_{\Omega_x M} \int_0^{\tau_+(x',v')} & \sigma_a(x,v) e^{-\int_0^{\tau_-(x,v)} \sigma(\vec{\gamma}_{(x,v)}(-s))ds} \\ & \times \delta_{(\vec{\gamma}_{(x',v')}(t))}(x,v)dtdv_x.\end{aligned}$$

Then

$$\begin{aligned} & \int_M \phi(x) \int_{\Gamma_-} \alpha_1(x,x',v')\psi(x',v')d\xi(x',v')dx \\ &= \int_M \phi(x) \int_{\Gamma_-} \int_{\Omega_x M} \int_0^{\tau_+(x',v')} \sigma_a(x,v) e^{-\int_0^{\tau_-(x,v)} \sigma(\vec{\gamma}_{(x,v)}(-s))ds} \\ & \quad \times \delta_{(\vec{\gamma}_{(x',v')}(t))}(x,v)dtdv_x \psi(x',v')d\xi(x',v')dx \\ &= \int_{\Gamma_-} \int_0^{\tau_+(x',v')} \phi(\gamma_{(x',v')}(t))\sigma_a(\vec{\gamma}_{(x',v')}(t)) e^{-\int_0^t \sigma(\vec{\gamma}_{(x',v')}(s))ds} \psi(x',v')dtd\xi(x',v'),\end{aligned}$$

so that

$$\Delta_0(\phi,\psi) = \int_{\Gamma_-} \int_0^{\tau_+(x',v')} \phi(\gamma_{(x',v')}(t))(\eta - \tilde{\eta})(t;x',v')dt\psi(x',v')d\xi(x',v') \qquad (4.2)$$

with

$$\begin{aligned}\eta(t;x',v') &= \sigma_a(\vec{\gamma}_{(x',v')}(t)) \exp\left(-\int_0^t \sigma(\vec{\gamma}_{(x',v')}(s))ds\right) \\ \tilde{\eta}(t;x',v') &= \tilde{\sigma}_a(\vec{\gamma}_{(x',v')}(t)) \exp\left(-\int_0^t \tilde{\sigma}(\vec{\gamma}_{(x',v')}(s))ds\right).\end{aligned}$$

Our next step is to make an appropriate choice for the functions ψ and ϕ in the expression (4.2). For ψ, we will refer to [26], lemma 5.1. The result is as follows:

There is a family of maps $\psi_{\epsilon,x_0',v_0'} \in L^1(\Gamma_-,d\xi)$, for $(x_0',v_0') \in \Gamma_-$ and $\epsilon > 0$, such that $\psi_{\epsilon,x_0',v_0'} \ge 0$, $||\psi_{\epsilon,x_0',v_0'}||_{L^1(\Gamma_-,d\xi)} = 1$ and for any given $f \in L^\infty(\Gamma_-,d\xi)$,

$$\lim_{\epsilon\to 0^+} \int_{\Gamma_-} f(x',v')\psi_{\epsilon,x_0',v_0'}(x',v')d\xi(x',v') = f(x_0',v_0') \qquad (4.3)$$

when (x_0', v_0') is in the Lebesgue set of f, which will be denoted $\mathcal{L}f$.

Next, we wish to define a sequence ϕ_m that will replace ϕ in (4.2). We begin by defining a family of functions

$$G_n(x', v') = \int_0^{\tau_+(x',v')} t^n(\eta - \tilde{\eta})(t; x', v')dt \qquad n \in \mathbb{N} \cup \{0\}. \tag{4.4}$$

Then set $\mathcal{L}_1 := \cap_{n\in\mathbb{N}\cup\{0\}}\mathcal{L}(G_n)$. We wish to define functions $\phi_m(x)$ with x expressed in terms of $\vec{\gamma}_{(x_0',v_0')}(t)$ for $(x_0', v_0') \in \mathcal{L}_1$.

