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Causal Effect Identification via Equivalence Classes of Acyclic Graphs and Data-Driven Adjustment

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Abstract

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Identifying a causal effect involves finding a function of observational densities that serves as an equivalent form of the interventional density of interest – denoted by $f(\mathbf{y} \mid do(\mathbf{x}))$ in the *unconditional* setting and $f(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z})$ in the *conditional setting*. This equivalency allows researchers to rely on observational data alone to estimate a causal effect. We consider the problem of identifying causal effects across three chapters.

In our first chapter, we focus on identifying conditional effects through covariate adjustment in a setting where the causal graph is known up to one of two types of graphs: a maximally oriented partially directed acyclic graph (MPDAG) or a partial ancestral graph (PAG). We provide a necessary and sufficient graphical criterion for finding these sets when conditioning on variables unaffected by treatment, and we provide explicit sets from the graph that satisfy this criterion.

In our second chapter, we continue exploring covariate adjustment but turn to focus on the unconditional setting where there is no prior knowledge of the underlying causal graph. We present two routes for finding adjustment sets that instead rely on in/dependencies in the data directly. One route applies a concept known as c-equivalence to extend the work of Entner et al. (2013) under a single treatment, and another provides sufficient criteria for finding adjustment sets under multiple treatments.

In our third chapter, we return to conditional identification where the causal graph is known up to an MPDAG. But rather than focusing on covariate adjustment, we consider identification more generally. We develop a conditional identification formula, based on graphical criteria, that extends beyond settings where conditional adjustment sets exist, and we pair this with a necessary and sufficient criterion for when this identification is possible. Further, we extend the well-known do calculus to the MPDAG setting and build a conditional identification algorithm based on this calculus that is complete for identifying these conditional effects.

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May the women who proceed me have an even softer blow.

DEDICATION

this adventure was for me

Chapter 1

INTRODUCTION

Researchers across academic fields, professional industries, and governmental organizations have an interest in understanding causal effects. They may want to know: Do newly developed endometriosis treatments decrease patient pain? Do electric car subsidies decrease reliance on international oil imports? To answer these questions, we can turn to an experimental setting where treatments are assigned with randomness and outcomes are measured. But there are many settings where experiments are too expensive, unethical, or logistically infeasible to perform.

In these cases, we turn to data that we can collect in the wild, without any external intervention. But the problem with asking causal questions using observational data is that the data cannot mimic the setting of an experiment. For example, perhaps we find that patients receiving a newly developed medical treatment reported much lower pain, on average, than patients receiving the current standard of care. This may imply that the treatment causes a decrease in reported pain. But perhaps those seeking and obtaining newly developed treatments are wealthier patients with liberal access to both peripheral medical care and professional safeguards that aid in the successful management of their pain. Therefore, had these patients instead received the current standard of care, they may have reported similar levels of pain (i.e., still lower, on average). This is the fundamental problem of causal inference [Holland, 1986] – that we cannot know the counterfactual for these patients. Thus, we need additional tools to measure a causal effect given only observational data. This dissertation focuses on finding such tools.

1.1 DO INTERVENTIONS

In statistical terms, consider a set of random variables $\mathbf{V}$ in a causal model with joint density $f(\mathbf{v})$. Suppose $\mathbf{V}$ contains a set of treatments $\mathbf{X}$ and a set of outcomes $\mathbf{Y}$. To model the causal effect of $\mathbf{X}$ on $\mathbf{Y}$, we need more than their joint density $f(\mathbf{y}, \mathbf{x})$. That is, we do not solely care how $\mathbf{X}$ and $\mathbf{Y}$ vary with each other in the model. Instead, we want to know how $\mathbf{Y}$ varies in the model given an intervention on $\mathbf{X}$. Researchers often refer to this intervention as a *do intervention* on $\mathbf{X}$ – abbreviated $do(\mathbf{X})$ – and the density under this intervention, or the *interventional density*, appears as $f(\mathbf{y} | do(\mathbf{x}))$ [Pearl, 1995]. We note that this does not designate the density of $\mathbf{Y}$ given $\mathbf{X}$ (i.e., $f(\mathbf{y} | \mathbf{x})$). Rather, it describes the behavior of $\mathbf{Y}$ under a do intervention – an entire procedure where we intervene to set $\mathbf{X}$ to a fixed value $\mathbf{x}$ and subsequently feed this value $\mathbf{x}$ to any relevant part of the data-generating mechanism [Pearl, 1995].

To concretize this concept, consider the following linear structural equation model (SEM) over $\mathbf{V} = \{X, Y\}$, where $\varepsilon_y \perp\!\!\!\perp \varepsilon_x$:

$$\begin{aligned} Y &\leftarrow \varepsilon_y, & \varepsilon_y &\sim \mathcal{N}(0, 1), \\ X &\leftarrow \frac{1}{2}Y + \varepsilon_x, & \varepsilon_x &\sim \mathcal{N}(0, \frac{3}{4}). \end{aligned} \tag{1.1}$$

In this SEM, note that $(Y, X)^T$ follows a multivariate normal distribution such that $Y | X \sim \mathcal{N}(\frac{1}{2}X, \frac{3}{4})$. Turn to consider a do intervention on X , where we set $X = x$. We define the interventional density by the conditional density under the following model:

$$\begin{aligned} Y &\leftarrow \varepsilon_y, & \varepsilon_y &\sim \mathcal{N}(0, 1), \\ X &\leftarrow x. \end{aligned}$$

Thus, $Y | do(X) \sim \mathcal{N}(0, 1)$ so that $f(y | do(x)) \neq f(y | x)$. That is, conditioning on $do(X)$ is not equivalent to conditioning on X in this model.

These interventional densities play a large role in causal effect research, because our goal for statistical inference is often a functional of this density. For example, with a discrete

treatment X taking values in $\{0, 1\}$, we can model the average treatment effect on Y as

$$\mathbb{E}[Y|do(X = 1)] - \mathbb{E}[Y|do(X = 0)]. \quad (1.2)$$

1.2 IDENTIFIABILITY

To estimate parameters that rely on interventional densities, such as the expression in (1.2), we can use data generated from the interventional distribution. And an experiment provides this type of data. In other words, the values of outcomes $\mathbf{Y}$ we measure in an experiment with treatments $\mathbf{X}$ are sampled values from the interventional distribution of $\mathbf{Y} | do(\mathbf{X})$. So, for example, we can estimate the expression in (1.2) by running an experiment, collecting data from the experiment, and using the following estimator: the average of Y values from the experimental data when $X = 1$ minus the average of Y values from the experimental data when $X = 0$.

But without an experiment, we only have access to the *observational density* $f(\mathbf{v})$ – and therefore to $f(\mathbf{y} | \mathbf{x})$. As we saw above, it does not hold universally that $f(\mathbf{y} | do(\mathbf{x})) = f(\mathbf{y} | \mathbf{x})$. So how do we estimate the causal effect?

In some cases, we can learn enough from observational data to find an equivalent form for $f(\mathbf{y} | do(\mathbf{x}))$ using only observational densities. For example, while it does not hold broadly, it may be the case that $f(\mathbf{y} | do(\mathbf{x})) = f(\mathbf{y} | \mathbf{x})$ in one specific causal model. Or it could hold in one specific model that $f(\mathbf{y} | do(\mathbf{x})) = f(\mathbf{y})$, as we saw in the SEM in (1.1). The key component is not the specific form that $f(\mathbf{y} | do(\mathbf{x}))$ takes, but the fact that the right hand side of the equation contains only observational densities. Logistically, this type of equivalency means that we can estimate any functional of $f(\mathbf{y} | do(\mathbf{x}))$ (e.g., the causal effect) by estimating its equivalent form using observational densities. And this estimation is possible with observational data alone. Thus, when we are able to find such a form, we say that the causal effect is *identifiable* from observational data. Identifying causal effects is the focus of this dissertation.

1.3 OUTLINE OF CHAPTERS

We consider causal effect identification in several settings. Chapters 2 and 4 consider when a graphical representation of a causal model is known and when the causal effect of interest is *conditional* (e.g., $f(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z})$). Chapter 2 allows for two types of these causal graphs: maximally oriented partially directed acyclic graphs (MPDAGs) and partial ancestral graphs (PAGs). Chapter 4 focuses in depth on the MPDAG setting. Both MPDAGs and PAGs represent equivalence classes of possible underlying causal models. Note that we review definitions related to these graphs in each of the chapters' preliminary sections. In contrast, Chapter 3 considers causal identification in a setting where graphical knowledge is unknown and where the causal effect of interest is a *total*, or *unconditional*, effect (e.g., $f(\mathbf{y} \mid do(\mathbf{x}))$).

We also consider two approaches to identification. Chapters 2 and 3 focus on the method of covariate adjustment, where conditioning on variables in $\mathbf{V} \setminus [\mathbf{Y} \cup \mathbf{X}]$ can account for issues like confounding in observational data. We review the history of this research at the beginning of Chapter 3 and we detail interesting nuances in a conditional setting at the beginning of Chapter 2. In contrast, Chapter 4 handles causal identification by finding a broader formula for the interventional density of interest. This identification formula applies in settings where covariate adjustment cannot identify the causal effect. We review further details of each chapter below.

In Chapter 2, as mentioned, we consider the problem of identifying a conditional causal effect through covariate adjustment. We focus on the setting where the causal graph is known up to one of two types of graphs: an MPDAG or a PAG. After defining adjustment sets in this setting, we provide a necessary and sufficient graphical criterion – the *conditional adjustment criterion* – for finding these sets under conditioning on variables unaffected by treatment. Further, we provide explicit sets from the graph that satisfy the conditional adjustment criterion, and therefore, can be used as adjustment sets for conditional causal effect identification.

We begin Chapter 3 by reviewing prior research on covariate adjustment, and we highlight

that much of this research assumes knowledge of an underlying causal graph. Then we turn to our results, where we present two routes for finding adjustment sets that do not require knowledge of a graph – and instead rely on dependencies and independencies in the data directly. We consider a setting where the adjustment set is unaffected by treatment or outcome. The first route shows how to extend prior research in this area using a concept known as *c-equivalence*. Our second route provides sufficient criteria for finding adjustment sets in the setting of multiple treatments.

In Chapter 4, we consider identifying a conditional causal effect when an MPDAG is known. We provide three results that address identification in this setting: an identification formula when the conditioning set is unaffected by treatment, a generalization of the well-known do calculus to the MPDAG setting, and an algorithm that is complete for identifying these conditional effects.

1.4 A NOTE ON APPENDICES

We briefly note that the appendices for Chapters 2-4 can be found at the end of this dissertation. These appendices contain the proofs for many of our main results, which we exclude from the text of Chapters 2-4 for clarity and brevity. Whenever possible in the main text, we provide a sketch of each proof as well as a specific location in the appendix where readers can find the full proofs. Appendices A-C also include results from existing research (for easy reference) and a handful of novel results tangential to the work of each chapter (but potentially useful for future research).

Chapter 2

CONDITIONAL ADJUSTMENT IN A MARKOV EQUIVALENCE CLASS

The following chapter is based on work published in the Proceedings of the International Conference on Artificial Intelligence and Statistics (see LaPlante and Perković, 2024).

2.1 INTRODUCTION

Many scientific disciplines have an interest in identifying and estimating causal effects for specific subgroups of a population. For instance, researchers may want to know if a medical treatment is beneficial for people with heart disease or if the treatment will harm older patients [Brand and Xie, 2010, Health, 2010]. Such causal effects are referred to as *conditional causal effects* or *heterogeneous causal effects*. The identification of these conditional causal effects from observational data is the subject of this work.

Much of the literature on estimating conditional causal effects from observational data focuses on the conditional average treatment effect (CATE; Athey and Imbens, 2016, Wager and Athey, 2018, Künzel et al., 2019, Nie and Wager, 2021, Kennedy et al., 2022). The CATE is represented as a contrast of means for a response Y under different do-interventions (see Section 2.2 for definition) of a treatment X when conditioning on a set of covariate values $\mathbf{z}$. These means take the form $\mathbb{E}[Y | do(X = x), \mathbf{Z} = \mathbf{z}]$.

Some results on CATE estimation assume that the conditioning set $\mathbf{Z}$ is rich enough to capture all relevant common causes of X and Y – meaning that X and Y are *unconfounded* given $\mathbf{Z}$. This implies

$$\mathbb{E}[Y | do(X = x), \mathbf{Z} = \mathbf{z}] = \mathbb{E}[Y | X = x, \mathbf{Z} = \mathbf{z}], \quad (2.1)$$

which allows the CATE to be estimated as a difference of means from observational data.

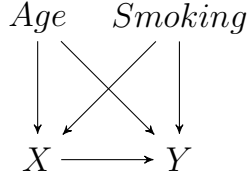


Figure 2.1: A causal DAG used in Section 2.1.

However, this assumption does not hold in all applications. Consider, for example, the setting depicted in the causal directed acyclic graph (DAG) of Figure 2.1, where we want to compute a causal effect of X on Y given some set $\mathbf{Z}$. In this setting, age and smoking status are common causes of X and Y , and therefore, X and Y are confounded unless we condition on both age and smoking status ($\mathbf{Z} = \{Age, Smoking\}$). But we may want to know the causal effect of X on Y conditional on age alone ($\mathbf{Z} = \{Age\}$).

To allow for estimation of the CATE in such cases, various recent works [Abrevaya et al., 2015, Fan et al., 2022, Chernozhukov et al., 2023, Smucler et al., 2022] have proposed estimation methods that rely on knowing an additional set of covariates $\mathbf{S}$ that – together with $\mathbf{Z}$ – leads to X and Y being unconfounded. We refer to this set of variables as a *conditional adjustment set* (Definition 1). For such a set $\mathbf{S}$,

$$\mathbb{E}[Y \mid do(X = x), \mathbf{Z} = \mathbf{z}] = \mathbb{E}_{\mathbf{S}} \left[\mathbb{E}[Y \mid X = x, \mathbf{Z}, \mathbf{S}] \mid \mathbf{Z} = \mathbf{z} \right]. \quad (2.2)$$

In the example above, if $\mathbf{Z} = \{Smoking\}$, then $\mathbf{S} = \{Age\}$.

Of course, not all conditional causal effect research focuses on estimation through the functional in Equation (2.2). Notably, other work has explored identifiability without limiting focus to a particular functional. For example, Shpitser and Pearl [2008] and Jaber et al. [2019, 2022] focus on the conditions under which the interventional distribution $f(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z})$ is identifiable given a causal graph. Though these results broaden the options for identification, estimators based on these results would have to rely on functionals that may prove difficult to estimate, such as $\frac{f(\mathbf{y}, \mathbf{z} \mid do(\mathbf{x}))}{f(\mathbf{z} \mid do(\mathbf{x}))}$ [Shpitser and Pearl, 2008, Jaber et al., 2019, 2022]. Our work addresses this by focusing on identification of the same interventional distribution given a

causal graph – but through the use of conditional adjustment sets, which may lead to more desirable estimators. To the best of our knowledge, this area of research is largely unexplored.

Our main contribution is the *conditional adjustment criterion* (Definitions 2 and 7), a graphical criterion that we show is necessary and sufficient for identifying a conditional adjustment set (Theorems 3 and 9). We additionally provide explicit sets that satisfy this criterion when any such set exists. We note, however, that these results are restricted to a setting where the conditioning set $\mathbf{Z}$ consists of variables known to be unaffected by treatment. While this restricted setting produces limitations (see the second example in the discussion, Section 2.5), our results are broadly applicable to a variety of research questions. For example, the restriction is met when the conditioning set includes exclusively pre-treatment variables.

In considering the problem of identifying a conditional adjustment set, we assume that the underlying causal system can be represented by a causal DAG. When we collect observational data on all variables in the system, we can attempt to learn this causal DAG by relying on the constraints present in the data [Spirtes et al., 1999, Chickering, 2002a, Zhang, 2008b, Hauser and Bühlmann, 2012, Mooij et al., 2020, Squires and Uhler, 2022]. However, this task is often impossible from observational data alone, regardless of the available sample size. And further, we cannot always observe every variable.

Thus, our work focuses on causal models that represent *Markov equivalence classes* of graphs that can be learned from observational data: a maximally oriented partially directed acyclic graph (MPDAG; Meek, 1995) and a maximally oriented partial ancestral graph (PAG; Richardson and Spirtes, 2002). An MPDAG represents a restriction of the Markov equivalence class of DAGs that can be learned from observational data and background knowledge when all variables are observed [Meek, 1995, Chickering, 2002a]. A PAG represents a Markov equivalence class of maximal ancestral graphs (MAGs; Richardson and Spirtes, 2002), which can be learned from observational data and which allows for unobserved variables [Spirtes et al., 2000, Zhang, 2008b, Ali et al., 2009]. A MAG, in turn, can be seen as a marginalization of a DAG containing only the observed variables [Richardson and Spirtes, 2002]. See Section

2.2 and App. A.1 for further definitions.

The structure of this paper is as follows: Section 2.2 provides preliminary definitions, with the remaining definitions given in App. A.1. Section 2.3 contains all results for the MPDAG setting. In particular, we introduce our conditional adjustment criterion in Section 2.3.1; Section 2.3.2 illustrates applications of our criterion with examples; Section 2.3.3 provides several methods for constructing conditional adjustment sets; and Section 2.3.4 includes a discussion of the similarities of our conditional adjustment criterion with both the adjustment criterion of Perković et al. [2017] and the $\mathbf{Z}$ -dependent dynamic adjustment criterion of Smucler et al. [2022]. We present some analogous results for PAGs in Section 2.4, and we discuss some limitations of our results and areas for future work in Section 2.5.

2.2 PRELIMINARIES

We use capital letters (e.g., X) to denote nodes in a graph as well as random variables that these nodes represent. Similarly, bold capital letters (e.g., $\mathbf{X}$) are used to denote node sets and random vectors.

Nodes, Edges, and Subgraphs. A graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ consists of a set of nodes (variables) $\mathbf{V} = \{V_1, \dots, V_p\}$, $p \geq 1$, and a set of edges $\mathbf{E}$. Edges can be *directed* ($\rightarrow$), *bi-directed* ($\leftrightarrow$), *undirected* ($\circ\circ$ or $-$), or *partially directed* ($\circ\rightarrow$). We use $\bullet$ as a stand in for any of the allowed edge marks. An edge is *into* (*out of*) a node X if the edge has an arrowhead (tail) at X . An *induced subgraph* $\mathcal{G}_{\mathbf{V}'} = (\mathbf{V}', \mathbf{E}')$ of $\mathcal{G}$ consists of $\mathbf{V}' \subseteq \mathbf{V}$ and $\mathbf{E}' \subseteq \mathbf{E}$ where $\mathbf{E}'$ are all edges in $\mathbf{E}$ between nodes in $\mathbf{V}'$.

Directed and Partially Directed Graphs. A *directed graph* contains only directed edges ($\rightarrow$). A *partially directed graph* may contain undirected edges ($-$) and directed edges ($\rightarrow$).

Mixed and Partially Directed Mixed Graphs. A *mixed graph* may contain directed and bi-directed edges. The *partially directed mixed graphs* we consider can contain any of the following edge types: $\circ\circ$, $\circ\rightarrow$, $\rightarrow$, and $\leftrightarrow$. Hence, an edge $\bullet\rightarrow$ in a partially directed graph can only refer to edge $\rightarrow$, whereas in a partially directed mixed graph, $\bullet\rightarrow$ can represent $\rightarrow$,

$\leftrightarrow$, or $\circ\rightarrow$.

Paths and Cycles. For disjoint node sets $\mathbf{X}$ and $\mathbf{Y}$, a path from $\mathbf{X}$ to $\mathbf{Y}$ is a sequence of distinct nodes $\langle X, \dots, Y \rangle$ from some $X \in \mathbf{X}$ to some $Y \in \mathbf{Y}$ for which every pair of successive nodes is adjacent. A path consisting of undirected edges ($-$ or $\circ\circ$) is an *undirected path*. A *directed path* from X to Y is a path of the form $X \rightarrow \dots \rightarrow Y$. A directed path from X to Y and the edge $Y \rightarrow X$ form a *directed cycle*. A directed path from X to Y and the edge $X \rightarrow Y$ form an *almost directed cycle*. A path $\langle V_1, \dots, V_k \rangle$, $k > 1$, in a graph $\mathcal{G}$ is a *possibly directed path* if no edge $V_i \leftarrow \bullet V_j$, $1 \leq i < j \leq k$, is in $\mathcal{G}$ (Perković et al., 2017, Zhang, 2008a).

A path from $\mathbf{X}$ to $\mathbf{Y}$ is *proper* (w.r.t. $\mathbf{X}$) if only its first node is in $\mathbf{X}$. A path from X to Y is a *back-door path* if does not begin with a visible edge out of X (see definition of *visible* below; Pearl, 2009, Maathuis and Colombo, 2015). For a path $p = \langle X_1, X_2, \dots, X_k \rangle$ and i, j, k such that $1 \leq i < j \leq k$, we define the *subpath* of p from X_i to X_j as the path $p(X_i, X_j) = \langle X_i, X_{i+1}, \dots, X_j \rangle$.

Colliders, Shields, and Definite Status Paths. If a path p contains $X_i \bullet \rightarrow X_j \leftarrow \bullet X_k$ as a subpath, then X_j is a *collider* on p . A path $\langle X_i, X_j, X_k \rangle$ is an *unshielded triple* if X_i and X_k are not adjacent. A path is *unshielded* if all successive triples on the path are unshielded. A node X_j is a *definite non-collider* on a path p if the edge $X_i \leftarrow X_j$ or $X_j \rightarrow X_k$ is on p , or if $\langle X_i, X_j, X_k \rangle$ is an undirected subpath of p and X_i is not adjacent to X_k . A node is of *definite status* on a path if it is a collider, a definite non-collider, or an endpoint on the path. A path p is of definite status if every node on p is of definite status.

Blocking, D-separation, and M-separation. Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ be pairwise disjoint node sets in a directed or partially directed graph $\mathcal{G}$. A definite-status path p from $\mathbf{X}$ to $\mathbf{Y}$ is *d-connecting* (*open*) given $\mathbf{Z}$ if every definite non-collider on p is not in $\mathbf{Z}$ and every collider on p has a descendant in $\mathbf{Z}$. Otherwise, $\mathbf{Z}$ *blocks* p . If $\mathbf{Z}$ blocks all definite status paths between $\mathbf{X}$ and $\mathbf{Y}$ in $\mathcal{G}$, then $\mathbf{X}$ is *d-separated* from $\mathbf{Y}$ given $\mathbf{Z}$ in $\mathcal{G}$ and we write $(\mathbf{X} \perp_d \mathbf{Y} | \mathbf{Z})_{\mathcal{G}}$ [Pearl, 2009]. If $\mathcal{G}$ is a mixed or partially directed mixed graph, the analogous terms to d-connection and d-separation are called *m-connection* and *m-separation* [Richardson and Spirtes, 2002]. We use the same notation $\perp_d$ to denote m-separation.

Ancestral Relationships. If $X \rightarrow Y$, then X is a *parent* of Y . If $X - Y$, $X \circ\!\!\!\circ Y$, $X \circ\!\!\!\rightarrow Y$, or $X \rightarrow Y$, then X is a *possible parent* of Y . If there is a directed path from X to Y , such as $X \rightarrow M_1 \rightarrow \cdots \rightarrow M_k$, $M_k = Y$, $k \geq 1$, then X is an *ancestor* of Y , Y is a *descendant* of X , and $M_1, \dots, M_k$ are *mediators* for X and Y . We use the convention that if Y is a descendant of X , then Y is also a mediator for X and Y . If there is a possibly directed path from X to Y , then X is a *possible ancestor* of Y , Y is a *possible descendant* of X , and any node on this path that is not X is a *possible mediator* of X and Y . We use the convention that if Y is a possible descendant of X , then Y is also a possible mediator for X and Y . We also use the convention that every node is an ancestor, descendant, possible ancestor, and possible descendant of itself. The sets of parents, possible parents, ancestors, descendants, possible ancestors, and possible descendants of X in $\mathcal{G}$ are denoted by $\text{Pa}(X, \mathcal{G})$, $\text{PossPa}(X, \mathcal{G})$, $\text{An}(X, \mathcal{G})$, $\text{De}(X, \mathcal{G})$, $\text{PossAn}(X, \mathcal{G})$, and $\text{PossDe}(X, \mathcal{G})$, respectively. Similarly, we denote the sets of mediators and possible mediators for X and Y in $\mathcal{G}$ by $\text{Med}(X, Y, \mathcal{G})$ and $\text{PossMed}(X, Y, \mathcal{G})$.

We let $\text{An}(\mathbf{X}, \mathcal{G}) = \cup_{X \in \mathbf{X}} \text{An}(X, \mathcal{G})$, with analogous definitions for $\text{De}(\mathbf{X}, \mathcal{G})$, $\text{PossAn}(\mathbf{X}, \mathcal{G})$, and $\text{PossDe}(\mathbf{X}, \mathcal{G})$. For disjoint node sets $\mathbf{X}$ and $\mathbf{Y}$, we let $\text{Med}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$ be the union of all mediators of $X \in \mathbf{X}$ and $Y \in \mathbf{Y}$ that lie on a proper directed path from $\mathbf{X}$ to $\mathbf{Y}$, with an analogous definition for $\text{PossMed}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$. Unconventionally, we define $\text{Pa}(\mathbf{X}, \mathcal{G}) = (\cup_{X \in \mathbf{X}} \text{Pa}(X, \mathcal{G})) \setminus \mathbf{X}$. We denote that X is adjacent to Y in $\mathcal{G}$ by $X \in \text{Adj}(Y, \mathcal{G})$.

DAGs and PDAGs. A directed graph without directed cycles is a *directed acyclic graph* (DAG). A *partially directed acyclic graph* (PDAG) is a partially directed graph without directed cycles.

MAGs. A mixed graph without directed or almost directed cycles is called *ancestral*. Note that we do not consider ancestral graphs that represent selection bias (see Zhang, 2008a, for details). A *maximal ancestral graph* (MAG) is an ancestral graph $\mathcal{M} = (\mathbf{V}, \mathbf{E})$ where every pair of non-adjacent nodes X and Y in $\mathcal{M}$ can be m-separated by a set $\mathbf{Z} \subseteq \mathbf{V} \setminus \{X, Y\}$. A DAG $\mathcal{D} = (\mathbf{V}, \mathbf{E})$ with unobserved variables $\mathbf{U} \subseteq \mathbf{V}$ can be uniquely *represented by* a MAG $\mathcal{M} = (\mathbf{V} \setminus \mathbf{U}, \mathbf{E}')$, which preserves the ancestry and m-separations among the observed

variables [Richardson and Spirtes, 2002].

MPDAGs and Markov Equivalence. All DAGs over a node set $\mathbf{V}$ with the same adjacencies and unshielded colliders can be uniquely *represented by a completed PDAG* (CPDAG). These DAGs form a Markov equivalence class with the same set of d-separations. A *maximally oriented PDAG* (MPDAG) is formed by taking a CPDAG, adding background knowledge (by directing undirected edges), and completing Meek [1995]’s orientation rules. We say a DAG is *represented by* an MPDAG $\mathcal{G}$ if it has the same nodes, adjacencies, and directed edges as $\mathcal{G}$. The set of such DAGs – denoted by $[\mathcal{G}]$ – forms a restriction of the Markov equivalence class so that all DAGs in $[\mathcal{G}]$ have same set of d-separations. Note that if $\mathcal{G}$ has the edge $A - B$, then $[\mathcal{G}]$ contains at least one DAG with $A \rightarrow B$ and one DAG with $A \leftarrow B$ [Meek, 1995]. Further, note that all DAGs and CPDAGs are MPDAGs.

PAGs and Markov Equivalence. All MAGs that encode the same set of m-separations form a Markov equivalence class, which can be uniquely *represented by a partial ancestral graph* (PAG; Richardson and Spirtes, 2002, Ali et al., 2009). $[\mathcal{G}]$ denotes all MAGs represented by a PAG $\mathcal{G}$. We say a DAG $\mathcal{D}$ is *represented by* a PAG $\mathcal{G}$ if there is a MAG $\mathcal{M} \in [\mathcal{G}]$ such that $\mathcal{D}$ is represented by $\mathcal{M}$.

We do not consider PAGs that represent selection bias (see Zhang, 2008b). Further, we only consider maximally informative PAGs [Zhang, 2008b]. That is, if a PAG $\mathcal{G}$ has the edge $A \bullet \circ B$, then $[\mathcal{G}]$ contains a MAG with $A \bullet \rightarrow B$ and a MAG with $A \leftarrow B$. (We preclude MAGs with $A - B$ by assuming no selection bias.) Any arrowhead or tail edge mark in a PAG $\mathcal{G}$ corresponds to that same arrowhead or tail edge mark in every MAG in $[\mathcal{G}]$. The edge orientations in every PAG we consider are completed with respect to orientation rules $R1 - R4$ and $R8 - R10$ of Zhang [2008b].

Visible and Invisible Edges. Given a MAG or PAG $\mathcal{G}$, a directed edge $X \rightarrow Y$ is *visible* in $\mathcal{G}$ if there is a node $V \notin \text{Adj}(Y, \mathcal{G})$ such that $\mathcal{G}$ contains either $V \bullet \rightarrow X$ or $V \bullet \rightarrow V_1 \leftrightarrow \dots \leftrightarrow V_k \leftrightarrow X$, where $k \geq 1$ and $V_1, \dots, V_k \in \text{Pa}(Y, \mathcal{G}) \setminus \{V, X, Y\}$ [Zhang, 2006]. A directed edge that is not visible in a MAG or PAG is said to be *invisible*. Every directed edge in an MPDAG is said to be visible.

Markov Compatibility and Positivity. An *observational density* $f(\mathbf{v})$ is *Markov compatible* with a DAG $\mathcal{D} = (\mathbf{V}, \mathbf{E})$ if $f(\mathbf{v}) = \prod_{V_i \in \mathbf{V}} f(v_i | \text{pa}(v_i, \mathcal{D}))$. If $f(\mathbf{v})$ is Markov compatible with a DAG $\mathcal{D}$, then it is Markov compatible with every DAG that is Markov equivalent to $\mathcal{D}$ [Pearl, 2009]. Hence, we say that a density is *Markov compatible* with an MPDAG, MAG, or PAG $\mathcal{G}$ if it is Markov compatible with a DAG represented by $\mathcal{G}$. Throughout, we assume positivity. That is, we only consider distributions that satisfy $f(\mathbf{v}) > 0$ for all valid values of $\mathbf{V}$ [Kivva et al., 2023].

Probabilistic Implications of Graph Separation. Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ be pairwise disjoint node sets in a DAG, MPDAG, MAG, or PAG $\mathcal{G}$. If $\mathbf{X}$ and $\mathbf{Y}$ are d-separated or m-separated given $\mathbf{Z}$ in $\mathcal{G}$, then $\mathbf{X}$ and $\mathbf{Y}$ are conditionally independent given $\mathbf{Z}$ in any *observational density* that is Markov compatible with $\mathcal{G}$ [Lauritzen et al., 1990, Zhang, 2008a, Henckel et al., 2022].

Causal Graphs. Let $\mathcal{G}$ be a graph with nodes V_i and V_j . When $\mathcal{G}$ is an MPDAG, it is a *causal MPDAG* if every edge $V_i \rightarrow V_j$ represents a direct causal effect of V_i on V_j and if every edge $V_i - V_j$ represents a direct causal effect of unknown direction (either V_i affects V_j or V_j affects V_i). Note that all DAGs are MPDAGs.

When $\mathcal{G}$ is a MAG or PAG, it is a *causal MAG* or *causal PAG*, respectively, if every edge $V_i \rightarrow V_j$ represents the presence of a causal path from V_i to V_j ; every edge $V_i \leftarrow \bullet V_j$ represents the absence of a causal path from V_i to V_j ; and every edge $V_i \circ - \circ V_j$ represents the presence of a causal path of unknown direction or a common cause in the underlying causal DAG.

Causal and Non-causal Paths. Note that any directed or possibly directed path in a causal graph is *causal* or *possibly causal*, respectively. However, since we focus on causal graphs, we will use this causal terminology for paths in any of our graphs. We will say a path is *non-causal* if it is not possibly causal.

Consistency. Let $f(\mathbf{v})$ be an observational density over $\mathbf{V}$. The notation $do(\mathbf{X} = \mathbf{x})$, or $do(\mathbf{x})$ for short, represents an outside intervention that sets $\mathbf{X} \subseteq \mathbf{V}$ to fixed values $\mathbf{x}$. An *interventional density* $f(\mathbf{v} | do(\mathbf{x}))$ is a density resulting from such an intervention.

Let $\mathbf{F}^*$ denote the set of all interventional densities $f(\mathbf{v} | do(\mathbf{x}))$ such that $\mathbf{X} \subseteq \mathbf{V}$ (including

$\mathbf{X} = \emptyset$). A causal DAG $\mathcal{D} = (\mathbf{V}, \mathbf{E})$ is a *causal Bayesian network compatible with $\mathbf{F}^*$* if and only if for all $f(\mathbf{v}|do(\mathbf{x})) \in \mathbf{F}^*$, the following *truncated factorization* holds:

$$f(\mathbf{v}|do(\mathbf{x})) = \prod_{V_i \in \mathbf{V} \setminus \mathbf{X}} f(v_i | \text{pa}(v_i, \mathcal{D})) \mathbf{1}(\mathbf{X} = \mathbf{x}) \quad (2.3)$$

[Pearl, 2009, Bareinboim et al., 2012]. We say an interventional density is *consistent* with a causal DAG $\mathcal{D}$ if it belongs to a set of interventional densities $\mathbf{F}^*$ such that $\mathcal{D}$ is compatible with $\mathbf{F}^*$. Note that any observational density that is Markov compatible with $\mathcal{D}$ is consistent with $\mathcal{D}$. We say an interventional density is *consistent* with a causal MPDAG, MAG, or PAG $\mathcal{G}$ if it is consistent with each DAG represented by $\mathcal{G}$ – were the DAG to be causal.

Identifiability. Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ be pairwise disjoint node sets in a causal MPDAG or PAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, and let $\mathbf{F}_i^* = \{f_i(\mathbf{v}|do(\mathbf{x}')) : \mathbf{X}' \subseteq \mathbf{V}\}$ be a set with which a DAG $\mathcal{D}_i$ represented by $\mathcal{G}$ is compatible – were $\mathcal{D}_i$ to be causal. We say the conditional causal effect of $\mathbf{X}$ on $\mathbf{Y}$ given $\mathbf{Z}$ is *identifiable* in $\mathcal{G}$ if for any $\mathbf{F}_1^*, \mathbf{F}_2^*$ where $f_1(\mathbf{v}) = f_2(\mathbf{v})$, we have $f_1(\mathbf{y}|do(\mathbf{x}), \mathbf{z}) = f_2(\mathbf{y}|do(\mathbf{x}), \mathbf{z})$ [Pearl, 2009].

Forbidden Set. Let $\mathbf{X}$ and $\mathbf{Y}$ be disjoint node sets in an MPDAG or PAG $\mathcal{G}$. Then the *forbidden set* relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{G}$ is

$$\text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) = \left\{ \begin{array}{l} \text{nodes in } \text{PossDe}(W, \mathcal{G}), \text{ where} \\ W \in \text{PossMed}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) \end{array} \right\}.$$

2.3 RESULTS - MPDAGS

In this section, we present our results on identifying a conditional causal effect via our conditional adjustment criterion in the setting of an MPDAG (Definition 2). Examples of how to use our criterion and explicit conditional adjustment sets based on our criterion follow these results. We remark here that our criterion shares similarities with the adjustment criterion for total effect identification of Perković et al. [2017] and with the $\mathbf{Z}$ -dependent dynamic adjustment criterion of Smucler et al. [2022], but we save these results and reflections for Section 2.3.4.

Note that the results of this section hold when a fully oriented DAG is known, since all DAGs are MPDAGs. Throughout, our goal is to identify the conditional causal effect of treatments $\mathbf{X}$ on responses $\mathbf{Y}$ conditional on covariates $\mathbf{Z}$ and given a known graph $\mathcal{G}$.

2.3.1 CONDITIONAL ADJUSTMENT CRITERION

We include our definition of a conditional adjustment set below (Definition 1). Note that, while this section focuses on MPDAGs, we write Definition 1 broadly for further use in Section 2.4. Our goal in this section is to find an equivalent graphical characterization of a conditional adjustment set. Theorem 3 establishes that Definition 2 provides such a graphical characterization, which we call the conditional adjustment criterion, under the assumption that the conditioning set does not contain variables affected by treatment ($\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$).

Definition 1 (Conditional Adjustment Set for MPDAGs, PAGs) *Let $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$, and $\mathbf{S}$ be pairwise disjoint node sets in a causal MPDAG or PAG $\mathcal{G}$. Then $\mathbf{S}$ is a conditional adjustment set relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{G}$ if for any density f consistent with $\mathcal{G}$*

$$f(\mathbf{y}|\text{do}(\mathbf{x}), \mathbf{z}) = \begin{cases} f(\mathbf{y}|\mathbf{x}, \mathbf{z}) & \mathbf{S} = \emptyset \\ \int f(\mathbf{y}|\mathbf{x}, \mathbf{z}, \mathbf{s})f(\mathbf{s}|\mathbf{z}) d\mathbf{s} & \mathbf{S} \neq \emptyset. \end{cases} \quad (2.4)$$

Definition 2 (Conditional Adjustment Criterion for MPDAGs) *Let $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$, and $\mathbf{S}$ be pairwise disjoint node sets in an MPDAG $\mathcal{G}$, where $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$ and where every proper possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$ starts with a directed edge. Then $\mathbf{S}$ satisfies the conditional adjustment criterion relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{G}$ if*

- (a) $\mathbf{S} \cap \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) = \emptyset$, and
- (b) $\mathbf{S} \cup \mathbf{Z}$ blocks all proper non-causal definite status paths from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$.

Theorem 3 (Completeness, Soundness of Conditional Adjustment Criterion for MPDAGs) *Let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$, and $\mathbf{S}$ be pairwise disjoint node sets in a causal MPDAG $\mathcal{G}$,*

where $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$. Then $\mathbf{S}$ is a conditional adjustment set relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{G}$ (Definition 1) if and only if $\mathbf{S}$ satisfies the conditional adjustment criterion relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{G}$ (Definition 2).

Proof of Theorem 3. First note the following facts.

- (i) Every proper possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$ starts with a directed edge.
- (ii) $\mathbf{Z} \cap \text{De}(\mathbf{X}, \mathcal{D}) = \emptyset$ in every DAG $\mathcal{D}$ in $[\mathcal{G}]$.
- (iii) $\mathbf{Z} \cap \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) = \emptyset$.

We have that (i) holds in either direction – by definition ($\Leftarrow$) or by Proposition 51 (App. A.3) ($\Rightarrow$). Then Lemmas 35 and 41 (App. A.2) imply (ii) and (iii), respectively, given $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$ and (i). Now consider the following statements.

- (a) $\mathbf{S}$ is a conditional adjustment set relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{G}$.
- (b) $\mathbf{S}$ is a conditional adjustment set relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in each DAG in $[\mathcal{G}]$ – were the DAG to be causal.
- (c) $\mathbf{S}$ satisfies the conditional adjustment criterion relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in each DAG in $[\mathcal{G}]$.
- (d) $\mathbf{S}$ satisfies the conditional adjustment criterion relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{G}$.

By definition, (a) $\Leftrightarrow$ (b). Then (b) $\Leftrightarrow$ (c) by Theorems 54 and 55 (App. A.4) and the fact that the conditional adjustment criterion does not require a causal DAG. Lastly, by the facts above and by applying Lemmas 36 and 37 (App. A.2) in turn, (c) $\Leftrightarrow$ (d). ■

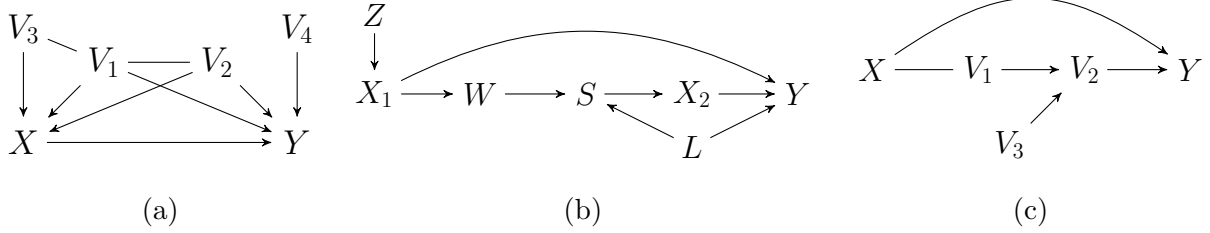


Figure 2.2: Causal MPDAGs used in Examples 1-4.

2.3.2 EXAMPLES

To illustrate the usefulness of the results above, we provide examples below where we aim to find $f(\mathbf{y}|do(\mathbf{x}), \mathbf{z})$ when $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$. Theorem 3 allows us to use the conditional adjustment criterion to (a) check whether a set can be used for conditional adjustment (Examples 1-3) or (b) determine if no such set exists (Example 4).

Example 1 (Empty Conditional Adjustment Set.) Let $\mathcal{G}$ be the causal MPDAG in Figure 2.2(a)¹, and let $\mathbf{X} = \{X\}$, $\mathbf{Y} = \{Y\}$, and $\mathbf{Z} = \{V_1, V_2\}$. Note that $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$ and that every possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$ starts with a directed edge.

Let $\mathbf{S} = \emptyset$. Note that $\mathbf{S} \cap (\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}) = \emptyset$, $\mathbf{S} \cap \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) = \emptyset$, and $\mathbf{S} \cup \mathbf{Z}$ blocks all non-causal definite status paths from $\mathbf{X}$ to $\mathbf{Y}$. Thus, $\mathbf{S}$ satisfies the conditional adjustment criterion relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{G}$, and by Theorem 3, $f(\mathbf{y}|do(\mathbf{x}), \mathbf{z}) = f(y|x, v_1, v_2)$.

Example 2 (Only Nonempty Conditional Adjustment Sets.) Again let $\mathcal{G}$ be the causal MPDAG in Figure 2.2(a), where $\mathbf{X} = \{X\}$ and $\mathbf{Y} = \{Y\}$. But now let $\mathbf{Z} = \{V_1\}$. We still have that $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$ and that every possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$ starts with a directed edge.

Note that if we let $\mathbf{S} = \emptyset$, $\mathbf{S} \cup \mathbf{Z}$ does not block the path $X \leftarrow V_2 \rightarrow Y$, which is a

¹Compare to Figure 5(a) of Perković [2020].

proper non-causal definite status path from $\mathbf{X}$ to $\mathbf{Y}$. Thus, the empty set is not a conditional adjustment set relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{G}$.

Consider, instead, the set $\mathbf{S} = \{V_2\}$. Note that $\mathbf{S} \cap (\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}) = \emptyset$, $\mathbf{S} \cap \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) = \emptyset$, and $\mathbf{S} \cup \mathbf{Z}$ blocks all non-causal definite status paths from $\mathbf{X}$ to $\mathbf{Y}$. Thus, $\mathbf{S}$ satisfies the conditional adjustment criterion relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{G}$, and by Theorem 3, $f(\mathbf{y}|\text{do}(\mathbf{x}), \mathbf{z}) = \int f(y|x, v_1, v_2)f(v_2|v_1) dv_2$.

Example 3 (Conditional Adjustment Set Contains Descendants of $\mathbf{X}$.) Let $\mathcal{G}$ be the causal DAG (and therefore, MPDAG) in Figure 2.2(b)², where we assume L is a variable that cannot be measured. Define $\mathbf{X} = \{X_1, X_2\}$, $\mathbf{Y} = \{Y\}$, and $\mathbf{Z} = \{Z\}$. Note that $\mathbf{Z} \cap \text{De}(\mathbf{X}, \mathcal{G}) = \emptyset$.

Consider the set $\mathbf{S} = \{S, W\}$. Note that $\mathbf{S} \cap (\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}) = \emptyset$, $\mathbf{S} \cap \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) = \emptyset$, and $\mathbf{S}$ blocks all proper non-causal paths from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$. Hence, $\mathbf{S}$ satisfies the conditional adjustment criterion relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{G}$, and by Theorem 3, $f(\mathbf{y}|\text{do}(\mathbf{x}), \mathbf{z}) = \int f(y|x_1, x_2, z, s, w)f(s, w|z) ds dw$.

Example 4 (No Conditional Adjustment Set, Effect Non-identifiable.) Let $\mathcal{G}$ be the causal MPDAG in Figure 2.2(c), and let $\mathbf{X} = \{X\}$, $\mathbf{Y} = \{Y\}$, and $\mathbf{Z} = \{V_3\}$. Note that $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$. However, $X - V_1 \rightarrow V_2 \rightarrow Y$ is a proper possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$ that starts with an undirected edge. Thus, by Theorem 3, there can be no conditional adjustment set relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{G}$. In fact, by Proposition 51 (App. A.3), $f(\mathbf{y}|\text{do}(\mathbf{x}), \mathbf{z})$ is not identifiable in $\mathcal{G}$ using any method.

2.3.3 CONSTRUCTING ADJUSTMENT SETS

The conditional adjustment criterion provides a way to check if a set can be used for conditional adjustment given an MPDAG $\mathcal{G}$, but it does not provide a way to construct a conditional adjustment set – a task that may be difficult when $\mathcal{G}$ is large. The results in

²Compare to Figure 6(a) of Perković et al. [2018].

this section provide such a roadmap under certain assumptions. The proofs can be found in App. A.6.

Lemma 4 *Let $\mathbf{X} = \{X\}$, $\mathbf{Y}$, and $\mathbf{Z}$ be pairwise disjoint node sets in a causal MPDAG $\mathcal{G}$, where $\mathbf{Z} \cap \text{PossDe}(X, \mathcal{G}) = \emptyset$ and where every possibly causal path from X to $\mathbf{Y}$ in $\mathcal{G}$ starts with a directed edge. If $\mathbf{Y} \cap \text{Pa}(X, \mathcal{G}) = \emptyset$, then the following is a conditional adjustment set relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{G}$:*

$$\text{Pa}(X, \mathcal{G}) \setminus \mathbf{Z}. \quad (2.5)$$

Theorem 5 *Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ be pairwise disjoint node sets in a causal MPDAG $\mathcal{G}$, where $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$ and where every proper possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$ starts with a directed edge.*

(a) *If there is any conditional adjustment set relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{G}$, then the following set is one:*

$$\begin{aligned} \text{Adjust}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathcal{G}) = & [\text{PossAn}(\mathbf{X} \cup \mathbf{Y}, \mathcal{G}) \cup \text{An}(\mathbf{Z}, \mathcal{G})] \\ & \setminus [\text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) \cup \mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}]. \end{aligned} \quad (2.6)$$

(b) *Suppose $\mathbf{Y} \subseteq \text{PossDe}(\mathbf{X}, \mathcal{G})$. If there is any conditional adjustment set relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{G}$, then the following set is one:*

$$\begin{aligned} \text{O}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) = & \text{Pa}\left(\text{PossMed}(\mathbf{X}, \mathbf{Y}, \mathcal{G}), \mathcal{G}\right) \\ & \setminus [\text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) \cup \mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}]. \end{aligned} \quad (2.7)$$

Example 5 *Consider again the causal MPDAG $\mathcal{G}$ in Figure 2.2(a), where $\mathbf{X} = \{X\}$, $\mathbf{Y} = \{Y\}$, and $\mathbf{Z} = \{V_1\}$. Note that the conditions of Lemma 4 and Theorem 5 are met, so we can construct three valid conditional adjustment sets using Equations (2.5), (2.6), and (2.7).*

$$\text{Pa}(X, \mathcal{G}) \setminus \mathbf{Z} = \{V_1, V_2, V_3\} \setminus \{V_1\}$$

$$\begin{aligned}
&= \{V_2, V_3\}. \\
\text{Adjust}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathcal{G}) &= \{X, Y, V_1, V_2, V_3, V_4\} \setminus \{X, Y, V_1\} \\
&= \{V_2, V_3, V_4\}. \\
\text{O}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) &= \{X, V_1, V_2, V_4\} \setminus \{X, Y, V_1\} \\
&= \{V_2, V_4\}.
\end{aligned}$$

2.3.4 COMPARISON OF CONTEXTS

In this section, we point out a bridge between our conditional adjustment results and prior literature on unconditional adjustment and adjustment under *dynamic treatment*. We begin by presenting Lemma 6, which provides an equivalence between our criterion and the criterion of Perković et al. [2017] used for unconditional adjustment given an MPDAG. Note that this lemma is used to prove Theorem 3 (see Figure A.1 in App. A.4). See App. A.4 for the lemma’s proof.

Lemma 6 *Let $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$, and $\mathbf{S}$ be pairwise disjoint node sets in an MPDAG $\mathcal{G}$, where $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$. Then we have the following.*

(a) ***Comparison of Adjustment Criteria:***

$\mathbf{S}$ satisfies the conditional adjustment criterion relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{G}$ (Definition 2) if and only if $\mathbf{S} \cup \mathbf{Z}$ satisfies the adjustment criterion relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{G}$ (Definition 27, App. A.1).

(b) ***Comparison of Adjustment Sets:***

$\mathbf{S}$ is a conditional adjustment set relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{G}$ (Definition 1) if and only if $\mathbf{S} \cup \mathbf{Z}$ is an adjustment set relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{G}$ (Definition 26, App. A.1).

Next we turn to the work of Smucler et al. [2022], where the authors consider causal effect estimation under a dynamic treatment. For this purpose, Smucler et al. [2022] define a dynamic adjustment set, which they then relate to the set used by Maathuis and Colombo

[2015] for unconditional adjustment (Definition 26, App. A.1). Lemma 6 allows us to connect this dynamic adjustment to our work.

Before making this connection, we briefly describe the context of these authors' work. Unlike a do-intervention that sets $\mathbf{X}$ to fixed values $\mathbf{x}$, a dynamic intervention sets $\mathbf{X}$ to values $\mathbf{x}$ with probability $\pi(\mathbf{x}|\mathbf{Z} = \mathbf{z})$. However, a do-intervention can be seen as a special case of a dynamic intervention where $\pi(\mathbf{x}|\mathbf{Z} = \mathbf{z}) = \mathbf{1}(\mathbf{X} = \mathbf{x})$. Dynamic interventions are often of interest in personalized medicine [Robins, 1993, Murphy et al., 2001, Chakraborty and Moodie, 2013].

Smucler et al. [2022] refer to a causal effect under a dynamic intervention, whose assignment probability depends on $\mathbf{Z}$, as a *$\mathbf{Z}$ -dependent dynamic causal effect* (also called a *single stage dynamic treatment effect* in Chakraborty and Moodie [2013]). They consider these causal effects in the setting where $\mathbf{X}$ and $\mathbf{Y}$ are nodes, the given graph $\mathcal{G}$ is a DAG, and the following assumption holds: $\mathbf{Z} \cap \text{De}(X, \mathcal{G}) = \emptyset$. They then define a *$\mathbf{Z}$ -dependent dynamic adjustment set* as a set $\mathbf{S}$ that satisfies

$$f(y|\pi(x|\mathbf{z})) = \begin{cases} \pi(x|\mathbf{z})f(y|x, \mathbf{z}) & \mathbf{S} = \emptyset, \\ \pi(x|\mathbf{z}) \int f(y|x, \mathbf{z}, \mathbf{s})f(\mathbf{s}|\mathbf{z}) d\mathbf{s} & \mathbf{S} \neq \emptyset. \end{cases}$$

To compare these sets to our conditional adjustment sets, we reference Proposition 1 of Smucler et al. [2022]. This result states that, under their assumptions, $\mathbf{S} \cup \mathbf{Z}$ is a $\mathbf{Z}$ -dependent dynamic adjustment set if and only if $\mathbf{S} \cup \mathbf{Z}$ is an adjustment set relative to (X, Y) in $\mathcal{G}$ (Definition 26, App. A.1). It follows from Lemma 6 that $\mathbf{S} \cup \mathbf{Z}$ is a $\mathbf{Z}$ -dependent dynamic adjustment set if and only if $\mathbf{S}$ is a conditional adjustment set relative to $(X, Y, \mathbf{Z})$ in $\mathcal{G}$ – when $\mathcal{G}$ is a DAG such that $\mathbf{Z} \cap \text{De}(X, \mathcal{G}) = \emptyset$. Thus, our results can be seen as generalizations of Smucler et al. [2022] for $|\mathbf{X}| > 1$ and, therefore, can be used for $\mathbf{Z}$ -dependent dynamic causal effect identification.

2.4 RESULTS - PAGS

We now extend our results on conditional adjustment to the setting of a PAG.

2.4.1 CONDITIONAL ADJUSTMENT CRITERION

We first introduce our conditional adjustment criterion for PAGs (Definition 7). Note that the difference between this criterion and the analogous criterion for MPDAGs is the use of a *visible* as opposed to a *directed* edge. Visibility is a stronger condition introduced by Zhang [2008a] (see App. A.1 for definition).

Following this, Lemma 8 provides an equivalence between our criterion and the criterion of Perković et al. [2018] used for unconditional adjustment given a PAG. Theorem 9 is our main result in this section. It establishes that, under restrictions on $\mathbf{Z}$, the conditional adjustment criterion is an equivalent graphical characterization of a conditional adjustment set in causal PAGs. Proofs of these results are given in App. A.7.

Definition 7 (Conditional Adjustment Criterion for PAGs) *Let $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$, and $\mathbf{S}$ be pairwise disjoint node sets in a PAG $\mathcal{G}$, where $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$ and where every proper possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$ starts with a **visible edge** out of $\mathbf{X}$. Then $\mathbf{S}$ satisfies the conditional adjustment criterion relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{G}$ if*

- (a) $\mathbf{S} \cap \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) = \emptyset$, and
- (b) $\mathbf{S} \cup \mathbf{Z}$ blocks all proper non-causal definite status paths from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$.

Lemma 8 *Let $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$, and $\mathbf{S}$ be pairwise disjoint node sets in a PAG $\mathcal{G}$, where $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$. Then we have the following.*

(a) **Comparison of Adjustment Criteria:**

$\mathbf{S}$ satisfies the conditional adjustment criterion relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{G}$ (Definition 7) if and only if $\mathbf{S} \cup \mathbf{Z}$ satisfies the adjustment criterion relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{G}$ (Definition 27, App. A.1).

(b) **Comparison of Adjustment Sets:**

$\mathbf{S}$ is a conditional adjustment set relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{G}$ (Definition 1) if and only if $\mathbf{S} \cup \mathbf{Z}$ is an adjustment set relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{G}$ (Definition 26, App. A.1).

Proof of Lemma 8. (a) Follows from the fact that $\text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) \subseteq \text{PossDe}(\mathbf{X}, \mathcal{G})$. (b) We start by noting the following fact. Since $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$, then $\mathbf{Z} \cap \text{De}(\mathbf{X}, \mathcal{D}) = \emptyset$ in every DAG represented by $\mathcal{G}$ (Lemma 64, App. A.7). Then consider the following statements.

- (a) $\mathbf{S}$ is a conditional adjustment set relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{G}$.
- (b) $\mathbf{S}$ is a conditional adjustment set relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in each DAG represented by $\mathcal{G}$ – were the DAG to be causal.
- (c) $\mathbf{S} \cup \mathbf{Z}$ is an adjustment set relative to $(\mathbf{X}, \mathbf{Y})$ in each DAG represented by $\mathcal{G}$ – were the DAG to be causal.
- (d) $\mathbf{S} \cup \mathbf{Z}$ is an adjustment set relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{G}$.

By definition, (a) $\Leftrightarrow$ (b). Then by Lemma 6(b) and the fact above, we have (b) $\Leftrightarrow$ (c). The statement (c) $\Leftrightarrow$ (d) follows again by definition. ■

Theorem 9 (Completeness, Soundness of Conditional Adjustment Criterion for PAGs) *Let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$, and $\mathbf{S}$ be pairwise disjoint node sets in a causal PAG $\mathcal{G}$, where $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$. Then $\mathbf{S}$ is a conditional adjustment set relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{G}$ (Definition 1) if and only if $\mathbf{S}$ satisfies the conditional adjustment criterion relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{G}$ (Definition 7).*

2.4.2 CONSTRUCTING ADJUSTMENT SETS

We now provide a method for constructing conditional adjustment sets given a causal PAG (Theorem 10). We illustrate this result in Example 6. The proof of Theorem 10 can be found in App. A.8.

Theorem 10 *Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ be pairwise disjoint node sets in a causal PAG $\mathcal{G}$, where $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$ and where every proper possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$ starts*

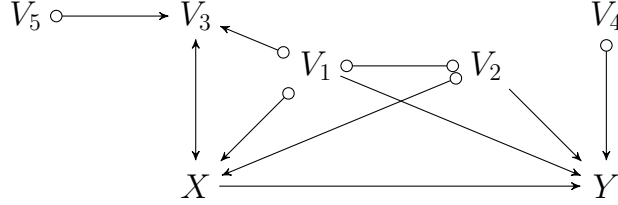


Figure 2.3: A causal PAG used in Example 6.

with a visible edge out of $\mathbf{X}$. If there is any conditional adjustment set relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{G}$, then the following set is one:

$$\begin{aligned} \text{Adjust}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathcal{G}) = & \left[\text{PossAn}(\mathbf{X} \cup \mathbf{Y}, \mathcal{G}) \cup \text{PossAn}(\mathbf{Z}, \mathcal{G}) \right] \\ & \setminus \left[\text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) \cup \mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z} \right]. \end{aligned} \quad (2.8)$$

Example 6 Let $\mathcal{G}$ be the causal PAG in Figure 2.3, and let $\mathbf{X} = \{X\}$, $\mathbf{Y} = \{Y\}$, and $\mathbf{Z} = \{V_1\}$. Note that $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$. Furthermore, the only possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$ is the edge $X \rightarrow Y$, which is visible due to the presence of $V_3 \leftrightarrow X$, where $V_3 \notin \text{Adj}(Y, \mathcal{G})$. If there is any conditional adjustment set relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{G}$, then the conditions of Theorem 10 are met. We consider the set from Equation (2.8).

$$\begin{aligned} \text{Adjust}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathcal{G}) &= \{X, Y, V_1, V_2, V_4\} \setminus \{X, Y, V_1\} \\ &= \{V_2, V_4\}. \end{aligned}$$

To see that this is a conditional adjustment set relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{G}$, we note that it fulfills the requirements of Definition 7. That is, $\text{Adjust}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathcal{G}) \cap \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) = \emptyset$ and $\text{Adjust}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathcal{G}) \cup \mathbf{Z} = \{V_1, V_2, V_4\}$ blocks all proper non-causal definite status paths from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$.

2.5 DISCUSSION

This paper defines a conditional adjustment set that can be used to identify a causal effect in a setting where a causal MPDAG or PAG is known (Definition 1). We give necessary and sufficient graphical conditions for identifying such a set when $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$ (Theorems 3 and 9). Further, we provide multiple methods for constructing these sets (Sections 2.3.3 and 2.4.2). While our results can be used to identify a broad class of conditional causal effects, we discuss some limitations below.

One such limitation is that there are conditional causal effects that can be identified but cannot be identified using conditional adjustment sets. As an example, consider the causal DAG (and therefore, MPDAG) $\mathcal{G}$ in Figure 2.4, and let $\mathbf{X} = \{X_1, X_2\}$, $\mathbf{Y} = \{Y\}$, and $\mathbf{Z} = \{V_2\}$. Note that the conditional causal effect of $\mathbf{X}$ on $\mathbf{Y}$ given $\mathbf{Z}$ is identifiable using do calculus rules (Pearl, 2009, see Equations (A.1)-(A.3) in App. A.2):

$$\begin{aligned} f(\mathbf{y}|do(\mathbf{x}), \mathbf{z}) &= \int_{v_1} f(y, v_1|do(\mathbf{x}), v_2) dv_1 \\ &= \int_{v_1} f(y|do(\mathbf{x}), v_1, v_2) f(v_1|do(\mathbf{x}), v_2) dv_1 \\ &= \int_{v_1} f(y|do(\mathbf{x}), v_1, v_2) f(v_1|do(\mathbf{x})) dv_1 \end{aligned} \tag{2.9}$$

$$= \int_{v_1} f(y|do(x_2), v_1, v_2) f(v_1|do(x_1)) dv_1 \tag{2.10}$$

$$= \int_{v_1} f(y|x_2, v_1, v_2) f(v_1|x_1) dv_1. \tag{2.11}$$

The first two equalities follow from basic probability rules. Equation (2.9) follows from Rule 1 of the do calculus, since $V_1 \perp_d V_2 | X_1, X_2$ in $\mathcal{G}_{\overline{\{X_1, X_2\}}}$. Equation (2.10) follows from Rule 3 of the do calculus, since $Y \perp_d X_1 | V_1, V_2, X_2$ in $\mathcal{G}_{\overline{X_2}}$ and $V_1 \perp_d X_2 | X_1$ in $\mathcal{G}_{\overline{\{X_1, X_2\}}}$. Equation (2.11) follows from Rule 2 of the do calculus, since $Y \perp_d X_2 | V_1, V_2$ in $\mathcal{G}_{\underline{X_2}}$ and $V_1 \perp_d X_1$ in $\mathcal{G}_{\underline{X_1}}$.

However, we can show that there is no conditional adjustment set relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{G}$ that could have been used to identify the effect above. To see this, note that since $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$, we can use Theorem 3 to state the following. A set $\mathbf{S}$ must satisfy

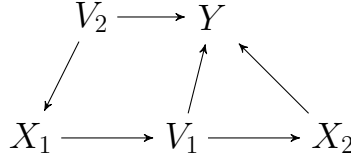


Figure 2.4: A causal DAG used in Section 2.5.

the conditional adjustment criterion relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{G}$ (Definition 2) in order to be a conditional adjustment set. Definition 2 requires that $\mathbf{S}$ block the path $X_2 \leftarrow V_1 \rightarrow Y$, since it is a proper non-causal definite status path from $\mathbf{X}$ to $\mathbf{Y}$. It follows that $\mathbf{S}$ must contain $V_1 \in \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$, which contradicts Definition 2's requirement that $\mathbf{S} \cap \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) = \emptyset$.

Adding to the limitation above, there are conditional causal effects that can be identified using conditional adjustment sets but where these conditional adjustment sets cannot be identified using our criterion. This can occur when $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) \neq \emptyset$, since our graphical criterion requires this restriction but our conditional adjustment set definition does not. As an example, consider again the causal DAG $\mathcal{G}$ given in Figure 2.2(b), and let $\mathbf{X} = \{X_1, X_2\}$, $\mathbf{Y} = \{Y\}$, and $\mathbf{Z} = \{Z, W\}$. Since $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) \neq \emptyset$, no set satisfies the conditional adjustment criterion. However, using do calculus rules [Pearl, 2009], we can show that $\mathbf{S} = \{S\}$ is a conditional adjustment set relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{G}$:

$$\begin{aligned}
 f(\mathbf{y}|\text{do}(\mathbf{x}), \mathbf{z}) &= \int_{\mathbf{s}} f(\mathbf{y}, \mathbf{s}|\text{do}(\mathbf{x}), \mathbf{z}) \, \text{d}\mathbf{s} \\
 &= \int_{\mathbf{s}} f(\mathbf{y}|\text{do}(\mathbf{x}), \mathbf{z}, \mathbf{s}) f(\mathbf{s}|\text{do}(\mathbf{x}), \mathbf{z}) \, \text{d}\mathbf{s} \\
 &= \int_{\mathbf{s}} f(\mathbf{y}|\mathbf{x}, \mathbf{z}, \mathbf{s}) f(\mathbf{s}|\mathbf{z}) \, \text{d}\mathbf{s}.
 \end{aligned}$$

The first and second equality follow from basic probability rules. The third follows by Rules 2 and 3 of the do calculus, since $\mathbf{Y} \perp_d \mathbf{X} | \mathbf{Z} \cup \mathbf{S}$ in $\mathcal{G}_{\underline{\mathbf{X}}}$ and $\mathbf{S} \perp_d \mathbf{X} | \mathbf{Z}$ in $\mathcal{G}_{\overline{\mathbf{X}(\mathbf{Z})}}$. Future work could address identification in this setting by expanding our graphical criterion to allow for arbitrary conditioning.

Chapter 3

DATA-DRIVEN ADJUSTMENT FOR MULTIPLE TREATMENTS

The following chapter is based on unpublished work that is available as a preprint on arXiv (see LaPlante et al., 2025).

3.1 INTRODUCTION

To identify a causal effect from observational data, researchers often turn to covariate adjustment, which can eliminate concerns of confounding bias. But choosing a set of adjustment variables that will accurately identify the causal effect of interest requires carefulness. Much of the literature has sought routes to finding such a set, and these routes typically include two steps: (1) knowing or learning a causal graph and (2) checking sets of variables in the graph against a list of graphical criteria.

For example, Pearl [1995] introduced graphical requirements known as the *back-door criterion* for use when a causal *directed acyclic graph* (DAG) is known. This criterion is sufficient for identifying the effect of a single treatment through adjustment. Subsequently, a graphical *adjustment criterion* for DAGs was developed that is both necessary and sufficient for identifying the effect of multiple treatments Shpitser et al. [2012], Perković et al. [2015], Perković et al. [2018]. Further results have considered settings where a full DAG is not known, including necessary and sufficient graphical criteria for *maximal ancestral graphs* (MAGs), which allow for latent confounding [Van der Zander et al., 2014], and extensions to equivalence classes of DAGs and MAGs known as *completed partially directed acyclic graphs* (CPDAGs) and *partial ancestral graphs* (PAGs) [Maathuis and Colombo, 2015, Perković et al., 2017, 2018], respectively. Notably, one of these extensions – the *generalized adjustment*

criterion Perković et al. [2018] – is necessary and sufficient for identification in all such graphs.

In their 2013 paper, Entner et al. also consider identifying causal effects through covariate adjustment. But unlike research that relies on graphical criteria, Entner et al. focus on identification through the observed data directly. Their paper’s main result is a pair of rules that they show are necessary and sufficient for discovering when a causal effect is identifiable. The first of these rules – reproduced as Theorem 11 below – provides an adjustment set for identifying a causal effect when one exists. The strength of this data-driven rule lies in its simplicity: the researcher only needs to find an observed variable that matches two conditional in/dependence criteria.

We consider extending the results of Entner et al. [2013] – first by reviewing the notion of c-equivalence [Pearl and Paz, 2014]. Notably, any set that is c-equivalent to an adjustment set must also be an adjustment set. So while the rules of Entner et al. [2013] guarantee finding one adjustment set when the causal effect is identifiable, subsequently finding c-equivalent sets will uncover additional sets for adjustment. Pearl and Paz [2014] provide criteria sufficient for finding c-equivalent sets. And since these criteria are based on in/dependencies in the data directly, they can be used to extend the Entner et al. [2013] criterion, without requiring additional graphical assumptions. We note briefly (for further discussion below) that having more than one adjustment set may seem unnecessary for practical research. However, this choice can be crucial in the process of causal effect estimation.

Our main contribution is an extension of Entner et al. [2013] to a setting with multiple treatments. That is, where Entner et al. [2013] consider only a single treatment X , we consider a set of treatments $\mathbf{X}$. We develop two data-driven rules, analogous to the first rule of Entner et al. [2013] (Theorem 11), that are sufficient for finding adjustment sets in this setting. Our first rule (Theorem 15) finds an adjustment set for $\mathbf{X}$ by finding a causal ordering of the treatments and building up from an adjustment set for the first treatment in the causal ordering. Our second rule (Theorem 17) finds an adjustment set for $\mathbf{X}$ by combining adjustment sets for each $X \in \mathbf{X}$ after paring off unnecessary variables. This process relies on the notion of c-equivalence.

This paper is organized as follows. Section 3.2 provides a set of definitions for graphical models. Section 3.3 explains how c-equivalence can be used to extend the results of Entner et al. [2013]. Section 3.4 details our extension of Entner et al. [2013] to multiple treatments. Then we measure the performance of our rules through a data simulation in Section 3.5, and we discuss the limitations of our work and suggestions for future research in Section 3.6.

3.2 PRELIMINARIES

Throughout this paper, we assume a causal model that induces a directed graph. The following are key definitions related to these graphs and their associated densities. We rely on the framework of Pearl [2009].

Nodes, Edges, and Graphs. We use capital letters (e.g., X) to denote nodes in a graph as well as the random variables these nodes represent. We use bold capital letters (e.g., $\mathbf{X}$) to denote node sets. A *graph* $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ consists of a set of nodes $\mathbf{V}$ and a set of edges $\mathbf{E}$. A *directed graph* contains only directed edges ($\rightarrow$).

Paths and DAGs. For disjoint node sets $\mathbf{X}$ and $\mathbf{Y}$, a *path* from $\mathbf{X}$ to $\mathbf{Y}$ is a sequence of distinct nodes $\langle X, \dots, Y \rangle$ from some $X \in \mathbf{X}$ to some $Y \in \mathbf{Y}$ for which every pair of successive nodes is adjacent. A *directed path* from X to Y is a path of the form $X \rightarrow \dots \rightarrow Y$. A directed path from X to Y and the edge $Y \rightarrow X$ form a *directed cycle*. A directed graph without directed cycles is a *directed acyclic graph* (DAG).

Colliders and Non-colliders. The *endpoints* of a path $p = \langle X_1, \dots, X_k \rangle$ are the nodes X_1 and X_k . For $1 < i < k$, the node X_i is a *collider* on p if p contains $X_{i-1} \rightarrow X_i \leftarrow X_{i+1}$, and X_i is a *non-collider* on p if p contains $X_{i-1} \leftarrow X_i$ or $X_i \rightarrow X_{i+1}$.

Ancestral Relationships. If $X \rightarrow Y$, then X is a *parent* of Y . If there is a directed path from X to Y , then X is an *ancestor* of Y and Y is a *descendant* of X . We use the convention that every node is an ancestor and descendant of itself. The sets of parents, ancestors, and descendants of X in $\mathcal{D}$ are denoted by $\text{Pa}(X, \mathcal{D})$, $\text{An}(X, \mathcal{D})$, and $\text{De}(X, \mathcal{D})$, respectively. We let $\text{An}(\mathbf{X}, \mathcal{D}) = \cup_{X \in \mathbf{X}} \text{An}(X, \mathcal{D})$ and $\text{De}(\mathbf{X}, \mathcal{D}) = \cup_{X \in \mathbf{X}} \text{De}(X, \mathcal{D})$. Unconventionally, we define $\text{Pa}(\mathbf{X}, \mathcal{D}) = [\cup_{X \in \mathbf{X}} \text{Pa}(X, \mathcal{D})] \setminus \mathbf{X}$.

Markov Compatibility and Faithfulness. An *observational density* $f(\mathbf{v})$ is *Markov compatible* with a DAG $\mathcal{D} = (\mathbf{V}, \mathbf{E})$ if $f(\mathbf{v}) = \prod_{V_i \in \mathbf{V}} f(v_i | \text{pa}(v_i, \mathcal{D}))$. It is *faithful* to $\mathcal{D}$ if $(\mathbf{X} \perp\!\!\!\perp \mathbf{Y} | \mathbf{Z})_f$ implies $(\mathbf{X} \perp_d \mathbf{Y} | \mathbf{Z})_{\mathcal{D}}$ (see definition of *d-separation* below). We require $f(\mathbf{v}) > 0$ for all valid values of $\mathbf{V}$.

D-connection, D-separation, and Probabilistic Implications. Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ be pairwise disjoint node sets in a DAG $\mathcal{D}$. A path p from $\mathbf{X}$ to $\mathbf{Y}$ is *d-connecting* (or *open*) given $\mathbf{Z}$ if every non-collider on p is not in $\mathbf{Z}$ and every collider on p has a descendant in $\mathbf{Z}$. Otherwise, p is *blocked* given $\mathbf{Z}$. If all paths between $\mathbf{X}$ and $\mathbf{Y}$ in $\mathcal{D}$ are blocked given $\mathbf{Z}$, then $\mathbf{X}$ is *d-separated* from $\mathbf{Y}$ given $\mathbf{Z}$ in $\mathcal{D}$ and we write $(\mathbf{X} \perp_d \mathbf{Y} | \mathbf{Z})_{\mathcal{D}}$. This d-separation implies that $\mathbf{X}$ and $\mathbf{Y}$ are independent given $\mathbf{Z}$ in any *observational density* that is Markov compatible with $\mathcal{D}$ [Lauritzen et al., 1990].

Causal Graphs. Let $\mathcal{D}$ be a DAG with nodes V_i and V_j . Then $\mathcal{D}$ is a *causal DAG* if every edge $V_i \rightarrow V_j$ represents a direct causal effect of V_i on V_j . In a causal DAG, any directed path is *causal*, and any other path is *non-causal*.

Consistency. Let $f(\mathbf{v})$ be an observational density over $\mathbf{V}$. The notation $do(\mathbf{X} = \mathbf{x})$, or $do(\mathbf{x})$ for short, represents an outside intervention that sets $\mathbf{X} \subseteq \mathbf{V}$ to fixed values $\mathbf{x}$. An *interventional density* $f(\mathbf{v} | do(\mathbf{x}))$ is a density resulting from such an intervention.

Let $\mathbf{F}^*$ denote the set of all interventional densities $f(\mathbf{v} | do(\mathbf{x}))$ such that $\mathbf{X} \subseteq \mathbf{V}$ (including $\mathbf{X} = \emptyset$). A causal DAG $\mathcal{D} = (\mathbf{V}, \mathbf{E})$ is a *causal Bayesian network compatible with $\mathbf{F}^*$* if and only if for all $f(\mathbf{v} | do(\mathbf{x})) \in \mathbf{F}^*$, the following *truncated factorization* holds:

$$f(\mathbf{v} | do(\mathbf{x})) = \prod_{V_i \in \mathbf{V} \setminus \mathbf{X}} f(v_i | \text{pa}(v_i, \mathcal{D})) \mathbf{1}(\mathbf{X} = \mathbf{x}).$$

We say an interventional density is *consistent* with a causal DAG $\mathcal{D}$ if it belongs to a set of interventional densities $\mathbf{F}^*$ such that $\mathcal{D}$ is compatible with $\mathbf{F}^*$. Note that any observational density that is Markov compatible with $\mathcal{D}$ is consistent with $\mathcal{D}$.

Causal Models. A *structural equation model* (SEM), or *causal model*, is a set of equations – one for each random variable $V \in \mathbf{V}$ that maps the causal determinants of V , along with random noise, to the values of V . This model induces a DAG $\mathcal{D}$ over $\mathbf{V}$ and a set of

interventional densities $\mathbf{F}^* = \{f(\mathbf{v}|do(\mathbf{x})) : \mathbf{X} \subseteq \mathbf{V}\}$ that are consistent with $\mathcal{D}$. The joint density $f(\mathbf{v}) \in \mathbf{F}^*$ is faithful to $\mathcal{D}$.

Identifiability. Let $\mathbf{X}$ and $\mathbf{Y}$ be disjoint node sets in a causal DAG $\mathcal{D} = (\mathbf{V}, \mathbf{E})$ that is compatible with $\mathbf{F}_i^* = \{f_i(\mathbf{v}|do(\mathbf{x}')) : \mathbf{X}' \subseteq \mathbf{V}\}$. We say the causal effect of $\mathbf{X}$ on $\mathbf{Y}$ is *identifiable* in $\mathcal{D}$ if for any $\mathbf{F}_1^*, \mathbf{F}_2^*$ where $f_1(\mathbf{v}) = f_2(\mathbf{v})$, we have $f_1(\mathbf{y}|do(\mathbf{x})) = f_2(\mathbf{y}|do(\mathbf{x}))$.

Adjustment Sets. Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ be pairwise disjoint node sets in a causal DAG $\mathcal{D}$. Then $\mathbf{Z}$ is an adjustment set relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{D}$ if and only if $f(\mathbf{y}|do(\mathbf{x})) = \int f(\mathbf{y}|\mathbf{x}, \mathbf{z})f(\mathbf{z})d\mathbf{z}$ for any f consistent with $\mathcal{D}$. We omit reference to $(\mathbf{X}, \mathbf{Y})$ or $\mathcal{D}$ when it can be assumed.

Causal Ordering. Let $\mathbf{V} = \{V_1, \dots, V_k\}$, $k \geq 1$, be a set of random variables in a causal model. We say $V_1 < \dots < V_k$ is a *causal ordering consistent with the model* if V_j is not a causal ancestor of V_i for $1 \leq i < j \leq k$. Note there can be more than one causal ordering. For example, a causal model that induces the DAG $X_i \leftarrow X_j \rightarrow X_k$, has consistent orderings $X_j < X_i < X_k$ and $X_j < X_k < X_i$ and $X_j < \{X_i, X_k\}$.

PAGs. We reference *partial ancestral graphs* (PAGs; Richardson and Spirtes, 2002) in several examples of Sections 3.3-3.4 and the simulations of Section 3.5. However, our results require no knowledge of PAGs directly. Thus, we suppress related definitions to App. B.1 and provide an informal overview below.

We can represent a causal model that has unmeasured variables by using a *maximal ancestral graph* (MAG) over the observed variables alone. Directed edges ($\rightarrow$) in a MAG denote causal ancestry, and bi-directed edges ($\leftrightarrow$) denote the presence of an unmeasured confounder. MAGs encode all the conditional in/dependencies among observed variables through a graphical criterion called *m-separation*.

PAGs represent an equivalence class of MAGs with the same m-separations. Directed and bi-directed edges in a PAG denote shared ancestry and confounding, respectively, among all represented MAGs. Circle edge marks denote disagreement among represented MAGs. For example, $X \circ \rightarrow Y$ denotes that at least one represented MAG has the edge $X \rightarrow Y$ and at least one represented MAG has $X \leftrightarrow Y$.

3.3 INSIGHTS FROM EXISTING WORK

In this section, we review the first rule of Entner et al. [2013] and suggest a known equivalency of adjustment sets as a direct extension. We close by providing a rationale for why this extension would be useful for causal effect estimation.

3.3.1 THE ORIGINAL RULE

As noted in Section 3.1, Entner et al. [2013] develop the following rule for finding an adjustment set when the causal effect of a single treatment is identifiable.

Theorem 11 (R1 Entner) *Let $\{X\}$, $\{Y\}$, and $\mathbf{W}$ be pairwise disjoint sets of observed random variables in a causal model. Suppose $\mathbf{W} < X < Y$ is a causal ordering consistent with the model. If there exists $W \in \mathbf{W}$ and $\mathbf{Z} \subseteq \mathbf{W} \setminus \{W\}$ such that*

$$(i) \ W \not\perp\!\!\!\perp Y \mid \mathbf{Z} \quad \text{and}$$

$$(ii) \ W \perp\!\!\!\perp Y \mid \mathbf{Z} \cup \{X\},$$

then X has a causal effect on Y that is identifiable through the adjustment set $\mathbf{Z}$.

Entner et al. [2013] show that the rule above is necessary for identification (see their Theorem 3). That is, if the causal effect of X on Y is identifiable and nonzero, then Theorem 11 will find an adjustment set. However, we note that Theorem 11 only guarantees finding one such set. In some cases, such as in Example 7 below, there may be additional adjustment sets that cannot be found using Theorem 11.

Example 7 (Limitations of R1 Entner) *Consider a causal model that induces the DAG in Figure 3.1(a). Suppose the DAG is unknown, but we have data on $\{X, Y, W, Z\}$ and expert knowledge that $\{W, Z\} < X < Y$.*

We want to know the effect of X on Y . We can learn from the data that $W \not\perp\!\!\!\perp Y \mid Z$ and $W \perp\!\!\!\perp Y \mid Z, X$, which implies $\{Z\}$ is an adjustment set relative to (X, Y) by Theorem 11.

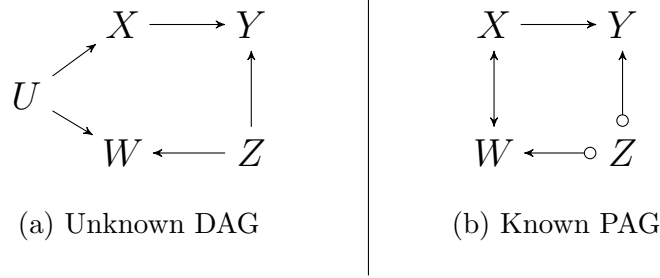


Figure 3.1: Graphs used in Examples 7-9

However, there are two adjustment sets Theorem 11 cannot find that we can find by building a graph from the data. To see this, let the PAG in Figure 3.1(b) represent all the in/dependencies we can learn from the data with the addition of our expert knowledge.¹ Using graphical criteria from prior research (see Theorem 69 in App. B.1), we can show $\emptyset$ and $\{W, Z\}$ are adjustment sets relative to (X, Y) .

3.3.2 AN EXTENSION

To find adjustment sets like those seen in Example 7, we note that one can extend Theorem 11 using the notion of *confounding equivalence* or *c-equivalence* [Pearl and Paz, 2014] shown in the definition below.

Definition 12 (c-equivalence) Let $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$, and $\mathbf{T}$ be pairwise disjoint sets of random variables with joint density f . Then $\mathbf{Z}$ and $\mathbf{T}$ are *c-equivalent* relative to $(\mathbf{X}, \mathbf{Y})$ if

$$\int f(\mathbf{y}|\mathbf{x}, \mathbf{z})f(\mathbf{z}) \, d\mathbf{z} = \int f(\mathbf{y}|\mathbf{x}, \mathbf{t})f(\mathbf{t}) \, d\mathbf{t}.$$

Pearl and Paz use this definition to find sets that, when used for adjustment, produce the same asymptotic bias for estimating a causal effect. For our purposes, we note that if $\mathbf{T}$ is c-equivalent to an adjustment set relative to $(\mathbf{X}, \mathbf{Y})$, then $\mathbf{T}$ is also an adjustment set relative to $(\mathbf{X}, \mathbf{Y})$.

¹Venkateswaran and Perković [2024] refer to 3.1(b) as a *restricted essential ancestral graph*.

In Theorem 13 below, we provide sufficient criteria from prior research for two sets of variables to be c-equivalent. These criteria do not require knowledge of a causal graph, and this will allow us to extend Theorem 11.

We note that while Theorem 13, as stated, mirrors Corollary 1 of Pearl and Paz [2014], its conditions can be found throughout prior research (e.g., Greenland et al., 1999, Kuroki and Miyakawa, 2003, Kuroki and Cai, 2004, De Luna et al., 2011), and across the literature, researchers have used these conditions for similar purposes. We will revisit these conditions in Section 3.3.3 in discussing the statistical efficiency of adjustment-based estimators.

Theorem 13 (Probabilistic Criteria for c-equivalence) *Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z} \cup \mathbf{T}$ be pairwise disjoint sets of random variables. Then $\mathbf{Z}$ and $\mathbf{T}$ are c-equivalent relative to $(\mathbf{X}, \mathbf{Y})$ if either of the following hold:*

$$(i) \mathbf{X} \perp\!\!\!\perp \mathbf{Z} \mid \mathbf{T} \text{ and } \mathbf{Y} \perp\!\!\!\perp \mathbf{T} \mid \mathbf{Z} \cup \mathbf{X}$$

$$(ii) \mathbf{X} \perp\!\!\!\perp \mathbf{T} \mid \mathbf{Z} \text{ and } \mathbf{Y} \perp\!\!\!\perp \mathbf{Z} \mid \mathbf{T} \cup \mathbf{X}.$$

We use Theorem 13 to extend the work of Entner et al. [2013] in the following way. When the causal effect of X on Y is identifiable, Theorem 11 will find at least one adjustment set $\mathbf{Z}$. Then, we can search for a set that is c-equivalent to $\mathbf{Z}$ relative to (X, Y) using Theorem 13. By definition, any such set will also be an adjustment set relative to (X, Y) . We show in Example 8 that this procedure can identify adjustment sets that Theorem 11 cannot.

Example 8 (Adjustment via c-equivalence) *Reconsider Example 7, where Theorem 11 could not find the adjustment sets $\emptyset$ and $\{W, Z\}$. Then turn to consider Theorem 13.*

From the data, we can learn that $X \perp\!\!\!\perp Z$. And trivially $Y \perp\!\!\!\perp \emptyset \mid Z, X$. Thus by (i), $\emptyset$ is c-equivalent to $\{Z\}$ and an adjustment set relative to (X, Y) . We already learned from the data that $Y \perp\!\!\!\perp W \mid Z, X$. And trivially $X \perp\!\!\!\perp Z \mid W, Z$. Thus by (i), $\{W, Z\}$ is c-equivalent to $\{Z\}$ and an adjustment set relative to (X, Y) .

3.3.3 RATIONALE

At first glance, it may seem unimportant to have a choice of sets to use for covariate adjustment. Entner et al. [2013] already have a data-driven method of finding one adjustment set when the causal effect of a single treatment is identifiable, and every adjustment set can be used to construct an unbiased estimator of the causal effect – given appropriate parametric assumptions, or in a discrete setting, given sufficient data. However, estimators constructed using different adjustment sets may have different statistical properties, such as their asymptotic variance.

Recent research considers adjustment sets that lead to asymptotically efficient estimators of a causal effect – called *efficient adjustment sets*. Broadly, this research takes two paths: (1) exploring the asymptotic variance of an estimator of the causal effect under assumptions of linearity [Henckel et al., 2022, Witte et al., 2020, Guo and Perković, 2022, Colnet et al., 2024], or (2) exploring the variance of the influence function of an asymptotically linear estimator of the causal effect in a semi-parametric setting with discrete treatment [Smucler et al., 2022].

In an effort to obtain more efficient estimators, both research paths use the conditions of Theorem 13 as guidance for adding or removing variables from an adjustment set. We provide a result for evaluating such variables below.

Lemma 14 (Precision and Overadjustment Variables, cf. Lemmas 4-5 of Smucler et al. [2022], Theorem 1 of Henckel et al., 2022) *Let $\mathbf{X}$, $\{Y\}$, $\mathbf{Z}$, and $\mathbf{T}$ be pairwise disjoint sets of random variables in a causal model, where both $\mathbf{Z}$ and $\mathbf{Z} \cup \mathbf{T}$ are adjustment sets relative to $(\mathbf{X}, Y)$. Then, $\mathbf{T}$ are precision variables and $\mathbf{Z} \cup \mathbf{T}$ is a more efficient adjustment set compared to $\mathbf{Z}$ if*

$$(i) \mathbf{X} \perp\!\!\!\perp \mathbf{T} \mid \mathbf{Z}.$$

$\mathbf{T}$ are overadjustment variables and $\mathbf{Z} \cup \mathbf{T}$ is a less efficient adjustment set compared to $\mathbf{Z}$ if

$$(ii) Y \perp\!\!\!\perp \mathbf{T} \mid \mathbf{Z} \cup \mathbf{X}.$$

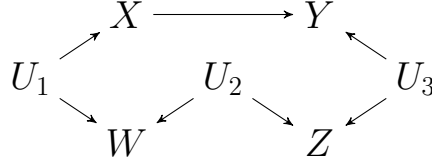


Figure 3.2: Unknown DAG used in Example 10

Note that in the result above, $\mathbf{Z} \cup \mathbf{T}$ being more (or less) efficient refers to the asymptotic properties of a causal effect estimator that relies on adjustment through $\mathbf{Z} \cup \mathbf{T}$. In Example 9 below, we use Lemma 14 to show that Theorem 11 may find an adjustment set that leads to asymptotically efficient estimation of the causal effect. But Example 10 shows this is not always the case.

Example 9 (Efficient Adjustment via R1 Entner) *Reconsider Examples 7-8, where Theorem 11 found $\{Z\}$ and Theorem 13 found $\emptyset$ and $\{W, Z\}$ as adjustment sets.*

We want to know which set leads to an asymptotically efficient estimator of the causal effect. By Lemma 14, we see that $\{Z\}$ is more efficient than $\emptyset$, since $X \perp\!\!\!\perp Z$. Similarly, $\{Z, W\}$ is less efficient than $\{Z\}$, since $Y \perp\!\!\!\perp W \mid Z, X$. Thus, $\{Z\}$ is the most efficient adjustment set – and one that Theorem 11 was able to find.

Example 10 (Inefficient Adjustment via R1 Entner) *Consider a causal model that induces the DAG in Figure 3.2. Suppose the DAG is unknown, but we have data on $\{X, Y, W, Z\}$ and expert knowledge that $\{W, Z\} < X < Y$.*

We want to know the effect of X on Y . By Theorem 11, $\emptyset$ is an adjustment set relative to (X, Y) , since we can learn from the data that $W \not\perp\!\!\!\perp Y$ and $W \perp\!\!\!\perp Y \mid X$. Theorem 11 finds no further sets, but Theorem 13 finds $\{Z\}$ and $\{W\}$ since we can learn from the data that $X \perp\!\!\!\perp Z$ and $Y \perp\!\!\!\perp W \mid X$.

By Lemma 14, we see that $\{Z\}$ adds precision and $\{W\}$ overadjusts compared to $\emptyset$, since $X \perp\!\!\!\perp Z$ and $Y \perp\!\!\!\perp W \mid X$. Thus $\{Z\}$ is again the most efficient adjustment set – but one that Theorem 11 was unable to find.

3.4 EXTENSION TO MULTIPLE TREATMENTS

In this section, we provide two paths (Theorems 15 and 17) to finding adjustment sets that rely on dependencies and independencies in the observed data directly. Both paths consider a setting with multiple treatments and thus, extend the work of Entner et al. [2013]. As in Theorem 11, our methods require that treatments cannot be causal ancestors of observed variables in the model, a condition satisfied when covariates are pre-treatment.

The adjustment set we offer in Theorem 15 is constructed from the ground up. That is, a researcher must find an adjustment set for one element of a set of treatments and then build, element by element, to an adjustment set for all treatments. The adjustment set we offer in Theorem 17 is constructed by carefully combining adjustment sets for each element in a set of treatments. Notably, our latter method relies on the notion of c-equivalence that we saw in Section 3.3.2.

3.4.1 BUILDING ON ADJUSTMENT SETS

Below we present our first path for extending Theorem 11 to multiple treatments. Example 11 illustrates its use.

Theorem 15 (R1 Build) *Let $\mathbf{X} = \{X_1, \dots, X_k\}$, $k \geq 1$; $\{Y\}$; and $\mathbf{W}$ be pairwise disjoint sets of observed random variables in a causal model, and for $i \in \{1, \dots, k\}$, define $\{X_1, \dots, X_i\}^{-i} = \{X_1, \dots, X_i\} \setminus \{X_i\}$.*

Suppose $\mathbf{W} < X_1 < \dots < X_k < Y$ is a causal ordering consistent with the model. If there exist $W_1, \dots, W_k \in \mathbf{W}$ and $\mathbf{Z} \subseteq \mathbf{W} \setminus \{W_1, \dots, W_k\}$ such that for all i ,

$$(i) \ W_i \not\perp\!\!\!\perp Y \mid \mathbf{Z} \cup \{X_1, \dots, X_i\}^{-i} \text{ and}$$

$$(ii) \ W_i \perp\!\!\!\perp Y \mid \mathbf{Z} \cup \{X_1, \dots, X_i\},$$

then $\mathbf{X}$ has a causal effect on Y that is identifiable through the adjustment set $\mathbf{Z}$.

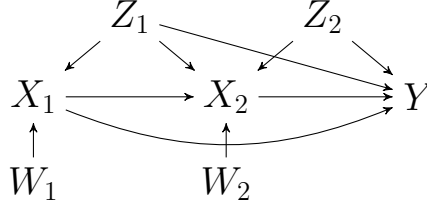


Figure 3.3: Unknown DAG used in Example 11

Proof Sketch. The proof for Theorem 15 can be found in App. B.2, but we provide an outline here for intuition. To see that $\mathbf{X}$ causes Y , note that (i)-(ii) require a path p_i from W_i to Y that is open given $\mathbf{Z} \cup \{X_1, \dots, X_i\}^{-i}$ and contains X_i as a non-collider. This combined with the causal ordering requires p_i to end $X_i \rightarrow \dots \rightarrow Y$. To show $\mathbf{Z}$ is an adjustment set, we only need $\mathbf{Z} \cup \mathbf{X}^{-i}$ to block all back-door paths from X_i to Y . We prove this holds for $i = k$ and proceed by induction. For contradiction in the base case, we assume a back-door path q_k from X_k to Y that is open given $\mathbf{Z} \cup \mathbf{X}^{-k}$. Then we define $r_k = p_k(W_k, A) \oplus q_k(A, Y)$ for the earliest shared node A . Showing r_k is open given $\mathbf{Z} \cup \mathbf{X}$ contradicts condition (ii). This holds for $p_k(W_k, A)$ and $q_k(A, Y)$ by definition. We complete the base case by showing it holds for r_k : when $A = X_k$, $A \in \mathbf{Z} \cup \mathbf{X}^{-k}$, and $A \notin \mathbf{Z} \cup \mathbf{X}$. The induction step follows a similar argument, where we solve two additional issues with the induction assumption. ■

Example 11 (Adjustment via R1 Build) Consider a causal model that induces the DAG in Figure 3.3. Suppose the DAG is unknown, but we have data on every variable and expert knowledge that $\{W_1, W_2, Z_1, Z_2\} < X_1 < X_2 < Y$.

To find the effect of $\mathbf{X}$ on Y , note that by Theorem 15, $\mathbf{Z} := \{Z_1, Z_2\}$ is an adjustment set relative to $(\mathbf{X}, Y)$, since we can learn from data that $W_1 \not\perp\!\!\!\perp Y \mid \mathbf{Z}$ and $W_1 \perp\!\!\!\perp Y \mid \mathbf{Z} \cup \{X_1\}$ as well as $W_2 \not\perp\!\!\!\perp Y \mid \mathbf{Z} \cup \{X_1\}$ and $W_2 \perp\!\!\!\perp Y \mid \mathbf{Z} \cup \{X_1, X_2\}$.

Theorem 15 is especially useful in settings where Theorem 11 has already found an

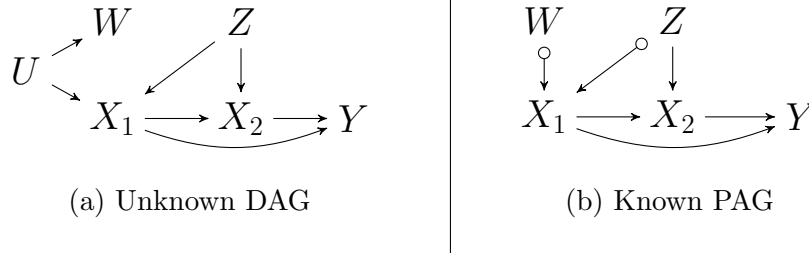


Figure 3.4: Graphs used in Examples 12-13

adjustment set for a causal effect on a single treatment, and a researcher would like to consider the addition of further treatments. But this method, while intuitive, has its limitations. We showcase this in the example below as motivation for our final extension of Theorem 11.

Example 12 (Limitations of R1 Build) *Consider a causal model that induces the DAG in Figure 3.4(a). Suppose the DAG is unknown, but we have data on $\{X_1, X_2, W, Y, Z\}$ and expert knowledge that $\{W, Z\} < X_1 < X_2 < Y$. We attempt to find the effect of $\mathbf{X}$ on Y using Theorem 15. To fulfill (i)-(ii), we must set $W_1 = W$ and $\mathbf{Z} = \{Z\}$. But then there is no W_2 that fulfills (i)-(ii). Thus, we cannot use Theorem 15 to find an adjustment set.*

However, we can find two adjustment sets relative to $(\mathbf{X}, Y)$ by building a graph from the data. To see this, let the PAG in Figure 3.4(b) represent all the in/dependencies we can learn from the data with the addition of our expert knowledge. Using graphical criteria from prior research (see Theorem 69 in App. B.1), we can show that $\emptyset$, $\{Z\}$, and $\{W, Z\}$ are adjustment sets relative to $(\mathbf{X}, Y)$.

3.4.2 COMBINING ADJUSTMENT SETS

Below we present our second path for extending Theorem 11 to multiple treatments. Informally, this method constructs an adjustment set for the full set of treatments by combining adjustment sets for the individual treatments – after first removing extraneous variables. We formalize this notion in the definition below before providing our result.

Definition 16 (Minimal Adjustment Set) A set $\mathbf{Z}$ is a minimal adjustment set relative to $(\mathbf{X}, Y)$ if $\mathbf{Z}$ is an adjustment set relative to $(\mathbf{X}, Y)$ and no proper subset of $\mathbf{Z}$ is an adjustment set relative to $(\mathbf{X}, Y)$.

Theorem 17 (R1 Combine) Let $\mathbf{X} = \{X_1, \dots, X_k\}$, $k \geq 1$; $\{Y\}$; and $\mathbf{W}$ be pairwise disjoint sets of observed random variables in a causal model, and for $i \in \{1, \dots, k\}$, define $\mathbf{X}_i^N$ to be the variables in $\mathbf{X}$ that are not causal descendants of X_i .

Suppose $\mathbf{W} < \mathbf{X} < Y$ is a causal ordering consistent with the model. If there exist $W_1, \dots, W_k \in \mathbf{W}$ and $\mathbf{T}_i \subseteq [\mathbf{W} \setminus \{W_i\}] \cup \mathbf{X}_i^N$ such that for all i ,

$$(i) \ W_i \not\perp\!\!\!\perp Y \mid \mathbf{T}_i \quad \text{and}$$

$$(ii) \ W_i \perp\!\!\!\perp Y \mid \mathbf{T}_i \cup \{X_i\},$$

then $\mathbf{X}$ has a causal effect on Y that is identifiable through the adjustment set $\mathbf{Z} := \cup_{i=1}^k \mathbf{Z}_i \setminus \mathbf{X}$, where $\mathbf{Z}_i$ is any minimal adjustment set relative to (X_i, Y) such that $\mathbf{Z}_i \subseteq \mathbf{T}_i$.

Proof Sketch. The proof for Theorem 17 can be found in App. B.3, but we provide an outline here for intuition. Note that (i), (ii), and Theorem 11 imply X_i causes Y and $\mathbf{T}_i$ is an adjustment set relative to (X_i, Y) . Thus if $\mathbf{T}_i$ exists, then the reduced adjustment set $\mathbf{Z}_i \subseteq \mathbf{T}_i$ is guaranteed. To show that $\mathbf{Z}$ is an adjustment set relative to $(\mathbf{X}, Y)$, we only need $\mathbf{Z} \cup \mathbf{X}^{-i}$ to block all back-door paths from X_i to Y , where we define $\mathbf{X}^{-i} = \mathbf{X} \setminus \{X_i\}$. Without loss of generality, we show this holds for $i = 1$. For contradiction, we assume a back-door path q from X_1 to Y that is open given $\mathbf{Z} \cup \mathbf{X}^{-1}$. Since $\mathbf{Z}_1$ is an adjustment set relative to (X_1, Y) , q must have a collider that is a causal ancestor of $\mathbf{Z} \cup \mathbf{X}^{-1}$ but not $\mathbf{Z}_1$. Using the minimality of each $\mathbf{Z}_i$, we show that all such colliders $C_1, \dots, C_\ell$ must have directed paths $t_1, \dots, t_\ell$ to Y , which we use to define a final back-door path u from X_1 to Y that is open given $\mathbf{Z}_1$. For example, when there is no directed path from $\{C_1, \dots, C_\ell\}$ to X_1 , then $u := q(X_1, C_1) \oplus t_1$. This path contradicts that $\mathbf{Z}_1$ is an adjustment set relative to (X_1, Y) . ■

Implementing Theorem 17 requires checking if each $\mathbf{T}_i$ is a minimal adjustment set, and if not, then finding such a set $\mathbf{Z}_i$. On its face, this involves knowledge of either the underlying graph or the underlying joint density of observed variables. However, a key appeal of the rules of Entner et al. [2013] – that we aim to replicate – is the lack of reliance on graphical criteria. To resolve this discrepancy, we present the lemma below, which provides a route for finding a minimal adjustment set through the testing of in/dependencies among observed variables in the data directly.

Lemma 18 (Probabilistic Criteria for Minimality) *Let $\mathbf{T}$ be an adjustment set relative to (X, Y) in a causal model where $\mathbf{T} < X$. Then $\mathbf{T}$ is a minimal adjustment set relative to (X, Y) if and only if for all $T \in \mathbf{T}$,*

$$(i) \ X \not\perp\!\!\!\perp T \mid \mathbf{T} \setminus \{T\}, \text{ and}$$

$$(ii) \ Y \not\perp\!\!\!\perp T \mid [\mathbf{T} \setminus \{T\}] \cup \{X\}.$$

Otherwise, $\mathbf{Z} \subset \mathbf{T}$ is a minimal adjustment set relative to (X, Y) if for all $Z \in \mathbf{Z}$,

$$(iii) \ X \not\perp\!\!\!\perp Z \mid \mathbf{Z} \setminus \{Z\},$$

$$(iv) \ Y \not\perp\!\!\!\perp Z \mid [\mathbf{Z} \setminus \{Z\}] \cup \{X\}, \text{ and}$$

$$(v) \ Y \perp\!\!\!\perp \mathbf{T} \mid \mathbf{Z} \cup \{X\} \quad \text{or} \quad X \perp\!\!\!\perp \mathbf{T} \mid \mathbf{Z}.$$

Proof Sketch. The proof for Lemma 18 can be found in App. B.3, but for intuition, we note that (i)-(ii) require a path from X to T that is open given $\mathbf{T} \setminus \{T\}$, and a path from T to Y that is open given $\{X\} \cup \mathbf{T} \setminus \{T\}$. Informally, combining these paths offers a non-causal path from X to Y that is open given $\mathbf{T} \setminus \{T\}$. Thus, $\mathbf{T}$ is *elementwise minimal*, meaning $\mathbf{T} \setminus \{T\}$ is not an adjustment set for any $T \in \mathbf{T}$, which we show implies minimality. When (i)-(ii) do not hold, (v) and Theorem 13 show $\mathbf{Z} \subset \mathbf{T}$ is an adjustment set, and the proof for the minimality of $\mathbf{Z}$ follows similarly from (iii)-(iv). ■

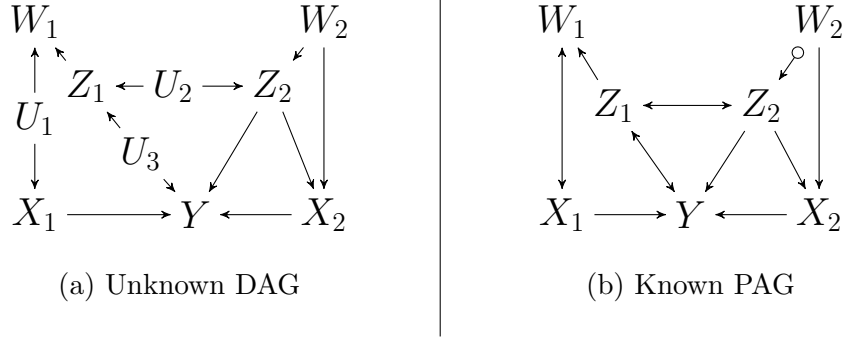


Figure 3.5: Graphs used in Example 14

We explore Theorem 17 and Lemma 18 in the examples below. Example 13 provides a straightforward demonstration of these results, and Example 14 shows why we require minimality.

Example 13 (Adjustment via R1 Combine) *Reconsider the setting of Example 12. While Theorem 15 could not find an adjustment set, we show Theorem 17 will. Let $W_1 = W$, $\mathbf{T}_1 = \{Z\}$, $W_2 = Z$, and $\mathbf{T}_2 = \{X_1\}$. Note that (i) and (ii) hold, since $W \not\perp\!\!\!\perp Y \mid Z$ and $W \perp\!\!\!\perp Y \mid Z, X_1$ as well as $Z \not\perp\!\!\!\perp Y \mid X_1$ and $Z \perp\!\!\!\perp Y \mid X_1, X_2$. By Lemma 18, we see that $\mathbf{T}_1$ is a minimal adjustment set relative to (X_1, Y) , since $X_1 \not\perp\!\!\!\perp Z$ and $Y \not\perp\!\!\!\perp Z \mid X_1$. The analogous claim holds for $\mathbf{T}_2$, since $X_2 \not\perp\!\!\!\perp X_1$ and $Y \not\perp\!\!\!\perp X_1 \mid X_2$. Thus by Theorem 17, $\{Z\}$ is an adjustment set relative to $(\mathbf{X}, Y)$.*

Example 14 (Limitations of Naive Combinations) *Consider a causal model that induces the DAG in Figure 3.5(a). Suppose the DAG is unknown, but we have data on every variable except $\{U_1, U_2, U_3\}$ and expert knowledge that $\{W_1, W_2, Z_1, Z_2\} < \{X_1, X_2\} < Y$.*

We consider constructing an adjustment set relative to $(\{X_1, X_2\}, Y)$ by combining one for (X_1, Y) with one for (X_2, Y) without requiring minimality. By Theorem 11, $\{Z_1\}$ and $\{Z_2\}$ are adjustment sets relative to (X_1, Y) and (X_2, Y) , respectively, since $W_1 \not\perp\!\!\!\perp Y \mid Z_1$; $W_1 \perp\!\!\!\perp Y \mid Z_1, X_1$; $W_2 \not\perp\!\!\!\perp Y \mid Z_2$; and $W_2 \perp\!\!\!\perp Y \mid Z_2, X_2$. However, we can show that $\{Z_1, Z_2\}$ is not an adjustment set relative to $(\{X_1, X_2\}, Y)$.

To see this, let the PAG in Figure 3.5(b) represent all the in/dependencies we can learn from the data with the addition of our expert knowledge. The claim follows by graphical criteria from prior research (see Theorem 69 in App. B.1). Had we required minimality, we would have found that $\{Z_1\}$ is not minimal. (This holds by Lemma 18, since $X_1 \perp\!\!\!\perp Z_1$.) Thus, Theorem 17 finds $\{Z_2\}$ is an adjustment set relative to $(\{X_1, X_2\}, Y)$, which Figure 3.5(b) confirms.

3.4.3 C-EQUIVALENCE AND EFFICIENCY

We close this section by showing how to extend Theorems 15 and 17 using the methods of Section 3.3.

Example 15 (Extending R1 Build, R1 Combine) *Revisit Example 13, where Theorem 17 found $\{Z\}$ as an adjustment set relative to $(\mathbf{X}, Y)$. This is, in fact, the only adjustment set Theorem 17 finds for this causal effect. However, as in Section 3.3.2, we can use c-equivalence to find $\emptyset$ and $\{W, Z\}$. (This holds by Theorem 13, since $Y \perp\!\!\!\perp \{W, Z\} \mid \mathbf{X}$.) Then as in Section 3.3.3, we can use Lemma 14 to see that $\{Z\}$ and $\{W, Z\}$ are both less efficient than $\emptyset$, since $Y \perp\!\!\!\perp \{W, Z\} \mid \mathbf{X}$. Thus, adjustment via $\emptyset$ may lead to more efficient estimation of the causal effect.*

3.5 SIMULATIONS

In this section, we use simulations to illustrate how our methods perform in settings with multiple treatments. To do this, we simulate data from random DAGs and then attempt to identify an adjustment set using Theorems 15 and 17 on “observed” variables in the generated data. We compare the performance of our data-driven methods against the performance of an existing graphical approach across three metrics: accuracy in identification, accuracy in estimation, and running time. We find that our methods outperform the graphical approach when a treatment effect exists – by all metrics except running time in high dimensions.

3.5.1 DATA GENERATION

To generate data, we start by building a random DAG. We assume a causal ordering of its nodes $V_1 < \dots < V_{15}$ and assign parents in the DAG as follows. For each V_i , we choose the size k of its parent set uniformly between zero and $\min(i - 1, 3)$. Then we choose k variables from $\{V_1, \dots, V_{i-1}\}$ at random. In this DAG, we designate

$V_1, \dots, V_5$: unobserved covariates,

$V_6, \dots, V_{12}$: observed covariates $\mathbf{W}$,

V_{13}, V_{14} : observed treatments X_1, X_2 , and

V_{15} : observed outcome Y .

Using this DAG, we consider two characteristics about the model. First, we note if the model contains a treatment effect by checking to see if the DAG contains $X_1 \rightarrow Y$ or $X_2 \rightarrow Y$. Second, we note if there is an adjustment set relative to $(\{X_1, X_2\}, Y)$ that we can learn from observed data generated by the model. To check this, we consider the PAG that represents in/dependencies from such data as well as knowledge that $\mathbf{W} < X_1 < X_2 < Y$. To confirm this PAG contains an adjustment set, we use graphical criteria from prior research (see Theorem 69 in App. B.1). These two characteristics are the basis for the settings we consider in our simulations.

Setting 1: Treatment effect exists, adjustment set exists.

Setting 2: Treatment effect exists, no adjustment set.

Setting 3: No treatment effect.

We continue generating graphs in this way until we have 100 models that fall into each of the three settings above. From each DAG, we generate three datasets – one with 100 observations; one with 1,000; and one with 10,000 – using the following linear Gaussian structural equation model (SEM). We let each random variable be a linear combination of its parents in the DAG $\mathcal{D}$ and an error term $\varepsilon_i \sim N(0, \sigma_i^2)$, where $\{\varepsilon_1, \dots, \varepsilon_{15}\}$ are mutually

independent:

$$V_i \longleftarrow \sum_{V_j \in \text{Pa}(V_i, \mathcal{D})} b_{ij} V_j + \varepsilon_i.$$

We choose each b_{ij} uniformly from $[-1.5, -0.1] \cup [0.1, 1.5]$ and each σ_i^2 uniformly from $[0.5, 1]$. This process gives us 100 datasets for each combination of setting and sample size.

3.5.2 METHOD APPLICATION

Using each dataset, we attempt to find an adjustment set for the causal effect of $\{X_1, X_2\}$ on Y by applying our data-driven methods (Theorems 15 and 17) to the observed data. To find variables W_i , $i \in \{1, \dots, k\}$, and a set $\mathbf{Z}$ that fulfill the in/dependencies of (i) and (ii) of each theorem, we run a brute-force search over all observed variables. For checking each in/dependency, we run a hypothesis test for non-zero partial correlation using Fisher’s z-transformation and a threshold of 0.05. Further, we allow both methods to assume $\mathbf{W} < X_1 < X_2 < Y$.

For comparison, we attempt to find an adjustment set for the same causal effect by applying an existing graphical approach – the *generalized adjustment criterion* (GAC; Perković et al., 2018; see Theorem 69 in App. B.1). In order to apply this method, we must first learn a causal PAG from the observed data. We do this using a causal discovery algorithm known as FCITIERS [Andrews et al., 2020] – a version of FCI [Spirtes et al., 2000] where variables cannot be causal descendants of downstream tiers. We run this algorithm using four tiers, based on knowledge that $\mathbf{W} < X_1 < X_2 < Y$. And we check for conditional independencies using hypothesis tests with a threshold of 0.05.

3.5.3 RESULTS

To compare how accurately all three methods (R1 Build, R1 Combine, FCI + GAC) identify the causal effect, consider the following outcomes.

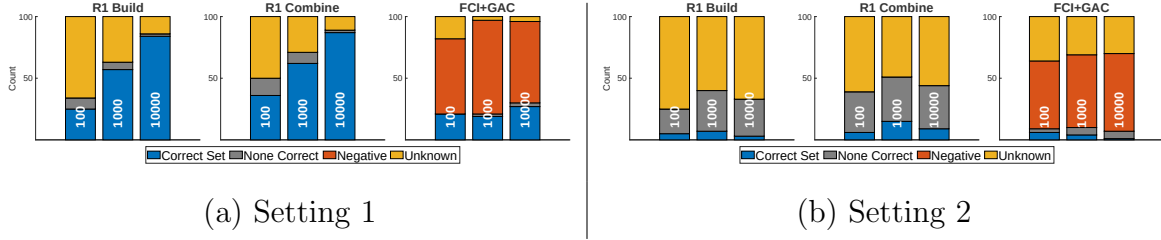


Figure 3.6: Outcomes of our data-driven methods (R1 Build, R1 Combine) and an existing graphical approach (FCI + GAC) on simulated data. Stacked bars show how often a method results in *correct set*, *none correct*, *negative*, or *unknown*.

Correct Set: Method concludes $\{X_1, X_2\}$ affects Y . At least one adjustment set it finds is correct in the underlying DAG.

None correct: Method concludes $\{X_1, X_2\}$ affects Y . No adjustment set it finds is correct in the underlying DAG.

Negative: Method concludes $\{X_1, X_2\}$ has no effect on Y .

Unknown: Method cannot find an adjustment set for the effect of $\{X_1, X_2\}$ on Y .

Figure 3.6 shows how often each outcome occurs for each method. The bars in each plot correspond to different dataset sizes (100; 1,000; and 10,000). The following outcomes are possible for all three methods: *correct set*, *none correct*, or *unknown*. FCI + GAC can additionally obtain a *negative* outcome, since FCI can learn a PAG that precludes a treatment effect.

Consider Setting 1, shown in Figure 3.6(a), where datasets are generated from models with both a treatment effect and adjustment set (among the observed variables). Successful performance in this setting is when a method finds a correct adjustment set. We see that R1 Build and R1 Combine make many accurate conclusions – particularly when applied to larger datasets – since they find a *correct set* in up to 85% of the datasets. In contrast,

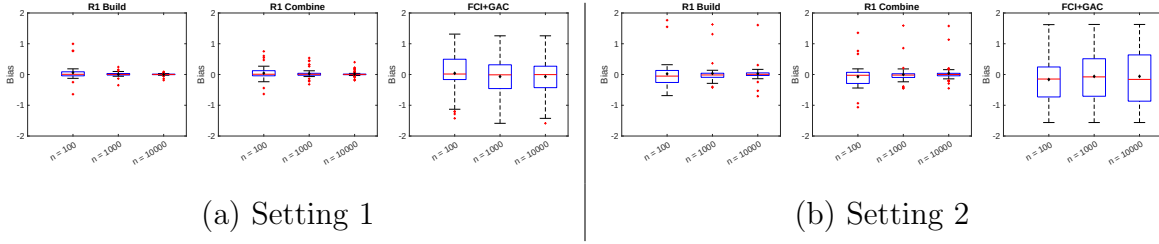


Figure 3.7: Differences between estimated and true causal effects. Results for our data-driven methods (R1 Build, R1 Combine) and an existing graphical approach (FCI + GAC) when applied to simulated data.

FCI + GAC makes fewer accurate conclusions, since it finds a *correct set* in under 25% of the datasets. Further, FCI + GAC inaccurately obtains a *negative* outcome in over 60% of the datasets. We conjecture that our methods outperform FCI + GAC in part because they do not attempt to identify the model’s entire causal structure and thus avoid errors in hypothesis testing for irrelevant areas of a graph.

Next, consider Setting 2, shown in Figure 3.6(b), where datasets are generated from models with a treatment effect but no adjustment set (among the observed variables). Successful performance in this setting is when a method does not find an adjustment set (i.e., the outcome is *unknown*). Although R1 Build and R1 Combine incorrectly find adjustment sets in 25-50% of the datasets, FCI + GAC incorrectly obtains a *negative* outcome for over 50% of the datasets.²

For Setting 3, where datasets are generated from models without a treatment effect, analogous plots can be found in App. B.4. But we note that FCI + GAC easily outperforms R1 Build and R1 Combine in this setting, since our methods cannot obtain a *negative* outcome.

We move beyond identification to compare the accuracy of estimating the causal effect

²For clarity in reading Figure 3.6(b), note that in a small percentage of datasets, each method obtains a *correct set* (based on the underlying DAG) even though models in Setting 2 do not have an adjustment set that we can learn from observed data (i.e., in the PAG that represents the model).

based on conclusions from each of the three methods. We do this estimation as follows. If a method finds an adjustment set $\mathbf{Z}$, we run a linear regression of Y on $\{X_1, X_2\} \cup \mathbf{Z}$. Then we estimate the causal effect using the sum of estimated coefficients for X_1 and X_2 from the regression. In the linear setting this corresponds to an estimate of $\mathbb{E}[Y|do(X_1 = x_1 + 1, X_2 = x_2 + 1)] - \mathbb{E}[Y|do(X_1 = x_1, X_2 = x_2)]$. Additionally for FCI + GAC, if the method obtains a *negative* outcome, then we set the causal effect to zero. If a method obtains an *unknown* outcome, then we do not estimate a causal effect.

Figure 3.7 shows the differences between these estimated values and the true causal effect across all three methods. Note that estimates based on our methods outperform FCI + GAC in both Settings 1 and 2, especially for larger datasets. Between our methods, estimates based on R1 Build perform slightly better than R1 Combine. We conjecture this is due to the larger number of hypothesis tests required to use R1 Combine. Further, estimation improves with increasing sample size – but only for our methods.

Though we want to temper these results, because the plots in Figure 3.7 hide a discrepancy between our methods and the existing graphical approach. Since we do not estimate a causal effect when a method obtains an *unknown* outcome, the estimates for each method in Figure 3.7 are based on a different subset of the 100 datasets in the simulation. To make this concrete, consider the estimates in Figure 3.7(b) from datasets with a sample size of 1,000. Here, R1 Build and R1 Combine find an adjustment set in 39 and 52 datasets, respectively. Thus, we estimate the causal effect based on these methods in 39 and 52 datasets, respectively. But FCI + GAC finds either an adjustment set or no treatment effect in 67 datasets, which include some but not all of the 39 and 52 datasets used in estimates based on our methods.³

For Setting 3, where datasets are generated from models without a treatment effect, analogous plots can be found in App. B.4. But we note that FCI + GAC easily outperforms

³To get a sense of these counts, return to Figure 3.6. We estimate a causal effect when a method obtains the following outcomes: *correct set*, *none correct*, *negative*. These correspond to the blue, gray, and red bars in Figure 3.6.

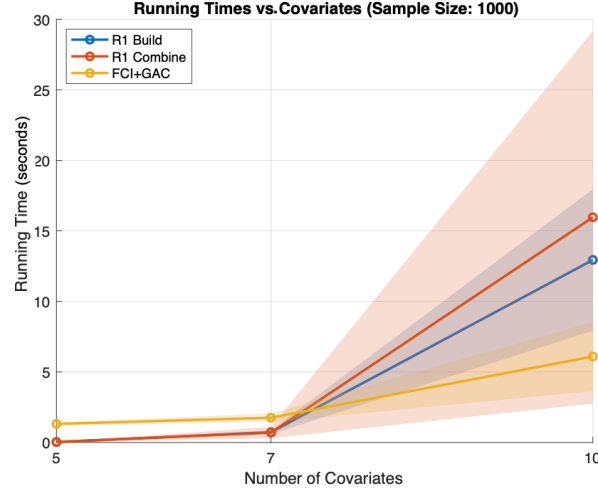


Figure 3.8: Running times for our data-driven methods (R1 Build, R1 Combine) and an existing graphical approach (FCI + GAC) when applied to simulated data.

R1 Build and R1 Combine in this setting, since our methods cannot obtain a *negative* outcome.

Finally, we compare all three methods' running times. Figure 3.8 shows that our data-driven methods perform similarly to FCI + GAC on models with five and seven observed covariates. However, FCI + GAC outperforms our methods, on average, in models with 10 observed covariates. See Section 3.6 below for further discussion.

3.6 DISCUSSION

This paper considers causal effect identification through covariate adjustment. While prior research in this area focuses on finding adjustment sets based on criteria from a causal graph [Pearl, 1995, Shpitser et al., 2012, Maathuis and Colombo, 2015, Perković et al., 2018], we consider instead a route that relies on conditional in/dependencies in the observed data directly. This extends the work of Entner et al. [2013].

We start by reviewing R1 of Entner et al. [2013] and explaining how c-equivalence can extend this rule. We provide a rationale for such an extension in the context of efficient

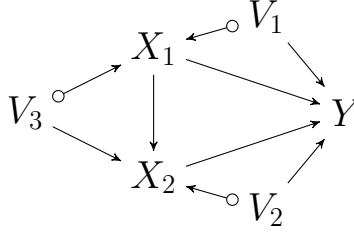


Figure 3.9: Known PAG used in Example 16

estimation. Then we present our main contributions: Theorems 15 and 17 (R1 Build and R1 Combine). These data-driven rules parallel R1 of Entner et al. [2013] but in a setting with multiple treatments. Using simulated data, we show that our rules outperform an existing graphical approach in settings where a treatment effect exists.

While the results of our simulations are encouraging, we discuss some limitations of our methods below. We begin by highlighting that Theorems 15 and 17 are sound but not complete for finding adjustment sets in the presence of multiple treatments. That is, any set these theorems find is a valid adjustment set, but these theorems cannot find every valid adjustment set. To see that the latter holds outside our restricted setting ($\mathbf{W} < \mathbf{X}$), note that there are models where every adjustment set includes descendants of $\mathbf{X}$ (see Examples 7-8 of Perković et al., 2018). But even within our restricted setting, Theorems 15 and 17 cannot find an adjustment set in every model. We show this for Theorem 15 in Example 12. For Theorem 17, see the example below.

Example 16 (Limitations of R1 Combine) *Consider an unknown causal model over $\{V_1, V_2, V_3, X_1, X_2, Y\}$, where we have data on every variable and expert knowledge that $\{V_1, V_2, V_3\} < \{X_1, X_2\} < Y$. Let the PAG in Figure 3.9 represent all the in/dependencies we can learn from the data with the addition of our expert knowledge. We want to know the effect of $\{X_1, X_2\}$ on Y .*

Using graphical criteria from prior research (see Theorem 69 in App. B.1), we can show there is an adjustment set relative to $(\{X_1, X_2\}, Y)$ – for example, $\{V_1, V_2\}$. But Theorem

17 cannot identify any such set, since there is no $W_1 \in \{V_1, V_2, V_3\}$ and $\mathbf{T}_1 \subseteq \{V_1, V_2, V_3\}$ such that $(W_1 \perp Y | \mathbf{T}_1, X_1)$.

A second limitation of our methods is computational. Theorems 15 and 17 require a search for variables W_i that fulfill specific in/dependencies, and further, Theorem 17 requires additional in/dependence testing in a search for minimal adjustment sets (see Lemma 18). To do this in our simulations, we performed brute-force searches over all possible variables. But these searches, which are expensive and require a large number of hypothesis tests, may hinder the applicability of our methods in high-dimensional settings.

Certainly, future work could address this by exploring greedy-search approaches for implementation. But we want to highlight an alternative – that an expensive brute-force search may not be necessary in practice, since expert domain knowledge could guide a search for an adjustment set that fits the criteria of Theorems 5 and 7. That is, researchers could begin such a search based on their experiences in the field, and thus, may not need to test all possible combinations of nodes before selecting an adjustment set that identifies their causal effect of interest. For a clear example of this, consider when an adjustment set for one treatment is already known and researchers are interested in modifying their analysis to account for additional treatments. Our methods are well-suited for this setting.

A third limitation of our work is that our methods cannot conclude that there is no treatment effect. Thus, we saw in our simulation study (Section 3.5) that an existing approach outperformed our methods in Setting 2, where datasets are generated from models without a treatment effect. Entner et al. [2013] address this setting with a second rule (R2) that can obtain a *negative* outcome. Future research could extend R2 to a setting with multiple treatments, and combining such an extension with our extensions of R1 could improve estimation of a “causal effect” of zero.

In addition to addressing the above limitations, future work could consider extensions to conditional adjustment [LaPlante and Perković, 2024], or combining experimental and observational datasets for estimation [Triantafillou and Cooper, 2021, Triantafillou et al.,

2021, 2023]. Further extensions that allow for inclusions of mediating variables may also be possible, for instance by considering sequential (time-dependent) back-door adjustment sets Pearl and Robins [1995], Murphy [2003], Pearl [2009], Smucler et al. [2022].

Chapter 4

IDENTIFYING CONDITIONAL CAUSAL EFFECTS IN MPDAGS

The following chapter is based on unpublished work that is available as a preprint on arXiv (see LaPlante and Perković, 2025).

4.1 INTRODUCTION

In finding causal effects, researchers may want to know the effect across an entire population, sometimes called a *total* or *unconditional causal effect*. For example, does free access to pre-kindergarten (pre-K) improve children’s socio-emotional skills throughout elementary school [Moffett et al., 2023]? However, researchers may want to know the effect within subgroups of the population, or a *conditional causal effect*. For instance, is there a subgroup of children who particularly benefit from free access to pre-K? Our research considers identifying these conditional effects from observational data.

We assume knowledge of a causal graph. Though, in general, observational data alone are insufficient for learning a full directed acyclic graph (DAG), even when all variables in the model are observed. Thus, we consider a setting where the causal graph is known only up to an equivalence class of DAGs that can be learned from observational data [Spirtes et al., 2000, Chickering, 2002b]. We allow the addition of expert knowledge as a way to restrict this class further. This restricted class of DAGs can be uniquely represented by a maximally oriented partially directed acyclic graph (MPDAG, Meek, 1995), which is the focus of our work. For illustration, see Figure 4.1.

Much of the literature focuses on identification in the unconditional setting, and many researchers consider identification through covariate adjustment. By definition, *adjustment*

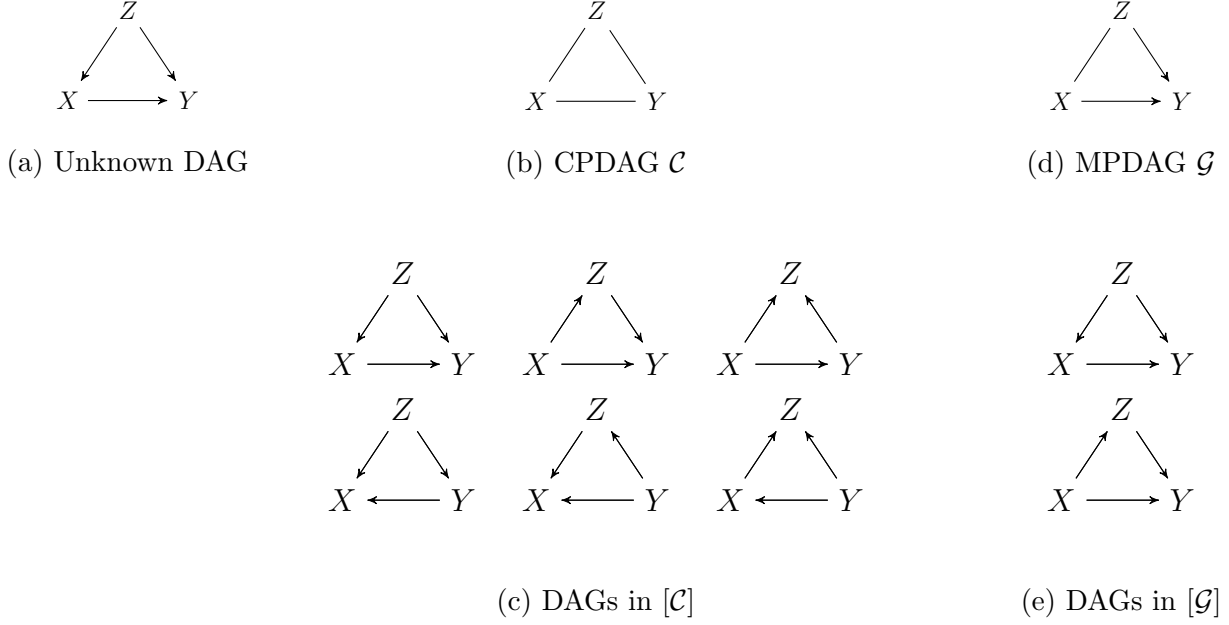


Figure 4.1: Illustrating an MPDAG. Let the DAG in ((a)) represent an unknown causal model. We cannot learn this DAG from observational data alone, but we can learn the CPDAG in ((b)), which represents the equivalence class of DAGs shown in ((c)). Adding background knowledge that X and Z precede Y produces the MPDAG in ((d)), which represents the restricted class of DAGs shown in ((e)).

sets identify the unconditional effect of $\mathbf{X}$ on $\mathbf{Y}$, since for any such set $\mathbf{S}$, it holds that $f(\mathbf{y} | do(\mathbf{x})) = \int f(\mathbf{y} | \mathbf{x}, \mathbf{s}) f(\mathbf{s}) d\mathbf{s}$, where f is any density consistent with the model. Prior research provides methods of finding these sets by checking graphical relationships in a known graph – for example, a DAG [Pearl, 1995, Shpitser et al., 2012], an MPDAG [Perković et al., 2017], or a *partial ancestral graph* (PAG) that allows for latent variables [Van der Zander et al., 2014, Maathuis and Colombo, 2015, Perković et al., 2018]. As an alternative to covariate adjustment, prior work also provides exact formulas for identifying unconditional effects when a DAG, MPDAG, or PAG is known [Perković, 2020, Jaber et al., 2018].

Other research focuses on identifying conditional effects, but in settings where graphs

other than an MPDAG is known. For example, the well-known do calculus of Pearl [2009] offers a tool for transforming *interventional densities* into observational densities based on *d-separations* in a known DAG. Zhang [2008a] extends this work by creating an analogous calculus for PAGs, and Jaber et al. [2022] update the extension to make it complete. Based on this calculus, Jaber et al. [2022] develop a sound and complete algorithm for conditional effect identification given a PAG.

To the best of our knowledge, the only work that considers the identification of conditional effects when an MPDAG is known is that of LaPlante and Perković [2024]. These authors offer a graphical criterion for finding *conditional adjustment sets*, where for any such set $\mathbf{S}$, it holds that $f(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z}) = \int f(\mathbf{y} \mid \mathbf{x}, \mathbf{z}, \mathbf{s})f(\mathbf{s} \mid \mathbf{z}) d\mathbf{s}$. But the authors note two limitations. First, there are identifiable effects where conditional adjustment sets do not exist (see Example 18 in Section 4.3.4 below). Further, there are conditional adjustment sets that LaPlante and Perković [2024]’s graphical criterion cannot find, since they require that the conditioning set must be unaffected by treatment (see Examples 23-24 Section 4.5.2 below).

Our results address this gap in the literature by providing three methods of identifying conditional causal effects in a setting where a causal MPDAG is known. We begin with an identification formula (Theorem 21), which provides an exact form of the interventional density in terms of observational densities. This result applies when the conditioning set is unaffected by treatment, which holds for the common case of pre-treatment covariates. Our second result (Theorem 24) generalizes the well-known do calculus of Pearl [2009] to the MPDAG setting. That is, we provide a set of rules based on *d-separations* in a causal graph with undirected edges that allow transformations of an interventional density under conditioning. We use this calculus in our final result: an identification algorithm (Algorithm 1) that we show is complete for identifying conditional causal effects.

We organize this paper in the following way. Section 4.2 provides relevant preliminaries on graphs and related densities. Section 4.3 presents our conditional identification formula. Then Sections 4.4 and 4.5 provide our do calculus for MPDAGs and conditional identification algorithm, respectively. We discuss the benefits, limitations, and possible usage of our results

in Section 4.6.

4.2 PRELIMINARIES

Nodes, Edges, and Graphs. We use capital letters (e.g., X) to denote nodes in a graph as well as random variables that these nodes represent. We use bold capital letters (e.g., $\mathbf{X}$) to denote node sets. A *graph* $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ consists of a set of nodes $\mathbf{V}$ and a set of edges $\mathbf{E}$. A *directed graph* contains only directed edges ($\rightarrow$). A *partially directed graph* may contain undirected edges ($-$) and directed edges ($\rightarrow$). An *induced subgraph* $\mathcal{G}_{\mathbf{V}'} = (\mathbf{V}', \mathbf{E}')$ of $\mathcal{G}$ consists of $\mathbf{V}' \subseteq \mathbf{V}$ and $\mathbf{E}' \subseteq \mathbf{E}$ where $\mathbf{E}'$ are all edges in $\mathbf{E}$ between nodes in $\mathbf{V}'$. In a partially directed graph, an edge is *into* (*out of*) a node X if the edge is directed and has an arrowhead (tail) at X .

Paths and Cycles. For disjoint node sets $\mathbf{X}$ and $\mathbf{Y}$, a *path* from $\mathbf{X}$ to $\mathbf{Y}$ is a sequence of distinct nodes $\langle X, \dots, Y \rangle$ from some $X \in \mathbf{X}$ to some $Y \in \mathbf{Y}$ for which every pair of successive nodes is adjacent. An *undirected path* is a path containing only undirected edges ($-$). A *directed path* from X to Y is a path of the form $X \rightarrow \dots \rightarrow Y$. A directed path from X to Y and the edge $Y \rightarrow X$ form a *directed cycle*. A path from $\mathbf{X}$ to $\mathbf{Y}$ is *proper* (w.r.t. $\mathbf{X}$) if only its first node is in $\mathbf{X}$. In a graph $\mathcal{G}$, the path $p := \langle V_0, \dots, V_k \rangle$, $k \geq 1$, is *possibly directed* if no edge $V_i \leftarrow V_j$, $0 \leq i < j \leq k$, is in $\mathcal{G}$ (Perković et al., 2017).

Subsequences, Subpaths, and Shields. A *subsequence* of a path p in a graph $\mathcal{G}$ is a path obtained by deleting a set of non-endpoint nodes from p without changing the order of the remaining nodes. For a path $p = \langle X_1, \dots, X_m \rangle$, the *subpath* from X_i to X_k , for $1 \leq i < k \leq m$, is the path $p(X_i, X_k) = \langle X_i, X_{i+1}, \dots, X_k \rangle$. We use the notation $(-p)(X_k, X_i)$ to denote the path $\langle X_k, X_{k-1}, \dots, X_i \rangle$. The subpath $\langle X_{j-1}, X_j, X_{j+1} \rangle$, for $1 < j < k$, is an *unshielded triple* if X_{j-1} and X_{j+1} are not adjacent in $\mathcal{G}$. A path is *unshielded* if all successive triples on the path are unshielded.

Colliders, Non-colliders, and Definite Status Paths. The *endpoints* of a path $p = \langle X_1, \dots, X_k \rangle$ in a graph $\mathcal{G}$ are the nodes X_1 and X_k . The node X_i , $1 < i < k$, is a *collider* on p if p contains $X_{i-1} \rightarrow X_i \leftarrow X_{i+1}$. The node X_i is a *definite non-collider* on p

if p contains $X_{i-1} \leftarrow X_i$ or $X_i \rightarrow X_{i+1}$, or if $\langle X_{i-1}, X_i, X_{i+1} \rangle$ is undirected and unshielded. A node is of *definite status* on p if it is an endpoint, collider, or definite non-collider on p . The path p is of definite status if every node on p is of definite status.

Ancestral Relationships. If a graph $\mathcal{G}$ contains the edge $X \rightarrow Y$, then X is a *parent* of Y in $\mathcal{G}$. If $\mathcal{G}$ contains $X - Y$ or $X \rightarrow Y$, then X is a *possible parent* of Y in $\mathcal{G}$. If $\mathcal{G}$ contains a directed path from X to Y , then X is an *ancestor* of Y and Y is a *descendant* of X in $\mathcal{G}$. If $\mathcal{G}$ contains a possibly directed path from X to Y , then X is a *possible ancestor* of Y and Y is a *possible descendant* of X in $\mathcal{G}$. We use the convention that every node is an ancestor, descendant, possible ancestor, and possible descendant of itself. The sets of parents, ancestors, and descendants and the sets of possible parents, ancestors, and descendants of X in $\mathcal{G}$ are denoted by $\text{Pa}(X, \mathcal{G})$, $\text{An}(X, \mathcal{G})$, $\text{De}(X, \mathcal{G})$, $\text{PossPa}(X, \mathcal{G})$, $\text{PossAn}(X, \mathcal{G})$, and $\text{PossDe}(X, \mathcal{G})$, respectively. For a set of nodes $\mathbf{X}$, we let $\text{An}(\mathbf{X}, \mathcal{G}) = \cup_{X \in \mathbf{X}} \text{An}(X, \mathcal{G})$, with analogous definitions for $\text{De}(\mathbf{X}, \mathcal{G})$, $\text{PossAn}(\mathbf{X}, \mathcal{G})$, and $\text{PossDe}(\mathbf{X}, \mathcal{G})$. Unconventionally, we define $\text{Pa}(\mathbf{X}, \mathcal{G}) = (\cup_{X \in \mathbf{X}} \text{Pa}(X, \mathcal{G})) \setminus \mathbf{X}$.

DAGs and MPDAGs. A directed graph without directed cycles is a *directed acyclic graph* (DAG). A partially directed graph without directed cycles is a *partially directed acyclic graph* (PDAG). All DAGs over a node set $\mathbf{V}$ with the same adjacencies and unshielded colliders can be uniquely *represented* by a *completed PDAG* (CPDAG). These DAGs form a Markov equivalence class with the same set of d-separations. A *maximally oriented PDAG* (MPDAG) is formed by taking a CPDAG, adding background knowledge (by directing undirected edges), and completing Meek [1995]’s orientation rules. We say a DAG is *represented* by an MPDAG $\mathcal{G}$ if it has the same nodes, adjacencies, and directed edges as $\mathcal{G}$ and if it has no additional unshielded colliders from those in $\mathcal{G}$. The set of such DAGs – denoted by $[\mathcal{G}]$ – forms a restriction of the Markov equivalence class so that all DAGs in $[\mathcal{G}]$ have same set of d-separations. Note that if $\mathcal{G}$ has the edge $A - B$, then $[\mathcal{G}]$ contains at least one DAG with $A \rightarrow B$ and one DAG with $A \leftarrow B$ [Meek, 1995]. Further, note that all DAGs and CPDAGs are MPDAGs.

Markov Compatibility and Positivity Assumption. An *observational density* $f(\mathbf{v})$

is *Markov compatible* with a DAG $\mathcal{D} = (\mathbf{V}, \mathbf{E})$ if $f(\mathbf{v}) = \prod_{V_i \in \mathbf{V}} f(v_i | \text{pa}(v_i, \mathcal{D}))$. If $f(\mathbf{v})$ is Markov compatible with a DAG $\mathcal{D}$, then it is Markov compatible with every DAG that is Markov equivalent to $\mathcal{D}$ [Pearl, 2009]. Hence, we say that a density is *Markov compatible* with an MPDAG $\mathcal{G}$ if it is Markov compatible with a DAG represented by $\mathcal{G}$. Throughout, we assume positivity. That is, we only consider densities that satisfy $f(\mathbf{v}) > 0$ for all valid values of $\mathbf{V}$ [Kivva et al., 2023]. Note that since, $f(\mathbf{v}) = \prod_{V_i \in \mathbf{V}} f(v_i | \text{pa}(v_i, \mathcal{D}))$, assuming $f(\mathbf{v}) > 0$ is equivalent to assuming $f(v_i | \text{pa}(v_i, \mathcal{G})) > 0$ for all $V_i \in \mathbf{V}$.

D-connection, D-separation, and Probabilistic Implications. Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ be pairwise disjoint node sets in a graph $\mathcal{G}$. A definite status path p from $\mathbf{X}$ to $\mathbf{Y}$ is *d-connecting* (or *open*) given $\mathbf{Z}$ if every definite non-collider on p is not in $\mathbf{Z}$ and every collider on p has a descendant in $\mathbf{Z}$. Otherwise, p is *blocked* given $\mathbf{Z}$. If all definite status paths between $\mathbf{X}$ and $\mathbf{Y}$ in $\mathcal{G}$ are blocked given $\mathbf{Z}$, then $\mathbf{X}$ is *d-separated* from $\mathbf{Y}$ given $\mathbf{Z}$ in $\mathcal{G}$ and we write $(\mathbf{X} \perp_d \mathbf{Y} | \mathbf{Z})_{\mathcal{G}}$. This d-separation implies that $\mathbf{X}$ and $\mathbf{Y}$ are conditionally independent given $\mathbf{Z}$ in any *observational density* that is Markov compatible with $\mathcal{G}$ [Lauritzen et al., 1990, Henckel et al., 2022].

Causal Graphs. An MPDAG $\mathcal{G}$ is a *causal MPDAG* if every edge $V_i \rightarrow V_j$ in $\mathcal{G}$ represents a direct causal effect of V_i on V_j and if every edge $V_i - V_j$ represents a direct causal effect of unknown direction (either V_i causes V_j or V_j causes V_i). In a causal MPDAG, any directed path is *causal*, any possibly directed path is *possibly causal*, and any other path is *non-causal*.

Causal Ordering. Let $\mathbf{X} = \{X_1, \dots, X_k\}$, $k \geq 1$, be a node set in a causal DAG $\mathcal{D}$. We say that $X_1 < \dots < X_k$ is a *total causal ordering* of $\mathbf{X}$ consistent with $\mathcal{D}$ if for every $X_i, X_j \in \mathbf{X}$ such that $X_i < X_j$ and such that X_i and X_j are adjacent in $\mathcal{D}$, then $\mathcal{D}$ contains $X_i \rightarrow X_j$. There can be more than one total causal ordering of a set of nodes in a DAG. For example, in the DAG $X_b \leftarrow X_a \rightarrow X_c$, both $X_a < X_b < X_c$ and $X_a < X_c < X_b$ are total causal orderings consistent with the DAG.

Now let $\mathbf{X}$ be a node set in a causal MPDAG $\mathcal{G}$. Since $\mathcal{G}$ may contain undirected edges, there is generally no total causal ordering of $\mathbf{X}$ consistent with $\mathcal{G}$. Instead, we define a

partial causal ordering of $\mathbf{X}$ consistent with $\mathcal{G}$ to be an ordering of pairwise disjoint node sets $\mathbf{A}_1, \dots, \mathbf{A}_k$, $k \geq 1$, $\cup_{i=1}^k \mathbf{A}_i = \mathbf{X}$, that satisfies the following: for every $i, j \in \{1, \dots, k\}$ such that $\mathbf{A}_i < \mathbf{A}_j$ and there is an edge between $A_i \in \mathbf{A}_i$ and $A_j \in \mathbf{A}_j$ in $\mathcal{G}$, then $\mathcal{G}$ contains $A_i \rightarrow A_j$.

Consistency. Let $f(\mathbf{v})$ be an observational density over a set of variables $\mathbf{V}$. The notation $do(\mathbf{X} = \mathbf{x})$, or $do(\mathbf{x})$ for short, represents an outside intervention that sets $\mathbf{X} \subseteq \mathbf{V}$ to fixed values $\mathbf{x}$. An *interventional density* $f(\mathbf{v}|do(\mathbf{x}))$ is a density resulting from such an intervention.

Let $\mathbf{F}^*$ denote the set of all interventional densities $f(\mathbf{v}|do(\mathbf{x}))$ such that $\mathbf{X} \subseteq \mathbf{V}$ (including $\mathbf{X} = \emptyset$). A causal DAG $\mathcal{D} = (\mathbf{V}, \mathbf{E})$ is a *causal Bayesian network compatible with $\mathbf{F}^*$* if and only if for all $f(\mathbf{v}|do(\mathbf{x})) \in \mathbf{F}^*$, the following *truncated factorization* holds:

$$f(\mathbf{v}|do(\mathbf{x})) = \prod_{V_i \in \mathbf{V} \setminus \mathbf{X}} f(v_i | \text{pa}(v_i, \mathcal{D})) \mathbb{1}(\mathbf{X} = \mathbf{x}) \quad (4.1)$$

[Pearl, 2009, Bareinboim et al., 2012]. We say an interventional density is *consistent* with a causal DAG $\mathcal{D}$ if it belongs to a set of interventional densities $\mathbf{F}^*$ such that $\mathcal{D}$ is compatible with $\mathbf{F}^*$. Note that any observational density that is Markov compatible with $\mathcal{D}$ is consistent with $\mathcal{D}$. We say an interventional density is *consistent* with a causal MPDAG $\mathcal{G}$ if it is consistent with each DAG in $[\mathcal{G}]$ – were the DAG to be causal. Following convention, we omit $\mathbb{1}(\mathbf{X} = \mathbf{x})$ from identifying expressions of interventional densities below.

Identifiability. Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ be pairwise disjoint node sets in a causal MPDAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, and let $\mathbf{F}_1^* = \{f_i(\mathbf{v}|do(\mathbf{x}')) : \mathbf{X}' \subseteq \mathbf{V}\}$ be a set with which a DAG $\mathcal{D}_i \in [\mathcal{G}]$ is compatible – were $\mathcal{D}_i$ to be causal. Then the conditional causal effect of $\mathbf{X}$ on $\mathbf{Y}$ given $\mathbf{Z}$ is *identifiable* in $\mathcal{G}$ if for any $\mathbf{F}_1^*, \mathbf{F}_2^*$ such that $f_1(\mathbf{v}) = f_2(\mathbf{v})$, we have $f_1(\mathbf{y}|do(\mathbf{x}), \mathbf{z}) = f_2(\mathbf{y}|do(\mathbf{x}), \mathbf{z})$ [Pearl, 2009].

4.3 IDENTIFICATION FORMULA

In this section, we provide our first result: a formula for identifying a conditional causal effect when a causal MPDAG is known and when the conditioning set is unaffected by

treatment. This formula relies on concepts found in prior research on MPDAGs, which we review briefly below. After presenting our formula, we explore its use through special cases and examples. We close this section by providing a necessary and sufficient condition for identifying conditional effects in this setting.

4.3.1 REVIEW OF THE PCO ALGORITHM

The identification formula we present in Section 4.3.2 relies on output from the *Partial Causal Ordering* (PCO) Algorithm of Perković [2020]. We provide the full algorithm in Appendix C.2, but offer a brief description here for broad understanding of our results. The PCO Algorithm inputs a set of nodes from an MPDAG and outputs an ordered partition of that node set, which we call an *ordered bucket decomposition*. That is, if $\mathbf{D}$ is a node set in an MPDAG $\mathcal{G}$, then $\text{PCO}(\mathbf{D}, \mathcal{G}) = (\mathbf{B}_1, \dots, \mathbf{B}_k)$, $k \geq 1$, where $\mathbf{B}_1, \dots, \mathbf{B}_k$ are pairwise disjoint subsets of $\mathbf{D}$ that follow a *partial causal ordering* consistent with $\mathcal{G}$ (see Section 4.2). For clarity, we formally define *buckets* and their ordered decompositions below.

Definition 19 (Bucket; Perković, 2020) *Let $\mathbf{B}$ and $\mathbf{D}$ be node sets in an MPDAG $\mathcal{G}$ such that $\mathbf{B} \subseteq \mathbf{D}$. Then $\mathbf{B}$ is a bucket in $\mathbf{D}$ given $\mathcal{G}$ if it is the intersection of $\mathbf{D}$ with a maximal undirected connected set in $\mathcal{G}$. That is,*

- $\mathcal{G}$ contains a undirected path from B_i to B_j for every $B_i, B_j \in \mathbf{B}$, and
- $\mathcal{G}$ contains no undirected path from $\mathbf{B}$ to any node in $\mathbf{D} \setminus \mathbf{B}$.

Definition 20 (Ordered Bucket Decomposition) *Let $\mathbf{D}$ be a node set in an MPDAG $\mathcal{G}$. Then $(\mathbf{B}_1, \dots, \mathbf{B}_k)$, $k \geq 1$, is an ordered bucket decomposition of $\mathbf{D}$ in $\mathcal{G}$ if*

- $\mathbf{B}_i$ is a bucket in $\mathbf{D}$ given $\mathcal{G}$ for all $i \in \{1, \dots, k\}$.
- $\mathbf{B}_1 < \dots < \mathbf{B}_k$ is a partial causal ordering of $\mathbf{D}$ consistent with $\mathcal{G}$ so that

$$- \cup_{i=1}^k \mathbf{B}_i = \mathbf{D}, \text{ and}$$

- any edge in $\mathcal{G}$ between $B_i \in \mathbf{B}_i$ and $B_j \in \mathbf{B}_j$ for $i, j \in \{1, \dots, k\}$, $i < j$, must take the form $B_i \rightarrow B_j$.

4.3.2 IDENTIFICATION FORMULA

We turn to our identification formula, where we consider the conditional causal effect of a set of treatments $\mathbf{X}$ on outcomes $\mathbf{Y}$ given covariates $\mathbf{Z}$ that are unaffected by treatment.

Theorem 21 (Conditional Identification Formula) *Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ be pairwise disjoint node sets in a causal MPDAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, where $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$ and where there is no proper possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$ that starts with an undirected edge. Then for any density f consistent with $\mathcal{G}$,*

$$f(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z}) = \int \prod_{i \in \mathbf{I}_E} f(\mathbf{b}_i \mid \text{pa}(\mathbf{b}_i, \mathcal{G})) \prod_{i \in \mathbf{I}_N} f(\mathbf{b}_i \mid \mathbf{b}_i^N, \mathbf{z}) \, d\mathbf{b}, \quad (4.2)$$

for $\mathbf{B} = \text{An}(\mathbf{Y}, \mathcal{G}_{\mathbf{V} \setminus \mathbf{X}}) \setminus (\mathbf{Z} \cup \mathbf{Y})$; $(\mathbf{B}_1, \dots, \mathbf{B}_k) = \text{PCO}(\text{An}(\mathbf{Y}, \mathcal{G}_{\mathbf{V} \setminus \mathbf{X}}) \setminus \mathbf{Z}, \mathcal{G})$; $\mathbf{B}_0 = \emptyset$; and $\mathbf{B}_i^N = \cup_{j=0}^{i-1} \mathbf{B}_j \setminus \text{PossDe}(\mathbf{X}, \mathcal{G})$. For $i \in \{1, \dots, k\}$, let $i \in \mathbf{I}_E$ when $\mathbf{Z} \cap \text{PossDe}(\mathbf{B}_i, \mathcal{G}) = \emptyset$ and let $i \in \mathbf{I}_N$ otherwise.

Proof of Theorem 21. The first two equalities below follow by the law of total probability and the chain rule, respectively. The third equality follows from our Lemma 103 (App. C.4), which we discuss following this proof.

$$\begin{aligned} f(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z}) &= \int f(\mathbf{b}, \mathbf{y} \mid do(\mathbf{x}), \mathbf{z}) \, d\mathbf{b} \\ &= \int \prod_{i=1}^k f(\mathbf{b}_i \mid \mathbf{b}_{i-1}, \dots, \mathbf{b}_0, do(\mathbf{x}), \mathbf{z}) \, d\mathbf{b} \\ &= \int \prod_{i \in \mathbf{I}_E} f(\mathbf{b}_i \mid \text{pa}(\mathbf{b}_i, \mathcal{G})) \prod_{i \in \mathbf{I}_N} f(\mathbf{b}_i \mid \mathbf{b}_i^N, \mathbf{z}) \, d\mathbf{b}. \quad \blacksquare \end{aligned}$$

Proof Sketch. Lemma 103 and its proof can be found in Appendix C.4, but we provide an overview here for intuition. This lemma requires the same assumptions as Theorem 21 – that

$\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$ and no proper possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$ starts undirected. Under these assumptions, the lemma claims

$$f(\mathbf{b}_i | \mathbf{b}_{i-1}, \dots, \mathbf{b}_0, do(\mathbf{x}), \mathbf{z}) = \begin{cases} f(\mathbf{b}_i | \text{pa}(\mathbf{b}_i, \mathcal{G})) & \mathbf{Z} \cap \text{PossDe}(\mathbf{B}_i, \mathcal{G}) = \emptyset \\ f(\mathbf{b}_i | \mathbf{b}_i^N, \mathbf{z}) & \mathbf{Z} \cap \text{PossDe}(\mathbf{B}_i, \mathcal{G}) \neq \emptyset. \end{cases} \quad (4.3)$$

The proof of this lemma relies on applying Rules 1-3 of Pearl's (2009) do calculus (Theorem 75, App. C.2) to an arbitrary DAG $\mathcal{D} \in [\mathcal{G}]$. To show the top equality of (4.3) holds, we use Rules 1 and 3 to restrict the conditioning set – including variables in the do intervention – to variables also in $\text{Pa}(\mathbf{B}_i, \mathcal{G})$. Applying Rule 2 leaves a conditioning set of $(\cup_{j=0}^{i-1} \mathbf{B}_j \cup \mathbf{Z} \cup \mathbf{X}) \cap \text{Pa}(\mathbf{B}_i, \mathcal{G})$, which is simply $\text{Pa}(\mathbf{B}_i, \mathcal{G})$. To show that the d-separations for these three rules hold, we assume, for sake of contradiction, that under a specific conditioning set, there is an open path in $\mathcal{D}$ between $\mathbf{B}_i$ and variables in $(\cup_{j=1}^{i-1} \mathbf{B}_j \cup \mathbf{Z}) \setminus \text{Pa}(\mathbf{B}_i, \mathcal{G})$ and $\mathbf{X}$. Informally, we rely on the fact that $\mathcal{G}$ cannot contain a possibly causal path from $\mathbf{B}_i$ to these variables. We use similar logic to show the lower equality of (4.3) holds, by applying Rules 1 and 3 to reduce the conditioning set to $\mathbf{B}_i^N \cup \mathbf{Z}$. ■

We want to highlight that Appendix C.3 includes broad results for MPDAGs used in the proofs of Lemma 103. To keep the focus of our paper on our three identification methods, we do not include these results in the main text. However, we recommend them to researchers in this area, since the results may be useful for future work on MPDAGs. As a brief review, Appendix C.3 includes an alternate definition for possibly causal paths, a claim about concatenating casual and possibly causal paths, examples when possibly causal paths in the MPDAG imply causal paths in the equivalence class of DAGs, and restrictions on possibly causal paths among buckets in the MPDAG.

4.3.3 SPECIAL CASES

Before providing examples, we return to the statement of Theorem 21 to consider two settings where Equation (4.2) can be replaced with a simplified form. For the first setting, we compare

our results to those of Perković [2020]. Under similar assumptions, the author provides the following identification formula for the *unconditional* causal effect of $\mathbf{X}$ on $\mathbf{Y}$:

$$f(\mathbf{y} \mid do(\mathbf{x})) = \int \prod_{i=1}^k f(\mathbf{b}_i \mid \text{pa}(\mathbf{b}_i, \mathcal{G})) d\mathbf{b}. \quad (4.4)$$

To compare Equations (4.2) and (4.4), we note that Theorem 21 holds for any conditioning set $\mathbf{Z}$, including when $\mathbf{Z} = \emptyset$. In this case, $\mathbf{Z} \cap \text{PossDe}(\mathbf{B}_i, \mathcal{G}) = \emptyset$ for all $i \in \{1, \dots, k\}$, and thus, Equation (4.2) reduces to the exact form of the unconditional identification formula shown in Equation (4.4).

For the second setting, suppose, as in Theorem 21, that $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$. But instead of excluding a subset of possibly causal paths from $\mathbf{X}$ to $\mathbf{Y}$, consider a stronger assumption that $\mathbf{Y} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$. In this setting, identification of the conditional effect reduces to an application of Rule 3 of the do calculus (Theorem 75, App. C.2). We show this formally below.

Theorem 22 *Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ be pairwise disjoint node sets in a causal MPDAG $\mathcal{G}$, where $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$ and $\mathbf{Y} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$. Then for any f consistent with $\mathcal{G}$,*

$$f(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z}) = f(\mathbf{y} \mid \mathbf{z}). \quad (4.5)$$

Proof of Theorem 22. Let $\mathcal{D}$ be an arbitrary DAG in $[\mathcal{G}]$. The result holds by Rule 3 of the do calculus (Theorem 75, App. C.2) if we show $(\mathbf{Y} \perp_d \mathbf{X} \mid \mathbf{Z})_{\mathcal{D}_{\overline{\mathbf{W}}}}$ for $\mathbf{W} := \mathbf{X} \setminus \text{An}(\mathbf{Z}, \mathcal{D})$. Note that $\text{De}(\mathbf{X}, \mathcal{D}) \subseteq \text{PossDe}(\mathbf{X}, \mathcal{G})$ (Lemma 93, App. C.3) so that $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$ implies $\mathbf{Z} \cap \text{De}(\mathbf{X}, \mathcal{D}) = \emptyset$. Thus, $\mathbf{W} = \mathbf{X}$ and we must show $(\mathbf{Y} \perp_d \mathbf{X} \mid \mathbf{Z})_{\mathcal{D}_{\overline{\mathbf{X}}}}$.

Let p be an arbitrary path in $\mathcal{D}_{\overline{\mathbf{X}}}$ from $X \in \mathbf{X}$ to $Y \in \mathbf{Y}$. By definition of $\mathcal{D}_{\overline{\mathbf{X}}}$, p begins $X \rightarrow$. But p cannot be causal, since this would imply $Y \in \text{De}(\mathbf{X}, \mathcal{D}) \subseteq \text{PossDe}(\mathbf{X}, \mathcal{G})$, which is a contradiction. Thus, p must have a collider. Form a set $\mathbf{S}$ with the collider on p that is closest to X on p and all of its descendants in $\mathcal{D}_{\overline{\mathbf{X}}}$. Since $\mathbf{S} \subseteq \text{De}(X, \mathcal{D}_{\overline{\mathbf{X}}}) \subseteq \text{De}(\mathbf{X}, \mathcal{D}) \subseteq \text{PossDe}(\mathbf{X}, \mathcal{G})$ and since $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$, then p must be d-separated given $\mathbf{Z}$. ■

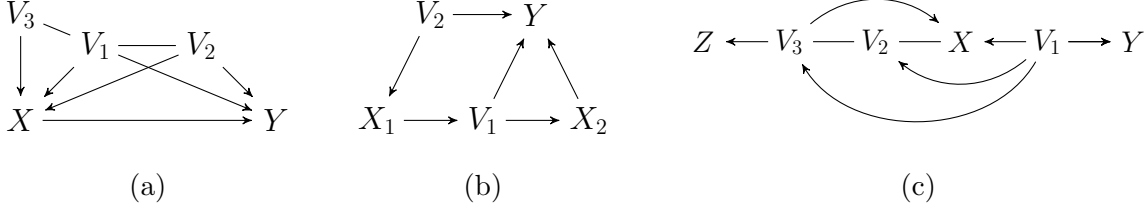


Figure 4.2: MPDAGs used in Examples 17-19

4.3.4 EXAMPLES

The examples below demonstrate how to use Theorems 21 and 22 to identify a conditional causal effect. As noted in Section 4.1, LaPlante and Perković [2024] already provide an identification method in the MPDAG setting using what the authors call *conditional adjustment sets*. But as they point out, these sets are not present in every MPDAG. Our identification formula fills in this gap when the conditioning set is unaffected by treatment. To highlight our contribution, Example 17 below considers a setting where a conditional adjustment set exists. We contrast this with Example 18, where there is no such set. Example 19 shows a straightforward application of Theorem 22.

Example 17 (Conditional Adjustment) *Let $\mathcal{G}$ be the causal MPDAG in Figure 4.2((a)), and let $\mathbf{X} = \{X\}$, $\mathbf{Y} = \{Y\}$, and $\mathbf{Z} = \{V_1\}$. To use Theorem 21, we confirm $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$ and no possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$ starts undirected. Then we find $\mathbf{B}_1 = \{V_2\}$ (where $1 \in \mathbf{I}_\mathbf{N}$), $\mathbf{B}_2 = \{Y\}$ (where $2 \in \mathbf{I}_\mathbf{E}$), and $\mathbf{B} = \{V_2\}$. Thus for f consistent with $\mathcal{G}$,*

$$\begin{aligned} f(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z}) &= \int f(\mathbf{b}_2 \mid \text{pa}(\mathbf{b}_2, \mathcal{G})) f(\mathbf{b}_1 \mid \mathbf{b}_1^\mathbf{N}, \mathbf{z}) d\mathbf{b} \\ &= \int f(y \mid x, v_1, v_2) f(v_2 \mid v_1) dv_2. \end{aligned}$$

Compare this to Example 2 of LaPlante and Perković [2024], where the authors find the

identical form of $f(\mathbf{y} | do(\mathbf{x}), \mathbf{z})$ by showing that $\mathbf{S} := \{V_2\}$ is a conditional adjustment set.

Example 18 (No Conditional Adjustment) Let $\mathcal{G}$ be the causal DAG (and therefore, MPDAG) in Figure 4.2((b)), and let $\mathbf{X} = \{X_1, X_2\}$, $\mathbf{Y} = \{Y\}$, and $\mathbf{Z} = \{V_2\}$. To use Theorem 21, we confirm $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$ and no possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$ starts undirected. Then we find $\mathbf{B}_1 = \{V_1\}$, $\mathbf{B}_2 = \{Y\}$ (where $1, 2 \in \mathbf{I}_{\mathbf{E}}$), and $\mathbf{B} = \{V_1\}$. Thus for f consistent with $\mathcal{G}$,

$$\begin{aligned} f(\mathbf{y} | do(\mathbf{x}), \mathbf{z}) &= \int f(\mathbf{b}_1 | \text{pa}(\mathbf{b}_1, \mathcal{G})) f(\mathbf{b}_2 | \text{pa}(\mathbf{b}_2, \mathcal{G})) d\mathbf{b} \\ &= \int f(v_1 | x_1) f(y | x_2, v_1, v_2) dv_1. \end{aligned}$$

Note that LaPlante and Perković [2024] show there is no conditional adjustment set in this case (see their Figure 4).

Example 19 (Using Theorem 22) Let $\mathcal{G}$ be the causal MPDAG in Figure 4.2((c)), and let $\mathbf{X} = \{X\}$, $\mathbf{Y} = \{Y\}$, and $\mathbf{Z} = \{Z\}$. To use Theorem 22, we confirm $\mathbf{Y} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$ and $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$. Thus for f consistent with $\mathcal{G}$,

$$f(\mathbf{y} | do(\mathbf{x}), \mathbf{z}) = f(y | z).$$

4.3.5 IDENTIFIABILITY CONDITION

We close the discussion of our identification formula (Theorem 21) by reflecting further on its second condition – the restriction on paths between $\mathbf{X}$ and $\mathbf{Y}$. The result below shows that this condition is necessary and sufficient for identification when the conditioning set is unaffected by treatment.

Proposition 23 (Identifiability Condition, Restricted $\mathbf{Z}$) Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ be pairwise disjoint node sets in a causal MPDAG $\mathcal{G}$ such that $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$. Then the conditional causal effect of $\mathbf{X}$ on $\mathbf{Y}$ given $\mathbf{Z}$ is identifiable in $\mathcal{G}$ if and only if there is no proper possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$ that starts with an undirected edge.

Proof Sketch. The proof of Proposition 23 can be found in Appendix C.5, but we provide an outline here for intuition. Note that $\Leftarrow$ follows from Theorem 21. We show $\Rightarrow$ through its contraposition. Thus, suppose there is a proper possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$ that starts undirected. We show in a separate result (Lemma 107, App. C.5) that this path implies there are DAGs $\mathcal{D}^1, \mathcal{D}^2 \in [\mathcal{G}]$ with corresponding paths $X \rightarrow \cdots \rightarrow Y$ and $X \leftarrow V_1 \rightarrow \cdots \rightarrow Y$. To show the conditional effect is not identifiable, we find $\{f_1(\mathbf{v}|do(\mathbf{x}')) : \mathbf{X}' \subseteq \mathbf{V}\}$ and $\{f_2(\mathbf{v}|do(\mathbf{x}')) : \mathbf{X}' \subseteq \mathbf{V}\}$ that are compatible with $\mathcal{D}^1$ and $\mathcal{D}^2$, where $f_1(\mathbf{v}) = f_2(\mathbf{v})$ but $f_1(\mathbf{y}|do(\mathbf{x}), \mathbf{z}) \neq f_2(\mathbf{y}|do(\mathbf{x}), \mathbf{z})$. We construct these sets using structural equation models (SEMs) based on DAGs with the same nodes as $\mathcal{D}^i$ but with only the edges from the path of interest. To complete the proof, we show $E_1[Y|do(\mathbf{X} = \mathbf{1}), \mathbf{Z}] \neq E_2[Y|do(\mathbf{X} = \mathbf{1}), \mathbf{Z}]$ using the relevant DAGs, the do calculus (Theorem 75, App. C.2), and Wright’s Rule (Lemma 77). ■

4.4 DO CALCULUS FOR MPDAGS

The do calculus of Pearl (2009, see Theorem 75, App. C.2) consists of three rules that justify transformations of an interventional density. These rules are based on d-separations in associated DAGs, and dictate, for example, when an interventional density is equivalent to an observational one. Thus, we often rely on these rules for causal identification.