For $(x_0', v_0') \in \mathcal{L}_1$ and $\phi \in C_0(0, \tau_+(x_0', v_0'))$, set

$$(\phi_m) \in (L^\infty(M))^{\mathbb{N}}$$
$$\phi_m(x) = \chi_{[0,\frac{1}{m+1})}(r)\phi(p)$$

for $x \in M$ and $m \in \mathbb{N}$ where

$$r = \text{dist}_g(\gamma_{(x_0',v_0')}, x) = \inf_{s\in(0,\tau_+(x_0',v_0'))} \text{dist}_g(\gamma_{(x_0',v_0')}(s), x)$$

and $p \in [0, \tau_+(x_0', v_0')]$ is such that

$$r = \text{dist}_g(\gamma_{(x_0',v_0')}(p), x).$$

The construction of the functions $\psi_{\epsilon,x_0',v_0'}$ insures that $\psi_{\epsilon,x_0',v_0'}$ is supported in an epsilon neighborhood of (x_0', v_0') in Γ_-. Then for epsilon sufficiently small, we can restrict our attention to $r \in [0, \frac{1}{m+1})$ so that $\chi_{[0,\frac{1}{m+1})}(r) = 1$. By applying the Stone-Weierstrass theorem to ϕ, we see that

$$\lim_{\epsilon\to 0^+} \Delta_0(\phi_m, \psi_{\epsilon,x_0',v_0'}) = \int_0^{\tau_+(x_0',v_0')} \phi_m(\gamma_{(x_0',v_0')}(t))(\eta - \tilde{\eta})(t; x_0', v_0')dt.$$

Furthermore, we have $\phi_m(\gamma_{(x_0',v_0')}(t)) = \phi(t)$, and so

$$\lim_{m\to\infty} \lim_{\epsilon\to 0^+} \Delta_0(\phi_m, \psi_{\epsilon,x_0',v_0'}) = \int_0^{\tau_+(x_0',v_0')} \phi(t)(\eta - \tilde{\eta})(t; x_0', v_0')dt. \tag{4.5}$$

Next, we need to analyze $\Delta_1(\phi_m, \psi_{\epsilon,x_0',v_0'})$. We will break this up into two pieces and first consider the expression

$$\left| \int_M \phi_m(x) \int_{\Gamma_-} \alpha_2(x, x', v')\psi_{\epsilon,x_0',v_0'}(x', v')d\xi(x', v')dx \right|.$$

To begin, suppose that $\beta(x) \in L^{\infty}(M)$ is a nonnegative function, and consider

$$\left|\int_M \beta(x) \int_{\Gamma_-} \alpha_2(x,x',v')\psi_{\epsilon,x_0',v_0'}(x',v')d\xi(x',v')dx\right|.$$

We have

$$\begin{aligned}&\left|\int_M \beta(x) \int_{\Gamma_-} \alpha_2(x,x',v')\psi_{\epsilon,x_0',v_0'}(x',v')d\xi(x',v')dx\right|\\ &\leq \int_{\Gamma_-} \psi_{\epsilon,x_0',v_0'}(x',v') \int_M |\alpha_2(x,x',v')|\beta(x)dxd\xi(x',v').\end{aligned}$$

We also have that

$$\int_M |\alpha_2(x,x',v')|\beta(x)dx \leq ||\beta||_{L^\infty(M)}||\alpha_2(\cdot,x',v')||_{L^1(M)} < \infty$$

for a.e. $(x',v') \in \Gamma_-$. Thus the lemma stated above from [26] gives us

$$\begin{aligned}&\lim_{\epsilon\to 0^+} \int_{\Gamma_-} \psi_{\epsilon,x_0',v_0'}(x',v') \int_M |\alpha_2(x,x',v')|\beta(x)dxd\xi(x',v')\\ &= \int_M |\alpha_2(x,x_0',v_0')|\beta(x)dx\end{aligned}$$

for $(x_0',v_0') \in \mathcal{L}(\int_M |\alpha_2(x,x',v')|\beta(x)dx)$. Then for $(x_0',v_0') \in \mathcal{L}_1$, we have ϕ_m defined as previously, and we consider the expression