But Pearl’s do calculus was designed for densities consistent with a known DAG, and researchers rarely have knowledge of every edge orientation in a causal graph. For this reason, Zhang [2008a] created a do calculus for the setting where the causal graph is known only up to a *maximal* or *partial ancestral graph* (MAG or PAG). Other extensions include Correa and Bareinboim’s (2020) σ -calculus for inference under non-static interventions. Following this branch of research, we offer Theorem 24: a do calculus for MPDAGs.

4.4.1 THE CALCULUS

Just as Pearl’s do calculus relies on manipulations of a known DAG (e.g., $\mathcal{D}_{\bar{\mathbf{X}}}$), our do calculus relies on manipulations of a known MPDAG (e.g., $\mathcal{G}_{\bar{\mathbf{X}}}$). But our graphs differ from Pearl’s in notable ways. In order to build intuition for our do calculus, we first pause to describe these graphs and their differences.

As a review, Pearl’s do calculus requires that interventional densities must be consistent with a DAG $\mathcal{D}$, and that d-separations must hold in specific manipulations of $\mathcal{D}$. For example, $\mathcal{D}_{\bar{\mathbf{X}}}$ denotes the graph obtained by taking $\mathcal{D}$ and removing all edges into $\mathbf{X}$. This process has an intuitive relationship with a do intervention on $\mathbf{X}$ in that intervening on $\mathbf{X}$ would override the causal effect of its parents in the original model. Note further that these mutilated graphs are still DAGs, and their d-separations capture probabilistic relationships in densities consistent with the mutilated DAGs.

In our do calculus, we assume knowledge of an MPDAG $\mathcal{G}$ and consider d-separations in mutilated versions of $\mathcal{G}$. For example, we use $\mathcal{G}_{\bar{\mathbf{X}}}$ to denote the graph obtained by taking $\mathcal{G}$ and removing all edges into $\mathbf{X}$. But this process differs from Pearl’s in several ways. First, removing edges from $\mathcal{G}$ into $\mathbf{X}$ does not have the same intuitive relationship with the intervention $do(\mathbf{x})$, since $\mathcal{G}_{\bar{\mathbf{X}}}$ may still contain edges adjacent to $\mathbf{X}$ that are undirected. In the true causal DAG, the corresponding edges may be directed into $\mathbf{X}$ and thus, represent a causal effect on $\mathbf{X}$. Second, these mutilated graphs may no longer be MPDAGs. For example, consider an MPDAG $\mathcal{G}$ that contains the paths $V_1 \rightarrow V_2 - X$ and $V_1 \rightarrow X$. The graph $\mathcal{G}_{\bar{\mathbf{X}}}$ only contains the path $V_1 \rightarrow V_2 - X$, and thus, is not closed under Meek’s (1995) orientation rules. In Appendix C.6, we note a third, more technical difference useful in the proofs of Theorem 24. With this discussion, we turn to our result.

Theorem 24 (Do Calculus for MPDAGs) *Let $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$, and $\mathbf{W}$ be pairwise disjoint node sets in a causal MPDAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$. Let $\mathcal{G}_{\bar{\mathbf{X}}\mathbf{Z}}$ denote the graph obtained by deleting all edges into $\mathbf{X}$ and all edges out of $\mathbf{Z}$ from $\mathcal{G}$. We write $\mathcal{G}_{\bar{\mathbf{X}}\mathbf{Z}}$ as $\mathcal{G}_{\mathbf{Z}}$ when $\mathbf{X}$ is empty and as $\mathcal{G}_{\bar{\mathbf{X}}}$ when $\mathbf{Z}$ is empty. Then the following hold for all densities f consistent with $\mathcal{G}$.*

Rule 1. If $(\mathbf{Y} \perp_d \mathbf{Z} | \mathbf{X}, \mathbf{W})_{\mathcal{G}_{\overline{\mathbf{X}}}}$, then

$$f(\mathbf{y} | do(\mathbf{x}), \mathbf{z}, \mathbf{w}) = f(\mathbf{y} | do(\mathbf{x}), \mathbf{w}). \quad (4.6)$$

Rule 2. If $(\mathbf{Y} \perp_d \mathbf{Z} | \mathbf{X}, \mathbf{W})_{\mathcal{G}_{\overline{\mathbf{X}\mathbf{Z}}}}$, then

$$f(\mathbf{y} | do(\mathbf{z}), \mathbf{w}, do(\mathbf{x})) = f(\mathbf{y} | \mathbf{z}, \mathbf{w}, do(\mathbf{x})). \quad (4.7)$$

Rule 3. If $(\mathbf{Y} \perp_d \mathbf{Z} | \mathbf{X}, \mathbf{W})_{\mathcal{G}_{\overline{\mathbf{X}}, \overline{\mathbf{Z}'(\mathbf{W})}}}$, then

$$f(\mathbf{y} | do(\mathbf{z}), \mathbf{w}, do(\mathbf{x})) = f(\mathbf{y} | \mathbf{w}, do(\mathbf{x})), \quad (4.8)$$

where $\mathbf{Z}'(\mathbf{W}) := \mathbf{Z} \setminus \text{PossAn}(\mathbf{W}, \mathcal{G}_{\mathbf{V} \setminus \mathbf{X}})$.

Proof Sketch. Theorem 24 follows from the do calculus of Pearl [2009] as well as Lemmas 114, 116, and 119. These lemmas and their proofs can be found in Appendix C.6, but we provide an overview here for intuition. Consider Rule 1 as an example. The result follows directly from Lemma 114, which claims that if the given d-separation – $\mathbf{Y} \perp_d \mathbf{Z} | \mathbf{X}, \mathbf{W}$ – holds in the mutilated graph $\mathcal{G}_{\overline{\mathbf{X}}}$, then the same d-separation will hold in $\mathcal{D}_{\overline{\mathbf{X}}}$ for any DAG $\mathcal{D}$ in the equivalence class $[\mathcal{G}]$. We prove this by contraposition. Thus, we start by assuming there is a DAG $\mathcal{D} \in [\mathcal{G}]$ such that $\mathcal{D}_{\overline{\mathbf{X}}}$ has a path $\bar{p}$ from $\mathbf{Y}$ to $\mathbf{Z}$ that is open given $\mathbf{X} \cup \mathbf{W}$. By definition, no non-collider on $\bar{p}$ is in $\mathbf{X} \cup \mathbf{W}$. We then show that every collider on $\bar{p}$ is in $\text{An}(\mathbf{X} \cup \mathbf{W}, \mathcal{G}_{\overline{\mathbf{X}}})$. To do this, we draw repeatedly on the relationships between $\mathcal{D}_{\overline{\mathbf{X}}}$, $\mathcal{D}$, $\mathcal{G}$, and $\mathcal{G}_{\overline{\mathbf{X}}}$. We prove this implies that the sequence of nodes in $\mathcal{G}_{\overline{\mathbf{X}}}$ corresponding to $\bar{p}$ forms a path from $\mathbf{Y}$ to $\mathbf{Z}$ that is open given $\mathbf{X} \cup \mathbf{W}$, which completes the proof. We rely on a similar contraposition for showing that Rules 2 and 3 hold. ■

Before providing examples, we pause to compare Theorem 24 with the do calculus of Pearl [2009]. Note that since all DAGs are MPDAGs, Theorem 24 holds when a full DAG is known. We want to highlight that our do calculus and Pearl’s are identical in this case. This claim follows without proof after considering a small detail found in Rule 3. Compare the definition

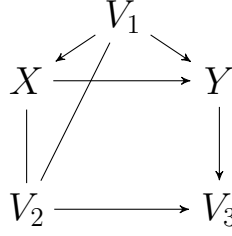


Figure 4.3: MPDAG used in Example 21

of $\mathbf{Z}'(\mathbf{W})$ from Theorem 24 with that of $\mathbf{Z}(\mathbf{W})$ from Theorem 75. To show these sets are equivalent when a DAG $\mathcal{G}$ is known, we must show that $\mathbf{Z} \setminus \text{An}(\mathbf{W}, \mathcal{G}_{\mathbf{V} \setminus \mathbf{X}}) = \mathbf{Z} \setminus \text{An}(\mathbf{W}, \mathcal{G}_{\overline{\mathbf{X}}})$. To see this, note that $\text{An}(\mathbf{W}, \mathcal{G}_{\mathbf{V} \setminus \mathbf{X}}) \subseteq \text{An}(\mathbf{W}, \mathcal{G}_{\overline{\mathbf{X}}})$, where the only variables in $\text{An}(\mathbf{W}, \mathcal{G}_{\overline{\mathbf{X}}})$ not in $\text{An}(\mathbf{W}, \mathcal{G}_{\mathbf{V} \setminus \mathbf{X}})$ are in $\mathbf{X}$. But $\mathbf{Z}$ and $\mathbf{X}$ are disjoint. Therefore, removing $\text{An}(\mathbf{W}, \mathcal{G}_{\mathbf{V} \setminus \mathbf{X}})$ from $\mathbf{Z}$ is equivalent to removing $\text{An}(\mathbf{W}, \mathcal{G}_{\overline{\mathbf{X}}})$ from $\mathbf{Z}$, and the claim holds.

4.4.2 EXAMPLES

The rules of Theorem 24 offer a broad tool for transforming interventional densities in the MPDAG setting. An appealing use of this tool is identifying a conditional causal effect. We demonstrate how to use Theorem 24 for this purpose in the examples below. Example 20 does this with the help of well-known probability rules, and it does this for an effect that our identification formula was able to find. We provide Examples 21 and 22 to highlight that our do calculus is able to find conditional effects that our identification formula cannot.

Example 20 (Multiple Routes) *Reconsider Example 17, where we found the form of a conditional effect using our identification formula. We could instead use Theorem 24:*

$$\begin{aligned}
 f(\mathbf{y}|\text{do}(\mathbf{x}), \mathbf{z}) &= \int f(y, v_2|\text{do}(x), v_1) dv_2 \\
 &= \int f(y|\text{do}(x), v_1, v_2) f(v_2|\text{do}(x), v_1) dv_2 \\
 &= \int f(y|x, v_1, v_2) f(v_2|v_1) dv_2,
 \end{aligned}$$

The first two equalities hold by the law of total probability and the chain rule, respectively. The third holds by Rules 2 and 3, since $Y \perp_d X \mid V_1, V_2$ in $\mathcal{G}_{\underline{\mathbf{X}}}$ and $V_2 \perp_d X \mid V_1$ in $\mathcal{G}_{\overline{\mathbf{X}}}$. This example provides a case where all three methods – conditional adjustment, the identification formula, and the do calculus for MPDAGs – find the same form for the conditional effect.

Example 21 (Using Rule 2) Let $\mathcal{G}$ be the causal MPDAG in Figure 4.3, and suppose we want to identify the causal effect of X on Y given $\{V_1, V_2\}$. Note that we cannot use Theorem 21, since $\{V_1, V_2\} \cap \text{PossDe}(X, \mathcal{G}) \neq \emptyset$. However, we can use Theorem 24 to show

$$f(y \mid \text{do}(x), v_1, v_2) = f(y \mid x, v_1, v_2).$$

The equality follows from Rule 2, since $Y \perp_d X \mid V_1, V_2$ in $\mathcal{G}_{\underline{\mathbf{X}}}$.

Example 22 (Using Rule 3) Let $\mathcal{G}$ be a causal MPDAG containing only $X - Z \rightarrow Y$, and suppose we want to identify the causal effect of X on Y given Z . Note that we cannot use Theorems 21 or 22, since $Z \in \text{PossDe}(X, \mathcal{G})$. However, we can use Theorem 24 to show

$$f(y \mid \text{do}(x), z) = f(y \mid z).$$

This holds by Rule 3, since $Y \perp_d X \mid Z$ in $\mathcal{G}_{\overline{\mathbf{X}}}$.

4.5 IDENTIFICATION ALGORITHM

In Section 4.3, we introduced our conditional identification formula for MPDAGs (Theorem 21), which is able to identify causal effects when the conditioning set is unaffected by treatment. In Section 4.4, we introduced our do calculus for MPDAGs (Theorem 24) – a broad tool that is able to identify further effects.

In this section, we introduce our conditional identification algorithm for MPDAGs (Algorithm 1), which combines Theorems 21 and 24. This algorithm does not restrict the conditioning set, and therefore, can identify further effects than our identification formula alone. In fact, we show below (Section 4.5.4) that Algorithm 1 is complete, and thus, can find any conditional effect that is identifiable given knowledge of an MPDAG.

Algorithm 1: Conditional Identification for MPDAGs (CIDM)

Input : Disjoint node sets $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \subseteq \mathbf{V}$ and causal MPDAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$

Output: Expression for $f(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z})$, for any f consistent with $\mathcal{G}$, or FAIL

```

1 Let  $\mathbf{Z}' = \mathbf{Z}$  and  $\mathbf{X}' = \mathbf{X}$ ;
2 While  $\exists$  a proper possibly causal path  $\mathbf{X}'$  to  $\mathbf{Y} \cup \mathbf{Z}'$  in  $\mathcal{G}$  that starts undirected
3   | Pick  $X \in \mathbf{X}'$  on such a path;
4   | If  $(\mathbf{Y} \perp_d X \mid \mathbf{X}' \setminus \{X\}, \mathbf{Z}')_{\mathcal{G}_{\mathbf{X}' \setminus \{X\}, \mathbf{X}}}$ 
5   |   | Add  $X$  to  $\mathbf{Z}'$ ;
6   |   | Remove  $X$  from  $\mathbf{X}'$ ;
7   | else
8   |   | Return FAIL;
9 If  $(\mathbf{Y} \perp_d \mathbf{X}' \mid \mathbf{Z}')_{\mathcal{G}_{\mathbf{X}'(\mathbf{Z}')}} where  $\mathbf{X}'(\mathbf{Z}') := \mathbf{X}' \setminus \text{PossAn}(\mathbf{Z}', \mathcal{G})$ 
10 |   | Return  $f(\mathbf{y} \mid \mathbf{z}')$ ;
11 else
12 |   | Let  $\mathbf{Z}^D = \mathbf{Z}' \cap \text{PossDe}(\mathbf{X}', \mathcal{G})$ ;
13 |   | Let  $\mathbf{Z}^N = \mathbf{Z}' \setminus \text{PossDe}(\mathbf{X}', \mathcal{G})$ ;
14 |   | Let  $A$  = identification formula for  $f(\mathbf{y}, \mathbf{z}^D \mid do(\mathbf{x}'), \mathbf{z}^N)$ ;
15 |   | Let  $B$  = identification formula for  $f(\mathbf{z}^D \mid do(\mathbf{x}'), \mathbf{z}^N)$ ;
16 |   | Return  $\frac{A}{B}$ ;$ 
```

Applies Rule 2 of
Theorem 24

Applies Rule 3 of
Theorem 24

Applies
Theorem 21

We start this section by describing the algorithm, and then we provide examples of how to use the algorithm for identifying conditional effects, when possible. We close this section with a discussion of the algorithm's completeness.

4.5.1 THE ALGORITHM

Our identification algorithm can be found in Algorithm 1. It begins with pairwise disjoint node sets $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ from a causal MPDAG $\mathcal{G}$. When possible, it outputs an identifying form for $f(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z})$, where f is any density consistent with $\mathcal{G}$. It does this in two broad steps: (1) manipulating $f(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z})$ in preparation for identification, and (2) identifying $f(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z})$. We describe these steps in more detail below.

Step (1) happens inside the **while** loop of lines 2-8. Here the algorithm successively transfers nodes from $do(\mathbf{X})$ into $\mathbf{Z}$ using Rule 2 of the do calculus for MPDAGs (Theorem 24). The **while** loop ends once there is no longer a proper possibly causal path from $\mathbf{X}'$ to $\mathbf{Y} \cup \mathbf{Z}'$ that starts undirected. In Section 4.5.4, we show the absence of such a path is necessary for identification. Thus, if there is a node $X \in \mathbf{X}'$ with a possibly causal path to $\mathbf{Y} \cup \mathbf{Z}'$ that starts undirected and which Rule 2 of the do calculus cannot transfer from $do(\mathbf{X})$ into $\mathbf{Z}$, then the algorithm outputs a **FAIL**.

Step (2) happens inside the **if-else** statement of lines 9-16. It begins by checking if the d-separation in Rule 3 of our do calculus holds. If so, then it provides the identifying expression $f(\mathbf{y} | do(\mathbf{x}), \mathbf{z}) = f(\mathbf{y} | \mathbf{z}')$. This step is not required for implementation, but we offer it as an off-ramp for a likely easy-to-estimate form of the density. If Rule 3 does not apply, then the algorithm relies on our identification formula (Theorem 21) in the following way. Lines 12-13 separate $\mathbf{Z}'$ into variables that *are* and *are not* possible descendants of $\mathbf{X}'$ – namely, $\mathbf{Z}^D$ and $\mathbf{Z}^N$. Then lines 14-16 apply Theorem 21 based on the following insight:

$$f(\mathbf{y} | do(\mathbf{x}'), \mathbf{z}') = \frac{f(\mathbf{y}, \mathbf{z}^D | do(\mathbf{x}'), \mathbf{z}^N)}{f(\mathbf{z}^D | do(\mathbf{x}'), \mathbf{z}^N)}. \quad (4.9)$$

This holds by the chain rule, where the denominator is non-zero since we assume positivity (see Section 4.2). Thus, the algorithm applies Theorem 21 to both the numerator and denominator of (4.9). To see that this is allowed, note that $\mathbf{Z}^N \cap \text{PossDe}(\mathbf{X}', \mathcal{G}) = \emptyset$, and by Step (1), there is no proper possibly causal path from $\mathbf{X}'$ to $\mathbf{Y} \cup \mathbf{Z}^D$ in $\mathcal{G}$ that starts undirected.

4.5.2 EXAMPLES

In the examples below, we use Algorithm 1 in an attempt to identify the conditional causal effect of $\mathbf{X}$ on $\mathbf{Y}$ given $\mathbf{Z}$. In Examples 23 and 24, Algorithm 1 succeeds in finding $f(\mathbf{y} | do(\mathbf{x}), \mathbf{z})$. But the algorithm outputs a **FAIL** in Example 25. Note that all three examples assume knowledge of an MPDAG $\mathcal{G}$ where $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) \neq \emptyset$ and thus, our identification formula (Theorem 21) cannot be applied directly.

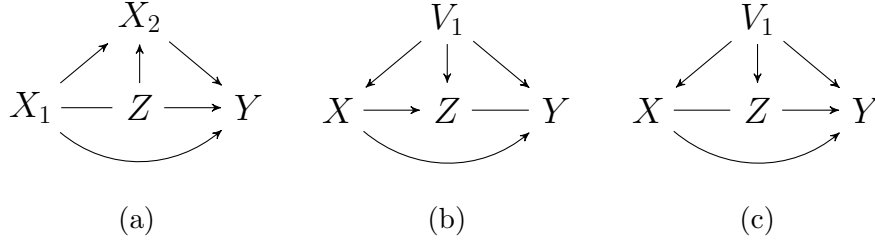


Figure 4.4: MPDAGs used in Examples 23-25

Example 23 (Non-fractional Form) Let $\mathcal{G}$ be the causal MPDAG in Figure 4.4((a)), and let $\mathbf{X} = \{X_1, X_2\}$, $\mathbf{Y} = \{Y\}$, and $\mathbf{Z} = \{Z\}$. To use Algorithm 1, we first consider the **while** loop of lines 2-8. Since there is a possibly causal path from $X_1 \in \mathbf{X}$ to $\mathbf{Y} \cup \mathbf{Z}$ that starts undirected and does not contain X_2 , we confirm that $\mathbf{Y} \perp_d X_1 \mid X_2, \mathbf{Z}$ in $\mathcal{G}_{\overline{X_2 X_1}}$. Thus, we define $\mathbf{X}' = \{X_2\}$ and $\mathbf{Z}' = \{X_1, Z\}$. Now that all possibly causal paths from $\mathbf{X}'$ to $\mathbf{Y} \cup \mathbf{Z}'$ start directed, we exit the **while** loop. Then we skip lines 9-10 since $\mathbf{Y} \not\perp_d \mathbf{X}' \mid \mathbf{Z}'$ in $\mathcal{G}_{\overline{\mathbf{X}'}}$. Finally, by lines 12-16,

$$\begin{aligned} f(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z}) &= f(y \mid do(x_2), x_1, z) \\ &= f(y \mid x_1, x_2, z). \end{aligned}$$

Example 24 (Fractional Form) Let $\mathcal{G}$ be the causal MPDAG in Figure 4.4((b)), and let $\mathbf{X} = \{X\}$, $\mathbf{Y} = \{Y\}$, and $\mathbf{Z} = \{Z\}$. To use Algorithm 1, we first bypass the **while** loop of lines 2-8, since all possibly causal paths from $\mathbf{X}$ to $\mathbf{Y} \cup \mathbf{Z}$ start directed. We also skip lines 9-10 since $\mathbf{Y} \not\perp_d \mathbf{X} \mid \mathbf{Z}$ in $\mathcal{G}$. Thus, by lines 12-16,

$$f(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z}) = \frac{\int f(y, z \mid v_1, x) f(v_1) dv_1}{\int f(z \mid v_1, x) f(v_1) dv_1}.$$

Example 25 (Not Identifiable) Let $\mathcal{G}$ be the causal MPDAG in Figure 4.4((c)), and let $\mathbf{X} = \{X\}$, $\mathbf{Y} = \{Y\}$, and $\mathbf{Z} = \{Z\}$. To use Algorithm 1, we first consider the **while** loop of lines 2-8. Note that there is a possibly causal path from $X \in \mathbf{X}$ to $\mathbf{Y} \cup \mathbf{Z}$ that starts undirected, but $\mathbf{Y} \not\perp_d X \mid \mathbf{Z}$ in $\mathcal{G}_{\underline{\mathbf{X}}}$. Thus, the algorithm returns a **FAIL**. In Section 4.5.4, we show this effect is not identifiable in $\mathcal{G}$ (see Example 29).

4.5.3 FURTHER EXAMPLES

The three examples above rely on our identification algorithm by walking through steps that our other methods would not. But we want to highlight that since Algorithm 1 combines our identification formula with our do calculus for MPDAGs, applying our algorithm sometimes amounts to no more than applying one of these methods alone. We demonstrate this in the examples below, which revisit examples from Sections 4.3 and 4.4.

Example 26 (Reduces to Identification Formula) *Reconsider Example 17, but now use Algorithm 1 to identify the causal effect. We skip lines 2-10, since the following hold: $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$, no proper possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$ starts undirected, and $\mathbf{Y} \not\perp_d \mathbf{X} | \mathbf{Z}$ in $\mathcal{G}_{\overline{\mathbf{X}}}$. Thus, the algorithm reduces to lines 12-16, and since $\mathbf{Z}^{\mathbf{N}} = \mathbf{Z}$, these lines reduce to the same application of our identification formula shown in Example 17. Note that this logic also applies to Example 18.*

Example 27 (Reduces to Rule 3) *Reconsider Example 19, but now use Algorithm 1 to identify the causal effect. We skip lines 2-8, since there is no possibly causal path from $\mathbf{X}$ to $\mathbf{Y} \cup \mathbf{Z}$. From lines 9-10, we confirm that $\mathbf{Y} \perp_d \mathbf{X} | \mathbf{Z}$ in $\mathcal{G}_{\overline{\mathbf{X}}}$. Thus, the algorithm reduces to an application of Rule 3 of our do calculus. In this case, note that this is equivalent to the application of Theorem 22 shown in Example 19.*

Example 28 (Reduces to Rule 2) *Reconsider Example 21. To use Algorithm 1, we first consider the **while** loop of lines 2-8. Since there is a possibly causal path from $X \in \mathbf{X}$ to $\mathbf{Y} \cup \mathbf{Z}$ that starts undirected, we confirm that $\mathbf{Y} \perp_d X | \mathbf{Z}$ in $\mathcal{G}_{\underline{\mathbf{X}}}$. Thus, we define $\mathbf{X}' = \emptyset$ and $\mathbf{Z}' = \{X, Z\}$. Formally, the algorithm must exit the **while** loop and consider lines 9-10, but note that it has already found an identifying form (that the additional steps will not change). Thus, the algorithm reduces to the same application of Rule 2 of our do calculus shown in Example 21. Additionally note that the same logic applies to Example 22.*

4.5.4 SOUNDNESS AND COMPLETENESS

Algorithm 1 is sound for identifying conditional causal effects given an MPDAG. That is, any expression the algorithm outputs will be an identifying form of $f(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z})$. This follows directly from Rules 2-3 of the do calculus for MPDAGs (Theorem 24), the chain rule, our positivity assumption (see Section 4.2), and our identification formula (Theorem 21).

Algorithm 1 is also complete for identifying conditional causal effects given an MPDAG. To prove this, we offer the theorem below. This result shows that when our algorithm outputs a **FAIL**, the effect is not identifiable. That is, our algorithm is able to find an identifying form for any identifiable effect.

Theorem 25 (Completeness of Algorithm 1) *Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ be pairwise disjoint node sets in a causal MPDAG $\mathcal{G}$. If there is a proper possibly causal path from $\mathbf{X}$ to $\mathbf{Y} \cup \mathbf{Z}$ in $\mathcal{G}$ that starts with an undirected edge and contains any $X \in \mathbf{X}$ such that $\mathbf{Y} \not\perp_d X \mid \mathbf{X} \setminus \{X\}, \mathbf{Z}$ in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}} \underline{X}}$, then the conditional causal effect of $\mathbf{X}$ on $\mathbf{Y}$ given $\mathbf{Z}$ is not identifiable in $\mathcal{G}$.*

Proof Sketch. The proof of Theorem 25 can be found in Appendix C.7, but we provide an overview here for intuition. We begin the proof by assuming that a possibly causal path – as given in the statement of Theorem 25 – exists in the MPDAG $\mathcal{G}$. Based on the given d-separation between $X \in \mathbf{X}$ and $\mathbf{Y}$, we consider the following non-empty sets:

$$\begin{aligned} \mathbf{S}_1 &= \left\{ \begin{array}{l} \text{definite status paths in } \mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}} \underline{X}} \text{ from } X \text{ to } \mathbf{Y} \text{ that are} \\ \text{d-connecting given } \mathbf{X} \setminus \{X\} \cup \mathbf{Z} \end{array} \right\} \text{ and} \\ \mathbf{S}_2 &= \{ \text{paths in } \mathbf{S}_1 \text{ with the fewest colliders} \}. \end{aligned} \tag{4.10}$$

The remainder of the proof falls into two cases: (a) when $\mathbf{S}_2$ only contains paths that start directed (i.e., of the form $X \leftarrow \dots Y$) and (b) when $\mathbf{S}_2$ contains paths that start undirected (i.e., of the form $X - \dots Y$). Rather than providing an overview of each case, we offer two examples below. Example 29 considers an MPDAG with a path that falls into (a), and Example 30 considers an MPDAG with a path that falls into (b). In both examples, we

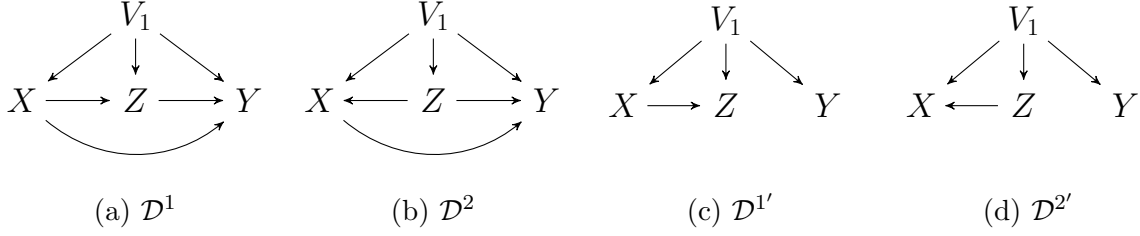


Figure 4.5: DAGs used in Example 29

show that the conditional causal effect is not identifiable. ■

Example 29 (Path in $\mathbf{S}_2$ that Starts Directed) *Reconsider Example 25, where we attempted to use Algorithm 1 to identify the conditional effect of $\mathbf{X}$ on $\mathbf{Y}$ given $\mathbf{Z}$ but the algorithm output a **FAIL**. Note that the MPDAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ in this example contains the path $X \leftarrow V_1 \rightarrow Y$, which belongs to the set $\mathbf{S}_2$ given in (4.10) and which starts directed.*

To show the conditional effect is not identifiable in $\mathcal{G}$, consider the DAGs $\mathcal{D}^1, \mathcal{D}^2 \in [\mathcal{G}]$ shown in Figure 4.5((a))-((b)). It suffices to show that each $\mathcal{D}^i$ is compatible with a family of interventional densities $\mathbf{F}_1^ = \{f_i(\mathbf{v} | do(\mathbf{x}')) : \mathbf{X}' \subseteq \mathbf{V}\}$ such that the following hold.*

$$f_1(\mathbf{v}) = f_2(\mathbf{v}). \quad (4.11)$$

$$f_1(\mathbf{y} | do(\mathbf{x}), \mathbf{z}) \neq f_2(\mathbf{y} | do(\mathbf{x}), \mathbf{z}). \quad (4.12)$$

To define such families, consider the DAG $\mathcal{D}^{1'}$ in Figure 4.5((c)), and note that $\mathcal{D}^{1'}$ is constructed by removing edges from $\mathcal{D}^1$. Below we build an $\mathbf{F}_1^$ such that $\mathcal{D}^{1'}$ is compatible with $\mathbf{F}_1^*$. It follows that $\mathcal{D}^1$ is compatible with $\mathbf{F}_1^*$ (see Lemma 108, App. C.5). Start by considering the following structural equation model (SEM) with independent errors:*

$$\begin{aligned} V_1 &\leftarrow \varepsilon_{v_1}; & \varepsilon_{v_1} &\sim \mathcal{N}(0, 1); \\ X &\leftarrow \frac{1}{2}V_1 + \varepsilon_x; & \varepsilon_x &\sim \mathcal{N}(0, \frac{3}{4}); \\ Z &\leftarrow \frac{1}{2}V_1 + \frac{1}{2}X + \varepsilon_z; & \varepsilon_z &\sim \mathcal{N}(0, \frac{1}{4}); \end{aligned} \quad (4.13)$$

$$Y \leftarrow \frac{1}{2}V_1 + \varepsilon_y; \quad \varepsilon_y \sim \mathcal{N}(0, \frac{3}{4}).$$

We build $\mathbf{F}_1^*$ by letting $f_1(\mathbf{v})$ be the density of the multivariate normal generated by the SEM in (4.13). For the remaining densities in $\mathbf{F}_1^*$, we let $f_1(\mathbf{v} \mid do(\mathbf{x}')) := f_*(\mathbf{v})$, where f_* is the density of the multivariate normal generated by taking the SEM in (4.13) and replacing $\mathbf{X}'$ with its interventional value $\mathbf{x}'$ [Pearl, 2009].

Now consider the DAG $\mathcal{D}^{2'}$ in Figure 4.5((d)), and note that $\mathcal{D}^{2'}$ is constructed by removing edges from $\mathcal{D}^2$. Next consider the following SEM with independent errors:

$$\begin{aligned} V_1 &\leftarrow \varepsilon_{v_1}; & \varepsilon_{v_1} &\sim \mathcal{N}(0, 1); \\ X &\leftarrow -\frac{1}{7}V_1 + \frac{6}{7}Z + \varepsilon_x; & \varepsilon_x &\sim \mathcal{N}(0, \frac{3}{7}); \\ Z &\leftarrow \frac{3}{4}V_1 + \varepsilon_z; & \varepsilon_z &\sim \mathcal{N}(0, \frac{7}{16}); \\ Y &\leftarrow \frac{1}{2}V_1 + \varepsilon_y; & \varepsilon_y &\sim \mathcal{N}(0, \frac{3}{4}), \end{aligned} \tag{4.14}$$

Just as we built $\mathbf{F}_1^*$ based on the SEM in (4.13), we build $\mathbf{F}_2^*$ based on the SEM in (4.14). It follows that $\mathcal{D}^{2'}$ – and therefore $\mathcal{D}^2$ – is compatible with $\mathbf{F}_2^*$.

Thus we have two families of interventional densities such that $\mathcal{D}^i$ is compatible with $\mathbf{F}_i^*$. Further, we can show that $f_1(\mathbf{v}) = f_2(\mathbf{v})$ so that (4.11) holds. To complete the proof, we show that (4.12) holds. It suffices to show that $E[Y \mid do(X = 1), Z = 0]$ is not the same under f_1 and f_2 . We start by calculating this expectation under f_2 :

$$\begin{aligned} \mathbb{E}_{f_2}[Y \mid do(X = 1), Z = 0] &= \mathbb{E}_{f_2}[Y \mid Z] \Big|_{Z=0} \\ &= \mathbb{E}_{f_2}[Y] + \frac{\text{Cov}_{f_2}(Y, Z)}{\text{Var}_{f_2}(Z)} \cdot \left(Z - \mathbb{E}_{f_2}[Z] \right) \Big|_{Z=0} \\ &= \frac{3}{8}Z \Big|_{Z=0} \\ &= 0. \end{aligned}$$

The first equality follows from Pearl's do calculus (Theorem 75), since $Y \perp_d X \mid Z$ in $\mathcal{D}_X^{2'}$ and since f_2 is consistent with $\mathcal{D}^{2'}$. The second equality follows from properties of multivariate normals (see Lemma 76).

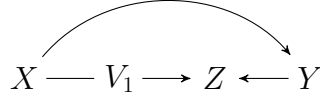


Figure 4.6: MPDAG used in Example 30

To calculate the same expectation under f_1 , recall that $f_1(\mathbf{v} \mid do(x)) := f_*(\mathbf{v})$, where $f_*(\mathbf{v})$ is the multivariate normal density generated by the following SEM with independent errors:

$$\begin{aligned}
 V_1 &\leftarrow \varepsilon_{v_1}; & \varepsilon_{v_1} &\sim \mathcal{N}(0, 1); \\
 X &\leftarrow x; \\
 Z &\leftarrow \frac{1}{2}V_1 + \frac{1}{2}x + \varepsilon_z; & \varepsilon_z &\sim \mathcal{N}(0, \frac{1}{4}); \\
 Y &\leftarrow \frac{1}{2}V_1 + \varepsilon_y; & \varepsilon_y &\sim \mathcal{N}(0, \frac{3}{4}),
 \end{aligned} \tag{4.15}$$

It follows that $f_1(y \mid do(x), z) := f_*(y \mid z)$ is the conditional density of $Y \mid Z$ under $f_*(\mathbf{v})$. Based on this and properties of multivariate normals (see Lemma 76),

$$\begin{aligned}
 \mathbb{E}_{f_1}[Y \mid do(X = 1), Z = 0] &= \mathbb{E}_{f_*}[Y \mid Z] \Big|_{x=1, Z=0} \\
 &= \mathbb{E}_{f_*}[Y] + \frac{\text{Cov}_{f_*}(Y, Z)}{\text{Var}_{f_*}(Z)} \cdot \left(Z - \mathbb{E}_{f_*}[Z] \right) \Big|_{x=1, Z=0} \\
 &= \frac{1}{2} \left(Z - \frac{1}{2}x \right) \Big|_{x=1, Z=0} \\
 &\neq 0.
 \end{aligned}$$

Example 30 (Path in $\mathbf{S}_2$ that Starts Undirected) Let $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ be the MPDAG in Figure 4.6. If we attempt to identify the conditional effect of X on Y given Z , Algorithm 1 outputs a **FAIL**. We want to show this effect is not identifiable in $\mathcal{G}$. To do this, we follow the same strategy as Example 29, and thus, exclude duplicate technical details. However, note that – unlike Example 29 – $\mathcal{G}$ contains the path $X - V_1 \rightarrow Z \leftarrow Y$, which belongs to the set $\mathbf{S}_2$ given in (4.10) and which starts undirected.

Let $\mathcal{D}^1, \mathcal{D}^2$ be the DAGs in $[\mathcal{G}]$ with edges $X \rightarrow V_1$ and $X \leftarrow V_1$, respectively. Then let $\mathcal{D}^{1'}, \mathcal{D}^{2'}$ be the DAGs constructed by removing the edge $X \rightarrow Y$ from $\mathcal{D}^1, \mathcal{D}^2$, so that

$$\mathcal{D}^{1'} : X \rightarrow V_1 \rightarrow Z \leftarrow Y, \text{ and}$$

$$\mathcal{D}^{2'} : X \leftarrow V_1 \rightarrow Z \leftarrow Y.$$

Define $\mathbf{F}_1^*$ (as in Example 29) based on the following SEM with independent errors:

$$\begin{aligned} X &\leftarrow \varepsilon_x; & \varepsilon_x &\sim \mathcal{N}(0, 1); \\ V_1 &\leftarrow \frac{1}{2}X + \varepsilon_{v_1}; & \varepsilon_{v_1} &\sim \mathcal{N}(0, \frac{3}{4}); \\ Z &\leftarrow \frac{1}{2}V_1 + \frac{1}{2}Y + \varepsilon_z; & \varepsilon_z &\sim \mathcal{N}(0, \frac{1}{2}); \\ Y &\leftarrow \varepsilon_y; & \varepsilon_y &\sim \mathcal{N}(0, 1). \end{aligned}$$

Analogously, define $\mathbf{F}_2^*$ based on the following SEM with independent errors:

$$\begin{aligned} X &\leftarrow \frac{1}{2}V_1 + \varepsilon_x; & \varepsilon_x &\sim \mathcal{N}(0, \frac{3}{4}); \\ V_1 &\leftarrow \varepsilon_{v_1}; & \varepsilon_{v_1} &\sim \mathcal{N}(0, 1); \\ Z &\leftarrow \frac{1}{2}V_1 + \frac{1}{2}Y + \varepsilon_z; & \varepsilon_z &\sim \mathcal{N}(0, \frac{1}{2}); \\ Y &\leftarrow \varepsilon_y; & \varepsilon_y &\sim \mathcal{N}(0, 1). \end{aligned}$$

Thus we have two families of interventional densities where $\mathcal{D}^i$ is compatible with $\mathbf{F}_i^*$ and where we can show $f_1(\mathbf{v}) = f_2(\mathbf{v})$. To complete the proof, we show that $E[Y \mid \text{do}(X = 1), Z = 0]$ is not the same under f_1 and f_2 .

$$\mathbb{E}_{f_1}[Y \mid \text{do}(X = 1), Z = 0] = \mathbb{E}_{f_1}[Y \mid X = 1, Z = 0] = -\frac{2}{15}X + \frac{8}{15}Z \Big|_{X=1, Z=0} \neq 0.$$

$$\mathbb{E}_{f_2}[Y \mid \text{do}(X = 1), Z = 0] = \mathbb{E}_{f_2}[Y \mid Z = 0] = \frac{1}{2}Z \Big|_{Z=0} = 0.$$

The first equality of each line follows from Rules 2 and 3 of Pearl's do calculus (Theorem 75), respectively, since $Y \perp_d X \mid Z$ in $\mathcal{D}_{\underline{X}}^{1'}$; $Y \perp_d X \mid Z$ in $\mathcal{D}_{\underline{X}}^{2'}$; and f_i is consistent with $\mathcal{D}^{i'}$. The remaining equalities follow from properties of multivariate normals (see Lemma 76).

4.6 DISCUSSION

In this paper, we propose strategies to identify a conditional causal effect from observational data. We assume knowledge of an MPDAG, a graph that can be learned from observational data and that allows for the addition of background knowledge. Our results include a/an

- identification formula (Theorem 21) that is sound and complete for identifying a conditional effect when the conditioning set is unaffected by treatment (Proposition 23),
- do calculus for MPDAGs (Theorem 24) that is sound for identifying a conditional effect, and
- identification algorithm (Algorithm 1) that is sound and complete for identifying a conditional effect (Theorem 25) and that combines our preceding results.

Our work builds on prior research in causal identification, and we detail these connections in the sections above. Most directly, our identification formula broadens the scope of Perković’s (2020) formula for unconditional effects given an MPDAG. Similarly, our do calculus is comparable to Zhang [2008a] and Jaber et al.’s (2022) calculus for PAGs. And our identification algorithm (**CIDM**) relates to a broad history of conditional identification algorithms, such as the IDC algorithm of Shpitser and Pearl [2006] and the **CIDP** algorithm of Jaber et al. [2022].

The main limitation of our work compared to the results of Shpitser and Pearl [2006], Zhang [2008a], and Jaber et al. [2022] is that we consider a setting that does not allow latent confounding. But we want to highlight that research on identification in these graphs (e.g., PAGs) does not cover the general MPDAG setting. For example, our **CIDM** algorithm cannot be seen as a simplification of the **CIDP** algorithm of Jaber et al. [2022]. That is, a naive translation of the **CIDP** algorithm to the MPDAG setting would not be complete for identification given an MPDAG (see Example 31, in App. C.8). This holds since the completeness of the **CIDM** algorithm relies on properties of PAGs (e.g., undirected components

must be chordal, undirected paths must be possibly causal) that do not hold generally for MPDAGs.

However, a key advantage of our work is that we consider a setting that allows the addition of expert knowledge, which follows research on the MPDAG setting seen in Perković [2020] and LaPlante and Perković [2024]. This process of adding expert knowledge to causal graphs is largely excluded from research on identification in causal PAGs (e.g., Zhang, 2008a, Jaber et al., 2022). Though we point out the work of Venkateswaran and Perković [2024], which considers a setting that allows for both latent variables and the addition of expert knowledge. We hope that our results in combination with those of Venkateswaran and Perković [2024] and LaPlante and Perković [2024] can be used toward future research on conditional identification in the presence of both latent variables and expert knowledge.

Other avenues for future work include estimating conditional effects that are not identifiable given an MPDAG – where our **CIDM** algorithm outputs a **FAIL** – by enumerating graphs represented by the MPDAG, similar to the work of Maathuis et al. [2009], Nandy et al. [2017], Guo and Perković [2021b], and Perković et al. [2017]. Additionally, researchers could explore strategies for efficient estimation of conditional effects given an MPDAG, similar to the work of Guo and Perković [2021a] and Guo et al. [2023].

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Appendix A

ADDITIONAL MATERIAL FOR CHAPTER 2 CONDITIONAL ADJUSTMENT

A.1 FURTHER PRELIMINARIES

Path Construction. A *subsequence* of a path p is a path obtained by deleting non-endpoint nodes from p without changing the order of the remaining nodes. Let $p = \langle X_1, X_2, \dots, X_k \rangle$ and i, j, k such that $1 \leq i < j \leq k$. We denote the concatenation of paths by the symbol $\oplus$, so that $p = p(X_1, X_i) \oplus p(X_i, X_k)$. We use the notation $(-p)(X_j, X_i)$ to denote the path $\langle X_j, X_{j-1}, \dots, X_i \rangle$.

Definition 26 (Adjustment Set for MPDAGs (PAGs); Perković et al., 2017, 2018, Perković et al., 2015; cf. Maathuis and Colombo, 2015) *Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{S}$ be pairwise disjoint node sets in a causal MPDAG (PAG) $\mathcal{G}$. Then $\mathbf{S}$ is an adjustment set relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{G}$ if for any density f consistent with $\mathcal{G}$*

$$f(\mathbf{y}|\text{do}(\mathbf{x})) = \begin{cases} f(\mathbf{y}|\mathbf{x}) & \mathbf{S} = \emptyset \\ \int f(\mathbf{y}|\mathbf{x}, \mathbf{s})f(\mathbf{s}) \, d\mathbf{s} & \mathbf{S} \neq \emptyset. \end{cases}$$

Definition 27 (Adjustment Criterion for MPDAGs (PAGs); Perković et al., 2017, 2018) *Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{S}$ be pairwise disjoint node sets in an MPDAG (PAG) $\mathcal{G}$, where every proper possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$ starts with a directed (visible) edge out of $\mathbf{X}$. Then $\mathbf{S}$ satisfies the adjustment criterion relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{G}$ if*

(a) $\mathbf{S} \cap \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) = \emptyset$, and

(b) $\mathbf{S}$ blocks all proper non-causal definite status paths from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$.

Definition 28 (Generalized Back-Door Criterion for DAGs; cf. Maathuis and Colombo, 2015) *Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{S}$ be pairwise disjoint node sets in a DAG $\mathcal{D}$. Then $\mathbf{S}$ satisfies the generalized back-door criterion relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{D}$ if*

- (a) $\mathbf{S} \cap \text{De}(\mathbf{X}, \mathcal{D}) = \emptyset$, and
- (b) $\mathbf{S} \cup \mathbf{X} \setminus \{X\}$ blocks all back-door paths from X to $\mathbf{Y}$ in $\mathcal{D}$, for every $X \in \mathbf{X}$.

Definition 29 (Proper Back-Door Graph for DAGs; cf. Perković et al., 2018) *Let $\mathbf{X}$ and $\mathbf{Y}$ be disjoint node sets in a DAG $\mathcal{D}$. The proper back-door graph $\mathcal{D}_{\mathbf{X}\mathbf{Y}}^{\text{pbd}}$ is obtained from $\mathcal{D}$ by removing all edges out of $\mathbf{X}$ that are on proper causal paths from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{D}$.*

Definition 30 (Moral Graph for DAGs; cf. Lauritzen and Spiegelhalter, 1988; cf. Perković et al., 2018) *Let $\mathcal{D} = (\mathbf{V}, \mathbf{E})$ be a DAG. The moral graph $\mathcal{D}^m$ is formed by adding the edge $A - B$ to any structure of the form $A \rightarrow C \leftarrow B$ for any $A, B, C \in \mathbf{V}$, with $A \notin \text{Adj}(B, \mathcal{D})$ (marrying unmarried parents) and subsequently making all edges in the resulting graph undirected.*

Definition 31 (Distance to $\mathbf{Z}$; Zhang, 2006; Perković et al., 2017) *Let $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$ be pairwise disjoint node sets in an MPDAG or PAG $\mathcal{G}$. Let p be a path between $\mathbf{X}$ and $\mathbf{Y}$ in $\mathcal{G}$ such that every collider C on p has a possibly directed path (possibly of length 0) to $\mathbf{Z}$. Define the distance to $\mathbf{Z}$ of C to be the length of a shortest possibly directed path (possibly of length 0) from C to $\mathbf{Z}$, and define the distance to $\mathbf{Z}$ of p to be the sum of the distances from $\mathbf{Z}$ of the colliders on p .*

A.2 EXISTING RESULTS

Rules of the Do Calculus [Pearl, 2009]. Let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$, and $\mathbf{W}$ be pairwise disjoint (possibly empty) node sets in a causal DAG $\mathcal{D}$. Let $\mathcal{D}_{\overline{\mathbf{X}}}$ denote the graph obtained by deleting all edges into $\mathbf{X}$ from $\mathcal{D}$. Similarly, let $\mathcal{D}_{\underline{\mathbf{X}}}$ denote the graph obtained by deleting all edges out of $\mathbf{X}$ in $\mathcal{D}$, and let $\mathcal{D}_{\overline{\mathbf{X}}\underline{\mathbf{Z}}}$ denote the graph obtained by deleting all edges into $\mathbf{X}$ and all edges out of $\mathbf{Z}$ in $\mathcal{D}$. The following rules hold for all densities consistent with $\mathcal{D}$.

Rule 1. If $(\mathbf{Y} \perp_d \mathbf{Z} \mid \mathbf{X} \cup \mathbf{W})_{\mathcal{D}_{\overline{\mathbf{X}}}}$, then

$$f(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z}, \mathbf{w}) = f(\mathbf{y} \mid do(\mathbf{x}), \mathbf{w}). \quad (\text{A.1})$$

Rule 2. If $(\mathbf{Y} \perp_d \mathbf{X} \mid \mathbf{Z} \cup \mathbf{W})_{\mathcal{D}_{\overline{\mathbf{XW}}}}$, then

$$f(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z}, do(\mathbf{w})) = f(\mathbf{y} \mid \mathbf{x}, \mathbf{z}, do(\mathbf{w})). \quad (\text{A.2})$$

Rule 3. If $(\mathbf{Y} \perp_d \mathbf{X} \mid \mathbf{Z} \cup \mathbf{W})_{\mathcal{D}_{\overline{\mathbf{X}(\mathbf{Z}) \cup \mathbf{W}}}}$, then

$$f(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z}, do(\mathbf{w})) = f(\mathbf{y} \mid \mathbf{z}, do(\mathbf{w})), \quad (\text{A.3})$$

where $\mathbf{X}(\mathbf{Z}) = \mathbf{X} \setminus \text{An}(\mathbf{Z}, \mathcal{D}_{\overline{\mathbf{W}}})$.

Lemma 32 (Wright's Rule of Wright, 1921) *Let $\mathbf{X} = \mathbf{A}\mathbf{X} + \epsilon$, where $\mathbf{Q} \in \mathbb{R}^{k \times k}$, $\mathbf{X} = (X_1, \dots, X_k)^T$ and $\epsilon = (\epsilon_1, \dots, \epsilon_k)^T$ is a vector of mutually independent errors with means zero. Moreover, let $\text{Var}(\mathbf{X}) = \mathbf{I}$. Let $\mathcal{D} = (\mathbf{X}, \mathbf{E})$, be the corresponding DAG such that $X_i \rightarrow X_j$ is in $\mathcal{D}$ if and only if $A_{ji} \neq 0$. A non-zero entry A_{ji} is called the edge coefficient of $X_i \rightarrow X_j$. For two distinct nodes $X_i, X_j \in \mathbf{X}$, let $p_1, \dots, p_r$ be all paths between X_i and X_j in $\mathcal{D}$ that do not contain a collider. Then $\text{Cov}(X_i, X_j) = \sum_{s=1}^r \pi_s$, where π_s is the product of all edge coefficients along path p_s , $s \in \{1, \dots, r\}$.*

Lemma 33 (Theorem 3.2.4 of Mardia et al., 1980) *Let $\mathbf{X} = (\mathbf{X}_1^T, \mathbf{X}_2^T)^T$ be a p -dimensional multivariate Gaussian random vector with mean vector $\mu = (\mu_1^T, \mu_2^T)^T$ and covariance matrix $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$, so that $\mathbf{X}_1$ is a q -dimensional multivariate Gaussian random vector with mean vector μ_1 and covariance matrix Σ_{11} and $\mathbf{X}_2$ is a $(p - q)$ -dimensional multivariate Gaussian random vector with mean vector μ_2 and covariance matrix Σ_{22} . Then $E[\mathbf{X}_2 \mid \mathbf{X}_1 = \mathbf{x}_1] = \mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(\mathbf{x}_1 - \mu_1)$.*

Lemma 34 (cf. Theorem 1 and Proposition 3 of Lauritzen et al., 1990) *Let $\mathcal{D} = (\mathbf{V}, \mathbf{E})$ be a DAG, and let f be an observational density over $\mathbf{V}$. Then f is Markov compatible with $\mathcal{D}$ if and only if*

$$V_i \perp\!\!\!\perp \left[\mathbf{V} \setminus (\text{De}(V_i, \mathcal{D}) \cup \text{Pa}(V_i, \mathcal{D})) \right] \mid \text{Pa}(V_i, \mathcal{D})$$

for all $V_i \in \mathbf{V}$, where $\perp\!\!\!\perp$ indicates independence with respect to f .

Lemma 35 (cf. Lemma 3.2 of Perković et al., 2017) *Let $\mathbf{X}$ and $\mathbf{Z}$ be disjoint node sets in an MPDAG $\mathcal{G}$. If $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$, then $\mathbf{Z} \cap \text{De}(\mathbf{X}, \mathcal{D}) = \emptyset$ in every DAG $\mathcal{D}$ in $[\mathcal{G}]$.*

Lemma 36 (Lemma C.2 of Perković et al., 2017, Lemma 9 of Perković et al., 2018) *Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{S}$ be pairwise disjoint node sets in an MPDAG (PAG) $\mathcal{G}$, where every proper possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$ starts with a directed (visible) edge out of $\mathbf{X}$. Then the following statements are equivalent.*

$$(i) \mathbf{S} \cap \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) = \emptyset.$$

$$(ii) \mathbf{S} \cap \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{D}) = \emptyset \text{ in every DAG (MAG) } \mathcal{D} \text{ in } [\mathcal{G}].$$

Lemma 37 (cf. Lemma C.3 of Perković et al., 2017, Lemma 10 of Perković et al., 2018) *Let $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{S}$ be pairwise disjoint node sets in an MPDAG (PAG) $\mathcal{G}$, where every proper possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$ starts with a directed (visible) edge out of $\mathbf{X}$ and where $\mathbf{S} \cap \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) = \emptyset$. Then the following statements are equivalent.*

$$(i) \mathbf{S} \text{ blocks all proper non-causal definite status paths from } \mathbf{X} \text{ to } \mathbf{Y} \text{ in } \mathcal{G}.$$

$$(ii) \mathbf{S} \text{ blocks all proper non-causal definite status paths from } \mathbf{X} \text{ to } \mathbf{Y} \text{ in } \mathcal{D} \text{ for every DAG (MAG) } \mathcal{D} \text{ in } [\mathcal{G}].$$

Theorem 38 (cf. Proposition 3 of Lauritzen et al. [1990], cf. Corollary 2 of Richardson [2003]) *Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ be pairwise disjoint node sets in a DAG $\mathcal{D}$. Further let $(\mathcal{D}_{\text{An}(\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}, \mathcal{D})})^m$ be the moral induced subgraph of $\mathcal{D}$ on nodes $\text{An}(\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}, \mathcal{D})$ (see Definition 30). Then $\mathbf{Z}$ d-separates $\mathbf{X}$ and $\mathbf{Y}$ in $\mathcal{D}$ if and only if all paths between $\mathbf{X}$ and $\mathbf{Y}$ in $(\mathcal{D}_{\text{An}(\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}, \mathcal{D})})^m$ contain at least one node in $\mathbf{Z}$.*

Theorem 39 (cf. Theorem 7 of Perković et al., 2018) *Consider the definition of the adjustment criterion for MPDAGs (Definition 27) in the specific setting of a DAG. In this setting, replacing condition (b) in Definition 27 with*

(b) $\mathbf{S}$ d -separates $\mathbf{X}$ and $\mathbf{Y}$ in $\mathcal{D}_{\mathbf{XY}}^{pbd}$ (see Definition 29)

results in a criterion that is equivalent to Definition 27 applied to a DAG.

Theorem 40 (cf. Theorem 3.1 of Maathuis and Colombo, 2015) *Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{S}$ be pairwise disjoint node sets in a causal DAG $\mathcal{D}$. If $\mathbf{S}$ satisfies the generalized back-door criterion relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{D}$ (Definition 28), then $\mathbf{S}$ is an adjustment set relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{D}$ (Definition 26).*

Lemma 41 (cf. Lemma E.6 of Henckel et al., 2022) *Let $\mathbf{X}, \mathbf{Y}$ be disjoint node sets in an MPDAG $\mathcal{G}$. If there is no proper possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$ that starts with an undirected edge in $\mathcal{G}$, then $\text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) \subseteq \text{De}(\mathbf{X}, \mathcal{G})$.*

Lemma 42 (cf. Lemma 3.5 of Perković et al., 2017) *Let $p = \langle V_1, \dots, V_k \rangle, k > 1$, be a definite status path in MPDAG $\mathcal{G}$. Then p is a possibly causal path in $\mathcal{G}$ if and only if there is no edge $V_i \leftarrow V_{i+1}, i \in \{1, \dots, k-1\}$ in $\mathcal{G}$.*

Lemma 43 (cf. Lemma 3.3.1 of Zhang, 2006) *Let X, Y , and Z be distinct nodes in a PAG $\mathcal{G}$. If $X \bullet \rightarrow Y \circ \bullet Z$, then there is an edge between X and Z with an arrowhead at Z . Furthermore, if the edge between X and Y is $X \rightarrow Y$, then the edge between X and Z is either $X \circ \rightarrow Z$ or $X \rightarrow Z$ (that is, not $X \leftrightarrow Z$).*

Lemma 44 (cf. Lemma 7.5 of Maathuis and Colombo, 2015) *Let X and Y be two distinct nodes in a MAG or PAG $\mathcal{G}$. Then $\mathcal{G}$ cannot have both an edge $Y \bullet \rightarrow X$ and a path $\langle X = V_1, \dots, V_k = Y \rangle, k > 2$ where each edge $\langle V_i, V_{i+1} \rangle, i \in \{1, \dots, k-1\}$, is of one of these forms: $V_i \rightarrow V_{i+1}$ or $V_i \circ \bullet V_{i+1}$.*

Lemma 45 (cf. Lemma 17 of Perković et al., 2018) *Let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ and $\mathbf{S}$ be pairwise disjoint node sets in a MAG or PAG $\mathcal{G}$. Suppose that every proper possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$ starts with a visible edge out of $\mathbf{X}$ and that $[\mathbf{S} \cup \mathbf{Z}] \cap \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) = \emptyset$. Suppose furthermore that there is a path p from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$ such that*

- (i) p is a proper definite status non-causal path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$,
- (ii) all colliders on p are in $\text{An}(\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z} \cup \mathbf{S}, \mathcal{G}) \setminus [\mathbf{X} \cup \mathbf{Y} \cup \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})]$, and
- (iii) no definite non-collider on p is in $\mathbf{S} \cup \mathbf{Z}$.

Then there is a proper definite status non-causal path from $\mathbf{X}$ to $\mathbf{Y}$ that is m -connecting given $\mathbf{S} \cup \mathbf{Z}$ in $\mathcal{G}$.

Theorem 46 (cf. Theorem 4.4 of Perković et al., 2017, Theorems 5 and 56 of Perković et al., 2018) *Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{S}$ be pairwise disjoint node sets in a causal MPDAG (PAG) $\mathcal{G}$. Then $\mathbf{S}$ is an adjustment set relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{G}$ (Definition 26) if and only if $\mathbf{S}$ satisfies the adjustment criterion relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{G}$ (Definition 27).*

Lemma 47 (cf. Lemma F.1 of Rothenhäusler et al., 2018) *Let X and Y be nodes in an MPDAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ such that $X - Y$ is in $\mathcal{G}$. Let $\mathcal{G}'$ be an MPDAG constructed from $\mathcal{G}$ by adding $X \rightarrow Y$ and completing the orientation rules R1 - R4 of Meek [1995]. For any $Z, W \in \mathbf{V}$, if $Z - W$ is in $\mathcal{G}$ and $Z \rightarrow W$ is in $\mathcal{G}'$, then $W \in \text{De}(Y, \mathcal{G}')$.*

Lemma 48 (cf. Lemma F.2 of Rothenhäusler et al., 2018) *Let X be a node in an MPDAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, and let $\mathbf{S}$ be a set such that for all $S \in \mathbf{S}$, $X - S$ is in $\mathcal{G}$. Then there is an MPDAG $\mathcal{G}' = (\mathbf{V}, \mathbf{E}')$ that is formed by taking $\mathcal{G}$, orienting $X \rightarrow S$ for all $S \in \mathbf{S}$, and completing R1-R4 of Meek [1995].*

Lemma 49 (cf. Lemma 59 of Perković et al., 2018) *Let $\mathbf{X}, \mathbf{Y}$ and $\mathbf{S}$ be pairwise disjoint node sets in a DAG $\mathcal{D}$ such that $\mathbf{S}$ satisfies the adjustment criterion relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{D}$ (Definition 27). Let $\mathbf{J} \subseteq \text{An}(\mathbf{X} \cup \mathbf{Y}, \mathcal{D}) \setminus (\text{De}(\mathbf{X}, \mathcal{D}) \cup \mathbf{Y})$ and $\tilde{\mathbf{S}} = \mathbf{S} \cup \mathbf{J}$. Then the following statements hold:*

- (i) $\tilde{\mathbf{S}}$ satisfies the adjustment criterion relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{D}$, and
- (ii) $\int_{\mathbf{s}} f(\mathbf{y} \mid \mathbf{x}, \mathbf{s}) f(\mathbf{s}) d\mathbf{s} = \int_{\tilde{\mathbf{s}}} f(\mathbf{y} \mid \mathbf{x}, \tilde{\mathbf{s}}) f(\tilde{\mathbf{s}}) d\tilde{\mathbf{s}}$, for any density f consistent with $\mathcal{D}$.

Lemma 50 (Lemma 60 of Perković et al., 2018) *Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{S}$ be pairwise disjoint node sets in a causal DAG $\mathcal{D}$ such that $\mathbf{S}$ satisfies the adjustment criterion relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{D}$. Let $\mathbf{J} = \text{An}(\mathbf{X} \cup \mathbf{Y}, \mathcal{D}) \setminus (\text{De}(\mathbf{X}, \mathcal{D}) \cup \mathbf{Y})$ and $\tilde{\mathbf{S}} = \mathbf{S} \cup \mathbf{J}$. Additionally, let $\tilde{\mathbf{S}}_{\mathbf{D}} = \tilde{\mathbf{S}} \cap \text{De}(\mathbf{X}, \mathcal{D})$, $\tilde{\mathbf{S}}_{\mathbf{N}} = \tilde{\mathbf{S}} \setminus \text{De}(\mathbf{X}, \mathcal{D})$, $\mathbf{Y}_{\mathbf{D}} = \mathbf{Y} \cap \text{De}(\mathbf{X}, \mathcal{D})$ and $\mathbf{Y}_{\mathbf{N}} = \mathbf{Y} \setminus \text{De}(\mathbf{X}, \mathcal{D})$. Then the following statements hold:*

- (i) $(\mathbf{X} \cup \mathbf{Y}_{\mathbf{N}} \cup \tilde{\mathbf{S}}) \cap \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{D}) = \emptyset$,
- (ii) if $p = \langle H, \dots, Y_D \rangle$ is a non-causal path from $H \in \mathbf{X} \cup \mathbf{Y}_{\mathbf{N}} \cup \tilde{\mathbf{S}}$ to $Y_D \in \mathbf{Y}_{\mathbf{D}}$, then p is blocked by $(\mathbf{X} \cup \mathbf{Y}_{\mathbf{N}} \cup \tilde{\mathbf{S}}_{\mathbf{N}}) \setminus \{H\}$ in $\mathcal{D}$,
- (iii) $\mathbf{Y}_{\mathbf{D}} \perp_d \tilde{\mathbf{S}}_{\mathbf{D}} \mid \mathbf{Y}_{\mathbf{N}} \cup \mathbf{X} \cup \tilde{\mathbf{S}}_{\mathbf{N}}$ in $\mathcal{D}$, where $\mathbf{Y}_{\mathbf{N}} = \emptyset$ is allowed,
- (iv) if $\mathbf{Y}_{\mathbf{N}} = \emptyset$ then $\tilde{\mathbf{S}}_{\mathbf{N}}$ satisfies the generalized back-door criterion relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{D}$ (Definition 28),
- (v) the empty set satisfies the generalized back-door criterion relative to $(\mathbf{X} \cup \mathbf{Y}_{\mathbf{N}} \cup \tilde{\mathbf{S}}_{\mathbf{N}}, \mathbf{Y}_{\mathbf{D}})$ in $\mathcal{D}$,
- (vi) $\mathbf{Y}_{\mathbf{D}} \perp_d (\mathbf{Y}_{\mathbf{N}} \cup \tilde{\mathbf{S}}_{\mathbf{N}}) \mid \mathbf{X}$ in $\mathcal{D}_{\overline{\mathbf{X} \cup \mathbf{Y}_{\mathbf{N}} \cup \tilde{\mathbf{S}}_{\mathbf{N}}}}$, and
- (vii) $\tilde{\mathbf{S}}_{\mathbf{N}} \perp_d \mathbf{X} \mid \mathbf{Y}_{\mathbf{N}}$ in $\mathcal{D}_{\overline{\mathbf{X}}}$.

A.3 A NECESSARY CONDITION FOR IDENTIFIABILITY

This section includes the proof of Proposition 51, which provides a necessary condition for the identifiability of the conditional causal effect given an MPDAG. This result is needed twice – once for the proof of Theorem 3 in Section 2.3.1 and once for Example 4 in Section 2.3.2. Below we also provide two supporting results for the proof of Proposition 51 – namely, Lemmas 52 and 53.

A.3.1 MAIN RESULT

Proposition 51 *Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ be pairwise disjoint node sets in a causal MPDAG $\mathcal{G}$. If there is a proper possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$ that starts with an undirected edge and does not contain any element of $\mathbf{Z}$, then the conditional causal effect of $\mathbf{X}$ on $\mathbf{Y}$ given $\mathbf{Z}$ is not identifiable in $\mathcal{G}$.*

Proof of Proposition 51. This lemma extends Proposition 3.2 of Perković [2020] and its proof follows similar logic to that of Perković [2020].

Suppose that there is a proper possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ that starts with an undirected edge and does not contain any element of $\mathbf{Z}$. Then by Lemma 52, there is one such path – call it $q = \langle X = V_0, \dots, V_k = Y \rangle$, $X \in \mathbf{X}$, $Y \in \mathbf{Y}$, $k \geq 1$ – where the corresponding paths in two DAGs in $[\mathcal{G}]$ take the forms $X \rightarrow \dots \rightarrow Y$ and $X \leftarrow V_1 \rightarrow \dots \rightarrow Y$ ($X \leftarrow Y$ when $k = 1$). Call these DAGs $\mathcal{D}^1$ and $\mathcal{D}^2$ with paths q_1 and q_2 , respectively.

To prove that the conditional causal effect of $\mathbf{X}$ on $\mathbf{Y}$ given $\mathbf{Z}$ is not identifiable in $\mathcal{G}$, it suffices to show that there are two families of interventional densities over $\mathbf{V}$ – call them $\mathbf{F}_1^*$ and $\mathbf{F}_2^*$, where for $i \in \{1, 2\}$, we define $\mathbf{F}_i^* = \{f_i(\mathbf{v} | do(\mathbf{x}')) : \mathbf{X}' \subseteq \mathbf{V}\}$ – such that the following properties hold.

- (i) $\mathcal{D}^1$ and $\mathcal{D}^2$ are compatible with $\mathbf{F}_1^*$ and $\mathbf{F}_2^*$, respectively.¹
- (ii) $f_1(\mathbf{v}) = f_2(\mathbf{v})$.
- (iii) $f_1(\mathbf{y} | do(\mathbf{x}), \mathbf{z}) \neq f_2(\mathbf{y} | do(\mathbf{x}), \mathbf{z})$.

To define such families, we start by introducing an additional DAG and an observational density $f(\mathbf{v})$. That is, let $\mathcal{D}^{1'}$ be a DAG constructed by removing every edge from $\mathcal{D}^1$ except for the edges on q_1 . Then let $f(\mathbf{v})$ be the multivariate normal distribution under the following

¹For brevity, we say a DAG is “compatible with” a set of interventional densities and an interventional density is “consistent with” a DAG as shorthand for these claims holding only were the DAG to be causal.

linear structural equation model (SEM). Each random variable $A \in \mathbf{V}$ has mean zero and is a linear combination of its parents in $\mathcal{D}^{1'}$ and $\epsilon_A \sim N(0, \sigma_A^2)$, where $\{\epsilon_A : A \in \mathbf{V}\}$ are mutually independent. The coefficients in this linear combination are defined by the edge coefficients of $\mathcal{D}^{1'}$. We pick these edge coefficients in conjunction with $\{\sigma_A^2 : A \in \mathbf{V}\}$ in such a way that each coefficient is in $(0, 1)$ and $\text{Var}(A) = 1$ for all $A \in \mathbf{V}$.

From this, we define $\mathbf{F}_1^* = \{f_1(\mathbf{v}|\text{do}(\mathbf{x}')) : \mathbf{X}' \subseteq \mathbf{V}\}$ such that $\mathcal{D}^{1'}$ is compatible with $\mathbf{F}_1^*$ and such that $f_1(\mathbf{v}) = f(\mathbf{v})$. Note that $f(\mathbf{v})$ is Markov compatible with $\mathcal{D}^{1'}$ by construction, and we build the interventional densities in $\mathbf{F}_1^*$ by replacing the intervening random variables in the SEM with their interventional values [Pearl, 2009].

To construct the second family of interventional densities, we introduce the DAG $\mathcal{D}^{2'}$, which we form by removing every edge from $\mathcal{D}^2$ except for the edges on q_2 . Then note that we could have defined $f(\mathbf{v})$ using a linear SEM based on the parents in $\mathcal{D}^{2'}$. In this case, the resulting observational density would again be a multivariate normal with mean vector zero and a covariance matrix with ones on the diagonal. The off-diagonal entries would be the covariances between the variables in $\mathcal{D}^{2'}$. But note that by Lemma 32, these values will equal the product of all edge coefficients between the relevant nodes in $\mathcal{D}^{2'}$. Since $\mathcal{D}^{1'}$ and $\mathcal{D}^{2'}$ contain no paths with colliders, the observational density $f(\mathbf{v})$ built using $\mathcal{D}^{2'}$ will be an identical distribution to that built under $\mathcal{D}^{1'}$. Thus, in an analogous way to $\mathbf{F}_1^*$, we define $\mathbf{F}_2^* = \{f_2(\mathbf{v}|\text{do}(\mathbf{x}')) : \mathbf{X}' \subseteq \mathbf{V}\}$ such that $\mathcal{D}^{2'}$ is compatible with $\mathbf{F}_2^*$ and such that $f_2(\mathbf{v}) = f(\mathbf{v})$.

Having defined $\mathbf{F}_1^*$ and $\mathbf{F}_2^*$, we check that their desired properties hold. Note that by construction, $\mathcal{D}^{1'}$ and $\mathcal{D}^{2'}$ are compatible with $\mathbf{F}_1^*$ and $\mathbf{F}_2^*$, respectively. Thus (i) holds by Lemma 53. Similarly by construction, (ii) holds. To show that (iii) holds, it suffices to show that $E[Y|\text{do}(\mathbf{X} = \mathbf{1}), \mathbf{Z}]$ is not the same under f_1 and f_2 .

To calculate these expectations, we first want to apply Rules 1-3 of the do calculus (Equations (A.1)-eq:rule3-do). Since $f_i(\mathbf{v}|\text{do}(\mathbf{x}'))$, $i \in \{1, 2\}$, is consistent with $\mathcal{D}^{i'}$, we apply these rules using graphical relationships in $\mathcal{D}^{i'}$. Because the path in $\mathcal{D}^{i'}$ corresponding to q_i , $i \in \{1, 2\}$, does not contain nodes in $\mathbf{Z}$ or $\mathbf{X} \setminus \{X\}$, then $Y \perp_d \mathbf{Z}|\mathbf{X}$ and $Y \perp_d \mathbf{X} \setminus \{X\}|X$.

in $\mathcal{D}_{\mathbf{X}}^{i'}$. Further, $Y \perp_d X$ in $\mathcal{D}_{\mathbf{X}}^{1'}$ and $Y \perp_d X$ in $\mathcal{D}_{\mathbf{X}}^{2'}$. Thus by Rules 1-3 of the do calculus (Equations (A.1)-(A.3)), the following hold.

$$E_1[Y|do(\mathbf{X} = \mathbf{1}), \mathbf{Z}] = E_1[Y|do(X = 1)] = E_1[Y|X = 1] := a.$$

$$E_2[Y|do(\mathbf{X} = \mathbf{1}), \mathbf{Z}] = E_2[Y|do(X = 1)] = E_2[Y] := b,$$

where $E_i, i \in \{1, 2\}$ is the expectation under f_i . To calculate a and b , we rely on the observational density $f(\mathbf{v})$, which was constructed using $\mathcal{D}^{1'}$. By Lemma 33, a equals the covariance of X and Y under $f(\mathbf{v})$, and by Lemma 32, $\text{Cov}(X, Y)$ equals the product of all edge coefficients in $\mathcal{D}^{1'}$, which were chosen to be in $(0, 1)$. Therefore, $a \neq 0$. But by definition of $f(\mathbf{v})$, $b = 0$. $\blacksquare$

A.3.2 SUPPORTING RESULT

Lemma 52 *Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ be pairwise disjoint node sets in an MPDAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$. Suppose that there is a proper possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$ that starts with an undirected edge and does not contain nodes in $\mathbf{Z}$. Then there is one such path $\langle X = V_0, \dots, V_k = Y \rangle$, $X \in \mathbf{X}$, $Y \in \mathbf{Y}$, $k \geq 1$, where the corresponding paths in two DAGs in $[\mathcal{G}]$ take the forms $X \rightarrow \dots \rightarrow Y$ and $X \leftarrow V_1 \rightarrow \dots \rightarrow Y$ ($X \leftarrow Y$ when $k = 1$), respectively.*

Proof of Lemma 52. This lemma is similar to Lemma A.3 of Perković [2020] and its proof borrows from the proof strategy of Lemma C.1 of Perković et al. [2017].

Let q^* be an arbitrary proper possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$ that starts with an undirected edge and does not contain nodes in $\mathbf{Z}$. Then let $q = \langle X = V_0, \dots, V_k = Y \rangle$, $X \in \mathbf{X}$, $Y \in \mathbf{Y}$, $k \geq 1$, be a shortest subsequence of q^* in $\mathcal{G}$ that also starts with an undirected edge. Note that q is a proper possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$ that starts with an undirected edge and does not contain nodes in $\mathbf{Z}$.

Consider when q is of definite status. Since q is possibly causal, all non-endpoints of q are definite non-colliders. Let $\mathcal{D}^1$ be a DAG in $[\mathcal{G}]$ that contains $X \rightarrow V_1$. Then since V_1 is either Y or a definite non-collider on q , the path corresponding to q in $\mathcal{D}^1$ takes the

form $X \rightarrow \cdots \rightarrow Y$ by induction. Let $\mathcal{D}^2$ be a DAG in $[\mathcal{G}]$ with no additional edges into V_1 compared to $\mathcal{G}$ (Lemma 48). Since $\mathcal{G}$ contains $X - V_1$, $\mathcal{D}^2$ contains $X \leftarrow V_1$. When $k > 1$, $\mathcal{G}$ contains either $V_1 - V_2$ or $V_1 \rightarrow V_2$, and so $\mathcal{D}^2$ contains $X \leftarrow V_1 \rightarrow V_2$. Thus by the same inductive reasoning as above, the path corresponding to q in $\mathcal{D}^2$ takes the form $X \leftarrow V_1 \rightarrow \cdots \rightarrow Y$ (or simply $X \leftarrow Y$ when $k = 1$).

Consider instead when q is not of definite status. Note that $k > 1$. To see that q contains $V_1 - V_2$, note that by the choice of q and the fact that q is possibly causal, $q(V_1, Y)$ is unshielded and possibly causal. Thus, $q(V_1, Y)$ is of definite status. However, q is not of definite status, so V_1 must not be of definite status on q , which implies that q cannot contain $V_1 \rightarrow V_2$. Since q is possibly causal, it also cannot contain $V_1 \leftarrow V_2$.

To find two DAGs in $[\mathcal{G}]$ with paths corresponding to q that fit our desired forms, we narrow our search to $[\mathcal{G}']$, where we let $\mathcal{G}'$ be an MPDAG constructed from $\mathcal{G}$ by adding $V_1 \rightarrow V_2$ and completing R1-R4 of Meek [1995]. We show below that the path corresponding to q in $\mathcal{G}'$ takes the form $X - V_1 \rightarrow \cdots \rightarrow Y$, and thus, there must be two DAGs in $[\mathcal{G}'] \subseteq [\mathcal{G}]$ with corresponding paths of the forms $X \rightarrow \cdots \rightarrow Y$ and $X \leftarrow V_1 \rightarrow \cdots \rightarrow Y$.

We first show that $\mathcal{G}'$ contains $X - V_1$ by the contraposition of Lemma 47. Note that we have already shown that $\mathcal{G}$ contains $V_1 - V_2$, that $\mathcal{G}'$ is formed by adding $V_1 \rightarrow V_2$ to $\mathcal{G}$, and that $\mathcal{G}$ contains $X - V_1$. It remains to show that $X, V_1 \notin \text{De}(V_2, \mathcal{G}')$. To see this, note that $\mathcal{G}$ must contain an edge $\langle X, V_2 \rangle$, because V_1 is not of definite status on q . This edge must take the form $X \rightarrow V_2$ by the choice of q and the fact that q is possibly causal. Thus, $\mathcal{G}'$ contains $X \rightarrow V_2$ and $V_1 \rightarrow V_2$. Therefore, $X, V_1 \notin \text{De}(V_2, \mathcal{G}')$. Finally, note that $\mathcal{G}'$ contains $V_1 \rightarrow \cdots \rightarrow Y$ by R1 of Meek [1995], since we constructed $\mathcal{G}'$ by adding $V_1 \rightarrow V_2$ to a path $q(V_1, Y)$ that is unshielded and possibly causal. ■

Lemma 53 *Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ be pairwise disjoint node sets in a causal DAG $\mathcal{D} = (\mathbf{V}, \mathbf{E})$. Then let $\mathcal{D}^* = (\mathbf{V}, \mathbf{E}')$ be a causal DAG constructed by removing edges from $\mathcal{D}$, and let $f(\mathbf{v}|\text{do}(\mathbf{x}))$ be an interventional density over $\mathbf{V}$. If $f(\mathbf{v}|\text{do}(\mathbf{x}))$ is consistent with $\mathcal{D}^*$, then it is consistent with $\mathcal{D}$.*

Proof of Lemma 53. Suppose that $f(\mathbf{v}|do(\mathbf{x}))$ is consistent with $\mathcal{D}^*$. Then by definition, there exists a set of interventional densities $\mathbf{F}^*$ such that $\mathcal{D}^*$ is compatible with $\mathbf{F}^*$. Let $f(\mathbf{v})$ be the density in $\mathbf{F}^*$ under a null intervention. Note that by the truncated factorization in Equation (2.3), $f(\mathbf{v})$ is Markov compatible with $\mathcal{D}^*$. Thus by Lemma 34,

$$V_i \perp\!\!\!\perp \left[\mathbf{V} \setminus (\text{De}(V_i, \mathcal{D}^*) \cup \text{Pa}(V_i, \mathcal{D}^*)) \right] \mid \text{Pa}(V_i, \mathcal{D}^*) \quad (\text{A.4})$$

for all $V_i \in \mathbf{V}$, where $\perp\!\!\!\perp$ indicates independence with respect to $f(\mathbf{v})$. Further, since $\text{De}(V_i, \mathcal{D}^*) \subseteq \text{De}(V_i, \mathcal{D})$, then $\text{De}(V_i, \mathcal{D}^*) \cap \text{Pa}(V_i, \mathcal{D}) = \emptyset$ and thus $\text{Pa}(V_i, \mathcal{D}) \subseteq \mathbf{V} \setminus \text{De}(V_i, \mathcal{D}^*)$. Therefore it follows from (A.4) that

$$V_i \perp\!\!\!\perp \left[\text{Pa}(V_i, \mathcal{D}) \setminus \text{Pa}(V_i, \mathcal{D}^*) \right] \mid \text{Pa}(V_i, \mathcal{D}^*). \quad (\text{A.5})$$

Let $f(\mathbf{v}|do(\mathbf{x}'))$, $\mathbf{X}' \subseteq \mathbf{V}$, be an arbitrary density in $\mathbf{F}^*$. Then by definition and (A.5)

$$\begin{aligned} f(\mathbf{v}|do(\mathbf{x}')) &= \prod_{V_i \in \mathbf{V} \setminus \mathbf{X}'} f(v_i | \text{pa}(v_i, \mathcal{D}^*)) \mathbb{1}(\mathbf{X}' = \mathbf{x}') \\ &= \prod_{V_i \in \mathbf{V} \setminus \mathbf{X}'} f(v_i | \text{pa}(v_i, \mathcal{D})) \mathbb{1}(\mathbf{X}' = \mathbf{x}'). \end{aligned}$$

Since $f(\mathbf{v}|do(\mathbf{x}'))$ was arbitrary, this holds for all densities in $\mathbf{F}^*$. Thus, $\mathcal{D}$ is compatible with $\mathbf{F}^*$. Since $f(\mathbf{v}|do(\mathbf{x})) \in \mathbf{F}^*$, then by definition, it is consistent with $\mathcal{D}$. $\blacksquare$

A.4 PROOFS FOR SECTION 2.3.1: MPDAGS - CONDITIONAL ADJUSTMENT CRITERION

The following results show the completeness and soundness of the conditional adjustment criterion for identifying conditional adjustment sets in DAGs. We rely on these results to show the analogous results for MPDAGs in Theorem 3 of Section 2.3.1. Figure A.1 shows how the results in this paper fit together to prove Theorem 3. Two supporting results needed for the proof of soundness in DAGs follow the main results below.

A.4.1 MAIN RESULTS

Proof of Lemma 6. (a) Follows from Lemma 41. (b) Holds by Theorem 3, Lemma 6(a), and Theorem 46. $\blacksquare$

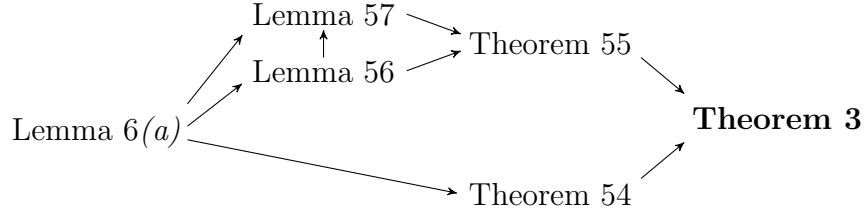


Figure A.1: Proof structure of Theorem 3.

Theorem 54 (Completeness of the Conditional Adjustment Criterion for DAGs)

Let $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$, and $\mathbf{S}$ be pairwise disjoint node sets in a causal DAG $\mathcal{D}$, where $\mathbf{Z} \cap \text{De}(\mathbf{X}, \mathcal{D}) = \emptyset$. If $\mathbf{S}$ is a conditional adjustment set relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{D}$ (Definition 1), then $\mathbf{S}$ satisfies the conditional adjustment criterion relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{D}$ (Definition 2).

Proof of Theorem 54. Let $\mathbf{S}$ be a conditional adjustment set relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{D}$, and let f be a density consistent with $\mathcal{D}$. We start by showing that $\mathbf{S} \cup \mathbf{Z}$ is an adjustment set relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{D}$. To do this, we calculate the following. (Justification for the numbered equations is below.)

$$\begin{aligned}
 f(\mathbf{y}|\text{do}(\mathbf{x})) &= \int_{\mathbf{z}} f(\mathbf{y}, \mathbf{z}|\text{do}(\mathbf{x})) \, d\mathbf{z} \\
 &= \int_{\mathbf{z}} f(\mathbf{z}|\text{do}(\mathbf{x})) f(\mathbf{y}|\text{do}(\mathbf{x}), \mathbf{z}) \, d\mathbf{z} \\
 &= \int_{\mathbf{z}} f(\mathbf{z}) f(\mathbf{y}|\text{do}(\mathbf{x}), \mathbf{z}) \, d\mathbf{z} \tag{A.6}
 \end{aligned}$$

$$\begin{aligned}
 &= \int_{\mathbf{z}} f(\mathbf{z}) \int_{\mathbf{s}} f(\mathbf{y}|\mathbf{x}, \mathbf{z}, \mathbf{s}) f(\mathbf{s}|\mathbf{z}) \, d\mathbf{s} \, d\mathbf{z} \tag{A.7} \\
 &= \int_{\mathbf{s}, \mathbf{z}} f(\mathbf{y}|\mathbf{x}, \mathbf{s}, \mathbf{z}) f(\mathbf{s}, \mathbf{z}) \, d\mathbf{s} \, d\mathbf{z}.
 \end{aligned}$$

Equation (A.6) follows from Rule 3 of the do calculus (Equation (A.3)). To show that this rule holds, let p be an arbitrary path from $\mathbf{X}$ to $\mathbf{Z}$ in $\mathcal{D}_{\overline{\mathbf{X}}}$. Note that p must begin with an edge out of $\mathbf{X}$. Since $\mathbf{Z} \cap \text{De}(\mathbf{X}, \mathcal{G}) = \emptyset$, p cannot be causal and, therefore, must have colliders. Thus, p is blocked, and so $(\mathbf{Z} \perp_d \mathbf{X})_{\mathcal{D}_{\overline{\mathbf{X}}}}$. Equation (A.7) follows from the fact

that $\mathbf{S}$ is a conditional adjustment set relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$. This shows that $\mathbf{S} \cup \mathbf{Z}$ is an adjustment set relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{D}$.

By Theorem 46, $\mathbf{S} \cup \mathbf{Z}$ satisfies the adjustment criterion relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{D}$. Then by Lemma 6(a), $\mathbf{S}$ satisfies the conditional adjustment criterion relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{D}$. ■

Theorem 55 (Soundness of the Conditional Adjustment Criterion for DAGs) *Let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$, and $\mathbf{S}$ be pairwise disjoint node sets in a causal DAG $\mathcal{D}$, where $\mathbf{Z} \cap \text{De}(\mathbf{X}, \mathcal{D}) = \emptyset$. If $\mathbf{S}$ satisfies the conditional adjustment criterion relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{D}$ (Definition 2), then $\mathbf{S}$ is a conditional adjustment set relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{D}$ (Definition 1).*

Proof of Theorem 55. This theorem is analogous to Theorem 58 of Perković et al. [2018] for the adjustment criterion. We use the same proof strategy and adapt the arguments to suit our needs.

Suppose that $\mathbf{S}$ satisfies the conditional adjustment criterion relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{D}$ and let f be a density consistent with $\mathcal{D}$. Our goal is to prove that

$$f(\mathbf{y} | do(\mathbf{x}), \mathbf{z}) = \int_{\mathbf{s}} f(\mathbf{y} | \mathbf{x}, \mathbf{z}, \mathbf{s}) f(\mathbf{s} | \mathbf{z}) d\mathbf{s}. \quad (\text{A.8})$$

We consider three cases below. Before this, we prove an equality that holds in all cases. Let $\mathbf{Y}_{\mathbf{D}} = \mathbf{Y} \cap \text{De}(\mathbf{X}, \mathcal{D})$ and $\mathbf{Y}_{\mathbf{N}} = \mathbf{Y} \setminus \text{De}(\mathbf{X}, \mathcal{D})$. Then $\mathbf{Y}_{\mathbf{N}} \perp_d \mathbf{X} | \mathbf{Z}$ in $\mathcal{D}_{\overline{\mathbf{X}}}$, since $\mathcal{D}_{\overline{\mathbf{X}}}$ does not contain edges into $\mathbf{X}$ and since all paths from $\mathbf{X}$ to $\mathbf{Y}_{\mathbf{N}}$ that start with an edge out of $\mathbf{X}$ in $\mathcal{D}_{\overline{\mathbf{X}}}$ contain a collider – a collider that cannot be an element of $\text{An}(\mathbf{Z}, \mathcal{D})$ since $\mathbf{Z} \cap \text{De}(\mathbf{X}, \mathcal{D}) = \emptyset$. Rule 3 of the do calculus (Equation (A.3)) then implies

$$f(\mathbf{y}_{\mathbf{N}} | do(\mathbf{x}), \mathbf{z}) = f(\mathbf{y}_{\mathbf{N}} | \mathbf{z}). \quad (\text{A.9})$$

Case 1: Assume that $\mathbf{Y}_{\mathbf{D}} = \emptyset$ so that $\mathbf{Y} = \mathbf{Y}_{\mathbf{N}}$. Then we have the following. (Justification for the numbered equations is below.)

$$f(\mathbf{y} | do(\mathbf{x}), \mathbf{z}) = f(\mathbf{y} | \mathbf{z}) \quad (\text{A.10})$$

$$\begin{aligned}
&= \int_{\mathbf{s}} f(\mathbf{y}|\mathbf{z}, \mathbf{s}) f(\mathbf{s}|\mathbf{z}) d\mathbf{s} \\
&= \int_{\mathbf{s}} f(\mathbf{y}|\mathbf{x}, \mathbf{z}, \mathbf{s}) f(\mathbf{s}|\mathbf{z}) d\mathbf{s}.
\end{aligned} \tag{A.11}$$

Equation (A.10) follows from Equation (A.9) and $\mathbf{Y} = \mathbf{Y}_{\mathbf{N}}$. Equation (A.11) follows from the following logic. Since $\mathbf{S}$ satisfies the conditional adjustment criterion relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{D}$ and since $\mathbf{Y} = \mathbf{Y}_{\mathbf{N}}$, it holds that $\mathbf{S} \cup \mathbf{Z}$ blocks all paths from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{D}$. Thus, $\mathbf{X} \perp_d \mathbf{Y} \mid \mathbf{S} \cup \mathbf{Z}$ in $\mathcal{D}$, which implies the analogous independence statement.

Case 2: Assume $\mathbf{Y}_{\mathbf{N}} = \emptyset$ so that $\mathbf{Y} = \mathbf{Y}_{\mathbf{D}}$. Define $\mathbf{H} = \text{An}(\mathbf{X} \cup \mathbf{Y}, \mathcal{D}) \setminus (\text{De}(\mathbf{X}, \mathcal{D}) \cup \mathbf{Y} \cup \mathbf{Z})$, $\tilde{\mathbf{S}} = \mathbf{S} \cup \mathbf{H}$, $\tilde{\mathbf{S}}_{\mathbf{D}} = \tilde{\mathbf{S}} \cap \text{De}(\mathbf{X}, \mathcal{D})$, and $\tilde{\mathbf{S}}_{\mathbf{N}} = \tilde{\mathbf{S}} \setminus \text{De}(\mathbf{X}, \mathcal{D})$. Then we have the following. (Justification for the numbered equations is below.)

$$f(\mathbf{y}|\text{do}(\mathbf{x}), \mathbf{z}) = \int_{\tilde{\mathbf{S}}_{\mathbf{N}}} f(\mathbf{y}|\mathbf{x}, \mathbf{z}, \tilde{\mathbf{S}}_{\mathbf{N}}) f(\tilde{\mathbf{S}}_{\mathbf{N}}|\mathbf{z}) d\tilde{\mathbf{S}}_{\mathbf{N}} \tag{A.12}$$

$$\begin{aligned}
&= \int_{\tilde{\mathbf{S}}_{\mathbf{N}}} f(\mathbf{y}|\mathbf{x}, \mathbf{z}, \tilde{\mathbf{S}}_{\mathbf{N}}) \int_{\tilde{\mathbf{S}}_{\mathbf{D}}} f(\tilde{\mathbf{S}}_{\mathbf{D}}, \tilde{\mathbf{S}}_{\mathbf{N}}|\mathbf{z}) d\tilde{\mathbf{S}}_{\mathbf{D}} d\tilde{\mathbf{S}}_{\mathbf{N}} \\
&= \int_{\tilde{\mathbf{S}}_{\mathbf{D}}, \tilde{\mathbf{S}}_{\mathbf{N}}} f(\mathbf{y}|\mathbf{x}, \mathbf{z}, \tilde{\mathbf{S}}_{\mathbf{N}}) f(\tilde{\mathbf{S}}_{\mathbf{D}}, \tilde{\mathbf{S}}_{\mathbf{N}}|\mathbf{z}) d\tilde{\mathbf{S}}_{\mathbf{D}} d\tilde{\mathbf{S}}_{\mathbf{N}}
\end{aligned} \tag{A.13}$$

$$= \int_{\tilde{\mathbf{S}}} f(\mathbf{y}|\mathbf{x}, \mathbf{z}, \tilde{\mathbf{S}}) f(\tilde{\mathbf{S}}|\mathbf{z}) d\tilde{\mathbf{S}} \tag{A.14}$$

$$= \int_{\mathbf{s}} f(\mathbf{y}|\mathbf{x}, \mathbf{z}, \mathbf{s}) f(\mathbf{s}|\mathbf{z}) d\mathbf{s}. \tag{A.15}$$

Equation (A.12) holds since by Lemma 57(iv), $\tilde{\mathbf{S}}_{\mathbf{N}}$ is a conditional adjustment set relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{D}$. Equation (A.13) holds since $\tilde{\mathbf{S}}_{\mathbf{D}}$ is disjoint from $\mathbf{Y} \cup \mathbf{X} \cup \tilde{\mathbf{S}}_{\mathbf{N}} \cup \mathbf{Z}$. Equation (A.14) holds since by Lemma 57(iii), we have $\mathbf{Y} \perp_d \tilde{\mathbf{S}}_{\mathbf{D}} \mid \mathbf{X} \cup \tilde{\mathbf{S}}_{\mathbf{N}} \cup \mathbf{Z}$ in $\mathcal{D}$, where the analogous independence statement follows. Finally, Equation (A.15) results from applying Lemma 56(ii).

Case 3: Assume $\mathbf{Y}_{\mathbf{D}} \neq \emptyset$ and $\mathbf{Y}_{\mathbf{N}} \neq \emptyset$ and define $\mathbf{H}, \tilde{\mathbf{S}}, \tilde{\mathbf{S}}_{\mathbf{D}}$, and $\tilde{\mathbf{S}}_{\mathbf{N}}$ as in Case 2 above. We start by showing two equalities that rely on the do calculus. First note that by Lemma 57(vi), $\mathbf{Y}_{\mathbf{D}} \perp_d \mathbf{Y}_{\mathbf{N}} \cup \tilde{\mathbf{S}}_{\mathbf{N}} \cup \mathbf{Z} \mid \mathbf{X}$ in $\mathcal{D}_{\overline{\mathbf{X}} \mathbf{Y}_{\mathbf{N}} \cup \tilde{\mathbf{S}}_{\mathbf{N}} \cup \mathbf{Z}}$. Thus by Rule 2 of the do calculus (Equation (A.2)), we have that

$$f(\mathbf{y}_{\mathbf{D}}|\text{do}(\mathbf{x}), \mathbf{y}_{\mathbf{N}}, \mathbf{z}, \tilde{\mathbf{S}}_{\mathbf{N}}) = f(\mathbf{y}_{\mathbf{D}}|\text{do}(\mathbf{x}, \mathbf{y}_{\mathbf{N}}, \mathbf{z}, \tilde{\mathbf{S}}_{\mathbf{N}})). \tag{A.16}$$

Second, note by Lemma 57(vii), $\tilde{\mathbf{S}}_{\mathbf{N}} \perp_d \mathbf{X} \mid \mathbf{Y}_{\mathbf{N}} \cup \mathbf{Z}$ in $\mathcal{D}_{\bar{\mathbf{X}}}$. Thus by Rule 3 of the do calculus (Equation (A.3)), we have that

$$f(\tilde{\mathbf{S}}_{\mathbf{N}} | do(\mathbf{x}), \mathbf{y}_{\mathbf{N}}, \mathbf{z}) = f(\tilde{\mathbf{S}}_{\mathbf{N}} | \mathbf{y}_{\mathbf{N}}, \mathbf{z}). \quad (\text{A.17})$$

Then we have the following. (Justification for the numbered equations is below.)

$$\begin{aligned} f(\mathbf{y} | do(\mathbf{x}), \mathbf{z}) &= \int_{\tilde{\mathbf{S}}_{\mathbf{N}}} f(\mathbf{y}, \tilde{\mathbf{S}}_{\mathbf{N}} | do(\mathbf{x}), \mathbf{z}) d\tilde{\mathbf{S}}_{\mathbf{N}} \\ &= \int_{\tilde{\mathbf{S}}_{\mathbf{N}}} f(\mathbf{y}_{\mathbf{D}} | \tilde{\mathbf{S}}_{\mathbf{N}}, \mathbf{y}_{\mathbf{N}}, do(\mathbf{x}), \mathbf{z}) f(\tilde{\mathbf{S}}_{\mathbf{N}} | \mathbf{y}_{\mathbf{N}}, do(\mathbf{x}), \mathbf{z}) f(\mathbf{y}_{\mathbf{N}} | do(\mathbf{x}), \mathbf{z}) d\tilde{\mathbf{S}}_{\mathbf{N}} \\ &= \int_{\tilde{\mathbf{S}}_{\mathbf{N}}} f(\mathbf{y}_{\mathbf{D}} | do(\mathbf{x}, \mathbf{y}_{\mathbf{N}}, \mathbf{z}, \tilde{\mathbf{S}}_{\mathbf{N}})) f(\tilde{\mathbf{S}}_{\mathbf{N}} | \mathbf{y}_{\mathbf{N}}, \mathbf{z}) f(\mathbf{y}_{\mathbf{N}} | \mathbf{z}) d\tilde{\mathbf{S}}_{\mathbf{N}} \end{aligned} \quad (\text{A.18})$$

$$\begin{aligned} &= \int_{\tilde{\mathbf{S}}_{\mathbf{N}}} f(\mathbf{y}_{\mathbf{D}} | do(\mathbf{x}, \mathbf{y}_{\mathbf{N}}, \mathbf{z}, \tilde{\mathbf{S}}_{\mathbf{N}})) \int_{\tilde{\mathbf{S}}_{\mathbf{D}}} f(\tilde{\mathbf{S}}_{\mathbf{N}}, \mathbf{y}_{\mathbf{N}}, \tilde{\mathbf{S}}_{\mathbf{D}} | \mathbf{z}) d\tilde{\mathbf{S}}_{\mathbf{D}} d\tilde{\mathbf{S}}_{\mathbf{N}} \\ &= \int_{\tilde{\mathbf{S}}_{\mathbf{N}}} f(\mathbf{y}_{\mathbf{D}} | do(\mathbf{x}, \mathbf{y}_{\mathbf{N}}, \mathbf{z}, \tilde{\mathbf{S}}_{\mathbf{N}})) \int_{\tilde{\mathbf{S}}_{\mathbf{D}}} f(\mathbf{y}_{\mathbf{N}} | \tilde{\mathbf{S}}, \mathbf{z}) f(\tilde{\mathbf{S}} | \mathbf{z}) d\tilde{\mathbf{S}}_{\mathbf{D}} d\tilde{\mathbf{S}}_{\mathbf{N}} \\ &= \int_{\tilde{\mathbf{S}}_{\mathbf{N}}} f(\mathbf{y}_{\mathbf{D}} | \mathbf{y}_{\mathbf{N}}, \mathbf{x}, \mathbf{z}, \tilde{\mathbf{S}}_{\mathbf{N}}) \int_{\tilde{\mathbf{S}}_{\mathbf{D}}} f(\mathbf{y}_{\mathbf{N}} | \mathbf{x}, \mathbf{z}, \tilde{\mathbf{S}}) f(\tilde{\mathbf{S}} | \mathbf{z}) d\tilde{\mathbf{S}}_{\mathbf{D}} d\tilde{\mathbf{S}}_{\mathbf{N}} \end{aligned} \quad (\text{A.19})$$

$$= \int_{\tilde{\mathbf{S}}} f(\mathbf{y}_{\mathbf{D}} | \mathbf{y}_{\mathbf{N}}, \mathbf{x}, \mathbf{z}, \tilde{\mathbf{S}}) f(\mathbf{y}_{\mathbf{N}} | \mathbf{x}, \mathbf{z}, \tilde{\mathbf{S}}) f(\tilde{\mathbf{S}} | \mathbf{z}) d\tilde{\mathbf{S}} \quad (\text{A.20})$$

$$\begin{aligned} &= \int_{\tilde{\mathbf{S}}} f(\mathbf{y} | \mathbf{x}, \mathbf{z}, \tilde{\mathbf{S}}) f(\tilde{\mathbf{S}} | \mathbf{z}) d\tilde{\mathbf{S}} \\ &= \int_{\mathbf{s}} f(\mathbf{y} | \mathbf{x}, \mathbf{z}, \mathbf{s}) f(\mathbf{s} | \mathbf{z}) d\mathbf{s}. \end{aligned} \quad (\text{A.21})$$

Equation (A.18) holds by the applying Equations (A.16), (A.17), and (A.9). Equation (A.19) holds by the following logic. By Lemma 57(v), the empty set is an adjustment set relative to $(\mathbf{X} \cup \mathbf{Y}_{\mathbf{N}} \cup \tilde{\mathbf{S}}_{\mathbf{N}} \cup \mathbf{Z}, \mathbf{Y}_{\mathbf{D}})$ in $\mathcal{D}$. Then by Lemma 56(i), $\tilde{\mathbf{S}}$ satisfies the conditional adjustment criterion relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{D}$, and so $\tilde{\mathbf{S}} \cup \mathbf{Z}$ blocks all paths from $\mathbf{X}$ to $\mathbf{Y}_{\mathbf{N}}$ in $\mathcal{D}$. Thus, $\mathbf{Y}_{\mathbf{N}} \perp_d \mathbf{X} \mid \tilde{\mathbf{S}} \cup \mathbf{Z}$ in $\mathcal{D}$, where the analogous independence statement follows.

Equation (A.20) holds since $\tilde{\mathbf{S}}_{\mathbf{D}}$ is disjoint from $\mathbf{Y} \cup \mathbf{X} \cup \tilde{\mathbf{S}}_{\mathbf{N}} \cup \mathbf{Z}$ and since by Lemma 57(iii), we have that $\mathbf{Y}_{\mathbf{D}} \perp_d \tilde{\mathbf{S}}_{\mathbf{D}} \mid \mathbf{Y}_{\mathbf{N}} \cup \mathbf{X} \cup \tilde{\mathbf{S}}_{\mathbf{N}} \cup \mathbf{Z}$ in $\mathcal{D}$, where the analogous independence statement follows. Finally, Equation (A.21) results from applying Lemma 56(ii). $\blacksquare$

A.4.2 SUPPORTING RESULTS

Lemma 56 *Let $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$, and $\mathbf{S}$ be pairwise disjoint node sets in a causal DAG $\mathcal{D}$, where $\mathbf{Z} \cap \text{De}(\mathbf{X}, \mathcal{D}) = \emptyset$ and where $\mathbf{S}$ satisfies the conditional adjustment criterion relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{D}$ (Definition 2). Let $\mathbf{H} \subseteq \text{An}(\mathbf{X} \cup \mathbf{Y}, \mathcal{D}) \setminus (\text{De}(\mathbf{X}, \mathcal{D}) \cup \mathbf{Y} \cup \mathbf{Z})$ and $\tilde{\mathbf{S}} = \mathbf{S} \cup \mathbf{H}$. Then:*

(i) $\tilde{\mathbf{S}}$ satisfies the conditional adjustment criterion relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{D}$, and

(ii) $\int_{\mathbf{s}} f(\mathbf{y}|\mathbf{x}, \mathbf{z}, \mathbf{s})f(\mathbf{s}|\mathbf{z}) \, d\mathbf{s} = \int_{\tilde{\mathbf{s}}} f(\mathbf{y}|\mathbf{x}, \mathbf{z}, \tilde{\mathbf{s}})f(\tilde{\mathbf{s}}|\mathbf{z}) \, d\tilde{\mathbf{s}}$, for any density f consistent with $\mathcal{D}$.

Proof of Lemma 56. This lemma is analogous to Lemma 59 of Perković et al. [2018] (Lemma 49). We use the same proof strategy and adapt the arguments to suit our needs.

(i) By Lemma 6(a), since $\mathbf{S}$ satisfies the conditional adjustment criterion relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{D}$, then $\mathbf{S} \cup \mathbf{Z}$ satisfies the adjustment criterion relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{D}$. Then by Lemma 49, $\tilde{\mathbf{S}} \cup \mathbf{Z}$ satisfies the adjustment criterion relative to $(\mathbf{X}, \mathbf{Y})$. The statement follows by a second use of Lemma 6(a).

(ii) Let f be an arbitrary density consistent with $\mathcal{D}$. We proceed with a proof by induction.

Base case: Suppose $\mathbf{H} = \{H\}$ so that $|\mathbf{H}| = 1$. When $H \in \mathbf{S}$, the claim clearly holds. Thus, we let $H \notin \mathbf{S}$. Note that the claim holds if either $\mathbf{Y} \perp_d H | \mathbf{X} \cup \mathbf{S} \cup \mathbf{Z}$ or $\mathbf{X} \perp_d H | \mathbf{S} \cup \mathbf{Z}$ in $\mathcal{D}$. To see this, we calculate the following.

(a) When $(\mathbf{Y} \perp_d H | \mathbf{X} \cup \mathbf{S} \cup \mathbf{Z})_{\mathcal{D}}$, then

$$\begin{aligned} \int_{\mathbf{s}} f(\mathbf{y}|\mathbf{x}, \mathbf{z}, \mathbf{s})f(\mathbf{s}|\mathbf{z}) \, d\mathbf{s} &= \int_{\mathbf{s}} f(\mathbf{y}|\mathbf{x}, \mathbf{z}, \mathbf{s}) \int_h f(\mathbf{s}, h|\mathbf{z}) \, dh \, d\mathbf{s} \\ &= \int_{\mathbf{s}, h} f(\mathbf{y}|\mathbf{x}, \mathbf{z}, \mathbf{s})f(\mathbf{s}, h|\mathbf{z}) \, d\mathbf{s} \, dh \\ &= \int_{\tilde{\mathbf{s}}} f(\mathbf{y}|\mathbf{x}, \mathbf{z}, \tilde{\mathbf{s}})f(\tilde{\mathbf{s}}|\mathbf{z}) \, d\tilde{\mathbf{s}}, \end{aligned}$$

where the second equality holds since $H \notin \mathbf{Y} \cup \mathbf{X} \cup \mathbf{S} \cup \mathbf{Z}$.

(b) When $(\mathbf{X} \perp_d H \mid \mathbf{S} \cup \mathbf{Z})_{\mathcal{D}}$, then

$$\begin{aligned}
\int_{\mathbf{s}} f(\mathbf{y}|\mathbf{x}, \mathbf{z}, \mathbf{s}) f(\mathbf{s}|\mathbf{z}) d\mathbf{s} &= \int_{\mathbf{s}} f(\mathbf{s}|\mathbf{z}) \int_h f(\mathbf{y}, h|\mathbf{x}, \mathbf{z}, \mathbf{s}) dh d\mathbf{s} \\
&= \int_{\mathbf{s}, h} f(\mathbf{y}, h|\mathbf{x}, \mathbf{z}, \mathbf{s}) f(\mathbf{s}|\mathbf{z}) d\mathbf{s} dh \\
&= \int_{\mathbf{s}, h} f(\mathbf{y}|\mathbf{x}, \mathbf{z}, \mathbf{s}, h) f(h|\mathbf{x}, \mathbf{z}, \mathbf{s}) f(\mathbf{s}|\mathbf{z}) d\mathbf{s} dh \\
&= \int_{\mathbf{s}, h} f(\mathbf{y}|\mathbf{x}, \mathbf{z}, \mathbf{s}, h) f(h|\mathbf{z}, \mathbf{s}) f(\mathbf{s}|\mathbf{z}) d\mathbf{s} dh \\
&= \int_{\tilde{\mathbf{s}}} f(\mathbf{y}|\mathbf{x}, \mathbf{z}, \tilde{\mathbf{s}}) f(\tilde{\mathbf{s}}|\mathbf{z}) d\tilde{\mathbf{s}},
\end{aligned}$$

where the second equality holds since $H \notin \mathbf{S} \cup \mathbf{Z}$.

We use the remainder of the base case to show that (a) or (b) must hold. For sake of contradiction, suppose that neither hold. This implies that there are two paths in $\mathcal{D}$: one from $\mathbf{X}$ to H that is d-connecting given $\mathbf{S} \cup \mathbf{Z}$ and one from $\mathbf{Y}$ to H that is d-connecting given $\mathbf{X} \cup \mathbf{S} \cup \mathbf{Z}$. Let $p = \langle X, \dots, H \rangle$, $X \in \mathbf{X}$, and $q = \langle H, \dots, Y \rangle$, $Y \in \mathbf{Y}$, be such paths, respectively, where p is proper. In the arguments below, we use paths related to p and q – in the proper back-door graph $\mathcal{D}_{\mathbf{XY}}^{pbd}$ (see Definition 29) and in four of its moral induced subgraphs (see Definition 30) – before applying Theorems 38 and 39 to reach our final contradiction (that $\mathbf{S}$ cannot satisfy the conditional adjustment criterion relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{D}$).

First, we claim that both p and q are d-connecting given $\mathbf{S} \cup \mathbf{Z}$. This holds for p by definition. For sake of contradiction, suppose that q is blocked by $\mathbf{S} \cup \mathbf{Z}$. Since q is d-connecting given $\mathbf{X} \cup \mathbf{S} \cup \mathbf{Z}$, it must contain a collider in $\text{An}(\mathbf{X}, \mathcal{D}) \setminus \text{An}(\mathbf{S} \cup \mathbf{Z}, \mathcal{D})$. Let C be the closest collider to Y on q such that $C \in (\text{An}(\mathbf{X}, \mathcal{D}) \setminus \text{An}(\mathbf{S} \cup \mathbf{Z}, \mathcal{D})) \cup \mathbf{X}$, and let $r = \langle C, \dots, X' \rangle$, $X' \in \mathbf{X}$, be a shortest causal path in $\mathcal{D}$ from C to $\mathbf{X}$. Then let V be the node closest to X' on r that is also on $q(C, Y)$, and define the path $t = (-r)(X', V) \oplus q(V, Y)$. Note that t is non-causal since either $(-r)(X', V)$ is of non-zero length or $X' = V = C$, so that t is a path into X' . Further, by the definitions of q , C , and r , we have that t is proper

non-causal path from $\mathbf{X}$ to $\mathbf{Y}$ that is d-connecting given $\mathbf{S} \cup \mathbf{Z}$. But this contradicts that $\mathbf{S}$ satisfies the conditional adjustment criterion relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{D}$.

Next, we prove that the sequence of nodes in $\mathcal{D}_{\mathbf{XY}}^{pbd}$ corresponding to p forms a path. Note that since p is proper, we only need to show that p does not start with an edge $X \rightarrow W$, where W is a node that lies on a proper causal path in $\mathcal{D}$ from X to $\mathbf{Y}$. For sake of contradiction, suppose that p starts with $X \rightarrow W$ for such a $W \in \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{D})$. Note that p cannot be causal from X to H , since $H \notin \text{De}(\mathbf{X}, \mathcal{D})$ by the definition of $\mathbf{H}$. Thus, p is non-causal and there is a collider C' on p such that $C' \in \text{De}(W, \mathcal{D})$. Since p is d-connecting given $\mathbf{S} \cup \mathbf{Z}$ and $\mathbf{Z} \cap \text{De}(\mathbf{X}, \mathcal{D}) = \emptyset$, then $\mathbf{S} \cap \text{De}(C', \mathcal{D}) \neq \emptyset$. Further, since $\text{De}(C', \mathcal{D}) \subseteq \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{D})$, this implies that $\mathbf{S} \cap \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{D}) \neq \emptyset$. But this contradicts that $\mathbf{S}$ satisfies the conditional adjustment criterion relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{D}$.

Similarly, we prove that the sequence of nodes in $\mathcal{D}_{\mathbf{XY}}^{pbd}$ corresponding to q also forms a path. For this, note that all nodes in $\mathbf{X}$ on q must be a colliders on q , since q is d-connecting given $\mathbf{X} \cup \mathbf{S} \cup \mathbf{Z}$. Thus, removing edges out of $\mathbf{X}$ from $\mathcal{D}$ in order to form $\mathcal{D}_{\mathbf{XY}}^{pbd}$ will not affect the edges on q .

Let $\tilde{p}$ and $\tilde{q}$ be the paths in $\mathcal{D}_{\mathbf{XY}}^{pbd}$ corresponding to p and q , respectively. Then for sake of contradiction, suppose either $\tilde{p}$ or $\tilde{q}$ is blocked given $\mathbf{S} \cup \mathbf{Z}$. Since p and q are d-connecting given $\mathbf{S} \cup \mathbf{Z}$, then there must be a node C on p or q where C is a collider on p or q and every causal path in $\mathcal{D}$ from C to $\mathbf{S} \cup \mathbf{Z}$ contains the first edge of a proper causal path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{D}$. Let d be an arbitrary such causal path in $\mathcal{D}$ from C to $\mathbf{S} \cup \mathbf{Z}$. Note that d is a path from C to $\mathbf{S}$, since d must contain a node in $\mathbf{X}$ and since $\mathbf{Z} \cap \text{De}(\mathbf{X}, \mathcal{D}) = \emptyset$. But since d contains the first edge of a proper causal path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{D}$, this implies that $\mathbf{S} \cap \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{D}) \neq \emptyset$, which contradicts that $\mathbf{S}$ satisfies the conditional adjustment criterion relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{D}$.

We continue the base case by reasoning with four moral induced subgraphs of $\mathcal{D}_{\mathbf{XY}}^{pbd}$ (see Definition 30). Start by defining the following.

$$\mathbf{A}_{\mathbf{XHYSZ}} = \text{An}(\mathbf{X} \cup \mathbf{H} \cup \mathbf{Y} \cup \mathbf{S} \cup \mathbf{Z}, \mathcal{D}_{\mathbf{XY}}^{pbd}).$$

$$\mathbf{A}_{\mathbf{XYSZ}} = \text{An}(\mathbf{X} \cup \mathbf{Y} \cup \mathbf{S} \cup \mathbf{Z}, \mathcal{D}_{\mathbf{XY}}^{pbd}).$$

$$\mathbf{A}_{\mathbf{XHSZ}} = \text{An}(\mathbf{X} \cup \mathbf{H} \cup \mathbf{S} \cup \mathbf{Z}, \mathcal{D}_{\mathbf{XY}}^{pbd}).$$

$$\mathbf{A}_{\mathbf{HYSZ}} = \text{An}(\mathbf{H} \cup \mathbf{Y} \cup \mathbf{S} \cup \mathbf{Z}, \mathcal{D}_{\mathbf{XY}}^{pbd}).$$

Then define $\mathcal{D}_{\mathbf{XHYSZ}}$, $\mathcal{D}_{\mathbf{XYSZ}}$, $\mathcal{D}_{\mathbf{XHSZ}}$, and $\mathcal{D}_{\mathbf{HYSZ}}$ to be the moral induced subgraphs of $\mathcal{D}_{\mathbf{XY}}^{pbd}$ on nodes $\mathbf{A}_{\mathbf{XHYSZ}}$, $\mathbf{A}_{\mathbf{XYSZ}}$, $\mathbf{A}_{\mathbf{XHSZ}}$, and $\mathbf{A}_{\mathbf{HYSZ}}$, respectively. In order to use Theorem 39, we want to show that $\mathcal{D}_{\mathbf{XYSZ}}$ contains a path from $\mathbf{X}$ to $\mathbf{Y}$ that does not contain a node in $\mathbf{S} \cup \mathbf{Z}$.

Since $\tilde{p}$ and $\tilde{q}$ are d-connecting given $\mathbf{S} \cup \mathbf{Z}$, then by Theorem 38, the following two paths must exist in $\mathcal{D}_{\mathbf{XHSZ}}$: path a from X to H and path b from H to Y , where neither path contains a node in $\mathbf{S} \cup \mathbf{Z}$. Note that since $\mathbf{A}_{\mathbf{XHSZ}} \subseteq \mathbf{A}_{\mathbf{XHYSZ}}$ and $\mathbf{A}_{\mathbf{HYSZ}} \subseteq \mathbf{A}_{\mathbf{XHYSZ}}$, any path in $\mathcal{D}_{\mathbf{XHSZ}}$ or $\mathcal{D}_{\mathbf{HYSZ}}$ will also be in $\mathcal{D}_{\mathbf{XHYSZ}}$. Further, since $H \in \text{An}(\mathbf{X} \cup \mathbf{Y}, \mathcal{D})$ by definition and since we form $\mathcal{D}_{\mathbf{XY}}^{pbd}$ by removing edges out of $\mathbf{X}$ from $\mathcal{D}$, then $H \in \text{An}(\mathbf{X} \cup \mathbf{Y}, \mathcal{D}_{\mathbf{XY}}^{pbd})$. Therefore, $\mathbf{A}_{\mathbf{XHYSZ}} = \mathbf{A}_{\mathbf{XYSZ}}$ and $\mathcal{D}_{\mathbf{XHYSZ}} = \mathcal{D}_{\mathbf{XYSZ}}$. Thus, a and b are both paths in $\mathcal{D}_{\mathbf{XYSZ}}$.

We complete the base case by applying Theorems 38 and 39 to show our necessary contradiction. Since we can combine subpaths of a and b to form a path c in $\mathcal{D}_{\mathbf{XYSZ}}$ from $\mathbf{X}$ to $\mathbf{Y}$ that does not contain a node in $\mathbf{S} \cup \mathbf{Z}$, then by Theorem 38, $\mathbf{X}$ and $\mathbf{Y}$ are d-connecting given $\mathbf{S} \cup \mathbf{Z}$ in $\mathcal{D}_{\mathbf{XY}}^{pbd}$. By Theorem 39, this implies that $\mathbf{S} \cup \mathbf{Z}$ does not satisfy the adjustment criterion relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{D}$ (see Definition 27). Therefore, by the contraposition of Lemma 6(a), $\mathbf{S}$ does not satisfy the conditional adjustment criterion relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{D}$, which is a contradiction.

Induction step: Assume that the result holds for $|\mathbf{H}| = k$, $k \in \mathbb{N}$, and let $|\mathbf{H}| = k + 1$. Take an arbitrary $H \in \mathbf{H}$, and define $\mathbf{S}' = \mathbf{S} \cup \{H\}$ and $\mathbf{H}' = \mathbf{H} \setminus \{H\}$. Since the base case holds and since $\{H\} \subseteq \text{An}(\mathbf{X} \cup \mathbf{Y}, \mathcal{D}) \setminus (\text{De}(\mathbf{X}, \mathcal{D}) \cup \mathbf{Y} \cup \mathbf{Z})$, then

$$\begin{aligned} \int_{\mathbf{s}} f(\mathbf{y}|\mathbf{x}, \mathbf{z}, \mathbf{s}) f(\mathbf{s}|\mathbf{z}) d\mathbf{s} &= \int_{\mathbf{s}, h} f(\mathbf{y}|\mathbf{x}, \mathbf{z}, \mathbf{s}, h) f(\mathbf{s}, h|\mathbf{z}) d\mathbf{s} dh \\ &= \int_{\mathbf{s}'} f(\mathbf{y}|\mathbf{x}, \mathbf{z}, \mathbf{s}') f(\mathbf{s}'|\mathbf{z}) d\mathbf{s}'. \end{aligned} \tag{A.22}$$

Further, by part (i), $\mathbf{S}'$ satisfies the conditional adjustment criterion relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{D}$. Since $\mathbf{H}' \subseteq \text{An}(\mathbf{X} \cup \mathbf{Y}, \mathcal{D}) \setminus (\text{De}(\mathbf{X}, \mathcal{D}) \cup \mathbf{Y} \cup \mathbf{Z})$ and $|\mathbf{H}'| = k$, then by the induction assumption,

$$\begin{aligned} \int_{\mathbf{s}'} f(\mathbf{y}|\mathbf{x}, \mathbf{z}, \mathbf{s}') f(\mathbf{s}'|\mathbf{z}) d\mathbf{s}' &= \int_{\mathbf{s}', \mathbf{h}'} f(\mathbf{y}|\mathbf{x}, \mathbf{z}, \mathbf{s}', \mathbf{h}') f(\mathbf{s}', \mathbf{h}'|\mathbf{z}) d\mathbf{s}' d\mathbf{h}' \\ &= \int_{\tilde{\mathbf{s}}} f(\mathbf{y}|\mathbf{x}, \mathbf{z}, \tilde{\mathbf{s}}) f(\tilde{\mathbf{s}}|\mathbf{z}) d\tilde{\mathbf{s}}. \end{aligned} \quad (\text{A.23})$$

Combining (A.22) and (A.23) yields the desired result. $\blacksquare$

Lemma 57 *Let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$, and $\mathbf{S}$ be pairwise disjoint node sets in a causal DAG $\mathcal{D}$, where $\mathbf{Z} \cap \text{De}(\mathbf{X}, \mathcal{D}) = \emptyset$ and where $\mathbf{S}$ satisfies the conditional adjustment criterion relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{D}$ (Definition 2). Let $\mathbf{H} = \text{An}(\mathbf{X} \cup \mathbf{Y}, \mathcal{D}) \setminus (\text{De}(\mathbf{X}, \mathcal{D}) \cup \mathbf{Y} \cup \mathbf{Z})$ and $\tilde{\mathbf{S}} = \mathbf{S} \cup \mathbf{H}$. Additionally, let $\tilde{\mathbf{S}}_{\mathbf{D}} = \tilde{\mathbf{S}} \cap \text{De}(\mathbf{X}, \mathcal{D})$, $\tilde{\mathbf{S}}_{\mathbf{N}} = \tilde{\mathbf{S}} \setminus \text{De}(\mathbf{X}, \mathcal{D})$, $\mathbf{Y}_{\mathbf{D}} = \mathbf{Y} \cap \text{De}(\mathbf{X}, \mathcal{D})$, and $\mathbf{Y}_{\mathbf{N}} = \mathbf{Y} \setminus \text{De}(\mathbf{X}, \mathcal{D})$. Then the following statements hold:*

- (i) $(\mathbf{X} \cup \mathbf{Y}_{\mathbf{N}} \cup \tilde{\mathbf{S}} \cup \mathbf{Z}) \cap \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{D}) = \emptyset$,
- (ii) if $p = \langle H, \dots, Y_D \rangle$ is a non-causal path in $\mathcal{D}$ from $H \in \mathbf{X} \cup \mathbf{Y}_{\mathbf{N}} \cup \tilde{\mathbf{S}} \cup \mathbf{Z}$ to a node $Y_D \in \mathbf{Y}_{\mathbf{D}}$, then p is blocked by $(\mathbf{X} \cup \mathbf{Y}_{\mathbf{N}} \cup \tilde{\mathbf{S}}_{\mathbf{N}} \cup \mathbf{Z}) \setminus \{H\}$,
- (iii) $\mathbf{Y}_{\mathbf{D}} \perp_d \tilde{\mathbf{S}}_{\mathbf{D}} \mid \mathbf{Y}_{\mathbf{N}} \cup \mathbf{X} \cup \tilde{\mathbf{S}}_{\mathbf{N}} \cup \mathbf{Z}$ in $\mathcal{D}$,
- (iv) if $\mathbf{Y}_{\mathbf{N}} = \emptyset$, then $\tilde{\mathbf{S}}_{\mathbf{N}}$ is a conditional adjustment set relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{D}$ (Definition 1),
- (v) the empty set is an adjustment set relative to $(\mathbf{X} \cup \mathbf{Y}_{\mathbf{N}} \cup \tilde{\mathbf{S}}_{\mathbf{N}} \cup \mathbf{Z}, \mathbf{Y}_{\mathbf{D}})$ in $\mathcal{D}$ (Definition 26),
- (vi) $\mathbf{Y}_{\mathbf{D}} \perp_d (\mathbf{Y}_{\mathbf{N}} \cup \tilde{\mathbf{S}}_{\mathbf{N}} \cup \mathbf{Z}) \mid \mathbf{X}$ in $\mathcal{D}_{\overline{\mathbf{X} \cup \mathbf{Y}_{\mathbf{N}} \cup \tilde{\mathbf{S}}_{\mathbf{N}} \cup \mathbf{Z}}}$, and
- (vii) $\tilde{\mathbf{S}}_{\mathbf{N}} \perp_d \mathbf{X} \mid \mathbf{Y}_{\mathbf{N}} \cup \mathbf{Z}$ in $\mathcal{D}_{\overline{\mathbf{X}}}$.

Proof of Lemma 57. This lemma is analogous to Lemma 60 of Perković et al. [2018] (Lemma 50), which is needed for adjustment in total effect identification. We rely on this result in the proof below.

Note that $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{S} \cup \mathbf{Z}$ are pairwise disjoint node sets in $\mathcal{D}$, where by Lemma 6(a), $\mathbf{S} \cup \mathbf{Z}$ satisfies the adjustment criterion relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{D}$. Results (i)-(iii) and (vi) follow directly from Lemma 50. Result (v) follows additionally from Theorem 40. Result (iv) follows additionally from Theorem 40 and Lemma 6(a).

(vii) Let p be an arbitrary path from $X \in \mathbf{X}$ to $\tilde{\mathbf{S}}_{\mathbf{N}}$ in $\mathcal{D}_{\bar{\mathbf{X}}}$. By definition of $\mathcal{D}_{\bar{\mathbf{X}}}$, p begins with an edge out of X . Since, by definition, $\tilde{\mathbf{S}}_{\mathbf{N}} \cap \text{De}(\mathbf{X}, \mathcal{D}) = \emptyset$, where $\text{De}(\mathbf{X}, \mathcal{D}_{\bar{\mathbf{X}}}) \subseteq \text{De}(\mathbf{X}, \mathcal{D})$, then p must contain at least one collider. Let $\mathbf{C}$ be the set containing the closest collider to X on p and its descendants in $\mathcal{D}_{\bar{\mathbf{X}}}$. Note that $\mathbf{C} \subseteq \text{De}(\mathbf{X}, \mathcal{D}_{\bar{\mathbf{X}}}) \subseteq \text{De}(\mathbf{X}, \mathcal{D})$. By definition of $\mathbf{Y}_{\mathbf{N}}$ and by assumption, $(\mathbf{Y}_{\mathbf{N}} \cup \mathbf{Z}) \cap \text{De}(\mathbf{X}, \mathcal{D}) = \emptyset$, and thus, p is blocked by $\mathbf{Y}_{\mathbf{N}} \cup \mathbf{Z}$. ■

A.5 CONDITIONAL BACK-DOOR CRITERION

This section extends Pearl’s back-door criterion (2009) to the context of estimating a conditional causal effect in a DAG. Definition 58 provides the extended criterion, and Lemma 59 establishes that this criterion is sufficient for conditional adjustment. Lemma 60 makes a comparison between this criterion and the generalized back-door criterion of Maathuis and Colombo [2015] (Definition 28).

Definition 58 (Conditional Back-door Criterion for DAGs) *Let $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$, and $\mathbf{S}$ be pairwise disjoint node sets in a DAG $\mathcal{D}$, where $\mathbf{Z} \cap \text{De}(\mathbf{X}, \mathcal{D}) = \emptyset$. Then $\mathbf{S}$ satisfies the conditional back-door criterion relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{D}$ if*

(a) $\mathbf{S} \cap \text{De}(\mathbf{X}, \mathcal{D}) = \emptyset$, and

(b) $\mathbf{S} \cup \mathbf{Z}$ blocks all proper back-door paths from $\mathbf{X}$ to $\mathbf{Y}$.

Lemma 59 *Let $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$, and $\mathbf{S}$ be pairwise disjoint node sets in a causal DAG $\mathcal{D}$, where $\mathbf{Z} \cap \text{De}(\mathbf{X}, \mathcal{D}) = \emptyset$. If $\mathbf{S}$ satisfies the conditional back-door criterion relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{D}$ (Definition 58), then $\mathbf{S}$ is a conditional adjustment set relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{D}$ (Definition 1).*

Proof of Lemma 59. Let $\mathbf{S}$ be a set that satisfies the conditional back-door criterion relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{D}$, and let f be a density consistent with $\mathcal{D}$. Then

$$\begin{aligned} f(\mathbf{y}|\text{do}(\mathbf{x}), \mathbf{z}) &= \int_{\mathbf{s}} f(\mathbf{y}, \mathbf{s}|\text{do}(\mathbf{x}), \mathbf{z}) \, d\mathbf{s} \\ &= \int_{\mathbf{s}} f(\mathbf{y}|\mathbf{s}, \text{do}(\mathbf{x}), \mathbf{z}) f(\mathbf{s}|\text{do}(\mathbf{x}), \mathbf{z}) \, d\mathbf{s} \\ &= \int_{\mathbf{s}} f(\mathbf{y}|\mathbf{s}, \mathbf{x}, \mathbf{z}) f(\mathbf{s}|\mathbf{z}) \, d\mathbf{s} \end{aligned}$$

The first two equalities follow from the law of total probability and the chain rule. The third equality follows from Rules 2 and 3 of the do calculus (Equations (A.2) and (A.3)) and the d-separations shown below.

In order to use Rule 2 to conclude that $f(\mathbf{y}|\mathbf{s}, \text{do}(\mathbf{x}), \mathbf{z}) = f(\mathbf{y}|\mathbf{s}, \mathbf{x}, \mathbf{z})$, we show that $(\mathbf{Y} \perp_d \mathbf{X} \mid \mathbf{S} \cup \mathbf{Z})_{\mathcal{D}_{\underline{\mathbf{X}}}}$. Note that $\mathcal{D}_{\underline{\mathbf{X}}}$ only contains back-door paths from $\mathbf{X}$ to $\mathbf{Y}$. So every path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{D}_{\underline{\mathbf{X}}}$ contains a proper back-door path from $\mathbf{X}$ to $\mathbf{Y}$ as a subpath. Since $\mathbf{S} \cup \mathbf{Z}$ blocks all proper back-door paths from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{D}$, the d-separation holds.

In order to use Rule 3 to conclude that $f(\mathbf{s}|\text{do}(\mathbf{x}), \mathbf{z}) = f(\mathbf{s}|\mathbf{z})$, we show that $(\mathbf{S} \perp_d \mathbf{X} \mid \mathbf{Z})_{\mathcal{D}_{\overline{\mathbf{X}(\mathbf{Z})}}}$. This follows from the assumptions that $\mathbf{Z} \cap \text{De}(\mathbf{X}, \mathcal{D}) = \emptyset$ and $\mathbf{S} \cap \text{De}(\mathbf{X}, \mathcal{D}) = \emptyset$.

■

Lemma 60 (Comparison of Back-door Criteria for DAGs) *Let $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$, and $\mathbf{S}$ be pairwise disjoint node sets in a DAG $\mathcal{D}$, where $\mathbf{Z} \cap \text{De}(\mathbf{X}, \mathcal{D}) = \emptyset$. Then $\mathbf{S}$ satisfies the conditional back-door criterion relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{D}$ (Definition 58) if and only if $\mathbf{S} \cup \mathbf{Z}$ satisfies the generalized back-door criterion relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{D}$ (Definition 28).*

Proof of Lemma 60. $\Leftarrow$: Follows immediately.

$\Rightarrow$: Since $\mathbf{S}$ satisfies the conditional back-door criterion relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{D}$, then $\mathbf{S} \cap \text{De}(\mathbf{X}, \mathcal{D}) = \emptyset$. Combining this with our assumptions gives us that $(\mathbf{S} \cup \mathbf{Z}) \cap \text{De}(\mathbf{X}, \mathcal{D}) = \emptyset$. In the remainder of the proof, we show that $\mathbf{S} \cup \mathbf{Z} \cup \mathbf{X} \setminus \{X\}$ blocks all back-door paths from $\mathbf{X}$ to $\mathbf{Y}$. The result follows by Definition 28.

Let p_1 be an arbitrary back-door path from $X \in \mathbf{X}$ to $Y \in \mathbf{Y}$ in $\mathcal{D}$. For sake of contradiction, suppose that p_1 is d-connecting given $\mathbf{S} \cup \mathbf{Z} \cup \mathbf{X} \setminus \{X\}$. Let X_C be the node in $\mathbf{X}$ closest to Y on p_1 , and let $p_2 = p_1(X_C, Y)$. Note that p_2 is proper. When $X_C = X$, then $p_2 = p_1$ is a back-door path. When $X_C \neq X$, then because p_1 is d-connecting given $\mathbf{S} \cup \mathbf{Z} \cup \mathbf{X} \setminus \{X\}$, we have that X_C is a collider on p_1 , and therefore, p_2 is again a back-door path. Thus, p_2 is a proper back-door path from $\mathbf{X}$ to $\mathbf{Y}$ that, by assumption, must be blocked by $\mathbf{S} \cup \mathbf{Z}$.

Let A be the node on p_2 immediately following X_C . That is, p_2 contains $X_C \leftarrow A$. Note that since p_2 is blocked given $\mathbf{S} \cup \mathbf{Z}$, then $A \neq Y$. Thus, we consider the path $p_3 = p_2(A, Y)$. Since p_1 is d-connecting given $\mathbf{S} \cup \mathbf{Z} \cup \mathbf{X} \setminus \{X\}$, where A is a non-collider on p_1 , then $A \notin \mathbf{S} \cup \mathbf{Z} \cup \mathbf{X} \setminus \{X\}$ and thus, p_3 is also d-connecting given $\mathbf{S} \cup \mathbf{Z} \cup \mathbf{X} \setminus \{X\}$. Similarly, since p_2 is blocked given $\mathbf{S} \cup \mathbf{Z}$, where A is not a collider on p_2 and $A \notin \mathbf{S} \cup \mathbf{Z} \cup \mathbf{X} \setminus \{X\}$, then p_3 is also blocked by $\mathbf{S} \cup \mathbf{Z}$.

Since p_3 is d-connecting given $\mathbf{S} \cup \mathbf{Z} \cup \mathbf{X} \setminus \{X\}$ and blocked given $\mathbf{S} \cup \mathbf{Z}$, then p_3 must contain at least one collider in $\text{An}(\mathbf{X} \setminus \{X\}, \mathcal{D}) \setminus \text{An}(\mathbf{S} \cup \mathbf{Z}, \mathcal{D})$. Let C be the closest such collider to Y on p_3 and let $r = \langle C, \dots, X' \rangle, X' \in \mathbf{X}$, be a shortest causal path from C to $\mathbf{X}$ in $\mathcal{D}$. While there must be a causal path from C to $\mathbf{X} \setminus \{X\}$ in $\mathcal{D}$, note that r need not be one, and thus, we allow for the possibility that $X' = X$.

Let B be the node closest to X' on r that is also on $p_3(C, Y)$, and define the path $t = (-r)(X', B) \oplus p_3(B, Y)$. Note that since p_2 is proper, $(-r)(X', B)$ is at least of length one, and therefore, t is a back-door path. Further, since p_3 is d-connecting given $\mathbf{S} \cup \mathbf{Z} \cup \mathbf{X} \setminus \{X\}$ and by the definition of C and r , we have that t is proper back-door path from $\mathbf{X}$ to $\mathbf{Y}$ that is d-connecting given $\mathbf{S} \cup \mathbf{Z}$. But this contradicts that $\mathbf{S}$ satisfies the conditional back-door criterion relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{D}$. ■

A.6 PROOFS FOR SECTION 2.3.3: MPDAGS - CONSTRUCTING CONDITIONAL ADJUSTMENT SETS

This section includes the proofs of two results from Section 2.3.3: Lemma 4 and Theorem 5. We also provide three supporting results needed for these proofs.

A.6.1 MAIN RESULTS

Proof of Lemma 4. By Lemma 41, $\text{Pa}(X, \mathcal{G})$ must satisfy condition (a) of Definition 2, so it suffices to show that $\text{Pa}(X, \mathcal{G}) \cup \mathbf{Z}$ blocks all non-causal definite status paths from X to $\mathbf{Y}$ in $\mathcal{G}$. Note that since $\mathbf{Y} \cap \text{Pa}(X, \mathcal{G}) = \emptyset$, any definite status path from X to $\mathbf{Y}$ in $\mathcal{G}$ that starts with an edge into X is blocked by $\text{Pa}(X, \mathcal{G}) \cup \mathbf{Z}$.

Further, any non-causal definite status path from X to $\mathbf{Y}$ in $\mathcal{G}$ that starts with an edge out of X or an undirected edge must contain a collider. Additionally, the closest collider to X on any such path and all of its descendants in $\mathcal{G}$ must be in $\text{PossDe}(X, \mathcal{G})$ by Lemma 63. Then since $[\text{Pa}(X, \mathcal{G}) \cup \mathbf{Z}] \cap \text{PossDe}(X, \mathcal{G}) = \emptyset$, these paths are also blocked by $\text{Pa}(X, \mathcal{G}) \cup \mathbf{Z}$.

■

Proof of Theorem 5. By Theorem 3, it suffices to show that $\text{Adjust}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathcal{G})$ and $\text{O}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$ separately satisfy the conditional adjustment criterion relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{G}$ (Definition 2). We start by noting that $\text{Adjust}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathcal{G})$ and $\text{O}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$ are both disjoint from $\text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) \cup \mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}$, so it suffices to prove that (a) $\text{Adjust}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathcal{G}) \cup \mathbf{Z}$ and (b) $\text{O}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) \cup \mathbf{Z}$ block all proper non-causal definite status paths from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$. We prove (a) and (b) below. For these proofs, note that $\mathbf{Z} \cap \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) = \emptyset$ by the assumption that $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$ and by Lemma 41.

(a) **Adjust($\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathcal{G}$) $\cup \mathbf{Z}$:** Suppose for sake of contradiction that there is a proper non-causal definite status path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$ that is d-connecting given $\text{Adjust}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathcal{G}) \cup \mathbf{Z}$. Let $p = \langle X, \dots, Y \rangle$ be a shortest such path.

Since p is proper, no non-endpoint on p is in $\mathbf{X}$. Suppose for sake of contradiction that there exists $Y' \in \mathbf{Y}$ that is a non-endpoint on p . By choice of p , this implies that $p(X, Y')$ is

possibly causal. Then by Lemma 42, since p is non-causal, $p(Y', Y)$ must contain a collider on p . Let C be the closest such collider to Y' (possibly $C = Y'$). Note that by Lemma 42, $C \in \text{PossDe}(Y', \mathcal{G})$, so by Lemma 63, $\text{De}(C, \mathcal{G}) \subseteq \text{PossDe}(Y', \mathcal{G})$, where $Y' \in \text{PossDe}(X, \mathcal{G})$. Thus, $\text{De}(C, \mathcal{G}) \subseteq \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$. However, this contradicts that p is d-connecting given $\text{Adjust}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathcal{G}) \cup \mathbf{Z}$. Therefore, no non-endpoint on p is in $\mathbf{X} \cup \mathbf{Y}$.

We now consider cases (1) and (2) below.

- (1) Consider when there is no collider on p . Since p is d-connecting given $\text{Adjust}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathcal{G}) \cup \mathbf{Z}$, no node on p is in $\text{Adjust}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathcal{G}) \cup \mathbf{Z}$. Then by Equation (2.6), no node on p is in $\text{PossAn}(\mathbf{X} \cup \mathbf{Y}, \mathcal{G}) \setminus [\text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) \cup \mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}]$. However, note that by Lemma 42, every non-endpoint on p is a possible ancestor of an endpoint on p and thus is in $\text{PossAn}(\mathbf{X} \cup \mathbf{Y}, \mathcal{G}) \setminus (\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z})$. Combining these, we have that all non-endpoints on p are in $\text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$. But this implies that there is no set that is both disjoint from $\text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$ and can block p . By Theorem 3, this contradicts our assumption that there is a conditional adjustment set relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{G}$.
- (2) Consider when there is at least one collider C on p . For sake of contradiction, suppose that there are more than three nodes on p . Then there is a non-collider $B \notin \mathbf{X} \cup \mathbf{Y}$ such that $C \leftarrow B$ or $B \rightarrow C$ is on p . Since p is d-connecting given $\text{Adjust}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathcal{G}) \cup \mathbf{Z}$, then $B \notin \mathbf{Z}$ and $B \in \text{An}(\text{Adjust}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathcal{G}) \cup \mathbf{Z}, \mathcal{G})$. By Equation (2.6) and Lemma 63, $B \in [\text{PossAn}(\mathbf{X} \cup \mathbf{Y}, \mathcal{G}) \cup \text{An}(\mathbf{Z}, \mathcal{G})] \setminus (\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z})$. Additionally, since p is d-connecting given $\text{Adjust}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathcal{G}) \cup \mathbf{Z}$, then $B \notin \text{Adjust}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathcal{G}) \equiv [\text{PossAn}(\mathbf{X} \cup \mathbf{Y}, \mathcal{G}) \cup \text{An}(\mathbf{Z}, \mathcal{G})] \setminus (\text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) \cup \mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z})$. Combining these, we have that $B \in \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$. Since there is a causal path in $\mathcal{G}$ from B to every node in $\text{De}(C, \mathcal{G})$, by Lemma 63, $\text{De}(C, \mathcal{G}) \subseteq \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$. However, this would contradict that p is d-connecting given $\text{Adjust}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathcal{G}) \cup \mathbf{Z}$.

Hence, p must be of the form $X \rightarrow C \leftarrow Y$, where $C \in \text{An}(\text{Adjust}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathcal{G}) \cup \mathbf{Z}, \mathcal{G})$ and thus by Equation (2.6) and Lemma 63, $C \in \text{PossAn}(\mathbf{X} \cup \mathbf{Y}, \mathcal{G}) \cup \text{An}(\mathbf{Z}, \mathcal{G})$. Note that $C \notin \text{An}(\mathbf{Z}, \mathcal{G})$, since otherwise, $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) \neq \emptyset$. Further, $C \notin$

$\text{PossAn}(\mathbf{Y}, \mathcal{G})$, because otherwise by Lemma 63, $C \in \text{PossMed}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$, which would imply $\text{De}(C, \mathcal{G}) \subseteq \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$ which we have shown is a contradiction. Therefore, $C \in \text{PossAn}(\mathbf{X}, \mathcal{G})$.

Let $q = \langle C = Q_1, \dots, Q_m = X' \rangle$, $m \geq 2$, be a shortest possibly causal path in $\mathcal{G}$ from C to $\mathbf{X}$. Further, define the node Q_j , $j \in \{1, \dots, m\}$, as follows. When q has no directed edges, let $Q_j = Q_m$. When q has at least one directed edge, let Q_j be the node on q closest to Q_1 such that $Q_j \rightarrow Q_{j+1}$ is on q . Note that by Lemma 61, q is unshielded. Thus by R1 of Meek [1995], q takes the form $Q_1 - \dots - Q_j \rightarrow \dots \rightarrow Q_m$.

Pause to consider the path $X \rightarrow Q_1 \leftarrow Y$. Note that $X \leftarrow Y$ cannot be in $\mathcal{G}$, because no set can block this proper non-causal definite status path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$. By Theorem 3, this would contradict our assumption that there is a conditional adjustment set relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{G}$. Similarly, $X \rightarrow Y$ and $X - Y$ are not in $\mathcal{G}$, because this would imply $\text{De}(C, \mathcal{G}) \subseteq \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$, which we have shown is a contradiction. Thus, $X \rightarrow Q_1 \leftarrow Y$ is an unshielded collider in $\mathcal{G}$.

We complete this case by showing that $\mathcal{G}$ contains $X \rightarrow Q_j \leftarrow Y$. If $j = 1$, we are done. If instead $j > 1$, then consider the node Q_2 . Since $X \rightarrow Q_1 - Q_2$ and $Y \rightarrow Q_1 - Q_2$ are in $\mathcal{G}$, so is a path $\langle X, Q_2, Y \rangle$ by R1 of Meek [1995]. The unshielded paths $X \rightarrow Q_2 - Y$ and $X - Q_2 \leftarrow Y$ contradict that R1 of Meek [1995] is completed in $\mathcal{G}$. Further, the path $Q_2 \rightarrow Y \rightarrow Q_1 - Q_2$ or $Q_2 \rightarrow X \rightarrow Q_1 - Q_2$ contradicts that R2 of Meek [1995] is completed in $\mathcal{G}$, and the path $X - Q_2 - Y$ contradicts that R3 of Meek [1995] is completed in $\mathcal{G}$. This leaves only one option for $\langle X, Q_2, Y \rangle$, and that is $X \rightarrow Q_2 \leftarrow Y$.

If $j = 2$, we are done. If instead $j > 2$, then we consider the node Q_3 . By identical logic to that above, we can show that $\mathcal{G}$ contains $X \rightarrow Q_3 \leftarrow Y$. Continuing in this way, we have that $\mathcal{G}$ contains $X \rightarrow Q_j \leftarrow Y$.

With this shown, we derive our final contradictions. When $j = m$, then $\mathcal{G}$ contains $X' \leftarrow Y$. But this is a proper non-causal definite status path from $\mathbf{X}$ to $\mathbf{Y}$ that no

set can block, which we have shown is a contradiction. When $j < m$, then $\mathcal{G}$ contains the following two paths: $X' \leftarrow \cdots \leftarrow Q_j \leftarrow Y$ and $X \rightarrow Q_j \leftarrow Y$. These paths are proper non-causal definite status paths from $\mathbf{X}$ to $\mathbf{Y}$ that cannot both be blocked by the same set, which again is a contradiction.

(b) $\mathbf{O}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) \cup \mathbf{Z}$: Let p' be an arbitrary proper non-causal definite status path from $X \in \mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$, and let Y be the node in $\mathbf{Y}$ closest to X on p' such that $p'(X, Y)$ is still a proper non-causal definite status path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$. Then let $p = p'(X, Y)$, where $p = \langle X = V_1, \dots, V_k = Y \rangle, k \geq 2$. Additionally, note that by assumption, $Y \in \text{PossMed}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$.

We now consider cases (1) and (2) below. In both cases, we show that p – and therefore p' – is blocked by $\mathbf{O}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) \cup \mathbf{Z}$.

- (1) Suppose that p ends with $V_{k-1} \leftarrow Y$ or $V_{k-1} - Y$. If p has no colliders, then by Lemma 42, $(-p)$ is a possibly causal path from Y to X . Since $Y \in \text{PossMed}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$, this implies that $V_2, \dots, V_{k-1} \in \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$. But then there is no set that is both disjoint from $\text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$ and can block p . By Theorem 3, this contradicts our assumption that there is a conditional adjustment set relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{G}$. Hence, there must be a collider on p .

Let C be the closest collider to Y on p . By Lemma 42, $C \in \text{PossDe}(Y, \mathcal{G})$. Thus by Lemma 63, $\text{De}(C, \mathcal{G}) \subseteq \text{PossDe}(Y, \mathcal{G})$. By assumption, $Y \in \text{PossMed}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$, which implies that $\text{De}(C, \mathcal{G}) \subseteq \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$. Since $[\mathbf{O}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) \cup \mathbf{Z}] \cap \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) = \emptyset$, p is blocked by $\mathbf{O}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) \cup \mathbf{Z}$.

- (2) Suppose that p ends with $V_{k-1} \rightarrow Y$. Note that p is not a possibly causal path from X to Y , so by Lemma 42, there must be an edge $V_{i-1} \leftarrow V_i, i \in \{2, \dots, k-1\}$, on p . In particular, let V_i be the closest node to Y on p such that $V_{i-1} \leftarrow V_i$ is on p .

In order to complete this proof, we want to show that either $\{V_i, \dots, V_{k-1}\} \cap [\mathbf{O}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) \cup \mathbf{Z}] \neq \emptyset$ or $\{V_i, \dots, V_{k-1}\} \subset \text{PossMed}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$. In both cases, we will

show that p is blocked by $O(\mathbf{X}, \mathbf{Y}, \mathcal{G}) \cup \mathbf{Z}$. To do this, we briefly note that by the choice of V_i , the path $p(V_i, Y)$ is possibly causal and every node in $\{V_i, \dots, V_{k-1}\}$ is a non-collider on p . Further by the choice of p , no node in $\{V_i, \dots, V_{k-1}\}$ is in $\mathbf{X} \cup \mathbf{Y}$. We turn to consider each node in $\{V_i, \dots, V_{k-1}\}$, working backward through the set.

Consider the node V_{k-1} . If $V_{k-1} \in O(\mathbf{X}, \mathbf{Y}, \mathcal{G}) \cup \mathbf{Z}$, then since V_{k-1} is a non-collider on p , p is blocked by $O(\mathbf{X}, \mathbf{Y}, \mathcal{G}) \cup \mathbf{Z}$, and we are done. Consider when $V_{k-1} \notin O(\mathbf{X}, \mathbf{Y}, \mathcal{G}) \cup \mathbf{Z}$. Since $Y \in \text{PossMed}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$ and since $V_{k-1} \rightarrow Y$ is in $\mathcal{G}$, then either $V_{k-1} \in \text{PossMed}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$ or $V_{k-1} \in \text{Pa}(\text{PossMed}(\mathbf{X}, \mathbf{Y}, \mathcal{G}), \mathcal{G})$. We show the latter is impossible. If $V_{k-1} \in \text{Pa}(\text{PossMed}(\mathbf{X}, \mathbf{Y}, \mathcal{G}), \mathcal{G})$ and $V_{k-1} \notin O(\mathbf{X}, \mathbf{Y}, \mathcal{G}) \cup \mathbf{Z}$, then by Equation (2.7), we have that $V_{k-1} \in \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$. But by Lemma 41, this implies that $V_{k-1} \in \text{De}(\mathbf{X}, \mathcal{G})$. Since $\mathcal{G}$ contains $V_{k-1} \rightarrow Y$, then $V_{k-1} \in \text{PossMed}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$. But this contradicts that $V_{k-1} \in \text{Pa}(\text{PossMed}(\mathbf{X}, \mathbf{Y}, \mathcal{G}), \mathcal{G})$ by the definition of a parent set. Therefore, either $V_{k-1} \in O(\mathbf{X}, \mathbf{Y}, \mathcal{G}) \cup \mathbf{Z}$ and we are done, or $V_{k-1} \in \text{PossMed}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$.

In the latter case, we turn to consider V_{k-2} if such a node exists. If p contains $V_{k-2} \rightarrow V_{k-1}$, then since $V_{k-1} \in \text{PossMed}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$, we can use the same logic as above to show that either $V_{k-2} \in O(\mathbf{X}, \mathbf{Y}, \mathcal{G}) \cup \mathbf{Z}$ and we are done, or $V_{k-2} \in \text{PossMed}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$. If p contains $V_{k-2} - V_{k-1}$, then since $V_{k-1} \in \text{PossMed}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$, we have that $V_{k-2} \in \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) \subseteq \text{De}(\mathbf{X}, \mathcal{G})$. Because $p(V_{k-2}, Y)$ is possibly causal, then by Lemma 63, $V_{k-2} \in \text{PossMed}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$.

Working backward in this way, either a node on $p(V_i, Y)$ is in $O(\mathbf{X}, \mathbf{Y}, \mathcal{G}) \cup \mathbf{Z}$ and we are done, or $V_j \in \text{PossMed}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$ for all $j \in \{i, \dots, k-1\}$. In the latter case, we have that $V_i \in \text{PossMed}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$ and that every node in $\{V_i, \dots, V_{k-1}\} \subseteq \text{PossMed}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) \subseteq \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$ is a non-collider on p . We can now apply the same argument as in (1) above to show that $p(X, V_i)$ – and therefore p – is blocked given $O(\mathbf{X}, \mathbf{Y}, \mathcal{G}) \cup \mathbf{Z}$. ■

A.6.2 SUPPORTING RESULTS

Lemma 61 *Let X and Y be distinct nodes in an MPDAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ and let p be a possibly causal path from X to Y in $\mathcal{G}$. Then any shortest subsequence of p forms an unshielded, possibly causal path from X to Y .*

Proof of Lemma 61. This result is similar to Lemma 3.6 of Perković et al. [2017], but we derive a slightly more general statement.

Let k be the number of nodes on p . Pick an arbitrary shortest subsequence of p and call it p^* , where $p^* = \langle X = V_0, \dots, V_\ell = Y \rangle$, $0 < \ell \leq k$. Note that there is no edge $V_i \leftarrow V_j$, $0 \leq i < j \leq k$ in $\mathcal{G}$, since this would contradict that p is possibly causal. Thus, p^* is also possibly causal by definition. Further note that p^* is unshielded, since if any triple on the path is shielded, it either contradicts that p^* is possibly causal (i.e., $V_i \leftarrow V_{i+2}$ cannot be in p^*) or that p^* is a shortest subsequence of p (i.e., $V_i \rightarrow V_{i+2}$ and $V_i - V_{i+2}$ cannot be in p^*). ■

Lemma 62 *Let $p = \langle P_0, \dots, P_k \rangle$ be a path in an MPDAG $\mathcal{G}$. Then p is possibly causal if and only if $\mathcal{G}$ does not contain any path $P_i \leftarrow \dots \leftarrow P_j$, $0 \leq i < j \leq k$.*

Proof of Lemma 62. Suppose that $\mathcal{G}$ does not contain any path $P_i \leftarrow \dots \leftarrow P_j$, $0 \leq i < j \leq k$. Then $\mathcal{G}$ does not contain any edge $P_i \leftarrow P_j$, $0 \leq i < j \leq k$. Therefore, by definition, p is possibly causal in $\mathcal{G}$.

Now suppose p is possibly causal in $\mathcal{G}$. For sake of contradiction, suppose $\mathcal{G}$ contains a path q from P_i to P_j , $0 \leq i < j \leq k$, of the form $P_i = Q_0 \leftarrow Q_1 \leftarrow \dots \leftarrow Q_{\ell-1} \leftarrow Q_\ell = P_j$.

Consider the subpath of p from P_i to P_j . Note that this subpath is a possibly causal path. Let $r = \langle P_i = R_0, R_1, \dots, R_m = P_j \rangle$ be a shortest subsequence of this subpath. By Lemma 61, r is an unshielded, possibly causal path.

Consider the edge $r(R_0, R_1)$. $R_0 \leftarrow R_1$ cannot be in r , since r is possibly causal. Neither is $R_0 \rightarrow R_1$ in r since r being unshielded would imply, by R1 of Meek [1995], that $\mathcal{G}$ contains

the cycle $P_i = R_0 \rightarrow R_1 \rightarrow \cdots \rightarrow R_m = P_j = Q_\ell \rightarrow Q_{\ell-1} \rightarrow \cdots \rightarrow Q_0 = P_i$. Thus r contains $R_0 - R_1$.

However, note that no DAG in $[\mathcal{G}]$ can contain the edge $R_0 \rightarrow R_1$, since r being unshielded would imply, by R1 of Meek [1995], that the DAG contains the cycle $P_i = R_0 \rightarrow R_1 \rightarrow \cdots \rightarrow R_m = P_j = Q_\ell \rightarrow Q_{\ell-1} \rightarrow \cdots \rightarrow Q_0 = P_i$. This contradicts that r contains $R_0 - R_1$. Thus we conclude that $\mathcal{G}$ does not contain any path $P_i \leftarrow \cdots \leftarrow P_j$, $0 \leq i < j \leq k$. ■

Lemma 63 *Let X , Y , and Z be distinct nodes in an MPDAG $\mathcal{G}$.*

- (i) *If p is a possibly causal path from X to Y and q is a causal path from Y to Z , then $p \oplus q$ is a possibly causal path from X to Z .*
- (ii) *If p is a causal path from X to Y and q is a possibly causal path from Y to Z , then $p \oplus q$ is a possibly causal path from X to Z .*

Proof of Lemma 63. Let $p = \langle X = P_0, P_1, \dots, P_k = Y \rangle$ and let $q = \langle Y = Q_0, Q_1, \dots, Q_r = Z \rangle$. Before beginning the main arguments, we note that p and q cannot share any nodes other than Y , and thus, we can define a path $p \oplus q$. To see this, for sake of contradiction, suppose p and q share at least one node other than Y . Let $\mathbf{S}$ denote the collection of such nodes, and consider the node in $\mathbf{S}$ with the lowest index on q . That is, consider $Q_j \in \mathbf{S}$ such that $j \leq \ell$ for all $Q_\ell \in \mathbf{S}$. Let $Q_j = P_i$ for some $P_i \neq Y$ on p . Note that since q or p is causal, $\mathcal{G}$ contains either $P_k = Q_0 \rightarrow Q_1 \rightarrow \cdots \rightarrow Q_j = P_i$ or $Q_j = P_i \rightarrow P_{i+1} \rightarrow \cdots \rightarrow Y = Q_0$. By Lemma 62, the first option contradicts that p is possibly causal and the second contradicts that q is possibly causal. Thus we conclude that p and q cannot share any nodes other than Y .

For $p \oplus q$ to be possibly causal in $\mathcal{G}$ we only need to show that there is no backward edge between any two nodes on $p \oplus q$. Note that there is no edge $P_{i_1} \leftarrow P_{j_1}$ for $0 \leq i_1 < j_1 \leq k$, or $Q_{i_2} \leftarrow Q_{j_2}$ for $0 \leq i_2 < j_2 \leq r$ in $\mathcal{G}$, by choice of p and q .

- (i) Assume for sake of contradiction that there exists an edge $P_i \leftarrow Q_j$ in $\mathcal{G}$ for $i \in \{0, \dots, k-1\}$ and $j \in \{1, \dots, r\}$. Note that P_i is on p and not q , and analogously, Q_j is on

q and not p , since we have shown p and q cannot share nodes other than Y . Also note that since q is causal, it contains $Y \rightarrow Q_1 \rightarrow \dots \rightarrow Q_j$.

Consider the subpath $p(P_i, Y)$. Since p is possibly causal, so is this subpath. Pick an arbitrary shortest subsequence of $p(P_i, Y)$ and call it t , where $t = \langle P_i = T_0, \dots, T_m = Y \rangle$, $m \geq 1$. By Lemma 61, t forms an unshielded, possibly causal path from P_i to Y .

Consider the edge $t(P_i, T_1)$. Edge $P_i \leftarrow T_1$ cannot be on t , since t is possibly causal. Then $P_i \rightarrow T_1$ or $P_i - T_1$ must be in $\mathcal{G}$. However, note that no DAG in $[\mathcal{G}]$ can contain the edge $P_i \rightarrow T_1$, since t being unshielded would imply, by R1 of Meek [1995], that the DAG contains the cycle $P_i \rightarrow T_1 \rightarrow \dots \rightarrow Y \rightarrow \dots \rightarrow Q_j \rightarrow P_i$. This contradicts that t contains $P_i - T_1$ or $P_i \rightarrow T_1$. Thus, there does not exist an edge $P_i \leftarrow Q_j$ in $\mathcal{G}$.

(ii) Assume for sake of contradiction that there exists an edge $P_i \leftarrow Q_j$ in $\mathcal{G}$ for $i \in \{0, \dots, k-1\}$ and $j \in \{1, \dots, r\}$. Note that P_i is on p and not q , and analogously, Q_j is on q and not p , since we have shown p and q cannot share nodes other than Y . Also note that since p is causal, it contains $P_i \rightarrow P_{i+1} \rightarrow \dots \rightarrow Y$.

Consider the subpath $q(Y, Q_j)$. Since q is possibly causal, so is this subpath. Pick an arbitrary shortest subsequence of $q(Y, Q_j)$ and call it t , where $t = \langle Y = T_0, \dots, T_m = Q_j \rangle$, $m \geq 1$. By Lemma 61, t forms an unshielded, possibly causal path from Y to Q_j .

Consider the edge $t(Y, T_1)$. Edge $Y \leftarrow T_1$ cannot be on t , since t is possibly causal. Then $Y \rightarrow T_1$ or $Y - T_1$ must be in $\mathcal{G}$. However, note that no DAG in $[\mathcal{G}]$ can contain the edge $Y \rightarrow T_1$, since t being unshielded would imply, by R1 of Meek [1995], that the DAG contains the cycle $Y \rightarrow T_1 \rightarrow \dots \rightarrow Q_j \rightarrow P_i \rightarrow P_{i+1} \rightarrow \dots \rightarrow Y$. This contradicts that t contains $Y - T_1$ or $Y \rightarrow T_1$. Thus, there does not exist an edge $P_i \leftarrow Q_j$ in $\mathcal{G}$. ■

A.7 PROOF FOR SECTION 2.4.1: PAGS - CONDITIONAL ADJUSTMENT CRITERION

This section includes the proof of Theorem 9 and one result (Lemma 64) needed for the proof of Lemma 8. The statements of Theorem 9 and Lemma 8 can be found in Section 2.4.1.

Figure A.2 shows how the results in this paper fit together to prove Theorem 9. Note that

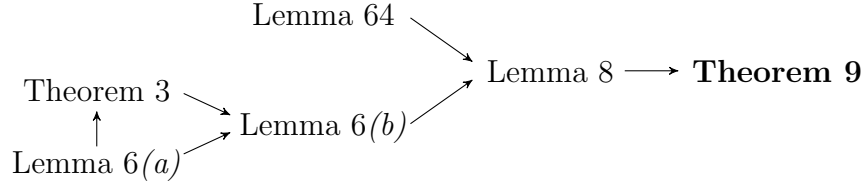


Figure A.2: Proof structure of Theorem 9.

Theorem 9 is an analogous result to Theorem 3 (Section 2.3.1), where the former applies to PAGs and the latter to MPDAGs. However, while the proof of Theorem 3 relies directly on completeness and soundness proofs for DAGs (see Figure A.1 in Appendix A.4), the proof of Theorem 9 relies on them indirectly through Theorem 3.

Proof of Theorem 9. Follows from Lemma 8 and Theorem 46. ■

Lemma 64 *Let $\mathbf{X}$ and $\mathbf{Z}$ be disjoint node sets in a PAG $\mathcal{G}$. Then the following statements are equivalent.*

(i) $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$.

(ii) $\mathbf{Z} \cap \text{De}(\mathbf{X}, \mathcal{D}) = \emptyset$ in every DAG $\mathcal{D}$ represented by $\mathcal{G}$.

Proof of Lemma 64. $\neg(i) \Rightarrow \neg(ii)$ Let p be a possibly causal path from $\mathbf{X}$ to $\mathbf{Z}$ in $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ and let $p^* = \langle X = V_0, \dots, V_k = Z \rangle$, $k \geq 1$, $X \in \mathbf{X}$, $Z \in \mathbf{Z}$, be an unshielded possibly causal subsequence of p in $\mathcal{G}$.

Since p^* contains $X \circ\!\!\circ V_1$, $X \circ\!\!\rightarrow V_1$ or $X \rightarrow V_1$, there must be some MAG $\mathcal{M}$ in $[\mathcal{G}]$ with the edge $X \rightarrow V_1$. Let p^{**} be the path in $\mathcal{M}$ corresponding to p^* in $\mathcal{G}$. Then since p^* is unshielded, so is p^{**} , and so p^{**} takes the form $X \rightarrow V_1 \rightarrow \dots \rightarrow V_k$. Let $\mathcal{D}$ be a DAG created from $\mathcal{M}$, by retaining all the nodes in $\mathcal{M}$ and all the directed edges in $\mathcal{M}$ and by adding a node L_{AB} and edges $L_{AB} \rightarrow B$ and $L_{AB} \rightarrow A$ for each bidirected edge $A \leftrightarrow B$ in $\mathcal{M}$ (this DAG is titled the canonical DAG by Richardson and Spirtes, 2002). Now, DAG $\mathcal{D}$ contains a causal path from X to Z .

$\neg(ii) \Rightarrow \neg(i)$ If there is a DAG $\mathcal{D}$ represented by $\mathcal{G}$ with a causal path from $X \in \mathbf{X}$ to $Z \in \mathbf{Z}$, then any MAG $\mathcal{M}$ of $\mathcal{D}$ that contains X and Z will contain a causal path from X to Z . This is due to the fact that a MAG of a DAG will preserve ancestral relationships between observed variables. Then the path in $\mathcal{G}$ that corresponds to q in $\mathcal{M}$ cannot have any arrowheads pointing in the direction of X , and so it must be possibly causal. ■

A.8 PROOFS FOR SECTION 2.4.2: PAGS - CONSTRUCTING CONDITIONAL ADJUSTMENT SETS

This section includes the proof of Theorem 10, which can be found in Section 2.4.2. We provide one supporting result needed for the proof of this theorem.

We make an important remark here on R software. Note that by Lemmas 6 and 8, any algorithms developed for checking the existence of an unconditional adjustment set (Definition 26) also apply to conditional adjustment sets – provided that $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$. First consider the R package `dagitty` [Textor et al., 2016]. Suppose the condition on $\mathbf{Z}$ is satisfied and let $\mathbf{S}$ be a set such that $\mathbf{S} \cap (\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}) = \emptyset$. Then, one can apply the function `isAdjustmentSet` of the package `dagitty` to a PAG $\mathcal{G}$, set $\mathbf{S} \cup \mathbf{Z}$, exposure $\mathbf{X}$, and outcome $\mathbf{Y}$ to learn whether $\mathbf{S}$ is a conditional adjustment set relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{G}$. Next consider the R package `pcalg` [Kalisch et al., 2012]. Suppose the condition on $\mathbf{Z}$ is satisfied and let $\mathbf{S}$ be a set such that $\mathbf{S} \cap (\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}) = \emptyset$. Then, one could apply the function `gac` of the package `pcalg` to the MPDAG or PAG $\mathcal{G}$ and to the node sets $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{S} \cup \mathbf{Z}$. These functions will return `TRUE` if and only if $\mathbf{S}$ is a conditional adjustment set relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{G}$, and `FALSE` otherwise.