$$\int_M |\alpha_2(x,x',v')|\phi_m(x)dx.$$

If we set $\mathcal{L}_2 = \cap_{m\in\mathbb{N}\cup 0}(\mathcal{L}(\int_M |\alpha_2(x,x',v')|\phi_m(x)dx) \cap \mathcal{L}_1$, then we have

$$\begin{aligned}&\lim_{\epsilon\to 0^+} \int_{\Gamma_-} \psi_{\epsilon,x_0',v_0'}(x',v') \int_M |\alpha_2(x,x',v')|\phi_m(x)dxd\xi(x',v')\\ &= \int_M |\alpha_2(x,x_0',v_0')|\phi_m(x)dx\end{aligned}$$

for $(x_0',v_0') \in \mathcal{L}_2$. An application of the dominated convergence theorem now shows us that

$$\lim_{m\to\infty} \int_M |\alpha_2(x,x_0',v_0')|\phi_m(x)dx = 0.$$

Finally, we consider

$$\begin{aligned}&\left|\int_M \phi_m(x) \int_{\Gamma_-} \alpha_3(x,x',v')\psi_{\epsilon,x_0',v_0'}(x',v')d\xi(x',v')dx\right|\\ &= \left|\int_M \phi_m(x) \int_{\Gamma_-} \int_{\Omega_x M} \sigma_a(x,v)f_3(x,v,x',v')\psi_{\epsilon,x_0',v_0'}(x',v')dvd\xi(x',v')dx\right|.\end{aligned}$$

We approach this as we did the previous term. Suppose β is any nonnegative function in $L^\infty(M)$. Then, recalling our assumption that $0 < \sigma_0 \le \sigma_a(x,v) \le \sigma_1$ on ΩM, we have

$$\begin{aligned}&\left|\int_M \beta(x) \int_{\Gamma_-} \int_{\Omega_x M} \sigma_a(x,v) f_3(x,v,x',v') \psi_{\epsilon,x_0',v_0'}(x',v') dv d\xi(x',v') dx\right| \\ \le& \sigma_1 \int_{\Gamma_-} \int_M \int_{\Omega_x M} \beta(x) |f_3(x,v,x',v')| \psi_{\epsilon,x_0',v_0'}(x',v') dv dx d\xi(x',v'),\end{aligned}$$

and we again apply the result from [26] to take the limit as $\epsilon \to 0^+$ to obtain

$$\begin{aligned}&\lim_{\epsilon\to 0^+} \int_{\Gamma_-} \int_M \int_{\Omega_x M} \beta(x) |f_3(x,v,x',v')| \psi_{\epsilon,x_0',v_0'}(x',v') dv dx d\xi(x',v') \\ =& \int_M \int_{\Omega_x M} \beta(x) |f_3(x,v,x_0',v_0')| dv dx\end{aligned}$$

when $(x_0', v_0') \in \mathcal{L}(\int_M \int_{\Omega_x M} \beta(x) |f_3(x,v,x',v')| dv dx)$. We consider the expression

$$\int_M \int_{\Omega_x M} \phi_m(x) |f_3(x,v,x',v')| dv dx,$$

and we set $\mathcal{L}_3 := \cap_{m\in\mathbb{N}\cup 0}(\mathcal{L}(\int_M \int_{\Omega_x M} \phi_m(x)|f_3(x,v,x',v')|dvdx)) \cap \mathcal{L}_1$. Then we conclude that

$$\begin{aligned}&\lim_{\epsilon\to 0^+} \left|\int_M \phi_m(x) \int_{\Gamma_-} \alpha_3(x,x',v') \psi_{\epsilon,x_0',v_0'}(x',v') d\xi(x',v') dx\right| \\ \le& \sigma_1 \int_M \int_{\Omega_x M} \phi_m(x) |f_3(x,v,x_0',v_0')| dv dx\end{aligned}$$

on $\mathcal{L}_3$. Another application of the dominated convergence theorem then shows us that

$$\lim_{m\to\infty} \int_M \int_{\Omega_x M} \phi_m(x) |f_3(x,v,x_0',v_0')| dv dx = 0.$$