A.8.1 MAIN RESULT

Proof of Theorem 10. Suppose that $\text{Adjust}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathcal{G})$ does not satisfy the conditional adjustment criterion relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{G}$. Since $\text{Adjust}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathcal{G}) \cap \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) = \emptyset$ by construction, it must be that there is a proper definite status non-causal path from $\mathbf{X}$ to $\mathbf{Y}$ that is m-connecting given $\text{Adjust}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathcal{G}) \cup \mathbf{Z}$. By Lemma 65, there is then a proper

definite status non-causal path p from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$ such that all definite non-colliders on p are in $\text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$ (case (ii) of Lemma 65) and all colliders on p are in $\text{An}(\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}, \mathcal{G})$ (cases (iii) and (vi) of Lemma 65). Since $\text{An}(\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}, \mathcal{G}) \subseteq \text{An}(\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z} \cup \mathbf{S}, \mathcal{G})$, for any set $\mathbf{S}$ that satisfies $[\mathbf{S} \cup \mathbf{Z}] \cap [\mathbf{X} \cup \mathbf{Y} \cup \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})] = \emptyset$, Lemma 45 implies that there is also a proper definite status non-causal path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$ that is open given $\mathbf{S}$. Since this is true for an arbitrary set $\mathbf{S}$ that satisfies condition (a) of Definition 7, it follows that there cannot be any set that satisfies the conditional adjustment criterion relative to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{G}$. $\blacksquare$

A.8.2 SUPPORTING RESULT

Lemma 65 *Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$, be pairwise disjoint node sets in a PAG $\mathcal{G}$, where $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$ and where every proper possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$ starts with a visible edge out of $\mathbf{X}$. Suppose furthermore, that there exists a set $\mathbf{S}$ that satisfies the conditional adjustment criterion for $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $\mathcal{G}$. If there is a proper definite status non-causal path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$ that is m -connecting given $\text{Adjust}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathcal{G}) \cup \mathbf{Z}$ (see definition in Theorem 10), then there is a path p from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$ such that the following hold.*

- (i) *Path p is a proper definite status non-causal path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$.*
- (ii) *All definite non-colliders on p are in $\text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$.*
- (iii) *There is at least one collider on p , and all colliders on p are in $\mathbf{C}_1 \cup \mathbf{C}_2$, where $\mathbf{C}_1$ and $\mathbf{C}_2$ are disjoint sets such that*

$$\begin{aligned} \mathbf{C}_1 &\subseteq \text{PossAn}(\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}, \mathcal{G}) \setminus [\text{An}(\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}, \mathcal{G}) \cup \mathbf{X} \cup \mathbf{Y} \cup \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})] \text{ and} \\ \mathbf{C}_2 &\subseteq \text{An}(\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}, \mathcal{G}) \setminus [\mathbf{X} \cup \mathbf{Y} \cup \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})]. \end{aligned}$$

- (iv) *None of the colliders on p can be possible descendants of a non-collider on p .*

- (v) For any collider $C \in \mathbf{C}_1$ on p there is an unshielded possibly directed path from C to $\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}$ that does not start with $\circ\circ$.
- (vi) $\mathbf{C}_1 = \emptyset$, that is for any collider $C \in \mathbf{C}_1$ on p there is an unshielded directed path from C to $\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}$.

Proof of Lemma 65. Consider the sets of all proper definite status non-causal paths from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$ that are m-connecting given $\text{Adjust}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathcal{G}) \cup \mathbf{Z}$ and choose among them a shortest path with a shortest distance to $\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}$ (Definition 31). Let this path be called p , where $p = \langle X = V_1, V_2, \dots, V_k = Y \rangle$, $X \in \mathbf{X}, Y \in \mathbf{Y}, k \geq 2$. By choice of p , (i) is satisfied. We will now show that p also satisfies properties (ii)-(vi) above.

First, consider properties (ii) and (iii). Since p is m-connecting given $\text{Adjust}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathcal{G}) \cup \mathbf{Z}$, any collider on p is in $\text{An}(\text{Adjust}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathcal{G}) \cup \mathbf{Z}, \mathcal{G})$. Furthermore, since $\text{Adjust}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathcal{G}) \cup \mathbf{Z} = \text{PossAn}(\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}, \mathcal{G}) \setminus [\mathbf{X} \cup \mathbf{Y} \cup \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})]$, and since in a PAG $\mathcal{G}$ for any set $\mathbf{W}$, $\text{An}(\text{PossAn}(\mathbf{W}, \mathcal{G})) = \text{PossAn}(\mathbf{W}, \mathcal{G})$, we have that any collider on p is in $\text{PossAn}(\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}, \mathcal{G})$. Furthermore, since by definition, $\text{De}(\text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G}), \mathcal{G}) = \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$, we have that no collider on p can be in $\text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$. Hence, all colliders on p are in $\text{PossAn}(\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}, \mathcal{G}) \setminus \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$.

Also, since p is proper, a node in $\mathbf{X}$ cannot be a non-endpoint node on p . Now, since p is additionally chosen as a shortest proper non-causal definite status path from $\mathbf{X}$ to $\mathbf{Y}$ that is m-connecting given $\text{Adjust}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathcal{G})$, it holds that either a node in $\mathbf{Y}$ is not a non-endpoint node on p , or there is a node $Y' \in \mathbf{Y} \setminus \{Y\}$ on p such that $p(X, Y')$ is a possibly causal path from X to Y' . Moreover, in this case $p(X, Y')$ must be a causal path in $\mathcal{G}$ (because p must start with a visible edge and because $A \bullet \rightarrow B \circ \bullet C$ cannot be a subpath of a definite status path). Since p itself is a non-causal path in $\mathcal{G}$, there is a collider on p that is a descendant of Y' . But since $Y' \in \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$, this collider would then also have to be in $\text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$, which we have ruled out as an option in the previous paragraph. Hence, a node on $\mathbf{Y}$ is also not a non-endpoint node on p .

Then all colliders on p are in $\text{PossAn}(\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}, \mathcal{G}) \setminus [\text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) \cup \mathbf{X} \cup \mathbf{Y}]$. Also,

any definite non-collider on p is a possible ancestor of a collider on p or of an endpoint on p . Hence, every definite non-collider on p is in $\text{PossAn}(\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}, \mathcal{G}) \setminus [\mathbf{X} \cup \mathbf{Y}]$. But, since p is m -connecting given $\text{Adjust}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathcal{G}) \cup \mathbf{Z}$, none of the definite non-colliders on p are in $\text{PossAn}(\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}, \mathcal{G}) \setminus [\mathbf{X} \cup \mathbf{Y} \cup \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})]$. Therefore, any definite non-collider on p is in $\text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$. This proves property (ii).

Next, consider property (iii). We have already shown that any collider on p is in $\text{PossAn}(\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}, \mathcal{G}) \setminus [\text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) \cup \mathbf{X} \cup \mathbf{Y}]$. So it is only left to show that at least one collider is on p . Since we know that p must be blocked by $\mathbf{S} \cup \mathbf{Z}$ for some set $\mathbf{S}$, where $\mathbf{S} \cap [\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z} \cup \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})] = \emptyset$, and since all definite non-colliders on p are in $\text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$, there is at least one collider C on p .

Property (iv) follows almost directly now, since by (ii), all definite non-colliders on p are in $\text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$ and by (iii), none of the colliders can be in $\text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$. The claim then holds since by definition of the $\text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$ in a PAG, $\text{PossDe}(\text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G}), \mathcal{G}) = \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$.

Next, we show properties (v) and (vi). Let $C \in \mathbf{C}_1$ be a collider on p . Then $C \notin [\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}]$ and that there is an unshielded possibly directed path $r = \langle C, Q, \dots, V \rangle$ from C to a node $V \in \mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}$.

(v) Suppose for a contradiction that edge $\langle C, Q \rangle$ on r is of type $C \circ \circ Q$ (possibly $Q = V$). We derive a contradiction by constructing a proper definite status non-causal path from $\mathbf{X}$ to $\mathbf{Y}$ that is m -connecting given $\text{Adjust}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathcal{G}) \cup \mathbf{Z}$ and shorter than p , or of the same length as p but with a shorter distance to $\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}$ (Definition 31).

Let A and B be nodes on p such that $A \bullet \rightarrow C \leftarrow \bullet B$ is a subpath of p (possibly $A = X$, $B = Y$). Then paths $A \bullet \rightarrow C \circ \circ Q$ and $B \bullet \rightarrow C \circ \circ Q$ together with Lemma 43 imply that $A \bullet \rightarrow Q \leftarrow \bullet B$ is in $\mathcal{G}$.

Suppose first that $A \neq X$, and $B \neq Y$. Note that by property (iv) above, if $A \neq X$, then $A \leftrightarrow C$ is in $\mathcal{G}$. Moreover, if $A \leftrightarrow C$ is in $\mathcal{G}$, then $A \leftrightarrow Q$ is in $\mathcal{G}$, otherwise path $\langle A, Q, C \rangle$ and edge $A \leftrightarrow C$ contradict Lemma 44. Hence, if $A \neq X$, the collider/definite non-collider status of A is the same on p and on $p(X, A) \oplus \langle A, Q \rangle$. Analogous reasoning can be employed

in the case when $B \neq Y$, to show that $B \leftrightarrow Q$, that is, the collider/definite non-collider status of B is the same on p and on $\langle Q, B \rangle \oplus p(B, Y)$.

Now, we return to the general case where we allow $A = X$ and $B = Y$. In each of the cases below we will derive the contradiction by finding a path s from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$ that is a proper non-causal definite status path in $\mathcal{G}$ and m-connecting given $\text{Adjust}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathcal{G}) \cup \mathbf{Z}$. Additionally, the path s will either be shorter than p or of the same length as p , but with a shorter distance to $\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}$ (Definition 31) which implies a contradiction with our choice of p .

Suppose first that Q is not a node on p .

- If $Q \notin \mathbf{X} \cup \mathbf{Y}$, then
 - if $A \neq X$ and $B \neq Y$, then let $s = p(X, A) \oplus \langle A, Q, B \rangle \oplus p(B, Y)$. By the reasoning above, this path transformation amounts to replacing $A \leftrightarrow C \leftrightarrow B$ on p with $A \leftrightarrow Q \leftrightarrow B$ on s thereby creating a path with the same properties as p but with a shorter distance to $\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}$ (Definition 31).
 - If $A = X$, and $B \neq Y$, then let $s = \langle A, Q, B \rangle \oplus p(B, Y)$. This path transformation amounts to replacing $X \bullet \rightarrow C \leftrightarrow B$ on p , with $X \bullet \rightarrow Q \leftrightarrow B$ on s , thereby creating a path with the same properties as p and of the same length as p but with a shorter distance to $\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}$ (Definition 31).
 - If $A \neq X$, and $B = Y$, then let $s = p(X, A) \oplus \langle A, Q, B \rangle$. This path transformation amounts to replacing $A \leftrightarrow C \leftarrow \bullet Y$ on p , with $A \leftrightarrow Q \leftarrow \bullet Y$ on s , thereby creating a path with the same properties as p , that is of the same length, but with a shorter distance to $\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}$ (Definition 31).
 - If $A = X$, and $B = Y$, then let $s \langle A, Q, B \rangle$. Now s is of the form $X \bullet \rightarrow Q \leftarrow \bullet Y$ and clearly satisfies all the same properties as p while being of the same length, but with a shorter distance to $\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}$ (Definition 31).

- If $Q \equiv X'$, $X' \in \mathbf{X}$, then:

- if $B \neq Y$, let $s = \langle Q, B \rangle \oplus p(B, Y)$. This path transformation amounts to replacing $X \dots C \leftrightarrow B$ on p , with $X' \leftrightarrow B$ on s , thereby creating a shorter path with the same properties as p .
- If $B = Y$, then let $s = \langle Q, B \rangle$. Due to the discussion above, s is of the form $X' \leftarrow \bullet Y$ in $\mathcal{G}$.
- Otherwise, $Q \equiv Y', Y' \in \mathbf{Y}$. If $Q \in \mathbf{Y} \cap \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$, this would imply that $C \in \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$, which contradicts (iii). So Q must be in $\mathbf{Y} \setminus \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$. Then:
 - if $A \neq X$, then let $s = p(X, A) \oplus \langle A, Q \rangle$. This path transformation amounts to replacing $A \leftrightarrow C \dots Y$ on p , with $A \leftrightarrow Y'$ on s , thereby creating a shorter path with the same properties as p .
 - If $A = X$, then let $s = \langle A, Q \rangle$. Due to the discussion above, s is of the form $X \leftarrow \bullet Y'$ in $\mathcal{G}$.

Otherwise, Q is on p . Therefore, $Q \notin \mathbf{X} \cup \mathbf{Y}$. Also, Q is a collider on p , otherwise $Q \in \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$ and $C \in \text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$, because of $C \circ \circ Q$.

- Suppose first that Q is on $p(C, Y)$. Then:
 - if $A \neq X$, then let $s = p(X, A) \oplus \langle A, Q \rangle \oplus p(Q, Y)$. This path transformation amounts to replacing $A \leftrightarrow C \leftrightarrow \dots \leftrightarrow Q$ on p , with $A \leftrightarrow Q$ on s , thereby creating a shorter path with the same properties as p .
 - If $A = X$, then let $s = \langle A, Q \rangle \oplus p(Q, Y)$. This path transformation amounts to replacing $X \leftrightarrow C \leftrightarrow \dots \leftrightarrow Q$ on p , with $X \leftrightarrow Q$ on s , thereby creating a shorter path with the same properties as p .
- Next, suppose that Q is on $p(X, C)$. Then depending on whether $B = Y$, we can choose one of the following paths as the path s :

- if $B \neq Y$, then let $s = p(X, Q) \oplus \langle Q, B \rangle \oplus p(B, Y)$. This path transformation amounts to replacing $Q \leftrightarrow \dots \leftrightarrow C \leftrightarrow B$ on p , with $Q \leftrightarrow B$ on s , thereby creating a shorter path with the same properties as p .
- If $B = Y$, then let $s = p(X, Q) \oplus \langle Q, B \rangle$. Similarly to above, this path transformation amounts to replacing $Q \leftrightarrow \dots \leftrightarrow C \leftrightarrow Y$ on p , with $Q \leftrightarrow Y$ on s , thereby creating a shorter path with the same properties as p .

(vi) Since we showed above that the starting edge $\langle C, Q \rangle$ on $r = \langle C, Q, \dots, V \rangle$ is not of the form $C \circ \rightarrow Q$, and since r is an unshielded possibly directed path from C to $V \in \mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}$, in order to prove property (vi) it is enough to show that $\langle C, Q \rangle$ is also not of the form $C \circ \rightarrow Q$ (since $P_1 \bullet \rightarrow P_2 \circ \bullet P_3$ cannot be a subpath of any unshielded possibly directed path in $\mathcal{G}$, Zhang, 2008b). Suppose for a contradiction that $\langle C, Q \rangle$ is exactly of that form. Since $A \bullet \rightarrow C \leftarrow \bullet B$ and $C \circ \rightarrow Q$ are in $\mathcal{G}$, by Lemma 43, $A \bullet \rightarrow Q \leftarrow \bullet B$ is in $\mathcal{G}$.

Now, our goal is to identify a nodes A' and B' on p that satisfy the following. Node A' is on $p(X, A)$, and edge $A' \bullet \rightarrow Q$ is in $\mathcal{G}$. Additionally, $A' = X$ or A' is a non-endpoint node on p that has the same definite non-collider/collider status on p and on $p(X, A') \oplus \langle A', Q \rangle$. Similarly, B' is on $p(B, Y)$, and edge $B' \bullet \rightarrow Q$ is in $\mathcal{G}$. Additionally, $B' = Y$ or B' is a non-endpoint node on p that has the same definite non-collider/collider status on p and on $\langle Q, B' \rangle \oplus p(B', Y)$. We only show how to find node A' on $p(X, A)$, since the argument for finding B' on $p(B, Y)$ is exactly symmetric.

- Consider the path $p(X, C) = \langle X = V_1, V_2, \dots, V_{i-1} = A, V_i = C \rangle$. Note that by (iv) and the properties of unshielded paths, $p(X, C)$ is of the form $X \bullet \rightarrow V_2 \leftrightarrow \dots \leftrightarrow A \leftrightarrow C$ or $X \leftarrow V_2 \leftarrow \dots \leftarrow V_j \leftrightarrow \dots \leftrightarrow C$, for some $V_j, j \in \{2, \dots, i-1\}$.

Hence, if there is any non-endpoint node W on $p(X, A)$ such that $W \leftrightarrow Q$, this node has the same definite collider / non-collider status on both p and on $p(X, W) \oplus \langle W, Q \rangle$. Then we choose $A' \equiv W$. Otherwise, if there is a non-endpoint node W on $p(X, A)$ such that $-p(W, X)$ is of the form $W \rightarrow \dots \rightarrow X$, and an edge $W \rightarrow Q$ or $W \circ \rightarrow Q$ is

in $\mathcal{G}$, then W is a definite non-collider on both p and $p(X, W) \oplus \langle W, Q \rangle$ and we choose $A' \equiv W$.

We will now show that if neither of the above choices for A' are possible in $\mathcal{G}$, then $p(X, C)$ is of the form $X \leftrightarrow V_2 \leftrightarrow \dots \leftrightarrow C$, and for every node V_j , $j \in \{1, \dots, i\}$ on $p(X, C)$, the edge $V_j \rightarrow Q$ or $V_j \circ \rightarrow Q$ is in $\mathcal{G}$. In this case, we choose $A' \equiv X$.

Hence, consider first node $V_{i-1} = A$ on p . By above $V_{i-1} \bullet \rightarrow Q$ is in $\mathcal{G}$. Also, by our assumption $V_{i-1} \leftrightarrow Q$ is not in $\mathcal{G}$, so we must have either $V_{i-1} \rightarrow Q$ or $V_{i-1} \circ \rightarrow Q$ is in $\mathcal{G}$. Similarly, by the assumption above we now know that edge $\langle V_{i-2}, V_{i-1} \rangle$ is not of the form $V_{i-2} \leftarrow V_{i-1}$, so we can conclude that $V_{i-2} \leftrightarrow V_{i-1}$ is in $\mathcal{G}$.

Now, $V_{i-2} \leftrightarrow V_{i-1} \leftrightarrow C \circ \rightarrow Q$ and either $V_{i-1} \rightarrow Q$ or $V_{i-1} \circ \rightarrow Q$ is in $\mathcal{G}$. If $V_{i-1} \rightarrow Q$ is in $\mathcal{G}$, then $R4$ of Zhang [2008b] would imply that $V_{i-2} \in \text{Adj}(Q, \mathcal{G})$. Moreover, since $V_{i-2} \leftrightarrow V_{i-1} \rightarrow Q$ is in $\mathcal{G}$, $R2$ of Zhang [2008b] would imply that $V_{i-2} \bullet \rightarrow Q$ is in $\mathcal{G}$, and our assumption further lets us conclude that $V_{i-2} \rightarrow Q$, or $V_{i-2} \circ \rightarrow Q$ is in $\mathcal{G}$.

If $V_{i-1} \circ \rightarrow Q$ is in $\mathcal{G}$, then $V_{i-2} \leftrightarrow V_{i-1} \circ \rightarrow Q$ and Lemma 43 imply that, $V_{i-2} \bullet \rightarrow Q$ is in $\mathcal{G}$. Hence, as above either $V_{i-2} \rightarrow Q$, or $V_{i-2} \circ \rightarrow Q$ is in $\mathcal{G}$.

If $V_{i-2} = X$ we are done. Otherwise, we can repeat the same argument as in the preceding three paragraphs to conclude that $V_{i-3} \leftrightarrow V_{i-2} \leftrightarrow V_{i-1} \leftrightarrow C$ is in $\mathcal{G}$, and either $V_{i-3} \rightarrow Q$ or $V_{i-3} \circ \rightarrow Q$ are in $\mathcal{G}$. If $X \neq V_{i-3}$, we can keep applying the same argument, until we reach X .

Now that we have chosen the appropriate A' and B' the remaining argument is very similar to case (v). In each of the cases below we will derive the contradiction by finding a path s from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$ that is a proper non-causal definite status path in $\mathcal{G}$ and m -connecting given $\text{Adjust}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathcal{G}) \cup \mathbf{Z}$. Additionally, the path s will either be shorter than p or of the same length as p , but with a shorter distance to $\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}$ (Definition 31) which implies a contradiction with our choice of p .

Suppose first that Q is not on p :

- If $Q \notin \mathbf{X} \cup \mathbf{Y}$ then:

- if $A' \neq X$ and $B' \neq Y$, then let $s = p(X, A') \oplus \langle A', Q, B' \rangle \oplus p(B', Y)$. By the reasoning above, this path transformation amounts to replacing $p(A', B')$ on p with $\langle A', Q, B' \rangle$ on s such that the collider / definite non-collider status of A' and B' is the same on both paths. Therefore, s is a path with the same properties as p , but either shorter than p or of the same length but with a shorter distance to $\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}$ (Definition 31).
- If $A' = X$, and $B' \neq Y$, then let $s = \langle A', Q, B' \rangle \oplus p(B', Y)$. By the reasoning above, this path transformation amounts to replacing $p(X, B')$ on p with $\langle X, Q, B' \rangle$ on s such that the collider / definite non-collider status of B' is the same on both paths, and s is a non-causal path because of $Q \leftarrow \bullet B'$ edge. Therefore, s is a path with the same properties as p but either shorter than p or of the same length but with a shorter distance to $\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}$ (Definition 31).
- If $A' \neq X$, and $B' = Y$, then let $s = p(X, A') \oplus \langle A', Q, B' \rangle$. This path transformation amounts to replacing $p(A', Y)$ on p with $\langle A', Q, Y \rangle$ on s such that the collider / definite non-collider status of B' is the same on both paths, and s is a non-causal path because of $Q \leftarrow \bullet Y$ edge. Therefore, s is a path with the same properties as p but either shorter than p or of the same length but with a shorter distance to $\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}$ (Definition 31).
- If $A' = X$ and $B' = Y$, $\langle A', Q, B' \rangle$. Then s is of the form $X \bullet \rightarrow Q \leftarrow \bullet Y$ and $Q \in \text{An}(\text{Adjust}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathcal{G}) \cup \mathbf{Z}, \mathcal{G})$ and Q has a shorter distance to $\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}$ than C .

- If $Q \equiv X'$, $X' \in \mathbf{X}$, then:

- if $B' \neq Y$, then let $s = \langle Q, B' \rangle \oplus p(B', Y)$. This path transformation amounts to replacing $p(X, B')$ on p with $\langle X', B' \rangle$ on s such that the collider / definite non-collider status of B' is the same on both paths, and s is a non-causal path

because of $X' \leftarrow \bullet B'$ edge. Therefore, s is a path with the same properties as p shorter than p .

- If $B' = Y$, then let $s = \langle Q, B' \rangle$, where based on the reasoning above, s is of the form $X' \leftarrow \bullet Y$.

- Otherwise, $Q \equiv Y'$, $Y' \in \mathbf{Y}$. Then

- if $A' \neq X$, then $s = p(X, A') \oplus \langle A', Q \rangle$. Note that in this case s is of the form $X \leftrightarrow \dots \leftrightarrow A' \leftrightarrow Y'$, or $X \leftarrow \dots \leftarrow A' \circ \rightarrow Y'$, or $X \leftarrow \dots \leftarrow A' \rightarrow Y'$. In all cases, s is a proper non-causal definite status path from $\mathbf{X}$ to $\mathbf{Y}$ that is m-connecting given $\text{Adjust}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathcal{G}) \cup \mathbf{Z}$.
- If $A' = X$, then let $s = \langle A', Q \rangle$. We now discuss why s is of the form $X \leftrightarrow Y'$ in $\mathcal{G}$.

Note that $X \circ \rightarrow Y'$ cannot be in $\mathcal{G}$, since there exists a set $\mathbf{S}$ that can satisfy the conditional adjustment criterion relative to $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ in $\mathcal{G}$. If instead $X \rightarrow Y'$ is a visible edge in $\mathcal{G}'$, then there is either a node $D \notin \text{Adj}(Y', \mathcal{G})$ such that $D \bullet \rightarrow X$ is in $\mathcal{G}$ or there is a collection of nodes $D_1, \dots, D_k$, such that $D_1 \notin \text{Adj}(Y', \mathcal{G})$, $D_2, \dots, D_k \in \text{Pa}(Y', \mathcal{G})$, and $D_1 \bullet \rightarrow D_2 \leftrightarrow \dots \leftrightarrow D_k \leftrightarrow X$ is in $\mathcal{G}$. Without loss of generality we will assume that we are in the first case, that is $D \bullet \rightarrow X$ is in $\mathcal{G}$ and $D \notin \text{Adj}(Y', \mathcal{G})$, since the latter case has an analogous proof to what follows.

By above, the only way way that $A' \equiv X$ is if $X \leftrightarrow V_2 \leftrightarrow \dots \leftrightarrow C$ is in $\mathcal{G}$ and if for all nodes $V_j \in \{V_2, \dots, V_{i-2}, V_{i-1}, V_i\}$, $V_j \rightarrow Y'$, or $V_j \circ \rightarrow Y'$ is in $\mathcal{G}$. Now since, $D \bullet \rightarrow X \leftrightarrow V_2 \leftrightarrow \dots \leftrightarrow V_{i-1} \leftrightarrow C$ is also in $\mathcal{G}$, and $D \notin \text{Adj}(Y', \mathcal{G})$, we can use *R4* of Zhang [2008b] iteratively to conclude that $V_j \rightarrow Y'$ is in $\mathcal{G}$ for all $j \in \{1, \dots, i\}$. However, as $V_i \equiv C$, this contradicts our assumption that $C \circ \rightarrow Y'$ is in $\mathcal{G}$, for $Y' = Q$.

Otherwise, Q is on p . Therefore, $Q \notin \mathbf{X} \cup \mathbf{Y}$.

- Suppose first that Q is on $p(C, Y)$. By (iii), (iv), and the definition of $\text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$, we have that $p(C, Y)$ is of one of the following forms:

- $C \leftrightarrow \dots \leftrightarrow Q \leftrightarrow \dots \leftrightarrow V_k \leftarrow \bullet Y$ for $k > i$, or
- $C \leftrightarrow \dots \leftrightarrow Q \leftrightarrow \dots \leftrightarrow T_1 \rightarrow \dots \rightarrow Y$, for some T_1 on $p(C, Y)$, or
- $C \leftrightarrow \dots \leftrightarrow T_2 \rightarrow \dots \rightarrow Q \rightarrow \dots \rightarrow Y$, for some T_2 on $p(C, Y)$, or
- $C \leftrightarrow \dots \leftrightarrow Q \rightarrow \dots \rightarrow Y$.

Then

- If $A' \neq X$, then $s = p(X, A') \oplus \langle A', Q \rangle \oplus p(Q, Y)$. Note that by above forms of $p(C, Y)$ s is always a proper non-causal path from $\mathbf{X}$ to $\mathbf{Y}$. Additionally, by above listed options for $p(C, Y)$ we know that Q has the same collider / definite non-collider status on both p and s . Hence, s is also an m-connecting path given $\text{Adjust}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathcal{G}) \cup \mathbf{Z}$. Since s is also shorter than p we obtain our contradiction.
- If $A' \equiv X$, we let $s = p(X, Q) \oplus p(Q, Y)$. Path s is proper, since p itself is proper and $Q \notin \mathbf{X} \cup \mathbf{Y}$. Furthermore, by the above listed options for $p(C, Y)$ we know that Q has the same collider / definite non-collider status on both p and s and that s is a definite status path. Hence, s is also an m-connecting path given $\text{Adjust}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathcal{G}) \cup \mathbf{Z}$. If s is a non-causal path in $\mathcal{G}$, we obtain a contradiction with the choice of p .

Hence, suppose for a contradiction that s is a possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$. By assumption, it must be that $X \rightarrow Q$ is a visible edge in $\mathcal{G}$. Now, similarly to the previous case, since $X \rightarrow Q$ is a visible edge in $\mathcal{G}$, there is either a node $D \notin \text{Adj}(Q, \mathcal{G})$ such that $D \bullet \rightarrow X$ is in $\mathcal{G}$ or there is a collection of nodes $D_1, \dots, D_k$ such that $D_1 \notin \text{Adj}(Q, \mathcal{G})$, $D_2, \dots, D_k \in \text{Pa}(Q, \mathcal{G})$, and $D_1 \bullet \rightarrow D_2 \leftrightarrow \dots \leftrightarrow D_k \leftrightarrow X$ is in $\mathcal{G}$. We again assume without loss of generality that we are in the former case, that is $D \bullet \rightarrow X$ is in $\mathcal{G}$ and $D \notin \text{Adj}(Q, \mathcal{G})$.

Since $A' \equiv X$, by the same reasoning as in the previous case above we know that $X \leftrightarrow V_2 \leftrightarrow \dots \leftrightarrow C$ is in $\mathcal{G}$ and that for all nodes $V_j \in \{V_1, \dots, V_{i-1}, V_i\}$, $V_j \rightarrow Q$, or $V_j \circ \rightarrow Q$ is in $\mathcal{G}$. Now since, $D \bullet \rightarrow X \leftrightarrow V_2 \leftrightarrow \dots \leftrightarrow C$ is in $\mathcal{G}$, and since $D \notin \text{Adj}(Q, \mathcal{G})$, we can use *R4* of Zhang [2008b] iteratively to conclude that $V_j \rightarrow Q$ is in $\mathcal{G}$ for all $j \in \{1, \dots, i\}$. However, as $V_i \equiv C$, this contradicts our assumption that $C \circ \rightarrow Q$ is in $\mathcal{G}$.

- Lastly, suppose that Q is on $p(X, C)$. Analogously to above, by (iii), (iv), and the definition of $\text{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$, we have that $p(X, C)$ is of one of the following forms:

- $X \bullet \rightarrow V_2 \leftrightarrow \dots \leftrightarrow Q \leftrightarrow \dots \leftrightarrow C$, or
- $X \leftarrow \dots \leftarrow T_1 \leftrightarrow \dots \leftrightarrow Q \leftrightarrow \dots \leftrightarrow C$, for some T_1 on $p(X, C)$, or
- $X \leftarrow \dots \leftarrow Q \leftarrow \dots \leftarrow T_2 \leftrightarrow \dots \leftrightarrow C$, for some T_2 on $p(X, C)$, or
- $X \leftarrow \dots \leftarrow Q \leftrightarrow \dots \leftrightarrow C$.

Then

- If $B' \neq Y$, we have that $s = p(X, Q) \oplus \langle Q, B' \rangle \oplus p(B, Y)$ is a proper non-causal path from $\mathbf{X}$ to $\mathbf{Y}$ that is shorter than p . Additionally, Q is of the same collider / definite non-collider status on both p and s and therefore, s is not only of definite status, but also m-connecting given $\text{Adjust}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathcal{G}) \cup \mathbf{Z}$ in $\mathcal{G}$ which leads to a contradiction.
- If $B' \equiv Y$, then $s = p(X, Q) \oplus \langle Q, B' \rangle$ is a proper definite status non-causal path that is m-connecting given $\text{Adjust}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathcal{G}) \cup \mathbf{Z}$ in $\mathcal{G}$ and shorter than p . ■

Appendix B

ADDITIONAL MATERIAL FOR CHAPTER 3 DATA-DRIVEN ADJUSTMENT

B.1 FURTHER PRELIMINARIES

B.1.1 DIRECTED GRAPHS

Proper and Back-door Paths. A path from $\mathbf{X}$ to $\mathbf{Y}$ is *proper* (with respect to $\mathbf{X}$) if only its first node is in $\mathbf{X}$. A path from X to Y that begins with the edge $X \leftarrow$ is said to be a path *into* X , or a *back-door path*.

Definition 66 (Back-door Adjustment Set for DAGs; Maathuis and Colombo, 2015) Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ be pairwise disjoint node sets in a DAG $\mathcal{D}$. Then $\mathbf{Z}$ is a back-door adjustment set relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{D}$ if and only if:

- (a) $\mathbf{Z} \cap \text{De}(\mathbf{X}, \mathcal{D}) = \emptyset$, and
- (b) $\mathbf{Z} \cup [\mathbf{X} \setminus \{X\}]$ blocks all back-door paths from X to $\mathbf{Y}$ in $\mathcal{D}$, for all $X \in \mathbf{X}$.

Theorem 67 (Adjustment Set for DAGs, Graphical Criteria; Shpitser et al., 2012, Perković et al., 2018) Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ be pairwise disjoint node sets in a causal DAG $\mathcal{D}$. Then $\mathbf{Z}$ is an adjustment set relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{D}$ if and only if:

- (a) $\mathbf{Z}$ contains no descendants of any $W \notin \mathbf{X}$ that lies on a proper causal path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{D}$, and
- (b) $\mathbf{Z}$ blocks all proper non-causal paths from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{D}$.

Lemma 68 (cf. Theorem 3.1 of Maathuis and Colombo, 2015) *Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ be pairwise disjoint node sets in a causal DAG $\mathcal{D}$. If $\mathbf{Z}$ is a back-door adjustment set relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{D}$, then $\mathbf{Z}$ is an adjustment set relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{D}$.*

B.1.2 ANCESTRAL GRAPHS

The following are key definitions related to ancestral graphs and their associated densities. We rely on the framework of Richardson and Spirtes [2002], Zhang [2008b], Ali et al. [2009].

Mixed and Partially Directed Mixed Graphs. A *mixed graph* may contain directed ($\rightarrow$) and bi-directed ($\leftrightarrow$) edges. The *partially directed mixed graphs* we consider may contain directed, bi-directed, undirected ($\circ\circ$), or partially directed ($\circ\rightarrow$) edges. We use $\bullet$ as a stand in for any edge mark.

Definite Status Paths. Let $\mathcal{G}$ be a mixed or partially directed mixed graph with a path $p := \langle X_1, \dots, X_k \rangle$, $k > 1$. If p contains $X_{j-1} \bullet \rightarrow X_j \leftarrow \bullet X_{j+1}$ for $1 < j < k$, then X_j is a *collider* on p . X_j is a *definite non-collider* on p if p contains $X_{j-1} \leftarrow X_j$ or $X_j \rightarrow X_{j+1}$, or if $\mathcal{G}$ contains $X_{j-1} \circ\circ X_j \circ\circ X_{j+1}$ but no edge $\langle X_{j-1}, X_{j+1} \rangle$. If every node on p is a collider, definite non-collider, or endpoint on p , then p is a *definite status path*.

M-connection and M-separation. Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ be pairwise disjoint node sets in a mixed or partially directed mixed graph $\mathcal{G}$. A definite-status path p from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$ is *open* given $\mathbf{Z}$ if every definite non-collider on p is not in $\mathbf{Z}$ and every collider on p has a descendant in $\mathbf{Z}$ in $\mathcal{G}$. Otherwise, p is *blocked* given $\mathbf{Z}$. If $\mathbf{Z}$ blocks all definite-status paths between $\mathbf{X}$ and $\mathbf{Y}$ in $\mathcal{G}$, then $\mathbf{X}$ is *m-separated* from $\mathbf{Y}$ given $\mathbf{Z}$ in $\mathcal{G}$ and we write $(\mathbf{X} \perp_m \mathbf{Y} \mid \mathbf{Z})_{\mathcal{G}}$. Otherwise, $\mathbf{X}$ is *m-connected* to $\mathbf{Y}$ given $\mathbf{Z}$ in $\mathcal{G}$ and we write $(\mathbf{X} \not\perp_m \mathbf{Y} \mid \mathbf{Z})_{\mathcal{G}}$.

MAGs. A directed path from X to Y and the edge $X \rightarrow Y$ form an *almost directed cycle*. A mixed graph without directed or almost directed cycles is called *ancestral*. Note that we do not consider ancestral graphs that represent selection bias. A *maximal ancestral graph* (MAG) is an ancestral graph $\mathcal{M} = (\mathbf{V}, \mathbf{E})$ where every pair of non-adjacent nodes X and Y in $\mathcal{M}$ can be m-separated by a set $\mathbf{Z} \subseteq \mathbf{V} \setminus \{X, Y\}$. A DAG $\mathcal{D} = (\mathbf{V}, \mathbf{E})$ with unobserved variables $\mathbf{U} \subseteq \mathbf{V}$ can be uniquely *represented by* a MAG $\mathcal{M} = (\mathbf{V} \setminus \mathbf{U}, \mathbf{E}')$, which

preserves the ancestry and m-separations among the observed variables.

PAGs. All MAGs that encode the same set of m-separations form a Markov equivalence class, which can be uniquely *represented by* a partially directed mixed graph called a *partial ancestral graph* (PAG). $[\mathcal{G}]$ denotes all MAGs represented by a PAG $\mathcal{G}$. We say a DAG $\mathcal{D}$ is *represented by* a PAG $\mathcal{G}$ if there is a MAG $\mathcal{M} \in [\mathcal{G}]$ such that $\mathcal{D}$ is represented by $\mathcal{M}$. Note that we only consider maximally informative PAGs that are complete with respect to orientation rules $R1 - R4$ and $R8 - R10$ of Zhang [2008b] and that do not represent selection bias.

Markov Compatibility and Faithfulness. We say an observational density is *Markov compatible* with a MAG or PAG $\mathcal{G}$ if it is Markov compatible with a DAG represented by $\mathcal{G}$. We say an observational density is *faithful* to a MAG or PAG $\mathcal{G}$ if it is faithful to a DAG represented by $\mathcal{G}$.

Probabilistic Implications of a Graph. Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ be pairwise disjoint node sets in a MAG or PAG $\mathcal{G}$. If $(\mathbf{X} \perp_m \mathbf{Y} \mid \mathbf{Z})_{\mathcal{G}}$, then $\mathbf{X}$ and $\mathbf{Y}$ are independent given $\mathbf{Z}$ in any observational density that is Markov compatible with $\mathcal{G}$. If $(\mathbf{X} \not\perp_m \mathbf{Y} \mid \mathbf{Z})_{\mathcal{G}}$, then $\mathbf{X}$ and $\mathbf{Y}$ are dependent given $\mathbf{Z}$ in any observational density that is faithful to $\mathcal{G}$.

Causal Graphs. Let $\mathcal{G}$ be a graph with nodes V_i and V_j . When $\mathcal{G}$ is a MAG or PAG, it is a *causal MAG* or *causal PAG* if every edge $V_i \rightarrow V_j$ represents the presence of a causal path from V_i to V_j ; every edge $V_i \leftarrow \bullet V_j$ represents the absence of a causal path from V_i to V_j ; and every edge $V_i \circ \rightarrow V_j$ represents the presence of a causal path of unknown direction or a common cause in the underlying causal DAG.

Possibly Causal Paths. Let $p := \langle X_1, \dots, X_k \rangle$, $k > 1$, be a path in a causal MAG or PAG $\mathcal{G}$. If $\mathcal{G}$ does not contain an edge $X_i \leftarrow \bullet X_j$, $1 \leq i < j \leq k$, then p is *possibly causal* and $X_2, \dots, X_k$ are *possible descendants* of X_1 . Otherwise, p is *non-causal*.

Visible Edges. Let $\mathcal{G}$ be a MAG or PAG. We denote that X is adjacent to Y in $\mathcal{G}$ by $X \in \text{Adj}(Y, \mathcal{G})$. A directed edge $X \rightarrow Y$ is *visible* in $\mathcal{G}$ if there is a node $V \notin \text{Adj}(Y, \mathcal{G})$ such that $\mathcal{G}$ contains either $V \bullet \rightarrow X$ or $V \bullet \rightarrow V_1 \leftrightarrow \dots \leftrightarrow V_k \leftrightarrow X$, where $k \geq 1$ and $V_1, \dots, V_k \in \text{Pa}(Y, \mathcal{G}) \setminus \{V, X, Y\}$.

Consistency. We say an interventional density is *consistent* with a causal MAG or PAG $\mathcal{G}$ if it is consistent with each DAG represented by $\mathcal{G}$ – were the DAG to be causal.

Adjustment Sets. Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ be pairwise disjoint node sets in a causal MAG or PAG $\mathcal{G}$. Then $\mathbf{Z}$ is an adjustment set relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{G}$ if and only if $f(\mathbf{y}|do(\mathbf{x})) = \int f(\mathbf{y}|\mathbf{x}, \mathbf{z})f(\mathbf{z}) d\mathbf{z}$ for any f consistent with $\mathcal{G}$. We omit reference to $(\mathbf{X}, \mathbf{Y})$ or $\mathcal{G}$ when it can be assumed.

Theorem 69 (Adjustment Set for PAGs, Graphical Criteria; cf. Theorem 5 of Perković et al., 2018) *Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ be pairwise disjoint node sets in a causal PAG $\mathcal{G}$. Then $\mathbf{Z}$ is an adjustment set relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{G}$ if and only if*

- (a) *every proper, possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$ starts with a visible edge,*
- (b) *$\mathbf{Z}$ contains no possible descendants of any $W \notin \mathbf{X}$ that lies on a proper, possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$, and*
- (c) *$\mathbf{Z}$ blocks all proper, definite status, non-causal paths from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$.*

B.2 PROOF FOR SECTION 3.4.1: R1 BUILD

Proof of Theorem 15 (R1 Build). The result holds for $k = 1$ by Theorem 11. Thus, let $k \geq 2$. Suppose there exist $W_1, \dots, W_k \in \mathbf{W}$ and $\mathbf{Z} \subseteq \mathbf{W} \setminus \{W_1, \dots, W_k\}$ such that (i) and (ii) hold for $i \in \{1, \dots, k\}$. Let $\mathcal{D}$ be the DAG induced by the causal model. Then for ease of notation, let $\mathbf{X}_i^N = \mathbf{X} \setminus \text{De}(X_i, \mathcal{D})$, and note that $\{X_1, \dots, X_i\}^{-i} \subseteq \mathbf{X}_i^N$ since $X_1 < \dots < X_k$ is a causal ordering consistent with the model.

We start by showing that $\mathbf{X}$ has a causal effect on Y . Consider an arbitrary $j \in \{1, \dots, k\}$. By (i), (ii), and faithfulness, there must be a path p_j from W_j to Y in $\mathcal{D}$ that is open given $\mathbf{Z} \cup \{X_1, \dots, X_j\}^{-j}$ and contains X_j as a non-collider. Therefore, either $p_j(W_j, X_j)$ ends $\leftarrow X_j$ or $p_j(X_j, Y)$ begins $X_j \rightarrow$. For sake of contradiction, suppose the former holds. Since $X_j \notin \text{An}(\mathbf{W}, \mathcal{D})$, then $p_j(W_j, X_j)$ must contain a collider. But since p_j is open given $\mathbf{Z} \cup \{X_1, \dots, X_j\}^{-j} \subseteq \mathbf{W} \cup \mathbf{X}_j^N$, the closest collider to X_j on $p_j(W_j, X_j)$ must be in $\text{An}(\mathbf{Z} \cup$

$\{X_1, \dots, X_j\}^{-j}, \mathcal{D})$, which contradicts that $X_j \notin \text{An}(\mathbf{W} \cup \mathbf{X}_j^{\mathbf{N}}, \mathcal{D})$. Thus, $p_j(X_j, Y)$ must begin $X_j \rightarrow$. By similar logic, $p_j(X_j, Y)$ cannot contain a collider, and so p_j must end $\rightarrow X_j \rightarrow \dots \rightarrow Y$.

We continue by showing that $\mathbf{Z}$ satisfies the conditions of Definition 66 and thus by Lemma 68, is an adjustment set relative to $(\mathbf{X}, Y)$. By assumption, no variable in $\mathbf{Z}$ is a descendant of $\mathbf{X}$. Thus, we only need to show the following for $i \in \{1, \dots, k\}$:

$$\begin{aligned} \mathbf{Z} \cup \mathbf{X}^{-i} \text{ blocks every back-door path} \\ \text{from } X_i \text{ to } Y \text{ in } \mathcal{D}. \end{aligned} \quad (*)$$

We begin with $i = k$, and proceed with a proof by induction for $i \in \{1, \dots, k-1\}$.

BASE CASE: Note from the discussion above that there must be a path p_k from W_k to Y in $\mathcal{D}$ that is open given $\mathbf{Z} \cup \mathbf{X}^{-k}$ and ends $\rightarrow X_k \rightarrow \dots \rightarrow Y$. Then for sake of contradiction, suppose there is a back-door path q_k from X_k to Y in $\mathcal{D}$ that is open given $\mathbf{Z} \cup \mathbf{X}^{-k}$. Let A be the node on both p_k and q_k that is closest to W_k on p_k , and define $r_k = p_k(W_k, A) \oplus q_k(A, Y)$. We show below that r_k must be open given $\mathbf{Z} \cup \mathbf{X}$ – that is, no non-collider on r_k is in $\mathbf{Z} \cup \mathbf{X}$ and every collider on r_k is in $\text{An}(\mathbf{Z} \cup \mathbf{X}, \mathcal{D})$ – which contradicts (ii) by faithfulness.

First consider the nodes on r_k other than A . Every collider on $r_k(W_k, A)$ and $r_k(A, Y)$ is in $\text{An}(\mathbf{Z} \cup \mathbf{X}, \mathcal{D})$, since p_k and q_k are open given $\mathbf{Z} \cup \mathbf{X}^{-k}$. By the same logic, no non-collider on $r_k(W_k, A)$ or $r_k(A, Y)$ is in $\mathbf{Z} \cup \mathbf{X}^{-k}$. Further, X_k is not a non-collider on $r_k(W_k, A)$ or $r_k(A, Y)$, since r_k does not contain X_k except possibly $A = X_k$.

Next consider the node A . When $A = W_k$, note that $r_k = r_k(A, Y)$, and the base case is done. When $A \neq W_k$, we show in the cases below that A is either a collider on r_k such that $A \in \text{An}(\mathbf{Z} \cup \mathbf{X}, \mathcal{D})$ or a non-collider on r_k such that $A \notin \mathbf{Z} \cup \mathbf{X}$, which completes the base case.

- Let $A = X_k$. Note that A is a collider on r_k (by definition of p_k and q_k) and $A \in \text{An}(\mathbf{Z} \cup \mathbf{X}, \mathcal{D})$.

- Let $A \in \mathbf{Z} \cup \mathbf{X}^{-\mathbf{k}}$. Since p_k and q_k are open given $\mathbf{Z} \cup \mathbf{X}^{-\mathbf{k}}$, then A must be a collider on both paths. Therefore, A is a collider on r_k and $A \in \text{An}(\mathbf{Z} \cup \mathbf{X}, \mathcal{D})$.
- Let $A \notin \mathbf{Z} \cup \mathbf{X}$. When A is a non-collider on r_k , the claim holds. When A is a collider on r_k , we only need to show that $A \in \text{An}(\mathbf{Z} \cup \mathbf{X}, \mathcal{D})$. This holds if A is a collider on p_k , since p_k is open given $\mathbf{Z} \cup \mathbf{X}^{-\mathbf{k}}$. Consider when A is a collider on r_k and a non-collider on p_k . Note that $p_k(W_k, X_k)$ begins $W_k \cdots \rightarrow A \rightarrow$ and ends $\rightarrow X_k$. When $p_k(A, X_k)$ is directed, then $A \in \text{An}(\mathbf{Z} \cup \mathbf{X}, \mathcal{D})$. When $p_k(A, X_k)$ contains a collider, the earliest such collider must be in $\text{An}(\mathbf{Z} \cup \mathbf{X}^{-\mathbf{k}}, \mathcal{D})$, since p_k is open given $\mathbf{Z} \cup \mathbf{X}^{-\mathbf{k}}$. Thus, $A \in \text{An}(\mathbf{Z} \cup \mathbf{X}, \mathcal{D})$.

INDUCTION: Pick an arbitrary $j \in \{1, \dots, k-1\}$, and for sake of induction, assume that $(*)$ holds for $i \in \{j+1, \dots, k\}$. We will show that $(*)$ also holds for $i = j$.

Recall that there must be a path p_j from W_j to Y in $\mathcal{D}$ that is open given $\mathbf{Z} \cup \{X_1, \dots, X_j\}^{-j}$ and ends $\rightarrow X_j \rightarrow \cdots \rightarrow Y$. Then for sake of contradiction, suppose there is a back-door path q_j from X_j to Y in $\mathcal{D}$ that is open given $\mathbf{Z} \cup \mathbf{X}^{\mathbf{j}}$. Let B be the node on both p_j and q_j that is closest to W_j on p_j , and define $r_j = p_j(W_j, B) \oplus q_j(B, Y)$. We show below that r_j must be open given $\mathbf{Z} \cup \{X_1, \dots, X_j\}$ – that is, no non-collider on r_j is in $\mathbf{Z} \cup \{X_1, \dots, X_j\}$ and every collider on r_j is in $\text{An}(\mathbf{Z} \cup \{X_1, \dots, X_j\}, \mathcal{D})$ – which contradicts (ii) by faithfulness.

First consider the nodes on $r_j(W_j, B)$. Every collider on $r_j(W_j, B)$ is in $\text{An}(\mathbf{Z} \cup \{X_1, \dots, X_j\}, \mathcal{D})$, since p_j is open given $\mathbf{Z} \cup \{X_1, \dots, X_j\}^{-j}$. By the same logic and the fact that r_j does not contain X_j except possibly $B = X_j$, no non-collider on $r_j(W_j, B)$ is in $\mathbf{Z} \cup \{X_1, \dots, X_j\}$.

Next consider the nodes on $r_j(B, Y)$. No non-collider on $r_j(B, Y)$ is in $\mathbf{Z} \cup \{X_1, \dots, X_j\}$, since q_j is open given $\mathbf{Z} \cup \mathbf{X}^{\mathbf{j}}$ and X_j is an endpoint on q_j . Then suppose for sake of contradiction that there is a collider on $r_j(B, Y)$ not in $\text{An}(\mathbf{Z} \cup \{X_1, \dots, X_j\}, \mathcal{D})$, and let C be the closest such collider to Y on $r_j(B, Y)$. Note that $C \in \text{An}(\{X_{j+1}, \dots, X_k\}, \mathcal{D})$, since q_j is open given $\mathbf{Z} \cup \mathbf{X}^{\mathbf{j}}$. Thus, let s be a shortest directed path in $\mathcal{D}$ from C to $\{X_{j+1}, \dots, X_k\}$,

where we denote the latter endpoint X_ℓ , $\ell \in \{j+1, \dots, k\}$. Then let E be the node on both s and r_j that is closest to Y on r_j .

Consider the path $t = s(X_\ell, E) \oplus r_j(E, Y)$. Start by noting that no non-collider on t is in $\mathbf{Z} \cup \mathbf{X}^{-\ell}$ due to the following: $C \notin \text{An}(\mathbf{Z} \cup \{X_1, \dots, X_j\}, \mathcal{D})$; s is a shortest path to $\{X_{j+1}, \dots, X_k\}$; q_j is open given $\mathbf{Z} \cup \mathbf{X}^{\mathbf{j}}$; and X_j is an endpoint on q_j . Further, every collider on t must be in $\text{An}(\mathbf{Z} \cup \mathbf{X}^{-\ell}, \mathcal{D})$, since t can only contain colliders that are colliders on $r_j(E, Y)$, where by definition of C , every collider on $r_j(E, Y)$ is in $\text{An}(\mathbf{Z} \cup \{X_1, \dots, X_j\}, \mathcal{D})$. Thus, t – a back-door path from X_ℓ to Y – is open given $\mathbf{Z} \cup \mathbf{X}^{-\ell}$, which contradicts our induction assumption that $(*)$ holds for $i = \ell$, $\ell \in \{j+1, \dots, k\}$.

Finally, consider the node B . When $B = W_j$, note that $r_j = r_j(B, Y)$, and the induction step is done. When $B \neq W_j$, we show in the cases below that B is either a collider on r_j such that $B \in \text{An}(\mathbf{Z} \cup \{X_1, \dots, X_j\}, \mathcal{D})$ or a non-collider on r_j such that $B \notin \mathbf{Z} \cup \{X_1, \dots, X_j\}$, which completes the proof.

- Let $B = X_j$. Note that B is a collider on r_j (by definition of p_j and q_j) and $B \in \text{An}(\mathbf{Z} \cup \{X_1, \dots, X_j\}, \mathcal{D})$.
- Let $B \in \mathbf{Z} \cup \{X_1, \dots, X_j\}^{-j}$. Since p_j is open given $\mathbf{Z} \cup \{X_1, \dots, X_j\}^{-j}$ and q_j is open given $\mathbf{Z} \cup \mathbf{X}^{\mathbf{j}}$, then B must be a collider on both paths. Therefore, B is a collider on r_j and $B \in \text{An}(\mathbf{Z} \cup \{X_1, \dots, X_j\}, \mathcal{D})$.
- For sake of contradiction, let $B = X_\ell$, $\ell \in \{j+1, \dots, k\}$. Note that B must be a collider on q_j , since q_j is open given $\mathbf{Z} \cup \mathbf{X}^{\mathbf{j}}$. Thus, $r_j(B, Y)$ is a back-door path from X_ℓ to Y . However, note that no non-collider on $r_j(B, Y)$ is in $\mathbf{Z} \cup \mathbf{X}^{-\ell}$, since q_j is open given $\mathbf{Z} \cup \mathbf{X}^{\mathbf{j}}$ and since X_j is an endpoint on q_j . Further, every collider on $r_j(B, Y)$ is in $\text{An}(\mathbf{Z} \cup \mathbf{X}^{-\ell}, \mathcal{D})$, since we have shown that every collider on $r_j(B, Y)$ is in $\text{An}(\mathbf{Z} \cup \{X_1, \dots, X_j\}, \mathcal{D})$. Therefore, $r_j(B, Y)$ is a back-door path from X_ℓ to Y that is open given $\mathbf{Z} \cup \mathbf{X}^{-\ell}$, which contradicts our induction assumption that $(*)$ holds for $i = \ell$, $\ell \in \{j+1, \dots, k\}$.

- Let $B \notin \mathbf{Z} \cup \mathbf{X}$. When B is a non-collider on r_j , the claim holds. When B is a collider on r_j , we only need to show that $B \in \text{An}(\mathbf{Z} \cup \{X_1, \dots, X_j\}, \mathcal{D})$. This holds if B is also a collider on p_j , since p_j is open given $\mathbf{Z} \cup \{X_1, \dots, X_j\}^{-j}$. Consider when B is a collider on r_j and a non-collider on p_j . Note that $p_j(W_j, X_j)$ begins $W_j \cdots \rightarrow B \rightarrow$ and ends $\rightarrow X_j$. When $p_j(B, X_j)$ is directed, then $B \in \text{An}(\mathbf{Z} \cup \{X_1, \dots, X_j\}, \mathcal{D})$. When $p_j(B, X_j)$ contains a collider, the earliest such collider must be in $\text{An}(\mathbf{Z} \cup \{X_1, \dots, X_j\}^{-j}, \mathcal{D})$, since p_j is open given $\mathbf{Z} \cup \{X_1, \dots, X_j\}^{-j}$. Thus, $B \in \text{An}(\mathbf{Z} \cup \{X_1, \dots, X_j\}, \mathcal{D})$. ■

B.3 PROOFS FOR SECTION 3.4.2: R1 COMBINE

B.3.1 MAIN RESULTS

Proof of Theorem 17 (R1 Combine). The result holds for $k = 1$ by Theorem 11. Thus, let $k \geq 2$. Suppose there exist $W_1, \dots, W_k \in \mathbf{W}$ and $\mathbf{T}_i \subseteq [\mathbf{W} \setminus \{W_i\}] \cup \mathbf{X}_i^{\mathbf{N}}$ such that (i)-(ii) hold for all $i \in \{1, \dots, k\}$. Note by (i), (ii), and Theorem 11 that X_i has a causal effect on Y that is identifiable through the adjustment set $\mathbf{T}_i$. Thus, $\mathbf{X}$ has a causal effect on Y , and we can define $\mathbf{Z}_i$ to be any minimal adjustment set relative to (X_i, Y) such that $\mathbf{Z}_i \subseteq \mathbf{T}_i$. Let $\mathcal{D}$ be the DAG induced by the causal model so that $\mathbf{X}_i^{\mathbf{N}} = \mathbf{X} \setminus \text{De}(X_i, \mathcal{D})$. Then for ease of notation, let $\mathbf{X}^{-i} = \mathbf{X} \setminus \{X_i\}$.

We show below that $\mathbf{Z} := \cup_{i=1}^k \mathbf{Z}_i \setminus \mathbf{X}$ satisfies the conditions of Definition 66 and thus by Lemma 68, is an adjustment set relative to $(\mathbf{X}, Y)$. Since $\mathbf{Z} \subseteq \mathbf{W}$ where $\mathbf{W} < \mathbf{X}$, we only need to show for every $i \in \{1, \dots, k\}$ that $\mathbf{Z} \cup \mathbf{X}^{-i}$ blocks every back-door path from X_i to Y in $\mathcal{D}$. Without loss of generality, we show this holds for $i = 1$.

For sake of contradiction, suppose there is a back-door path q from X_1 to Y in $\mathcal{D}$ that is open given $\mathbf{Z} \cup \mathbf{X}^{-1}$. Since $\mathbf{Z}_1$ is an adjustment set relative to (X_1, Y) , then by Theorem 67, q must be blocked given $\mathbf{Z}_1 \subseteq \mathbf{Z} \cup \mathbf{X}^{-1}$. This implies that q must contain a collider in $\text{An}(\mathbf{Z} \cup \mathbf{X}^{-1}, \mathcal{D}) \setminus \text{An}(\mathbf{Z}_1, \mathcal{D})$. Let $C_1, \dots, C_\ell$, $\ell \geq 1$, be the set of all such colliders – ordered so that C_1 is the closest such collider to X_1 on q and C_ℓ is the furthest such collider from X_1 on q .

We pause to define a path t_i in $\mathcal{D}$ that is directed from C_i to Y for any $i \in \{1, \dots, \ell\}$ such that $C_i \notin \text{An}(X_1, \mathcal{D})$. This path will be useful in showing a contradiction. First pick such an $i \in \{1, \dots, \ell\}$ (supposing one exists), and define r_i as a longest directed path from C_i to $(\mathbf{Z} \cup \mathbf{X}^{-1}) \setminus \text{An}(\mathbf{Z}_1, \mathcal{D})$ in $\mathcal{D}$. Then define t_i based on how r_i ends.

When r_i ends with $X_j \in \mathbf{X}^{-1} \setminus \text{An}(\mathbf{Z}_1, \mathcal{D})$ for some $j \in \{2, \dots, k\}$, note that by (i), (ii), and faithfulness, there must be a path p_j from W_j to Y in $\mathcal{D}$ that is open given $\mathbf{T}_j$ and contains X_j as a non-collider. Therefore, either $p_j(W_j, X_j)$ ends $\leftarrow X_j$ or $p_j(X_j, Y)$ begins $X_j \rightarrow$. For sake of contradiction, suppose the former holds. Since $X_j \notin \text{An}(\mathbf{W}, \mathcal{D})$, then $p_j(W_j, X_j)$ must contain a collider. But since p_j is open given $\mathbf{T}_j \subseteq \mathbf{W} \cup \mathbf{X}_j^{\mathbf{N}}$, the closest collider to X_j on $p_j(W_j, X_j)$ must be in $\text{An}(\mathbf{T}_j, \mathcal{D})$, which contradicts that $X_j \notin \text{An}(\mathbf{W} \cup \mathbf{X}_j^{\mathbf{N}}, \mathcal{D})$. Thus, $p_j(X_j, Y)$ must begin $X_j \rightarrow$. By similar logic, $p_j(X_j, Y)$ cannot contain a collider, and so $p_j(X_j, Y)$ must take the form $X_j \rightarrow \dots \rightarrow Y$. Thus, we can define $t_i = r_i \oplus p_j(X_j, Y)$.

Next, consider when r_i ends with $Z_j \in \mathbf{Z} \setminus \text{An}(\mathbf{Z}_1, \mathcal{D})$ so that $Z_j \in \mathbf{Z}_j \setminus \mathbf{X}$ for some $j \in \{2, \dots, k\}$. By definition, $\mathbf{Z}_j$ is an adjustment set relative to (X_j, Y) , but this does not hold for any subset $\mathbf{Z}_j \setminus \{Z_j\}$, where $Z_j \in \mathbf{Z}_j$. Thus by Theorem 67, there must be a non-causal path from X_j to Y in $\mathcal{D}$ that contains Z_j as a non-collider and that is open given $\mathbf{Z}_j \setminus \{Z_j\}$. Let s_j be one such path. For sake of contradiction, suppose $s_j(Z_j, Y)$ begins $Z_j \leftarrow$. Since Z_j is a non-collider on s_j , then $s_j(X_j, Z_j)$ must end with $\leftarrow Z_j$. If $s_j(X_j, Z_j)$ contains a collider, then by the definition of s_j , the closest such collider to Z_j on s_j must be in $\text{An}(\mathbf{Z}_j \setminus \{Z_j\}, \mathcal{D})$, which contradicts either the definition of r_i as a longest path to $(\mathbf{Z} \cup \mathbf{X}^{-1}) \setminus \text{An}(\mathbf{Z}_1, \mathcal{D})$ or the definition of $C_i \notin \text{An}(\mathbf{Z}_1 \cup \{X_1\}, \mathcal{D})$. The same contradiction holds if instead $s_j(X_j, Z_j)$ takes the form $X_j \leftarrow \dots \leftarrow Z_j$. Therefore, $s_j(Z_j, Y)$ must begin $Z_j \rightarrow$. Further, $s_j(Z_j, Y)$ must take the form $Z_j \rightarrow \dots \rightarrow Y$, since by the same logic, $s_j(Z_j, Y)$ cannot contain a collider. Thus, here we define $t_i = r_i \oplus s_j(Z_j, Y)$.

We proceed to cases with a directed path from C_i to Y defined as follows:

$$t_i := \begin{cases} r_i \oplus p_j(X_j, Y) & r_i \text{ ends } X_j \in \mathbf{X}^{-1} \setminus \text{An}(\mathbf{Z}_1, \mathcal{D}) \\ r_i \oplus s_j(Z_j, Y) & r_i \text{ ends } Z_j \in \mathbf{Z} \setminus \text{An}(\mathbf{Z}_1, \mathcal{D}). \end{cases}$$

We use this path in the cases below to show there must be a non-causal path u from X_1 to Y in $\mathcal{D}$ that is open given $\mathbf{Z}_1$, which by Theorem 67, contradicts that $\mathbf{Z}_1$ is an adjustment set relative to (X_1, Y) .

CASE 1: Suppose there is no directed path from $\{C_1, \dots, C_\ell\}$ to X_1 in $\mathcal{D}$. Therefore, consider C_1 and its corresponding path t_1 . Note that $q(X_1, C_1)$ and t_1 cannot share a node other than C_1 , since any such node would have a directed path (possibly of length zero) in $\mathcal{D}$ to C_1 , X_1 , or a collider on $q(X_1, C_1)$. And this would contradict the acyclicity of $\mathcal{D}$, the fact that there is no directed path from $\{C_1, \dots, C_\ell\}$ to X_1 in $\mathcal{D}$, or the definition of $C_1 \notin \text{An}(\mathbf{Z}_1, \mathcal{D})$ as the closest such collider to X_1 on q . Thus, we can consider the path $u := q(X_1, C_1) \oplus t_1$. Note that $q(X_1, C_1)$ is open given $\mathbf{Z}_1$ by the definition of q and C_1 . Further by the definition of C_1 and t_1 , no node on t_1 is in $\mathbf{Z}_1$ and every node on t_1 is a non-collider on u . Therefore, u – a non-causal path in $\mathcal{D}$ from X_1 to Y – is open given $\mathbf{Z}_1$, which is a contradiction.

CASE 2: Suppose there is a directed path from $\{C_1, \dots, C_\ell\}$ to X_1 in $\mathcal{D}$. Let C_a , $a \in \{1, \dots, \ell\}$, be the closest such collider to Y on q , let w be an arbitrary directed path from C_a to X_1 in $\mathcal{D}$, and let A be the node on both q and w that is closest to Y on q .

When A is a node on $q(C_\ell, Y)$, consider the path $u := (-w)(X_1, A) \oplus q(A, Y)$. Note by the definition of C_a and w that no node on $(-w)(X_1, A)$ is in $\mathbf{Z}_1$ and every node on $(-w)(X_1, A)$ is a non-collider on u . Further note that $q(A, Y)$ is open given $\mathbf{Z}_1$ by the definition of q and C_ℓ . Therefore, u – a non-causal path in $\mathcal{D}$ from X_1 to Y – is open given $\mathbf{Z}_1$, which is a contradiction.

When A precedes C_ℓ on q , let C_b be the collider in $\{C_{a+1}, \dots, C_\ell\}$ closest to A on $q(A, Y)$. Note by the definition of C_a that $C_b \notin \text{An}(X_1, \mathcal{D})$. Therefore, we can consider the path t_b , which is directed from C_b to Y . Note that $(-w)(X_1, A) \oplus q(A, C_b)$ and t_b cannot share a node other than C_b , since any such node would have a directed path (possibly of length zero) in $\mathcal{D}$ to C_b , X_1 , or a collider on $q(A, C_b)$. And this would contradict the acyclicity of $\mathcal{D}$, the definition of C_a as the closest collider in $\{C_1, \dots, C_\ell\}$ to Y on q with a directed path to X_1 , or the definition of $C_b \notin \text{An}(\mathbf{Z}_1, \mathcal{D})$ as the closest such collider to A on $q(A, Y)$. Thus, we

can consider the path $u := (-w)(X_1, A) \oplus q(A, C_b) \oplus t_b$. Note by the definition of C_a and w that no node on $(-w)(X_1, A)$ is in $\mathbf{Z}_1$ and every node on $(-w)(X_1, A)$ is a non-collider on u . The same holds for t_b by analogous logic. Further, note that $q(A, C_b)$ is open given $\mathbf{Z}_1$ by the definition of q , A , and C_b . Therefore, u – a non-causal path in $\mathcal{D}$ from X_1 to Y – is open given $\mathbf{Z}_1$, which is a contradiction. ■

Proof of Lemma 18 (Probabilistic Criteria for Minimality). Let $\mathbf{T}$ be an adjustment set relative to (X, Y) in a causal model where $\mathbf{T} < X$. Then by Lemmas 71-72, $\mathbf{T}$ is a minimal adjustment set relative to (X, Y) if and only if (i)-(ii) hold for all $T \in \mathbf{T}$.

When (i) or (ii) do not hold for some $T \in \mathbf{T}$, let $\mathbf{Z}$ be a proper subset of $\mathbf{T}$ such that (iii)-(v) hold for all $Z \in \mathbf{Z}$. In addition to (v), note that trivially $X \perp\!\!\!\perp \mathbf{Z} \mid \mathbf{T}$ and $Y \perp\!\!\!\perp \mathbf{Z} \mid \mathbf{T} \cup \{X\}$, since $\mathbf{Z} \subset \mathbf{T}$. Therefore, by Theorem 13, $\mathbf{Z}$ is c-equivalent to $\mathbf{T}$ and thus, is an adjustment set relative to (X, Y) . Then, as above, $\mathbf{Z}$ is minimal by (iii)-(iv) and Lemmas 71-72. ■

B.3.2 SUPPORTING RESULTS

Definition 70 (Elementwise Minimal Adjustment Set) *A set $\mathbf{Z}$ is an elementwise minimal adjustment set relative to $(\mathbf{X}, \mathbf{Y})$ if $\mathbf{Z}$ is an adjustment set relative to $(\mathbf{X}, \mathbf{Y})$ and if $\mathbf{Z} \setminus \{Z\}$ is not an adjustment set relative to $(\mathbf{X}, \mathbf{Y})$ for any $Z \in \mathbf{Z}$.*

Lemma 71 (Criteria for Elementwise Minimality) *Let $\mathbf{Z}$ be an adjustment set relative to $(\mathbf{X}, \mathbf{Y})$ in a causal model where $\mathbf{Z} < \mathbf{X}$. Then $\mathbf{Z}$ is an elementwise minimal adjustment set relative to $(\mathbf{X}, \mathbf{Y})$ if and only if the following hold for some $X \in \mathbf{X}$, $Y \in \mathbf{Y}$, and for all $Z \in \mathbf{Z}$:*

$$(i) \ X \not\perp\!\!\!\perp Z \mid \mathbf{Z} \setminus \{Z\}, \quad \text{and}$$

$$(ii) \ Y \not\perp\!\!\!\perp Z \mid [\mathbf{Z} \setminus \{Z\}] \cup \{X\}.$$

Proof of Lemma 71. Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ be pairwise disjoint node sets in a causal DAG $\mathcal{D}$, where $\mathbf{Z}$ is an adjustment set relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{D}$ and where $\mathbf{Z} < \mathbf{X}$.

$\Rightarrow$ Let $\mathbf{Z}$ be an elementwise minimal adjustment set relative to $(\mathbf{X}, \mathbf{Y})$, and consider an arbitrary $Z \in \mathbf{Z}$. Since $\mathbf{Z}$ is an adjustment set but $\mathbf{Z} \setminus \{Z\}$ is not an adjustment set relative to $(\mathbf{X}, \mathbf{Y})$, then by Theorem 67, there must be a non-causal path p from some $X \in \mathbf{X}$ to some $Y \in \mathbf{Y}$ in $\mathcal{D}$ that contains Z as a non-collider and is open given $\mathbf{Z} \setminus \{Z\}$. Note that (i) holds since $p(X, Z)$ is open given $\mathbf{Z} \setminus \{Z\}$. Further, (ii) holds since $p(Z, Y)$ is open given $\mathbf{Z} \setminus \{Z\}$ and does not contain X so that $p(Z, Y)$ is open given $[\mathbf{Z} \setminus \{Z\}] \cup \{X\}$.

$\Leftarrow$ Let (i) and (ii) hold for some $X \in \mathbf{X}$, $Y \in \mathbf{Y}$, and for all $Z \in \mathbf{Z}$. Then consider an arbitrary $Z \in \mathbf{Z}$. Note that by (i), (ii), and faithfulness, the following two paths must exist in $\mathcal{D}$:

p : a path from X to Z that is open given $\mathbf{Z} \setminus \{Z\}$ and

q : a path from Z to Y that is open given $[\mathbf{Z} \setminus \{Z\}] \cup \{X\}$

with the fewest colliders of all such paths.

Note that p and q share at least one node, since Z is an endpoint on both paths. Therefore, let M be the node on both p and q that is closest to X on p , and define $s = p(X, M) \oplus q(M, Y)$. We show below that s is non-causal, q is open given $\mathbf{Z} \setminus \{Z\}$, and M is a non-collider on s , where $M \notin \mathbf{Z} \setminus \{Z\}$. Since p is open given $\mathbf{Z} \setminus \{Z\}$, it will immediately follow that s – a non-causal path from X to Y – is open given $\mathbf{Z} \setminus \{Z\}$. Thus by Theorem 67, $\mathbf{Z} \setminus \{Z\}$ is not an adjustment set relative to $(\mathbf{X}, \mathbf{Y})$. Since Z was arbitrary, this will complete the proof.

We start by showing that s is non-causal. When $M = X$, then X is a node on q . Since q is open given $[\mathbf{Z} \setminus \{Z\}] \cup \{X\}$, then X must be a collider on q so that $s = q(X, Y)$ is non-causal. When instead $M \neq X$, suppose for sake of contradiction that p – and therefore s – begins $X \rightarrow$. Since $\mathbf{Z} < \mathbf{X}$, then p cannot be directed. Thus, p contains a collider. But by the definition of p , the closest such collider to X on p must be in $\text{An}(\mathbf{Z} \setminus \{Z\}, \mathcal{D})$, which contradicts that $\mathbf{Z} < \mathbf{X}$.

Next, suppose for sake of contradiction that q is blocked given $\mathbf{Z} \setminus \{Z\}$. Since q is open given $[\mathbf{Z} \setminus \{Z\}] \cup \{X\}$, there must be a collider on q in $\text{An}(X, \mathcal{D}) \setminus \text{An}(\mathbf{Z} \setminus \{Z\}, \mathcal{D})$. Let C be the closest such collider to Y on q , and let r be a directed path (possibly of length

zero) from C to X in $\mathcal{D}$. Note that r and $q(C, Y)$ cannot share a node other than C , since any such node would have a directed path (possibly of length zero) to C , Y , or a collider on $q(C, Y)$. And this would contradict the acyclicity of $\mathcal{D}$, the definition of q as a path that is open given $[\mathbf{Z} \setminus \{Z\}] \cup \{X\}$ with the fewest colliders, or the definition of $C \notin \text{An}(\mathbf{Z} \setminus \{Z\}, \mathcal{D})$ as the closest such collider to Y on q .

Thus, consider the path $t := (-r)(X, C) \oplus q(C, Y)$. Note by the definition of q , C , and r that t is a non-causal path from X to Y in $\mathcal{D}$ that is open given $\mathbf{Z} \setminus \{Z\}$. By Theorem 67 and the fact that $\mathbf{Z}$ is an adjustment set relative to $(\mathbf{X}, \mathbf{Y})$, t must be blocked given $\mathbf{Z}$, and therefore, t must contain Z as a non-collider. Note that $q(C, Y)$ cannot contain Z by the definition of q as a path from Z to Y , and so $(-r)(X, C)$ must contain Z . But then $(-r)(Z, C) \oplus q(C, Y)$ is a path from Z to Y that is open given $[\mathbf{Z} \setminus \{Z\}] \cup \{X\}$ with fewer colliders than q , which contradicts the definition of q . Therefore, q is open given $\mathbf{Z} \setminus \{Z\}$.

Finally, we show that M is a non-collider on s . It will follow that M is a non-collider on either p or q . Since p and q are open given $\mathbf{Z} \setminus \{Z\}$, then $M \notin \mathbf{Z} \setminus \{Z\}$. Thus, for sake of contradiction, suppose M is not a non-collider on s . In each of the cases below, we find a non-causal path from X to Y that is open given $\mathbf{Z}$, which by Theorem 67 contradicts that $\mathbf{Z}$ is an adjustment set relative to $(\mathbf{X}, \mathbf{Y})$.

- Let $M = X$. When $M = X$, we have already shown that $q(X, Y)$ is a non-causal path from X to Y . Then note that $q(X, Y)$ is open given $\mathbf{Z}$, since q is open given $\mathbf{Z} \setminus \{Z\}$ and $q(X, Y)$ does not contain Z .
- Let $M = Y$. When $M \neq X$, we have already shown that p – and therefore $p(X, Y)$ – is a non-causal path from X to Y . Then note that $p(X, Y)$ is open given $\mathbf{Z}$, since p is open given $\mathbf{Z} \setminus \{Z\}$ and $p(X, Y)$ does not contain Z .
- Let M be a collider on s . We have shown that s is a non-causal path from X to Y . To see that s is open given $\mathbf{Z}$, note since p and q are open given $\mathbf{Z} \setminus \{Z\}$ that every collider on $s(X, M)$ and $s(M, Y)$ is in $\text{An}(\mathbf{Z} \setminus \{Z\}, \mathcal{D})$ and no non-collider on $s(X, M)$

or $s(M, Y)$ is in $\mathbf{Z} \setminus \{Z\}$. Further, s does not contain Z except possibly $M = Z$. It remains to show that $M \in \text{An}(\mathbf{Z}, \mathcal{D})$. This clearly holds when $M = Z$ or when M is a collider on p , since p is open given $\mathbf{Z} \setminus \{Z\}$. Finally, consider when $M \neq Z$ and M is a non-collider on p . Since M is a collider on s , then $p(M, Z)$ begins $M \rightarrow$. Thus, either $p(M, Z)$ is directed or contains a collider in $\text{An}(\mathbf{Z} \setminus \{Z\}, \mathcal{D})$. ■

Lemma 72 (Equivalency of Definitions 16 and 70) *A set is an elementwise minimal adjustment set relative to (X, Y) if and only if it is a minimal adjustment set relative to (X, Y) .*

Proof of Lemma 72. $\Leftarrow$ Holds by definition. $\Rightarrow$ Let $\{X, Y\}$ and $\mathbf{Z}$ be disjoint node sets in a causal DAG $\mathcal{D}$, where $\mathbf{Z}$ is an elementwise minimal adjustment set relative to (X, Y) in $\mathcal{D}$. Let $|\mathbf{Z}| = n$, and note that the lemma holds trivially when $n \in \{0, 1\}$. Thus, let $n \geq 2$. For sake of contradiction, suppose that $\mathbf{Z}$ is not a minimal adjustment set relative to (X, Y) , and let $\mathbf{Z}_{\mathbf{k}}$ be a largest proper subset of $\mathbf{Z}$ such that $\mathbf{Z}_{\mathbf{k}}$ is an adjustment set relative to (X, Y) . Let $|\mathbf{Z}_{\mathbf{k}}| = k$, and note that $0 \leq k \leq n - 2$. Then order the nodes in $\mathbf{Z} := \{Z_1, \dots, Z_n\}$ so that $\mathbf{Z} \setminus \mathbf{Z}_{\mathbf{k}} = \{Z_{k+1}, \dots, Z_n\}$.

Consider the set $\mathbf{Z}_{\mathbf{k}+1} := \{Z_1, \dots, Z_{k+1}\}$. By the definition of $\mathbf{Z}_{\mathbf{k}}$ as a largest subset, $\mathbf{Z}_{\mathbf{k}+1}$ is not an adjustment set relative to (X, Y) . Thus by Theorem 67 and the fact that $\mathbf{Z}$ is an adjustment set, $\mathcal{D}$ must contain a non-causal path p from X to Y that is open given $\mathbf{Z}_{\mathbf{k}+1}$. Note however that p is blocked given $\mathbf{Z}_{\mathbf{k}}$ by Theorem 67 and the definition of $\mathbf{Z}_{\mathbf{k}}$ as an adjustment set. Therefore, p must contain a collider in $\text{An}(Z_{k+1}, \mathcal{D}) \setminus \text{An}(\mathbf{Z}_{\mathbf{k}}, \mathcal{D})$, where we use the convention that $\text{An}(\emptyset, \mathcal{D}) = \emptyset$. Of all such colliders, let C_X be the closest to X on p , and C_Y , the closest to Y on p , where possibly $C_X = C_Y$. Define r_x to be a directed path in $\mathcal{D}$ (possibly of length zero) from C_X to Z_{k+1} , and define r_y analogously from C_Y to Z_{k+1} . Then define r_{xy} to be a longest directed path in $\mathcal{D}$ (possibly of length zero) from Z_{k+1} to $\{Z_{k+1}, \dots, Z_n\}$, and let r_{xy} end with the node Z_ℓ , $\ell \in \{k+1, \dots, n\}$.

Since $\mathbf{Z}$ is elementwise minimal, then by Theorem 67, we can define a non-causal path q from X to Y that is open given $\mathbf{Z} \setminus \{Z_\ell\}$ and that contains Z_ℓ as a non-collider. Therefore,

either $q(X, Z_\ell)$ ends $\leftarrow Z_\ell$ or $q(Z_\ell, Y)$ begins $Z_\ell \rightarrow$. We show below that in both cases, there is a non-causal path t from X to Y in $\mathcal{D}$ that is open given $\mathbf{Z}_k$, which by Theorem 67 contradicts that $\mathbf{Z}_k$ is an adjustment set relative to (X, Y) and completes the proof.

CASE 1: Suppose $q(X, Z_\ell)$ ends $\leftarrow Z_\ell$. To form t , we want to combine $q(X, Z_\ell)$, $(-r_{xy})(Z_\ell, Z_{k+1}) \oplus (-r_y)(Z_{k+1}, C_Y)$, and $p(C_Y, Y)$.

We start by supposing, for sake of contradiction, that $q(X, Z_\ell)$ contains a collider. Since $q(X, Z_\ell)$ ends $\leftarrow Z_\ell$ and q is open given $\mathbf{Z} \setminus \{Z_\ell\}$, it follows that $\mathcal{D}$ contains a causal path s from Z_ℓ to $\mathbf{Z} \setminus \{Z_\ell\}$. But then either $r_{xy} \oplus s$ contradicts the definition of r_{xy} as a longest path from Z_{k+1} to $\{Z_{k+1}, \dots, Z_n\}$, or $r_x \oplus r_{xy} \oplus s$ contradicts the definition of $C_X \notin \text{An}(\mathbf{Z}_k, \mathcal{D})$.

Thus, $q(X, Z_\ell)$ takes the form $X \leftarrow \dots \leftarrow Z_\ell$ so that $r := q(X, Z_\ell) \oplus (-r_{xy})(Z_\ell, Z_{k+1}) \oplus (-r_y)(Z_{k+1}, C_Y)$ takes the form $X \leftarrow \dots \leftarrow C_Y$. If we let B be the node on both r and $p(C_Y, Y)$ that is closest to Y on p , then we can define $t = r(X, B) \oplus p(B, Y)$.

Note that t is a non-causal path from X to Y . To see that t is open given $\mathbf{Z}_k$, note that no node on $t(X, B)$ is a collider on t . And no node on $t(X, B)$ is in $\mathbf{Z}_k$, since C_Y is an ancestor of every node on $t(X, B)$ and $C_Y \notin \text{An}(\mathbf{Z}_k, \mathcal{D})$. Finally, note by the definition of p , C_Y , and B that every collider on $t(B, Y)$ is in $\text{An}(\mathbf{Z}_k, \mathcal{D})$, and no non-collider on $t(B, Y)$ is in $\mathbf{Z}_k$.

CASE 2: Suppose $q(Z_\ell, Y)$ begins $Z_\ell \rightarrow$. To form t , we want to combine $p(X, C_X)$, $r_x(C_X, Z_{k+1}) \oplus r_{xy}(Z_{k+1}, Z_\ell)$, and $q(Z_\ell, Y)$.