Thus, in total we have:

$$\left|\int_0^{\tau_+(x',v')} \phi(t)(\eta - \tilde{\eta})(t;x',v') dt\right| \le ||A - \tilde{A}||_{\mathcal{L}(L^1(\Gamma_-,d\xi),L^1(X))}$$

for a.e. $(x',v') \in \mathcal{L}_1 \cap \mathcal{L}_2 \cap \mathcal{L}_3$ (which is a set with negligible complement) and $\phi \in C_0(0,\tau_+(x',v'))$ satisfying $||\phi||_\infty \le 1$. As in [2], we can see that in order to obtain the desired result, we should choose $\phi(t) = \text{sgn}(\eta - \tilde{\eta})$, however this is not in C_0. Fortunately, because we know that $C_0(0,\tau_+(x',v'))$ is dense in $L^1(0,\tau_+(x',v'))$, it follows that the expression also holds for $\phi \in (L^\infty \cap L^1)(0,\tau_+(x',v'))$, $||\phi||_\infty \le 1$. Thus the theorem follows. □

Finally, we conclude by demonstrating that, in certain cases, the above stability estimate allows for the unique determination of σ and σ_a. This is again done following the method of [2], and we include it here for completeness.

The situation we will consider is the case in which

$$\sigma_a(x,v) = \sigma_a(x,-v) \qquad \text{and} \qquad \sigma(x,v) = \sigma(x,-v).$$

This is generally considered to be a physically reasonable assumption, and is in fact less restrictive than assumptions that are typically made for recovery the absorption coefficient in the transport equation, see [8] and [23] for example.

We know that we can uniquely recover $\sigma_a(\vec{\gamma}_{(x',v')}(t))e^{-\int_0^t \sigma(\vec{\gamma}_{(x',v')}(s))ds}$ for a.e. $(x',v') \in \Gamma_-$ and $t > 0$. Note that the point $(\gamma_{(x',v')}(\tau_+(x',v')), -\dot{\gamma}_{(x',v')}(\tau_+(x',v')))$ is in Γ_-. We make the following observations:

$$\begin{aligned}
&\sigma_a(\vec{\gamma}_{(\gamma_{(x',v')}(\tau_+(x',v')), -\dot{\gamma}_{(x',v')}(\tau_+(x',v')))}(\tau_+(x',v') - t)) \\
=&\sigma_a(\gamma_{(\vec{\gamma}_{(x',v')}(\tau_+(x',v')))}(t - \tau_+(x',v')), -\dot{\gamma}_{(\vec{\gamma}_{(x',v')}(\tau_+(x',v')))}(t - \tau_+(x',v'))) \\
=&\sigma_a(\gamma_{(x',v')}(t), -\dot{\gamma}_{(x',v')}(t)) = \sigma_a(\vec{\gamma}_{(x',v')}(t))
\end{aligned}$$

since $\sigma_a(x,-v) = \sigma_a(x,v)$, and furthermore,

$$\begin{aligned}
&e^{-\int_0^{\tau_+(x',v')-t} \sigma(\vec{\gamma}_{(\gamma_{(x',v')}(\tau_+(x',v')), -\dot{\gamma}_{(x',v')}(\tau_+(x',v')))}(s))ds} \\
=&e^{-\int_0^{\tau_+(x',v')-t} \sigma(\vec{\gamma}_{(x',v')}(\tau_+(x',v')-s))ds} \\
=&e^{-\int_t^{\tau_+(x',v')} \sigma(\vec{\gamma}_{(x',v')}(s))ds}.
\end{aligned}$$

Here we again used the assumption $\sigma(x,-v) = \sigma(x,v)$, and the final step follows by making the change of variables $\tau_+(x',v') - s \to s$. We thus conclude that the quantity

$$\sigma_a(\vec{\gamma}_{(x',v')}(t))e^{-\int_t^{\tau_+(x',v')} \sigma(\vec{\gamma}_{(x',v')}(s))ds}$$

is uniquely determined.