By analogous logic to CASE 1, $q(Z_\ell, Y)$ cannot contain colliders and thus, takes the form $Z_\ell \rightarrow \dots \rightarrow Y$. Then $r := r_x(C_X, Z_{k+1}) \oplus r_{xy}(Z_{k+1}, Z_\ell) \oplus q(Z_\ell, Y)$ takes the form $C_X \rightarrow \dots \rightarrow Y$. If we let B be the node on both $p(X, C_X)$ and r that is closest to X on p , then we can define $t = p(X, B) \oplus r(B, Y)$.

For sake of contradiction, suppose that t is causal, and consider where B falls on r . If B were to fall on $r(C_X, Z_\ell)$, then t would contain $Z_\ell \in \mathbf{Z}$. But by Theorem 67, this contradicts that $\mathbf{Z}$ is an adjustment set relative to (X, Y) . Thus, B falls on $r(Z_\ell, Y)$. Note that $B \neq X$, since $r(Z_\ell, Y) = q(Z_\ell, Y)$, where q is a path from X to Y . This implies B is a non-endpoint

node on $p(X, C_X)$ and $p(X, C_X)$ begins $X \rightarrow$. But then $p(X, C_X)$ must contain a directed path from B to either C_X or a collider on $p(X, C_X)$. This path in combination with $r(C_X, B)$ contradicts either the acyclicity of $\mathcal{D}$ or the definition of $C_X \notin \text{An}(\mathbf{Z}_k, \mathcal{D})$ as the earliest such collider on p .

To see that t is open given $\mathbf{Z}_k$, note that no node on $t(B, Y)$ is a collider on t . And no node on $t(B, Y)$ is in $\mathbf{Z}_k$, since C_X is an ancestor of every node on $t(B, Y)$ and $C_X \notin \text{An}(\mathbf{Z}_k, \mathcal{D})$. Finally, note by the definition of p , C_X , and B that every collider on $t(X, B)$ is in $\text{An}(\mathbf{Z}_k, \mathcal{D})$, and no non-collider on $t(X, B)$ is in $\mathbf{Z}_k$. ■

B.4 Additional Simulation Results

The following are two additional plots from our simulation results that we excluded from Section 3.5. They consider Setting 3, where datasets are generated from models without a treatment effect. Compare these plots to Figures 3.6-3.7 that consider Settings 1-2, where datasets are generated from models with a treatment effect.

Figure B.1 shows how often all three methods (R1 Build, R1 Combine, FCI + GAC) obtain one of the following outcomes.

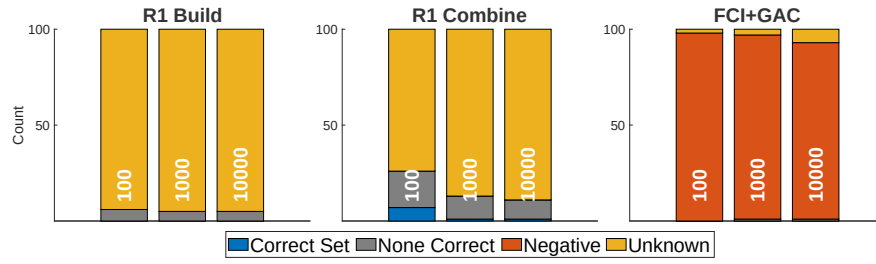
Correct Set: Method concludes $\{X_1, X_2\}$ affects Y . At least one adjustment set it finds is correct in the underlying DAG.

None correct: Method concludes $\{X_1, X_2\}$ affects Y . No adjustment set it finds is correct in the underlying DAG.

Negative: Method concludes $\{X_1, X_2\}$ has no effect on Y .

Unknown: Method cannot find an adjustment set for the effect of $\{X_1, X_2\}$ on Y .

Successful performance in Setting 3 is when a method obtains a *negative* outcome. As noted in the main text, FCI + GAC easily outperforms R1 Build and R1 Combine in this setting,



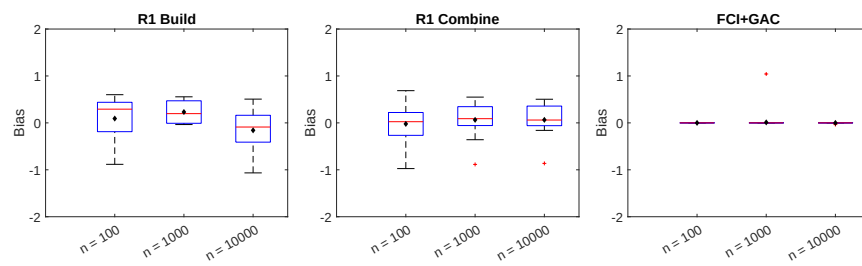
(a) Setting 3

Figure B.1: Outcomes of our data-driven methods (R1 Build, R1 Combine) and an existing graphical approach (FCI + GAC) on simulated data. Stacked bars show how often a method results in *correct set*, *none correct*, *negative*, or *unknown*.

since our methods cannot obtain a *negative* outcome.¹

Figure B.2 shows the differences between causal effect estimates (based on conclusions from each method) and the true causal effect. As noted in the main text, FCI + GAC easily outperforms R1 Build and R1 Combine in this setting, since our methods cannot obtain a *negative* outcome.

¹For clarity in reading Figure B.1, note that in a small percentage of datasets, R1 Combine incorrectly concludes that there is a non-zero treatment effect, but the adjustment set it finds is accurate (for a “treatment effect” of zero).



(a) Setting 3

Figure B.2: Differences between estimated and true causal effects. Results for our data-driven methods (R1 Build, R1 Combine) and an existing graphical approach (FCI + GAC) when applied to simulated data.

Appendix C

ADDITIONAL MATERIAL FOR CHAPTER 4 IDENTIFYING CONDITIONAL EFFECTS

C.1 FURTHER PRELIMINARIES

Concatenation. We denote the concatenation of paths by the symbol $\oplus$, so that for a path $p = \langle X_1, X_2, \dots, X_m \rangle$, $p = p(X_1, X_r) \oplus p(X_r, X_m)$, for $1 \leq r \leq m$.

Definition 73 (Bucket Decomposition; Perković, 2020) *Let $\mathbf{D}$ be a node set in an MPDAG $\mathcal{G}$. Then $\mathbf{B}_1, \dots, \mathbf{B}_k$, $k \geq 1$, is the bucket decomposition of $\mathbf{D}$ in $\mathcal{G}$ if $\mathbf{B}_i$, $i \in \{1, \dots, k\}$, is a bucket in $\mathbf{D}$ and $\cup_{i=1}^k \mathbf{B}_i = \mathbf{D}$.*

Definition 74 (Distance to $\mathbf{Z}$; Perković et al., 2017) *Let $\{X, Y\}$ and $\mathbf{Z}$ be disjoint node sets in an MPDAG $\mathcal{G}$, and let p be a path in $\mathcal{G}$ from X to Y such that $\mathcal{G}$ has a possibly directed path (possibly of zero length) from every collider on p to $\mathbf{Z}$. Further, let $\{C_1, \dots, C_k\}$ be the set of colliders on p , and let ℓ_i , $i \in \{1, \dots, k\}$, be the length of a shortest possibly directed path in $\mathcal{G}$ from C_i to $\mathbf{Z}$. Then the distance to $\mathbf{Z}$ for p is $\sum_{i=1}^k \ell_i$.*

C.2 EXISTING RESULTS

Theorem 75 (Rules of the do Calculus; Theorem 3.4.1 of Pearl [2009]) *Let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$, and $\mathbf{W}$ be pairwise disjoint (possibly empty) node sets in a causal DAG $\mathcal{D}$. Let $\mathcal{D}_{\overline{\mathbf{X}}}$ denote the graph obtained by deleting all edges into $\mathbf{X}$ from $\mathcal{D}$. Similarly, let $\mathcal{D}_{\underline{\mathbf{X}}}$ denote the graph obtained by deleting all edges out of $\mathbf{X}$ in $\mathcal{D}$, and let $\mathcal{D}_{\overline{\mathbf{X}}\underline{\mathbf{Z}}}$ denote the graph obtained by deleting all edges into $\mathbf{X}$ and all edges out of $\mathbf{Z}$ in $\mathcal{D}$. The following rules hold for all densities consistent with $\mathcal{D}$.*

Rule 1. If $(Y \perp_d Z \mid X, W)_{\mathcal{D}_{\overline{X}}}$, then

$$f(y|do(\mathbf{x}), \mathbf{z}, \mathbf{w}) = f(y|do(\mathbf{x}), \mathbf{w}). \quad (\text{C.1})$$

Rule 2. If $(Y \perp_d X \mid Z, W)_{\mathcal{D}_{\underline{X}\overline{W}}}$, then

$$f(y|do(\mathbf{x}), \mathbf{z}, do(\mathbf{w})) = f(y|\mathbf{x}, \mathbf{z}, do(\mathbf{w})). \quad (\text{C.2})$$

Rule 3. If $(Y \perp_d X \mid Z, W)_{\mathcal{D}_{\overline{W}, \overline{X(Z)}}}$, then

$$f(y|do(\mathbf{x}), \mathbf{z}, do(\mathbf{w})) = f(y|\mathbf{z}, do(\mathbf{w})), \quad (\text{C.3})$$

where $\mathbf{X}(\mathbf{Z}) := \mathbf{X} \setminus \text{An}(\mathbf{Z}, \mathcal{D}_{\overline{W}})$.

Lemma 76 (Theorem 3.2.4 of Mardia et al., 1980) *Let $\mathbf{X} = (\mathbf{X}_1^T, \mathbf{X}_2^T)^T$ be a p -dimensional multivariate Gaussian random vector with mean vector $\mu = (\mu_1^T, \mu_2^T)^T$ and covariance matrix $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$, so that $\mathbf{X}_1$ is a q -dimensional multivariate Gaussian random vector with mean vector μ_1 and covariance matrix Σ_{11} and $\mathbf{X}_2$ is a $(p - q)$ -dimensional multivariate Gaussian random vector with mean vector μ_2 and covariance matrix Σ_{22} . Then $E[\mathbf{X}_2 | \mathbf{X}_1 = \mathbf{x}_1] = \mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(\mathbf{x}_1 - \mu_1)$.*

Lemma 77 (Wright's Rule; Wright, 1921, 1934) *Consider the following structural equation model (SEM) over a vector of random variables $\mathbf{X} = (X_1, \dots, X_k)^T$. The set of equations is given by $\mathbf{X} = \mathbf{A}\mathbf{X} + \varepsilon$, where $\mathbf{A} \in \mathbb{R}^{k \times k}$ is a matrix with zeroes on the diagonal and coefficients from the SEM on the off-diagonals, $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k)^T$ is a vector of mutually independent errors with finite means, and $\text{Var}(X_i) = 1$ for all $i \in \{1, \dots, k\}$. Let $\mathcal{D}$ be a DAG over $\mathbf{X}$ such that $X_i \rightarrow X_j$ is in $\mathcal{D}$ if and only if $A_{ji} \neq 0$ for $i, j \in \{1, \dots, k\}$, $i \neq j$. (We call A_{ji} the edge coefficient of $X_i \rightarrow X_j$ when $A_{ji} \neq 0$.) Further, let $\{p_1, \dots, p_s\}$ be the set of paths between X_i and X_j in $\mathcal{D}$ that do not contain a collider, and let π_r be the product of all edge coefficients along the path p_r , $r \in \{1, \dots, s\}$. Then $\text{Cov}(X_i, X_j) = \sum_{r=1}^s \pi_r$.*

Lemma 78 (cf. proof of Lemma B.4, Henckel et al. [2022]) *Consider the following structural equation model (SEM) over a vector of random variables $\mathbf{X} = (X_1, \dots, X_k)^T$. The set of equations is given by $\mathbf{X} = \mathbf{A}\mathbf{X} + \varepsilon$, where $\mathbf{A} \in \mathbb{R}^{k \times k}$ is a matrix with zeroes on the diagonal and coefficients from the SEM on the off-diagonals, $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k)^T$ is a vector of mutually independent errors with finite means, and finite variances. Let $\mathcal{D}$ be a DAG over $\mathbf{X}$ such that $X_i \rightarrow X_j$ is in $\mathcal{D}$ if and only if $A_{ji} \neq 0$ for $i, j \in \{1, \dots, k\}$, $i \neq j$. (We call A_{ji} the edge coefficient of $X_i \rightarrow X_j$ when $A_{ji} \neq 0$.) Then we can write*

$$X_i = \sum_{j: X_j \in \text{An}(X_i, \mathcal{D})} \tau_{ji} \varepsilon_j$$

where we define τ_{ji} as follows. Let $\tau_{ii} = 1$. For the remaining τ_{ji} , consider the set of all causal paths $\{p_1, \dots, p_s\}$ from X_j to X_i in $\mathcal{D}$, and let π_r be the product of all edge coefficients along the path p_r , $r \in \{1, \dots, s\}$. Then let $\tau_{ji} = \sum_{r=1}^s \pi_r$. Further, we have

$$\text{Cov}(X_i, X_j) = \sum_{k: X_k \in \text{An}(X_i, \mathcal{D}) \cap \text{An}(X_j, \mathcal{D})} \tau_{ki} \tau_{kj} \text{Var}(\varepsilon_k),$$

Lemma 79 (cf. Theorem 1 and Proposition 3 of Lauritzen et al., 1990) *Let $\mathcal{D} = (\mathbf{V}, \mathbf{E})$ be a DAG, and let f be an observational density over $\mathbf{V}$. Then f is Markov compatible with $\mathcal{D}$ if and only if*

$$V_i \perp\!\!\!\perp \left[\mathbf{V} \setminus (\text{De}(V_i, \mathcal{D}) \cup \text{Pa}(V_i, \mathcal{D})) \right] \mid \text{Pa}(V_i, \mathcal{D})$$

for all $V_i \in \mathbf{V}$, where $\perp\!\!\!\perp$ indicates independence with respect to f .

Lemma 80 (cf. Lemma F.1 of Rothenhäusler et al., 2018) *Let X and Y be nodes in an MPDAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ such that $X - Y$ is in $\mathcal{G}$. Let $\mathcal{G}'$ be an MPDAG constructed from $\mathcal{G}$ by adding $X \rightarrow Y$ and completing the orientation rules R1 - R4 of Meek [1995]. For any $Z, W \in \mathbf{V}$, if $Z - W$ is in $\mathcal{G}$ and $Z \rightarrow W$ is in $\mathcal{G}'$, then $W \in \text{De}(Y, \mathcal{G}')$.*

Lemma 81 (cf. Lemma F.2 of Rothenhäusler et al., 2018) *Let X be a node in an MPDAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, and let $\mathbf{S}$ be a set such that for all $S \in \mathbf{S}$, $X - S$ is in $\mathcal{G}$. Then there is an MPDAG $\mathcal{G}' = (\mathbf{V}, \mathbf{E}')$ that is formed by taking $\mathcal{G}$, orienting $X \rightarrow S$ for all $S \in \mathbf{S}$, and completing R1-R4 of Meek [1995].*

Algorithm 2: PCO Algorithm of Perković [2020]

Inputs : Node set $\mathbf{D} \subseteq \mathbf{V}$ and MPDAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$

Output: List of buckets $\mathbf{B} = (\mathbf{B}_1, \dots, \mathbf{B}_k), k \geq 1$, in $\mathbf{D}$ given $\mathcal{G}$

```

1 Let ConComp be the bucket decomposition of  $\mathbf{V}$  in  $\mathcal{G}$  (Definition 73);
2 Let  $\mathbf{B}$  be an empty list;
3 While ConComp  $\neq \emptyset$ 
4   | Let  $\mathbf{C} \in \mathbf{ConComp}$ ;
5   | Let  $\overline{\mathbf{C}}$  be the set of nodes in ConComp that are not in  $\mathbf{C}$ ;
6   | If all edges between  $\mathbf{C}$  and  $\overline{\mathbf{C}}$  are into  $\mathbf{C}$  in  $\mathcal{G}$ 
7   |   | Remove  $\mathbf{C}$  from ConComp;
8   |   | Let  $\mathbf{B}_* = \mathbf{C} \cap \mathbf{D}$ ;
9   |   | If  $\mathbf{B}_* \neq \emptyset$ 
10  |   |   | Add  $\mathbf{B}_*$  to the beginning of  $\mathbf{B}$ ;
11 Return  $\mathbf{B}$ ;
```

Lemma 82 (cf. Lemma 1 of Meek, 1995) *Let X , Y , and Z be nodes in a CPDAG $\mathcal{C}$. If $\mathcal{C}$ contains $X \rightarrow Y - Z$, then $\mathcal{C}$ contains $X \rightarrow Y$.*

Lemma 83 (Lemma 3.2 of Perković et al., 2017) *Let X and Y be nodes in an MPDAG $\mathcal{G}$ and let p be a path from X to Y in $\mathcal{G}$. If there exists a DAG $\mathcal{D} \in [\mathcal{G}]$ such that the path in $\mathcal{D}$ corresponding to p in $\mathcal{G}$ is causal, then p is possibly causal.*

Lemma 84 (Lemma 3.5 of Perković et al., 2017) *Let $p = \langle V_1, \dots, V_k \rangle$, $k \geq 2$, be a definite status path in an MPDAG. Then p is possibly causal if and only if p has no edge $V_i \leftarrow V_{i+1}$ for any $i \in \{1, \dots, k-1\}$.*

Lemma 85 (Lemma 3.5 of Perković, 2020) *Let $\mathbf{D}$ be a node set in an MPDAG $\mathcal{G}$ and let $\text{PCO}(\mathbf{D}, \mathcal{G}) = (\mathbf{B}_1, \dots, \mathbf{B}_k)$, $k \geq 1$, be the output of the PCO Algorithm (Algorithm 2). Then for $i, j \in \{1, \dots, k\}$, $\mathbf{B}_i$ and $\mathbf{B}_j$ are buckets in $\mathbf{D}$, and if $i < j$, then $\mathbf{B}_i < \mathbf{B}_j$ in $\mathcal{G}$.*

Lemma 86 (Lemma C.2 of Perković, 2020) *Let $\mathbf{C}$ be a bucket in $\mathbf{V}$ in an MPDAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ and let $X \in \mathbf{V} \setminus \mathbf{C}$. If there is a causal path from X to $\mathbf{C}$ in $\mathcal{G}$, then for every node $C \in \mathbf{C}$, there is a causal path from X to C in $\mathcal{G}$.*

Lemma 87 (cf. Proof of Lemma C.2 of Perković, 2020) *Let $\mathbf{C}$ be a bucket in $\mathbf{V}$ in an MPDAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ and let $X \in \mathbf{V} \setminus \mathbf{C}$. If $X \in \text{Pa}(\mathbf{C}, \mathcal{G})$, then $X \in \text{Pa}(C, \mathcal{G})$ for every node $C \in \mathbf{C}$.*

Lemma 88 (Lemma D.1(i) of Perković, 2020) *Let $\mathbf{X}$ and $\mathbf{Y}$ be disjoint node sets in an MPDAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, where there is no proper possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$ that starts with an undirected edge in $\mathcal{G}$. Then there is no proper possibly causal path from $\mathbf{X}$ to $\text{An}(\mathbf{Y}, \mathcal{G}_{\mathbf{V} \setminus \mathbf{X}})$ that starts with an undirected edge in $\mathcal{G}$.*

C.3 BROAD MPDAG RESULTS

This section presents broad results for MPDAGs used in the proofs of Appendix C.4. We include these results here rather than in Appendix C.4, because they are likely useful beyond the scope of this paper.

C.3.1 RESULTS ON POSSIBLY CAUSAL PATHS

Lemma 89 (cf. Lemma 3.6 of Perković et al., 2017) *Let X and Y be distinct nodes in an MPDAG $\mathcal{G}$ and let p be a possibly causal path from X to Y in $\mathcal{G}$. Then any shortest subsequence of p in $\mathcal{G}$ forms an unshielded, possibly causal path from X to Y .*

Proof of Lemma 89. Let k be the number of nodes on p . Pick an arbitrary shortest subsequence of p and call it $p^* := \langle X = V_0, \dots, V_\ell = Y \rangle$, $0 < \ell \leq k$. Note that there is no edge $V_i \leftarrow V_j$, $0 \leq i < j \leq k$ in $\mathcal{G}$, since this would contradict that p is possibly causal. Thus, p^* is also possibly causal. Further, note that p^* is unshielded, since if any triple on the path is shielded, it either contradicts that p^* is possibly causal (i.e., $V_i \leftarrow V_{i+2}$ cannot be in p^*) or that p^* is a shortest subsequence of p (i.e., $V_i \rightarrow V_{i+2}$ and $V_i - V_{i+2}$ cannot be in p^*). ■

Lemma 90 *Let $p = \langle P_0, \dots, P_k \rangle$ be a path in an MPDAG $\mathcal{G}$. Then p is possibly causal if and only if $\mathcal{G}$ does not contain any path $P_i \leftarrow \dots \leftarrow P_j$, $0 \leq i < j \leq k$.*

Proof of Lemma 90. $\Leftarrow$ Follows immediately. $\Rightarrow$ Let p be possibly causal, and for sake of contradiction, suppose $\mathcal{G}$ contains a path q from P_i to P_j , $0 \leq i < j \leq k$, of the form $P_i = Q_0 \leftarrow \cdots \leftarrow Q_\ell = P_j$. Then define $r = \langle P_i = R_0, \dots, R_m = P_j \rangle$ to be a shortest subsequence of $p(P_i, P_j)$ in $\mathcal{G}$, and note that by Lemma 89, r is an unshielded, possibly causal path.

To see that $r(R_0, R_1)$ takes the form $R_0 - R_1$, note that $\mathcal{G}$ cannot contain $R_0 \leftarrow R_1$, since r is possibly causal. Further, $\mathcal{G}$ cannot contain $R_0 \rightarrow R_1$, since r being unshielded would imply, by R1 of Meek [1995], that $\mathcal{G}$ contains the cycle $P_i = R_0 \rightarrow \cdots \rightarrow R_m = P_j = Q_\ell \rightarrow \cdots \rightarrow Q_0 = P_i$. Then since $\mathcal{G}$ contains $R_0 - R_1$, there must be a DAG $\mathcal{D} \in [\mathcal{G}]$ that contains $R_0 \rightarrow R_1$. But since r is unshielded, this implies, by R1 of Meek [1995], that $\mathcal{D}$ contains the cycle $P_i = R_0 \rightarrow \cdots \rightarrow R_m = P_j = Q_\ell \rightarrow \cdots \rightarrow Q_0 = P_i$, which is a contradiction. $\blacksquare$

Lemma 91 *Let X , Y , and Z be distinct nodes in an MPDAG $\mathcal{G}$.*

- (i) *If p is a possibly causal path from X to Y and q is a causal path from Y to Z , then $p \oplus q$ is a possibly causal path from X to Z .*
- (ii) *If p is a causal path from X to Y and q is a possibly causal path from Y to Z , then $p \oplus q$ is a possibly causal path from X to Z .*

Proof of Lemma 91. Let $p = \langle X = P_0, \dots, P_k = Y \rangle$ and $q = \langle Y = Q_0, \dots, Q_r = Z \rangle$, where either (i) p is possibly causal and q is causal, or (ii) q is possibly causal and p is causal. To see that $p \oplus q$ exists, suppose for sake of contradiction that p and q share at least one node other than Y . Let $\mathbf{S}$ denote the collection of such nodes, and consider the node in $\mathbf{S}$ with the lowest index on q . That is, consider $Q_j \in \mathbf{S}$ such that $j \leq \ell$ for all $Q_\ell \in \mathbf{S}$. Let $Q_j = P_i$ for some $P_i \neq Y$ on p . Under (i), q is causal so that $\mathcal{G}$ contains $P_k = Q_0 \rightarrow \cdots \rightarrow Q_j = P_i$. But by Lemma 90, this contradicts that p is possibly causal. Similarly, under (ii), p is causal so that $\mathcal{G}$ contains $Q_j = P_i \rightarrow \cdots \rightarrow P_k = Q_0$, which contradicts that q is possibly causal.

To complete the proof, we show that there is no backward edge between any two nodes on $p \oplus q$. By the choice of p and q , note that there is no edge $P_{i_1} \leftarrow P_{j_1}$ for $0 \leq i_1 < j_1 \leq k$

in $\mathcal{G}$, and there is no edge $Q_{i_2} \leftarrow Q_{j_2}$ for $0 \leq i_2 < j_2 \leq r$ in $\mathcal{G}$. Thus, suppose for sake of contradiction that there exists an edge $P_i \leftarrow Q_j$ in $\mathcal{G}$ for $i \in \{0, \dots, k-1\}$ and $j \in \{1, \dots, r\}$. Note that P_i is on p and not q , and analogously, Q_j is on q and not p , since we have shown p and q cannot share nodes other than Y .

CASE (i): Note that $p(P_i, Y)$ is possibly causal. Then pick an arbitrary shortest subsequence of $p(P_i, Y)$ and call it $t := \langle P_i = T_0, \dots, T_m = Y \rangle$, $m \geq 1$. By Lemma 89, t forms an unshielded, possibly causal path. Since t is possibly causal, $\mathcal{G}$ must contain $T_0 \rightarrow T_1$ or $T_0 - T_1$. Thus, there is a DAG $\mathcal{D} \in [\mathcal{G}]$ that contains the edge $T_0 \rightarrow T_1$. Further, since t is unshielded, $\mathcal{D}$ contains $T_0 \rightarrow \dots \rightarrow T_m$ by R1 of Meek [1995]. But this is a contradiction, since q is causal and thus $\mathcal{D}$ contains the cycle $P_i = T_0 \rightarrow \dots \rightarrow T_m = Y = Q_0 \rightarrow \dots \rightarrow Q_j \rightarrow P_i$.

CASE (ii): Note that $q(Y, Q_j)$ is possibly causal. Then pick an arbitrary shortest subsequence of $q(Y, Q_j)$ and call it $t := \langle Y = T_0, \dots, T_m = Q_j \rangle$, $m \geq 1$. By Lemma 89, t forms an unshielded, possibly causal path. Since t is possibly causal, $\mathcal{G}$ must contain $T_0 \rightarrow T_1$ or $T_0 - T_1$. Thus, there is a DAG $\mathcal{D} \in [\mathcal{G}]$ that contains the edge $T_0 \rightarrow T_1$. Further, since t is unshielded, $\mathcal{D}$ contains $T_0 \rightarrow \dots \rightarrow T_m$ by R1 of Meek [1995]. But this is a contradiction, since p is causal and thus $\mathcal{D}$ contains the cycle $Y = T_0 \rightarrow \dots \rightarrow T_m = Q_j \rightarrow P_i \rightarrow \dots \rightarrow P_k = Y$. ■

Remark 92 *Perhaps contrary to intuition, even if a path is possibly causal in an MPDAG $\mathcal{G}$, there may not exist a DAG in $[\mathcal{G}]$ where the corresponding path is causal.*

To see this, consider the MPDAG $\mathcal{G}$ in Figure C.1 and let $p = \langle X, V_1, V_2, Y \rangle$. Note that p is possibly causal. Let $\mathcal{D}$ be an arbitrary DAG in $\mathcal{G}$ and define p^ to be the path in $\mathcal{D}$ corresponding to p in $\mathcal{G}$. If p^* begins $X \leftarrow V_1$, it is non-causal. Thus, consider when p^* begins $X \rightarrow V_1$. It follows that $\mathcal{D}$ contains $V_1 \rightarrow V_2$ by R1 of Meek [1995] as well as $V_2 \leftarrow Y$ by R4 of Meek [1995], so that again, p^* is non-causal.*

Lemma 93 *Let X and Y be nodes in an MPDAG $\mathcal{G}$. Then there is a possibly causal path from X to Y in $\mathcal{G}$ if and only if there is at least one DAG in $[\mathcal{G}]$ with a causal path from X to Y .*

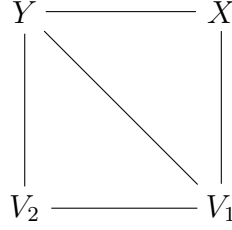


Figure C.1: An MPDAG used in Remark 92.

Proof of Lemma 93. $\Leftarrow$ Holds by Lemma 83. $\Rightarrow$ Let p be a possibly causal path from X to Y in $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ and let $p^* = \langle X = V_0, \dots, V_k = Y \rangle$, $k \geq 1$, be a shortest subsequence of p in $\mathcal{G}$. By Lemma 89, p^* forms an unshielded, possibly causal path. Since p^* is possibly causal, $\mathcal{G}$ must contain $V_0 \rightarrow V_1$ or $V_0 - V_1$. Thus, there is a DAG $\mathcal{D} \in [\mathcal{G}]$ that contains the edge $V_0 \rightarrow V_1$. Further, since p^* is unshielded, $\mathcal{D}$ contains $X = V_0 \rightarrow \dots \rightarrow V_k = Y$ by R1 of Meek [1995]. $\blacksquare$

C.3.2 REMARKS ON THE ORDERING OF BUCKETS

Remark 94 *Note that the output of the PCO Algorithm (Algorithm 2) is an ordered bucket decomposition (Definition 20) of the input node set in the input MPDAG. This follows from the steps of Algorithm 2 and Lemma 85.*

Definition 95 (Bucket Decomposition Matching) *Let $\mathbf{D}$ be a node set in an MPDAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$. Let $(\mathbf{B}_1, \dots, \mathbf{B}_k)$, $k \geq 1$, be an ordered bucket decomposition of $\mathbf{D}$ in $\mathcal{G}$ (Definition 20), and let $(\mathbf{C}_1, \dots, \mathbf{C}_\ell)$, $\ell \geq k$, be an ordered bucket decomposition of $\mathbf{V}$ in $\mathcal{G}$. We say that $(\mathbf{C}_1, \dots, \mathbf{C}_\ell)$ is an ordered bucket decomposition that matches the ordering of $(\mathbf{B}_1, \dots, \mathbf{B}_k)$ if the following holds. For any $i, j \in \{1, \dots, k\}$ and $m, n \in \{1, \dots, \ell\}$ such that $\mathbf{B}_i \cap \mathbf{C}_m \neq \emptyset$ and $\mathbf{B}_j \cap \mathbf{C}_n \neq \emptyset$, if $\mathbf{B}_i < \mathbf{B}_j$, then $\mathbf{C}_m < \mathbf{C}_n$.*

Remark 96 *Let $\mathbf{D}$ be a node set in the MPDAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, and let $\text{PCO}(\mathbf{D}, \mathcal{G}) = (\mathbf{B}_1, \dots, \mathbf{B}_k)$, $k \geq 1$, be the output of the PCO Algorithm (Algorithm 2). Since the algorithm relies on the intersection of $\mathbf{D}$ with an ordering of the bucket decomposition of $\mathbf{V}$ in*

$\mathcal{G}$, there must exist an ordered bucket decomposition of $\mathbf{V}$ in $\mathcal{G}$ (Definition 20) that matches the ordering (Definition 95) of $(\mathbf{B}_1, \dots, \mathbf{B}_k)$.

C.3.3 RESULTS ON PATHS AMONG BUCKETS

Lemma 97 *Let $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ be an MPDAG, and let $\mathbf{C}_1, \dots, \mathbf{C}_\ell$, $\ell \geq 1$, be the bucket decomposition of $\mathbf{V}$ in $\mathcal{G}$ (Definition 73). Further, let X and Y be nodes in $\mathcal{G}$ such that $X \in \mathbf{C}_x$ and $Y \in \mathbf{C}_y$ for some $x, y \in \{1, \dots, \ell\}$, $x \neq y$. If $p = \langle X = P_0, \dots, P_k = Y \rangle$, $k \geq 1$, is a proper possibly causal path from $\mathbf{C}_x$ to $\mathbf{C}_y$ in $\mathcal{G}$, then $\mathcal{G}$ contains a subsequence of p that is a causal path from X to Y .*

Proof of Lemma 97. Let $p = \langle X = P_0, \dots, P_k = Y \rangle$, $k \geq 1$, be a proper possibly causal path from $\mathbf{C}_x$ to $\mathbf{C}_y$ in $\mathcal{G}$. Since p is proper, X is the only node in $\mathbf{C}_x$ on p . Thus P_1 is the first node in the bucket $\mathbf{C}_a$ where $a \in \{1, \dots, \ell\} \setminus \{x\}$. By the definition of buckets and the fact that p is possibly causal, p contains $X \rightarrow P_1$. Then by Lemma 87, $\mathcal{G}$ contains $X \rightarrow A$ for all $A \in \mathbf{C}_a$.

When $Y \in \mathbf{C}_a$, $\mathcal{G}$ contains the subsequence $X \rightarrow Y$ and we are done. When $Y \notin \mathbf{C}_a$, define P_q , $q \in \{1, \dots, k\}$, to be the last node on p in $\mathbf{C}_a$. Note that $\mathcal{G}$ contains $X \rightarrow P_q$ and consider the node P_{q+1} . By Lemma 98, $p(P_1, P_q)$ is entirely contained in $\mathbf{C}_a$. Thus, P_{q+1} is the first node in the bucket $\mathbf{C}_b$ where $b \in \{1, \dots, \ell\} \setminus \{a, x\}$. By the definition of buckets and the fact that p is possibly causal, p contains $P_q \rightarrow P_{q+1}$. Then by Lemma 87, $\mathcal{G}$ contains $P_q \rightarrow B$ for all $B \in \mathbf{C}_b$.

When $Y \in \mathbf{C}_b$, $\mathcal{G}$ contains the subsequence $X \rightarrow P_q \rightarrow Y$ and we are done. When $Y \notin \mathbf{C}_b$, we can continue in this way until we reach the first node in $\mathbf{C}_y$ – call it P_{r+1} . By the same logic as above, p must contain $P_r \rightarrow P_{r+1}$, where $P_r \notin \mathbf{C}_y$ and where $\mathcal{G}$ contains $X \rightarrow P_q \rightarrow \dots \rightarrow P_r$. So again by Lemma 87, $\mathcal{G}$ contains $P_r \rightarrow W$ for all $W \in \mathbf{C}_y$ and thus $\mathcal{G}$ contains the subsequence $X \rightarrow P_q \rightarrow \dots \rightarrow P_r \rightarrow Y$. ■

Lemma 98 *Let $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ be an MPDAG, and let $\mathbf{C}_1, \dots, \mathbf{C}_k$, $k \geq 1$, be the bucket decomposition of $\mathbf{V}$ in $\mathcal{G}$ (Definition 73). If $X, Y \in \mathbf{C}_i$, $i \in \{1, \dots, k\}$, then there is no possibly*

causal path from X to Y in $\mathcal{G}$ containing nodes in $\mathbf{V} \setminus \mathbf{C}_i$.

Proof of Lemma 98. For sake of contradiction, suppose that there is a possibly causal path from X to Y in $\mathcal{G}$ containing nodes in $\mathbf{V} \setminus \mathbf{C}_i$ – call this path $p = \langle X = P_0, \dots, P_u = Y \rangle$, $u \geq 1$. Let P_s , $s \in \{1, \dots, u-1\}$, be an arbitrary node on p such that $P_s \in \mathbf{V} \setminus \mathbf{C}_i$. Let P_r , $r \in \{0, \dots, s-1\}$ be the node on p closest to P_s such that $P_r \in \mathbf{C}_i$. And let P_t , $t \in \{s+1, \dots, u\}$ be the node on p closest to P_s such that $P_t \in \mathbf{C}_i$.

By the definition of P_t , we have that $P_{t-1} \notin \mathbf{C}_i$ and $P_t \in \mathbf{C}_i$. Consider the edge between P_{t-1} and P_t . Since $\mathbf{C}_i$ is a maximal undirected connected subset of $\mathbf{V}$, then p does not contain $P_{t-1} - P_t$. Further since p is possibly causal, it does not contain $P_{t-1} \leftarrow P_t$. Therefore p must contain $P_{t-1} \rightarrow P_t$. But then by Lemma 87, $\mathcal{G}$ must contain $P_{t-1} \rightarrow P_r$, which contradicts that p is possibly causal. $\blacksquare$

Lemma 99 *Let $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ be an MPDAG and let $(\mathbf{C}_1, \dots, \mathbf{C}_k)$, $k \geq 1$, be an ordered bucket decomposition of $\mathbf{V}$ in $\mathcal{G}$ (Definition 20). Then for any $i, j \in \{1, \dots, k\}$, $i < j$, there is no possibly causal path from $\mathbf{C}_j$ to $\mathbf{C}_i$ in $\mathcal{G}$.*

Proof of Lemma 99. For sake of contradiction, let there be a possibly causal path from $\mathbf{C}_j$ to $\mathbf{C}_i$ in $\mathcal{G}$. Choose a shortest such path $p := \langle P_0, \dots, P_u \rangle$, $u \geq 1$, and let $\mathbf{S}$ be the smallest collection of buckets in $\mathbf{V}$ such that every node on p is in one bucket in $\mathbf{S}$. Consider the order in which p enters and exits each bucket in $\mathbf{S}$. p begins in $\mathbf{C}_j$, but once it exits $\mathbf{C}_j$, it never returns. This holds by Lemma 98 and the fact that p is possibly causal. By the same logic, once p enters any other bucket in $\mathbf{S}$, the first time it exits that bucket, it will never return. Thus, let $\mathbf{C}_{\ell_1}, \dots, \mathbf{C}_{\ell_m}$, $1 < m \leq k$, be an arrangement of the buckets in $\mathbf{S}$ in order of their appearance on p .

Note that P_0 is the only node on p in $\mathbf{C}_j$ by the choice of p as a shortest path. Therefore, p is a proper possibly causal path from $\mathbf{C}_j$ to $\mathbf{C}_i$. It follows by Lemma 97 and the choice of p as a shortest path that p is causal. Based on the existence of this causal path, any partial causal ordering of $\mathbf{V}$ consistent with $\mathcal{G}$ must require $\mathbf{C}_{\ell_1} < \dots < \mathbf{C}_{\ell_m}$. Since $\ell_1 = j$ and

$\ell_m = i$ by the definition of p , then any partial causal ordering of $\mathbf{V}$ consistent with $\mathcal{G}$ must require $\mathbf{C}_j < \mathbf{C}_i$. But this contradicts the definition of $(\mathbf{C}_1, \dots, \mathbf{C}_k)$ as an ordered bucket decomposition of $\mathbf{V}$ in $\mathcal{G}$, where $\mathbf{C}_i < \mathbf{C}_j$. ■

Corollary 100 *Let $\mathbf{D}$ be a node set in an MPDAG $\mathcal{G}$ and let $(\mathbf{B}_1, \dots, \mathbf{B}_k), k \geq 1$, be the output of $\text{PCO}(\mathbf{D}, \mathcal{G})$. Then for any $i, j \in \{1, \dots, k\}, i < j$, there is no possibly causal path from $\mathbf{B}_j$ to $\mathbf{B}_i$ in $\mathcal{G}$.*

Proof of Corollary 100. Let $(\mathbf{C}_1, \dots, \mathbf{C}_\ell), \ell \geq k$, be the ordered bucket decomposition of $\mathbf{V}$ in $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ that matches the ordering of $(\mathbf{B}_1, \dots, \mathbf{B}_k)$ (see Remark 96). Let $\mathbf{B}_i \cap \mathbf{C}_m \neq \emptyset$ and $\mathbf{B}_j \cap \mathbf{C}_n \neq \emptyset$, where $m, n \in \{1, \dots, \ell\}$. By construction, $m < n$. If there were a possibly causal path from $B_j \in \mathbf{B}_j$ to $B_i \in \mathbf{B}_i$ in $\mathcal{G}$, then there would be a possibly causal path from $B_j \in \mathbf{C}_n$ to $B_i \in \mathbf{C}_m$ in $\mathcal{G}$, which contradicts Lemma 99. ■

Corollary 101 *Let $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ be an MPDAG and let $(\mathbf{C}_1, \dots, \mathbf{C}_k), k \geq 1$, be an ordered bucket decomposition of $\mathbf{V}$ in $\mathcal{G}$. Then for any $i, j \in \{1, \dots, k\}, i < j$, there is no DAG in $[\mathcal{G}]$ with a causal path from $\mathbf{C}_j$ to $\mathbf{C}_i$.*

Proof of Corollary 101. For sake of contradiction, suppose there is a DAG in $[\mathcal{G}]$ with a causal path p from $\mathbf{C}_j$ to $\mathbf{C}_i$. Then by Lemma 83, the path in $\mathcal{G}$ corresponding to p is possibly causal from $\mathbf{C}_j$ to $\mathbf{C}_i$, which contradicts Lemma 99. ■

Corollary 102 *Let $\mathbf{D}$ be a node set in an MPDAG $\mathcal{G}$ and let $(\mathbf{B}_1, \dots, \mathbf{B}_k), k \geq 1$, be the output of $\text{PCO}(\mathbf{D}, \mathcal{G})$. Then for any $i, j \in \{1, \dots, k\}, i < j$, there is no DAG in $[\mathcal{G}]$ with a causal path from $\mathbf{B}_j$ to $\mathbf{B}_i$.*

Proof of Corollary 102. For sake of contradiction, suppose there is a DAG in $[\mathcal{G}]$ with a causal path p from $\mathbf{B}_j$ to $\mathbf{B}_i$. Then by Lemma 83, the path in $\mathcal{G}$ corresponding to p is possibly causal from $\mathbf{B}_j$ to $\mathbf{B}_i$, which contradicts Corollary 100. ■

C.4 PROOFS FOR SECTION 4.3.2: IDENTIFICATION FORMULA

This section includes the statement and proof of Lemma 103, which provides the core justification for the proof of Theorem 21 found in Section 4.3.2. Three supporting results needed for the proof of Lemma 103 follow.

C.4.1 MAIN RESULT

Lemma 103 (Setup for Theorem 21) *Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ be pairwise disjoint node sets in a causal MPDAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, where $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$ and where there is no proper possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$ that starts with an undirected edge in $\mathcal{G}$. Let $\mathbf{D} = \text{An}(\mathbf{Y}, \mathcal{G}_{\mathbf{V} \setminus \mathbf{X}}) \setminus \mathbf{Z}$, let $(\mathbf{B}_1, \dots, \mathbf{B}_k) = \text{PCO}(\mathbf{D}, \mathcal{G})$, $k \geq 1$, and let $\mathbf{B}_0 = \emptyset$. Then the following hold for every density f consistent with $\mathcal{G}$, and every $i \in \{1, \dots, k\}$.*

(i) *If $\mathbf{Z} \cap \text{PossDe}(\mathbf{B}_i, \mathcal{G}) = \emptyset$, then*

$$f(\mathbf{b}_i | \mathbf{b}_{i-1}, \dots, \mathbf{b}_0, \text{do}(\mathbf{x}), \mathbf{z}) = f(\mathbf{b}_i | \text{pa}(\mathbf{b}_i, \mathcal{G})) \quad (\text{C.4})$$

for values $\text{pa}(\mathbf{b}_i, \mathcal{G})$ of $\text{Pa}(\mathbf{B}_i, \mathcal{G})$ that are in agreement with $\mathbf{x}$.

(ii) *If $\mathbf{Z} \cap \text{PossDe}(\mathbf{B}_i, \mathcal{G}) \neq \emptyset$, then*

$$f(\mathbf{b}_i | \mathbf{b}_{i-1}, \dots, \mathbf{b}_0, \text{do}(\mathbf{x}), \mathbf{z}) = f(\mathbf{b}_i | \mathbf{b}_i^{\mathbf{N}}, \mathbf{z}), \quad (\text{C.5})$$

where $\mathbf{B}_i^{\mathbf{N}} = \cup_{j=0}^{i-1} \mathbf{B}_j \setminus \text{PossDe}(\mathbf{X}, \mathcal{G})$.

Proof of Lemma 103. Let $\mathcal{D}$ be an arbitrary DAG in $[\mathcal{G}]$, let f be an arbitrary density consistent with $\mathcal{G}$, and pick $i \in \{1, \dots, k\}$. Then let $(\mathbf{C}_1, \dots, \mathbf{C}_\ell), \ell \geq k$, be the ordered bucket decomposition of $\mathbf{V}$ in $\mathcal{G}$ that matches the ordering of $(\mathbf{B}_1, \dots, \mathbf{B}_k)$ (see Remark 96). Further, let $\mathbf{C}_n \cap \mathbf{B}_i \neq \emptyset$, $n \in \{1, \dots, \ell\}$, so that $\mathbf{B}_i \subseteq \mathbf{C}_n$ and $\cup_{j=0}^{i-1} \mathbf{B}_j \subseteq \cup_{j=1}^{n-1} \mathbf{C}_j$.

PART (i) Note that this result is analogous to Lemma D.1 of Perković [2020] for the unconditional causal effect setting. Start by defining the following sets.

$$\mathbf{P}_i = (\cup_{j=0}^{i-1} \mathbf{B}_j \cup \mathbf{Z}) \cap \text{Pa}(\mathbf{B}_i, \mathcal{G}). \quad \mathbf{X}_{\mathbf{P}_i} = \mathbf{X} \cap \text{Pa}(\mathbf{B}_i, \mathcal{G}).$$

$$\begin{aligned}\mathbf{N}_i &= (\cup_{j=0}^{i-1} \mathbf{B}_j \cup \mathbf{Z}) \setminus \text{Pa}(\mathbf{B}_i, \mathcal{G}). & \mathbf{X}_{n_i} &= \mathbf{X} \setminus \text{Pa}(\mathbf{B}_i, \mathcal{G}). \\ \mathbf{X}'_{n_i} &= \mathbf{X}_{n_i} \setminus \text{An}(\mathbf{P}_i, \mathcal{D}_{\overline{\mathbf{X}_{P_i}}}).\end{aligned}$$

We show the result in the three steps below.

$$f(\mathbf{b}_i | \mathbf{b}_{i-1}, \dots, \mathbf{b}_0, do(\mathbf{x}), \mathbf{z}) = f(\mathbf{b}_i | \mathbf{p}_i, do(\mathbf{x})) \quad (\text{C.6})$$

$$= f(\mathbf{b}_i | \mathbf{p}_i, do(\mathbf{x}_{P_i})) \quad (\text{C.7})$$

$$= f(\mathbf{b}_i | \text{pa}(\mathbf{b}_i, \mathcal{G})). \quad (\text{C.8})$$

In order to use Rules 1, 3, and 2 of the do calculus (Theorem 75) to show the equalities in (C.6), (C.7), and (C.8), respectively, we must show the following.

$$(\mathbf{B}_i \perp_d \mathbf{N}_i \mid \mathbf{X}, \mathbf{P}_i)_{\mathcal{D}_{\overline{\mathbf{X}}}}. \quad (\text{C.9})$$

$$(\mathbf{B}_i \perp_d \mathbf{X}_{n_i} \mid \mathbf{X}_{P_i}, \mathbf{P}_i)_{\mathcal{D}_{\overline{\mathbf{X}_{P_i} \mathbf{X}'_{n_i}}}}. \quad (\text{C.10})$$

$$(\mathbf{B}_i \perp_d \mathbf{X}_{P_i} \mid \mathbf{P}_i)_{\mathcal{D}_{\overline{\mathbf{X}_{P_i}}}}. \quad (\text{C.11})$$

We first show a broader independence claim and complete the proof by showing that each of the independence statements above is a special case of the broader claim.

Broader Claim: Define the sets $\mathbf{N}$, $\mathbf{H}$, $\mathbf{X}'$, and $\mathbf{N}'$ such that $\mathbf{N}$, $\mathbf{H}$, $\mathbf{X}'$, $\mathbf{B}_i$, and $\mathbf{P}_i$ are pairwise disjoint and such that the following hold.

$$\mathbf{N} \subseteq \mathbf{N}_i \cup \mathbf{X}_{n_i} = (\cup_{j=1}^{i-1} \mathbf{B}_j \cup \mathbf{Z} \cup \mathbf{X}) \setminus \text{Pa}(\mathbf{B}_i, \mathcal{G}).$$

$$\mathbf{H} \subseteq \mathbf{X}_{P_i} = \mathbf{X} \cap \text{Pa}(\mathbf{B}_i, \mathcal{G}).$$

$$\mathbf{X}' \subseteq \mathbf{X}.$$

$$\mathbf{N}' \subseteq \mathbf{X}_{n_i} \setminus \text{An}(\mathbf{P}_i, \mathcal{D}_{\overline{\mathbf{X}}}).$$

$$\text{Pa}(\mathbf{B}_i, \mathcal{G}) \subseteq \mathbf{X}' \cup \mathbf{P}_i \cup \mathbf{H}.$$

$$\mathbf{N} \setminus \mathbf{N}' \subseteq \cup_{j=1}^{n-1} \mathbf{C}_j \cup \mathbf{Z}. \quad (\text{C.12})$$

The broader claim we will show is

$$(\mathbf{B}_i \perp_d [\mathbf{N} \cup \mathbf{H}] \mid \mathbf{X}', \mathbf{P}_i)_{\mathcal{D}_{\overline{\mathbf{X}' \mathbf{N}' \mathbf{H}}}}. \quad (\text{C.13})$$

Before beginning the proof of this broader claim, note that from the assumptions above, we also have the following.

(*)

There is no possibly causal path in $\mathcal{G}$ from

$$\mathbf{B}_i \text{ to } (\mathbf{N} \cup \mathbf{H} \cup \mathbf{P}_i) \setminus \mathbf{N}'.$$

To see this, note that $\mathbf{H} \subseteq \mathbf{X} \cap \text{Pa}(\mathbf{B}_i, \mathcal{G})$, and by Lemmas 99 and 105(i), $[\mathbf{X} \cap \text{Pa}(\mathbf{B}_i, \mathcal{G})] \cap (\cup_{j=n}^{\ell} \mathbf{C}_j) = \emptyset$. Thus, $\mathbf{H} \subseteq \cup_{j=1}^{n-1} \mathbf{C}_j$. Then by (C.12) and the definition of $\mathbf{P}_i$, we have $(\mathbf{N} \cup \mathbf{H} \cup \mathbf{P}_i) \setminus \mathbf{N}' \subseteq \cup_{j=1}^{n-1} \mathbf{C}_j \cup \mathbf{Z}$. The result follow by the fact that $\mathbf{B}_i \subseteq \mathbf{C}_n$, Lemma 99, and the fact that $\mathbf{Z} \cap \text{PossDe}(\mathbf{B}_i, \mathcal{G}) = \emptyset$.

We turn to prove the broader claim using a strategy similar to that of the proof of Lemma D.1 of Perković [2020]. For sake of contradiction, suppose that there is a path from $\mathbf{B}_i$ to $\mathbf{N} \cup \mathbf{H}$ in $\mathcal{D}_{\overline{\mathbf{X}'\mathbf{N}'\mathbf{H}}}$ that is d-connecting given $\mathbf{X}' \cup \mathbf{P}_i$. Let $p = \langle B_i, \dots, N \rangle$, $B_i \in \mathbf{B}_i$, $N \in \mathbf{N} \cup \mathbf{H}$, be a shortest such path, and let p^* be the corresponding path in $\mathcal{G}$.

Consider when p begins with an arrow out of B_i . If p were causal, then $N \notin \mathbf{N}'$ since p is in $\mathcal{D}_{\overline{\mathbf{X}'\mathbf{N}'\mathbf{H}}}$. Further, the corresponding path in $\mathcal{D}$ would also be causal, which by Lemma 83 would imply that p^* is a possibly causal path from $\mathbf{B}_i$ to $(\mathbf{N} \cup \mathbf{H}) \setminus \mathbf{N}'$. But this contradicts (*). Thus, let C be the closest collider to B_i on p . Since p is d-connecting given $\mathbf{X}' \cup \mathbf{P}_i$ in $\mathcal{D}_{\overline{\mathbf{X}'\mathbf{N}'\mathbf{H}}}$, then $C \in \text{An}(\mathbf{P}_i, \mathcal{D}_{\overline{\mathbf{X}'\mathbf{N}'\mathbf{H}}})$. But this implies that there is a causal path in $\mathcal{D}_{\overline{\mathbf{X}'\mathbf{N}'\mathbf{H}}}$ – and therefore in $\mathcal{D}$ – from $\mathbf{B}_i$ to $\mathbf{P}_i$. By Lemma 83, the corresponding path in $\mathcal{G}$ is possibly causal, which again contradicts (*).

Consider next when p begins with $B_i \leftarrow A$. By choice of p , $A \notin \mathbf{B}_i$ and so $A \in \text{Pa}(\mathbf{B}_i, \mathcal{D}_{\overline{\mathbf{X}'\mathbf{N}'\mathbf{H}}})$. But $A \notin \text{Pa}(\mathbf{B}_i, \mathcal{G})$, since $\text{Pa}(\mathbf{B}_i, \mathcal{G}) \subseteq \mathbf{X}' \cup \mathbf{P}_i \cup \mathbf{H}$ and p is d-connecting given $\mathbf{X}' \cup \mathbf{P}_i$ in $\mathcal{D}_{\overline{\mathbf{X}'\mathbf{N}'\mathbf{H}}}$. Thus, p^* begins with $B_i - A$. If p^* were undirected, then $N \in \mathbf{C}_n$ and so $N \notin \mathbf{H} \subseteq \cup_{j=1}^{n-1} \mathbf{C}_j$. Thus by Lemma 104(ii), p^* would be a possibly causal path from $\mathbf{B}_i$ to $N \in \mathbf{N}$ and $(-p^*)$ would be a possibly causal path from $N \in \mathbf{N}$ to $\mathbf{B}_i$. The former contradicts (*) when $N \in \mathbf{N} \setminus \mathbf{N}'$. The latter contradicts Lemma 105(i) when $N \in \mathbf{N}' \subseteq \mathbf{X}$.

Since p^* must contain a directed edge, let T be the node closest to B_i on p^* such that $p^*(T, N)$ starts with such an edge. Let S and U be the nodes on p^* immediately preceding

and following T , respectively. Note that $T \notin \mathbf{H}$ by choice of p .

Consider when p^* contains $T \rightarrow U$. Note that $p(T, N)$ also contains $T \rightarrow U$. When $p(T, N)$ contains at least one collider, then by Lemma 104(iv), $\mathcal{G}$ contains a causal path from T to $\mathbf{P}_i$. When instead $p(T, N)$ is causal, then $N \notin \mathbf{N}'$ since p is in $\mathcal{D}_{\overline{\mathbf{X}'\mathbf{N}'\mathbf{H}}}$. Thus by Lemma 104(iii), $\mathcal{G}$ contains a causal path from T to $(\mathbf{N} \cup \mathbf{H}) \setminus \mathbf{N}'$. In either case, $\mathcal{G}$ contains a causal path from T to $(\mathbf{N} \cup \mathbf{H} \cup \mathbf{P}_i) \setminus \mathbf{N}'$. Since $\mathcal{G}$ also contains $p^*(B_i, T)$ – which is possibly causal by Lemma 104(ii) and the fact that $T \notin \mathbf{H}$ – then by Lemma 91, $\mathcal{G}$ contains a possibly causal path from $B_i \in \mathbf{B}_i$ to $(\mathbf{N} \cup \mathbf{H} \cup \mathbf{P}_i) \setminus \mathbf{N}'$, which contradicts (*).

Finally, consider when p^* contains $T \leftarrow U$. Then by R1 of Meek [1995], $\mathcal{G}$ – and therefore $\mathcal{D}$ – contains the edge $\langle S, U \rangle$. Note that $\mathcal{D}$ also contains $T \leftarrow U$. By Lemma 104(i), the path in $\mathcal{D}$ corresponding to $p(B_i, T)$ cannot contain colliders on the path in $\mathcal{D}$ corresponding to p . Thus, $\mathcal{D}$ contains $S \leftarrow T \leftarrow U$ and therefore $S \leftarrow U$. Further since $\mathcal{D}_{\overline{\mathbf{X}'\mathbf{N}'\mathbf{H}}}$ must also contain $S \leftarrow T \leftarrow U$, then $S \notin \mathbf{X}' \cup \mathbf{N}'$ and $U \notin \mathbf{H}$, and so $\mathcal{D}_{\overline{\mathbf{X}'\mathbf{N}'\mathbf{H}}}$ contains $S \leftarrow U$. But then $p(B_i, S) \oplus \langle S, U \rangle \oplus p(U, N)$ contradicts the choice of p .

Special Cases: With the broader claim shown, we now complete the proof by showing that the independence statements in (C.9), (C.10), and (C.11) are special cases of (C.13).

(C.9) Let $\mathbf{N} = \mathbf{N}_i$, $\mathbf{H} = \emptyset$, $\mathbf{X}' = \mathbf{X}$, and $\mathbf{N}' = \emptyset$. Since $\mathbf{X}$, $\mathbf{Z}$, and $\mathbf{D}$ are pairwise disjoint, then $\mathbf{N}$, $\mathbf{X}'$, $\mathbf{B}_i$, and $\mathbf{P}_i$ are pairwise disjoint by definition. Clearly, $\mathbf{N} \subseteq \mathbf{N}_i \cup \mathbf{X}_{\mathbf{n}_i}$ and $\mathbf{X}' \subseteq \mathbf{X}$. Then note that $\text{Pa}(\mathbf{B}_i, \mathcal{G}) = \mathbf{X}_{\mathbf{p}_i} \cup \mathbf{P}_i \subseteq \mathbf{X}' \cup \mathbf{P}_i \cup \mathbf{H}$. Finally, note that $\mathbf{N} \setminus \mathbf{N}' = \mathbf{N}_i$, where $\mathbf{N}_i \subseteq \cup_{j=1}^{i-1} \mathbf{B}_j \cup \mathbf{Z} \subseteq \cup_{j=1}^{n-1} \mathbf{C}_j \cup \mathbf{Z}$ by definition.

(C.10) Let $\mathbf{N} = \mathbf{X}_{\mathbf{n}_i}$, $\mathbf{H} = \emptyset$, $\mathbf{X}' = \mathbf{X}_{\mathbf{p}_i}$, and $\mathbf{N}' = \mathbf{X}'_{\mathbf{n}_i}$. Since $\mathbf{X}$, $\mathbf{Z}$, and $\mathbf{D}$ are pairwise disjoint, then $\mathbf{N}$, $\mathbf{X}'$, $\mathbf{B}_i$, and $\mathbf{P}_i$ are pairwise disjoint by definition. Clearly, $\mathbf{N} \subseteq \mathbf{N}_i \cup \mathbf{X}_{\mathbf{n}_i}$ and $\mathbf{X}' \subseteq \mathbf{X}$. Further, $\mathbf{N}' = \mathbf{X}_{\mathbf{n}_i} \setminus \text{An}(\mathbf{P}_i, \mathcal{D}_{\overline{\mathbf{X}_{\mathbf{p}_i}}}) \subseteq \mathbf{X}_{\mathbf{n}_i} \setminus \text{An}(\mathbf{P}_i, \mathcal{D}_{\overline{\mathbf{X}'}})$. Then note that $\text{Pa}(\mathbf{B}_i, \mathcal{G}) = \mathbf{X}_{\mathbf{p}_i} \cup \mathbf{P}_i \subseteq \mathbf{X}' \cup \mathbf{P}_i \cup \mathbf{H}$.

Finally, to see that $\mathbf{N} \setminus \mathbf{N}' \subseteq \cup_{j=1}^{n-1} \mathbf{C}_j$, let N be an arbitrary node in $\mathbf{N} \setminus \mathbf{N}' = \mathbf{X}_{\mathbf{n}_i} \cap \text{An}(\mathbf{P}_i, \mathcal{D}_{\overline{\mathbf{X}_{\mathbf{p}_i}}})$. This implies that there is a causal path in $\mathcal{D}_{\overline{\mathbf{X}_{\mathbf{p}_i}}}$ from N to some $P_i \in \mathbf{P}_i \subseteq \text{Pa}(\mathbf{B}_i, \mathcal{G})$. This path is also in $\mathcal{D}$, and thus by Lemma 83, its corresponding path in $\mathcal{G}$ is

possibly causal from N to $\text{Pa}(\mathbf{B}_i, \mathcal{G})$. Therefore by Lemma 91, there is a possibly causal path in $\mathcal{G}$ from N to $\mathbf{B}_i$. Since $\mathbf{B}_i \subseteq \mathbf{C}_n$, then $N \notin \cup_{j=n+1}^k \mathbf{C}_j$ by Lemma 99. Further, since $N \in \mathbf{X}$ and $\mathbf{B}_i \subseteq \mathbf{D}$, then $N \notin \mathbf{C}_n$ by Lemma 105(i).

(C.11) Let $\mathbf{N} = \emptyset$, $\mathbf{H} = \mathbf{X}_{\mathbf{p}_i}$, $\mathbf{X}' = \emptyset$, and $\mathbf{N}' = \emptyset$. Since $\mathbf{X}$, $\mathbf{Z}$, and $\mathbf{D}$ are pairwise disjoint, then $\mathbf{H}$, $\mathbf{B}_i$, and $\mathbf{P}_i$ are pairwise disjoint by definition. Clearly, $\mathbf{H} \subseteq \mathbf{X}_{\mathbf{p}_i}$. Then note that $\text{Pa}(\mathbf{B}_i, \mathcal{G}) = \mathbf{X}_{\mathbf{p}_i} \cup \mathbf{P}_i \subseteq \mathbf{X}' \cup \mathbf{P}_i \cup \mathbf{H}$. Finally, note that $\mathbf{N} \setminus \mathbf{N}' = \emptyset \subseteq \cup_{j=1}^{n-1} \mathbf{C}_j \cup \mathbf{Z}$.

PART (ii) Let $\mathbf{B}_i^D = \cup_{j=0}^{i-1} \mathbf{B}_j \cap \text{PossDe}(\mathbf{X}, \mathcal{G})$. We show the result in the steps below.

$$f(\mathbf{b}_i | \mathbf{b}_{i-1}, \dots, \mathbf{b}_0, do(\mathbf{x}), \mathbf{z}) = f(\mathbf{b}_i | \mathbf{b}_i^N, do(\mathbf{x}), \mathbf{z}) \quad (\text{C.14})$$

$$= f(\mathbf{b}_i | \mathbf{b}_i^N, \mathbf{z}). \quad (\text{C.15})$$

In order to use Rules 1 and 3 of the do calculus (Theorem 75) to show the equalities in (C.14) and (C.15), respectively, we must show the following.

$$(\mathbf{B}_i \perp_d \mathbf{B}_i^D | \mathbf{X}, \mathbf{Z}, \mathbf{B}_i^N)_{\mathcal{D}_{\overline{\mathbf{X}}}}. \quad (\text{C.16})$$

$$(\mathbf{B}_i \perp_d \mathbf{X} \mid \mathbf{B}_i^N, \mathbf{Z})_{\mathcal{D}_{\overline{\mathbf{W}}}}, \quad (\text{C.17})$$

where we let $\mathbf{W} = \mathbf{X} \setminus \text{An}(\mathbf{B}_i^N \cup \mathbf{Z}, \mathcal{D})$. We prove these independencies below.

(C.16) Suppose for sake of contradiction that there is a path in $\mathcal{D}_{\overline{\mathbf{X}}}$ from $\mathbf{B}_i$ to $\mathbf{B}_i^D$ that is d-connecting given $\mathbf{X} \cup \mathbf{Z} \cup \mathbf{B}_i^N$. Let $q = \langle B_i = Q_0, \dots, Q_m = B_i^D \rangle$, $m \geq 1$, be a shortest such path and let q^* be the corresponding path in $\mathcal{G}$. Further, let $\mathbf{C}_h$, $h \in \{1, \dots, n-1\}$, be the bucket in $\mathbf{V}$ such that $B_i^D \in \mathbf{C}_h$.

Before beginning the proof, note the following.

There is no causal path in $\mathcal{D}_{\overline{\mathbf{X}}}$ from (**)

$\mathbf{C}_h$ to $\mathbf{B}_i \cup \mathbf{Z} \cup \mathbf{B}_i^N$.

To see this, suppose for sake of contradiction that such a path p did exist, and let p^* be the corresponding path in $\mathcal{G}$. Since the corresponding path in $\mathcal{D}$ is also causal, then by

Lemma 83, p^* is a possibly causal path from $\mathbf{C}_h$ to $\mathbf{B}_i \cup \mathbf{Z} \cup \mathbf{B}_i^N$, where by Lemma 105(i), $\mathbf{C}_h \subseteq \text{De}(\mathbf{X}, \mathcal{G})$. Thus by Lemma 91, $\mathcal{G}$ contains a possibly causal path from $\mathbf{X}$ to $\mathbf{B}_i \cup \mathbf{Z} \cup \mathbf{B}_i^N$. But by assumption, definition, and Lemma 105(ii), $(\mathbf{B}_i \cup \mathbf{Z} \cup \mathbf{B}_i^N) \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$. With the claim shown, we turn to find a contradiction in the two cases below.

Case 1: Consider when q contains $Q_{m-1} \leftarrow B_i^D$. Note that $(-q)$ cannot be causal from $B_i^D \in \mathbf{C}_h$ to $B_i \in \mathbf{B}_i$ by $(**)$ and therefore, must contain a collider. Let $\mathbf{L}$ be the set containing the collider closest to B_i^D on q and all of its descendants in $\mathcal{D}_{\overline{\mathbf{X}}}$. Since q is d-connecting given $\mathbf{X} \cup \mathbf{Z} \cup \mathbf{B}_i^N$ in $\mathcal{D}_{\overline{\mathbf{X}}}$, then $\mathbf{L}$ must contain an element in $\mathbf{Z} \cup \mathbf{B}_i^N$. But since $\mathbf{L} \subseteq \text{De}(\mathbf{C}_h, \mathcal{D}_{\overline{\mathbf{X}}})$, this contradicts $(**)$.

Case 2: Consider instead when q contains $Q_{m-1} \rightarrow B_i^D$. Note that q is not causal from $B_i \in \mathbf{C}_h$ to $B_i^D \in \mathbf{C}_h$, since otherwise the corresponding path in $\mathcal{D}$ would also be causal, which would contradict Corollary 101. Thus, q contains a node Q_c , $c \in \{1, \dots, m-1\}$, such that q contains $Q_{c-1} \leftarrow Q_c \rightarrow \dots \rightarrow B_i^D$. Let $\mathbf{C}_e$, $e \in \{1, \dots, n-1\}$, be the bucket in $\mathbf{V}$ such that $Q_c \in \mathbf{C}_e$.

For sake of contradiction, suppose $e \neq h$. That is, Q_c and B_i^D are in the distinct buckets $\mathbf{C}_e$ and $\mathbf{C}_h$, respectively. Define Q_d , where $d \in \{1, \dots, m-1\}$, $d \geq c$, to be the last node on q in $\mathbf{C}_e$. Note that since $q(Q_d, B_i^D)$ is causal, the corresponding path in $\mathcal{D}$ is also causal. Therefore, by Lemma 83, $q^*(Q_d, B_i^D)$ is a proper possibly causal path from $\mathbf{C}_e$ to $\mathbf{C}_h$ and so by Lemma 97, $\mathcal{G}$ contains at least one subsequence of $q^*(Q_d, B_i^D)$ that is causal from Q_d to B_i^D . Let r^* be an arbitrary such path.

Note that since q is d-connecting given $\mathbf{X} \cup \mathbf{Z} \cup \mathbf{B}_i^N$, no node on $q(Q_d, B_i^D)$ – and therefore no node on r^* – is in $\mathbf{X}$. That is, r^* is a causal path in $\mathcal{G}$ from Q_d to $B_i^D \in \text{An}(\mathbf{Y}, \mathcal{G}_{\mathbf{V} \setminus \mathbf{X}}) \setminus \mathbf{Z}$ that does not contain any nodes in $\mathbf{X}$. Thus, $Q_d \in \mathbf{Z} \cup \mathbf{D}$. By the same logic, $Q_d \notin \mathbf{Z} \cup \mathbf{B}_i^N$, and further, by the choice of q as a shortest path, $Q_d \notin \mathbf{B}_i^D$. But this implies that $Q_d \in \cup_{j=i}^k \mathbf{B}_j$, which contradicts Corollary 100.

Note that we have shown q contains $Q_{c-1} \leftarrow Q_c$, where $Q_c \in \mathbf{C}_h$. We can derive our final contradictions by using identical logic to that of Case 1 above.

(C.17) Note that by Lemma 93, $\text{De}(\mathbf{X}, \mathcal{D}) \subseteq \text{PossDe}(\mathbf{X}, \mathcal{G})$. From this and since $(\mathbf{B}_i^{\mathbf{N}} \cup \mathbf{Z}) \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$, it follows that $(\mathbf{B}_i^{\mathbf{N}} \cup \mathbf{Z}) \cap \text{De}(\mathbf{X}, \mathcal{D}) = \emptyset$ and thus $\mathbf{W} = \mathbf{X}$. Therefore, we want to show $(\mathbf{B}_i \perp_d \mathbf{X} | \mathbf{B}_i^{\mathbf{N}}, \mathbf{Z})_{\mathcal{D}_{\overline{\mathbf{X}}}}$.

Let p be an arbitrary path in $\mathcal{D}_{\overline{\mathbf{X}}}$ from $X \in \mathbf{X}$ to $B_i \in \mathbf{B}_i$. By the definition of $\mathcal{D}_{\overline{\mathbf{X}}}$, p begins with $X \rightarrow$. But note that p cannot be causal, since this would imply that $B_i \in \text{De}(\mathbf{X}, \mathcal{D}) \subseteq \text{PossDe}(\mathbf{X}, \mathcal{G})$, which would contradict Lemma 105(ii). Thus, p must have at least one collider.

Let $\mathbf{S}$ be the set containing the collider that is closest to X on p and all of its descendants in $\mathcal{D}_{\overline{\mathbf{X}}}$. Since $\mathbf{S} \subseteq \text{De}(X, \mathcal{D}_{\overline{\mathbf{X}}}) \subseteq \text{De}(\mathbf{X}, \mathcal{D}) \subseteq \text{PossDe}(\mathbf{X}, \mathcal{G})$ and since $(\mathbf{B}_i^{\mathbf{N}} \cup \mathbf{Z}) \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$, then p must be d-separated given $\mathbf{B}_i^{\mathbf{N}} \cup \mathbf{Z}$. ■

C.4.2 SUPPORTING RESULTS

Lemma 104 (Setup for Proof of Lemma 103(i)) *Let $\mathbf{N}$, $\mathbf{H}$, $\mathbf{X}'$, $\mathbf{B}$, and $\mathbf{P}$ be pairwise disjoint node sets in an MPDAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, and let $\mathcal{D}$ be a DAG in $[\mathcal{G}]$. Suppose there is a path from $\mathbf{B}$ to $\mathbf{N} \cup \mathbf{H}$ that is d-connecting given $\mathbf{X}' \cup \mathbf{P}$ in $\mathcal{D}_{\overline{\mathbf{X}'\mathbf{N}'\mathbf{H}}}$, where $\mathbf{N}'$ is a node set in $\mathcal{G}$ such that $\mathbf{N}' \subseteq \mathbf{N}$ and $\mathbf{N}' \cap \text{An}(\mathbf{P}, \mathcal{D}_{\overline{\mathbf{X}'}})) = \emptyset$. Let $p_1 = \langle B, \dots, N \rangle$, $B \in \mathbf{B}$, $N \in \mathbf{N} \cup \mathbf{H}$, be a shortest such path, and let p^* and p_2 be the corresponding paths in $\mathcal{G}$ and $\mathcal{D}$, respectively. Finally, let $q_1 = \langle Q_0, \dots, Q_k \rangle$, $k \geq 1$, be a subpath of p_1 , and let q^* and q_2 be the corresponding paths in $\mathcal{G}$ and $\mathcal{D}$, respectively.*

- (i) *If q^* is undirected, then q_2 does not contain a collider on p_2 .*
- (ii) *If q^* is undirected and $Q_k \notin \mathbf{H}$, then q^* and $(-q^*)$ are possibly causal.*
- (iii) *If q_1 is causal and q^* begins with $Q_0 \rightarrow Q_1$, then q^* is causal.*
- (iv) *If q_1 contains at least one collider and q^* begins with $Q_0 \rightarrow Q_1$, then $\mathcal{G}$ contains a causal path from Q_0 to $\mathbf{P}$.*

Proof of Lemma 104. Let Q_{-1} designate the node immediately preceding Q_0 on p_1 when $Q_0 \neq B$, and let Q_{k+1} designate the node immediately following Q_k on p_1 when $Q_k \neq N$.

(i) Let q^* be undirected. For sake of contradiction, suppose that q_2 contains a collider on p_2 at some node Q_j , $0 \leq j \leq k$. But since q^* is undirected in $\mathcal{G}$, then Q_j is not a collider on p^* . Thus by properties of MPDAGs, $\mathcal{G}$ – and therefore $\mathcal{D}$ – must contain an edge $\langle Q_{j-1}, Q_{j+1} \rangle$. We show below that this edge also exists in $\mathcal{D}_{\overline{\mathbf{X}'\mathbf{N}'\mathbf{H}}}$.

Note that Q_j is also a collider on p_1 . Since p_1 is d-connecting given $\mathbf{X}' \cup \mathbf{P}$, then Q_j and therefore Q_{j-1} and Q_{j+1} are in $\text{An}(\mathbf{P}, \mathcal{D}_{\overline{\mathbf{X}'\mathbf{N}'\mathbf{H}}}) \subseteq \text{An}(\mathbf{P}, \mathcal{D}_{\overline{\mathbf{X}'}})$. So since $\mathbf{N}' \cap \text{An}(\mathbf{P}, \mathcal{D}_{\overline{\mathbf{X}'}}) = \emptyset$, then $Q_{j-1}, Q_{j+1} \notin \mathbf{N}'$. Further, since Q_{j-1} and Q_{j+1} are not colliders on p_1 , they cannot be nodes in $\mathbf{X}'$ (or $\mathbf{P}$). Finally, since $\mathcal{D}_{\overline{\mathbf{X}'\mathbf{N}'\mathbf{H}}}$ contains $Q_{j-1} \rightarrow Q_j \leftarrow Q_{j+1}$, then $Q_{j-1}, Q_{j+1} \notin \mathbf{H}$. Therefore $\mathcal{D}_{\overline{\mathbf{X}'\mathbf{N}'\mathbf{H}}}$ must also contain an edge $\langle Q_{j-1}, Q_{j+1} \rangle$.

Define the path $m_1 = p_1(B, Q_{j-1}) \oplus \langle Q_{j-1}, Q_{j+1} \rangle \oplus p_1(Q_{j+1}, N)$ in $\mathcal{D}_{\overline{\mathbf{X}'\mathbf{N}'\mathbf{H}}}$. By the choice of p_1 , m_1 must be blocked given $\mathbf{X}' \cup \mathbf{P}$. Since $Q_{j-1}, Q_{j+1} \notin \mathbf{X}' \cup \mathbf{P}$, then m_1 must be blocked given $\mathbf{X}' \cup \mathbf{P}$ by a collider at Q_{j-1} or Q_{j+1} . But this contradicts that $Q_{j-1}, Q_{j+1} \in \text{An}(\mathbf{P}, \mathcal{D}_{\overline{\mathbf{X}'\mathbf{N}'\mathbf{H}}})$.

(ii) Let q^* be undirected and $Q_k \notin \mathbf{H}$. We show that q^* is possibly causal. The proof for $(-q^*)$ follows analogous logic. Note that by part (i), q_2 does not contain a collider on p_2 , and therefore, q_1 does not contain a collider on p_1 .

For sake of contradiction, suppose that q^* is non-causal. Then there exists an edge $Q_g \leftarrow Q_h$ in $\mathcal{G}$ – and therefore in $\mathcal{D}$ – for some $g, h \in \{0, \dots, k\}$, $h > g + 1$. To see that $\mathcal{D}_{\overline{\mathbf{X}'\mathbf{N}'\mathbf{H}}}$ also contains $Q_g \leftarrow Q_h$, note that $Q_h \notin \mathbf{H}$ by assumption and choice of p_1 . Additionally, $Q_g \notin \mathbf{X}' \cup \mathbf{N}'$, since q_2 does not contain a collider on p_2 and $\mathcal{D}$ does not contain a cycle, which implies that $\mathcal{D}$ – and therefore $\mathcal{D}_{\overline{\mathbf{X}'\mathbf{N}'\mathbf{H}}}$ – must contain $Q_g \leftarrow Q_{g+1}$.

Define a path $u_1 = p_1(B, Q_g) \oplus \langle Q_g, Q_h \rangle \oplus p_1(Q_h, N)$ in $\mathcal{D}_{\overline{\mathbf{X}'\mathbf{N}'\mathbf{H}}}$. By the choice of p_1 , u_1 must be blocked given $\mathbf{X}' \cup \mathbf{P}$ at Q_g or Q_h . However, note that since q_1 cannot contain colliders on p_1 , then Q_g and Q_h are non-colliders on p_1 . Further, since $\mathcal{D}_{\overline{\mathbf{X}'\mathbf{N}'\mathbf{H}}}$ contains $Q_g \leftarrow Q_h$ and $Q_g \leftarrow Q_{g+1}$, then Q_g and Q_h are also non-colliders on u_1 . So if u_1 is blocked

given $\mathbf{X}' \cup \mathbf{P}$ at Q_g or Q_h , then so is p_1 , which is a contradiction.

(iii) Let q_1 be causal and let q^* begin with $Q_0 \rightarrow Q_1$. Note that q_2 is also causal. Suppose for sake of contradiction that q^* is shielded. That is, $\mathcal{G}$ – and therefore $\mathcal{D}$ – contains at least one edge $\langle Q_d, Q_{d+2} \rangle$ for some $d \in \{0, \dots, k-2\}$. Since q_2 is causal, $\mathcal{D}$ must contain $Q_d \rightarrow Q_{d+2}$. Further, because $\mathcal{D}_{\overline{\mathbf{X}'\mathbf{N}'\mathbf{H}}}$ contains $Q_d \rightarrow Q_{d+1} \rightarrow Q_{d+2}$, then $Q_{d+2} \notin \mathbf{X}' \cup \mathbf{N}'$ and $Q_d \notin \mathbf{H}$, and so $\mathcal{D}_{\overline{\mathbf{X}'\mathbf{N}'\mathbf{H}}}$ must also contain $Q_d \rightarrow Q_{d+2}$. But then the path $p_1(B, Q_d) \oplus \langle Q_d, Q_{d+2} \rangle \oplus p_1(Q_{d+2}, N)$ contradicts the choice of p_1 . Thus, q^* is unshielded.

Since q_2 is causal, q^* can only contain edges of the form $Q_i - Q_{i+1}$ and $Q_i \rightarrow Q_{i+1}$ for all $i \in \{0, \dots, k-1\}$. Since additionally q^* begins with $Q_0 \rightarrow Q_1$ and is unshielded, the result follows by R1 of Meek [1995].

(iv) Let q_1 contain at least one collider and q^* begin with $Q_0 \rightarrow Q_1$. Note that q_1 also begins with $Q_0 \rightarrow Q_1$. Then let T_0 be the closest collider to Q_0 on q_1 . Note that T_0 is also a collider on p_1 . Since p_1 is d-connecting given $\mathbf{X}' \cup \mathbf{P}$ in $\mathcal{D}_{\overline{\mathbf{X}'\mathbf{N}'\mathbf{H}}}$, then $T_0 \in \text{An}(\mathbf{P}, \mathcal{D}_{\overline{\mathbf{X}'\mathbf{N}'\mathbf{H}}})$. Thus let $t_1 = \langle T_0, \dots, T_w \rangle$, $w \geq 0$, $T_w \in \mathbf{P}$, be a shortest causal path in $\mathcal{D}_{\overline{\mathbf{X}'\mathbf{N}'\mathbf{H}}}$ from T_0 to $\mathbf{P}$, and let t^* and t_2 be the corresponding paths in $\mathcal{G}$ and $\mathcal{D}$, respectively.

Note that $q_1(Q_0, T_0)$ is causal, and therefore, by part (iii), the subpath $q^*(Q_0, T_0)$ is causal. We want to show that there is a causal path from Q_0 to T_w in $\mathcal{G}$. If t^* is causal, then $q^*(Q_0, T_0) \oplus t^*$ is such a path and we are done.

Consider when t^* is not causal, and focus on the nodes adjacent to T_0 on q_1 . Let S be the node immediately preceding and U the node immediately following T_0 on q_1 . Note that since S and U are non-colliders on p_1 and p_1 is d-connecting given $\mathbf{X}' \cup \mathbf{P}$, then $S, U \notin \mathbf{X}' \cup \mathbf{P}$. Also since $\mathcal{D}_{\overline{\mathbf{X}'\mathbf{N}'\mathbf{H}}}$ contains $S \rightarrow T_0 \leftarrow U$, then $S, U \notin \mathbf{H}$. Further since $T_0 \in \text{An}(\mathbf{P}, \mathcal{D}_{\overline{\mathbf{X}'\mathbf{N}'\mathbf{H}}})$, then $S, U \in \text{An}(\mathbf{P}, \mathcal{D}_{\overline{\mathbf{X}'\mathbf{N}'\mathbf{H}}}) \subseteq \text{An}(\mathbf{P}, \mathcal{D}_{\overline{\mathbf{X}'}})$. Therefore, $S, U \notin \mathbf{N}'$. Putting this together, we have that $S, U \notin \mathbf{X}' \cup \mathbf{P} \cup \mathbf{H} \cup \mathbf{N}'$.

For sake of contradiction, suppose $\langle S, T_0, U \rangle$ is shielded in $\mathcal{G}$ – and therefore in $\mathcal{D}$. Since $S, U \notin \mathbf{X}' \cup \mathbf{N}' \cup \mathbf{H}$, then $\mathcal{D}_{\overline{\mathbf{X}'\mathbf{N}'\mathbf{H}}}$ also contains $\langle S, U \rangle$. Define a path $m_1 = p_1(B, S) \oplus \langle S, U \rangle \oplus p(U, N)$ in $\mathcal{D}_{\overline{\mathbf{X}'\mathbf{N}'\mathbf{H}}}$. By the choice of p_1 , m_1 must be blocked given $\mathbf{X}' \cup \mathbf{P}$. Since

$S, U \notin \mathbf{X}' \cup \mathbf{P}$, then either S or U must be a collider on m_1 that is not in $\text{An}(\mathbf{P}, \mathcal{D}_{\overline{\mathbf{X}'\mathbf{N}'\mathbf{H}}})$, which is a contradiction. Thus, $\langle S, T_0, U \rangle$ is unshielded in $\mathcal{G}$. Since $\mathcal{D}$ must also contain $S \rightarrow T_0 \leftarrow U$, then by properties of MPDAGs, so does $\mathcal{G}$.

Turn to consider t^* . Suppose for sake of contradiction that $\mathcal{G}$ – and therefore $\mathcal{D}$ – contains at least one edge $\langle T_f, T_{f+2} \rangle$ for some $f \in \{0, \dots, w-2\}$. Note that t_2 is causal, and so $\mathcal{D}$ must contain $T_f \rightarrow T_{f+2}$. Since $\mathcal{D}_{\overline{\mathbf{X}'\mathbf{N}'\mathbf{H}}}$ contains an edge into and out of each node on t_1 except for $T_w \in \mathbf{P}$, then no node on t_1 is in $\mathbf{X}' \cup \mathbf{N}' \cup \mathbf{H}$ and so $\mathcal{D}_{\overline{\mathbf{X}'\mathbf{N}'\mathbf{H}}}$ must also contain $T_f \rightarrow T_{f+2}$. But then the path $t_1(T_0, T_f) \oplus \langle T_f, T_{f+2} \rangle \oplus t_1(T_{f+2}, T_w)$ contradicts the choice of t_1 , and thus, t^* is unshielded.

Next, define a node $T_v, v \in \{0, \dots, w\}$, as follows. When t^* is entirely undirected, let $T_v = T_w$. Otherwise, let T_v be the first node on t^* followed by a directed arrow. Note that since t_2 is causal, t^* can only contain edges of the form $T_i - T_{i+1}$ or $T_i \rightarrow T_{i+1}$ for $i \in \{0, \dots, w-1\}$. Thus, when $T_v \neq T_w$, t^* contains $T_v \rightarrow T_{v+1}$. Then since t^* is unshielded, by R1 of Meek [1995], t^* takes the form $T_0 - \dots - T_v \rightarrow \dots \rightarrow T_w$.

We now show that $\mathcal{G}$ contains the edge $S \rightarrow T_v$, which will make $q^*(Q_0, S) \oplus \langle S, T_v \rangle \oplus t^*(T_v, T_w)$ a causal path in $\mathcal{G}$ from Q_0 to $\mathbf{P}$. We have already shown that $\mathcal{G}$ and $\mathcal{D}$ contain $S \rightarrow T_0 \leftarrow U$. For sake of induction, suppose that $\mathcal{G}$ – and therefore $\mathcal{D}$ – contains $S \rightarrow T_d \leftarrow U$ for some $d \in \{0, \dots, v-1\}$. Since $\mathcal{G}$ contains the paths $S \rightarrow T_d - T_{d+1}$ and $U \rightarrow T_d - T_{d+1}$, then $\mathcal{G}$ – and therefore $\mathcal{D}$ – must contain the edges $\langle S, T_{d+1} \rangle$ and $\langle U, T_{d+1} \rangle$ by R1 of Meek [1995]. Since $\mathcal{D}$ contains $S \rightarrow T_0 \rightarrow \dots \rightarrow T_{d+1}$ and $U \rightarrow T_0 \rightarrow \dots \rightarrow T_{d+1}$, it must also contain $S \rightarrow T_{d+1} \leftarrow U$. Then because $\langle S, T_0, U \rangle$ is unshielded in $\mathcal{G}$, so is $\langle S, T_{d+1}, U \rangle$. Thus, by properties of MPDAGs, $\mathcal{G}$ must also contain $S \rightarrow T_{d+1} \leftarrow U$. ■

Lemma 105 (Setup for Proof of Lemma 103) *Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ be pairwise disjoint node sets in an MPDAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, where $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$ and where there is no proper possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$ that starts with an undirected edge in $\mathcal{G}$. Further, let $\mathbf{D} = \text{An}(\mathbf{Y}, \mathcal{G}_{\mathbf{V} \setminus \mathbf{X}}) \setminus \mathbf{Z}$.*

(i) *Let $X \in \mathbf{X}$, $D \in \mathbf{D} \cap \text{PossDe}(X, \mathcal{G})$, and $\mathbf{C}_d$ be the bucket in $\mathbf{V}$ such that $D \in \mathbf{C}_d$.*

Then $\mathbf{C_d} \subseteq \text{De}(X, \mathcal{G})$.

(ii) Let $(\mathbf{B}_1, \dots, \mathbf{B}_k) = \text{PCO}(\mathbf{D}, \mathcal{G})$, $k \geq 1$, and let $i \in \{1, \dots, k\}$ such that $\mathbf{Z} \cap \text{PossDe}(\mathbf{B}_i, \mathcal{G}) \neq \emptyset$. Then $\mathbf{B}_i \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$.

Proof of Lemma 105.

(i) We start by showing that $D \in \text{De}(X, \mathcal{G})$. To see this, pick an arbitrary possibly causal path in $\mathcal{G}$ from X to D , and let $p = \langle X = P_0, \dots, P_r = D \rangle, r \geq 1$, be a shortest subsequence in $\mathcal{G}$ of that path. By Lemma 89, p is a possibly causal, unshielded path from X to D . Since $\mathbf{D} \subseteq \text{An}(\mathbf{Y}, \mathcal{G}_{\mathbf{V} \setminus \mathbf{X}})$, then by Lemma 88, there is no proper possibly causal path from X to D that starts with an undirected edge in $\mathcal{G}$, and so p must contain $P_0 \rightarrow P_1$. Then by R1 of Meek [1995], p takes the form $P_0 \rightarrow \dots \rightarrow P_r$.

Now suppose for sake of contradiction that $X \in \mathbf{C_d}$. Let X' be the last node on p in $\mathbf{X}$, and let H be the node on p immediately following X' . That is, $\mathcal{G}$ contains $X \rightarrow \dots \rightarrow X' \rightarrow H \rightarrow \dots \rightarrow D$, where it may be that $X' = X$ and $H = D$. Note that by Lemma 98, p is entirely contained in $\mathbf{C_d}$ and so $X', H \in \mathbf{C_d}$.

By the definition of $\mathbf{C_d}$ as a bucket in $\mathbf{V}$, $\mathcal{G}$ must contain an undirected path from X' to H . Let $u = \langle X' = U_0, \dots, U_r = H \rangle, r \geq 2$ be a shortest such path. We show by induction that $\mathcal{G}$ contains $X' \rightarrow U_c$ for all $c \in \{2, \dots, r\}$. We already have that $\mathcal{G}$ contains $X' \rightarrow U_r$. So suppose $\mathcal{G}$ contains $X' \rightarrow U_{d+1}$ for some $d \in \{2, \dots, r-1\}$. Since $\mathcal{G}$ contains $X' \rightarrow U_{d+1} - U_d$, $\mathcal{G}$ must contain an edge $\langle X', U_d \rangle$. By the choice of u , $\mathcal{G}$ cannot contain $X' - U_d$. Neither can $\mathcal{G}$ contain $X' \leftarrow U_d$, since by R2 of Meek [1995] this would force u to contain $U_d \rightarrow U_{d+1}$. Therefore $\mathcal{G}$ contains $X' \rightarrow U_d$.

Turn to note that, by definition of H , $p(H, D)$ is a causal path with no nodes in $\mathbf{X}$, where $D \in \text{An}(\mathbf{Y}, \mathcal{G}_{\mathbf{V} \setminus \mathbf{X}}) \setminus \mathbf{Z}$. Thus, $\mathcal{G}$ contains a causal path from H to $\mathbf{Y}$ with no nodes in $\mathbf{X}$. Since $\mathcal{G}$ contains $X' \rightarrow H$ and $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$, then $H \notin \mathbf{Z}$. Thus, $H \in \mathbf{D}$, and by Lemma 88, there is no proper possibly causal path from $\mathbf{X}$ to H in $\mathcal{G}$ that starts with an undirected edge. Note that u may not be proper, but there is a subpath of u that is a proper path from $\mathbf{X}$ to H , which therefore cannot be possibly causal. Thus, $\mathcal{G}$ must contain

$U_a \leftarrow U_b$ for some $a, b \in \{0, \dots, r\}$, $a + 1 < b$. Pick the earliest such U_a on u .

Suppose for sake of contradiction that $U_a \neq X'$. Then $\mathcal{G}$ contains $U_{a-1} - U_a \leftarrow U_b$ and so R1 of Meek [1995], $\mathcal{G}$ must contain an edge $\langle U_{a-1}, U_b \rangle$. $\mathcal{G}$ cannot contain $U_{a-1} - U_b$ since this would contradict the choice of u . Neither can $\mathcal{G}$ contain $U_{a-1} \rightarrow U_b$ since by R2 of Meek [1995], this would force u to contain the directed edge $U_{a-1} \rightarrow U_a$. Thus $\mathcal{G}$ contains $U_{a-1} \leftarrow U_b$, but this contradicts the choice of U_a . Therefore $\mathcal{G}$ contains $X' \leftarrow U_b$, $b \in \{2, \dots, r\}$, but this contradicts that $\mathcal{G}$ contains $X' \rightarrow U_c$ for all $c \in \{2, \dots, r\}$.

We have shown that $X \notin \mathbf{C}_d$, where $\mathcal{G}$ contains a causal path from X to $D \in \mathbf{C}_d$. The result follows from Lemma 86.

(ii) Suppose for sake of contradiction that $\mathbf{B}_i \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) \neq \emptyset$. By part (i), there is a causal path in $\mathcal{G}$ from $\mathbf{X}$ to every node in $\mathbf{B}_i$. Since $\mathbf{Z} \cap \text{PossDe}(\mathbf{B}_i, \mathcal{G}) \neq \emptyset$, then there is also a possibly causal path in $\mathcal{G}$ from some $B_i \in \mathbf{B}_i$ to $\mathbf{Z}$. Thus by Lemma 91, there is a possibly causal path in $\mathcal{G}$ from $\mathbf{X}$ to $\mathbf{Z}$, which contradicts that $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$. ■

C.5 PROOFS FOR SECTION 4.3.5: IDENTIFIABILITY CONDITION

This section includes the proof of Proposition 23 found in Section 4.3.5. We note that one direction of the proof ($\Leftarrow$) relies heavily on a stronger claim that does not require $\mathbf{Z} \cap \text{PossDe}(\mathbf{X}, \mathcal{G}) = \emptyset$ (see Proposition 106). But we do not include this claim in the main text, since its converse does not hold (see Examples 29-30). The statement and proof of this claim are in the main results below. Two supporting results follow.

C.5.1 MAIN RESULTS

Proof of Proposition 23. Follows from Theorem 21 and Proposition 106. ■

Proposition 106 (Necessity Condition) *Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ be pairwise disjoint node sets in a causal MPDAG $\mathcal{G}$. If the conditional causal effect of $\mathbf{X}$ on $\mathbf{Y}$ given $\mathbf{Z}$ is identifiable in $\mathcal{G}$, then there is no proper possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$ that starts with an undirected edge and does not contain nodes in $\mathbf{Z}$.*

Proof of Proposition 106. We prove the contrapositive using similar logic to the proof of Proposition 3.2 in Perković [2020]. Thus, suppose there is a proper possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ that starts with an undirected edge and does not contain nodes in $\mathbf{Z}$. Then by Lemma 107, there is one such path – call it $q = \langle X = V_0, \dots, V_k = Y \rangle$, $X \in \mathbf{X}$, $Y \in \mathbf{Y}$, $k \geq 1$ – where the corresponding paths in two DAGs in $[\mathcal{G}]$ take the forms $X \rightarrow \dots \rightarrow Y$ and $X \leftarrow V_1 \rightarrow \dots \rightarrow Y$ ($X \leftarrow Y$ when $k = 1$). Call these DAGs $\mathcal{D}^1$ and $\mathcal{D}^2$ with paths q_1 and q_2 , respectively.

To prove that the conditional causal effect of $\mathbf{X}$ on $\mathbf{Y}$ given $\mathbf{Z}$ is not identifiable in $\mathcal{G}$, it suffices to show that there are two families of interventional densities over $\mathbf{V}$ – call them $\mathbf{F}_1^*$ and $\mathbf{F}_2^*$, where for $i \in \{1, 2\}$, we define $\mathbf{F}_i^* = \{f_i(\mathbf{v}|\text{do}(\mathbf{x}')) : \mathbf{X}' \subseteq \mathbf{V}\}$ – such that the following properties hold.

- (i) $\mathcal{D}^1$ and $\mathcal{D}^2$ are compatible with $\mathbf{F}_1^*$ and $\mathbf{F}_2^*$, respectively.¹
- (ii) $f_1(\mathbf{v}) = f_2(\mathbf{v})$.
- (iii) $f_1(\mathbf{y}|\text{do}(\mathbf{x}), \mathbf{z}) \neq f_2(\mathbf{y}|\text{do}(\mathbf{x}), \mathbf{z})$.

To define such families, we start by introducing an additional DAG and an observational density $f(\mathbf{v})$. That is, let $\mathcal{D}^{1'}$ be a DAG constructed by removing every edge from $\mathcal{D}^1$ except for the edges on q_1 . Then let $f(\mathbf{v})$ be the multivariate normal density under the following linear structural equation model (SEM). Each random variable $A \in \mathbf{V}$ has mean zero and is a linear combination of its parents in $\mathcal{D}^{1'}$ and $\varepsilon_A \sim N(0, \sigma_A^2)$, where $\{\varepsilon_A : A \in \mathbf{V}\}$ are mutually independent. The coefficients in this linear combination are defined by the edge coefficients of $\mathcal{D}^{1'}$. We pick these edge coefficients in conjunction with $\{\sigma_A^2 : A \in \mathbf{V}\}$ in such a way that each coefficient is in $(0, 1)$ and $\text{Var}(A) = 1$ for all $A \in \mathbf{V}$.

From this, we define $\mathbf{F}_1^* = \{f_1(\mathbf{v}|\text{do}(\mathbf{x}')) : \mathbf{X}' \subseteq \mathbf{V}\}$ such that $\mathcal{D}^{1'}$ is compatible with $\mathbf{F}_1^*$ and such that $f_1(\mathbf{v}) = f(\mathbf{v})$. Note that $f(\mathbf{v})$ is Markov compatible with $\mathcal{D}^{1'}$ by construction,

¹For brevity, we say a DAG is “compatible with” a set of interventional densities and an interventional density is “consistent with” a DAG as shorthand for these claims holding only were the DAG to be causal.

and we build the interventional densities in $\mathbf{F}_1^*$ by replacing the intervening random variables in the SEM with their interventional values [Pearl, 2009].

To construct the second family of interventional densities, we introduce the DAG $\mathcal{D}^{2'}$, which we form by removing every edge from $\mathcal{D}^2$ except for the edges on q_2 . Then note that we could have defined $f(\mathbf{v})$ using a linear SEM based on the parents in $\mathcal{D}^{2'}$. In this case, the resulting observational density would again be a multivariate normal with mean vector zero and a covariance matrix with ones on the diagonal. The off-diagonal entries would be the covariances between the variables in $\mathcal{D}^{2'}$. But note that by Lemma 77, these values will equal the product of all edge coefficients between the relevant nodes in $\mathcal{D}^{2'}$. Since $\mathcal{D}^{1'}$ and $\mathcal{D}^{2'}$ contain no paths with colliders, the observational density $f(\mathbf{v})$ built using $\mathcal{D}^{2'}$ will be an identical density to that built under $\mathcal{D}^{1'}$. Thus, in an analogous way to $\mathbf{F}_1^*$, we define $\mathbf{F}_2^* = \{f_2(\mathbf{v}|do(\mathbf{x}')) : \mathbf{X}' \subseteq \mathbf{V}\}$ such that $\mathcal{D}^{2'}$ is compatible with $\mathbf{F}_2^*$ and such that $f_2(\mathbf{v}) = f(\mathbf{v})$.

Having defined $\mathbf{F}_1^*$ and $\mathbf{F}_2^*$, we check that their desired properties hold. Note that by construction, $\mathcal{D}^{1'}$ and $\mathcal{D}^{2'}$ are compatible with $\mathbf{F}_1^*$ and $\mathbf{F}_2^*$, respectively. Thus (i) holds by Lemma 108. Similarly by construction, (ii) holds. To show that (iii) holds, it suffices to show that $E[Y|do(\mathbf{X} = \mathbf{1}), \mathbf{Z}]$ is not the same under f_1 and f_2 .

To calculate these expectations, we first want to apply Rules 1-3 of the do calculus (Theorem 75). Since $f_i(\mathbf{v}|do(\mathbf{x}))$, $i \in \{1, 2\}$, is consistent with $\mathcal{D}^{i'}$, we apply these rules using graphical relationships in $\mathcal{D}^{i'}$. Because the path in $\mathcal{D}^{i'}$ corresponding to q_i , $i \in \{1, 2\}$, does not contain nodes in $\mathbf{Z}$ or $\mathbf{X} \setminus \{X\}$, then $Y \perp_d \mathbf{Z}|\mathbf{X}$ and $Y \perp_d \mathbf{X} \setminus \{X\}|X$ in $\mathcal{D}_{\mathbf{X}}^{i'}$. Further, $Y \perp_d X$ in $\mathcal{D}_{\mathbf{X}}^{1'}$ and $Y \perp_d X$ in $\mathcal{D}_{\mathbf{X}}^{2'}$. Thus by Rules 1-3 of the do calculus (Theorem 75), the following hold.

$$E_1[Y|do(\mathbf{X} = \mathbf{1}), \mathbf{Z}] = E_1[Y|do(X = 1)] = E_1[Y|X = 1] := a.$$

$$E_2[Y|do(\mathbf{X} = \mathbf{1}), \mathbf{Z}] = E_2[Y|do(X = 1)] = E_2[Y] := b,$$

where $E_i, i \in \{1, 2\}$ is the expectation under f_i . To calculate a and b , we rely on the observational density $f(\mathbf{v})$, which was constructed using $\mathcal{D}^{1'}$. By Lemma 76, a equals the

covariance of X and Y under $f(\mathbf{v})$, and by Lemma 77, $\text{Cov}(X, Y)$ equals the product of all edge coefficients in $\mathcal{D}'$, which were chosen to be in $(0, 1)$. Therefore, $a \neq 0$. But by definition of $f(\mathbf{v})$, $b = 0$. ■

C.5.2 SUPPORTING RESULTS

Lemma 107 *Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ be pairwise disjoint node sets in an MPDAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$. Suppose that there is a proper possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$ that starts with an undirected edge and does not contain nodes in $\mathbf{Z}$. Then there is one such path $\langle X = V_0, \dots, V_k = Y \rangle$, $X \in \mathbf{X}$, $Y \in \mathbf{Y}$, $k \geq 1$, where the corresponding paths in two DAGs in $[\mathcal{G}]$ take the forms $X \rightarrow \dots \rightarrow Y$ and $X \leftarrow V_1 \rightarrow \dots \rightarrow Y$ ($X \leftarrow Y$ when $k = 1$), respectively.*

Proof of Lemma 107. This lemma is similar to Lemma A.3 of Perković [2020] and its proof borrows from the proof strategy of Lemma C.1 of Perković et al. [2017].

Let q^* be an arbitrary proper possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$ that starts with an undirected edge and does not contain nodes in $\mathbf{Z}$. Then let $q = \langle X = V_0, \dots, V_k = Y \rangle$, $X \in \mathbf{X}$, $Y \in \mathbf{Y}$, $k \geq 1$, be a shortest subsequence of q^* in $\mathcal{G}$ that also starts with an undirected edge. Note that q is a proper possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$ that starts with an undirected edge and does not contain nodes in $\mathbf{Z}$.

Consider when q is of definite status. Since q is possibly causal, all non-endpoints of q are definite non-colliders. Let $\mathcal{D}^1$ be a DAG in $[\mathcal{G}]$ that contains $X \rightarrow V_1$. Then since V_1 is either Y or a definite non-collider on q , the path corresponding to q in $\mathcal{D}^1$ takes the form $X \rightarrow \dots \rightarrow Y$ by induction. Let $\mathcal{D}^2$ be a DAG in $[\mathcal{G}]$ with no additional edges into V_1 compared to $\mathcal{G}$ (Lemma 81). Since $\mathcal{G}$ contains $X - V_1$, $\mathcal{D}^2$ contains $X \leftarrow V_1$. When $k > 1$, $\mathcal{G}$ contains either $V_1 - V_2$ or $V_1 \rightarrow V_2$, and so $\mathcal{D}^2$ contains $X \leftarrow V_1 \rightarrow V_2$. Thus by the same inductive reasoning as above, the path corresponding to q in $\mathcal{D}^2$ takes the form $X \leftarrow V_1 \rightarrow \dots \rightarrow Y$ (or simply $X \leftarrow Y$ when $k = 1$).

Consider instead when q is not of definite status. Note that $k > 1$. To see that q contains $V_1 - V_2$, note that by the choice of q and the fact that q is possibly causal, $q(V_1, Y)$ is

unshielded and possibly causal. Thus, $q(V_1, Y)$ is of definite status. However, q is not of definite status, so V_1 must not be of definite status on q , which implies that q cannot contain $V_1 \rightarrow V_2$. Since q is possibly causal, it also cannot contain $V_1 \leftarrow V_2$.

To find two DAGs in $[\mathcal{G}]$ with paths corresponding to q that fit our desired forms, we narrow our search to $[\mathcal{G}']$, where we let $\mathcal{G}'$ be an MPDAG constructed from $\mathcal{G}$ by adding $V_1 \rightarrow V_2$ and completing R1-R4 of Meek [1995]. We show below that the path corresponding to q in $\mathcal{G}'$ takes the form $X - V_1 \rightarrow \dots \rightarrow Y$, and thus, there must be two DAGs in $[\mathcal{G}'] \subseteq [\mathcal{G}]$ with corresponding paths of the forms $X \rightarrow \dots \rightarrow Y$ and $X \leftarrow V_1 \rightarrow \dots \rightarrow Y$.

We first show that $\mathcal{G}'$ contains $X - V_1$ by the contraposition of Lemma 80. Note that we have already shown that $\mathcal{G}$ contains $V_1 - V_2$, that $\mathcal{G}'$ is formed by adding $V_1 \rightarrow V_2$ to $\mathcal{G}$, and that $\mathcal{G}$ contains $X - V_1$. It remains to show that $X, V_1 \notin \text{De}(V_2, \mathcal{G}')$. To see this, note that $\mathcal{G}$ must contain an edge $\langle X, V_2 \rangle$, because V_1 is not of definite status on q . This edge must take the form $X \rightarrow V_2$ by the choice of q and the fact that q is possibly causal. Thus, $\mathcal{G}'$ contains $X \rightarrow V_2$ and $V_1 \rightarrow V_2$. Therefore, $X, V_1 \notin \text{De}(V_2, \mathcal{G}')$. Finally, note that $\mathcal{G}'$ contains $V_1 \rightarrow \dots \rightarrow Y$ by R1 of Meek [1995], since we constructed $\mathcal{G}'$ by adding $V_1 \rightarrow V_2$ to a path $q(V_1, Y)$ that is unshielded and possibly causal. ■

Lemma 108 *Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ be pairwise disjoint node sets in a causal DAG $\mathcal{D} = (\mathbf{V}, \mathbf{E})$. Then let $\mathcal{D}^* = (\mathbf{V}, \mathbf{E}')$ be a causal DAG constructed by removing edges from $\mathcal{D}$, and let $f(\mathbf{v}|\text{do}(\mathbf{x}))$ be an interventional density over $\mathbf{V}$. If $f(\mathbf{v}|\text{do}(\mathbf{x}))$ is consistent with $\mathcal{D}^*$, then it is consistent with $\mathcal{D}$.*

Proof of Lemma 108. Suppose that $f(\mathbf{v}|\text{do}(\mathbf{x}))$ is consistent with $\mathcal{D}^*$. Then by definition, there exists a set of interventional densities $\mathbf{F}^*$ such that $\mathcal{D}^*$ is compatible with $\mathbf{F}^*$. Let $f(\mathbf{v})$ be the density in $\mathbf{F}^*$ under a null intervention. Note that by the truncated factorization in Equation (4.1), $f(\mathbf{v})$ is Markov compatible with $\mathcal{D}^*$. Thus by Lemma 79,

$$V_i \perp\!\!\!\perp \left[\mathbf{V} \setminus (\text{De}(V_i, \mathcal{D}^*) \cup \text{Pa}(V_i, \mathcal{D}^*)) \right] \mid \text{Pa}(V_i, \mathcal{D}^*) \quad (\text{C.18})$$

for all $V_i \in \mathbf{V}$, where $\perp\!\!\!\perp$ indicates independence with respect to $f(\mathbf{v})$. Further, since $\text{De}(V_i, \mathcal{D}^*) \subseteq \text{De}(V_i, \mathcal{D})$, then $\text{De}(V_i, \mathcal{D}^*) \cap \text{Pa}(V_i, \mathcal{D}) = \emptyset$ and thus $\text{Pa}(V_i, \mathcal{D}) \subseteq \mathbf{V} \setminus \text{De}(V_i, \mathcal{D}^*)$.

Therefore it follows from (C.18) that

$$V_i \perp\!\!\!\perp \left[\text{Pa}(V_i, \mathcal{D}) \setminus \text{Pa}(V_i, \mathcal{D}^*) \right] \mid \text{Pa}(V_i, \mathcal{D}^*). \quad (\text{C.19})$$

Let $f(\mathbf{v}|\text{do}(\mathbf{x}'))$, $\mathbf{X}' \subseteq \mathbf{V}$, be an arbitrary density in $\mathbf{F}^*$. Then by definition and (C.19)

$$\begin{aligned} f(\mathbf{v}|\text{do}(\mathbf{x}')) &= \prod_{V_i \in \mathbf{V} \setminus \mathbf{X}'} f(v_i | \text{pa}(v_i, \mathcal{D}^*)) \mathbb{1}(\mathbf{X}' = \mathbf{x}') \\ &= \prod_{V_i \in \mathbf{V} \setminus \mathbf{X}'} f(v_i | \text{pa}(v_i, \mathcal{D})) \mathbb{1}(\mathbf{X}' = \mathbf{x}'). \end{aligned}$$

Since $f(\mathbf{v}|\text{do}(\mathbf{x}'))$ was arbitrary, this holds for all densities in $\mathbf{F}^*$. Thus, $\mathcal{D}$ is compatible with $\mathbf{F}^*$. Since $f(\mathbf{v}|\text{do}(\mathbf{x})) \in \mathbf{F}^*$, then by definition, it is consistent with $\mathcal{D}$. ■

C.6 PROOFS FOR SECTION 4.4: DO CALCULUS FOR MPDAGS

This section includes lemmas needed for the proof of our do calculus (Theorem 24) found in Section 4.4. We divide these results into those needed for all of our do rules (Lemmas 109-112) and those needed for Rule 1 (Lemmas 113-114), Rule 2 (Lemmas 113-114), and Rule 3 (Lemmas 117-119). Figure C.2 provides a map of these results.

We began Section 4.4 with a discussion of the mutilated graphs seen in Theorem 24, and we noted two differences between these graphs and their counterparts in Pearl's do calculus. Here, we highlight a third, more technical difference that will be useful in navigating the proofs below. Because the mutilated graphs in Theorem 24 might not be MPDAGs, the *definite status* of a path in these graphs no longer conveys a certainty about the corresponding paths in related DAGs. To see this, consider an MPDAG $\mathcal{G}$ that contains the paths $V_1 - V_2 - X$ and $V_1 \rightarrow X$. The graph $\mathcal{G}_{\overline{X}}$ only contains the path $V_1 - V_2 - X$, where V_2 is a definite non-collider by definition. However, note that $[\mathcal{G}]$ contains a DAG with the path $V_1 \rightarrow V_2 \leftarrow X$.

C.6.1 Results for All do Rules

Lemma 109 (cf. Lemma 1 of Zhang [2006]) *Let $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$, and $\mathbf{W}$ be pairwise disjoint node sets in a DAG $\mathcal{D} = (\mathbf{V}, \mathbf{E})$, and let $\mathcal{G}$ be an MPDAG such that $\mathcal{D} \in [\mathcal{G}]$. Additionally, let p*

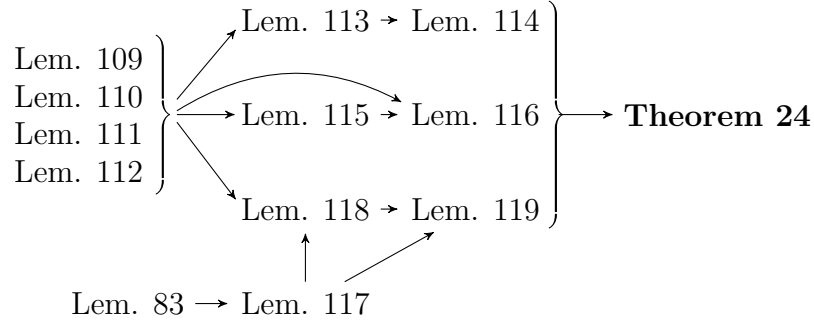


Figure C.2: Proof structure of Theorem 24.

be a path in $\mathcal{D}$ from $\mathbf{Z}$ to $\mathbf{Y}$ such that p is d -connecting given $\mathbf{X} \cup \mathbf{W}$ but no subsequence of p in $\mathcal{D}$ is d -connecting given $\mathbf{X} \cup \mathbf{W}$. Then the following hold.

(i) All colliders on p are unshielded.

(ii) The path in $\mathcal{G}$ corresponding to p is of definite status.

Proof of Lemma 109. Let $p = \langle Z = V_1, \dots, V_k = Y \rangle$, $Z \in \mathbf{Z}$, $Y \in \mathbf{Y}$, $k > 1$.

(i) Let V_i , $i \in \{2, \dots, k-1\}$, be an arbitrary collider on p . For sake of contradiction, suppose $\mathcal{D}$ contains the edge $\langle V_{i-1}, V_{i+1} \rangle$. Define $q = p(Z, V_{i-1}) \oplus \langle V_{i-1}, V_{i+1} \rangle \oplus p(V_{i+1}, Y)$. By the choice of p , q must be blocked by $\mathbf{X} \cup \mathbf{W}$. Since V_{i-1} and V_{i+1} are non-colliders on p , q must be blocked by a collider at either V_{i-1} or V_{i+1} . Without loss of generality, pick V_{i-1} so that $\mathcal{D}$ contains $V_{i-2} \rightarrow V_{i-1} \leftarrow V_{i+1}$, where no descendant of V_{i-1} is in $\mathbf{X} \cup \mathbf{W}$. However, since p is d -connecting given $\mathbf{X} \cup \mathbf{W}$ and p contains $V_{i-1} \rightarrow V_i \leftarrow V_{i+1}$, at least one descendant of V_i – and therefore of V_{i-1} – in $\mathcal{D}$ must be in $\mathbf{X} \cup \mathbf{W}$.

(ii) Let p^* be the path in $\mathcal{G}$ corresponding to p . For sake of contradiction, suppose there is a node V_i , $i \in \{2, \dots, k-1\}$, that is not of definite status on p^* . This implies that $\mathcal{G}$ – and therefore $\mathcal{D}$ – must contain the edge $\langle V_{i-1}, V_{i+1} \rangle$.

Define $q = p(Z, V_{i-1}) \oplus \langle V_{i-1}, V_{i+1} \rangle \oplus p(V_{i+1}, Y)$ and consider the form of $p(V_{i-1}, V_{i+1})$. Since $\mathcal{D}$ contains $\langle V_{i-1}, V_{i+1} \rangle$, then by part (i), it cannot contain $V_{i-1} \rightarrow V_i \leftarrow V_{i+1}$. For

sake of contradiction, suppose that $\mathcal{D}$ contains $V_{i-1} \rightarrow V_i \rightarrow V_{i+1}$. This forces $\mathcal{D}$ to contain $V_{i-1} \rightarrow V_{i+1}$. But since p is d-connected given $\mathbf{X} \cup \mathbf{W}$, this implies that q is d-connected given $\mathbf{X} \cup \mathbf{W}$, which contradicts the choice of p . By the same logic, $\mathcal{D}$ cannot contain $V_{i-1} \leftarrow V_i \leftarrow V_{i+1}$. Therefore, $\mathcal{D}$ must contain $V_{i-1} \leftarrow V_i \rightarrow V_{i+1}$.

Without loss of generality, let $\langle V_{i-1}, V_{i+1} \rangle$ take the form $V_{i-1} \rightarrow V_{i+1}$ in $\mathcal{D}$. Note that for p to be d-connected and q to be blocked given $\mathbf{X} \cup \mathbf{W}$, V_{i-1} must be a collider on p and a node in $\mathbf{X} \cup \mathbf{W}$. Then by part (i), V_{i-1} is unshielded on p . So by properties of MPDAGs, V_{i-1} must also be a collider on p^* . Thus, $\mathcal{G}$ contains $V_{i-1} \leftarrow V_i$. But this contradicts that V_i is not of definite status on p^* . ■

Lemma 110 *Let $\mathbf{X}$ and $\mathbf{W}$ be disjoint node sets in a DAG $\mathcal{D}$ that has no edges into $\mathbf{X}$. If a path in $\mathcal{D}$ is d-connecting given $\mathbf{X} \cup \mathbf{W}$, then it does not contain nodes in $\mathbf{X}$ and is d-connecting given $\mathbf{W}$.*

Proof of Lemma 110. Any path in $\mathcal{D}$ that is d-connecting given $\mathbf{X} \cup \mathbf{W}$ cannot have non-colliders in $\mathbf{X}$ or $\mathbf{W}$. Further, since there are no edges into $\mathbf{X}$ in $\mathcal{D}$, all of the colliders on such a path cannot be in $\mathbf{X}$ and must have descendants in $\mathbf{W}$. ■

Lemma 111 *Let $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$, and $\mathbf{W}$ be pairwise disjoint node sets in a DAG $\mathcal{D}$, let p be a path in $\mathcal{D}$ from $\mathbf{Z}$ to $\mathbf{Y}$ that is d-connecting given $\mathbf{X} \cup \mathbf{W}$, and let q be a subsequence of p in $\mathcal{D}$. Then all colliders on q are in $\text{An}(\mathbf{X} \cup \mathbf{W}, \mathcal{D})$.*

Proof of Lemma 111. Let $p = \langle Z = V_1, \dots, V_k = Y \rangle$, $Z \in \mathbf{Z}$, $Y \in \mathbf{Y}$, $k > 1$, and let V_i , $i \in \{2, \dots, k-1\}$, be an arbitrary collider on q .

When V_i is also a collider on p , then $V_i \in \text{An}(\mathbf{X} \cup \mathbf{W}, \mathcal{D})$, since p is d-connecting given $\mathbf{X} \cup \mathbf{W}$. Consider when V_i is not a collider on p . Then p contains $V_{i-1} \leftarrow V_i$ or $V_i \rightarrow V_{i+1}$. Without loss of generality, suppose p contains $V_i \rightarrow V_{i+1}$. Note that V_{i+1} is not on q , since V_i is a collider on q . Therefore, let V_j , $j \in \{i+2, \dots, k\}$, be the node following V_i on q so that q contains $V_i \leftarrow V_j$. Since $\mathcal{D}$ is acyclic, p cannot contain $V_i \rightarrow \dots \rightarrow V_j$. Hence, let C be the collider on $p(V_i, V_j)$ closest to V_i so that $V_i \in \text{An}(C, \mathcal{D})$. Since p is d-connecting given $\mathbf{X} \cup \mathbf{W}$, then $C \in \text{An}(\mathbf{X} \cup \mathbf{W}, \mathcal{D})$ and thus, $V_i \in \text{An}(\mathbf{X} \cup \mathbf{W}, \mathcal{D})$. ■

Lemma 112 *Let $\mathbf{X}$ and $\mathbf{Z}$ be disjoint node sets in an MPDAG $\mathcal{G}$, let $\mathbf{X}' \subseteq \mathbf{X}$, let $\mathcal{D}$ be a DAG in $[\mathcal{G}]$, and let $\bar{p}$ be a path in $\mathcal{D}_{\bar{\mathbf{X}}\mathbf{Z}}$. If the path in $\mathcal{G}$ corresponding to $\bar{p}$ is of definite status, then the sequence of nodes in $\mathcal{G}_{\bar{\mathbf{X}}'\mathbf{Z}}$ corresponding to $\bar{p}$ forms a path where a node is a definite non-collider or collider if and only if it is a non-collider or collider, respectively, on $\bar{p}$.*

Proof of Lemma 112. To see that the sequence of nodes in $\mathcal{G}_{\bar{\mathbf{X}}'\mathbf{Z}}$ corresponding to $\bar{p}$ forms a path, note that the set of adjacencies in $\mathcal{D}_{\bar{\mathbf{X}}\mathbf{Z}}$ is a subset of the set of adjacencies in $\mathcal{G}_{\bar{\mathbf{X}}'\mathbf{Z}}$. Then let V be an arbitrary node on $\bar{p}$, and let p , p^* , and $\bar{p}^*$ be the paths in $\mathcal{D}$, $\mathcal{G}$, and $\mathcal{G}_{\bar{\mathbf{X}}'\mathbf{Z}}$, respectively, corresponding to $\bar{p}$. Consider the following.

- (a) V is a non-collider/collider on $\bar{p}$.
- (b) V is a non-collider/collider on p .
- (c) V is a definite non-collider/collider on p^* .
- (d) V is a definite non-collider/collider on $\bar{p}^*$.

By definition of the graphs, (a) $\Leftrightarrow$ (b) and (b) $\Leftarrow$ (c). Further, if p^* is of definite status, (b) $\Rightarrow$ (c) and (c) $\Leftarrow$ (d). Then since removing edges from $\mathcal{G}$ outside of p^* cannot change a node of definite status on p^* , (c) $\Rightarrow$ (d). ■

C.6.2 Results for do Rule 1

Lemma 113 *Let $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$, and $\mathbf{W}$ be pairwise disjoint node sets in an MPDAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, and let $\mathcal{D}$ be a DAG in $[\mathcal{G}]$. Additionally, let $\bar{p}$ be a path in $\mathcal{D}_{\bar{\mathbf{X}}}$ from $\mathbf{Z}$ to $\mathbf{Y}$ such that $\bar{p}$ is d -connecting given $\mathbf{X} \cup \mathbf{W}$ but no subsequence of $\bar{p}$ in $\mathcal{D}_{\bar{\mathbf{X}}}$ is d -connecting given $\mathbf{X} \cup \mathbf{W}$. Then the sequence of nodes in $\mathcal{G}_{\bar{\mathbf{X}}}$ corresponding to $\bar{p}$ forms a path where a node is a definite non-collider or collider if and only if it is a non-collider or collider, respectively, on $\bar{p}$.*

Proof of Lemma 113. Let p be the path in $\mathcal{D}$ corresponding to $\bar{p}$. If p satisfies the assumptions of Lemma 109, then the path in $\mathcal{G}$ corresponding to $\bar{p}$ is of definite status, and the claim follows by Lemma 112. We show these assumptions hold below.

First note that p must be d-connecting given $\mathbf{X} \cup \mathbf{W}$, since $\bar{p}$ is d-connecting given $\mathbf{X} \cup \mathbf{W}$ and adding edges to a graph cannot block existing d-connecting paths. Next, for sake of contradiction, suppose that there is a subsequence of p in $\mathcal{D}$ – call it q – that is d-connecting given $\mathbf{X} \cup \mathbf{W}$. By Lemma 110, $\bar{p}$ – and therefore q – does not contain nodes in $\mathbf{X}$. Thus, the sequence of nodes in $\mathcal{D}_{\bar{\mathbf{X}}}$ corresponding to q forms a path – call it $\bar{q}$. Since q is d-connecting given $\mathbf{X} \cup \mathbf{W}$, none of the non-colliders on q – and therefore on $\bar{q}$ – are in $\mathbf{X} \cup \mathbf{W}$. Further, since $\bar{p}$ is d-connecting given $\mathbf{X} \cup \mathbf{W}$, then by Lemma 111, all colliders on $\bar{q}$ are in $\text{An}(\mathbf{X} \cup \mathbf{W}, \mathcal{D}_{\bar{\mathbf{X}}})$. But this implies that $\bar{q}$, which is a subsequence of $\bar{p}$, is d-connecting given $\mathbf{X} \cup \mathbf{W}$, which contradicts the choice of $\bar{p}$. ■

Lemma 114 *Let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$, and $\mathbf{W}$ be pairwise disjoint node sets in an MPDAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$. If $(\mathbf{Z} \perp_d \mathbf{Y} | \mathbf{X}, \mathbf{W})_{\mathcal{G}_{\bar{\mathbf{X}}}}$, then $(\mathbf{Z} \perp_d \mathbf{Y} | \mathbf{X}, \mathbf{W})_{\mathcal{D}_{\bar{\mathbf{X}}}}$ for all DAGs $\mathcal{D} \in [\mathcal{G}]$.*

Proof of Lemma 114. We prove the contrapositive. Suppose there exists a DAG $\mathcal{D} \in [\mathcal{G}]$ such that $\mathcal{D}_{\bar{\mathbf{X}}}$ contains a path from $\mathbf{Z}$ to $\mathbf{Y}$ that is d-connecting given $\mathbf{X} \cup \mathbf{W}$. Let $\mathbf{S}$ be the set of all shortest such paths, and let $\bar{p} = \langle Z = V_1, \dots, V_k = Y \rangle$, $Z \in \mathbf{Z}$, $Y \in \mathbf{Y}$, $k > 1$, be a path in $\mathbf{S}$ with the shortest distance to $\mathbf{X} \cup \mathbf{W}$ (Definition 74).

To complete the proof, we want to show that the sequence of nodes in $\mathcal{G}_{\bar{\mathbf{X}}}$ corresponding to $\bar{p}$ forms a path that is d-connecting given $\mathbf{X} \cup \mathbf{W}$. Note that by definition, $\bar{p}$ is d-connecting given $\mathbf{X} \cup \mathbf{W}$ and no subsequence of $\bar{p}$ in $\mathcal{D}_{\bar{\mathbf{X}}}$ is d-connecting given $\mathbf{X} \cup \mathbf{W}$. Thus by Lemma 113, we only need to show the following.

- (i) No non-collider on $\bar{p}$ is in $\mathbf{X} \cup \mathbf{W}$.
- (ii) Every collider on $\bar{p}$ is in $\text{An}(\mathbf{X} \cup \mathbf{W}, \mathcal{G}_{\bar{\mathbf{X}}})$.

Claim (i) holds since $\bar{p}$ is d-connecting given $\mathbf{X} \cup \mathbf{W}$. The remainder of the proof shows that claim (ii) holds.

Let V_i , $i \in \{2, \dots, k-1\}$, be an arbitrary collider on $\bar{p}$. Since $\bar{p}$ is d-connecting given $\mathbf{X} \cup \mathbf{W}$ in $\mathcal{D}_{\bar{\mathbf{X}}}$, then $V_i \in \text{An}(\mathbf{W}, \mathcal{D}_{\bar{\mathbf{X}}})$. When $V_i \in \mathbf{W}$, then $V_i \in \text{An}(\mathbf{X} \cup \mathbf{W}, \mathcal{G}_{\bar{\mathbf{X}}})$ and we are done.

When $V_i \notin \mathbf{W}$, then let $\bar{q} = \langle V_i = Q_1, \dots, Q_m = W \rangle, W \in \mathbf{W}, m > 1$, be a shortest directed path in $\mathcal{D}_{\bar{\mathbf{X}}}$ from V_i to $\mathbf{W}$, and let q^* be the path in $\mathcal{G}$ corresponding to $\bar{q}$. Note that no node on q^* is in $\mathbf{X}$, since $\mathcal{D}_{\bar{\mathbf{X}}}$ contains $V_{i-1} \rightarrow Q_1 \rightarrow \dots \rightarrow Q_m$. Thus if q^* is directed, then the corresponding sequence of nodes in $\mathcal{G}_{\bar{\mathbf{X}}}$ will also form a directed path, which again implies that $V_i \in \text{An}(\mathbf{X} \cup \mathbf{W}, \mathcal{G}_{\bar{\mathbf{X}}})$ and the proof is complete. We show below that q^* is directed.

We start by showing that q^* contains $Q_1 \rightarrow Q_2$. Note that by Lemma 113, $\mathcal{G}_{\bar{\mathbf{X}}}$ – and therefore $\mathcal{G}$ – contains $V_{i-1} \rightarrow Q_1 \leftarrow V_{i+1}$. For sake of contradiction, suppose that $\mathcal{G}$ – and therefore $\mathcal{D}$ – contains the edges $\langle V_{i-1}, Q_2 \rangle$ and $\langle V_{i+1}, Q_2 \rangle$. Since $\mathcal{D}$ must contain $V_{i-1} \rightarrow Q_1 \rightarrow Q_2$ and $V_{i+1} \rightarrow Q_1 \rightarrow Q_2$, it must contain $V_{i-1} \rightarrow Q_2 \leftarrow V_{i+1}$. And since no node on q^* is in $\mathbf{X}$, then $\mathcal{D}_{\bar{\mathbf{X}}}$ must also contain $V_{i-1} \rightarrow Q_2 \leftarrow V_{i+1}$.

When Q_2 is not on $\bar{p}$, consider the path $\bar{p}(Z, V_{i-1}) \oplus \langle V_{i-1}, Q_2, V_{i+1} \rangle \oplus \bar{p}(V_{i+1}, Y)$ in $\mathcal{D}_{\bar{\mathbf{X}}}$. Note that this is a path from $\mathbf{Z}$ to $\mathbf{Y}$ that is d-connecting given $\mathbf{X} \cup \mathbf{W}$, is the same length as $\bar{p}$, and has a shorter distance than $\bar{p}$ to $\mathbf{X} \cup \mathbf{W}$, which contradicts the choice of $\bar{p}$.

We complete the contradiction by noting that there are no further cases – that is, Q_2 cannot be a node on $\bar{p}$. Suppose for sake of contradiction that it is, and without loss of generality, let Q_2 follow V_i on $\bar{p}$. By the choice of $\bar{p}$ as a shortest path, the path $\bar{r} = \bar{p}(Z, Q_1) \oplus \langle Q_1, Q_2 \rangle \oplus \bar{p}(Q_2, Y)$ in $\mathcal{D}_{\bar{\mathbf{X}}}$ must be blocked given $\mathbf{X} \cup \mathbf{W}$. Since $\bar{p}$ is d-connecting given $\mathbf{X} \cup \mathbf{W}$, $V_i \notin \mathbf{X} \cup \mathbf{W}$, and $\bar{r}$ contains $Q_1 \rightarrow Q_2$, then Q_2 must be a non-collider on $\bar{p}$ and a collider on $\bar{r}$ such that $Q_2 \notin \text{An}(\mathbf{W}, \mathcal{D}_{\bar{\mathbf{X}}})$. But since $V_i \in \text{An}(\mathbf{W}, \mathcal{D}_{\bar{\mathbf{X}}})$ and $V_i \notin \mathbf{W}$, then $Q_2 \in \text{An}(\mathbf{W}, \mathcal{D}_{\bar{\mathbf{X}}})$, which is a contradiction.

Thus, $\mathcal{G}$ does not contain both edges $\langle V_{i-1}, Q_2 \rangle$ and $\langle V_{i+1}, Q_2 \rangle$. Since $\mathcal{G}$ contains $V_{i-1} \rightarrow Q_1 \leftarrow V_{i+1}$, then by Rule 1 of Meek [1995], q^* contains $Q_1 \rightarrow Q_2$. If $m = 2$, then q^* is directed and we are done.

Consider when $m > 2$, and suppose for sake of contradiction that q^* is shielded. That is,

there exists an edge $\langle Q_j, Q_{j+2} \rangle$, $1 \leq j \leq m-2$, in $\mathcal{G}$. Since no node on q^* is in $\mathbf{X}$, then $\mathcal{D}_{\bar{\mathbf{X}}}$ also contains $\langle Q_j, Q_{j+2} \rangle$. Because $\mathcal{D}_{\bar{\mathbf{X}}}$ contains $Q_j \rightarrow Q_{j+1} \rightarrow Q_{j+2}$, then it must contain $Q_j \rightarrow Q_{j+2}$. But then the path $\bar{q}(Q_1, Q_j) \oplus \langle Q_j, Q_{j+2} \rangle \oplus \bar{q}(Q_{j+2}, Q_m)$ contradicts the choice of $\bar{q}$. Thus when $m > 2$, q^* is unshielded and begins with $Q_1 \rightarrow Q_2$. So by Rule 1 of Meek [1995], q^* is again directed and we are done. $\blacksquare$

C.6.3 Results for do Rule 2

Lemma 115 *Let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$, and $\mathbf{W}$ be pairwise disjoint node sets in an MPDAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ and let $\mathcal{D}$ be a DAG in $[\mathcal{G}]$. Additionally, let $\bar{p} = \langle Z = V_1, \dots, V_k = Y \rangle$, $Z \in \mathbf{Z}$, $Y \in \mathbf{Y}$, $k > 1$, be a proper path from $\mathbf{Z}$ to $\mathbf{Y}$ in $\mathcal{D}_{\bar{\mathbf{X}}\mathbf{Z}}$ such that $\bar{p}$ is d-connecting given $\mathbf{X} \cup \mathbf{W}$ and no subsequence of $\bar{p}$ in $\mathcal{D}_{\bar{\mathbf{X}}\mathbf{Z}}$ is d-connecting given $\mathbf{X} \cup \mathbf{W}$. Let p^* be the corresponding path in $\mathcal{G}$.*

- (i) *If $\mathcal{G}$ has no edge $Z - V_i$, $i \in \{2, \dots, k\}$, then the sequence of nodes in $\mathcal{G}_{\bar{\mathbf{X}}\mathbf{Z}}$ corresponding to $\bar{p}$ forms a path where a node is a definite non-collider or collider if and only if it is a non-collider or collider, respectively, on $\bar{p}$.*
- (ii) *If $\mathcal{G}$ has an edge $Z - V_i$, $i \in \{2, \dots, k\}$, let V_i be the closest such node to Y on p^* . Then the sequence of nodes in $\mathcal{G}_{\bar{\mathbf{X}}\mathbf{Z}}$ corresponding to $\langle Z, V_i \rangle \oplus \bar{p}(V_i, Y)$ forms a path where a node is a definite non-collider or collider if and only if it is a non-collider or collider, respectively, on $\bar{p}$.*

Proof of Lemma 115. Let p be the path in $\mathcal{D}$ corresponding to $\bar{p}$. Before proving parts (i) and (ii), we first show that V_1 and $V_3, \dots, V_k$ are nodes of definite status on p^* . This clearly holds for the endpoints V_1 and V_k . Next, pick an arbitrary V_j , $j \in \{3, \dots, k-1\}$, and for sake of contradiction, suppose V_j is not of definite status on p^* . This implies that $\mathcal{G}$ contains an edge $\langle V_{j-1}, V_{j+1} \rangle$. Since $\bar{p}$ is proper and has no nodes in $\mathbf{X}$ (Lemma 110), then $\mathcal{D}_{\bar{\mathbf{X}}\mathbf{Z}}$ must contain $\langle V_{j-1}, V_{j+1} \rangle$.

Define the path $\bar{r} = \bar{p}(Z, V_{j-1}) \oplus \langle V_{j-1}, V_{j+1} \rangle \oplus \bar{p}(V_{j+1}, Y)$ in $\mathcal{D}_{\bar{\mathbf{X}}\mathbf{Z}}$. Note that by the choice of $\bar{p}$, $\bar{r}$ cannot be d-connecting given $\mathbf{X} \cup \mathbf{W}$. But by Lemma 111, any collider on $\bar{r}$

is in $\text{An}(\mathbf{X} \cup \mathbf{W}, \mathcal{D}_{\overline{\mathbf{XZ}}})$. Hence, the only way for $\bar{p}$ to be d-connecting but $\bar{r}$ to be blocked given $\mathbf{X} \cup \mathbf{W}$ is if V_{j-1} or V_{j+1} is a node in $\mathbf{X} \cup \mathbf{W}$, a collider on $\bar{p}$, and a non-collider on $\bar{r}$.

Without loss of generality, pick V_{j-1} . By Lemma 109(i), V_{j-1} is unshielded on $\bar{p}$. Further, since $\bar{p}$ has no nodes in $\mathbf{X}$ and $\mathcal{D}_{\overline{\mathbf{XZ}}}$ contains $V_{j-2} \rightarrow V_{j-1} \leftarrow V_j$, then $V_{j-2}, V_j \notin \mathbf{X} \cup \mathbf{Z}$. Therefore, V_j is also an unshielded collider on p , and by properties of MPDAGs, $\mathcal{G}$ must contain $V_{j-2} \rightarrow V_{j-1} \leftarrow V_j$. But this contradicts that V_j is not of definite status on p^* .

(i) Suppose $\mathcal{G}$ has no edge $Z - V_i$, $i \in \{2, \dots, k\}$, so that p^* cannot start undirected. Further, note that p^* cannot start with $Z \rightarrow V_2$, since this would imply $\mathcal{D}_{\overline{\mathbf{XZ}}}$ contains $Z \rightarrow V_2$, which is a contradiction. Thus, p^* must start with $Z \leftarrow V_2$, which makes V_2 a node of definite status on p^* . Since we have already shown V_1 and $V_3, \dots, V_k$ are nodes of definite status on p^* , we have that p^* is of definite status. The claim follows by Lemma 112.

(ii) Suppose there is an edge $Z - V_i$, $i \in \{2, \dots, k\}$, in $\mathcal{G}$ where V_i is the closest such node to Y on p^* . Then let q^* be the the path $\langle Z, V_i \rangle \oplus p^*(V_i, Y)$ in $\mathcal{G}$. Note that we have already shown $V_{i+1}, \dots, V_k$ are nodes of definite status on p^* . So by Lemma 112, the sequence of nodes in $\mathcal{G}_{\overline{\mathbf{XZ}}}$ corresponding to $\bar{p}(V_i, Y)$ forms a path where a node is a definite non-collider or collider if and only if it is a non-collider or collider, respectively, on $\bar{p}$.

Note that $\mathcal{G}_{\overline{\mathbf{XZ}}}$ also contains $Z - V_i$, and consider the path in $\mathcal{G}_{\overline{\mathbf{XZ}}}$ corresponding to $\langle Z, V_i \rangle \oplus \bar{p}(V_i, Y)$ – call this $\bar{q}^*$. We have shown the claim holds for $\bar{q}^*(V_i, Y)$. Then since V_1 is an endpoint on $\bar{q}^*$, it remains to consider the status of V_i on $\bar{q}^*$.

For sake of contradiction, suppose that V_i is not of definite status on $\bar{q}^*$. Then note that V_i cannot be of definite status on q^* , since removing edges from $\mathcal{G}$ outside of q^* cannot change a node of definite status of q^* . It follows that $\mathcal{G}$ must contain an edge $\langle Z, V_{i+1} \rangle$, which by choice of V_i , must be directed.

Consider when $\mathcal{G}$ contains $Z \rightarrow V_{i+1}$ so that $\mathcal{G}_{\overline{\mathbf{XZ}}}$ contains $Z - V_i$ but not the edge $\langle Z, V_{i+1} \rangle$. Since V_i is not of definite status on $\bar{q}^*$, then $\mathcal{G}_{\overline{\mathbf{XZ}}}$ cannot contain $Z - V_i - V_{i+1}$ or $Z - V_i \rightarrow V_{i+1}$. Thus, $\mathcal{G}_{\overline{\mathbf{XZ}}}$ contains $Z - V_i \leftarrow V_{i+1}$. But this implies that $\mathcal{G}$ contains $Z - V_i \leftarrow V_{i+1}$ and $Z \rightarrow V_{i+1}$, which contradicts Rule 2 of Meek [1995].

Consider instead when $\mathcal{G}$ contains $Z \leftarrow V_{i+1}$. Then $\mathcal{D}_{\overline{\mathbf{X}}\mathbf{Z}}$ also contains $Z \leftarrow V_{i+1}$, since $Z \notin \mathbf{X}$ and $\bar{p}$ is proper. By choice of $\bar{p}$, the path $\bar{t} = \langle Z, V_{i+1} \rangle \oplus \bar{p}(V_{i+1}, Y)$ in $\mathcal{D}_{\overline{\mathbf{X}}\mathbf{Z}}$ must be blocked by $\mathbf{X} \cup \mathbf{W}$. Since $\bar{p}$ is d-connecting given $\mathbf{X} \cup \mathbf{W}$ and V_{i+1} is a non-collider on $\bar{t}$, then V_{i+1} must be a node in $\mathbf{X} \cup \mathbf{W}$ and a collider on $\bar{p}$. By Lemma 109(i), V_{i+1} must be an unshielded collider on $\bar{p}$. Since $\bar{p}$ has no nodes in $\mathbf{X}$ (Lemma 110) and $\mathcal{D}_{\overline{\mathbf{X}}\mathbf{Z}}$ contains $V_i \rightarrow V_{i+1} \leftarrow V_{i+2}$, then $V_i, V_{i+2} \notin \mathbf{X} \cup \mathbf{Z}$. Therefore, V_{i+1} is also an unshielded collider on p , and by properties of MPDAGs, $\mathcal{G}$ must contain $V_i \rightarrow V_{i+1} \leftarrow V_{i+2}$. But this contradicts that V_i is not of definite status on q^* .

Thus, V_i is of definite status on $\bar{q}^*$. Finally, we show that V_i is a definite non-collider on both $\bar{q}^*$ and $\bar{p}$. Since $\bar{q}^*$ starts undirected and V_i is of definite status on $\bar{q}^*$, then V_i is a definite non-collider on $\bar{q}^*$. For sake of contradiction, suppose that V_i is a collider on $\bar{p}$. By Lemma 109(i), V_i must be an unshielded collider on $\bar{p}$. Since $\bar{p}$ has no nodes in $\mathbf{X}$ and $\mathcal{D}_{\overline{\mathbf{X}}\mathbf{Z}}$ contains $V_{i-1} \rightarrow V_i \leftarrow V_{i+1}$, then $V_{i-1}, V_{i+1} \notin \mathbf{X} \cup \mathbf{Z}$. Therefore, V_i is also an unshielded collider on p , and by properties of MPDAGs, $\mathcal{G}$ – and therefore $\mathcal{G}_{\overline{\mathbf{X}}\mathbf{Z}}$ – must contain $V_{i-1} \rightarrow V_i \leftarrow V_{i+1}$. But then $\mathcal{G}_{\overline{\mathbf{X}}\mathbf{Z}}$ contains $Z - V_i \leftarrow V_{i+1}$, which contradicts that V_i is of definite status on $\bar{q}^*$. ■

Lemma 116 *Let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$, and $\mathbf{W}$ be pairwise disjoint node sets in an MPDAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$. If $(\mathbf{Z} \perp_d \mathbf{Y} | \mathbf{X}, \mathbf{W})_{\mathcal{G}_{\overline{\mathbf{X}}\mathbf{Z}}}$, then $(\mathbf{Z} \perp_d \mathbf{Y} | \mathbf{X}, \mathbf{W})_{\mathcal{D}_{\overline{\mathbf{X}}\mathbf{Z}}}$ for all DAGs $\mathcal{D} \in [\mathcal{G}]$.*

Proof of Lemma 116. We prove the contrapositive. Suppose there exists a DAG $\mathcal{D} \in [\mathcal{G}]$ such that $\mathcal{D}_{\overline{\mathbf{X}}\mathbf{Z}}$ contains a path from $\mathbf{Z}$ to $\mathbf{Y}$ that is d-connecting given $\mathbf{X} \cup \mathbf{W}$. Let $\mathbf{S}$ be the set of all shortest such paths, and let $\bar{p} = \langle Z = V_1, \dots, V_k = Y \rangle$, $Z \in \mathbf{Z}$, $Y \in \mathbf{Y}$, $k > 1$, be a path in $\mathbf{S}$ with the shortest distance to $\mathbf{X} \cup \mathbf{W}$ (Definition 74).

Note that by definition, $\bar{p}$ is proper, $\bar{p}$ is d-connecting given $\mathbf{X} \cup \mathbf{W}$, and no subsequence of $\bar{p}$ in $\mathcal{D}_{\overline{\mathbf{X}}\mathbf{Z}}$ is d-connecting given $\mathbf{X} \cup \mathbf{W}$. Thus by Lemma 115, $\mathcal{G}_{\overline{\mathbf{X}}\mathbf{Z}}$ contains a definite status path from $\mathbf{Z}$ to $\mathbf{Y}$ where a node is a definite non-collider or collider if and only if it is a non-collider or collider, respectively, on $\bar{p}$. To complete the proof, we want to show that this path is d-connecting given $\mathbf{X} \cup \mathbf{W}$. To do this, we only need to show the following.

- (i) No non-collider on $\bar{p}$ is in $\mathbf{X} \cup \mathbf{W}$.
- (ii) Every collider on $\bar{p}$ is in $\text{An}(\mathbf{X} \cup \mathbf{W}, \mathcal{G}_{\bar{\mathbf{XZ}}})$.

Claim (i) holds since $\bar{p}$ is d-connecting given $\mathbf{X} \cup \mathbf{W}$. The remainder of the proof shows that claim (ii) holds.

Let V_i , $i \in \{2, \dots, k-1\}$, be an arbitrary collider on $\bar{p}$. Since $\bar{p}$ is d-connecting given $\mathbf{X} \cup \mathbf{W}$ in $\mathcal{D}_{\bar{\mathbf{XZ}}}$, then $V_i \in \text{An}(\mathbf{W}, \mathcal{D}_{\bar{\mathbf{XZ}}})$. When $V_i \in \mathbf{W}$, then $V_i \in \text{An}(\mathbf{X} \cup \mathbf{W}, \mathcal{G}_{\bar{\mathbf{XZ}}})$ and we are done.

When $V_i \notin \mathbf{W}$, then let $\bar{q} = \langle V_i = Q_1, \dots, Q_m = W \rangle$, $W \in \mathbf{W}$, $m > 1$, be a shortest directed path in $\mathcal{D}_{\bar{\mathbf{XZ}}}$ from V_i to $\mathbf{W}$, and let q^* be the path in $\mathcal{G}$ corresponding to $\bar{q}$. Note that no node on q^* is in $\mathbf{X} \cup \mathbf{Z}$, since $\mathcal{D}_{\bar{\mathbf{XZ}}}$ contains $V_{i-1} \rightarrow Q_1 \rightarrow \dots \rightarrow Q_m$ and $Q_m \in \mathbf{W}$. Thus if q^* is directed, then the corresponding sequence of nodes in $\mathcal{G}_{\bar{\mathbf{XZ}}}$ will also form a directed path, which again implies that $V_i \in \text{An}(\mathbf{X} \cup \mathbf{W}, \mathcal{G}_{\bar{\mathbf{XZ}}})$ and the proof is complete. We show below that q^* is directed.

We start by showing that q^* contains $Q_1 \rightarrow Q_2$. Note that V_i is an unshielded collider on $\bar{p}$ by Lemma 109(i). Since $\bar{p}$ has no nodes in $\mathbf{X}$ (Lemma 110) and $\mathcal{D}_{\bar{\mathbf{XZ}}}$ contains $V_{i-1} \rightarrow V_i \leftarrow V_{i+1}$, then $V_{i-1}, V_{i+1} \notin \mathbf{X} \cup \mathbf{Z}$. Therefore, V_i is also an unshielded collider on the path in $\mathcal{D}$ corresponding to $\bar{p}$, and by properties of MPDAGs, $\mathcal{G}$ contains $V_{i-1} \rightarrow Q_1 \leftarrow V_{i+1}$.

For sake of contradiction, suppose that $\mathcal{G}$ – and therefore $\mathcal{D}$ – contains the edges $\langle V_{i-1}, Q_2 \rangle$ and $\langle V_{i+1}, Q_2 \rangle$. Since $\mathcal{D}$ must contain $V_{i-1} \rightarrow Q_1 \rightarrow Q_2$ and $V_{i+1} \rightarrow Q_1 \rightarrow Q_2$, it must contain $V_{i-1} \rightarrow Q_2 \leftarrow V_{i+1}$. And since no node on q^* is in $\mathbf{X}$ and $V_{i-1}, V_{i+1} \notin \mathbf{Z}$, then $\mathcal{D}_{\bar{\mathbf{XZ}}}$ must also contain $V_{i-1} \rightarrow Q_2 \leftarrow V_{i+1}$. When Q_2 is not on $\bar{p}$, consider the path $\bar{p}(Z, V_{i-1}) \oplus \langle V_{i-1}, Q_2, V_{i+1} \rangle \oplus \bar{p}(V_{i+1}, Y)$ in $\mathcal{D}_{\bar{\mathbf{XZ}}}$. Note that this is a path from $\mathbf{Z}$ to $\mathbf{Y}$ that is d-connecting given $\mathbf{X} \cup \mathbf{W}$, is the same length as $\bar{p}$, and has a shorter distance than $\bar{p}$ to $\mathbf{X} \cup \mathbf{W}$, which contradicts the choice of $\bar{p}$.

We complete the contradiction by noting that there are no further cases – that is, Q_2 cannot be a node on $\bar{p}$. Suppose for sake of contradiction that it is, and without loss of generality, let Q_2 follow V_i on $\bar{p}$. By the choice of $\bar{p}$ as a shortest path, the path $\bar{r} =$

$\bar{p}(Z, Q_1) \oplus \langle Q_1, Q_2 \rangle \oplus \bar{p}(Q_2, Y)$ in $\mathcal{D}_{\bar{\mathbf{X}}\mathbf{Z}}$ must be blocked given $\mathbf{X} \cup \mathbf{W}$. Since $\bar{p}$ is d-connecting given $\mathbf{X} \cup \mathbf{W}$, $V_i \notin \mathbf{X} \cup \mathbf{W}$, and $\bar{r}$ contains $Q_1 \rightarrow Q_2$, then Q_2 must be a non-collider on $\bar{p}$ and a collider on $\bar{r}$ such that $Q_2 \notin \text{An}(\mathbf{W}, \mathcal{D}_{\bar{\mathbf{X}}\mathbf{Z}})$. But since $V_i \in \text{An}(\mathbf{W}, \mathcal{D}_{\bar{\mathbf{X}}\mathbf{Z}})$ and $V_i \notin \mathbf{W}$, then $Q_2 \in \text{An}(\mathbf{W}, \mathcal{D}_{\bar{\mathbf{X}}\mathbf{Z}})$, which is a contradiction.

Thus, $\mathcal{G}$ does not contain both edges $\langle V_{i-1}, Q_2 \rangle$ and $\langle V_{i+1}, Q_2 \rangle$. Since $\mathcal{G}$ contains $V_{i-1} \rightarrow Q_1 \leftarrow V_{i+1}$, then by Rule 1 of Meek [1995], q^* contains $Q_1 \rightarrow Q_2$. If $m = 2$, then q^* is directed and we are done.

Consider when $m > 2$, and suppose for sake of contradiction that q^* is shielded. That is, there exists an edge $\langle Q_j, Q_{j+2} \rangle$, $1 \leq j \leq m - 2$, in $\mathcal{G}$. Since no node on q^* is in $\mathbf{X} \cup \mathbf{Z}$, then $\mathcal{D}_{\bar{\mathbf{X}}\mathbf{Z}}$ also contains $\langle Q_j, Q_{j+2} \rangle$. Because $\mathcal{D}_{\bar{\mathbf{X}}\mathbf{Z}}$ contains $Q_j \rightarrow Q_{j+1} \rightarrow Q_{j+2}$, then it must contain $Q_j \rightarrow Q_{j+2}$. But then the path $\bar{q}(Q_1, Q_j) \oplus \langle Q_j, Q_{j+2} \rangle \oplus \bar{q}(Q_{j+2}, Q_m)$ contradicts the choice of $\bar{q}$. Thus when $m > 2$, q^* is unshielded and begins with $Q_1 \rightarrow Q_2$. So by Rule 1 of Meek [1995], q^* is again directed and we are done. ■

C.6.4 Results for do Rule 3

Lemma 117 *Let $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$, and $\mathbf{W}$ be pairwise disjoint node sets in an MPDAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ and let $\mathcal{D}$ be a DAG in $[\mathcal{G}]$. Define $\mathbf{Z}(\mathbf{W}) = \mathbf{Z} \setminus \text{An}(\mathbf{W}, \mathcal{D}_{\bar{\mathbf{X}}})$ and $\mathbf{Z}'(\mathbf{W}) = \mathbf{Z} \setminus \text{PossAn}(\mathbf{W}, \mathcal{G}_{\mathbf{V} \setminus \mathbf{X}})$. Then $\mathbf{Z}'(\mathbf{W}) \subseteq \mathbf{Z}(\mathbf{W})$.*

Proof of Lemma 117. Note that we can write $\mathbf{Z}(\mathbf{W}) = \mathbf{Z} \setminus [\text{An}(\mathbf{W}, \mathcal{D}_{\bar{\mathbf{X}}}) \setminus \mathbf{X}]$. Thus to prove this claim, we show that $\text{An}(\mathbf{W}, \mathcal{D}_{\bar{\mathbf{X}}}) \setminus \mathbf{X} \subseteq \text{PossAn}(\mathbf{W}, \mathcal{G}_{\mathbf{V} \setminus \mathbf{X}})$.

Pick any $A \in \text{An}(\mathbf{W}, \mathcal{D}_{\bar{\mathbf{X}}}) \setminus \mathbf{X}$ and let $\bar{p}$ be an arbitrary causal path in $\mathcal{D}_{\bar{\mathbf{X}}}$ from A to $\mathbf{W}$. Since the path in $\mathcal{D}$ corresponding to $\bar{p}$ must be causal, then by Lemma 83, the corresponding path in $\mathcal{G}$ must be possibly causal. Note that no node on $\bar{p}$ is in $\mathbf{X}$. Thus, the sequence of nodes in $\mathcal{G}_{\mathbf{V} \setminus \mathbf{X}}$ corresponding to $\bar{p}$ forms a path. Further, since removing edges from $\mathcal{G}$ that are outside of a possibly causal path cannot make that path non-causal, then the path in $\mathcal{G}_{\mathbf{V} \setminus \mathbf{X}}$ corresponding to $\bar{p}$ must be possibly causal. Therefore, $A \in \text{PossAn}(\mathbf{W}, \mathcal{G}_{\mathbf{V} \setminus \mathbf{X}})$. ■

Lemma 118 *Let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$, and $\mathbf{W}$ be pairwise disjoint node sets in an MPDAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ and let $\mathcal{D}$ be a DAG in $[\mathcal{G}]$. Define $\mathbf{Z}(\mathbf{W}) = \mathbf{Z} \setminus \text{An}(\mathbf{W}, \mathcal{D}_{\overline{\mathbf{X}}})$. Then let $\bar{p}$ be a proper path from $\mathbf{Z}$ to $\mathbf{Y}$ in $\mathcal{D}_{\overline{\mathbf{XZ}(\mathbf{W})}}$ such that $\bar{p}$ is d-connecting given $\mathbf{X} \cup \mathbf{W}$ but no subsequence of $\bar{p}$ in $\mathcal{D}_{\overline{\mathbf{XZ}(\mathbf{W})}}$ is d-connecting given $\mathbf{X} \cup \mathbf{W}$. Define $\mathbf{Z}'(\mathbf{W}) = \mathbf{Z} \setminus \text{PossAn}(\mathbf{W}, \mathcal{G}_{\mathbf{V} \setminus \mathbf{X}})$. Then the sequence of nodes in $\mathcal{G}_{\overline{\mathbf{XZ}'(\mathbf{W})}}$ corresponding to $\bar{p}$ forms a path where a node is a definite non-collider or collider if and only if it is a non-collider or collider, respectively, on $\bar{p}$.*

Proof of Lemma 118. Let p be the path in $\mathcal{D}$ corresponding to $\bar{p}$. If p satisfies the assumptions of Lemma 109, then the path in $\mathcal{G}$ corresponding to $\bar{p}$ is of definite status, and the claim follows by Lemmas 117 and 112. We show these assumptions hold below.

First note that p must be d-connecting given $\mathbf{X} \cup \mathbf{W}$, since $\bar{p}$ is d-connecting given $\mathbf{X} \cup \mathbf{W}$ and adding edges to a graph cannot block existing d-connecting paths. Next, for sake of contradiction, suppose that there is a subsequence of p in $\mathcal{D}$ – call it q – that is d-connecting given $\mathbf{X} \cup \mathbf{W}$.

Consider when the sequence of nodes in $\mathcal{D}_{\overline{\mathbf{XZ}(\mathbf{W})}}$ corresponding to q forms a path – call it $\bar{q}$. Since q is d-connecting given $\mathbf{X} \cup \mathbf{W}$, none of the non-colliders on q – and therefore on $\bar{q}$ – are in $\mathbf{X} \cup \mathbf{W}$. Further, since $\bar{p}$ is d-connecting given $\mathbf{X} \cup \mathbf{W}$, then by Lemma 111, all colliders on $\bar{q}$ are in $\text{An}(\mathbf{X} \cup \mathbf{W}, \mathcal{D}_{\overline{\mathbf{XZ}(\mathbf{W})}})$. But this implies that $\bar{q}$, which is a subsequence of $\bar{p}$, is d-connecting given $\mathbf{X} \cup \mathbf{W}$, which contradicts the choice of $\bar{p}$.

Consider instead when the sequence of nodes in $\mathcal{D}_{\overline{\mathbf{XZ}(\mathbf{W})}}$ corresponding to q does not form a path. This implies that q must contain an edge into $\mathbf{X} \cup \mathbf{Z}(\mathbf{W})$. By Lemma 110, $\bar{p}$ – and therefore q – is proper and has no nodes in $\mathbf{X}$. Thus, q must start with an edge into Z and Z must be a node in $\mathbf{Z}(\mathbf{W})$. But since $\bar{p}$ is a path in $\mathcal{D}_{\overline{\mathbf{XZ}(\mathbf{W})}}$, then $\bar{p}$ – and therefore p – must start with an edge out of Z . Let Q_2 and V_2 be the distinct second nodes on q and p , respectively, so that $\mathcal{D}$ contains $Z \leftarrow Q_2$ and $Z \rightarrow V_2 \dots Q_2$. By the acyclicity of $\mathcal{D}$, there must be a collider on $p(Z, Q_2)$ – and therefore on $\bar{p}(Z, Q_2)$.

Let C be the closest collider to Z on $\bar{p}$. Since $\bar{p}$ is d-connecting given $\mathbf{X} \cup \mathbf{W}$ in $\mathcal{D}_{\overline{\mathbf{XZ}(\mathbf{W})}}$, then $C \in \text{An}(\mathbf{W}, \mathcal{D}_{\overline{\mathbf{XZ}(\mathbf{W})}})$. Further, since $\mathcal{D}$ contains $Z \rightarrow \dots \rightarrow C$, then

$Z \in \text{An}(\mathbf{W}, \mathcal{D}_{\overline{\mathbf{XZ}(\mathbf{W})}}) \subseteq \text{An}(\mathbf{W}, \mathcal{D}_{\overline{\mathbf{X}}})$, which contradicts that $Z \in \mathbf{Z}(\mathbf{W})$. $\blacksquare$

Lemma 119 *Let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$, and $\mathbf{W}$ be pairwise disjoint node sets in an MPDAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$. If $(\mathbf{Z} \perp_d \mathbf{Y} | \mathbf{X}, \mathbf{W})_{\mathcal{G}_{\overline{\mathbf{XZ}'(\mathbf{W})}}}$, then $(\mathbf{Z} \perp_d \mathbf{Y} | \mathbf{X}, \mathbf{W})_{\mathcal{D}_{\overline{\mathbf{XZ}(\mathbf{W})}}}$ for all DAGs $\mathcal{D} \in [\mathcal{G}]$, where we define $\mathbf{Z}'(\mathbf{W}) = \mathbf{Z} \setminus \text{PossAn}(\mathbf{W}, \mathcal{G}_{\mathbf{V} \setminus \mathbf{X}})$ and $\mathbf{Z}(\mathbf{W}) = \mathbf{Z} \setminus \text{An}(\mathbf{W}, \mathcal{D}_{\overline{\mathbf{X}}})$.*

Proof of Lemma 119. We prove the contrapositive. Suppose there exists a DAG $\mathcal{D} \in [\mathcal{G}]$ such that $\mathcal{D}_{\overline{\mathbf{XZ}(\mathbf{W})}}$ contains a path from $\mathbf{Z}$ to $\mathbf{Y}$ that is d-connecting given $\mathbf{X} \cup \mathbf{W}$. Let $\mathbf{S}$ be the set of all shortest such paths, and let $\bar{p} = \langle Z = V_1, \dots, V_k = Y \rangle$, $Z \in \mathbf{Z}$, $Y \in \mathbf{Y}$, $k > 1$, be a path in $\mathbf{S}$ with the shortest distance to $\mathbf{X} \cup \mathbf{W}$ (Definition 74).

To complete the proof, we want to show that the sequence of nodes in $\mathcal{G}_{\overline{\mathbf{XZ}'(\mathbf{W})}}$ corresponding to $\bar{p}$ forms a path that is d-connecting given $\mathbf{X} \cup \mathbf{W}$. Note that by definition, $\bar{p}$ is proper, $\bar{p}$ is d-connecting given $\mathbf{X} \cup \mathbf{W}$, and no subsequence of $\bar{p}$ in $\mathcal{D}_{\overline{\mathbf{XZ}(\mathbf{W})}}$ is d-connecting given $\mathbf{X} \cup \mathbf{W}$. Thus by Lemma 118, we only need to show the following.

- (i) No non-collider on $\bar{p}$ is in $\mathbf{X} \cup \mathbf{W}$.
- (ii) Every collider on $\bar{p}$ is in $\text{An}(\mathbf{X} \cup \mathbf{W}, \mathcal{G}_{\overline{\mathbf{XZ}'(\mathbf{W})}})$.

Claim (i) holds since $\bar{p}$ is d-connecting given $\mathbf{X} \cup \mathbf{W}$. The remainder of the proof shows that claim (ii) holds.

Let $V_i, i \in \{2, \dots, k-1\}$, be an arbitrary collider on $\bar{p}$. Since $\bar{p}$ is d-connecting given $\mathbf{X} \cup \mathbf{W}$ in $\mathcal{D}_{\overline{\mathbf{XZ}(\mathbf{W})}}$, then $V_i \in \text{An}(\mathbf{W}, \mathcal{D}_{\overline{\mathbf{XZ}(\mathbf{W})}})$. When $V_i \in \mathbf{W}$, then $V_i \in \text{An}(\mathbf{X} \cup \mathbf{W}, \mathcal{G}_{\overline{\mathbf{XZ}'(\mathbf{W})}})$ and we are done.

When $V_i \notin \mathbf{W}$, then let $\bar{q} = \langle V_i = Q_1, \dots, Q_m = W \rangle, W \in \mathbf{W}, m > 1$, be a shortest directed path in $\mathcal{D}_{\overline{\mathbf{XZ}(\mathbf{W})}}$ from V_i to $\mathbf{W}$, and let q^* be the path in $\mathcal{G}$ corresponding to $\bar{q}$. Note that no node on q^* is in $\mathbf{X} \cup \mathbf{Z}(\mathbf{W})$, since $\mathcal{D}_{\overline{\mathbf{XZ}(\mathbf{W})}}$ contains $V_{i-1} \rightarrow Q_1 \rightarrow \dots \rightarrow Q_m$. By Lemma 117, this implies that no node on q^* is in $\mathbf{X} \cup \mathbf{Z}'(\mathbf{W})$. Thus, if q^* is directed, then the corresponding sequence of nodes in $\mathcal{G}_{\overline{\mathbf{XZ}'(\mathbf{W})}}$ will also form a directed path, which again implies that $V_i \in \text{An}(\mathbf{X} \cup \mathbf{W}, \mathcal{G}_{\overline{\mathbf{XZ}'(\mathbf{W})}})$ and the proof is complete. We show below that q^* is directed.

We start by showing that q^* contains $Q_1 \rightarrow Q_2$. Note that by Lemma 118, $\mathcal{G}_{\overline{\mathbf{XZ}(\mathbf{W})}}$ – and therefore $\mathcal{G}$ – contains $V_{i-1} \rightarrow Q_1 \leftarrow V_{i+1}$. For sake of contradiction, suppose that $\mathcal{G}$ – and therefore $\mathcal{D}$ – contains the edges $\langle V_{i-1}, Q_2 \rangle$ and $\langle V_{i+1}, Q_2 \rangle$. Since $\mathcal{D}$ must contain $V_{i-1} \rightarrow Q_1 \rightarrow Q_2$ and $V_{i+1} \rightarrow Q_1 \rightarrow Q_2$, it must contain $V_{i-1} \rightarrow Q_2 \leftarrow V_{i+1}$. And since no node on q^* is in $\mathbf{X} \cup \mathbf{Z}(\mathbf{W})$, then $\mathcal{D}_{\overline{\mathbf{XZ}(\mathbf{W})}}$ must also contain $V_{i-1} \rightarrow Q_2 \leftarrow V_{i+1}$.

When Q_2 is not on $\bar{p}$, consider the path $\bar{p}(Z, V_{i-1}) \oplus \langle V_{i-1}, Q_2, V_{i+1} \rangle \oplus \bar{p}(V_{i+1}, Y)$ in $\mathcal{D}_{\overline{\mathbf{XZ}(\mathbf{W})}}$. Note that this is a path from $\mathbf{Z}$ to $\mathbf{Y}$ that is d-connecting given $\mathbf{X} \cup \mathbf{W}$, is the same length as $\bar{p}$, and has a shorter distance than $\bar{p}$ to $\mathbf{X} \cup \mathbf{W}$, which contradicts the choice of $\bar{p}$.