By considering

$$\frac{\sigma_a(\vec{\gamma}_{(x',v')}(t))e^{-\int_0^t \sigma(\vec{\gamma}_{(x',v')}(s))ds}}{\sigma_a(\vec{\gamma}_{(x',v')}(t))e^{-\int_t^{\tau_+(x',v')} \sigma(\vec{\gamma}_{(x',v')}(s))ds}}$$

we see that we can recover

$$e^{-\int_0^t \sigma(\vec{\gamma}_{(x',v')}(s))ds + \int_t^{\tau_+(x',v')} \sigma(\vec{\gamma}_{(x',v')}(s))ds}.$$

Then we have

$$2\sigma(\vec{\gamma}_{(x',v')}(t)) = \frac{d}{dt}\left(-\ln e^{-\int_0^t \sigma(\vec{\gamma}_{(x',v')}(s))ds + \int_t^{\tau_+(x',v')} \sigma(\vec{\gamma}_{(x',v')}(s))ds}\right)$$

for a.e. $(x', v') \in \Gamma_-$, $t > 0$. Thus we have recovered $\sigma(x, v)$ for a.e. $(x, v) \in \Omega M$. Once σ is known, σ_a can be recovered from the original expression

$$\sigma_a(\vec{\gamma}_{(x',v')}(t))e^{-\int_0^t \sigma(\vec{\gamma}_{(x',v')}(s))ds}.$$

4.3 Conclusions

Our results show that in the case of smooth, variable index of refraction, one can recover the absorption coefficient in the transport equation from internal photoacoustic data, provided that $\sigma(x, v) = \sigma(x, -v)$ and $\sigma_a(x, v) = \sigma_a(x, -v)$. This requirement on σ and σ_a is weaker than what has typically been required in previous results for the transport equation (namely that $\sigma(x, v) = \sigma(x, |v|)$), and is considered to be physically reasonable.

We conclude by mentioning some open problems in transport theory and photoacoustics. Although we have demonstrated how to recover the absorption information, another problem of interest would be to see what information can be recovered about the scattering kernel, k. As noted previously, in the transport model of photoacoustics (where we only have angularly averaged data), we recover very limited information about k, even in the Euclidean case. Further analysis is needed to attempt to recover more information.

Another problem that would be a natural next step is the partial data problem. Recall that for the diffusion approximation, unique recovery of the coefficients was obtained with a finite number of measurements. In the transport case, we are assuming that all measurements are available. This is, of course, never what happens when imaging methods are implemented, since measurements can only be taken discretely in space and over a finite length of time. It would be of practical interest to see what information could be recovered in the transport model with a finite number of measurement or with measurements taken on only a portion of the observation surface.

In addition, there are several problems related to variable sound speed and index of refraction. Most of the work for inhomogeneous media has been done for recovery of the thermal deposition map where there are results for both the smooth sound speed case ([33]) and in certain cases of discontinuous sound speed ([34]). Analysis for QTAT and the diffusion model of QPAT in inhomogeneous media is still lacking. Furthermore, there is still significant work that can be done for the transport model in inhomogeneous media, including the case of discontinuous index of refraction and results for the scattering coefficient.

Another issue that we have not addressed in this paper is the knowledge of the sound speed and the index of refraction. All of our analysis has assumed that the index of refraction is known, which is also assumed to be the case in most other papers dealing with inhomogeneous media. In particular, one often assumes that sound speed has been determined by a prior transmission ultrasound scan, however, recovery of the sound speed is an interesting problem in itself, and it would be useful to find a way to recover the sound speed and the thermal deposition map simultaneously ([20]).

Overall, photoacoustic and thermoacoustic tomography provide not only a safe non-ionizing imaging method with excellent image quality, but also provide a wealth of mathematically interesting problems for further study.

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