We complete the contradiction by noting that there are no further cases – that is, Q_2 cannot be a node on $\bar{p}$. Suppose for sake of contradiction that it is, and without loss of generality, let Q_2 follow V_i on $\bar{p}$. By the choice of $\bar{p}$ as a shortest path, the path $\bar{r} = \bar{p}(Z, Q_1) \oplus \langle Q_1, Q_2 \rangle \oplus \bar{p}(Q_2, Y)$ in $\mathcal{D}_{\overline{\mathbf{XZ}(\mathbf{W})}}$ must be blocked given $\mathbf{X} \cup \mathbf{W}$. Since $\bar{p}$ is d-connecting given $\mathbf{X} \cup \mathbf{W}$, $V_i \notin \mathbf{X} \cup \mathbf{W}$, and $\bar{r}$ contains $Q_1 \rightarrow Q_2$, then Q_2 must be a non-collider on $\bar{p}$ and a collider on $\bar{r}$ such that $Q_2 \notin \text{An}(\mathbf{W}, \mathcal{D}_{\overline{\mathbf{XZ}(\mathbf{W})}})$. But since $V_i \in \text{An}(\mathbf{W}, \mathcal{D}_{\overline{\mathbf{XZ}(\mathbf{W})}})$ and $V_i \notin \mathbf{W}$, then $Q_2 \in \text{An}(\mathbf{W}, \mathcal{D}_{\overline{\mathbf{XZ}(\mathbf{W})}})$, which is a contradiction.

Thus, $\mathcal{G}$ does not contain both edges $\langle V_{i-1}, Q_2 \rangle$ and $\langle V_{i+1}, Q_2 \rangle$. Since $\mathcal{G}$ contains $V_{i-1} \rightarrow Q_1 \leftarrow V_{i+1}$, then by Rule 1 of Meek [1995], q^* contains $Q_1 \rightarrow Q_2$. If $m = 2$, then q^* is directed and we are done.

Consider when $m > 2$, and suppose for sake of contradiction that q^* is shielded. That is, there exists an edge $\langle Q_j, Q_{j+2} \rangle$, $1 \leq j \leq m - 2$, in $\mathcal{G}$. Since no node on q^* is in $\mathbf{X} \cup \mathbf{Z}(\mathbf{W})$, then $\mathcal{D}_{\overline{\mathbf{XZ}(\mathbf{W})}}$ also contains $\langle Q_j, Q_{j+2} \rangle$. Because $\mathcal{D}_{\overline{\mathbf{XZ}(\mathbf{W})}}$ contains $Q_j \rightarrow Q_{j+1} \rightarrow Q_{j+2}$, then it must contain $Q_j \rightarrow Q_{j+2}$. But then the path $\bar{q}(Q_1, Q_j) \oplus \langle Q_j, Q_{j+2} \rangle \oplus \bar{q}(Q_{j+2}, Q_m)$ contradicts the choice of $\bar{q}$. Thus when $m > 2$, q^* is unshielded and begins with $Q_1 \rightarrow Q_2$. So by Rule 1 of Meek [1995], q^* is again directed and we are done. $\blacksquare$

C.7 PROOFS FOR SECTION 4.5.4: IDENTIFICATION ALGORITHM

This section includes the proof of Theorem 25 found in Section 4.5.4. Four supporting results needed for this proof follow.

C.7.1 Main Result

Proof of Theorem 25. Suppose there is a path in $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ from $X \in \mathbf{X}$ to $\mathbf{Y} \cup \mathbf{Z}$ that does not contain $\mathbf{X} \setminus \{X\}$, is possibly causal, starts undirected – and where $\mathbf{Y} \not\perp_d X \mid \mathbf{X} \setminus \{X\}, \mathbf{Z}$ in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}} \underline{\mathbf{X}}}$. We want to show that the conditional effect of $\mathbf{X}$ on $\mathbf{Y}$ given $\mathbf{Z}$ is not identifiable in $\mathcal{G}$. Since $\mathbf{Y} \not\perp_d X \mid \mathbf{X} \setminus \{X\}, \mathbf{Z}$ in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}} \underline{\mathbf{X}}}$, the following sets must be non-empty:

$$\mathbf{S}_1 = \left\{ \begin{array}{l} \text{definite status paths in } \mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}} \underline{\mathbf{X}}} \text{ from } X \text{ to } \mathbf{Y} \text{ that are} \\ \text{d-connecting given } \mathbf{X} \setminus \{X\} \cup \mathbf{Z} \end{array} \right\} \text{ and}$$

$$\mathbf{S}_2 = \{ \text{paths in } \mathbf{S}_1 \text{ with the fewest colliders} \}.$$

Then consider the following set:

$$\mathbf{S}_3 = \{ \text{paths in } \mathbf{S}_2 \text{ that start undirected} \}.$$

We divide the remainder of the proof into two cases based on whether $\mathbf{S}_3 = \emptyset$. In both cases, we consider DAGs $\mathcal{D}^1, \mathcal{D}^2 \in [\mathcal{G}]$ with paths in the sets above. To show the conditional effect is not identifiable in $\mathcal{G}$, it suffices to show that each $\mathcal{D}^i$ is compatible with a family of interventional densities $\mathbf{F}_1^* = \{f_i(\mathbf{v} \mid do(\mathbf{x}')) : \mathbf{X}' \subseteq \mathbf{V}\}$ such that the following hold.

$$f_1(\mathbf{v}) = f_2(\mathbf{v}). \tag{C.20}$$

$$f_1(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z}) \neq f_2(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z}). \tag{C.21}$$

Before proceeding to cases, we briefly show that all paths in $\mathbf{S}_1$ are proper paths from $\mathbf{X}$ to $\mathbf{Y}$ that are open given $\mathbf{Z}$. To see this, note that since every path in $\mathbf{S}_1$ is of definite status and open given $\mathbf{X} \setminus \{X\} \cup \mathbf{Z}$, it cannot have definite non-colliders in $\mathbf{X} \setminus \{X\} \cup \mathbf{Z}$. Further, since there are no edges into $\mathbf{X} \setminus \{X\}$ in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}} \underline{\mathbf{X}}}$, all of the colliders on any path in $\mathbf{S}_1$ cannot be in $\mathbf{X} \setminus \{X\}$ and must have descendants in $\mathbf{Z}$.

CASE 1: Consider when $\mathbf{S}_3 \neq \emptyset$, and define the following sets.

$$\mathbf{S}_4 = \{ \text{paths in } \mathbf{S}_3 \text{ with the fewest edges} \}.$$

$$\mathbf{S}_5 = \{\text{paths in } \mathbf{S}_4 \text{ with a shortest distance to } \mathbf{Z} \text{ in } \mathcal{G}_{\overline{\mathbf{X} \setminus \{\mathbf{X}\}} \underline{\mathbf{X}}}\}.$$

Pick an arbitrary path $\bar{p} := \langle X = P_1, \dots, P_k = Y \rangle$, $k > 1$, $Y \in \mathbf{Y}$, in $\mathbf{S}_5 \subseteq \mathbf{S}_1$. Note that the sequence of nodes in $\mathcal{G}$ corresponding to $\bar{p}$ in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{\mathbf{X}\}} \underline{\mathbf{X}}}$ forms a path p in $\mathcal{G}$ that contains the same edge orientations as $\bar{p}$. Then note that since $\bar{p} \in \mathbf{S}_1$, we have shown that $\bar{p} -$ and therefore $p -$ is a proper path from $\mathbf{X}$ to $\mathbf{Y}$. Further, both paths start undirected, since $\bar{p} \in \mathbf{S}_3$.

Consider first when p is possibly causal so that $p -$ and therefore $\bar{p} -$ has no colliders. Since $\bar{p} \in \mathbf{S}_1$ is of definite status and we have shown that paths in $\mathbf{S}_1$ are open given $\mathbf{Z}$, then $\bar{p} -$ and therefore $p -$ does not contain nodes in $\mathbf{Z}$. Thus, p is a proper, possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$ that starts undirected and does not contain nodes in $\mathbf{Z}$. It follows by Proposition 106 that the conditional effect of $\mathbf{X}$ on $\mathbf{Y}$ given $\mathbf{Z}$ is not identifiable in $\mathcal{G}$, and we are done.

Hence, we consider when p is not possibly causal for the remainder of this case. For sake of contradiction, suppose $\bar{p}$ does not contain colliders. Since $\bar{p}$ is of definite status and starts undirected, then $\bar{p} -$ and therefore $p -$ cannot contain an edge $P_i \leftarrow P_{i+1}$, $i \in \{1, \dots, k-1\}$. Thus by Lemma 84, p is not of definite status. The only way for $\bar{p}$ to be of definite status while p is not is for the paths to contain at least one subpath $P_i - P_{i+1} - P_{i+2}$ that is unshielded in $\bar{p}$ but shielded in p . This forces $i = 1$, since $\bar{p}$ is a proper path in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{\mathbf{X}\}} \underline{\mathbf{X}}}$. That is, $\bar{p}$ begins $X - P_2 - P_3$ and $\mathcal{G}$ contains $X \rightarrow P_3$. It follows that $p(P_2, P_k)$ is of definite status. Since $p -$ and therefore $p(P_2, P_k) -$ does not contain an edge $P_i \leftarrow P_{i+1}$, then by Lemma 84, $p(P_2, P_k)$ is possibly causal. However, p is not possibly causal, so $\mathcal{G}$ must contain an edge $X \leftarrow P_j$, $j \in \{4, \dots, k\}$, so that $\mathcal{G}$ contains $P_3 \leftarrow X \leftarrow P_j$. But this contradicts that $p(P_2, P_k)$ is possibly causal (Lemma 90).

Thus, $\bar{p}$ has at least one collider. Let $\mathbf{C} = \{C_1, \dots, C_m\}$, $1 \leq m \leq k-3$, be the set of all colliders on $\bar{p}$ in order of their appearance on $\bar{p}$. Since we have shown that paths in $\mathbf{S}_1$ are open given $\mathbf{Z}$, every collider in $\mathbf{C}$ is in $\text{An}(\mathbf{Z}, \mathcal{G}_{\overline{\mathbf{X} \setminus \{\mathbf{X}\}} \underline{\mathbf{X}}})$. Then for each $i \in \{1, \dots, m\}$, let $\bar{q}_i = \langle C_i = Q_{i_1}, \dots, Q_{i_{k_i}} = Z_i \rangle$ be a shortest causal path from $C_i \in \mathbf{C}$ to $\mathbf{Z}$ in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{\mathbf{X}\}} \underline{\mathbf{X}}}$.

Using these paths, we turn to the task of finding DAGs in $[\mathcal{G}]$ that are compatible with families of interventional densities such that (C.20)-(C.21) hold. To do this, we consider the subcases below.

- **SUBCASE 1:** Let p be of definite status. Consider any two DAGs $\mathcal{D}^1, \mathcal{D}^2 \in [\mathcal{G}]$ that include the edges $X \rightarrow P_2$ and $X \leftarrow P_2$, respectively. Then construct $\mathcal{D}^{1'}$ and $\mathcal{D}^{2'}$ by removing every edge from $\mathcal{D}^1$ and $\mathcal{D}^2$ other than those on paths corresponding to $\bar{p}$ and $\bar{q}_1, \dots, \bar{q}_m$. Since p is of definite status, note that the paths in $\mathcal{D}^{1'}$ and $\mathcal{D}^{2'}$ corresponding to p begin $X \rightarrow P_2 \rightarrow \dots \rightarrow C_1$, and $X \leftarrow P_2 \dots C_1$, respectively.
- **SUBCASE 2:** Let p be of non-definite status. As before, this forces $\bar{p}$ to begin $X - P_2 - P_3$; $\mathcal{G}$ to contain $X \rightarrow P_3$; and P_2 to be the only node on p not of definite status. To choose DAGs in $[\mathcal{G}]$, we first restrict the class. Thus, construct the MPDAG $\mathcal{G}^*$ by adding $P_2 \rightarrow P_3$ to $\mathcal{G}$ and completing Meek's (1995) orientation rules. Note that by Lemma 80, $\mathcal{G}^*$ contains $X - P_2$. Now consider any two DAGs $\mathcal{D}^1, \mathcal{D}^2 \in [\mathcal{G}^*] \subseteq [\mathcal{G}]$ that include the edges $X \rightarrow P_2$ and $X \leftarrow P_2$, respectively. And construct $\mathcal{D}^{1'}$ and $\mathcal{D}^{2'}$ by removing every edge from $\mathcal{D}^1$ and $\mathcal{D}^2$ other than those on paths corresponding to $\bar{p}$ and $\bar{q}_1, \dots, \bar{q}_m$. Note that the paths in $\mathcal{D}^{1'}$ and $\mathcal{D}^{2'}$ corresponding to p begin $X \rightarrow P_2 \rightarrow \dots \rightarrow C_1$, and $X \leftarrow P_2 \rightarrow \dots \rightarrow C_1$, respectively.

Now that we have chosen $\mathcal{D}^1, \mathcal{D}^2 \in [\mathcal{G}]$, we construct a family of interventional densities $\mathbf{F}_i^* = \{f_i(\mathbf{v} | do(\mathbf{x}')) : \mathbf{X}' \subseteq \mathbf{V}\}$ compatible with each DAG. Start by considering the following linear structural equation model (SEM). Each random variable $V_i \in \mathbf{V}$ is a linear combination of its parents in $\mathcal{D}^{1'}$ and $\varepsilon_{v_i} \sim N(0, \sigma_{v_i}^2)$, where $\sigma_{v_i}^2 \in (0, 1]$ and $\{\varepsilon_{v_i} : V_i \in \mathbf{V}\}$ are mutually independent:

$$V_i \leftarrow \sum_{V_j \in \text{Pa}(V_i, \mathcal{D}^{1'})} A_{ij} V_j + \varepsilon_{v_i}. \quad (\text{C.22})$$

Each coefficient A_{ij} in this linear combination is defined by the edge coefficient of each $V_i \leftarrow V_j$ in $\mathcal{D}^{1'}$. We pick these edge coefficients in conjunction with $\{\sigma_{v_i}^2 : V_i \in \mathbf{V}\}$ in such a way that each $A_{ij} \in (0, 1)$ and $\text{Var}(V_i) = 1$.

We build $\mathbf{F}_1^*$ by letting $f_1(\mathbf{v})$ be the density of the multivariate normal generated by the SEM in (C.22). For the remaining densities in $\mathbf{F}_1^*$, we let $f_1(\mathbf{v} \mid do(\mathbf{x}')) := f_*(\mathbf{v})$, where f_* is the density of the multivariate normal generated by taking the SEM in (C.22) and replacing $\mathbf{X}'$ with its interventional value $\mathbf{x}'$ [Pearl, 2009]. Note that $\mathcal{D}^{1'}$ is compatible with $\mathbf{F}_1^*$ by construction – and therefore, so is $\mathcal{D}^1$ (Lemma 108).

To construct $\mathbf{F}_2^*$, define a linear SEM identical to the SEM in (C.22) but based on the parents in $\mathcal{D}^{2'}$. Let $f_2(\mathbf{v})$ be the density of the resulting distribution, which is another multivariate normal with mean vector zero and a covariance matrix with ones on the diagonal. We show below that the off-diagonal entries – which house the variables' covariances – are identical to the covariances from the SEM in (C.22). Thus, in an analogous way to $\mathbf{F}_1^*$, we can define $\mathbf{F}_2^* = \{f_2(\mathbf{v} \mid do(\mathbf{x}')) : \mathbf{X}' \subseteq \mathbf{V}\}$ such that $\mathcal{D}^{2'}$ – and therefore $\mathcal{D}^2$ – is compatible with $\mathbf{F}_2^*$ and such that $f_2(\mathbf{v}) = f_1(\mathbf{v})$.

To see that the covariances are identical, note that by Lemma 77, the covariances based on $\mathcal{D}' \in \{\mathcal{D}^{1'}, \mathcal{D}^{2'}\}$ will equal the product of edge coefficients along non-collider paths in $\mathcal{D}'$. Since $\mathcal{D}^{1'}$ and $\mathcal{D}^{2'}$ have identical adjacencies, we only need to show that these DAGs have an identical set of colliders. Let $q'_1, \dots, q'_m$, and p' be the paths in $\mathcal{D}'$ corresponding to $\bar{p}$ and $\bar{q}_1, \dots, \bar{q}_m$ in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}}_X}$. Recall that $\mathbf{C}$ is the set of all colliders on $\bar{p}$. Then note that $\mathbf{C}$ is also the set of all colliders on p' , since $P_1, P_3, \dots, P_k$ are nodes of definite status on p in $\mathcal{G}$ and since P_2 is always a non-collider on p' . Further, by Lemma 122, C_i is the only node on both p' and q'_i , $i \in \{1, \dots, m\}$, and there is no shared node between the causal paths q'_i and q'_j , $i \neq j$. Therefore, $\mathbf{C}$ is the set of all colliders on any path in either $\mathcal{D}^{1'}$ or $\mathcal{D}^{2'}$.

Thus we have two families of interventional densities such that $\mathcal{D}^i$ is compatible with $\mathbf{F}_i^*$ and such that $f_1(\mathbf{v}) = f_2(\mathbf{v})$ so that (C.20) holds. To complete Case 1, we show that (C.21) holds. It suffices to show that $E[Y \mid do(\mathbf{X} = \mathbf{1}), \mathbf{Z} = \mathbf{0}]$ is not the same under f_1 and f_2 . We start by calculating this expectation under f_2 :

$$\begin{aligned} \mathbb{E}_{f_2}[Y \mid do(\mathbf{X} = \mathbf{1}), \mathbf{Z} = \mathbf{0}] \\ = \mathbb{E}_{f_2}[Y \mid Z_1, \dots, Z_m] \mid_{\mathbf{Z}=\mathbf{0}} \end{aligned}$$

$$\begin{aligned}
&= \left[\text{Cov}_{f_2}(Y, Z_1) \cdots \text{Cov}_{f_2}(Y, Z_m) \right] \Sigma^{-1} \mathbf{Z} \Big|_{\mathbf{Z}=\mathbf{0}} \\
&= 0,
\end{aligned}$$

where Σ is the covariance matrix of $\mathbf{Z}$ under f_2 (and where Σ is invertible since $\mathbf{V}$ is non-degenerate). The first equality follows from Rule 3 of Pearl's do calculus (Theorem 75), since $Y \perp_d \mathbf{X} \mid \mathbf{Z}$ in $\mathcal{D}_{\underline{\mathbf{X}}}^{2'}$ and since f_2 is consistent with $\mathcal{D}^{2'}$. The second equality follows from properties of multivariate normals (see Lemma 76).

Next, we calculate the same expectation under f_1 :

$$\begin{aligned}
&\mathbb{E}_{f_1}[Y \mid \text{do}(\mathbf{X} = \mathbf{1}), \mathbf{Z} = \mathbf{0}] \\
&= \mathbb{E}_{f_1}[Y \mid X, Z_1, \dots, Z_m] \Big|_{X=1, \mathbf{Z}=\mathbf{0}} \\
&= \left[\text{Cov}_{f_1}(Y, X) \quad \text{Cov}_{f_1}(Y, Z_1) \cdots \text{Cov}_{f_1}(Y, Z_m) \right] \Sigma^{-1} \begin{bmatrix} X \\ \mathbf{Z} \end{bmatrix} \Big|_{X=1, \mathbf{Z}=\mathbf{0}} \\
&= \begin{bmatrix} 0 & \cdots & 0 & \text{Cov}_{f_1}(Y, Z_m) \end{bmatrix} \Sigma_1^{-1} \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix} \\
&= \Sigma_{m+1,1} \cdot \text{Cov}_{f_1}(Y, Z_m),
\end{aligned}$$

where Σ is the invertible covariance matrix of $(X, Z_1, \dots, Z_m)^T$ under f_1 and where $\Sigma_{m+1,1}$ is the $(m+1, 1)^{th}$ entry of Σ^{-1} . The first equality follows from Rules 3 and 2 of Pearl's do calculus (Theorem 75), since $Y \perp_d \mathbf{X} \setminus \{X\} \mid X, \mathbf{Z}$ in $\mathcal{D}_{\underline{\mathbf{X}}}^{1'}$, since $Y \perp_d X \mid \mathbf{Z}$ in $\mathcal{D}_{\underline{\mathbf{X}}}^{1'}$, and since f_1 is consistent with $\mathcal{D}^{1'}$. The second equality follows from properties of multivariate normals (see Lemma 76), and the third follows from applying Wright's Rule (Lemma 77) to $\mathcal{D}^{1'}$.

To complete Case 1, we show this expectation is non-zero so that (C.21) holds. Note that $\text{Cov}_{f_1}(Y, Z_m) \neq 0$, since by Lemma 77, it equals the product of all edge coefficients on $-(q'_m) \oplus p'(C_m, Y)$ in $\mathcal{D}^{1'}$, which were chosen to be in $(0, 1)$. To show that $\Sigma_{m+1,1} \neq 0$, we

find the following.

$$\Sigma = \begin{bmatrix} 1 & \text{Cov}_{f_1}(X, Z_1) & & & 0 \\ \text{Cov}_{f_1}(X, Z_1) & 1 & \ddots & & \\ & \text{Cov}_{f_1}(Z_1, Z_2) & \ddots & & \\ 0 & & \ddots & 1 & \text{Cov}_{f_1}(Z_{m-1}, Z_m) \\ & & & \text{Cov}_{f_1}(Z_{m-1}, Z_m) & 1 \end{bmatrix}.$$

$$\Sigma_{m+1,1} = \frac{(-1)^{m+2}}{\text{Det}(\Sigma)} \cdot \text{Cov}_{f_1}(X, Z_1) \cdot \text{Cov}_{f_1}(Z_1, Z_2) \cdots \text{Cov}_{f_1}(Z_{m-1}, Z_m) \neq 0.$$

The final equality follows by applying Wright's Rule (Lemma 77) to $\mathcal{D}^{1'}$ and noting that the paths between X and Z_1 as well as Z_i and Z_{i+1} , $i \in \{1, \dots, m-1\}$, have edge coefficients in $(0, 1)$ and no colliders.

CASE 2: Consider when $\mathbf{S}_3 = \emptyset$, and define the following sets.

$$\mathbf{S}_6 = \{\text{paths in } \mathbf{S}_2 \text{ with the fewest edges}\}.$$

$$\mathbf{S}_7 = \{\text{paths in } \mathbf{S}_6 \text{ with a shortest distance to } \mathbf{Z} \text{ in } \mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}}_{\underline{X}}}\}.$$

Pick an arbitrary path $\bar{p} := \langle X = P_1, \dots, P_k = Y \rangle$, $k > 1$, $Y \in \mathbf{Y}$, in $\mathbf{S}_7$. Note that $\bar{p}$ begins $X \leftarrow P_2$, since $\bar{p} \in \mathbf{S}_2$ is a path in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}}_{\underline{X}}}$ and $\mathbf{S}_3 = \emptyset$. Further, $\bar{p}$ is a proper path from $\mathbf{X}$ to $\mathbf{Y}$ that is open given $\mathbf{Z}$, since we have shown this holds for all paths in $\mathbf{S}_1$. Let $\mathbf{C} = \{C_1, \dots, C_m\}$, $1 \leq m \leq k-3$, be the set of all colliders on $\bar{p}$ in order of their appearance on $\bar{p}$, where possibly $\mathbf{C} = \emptyset$. Note that every node in $\mathbf{C}$ is in $\text{An}(\mathbf{Z}, \mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}}_{\underline{X}}})$, since $\bar{p}$ is open given $\mathbf{Z}$. Then let $\bar{q}_i = \langle Q_{i_1}, \dots, Q_{i_{k_i}} \rangle$, $i \in \{1, \dots, m\}$, be a shortest causal path from C_i to $\mathbf{Z}$ in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}}_{\underline{X}}}$. Let $p, q_1, \dots, q_m$ be the paths in $\mathcal{G}$ corresponding to $\bar{p}, \bar{q}_1, \dots, \bar{q}_m$ in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}}_{\underline{X}}}$.

Next, recall from the beginning of the proof that there is a path in $\mathcal{G}$ from X to $\mathbf{Y} \cup \mathbf{Z}$ that does not contain $\mathbf{X} \setminus \{X\}$, is possibly causal, and starts undirected. Let $r = \langle X = R_1, \dots, R_n \rangle$, $n > 1$, be a shortest such path. When r ends in $\mathbf{Y}$, it is a proper, possibly

causal path from $\mathbf{X}$ to $\mathbf{Y}$ that starts undirected and does not contain nodes in $\mathbf{Z}$. It follows by Proposition 106 that the conditional effect of $\mathbf{X}$ on $\mathbf{Y}$ given $\mathbf{Z}$ is not identifiable in $\mathcal{G}$, and we are done. Hence, suppose r ends in $\mathbf{Z}$ for the remainder of this proof.

Using these paths, we turn to the task of finding DAGs in $[\mathcal{G}]$ that are compatible with families of interventional densities such that (C.20)-(C.21) hold. We start by considering a restriction of the class $[\mathcal{G}]$ using an MPDAG $\mathcal{G}^*$. When $X \notin \text{Adj}(R_3, \mathcal{G})$, then let $\mathcal{G}^* = \mathcal{G}$. When $X \in \text{Adj}(R_3, \mathcal{G})$, note that $\mathcal{G}$ cannot contain $X \leftarrow R_3$, since r is possibly causal. Neither can $\mathcal{G}$ contain $X - R_3$, since $\langle X, R_3 \rangle \oplus r(R_3, R_n)$ would contradict the definition of r as a shortest proper possibly causal path in $\mathcal{G}$ from X to $\mathbf{Z}$ that starts undirected. Thus, $\mathcal{G}$ contains $X \rightarrow R_3$. In this case, construct the MPDAG $\mathcal{G}^*$ by adding $R_2 \rightarrow R_3$ to $\mathcal{G}$ (if it does not already exist) and completing Meek's (1995) orientation rules. Note by Lemma 80 that $\mathcal{G}^*$ still contains $X - R_2$, since $X, R_2 \in \text{Pa}(R_3, \mathcal{G}^*)$.

With $\mathcal{G}^*$ defined, we let $\mathcal{D}^1$ be a DAG in $[\mathcal{G}^*] \subseteq [\mathcal{G}]$ that contains $X \rightarrow R_2$. To see that the path in $\mathcal{D}^1$ corresponding to r in $\mathcal{G}$ takes the form $X \rightarrow R_2 \rightarrow \cdots \rightarrow R_n$, note that $r(R_2, R_n)$ is unshielded by the definition of r as a shortest possibly causal path. So when $X \notin \text{Adj}(R_3, \mathcal{G})$, the claim holds by R1 of Meek [1995] since we formed $\mathcal{D}^1$ by adding $X \rightarrow R_2$ to $\mathcal{G}$. When $X \in \text{Adj}(R_3, \mathcal{G})$, then the same logic holds since we formed $\mathcal{D}^1$ by adding $X \rightarrow R_2 \rightarrow R_3$ to $\mathcal{G}$. Further, note that by Lemma 123, $\mathcal{G}$ – and therefore $\mathcal{D}^1$ – contains the edge $\langle P_2, R_2 \rangle$. Since $\mathcal{D}^1$ is acyclic and has the path $P_2 \rightarrow X \rightarrow R_2$, then it contains $P_2 \rightarrow R_2$ so that P_2 is a non-collider on $\langle P_3, P_2, R_2 \rangle$ in $\mathcal{D}^1$.

Similarly, we let $\mathcal{D}^2$ be a DAG in $[\mathcal{G}^*] \subseteq [\mathcal{G}]$ that contains $X \leftarrow R_2 \rightarrow R_3$. (This DAG clearly exists when $\mathcal{G}^*$ contains $X - R_2 \rightarrow R_3$. When instead $\mathcal{G}^*$ contains $X - R_2 - R_3$, this DAG exists by Lemma 81.) Then note that the path in $\mathcal{D}^2$ corresponding to r in $\mathcal{G}$ takes the form $X \leftarrow R_2 \rightarrow \cdots \rightarrow R_n$ by R1 of Meek [1995], since $r(R_2, R_n)$ is unshielded and possibly causal. To see that P_2 is a non-collider on $\langle P_3, P_2, R_2 \rangle$ in $\mathcal{D}^1$, note that by Lemma 123, $\mathcal{G}$ must contain $P_2 \rightarrow R_2$; or $P_3 \leftarrow P_2$; or $P_3 - P_2 - R_2$ such that $P_3 \notin \text{Adj}(R_2, \mathcal{G})$.

Now that we have chosen $\mathcal{D}^1, \mathcal{D}^2 \in [\mathcal{G}]$, we construct a family of interventional densities $\mathbf{F}_i^* = \{f_i(\mathbf{v}|\text{do}(\mathbf{x}')) : \mathbf{X}' \subseteq \mathbf{V}\}$ compatible with each DAG. Start by constructing a DAG $\mathcal{D}^{1'}$

by removing every edge from $\mathcal{D}^1$ except those on the paths corresponding to $p, q_1, \dots, q_m, r$, and $\langle P_2, R_2 \rangle$. Then consider the following linear SEM. Each random variable $V_i \in \mathbf{V}$ is a linear combination of its parents in $\mathcal{D}^{1'}$ and $\varepsilon_{v_i} \sim N(0, \sigma_{v_i}^2)$, where $\{\varepsilon_{v_i} : V_i \in \mathbf{V}\}$ are mutually independent; where the coefficients in the SEM are all $\frac{1}{2}$; and where we choose $\sigma_{v_i}^2$ such that $\text{Var}(V_i) = 1$ for all $V_i \in \mathbf{V}$:

$$V_i \leftarrow \sum_{V_j \in \text{Pa}(V_i, \mathcal{D}^{1'})} \frac{1}{2} V_j + \varepsilon_{v_i}. \quad (\text{C.23})$$

We build $\mathbf{F}_1^*$ by letting $f_1(\mathbf{v})$ be the density of the multivariate normal generated by this SEM. For the remaining densities in $\mathbf{F}_1^*$, we let $f_1(\mathbf{v} | do(\mathbf{x}')) := f_*(\mathbf{v})$, where f_* is the density of the multivariate normal generated by taking the SEM in (C.23) and replacing $\mathbf{X}'$ with its interventional value $\mathbf{x}'$ [Pearl, 2009]. Note that $\mathcal{D}^{1'}$ is compatible with $\mathbf{F}_1^*$ by construction – and therefore, so is $\mathcal{D}^1$ (Lemma 108).

To construct $\mathbf{F}_2^*$ consider the following linear SEM. Each random variable $V_i \in \mathbf{V}$ is a linear combination of its parents in $\mathcal{D}^{2'}$ and $\varepsilon_{v_i} \sim N(0, \sigma_{v_i}^2)$, where $\{\varepsilon_{v_i} : V_i \in \mathbf{V}\}$ are mutually independent:

$$V_i \leftarrow \sum_{V_j \in \text{Pa}(V_i, \mathcal{D}^{2'})} A_{ij} V_j + \varepsilon_{v_i}. \quad (\text{C.24})$$

Each coefficient A_{ij} in this linear combination is defined by the edge coefficient of $\langle V_i, V_j \rangle$ in $\mathcal{D}^{2'}$. We set the edge coefficients for $\langle X, P_2 \rangle$, $\langle X, R_2 \rangle$, $\langle P_2, R_2 \rangle$ to $-\frac{1}{7}$, $\frac{6}{7}$, $\frac{3}{4}$, respectively, and we set the remaining $A_{ij} = \frac{1}{2}$. Further, we choose $\{\sigma_{v_i}^2 : V_i \in \mathbf{V}\}$ so that $\text{Var}(V_i) = 1$. Then let $f_2(\mathbf{v})$ be the density of the resulting distribution, which is another multivariate normal with mean vector zero and a covariance matrix with ones on the diagonal. We show below that the off-diagonal entries – which house the variables' covariances – are identical to the covariances from the SEM in (C.23). Thus, in an analogous way to $\mathbf{F}_1^*$, we can define $\mathbf{F}_2^* = \{f_2(\mathbf{v} | do(\mathbf{x}')) : \mathbf{X}' \subseteq \mathbf{V}\}$ such that $\mathcal{D}^{2'}$ – and therefore $\mathcal{D}^2$ – is compatible with $\mathbf{F}_2^*$ and such that $f_2(\mathbf{v}) = f_1(\mathbf{v})$.

To see that the covariances are identical, note that by Lemma 77, the covariances based on $\mathcal{D}' \in \{\mathcal{D}^{1'}, \mathcal{D}^{2'}\}$ will equal the product of edge coefficients along non-collider paths in

$\mathcal{D}'$. Consider the paths in $\mathcal{D}'$ and their colliders. By Lemmas 122-123, the paths in $\mathcal{D}'$ corresponding to $p, q_1, \dots, q_m, r$ do not overlap anywhere except at X and $C_i, i \in \{1, \dots, m\}$. Then note that $\mathcal{D}'$ contains no colliders on $q_1, \dots, q_m, \langle R_1, \dots, R_n \rangle$, or $\langle P_3, P_2, R_2, R_3 \rangle$. To see that that $\mathbf{C}$ is the set of all colliders on $\langle P_1, \dots, P_k \rangle$, note that p is of definite status in $\mathcal{G}$, since $\bar{p} \in \mathbf{S}_1$ is a definite status path in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}} \underline{X}}$ that begins $X \leftarrow P_2$ and has no nodes in $\mathbf{X} \setminus \{X\}$. However, note that $\mathcal{D}^{1'}$ contains $X \rightarrow R_2 \leftarrow P_2$, whereas $\mathcal{D}^{2'}$ contains $P_2 \rightarrow X \leftarrow R_2$. Therefore, the claim that the covariances from (C.23) and (C.24) are identical holds after calculating the following for both models: $\text{Cov}(P_2, X) = \frac{1}{2}$, $\text{Cov}(P_2, R_2) = \frac{3}{4}$, $\text{Cov}(P_2, R_3) = \frac{3}{8}$, $\text{Cov}(X, R_2) = \frac{3}{4}$.

Thus we have two families of interventional densities such that $\mathcal{D}^i$ is compatible with $\mathbf{F}_i^*$ and such that (C.20) holds. To complete Case 2, we show that (C.21) holds. It suffices to show that $E[Y \mid do(\mathbf{X} = \mathbf{1}), \mathbf{Z} = \mathbf{0}]$ is not the same under f_1 and f_2 . We start by calculating this expectation under f_2 . For ease of notation, let $Z_0 = R_n$ and let $Z_i = Q_{i_{k_i}}$ for all $i \in \{1, \dots, m\}$.

$$\begin{aligned}
\mathbb{E}_{f_2}[Y \mid do(\mathbf{X} = \mathbf{1}), \mathbf{Z} = \mathbf{0}] &= \mathbb{E}_{f_2}[Y \mid Z_0, \dots, Z_m] \Big|_{\{Z_0, \dots, Z_m\}=\mathbf{0}} \\
&= \left[\text{Cov}_{f_2}(Y, Z_0) \ \cdots \ \text{Cov}_{f_2}(Y, Z_m) \right] \boldsymbol{\Sigma}^{-1} \begin{bmatrix} Z_0 \\ \vdots \\ Z_m \end{bmatrix} \Big|_{\{Z_0, \dots, Z_m\}=\mathbf{0}} \\
&= 0,
\end{aligned}$$

where $\boldsymbol{\Sigma}$ is the covariance matrix of $\{Z_0, \dots, Z_m\}^T$ under f_2 (and where $\boldsymbol{\Sigma}$ is invertible since $\mathbf{V}$ is non-degenerate). The first equality follows from Rules 3 and 1 of Pearl's do calculus (Theorem 75), since $Y \perp_d \mathbf{X} \mid \mathbf{Z}$ in $\mathcal{D}_{\mathbf{X}}^{2'}$; since $Y \perp_d \mathbf{Z} \setminus \{Z_0, \dots, Z_m\} \mid Z_0, \dots, Z_m$ in $\mathcal{D}^{2'}$; and since f_2 is consistent with $\mathcal{D}^{2'}$. The second equality follows from properties of multivariate normals (see Lemma 76).

Next, we calculate the same expectation under f_1 . In each of the subcases below, we show that $\mathbb{E}_{f_1}[Y \mid do(\mathbf{X} = \mathbf{1}), \mathbf{Z} = \mathbf{0}]$ is non-zero so that (C.21) holds. For these calculations, note that $f_1(y \mid do(\mathbf{x}), \mathbf{z})$ is the conditional density of $Y \mid \mathbf{Z}$ under $f_*(\mathbf{v})$.

- **SUBCASE 1:** When p' has no colliders in $\mathcal{D}^{1'}$,

$$\begin{aligned}
\mathbb{E}_{f_1}[Y \mid do(\mathbf{X} = \mathbf{1}), \mathbf{Z} = \mathbf{0}] &= \mathbb{E}_{f_*}[Y \mid \mathbf{Z}] \Big|_{\mathbf{x}=\mathbf{1}, \mathbf{Z}=\mathbf{0}} \\
&= \mathbb{E}_{f_*}[Y \mid Z_0] \Big|_{x=1, Z_0=0} \\
&= \mathbb{E}_{f_*}[Y] + \frac{\text{Cov}_{f_*}(Y, Z_0)}{\text{Var}_{f_*}(Z_0)} \cdot \left(Z_0 - \mathbb{E}_{f_*}[Z_0] \right) \Big|_{x=1, Z_0=0} \\
&= \frac{\text{Cov}_{f_*}(Y, Z_0)}{\text{Var}_{f_*}(Z_0)} \cdot \left(-\frac{1}{2}^{n-1} \right) \\
&\neq 0.
\end{aligned}$$

The third equality follows from properties of multivariate normals (see Lemma 76). For the final equality, consider Lemma 78, and note that the SEM that defines $f_*(\mathbf{v})$ under $do(\mathbf{X})$ has a matrix of coefficients that are non-zero exactly when $\mathcal{D}_{\mathbf{X}}^{1'}$ contains an edge. It follows that $\text{Cov}_{f_*}(Y, Z_0) \neq 0$, since $\mathcal{D}_{\mathbf{X}}^{1'}$ contains a node in $\{P_2, \dots, P_k\}$ with causal paths to Y and Z_0 .

- **SUBCASE 2:** When p' has at least one collider,

$$\begin{aligned}
\mathbb{E}_{f_1}[Y \mid do(\mathbf{X} = \mathbf{1}), \mathbf{Z} = \mathbf{0}] &= \mathbb{E}_{f_*}[Y \mid \mathbf{Z}] \Big|_{\mathbf{x}=\mathbf{1}, \mathbf{Z}=\mathbf{0}} \\
&= \mathbb{E}_{f_*}[Y \mid Z_0, \dots, Z_m] \Big|_{x=1, \{Z_0, \dots, Z_m\}=\mathbf{0}} \\
&= \mathbb{E}_{f_*}[Y] + \begin{bmatrix} \text{Cov}_{f_*}(Y, Z_0) & \cdots & \text{Cov}_{f_*}(Y, Z_m) \end{bmatrix} \Sigma^{-1} \begin{bmatrix} Z_0 - \mathbb{E}_{f_*}[Z_0] \\ \vdots \\ Z_m - \mathbb{E}_{f_*}[Z_m] \end{bmatrix} \Big|_{x=1, \{Z_0, \dots, Z_m\}=\mathbf{0}} \\
&= \begin{bmatrix} 0 & \cdots & 0 & \text{Cov}_{f_*}(Y, Z_m) \end{bmatrix} \Sigma^{-1} \begin{bmatrix} -\mathbb{E}_{f_*}[Z_0] \\ \mathbf{0} \end{bmatrix} \\
&= \frac{1}{2}^{k'_m} \cdot \Sigma_{m+1,1} \cdot \left(-\frac{1}{2}^{n-1} \right),
\end{aligned}$$

where k'_m is the number of edges on $(-q'_m)(Z_m, C_m) \oplus p'(C_m, Y)$; where Σ is the invertible covariance matrix of $(Z_0, \dots, Z_m)^T$ under f_* ; and where $\Sigma_{m+1,1}$ is the $(m+1, 1)^{th}$ entry of Σ^{-1} . The third equality follows from properties of multivariate normals (see

Lemma 76). The last two equalities follow by Lemma 78, since $\mathcal{D}_{\mathbf{X}}^{1'}$ contains no node with causal paths to Y and Z_i , $i \in \{0, \dots, m-1\}$, but $\mathcal{D}_{\mathbf{X}}^{1'}$ does contain a node with causal paths to Y and Z_0 .

To complete Subcase 2, we show that $\Sigma_{m+1,1} \neq 0$.

$$\Sigma = \begin{bmatrix} 1 & \text{Cov}_{f_*}(Z_0, Z_1) & & & & \\ \text{Cov}_{f_*}(Z_0, Z_1) & 1 & \ddots & & & \\ & \text{Cov}_{f_*}(Z_1, Z_2) & \ddots & & & \\ 0 & & \ddots & \text{Cov}_{f_*}(Z_{m-1}, Z_m) & & \\ & & & 1 & \text{Cov}_{f_*}(Z_m, Y) & \\ & & & \text{Cov}_{f_*}(Z_m, Y) & 1 & \end{bmatrix}.$$

$$\Sigma_{m+1,1} = \frac{(-1)^{m+2}}{\text{Det}(\Sigma)} \text{Cov}_{f_*}(Z_0, Z_1) \cdots \text{Cov}_{f_*}(Z_{m-1}, Z_m) \neq 0.$$

The final equality follows from Lemma 78, since for each $i \in \{0, \dots, m-1\}$, $\mathcal{D}_{\mathbf{X}}^{1'}$ contains a node in $\{P_2, \dots, P_k\}$ with causal paths to Z_i and Z_{i+1} . $\blacksquare$

C.7.2 SUPPORTING RESULTS

Lemma 120 *Let $p = \langle V_1, \dots, V_k \rangle$, $k > 1$, be a definite status path in $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, where $\mathcal{G}$ is either a causal MPDAG or there exists a causal MPDAG $\mathcal{G}'$ and node sets $\mathbf{A} \subseteq \mathbf{V}$, and $\mathbf{B} \subseteq \mathbf{V} \setminus \mathbf{A}$ such that $\mathcal{G} \equiv \mathcal{G}'_{\overline{\mathbf{A}\mathbf{B}}}$. Further, let $\mathbf{C} \subseteq \mathbf{V}$.*

Suppose p is d -connecting given $\mathbf{C}$ in $\mathcal{G}$ and suppose $\mathcal{G}$ contains an edge $\langle V_i, V_j \rangle$, $1 \leq i < j \leq k$, such that $t = p(V_1, \dots, V_i) \oplus \langle V_i, V_j \rangle \oplus p(V_j, V_k)$ is also a definite status path in $\mathcal{G}$. If V_i and V_j are definite non-colliders or endpoints on p , then t is also d -connecting given $\mathbf{C}$.

Proof of Lemma 120. If V_i and V_j are still definite non-colliders (or endpoints) on t , the statement clearly holds. Hence, suppose that one of V_i or V_j is a collider on t . Without loss

of generality, we will assume that it is V_i that is a collider on t . Then $V_{i-1} \rightarrow V_i \rightarrow V_{i+1}$ must be on p , whereas $V_{i-1} \rightarrow V_i \leftarrow V_j$ is on $\mathcal{G}$.

Consider $p(V_i, V_j)$. This path cannot take the form $V_i \rightarrow \cdots \rightarrow V_j$ by the acyclicity of $\mathcal{G}'$. Further, p cannot contain $V_l \rightarrow V_{l+1} \leftarrow V_{l+1}$, it is of definite status. Hence, there must be a collider on $p(V_i, V_j)$ in $\text{De}(V_i, \mathcal{G})$. Since this collider is in $\text{An}(\mathbf{C}, \mathcal{G})$, it also holds that $V_i \in \text{An}(\mathbf{C}, \mathcal{G})$. Therefore, t is still d-connecting given $\mathbf{C}$. $\blacksquare$

Lemma 121 *Let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ be pairwise disjoint node sets in a causal MPDAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$. Suppose there is no possibly causal path in $\mathcal{G}$ from X to $\mathbf{Y}$ that starts undirected and does not contain $\mathbf{X} \setminus \{X\} \cup \mathbf{Z}$. Further, suppose there is a possibly causal path in $\mathcal{G}$ from X to $\mathbf{Z}$ that starts undirected and does not contain $\mathbf{X} \setminus \{X\} \cup \mathbf{Y}$, where $(X \not\perp_d \mathbf{Y} | \mathbf{X} \setminus \{X\}, \mathbf{Z})_{\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}} \underline{X}}}$. Then consider the following sets.*

$$\mathbf{S}_1 = \left\{ \begin{array}{l} \text{definite status paths in } \mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}} \underline{X}} \text{ from } X \text{ to } \mathbf{Y} \text{ that are} \\ \text{d-connecting given } \mathbf{X} \setminus \{X\} \cup \mathbf{Z} \end{array} \right\}.$$

$$\mathbf{S}_2 = \{\text{paths in } \mathbf{S}_1 \text{ with the fewest colliders}\}.$$

$$\mathbf{S}_3 = \{\text{paths in } \mathbf{S}_2 \text{ that start undirected}\}.$$

$$\mathbf{S}_4 = \{\text{paths in } \mathbf{S}_3 \text{ with the fewest edges}\}.$$

$$\mathbf{S}_5 = \{\text{paths in } \mathbf{S}_4 \text{ with a shortest distance to } \mathbf{Z} \text{ in } \mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}} \underline{X}}\}.$$

$$\mathbf{S}_6 = \{\text{paths in } \mathbf{S}_2 \text{ with the fewest edges}\}.$$

$$\mathbf{S}_7 = \{\text{paths in } \mathbf{S}_6 \text{ with a shortest distance to } \mathbf{Z} \text{ in } \mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}} \underline{X}}\}.$$

When $\mathbf{S}_3 \neq \emptyset$, pick an arbitrary path in $\mathbf{S}_5$. When $\mathbf{S}_3 = \emptyset$, pick an arbitrary path in $\mathbf{S}_7$. Call this path $p := \langle X = P_1, \dots, P_k = Y \rangle$, $k > 1$, $Y \in \mathbf{Y}$, and let p^* be the path in $\mathcal{G}$ corresponding to p in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}} \underline{X}}$. Then the following hold.

(i) p contains no nodes in $\mathbf{X} \setminus \{X\}$ or $\mathbf{Y} \setminus \{Y\}$.

(ii) $\mathcal{G}$ does not contain $X - P_i$ for any $i > 2$.

(iii) Every collider on p or p^* is unshielded in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}}_{\underline{X}}}$ or $\mathcal{G}$, respectively.

Proof of Lemma 121.

- (i) Since p is a d-connecting given $\mathbf{X} \setminus \{X\} \cup \mathbf{Z}$, no definite non-collider on p is in $\mathbf{X} \setminus \{X\}$. Also, since all edges into $\mathbf{X} \setminus \{X\}$ are removed in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}}_{\underline{X}}}$, no collider on p is in $\mathbf{X} \setminus \{X\}$. To show that $P_k = Y$ is the only node from $\mathbf{Y}$ on p , it is enough to notice that otherwise, for $P_l \in \mathbf{Y}, l \in \{2, \dots, k-1\}$ we could choose $p(X, P_l)$ instead of p .
- (ii) We show that $X - P_i$ is not in $\mathcal{G}$ for any $i > 2$ by contradiction. Hence, let $j \in \{3, \dots, k\}$ be the largest index such that $X - P_j$ is in $\mathcal{G}$. Now, $j \neq k$ since otherwise, $X - Y$ is in $\mathcal{G}$, which contradicts our assumptions.

Next, suppose that $2 < j < k$ and $X - P_j$ is in $\mathcal{G}$. If $P_j \rightarrow P_{j+1}$ is on p , we have a contradiction with the choice of p , since we could have chosen $\langle X, P_j \rangle \oplus p(P_j, Y)$. We get the same contradiction if $P_j - P_{j+1}$ is in $\mathcal{G}$ and $X \notin \text{Adj}(P_{j+1}, \mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}}_{\underline{X}}})$.

Two cases are left to consider, the case when $P_j - P_{j+1}$ and $X \leftarrow P_{j+1}$ are in $\mathcal{G}$ and $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}}_{\underline{X}}}$, and the case when $P_j \leftarrow P_{j+1}$ and $X \leftarrow P_{j+1}$ are in $\mathcal{G}$ and $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}}_{\underline{X}}}$ (by choice of j , we cannot have $X - P_{j+1}$ in $\mathcal{G}$). In both cases, $s = \langle X, P_{j+1} \rangle \oplus p(P_{j+1}, Y)$, $s \in \mathbf{S}_1$ (as P_{j+1} is a definite non-collider on both p and s). Furthermore, s cannot have more colliders than p , so $s \in \mathbf{S}_2$.

Suppose next that $\mathbf{S}_3 = \emptyset$, that is $p \in \mathbf{S}_7$. Since s has fewer edges than p , we have a contradiction with the choice of p .

Otherwise, $\mathbf{S}_3 \neq \emptyset$. Now, for p to be in $\mathbf{S}_5 \subseteq \mathbf{S}_3 \subseteq \mathbf{S}_2$, we need to have that $p(X, P_{j+1})$ contains no colliders and that p starts undirected. Furthermore, since p is of definite status, we have that $P_{i-1} - P_i \leftarrow P_{i+1}$, $i \in \{2, \dots, j\}$ cannot be on $p(X, P_{j+1})$. Combining the information that $P_i \rightarrow P_{i+1} \leftarrow P_{i+2}$ and $P_i - P_{i+1} \leftarrow P_{i+2}$ cannot be on $p(X, P_{j+1})$, for $i \in \{1, j-1\}$ and that $X - P_2$ is on $p(X, P_{j+1})$ leads us to conclude that $P_l \leftarrow P_{l+1}$, $l \in \{1, \dots, j\}$ cannot be on $p(X, P_{j+1})$, either.

If p^* is of definite status, then since $P_l \leftarrow P_{l+1}$, $l \in \{1, \dots, j\}$ is not on $p^*(X, P_{j+1})$, we have that $p^*(X, P_{j+1})$ is possibly causal in $\mathcal{G}$ (Lemma 84). Therefore, $X \leftarrow P_{j+1}$ being in $\mathcal{G}$ is a contradiction.

Otherwise, p^* is not of definite status in $\mathcal{G}$. This implies that $X \rightarrow P_3$ is in $\mathcal{G}$ (and not in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}} \underline{X}}$). Moreover, case (i) implies that $p^*(P_2, P_{j+1})$ is of definite status. Hence, since $P_l \leftarrow P_{l+1}$, $l \in \{1, \dots, j\}$ is not on $p^*(X, P_{j+1})$, we have that $p^*(P_2, P_{j+1})$ is possibly causal in $\mathcal{G}$ (Lemma 84). But then the fact that $p^*(P_3, P_{j+1})$ is possibly causal and $P_{j+1} \rightarrow X \rightarrow P_3$ is in $\mathcal{G}$, contradicts Lemma 90.

- (iii) Next, we show that every collider on p is unshielded. Hence, for the rest of the proof we will assume that there is at least one collider on p . By case (i), p does not contain any node in $\mathbf{X} \setminus \{X\}$. So for the colliders on p^* to be unshielded, it is enough to show that colliders on p are unshielded. Also, note that any path in $\mathbf{S}_1$ must be d-connecting given $\mathbf{Z}$ (since all edges into $\mathbf{X} \setminus \{X\}$ are removed in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}} \underline{X}}$).

Let $\mathbf{C} = \{C_1, \dots, C_m\}$, $1 \leq m \leq k-3$, be the set of all colliders on p , labeled such that if $1 \leq i < j \leq m$, $C_i \equiv P_l$, $C_j \equiv P_r$, $2 \leq l < r \leq k-1$. Then each C_i , $i \in \{1, \dots, m\}$ is in $\text{An}(\mathbf{Z}, \mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}} \underline{X}})$. Let $q_i = \langle C_i = Q_{i_1}, \dots, Q_{i_{k_i}} \rangle$, be a shortest directed path from C_i to $\mathbf{Z}$ in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}} \underline{X}}$. Note that $Q_{i_{k_i}}$ is the only node in $\mathbf{Z}$ on q_i and that q_i cannot contain nodes in $\mathbf{X}$.

Suppose for a contradiction that a collider C_i on p is shielded, that is for $C_i \equiv P_j$, $j \in \{3, \dots, k-1\}$, we have that $\langle P_{j-1}, P_{j+1} \rangle$ is in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}} \underline{X}}$. Then consider the path $s = p(X, P_{j-1}) \oplus \langle P_{j-1}, P_{j+1} \rangle \oplus p(P_{j+1}, Y)$ in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}} \underline{X}}$. If s is of definite status in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}} \underline{X}}$, then by Lemma 120, s must be d-connecting given $\mathbf{Z}$. Then since s starts with the same edge as p , but is shorter than p , we have a contradiction with the choice of p .

Otherwise, s is not of definite status, meaning that $\langle P_{j-2}, P_{j+1} \rangle$ or $\langle P_{j-1}, P_{j+2} \rangle$, or both, are in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}} \underline{X}}$. We now split the proof into four parts depending on the position of

collider C_i on p and the number of colliders on p .

- (1) Let $i = 1 = k$. That means $C_i = P_j$ is the first and only collider on p . Hence, let $P_l, l \geq 1$ be the closest node to X on $p(X, P_{j-1})$ that is adjacent to a node on $p(P_{j+1}, Y)$. Furthermore, let P_r , be the closest node to Y on $p(P_{j+1}, Y)$ such that $\langle P_l, P_r \rangle$ is in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}} \underline{X}}$. Furthermore, consider the path $t = p(X, P_l) \oplus \langle P_l, P_r \rangle \oplus p(P_r, Y)$. Note that it is possible that $l = 1$ and $r = k$, meaning it is possible that $p(X, P_l), p(P_r, Y)$ are of length zero.

Then t must be of definite status path by construction. Also, since neither P_l nor P_r are colliders on p , by Lemma 120, t must also be d-connecting given $\mathbf{X} \setminus \{X\} \cup \mathbf{Z}$. Now, if $l = 1$, then t starts with an edge into X (by case (ii)) and has fewer colliders than p , so we have a contradiction. If $l > 1$, then we again have a contradiction with the choice of p , because t and p start with the same edge, and t either has fewer colliders than p , or the same number of colliders but fewer edges than p .

- (2) Let $i = 1 < k$. In this case, we assume $i = 1 < k$, so $C_i = P_j$ is the collider closest to X on p , but not the only collider on p . We will now proceed similarly to the case (1).

First, let $C_2 \equiv P_w$ for some $j + 1 < w < k$. Then, let P_l be the closest node to X on $p(X, P_{j-1})$ that is adjacent to a node on $p(P_{j+1}, P_{w-1})$. Furthermore, let P_r , be the closest node to P_{w-1} on $p(P_{j+1}, P_{w-1})$ such that $\langle P_l, P_r \rangle$ is in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}} \underline{X}}$. Furthermore, consider the path $t = p(X, P_l) \oplus \langle P_l, P_r \rangle \oplus p(P_r, Y)$.

Note that t must be of definite status by construction, even in the case where $P_r \equiv P_{w-1}$, simply because $p(P_{w-1}, Y)$ is out of P_{w-1} . Furthermore, since both P_l and P_r are non-colliders on p , we have that t is d-connecting given $\mathbf{X} \setminus \{X\} \cup \mathbf{Z}$ by Lemma 120. Then if $l = 1$, t starts with an edge into X (by case (ii)) and has one fewer collider than p , which leads us to a contradiction. Otherwise, $l \neq 1$, and t starts with the same edge as p , but t either has fewer colliders than p , or the

same number of colliders as p but fewer edges than p which gives us the desired contradiction.

- (3) Let $1 < i < k$. That is, C_i is not the closest collider to X or to Y on p . Let $P_u = C_{i-1}$ and let $P_w = C_{i+1}$ and note that, $1 < u < j - 1$ and $j + 1 < w < k$. Then, let P_l be the closest node to P_{u+1} on $p(P_{u+1}, P_{j-1})$ that is adjacent to a node on $p(P_{j+1}, P_{w-1})$. Furthermore, let P_r be the closest node to P_{w-1} on $p(P_{j+1}, P_{w-1})$ such that $\langle P_l, P_r \rangle$ is in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}} \underline{X}}$. Furthermore, consider the path $t = p(X, P_l) \oplus \langle P_l, P_r \rangle \oplus p(P_r, Y)$.

As in the cases above, t must be of definite status by construction, even in the case where $P_r = P_{u+1}$, as $-p(P_{u+1}, X)$ is out of P_{u+1} and even if $P_l \equiv P_{w-1}$, $p(P_{w-1}, Y)$ is out of P_{w-1} . Now, similarly to the above cases, we can show that t must also be d-connecting given $\mathbf{X} \setminus \{X\} \cup \mathbf{Z}$ (Lemma 120). And similarly to the above cases t then contradicts the choice of p .

- (4) Let $1 < i = k$. That is, C_i is the closest collider to Y on p . Let $P_u = C_{i-1}$ and note that, $1 < u < j - 1$. Then, let P_l be the closest node to P_{u+1} on $p(P_{u+1}, P_{j-1})$ that is adjacent to a node on $p(P_{j+1}, Y)$. Furthermore, let P_r be the closest node to Y on $p(P_{j+1}, Y)$ such that $\langle P_l, P_r \rangle$ is in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}} \underline{X}}$. Furthermore, consider the path $t = p(X, P_l) \oplus \langle P_l, P_r \rangle \oplus p(P_r, Y)$. Now, using Lemma 120, path t leads us to a contradiction similar to above. ■

Lemma 122 *Let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ be pairwise disjoint node sets in a causal MPDAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$. Suppose there is no possibly causal path in $\mathcal{G}$ from X to $\mathbf{Y}$ that starts undirected and does not contain $\mathbf{X} \setminus \{X\} \cup \mathbf{Z}$. Further, suppose there is a possibly causal path in $\mathcal{G}$ from X to $\mathbf{Z}$ that starts undirected and does not contain $\mathbf{X} \setminus \{X\} \cup \mathbf{Y}$, where $(X \not\perp_d \mathbf{Y} | \mathbf{X} \setminus \{X\}, \mathbf{Z})_{\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}} \underline{X}}}$. Then consider the following sets.*

$$\mathbf{S}_1 = \left\{ \begin{array}{l} \text{definite status paths in } \mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}} \underline{X}} \text{ from } X \text{ to } \mathbf{Y} \text{ that are} \\ \text{d-connecting given } \mathbf{X} \setminus \{X\} \cup \mathbf{Z} \end{array} \right\}.$$

$$\mathbf{S}_2 = \{\text{paths in } \mathbf{S}_1 \text{ with the fewest colliders}\}.$$

$\mathbf{S}_3 = \{\text{paths in } \mathbf{S}_2 \text{ that start undirected}\}.$

$\mathbf{S}_4 = \{\text{paths in } \mathbf{S}_3 \text{ with the fewest edges}\}.$

$\mathbf{S}_5 = \{\text{paths in } \mathbf{S}_4 \text{ with a shortest distance to } \mathbf{Z} \text{ in } \mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}} \underline{X}}\}.$

$\mathbf{S}_6 = \{\text{paths in } \mathbf{S}_2 \text{ with the fewest edges}\}.$

$\mathbf{S}_7 = \{\text{paths in } \mathbf{S}_6 \text{ with a shortest distance to } \mathbf{Z} \text{ in } \mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}} \underline{X}}\}.$

When $\mathbf{S}_3 \neq \emptyset$, pick an arbitrary path in $\mathbf{S}_5$. When $\mathbf{S}_3 = \emptyset$, pick an arbitrary path in $\mathbf{S}_7$. Call this path $p := \langle X = P_1, \dots, P_k = Y \rangle$, $k > 1$, $Y \in \mathbf{Y}$. Let $\mathbf{C} = \{C_1, \dots, C_m\}$, $1 \leq m \leq k - 3$, be the set of all colliders on p in order of their appearance on p , and let $q_i = \langle C_i = Q_{i_1}, \dots, Q_{i_{k_i}} \rangle$ be a shortest causal path from C_i to $\mathbf{Z}$ in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}} \underline{X}}$. Further, let $\mathcal{C}$ be the CPDAG that represents all the DAGs in $[\mathcal{G}]$, and let p^* and q_i^* , $i \in \{1, \dots, m\}$, be the paths in $\mathcal{C}$ corresponding to p and q_i in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}} \underline{X}}$.

Then if $\mathbf{C} \neq \emptyset$, the following hold.

- (i) Each q_i^* is causal.
- (ii) C_i is the only node on both p and q_i .
- (iii) q_i and q_j do not share any nodes, where $i \neq j$.

Proof of Lemma 122.

- (i) Note that none of the nodes on q_i are in $\mathbf{X}$, as q_i is a path in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}} \underline{X}}$. Also, the only node on q_i that is in $\mathbf{Z}$ is $Q_{i_{k_i}}$ (otherwise, we can choose a shorter path). Therefore, q_i^* must be a possibly directed unshielded path from C_i to $\mathbf{Z}$ in $\mathcal{C}$.

Furthermore, by Lemma 121, all colliders on p are unshielded and none of nodes on p are in $\mathbf{X} \setminus \{X\}$. Hence, let $P_j = C_i$, $j > 2$, then $P_{j-1} \rightarrow P_j \leftarrow P_{j+1}$ is an unshielded collider on p^* in $\mathcal{C}$. Then if either $P_{j-1} \notin \text{Adj}(Q_{i_2}, \mathcal{C})$, or $P_{j+1} \notin \text{Adj}(Q_{i_2}, \mathcal{C})$, $Q_{i_1} \rightarrow Q_{i_2}$ is in $\mathcal{C}$ by Lemma 82. Then in turn, q_i^* must be a directed path in $\mathcal{C}$.

Otherwise, $P_{j-1} \rightarrow Q_{i_2} \leftarrow P_{j+1}$ must be in $\mathcal{C}$ (and $\mathcal{G}$). But in this case, depending on whether Q_{i_2} is on p , we can derive a contradiction with the choice of p .

For instance, if Q_{i_2} is not a node on p , then clearly $p(X, P_{j-1}) \oplus \langle P_{j-1}, Q_{i_2}, P_{j+1} \rangle \oplus p(P_{j+1}, Y)$ is a path with exactly the same properties as p , but with a shorter distance to $\mathbf{Z}$, which leads us to a contradiction with the choice for p . Otherwise, Q_{i_2} is on p , so either (a) Q_{i_2} is on $p(X, P_{j-1})$, or (b) Q_{i_2} is on $p(P_{j+1}, Y)$

- (a) If Q_{i_2} is a collider on $p(X, P_{j-1})$, then $p(X, Q_{i_2}) \oplus \langle Q_{i_2}, P_{j+1} \rangle \oplus p(P_{j+1}, Y)$ gives us the desired path to derive the contradiction. If Q_{i_2} is a definite non-collider on p and if $-p(Q_{i_2}, X)$ is out of Q_{i_2} , we again have that $p(X, Q_{i_2}) \oplus \langle Q_{i_2}, P_{j+1} \rangle \oplus p(P_{j+1}, Y)$ gives us the desired contradiction. Otherwise, Q_{i_2} is a definite non-collider on p and $-p(Q_{i_2}, X)$ starts with an undirected edge, that is $Q_{i_2} - P_u \dots X$, $1 < u << j - 1$. But then since $P_{j+1} \rightarrow Q_{i_2}$ is in $\mathcal{C}$, Lemma 82, lets us conclude that $P_{j+1} \rightarrow P_l$ for every $1 \leq l < u$ such that $-p^*(P_u, P_l)$ is an undirected path. Then either $P_{j+1} \rightarrow X$ is in $\mathcal{C}$, or there exists an $1 < r < u$, such that $-p^*(P_r, X)$ starts with an edge out of P_r , and $P_{j+1} \rightarrow P_r$ is in $\mathcal{C}$. Then we have that either $\langle X, P_{j+1} \rangle \oplus p(P_{j+1}, Y)$ (which has one fewer collider than p), or $p(X, P_r) \oplus \langle P_r, P_{j+1} \rangle \oplus p(P_{j+1}, Y)$ that leads us to the desired contradiction.
- (b) If Q_{i_2} is a collider on $p(P_{j+1}, Y)$, then $p(X, P_{j-1}) \oplus \langle P_{j-1}, Q_{i_2} \rangle \oplus p(Q_{i_2}, Y)$ gives us the desired path to derive the contradiction. If Q_{i_2} is a definite non-collider on $p(P_{j+1}, Y)$ and if $p(Q_{i_2}, Y)$ is out of Q_{i_2} , we have that $p(X, P_{j-1}) \oplus \langle P_{j-1}, Q_{i_2} \rangle \oplus p(Q_{i_2}, Y)$ gives us the desired contradiction. Otherwise, Q_{i_2} is a definite non-collider on $p(P_{j+1}, Y)$ and $p(Q_{i_2}, Y)$ starts with an undirected edge, that is $Q_{i_2} - P_u \dots Y$, $j + 1 < u < k$. But then since $P_{j-1} \rightarrow Q_{i_2}$ is in $\mathcal{C}$, Lemma 82, lets us conclude that $P_{j-1} \rightarrow P_l$ for every $u < l \leq k$ such that $p^*(P_u, P_l)$ is an undirected path. Then either $P_{j-1} \rightarrow Y$ is in $\mathcal{C}$, or there exists an $u < r < k$, such that $p^*(P_r, Y)$ starts with an edge out of P_r , and $P_{j-1} \rightarrow P_r$ is in $\mathcal{C}$. Then we have that either $p(X, P_{j-1}) \oplus \langle P_{j-1}, Y \rangle$, or $p(X, P_{j-1}) \oplus \langle P_{j-1}, P_r \rangle \oplus p(P_r, Y)$ that leads us

to the desired contradiction.

(ii) Next, we show that the only node that q_i and p have in common is C_i . Hence, suppose for a contradiction that Q_{i_d} , $d \in \{2, \dots, Q_{i_{k_i}}\}$ is the closest node to C_i on q that is also p . Then Q_{i_d} is either (a) on $p(X, P_{j-1})$, or (b) on $p(P_{j+1}, Y)$.

(a) In this case, Q_{i_d} is either a collider or a definite non-collider on $p(X, P_{j-1})$. If Q_{i_d} is a collider on $p(X, P_{j-1})$, then $p(X, Q_{i_d}) \oplus (-q_i(Q_{i_d}, P_j)) \oplus p(P_j, Y)$ gives us the desired path to derive the contradiction. If Q_{i_d} is a definite non-collider on $p(X, P_{j-1})$ and $-p(Q_{i_d}, X)$ is a path that starts with a directed edge out of Q_{i_d} , we again have that $p(X, Q_{i_d}) \oplus (-q_i(Q_{i_d}, P_j)) \oplus p(P_j, Y)$ gives us the desired contradiction. Otherwise, Q_{i_d} is a definite non-collider on $p(X, P_{j-1})$ and $-p(Q_{i_d}, X)$ starts with an undirected edge, that is $Q_{i_d} - P_u \dots X$, for some $u \in \{1, \dots, j-2\}$. But then, we have by case (i) that $Q_{i_{d-1}} \rightarrow Q_{i_d}$ is in $\mathcal{C}$. Then Lemma 82, lets us conclude that $Q_{i_{d-1}} \rightarrow P_l$ for every $1 \leq l < u$ such that $-p^*(P_u, P_l)$ is an undirected path. Then either $Q_{i_{d-1}} \rightarrow X$ is in $\mathcal{C}$, or there exists an $1 < r < u$, such that $-p^*(P_r, X)$ starts with an edge out of P_r , and $Q_{i_{d-1}} \rightarrow P_r$ is in $\mathcal{C}$. Then we have that either $\langle X, Q_{i_{d-1}} \rangle \oplus (-q(Q_{i_{d-1}}, P_j)) \oplus p(P_j, Y)$, or $p(X, P_r) \oplus \langle P_r, Q_{i_2} \rangle \oplus (-q(Q_{i_{d-1}}, P_j)) \oplus p(P_j, Y)$ contradict the choice of p , since both of these paths contain one fewer collider compared to p .

(b) In this case, Q_{i_d} is either a collider or a definite non-collider on $p(P_{j+1}, Y)$. If Q_{i_d} is a collider on $p(P_{j+1}, Y)$, then $p(X, P_{j-1}) \oplus q_i(P_{j-1}, Q_{i_d}) \oplus p(Q_{i_d}, Y)$ gives us the desired path to derive the contradiction. If Q_{i_d} is a definite non-collider on $p(P_{j+1}, Y)$ and $p(Q_{i_d}, Y)$ is a path that starts with a directed edge out of Q_{i_d} , we again have that $p(X, P_j) \oplus q_i(P_j, Q_{i_d}) \oplus p(Q_{i_d}, Y)$ gives us the desired contradiction. Otherwise, Q_{i_d} is a definite non-collider on $p(P_{j+1}, Y)$ and $p(Q_{i_d}, Y)$ starts with an undirected edge, that is $Q_{i_d} - P_u \dots Y$, for some $u \in \{j+2, \dots, k\}$. But then, since by case (i) $Q_{i_{d-1}} \rightarrow Q_{i_d}$ is in $\mathcal{C}$, Lemma 82, lets us conclude that $Q_{i_{d-1}} \rightarrow P_l$ for every $u \leq l < k$ such that $p^*(P_u, P_l)$ is an undirected path. Then either $Q_{i_{d-1}} \rightarrow Y$

is in $\mathcal{C}$, or there exists an $u < r < k$, such that $p^*(P_r, Y)$ starts with a directed edge out of P_r , and $Q_{i_{d-1}} \rightarrow P_r$ is in $\mathcal{C}$. Then we have that either $p(X, P_{j-1}) \oplus q(P_{j-1}, Q_{i_{d-1}}) \oplus \langle Q_{i_{d-1}}, Y \rangle$, or $p(X, P_{j-1}) \oplus q(P_{j-1}, Q_{i_{d-1}}) \oplus \langle Q_{i_{d-1}}, P_r \rangle \oplus p(P_r, Y)$ contradict the choice of p , since both of these paths contain one fewer collider compared to p .

- (iii) Lastly, we show that q_i and q_j such that $i, j \in \{1, \dots, m\}$, $i < j$ cannot contain a node in common by contradiction. Hence, let Q_{i_d} , $d > 1$, be the closest node to Q_{i_1} on q_i that is also on q_j . Then the path $p(X, C_i) \oplus q_i(C_i, Q_{i_d}) \oplus (-q_j(Q_{i_d}, C_j)) \oplus p(C_j, Y)$ contradicts the choice of p as it contains one fewer collider than p . ■

Lemma 123 *Let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ be pairwise disjoint node sets in a causal MPDAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$. Suppose there is no possibly causal path in $\mathcal{G}$ from X to $\mathbf{Y}$ that starts undirected and does not contain $\mathbf{X} \setminus \{X\} \cup \mathbf{Z}$. Further, suppose there is a possibly causal path in $\mathcal{G}$ from X to $\mathbf{Z}$ that starts undirected and does not contain $\mathbf{X} \setminus \{X\} \cup \mathbf{Y}$, where $(X \not\perp_d \mathbf{Y} | \mathbf{X} \setminus \{X\}, \mathbf{Z})_{\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}} \underline{X}}}$. Let $r^* = \langle X = R_1, \dots, R_{k_0} \rangle$, $k_0 > 1$, be a shortest such path, and let r be the corresponding sequence of nodes in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}} \underline{X}}$. Then consider the following sets.*

$$\mathbf{S}_1 = \left\{ \begin{array}{l} \text{definite status paths in } \mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}} \underline{X}} \text{ from } X \text{ to } \mathbf{Y} \text{ that are} \\ d\text{-connecting given } \mathbf{X} \setminus \{X\} \cup \mathbf{Z} \end{array} \right\}.$$

$$\mathbf{S}_2 = \{\text{paths in } \mathbf{S}_1 \text{ with the fewest colliders}\}.$$

$$\mathbf{S}_3 = \{\text{paths in } \mathbf{S}_2 \text{ that start undirected}\}.$$

$$\mathbf{S}_6 = \{\text{paths in } \mathbf{S}_2 \text{ with the fewest edges}\}.$$

$$\mathbf{S}_7 = \{\text{paths in } \mathbf{S}_6 \text{ with a shortest distance to } \mathbf{Z} \text{ in } \mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}} \underline{X}}\}.$$

Suppose $\mathbf{S}_3 = \emptyset$, and pick an arbitrary path $p := \langle X = P_1, \dots, P_k = Y \rangle$, $k > 1$, $Y \in \mathbf{Y}$ in $\mathbf{S}_7$. Let $\mathbf{C} = \{C_1, \dots, C_m\}$, $1 \leq m \leq k - 3$, be the set of all colliders on p in order of their appearance on p , and let $q_i := \langle C_i = Q_{i_1}, \dots, Q_{i_{k_i}} \rangle$ be a shortest causal path from C_i to $\mathbf{Z}$ in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}} \underline{X}}$. Then the following hold.

- (i) r is a path in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}}_X}$.
- (ii) r^* is unshielded in $\mathcal{G}$ and does not contain nodes in $\mathbf{Z} \setminus \{R_{k_0}\}$.
- (iii) $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}}_X}$ does not contain $X - R_i$ for any $i \neq 2$.
- (iv) Either $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}}_X}$ contains $P_2 \rightarrow R_2$, or it contains $P_2 - R_2$ where $P_2 \notin \{R_2, \dots, R_{k_0}\}$.
Further, if $P_2 \in \text{Adj}(R_i, \mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}}_X})$, $i > 2$, then $\mathcal{G}$ contains $P_2 \rightarrow R_i$ or $P_2 - R_i$.
- (v) If P_i is a definite non-collider on p , then $R_2 \notin \text{Adj}(P_i, \mathcal{G})$ for any $i \neq 2$.
- (vi) X is the only node on both r and p .
- (vii) r and q_i do not share any nodes.

Proof of Lemma 123.

- (i) Since r does not contain a node from $\mathbf{X} \setminus \{X\}$ and since $X - R_2$ is in $\mathcal{G}$, r is a path in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}}_X}$.
- (ii) By choice of r , R_{k_0} is the only node on r that is in $\mathbf{Z}$ and r must be unshielded $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}}_X}$ otherwise, we can choose a shorter path as r^* .
- (iii) $X - R_i$ is not in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}}_X}$ for any $i \neq 2$, otherwise we can choose a shorter path r^* .
- (iv) Since r^* is a possibly causal path, $P_2 \notin \{R_2, \dots, R_{k_0}\}$. Also, since $P_2 \rightarrow X - R_2$ is in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}}_X}$ and since $P_2 \notin \mathbf{X}$, we have that $P_2 \in \text{Adj}(R_2, \mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}}_X})$ (since otherwise, $\mathcal{G}$ is not an MPDAG). We also cannot have $R_2 \rightarrow P_2 \rightarrow X$ and $X - R_2$ in $\mathcal{G}$, due to Lemma 90. Then $P_2 \rightarrow R_2$ or $P_2 - R_2$ is in $\mathcal{G}$.

Similarly if $P_2 \in \text{Adj}(R_i, \mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}}_X})$ for any $i \in \{3, \dots, k_0\}$, we know that $R_i \rightarrow P_2$ cannot be in $\mathcal{G}$ otherwise, $R_i \rightarrow P_2 \rightarrow X$ and Lemma 90 would contradict that r^* is a possibly causal path.

(v) Suppose for a contradiction that $R_2 \in \text{Adj}(P_i, \mathcal{G})$ for some definite non-collider P_i on p , and $i > 2$. Choose the definite non-collider P_j on p with the largest index $j \in \{3, \dots, k\}$ such that $R_2 \in \text{Adj}(P_j, \mathcal{G})$. Then consider path $t = \langle X, R_2, P_j \rangle \oplus p(P_j, Y)$. Note that R_2 is of definite status on t in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}}_X}$, since otherwise $X \leftarrow P_j$ is in $\mathcal{G}$ (due to Lemma 121) and $\langle X, P_j \rangle \oplus p(P_j, Y)$ contradicts the choice of p . Furthermore, P_j must be of definite status on t , otherwise, $R_2 \in \text{Adj}(P_{j+1}, \mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}}_X})$ and $R_2 - P_j - P_{j+1}$ or $R_2 - P_j \leftarrow P_{j+1}$ is in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}}_X}$, but that contradicts the choice of P_j on p . Hence, t is of definite status in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}}_X}$. Now, it is enough to show that t is d-connecting given $\mathbf{Z}$ to derive the contradiction. Hence, if P_j is a non-collider on t , we have our contradiction. Otherwise, P_j is a collider on t , which further means that $P_{j-1} \leftarrow P_j \leftarrow P_{j+1}$ is on p and $R_2 \rightarrow P_j$ is on t . Note that $X \leftarrow \dots \leftarrow P_j$ cannot be the form of $-p(P_j, X)$, otherwise $\langle R_2, P_j \rangle \oplus (-p(P_j, R_2))$ and $X - R_2$ contradict Lemma 90. Hence, there must be collider that is a descendant of P_j on $p(X, P_j)$, which further implies that $P_j \in \text{An}(\mathbf{Z}, \mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}}_X})$. Therefore, t is d-connecting given $\mathbf{Z}$.

(vi) Next, we show that the only node on both p and r is X . We will assume for a contradiction that there is a node on p and r that is not X . Hence let P_u be chosen as the node on p with the largest index $u \in \{3, \dots, k\}$ that is also on r , let $P_u \equiv R_l, l \in \{3, \dots, k_0\}$. Then consider path $t = r(X, P_u) \oplus p(P_u, Y)$. If t is a path of definite status and P_u is a definite non-collider on t , then as $t \in \mathbf{S}_3$, we have a contradiction with the assumption that $\mathbf{S}_3 = \emptyset$. If t is a path of definite status and P_u is a collider on t , we still obtain a contradiction as long as there is a collider on $p(X, P_u)$. Note that there must be such a collider on $p(X, P_u)$ otherwise, $p(X, P_u)$ is of the form $X \leftarrow \dots \leftarrow P_u$ and since $P_u \equiv R_l$ and $r^*(X, R_l)$ is a possibly causal path from X to R_l we would have a contradiction with Lemma 90.

Otherwise, t is not of definite status due to P_u , and $R_{l-1} - P_u - P_{u+1}$ or $R_{l-1} - P_u \leftarrow P_{u+1}$ and $\langle R_{l-1}, P_u \rangle$ is in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}}_X}$. Let $P_d, d \in \{u+1, \dots, k\}$ be chosen so that P_{d+1} is the closest collider to P_u on $p(P_u, Y)$ or if there is no collider on $p(P_u, Y)$, then let, $P_d = Y$.

Consider that none of the nodes among $P_{u+1}, \dots, P_d$ are adjacent to X as that would imply that $X - P_h$ or $X \leftarrow P_h$, for $h \in \{u+1, \dots, d\}$ is in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}}_X}$ and neither of that is possible (the former due to Lemma 121, and the latter due to the choice of p , as $\langle X, P_h \rangle \oplus p(P_h, Y)$ has all the same properties as p but is shorter).

Hence, choose as P_h $h \in \{u+1, \dots, d\}$ a node with the largest index that is adjacent to a node on $r(X, R_{l-1})$. Next, choose as R_o , $o \in \{1, \dots, l-1\}$ a node with the smallest index such that $R_o \in \text{Adj}(P_h, \mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}}_X})$. Then consider path $t = r(X, R_o) \langle R_o, P_h \rangle \oplus p(P_h, Y)$. Path t has the same properties as p but starts with an undirected edge, which contradicts that $\mathbf{S}_3 = \emptyset$.

(vii) Lastly we show that there is no node that is common to r and q_i for any $i \in \{1, \dots, m\}$.

Suppose for a contradiction that there exists $i \in \{1, \dots, m\}$ such that q_i and r have a node in common and choose the smallest such i . By above, we have that Q_{i_1} is not on r , so let Q_{i_l} be the closest node to Q_{i_1} on q_i that is on both r and q_i , $Q_{i_l} \equiv R_d$, $d \in \{2, \dots, k_0\}$. Consider path $t = r(X, Q_{i_l}) \oplus (-q(Q_{i_l}, Q_{i_1})) \oplus p(Q_{i_1}, Y)$. If t is of definite status, then $t \in \mathbf{S}_3$ which contradicts our assumptions about $\mathbf{S}_3 = \emptyset$. Otherwise, $R_{d-1} - Q_{i_l} \leftarrow Q_{i_{l-1}}$ is in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}}_X}$ and by Lemma 122 also in the CPDAG $\mathcal{C}$. Hence, by Lemma 82, $R_{d-1} \leftarrow Q_{i_{l-1}}$. Furthermore, since r^* is a possibly causal path and r is an unshielded path $X - \dots - R_{d-1}$ is in $\mathcal{G}_{\overline{\mathbf{X} \setminus \{X\}}_X}$. But this would then imply that $X \leftarrow Q_{i_{l-1}}$ by successive applications of Lemma 82. However, now we have that, $\langle X, Q_{i_{l-1}} \rangle \oplus (-q(Q_{i_{l-1}}, Q_{i_1})) \oplus p(Q_{i_1}, Y)$ contradicts the choice of p . ■

C.8 A NOTE ON JABER'S (2022) CIDP ALGORITHM

Despite the generality of PAGs, the CIDP algorithm of Jaber et al. [2022] does not hold in the general MPDAG setting. We demonstrate this in the example below, where the CIDP algorithm cannot identify an identifiable conditional effect given an MPDAG.

Example 31 *Reconsider Example 19, where a causal MPDAG $\mathcal{G}$ is known. We will attempt to use a naive translation of Jaber et al.'s CIDP algorithm to identify the conditional effect*

of X on Y given Z in this setting. Start by defining $\mathbf{D} = \text{PossAn}(\mathbf{Y} \cup \mathbf{Z}, \mathcal{G}_{\mathbf{V} \setminus \mathbf{X}})$ so that $\mathbf{D} = \{V_1, V_2, V_3, Y, Z\}$. Then define the ordered bucket decomposition (Definition 20) of $\mathbf{V}$ in $\mathcal{G}$ as $(\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \mathbf{B}_4) = (\{V_1\}, \{Y\}, \{X, V_2, V_3\}, \{Z\})$. Since $\mathbf{B}_3 \cap \mathbf{D} \neq \emptyset$ and $\mathbf{B}_3 \not\subseteq \mathbf{D}$, we enter the *while* loop of the *CIDP* algorithm. But $\mathbf{B}_3 \cap \{X\} = \{X\}$, where $X \not\perp_d Y|Z$ in $\mathcal{G}_{\underline{X}}$. Thus, the algorithm outputs a **FAIL**, implying the conditional effect is not identifiable in $\mathcal{G}$. However, we showed in Example 19 that $f(y | do(x), z) = f(y | z)$.